

Mathematics

Robin Adams

November 1, 2023

Contents

I	Set Theory	9
1	Primitive Terms and Axioms	11
1.1	Primitive Terms	11
1.2	Injections, Surjections and Bijections	11
1.3	Axioms	12
2	Sets and Functions	15
2.1	Composition	15
2.1.1	Injections	15
2.1.2	Surjections	17
2.1.3	Bijections	17
2.1.4	Equinumerosity	18
2.2	Domination	18
2.3	Identity Function	19
2.3.1	Injections, Surjections, Bijections	19
2.3.2	Composition	20
2.4	The Empty Set	21
2.5	The Singleton	22
2.5.1	Injections	22
2.6	The Set Two	22
2.7	Subsets	23
2.8	Saturated Set	24
2.9	Union	24
2.9.1	Intersection	25
2.9.2	Direct Image	25
2.10	Inverse Image	25
2.10.1	Saturated Sets	26
2.11	Relations	26
2.11.1	Equivalence Relations	27
2.12	Power Set	27
2.12.1	Partitions	27
2.13	Cartesian Product	27
2.14	Quotient Sets	27
2.15	Partitions	28

2.16	Disjoint Union	28
2.17	Natural Numbers	28
2.18	Finite and Infinite Sets	29
2.19	Countable Sets	30
2.20	Fixed Points	32
3	Relations	33
4	Order Theory	35
4.1	Strict Partial Orders	35
4.1.1	Linear Orders	35
4.1.2	Sets of Finite Type	37
4.2	Linear Continua	37
4.3	Well Orders	38
II	Category Theory	43
5	Category Theory	45
5.1	Categories	45
5.1.1	Monomorphisms	49
5.1.2	Epimorphisms	49
5.1.3	Sections and Retractions	50
5.1.4	Isomorphisms	50
5.1.5	Initial Objects	51
5.1.6	Terminal Objects	51
5.1.7	Zero Objects	52
5.1.8	Triads	52
5.1.9	Cotriads	52
5.1.10	Pullbacks	52
5.1.11	Pushouts	55
5.1.12	Subcategories	58
5.1.13	Opposite Category	58
5.1.14	Groupoids	59
5.1.15	Concrete Categories	59
5.1.16	Power of Categories	59
5.1.17	Arrow Category	59
5.1.18	Slice Category	59
5.2	Functors	62
5.3	Natural Transformations	65
5.4	Bifunctors	66
5.5	Functor Categories	67

<i>CONTENTS</i>	5
-----------------	---

III Number Systems 69

6 The Real Numbers	71
6.1 Subtraction	73
6.2 The Ordered Square	79
6.3 Punctured Euclidean Space	80
6.4 Topologist's Sine Curve	80
6.5 The Long Line	80
7 Integers and Rationals	81
7.1 Positive Integers	81
7.1.1 Exponentiation	82
7.2 Integers	83
7.3 Rational Numbers	85
7.4 Algebraic Numbers	86

IV Algebra 87

8 Monoid Theory	89
9 Group Theory	91
9.1 Category of Small Groups	91
10 Ring Theory	93
11 Field Theory	95
12 Linear Algebra	97

V Topology 99

13 Topology	101
13.1 Topological Spaces	101
13.2 Bases	105
13.3 Order Topology	106
13.3.1 Subspaces	107
13.3.2 Product Topology	108
13.4 Subbases	108
13.5 Neighbourhood Bases	109
13.6 First Countable Spaces	109
13.7 Second Countable Spaces	110
13.8 Interior	110
13.9 Closure	110
13.9.1 Bases	111
13.9.2 Subspaces	112

13.9.3 Product Topology	112
13.9.4 Interior	112
13.10 Boundary	113
13.11 Limit Points	114
13.12 Continuous Functions	115
13.12.1 Order Topology	119
13.12.2 Paths	120
13.12.3 Loops	120
13.13 Convergence	120
13.13.1 Closure	121
13.13.2 Continuous Functions	121
13.13.3 Infinite Series	121
13.14 Strong Continuity	121
13.15 Subspaces	122
13.15.1 Product Topology	124
13.16 Embedding	124
13.17 Open Maps	125
13.17.1 Subspaces	125
13.18 Locally Finite	125
13.19 Closed Maps	126
13.20 Product Topology	126
13.20.1 Closed Sets	126
13.20.2 Closure	128
13.20.3 Convergence	129
13.21 Topological Disjoint Union	129
13.22 Quotient Spaces	131
13.22.1 Quotient Maps	133
13.23 Box Topology	137
13.23.1 Bases	137
13.23.2 Subspaces	138
13.23.3 Closure	138
13.24 Separations	139
13.25 Connected Spaces	139
13.25.1 The Real Numbers	139
13.25.2 The Indiscrete Topology	139
13.25.3 The Cofinite Topology	140
13.25.4 Finer and Coarser	140
13.25.5 Boundary	140
13.25.6 Continuous Functions	140
13.25.7 Subspaces	141
13.25.8 Order Topology	142
13.25.9 Product Topology	144
13.25.10 Quotient Spaces	146
13.26 T_1 Spaces	146
13.26.1 Limit Points	147
13.27 Hausdorff Spaces	147

13.27.1 Product Topology	149
13.27.2 Box Topology	149
13.27.3 T_1 Spaces	149
13.28 Separable Spaces	150
13.29 Sequential Compactness	150
13.30 Compactness	150
13.31 Gluing	151
13.32 Homogeneous Spaces	151
13.33 Regular Spaces	151
13.34 Totally Disconnected Spaces	152
13.35 Path Connected Spaces	152
13.35.1 The Ordered Square	152
13.35.2 Punctured Euclidean Space	152
13.35.3 The Topologist's Sine Curve	152
13.35.4 Continuous Functions	153
13.35.5 Subspaces	153
13.35.6 Product Topology	153
13.35.7 Connected Spaces	154
14 Metric Spaces	157
14.0.1 Balls	158
14.0.2 Subspaces	163
14.0.3 Convergence	164
14.0.4 Continuous Functions	165
14.0.5 First Countable Spaces	166
14.0.6 Hausdorff Spaces	166
14.0.7 Bounded Sets	167
14.0.8 Uniform Convergence	167
14.0.9 Standard Bounded Metric	169
14.0.10 Product Spaces	169
14.1 Uniform Metric	170
14.1.1 Products	173
14.1.2 Connected Spaces	174
14.2 Isometric Embeddings	175
14.3 Complete Metric Spaces	175
14.4 Manifolds	176
15 Homotopy Theory	177
15.1 Homotopies	177
15.2 Homotopy Equivalence	177
16 Simplicial Complexes	179
16.1 Cell Decompositions	179
16.2 CW-complexes	179

17 Topological Groups	181
17.1 Topological Groups	181
17.1.1 Subgroups	182
17.1.2 Left Cosets	183
17.1.3 Homogeneous Spaces	185
17.2 Symmetric Neighbourhoods	185
17.3 Continuous Actions	186
18 Topological Vector Spaces	189
18.1 Cauchy Sequences	189
18.2 Seminorms	190
18.3 Fréchet Spaces	190
18.4 Normed Spaces	190
18.5 Unit Ball	196
18.6 Unit Sphere	197
18.7 Inner Product Spaces	197
18.8 Banach Spaces	198
18.9 Hilbert Spaces	198
18.10 Locally Convex Spaces	199

Part I

Set Theory

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*.

For any set A , let there be *elements* of A . We write $a \in A$ for: a is an element of A .

For any sets A and B , let there be a set B^A , whose elements are called *functions* from A to B . We write $f : A \rightarrow B$ for $f \in B^A$.

For any function $f : A \rightarrow B$ and element $a \in A$, let there be an element $f(a) \in B$, the *value* of the function f at the *argument* a .

1.2 Injections, Surjections and Bijections

Definition 1.2.1 (Injective). A function $f : A \rightarrow B$ is *injective* or an *injection* iff, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

Definition 1.2.2 (Surjective). A function $f : A \rightarrow B$ is *surjective* or a *surjection* iff, for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Definition 1.2.3 (Bijective). A function $f : A \rightarrow B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$ ".

1.3 Axioms

Axiom Schema 1.3.1 (Choice). *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x \in X$ and $y \in Y$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a \in A$, there exists $b \in B$ such that $P[A, B, a, b]$. Then there exists a function $f : A \rightarrow B$ such that $\forall a \in A. P[A, B, a, f(a)]$.

Axiom 1.3.2 (Extensionality). *Let $f, g : A \rightarrow B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Definition 1.3.3 (Composition). *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. The composite $g \circ f : A \rightarrow C$ is the function such that, for all $a \in A$, we have*

$$(g \circ f)(a) = g(f(a)) .$$

Axiom 1.3.4 (Pairing). *For any sets A and B , there exists a set $A \times B$, the Cartesian product of A and B , and functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ such that, for all $a \in A$ and $b \in B$, there exists a unique $(a, b) \in A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Axiom Schema 1.3.5 (Separation). *For every property $P[X, x]$ where X is a set variable and $x \in X$, the following is an axiom:*

For every set A , there exists a set $S = \{x \in A : P[A, x]\}$ and an injection $i : S \rightarrow A$ such that, for all $x \in A$, we have

$$(\exists y \in S. i(y) = x) \Leftrightarrow P[A, x] .$$

Axiom 1.3.6 (Infinity). *There exists a set \mathbb{N} , an element $0 \in \mathbb{N}$, and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that:*

- $\forall n \in \mathbb{N}. s(n) \neq 0$
- $\forall m, n \in \mathbb{N}. s(m) = s(n) \Rightarrow m = n$.

Axiom Schema 1.3.7 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A , there exist sets B and Y and functions $p : B \rightarrow A$, and $m : B \times Y \Rightarrow \mathbb{N}$ such that:

- m is injective.
- $\forall b \in B. P[A, \{y \in Y : m(b, y) = 0\}, p(b)]$
- For all $a \in A$, if $\exists Y. P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.

Axiom 1.3.8 (Universe). *There exists a set E , a set U and a function $el : E \rightarrow U$ such that the following holds.*

Let us say that a set A is small iff there exists $u \in U$ such that $A \approx \{e \in E : el(e) = u\}$.

- \mathbb{N} is small.
- For any U -small sets A and B , the set B^A is small.
- For any U -small sets A and B , the set $A \times B$ is small.
- Let $f : A \rightarrow B$ be a function. If B is small and $\{a \in A : f(a) = b\}$ is small for all $b \in B$, then A is small.
- If $p : B \rightarrow A$ is a surjective function such that A is small, then there exists a U -small set C , a surjection $q : C \rightarrow A$, and a function $f : C \rightarrow B$ such that $q = p \circ f$.

Chapter 2

Sets and Functions

2.1 Composition

Proposition 2.1.1. *Given functions $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f .$$

PROOF:

$\langle 1 \rangle 1$. For all $x \in A$ we have $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$.

$\langle 2 \rangle 1$. LET: $x \in A$

$\langle 2 \rangle 2$. $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$

PROOF:

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) \quad (\text{Definition of composition})$$

$$= h(g(f(x))) \quad (\text{Definition of composition})$$

$$= (h \circ g)(f(x)) \quad (\text{Definition of composition})$$

$$= ((h \circ g) \circ f)(x) \quad (\text{Definition of composition})$$

$\langle 1 \rangle 2$. Q.E.D.

PROOF: By the Axiom of Extensionality.

□

2.1.1 Injections

Proposition 2.1.2. *The composite of injective functions is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: A , B and C be sets.

$\langle 1 \rangle 2$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 3$. LET: $g : B \rightarrow C$

$\langle 1 \rangle 4$. ASSUME: g is injective.

$\langle 1 \rangle 5$. ASSUME: f is injective.

$\langle 1 \rangle 6$. LET: $x, y \in A$

$\langle 1 \rangle 7$. ASSUME: $(g \circ f)(x) = (g \circ f)(y)$

PROVE: $x = y$
 $\langle 1 \rangle 8. g(f(x)) = g(f(y))$
 PROOF:

$$\begin{aligned} g(f(x)) &= (g \circ f)(x) && \text{(definition of composition)} \\ &= (g \circ f)(y) && (\langle 1 \rangle 7) \\ &= g(f(y)) && \text{(definition of composition)} \end{aligned}$$

 $\langle 1 \rangle 9. f(x) = f(y)$
 PROOF: $\langle 1 \rangle 4, \langle 1 \rangle 8$
 $\langle 1 \rangle 10. x = y$
 PROOF: $\langle 1 \rangle 5, \langle 1 \rangle 9$
 \square

Proposition 2.1.3. *For functions $f : A \rightarrow B$ and $g : B \rightarrow C$, if $g \circ f$ is injective then f is injective.*

PROOF:
 $\langle 1 \rangle 1.$ LET: A, B and C be sets.
 $\langle 1 \rangle 2.$ LET: $f : A \rightarrow B$
 $\langle 1 \rangle 3.$ LET: $g : B \rightarrow C$
 $\langle 1 \rangle 4.$ ASSUME: $g \circ f$ is injective.
 $\langle 1 \rangle 5.$ LET: $x, y \in A$
 $\langle 1 \rangle 6.$ ASSUME: $f(x) = f(y)$
 $\langle 1 \rangle 7. (g \circ f)(x) = (g \circ f)(y)$
 PROOF:

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{(definition of composition)} \\ &= g(f(y)) && (\langle 1 \rangle 6) \\ &= (g \circ f)(y) && \text{(definition of composition)} \end{aligned}$$

 $\langle 1 \rangle 8. x = y$
 PROOF: $\langle 1 \rangle 4, \langle 1 \rangle 7$
 \square

Proposition 2.1.4. *Let $f : A \rightarrow B$ be injective. For every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.*

PROOF:
 $\langle 1 \rangle 1.$ ASSUME: f is injective.
 $\langle 1 \rangle 2.$ LET: X be a set.
 $\langle 1 \rangle 3.$ LET: $x, y : X \rightarrow A$
 $\langle 1 \rangle 4.$ ASSUME: $f \circ x = f \circ y$
 $\langle 1 \rangle 5. \forall t \in X. x(t) = y(t)$
 $\langle 2 \rangle 1.$ LET: $t \in X$
 $\langle 2 \rangle 2. f(x(t)) = f(y(t))$
 PROOF:

$$\begin{aligned} f(x(t)) &= (f \circ x)(t) && \text{(definition of composition)} \\ &= (f \circ y)(t) && (\langle 1 \rangle 4) \\ &= f(y(t)) && \text{(definition of composition)} \end{aligned}$$

$\langle 2 \rangle 3. x(t) = y(t)$

PROOF: $\langle 1 \rangle 1, \langle 2 \rangle 2$

$\langle 1 \rangle 6. x = y$

PROOF: Axiom of Extensionality, $\langle 1 \rangle 5$

□

We will prove the converse as Proposition 2.5.4.

2.1.2 Surjections

Proposition 2.1.5. *The composite of surjective functions is surjective.*

PROOF:

$\langle 1 \rangle 1.$ LET: A, B and C be sets.

$\langle 1 \rangle 2.$ LET: $f : A \rightarrow B$ and $g : B \rightarrow C$

$\langle 1 \rangle 3.$ ASSUME: g is surjective.

$\langle 1 \rangle 4.$ ASSUME: f is surjective.

$\langle 1 \rangle 5.$ LET: $c \in C$

$\langle 1 \rangle 6.$ PICK $b \in B$ such that $g(b) = c$.

PROOF: $\langle 1 \rangle 3$

$\langle 1 \rangle 7.$ PICK $a \in A$ such that $f(a) = b$.

PROOF: $\langle 1 \rangle 4$

$\langle 1 \rangle 8. (g \circ f)(a) = c$

PROOF:

$$\begin{aligned} (g \circ f)(a) &= g(f(a)) && \text{(definition of composition)} \\ &= g(b) && (\langle 1 \rangle 7) \\ &= c && (\langle 1 \rangle 6) \end{aligned}$$

□

Proposition 2.1.6. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is surjective then g is surjective.*

PROOF:

$\langle 1 \rangle 1.$ LET: A, B and C be sets.

$\langle 1 \rangle 2.$ LET: $f : A \rightarrow B$ and $g : B \rightarrow C$.

$\langle 1 \rangle 3.$ ASSUME: $g \circ f$ is surjective.

$\langle 1 \rangle 4.$ LET: $c \in C$

$\langle 1 \rangle 5.$ PICK $a \in A$ such that $(g \circ f)(a) = c$

PROOF: $\langle 1 \rangle 3$

$\langle 1 \rangle 6. g(f(a)) = c$

PROOF: From $\langle 1 \rangle 5$ and the definition of composition.

$\langle 1 \rangle 7.$ Q.E.D.

PROOF: There exists $b \in B$ such that $g(b) = c$, namely $b = f(a)$.

□

2.1.3 Bijections

Proposition 2.1.7. *The composite of bijections is a bijection.*

PROOF:

- ⟨1⟩1. LET: A, B and C be sets.
- ⟨1⟩2. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$
- ⟨1⟩3. ASSUME: g is bijective.
- ⟨1⟩4. ASSUME: f is bijective.
- ⟨1⟩5. g is injective.
PROOF: From ⟨1⟩3.
- ⟨1⟩6. g is surjective.
PROOF: From ⟨1⟩3.
- ⟨1⟩7. f is injective.
PROOF: From ⟨1⟩4.
- ⟨1⟩8. f is surjective.
PROOF: From ⟨1⟩4.
- ⟨1⟩9. $g \circ f$ is injective.
PROOF: Proposition 2.1.2, ⟨1⟩5, ⟨1⟩7.
- ⟨1⟩10. $g \circ f$ is surjective.
PROOF: Proposition 2.1.5, ⟨1⟩6, ⟨1⟩8.
- ⟨1⟩11. $g \circ f$ is bijective.
PROOF: ⟨1⟩9, ⟨1⟩10

□

2.1.4 Equinumerosity

Proposition 2.1.8.

$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps f to $(\pi_1 \circ f, \pi_2 \circ f)$ is a bijection. □

Proposition 2.1.9.

$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function Φ such that $\Phi(f)(c)(b) = f(b, c)$ is a bijection. □

2.2 Domination

Definition 2.2.1 (Dominate). Let A and B be sets. We say that B *dominates* A , and write $A \leqslant B$, iff there exists an injective function $A \rightarrow B$.

Theorem 2.2.2 (Schroeder-Bernstein). *Let A and B be sets. If $A \leqslant B$ and $B \leqslant A$ then $A \approx B$.*

PROOF:

- ⟨1⟩1. LET: $f : A \rightarrow B$ and $g : B \rightarrow A$ be injections.
- ⟨1⟩2. Define the subsets A_n of A by

$$\begin{aligned} A_0 &:= A - g(B) \\ A_{n+1} &:= g(f(A_n)) \end{aligned}$$

⟨1⟩3. Define $h : A \rightarrow B$ by

$$h(x) = \begin{cases} f(x) & \text{if } \exists n. x \in A_n \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

⟨1⟩4. h is injective.

⟨2⟩1. LET: $x, y \in A$

⟨2⟩2. ASSUME: $h(x) = h(y)$

⟨2⟩3. CASE: $x \in A_m$ and $y \in A_n$.

PROOF: Then $f(x) = f(y)$ so $x = y$ since f is injective.

⟨2⟩4. CASE: $x \in A_m$ and there is no y such that $y \in A_n$.

⟨3⟩1. $f(x) = g^{-1}(y)$

⟨3⟩2. $y = g(f(x))$

⟨3⟩3. $y \in A_{m+1}$

⟨3⟩4. Q.E.D.

PROOF: This is a contradiction.

⟨2⟩5. CASE: $y \in A_n$ and there is no m such that $x \in A_m$.

PROOF: Similar.

⟨2⟩6. CASE: There is no m such that $x \in A_m$ and there is no n such that $y \in A_n$.

PROOF: Then $g^{-1}(x) = g^{-1}(y)$ and so $x = y$.

⟨1⟩5. h is surjective.

⟨2⟩1. LET: $y \in B$

⟨2⟩2. CASE: $g(y) \in A_n$

⟨3⟩1. $n \neq 0$

⟨3⟩2. PICK $x \in A_{n-1}$ such that $g(y) = g(f(x))$

⟨3⟩3. $y = f(x)$

⟨3⟩4. $y = h(x)$

⟨2⟩3. CASE: There is no n such that $g(y) \in A_n$.

PROOF: Then $h(g(y)) = y$.

□

2.3 Identity Function

Definition 2.3.1 (Identity). For any set A , the *identity* function $\text{id}_A : A \rightarrow A$ is the function defined by $\text{id}_A(a) = a$.

2.3.1 Injections, Surjections, Bijections

Proposition 2.3.2. For any set A , the identity function id_A is a bijection.

PROOF:

⟨1⟩1. LET: A be a set.

⟨1⟩2. id_A is injective.

PROOF: If $\text{id}_A(x) = \text{id}_A(y)$ then $x = y$.

⟨1⟩3. id_A is surjective.

PROOF: For any $y \in A$, there exists $x \in A$ such that $\text{id}_A(x) = y$, namely $x = y$.

□

2.3.2 Composition

Proposition 2.3.3. *Let $f : A \rightarrow B$. Then $\text{id}_B \circ f = f = f \circ \text{id}_A$.*

PROOF: Each is the function that maps a to $f(a)$. □

Proposition 2.3.4. *Let $f : A \rightarrow B$.*

1. *If there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$ then f is injective.*
2. *If f is injective and A is nonempty, then there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$.*

PROOF:

⟨1⟩1. If there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$ then f is injective.

PROOF: If $f(x) = f(y)$ then $x = g(f(x)) = g(f(y)) = y$.

⟨1⟩2. If f is injective and A is nonempty, then there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$.

⟨2⟩1. ASSUME: f is injective and A is nonempty.

⟨2⟩2. PICK $a \in A$

⟨2⟩3. Choose a function $g : B \rightarrow A$ such that $f(g(x)) = x$ if there exists $y \in A$ such that $f(y) = x$, otherwise $g(x) = a$.

⟨2⟩4. LET: $x \in A$

PROVE: $g(f(x)) = x$

⟨2⟩5. $f(g(f(x))) = f(x)$

⟨2⟩6. $g(f(x)) = x$

□

Proposition 2.3.5. *Let $f : A \rightarrow B$. Then f is surjective if and only if there exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$.*

PROOF:

⟨1⟩1. If f is surjective then there exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$.

⟨2⟩1. ASSUME: f is surjective.

⟨2⟩2. PICK $g : B \rightarrow A$ such that, for all $b \in B$, we have $f(g(b)) = b$.

PROOF: Axiom of Choice.

⟨2⟩3. $f \circ g = \text{id}_B$.

⟨1⟩2. If there exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$ then f is surjective.

⟨2⟩1. LET: $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$

⟨2⟩2. LET: X be a set.

⟨2⟩3. LET: $h, k : B \rightarrow X$

⟨2⟩4. ASSUME: $h \circ f = k \circ f$

⟨2⟩5. $h = k$

PROOF: $h = h \circ f \circ g = k \circ f \circ g = k$

□

Corollary 2.3.5.1. *Let A and B be sets.*

1. *If there exists a surjective function $A \rightarrow B$ then there exists an injective function $B \rightarrow A$.*
2. *If there exists an injective function $A \rightarrow B$ and A is nonempty then there exists a surjective function $B \rightarrow A$.*

Proposition 2.3.6. *Let $f : A \rightarrow B$. Then f is bijective if and only if there exists a function $f^{-1} : B \rightarrow A$, the inverse of f , such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$, in which case the inverse is unique.*

PROOF:

- ⟨1⟩1. If f is bijective then there exists $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$.
- ⟨2⟩1. ASSUME: f is bijective.
- ⟨2⟩2. PICK $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$
- PROOF: Proposition 2.6.2.
- ⟨2⟩3. $f \circ g \circ f = f$
- ⟨2⟩4. $g \circ f = \text{id}_A$
- PROOF: Proposition 2.1.4.
- ⟨1⟩2. If there exists $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$, then f is bijective.
- ⟨2⟩1. LET: $f^{-1} : B \rightarrow A$ satisfy $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$
- ⟨2⟩2. f is injective.
- PROOF: If $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$.
- ⟨2⟩3. f is surjective.
- PROOF: Proposition 2.6.2.
- ⟨1⟩3. If $g, h : B \rightarrow A$ satisfy $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$ and $f \circ h = \text{id}_B$ and $h \circ f = \text{id}_A$ then $g = h$.
- PROOF: We have $g = g \circ f \circ h = h$.
-

2.4 The Empty Set

Theorem 2.4.1. *There exists a set which has no elements.*

PROOF: Take $\{x \in \mathbb{N} : \perp\}$. □

Theorem 2.4.2. *If E and E' have no elements then $E \approx E'$.*

PROOF:

- ⟨1⟩1. LET: E and E' have no elements.
- ⟨1⟩2. PICK a function $F : E \rightarrow E'$.
- PROOF: Axiom of Choice since vacuously $\forall x \in E. \exists y \in E'. \top$.
- ⟨1⟩3. F is injective.

PROOF: Vacuously, for all $x, y \in E$, if $F(x) = F(y)$ then $x = y$.

$\langle 1 \rangle 4$. F is surjective.

PROOF: Vacuously, for all $y \in E$, there exists $x \in E$ such that $F(x) = y$.

□

Definition 2.4.3 (Empty Set). The *empty set* \emptyset is the set with no elements.

2.5 The Singleton

Theorem 2.5.1. *There exists a set that has exactly one element.*

PROOF: The set $\{x \in \mathbb{N} : x = 0\}$ has exactly one element. □

Theorem 2.5.2. *If A and B both have exactly one element then $A \approx B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B both have exactly one element a and b respectively.

$\langle 1 \rangle 2$. LET: $F : A \rightarrow B$ be the function such that, for all $x \in A$, we have
 $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$. F is a bijection.

□

Definition 2.5.3 (Singleton). Let 1 be the set that has exactly one element. Let $*$ be its element.

2.5.1 Injections

Proposition 2.5.4. *Let $f : A \rightarrow B$. Assume that, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$. Then f is injective.*

PROOF: Take $X = 1$. □

2.6 The Set Two

Definition 2.6.1 (The Set Two). Let $2 = \{x \in \mathbb{N} : x = 0 \vee x = 1\}$.

Proposition 2.6.2. *Let $f : A \rightarrow B$. Then f is surjective if and only if, for any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$.*

PROOF:

$\langle 1 \rangle 1$. If f is surjective then, for any set X and functions $g, h : B \rightarrow X$, if
 $g \circ f = h \circ f$ then $g = h$.

$\langle 2 \rangle 1$. ASSUME: f is surjective.

$\langle 2 \rangle 2$. LET: X be a set.

$\langle 2 \rangle 3$. LET: $g, h : B \rightarrow X$

$\langle 2 \rangle 4$. ASSUME: $g \circ f = h \circ f$

$\langle 2 \rangle 5$. LET: $b \in B$

PROVE: $g(b) = h(b)$

- $\langle 2 \rangle 6$. PICK $a \in A$ such that $f(a) = b$
 $\langle 2 \rangle 7$. $g(b) = h(b)$
 PROOF: $g(b) = g(f(a)) = h(f(a)) = h(b)$
 $\langle 1 \rangle 2$. If, for any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$, then f is surjective.
 $\langle 2 \rangle 1$. ASSUME: For any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$.
 $\langle 2 \rangle 2$. LET: $b \in B$
 $\langle 2 \rangle 3$. LET: $h : B \rightarrow 2$ be the function that maps everything to 1.
 $\langle 2 \rangle 4$. LET: $k : B \rightarrow 2$ be the function that maps b to 0 and everything else to 1.
 $\langle 2 \rangle 5$. $h \neq k$
 $\langle 2 \rangle 6$. $h \circ f \neq k \circ f$
 $\langle 2 \rangle 7$. PICK $a \in A$ such that $h(f(a)) \neq k(f(a))$
 $\langle 2 \rangle 8$. $f(a) = b$
 \square

2.7 Subsets

Definition 2.7.1 (Subset). A *subset* of a set A consists of a set S and an injection $i : S \rightarrow A$. We write $(S, i) \subseteq A$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Proposition 2.7.2. For any subset (S, i) of A we have $(S, i) = (S, i)$.

PROOF: We have $\text{id}_S : S \approx S$ and $i \circ \text{id}_S = i$.

Proposition 2.7.3. If $(S, i) = (T, j)$ then $(T, j) = (S, i)$.

PROOF: If $\phi : S \approx T$ and $j \circ \phi = i$ then $\phi^{-1} : T \approx S$ and $i \circ \phi^{-1} = j$. \square

Proposition 2.7.4. If $(R, i) = (S, j)$ and $(S, j) = (T, k)$ then $(R, i) = (T, k)$.

PROOF: If $\phi : R \approx S$ and $j \circ \phi = i$, and $\psi : S \approx T$ and $k \circ \psi = j$, then $\psi \circ \phi : R \approx T$ and $k \circ \psi \circ \phi = i$. \square

Definition 2.7.5 (Membership). Given $(S, i) \subseteq A$ and $a \in A$, we write $a \in (S, i)$ for $\exists s \in S. i(s) = a$.

Proposition 2.7.6. If $a \in (S, i)$ and $(S, i) = (T, j)$ then $a \in (T, j)$.

PROOF: If $i(s) = a$ then $j(\phi(s)) = a$. \square

Definition 2.7.7 (Union). Given subsets S and T of A , the *union* is the subset $\{x \in A : x \in S \vee x \in T\}$.

Definition 2.7.8 (Intersection). Given subsets S and T of A , the *intersection* is the subset $\{x \in A : x \in S \wedge x \in T\}$.

Proposition 2.7.9 (Distributive Law).

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

Proposition 2.7.10 (Distributive Law).

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

Definition 2.7.11. Given a set A , we write \emptyset for the subset $(\emptyset, !)$ where $!$ is the unique function $\emptyset \rightarrow A$.

Proposition 2.7.12.

$$S \cup \emptyset = S$$

Proposition 2.7.13.

$$S \cap \emptyset = \emptyset$$

Definition 2.7.14 (Inclusion). Given subsets (S, i) and (T, j) of a set A , we write $(S, i) \subseteq (T, j)$ iff there exists $f : S \rightarrow T$ such that $j \circ f = i$.

Proposition 2.7.15.

$$\emptyset \subseteq S$$

Definition 2.7.16 (Disjoint). Subsets S and T of A are *disjoint* iff $S \cap T = \emptyset$.

Definition 2.7.17 (Difference). Given subsets S and T of A , the *difference* of S and T is $S - T = \{x \in A : x \in S \wedge x \notin T\}$.

Proposition 2.7.18 (De Morgan's Law).

$$R - (S \cup T) = (R - S) \cap (R - T)$$

Proposition 2.7.19 (De Morgan's Law).

$$R - (S \cap T) = (R - S) \cup (R - T)$$

2.8 Saturated Set

Definition 2.8.1 (Saturated). Let A and B be sets. Let $f : A \rightarrow B$ be surjective. Let $C \subseteq A$. Then C is *saturated* with respect to f iff, for all $x \in C$ and $y \in A$, if $f(x) = f(y)$ then $y \in C$.

2.9 Union

Definition 2.9.1 (Union). Given $\mathcal{A} \in \mathcal{P}\mathcal{P}X$, its *union* is

$$\bigcup \mathcal{A} := \{x \in X : \exists S \in \mathcal{A}. x \in S\} \in \mathcal{P}X .$$

2.9.1 Intersection

Definition 2.9.2 (Intersection). Given $\mathcal{A} \in \mathcal{PP}X$, its *intersection* is

$$\bigcap \mathcal{A} := \{x \in X : \forall S \in \mathcal{A}. x \in S\} \in \mathcal{P}X .$$

2.9.2 Direct Image

Definition 2.9.3 (Direct Image). Let $f : A \rightarrow B$. Let S be a subset of A . The *(direct) image* of S under f is the subset of B given by

$$f(S) := \{f(a) : a \in S\} .$$

Proposition 2.9.4.

1. If $S \subseteq T$ then $f(S) \subseteq f(T)$
2. $f(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f(S)$

Example 2.9.5. It is not true in general that $f(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f(S)$. Take f to be the only function $\{0, 1\} \rightarrow \{0\}$, and $\mathcal{S} = \{\{0\}, \{1\}\}$. Then $f(\bigcap \mathcal{S}) = \emptyset$ but $\bigcap_{S \in \mathcal{S}} f(S) = \{0\}$.

Example 2.9.6. It is not true in general that $f(S - T) = f(S) - f(T)$. Take f to be the only function $\{0, 1\} \rightarrow \{0\}$, $S = \{0\}$ and $T = \{1\}$. Then $f(S - T) = \{0\}$ but $f(S) - f(T) = \emptyset$.

2.10 Inverse Image

Definition 2.10.1 (Inverse Image). Let $f : A \rightarrow B$. Let S be a subset of B . The *inverse image* or *preimage* of S under f is the subset of A given by

$$f^{-1}(S) := \{x \in A : f(x) \in S\} .$$

Proposition 2.10.2. 1. If $S \subseteq T$ then $f^{-1}(S) \subseteq f^{-1}(T)$

2. $f^{-1}(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f^{-1}(S)$
3. $f^{-1}(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f^{-1}(S)$
4. $f^{-1}(S - T) = f^{-1}(S) - f^{-1}(T)$
5. $S \subseteq f^{-1}(f(S))$. Equality holds if f is injective.
6. $f(f^{-1}(T)) \subseteq T$. Equality holds if f is surjective.
7. $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$

2.10.1 Saturated Sets

Proposition 2.10.3. *Let A and B be sets. Let $f : A \rightarrow B$ be surjective. Let $C \subseteq A$. Then C is saturated if and only if there exists $D \subseteq B$ such that $C = f^{-1}(D)$.*

PROOF:

$\langle 1 \rangle 1$. If C is saturated then there exists $D \subseteq B$ such that $C = f^{-1}(D)$.

$\langle 2 \rangle 1$. ASSUME: C is saturated.

$\langle 2 \rangle 2$. LET: $D = f(C)$

$\langle 2 \rangle 3$. $C \subseteq f^{-1}(D)$

$\langle 3 \rangle 1$. LET: $x \in C$

$\langle 3 \rangle 2$. $f(x) \in D$

PROOF: $\langle 2 \rangle 2$

$\langle 3 \rangle 3$. $x \in f^{-1}(D)$

$\langle 2 \rangle 4$. $f^{-1}(D) \subseteq C$

$\langle 3 \rangle 1$. LET: $x \in f^{-1}(D)$

$\langle 3 \rangle 2$. $f(x) \in D$

$\langle 3 \rangle 3$. PICK $y \in C$ such that $f(x) = f(y)$

PROOF: $\langle 2 \rangle 2$

$\langle 3 \rangle 4$. $x \in C$

PROOF: $\langle 2 \rangle 1$

$\langle 1 \rangle 2$. If there exists $D \subseteq B$ such that $C = f^{-1}(D)$ then C is saturated.

$\langle 2 \rangle 1$. LET: $D \subseteq B$ be such that $C = f^{-1}(D)$.

$\langle 2 \rangle 2$. LET: $x \in C$ and $y \in A$

$\langle 2 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 4$. $f(x) \in D$

$\langle 2 \rangle 5$. $f(y) \in D$

$\langle 2 \rangle 6$. $y \in C$

□

2.11 Relations

Definition 2.11.1 (Relation). Let A and B be sets. A *relation* R between A and B , $R : A \rightarrow B$, is a subset of $A \times B$.

Given $a \in A$ and $b \in B$, we write aRb for $(a, b) \in R$.

A relation *on* a set A is a relation between A and A .

Definition 2.11.2 (Reflexive). A relation R on a set A is *reflexive* iff $\forall a \in A. aRa$.

Definition 2.11.3 (Symmetric). A relation R on a set A is *symmetric* iff, whenever xRy , then yRx .

Definition 2.11.4 (Transitive). A relation R on a set A is *transitive* iff, whenever xRy and yRz , then xRz .

2.11.1 Equivalence Relations

Definition 2.11.5 (Equivalence Relation). A relation R on a set A is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Definition 2.11.6 (Equivalence Class). Let R be an equivalence relation on a set A and $a \in A$. The *equivalence class* of a with respect to R is

$$\{x \in A : xRa\} .$$

Proposition 2.11.7. *Two equivalence classes are either disjoint or equal.*

2.12 Power Set

Definition 2.12.1 (Power Set). The *power set* of a set A is $\mathcal{P}A := 2^A$.

Given $S \in \mathcal{P}A$ and $a \in A$, we write $a \in S$ for $S(a) = 1$.

Definition 2.12.2 (Pairwise Disjoint). Let $P \subseteq \mathcal{P}A$. We say the members of P are *pairwise disjoint* iff, for all $S, T \in P$, if $S \neq T$ then $S \cap T = \emptyset$.

2.12.1 Partitions

Definition 2.12.3 (Partition). Let A be a set. A *partition* of A is a set $P \in \mathcal{P}\mathcal{P}A$ such that:

- $\bigcup P = A$
- Every member of P is nonempty.
- The members of P are pairwise disjoint.

2.13 Cartesian Product

Definition 2.13.1 (Cartesian Product). Let A and B be sets. The *Cartesian product* of A and B , $A \times B$, is the tabulation of the relation $A \bowtie B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

2.14 Quotient Sets

Proposition 2.14.1. *Let \sim be an equivalence relation on X . Then there exists a set X/\sim , the quotient set of X with respect to \sim , and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the canonical projection, such that, for all $x, y \in X$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to \sim , then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

2.15 Partitions

Definition 2.15.1 (Partition). A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X .

2.16 Disjoint Union

Theorem 2.16.1. For any sets A and B , there exists a set $A + B$, the disjoint union of A and B , and functions $\kappa_1 : A \rightarrow A + B$ and $\kappa_2 : B \rightarrow A + B$, the injections, such that, for every set X and functions $f : A \rightarrow X$ and $g : B \rightarrow X$, there exists a unique function $[f, g] : A + B \rightarrow X$ such that $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$.

PROOF:

$\langle 1 \rangle 1$. LET: $A + B := \{p \in \mathcal{P}A \times \mathcal{P}B : \exists a \in A. p = (\{a\}, \emptyset) \vee \exists b \in B. p = (\emptyset, \{b\})\}$

Definition 2.16.2 (Restriction). Let $f : A \rightarrow B$ and let (S, i) be a subset of A . The *restriction* of f to S is the function $f \upharpoonright S : S \rightarrow B$ defined by $f \upharpoonright S = f \circ i$.

2.17 Natural Numbers

Theorem 2.17.1 (Principle of Recursive Definition). Let A be a set. Let F be the set of all functions $\{m \in \mathbb{N} : m < n\} \rightarrow A$ for some n . Let $\rho : F \rightarrow A$. Then there exists a unique $g : \mathbb{N} \rightarrow A$ such that, for all $n \in \mathbb{N}$, we have

$$g(n) = \rho(g \upharpoonright \{m \in \mathbb{N} : m < n\}) .$$

PROOF:

$\langle 1 \rangle 1$. Given a subset $B \subseteq \mathbb{N}$, let us say that a function $g : B \rightarrow A$ is *acceptable* iff, for all $n \in B$, we have

$$\forall m < n. m \in B$$

and

$$g(n) = \rho(g \upharpoonright \{m \in \mathbb{N} : m < n\}) .$$

$\langle 1 \rangle 2$. For all $n \in \mathbb{N}$, there exists an acceptable function $\{m \in \mathbb{N} : m < n\} \rightarrow A$.

$\langle 2 \rangle 1$. LET: $P[n]$ be the property: There exists an acceptable function $\{m \in \mathbb{N} : m < n\} \rightarrow A$.

$\langle 2 \rangle 2$. $P[0]$

PROOF: The unique function $\emptyset \rightarrow A$ is acceptable.

$\langle 2 \rangle 3$. For any natural number n , if $P[n]$ then $P[n + 1]$.

$\langle 3 \rangle 1$. ASSUME: $P[n]$

$\langle 3 \rangle 2$. PICK an acceptable $f : \{m \in \mathbb{N} : m < n\} \rightarrow A$.

$\langle 3 \rangle 3$. LET: $g : \{m \in \mathbb{N} : m < n + 1\} \rightarrow A$ be the function

$$g(m) = \begin{cases} f(m) & \text{if } m < n \\ \rho(f) & \text{if } m = n \end{cases}$$

$\langle 3 \rangle 4$. g is acceptable.

- ⟨1⟩3. If $g : B \rightarrow A$ and $h : C \rightarrow A$ are acceptable, then g and h agree on $B \cap C$.
 - ⟨1⟩4. Define $g : \mathbb{N} \rightarrow A$ by: $g(n) = a$ iff there exists an acceptable $h : \{m \in \mathbb{N} : m < n + 1\}$ such that $h(n) = a$.
 - ⟨1⟩5. g is acceptable.
 - ⟨1⟩6. If $g' : \mathbb{N} \rightarrow A$ is acceptable then $g' = g$.
-

2.18 Finite and Infinite Sets

Definition 2.18.1 (Finite). A set A is *finite* iff there exists $n \in \mathbb{N}$ such that $A \approx \{m \in \mathbb{N} : m < n\}$. In this case, we say A has *cardinality* n .

Proposition 2.18.2. Let $n \in \mathbb{N}$. Let A be a set. Let $a_0 \in A$. Then $A \approx \{m \in \mathbb{N} : m < n + 1\}$ if and only if $A - \{a_0\} \approx \{m \in \mathbb{N} : m < n\}$.

Theorem 2.18.3. Let A be a set. Suppose that $A \approx \{m \in \mathbb{N} : m < n\}$. Let B be a proper subset of A . Then $B \not\approx \{m \in \mathbb{N} : m < n\}$ but there exists $m < n$ such that $B \approx \{k \in \mathbb{N} : k < m\}$.

PROOF:

- ⟨1⟩1. LET: $P[n]$ be the property: for every set A , if $A \approx \{m \in \mathbb{N} : m < n\}$, then for every proper subset B of A , we have $B \not\approx \{m \in \mathbb{N} : m < n\}$ but there exists $m < n$ such that $B \approx \{k \in \mathbb{N} : k < m\}$.
 - ⟨1⟩2. $P[0]$
 PROOF: If $A \approx \{m \in \mathbb{N} : m < 0\}$ then A is empty and so has no proper subset.
 - ⟨1⟩3. For every natural number n , if $P[n]$ then $P[n + 1]$.
 - ⟨2⟩1. LET: n be a natural number.
 - ⟨2⟩2. ASSUME: $P[n]$
 - ⟨2⟩3. LET: A be a set.
 - ⟨2⟩4. ASSUME: $A \approx \{m \in \mathbb{N} : m < n + 1\}$
 - ⟨2⟩5. LET: B be a proper subset of A .
 - ⟨2⟩6. CASE: $B = \emptyset$
 PROOF: Then $B \not\approx \{m \in \mathbb{N} : m < n + 1\}$ but $B \approx \{k \in \mathbb{N} : k < 0\}$.
 - ⟨2⟩7. CASE: $B \neq \emptyset$
 - ⟨3⟩1. PICK $b_0 \in B$
 - ⟨3⟩2. $A - \{b_0\} \approx \{m \in \mathbb{N} : m < n\}$
 - ⟨3⟩3. $B - \{b_0\}$ is a proper subset of $A - \{b_0\}$
 - ⟨3⟩4. $B - \{b_0\} \not\approx \{m \in \mathbb{N} : m < n\}$
 - ⟨3⟩5. $B \approx \{m \in \mathbb{N} : m < n + 1\}$
 - ⟨3⟩6. PICK $m < n$ such that $B - \{b_0\} \approx \{k \in \mathbb{N} : k < m\}$
 - ⟨3⟩7. $m + 1 < n + 1$
 - ⟨3⟩8. $B \approx \{k \in \mathbb{N} : k < m + 1\}$
-

Corollary 2.18.3.1. If A is finite then there is no bijection between A and a proper subset of A .

Corollary 2.18.3.2. \mathbb{N} is infinite.

Corollary 2.18.3.3. The cardinality of a finite set is unique.

Corollary 2.18.3.4. A subset of a finite set is finite.

Corollary 2.18.3.5. If A is finite and B is a proper subset of A then $|B| < |A|$.

Corollary 2.18.3.6. Let A be a set. Then the following are equivalent:

1. A is finite.
2. There exists a surjection from an initial segment of \mathbb{N} onto A .
3. There exists an injection from A to an initial segment of \mathbb{N} .

Corollary 2.18.3.7. A finite union of finite sets is finite.

Corollary 2.18.3.8. A finite Cartesian product of finite sets is finite.

Theorem 2.18.4. Let A be a set. The following are equivalent:

1. There exists an injective function $\mathbb{N} \rightarrow A$.
2. There exists a bijection between A and a proper subset of A .
3. A is infinite.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ LET: $f : \mathbb{N} \rightarrow A$ be injective.

$\langle 2 \rangle 2.$ LET: $s : \mathbb{N} \approx \mathbb{N} - \{0\}$ be the function $s(n) = n + 1$.

$\langle 2 \rangle 3.$ $f \circ s \circ f^{-1} : A \approx A - \{f(0)\}$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: Corollary 2.18.3.1.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: Choose a function $f : \mathbb{N} \rightarrow A$ such that $f(n) \in A - \{f(m) : m < n\}$ for all n .

□

2.19 Countable Sets

Definition 2.19.1 (Countable). A set A is *countably infinite* iff $A \approx \mathbb{N}$.

Proposition 2.19.2. $\mathbb{N} \times \mathbb{N}$ is countably infinite.

PROOF: Define $f : \mathbb{N} \times \mathbb{N} \approx \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x\}$ by

$$f(x, y) = (x + y, y)$$

Define $g : \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x\} \approx \mathbb{N}$ by

$$g(x, y) = x(x - 1)/2 + y \quad \square$$

Proposition 2.19.3. *Every infinite subset of \mathbb{N} is countably infinite.*

PROOF:

$\langle 1 \rangle 1$. LET: C be an infinite subset of \mathbb{N}

$\langle 1 \rangle 2$. Define $h : \mathbb{Z} \rightarrow C$ by recursion thus: $h(n)$ is the smallest element of $C - \{h(m) : m < n\}$.

$\langle 1 \rangle 3$. h is injective.

PROOF: If $m < n$ then $h(m) \neq h(n)$ because $h(n) \in C - \{h(m) : m < n\}$.

$\langle 1 \rangle 4$. h is surjective.

$\langle 2 \rangle 1$. For all $n \in \mathbb{N}$ we have $n \leq h(n)$.

$\langle 2 \rangle 2$. LET: $c \in C$

$\langle 2 \rangle 3$. $c \leq h(c)$

$\langle 2 \rangle 4$. LET: n be least such that $c \leq h(n)$

$\langle 2 \rangle 5$. $c \in C - \{h(m) : m < n\}$

$\langle 2 \rangle 6$. $h(n) \leq c$

$\langle 2 \rangle 7$. $h(n) = c$

□

Definition 2.19.4 (Countable). A set is *countable* iff it is either finite or countably infinite; otherwise it is *uncountable*.

Proposition 2.19.5. *Let B be a nonempty set. Then the following are equivalent.*

1. B is countable.
2. There exists a surjection $\mathbb{N} \twoheadrightarrow B$.
3. There exists an injection $B \hookrightarrow \mathbb{N}$.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: B is countable.

$\langle 2 \rangle 2$. CASE: B is finite.

$\langle 3 \rangle 1$. PICK a natural number n and bijection $f : \{m \in \mathbb{N} : m < n\} \approx B$

$\langle 3 \rangle 2$. PICK $b \in B$

$\langle 3 \rangle 3$. Extend f to a surjection $g : \mathbb{N} \twoheadrightarrow B$ by setting $g(m) = b$ for $m \geq n$.

$\langle 2 \rangle 3$. CASE: B is countably infinite.

PROOF: Then there exists a bijection $\mathbb{N} \approx B$.

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

PROOF: Given a surjection $f : \mathbb{N} \twoheadrightarrow B$, define $g : B \hookrightarrow \mathbb{N}$ by $g(b)$ is the smallest number such that $f(g(b)) = b$.

$\langle 1 \rangle 3$. $3 \Rightarrow 1$

$\langle 2 \rangle 1$. LET: $f : B \hookrightarrow \mathbb{N}$ be injective.

$\langle 2 \rangle 2$. $f(B)$ is countable.

$\langle 2 \rangle 3$. $B \approx f(B)$

$\langle 2 \rangle 4$. B is countable.

□

Corollary 2.19.5.1. *A subset of a countable set is countable.*

Corollary 2.19.5.2. $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

PROOF: The function that maps (m, n) to $2^m 3^n$ is injective. \square

Corollary 2.19.5.3. *The Cartesian product of two countable sets is countable.*

Theorem 2.19.6. *A countable union of countable sets is countable.*

PROOF:

$\langle 1 \rangle 1$. LET: A be a set.

$\langle 1 \rangle 2$. LET: $\mathcal{B} \subseteq \mathcal{P}A$ be a countable set of countable sets such that $\bigcup \mathcal{B} = A$

$\langle 1 \rangle 3$. PICK a surjection $B : \mathbb{N} \twoheadrightarrow \mathcal{B}$

$\langle 1 \rangle 4$. ASSUME: w.l.o.g. each $B(n)$ is nonempty.

$\langle 1 \rangle 5$. For $n \in \mathbb{N}$, PICK a surjective function $g_n : \mathbb{N} \twoheadrightarrow B(n)$

$\langle 1 \rangle 6$. LET: $h : \mathbb{N} \times \mathbb{N} \rightarrow A$ be the function $h(m, n) = g_m(n)$

$\langle 1 \rangle 7$. h is surjective.

\square

Theorem 2.19.7. $2^{\mathbb{N}}$ *is uncountable.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : \mathbb{N} \rightarrow 2^{\mathbb{N}}$

PROVE: f is not surjective.

$\langle 1 \rangle 2$. Define $g : \mathbb{N} \rightarrow 2$ by $g(n) = 1 - f(n)(n)$.

$\langle 1 \rangle 3$. For all $n \in \mathbb{N}$ we have $g(n) \neq f(n)(n)$.

$\langle 1 \rangle 4$. For all $n \in \mathbb{N}$ we have $g \neq f(n)$.

\square

Theorem 2.19.8. *For any set A , there is no surjective function $A \rightarrow \mathcal{P}A$.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow \mathcal{P}A$

$\langle 1 \rangle 2$. LET: $S = \{x \in A : x \notin f(x)\}$

$\langle 1 \rangle 3$. For all $a \in A$ we have $S \neq f(a)$

PROOF: We have $a \in S$ if and only if $a \notin f(a)$.

\square

Corollary 2.19.8.1. *For any set A , there is no injective function $\mathcal{P}A \rightarrow A$.*

2.20 Fixed Points

Definition 2.20.1 (Fixed Point). Let A be a set and $f : A \rightarrow A$. A *fixed point* of f is an element $a \in A$ such that $f(a) = a$.

Chapter 3

Relations

Definition 3.0.1 (Reflexive). A relation $R \subseteq A \times A$ is *reflexive* iff, for all $a \in A$, we have $(a, a) \in R$.

Definition 3.0.2 (Antisymmetric). A relation $R \subseteq A \times A$ is *antisymmetric* iff, for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Definition 3.0.3 (Transitive). A relation $R \subseteq A \times A$ is *transitive* iff, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Definition 3.0.4 (Partial Order). A *partial order* on a set A is a relation on A that is reflexive, antisymmetric and transitive.

We say (A, \leq) is a *partially ordered set* or *poset* iff \leq is a partial order on A .

Definition 3.0.5 (Greatest). Let A be a poset and $a \in A$. Then a is the *greatest* element iff $\forall x \in A. x \leq a$.

Definition 3.0.6 (Least). Let A be a poset and $a \in A$. Then a is the *least* element iff $\forall x \in A. a \leq x$.

Definition 3.0.7 (Upper Bound). Let A be a poset, $S \subseteq A$, and $u \in A$. Then u is an *upper bound* for S iff $\forall x \in S. x \leq u$. We say S is *bounded above* iff it has an upper bound.

Definition 3.0.8 (Lower Bound). Let A be a poset, $S \subseteq A$, and $l \in A$. Then l is a *lower bound* for S iff $\forall x \in S. l \leq x$. We say S is *bounded below* iff it has a lower bound.

Definition 3.0.9 (Supremum). Let A be a poset, $S \subseteq A$ and $s \in A$. Then s is the *supremum* or *least upper bound* for S iff s is the least element in the sub-poset of upper bounds for A .

Definition 3.0.10 (Infimum). Let A be a poset, $S \subseteq A$ and $i \in A$. Then i is the *infimum* or *greatest lower bound* for S iff i is the greatest element in the sub-poset of lower bounds for A .

Definition 3.0.11 (Least Upper Bound Property). A poset A has the *least upper bound property* iff every nonempty subset of A that is bounded above has a least upper bound.

Proposition 3.0.12. *Let A be a poset. Then A has the least upper bound property if and only if every nonempty subset of A that is bounded below has a greatest lower bound.*

PROOF:

$\langle 1 \rangle 1$. If A has the least upper bound property then every subset of A that is bounded below has a greatest lower bound.

$\langle 2 \rangle 1$. ASSUME: A has the least upper bound property.

$\langle 2 \rangle 2$. LET: $S \subseteq A$ be nonempty and bounded below.

$\langle 2 \rangle 3$. LET: L be the set of lower bounds of S .

$\langle 2 \rangle 4$. L is nonempty.

PROOF: Because S is bounded below.

$\langle 2 \rangle 5$. L is bounded above.

PROOF: Pick an element $s \in S$. Then s is an upper bound for L .

$\langle 2 \rangle 6$. LET: s be the supremum of L .

$\langle 2 \rangle 7$. s is the greatest lower bound of S .

$\langle 3 \rangle 1$. s is a lower bound of S .

$\langle 4 \rangle 1$. LET: $x \in S$

$\langle 4 \rangle 2$. x is an upper bound for L .

$\langle 4 \rangle 3$. $s \leq x$

$\langle 3 \rangle 2$. For any lower bound l of S we have $l \leq s$.

PROOF: Immediate from $\langle 2 \rangle 6$.

$\langle 1 \rangle 2$. If every subset of A that is bounded below has a greatest lower bound, then A has the least upper bound property.

PROOF: Dual.

□

Chapter 4

Order Theory

4.1 Strict Partial Orders

Definition 4.1.1 (Strict Partial Order). A *strict partial order* on a set A is a relation on A that is irreflexive and transitive.

Proposition 4.1.2. 1. If \leq is a partial order on A then $<$ is a strict partial order on A , where $x < y$ iff $x \leq y \wedge x \neq y$.

2. If $<$ is a strict partial order on A then \leq is a partial order on A , where $x \leq y$ iff $x < y \vee x = y$.

3. These two relations are inverses of one another.

4.1.1 Linear Orders

Definition 4.1.3 (Linear Order). A *linear order* on a set A is a partial order \leq on A such that, for all $x, y \in A$, we have $x \leq y$ or $y \leq x$.

A *linearly ordered set* is a pair (X, \leq) such that X is a set and \leq is a linear order on X .

Definition 4.1.4 (Open Interval). Let X be a linearly ordered set and $a, b \in X$. The *open interval* (a, b) is the set

$$\{x \in X : a < x < b\} .$$

Definition 4.1.5 (Immediate Predecessor, Immediate Successor). Let X be a linearly ordered set and $a, b \in X$. Then b is the (*immediate*) *successor* of a , and a is the (*immediate*) *predecessor* of b , iff $a < b$ and there is no x such that $a < x < b$.

Definition 4.1.6 (Dictionary Order). Let A and B be linearly ordered sets. The *dictionary order* on $A \times B$ is the order defined by

$$(a, b) < (a', b') \Leftrightarrow a < a' \vee (a = a' \wedge b < b') .$$

Theorem 4.1.7 (Maximum Principle). *Every poset has a maximal linearly ordered subset.*

PROOF:

⟨1⟩1. LET: (A, \leq) be a poset.

⟨1⟩2. PICK a well ordering \preceq of A .

PROOF: Well Ordering Theorem.

⟨1⟩3. LET: $h : A \rightarrow 2$ be the function defined by \preceq -recursion thus:

$$h(a) = \begin{cases} 1 & \text{if } a \text{ is } \leq\text{-comparable with every } b < a \text{ such that } h(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

⟨1⟩4. LET: $B = \{x \in A : h(x) = 1\}$

PROVE: B is a maximal subset linearly ordered by \leq .

⟨1⟩5. B is linearly ordered by \leq .

⟨2⟩1. LET: $x, y \in B$

⟨2⟩2. ASSUME: w.l.o.g. $x \preceq y$

⟨2⟩3. y is \leq -comparable with x

⟨1⟩6. For any subset $C \subseteq A$ linearly ordered by \leq , if $B \subseteq C$ then $B = C$.

⟨2⟩1. LET: $x \in C$

⟨2⟩2. x is comparable with every $y \preceq x$ such that $h(y) = 1$

⟨2⟩3. $x \in B$

□

Theorem 4.1.8 (Zorn's Lemma). *Let A be a poset. If every linearly ordered subset of A is bounded above, then A has a maximal element.*

PROOF:

⟨1⟩1. PICK a maximal linearly ordered subset B of A .

PROOF: Maximal Principle

⟨1⟩2. PICK an upper bound c for B .

PROVE: c is maximal.

⟨1⟩3. LET: $x \in A$

⟨1⟩4. ASSUME: $c \leq x$

PROVE: $x = c$

⟨1⟩5. x is an upper bound for B .

⟨1⟩6. $x \in B$

PROOF: By the maximality of B , since $B \cup \{x\}$ is linearly ordered.

⟨1⟩7. $x \leq c$

PROOF: ⟨1⟩2

⟨1⟩8. $x = c$

□

Corollary 4.1.8.1 (Kuratowski's Lemma). *Let $\mathcal{A} \subseteq \mathcal{P}X$. Suppose that, for every subset $\mathcal{B} \subseteq \mathcal{A}$ that is linearly ordered by inclusion, we have $\bigcup \mathcal{B} \in \mathcal{A}$. Then \mathcal{A} has a maximal element.*

Definition 4.1.9 (Closed Interval). Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *closed interval* $[a, b]$ is

$$[a, b] := \{x \in X : a \leq x \leq b\} .$$

Definition 4.1.10 (Half-Open Interval). Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *half-open intervals* $(a, b]$ and $[a, b)$ are defined by

$$\begin{aligned}(a, b] &:= \{x \in X : a < x \leq b\} \\ [a, b) &:= \{x \in X : a \leq x < b\}\end{aligned}$$

Definition 4.1.11 (Open Ray). Let X be a linearly ordered set and $a \in X$. The *open rays* $(a, +\infty)$ and $(-\infty, a)$ are defined by:

$$\begin{aligned}(a, +\infty) &:= \{x \in X : a < x\} \\ (-\infty, a) &:= \{x \in X : x < a\}\end{aligned}$$

Definition 4.1.12 (Closed Ray). Let X be a linearly ordered set and $a \in X$. The *closed rays* $[a, +\infty)$ and $(-\infty, a]$ are defined by:

$$\begin{aligned}[a, +\infty) &:= \{x \in X : a \leq x\} \\ (-\infty, a] &:= \{x \in X : x \leq a\}\end{aligned}$$

Definition 4.1.13 (Convex). Let X be a linearly ordered set and $Y \subseteq X$. Then Y is *convex* iff, for all $a, b \in Y$ and $c \in X$, if $a < c < b$ then $c \in Y$.

4.1.2 Sets of Finite Type

Definition 4.1.14 (Finite Type). Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then \mathcal{A} is of *finite type* if and only if, for any $B \subseteq X$, we have $B \in \mathcal{A}$ if and only if every finite subset of B is in \mathcal{A} .

Proposition 4.1.15 (Tukey's Lemma). *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$. If \mathcal{A} is of finite type, then \mathcal{A} has a maximal element.*

PROOF:

$\langle 1 \rangle 1$. For every subset $\mathcal{B} \subseteq \mathcal{A}$ that is linearly ordered by inclusion, we have $\bigcup \mathcal{B} \in \mathcal{A}$.

$\langle 2 \rangle 1$. LET: $\mathcal{B} \subseteq \mathcal{A}$

$\langle 2 \rangle 2$. ASSUME: \mathcal{B} is linearly ordered by inclusion.

$\langle 2 \rangle 3$. Every finite subset of $\bigcup \mathcal{B}$ is in \mathcal{A}

$\langle 2 \rangle 4$. $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Kuratowski's Lemma.

□

4.2 Linear Continua

Definition 4.2.1 (Linear Continuum). A *linear continuum* is a linearly ordered set with more than one element that is dense and has the least upper bound property.

Proposition 4.2.2. *Every convex subset of a linear continuum with more than one element is a linear continuum.*

PROOF: Easy. \square

Corollary 4.2.2.1. *Every interval and ray in a linear continuum is a linear continuum.*

4.3 Well Orders

Definition 4.3.1 (Well Ordered Set). A *well ordered set* is a linearly ordered set such that every nonempty subset has a least element.

Proposition 4.3.2. *Any subset of a well ordered set is well ordered.*

Proposition 4.3.3. *The product of two well ordered sets is well ordered under the dictionary order.*

Theorem 4.3.4 (Well Ordering Theorem). *Every set has a well ordering.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a set.

$\langle 1 \rangle 2$. PICK a choice function $c : \mathcal{P}X - \{\emptyset\} \rightarrow X$

$\langle 1 \rangle 3$. Define a *tower* to be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well ordering of T , and

$$\forall x \in T. x = c(X - \{y \in T : y < x\}) .$$

$\langle 1 \rangle 4$. Given two towers, either they are equal or one is a section of the other.

$\langle 2 \rangle 1$. LET: $(T_1, <_1)$ and $(T_2, <_2)$ be towers.

$\langle 2 \rangle 2$. ASSUME: w.l.o.g. there exists a strictly monotone function $h : T_1 \rightarrow T_2$

$\langle 2 \rangle 3$. $h(T_1)$ is either T_2 or a section of T_2

PROOF: Proposition 4.3.11.

$\langle 2 \rangle 4$. $\forall x \in T_1. h(x) = x$

$\langle 3 \rangle 1$. LET: $x \in T_1$

$\langle 3 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall y < x. h(y) = y$

$\langle 3 \rangle 3$. $h(x)$ is the least element of $T_2 - \{h(y) \in T_1 : y < x\}$

$\langle 3 \rangle 4$. $h(x)$ is the least element of $T_2 - \{y \in T_1 : y < x\}$

PROOF: $\langle 3 \rangle 2$

$\langle 3 \rangle 5$. $h(x) = x$

PROOF:

$$h(x) = c(X - \{y \in T_2 : y < h(x)\}) \quad (\langle 1 \rangle 3)$$

$$= c(X - \{y \in T_2 : y < x\}) \quad (\langle 3 \rangle 4)$$

$$= c(X - \{y \in T_1 : y < x\}) \quad (\langle 3 \rangle 2)$$

$$= x \quad (\langle 1 \rangle 3)$$

$\langle 1 \rangle 5$. If $(T, <)$ is a tower and $T \neq X$, then there exists a tower of which $(T, <)$ is a section.

PROOF: Let $T_1 = T \cup \{c(T)\}$ and $<_1$ be the extension of $<$ such that $x < c(T)$ for all $x \in T$.

- $\langle 1 \rangle 6$. LET: $\mathbf{T} = \bigcup \{T : \exists R. (T, R) \text{ is a tower}\}$ and $\mathbf{R} = \bigcup \{R : \exists T. (T, R) \text{ is a tower}\}$
 $\langle 1 \rangle 7$. (\mathbf{T}, \mathbf{R}) is a tower.
 $\langle 2 \rangle 1$. \mathbf{R} is irreflexive.
 PROOF: Since for every tower $(T, <)$ we have $<$ is irreflexive.
 $\langle 2 \rangle 2$. \mathbf{R} is transitive.
 $\langle 3 \rangle 1$. ASSUME: $x\mathbf{R}y$ and $y\mathbf{R}z$
 $\langle 3 \rangle 2$. PICK towers $(T_1, <_1)$ and $(T_2, <_2)$ such that $x <_1 y$ and $y <_2 z$
 $\langle 3 \rangle 3$. ASSUME: w.l.o.g. $(T_1, <_1)$ is either $(T_2, <_2)$ or a section of $(T_2, <_2)$
 $\langle 3 \rangle 4$. $x <_2 y <_2 z$
 $\langle 3 \rangle 5$. $x <_2 z$
 $\langle 3 \rangle 6$. $x\mathbf{R}z$
 $\langle 2 \rangle 3$. For all $x, y \in \mathbf{T}$, either $x\mathbf{R}y$ or $x = y$ or $y\mathbf{R}x$
 PROOF: There exists a tower that has both x and y .
 $\langle 2 \rangle 4$. Every nonempty subset of \mathbf{T} has an \mathbf{R} -least element.
 $\langle 3 \rangle 1$. LET: $A \subseteq \mathbf{T}$ be nonempty.
 $\langle 3 \rangle 2$. PICK $a \in A$
 $\langle 3 \rangle 3$. PICK a tower $(T, <)$ such that $a \in T$.
 $\langle 3 \rangle 4$. LET: b be the $<$ -least element of $A \cap T$
 PROVE: b is \mathbf{R} -least in A .
 $\langle 3 \rangle 5$. LET: $x \in A$
 $\langle 3 \rangle 6$. Etc.
 $\langle 2 \rangle 5$. $\forall x \in \mathbf{T}. x = c(X - \{y \in \mathbf{T} : y\mathbf{R}x\})$
 $\langle 1 \rangle 8$. $\mathbf{T} = X$
 $\langle 1 \rangle 9$. \mathbf{R} is a well ordering of X .
 \square

Proposition 4.3.5. *There exists a well-ordered set with a largest element Ω such that $(-\infty, \Omega)$ is uncountable but, for all $\alpha < \Omega$, we have $(-\infty, \alpha)$ is countable.*

PROOF:

- $\langle 1 \rangle 1$. PICK an uncountable well ordered set B .
 $\langle 1 \rangle 2$. LET: $C = 2 \times B$ under the dictionary order.
 $\langle 1 \rangle 3$. LET: Ω be the least element of C such that $(-\infty, \Omega)$ is uncountable.
 $\langle 1 \rangle 4$. LET: $A = (-\infty, \Omega]$
 $\langle 1 \rangle 5$. A is a well ordered set with largest element Ω such that $(-\infty, \Omega)$ is uncountable but, for all $\alpha < \Omega$, we have $(-\infty, \alpha)$ is countable.
 \square

Proposition 4.3.6. *Every well ordered set has the least upper bound property.*

PROOF: For any subset that is bounded above, the set of upper bounds is nonempty, hence has a least element. \square

Proposition 4.3.7. *In a well ordered set, every element that is not greatest has a successor.*

PROOF: If a is not greatest, then $\{x : x > a\}$ is nonempty, hence has a least element. \square

Theorem 4.3.8 (Transfinite Induction). *Let J be a well ordered set. Let $S \subseteq J$. Assume that, for every $\alpha \in J$, if $\forall x < \alpha. x \in S$ then $\alpha \in S$. Then $S = J$.*

PROOF: Otherwise $J - S$ would be a nonempty subset of J with no least element. \square

Proposition 4.3.9. *Let I be a well ordered set. Let $\{A_i\}_{i \in I}$ be a family of well ordered sets. Define $<$ on $\coprod_{i \in I} A_i$ by: $\kappa_i(a) < \kappa_j(b)$ iff either $i < j$, or $i = j$ and $a < b$ in A_i . Then $<$ well orders $\coprod_{i \in I} A_i$.*

PROOF: Easy. \square

Theorem 4.3.10 (Principle of Transfinite Recursion). *Let J be a well ordered set. Let C be a set. Let \mathcal{F} be the set of all functions from a section of J into C . Let $\rho : \mathcal{F} \rightarrow C$. Then there exists a unique function $h : J \rightarrow C$ such that, for all $\alpha \in J$, we have*

$$h(\alpha) = \rho(h \upharpoonright (-\infty, \alpha)) .$$

PROOF:

$\langle 1 \rangle 1$. For a function h mapping either a section of J or all of J into C , let us say h is *acceptable* iff, for all $x \in \text{dom } h$, we have $(-\infty, x) \subseteq \text{dom } h$ and $h(x) = \rho(h \upharpoonright (-\infty, x))$.

$\langle 1 \rangle 2$. If h and k are acceptable functions then $h(x) = k(x)$ for all x in both domains.

$\langle 2 \rangle 1$. LET: $x \in J$

$\langle 2 \rangle 2$. ASSUME: as transfinite induction hypothesis that, for all $y < x$ and any acceptable functions h and k with $y \in \text{dom } h \cap \text{dom } k$, we have $h(y) = k(y)$

$\langle 2 \rangle 3$. LET: h and k be acceptable functions with $x \in \text{dom } h \cap \text{dom } k$

$\langle 2 \rangle 4$. $h \upharpoonright (-\infty, x) = k \upharpoonright (-\infty, x)$

PROOF: By $\langle 2 \rangle 2$.

$\langle 2 \rangle 5$. $h(x) = k(x)$

PROOF: By $\langle 2 \rangle 3$, each is the least element of the set in $\langle 2 \rangle 4$.

$\langle 1 \rangle 3$. For $\alpha \in J$, if there exists an acceptable function $(-\infty, \alpha) \rightarrow C$, then there exists an acceptable function $(-\infty, \alpha] \rightarrow C$.

$\langle 2 \rangle 1$. LET: $\alpha \in J$

$\langle 2 \rangle 2$. LET: $f : (-\infty, \alpha) \rightarrow C$ be acceptable.

$\langle 2 \rangle 3$. LET: $g : (-\infty, \alpha] \rightarrow C$ be the function given by

$$g(x) = \begin{cases} f(x) & \text{if } x < \alpha \\ \rho(f) & \text{if } x = \alpha \end{cases}$$

$\langle 2 \rangle 4$. g is acceptable.

$\langle 1 \rangle 4$. Let $K \subseteq J$. Assume that, for all $\alpha \in K$, there exists an acceptable function $(-\infty, \alpha) \rightarrow C$. Then there exists an acceptable function $\bigcup_{\alpha \in K} (-\infty, \alpha) \rightarrow C$.

$\langle 2 \rangle 1$. Define $f : \bigcup_{\alpha \in K} (-\infty, \alpha) \rightarrow C$ by: $f(x) = y$ iff there exists $\alpha \in K$ and $g : (-\infty, \alpha) \rightarrow C$ acceptable such that $g(x) = y$.

$\langle 1 \rangle 5$. For every $\beta \in J$, there exists an acceptable function $(-\infty, \beta) \rightarrow C$

- ⟨2⟩1. LET: $\beta \in J$
- ⟨2⟩2. ASSUME: as transfinite induction hypothesis that, for all $\alpha < \beta$, there exists an acceptable function $(-\infty, \alpha) \rightarrow C$
- ⟨2⟩3. CASE: β has a predecessor
 - ⟨3⟩1. LET: α be the predecessor of β .
 - ⟨3⟩2. There exists an acceptable function $(-\infty, \alpha) \rightarrow C$.
 - ⟨3⟩3. There exists an acceptable function $(-\infty, \beta) \rightarrow C$.
 PROOF: By ⟨1⟩3 since $(-\infty, \beta) = (-\infty, \alpha]$.
- ⟨2⟩4. CASE: β has no predecessor.
 PROOF: The result follows by ⟨1⟩4 since $(-\infty, \beta) = \bigcup_{\alpha < \beta} (-\infty, \alpha)$.
- ⟨1⟩6. There exists an acceptable function $J \rightarrow C$.
 - ⟨2⟩1. CASE: J has a greatest element.
 - ⟨3⟩1. LET: g be greatest.
 - ⟨3⟩2. There exists an acceptable function $(-\infty, g) \rightarrow C$.
 PROOF: ⟨1⟩5
 - ⟨3⟩3. There exists an acceptable function $J \rightarrow C$.
 PROOF: By ⟨1⟩3 since $J = (-\infty, g]$.
 - ⟨2⟩2. CASE: J has no greatest element.
 PROOF: By ⟨1⟩4 since $J = \bigcup_{\alpha \in J} (-\infty, \alpha)$.

□

Corollary 4.3.10.1 (Cardinal Comparability). *Let A and B be sets. Then either $A \leq B$ or $B \leq A$.*

PROOF: Choose well orderings of A and B . Then either there exists a surjection $A \twoheadrightarrow B$, or there exists an injective function $h : A \rightarrow B$ defined by transfinite recursion by $h(x)$ is the least element of $B - h((-\infty, x))$. □

Proposition 4.3.11. *Let J and E be well ordered sets. Let $h : J \rightarrow E$. Then the following are equivalent.*

1. h is strictly monotone and $h(J)$ is either E or a section of E .
2. For all $\alpha \in J$, we have $h(\alpha)$ is the least element of $E - h((-\infty, \alpha))$.

PROOF:

- ⟨1⟩1. $1 \Rightarrow 2$
 - ⟨2⟩1. ASSUME: 1
 - ⟨2⟩2. $h(J)$ is closed downwards.
 - ⟨2⟩3. LET: $\alpha \in J$
 - ⟨2⟩4. $h(\alpha) \in E - h((-\infty, \alpha))$
 PROOF: If $\beta < \alpha$ then $h(\beta) < h(\alpha)$.
 - ⟨2⟩5. For all $y \in E - h((-\infty, \alpha))$ we have $h(\alpha) \leq y$
 - ⟨3⟩1. ASSUME: for a contradiction $y < h(\alpha)$
 - ⟨3⟩2. $y \in h(J)$
 - ⟨3⟩3. PICK $\beta \in J$ such that $h(\beta) = y$
 - ⟨3⟩4. $h(\beta) < h(\alpha)$
 - ⟨3⟩5. $\beta < \alpha$

⟨3⟩6. Q.E.D.

PROOF: This contradicts the fact that $y \notin h((-\infty, \alpha))$.

⟨1⟩2. $2 \Rightarrow 1$

⟨2⟩1. ASSUME: 2

⟨2⟩2. h is strictly monotone.

⟨3⟩1. LET: $\alpha, \beta \in J$ with $\alpha < \beta$

⟨3⟩2. $h(\alpha) \neq h(\beta)$

PROOF: Because $h(\beta) \in E - h((-\infty, \beta))$.

⟨3⟩3. $h(\alpha) \leq h(\beta)$

PROOF: Because $h(\alpha)$ is least in $E - h((-\infty, \alpha))$.

⟨3⟩4. $h(\alpha) < h(\beta)$

⟨2⟩3. $h(J)$ is either E or a section of E .

⟨3⟩1. ASSUME: $h(J) \neq E$

⟨3⟩2. LET: e be least in $E - h(J)$

PROVE: $h(J) = (-\infty, e)$

⟨3⟩3. $h(J) \subseteq (-\infty, e)$

⟨4⟩1. LET: $\alpha \in J$

⟨4⟩2. $h(\alpha) \neq e$

PROOF: $e \notin h(J)$

⟨4⟩3. $h(\alpha) \leq e$

PROOF: Since $h(\alpha)$ is least in $E - h((-\infty, \alpha))$.

⟨4⟩4. $h(\alpha) < e$

⟨3⟩4. $(-\infty, e) \subseteq h(J)$

PROOF: If $e' < e$ then $e' \in h(J)$ by leastness of e .

□

Part II

Category Theory

Chapter 5

Category Theory

5.1 Categories

Definition 5.1.1. A *category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \text{Ob}(\mathcal{C})$.
- for any objects X and Y , a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y . We write $f : X \rightarrow Y$ for $f \in \mathcal{C}[X, Y]$.
- for any objects X, Y and Z , a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \rightarrow \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \rightarrow Y, g : Y \rightarrow Z$ and $h : Z \rightarrow W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X , there exists a morphism $\text{id}_X : X \rightarrow X$, the *identity morphism* on X , such that:
 - for any object Y and morphism $f : Y \rightarrow X$ we have $\text{id}_X \circ f = f$
 - for any object Y and morphism $f : X \rightarrow Y$ we have $f \circ \text{id}_X = f$

We write the composite of morphism f_1, \dots, f_n as $f_n \circ \dots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 5.1.2. Let **Set** be the category of small sets and functions.

Definition 5.1.3. Let **LPos** be the category of linearly ordered sets and monotone functions.

Proposition 5.1.4. Any finite linearly ordered set is isomorphic to $\{m \in \mathbb{N} : m < n\}$ for some n .

PROOF:

$\langle 1 \rangle$ 1. Every finite nonempty linearly ordered set has a greatest element.

- $\langle 2 \rangle 1$. LET: $P[n]$ be the property: for any linearly ordered set A , if there exists a bijection $A \approx \{m \in \mathbb{N} : m < n\}$ and A is nonempty then A has a greatest element.
- $\langle 2 \rangle 2$. $P[0]$
 PROOF: Vacuous.
- $\langle 2 \rangle 3$. $\forall n \in \mathbb{N}. P[n] \Rightarrow P[n+1]$
- $\langle 3 \rangle 1$. LET: $n \in \mathbb{N}$
- $\langle 3 \rangle 2$. ASSUME: $P[n]$
- $\langle 3 \rangle 3$. LET: A be a nonempty linearly ordered set.
- $\langle 3 \rangle 4$. LET: $f : A \approx \{m \in \mathbb{N} : m < n+1\}$
- $\langle 3 \rangle 5$. LET: $a = f^{-1}(n)$
- $\langle 3 \rangle 6$. $f \upharpoonright (A - \{a\}) : A - \{a\} \approx \{m \in \mathbb{N} : m < n\}$
- $\langle 3 \rangle 7$. ASSUME: w.l.o.g. a is not greatest in A .
- $\langle 3 \rangle 8$. LET: b be greatest in $A - \{a\}$
 PROOF: $\langle 3 \rangle 2$
- $\langle 3 \rangle 9$. b is greatest in A .
- $\langle 1 \rangle 2$. LET: $P[n]$ be the property: for any linearly ordered set A , if there exists a bijection $A \approx \{m \in \mathbb{N} : m < n\}$ then there exists an isomorphism in **LPos** $A \cong \{m \in \mathbb{N} : m < n\}$.
- $\langle 1 \rangle 3$. $P[0]$
 PROOF: If there exists a bijection $A \approx \emptyset$ then A is empty and so the unique function $A \rightarrow \emptyset$ is an order isomorphism.
- $\langle 1 \rangle 4$. For every natural number n , if $P[n]$ then $P[n+1]$.
- $\langle 2 \rangle 1$. LET: n be a natural number.
- $\langle 2 \rangle 2$. ASSUME: $P[n]$
- $\langle 2 \rangle 3$. LET: A be a linearly ordered set.
- $\langle 2 \rangle 4$. ASSUME: A has $n+1$ elements.
- $\langle 2 \rangle 5$. LET: a be the greatest element in A .
- $\langle 2 \rangle 6$. LET: $f : A - \{a\} \cong \{m \in \mathbb{N} : m < n\}$ be an order isomorphism.
 PROOF: $\langle 2 \rangle 2$
- $\langle 2 \rangle 7$. Define $g : A \rightarrow \{m \in \mathbb{N} : m < n+1\}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ n & \text{if } x = a \end{cases}$$
- $\langle 2 \rangle 8$. g is an order isomorphism.
- $\langle 1 \rangle 5$. $\forall n \in \mathbb{N}. P[n]$
 \square

Corollary 5.1.4.1. *Any finite linearly ordered set is well ordered.*

Proposition 5.1.5. *Let J and E be well ordered sets. Suppose there is a strictly monotone map $J \rightarrow E$. Then J is isomorphic either to E or a section of E .*

PROOF:

- $\langle 1 \rangle 1$. LET: $k : J \rightarrow E$ be strictly monotone.
- $\langle 1 \rangle 2$. ASSUME: w.l.o.g. E is nonempty.
- $\langle 1 \rangle 3$. PICK $e_0 \in E$

⟨1⟩4. LET: $h : J \rightarrow E$ be the function defined by transfinite recursion thus:

$$h(\alpha) = \begin{cases} \text{the least element in } E - h((-\infty, \alpha)) & \text{if } h((-\infty, \alpha)) \neq E \\ e_0 & \text{if } h((-\infty, \alpha)) = E \end{cases}$$

⟨1⟩5. $\forall \alpha \in J, h(\alpha) \leq k(\alpha)$

⟨2⟩1. LET: $\alpha \in J$

⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha, h(\beta) \leq k(\beta)$.

⟨2⟩3. $\forall \beta < \alpha, h(\beta) < k(\alpha)$

⟨2⟩4. $h((-\infty, \alpha)) \neq E$

⟨2⟩5. $h(\alpha)$ is the least element in $E - h((-\infty, \alpha))$.

⟨2⟩6. $k(\alpha) \in E - h((-\infty, \alpha))$

⟨2⟩7. $h(\alpha) \leq k(\alpha)$

⟨1⟩6. $\forall \alpha \in J, h((-\infty, \alpha)) \neq E$

PROOF: For $\beta < \alpha$ we have $h(\beta) \leq k(\beta) < k(\alpha)$ so $k(\alpha) \notin h((-\infty, \alpha))$.

⟨1⟩7. For all $\alpha \in J$, we have $h(\alpha)$ is the least element of $E - h((-\infty, \alpha))$.

⟨1⟩8. h is strictly monotone and $h(J)$ is either E or a section of E .

PROOF: Proposition 4.3.11.

□

Proposition 5.1.6. *If A and B are well ordered sets, then exactly one of the following conditions hold: $A \cong B$, or A is isomorphic to a section of B , or B is isomorphic to a section of A .*

PROOF:

⟨1⟩1. At least one of the conditions holds.

⟨2⟩1. B is isomorphic to either $A + B$ or a section of $A + B$.

⟨2⟩2. CASE: $B \cong A + B$

⟨3⟩1. LET: ϕ be the isomorphism $B \cong A + B$

⟨3⟩2. LET: b_0 be the least element in B .

⟨3⟩3. A is isomorphic to the section $(-\infty, \phi^{-1}(\kappa_2(b_0)))$ of B .

⟨2⟩3. CASE: $a \in A$ and $B \cong (-\infty, \kappa_1(a))$

PROOF: Then B is isomorphic to the section $(-\infty, a)$ of A .

⟨2⟩4. CASE: $b \in B$ and $\phi : B \cong (-\infty, \kappa_2(b))$

⟨3⟩1. CASE: b is least in B .

PROOF: Then $A \cong B$.

⟨3⟩2. CASE: b is not least in B .

⟨4⟩1. LET: b_0 be least in B .

⟨4⟩2. A is isomorphic to the section $(-\infty, \phi^{-1}(\kappa_2(b_0)))$ of B .

⟨1⟩2. At most one of the conditions holds.

PROOF: Since a well ordered set cannot be isomorphic to a section of itself.

□

Theorem 5.1.7. *There exists a well ordered set, unique up to order isomorphism, that is uncountable but such that every section is countable.*

PROOF:

⟨1⟩1. There exists a well ordered set that is uncountable but such that every section is countable.

- $\langle 2 \rangle 1$. PICK a well ordered set A with an element $\Omega \in A$ such that $(-\infty, \Omega)$ is uncountable but $\forall \alpha < \Omega. (-\infty, \alpha)$ is countable.
 $\langle 2 \rangle 2$. LET: $(-\infty, \Omega)$ is uncountable but every section is countable.
 $\langle 1 \rangle 2$. If A and B are uncountable well ordered sets such that every section is countable, then $A \cong B$.

PROOF: Since it cannot be that one of A and B is isomorphic to a section of the other.

□

Definition 5.1.8 (Minimal Uncountable Well Ordered Set). The *minimal uncountable well ordered set* Ω is the well ordered set that is uncountable but such that every section is countable.

We write $\bar{\Omega}$ for the well ordered set $\Omega \cup \{\Omega\}$ where Ω is greatest.

Proposition 5.1.9. *Every countable subset of Ω is bounded above.*

PROOF:

- $\langle 1 \rangle 1$. LET: A be a countable subset of Ω .
 $\langle 1 \rangle 2$. For all $a \in A$ we have $(-\infty, a)$ is countable.
 $\langle 1 \rangle 3$. $\bigcup_{a \in A} (-\infty, a)$ is countable.
 $\langle 1 \rangle 4$. $\bigcup_{a \in A} (-\infty, a) \neq \Omega$
 $\langle 1 \rangle 5$. PICK $x \in \Omega - \bigcup_{a \in A} (-\infty, a)$
 $\langle 1 \rangle 6$. x is an upper bound for A .

□

Proposition 5.1.10. *Ω has no greatest element.*

PROOF: For any $\alpha \in \Omega$ we have $(-\infty, \alpha]$ is countable and hence not the whole of Ω . □

Proposition 5.1.11. *There are uncountably many elements of Ω that have no predecessor.*

PROOF:

- $\langle 1 \rangle 1$. LET: A be the set of all elements of Ω that have no predecessor.
 $\langle 1 \rangle 2$. LET: $f : A \times \mathbb{N} \rightarrow \Omega$ be the function that maps (a, n) to the n th successor of a .
 $\langle 1 \rangle 3$. f is surjective.
 $\langle 2 \rangle 1$. ASSUME: for a contradiction $x \in \Omega$ and there is no element $a \in A$ and $n \in \mathbb{N}$ such that x is the n th successor of a .
 $\langle 2 \rangle 2$. LET: x_n be the n th predecessor of x for $n \in \mathbb{N}$.
 $\langle 2 \rangle 3$. $\{x_n : n \in \mathbb{N}\}$ is a nonempty subset of Ω with no least element.
 $\langle 1 \rangle 4$. $A \times \mathbb{N}$ is uncountable.
 $\langle 1 \rangle 5$. A is uncountable.

□

Definition 5.1.12. We identify a poset (A, \leq) with the category with:

- set of objects A

- for $a, b \in A$, the set of homomorphisms is $\{x \in 1 : a \leq b\}$

Proposition 5.1.13. *A category is a poset iff, for any two objects, there exists at most one morphism between them.*

Proposition 5.1.14. *The identity morphism on an object is unique.*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{C} be a category.

$\langle 1 \rangle 2$. LET: $A \in \mathcal{C}$

$\langle 1 \rangle 3$. LET: $i, j : A \rightarrow A$ be identity morphisms on A .

$\langle 1 \rangle 4$. $i = j$

PROOF:

$$\begin{aligned} i &= i \circ j & (j \text{ is an identity on } A) \\ &= j & (i \text{ is an identity on } A) \end{aligned}$$

□

Proposition 5.1.15. *Let A be a linearly ordered set. Then A is well ordered if and only if it does not contain a subset of order type \mathbb{N}^{op} .*

PROOF:

$\langle 1 \rangle 1$. If A is well ordered then it does not contain a subset of order type \mathbb{N}^{op} .

PROOF: A subset of order type \mathbb{N}^{op} would be a subset with no least element.

$\langle 1 \rangle 2$. If A is not well ordered then it contains a subset of order type \mathbb{N}^{op} .

$\langle 2 \rangle 1$. ASSUME: A is not well ordered.

$\langle 2 \rangle 2$. PICK a nonempty subset S with no least element.

$\langle 2 \rangle 3$. PICK $a_0 \in S$

$\langle 2 \rangle 4$. Extend to a sequence (a_n) in S such that $a_{n+1} < a_n$ for all n .

$\langle 2 \rangle 5$. $\{a_n : n \in \mathbb{N}\}$ has order type \mathbb{N}^{op} .

□

Corollary 5.1.15.1. *Let A be a linearly ordered set. If every countable subset of A is well ordered, then A is well ordered.*

Definition 5.1.16. Given $f : A \rightarrow B$ and an object C , define the function $f^* : \mathcal{C}[B, C] \rightarrow \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 5.1.17. Given $f : A \rightarrow B$ and an object C , define the function $f_* : \mathcal{C}[C, A] \rightarrow \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

5.1.1 Monomorphisms

Definition 5.1.18 (Monomorphism). Let $f : A \rightarrow B$. Then f is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for any object X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

5.1.2 Epimorphisms

Definition 5.1.19 (Epimorphism). Let $f : A \rightarrow B$. Then f is *epic* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for any object X and functions $x, y : B \rightarrow X$, if $x \circ f = y \circ f$ then $x = y$.

5.1.3 Sections and Retractions

Definition 5.1.20 (Section, Retraction). Let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $rs = \text{id}_B$.

Proposition 5.1.21. *Let $f : A \rightarrow B$ and $r, s : B \rightarrow A$. If r is a retraction of f and s is a section of f then $r = s$.*

PROOF:

$$\begin{aligned}
 r &= r \text{id}_B && \text{(Unit Law)} \\
 &= rfs && (s \text{ is a section of } f) \\
 &= \text{id}_A s && (r \text{ is a retraction of } f) \\
 &= s && \text{(Unit Law)} \square
 \end{aligned}$$

Proposition 5.1.22. *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $s : B \rightarrow A$ be a section of $r : A \rightarrow B$.

$\langle 1 \rangle 2$. LET: X be an object and $x, y : X \rightarrow B$

$\langle 1 \rangle 3$. ASSUME: $s \circ x = s \circ y$

$\langle 1 \rangle 4$. $x = y$

PROOF: $x = r \circ s \circ x = r \circ s \circ y = y$.

\square

Proposition 5.1.23. *Every retraction is epic.*

PROOF: Dual. \square

5.1.4 Isomorphisms

Definition 5.1.24 (Isomorphism). A morphism $f : A \rightarrow B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$ that is both a retraction and section of f .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 5.1.25. *The inverse of an isomorphism is unique.*

PROOF: From Proposition 5.1.21. \square

Proposition 5.1.26. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $ff^{-1} = \text{id}_B$ and $f^{-1}f = \text{id}_A$. \square

Isomorphism.

Define the opposite category.

Slice categories

Definition 5.1.27. Let \mathcal{C} be a category and $B \in \mathcal{C}$. The category \mathcal{C}_B^B of objects *over and under* B is the category with:

- objects all triples (X, u, p) such that $u : B \rightarrow X$ and $p : X \rightarrow B$
- morphisms $f : (X, u, p) \rightarrow (Y, u', p')$ all morphisms $f : X \rightarrow Y$ such that $fu = u'$ and $p'f = p$.

Proposition 5.1.28.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B) \backslash \text{id}_B \cong (\mathcal{C} \backslash B) / \text{id}_B$$

$(B, \text{id}_B, \text{id}_B)$ is the zero object in \mathcal{C}_B^B .

5.1.5 Initial Objects

Definition 5.1.29 (Initial Object). An object I is *initial* iff, for any object X , there exists exactly one morphism $I \rightarrow X$.

Proposition 5.1.30. *The empty set is initial in Set.*

PROOF: For any set A , the nowhere-defined function is the unique function $\emptyset \rightarrow A$. \square

Proposition 5.1.31. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : I \rightarrow I'$ be the unique morphism $I \rightarrow I'$.

$\langle 1 \rangle 2$. LET: $i^{-1} : I' \rightarrow I$ be the unique morphism $I' \rightarrow I$.

$\langle 1 \rangle 3$. $ii^{-1} = \text{id}_{I'}$

PROOF: There is only one morphism $I' \rightarrow I'$.

$\langle 1 \rangle 4$. $i^{-1}i = \text{id}_I$

PROOF: There is only one morphism $I \rightarrow I$.

\square

5.1.6 Terminal Objects

Definition 5.1.32 (Terminal Object). An object T is *terminal* iff, for any object X , there exists exactly one morphism $X \rightarrow T$.

Proposition 5.1.33. *1 is terminal in Set.*

PROOF: For any set A , the constant function to $*$ is the only function $A \rightarrow 1$. \square

Proposition 5.1.34. *If T and T' are terminal objects, then there exists a unique isomorphism $T \cong T'$.*

PROOF: Dual to Proposition 5.1.31. \square

5.1.7 Zero Objects

Definition 5.1.35 (Zero Object). An object Z is a *zero object* iff it is an initial object and a terminal object.

Definition 5.1.36 (Zero Morphism). Let \mathcal{C} be a category with a zero object Z . Let $A, B \in \mathcal{C}$. The *zero morphism* $A \rightarrow B$ is the unique morphism $A \rightarrow Z \rightarrow B$.

Proposition 5.1.37. *There is no zero object in **Set**.*

PROOF: Since $\emptyset \not\approx 1$. \square

5.1.8 Triads

Definition 5.1.38 (Triad). Let \mathcal{C} be a category. A *triad* consists of objects X, Y, M and morphisms $\alpha : X \rightarrow M, \beta : Y \rightarrow M$. We call M the *codomain* of the triad.

5.1.9 Cotriads

Definition 5.1.39 (Cotriad). Let \mathcal{C} be a category. A *cotriad* consists of objects X, Y, W and morphisms $\xi : W \rightarrow X, \eta : W \rightarrow Y$. We call W the *domain* of the triad.

5.1.10 Pullbacks

Definition 5.1.40 (Pullback). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

is a *pullback* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$.

In this case we also say that η is the *pullback* of β along α .

Proposition 5.1.41. *If $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$ form a pullback of $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$, and $\xi' : W' \rightarrow X$ and $\eta' : W' \rightarrow Y$ also form the pullback of α and β , then there exists a unique isomorphism $\phi : W \cong W'$ such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : W \rightarrow W'$ be the unique morphism such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.

$\langle 1 \rangle 2$. LET: $\phi^{-1} : W' \rightarrow W$ be the unique morphism such that $\eta\phi^{-1} = \eta'$ and $\xi\phi^{-1} = \xi'$.

$\langle 1 \rangle 3$. $\phi\phi^{-1} = \text{id}_{W'}$

PROOF: Each is the unique $x : W' \rightarrow W'$ such that $\eta'x = \eta'$ and $\xi'x = \xi'$.

$\langle 1 \rangle 4$. $\phi^{-1}\phi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\eta x = \eta$ and $\xi x = \xi$.

□

Proposition 5.1.42. *For any morphism $h : A \rightarrow B$, the following diagram is a pullback diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: Z be an object.

$\langle 1 \rangle 2$. LET: $f : Z \rightarrow B$ and $g : Z \rightarrow A$ satisfy $\text{id}_B f = hg$

$\langle 1 \rangle 3$. $g : Z \rightarrow A$ is the unique morphism such that $\text{id}_A g = g$ and $hg = f$.

□

Proposition 5.1.43. *The pullback of an isomorphism is an isomorphism.*

PROOF:

$\langle 1 \rangle 1$. LET:

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback diagram.

$\langle 1 \rangle 2$. ASSUME: β is an isomorphism.

$\langle 1 \rangle 3$. LET: ξ^{-1} be the unique morphism $X \rightarrow W$ such that $\xi\xi^{-1} = \text{id}_X$ and $\eta\xi^{-1} = \beta^{-1}\alpha$.

PROOF: This exists since $\alpha\text{id}_X = \beta\beta^{-1}\alpha = \alpha$.

$\langle 1 \rangle 4$. $\xi^{-1}\xi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\xi x = \xi$ and $\eta x = \eta$.

□

Proposition 5.1.44. *Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w : A \rightarrow W$ be the unique morphism such that $\xi w = x$ and $\eta w = y$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ is the pullback of β and α in $\mathcal{C} \setminus A$.

PROOF:

$\langle 1 \rangle 1$. LET: $(Z, z) \in \mathcal{C} \backslash A$

$\langle 1 \rangle 2$. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle 3$. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle 4$. $hz = w$

$\langle 2 \rangle 1$. $\xi h z = \xi w$

PROOF:

$$\xi h z = f z \quad (\langle 1 \rangle 3)$$

$$= x \quad (\langle 1 \rangle 2)$$

$$= \xi w$$

$\langle 2 \rangle 2$. $\eta h z = \eta w$

PROOF: Similar.

$\langle 1 \rangle 5$. $h : (Z, z) \rightarrow (W, w)$

□

Proposition 5.1.45. Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in \mathcal{C}/A . Let

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w = x\xi : W \rightarrow A$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ form a pullback of α and β in \mathcal{C}/A .

PROOF:

$\langle 1 \rangle 1$. $\eta : (W, w) \rightarrow (Y, y)$

PROOF:

$$y\eta = m\beta\eta$$

$$= m\alpha\xi$$

$$= x\xi$$

$$= w$$

$\langle 1 \rangle 2$. LET: $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 3$. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle 4$. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle 5$. $h : (Z, z) \rightarrow (W, w)$

PROOF:

$$wh = x\xi h$$

$$= xf \quad (\langle 1 \rangle 4)$$

$$= z \quad (\langle 1 \rangle 3)$$

□

Proposition 5.1.46. In **Set**, let $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$. Let $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$ with inclusion $i : W \rightarrow X \times Y$. Let $\xi = \pi_1 i : W \rightarrow X$ and $\eta = \pi_2 i : W \rightarrow Y$. Then ξ and η form the pullback of α and β .

PROOF:

$\langle 1 \rangle 1. \alpha\xi = \beta\eta$

PROOF: For $w \in W$, if $i(w) = (x, y)$ then $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$.

$\langle 1 \rangle 2.$ For every set Z and functions $f : Z \rightarrow X, g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$

PROOF: For $z \in Z$, let $h(z)$ be the unique element of W such that $i(h(z)) = (f(z), g(z))$.

□

Pullback lemma

5.1.11 Pushouts

Definition 5.1.47 (Pushout). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array} \quad (5.1)$$

is a *pushout* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ such that $f\xi = g\eta$, there exists a unique $h : M \rightarrow Z$ such that $h\alpha = f$ and $h\beta = g$.

We also say that β is the *pushout* of ξ along η .

Proposition 5.1.48. If $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$ form a pushout of $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$, and $\alpha' : X \rightarrow M'$ and $\beta' : Y \rightarrow M'$ also form a pushout of ξ and η , then there exists a unique isomorphism $\phi : M \cong M'$ such that $\phi\alpha = \alpha'$ and $\phi\beta = \beta'$.

PROOF: Dual to Proposition 5.1.41. □

Proposition 5.1.49. For any morphism $h : A \rightarrow B$, the following diagram is a pushout diagram.

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF: Dual to Proposition 5.1.42.

Proposition 5.1.50. The diagram (5.1) is a pushout in \mathcal{C} iff it is a pullback in \mathcal{C}^{op} .

PROOF: Immediate from definitions. □

Proposition 5.1.51. The pushout of an isomorphism is an isomorphism.

PROOF: Dual to Proposition 5.1.43. \square

Proposition 5.1.52. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m := \alpha x : A \rightarrow M$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in $\mathcal{C} \setminus A$.

PROOF: Dual to Proposition 5.1.45. \square

Proposition 5.1.53. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in \mathcal{C}/A . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m : M \rightarrow A$ be the unique morphism such that $m\alpha = x$ and $m\beta = y$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in $\mathcal{C} \setminus A$.

PROOF: Dual to Proposition 5.1.44. \square

Proposition 5.1.54. *Set has pushouts.*

PROOF:

$\langle 1 \rangle 1$. LET: $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$.

$\langle 1 \rangle 2$. LET: \sim be the equivalence relation on $X + Y$ generated by $\xi(w) \sim \eta(w)$ for all $w \in W$

$\langle 1 \rangle 3$. LET: $M = (X + Y)/\sim$ with canonical projection $\pi : X + Y \rightarrow M$.

$\langle 1 \rangle 4$. LET: $\alpha = \pi \circ \kappa_1 : X \rightarrow M$

$\langle 1 \rangle 5$. LET: $\beta = \pi \circ \kappa_2 : Y \rightarrow M$

$\langle 1 \rangle 6$. LET: Z be any set, $f : X \rightarrow Z$ and $g : Y \rightarrow Z$.

$\langle 1 \rangle 7$. ASSUME: $f\xi = g\eta$

$\langle 1 \rangle 8$. LET: $h : X + Y \rightarrow Z$ be the function defined by $h(x) = f(x)$ and $h(y) = g(y)$ for $x \in X$ and $y \in Y$

$\langle 1 \rangle 9$. h respects \sim

PROOF: For $w \in W$ we have

$$h(\xi(w)) = f(\xi(w)) \quad (\langle 1 \rangle 8)$$

$$= g(\eta(w)) \quad (\langle 1 \rangle 7)$$

$$= h(\eta(w)) \quad (\langle 1 \rangle 8)$$

$\langle 1 \rangle 10$. LET: $\bar{h} : M \rightarrow Z$ be the induced function.

$\langle 1 \rangle 11$. $\bar{h}\alpha = f$

PROOF:

$$\begin{aligned}\bar{h}(\alpha(x)) &= \bar{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x)\end{aligned}$$

$\langle 1 \rangle 12.$ $\bar{h}\beta = g$

PROOF: Similar.

$\langle 1 \rangle 13.$ For all $k : M \rightarrow Z$, if $k\alpha = f$ and $k\beta = g$ then $k = \bar{h}$.

PROOF:

$$\begin{aligned}k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\ &= f(x) \\ k(\pi(\kappa_2(y))) &= k(\beta(y)) \\ &= g(y) \\ \therefore k \circ \pi &= h \\ \therefore k &= \bar{h}\end{aligned}$$

□

Definition 5.1.55. Let $u : A \rightarrow X$ be an injection. The *pointed set obtained from X by collapsing (A, u)* , denoted $X/(A, u)$, is the pushout

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ \downarrow u & & \downarrow * \\ X & \longrightarrow & X/(A, u) \end{array}$$

Proposition 5.1.56. In \mathbf{Set}_* , any two morphisms $1 \rightarrow X$ and $1 \rightarrow Y$ have a pushout.

PROOF: The pushout of $a : (1, *) \rightarrow (X, x)$ and $b : (1, *) \rightarrow (Y, y)$ is $(X+Y/\sim, x)$ where \sim is the equivalence relation generated by $x \sim y$. □

Definition 5.1.57 (Wedge). The *wedge* of pointed sets X and Y , $X \vee Y$, is the pushout of the unique morphism $1 \rightarrow X$ and $1 \rightarrow Y$.

Definition 5.1.58 (Smash). Let X and Y be pointed sets. Let $\xi : X \vee Y \rightarrow X$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc} 1 & \longrightarrow & X & & \\ \downarrow & & \downarrow & \searrow & \\ Y & \longrightarrow & X \vee Y & \xrightarrow{\xi} & X \\ & \searrow 0 & & & \end{array}$$

Let $\eta : X \vee Y \rightarrow Y$ be the unique morphism such that the following diagram

commutes.



Let $\zeta = \langle \xi, \eta \rangle : X \vee Y \rightarrow X \times Y$. The *smash* of X and Y , $X \wedge Y$, is the result of collapsing $X \times Y$ with respect to ζ .

Pushout lemma

5.1.12 Subcategories

Definition 5.1.59 (Subcategory). A *subcategory* \mathcal{C}' of a category \mathcal{C} consists of:

- a subset $\text{Ob}(\mathcal{C}')$ of \mathcal{C}
- for all $A, B \in \text{Ob}(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \text{Ob}(\mathcal{C}')$, we have $\text{id}_A \in \mathcal{C}'[A, A]$
- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \text{Ob}(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

5.1.13 Opposite Category

Definition 5.1.60 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- $\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$
- Given $f \in \mathcal{C}^{\text{op}}[A, B]$ and $g \in \mathcal{C}^{\text{op}}[B, C]$, their composite in \mathcal{C}^{op} is $f \circ g$, where \circ is composition in \mathcal{C} .

Proposition 5.1.61. An object is initial in \mathcal{C} iff it is terminal in \mathcal{C}^{op} .

PROOF: Immediate from definitions. \square

Proposition 5.1.62. An object is terminal in \mathcal{C} iff it is initial in \mathcal{C}^{op} .

PROOF: Immediate from definitions. \square

Corollary 5.1.62.1. If T and T' are terminal objects in \mathcal{C} then there exists a unique isomorphism $T \cong T'$.

5.1.14 Groupoids

Definition 5.1.63 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

5.1.15 Concrete Categories

Definition 5.1.64 (Concrete Category). A *concrete category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*
- for any object $A \in \text{Ob}(\mathcal{C})$, a set $|A|$
- for any objects $A, B \in \text{Ob}(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$
- for any object A we have $\text{id}_{|A|} \in \mathcal{C}[A, A]$.

5.1.16 Power of Categories

Definition 5.1.65. Let \mathcal{C} be a category and J a set. The category \mathcal{C}^J is the category with:

- objects all J -indexed families of objects of \mathcal{C}
- morphisms $\{X_j\}_{j \in J} \rightarrow \{Y_j\}_{j \in J}$ all families $\{f_j\}_{j \in J}$ where $f_j : X_j \rightarrow Y_j$

5.1.17 Arrow Category

Definition 5.1.66 (Arrow Category). Let \mathcal{C} be a category. The *arrow category* \mathcal{C}^\rightarrow is the category with:

- objects all triples (A, B, f) where $f : A \rightarrow B$ in \mathcal{C}
- morphisms $(A, B, f) \rightarrow (C, D, g)$ all pairs $(u : A \rightarrow C, v : B \rightarrow D)$ such that $vf = gu$.

5.1.18 Slice Category

Definition 5.1.67 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category under A* , $\mathcal{C}_{\backslash A}$, is the category with:

- objects all pairs (B, f) where $B \in \mathcal{C}$ and $f : A \rightarrow B$
- morphisms $(B, f) \rightarrow (C, g)$ are morphisms $u : B \rightarrow C$ such that $uf = g$.

We identify this with the subcategory of \mathcal{C}^\rightarrow formed by mapping (B, f) to (A, B, f) and u to (id_A, u) .

Proposition 5.1.68. *If $s : (B, f) \rightarrow (C, g)$ in $\mathcal{C} \setminus A$, then any retraction of s in \mathcal{C} is a retraction of s in $\mathcal{C} \setminus A$.*

PROOF:

$\langle 1 \rangle 1$. LET: $r : C \rightarrow B$ be a retraction of s in \mathcal{C} .

$\langle 1 \rangle 2$. $rg = f$

PROOF: $rg = rsf = f$.

$\langle 1 \rangle 3$. $r : (C, g) \rightarrow (B, f)$ in $\mathcal{C} \setminus A$

$\langle 1 \rangle 4$. $rs = \text{id}_{(B, f)}$

PROOF: Because composition is inherited from \mathcal{C} .

□

Proposition 5.1.69. *id_A is the initial object in $\mathcal{C} \setminus A$.*

PROOF: For any $(B, f) \in \mathcal{C} \setminus A$, we have f is the only morphism $A \rightarrow B$ such that $f\text{id}_A = f$. □

Proposition 5.1.70. *If A is terminal in \mathcal{C} then id_A is the zero object in $\mathcal{C} \setminus A$.*

PROOF: For any $(B, f) \in \mathcal{C} \setminus A$, the unique morphism $! : B \rightarrow A$ is the unique morphism such that $!f = \text{id}_A$. □

Definition 5.1.71 (Pointed Sets). The *category of pointed sets* is **Set** \ 1.

Definition 5.1.72. Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category over A* , \mathcal{C}/A , is the category with:

- objects all pairs (B, f) with $f : B \rightarrow A$
- morphisms $u : (B, f) \rightarrow (C, g)$ all morphisms $u : B \rightarrow C$ such that $gu = f$.

Proposition 5.1.73. *Let $u : (B, f) \rightarrow (C, g) : \mathcal{C}/A$. Any section of u in \mathcal{C} is a section of u in \mathcal{C}/A .*

PROOF: Dual to Proposition 5.1.68. □

Proposition 5.1.74. *id_A is terminal in \mathcal{C}/A .*

PROOF: Dual to Proposition 5.1.69. □

Proposition 5.1.75. *If A is initial in \mathcal{C} then id_A is the zero object in \mathcal{C}/A .*

PROOF: Dual to Proposition 5.1.70. □

Definition 5.1.76. Let $A \in \mathcal{C}$. The category of objects *over and under* A , written \mathcal{C}_A^A , is the category with:

- objects all triples (X, u, p) where $u : A \rightarrow X$, $p : X \rightarrow A$ and $pu = \text{id}_A$
- morphism $f : (X, u, p) \rightarrow (Y, v, q)$ all morphisms $f : X \rightarrow Y$ such that $fu = v$ and $qf = p$

Proposition 5.1.77. *$(A, \text{id}_A, \text{id}_A)$ is the zero object in \mathcal{C}_A^A .*

PROOF: For any object (X, u, p) , we have p is the unique morphism $(X, u, p) \rightarrow (A, \text{id}_A, \text{id}_A)$, and u is the unique morphism $(A, \text{id}_A, \text{id}_A) \rightarrow (X, u, p)$. \square

Definition 5.1.78 (Fibre Collapsing). Let B be a set. Let $u : (A, a) \rightarrow (X, x)$ in \mathbf{Set}/B . Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow u & & \downarrow j \\ X & \xrightarrow{i} & C \end{array}$$

Let $c : C \rightarrow B$ be the unique morphism such that $cj = \text{id}_B$ and $ci = x$. Then $(C, j, c) \in \mathbf{Set}_B^B$ is called the set over and under B obtained from X by *fibre collapsing* with respect to u . If (A, u) is a subset of X , we denote this set over and under B by $X/_B(A, u)$.

Definition 5.1.79 (Fibre Wedge). Let B be a small set. Let $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$. The *fibre wedge* of X and Y is the pushout of u_X and u_Y :

$$\begin{array}{ccc} B & \xrightarrow{u_X} & X \\ \downarrow u_Y & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \end{array}$$

Definition 5.1.80 (Fibre Smash). Let $X, Y \in \mathbf{Set}_B^B$. Let $\xi : X \vee_B Y \rightarrow X$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \xi & \\ & & X \end{array}$$

0

Let $\eta : X \vee_B Y \rightarrow Y$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \eta & \\ & & Y \end{array}$$

0

Let $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \rightarrow X \times Y$. The *fibre smash* of X and Y , $X \wedge_B Y$, is the result of collapsing $X \times Y$ with respect to ζ .

Proposition 5.1.81. *Set has products and coproducts.*

Proposition 5.1.82. *Let \mathcal{C} be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in \mathcal{C} and $Z \in \mathcal{C}$. Let $\coprod_{\alpha \in I} X_\alpha$ be the coproduct of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[\coprod_{\alpha \in I} X_\alpha, Z] \approx \prod_{\alpha \in I} \mathcal{C}[X_\alpha, Z] .$$

Proposition 5.1.83. *Let \mathcal{C} be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in \mathcal{C} and $Z \in \mathcal{C}$. Let $\prod_{\alpha \in I} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[Z, \prod_{\alpha \in I} X_\alpha] \approx \prod_{\alpha \in I} \mathcal{C}[Z, X_\alpha] .$$

Proposition 5.1.84. *A product in \mathcal{C} constitutes a product in \mathcal{C}/A .*

Proposition 5.1.85. *A coproduct in \mathcal{C} constitutes a product in \mathcal{C}/A .*

5.2 Functors

Definition 5.2.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- a function $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism $f : A \rightarrow B$ in \mathcal{C} , a morphism $Ff : FA \rightarrow FB$ in \mathcal{D}

such that:

- for all $A \in \text{Ob}(\mathcal{C})$ we have $F\text{id}_A = \text{id}_{FA}$
- for any morphism $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathcal{C} , we have $F(g \circ f) = Fg \circ Ff$

Proposition 5.2.2. *Functors preserve isomorphisms.*

PROOF:

$\langle 1 \rangle 1$. LET: $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

$\langle 1 \rangle 2$. LET: $f : A \cong B$ in \mathcal{C}

$\langle 1 \rangle 3$. $Ff^{-1} \circ Ff = \text{id}_{FA}$

PROOF:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\text{id}_A \\ &= \text{id}_{FA} \end{aligned}$$

$\langle 1 \rangle 4$. $Ff \circ Ff^{-1} = \text{id}_{FB}$

PROOF:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

Definition 5.2.3 (Identity Functor). For any category \mathcal{C} , the *identity* functor on \mathcal{C} is the functor $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$\begin{aligned} I_{\mathcal{C}}A &:= A & (A \in \mathcal{C}) \\ I_{\mathcal{C}}f &:= f & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

Proposition 5.2.4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $r : A \rightarrow B$ is a retraction of $s : B \rightarrow A$ in \mathcal{C} then Fr is a retraction of Fs .

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

Corollary 5.2.4.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $\phi : A \cong B$ is an isomorphism in \mathcal{C} then $F\phi : FA \cong FB$ is an isomorphism in \mathcal{D} with $(F\phi)^{-1} = F\phi^{-1}$.

Definition 5.2.5 (Composition of Functors). Given functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$, the *composite* functor $GF : \mathcal{C} \rightarrow \mathcal{E}$ is defined by

$$\begin{aligned} (GF)A &= G(FA) & (A \in \mathcal{C}) \\ (GF)f &= G(Ff) & (f : A \rightarrow B : \mathcal{C}) \end{aligned}$$

Definition 5.2.6 (Category of Categories). Let **Cat** be the category of small categories and functors.

Definition 5.2.7 (Isomorphism of Categories). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F is an *isomorphism of categories* iff there exists a functor $F^{-1} : \mathcal{D} \rightarrow \mathcal{C}$, the *inverse* of F , such that $FF^{-1} = I_{\mathcal{D}}$ and $F^{-1}F = I_{\mathcal{C}}$.

Categories \mathcal{C} and \mathcal{D} are *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, iff there exists an isomorphism between them.

Proposition 5.2.8. If A is initial in \mathcal{C} then $\mathcal{C} \setminus A \cong \mathcal{C}$.

PROOF:

⟨1⟩1. Define $F : \mathcal{C} \setminus A \rightarrow \mathcal{C}$ by

$$F(B, f) = B$$

$$F(u : (B, f) \rightarrow (C, g)) = u$$

⟨1⟩2. Define $G : \mathcal{C} \rightarrow \mathcal{C} \setminus A$ by

$$GB = (B, !_B)$$

where $!_B$ is the unique morphism $A \rightarrow B$

$$G(u : B \rightarrow C) = u : (B, !_B) \rightarrow (C, !_C)$$

⟨1⟩3. $FG = \text{id}_{\mathcal{C}}$

⟨1⟩4. $GF = \text{id}_{\mathcal{C} \setminus A}$

PROOF: Since $GF(B, f) = (B, !_B) = (B, f)$ because the morphism $A \rightarrow B$ is unique.

□

Proposition 5.2.9. *If A is terminal in \mathcal{C} then $\mathcal{C}/A \cong \mathcal{C}$.*

PROOF: Dual. \square

Proposition 5.2.10.

$$\mathcal{C}_A^A \cong (\mathcal{C}/A) \backslash (A, \text{id}_A) \cong (\mathcal{C} \backslash A) / (A, \text{id}_A)$$

PROOF:

- $\langle 1 \rangle 1$. Define a functor $F : \mathcal{C}_A^A \rightarrow (\mathcal{C}/A) \backslash (A, \text{id}_A)$.
 $\langle 2 \rangle 1$. Given $A \xrightarrow{u} X \xrightarrow{p} A$ in \mathcal{C}_A^A , let $F(X, u, p) = ((X, p), u)$
 $\langle 2 \rangle 2$. Given $f : (A \xrightarrow{u} X \xrightarrow{p} A) \rightarrow (A \xrightarrow{v} Y \xrightarrow{q} A)$, let $Ff = f$.
 $\langle 1 \rangle 2$. Define a functor $G : (\mathcal{C}/A) \backslash (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.
 $\langle 1 \rangle 3$. Define a functor $H : \mathcal{C}_A^A \rightarrow (\mathcal{C} \backslash A) / (A, \text{id}_A)$.
 $\langle 1 \rangle 4$. Define a functor $K : (\mathcal{C} \backslash A) / (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.
 \square

Definition 5.2.11 (Forgetful Functor). For any concrete category \mathcal{C} , define the *forgetful* functor $U : \mathcal{C} \rightarrow \mathbf{Set}$ by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

Definition 5.2.12 (Switching Functor). For any category \mathcal{C} , define the *switching* functor $T : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

Definition 5.2.13 (Reduction). Let $\Phi : \mathbf{Set} \rightarrow \mathbf{Set}$ be a functor. The *reduction* of Φ is the functor $\Phi^* : \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ defined by: $\Phi^*(X, a)$ is the collapse of $\Phi(X)$ with respect to $\Phi(a) : \Phi(1) \rightarrow \Phi(X)$.

Definition 5.2.14. Extend the wedge \vee to a functor $\mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ by defining, given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, then $f \vee g$ is the unique morphism that makes the following diagram commute.

$$\begin{array}{ccccc} 1 & \longrightarrow & X & & \\ \downarrow & & \downarrow & \searrow f & \\ Y & \longrightarrow & X \vee Y & & X' \\ & \searrow g & \searrow f \vee g & & \downarrow \\ & & Y' & \longrightarrow & X' \vee Y' \end{array}$$

Definition 5.2.15. Extend smash to a functor $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ as follows. Given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, let $f \wedge g : X \wedge Y \rightarrow X' \wedge Y'$ be the

unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc}
 X \vee Y & \longrightarrow & 1 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 X \times Y & \longrightarrow & X \wedge Y & & \\
 & \searrow & \downarrow & \searrow & \\
 & & X' \vee Y' & \longrightarrow & 1 \\
 & \searrow & \downarrow & \searrow & \\
 & & X' \times Y' & \longrightarrow & X' \wedge Y'
 \end{array}$$

$f \times g$

Definition 5.2.16 (Reduction). Let B be a small set. Let $\Phi_B : \mathbf{Set}/B \rightarrow \mathbf{Set}/B$ be a functor. The *reduction* of Φ_B is the functor $\Phi_B^B : \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ defined as follows.

For $(X, u : B \rightarrow X, p : X \rightarrow B) \in \mathbf{Set}_B^B$, let $\Phi_B^B(X)$ be the set over and under B obtained from $\Phi_B(X)$ by collapsing with respect to $\Phi_B(u) : \Phi_B(B) \rightarrow \Phi_B(X)$.

Definition 5.2.17. Extend \vee_B to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$.

Definition 5.2.18. Extend \wedge_B to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$.

Definition 5.2.19 (Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *faithful* iff, for any objects $A, B \in \mathcal{C}$ and morphisms $f, g : A \rightarrow B : \mathcal{C}$, if $Ff = Fg$ then $f = g$.

Definition 5.2.20 (Full). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *full* iff, for any objects $A, B \in \mathcal{C}$ and morphism $g : FA \rightarrow FB : \mathcal{D}$, there exists $f : A \rightarrow B : \mathcal{C}$ such that $Ff = g$.

Definition 5.2.21 (Fully Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *fully faithful* iff it is full and faithful.

Definition 5.2.22 (Full Embedding). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *full embedding* iff it is fully faithful and injective on objects.

5.3 Natural Transformations

Definition 5.3.1 (Natural Transformation). Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$. A *natural transformation* $\tau : F \Rightarrow G$ is a family of morphisms $\{\tau_X : FX \rightarrow GX\}_{X \in \mathcal{C}}$ such that, for every morphism $f : X \rightarrow Y : \mathcal{C}$, we have $Gf \circ \tau_X = \tau_Y \circ Ff$.

$$\begin{array}{ccc}
 FX & \xrightarrow{Ff} & FY \\
 \tau_X \downarrow & & \downarrow \tau_Y \\
 GX & \xrightarrow{Gf} & GY
 \end{array}$$

Definition 5.3.2 (Natural Isomorphism). A natural transformation $\tau : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$ is a *natural isomorphism*, $\tau : F \cong G$, iff for all $X \in \mathcal{C}$, τ_X is an isomorphism $FX \cong GX$.

Functors F and G are *naturally isomorphic*, $F \cong G$, iff there exists a natural isomorphism between them.

Definition 5.3.3 (Inverse). Let $\tau : F \cong G$. The *inverse* natural isomorphism $\tau^{-1} : G \cong F$ is defined by $(\tau^{-1})_X = \tau_X^{-1}$.

5.4 Bifunctors

Definition 5.4.1 (Commutative). A bifunctor $\square : \mathcal{C}^2 \rightarrow \mathcal{C}$ is *commutative* iff $\square \cong \square \circ T$, where $T : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ is the swap functor.

Proposition 5.4.2. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is commutative.

PROOF: Since the pushout of f and g is the pushout of g and f . \square

Proposition 5.4.3. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is commutative.

PROOF: In the diagram defining $X \wedge Y$, construct the isomorphism between the version with X and Y and the version with Y with X for every object. \square

Proposition 5.4.4. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is commutative.

Proposition 5.4.5. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is commutative.

Definition 5.4.6 (Associative). A bifunctor \square is *associative* iff $\square \circ (\square \times \text{id}) \cong \square \circ (\text{id} \times \square)$.

Proposition 5.4.7. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is associative.

PROOF: Since $X \vee (Y \vee Z)$ and $(X \vee Y) \vee Z$ are both the pushout of the unique morphisms $1 \rightarrow X$, $1 \rightarrow Y$ and $1 \rightarrow Z$. \square

Proposition 5.4.8. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is associative.

PROOF: Draw isomorphisms between the diagrams for $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$. \square

Product and coproduct are commutative and associative.

Proposition 5.4.9. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is associative.

Proposition 5.4.10. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is associative.

Proposition 5.4.11. Let \mathcal{C} be a category with binary coproducts. Let $\square : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ be a bifunctor. Then \square distributes over $+$ iff the canonical morphism

$$(X \square Z) + (Y \square Z) \rightarrow (X + Y) \square Z$$

is an isomorphism for all X, Y, Z .

Proposition 5.4.12. *In a category with binary products and binary coproducts, then \times distributes over $+$.*

Proposition 5.4.13. *In $\mathbf{Set}/*$, we have \times does not distribute over \vee .*

Proposition 5.4.14. *In $\mathbf{Set}/*$, we have \wedge distributes over \vee .*

Proposition 5.4.15. *In \mathbf{Set}/B , we have \times_B distributes over $+_B$.*

Proposition 5.4.16. *In \mathbf{Set}/B^B , we have \wedge_B distributes over \vee_B .*

5.5 Functor Categories

Definition 5.5.1 (Functor Category). Given categories \mathcal{C} and \mathcal{D} , define the *functor category* $\mathcal{C}^{\mathcal{D}}$ to be the category with objects the functors from \mathcal{D} to \mathcal{C} and morphisms the natural transformations.

Definition 5.5.2 (Yoneda Embedding). Let \mathcal{C} be a category. The *Yoneda embedding* $Y : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ is the functor that maps an object A to $\mathcal{C}[-, A]$ and morphisms similarly.

Theorem 5.5.3 (Yoneda Lemma). *Let \mathcal{C} be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\text{op}}}[\mathcal{C}[-, X], F] \cong FX$$

that maps $\tau : \mathcal{C}[-, X] \Rightarrow F$ to $\tau_X(\text{id}_X)$.

PROOF:

$\langle 1 \rangle 1$. ϕ is natural in X .

PROOF:

$\langle 2 \rangle 1$. LET: $f : X \rightarrow Y : \mathcal{C}$

$\langle 2 \rangle 2$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$. $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$

PROOF:

$$\begin{aligned} \phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\text{id}_Y \circ f) \\ &= \tau_Y(f) \\ &= \tau_Y(f \circ \text{id}_X) \\ &= Ff(\tau_X(\text{id}_X)) && (\tau \text{ natural}) \\ &= Ff(\phi(\tau)) \end{aligned}$$

$\langle 1 \rangle 2$. ϕ is natural in F .

$\langle 2 \rangle 1$. LET: $\alpha : F \Rightarrow G : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$

$\langle 2 \rangle 2$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$. $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$

PROOF: $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\text{id}_X)) = \alpha_X(\phi(\tau))$

$\langle 1 \rangle 3$. Each ϕ_{XF} is injective.

$\langle 2 \rangle 1$. LET: $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 2$. ASSUME: $\phi(\sigma) = \phi(\tau)$

$\langle 2 \rangle 3$. LET: $f : Y \rightarrow X$

$\langle 2 \rangle 4$. $\sigma_Y(f) = \tau_Y(f)$

PROOF:

$$\begin{aligned}
 \sigma_Y(f) &= \sigma_Y(\text{id}_X \circ f) \\
 &= Ff(\sigma_X(\text{id}_X)) && (\sigma \text{ is natural}) \\
 &= Ff(\tau_X(\text{id}_X)) && (\langle 2 \rangle 2) \\
 &= \tau_Y(\text{id}_X \circ f) && (\tau \text{ is natural}) \\
 &= \tau_Y(f)
 \end{aligned}$$

$\langle 1 \rangle 4$. Each ϕ_{XF} is surjective.

$\langle 2 \rangle 1$. LET: $X \in \mathcal{C}$ and $F : \mathcal{C} \rightarrow \mathcal{D}$

$\langle 2 \rangle 2$. LET: $a \in FX$

$\langle 2 \rangle 3$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$ be given by $\tau_Y(g) = Fg(a)$ for $g : Y \rightarrow X$

$\langle 2 \rangle 4$. τ is natural.

$\langle 3 \rangle 1$. LET: $h : Y \rightarrow Z : \mathcal{C}$

PROVE: $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \text{id}_X]$

$\langle 3 \rangle 2$. LET: $g : Z \rightarrow X$

$\langle 3 \rangle 3$. $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$

PROOF:

$$\begin{aligned}
 \tau_Y(g \circ h) &= F(g \circ h)(a) \\
 &= Fh(Fg(a)) \\
 &= Fh(\tau_Z(g))
 \end{aligned}$$

$\langle 2 \rangle 5$. $\phi(\tau) = a$

PROOF:

$$\begin{aligned}
 \phi_X(\tau) &= \tau_X(\text{id}_X) \\
 &= F\text{id}_X(a) \\
 &= a
 \end{aligned}$$

□

Corollary 5.5.3.1. *The Yoneda embedding is fully faithful.*

Corollary 5.5.3.2. *Given objects A and B in \mathcal{C} , we have $A \cong B$ if and only if $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$.*

Part III

Number Systems

Chapter 6

The Real Numbers

Theorem 6.0.1. *The following hold in the real numbers:*

1. $x + (y + z) = (x + y) + z$
2. $x(yz) = (xy)z$
3. $x + y = y + x$
4. $xy = yx$
5. $x + 0 = x$
6. $x1 = x$
7. $x + (-x) = 0$
8. *If $x \neq 0$ then $x \cdot (1/x) = 1$*
9. $x(y + z) = xy + xz$
10. *If $x > y$ then $x + z > y + z$.*
11. *If $x > y$ and $z > 0$ then $xz > yz$.*
12. \mathbb{R} *has the least upper bound property.*
13. *If $x < y$ then there exists z such that $x < z < y$.*

Definition 6.0.2. Given real numbers x and y with $y \neq 0$, we write x/y for xy^{-1} .

Theorem 6.0.3. *For any real numbers x and y , if $x + y = x$ then $y = 0$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 2.$ ASSUME: $x + y = x$

$\langle 1 \rangle 3.$ $y = 0$

PROOF:

$$\begin{aligned}
y &= y + 0 && \text{(Definition of zero)} \\
&= y + (x + (-x)) && \text{(Definition of } -x) \\
&= (y + x) + (-x) && \text{(Associativity of Addition)} \\
&= (x + y) + (-x) && \text{(Commutativity of Addition)} \\
&= x + (-x) && (\langle 1 \rangle 2) \\
&= 0 && \text{(Definition of } -x)
\end{aligned}$$

□

Theorem 6.0.4.

$$\forall x \in \mathbb{R}. 0x = 0$$

PROOF:

 $\langle 1 \rangle 1$. LET: $x \in \mathbb{R}$ $\langle 1 \rangle 2$. $xx + 0x = xx$

PROOF:

$$\begin{aligned}
xx + 0x &= (x + 0)x && \text{(Distributive Law)} \\
&= xx && \text{(Definition of 0)}
\end{aligned}$$

 $\langle 1 \rangle 3$. $0x = 0$ PROOF: Theorem 6.0.4, $\langle 1 \rangle 2$.

□

Theorem 6.0.5.

$$-0 = 0$$

PROOF: Since $0 + 0 = 0$. □**Theorem 6.0.6.**

$$\forall x \in \mathbb{R}. -(-x) = x$$

PROOF: Since $-x + x = 0$. □**Theorem 6.0.7.**

$$\forall x, y \in \mathbb{R}. x(-y) = -(xy)$$

PROOF:

$$\begin{aligned}
x(-y) + xy &= x((-y) + y) && \text{(Distributive Law)} \\
&= x0 && \text{(Definition of } -y) \\
&= 0 && \text{(Theorem 6.0.5) } \square
\end{aligned}$$

Theorem 6.0.8.

$$\forall x \in \mathbb{R}. (-1)x = -x$$

PROOF:

$$\begin{aligned}
(-1)x &= -(1 \cdot x) && \text{(Theorem 6.0.8)} \\
&= -x && \text{(Definition of 1) } \square
\end{aligned}$$

Proposition 6.0.9. *Let X be a linearly ordered set. Let $a, b, c \in X$ with $a < b < c$. Then $[a, c] \cong [0, 1]$ if and only if $[a, b] \cong [0, 1]$ and $[b, c] \cong [0, 1]$.*

PROOF:

$\langle 1 \rangle 1$. For all $x \in (0, 1)$ we have $[0, x] \cong [0, 1]$.

PROOF: The function that maps t to t/x is an order isomorphism.

$\langle 1 \rangle 2$. For all $x \in (0, 1)$ we have $[x, 1] \cong [0, 1]$.

PROOF: The function that maps t to $(t - x)/(1 - x)$ is an order isomorphism.

$\langle 1 \rangle 3$. We have $[0, 2] \cong [0, 1]$.

PROOF: The function that maps t to $t/2$ is an order isomorphism.

□

6.1 Subtraction

Definition 6.1.1 (Subtraction). We write $x - y$ for $x + (-y)$.

Theorem 6.1.2.

$$\forall x, y, z \in \mathbb{R}. x(y - z) = xy - xz$$

PROOF:

$$\begin{aligned} x(y - z) &= x(y + (-z)) && \text{(Definition of subtraction)} \\ &= xy + x(-z) && \text{(Distributive Law)} \\ &= xy + (-(xz)) && \text{(Theorem 6.0.8)} \\ &= xy - xz && \text{(Definition of subtraction)} \quad \square \end{aligned}$$

Theorem 6.1.3.

$$\forall x, y \in \mathbb{R}. -(x + y) = -x - y$$

PROOF:

$$\begin{aligned} -(x + y) &= (-1)(x + y) && \text{(Theorem 6.0.9)} \\ &= (-1)x + (-1)y && \text{(Distributive Law)} \\ &= -x + (-y) && \text{(Theorem 6.0.9)} \\ &= -x - y && \text{(Definition of subtraction)} \quad \square \end{aligned}$$

Theorem 6.1.4.

$$\forall x, y \in \mathbb{R}. -(x - y) = -x + y$$

PROOF:

$$\begin{aligned} -(x - y) &= -(x + (-y)) && \text{(Definition of subtraction)} \\ &= -x - (-y) && \text{(Theorem 6.1.2)} \\ &= -x + (-(-y)) && \text{(Definition of subtraction)} \\ &= -x + y && \text{(Theorem 6.0.7)} \quad \square \end{aligned}$$

Definition 6.1.5 (Reciprocal). Given $x \in \mathbb{R}$ with $x \neq 0$, the *reciprocal* of x , $1/x$, is the unique real number such that $x \cdot 1/x = 1$.

Theorem 6.1.6. *For any real numbers x and y , if $x \neq 0$ and $xy = x$ then $y = 1$.*

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 2$. ASSUME: $x \neq 0$

$\langle 1 \rangle 3$. ASSUME: $xy = x$

$\langle 1 \rangle 4$. $y = 1$

PROOF:

$$\begin{aligned}
 y &= y1 && \text{(Definition of 1)} \\
 &= y(x \cdot 1/x) && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\
 &= (yx)1/x && \text{(Associativity of Multiplication)} \\
 &= (xy)1/x && \text{(Commutativity of Multiplication)} \\
 &= x \cdot 1/x && (\langle 1 \rangle 3) \\
 &= 1 && \text{(Definition of } 1/x, \langle 1 \rangle 2)
 \end{aligned}$$

□

Definition 6.1.7 (Quotient). Given real numbers x and y with $y \neq 0$, the quotient x/y is defined by

$$x/y = x \cdot 1/y .$$

Theorem 6.1.8. *For any real number x , if $x \neq 0$ then $x/x = 1$.*

PROOF: Immediate from definitions. □

Theorem 6.1.9.

$$\forall x \in \mathbb{R}. x/1 = x$$

PROOF:

$\langle 1 \rangle 1$. LET: $x \in \mathbb{R}$

$\langle 1 \rangle 2$. $1/1 = 1$

PROOF: Since $1 \cdot 1 = 1$.

$\langle 1 \rangle 3$. $x/1 = x$

PROOF: Since $x/1 = x \cdot 1/1 = x \cdot 1 = x$.

□

Theorem 6.1.10. *For any real numbers x and y , if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.*

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 2$. ASSUME: $xy = 0$ and $x \neq 0$

PROVE: $y = 0$

$\langle 1 \rangle 3$. $y = 0$

PROOF:

$$\begin{aligned}
 y &= 1y && \text{(Definition of 1)} \\
 &= (1/x)xy && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\
 &= (1/x)0 && (\langle 1 \rangle 2) \\
 &= 0 && \text{(Theorem 6.0.5)}
 \end{aligned}$$

□

Theorem 6.1.11. *For any real numbers y and z , if $y \neq 0$ and $z \neq 0$ then $(1/y)(1/z) = 1/(yz)$.*

PROOF: Since $yz(1/y)(1/z) = 1 \cdot 1 = 1$. □

Corollary 6.1.11.1. *For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have $(x/y)(z/w) = (xz)/(yw)$.*

Theorem 6.1.12. *For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have*

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + yz}{yw}$$

PROOF:

$$\begin{aligned}
 yw \left(\frac{x}{y} + \frac{z}{w} \right) &= yw \frac{x}{y} + yw \frac{z}{w} \\
 &= wx + yz
 \end{aligned}$$

□

Theorem 6.1.13. *For any real number x , if $x \neq 0$ then $1/x \neq 0$.*

PROOF: Since $x \cdot 1/x = 1 \neq 0$. □

Theorem 6.1.14. *For any real numbers w, z , if $w \neq 0 \neq z$ then $1/(w/z) = z/w$.*

PROOF: Since $(z/w)(w/z) = (wz)/(wz) = 1$. □

Theorem 6.1.15. *For any real numbers a, x and y , if $y \neq 0$ then $(ax)/y = a(x/y)$*

PROOF: Since $ya(x/y) = ax$. □

Theorem 6.1.16. *For any real numbers x and y , if $y \neq 0$ then $(-x)/y = x/(-y) = -(x/y)$.*

PROOF:

$$\langle 1 \rangle 1. \quad (-x)/y = -(x/y)$$

PROOF: Take $a = -1$ in Theorem 6.1.14.

$$\langle 1 \rangle 2. \quad x/(-y) = -(x/y)$$

PROOF: Since $(-y)(-(x/y)) = y(x/y) = x$.

□

Theorem 6.1.17. *For any real numbers x, y, z and w , if $x > y$ and $w > z$ then $x + w > y + z$.*

PROOF: We have $y + z < x + z < x + w$ by Monotonicity of Addition twice. \square

Corollary 6.1.17.1. *For any real numbers x and y , if $x > 0$ and $y > 0$ then $x + y > 0$.*

Theorem 6.1.18. *For any real numbers x and y , if $x > 0$ and $y > 0$ then $xy > 0$.*

PROOF:

$$\begin{aligned} xy &> 0y && \text{(Monotonicity of Multiplication)} \\ &= 0 && \text{(Theorem 6.0.5)} \quad \square \end{aligned}$$

Theorem 6.1.19. *For any real number x , we have $x > 0$ iff $-x < 0$.*

PROOF:

$\langle 1 \rangle 1$. If $0 < x$ then $-x < 0$

PROOF: By Monotonicity of Addition adding $-x$ to both sides.

$\langle 1 \rangle 2$. If $-x < 0$ then $0 < x$

PROOF: By Monotonicity of Addition adding x to both sides.

\square

Theorem 6.1.20. *For any real numbers x and y , we have $x > y$ iff $-x < -y$.*

PROOF:

$\langle 1 \rangle 1$. If $y < x$ then $-x < -y$.

PROOF: By Monotonicity of Addition adding $-x - y$ to both sides.

$\langle 1 \rangle 2$. If $-x < -y$ then $y < x$.

PROOF: By Monotonicity of Addition adding $x + y$ to both sides.

\square

Theorem 6.1.21. *For any real numbers x , y and z , if $x > y$ and $z < 0$ then $xz < yz$.*

PROOF:

$\langle 1 \rangle 1$. LET: x , y and z be real numbers.

$\langle 1 \rangle 2$. ASSUME: $x > y$

$\langle 1 \rangle 3$. ASSUME: $z < 0$

$\langle 1 \rangle 4$. $-z > 0$

PROOF: Theorem 6.1.18, $\langle 1 \rangle 3$.

$\langle 1 \rangle 5$. $x(-z) > y(-z)$

PROOF: $\langle 1 \rangle 2$, $\langle 1 \rangle 4$, Monotonicity of Multiplication.

$\langle 1 \rangle 6$. $-(xz) > -(yz)$

PROOF: Theorem 6.0.8, $\langle 1 \rangle 5$.

$\langle 1 \rangle 7$. $xz < yz$

PROOF: Theorem 6.1.18, $\langle 1 \rangle 6$.

\square

Theorem 6.1.22. *For any real number x , if $x \neq 0$ then $xx > 0$.*

PROOF:

⟨1⟩1. If $x > 0$ then $xx > 0$

PROOF: By Monotonicity of Multiplication.

⟨1⟩2. If $x < 0$ then $xx > 0$

PROOF: Theorem 6.1.20.

□

Theorem 6.1.23.

$$0 < 1$$

PROOF: By Theorem 6.1.21 since $1 = 1 \cdot 1$. □

Definition 6.1.24 (Positive). A real number x is *positive* iff $x > 0$.

We write \mathbb{R}_+ for the set of positive reals.

Theorem 6.1.25. For any real numbers x and y , we have xy is positive if and only if x and y are both positive or both negative.

PROOF: By the Monotonicity of Multiplication and Theorem 6.1.20. □

Corollary 6.1.25.1. For any real number x , if $x > 0$ then $1/x > 0$.

PROOF: Since $x \cdot 1/x = 1$ is positive. □

Theorem 6.1.26. For any real numbers x and y , if $x > y > 0$ then $1/x < 1/y$.

PROOF: If $1/y \leq 1/x$ then $1 < 1$ by Monotonicity of Multiplication. □

Theorem 6.1.27. For any real numbers x and y , if $x < y$ then $x < (x+y)/2 < y$.

PROOF: We have $2x < x+y$ and $x+y < 2y$ by Monotonicity of Addition, hence $x < (x+y)/2 < y$ by Monotonicity of Multiplication since $1/2 > 0$. □

Corollary 6.1.27.1. \mathbb{R} is a linear continuum.

Definition 6.1.28 (Negative). A real number x is *negative* iff $x < 0$.

We write $\overline{\mathbb{R}_+}$ for the set of nonnegative reals.

Theorem 6.1.29. For every positive real number a , there exists a unique positive real \sqrt{a} such that $\sqrt{a}^2 = a$.

PROOF:

⟨1⟩1. LET: a be a positive real.

⟨1⟩2. For any real numbers x and h , if $0 \leq h < 1$, then

$$(x+h)^2 < x^2 + h(2x+1) .$$

⟨2⟩1. LET: x and h be real numbers.

⟨2⟩2. ASSUME: $0 \leq h < 1$

⟨2⟩3. $(x+h)^2 < x^2 + h(2x+1)$

PROOF:

$$\begin{aligned} (x+h)^2 &= x^2 + 2hx + h^2 \\ &< x^2 + 2hx + h & (\langle 2 \rangle 2) \\ &= x^2 + h(2x+1) \end{aligned}$$

⟨1⟩3. For any real numbers x and h , if $h > 0$ then
 $(x - h)^2 > x^2 - 2hx$.

⟨2⟩1. LET: x and h be real numbers.

⟨2⟩2. ASSUME: $h > 0$

⟨2⟩3. $(x - h)^2 > x^2 - 2hx$

PROOF:

$$\begin{aligned} (x - h)^2 &= x^2 - 2hx + h^2 \\ &> x^2 - 2hx \end{aligned} \quad (\langle 2 \rangle 2)$$

⟨1⟩4. For any positive real x , if $x^2 < a$ then there exists $h > 0$ such that
 $(x + h)^2 < a$.

⟨2⟩1. LET: x be a positive real.

⟨2⟩2. ASSUME: $x^2 < a$

⟨2⟩3. LET: $h = \min((a - x^2)/(2x + 1), 1/2)$

⟨2⟩4. $0 < h < 1$

⟨2⟩5. $(x + h)^2 < a$

PROOF:

$$\begin{aligned} (x + h)^2 &< x^2 + h(2x + 1) \\ &\leq a \end{aligned} \quad (\langle 1 \rangle 2)$$

⟨1⟩5. For any positive real x , if $x^2 > a$ then there exists $h > 0$ such that
 $(x - h)^2 > a$.

⟨2⟩1. LET: x be a positive real.

⟨2⟩2. ASSUME: $x^2 > a$

⟨2⟩3. LET: $h = (x^2 - a)/2x$

⟨2⟩4. $h > 0$

⟨2⟩5. $(x - h)^2 > a$

PROOF:

$$\begin{aligned} (x - h)^2 &> x^2 - 2hx \\ &= a \end{aligned} \quad (\langle 2 \rangle 3)$$

⟨1⟩6. LET: $B = \{x \in \mathbb{R} : x^2 < a\}$

⟨1⟩7. B is bounded above.

PROOF: If $a \geq 1$ then a is an upper bound. If $a < 1$ then 1 is an upper bound.

⟨1⟩8. B contains at least one positive real.

PROOF: If $a \geq 1$ then $1 \in B$. If $a < 1$ then $a \in B$.

⟨1⟩9. LET: $b = \sup B$

⟨1⟩10. $b^2 = a$

⟨2⟩1. $b^2 \geq a$

⟨3⟩1. ASSUME: for a contradiction $b^2 < a$

⟨3⟩2. PICK $h > 0$ such that $(b + h)^2 < a$

PROOF: ⟨1⟩4

⟨3⟩3. $b + h \in B$

⟨3⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩9.

⟨2⟩2. $b^2 \leq a$

⟨3⟩1. ASSUME: for a contradiction $b^2 > a$

⟨3⟩2. PICK $h > 0$ such that $(b - h)^2 > a$

PROOF: $\langle 1 \rangle 5$

$\langle 3 \rangle 3$. PICK $x \in B$ such that $b - h < x$

PROOF: $\langle 1 \rangle 9$

$\langle 3 \rangle 4$. $(b - h)^2 < x^2 < a$

$\langle 3 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 3 \rangle 2$

$\langle 1 \rangle 11$. For any positive reals b and c , if $b^2 = c^2$ then $b = c$.

$\langle 2 \rangle 1$. LET: b and c be positive reals.

$\langle 2 \rangle 2$. ASSUME: $b^2 = c^2$

$\langle 2 \rangle 3$. $b^2 - c^2 = 0$

$\langle 2 \rangle 4$. $(b - c)(b + c) = 0$

$\langle 2 \rangle 5$. $b - c = 0$ or $b + c = 0$

$\langle 2 \rangle 6$. $b + c \neq 0$

PROOF: Since $b + c > 0$

$\langle 2 \rangle 7$. $b - c = 0$

$\langle 2 \rangle 8$. $b = c$

□

Theorem 6.1.30. *The set of real numbers is uncountable.*

Definition 6.1.31. We write \mathbb{R}^ω for the set of sequences in \mathbb{R} that are eventually zero.

Definition 6.1.32 (Hilbert Cube). The *Hilbert cube* is $\prod_{n=0}^\infty [0, 1/(n+1)]$.

6.2 The Ordered Square

Definition 6.2.1 (Ordered Square). The *ordered square* I_o^2 is the set $[0, 1]^2$ under the dictionary order.

Proposition 6.2.2. *The ordered square is a linear continuum.*

PROOF:

$\langle 1 \rangle 1$. I_o^2 has the least upper bound property.

$\langle 2 \rangle 1$. LET: S be a nonempty subset of I_o^2 .

$\langle 2 \rangle 2$. LET: a be the supremum of $\pi_1(S)$

$\langle 2 \rangle 3$. CASE: $a \in \pi_1(S)$

$\langle 3 \rangle 1$. LET: b be the supremum of $\{y \in [0, 1] : (a, y) \in S\}$

$\langle 3 \rangle 2$. (a, b) is the supremum of S .

$\langle 2 \rangle 4$. CASE: $a \notin \pi_1(S)$

PROOF: $(a, 0)$ is the supremum of S .

$\langle 1 \rangle 2$. I_o^2 is dense.

$\langle 2 \rangle 1$. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$

PROVE: There exists $(x_3, y_3) \in I_o^2$ such that $(x_1, y_1) < (x_3, y_3) < (x_2, y_2)$

$\langle 2 \rangle 2$. CASE: $x_1 < x_2$

$\langle 3 \rangle 1$. PICK x_3 such that $x_1 < x_3 < x_2$

- $\langle 3 \rangle 2.$ $(x_1, y_1) < (x_3, 0) < (x_2, y_2)$
 $\langle 2 \rangle 3.$ CASE: $x_1 = x_2$ and $y_1 < y_2$
 $\langle 3 \rangle 1.$ PICK y_3 such that $y_1 < y_3 < y_2$
 $\langle 3 \rangle 2.$ $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

6.3 Punctured Euclidean Space

Definition 6.3.1 (Punctured Euclidean Space). Let n be a positive integer. The *punctured Euclidean space* is $\mathbb{R}^n - \{\vec{0}\}$.

6.4 Topologist's Sine Curve

Definition 6.4.1 (Topologist's Sine Curve). The *topologist's sine curve* is

$$(\{0\} \times [-1, 1]) \cup \{(x, \sin 1/x) : 0 < x \leq 1\} .$$

6.5 The Long Line

Definition 6.5.1 (Long Line). The *long line* is $S_\Omega \times [0, 1)$ in the dictionary order.

Chapter 7

Integers and Rationals

7.1 Positive Integers

Definition 7.1.1 (Inductive). A set of real numbers A is *inductive* iff $1 \in A$ and $\forall x \in A. x + 1 \in A$.

Definition 7.1.2 (Positive Integer). The set \mathbb{Z}_+ of *positive integers* is the intersection of the set of inductive sets.

Proposition 7.1.3. *Every positive integer is positive.*

PROOF: The set of positive reals is inductive. \square

Proposition 7.1.4. *1 is the least element of \mathbb{Z}_+ .*

PROOF: Since $\{x \in \mathbb{R} : x \geq 1\}$ is inductive. \square

Proposition 7.1.5. *\mathbb{Z}_+ is inductive.*

PROOF: 1 is an element of every inductive set, and for all $x \in \mathbb{R}$, if x is an element of every inductive set then so is $x + 1$. \square

Theorem 7.1.6 (Principle of Induction). *If A is an inductive set of positive integers then $A = \mathbb{Z}_+$.*

PROOF: Immediate from definitions. \square

Theorem 7.1.7 (Well-Ordering Property). *\mathbb{Z}_+ is well ordered.*

PROOF: Construct the obvious order isomorphism $\omega \cong \mathbb{Z}_+$. \square

Theorem 7.1.8 (Archimedean Ordering Property). *The set \mathbb{Z}_+ is unbounded above.*

PROOF:

$\langle 1 \rangle$ 1. ASSUME: for a contradiction \mathbb{Z}_+ is bounded above.

⟨1⟩2. LET:

$$s = \sup \mathbb{Z}_+$$

⟨1⟩3. PICK $n \in \mathbb{Z}_+$ such that $s - 1 < n$

⟨1⟩4. $s < n + 1$

⟨1⟩5. Q.E.D.

PROOF: ⟨1⟩2 and ⟨1⟩4 form a contradiction.

□

7.1.1 Exponentiation

Definition 7.1.9. For a a real number and n a positive integer, define the real number a^n recursively as follows:

$$\begin{aligned} a^1 &= a \\ a^{n+1} &= a^n a \end{aligned}$$

Theorem 7.1.10. For all $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}_+$, we have

$$a^n a^m = a^{n+m}$$

PROOF:

⟨1⟩1. LET: $P(m)$ be the property $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. a^n a^m = a^{n+m}$

⟨1⟩2. $P(1)$

PROOF: $a^n a^1 = a^n a = a^{n+1}$.

⟨1⟩3. $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$

⟨2⟩1. LET: m be a positive integer.

⟨2⟩2. ASSUME: $P(m)$

⟨2⟩3. LET: $a \in \mathbb{R}$

⟨2⟩4. LET: $n \in \mathbb{Z}_+$

⟨2⟩5. $a^n a^{m+1} = a^{n+m+1}$

PROOF:

$$\begin{aligned} a^n a^{m+1} &= a^n a^m a \\ &= a^{n+m} a && (\langle 2 \rangle 2) \\ &= a^{n+m+1} \end{aligned}$$

⟨1⟩4. Q.E.D.

PROOF: By induction.

□

Theorem 7.1.11. For all $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}_+$,

$$(a^n)^m = a^{nm}.$$

PROOF:

⟨1⟩1. LET: $P(m)$ be the property $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. (a^n)^m = a^{nm}$.

⟨1⟩2. $P(1)$

PROOF: $(a^n)^1 = a^n = a^{n \cdot 1}$

⟨1⟩3. $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$

PROOF:

$$\begin{aligned} (a^n)^{m+1} &= (a^n)^m a^n \\ &= a^{nm} a^n \\ &= a^{nm+n} && (\text{Theorem 7.1.10}) \\ &= a^{n(m+1)} \end{aligned}$$

□

Theorem 7.1.12. *For any real numbers a and b and positive integer m ,*

$$a^m b^m = (ab)^m .$$

PROOF: Induction on m . □

7.2 Integers

Definition 7.2.1 (Integer). The set \mathbb{Z} of *integers* is

$$\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \{-x : x \in \mathbb{Z}_+\} .$$

Proposition 7.2.2. *The sum, difference and product of two integers is an integer.*

PROOF: Easy. □

Example 7.2.3. $1/2$ is not an integer.

Proposition 7.2.4. *For any integer n , there is no integer a such that $n < a < n+1$.*

PROOF:

⟨1⟩1. For any positive integer n , there is no integer a such that $n < a < n+1$.

⟨2⟩1. There is no integer a such that $1 < a < 2$.

⟨3⟩1. There is no positive integer a such that $1 < a < 2$.

⟨4⟩1. We do not have $1 < 1 < 2$.

⟨4⟩2. For any positive integer n , we do not have $1 < n+1 < 2$.

PROOF: Since $n \geq 1$ so $n+1 \geq 2$.

⟨3⟩2. We do not have $1 < 0 < 2$.

⟨3⟩3. For any positive integer a , we do not have $1 < -a < 2$.

PROOF: Since $-a < 0 < 1$.

⟨2⟩2. For any positive integer n , if there is no integer a such that $n < a < n+1$, then there is no integer a such that $n+1 < a < n+2$.

PROOF: If $n+1 < a < n+2$ then $n < a-1 < n+1$.

⟨1⟩2. There is no integer a such that $0 < a < 1$.

PROOF: If $0 < a < 1$ then $1 < a+1 < 2$.

⟨1⟩3. For any positive integer n , there is no integer a such that $-n < a < -n+1$.

PROOF: If $-n < a < -n+1$ then $n-1 < -a < n$.

□

Theorem 7.2.5. *Every nonempty subset of \mathbb{Z} bounded above has a largest element.*

PROOF:

⟨1⟩1. LET: S be a nonempty subset of \mathbb{Z} bounded above.

⟨1⟩2. LET: u be an upper bound for S .

⟨1⟩3. PICK an integer $n > u$

PROOF: Archimedean property.

⟨1⟩4. LET: k be the least positive integer such that $n - k \in S$.

⟨2⟩1. PICK $m \in S$

⟨2⟩2. $n - m$ is a positive integer.

⟨2⟩3. There exists a positive integer k such that $n - k \in S$.

⟨1⟩5. $n - k$ is the greatest element in S .

⟨2⟩1. LET: $m \in S$

⟨2⟩2. $n - m \geq k$

⟨2⟩3. $m \leq n - k$

□

Theorem 7.2.6. *For any real number x , if x is not an integer then there exists a unique integer n such that $n < x < n + 1$.*

PROOF:

⟨1⟩1. $\{n \in \mathbb{Z} : n < x\}$ is a nonempty set of integers bounded above.

⟨2⟩1. PICK $m > -x$

PROOF: Archimedean property.

⟨2⟩2. $-m < x$

⟨2⟩3. $\{n \in \mathbb{Z} : n < x\}$ is nonempty.

⟨1⟩2. LET: n be the greatest integer such that $n < x$

⟨1⟩3. $x < n + 1$

⟨1⟩4. If n' is an integer with $n' < x < n' + 1$ then $n' = n$.

PROOF: We have $n' < n + 1$ so $n' \leq n$, and $n < n' + 1$ so $n \leq n'$.

□

Definition 7.2.7 (Even). An integer n is *even* iff $n/2$ is an integer; otherwise, n is *odd*.

Theorem 7.2.8. *If the integer m is odd then there exists an integer n such that $m = 2n + 1$.*

PROOF:

⟨1⟩1. LET: n be the integer such that $n < m/2 < n + 1$

PROOF: Theorem 7.2.6.

⟨1⟩2. $2n < m < 2n + 2$

⟨1⟩3. $m = 2n + 1$

□

Theorem 7.2.9. *The product of two odd integers is odd.*

PROOF: $(2m + 1)(2n + 1) = 2(2mn + m + n) + 1$. \square

Corollary 7.2.9.1. *If p is an odd integer and n is a positive integer then p^n is an odd integer.*

Definition 7.2.10 (Exponentiation). Extend the definition of exponentiation so a^n is defined for:

- all real numbers a and non-negative integers n
- all non-zero real numbers a and integers n

as follows:

$$\begin{aligned} a^0 &= 1 \\ a^{-n} &= 1/a^n \end{aligned} \quad (n \text{ a positive integer})$$

Theorem 7.2.11 (Laws of Exponents). *For all non-zero reals a and b and integers m and n ,*

$$\begin{aligned} a^n a^m &= a^{n+m} \\ (a^n)^m &= a^{nm} \\ a^m b^m &= (ab)^m \end{aligned}$$

PROOF: Easy. \square

Theorem 7.2.12. \mathbb{Z} is countable.

PROOF: The function that maps an integer n to $2n$ if $n \geq 0$ and $-1 - 2n$ if $n < 0$ is a bijection $\mathbb{Z} \approx \mathbb{N}$. \square

7.3 Rational Numbers

Definition 7.3.1 (Rational Number). The set \mathbb{Q} of *rational numbers* is the set of all real numbers that are the quotient of two integers. A real that is not rational is *irrational*.

Theorem 7.3.2. $\sqrt{2}$ is irrational.

PROOF:

- $\langle 1 \rangle$ 1. For any positive rational a , there exist positive integers m and n not both even such that $a = m/n$.
- $\langle 2 \rangle$ 1. LET: a be a positive rational.
- $\langle 2 \rangle$ 2. LET: n be the least positive integer such that na is a positive integer.
- $\langle 2 \rangle$ 3. LET: $m = na$
- $\langle 2 \rangle$ 4. ASSUME: for a contradiction m and n are both even.
- $\langle 2 \rangle$ 5. $m/2 = (n/2)a$
- $\langle 2 \rangle$ 6. Q.E.D.

PROOF: This contradicts the leastness of n ($\langle 2 \rangle 2$).

$\langle 1 \rangle 2$. ASSUME: for a contradiction $\sqrt{2}$ is rational.

$\langle 1 \rangle 3$. PICK positive integers m and n not both even such that $\sqrt{2} = m/n$.

$\langle 1 \rangle 4$. $m^2 = 2n^2$

$\langle 1 \rangle 5$. m^2 is even.

$\langle 1 \rangle 6$. m is even.

PROOF: Theorem 7.2.9.

$\langle 1 \rangle 7$. LET: $k = m/2$

$\langle 1 \rangle 8$. $4k^2 = 2n^2$

$\langle 1 \rangle 9$. $n^2 = 2k^2$

$\langle 1 \rangle 10$. n^2 is even.

$\langle 1 \rangle 11$. n is even.

PROOF: Theorem 7.2.9.

$\langle 1 \rangle 12$. Q.E.D.

PROOF: $\langle 1 \rangle 3$, $\langle 1 \rangle 6$ and $\langle 1 \rangle 11$ form a contradiction.

□

Theorem 7.3.3. \mathbb{Q} is countably infinite.

PROOF: The function $\mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$ that maps (m, n) to $m/(n+1)$ is a surjection.

□

7.4 Algebraic Numbers

Definition 7.4.1 (Algebraic Number). A real number r is *algebraic* iff there exists a natural number n and rational numbers a_0, a_1, \dots, a_{n-1} such that

$$r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0$$

Otherwise, r is *transcendental*.

Proposition 7.4.2. The set of algebraic numbers is countably infinite.

PROOF: There are countably many finite sequences of rational numbers, and each corresponding polynomial has only finitely many roots. □

Corollary 7.4.2.1. The set of transcendental numbers is uncountable.

Part IV

Algebra

Chapter 8

Monoid Theory

Definition 8.0.1 (Monoid). A *monoid* is a category with one object.

Definition 8.0.2. Let \mathcal{C} be a category and $X \in \mathcal{C}$. The monoid $\text{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \rightarrow X$ under composition.

Proposition 8.0.3. *For any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and $X \in \mathcal{C}$, we have that $F : \text{End}_{\mathcal{C}}(X) \rightarrow \text{End}_{\mathcal{D}}(FX)$ is a monoid homomorphism.*

PROOF: Since $F\text{id}_X = \text{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$. \square

Chapter 9

Group Theory

9.1 Category of Small Groups

Definition 9.1.1. Let **Grp** be the category of small groups and group homomorphisms.

Definition 9.1.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G .

Proposition 9.1.3. *The trivial group is a zero object in **Grp**.*

PROOF: Easy. \square

The zero morphism $G \rightarrow H$ maps every element in G to e .

Definition 9.1.4. Let \mathcal{C} be a category and $X \in \mathcal{C}$. We write $\text{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Proposition 9.1.5. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $X \in \mathcal{C}$. Then $F : \text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(FX)$ is a group homomorphism.*

PROOF: Since $F \text{id}_X = \text{id}_{FX}$, $F(g \circ f) = Fg \circ Ff$, and $Ff^{-1} = (Ff)^{-1}$. \square

Proposition 9.1.6. **Grp** has products.

Definition 9.1.7 (Free Product). The product of a family of groups in **Grp** is called the *free product*.

Proposition 9.1.8. **Ab** has products given by direct sums.

Definition 9.1.9 (Left Coset). Let G be a group and H a subgroup of G . The *left cosets* of H are the sets of the form

$$xH := \{xh : h \in H\}$$

We write G/H for the set of left cosets of H in G .

Proposition 9.1.10. *Let G be a group and H a subgroup of G . Then G/H is a partition of G .*

PROOF:

$\langle 1 \rangle 1$. $\bigcup (G/H) = G$

PROOF: Since $x = xe$ and so $x \in xH$.

$\langle 1 \rangle 2$. Any two distinct left cosets of H are disjoint.

PROOF: Since if $z \in xH$ and $z \in yH$ then $xH = yH = zH$.

□

Definition 9.1.11. Let G be a group. Let A and B be subsets of G . Then

$$AB := \{ab : a \in A, b \in B\} .$$

Definition 9.1.12. Let G be a group. Let A be a subset of G . Then

$$A^{-1} := \{a^{-1} : a \in A\} .$$

Chapter 10

Ring Theory

Definition 10.0.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Definition 10.0.2 (Spectrum). Let R be a commutative ring. The *spectrum* of R , $\text{spec } R$, is the set of all prime ideals of R .

Definition 10.0.3 (Zariski Topology). Let R be a commutative ring. The *Zariski topology* on $\text{spec } R$ is the topology where the closed sets are the sets of the form

$$VE := \{p \in \text{spec } R : E \subseteq p\}$$

for any $E \in \mathcal{P}R$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$

$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{A} \in \mathcal{C}$

$\langle 2 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{C}$

$\langle 2 \rangle 2$. LET: $E = \bigcup \{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$

PROVE: $VE = \bigcap \mathcal{A}$

$\langle 2 \rangle 3$. For all $p \in \text{spec } R$, if $E \subseteq p$ then $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 1$. LET: $p \in \text{spec } R$

$\langle 3 \rangle 2$. ASSUME: $E \subseteq p$

$\langle 3 \rangle 3$. LET: $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$

$\langle 3 \rangle 4$. $E' \subseteq E$

$\langle 3 \rangle 5$. $E' \subseteq p$

$\langle 3 \rangle 6$. $p \in VE'$

$\langle 2 \rangle 4$. For all $p \in \text{spec } R$, if $p \in \bigcap \mathcal{A}$ then $E \subseteq p$

$\langle 3 \rangle 1$. LET: $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 2$. For all $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$ we have $E' \subseteq p$

$\langle 3 \rangle 3$. $E \subseteq p$

$\langle 1 \rangle 3$. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

PROOF: Since $VE \cup VE' = V(E \cap E')$

$\langle 1 \rangle 4. \emptyset \in \mathcal{C}$

$\langle 2 \rangle 1. VR = \emptyset$

PROOF: If $p \in VR$ then $R \subseteq p$ contradicting the fact that p is a prime ideal.

□

Definition 10.0.4. For any ring R , let $R - \mathbf{Mod}$ be the category of small R -modules and R -module homomorphisms.

Proposition 10.0.5. $R - \mathbf{Mod}$ has products and coproducts.

Chapter 11

Field Theory

Proposition 11.0.1. *Field does not have binary products.*

PROOF: There cannot be a field K with field homomorphisms $K \rightarrow \mathbb{Z}_2$ and $K \rightarrow \mathbb{Z}_3$, because its characteristic would be both 2 and 3. \square

Chapter 12

Linear Algebra

Definition 12.0.1 (Span). Let V be a vector space and $A \subseteq V$. The *span* of A is the set of all linear combinations of elements of A .

Definition 12.0.2 (Independent). Let V be a vector space and $A \subseteq V$. Then A is *linearly independent* iff, whenever

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

where $v_1, \dots, v_n \in A$, then

$$\alpha_1 = \cdots = \alpha_n = 0 .$$

Proposition 12.0.3. *Let V be a vector space, $A \subseteq V$ and $v \in V$. If A is linearly independent and $v \notin \text{span } A$, then $A \cup \{v\}$ is independent.*

PROOF:

$\langle 1 \rangle 1$. LET: $\alpha_1 v_1 + \cdots + \alpha_n v_n + \beta v = 0$ where $v_1, \dots, v_n \in A$

$\langle 1 \rangle 2$. $\beta = 0$

PROOF: Otherwise $v = (\alpha_1/\beta)v_1 + \cdots + (\alpha_n/\beta)v_n \in \text{span } A$.

$\langle 1 \rangle 3$. $\alpha_1 = \cdots = \alpha_n = 0$

PROOF: Since A is linearly independent.

□

Theorem 12.0.4. *Every vector space has a basis.*

PROOF:

$\langle 1 \rangle 1$. LET: V be a vector space.

$\langle 1 \rangle 2$. PICK a maximal linearly independent set \mathcal{B} .

PROOF: By Tukey's Lemma.

$\langle 1 \rangle 3$. $\text{span } \mathcal{B} = V$

PROOF: Proposition 12.0.3.

□

Definition 12.0.5. For any field K , we write \mathbf{Vect}_K for $K - \mathbf{Mod}$.

Dual space functor $\mathbf{Vect}_K^{\text{op}} \rightarrow \mathbf{Vect}_K$.

Part V

Topology

Chapter 13

Topology

13.1 Topological Spaces

Definition 13.1.1 (Topological Space). Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say (X, \mathcal{O}) is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call \mathcal{O} the *topology* of the topological space, and call its elements *open* sets. We shall often write X for the topological space (X, \mathcal{O}) .

Definition 13.1.2 (Discrete Topology). For any set X , the power set $\mathcal{P}X$ is called the *discrete* topology on X .

Proposition 13.1.3. *For any set X , the discrete topology on X is a topology on X .*

Definition 13.1.4 (Indiscrete Topology). For any set X , the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Proposition 13.1.5. *For any set X , the indiscrete topology on X is a topology on X .*

Definition 13.1.6 (Cofinite Topology). For any set X , the *cofinite* topology is $\{X - U : U \subseteq X \text{ is finite}\}$.

Definition 13.1.7 (Cocountable Topology). For any set X , the *cocountable* topology is $\{X - U : U \subseteq X \text{ is countable}\}$.

Definition 13.1.8 (Sierpiński Two-Point Space). The *Sierpiński two-point space* is $\{0, 1\}$ under the topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Proposition 13.1.9. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

Proposition 13.1.10. *The intersection of a set of topologies on a set X is a topology on X .*

Definition 13.1.11 (Closed Set). *Let X be a topological space and $A \subseteq X$. Then A is closed iff $X - A$ is open.*

Proposition 13.1.12. *A set B is open if and only if $X - B$ is closed.*

Proposition 13.1.13. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that \mathcal{C} is the set of closed sets if and only if:*

- For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$
- For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.
- $\emptyset \in \mathcal{C}$

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Theorem 13.1.14. *Let X be a set. Let $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology on X such that \mathcal{C} is the set of closed sets if and only if:*

1. $\emptyset \in \mathcal{C}$
2. $\forall \mathcal{A} \subseteq \mathcal{C}. \bigcap \mathcal{A} \in \mathcal{C}$
3. $\forall C, D \in \mathcal{C}. C \cup D \in \mathcal{C}$

In this case, the topology is unique, and is $\{X - C : C \in \mathcal{C}\}$.

PROOF: Straightforward.

Theorem 13.1.15. *There are infinitely many primes.*

Furstenberg's proof:

PROOF:

- $\langle 1 \rangle 1$. For $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$,
 LET: $S(a, b) := \{an + b : n \in \mathbb{N}\}$
- $\langle 1 \rangle 2$. LET: \mathcal{T} be the topology generated by the basis $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$
- $\langle 2 \rangle 1$. For every $n \in \mathbb{Z}$, there exist a, b such that $n \in S(a, b)$.
 PROOF: $n \in S(n, 0)$
- $\langle 2 \rangle 2$. If $n \in S(a_1, b_1) \cap S(a_2, b_2)$ then there exist a_3, b_3 such that $n \in S(a_3, b_3) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$
- $\langle 3 \rangle 1$. LET: $d = \text{lcm}(a_1, a_2)$
 PROVE: $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$
- $\langle 3 \rangle 2$. LET: $d = a_1k = a_2l$
- $\langle 3 \rangle 3$. LET: $n = a_1c + b_1 = a_2d + b_2$
- $\langle 3 \rangle 4$. LET: $z \in \mathbb{Z}$
 PROVE: $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$

⟨3⟩5. $dz + n \in S(a_1, b_1)$

PROOF:

$$\begin{aligned} dz + n &= a_1 kz + a_1 c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

⟨3⟩6. $dz + n \in S(a_2, b_2)$

PROOF: Similar.

⟨1⟩3. For all $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$ we have $S(a, b)$ is closed.

⟨2⟩1. LET: $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$

⟨2⟩2. LET: $n \in \mathbb{Z} - S(a, b)$

⟨2⟩3. $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$

⟨3⟩1. LET: $x \in S(a, n)$

⟨3⟩2. ASSUME: for a contradiction $x \in S(a, b)$

⟨3⟩3. PICK m such that $x = am + b$

⟨3⟩4. PICK l such that $x = al + n$

⟨3⟩5. $n = a(m - l) + b$

⟨3⟩6. $n \in S(a, b)$

⟨3⟩7. Q.E.D.

PROOF: This contradicts ⟨2⟩2.

⟨1⟩4.

$$\mathbb{Z} - \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$$

PROOF: Since every integer except 1 and -1 is divisible by a prime.

⟨1⟩5. No nonempty finite set is open.

⟨2⟩1. LET: U be a nonempty open set

⟨2⟩2. PICK $n \in U$

⟨2⟩3. There exist a, b such that $n \in S(a, b) \subseteq U$

⟨2⟩4. U is infinite.

⟨1⟩6. $\mathbb{Z} - \{1, -1\}$ is not closed.

⟨1⟩7. $\bigcup_{p \text{ prime}} S(p, 0)$ is not closed.

⟨1⟩8. The union of finitely many closed sets is closed.

⟨1⟩9. There are infinitely many primes.

□

Proposition 13.1.16. *In a discrete topological space, every set is closed.*

PROOF: Immediate from definitions. □

Proposition 13.1.17. *In a linearly ordered set under the order topology, every closed interval and closed ray is closed.*

PROOF:

⟨1⟩1. LET: X be a linearly ordered set under the order topology.

⟨1⟩2. Every closed interval in X is closed.

PROOF: Since $X - [a, b] = (-\infty, a) \cup (b, +\infty)$.

⟨1⟩3. Every closed ray in X is closed.

PROOF: Since $X - [a, +\infty) = (-\infty, a)$ and $X - (-\infty, a] = (a, +\infty)$.

□

Proposition 13.1.18. *Let X be a topological space and Y a subspace of X . Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set B in X such that $A = B \cap Y$.*

PROOF:

$$\begin{aligned}
 A \text{ is closed in } Y &\Leftrightarrow Y - A \text{ is open in } Y \\
 &\Leftrightarrow \exists U \text{ open in } X. Y - A = U \cap Y \\
 &\Leftrightarrow \exists C \text{ closed in } X. Y - A = Y - C \\
 &\Leftrightarrow \exists C \text{ closed in } X. A = Y \cap C \quad \square
 \end{aligned}$$

Proposition 13.1.19. *Let X be a topological space and Y a subspace of X . Let $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X .*

PROOF:

$\langle 1 \rangle 1$. PICK C closed in X such that $A = C \cap Y$.

$\langle 1 \rangle 2$. A is closed in X .

PROOF: It is the intersection of two closed sets in X .

\square

Definition 13.1.20 (Neighbourhood). Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x , and x is an *interior* point of U , iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 13.1.21. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 13.1.22. *Let X be a set and $\mathcal{N} : X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that, for all $x \in X$, we have \mathcal{N}_x is the set of neighbourhoods of x , if and only if:*

- For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$
- For all $x \in X$ we have $X \in \mathcal{N}_x$
- For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$
- For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$
- For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M. M \in \mathcal{N}_y$.

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$.

Definition 13.1.23 (Exterior Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x .

Definition 13.1.24 (Boundary Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B .

Definition 13.1.25 (Interior). Let X be a topological space and $B \subseteq X$. The *interior* of B , B° , is the set of all interior points of B .

Proposition 13.1.26. *The interior of B is the union of all the open sets included in B .*

Definition 13.1.27 (Closure). Let X be a topological space and $B \subseteq X$. The *closure* of B , \overline{B} , is the set of all points that are not exterior points of B .

Proposition 13.1.28. *The closure of B is the intersection of all the closed sets that include B .*

Proposition 13.1.29. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 13.1.30 (Kuratowski Closure Axioms). *Let X be a set and $- : \mathcal{P}X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} such that, for all $B \subseteq X$, \overline{B} is the closure of B , if and only if:*

- $\overline{\emptyset} = \emptyset$
- For all $A \subseteq X$ we have $A \subseteq \overline{A}$
- For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$
- For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$

In this case, \mathcal{O} is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

Definition 13.1.31 (Finer, Coarser). Let \mathcal{T} and \mathcal{T}' be topologies on the set X . Then \mathcal{T} is *coarser*, *smaller* or *weaker* than \mathcal{T}' , or \mathcal{T}' is *finer*, *larger* or *stronger* than \mathcal{T} , iff $\mathcal{T} \subseteq \mathcal{T}'$.

13.2 Bases

Definition 13.2.1 (Basis). Let X be a topological space. A *basis* for the topology on X is a set of open sets \mathcal{B} such that every open set is the union of a subset of \mathcal{B} . The elements of \mathcal{B} are called *basic open neighbourhoods* of their elements.

Proposition 13.2.2. *Let X be a set. The set of all one-element subsets of X is a basis for the discrete topology on X .*

Proposition 13.2.3. *Let X be a topological space. Let \mathcal{B} be a basis for the topology on X . Then the topology on X is the coarsest topology that includes \mathcal{B} .*

Proposition 13.2.4. *Let X and Y be topological spaces. Let \mathcal{B} be a basis for the topology on X and \mathcal{C} a basis for the topology on Y . Then*

$$\{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

13.3 Order Topology

Definition 13.3.1 (Order Topology). Let X be a linearly ordered set. The *order topology* on X is the topology generated by the open interval (a, b) as well as the open rays $(a, +\infty)$ and $(-\infty, b)$ for $a, b \in X$.

The *standard topology* on \mathbb{R} is the order topology.

Proposition 13.3.2. *Let X be a linearly ordered set. Then the order topology is generated by the basis consisting of:*

- all open intervals (a, b)
- all intervals of the form $[\perp, b)$ where \perp is the least element of X , if any
- all intervals of the form $(a, \top]$ where \top is the greatest element of X , if any.

Proposition 13.3.3. *Let X be a linearly ordered set. The open rays in X form a subbasis for the order topology.*

Definition 13.3.4 (Lower Limit Topology). The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology on \mathbb{R} generated by the basis consisting of all half-open intervals $[a, b)$.

We write \mathbb{R}_l for \mathbb{R} under the lower limit topology.

Definition 13.3.5 (K -topology). Let $K = \{1/n : n \in \mathbb{Z}_+\}$. The K -topology on \mathbb{R} is the topology generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) - K$.

We write \mathbb{R}_K for \mathbb{R} under the K -topology.

Proposition 13.3.6. *Let X be a linearly ordered set under the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology.*

PROOF:

⟨1⟩1. The order topology is coarser than the subspace topology.

⟨2⟩1. For all $a \in Y$, the open ray $\{y \in Y : a < y\}$ is open in the subspace topology.

PROOF: It is $(a, +\infty) \cap Y$.

⟨2⟩2. For all $a \in Y$, the open ray $\{y \in Y : y < a\}$ is open in the subspace topology.

PROOF: It is $(-\infty, a) \cap Y$.

⟨1⟩2. The subspace topology is coarser than the order topology.

⟨2⟩1. For all $a \in X$, the set $(-\infty, a) \cap Y$ is open in the order topology.

⟨3⟩1. CASE: $a \in Y$

PROOF: Then $(-\infty, a) \cap Y = \{y \in Y : y < a\}$ is an open ray in Y .

⟨3⟩2. CASE: a is an upper bound for Y

PROOF: Then $(-\infty, a) \cap Y = Y$.

⟨3⟩3. CASE: a is a lower bound for Y

PROOF: Then $(-\infty, a) \cap Y = \emptyset$.

$\langle 3 \rangle 4$. Q.E.D.

PROOF: These are the only three cases because Y is convex.

$\langle 2 \rangle 2$. For all $a \in X$, the set $(a, +\infty) \cap Y$ is open in the order topology.

PROOF: Similar.

□

Example 13.3.7. We cannot remove the hypothesis that the set Y is convex.

Let $X = \mathbb{R}$ and $Y = [0, 1) \cup \{2\}$. Then $\{2\}$ is open in the subspace topology but not in the order topology on Y .

Proposition 13.3.8. Let X be a topological space. Let \mathcal{B} be a basis for the topology on X and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proposition 13.3.9. Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Assume that, for every open set U and element $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then \mathcal{B} is a basis for the topology on X .

Proposition 13.3.10. Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is a basis for a topology on X if and only if:

1. $\bigcup \mathcal{B} = X$
2. For all $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

In this case, the topology is unique and is the set of all unions of subsets of \mathcal{B} . We call it the topology generated by \mathcal{B} .

Proposition 13.3.11. Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' , respectively, on X . Then \mathcal{T}' is finer than \mathcal{T} if and only if, for every $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Corollary 13.3.11.1. The topologies of \mathbb{R}_l and \mathbb{R}_K are strictly finer than the standard topology on \mathbb{R} but are not comparable to one another.

13.3.1 Subspaces

Proposition 13.3.12. Let X be a topological space. Let Y be a subspace of X . Let \mathcal{B} be a basis for the topology on X . Then $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for the topology on Y .

PROOF:

$\langle 1 \rangle 1$. For all $B \in \mathcal{B}$ we have $B \cap Y$ is open in Y .

PROOF: Since B is open in X .

$\langle 1 \rangle 2$. For any open set V in Y and $y \in V$, there exists $B \in \mathcal{B}$ such that $y \in B \cap Y \subseteq V$.

$\langle 2 \rangle 1$. LET: V be open in Y .

$\langle 2 \rangle 2$. LET: $y \in V$

$\langle 2 \rangle 3$. PICK U open in X such that $V = U \cap Y$.

- $\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$.
 $\langle 2 \rangle 5$. $y \in B \cap Y \subseteq V$

□

13.3.2 Product Topology

Proposition 13.3.13. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. For all $i \in I$, let \mathcal{B}_i be a basis for the topology on X_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i : \text{for finitely many } i \in I \text{ we have } B_i \in \mathcal{B}_i, \text{ and } B_i = X_i \text{ for all other } i\}$ is a basis for the product topology on $\prod_{i \in I} X_i$.*

PROOF:

- $\langle 1 \rangle 1$. Every $B \in \mathcal{B}$ is open in the product topology.
 PROOF: Since every element of \mathcal{B}_i is open in X_i .
 $\langle 1 \rangle 2$. For any open set U in the product topology and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
 $\langle 2 \rangle 1$. LET: U be a set open in the box topology.
 $\langle 2 \rangle 2$. LET: $x \in U$
 $\langle 2 \rangle 3$. PICK a family $\{U_i\}_{i \in I}$ where U_i is open in X_i for $i = i_1, \dots, i_n$, and $U_i = X_i$ for all other i , such that $x \in \prod_{i \in I} U_i \subseteq U$
 $\langle 2 \rangle 4$. For $i = i_1, \dots, i_n$, choose $B_i \in \mathcal{B}_i$ such that $x_i \in B_i \subseteq U_i$. Let $B_i = X_i$ for all other i .
 $\langle 2 \rangle 5$. $\prod_{i \in I} B_i \in \mathcal{B}$
 $\langle 2 \rangle 6$. $x \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} U_i \subseteq U$

□

13.4 Subbases

Definition 13.4.1 (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a set \mathcal{S} of open sets such that every open set is a union of finite intersections of \mathcal{S} .

Proposition 13.4.2. *Let X be a set and $\mathcal{S} \subseteq X$. Then \mathcal{S} is a subbasis for a topology on X if and only if $\bigcup \mathcal{S} = X$, in which case the topology is unique and is the set of all unions of finite intersections of elements of \mathcal{S} .*

Proposition 13.4.3. *Let X be a topological space. Let \mathcal{S} be a subbasis for the topology on X . Then the topology on X is the coarsest topology that includes \mathcal{S} .*

Proposition 13.4.4. *Let X and Y be topological spaces. Then*

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

PROOF:

- $\langle 1 \rangle 1$. Every element of \mathcal{S} is open.
 PROOF: Since $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$.

⟨1⟩2. Every open set is a union of finite intersections of elements of \mathcal{S} .

PROOF: Since, for U open in X and V open in Y , we have $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$.

□

Definition 13.4.5 (Space with Basepoint). A *space with basepoint* is a pair (X, x) where X is a topological space and $x \in X$.

13.5 Neighbourhood Bases

Definition 13.5.1 (Neighbourhood Basis). Let X be a topological space and $x_0 \in X$. A *neighbourhood basis* of x_0 is a set \mathcal{U} of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of \mathcal{U} .

13.6 First Countable Spaces

Definition 13.6.1 (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

Proposition 13.6.2. \mathbb{R}_l is first countable.

PROOF: For any $x \in \mathbb{R}$ we have $\{[x, x + 1/n) : n \in \mathbb{Z}_+\}$ is a countable local basis.

□

Proposition 13.6.3. The ordered square is first countable.

PROOF:

⟨1⟩1. Every point (a, b) with $0 < b < 1$ has a countable local basis.

PROOF: The set of all intervals $((a, q), (a, r))$ where q and r are rational and $0 \leq q < b < r \leq 1$ is a countable local basis.

⟨1⟩2. Every point $(a, 0)$ has a countable local basis with $a > 0$.

PROOF: The set of all intervals $((q, 0), (a, r))$ where q and r are rational with $0 \leq q < a$ and $0 < r \leq 1$ is a countable local basis.

⟨1⟩3. Every point $(a, 1)$ has a countable local basis with $a < 1$.

PROOF: The set of all intervals $((a, q), (r, 1))$ with q and r rational and $0 \leq q < 1, a < r \leq 1$ is a countable local basis.

⟨1⟩4. $(0, 0)$ has a countable local basis.

PROOF: The set of all intervals $[(0, 0), (0, r))$ with r rational and $0 < r \leq 1$ is a countable local basis.

⟨1⟩5. $(1, 1)$ has a countable local basis.

PROOF: The set of all intervals $((1, q), (1, 1])$ with q rational and $0 \leq q < 1$ is a countable local basis.

□

13.7 Second Countable Spaces

Definition 13.7.1 (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

\mathbb{R}^n is second countable.

An uncountable discrete space is first countable but not second countable.

Proposition 13.7.2. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.*

PROOF:

$\langle 1 \rangle 1$. For all $\lambda \in \Lambda$, PICK U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.

$\langle 1 \rangle 2$. For all $\lambda \in \Lambda$, PICK $x_\lambda \in U_\lambda$.

$\langle 1 \rangle 3$. ASSUME: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda \in \Lambda}$.

$\langle 1 \rangle 4$. PICK $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$

$\langle 1 \rangle 5$. There is no $U \in B$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

□

13.8 Interior

Definition 13.8.1 (Interior). Let X be a topological space. Let $A \subseteq X$. The *interior* of A , A° , is the union of all the open sets included in A .

13.9 Closure

Definition 13.9.1 (Closure). Let X be a topological space. Let $A \subseteq X$. The *closure* of A , \bar{A} , is the intersection of all the closed sets that include A .

Proposition 13.9.2. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \bar{A}$ if and only if every open set that contains x intersects A .*

PROOF:

$x \in \bar{A} \Leftrightarrow$ for every closed set C , if $A \subseteq C$ then $x \in C$

\Leftrightarrow for every open set U , if $A \subseteq X - U$ then $x \in X - U$

\Leftrightarrow for every open set U , if $A \cap U = \emptyset$ then $x \notin U$

\Leftrightarrow for every open set U , if $x \in U$ then A intersects U □

Proposition 13.9.3. *Let X be a topological space. Let $A \subseteq B \subseteq X$. Then $\bar{A} \subseteq \bar{B}$.*

PROOF: Since every closed set that includes B is a closed set that includes A . \square

Proposition 13.9.4. *Let X be a topological space. Let $A, B \subseteq X$. Then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

PROOF:

$\langle 1 \rangle 1$. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$

PROOF: Since $\overline{A \cup B}$ is a closed set that includes $A \cup B$.

$\langle 1 \rangle 2$. $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$

PROOF: Since $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ by Proposition 13.9.3.

\square

Proposition 13.9.5. *Let X be a topological space. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then*

$$\bigcup \{\overline{A} : A \in \mathcal{A}\} \subseteq \overline{\bigcup \mathcal{A}}.$$

PROOF: For all $A \in \mathcal{A}$ we have $\overline{A} \subseteq \overline{\bigcup \mathcal{A}}$ by Proposition 13.9.3. \square

Example 13.9.6. The converse does not always hold. In \mathbb{R} , let $\mathcal{A} = \{\{x\} : 0 < x < 1\}$. Then $\bigcup \{\overline{A} : A \in \mathcal{A}\} = (0, 1)$ but $\overline{\bigcup \mathcal{A}} = [0, 1]$.

Proposition 13.9.7. *Let X be a topological space. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then $\overline{\bigcap \mathcal{A}} \subseteq \bigcap \{\overline{A} : A \in \mathcal{A}\}$.*

PROOF: Since $\overline{\bigcap \mathcal{A}} \subseteq \overline{A}$ for all $A \in \mathcal{A}$ by Proposition 13.9.3. \square

Example 13.9.8. The converse does not always hold. In \mathbb{R} , if A is the set of all rational numbers and B is the set of all irrational numbers then $\bigcap A \cap B = \emptyset$ but $\bigcap A \cap \bigcap B = \mathbb{R}$.

13.9.1 Bases

Proposition 13.9.9. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let \mathcal{B} be a basis for the topology on X . Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .*

PROOF:

$\langle 1 \rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

PROOF: Proposition 13.9.2 since every element of \mathcal{B} is open.

$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A , then $x \in \overline{A}$.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

$\langle 2 \rangle 2$. LET: U be an open set that contains x .

$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 4$. B intersects A .

PROOF: $\langle 2 \rangle 1$

$\langle 2 \rangle 5$. U intersects A .

\square

13.9.2 Subspaces

Proposition 13.9.10. *Let X be a topological space. Let Y be a subspace of X . Let $A \subseteq Y$. Let \overline{A} be the closure of A in X . Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF:

$\langle 1 \rangle 1$. $\overline{A} \cap Y$ is the closed in Y .

PROOF: Since \overline{A} is closed in X .

$\langle 1 \rangle 2$. For any closed set B in Y , if $A \subseteq B$ then $\overline{A} \cap Y \subseteq B$.

$\langle 2 \rangle 1$. LET: B be closed in Y .

$\langle 2 \rangle 2$. ASSUME: $A \subseteq B$

$\langle 2 \rangle 3$. PICK C closed in X such that $B = C \cap Y$.

$\langle 2 \rangle 4$. $A \subseteq C$

$\langle 2 \rangle 5$. $\overline{A} \subseteq C$

$\langle 2 \rangle 6$. $\overline{A} \cap Y \subseteq B$

□

13.9.3 Product Topology

Proposition 13.9.11. *Let X and Y be topological spaces. Let $A \subseteq X$ and $B \subseteq Y$. Then $\overline{A \times B} = \overline{A} \times \overline{B}$.*

PROOF:

$\langle 1 \rangle 1$. $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$

PROOF: Since $\overline{A} \times \overline{B}$ is a closed set that includes $A \times B$ by Proposition 13.20.2.

$\langle 1 \rangle 2$. $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$

$\langle 2 \rangle 1$. LET: $x \in \overline{A}$ and $y \in \overline{B}$.

$\langle 2 \rangle 2$. LET: U be an open set that contains (x, y) .

$\langle 2 \rangle 3$. PICK open sets V in X and W in Y such that $(x, y) \in V \times W \subseteq U$.

$\langle 2 \rangle 4$. V intersects A and W intersects B .

$\langle 2 \rangle 5$. U intersects $A \times B$.

□

13.9.4 Interior

Proposition 13.9.12. *Let X be a topological space and $A \subseteq X$. Then*

$$X - A^\circ = \overline{X - A}$$

PROOF:

$$\begin{aligned} X - A^\circ &= X - \bigcup \{U \text{ open in } X : U \subseteq A\} \\ &= \bigcap \{X - U : U \text{ open in } X, U \subseteq A\} && \text{(De Morgan's Law)} \\ &= \bigcap \{C : C \text{ closed in } X, X - A \subseteq C\} \\ &= \overline{X - A} \end{aligned}$$

□

Proposition 13.9.13. *Let X be a topological space and $A \subseteq X$. Then*

$$X - \overline{A} = (X - A)^\circ$$

PROOF: Dual. \square

13.10 Boundary

Definition 13.10.1 (Boundary). Let X be a topological space. Let $A \subseteq X$. The *boundary* of A is

$$\partial A := \overline{A} \cap \overline{X - A}.$$

Proposition 13.10.2. *Let X be a topological space. Let $A \subseteq X$. Then*

$$A^\circ \cap \partial A = \emptyset.$$

PROOF:

$\langle 1 \rangle 1.$ $A^\circ \subseteq A$

$\langle 1 \rangle 2.$ $X - A \subseteq X - A^\circ$

$\langle 1 \rangle 3.$ $\overline{X - A} \subseteq \overline{X - A^\circ}$

$\langle 1 \rangle 4.$ $\partial A \subseteq X - A^\circ$

\square

Proposition 13.10.3. *Let X be a topological space. Let $A \subseteq X$. Then*

$$\overline{A} = A^\circ \cup \partial A$$

$\langle 1 \rangle 1.$ $A^\circ \subseteq \overline{A}$

PROOF: Since $A^\circ \subseteq A \subseteq \overline{A}$.

$\langle 1 \rangle 2.$ $\partial A \subseteq \overline{A}$

PROOF: Definition of ∂A .

$\langle 1 \rangle 3.$ $\overline{A} \subseteq A^\circ \cup \partial A$

$\langle 2 \rangle 1.$ LET: $x \in \overline{A}$

$\langle 2 \rangle 2.$ ASSUME: $x \notin A^\circ$

PROVE: $x \in \partial A$

$\langle 2 \rangle 3.$ $x \in \overline{X - A}$

PROOF: Since $\overline{X - A} = X - A^\circ$.

$\langle 2 \rangle 4.$ $x \in \partial A$

PROOF: Since $\partial A = \overline{A} \cap \overline{X - A}$.

\square

Proposition 13.10.4. *Let X be a topological space. Let $A \subseteq X$. Then $\partial A = \emptyset$ if and only if A is both open and closed.*

PROOF:

$\langle 1 \rangle 1.$ If $\partial A = \emptyset$ then A is open and closed.

$\langle 2 \rangle 1.$ ASSUME: $\partial A = \emptyset$

$\langle 2 \rangle 2.$ $\overline{A} = A^\circ$

PROOF: Proposition 13.10.3.

$\langle 2 \rangle 3$. $\bar{A} = A = A^\circ$

$\langle 1 \rangle 2$. If A is open and closed then $\partial A = \emptyset$.

PROOF: If A is open and closed then

$$\begin{aligned}\partial A &= \bar{A} \cap \overline{X - A} \\ &= \bar{A} \cap (X - A^\circ) \\ &= A \cap (X - A) \\ &= \emptyset\end{aligned}$$

□

Proposition 13.10.5. *Let X be a topological space. Let $U \subseteq X$. Then U is open if and only if $\partial U = \bar{U} - U$.*

PROOF:

$\langle 1 \rangle 1$. If U is open then $\partial U = \bar{U} - U$

PROOF: If U is open then

$$\begin{aligned}\partial U &= \bar{U} \cap \overline{X - U} \\ &= \bar{U} \cap (X - U^\circ) \\ &= \bar{U} - U^\circ \\ &= \bar{U} - U\end{aligned}$$

$\langle 1 \rangle 2$. If $\partial U = \bar{U} - U$ then U is open.

$\langle 2 \rangle 1$. ASSUME: $\partial U = \bar{U} - U$

$\langle 2 \rangle 2$. $\bar{U} - U^\circ = \bar{U} - U$

$\langle 2 \rangle 3$. $U \subseteq U^\circ$

$\langle 2 \rangle 4$. $U = U^\circ$

□

13.11 Limit Points

Definition 13.11.1 (Limit Point). Let X be a topological space, $x \in X$ and $A \subseteq X$. Then x is a *limit point*, *cluster point* or *point of accumulation* of A iff every neighbourhood of x intersects $A - \{x\}$.

Proposition 13.11.2. *Let X be a topological space. Let $A \subseteq X$. Let A' be the set of limit points of A . Then*

$$\bar{A} = A \cup A'$$

PROOF:

$\langle 1 \rangle 1$. $\bar{A} \subseteq A \cup A'$

$\langle 2 \rangle 1$. LET: $x \in \bar{A}$

$\langle 2 \rangle 2$. ASSUME: $x \notin A$

PROVE: $x \in A'$

$\langle 2 \rangle 3$. LET: U be a neighbourhood of x .

$\langle 2 \rangle 4$. PICK $y \in U \cap A$

PROOF: Proposition 13.9.2.

$\langle 2 \rangle 5$. $y \neq x$

$\langle 1 \rangle 2$. $A \subseteq \bar{A}$

PROOF: Immediate from the definition of \bar{A} .

$\langle 1 \rangle 3$. $A' \subseteq \bar{A}$

PROOF: From Proposition 13.9.2.

□

Corollary 13.11.2.1. *A set is closed if and only if it contains all its limit points.*

13.12 Continuous Functions

Definition 13.12.1 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* iff, for every open set V in Y , the inverse image $f^{-1}(V)$ is open in X .

Proposition 13.12.2. *The composite of two continuous functions is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous.

$\langle 1 \rangle 2$. LET: U be open in Z .

$\langle 1 \rangle 3$. $g^{-1}(U)$ is open in Y .

$\langle 1 \rangle 4$. $\inf f(g^{-1}(U))$ is open in X .

□

Proposition 13.12.3. 1. id_X is continuous

2. If $f : X \rightarrow Y$ is continuous and $X_0 \subseteq X$ then $f \upharpoonright X_0 : X_0 \rightarrow Y$ is continuous.

3. If $f : X + Y \rightarrow Z$, then f is continuous iff $f \circ \kappa_1 : X \rightarrow Z$ and $f \circ \kappa_2 : Y \rightarrow Z$ are continuous.

4. If $f : Z \rightarrow X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Proposition 13.12.4. Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then the following are equivalent.

1. f is continuous.

2. For all $A \subseteq X$ we have $f(\bar{A}) \subseteq \overline{f(A)}$.

3. For every closed B in Y , we have $f^{-1}(B)$ is closed in X .

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

- $\langle 2 \rangle 1$. ASSUME: f is continuous.
 $\langle 2 \rangle 2$. LET: $A \subseteq X$
 $\langle 2 \rangle 3$. LET: $x \in \bar{A}$
 PROVE: $f(x) \in \overline{f(A)}$
 $\langle 2 \rangle 4$. LET: V be a neighbourhood of $f(x)$.
 PROVE: V intersects $f(A)$.
 $\langle 2 \rangle 5$. $f^{-1}(V)$ is a neighbourhood of x .
 $\langle 2 \rangle 6$. PICK $y \in f^{-1}(V) \cap A$
 $\langle 2 \rangle 7$. $f(y) \in V \cap f(A)$
 $\langle 1 \rangle 2$. $2 \Rightarrow 3$
 $\langle 2 \rangle 1$. ASSUME: 2
 $\langle 2 \rangle 2$. LET: B be closed in Y
 $\langle 2 \rangle 3$. LET: $A = f^{-1}(B)$
 PROVE: $\bar{A} = A$
 $\langle 2 \rangle 4$. $f(A) \subseteq B$
 $\langle 2 \rangle 5$. $\bar{A} \subseteq A$
 $\langle 3 \rangle 1$. LET: $x \in \bar{A}$
 $\langle 3 \rangle 2$. $f(x) \in B$
 PROOF:

$$f(x) \in f(\bar{A})$$

$$\subseteq \overline{f(A)} \quad (\langle 2 \rangle 1)$$

$$\subseteq \bar{B} \quad (\langle 2 \rangle 4)$$

$$= B \quad (\langle 2 \rangle 2)$$

- $\langle 1 \rangle 3$. $3 \Rightarrow 1$
 $\langle 2 \rangle 1$. ASSUME: 3
 $\langle 2 \rangle 2$. LET: V be open in Y .
 $\langle 2 \rangle 3$. $f^{-1}(Y - V)$ is closed in X .
 $\langle 2 \rangle 4$. $X - f^{-1}(V)$ is closed in X .
 $\langle 2 \rangle 5$. $f^{-1}(V)$ is open in X .

□

Proposition 13.12.5. *Let X and Y be topological spaces. Any constant function $X \rightarrow Y$ is continuous.*

PROOF:

- $\langle 1 \rangle 1$. LET: $b \in Y$
 $\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$ be the constant function with value b .
 $\langle 1 \rangle 3$. LET: $V \subseteq Y$ be open.
 $\langle 1 \rangle 4$. $f^{-1}(V)$ is either \emptyset or X .
 $\langle 1 \rangle 5$. $f^{-1}(V)$ is open.

□

Proposition 13.12.6. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{B} be a basis for Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .*

PROOF:

⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF: Since every element of \mathcal{B} is open in Y .

⟨1⟩2. If, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X , then f is continuous.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

⟨2⟩2. LET: U be open in Y .

⟨2⟩3. LET: $x \in f^{-1}(U)$

⟨2⟩4. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq U$.

⟨2⟩5. $x \in f^{-1}(B) \subseteq f^{-1}(U)$

□

Proposition 13.12.7. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{S} be a subbasis for the topology on Y . Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .*

PROOF:

⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

PROOF: Immediate from definitions.

⟨1⟩2. If, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X , then f is continuous.

⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

⟨2⟩2. For all $V_1, \dots, V_n \in \mathcal{S}$ we have $f^{-1}(V_1 \cap \dots \cap V_n)$ is open in X .

PROOF: Since $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$.

⟨2⟩3. Q.E.D.

PROOF: By Proposition 13.12.6 since the set of all finite intersections of elements of \mathcal{S} forms a basis for the topology on Y .

□

Proposition 13.12.8. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Then f is continuous if and only if, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.*

PROOF:

⟨1⟩1. If f is continuous then, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.

⟨2⟩1. ASSUME: f is continuous.

⟨2⟩2. LET: $x \in \mathbb{R}$

⟨2⟩3. LET: $\epsilon > 0$

⟨2⟩4. $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open in X .

⟨2⟩5. PICK a, b such that $x \in (a, b) \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$.

⟨2⟩6. LET: $\delta = \min(x - a, b - x)$

⟨2⟩7. LET: $y \in \mathbb{R}$

⟨2⟩8. ASSUME: $|y - x| < \delta$

⟨2⟩9. $y \in (a, b)$

⟨2⟩10. $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$

⟨2⟩11. $|f(y) - f(x)| < \epsilon$

⟨1⟩2. If, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$, then f is continuous.

⟨2⟩1. ASSUME: For all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.

- ⟨2⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.
 ⟨3⟩1. LET: $a \in \mathbb{R}$
 ⟨3⟩2. LET: $x \in f^{-1}((a, +\infty))$
 ⟨3⟩3. LET: $\epsilon = f(x) - a$
 ⟨3⟩4. PICK $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$
 ⟨3⟩5. $x \in (x - \delta, x + \delta) \subseteq f^{-1}((a, +\infty))$
 ⟨2⟩3. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.
 PROOF: Similar.
 ⟨2⟩4. Q.E.D.
 PROOF: Proposition 13.12.8.

□

Definition 13.12.9 (Continuity at a Point). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let $a \in X$. Then f is *continuous at a* iff, for every neighbourhood V of $f(a)$, there exists a neighbourhood U of a such that $f(U) \subseteq V$.

Proposition 13.12.10. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is continuous if and only if f is continuous at every point in X .*

- ⟨1⟩1. If f is continuous then f is continuous at every point in X .
 ⟨2⟩1. ASSUME: f is continuous.
 ⟨2⟩2. LET: $a \in X$
 ⟨2⟩3. LET: V be a neighbourhood of $f(a)$
 ⟨2⟩4. LET: $U = f^{-1}(V)$
 ⟨2⟩5. U is a neighbourhood of a .
 ⟨2⟩6. $f(U) \subseteq V$
 ⟨1⟩2. If f is continuous at every point in X then f is continuous.
 ⟨2⟩1. ASSUME: f is continuous at every point in X .
 ⟨2⟩2. LET: V be open in Y .
 ⟨2⟩3. LET: $x \in f^{-1}(V)$
 ⟨2⟩4. V is a neighbourhood of $f(x)$
 ⟨2⟩5. PICK a neighbourhood U of x such that $f(U) \subseteq V$
 ⟨2⟩6. $x \in U \subseteq f^{-1}(V)$

□

Definition 13.12.11 (Homeomorphism). Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Proposition 13.12.12. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is a homeomorphism iff f is bijective and, for all $U \subseteq X$, we have $f(U)$ is open if and only if U is open.*

PROOF: Immediate from definitions. □

Definition 13.12.13 (Topological Property). A property P of topological spaces is a *topological* property iff, for any topological spaces X and Y , if $P[X]$ and $X \cong Y$ then $P[Y]$.

Definition 13.12.14 (Retraction). Let X be a topological space and A a subspace of X . A continuous function $\rho : X \rightarrow A$ is a *retraction* iff $\rho \upharpoonright A = \text{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 13.12.15. Let **Top** be the category of small topological spaces and continuous functions.

Proposition 13.12.16. \emptyset is initial in **Top**.

Proposition 13.12.17. 1 is terminal in **Top**.

Forgetful functor **Top** \rightarrow **Set**.

Basepoint preserving continuous functor.

Proposition 13.12.18. Let (X, \mathcal{T}) be a topological space. Let S be the Sierpiński two-point space. Define $\Phi : \mathcal{T} \rightarrow \mathbf{Top}[X, S]$ by $\Phi(U)(x) = 1$ iff $x \in U$. Then Φ is a bijection.

PROOF:

$\langle 1 \rangle 1$. For all $U \in \mathcal{T}$ we have $\Phi(U)$ is continuous.

$\langle 2 \rangle 1$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 2$. $\Phi(U)(\{1\})$ is open.

PROOF: Since $\Phi(U)(\{1\}) = U$.

$\langle 1 \rangle 2$. Φ is injective.

PROOF: If $\Phi(U) = \Phi(V)$ then we have $\forall x(x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$.

$\langle 1 \rangle 3$. Φ is surjective.

PROOF: Given $f : X \rightarrow S$ continuous we have $\Phi(f^{-1}(1)) = f$.

□

13.12.1 Order Topology

Proposition 13.12.19. Let X and Y be linearly ordered sets under the order topology. Let $f : X \rightarrow Y$ be strictly monotone and surjective. Then f is a homeomorphism.

PROOF:

$\langle 1 \rangle 1$. f is continuous.

$\langle 2 \rangle 1$. For all $b \in Y$ we have $f^{-1}((b, +\infty))$ is open in X .

$\langle 3 \rangle 1$. LET: $b \in Y$

$\langle 3 \rangle 2$. LET: a be the element of X such that $f(a) = b$.

$\langle 3 \rangle 3$. $f^{-1}((b, +\infty)) = (a, +\infty)$

$\langle 2 \rangle 2$. For all $b \in Y$ we have $f^{-1}((-\infty, b))$ is open in X .

PROOF: Similar.

$\langle 1 \rangle 2$. f^{-1} is continuous.

PROOF: Similar.

□

Corollary 13.12.19.1. For n a positive integer, the n th root function $\overline{\mathbb{R}_+} \rightarrow \overline{\mathbb{R}_+}$ is continuous.

13.12.2 Paths

Definition 13.12.20 (Path). A *path* in a topological space X is a continuous function $[0, 1] \rightarrow X$.

13.12.3 Loops

Definition 13.12.21 (Loop). A *loop* in a topological space X is a path $\alpha : [0, 1] \rightarrow X$ such that $\alpha(0) = \alpha(1)$.

13.13 Convergence

Definition 13.13.1 (Convergence). Let X be a topological space. Let (x_n) be a sequence in X . A point $a \in X$ is a *limit* of the sequence iff, for every neighbourhood U of a , there exists n_0 such that $\forall n \geq n_0, x_n \in U$.

Proposition 13.13.2. *If $f : X \rightarrow Y$ is continuous and $x_n \rightarrow l$ in X then $f(x_n) \rightarrow f(l)$ in Y .*

Example 13.13.3. The converse does not hold.

Let X be the set of all continuous functions $[0, 1] \rightarrow [-1, 1]$ under the product topology. Let $i : X \rightarrow L^2([0, 1])$ be the inclusion.

If $f_n \rightarrow f$ then $i(f_n) \rightarrow i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K, \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0, 1]} U_\lambda$ where $U_\lambda = [-1, 1]$ except for $\lambda = \lambda_1, \dots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \dots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 13.13.4. *The converse does hold for first countable spaces. If $f : X \rightarrow Y$ where X is first countable, and Y is a topological space, and whenever $x_n \rightarrow x$ then $f(x_n) \rightarrow f(x)$, then f is continuous.*

Proposition 13.13.5. *If (s_n) is an increasing sequence of real numbers bounded above, then (s_n) converges.*

PROOF:

<1>1. LET: s be the supremum of $\{s_n : n \in \mathbb{N}\}$.

PROVE: $s_n \rightarrow s$ as $n \rightarrow \infty$.

<1>2. LET: $\epsilon > 0$

<1>3. PICK N such that $s_N > s - \epsilon$.

<1>4. $\forall n \geq N, s - \epsilon \leq s_n \leq s$

<1>5. $\forall n \geq N, |s_n - s| < \epsilon$

□

13.13.1 Closure

Proposition 13.13.6. *Let X be a topological space. Let $A \subseteq X$. Let (a_n) be a sequence in A and $l \in X$. If $a_n \rightarrow l$ as $n \rightarrow \infty$, then $l \in \overline{A}$.*

PROOF:

$\langle 1 \rangle 1$. LET: U be a neighbourhood of l .

$\langle 1 \rangle 2$. PICK N such that $\forall n \in N. a_n \in U$

$\langle 1 \rangle 3$. $a_N \in A \cap U$

□

13.13.2 Continuous Functions

Proposition 13.13.7. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$ be continuous. Let $x_n \rightarrow x$ as $n \rightarrow \infty$ in X . Then $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$ in Y .*

PROOF:

$\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(x)$.

$\langle 1 \rangle 2$. PICK N such that $\forall n \geq N. x_n \in f^{-1}(V)$

$\langle 1 \rangle 3$. $\forall n \geq N. f(x_n) \in V$

□

13.13.3 Infinite Series

Definition 13.13.8 (Series). Let (a_n) be a sequence of real numbers. We say that the infinite series $\sum_{n=0}^{\infty} a_n$ *converges* to s , and write

$$\sum_{n=0}^{\infty} a_n = s$$

iff $\sum_{n=0}^N a_n \rightarrow s$ as $N \rightarrow \infty$.

13.14 Strong Continuity

Definition 13.14.1 (Strong Continuity). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is *strongly continuous* iff, for every $V \subseteq Y$, we have V is open in Y if and only if $f^{-1}(V)$ is open in X .

Proposition 13.14.2. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is strongly continuous if and only if, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X .*

PROOF:

f is continuous $\Leftrightarrow \forall V \subseteq Y (V \text{ is open in } Y \Leftrightarrow f^{-1}(V) \text{ is open in } X)$

$\Leftrightarrow \forall C \subseteq Y (Y - C \text{ is open in } Y \Leftrightarrow f^{-1}(Y - C) \text{ is open in } X)$

$\Leftrightarrow \forall C \subseteq Y (C \text{ is closed in } Y \Leftrightarrow f^{-1}(C) \text{ is closed in } X)$ □

13.15 Subspaces

Definition 13.15.1 (Subspace). Let X be a topological space, Y a set, and $f : Y \rightarrow X$. The *subspace topology* on Y induced by f is $\mathcal{T} = \{f^{-1}(U) : U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF: Since $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$.

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: Since $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.

$\langle 1 \rangle 3$. $Y \in \mathcal{T}$

PROOF: Since $Y = f^{-1}(X)$.

□

Proposition 13.15.2. Let X be a topological space, Y a set and $f : Y \rightarrow X$ a function. Then the subspace topology on Y is the coarsest topology such that f is continuous.

PROOF: Immediate from definition. □

Proposition 13.15.3 (Local Formulation of Continuity). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{U} be a set of open subspaces of X such that $X = \bigcup \mathcal{U}$. If $f|_U : U \rightarrow Y$ is continuous for all $U \in \mathcal{U}$, then f is continuous.

PROOF:

$\langle 1 \rangle 1$. LET: $x \in X$

PROVE: f is continuous at x .

$\langle 1 \rangle 2$. LET: V be a neighbourhood of $f(x)$.

$\langle 1 \rangle 3$. PICK $U \in \mathcal{U}$ such that $x \in U$.

$\langle 1 \rangle 4$. PICK W open in U such that $x \in W$ and $f(W) \subseteq V$.

$\langle 1 \rangle 5$. W is open in X .

□

Theorem 13.15.4. Let X be a topological space and (Y, i) a subset of X . Then the subspace topology on Y is the unique topology such that, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f : Z \rightarrow X$ is continuous.

PROOF:

$\langle 1 \rangle 1$. If we give Y the subspace topology then, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.

$\langle 2 \rangle 1$. Given Y the subspace topology.

$\langle 2 \rangle 2$. LET: Z be a topological space.

$\langle 2 \rangle 3$. LET: $f : Z \rightarrow Y$

$\langle 2 \rangle 4$. If f is continuous then $i \circ f$ is continuous.

PROOF: Since i is continuous.

- ⟨2⟩5. If $i \circ f$ is continuous then f is continuous.
- ⟨3⟩1. ASSUME: $i \circ f$ is continuous.
- ⟨3⟩2. LET: U be open in Y .
- ⟨3⟩3. $f^{-1}(i^{-1}(i(U)))$ is open in Z .
- ⟨3⟩4. $f^{-1}(U)$ is open in Z .
- ⟨1⟩2. If, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.
- ⟨2⟩1. ASSUME: For every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.
- ⟨2⟩2. i is continuous.
- ⟨2⟩3. For every open set U in X , we have $i^{-1}(X)$ is open in Y .
- ⟨2⟩4. LET: Z be the set Y under the subspace topology and $f : Z \rightarrow Y$ the identity function.
- ⟨2⟩5. $i \circ f$ is continuous.
- ⟨2⟩6. f is continuous.
- ⟨2⟩7. Every set open in Y is open in Z .

□

Proposition 13.15.5. *Let X be a topological space, Y a subspace of X and $U \subseteq Y$. If Y is open in X and U is open in Y then U is open in X .*

PROOF:

- ⟨1⟩1. PICK V open in X such that $U = V \cap Y$
- ⟨1⟩2. U is open in X .

PROOF: It is the intersection of two open sets in X .

□

Proposition 13.15.6. *Let Y be a subspace of X and $A \subseteq Y$. Then the subspace topology on A as a subspace of Y is the same as the subspace topology on A as a subspace of X .*

PROOF:

- ⟨1⟩1. LET: \mathcal{T}_Y be the subspace topology on A as a subspace of Y .
- ⟨1⟩2. LET: \mathcal{T}_X be the subspace topology on A as a subspace of X .
- ⟨1⟩3. LET: $U \subseteq A$
- ⟨1⟩4. $U \in \mathcal{T}_Y \Leftrightarrow U \in \mathcal{T}_X$

PROOF:

$$\begin{aligned}
 U \in \mathcal{T}_Y &\Leftrightarrow \exists V \text{ open in } Y. U = V \cap A \\
 &\Leftrightarrow \exists V. \exists W \text{ open in } X. (V = Y \cap W \wedge U = V \cap A) \\
 &\Leftrightarrow \exists W \text{ open in } X. U = Y \cap W \cap A \\
 &\Leftrightarrow \exists W \text{ open in } X. U = W \cap A \\
 &\Leftrightarrow U \in \mathcal{T}_X
 \end{aligned}$$

□

Proposition 13.15.7. *Let X be a topological space. Let \mathcal{B} be a basis for the topology on X . Let $Y \subseteq X$. Then $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the topology on Y .*

PROOF:

$\langle 1 \rangle 1$. Every element of \mathcal{B}' is open.

PROOF: For all $B \in \mathcal{B}$, we have B is open in X , so $B \cap Y$ is open in Y .

$\langle 1 \rangle 2$. For any open set V in Y and $y \in V$, there exists $B' \in \mathcal{B}'$ such that
 $y \in B' \subseteq V$

$\langle 2 \rangle 1$. LET: V be open in Y .

$\langle 2 \rangle 2$. LET: $y \in V$

$\langle 2 \rangle 3$. PICK U open in X such that $V = U \cap Y$.

$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$

$\langle 2 \rangle 5$. $B \cap Y \in \mathcal{B}'$ and $y \in B \cap Y \subseteq V$

□

13.15.1 Product Topology

Proposition 13.15.8. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology on $\prod_{i \in I} Y_i$ as a subspace of $\prod_{i \in I} X_i$.*

PROOF:

$\langle 1 \rangle 1$. Given $\prod_{i \in I} Y_i$ the subspace topology.

$\langle 1 \rangle 2$. LET: $\iota : \prod_{i \in I} Y_i$ be the inclusion.

$\langle 1 \rangle 3$. LET: Z be any topological space.

$\langle 1 \rangle 4$. LET: $f : Z \rightarrow \prod_{i \in I} Y_i$

$\langle 1 \rangle 5$. f is continuous if and only if, for all $i \in I$, we have $\pi_i \circ f$ is continuous.

PROOF:

$$f \text{ is continuous} \Leftrightarrow \iota \circ f : Z \rightarrow \prod_{i \in I} X_i \text{ is continuous} \quad (\text{Theorem 13.15.4})$$

$$\Leftrightarrow \forall i \in I. \pi_i \circ \iota \circ f : Z \rightarrow X_i \text{ is continuous} \quad (\text{Theorem 13.20.4})$$

$$\Leftrightarrow \forall i \in I. \iota_i \circ \pi_i \circ f : Z \rightarrow X_i \text{ is continuous}$$

$$\Leftrightarrow \forall i \in I. \pi_i \circ f : Z \rightarrow Y_i \text{ is continuous} \quad (\text{Theorem 13.15.4})$$

where ι_i is the inclusion $Y_i \rightarrow X_i$.

□

13.16 Embedding

Definition 13.16.1 (Embedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *embedding* iff f is injective and the topology on X is the subspace induced by f .

Proposition 13.16.2. *Every embedding is continuous.*

PROOF: Theorem 13.15.4. □

Proposition 13.16.3. *Let X and Y be topological spaces. Let $b \in Y$. The function $\kappa : X \rightarrow X \times Y$ that maps x to (x, b) is an embedding.*

PROOF:

$\langle 1 \rangle 1$. For all U open in X , we have $U = \kappa^{-1}(V)$ for some V open in $X \times Y$.

PROOF: Take $V = U \times Y$.

$\langle 1 \rangle 2$. For all V open in $X \times Y$ we have $\kappa^{-1}(V)$ is open in X .

PROOF: Since $\pi_1 \circ \kappa = \text{id}_X$ and $\pi_2 \circ \kappa$ (which is the constant function with value b) are both continuous, hence κ is continuous.

□

13.17 Open Maps

Definition 13.17.1 (Open Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *open map* iff, for all U open in X , we have $f(U)$ is open in Y .

Proposition 13.17.2. *Let X and Y be topological spaces. The projections $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are open maps.*

PROOF:

$\langle 1 \rangle 1$. π_1 is an open map.

$\langle 2 \rangle 1$. LET: U be open in $X \times Y$.

$\langle 2 \rangle 2$. LET: $x \in \pi_1(U)$

$\langle 2 \rangle 3$. PICK y such that $(x, y) \in U$

$\langle 2 \rangle 4$. PICK V and W open in X and Y respectively such that $(x, y) \in V \times W \subseteq U$

$\langle 2 \rangle 5$. $x \in V \subseteq \pi_1(U)$

$\langle 1 \rangle 2$. π_2 is an open map.

PROOF: Similar.

□

13.17.1 Subspaces

Proposition 13.17.3. *Let X and Y be topological spaces. Let $p : X \rightarrow Y$ be an open map. Let A be an open set in X . Then $p|_A : A \rightarrow p(A)$ is an open map.*

PROOF:

$\langle 1 \rangle 1$. LET: U be open in A .

$\langle 1 \rangle 2$. U is open in X .

PROOF: Proposition 13.15.5.

$\langle 1 \rangle 3$. $p(U)$ is open in Y .

$\langle 1 \rangle 4$. $p(U)$ is open in $p(A)$.

PROOF: Since $p(U) = p(U) \cap p(A)$.

□

13.18 Locally Finite

Definition 13.18.1 (Locally Finite). Let X be a topological space. Let $\{A_i\}_{i \in I}$ be a family of subsets of X . Then $\{A_i\}_{i \in I}$ is *locally finite* iff, for every $x \in X$,

there exist only finitely many $i \in I$ such that $x \in A_i$.

Theorem 13.18.2 (Pasting Lemma). *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let $\{A_i\}_{i \in I}$ be a locally finite family of closed subspaces of X such that $X = \bigcup_{i \in I} A_i$. If $f|_{A_i} : A_i \rightarrow Y$ is continuous for all $i \in I$, then f is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: B be closed in Y .

$\langle 1 \rangle 2$. LET: $A = f^{-1}(B)$

PROVE: A is closed in X .

$\langle 1 \rangle 3$. $A = \bigcup_{i \in I} f|_{A_i}^{-1}(B)$

$\langle 1 \rangle 4$. LET: $x \in X - A$

PROVE: There exists a neighbourhood U' of x such that $U' \subseteq X - A$.

$\langle 1 \rangle 5$. PICK a neighbourhood U of x such that U intersects A_i for only finitely many $i \in I$.

$\langle 1 \rangle 6$. LET: i_1, \dots, i_n be the elements of I such that U intersects A_{i_1}, \dots, A_{i_n} .

$\langle 1 \rangle 7$. For $j = 1, \dots, n$,

LET: $S_j = f|_{A_{i_j}}^{-1}(B)$

$\langle 1 \rangle 8$. For $j = 1, \dots, n$, we have S_j is closed in X .

$\langle 1 \rangle 9$. For $j = 1, \dots, n$, we have $x \notin S_j$.

$\langle 1 \rangle 10$. LET: $U' = U \cap \bigcap_{j=1}^n (X - S_j)$

$\langle 1 \rangle 11$. U' is a neighbourhood of x .

$\langle 1 \rangle 12$. $U' \subseteq X - A$

□

13.19 Closed Maps

Definition 13.19.1 (Closed Map). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is a *closed map* iff, for every closed set C in X , we have $f(C)$ is closed in Y .

13.20 Product Topology

Definition 13.20.1 (Product Topology). Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

13.20.1 Closed Sets

Proposition 13.20.2. *Let X and Y be topological spaces. Let A be a closed set in X and B a closed set in Y . Then $A \times B$ is closed in $X \times Y$.*

PROOF: Since $(X \times Y) - (A \times B) = ((X - A) \times Y) \cup (X \times (Y - B))$. □

Proposition 13.20.3. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The product topology on $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis $\mathcal{B} = \{\prod_{\alpha \in A} U_\alpha : \text{for all } \alpha \in A, U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in A\}$.*

PROOF:

- $\langle 1 \rangle 1.$ \mathcal{B} is a basis for a topology.
- $\langle 1 \rangle 2.$ LET: \mathcal{T} be the topology generated by \mathcal{B} .
- $\langle 1 \rangle 3.$ LET: \mathcal{T}_p be the product topology.
- $\langle 1 \rangle 4.$ $\mathcal{T} \subseteq \mathcal{T}_p$
 - $\langle 2 \rangle 1.$ LET: $B \in \mathcal{B}$
 - $\langle 2 \rangle 2.$ LET: $B = \prod_{\alpha \in A} U_\alpha$ with each U_α open in X_α and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$
 - $\langle 2 \rangle 3.$ $B = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$
 - $\langle 2 \rangle 4.$ $B \in \mathcal{T}_p$
- $\langle 1 \rangle 5.$ $\mathcal{T}_p \subseteq \mathcal{T}$
 - $\langle 2 \rangle 1.$ For every $\alpha \in A$ we have π_α is continuous.

PROOF: Since $\pi^{-1}(U)$ is open for every U open in X_α .

□

Theorem 13.20.4. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Then the product topology on $\prod_{\alpha \in A} X_\alpha$ is the unique topology such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f : Z \rightarrow X_\alpha$ is continuous.*

PROOF:

- $\langle 1 \rangle 1.$ If we give $\prod_{\alpha \in A} X_\alpha$ the product topology, then for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
- $\langle 2 \rangle 1.$ Give $\prod_{\alpha \in A} X_\alpha$ the product topology.
- $\langle 2 \rangle 2.$ LET: Z be a topological space.
- $\langle 2 \rangle 3.$ LET: $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$
- $\langle 2 \rangle 4.$ If f is continuous then, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
- PROOF: Since the composite of two continuous functions is continuous.
- $\langle 2 \rangle 5.$ If, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then f is continuous.
- $\langle 3 \rangle 1.$ ASSUME: For all $\alpha \in A$ we have $\pi_\alpha \circ f$ is continuous.
- $\langle 3 \rangle 2.$ LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with U_α open in X_α such that $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \dots, \alpha_n$.
- $\langle 3 \rangle 3.$ For all α we have $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ is open in Z .
- $\langle 3 \rangle 4.$ $f^{-1}(\prod_{\alpha} U_\alpha)$ is open in Z
- PROOF: Since $f^{-1}(\prod_{\alpha} U_\alpha) = f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \dots \cap f^{-1}(\pi_{\alpha_n}^{-1}(U_{\alpha_n}))$.
- $\langle 1 \rangle 2.$ If \mathcal{T} is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then \mathcal{T} is the product topology.
- $\langle 2 \rangle 1.$ ASSUME: \mathcal{T} is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.

⟨2⟩2. LET: \mathcal{T}_p be the product topology.

⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_p$

⟨3⟩1. LET: $Z = (\prod_{\alpha} X_{\alpha}, \mathcal{T}_p)$

⟨3⟩2. LET: $f : Z \rightarrow \prod_{\alpha} X_{\alpha}$ be the identity function

⟨3⟩3. For all α we have $\pi_{\alpha} \circ f$ is continuous.

⟨3⟩4. f is continuous.

PROOF: ⟨2⟩1

⟨3⟩5. Every set open in \mathcal{T} is open in \mathcal{T}_p

⟨2⟩4. $\mathcal{T}_p \subseteq \mathcal{T}$

⟨3⟩1. $\text{id}_{\prod_{\alpha} X_{\alpha}}$ is continuous.

⟨3⟩2. For all α we have π_{α} is continuous.

PROOF: ⟨2⟩1

⟨3⟩3. $\mathcal{T}_p \subseteq \mathcal{T}$

PROOF: Since \mathcal{T}_p is the coarsest topology such that every π_{α} is continuous.

□

Example 13.20.5. It is not true that, for any function $f : \prod_{\alpha \in A} X_{\alpha} \rightarrow Y$, if f is continuous in every variable separately then f is continuous.

Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

Then f is continuous in x and in y , but is not continuous.

Proposition 13.20.6. Let $\{X_i\}_{i \in I}$ be a nonempty family of topological spaces. The product topology on $\prod_{i \in I} X_i$ is the topology generated by the subbasis $\{\pi_i^{-1}(U) : i \in I, U \text{ is open in } X_i\}$.

PROOF:

⟨1⟩1. $\{\pi_i^{-1}(U) : i \in I, U \text{ is open in } X_i\}$ is a subbasis for a topology on $\prod_{i \in I} X_i$.

⟨2⟩1. PICK $i_0 \in I$

⟨2⟩2. $\prod_{i \in I} X_i = \pi_{i_0}^{-1}(X_{i_0})$

⟨1⟩2. The topology generated by this subbasis is the product topology.

PROOF: Since the basis in Proposition 13.20.3 is the set of all finite intersections of elements of this subbasis.

□

13.20.2 Closure

Proposition 13.20.7. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

⟨1⟩1. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

- $\langle 2 \rangle 1$. LET: $x \in \prod_{i \in I} \overline{A_i}$
 $\langle 2 \rangle 2$. For any family $\{U_i\}_{i \in I}$ where each U_i is open in X_i , and $U_i = X_i$ for all but finitely many $i \in I$, if $x \in \prod_{i \in I} U_i$ then $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$.
 $\langle 3 \rangle 1$. LET: $\{U_i\}_{i \in I}$ be a family where each U_i is open in X_i , and $U_i = X_i$ for all but finitely many i .
 $\langle 3 \rangle 2$. ASSUME: $x \in \prod_{i \in I}$
 $\langle 3 \rangle 3$. For all $i \in I$ we have U_i intersects A_i
 PROOF: Since $\pi_i(x) \in \overline{A_i}$ and U_i is a neighbourhood of $\pi_i(x)$.
 $\langle 3 \rangle 4$. $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$
 $\langle 2 \rangle 3$. $x \in \overline{\prod_{i \in I} A_i}$
 PROOF: Proposition 13.9.9.
 $\langle 1 \rangle 2$. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
 $\langle 2 \rangle 1$. LET: $x \in \overline{\prod_{i \in I} A_i}$
 $\langle 2 \rangle 2$. LET: $i \in I$
 PROVE: $\pi_i(x) \in \overline{A_i}$
 $\langle 2 \rangle 3$. LET: U be a neighbourhood of $\pi_i(x)$ in X_i
 $\langle 2 \rangle 4$. $\pi_i^{-1}(U)$ is a neighbourhood of x in $\prod_{i \in I} X_i$
 $\langle 2 \rangle 5$. PICK $y \in \pi_i^{-1}(U) \cap \prod_{i \in I} A_i$
 $\langle 2 \rangle 6$. $\pi_i(y) \in U \cap A_i$
 \square

13.20.3 Convergence

Proposition 13.20.8. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (x_n) be a sequence of points in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $x_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for all $i \in I$, we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.*

PROOF:

- $\langle 1 \rangle 1$. If $x_n \rightarrow l$ as $n \rightarrow \infty$ then, for all $i \in I$, we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.
 PROOF: Proposition 13.13.2.
 $\langle 1 \rangle 2$. If, for all $i \in I$, we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$, then $x_n \rightarrow l$ as $n \rightarrow \infty$.
 $\langle 2 \rangle 1$. ASSUME: For all $i \in I$ we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.
 $\langle 2 \rangle 2$. LET: U be a neighbourhood of l .
 $\langle 2 \rangle 3$. PICK $i_1, \dots, i_n \in I$ and open sets U_j in X_{i_j} for $j = 1, \dots, n$ such that $l \in \pi_{i_1}^{-1}(U_1) \cap \dots \cap \pi_{i_n}^{-1}(U_n) \subseteq U$
 $\langle 2 \rangle 4$. For $j = 1, \dots, n$ we have $\pi_{i_j}(l) \in U_j$
 $\langle 2 \rangle 5$. PICK N such that, for all $m \geq N$, we have $\pi_{i_j}(x_m) \in U_j$
 $\langle 2 \rangle 6$. $\forall m \geq N. x_m \in U$
 \square

13.21 Topological Disjoint Union

Definition 13.21.1 (Coproduct Topology). Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The *coproduct topology* on $\coprod_{\alpha \in A} X_\alpha$ is

$$\mathcal{T} = \left\{ \coprod_{\alpha \in A} U_{\alpha} : \{U_{\alpha}\}_{\alpha \in A} \text{ is a family with } U_{\alpha} \text{ open in } X_{\alpha} \text{ for all } \alpha \right\}.$$

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF:

$$\bigcup_{i \in I} \coprod_{\alpha \in A} U_{i\alpha} = \coprod_{\alpha \in A} \bigcup_{i \in I} U_{i\alpha}$$

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF:

$$\coprod_{\alpha \in A} U_{\alpha} \cap \coprod_{\alpha \in A} V_{\alpha} = \coprod_{\alpha \in A} (U_{\alpha} \cap V_{\alpha})$$

$\langle 1 \rangle 3$. $\coprod_{\alpha \in A} X_{\alpha} \in \mathcal{T}$

PROOF: Since every X_{α} is open in X_{α} .

□

Proposition 13.21.2. *The coproduct topology is the finest topology on $\coprod_{\alpha \in A} X_{\alpha}$ such that every injection $\kappa_{\alpha} : X_{\alpha} \rightarrow \coprod_{\alpha \in A} X_{\alpha}$ is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $P = \coprod_{\alpha \in A} X_{\alpha}$

$\langle 1 \rangle 2$. LET: \mathcal{T}_c be the coproduct topology.

$\langle 1 \rangle 3$. LET: \mathcal{T} be any topology on P

$\langle 1 \rangle 4$. For all $\alpha \in A$, the injection $\kappa_{\alpha} : X_{\alpha} \rightarrow (P, \mathcal{T}_c)$ is continuous.

$\langle 2 \rangle 1$. LET: $\alpha \in A$

$\langle 2 \rangle 2$. LET: $\{U_{\alpha}\}_{\alpha \in A}$ be a family with each U_{α} open in X_{α} .

$\langle 2 \rangle 3$. For all $\alpha \in A$, we have $\kappa_{\alpha}^{-1}(\coprod_{\alpha \in A} U_{\alpha})$ is open in X_{α} .

PROOF: Since $\kappa_{\alpha}^{-1}(\coprod_{\alpha \in A} U_{\alpha}) = U_{\alpha}$.

$\langle 1 \rangle 5$. If, for all $\alpha \in A$, the injection $\kappa_{\alpha} : X_{\alpha} \rightarrow (P, \mathcal{T})$ is continuous, then $\mathcal{T} \subseteq \mathcal{T}_c$.

$\langle 2 \rangle 1$. ASSUME: For all $\alpha \in A$, the injection $\kappa_{\alpha} : X_{\alpha} \rightarrow (P, \mathcal{T})$ is continuous.

$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 3$. For all $\alpha \in A$, we have $\kappa_{\alpha}^{-1}(U)$ is open in X_{α} .

$\langle 2 \rangle 4$. $U = \coprod_{\alpha \in A} \kappa_{\alpha}^{-1}(U) \in \mathcal{T}_c$

□

Theorem 13.21.3. *Let $\{X_{\alpha}\}_{\alpha \in A}$ be a family of topological spaces. The coproduct topology is the unique topology on $\coprod_{\alpha \in A} X_{\alpha}$ such that, for every topological space Z and function $f : \coprod_{\alpha \in A} X_{\alpha} \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_{\alpha}$ is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $X = \coprod_{\alpha \in A} X_{\alpha}$

$\langle 1 \rangle 2$. LET: \mathcal{T}_c be the coproduct topology.

- $\langle 1 \rangle 3$. For every topological space Z and function $f : (X, \mathcal{T}_c) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
 $\langle 2 \rangle 1$. LET: Z be a topological space.
 $\langle 2 \rangle 2$. LET: $f : X \rightarrow Z$
 $\langle 2 \rangle 3$. If f is continuous then $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
 PROOF: Because the composite of two continuous functions is continuous.
 $\langle 2 \rangle 4$. If $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous then f is continuous.
 $\langle 3 \rangle 1$. ASSUME: $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
 $\langle 3 \rangle 2$. LET: U be open in Z
 $\langle 3 \rangle 3$. For all $\alpha \in A$ we have $\kappa_\alpha^{-1}(f^{-1}(U))$ is open in X_α
 $\langle 3 \rangle 4$. $f^{-1}(U) = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(f^{-1}(U))$
 $\langle 3 \rangle 5$. $f^{-1}(U)$ is open in X
 $\langle 1 \rangle 4$. For any topology \mathcal{T} on X , if for every topological space Z and function $f : (X, \mathcal{T}) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous, then $\mathcal{T} = \mathcal{T}_c$.
 $\langle 2 \rangle 1$. LET: \mathcal{T} be a topology on X .
 $\langle 2 \rangle 2$. ASSUME: For every topological space Z and function $f : (X, \mathcal{T}) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
 $\langle 2 \rangle 3$. $\mathcal{T} \subseteq \mathcal{T}_c$
 $\langle 3 \rangle 1$. For all $\alpha \in A$ we have $\kappa_\alpha : X_\alpha \rightarrow (X, \mathcal{T})$ is continuous.
 PROOF: From $\langle 2 \rangle 1$ since id_X is continuous.
 $\langle 3 \rangle 2$. $\mathcal{T} \subseteq \mathcal{T}_c$
 PROOF: Proposition 13.21.2.
 $\langle 2 \rangle 4$. $\mathcal{T}_c \subseteq \mathcal{T}$
 $\langle 3 \rangle 1$. LET: $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}_c)$ be the identity function.
 $\langle 3 \rangle 2$. $f \circ \kappa_\alpha$ is continuous for all α .
 $\langle 3 \rangle 3$. f is continuous.
 PROOF: $\langle 2 \rangle 1$
 $\langle 3 \rangle 4$. $\mathcal{T}_c \subseteq \mathcal{T}$

□

13.22 Quotient Spaces

Definition 13.22.1 (Quotient Topology). Let X be a topological space, S a set, and $\pi : X \twoheadrightarrow S$ be a surjection. The *quotient topology* on S induced by π is $\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

- $\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
 PROOF: Since $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}$.
 $\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.
 PROOF: Since $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$.
 $\langle 1 \rangle 3$. $X \in \mathcal{T}$
 PROOF: Since $X = \pi^{-1}(Y)$.

□

Proposition 13.22.2. *Let X be a topological space, S a set and $\pi : X \twoheadrightarrow S$ a surjection. Then the quotient topology on S is the finest topology such that π is continuous.*

PROOF: Immediate from definitions. □

Theorem 13.22.3. *Let X be a topological space, let S be a set, and let $\pi : X \twoheadrightarrow S$ be surjective. Then the quotient topology on S is the unique topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.*

PROOF:

- ⟨1⟩1. If S is given the quotient topology, then for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.
- ⟨2⟩1. Give S the quotient topology.
- ⟨2⟩2. LET: Z be a topological space.
- ⟨2⟩3. LET: $f : S \rightarrow Z$
- ⟨2⟩4. If f is continuous then $f \circ \pi$ is continuous.
- PROOF: The composite of two continuous functions is continuous.
- ⟨2⟩5. If $f \circ \pi$ is continuous then f is continuous.
- ⟨3⟩1. ASSUME: $f \circ \pi$ is continuous.
- ⟨3⟩2. LET: U be open in Z .
- ⟨3⟩3. $\pi^{-1}(f^{-1}(U))$ is open in X .
- ⟨3⟩4. $f^{-1}(U)$ is open in S .
- ⟨1⟩2. If S is given a topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous, then that topology is the quotient topology.
- ⟨2⟩1. Give S a topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.
- ⟨2⟩2. LET: $U \subseteq S$
- ⟨2⟩3. If $\pi^{-1}(U)$ is open in X then U is open in S .
- ⟨3⟩1. LET: Z be S under the quotient topology induced by π .
- ⟨3⟩2. LET: $f : S \rightarrow Z$ be the identity function.
- ⟨3⟩3. $f \circ \pi$ is continuous.
- ⟨3⟩4. f is continuous.
- PROOF: ⟨2⟩1
- ⟨3⟩5. U is open in Z .
- ⟨3⟩6. U is open in X .
- ⟨2⟩4. If U is open in S then $\pi^{-1}(U)$ is open in X .
- PROOF: Since π is continuous (taking $Z = S$ and $f = \text{id}_S$ in ⟨2⟩1).

□

13.22.1 Quotient Maps

Definition 13.22.4 (Quotient Map). Let X and S be topological spaces and $\pi : X \rightarrow S$. Then π is a *quotient map* iff π is surjective and the topology on S is the quotient topology induced by π .

Proposition 13.22.5. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is a quotient map if and only if f is surjective and strongly continuous.*

PROOF: Immediate from definition. \square

Proposition 13.22.6. *Let X and Y be topological spaces. Let $p : X \twoheadrightarrow Y$ be surjective. Then the following are equivalent.*

1. p is a quotient map.
2. p is continuous and maps saturated open sets to open sets.
3. p is continuous and maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

- $\langle 2 \rangle 1.$ ASSUME: p is a quotient map.
- $\langle 2 \rangle 2.$ p is continuous.
- $\langle 2 \rangle 3.$ p maps saturated open sets to open sets.
- $\langle 3 \rangle 1.$ LET: $U \subseteq X$ be a saturated open set.
- $\langle 3 \rangle 2.$ $p^{-1}(p(U)) = U$
- $\langle 3 \rangle 3.$ $p^{-1}(p(U))$ is open in X .
- $\langle 3 \rangle 4.$ $p(U)$ is open in Y .

$\langle 1 \rangle 2. 2 \Rightarrow 3$

- $\langle 2 \rangle 1.$ ASSUME: p is continuous and maps saturated open sets to open sets.
- $\langle 2 \rangle 2.$ LET: C be a saturated closed set in X .
- $\langle 2 \rangle 3.$ $X - C$ is a saturated open set.
- $\langle 2 \rangle 4.$ $Y - p(C)$ is open.
- $\langle 2 \rangle 5.$ $p(C)$ is closed.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

- $\langle 2 \rangle 1.$ ASSUME: p is continuous and maps closed sets to closed sets.
- $\langle 2 \rangle 2.$ LET: $C \subseteq Y$
- $\langle 2 \rangle 3.$ ASSUME: $p^{-1}(C)$ is closed in X .
PROVE: C is closed in Y .
- $\langle 2 \rangle 4.$ $p^{-1}(C)$ is saturated.
- $\langle 2 \rangle 5.$ $p(p^{-1}(C))$ is closed.
- $\langle 2 \rangle 6.$ C is closed.

\square

Corollary 13.22.6.1. *Let X and Y be topological spaces. Let $p : X \twoheadrightarrow Y$ be continuous and surjective. If p is either an open map or a closed map, then p is a quotient map.*

Example 13.22.7. The converse does not hold.

Let $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \vee y = 0\}$. Then the first projection $\pi_1 : A \rightarrow \mathbb{R}$ is a quotient map that is neither an open map nor a closed map.

PROOF:

$\langle 1 \rangle 1.$ π_1 is a quotient map.

$\langle 2 \rangle 1.$ LET: $U \subseteq \mathbb{R}$

$\langle 2 \rangle 2.$ If U is open then $\pi_1^{-1}(U)$ is open.

PROOF: Since $\pi_1^{-1}(U) = (U \times \mathbb{R}) \cap A$.

$\langle 2 \rangle 3.$ If $\pi_1^{-1}(U)$ is open then U is open.

$\langle 3 \rangle 1.$ ASSUME: $\pi_1^{-1}(U)$ is open.

$\langle 3 \rangle 2.$ LET: $x \in U$

$\langle 3 \rangle 3.$ $(x, 0) \in \pi_1^{-1}(U)$

$\langle 3 \rangle 4.$ PICK open neighbourhoods V of x and W of 0 such that $V \times W \subseteq \pi_1^{-1}(U)$

$\langle 3 \rangle 5.$ $V \subseteq U$

PROOF: For all $x' \in V$ we have $(x', 0) \in V \times W \subseteq \pi_1^{-1}(U)$.

$\langle 1 \rangle 2.$ π_1 is not an open map.

PROOF: $\pi_1(((-1, 1) \times (1, 2)) \cap A) = [0, 1)$ which is not open in \mathbb{R} .

$\langle 1 \rangle 3.$ π_1 is not a closed map.

PROOF: $\pi_1(\{(x, 1/x) \in \mathbb{R}^2 : x > 0\}) = (0, +\infty)$ is not closed in \mathbb{R} .

□

Corollary 13.22.7.1. Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be families of topological spaces and $p_i : X_i \rightarrow Y_i$ for all $i \in I$.

1. If every p_i is an open quotient map, then $\prod_{i \in I} p_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$ is an open quotient map.

2. If every p_i is a closed quotient map, then $\prod_{i \in I} p_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$ is a closed quotient map.

Example 13.22.8. The product of two quotient maps is not necessarily a quotient map.

Let Y be the quotient space of \mathbb{R}_K obtained by collapsing the set K to a point. Let $p : \mathbb{R}_K \rightarrow Y$ be the quotient map. Then $q \times q : \mathbb{R}_K^2 \rightarrow Y^2$ is not a quotient map.

PROOF:

$\langle 1 \rangle 1.$ LET: $\Delta = \{(y, y) : y \in Y\}$

$\langle 1 \rangle 2.$ Y is not Hausdorff.

$\langle 2 \rangle 1.$ LET: $*_K \in Y$ be the point such that $q(K) = \{*_K\}$

$\langle 2 \rangle 2.$ ASSUME: for a contradiction U and V are disjoint neighbourhoods of 0 and $*_K$

$\langle 2 \rangle 3.$ $q^{-1}(U)$ and $q^{-1}(V)$ are disjoint open sets with $0 \in q^{-1}(U)$ and $K \subseteq q^{-1}(V)$

$\langle 2 \rangle 4.$ Q.E.D.

PROOF: This is a contradiction.

⟨1⟩3. Δ is not closed in Y^2 .

⟨1⟩4. $(q \times q)^{-1}(\Delta)$ is closed in \mathbb{R}_K^2 .

PROOF: It is $\{(x, x) : x \in \mathbb{R}\} \cup K^2$.

□

Proposition 13.22.9. *Let $\pi : X \twoheadrightarrow S$ be a quotient map. Let Z be a topological space. Let $f : X \rightarrow Z$ be continuous. Then there exists a continuous map $g : S \rightarrow Z$ such that $f = g \circ \pi$ if and only if, for all $s \in S$, we have f is constant on $\pi^{-1}(s)$.*

PROOF: From Theorem 13.22.3. □

Proposition 13.22.10. *Let Z be a topological space. Define $\pi : [0, 1] \rightarrow S^1$ by $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$. Given any continuous function $f : S^1 \rightarrow Z$, we have $f \circ \pi$ is a loop in Z . This defines a bijection between $\mathbf{Top}[S^1, Z]$ and the set of loops in Z .*

PROOF: Since π is a quotient map. □

Definition 13.22.11 (Projective Space). The *projective space* \mathbb{RP}^n is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by \sim where $x \sim \lambda x$ for all $x \in \mathbb{R}^{n+1} - \{0\}$ and $\lambda \in \mathbb{R}$.

Definition 13.22.12 (Torus). The *torus* T is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$.

Definition 13.22.13 (Möbius Band). The *Möbius band* is the quotient of $[0, 1]^2$ by \sim where $(0, y) \sim (1, 1 - y)$.

Definition 13.22.14 (Klein Bottle). The *Klein bottle* is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$.

Proposition 13.22.15. \mathbb{RP}^2 is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (1 - x, 1)$ and $(0, y) \sim (1, 1 - y)$.

PROOF: TODO

Example 13.22.16. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and $\{Y_i\}_{i \in I}$ a family of sets. Let $q_i : X_i \twoheadrightarrow Y_i$ be a surjective function for all $i \in I$. Give each Y_i the quotient topology. It is not true in general that the product topology on $\prod_{i \in I} Y_i$ is the same as the quotient topology induced by $\prod_{i \in I} q_i : \prod_{i \in I} X_i \twoheadrightarrow \prod_{i \in I} Y_i$.

PROOF:

⟨1⟩1. LET: $X^* = \mathbb{R} - \mathbb{Z}_+ + \{b\}$ be the quotient space obtained from \mathbb{R} by identifying the subset \mathbb{Z}_+ to the point b .

⟨1⟩2. LET: $p : \mathbb{R} \rightarrow X^*$ be the quotient map.

PROVE: $p \times \text{id}_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$ is not a quotient map.

⟨1⟩3. For $n \in \mathbb{Z}_+$,

LET: $c_n = \sqrt{2}/n$

⟨1⟩4. For $n \in \mathbb{Z}_+$,

- LET: $U_n = \{(x, y) \in \mathbb{Q} \times \mathbb{R} : n - 1/4 < x < n + 1/4 \text{ and } ((y > x + c_n - n \text{ and } y > -x + c_n + n) \text{ or } (y < x + c_n - n \text{ and } y < -x + c_n + n))\}$
- $\langle 1 \rangle 5$. For all $n \in \mathbb{Z}_+$, U_n is open in $\mathbb{R} \times \mathbb{Q}$
- $\langle 1 \rangle 6$. For all $n \in \mathbb{Z}_+$ we have $\{n\} \times \mathbb{Q} \subseteq U_n$
- $\langle 1 \rangle 7$. LET: $U = \bigcup_{n \in \mathbb{Z}_+} U_n$
- $\langle 1 \rangle 8$. U is open in $\mathbb{R} \times \mathbb{Q}$.
- $\langle 1 \rangle 9$. U is saturated with respect to $p \times \text{id}_{\mathbb{Q}}$.
- $\langle 1 \rangle 10$. LET: $U' = (p \times \text{id}_{\mathbb{Q}})(U)$
- $\langle 1 \rangle 11$. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$.

Proposition 13.22.17. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $\phi : Y \rightarrow X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \rightarrow X$ such that $\pi \circ \Phi = \phi|_U$. Then ϕ is continuous.

Proposition 13.22.18. *Let X be a topological space and \sim an equivalence relation on X . If X/\sim is Hausdorff then every equivalence class of \sim is closed in X .*

Definition 13.22.19. Let X be a topological space and $A_1, \dots, A_r \subseteq X$. Then $X/A_1, \dots, A_r$ is the quotient space of X with respect to \sim where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 13.22.20 (Cone). Let X be a topological space. The *cone over X* is the space $(X \times [0, 1])/(X \times \{1\})$.

Definition 13.22.21 (Suspension). Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 13.22.22 (Wedge Product). Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 13.22.23 (Smash Product). Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 13.22.24. $D^n/S^{n-1} \cong S^n$

PROOF:

- $\langle 1 \rangle 1$. LET: $\phi : D^n/S^{n-1} \rightarrow S^n$ be the function induced by the map $D^n \rightarrow S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.
- $\langle 1 \rangle 2$. ϕ is a bijection.
- $\langle 1 \rangle 3$. ϕ is a homeomorphism.

PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.

□

13.23 Box Topology

Definition 13.23.1 (Box Topology). Let $\{X_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $X = \prod_{i \in I} X_i$ is the topology generated by the basis $\mathcal{B} = \{\prod_{i \in I} U_i : \{U_i\}_{i \in I} \text{ is a family with each } U_i \text{ an open set in } X_i\}$.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1. \bigcup \mathcal{B} = X$

PROOF: Since $\prod_{i \in I} X_i \in \mathcal{B}$.

$\langle 1 \rangle 2.$ For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

$\langle 2 \rangle 1.$ LET: $B_1, B_2 \in \mathcal{B}$

$\langle 2 \rangle 2.$ LET: $x \in B_1 \cap B_2$

$\langle 2 \rangle 3.$ PICK a family $\{U_i\}_{i \in I}$ such that $B_1 = \prod_{i \in I} U_i$.

$\langle 2 \rangle 4.$ PICK a family $\{V_i\}_{i \in I}$ such that $B_2 = \prod_{i \in I} V_i$.

$\langle 2 \rangle 5.$ LET: $B_3 = \prod_{i \in I} (U_i \cap V_i)$

$\langle 2 \rangle 6.$ $x \in B_3 \subseteq B_1 \cap B_2$

□

Proposition 13.23.2. *The box topology is finer than the product topology.*

PROOF: Immediate from definitions. □

Proposition 13.23.3. *On a finite family of topological spaces, the box topology and the product topology are the same.*

PROOF: Immediate from definitions. □

Proposition 13.23.4. *The box topology is strictly finer than the product topology on the Hilbert cube.*

PROOF: The set $\prod_{n=0}^{\infty} (0, 1/(n+1)^2)$ is open in the box topology but not in the product topology. □

13.23.1 Bases

Proposition 13.23.5. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. For all $i \in I$, let \mathcal{B}_i be a basis for the topology on X_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i : \forall i \in I, B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} X_i$.*

PROOF:

$\langle 1 \rangle 1.$ For every family $\{B_i\}_{i \in I}$ where $\forall i \in I, B_i \in \mathcal{B}_i$, we have $\prod_{i \in I} B_i$ is open in the box topology.

PROOF: Since each B_i is open in X_i .

$\langle 1 \rangle 2.$ For any open set U in the box topology and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 1.$ LET: U be a set open in the box topology.

$\langle 2 \rangle 2.$ LET: $x \in U$

- $\langle 2 \rangle 3$. PICK a family $\{U_i\}_{i \in I}$ where each U_i is open in X_i such that $x \in \prod_{i \in I} U_i \subseteq U$
 $\langle 2 \rangle 4$. For $i \in I$, choose $B_i \in \mathcal{B}_i$ such that $x_i \in B_i \subseteq U_i$.
 $\langle 2 \rangle 5$. $\prod_{i \in I} B_i \in \mathcal{B}$
 $\langle 2 \rangle 6$. $x \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} U_i \subseteq U$

□

13.23.2 Subspaces

Proposition 13.23.6. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the box topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology that $\prod_{i \in I} Y_i$ inherits as a subspace of $\prod_{i \in I} X_i$ under the box topology.*

PROOF: A basis for the box topology is

$$\begin{aligned}
 & \left\{ \prod_{i \in I} V_i : V_i \text{ open in } Y_i \right\} \\
 &= \left\{ \prod_{i \in I} (U_i \cap Y_i) : U_i \text{ open in } X_i \right\} \\
 &= \left\{ \prod_{i \in I} U_i \cap \prod_{i \in I} Y_i : U_i \text{ open in } X_i \right\}
 \end{aligned}$$

which is a basis for the subspace topology by Proposition 13.3.12. □

13.23.3 Closure

Proposition 13.23.7. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

- $\langle 1 \rangle 1$. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
 $\langle 2 \rangle 1$. LET: $x \in \prod_{i \in I} \overline{A_i}$
 $\langle 2 \rangle 2$. For any family $\{U_i\}_{i \in I}$ where each U_i is open in X_i , if $x \in \prod_{i \in I} U_i$ then $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$.
 $\langle 3 \rangle 1$. LET: $\{U_i\}_{i \in I}$ be a family where each U_i is open in X_i .
 $\langle 3 \rangle 2$. ASSUME: $x \in \prod_{i \in I} U_i$
 $\langle 3 \rangle 3$. For all $i \in I$ we have U_i intersects A_i
 PROOF: Since $\pi_i(x) \in U_i$ and U_i is a neighbourhood of $\pi_i(x)$.
 $\langle 3 \rangle 4$. $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$
 $\langle 2 \rangle 3$. $x \in \overline{\prod_{i \in I} A_i}$
 PROOF: Proposition 13.9.9.
 $\langle 1 \rangle 2$. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
 $\langle 2 \rangle 1$. LET: $x \in \overline{\prod_{i \in I} A_i}$
 $\langle 2 \rangle 2$. LET: $i \in I$

PROVE: $\pi_i(x) \in \overline{A_i}$

- $\langle 2 \rangle 3$. LET: U be a neighbourhood of $\pi_i(x)$ in X_i
 $\langle 2 \rangle 4$. $\pi_i^{-1}(U)$ is a neighbourhood of x in $\prod_{i \in I} X_i$
 $\langle 2 \rangle 5$. PICK $y \in \pi_i^{-1}(U) \cap \prod_{i \in I} A_i$
 $\langle 2 \rangle 6$. $\pi_i(y) \in U \cap A_i$

□

13.24 Separations

Definition 13.24.1 (Separation). Let X be a topological space. A *separation* of X is a pair (U, V) of disjoint nonempty oped subsets in X such that $U \cup V = X$.

Subspaces

Proposition 13.24.2. Let X be a topological space and Y a subspace of X . Then a separation of Y is a pair (A, B) of disjoint nonempty subsets of Y , neither of which contains a limit point of the other, such that $A \cup B = Y$.

PROOF: Since the following are equivalent:

- Neither of A and B contains a limit point of the other.
- A contains all its own limit points in Y , and B contains all its own limit points in Y .
- A and B are closed in Y .

□

13.25 Connected Spaces

Definition 13.25.1 (Connected). A topological space is *connected* iff it has no separation.

13.25.1 The Real Numbers

Example 13.25.2. The space \mathbb{R}_l is disconnected. The sets $(-\infty, 0)$ and $[0, +\infty)$ form a separation.

13.25.2 The Indiscrete Topology

Example 13.25.3. Any indiscrete space is connected.

13.25.3 The Cofinite Topology

Example 13.25.4. Any infinite set under the cofinite topology is connected.

PROOF:

⟨1⟩1. LET: X be an infinite set under the cofinite topology.

⟨1⟩2. ASSUME: for a contradiction (C, D) is a separation of X .

⟨1⟩3. $X = (X - C) \cup (X - D) \cup (C \cap D)$

⟨1⟩4. Q.E.D.

PROOF: This is a contradiction since X is infinite, $X - C$ and $X - D$ are finite, and $C \cap D = \emptyset$.

□

Example 13.25.5. The rationals are disconnected. For any irrational a , we have $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of \mathbb{Q} .

Example 13.25.6. \mathbb{R}^ω under the box topology is not connected. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 13.25.7. *A topological space X is connected if and only if the only sets that are both open and closed are \emptyset and X .*

PROOF: Since (U, V) is a separation of X iff U is both open and closed and $V = X - U$. □

13.25.4 Finer and Coarser

Proposition 13.25.8. *Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . Assume $\mathcal{T} \subseteq \mathcal{T}'$. If \mathcal{T}' is connected then \mathcal{T} is connected.*

PROOF: If (C, D) is a separation of (X, \mathcal{T}) then it is a separation of (X, \mathcal{T}') . □

13.25.5 Boundary

Proposition 13.25.9. *Let X be a topological space. Let $A \subseteq X$. Let C be a connected subspace of X . If C intersects A and $X - A$ then C intersects ∂A .*

PROOF: Otherwise $(C \cap \overline{A}, C \cap \overline{X - A})$ would be a separation of C . □

13.25.6 Continuous Functions

Proposition 13.25.10. *The continuous image of a connected space is connected.*

PROOF:

⟨1⟩1. LET: X and Y be topological spaces.

⟨1⟩2. LET: $f : X \rightarrow Y$ be a surjective continuous function.

⟨1⟩3. LET: (C, D) be a separation of Y .

⟨1⟩4. $(f^{-1}(C), f^{-1}(D))$ is a separation of X .

□

13.25.7 Subspaces

Proposition 13.25.11. *Let X be a topological space. Let (C, D) be a separation of X . Let Y be a connected subspace of X . Then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $(Y \cap C, Y \cap D)$ would be a separation of Y . \square

Proposition 13.25.12. *Let X be a topological space. Let \mathcal{A} be a set of connected subspaces of X and B a connected subspace of X . Assume that, for all $A \in \mathcal{A}$, we have $A \cap B \neq \emptyset$. Then $\bigcup \mathcal{A} \cup B$ is connected.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction (C, D) is a separation of $\bigcup \mathcal{A} \cup B$.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $B \subseteq C$

PROOF: Proposition 13.25.11.

$\langle 1 \rangle 3$. For all $A \in \mathcal{A}$ we have $A \subseteq C$

PROOF: Proposition 13.25.11.

$\langle 1 \rangle 4$. $D = \emptyset$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

\square

Proposition 13.25.13. *Let X be a topological space. Let A be a connected subspace of X . Let B be a subspace of X . If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction (C, D) is a separation of B .

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $A \subseteq C$

PROOF: Proposition 13.25.11.

$\langle 1 \rangle 3$. $\overline{A} \subseteq \overline{C}$

$\langle 1 \rangle 4$. $\overline{C} \cap D = \emptyset$

$\langle 1 \rangle 5$. $B \cap D = \emptyset$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

\square

Corollary 13.25.13.1. *The topologist's sine curve is connected.*

PROOF: The set $\{(x, \sin 1/x) : 0 < x \leq 1\}$ is connected, since it is the continuous image of the connected set $(0, 1]$. The topologist's sine curve is its closure, hence connected by Proposition 13.25.13. \square

Proposition 13.25.14. *Let X be a topological space. Let (A_n) be a sequence of connected subspaces of X such that, for all n , we have $A_n \cap A_{n+1} \neq \emptyset$. Then $\bigcup_n A_n$ is connected.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction (C, D) is a separation of $\bigcup_n A_n$

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $A_0 \subseteq C$

PROOF: Proposition 13.25.11.

- ⟨1⟩3. $\forall n. A_n \subseteq C$
- ⟨2⟩1. ASSUME: as induction hypothesis $A_n \subseteq C$
- ⟨2⟩2. PICK $x \in A_n \cap A_{n+1}$
- ⟨2⟩3. $x \in C$
- ⟨2⟩4. $A_{n+1} \subseteq C$

PROOF: Proposition 13.25.11.

- ⟨1⟩4. $\bigcup_n A_n \subseteq C$
- ⟨1⟩5. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 13.25.15. *Let X be a connected topological space. Let $Y \subseteq X$ be connected. Let (A, B) be a separation of $X - Y$. Then $Y \cup A$ and $Y \cup B$ are connected.*

PROOF:

- ⟨1⟩1. $Y \cup A$ is connected.
- ⟨2⟩1. ASSUME: for a contradiction (C, D) is a separation of $Y \cup A$
- ⟨2⟩2. ASSUME: w.l.o.g. $Y \subseteq C$
- ⟨2⟩3. PICK C' and D' open in X such that $C = C' \cap (Y \cup A)$ and $D = D' \cap (Y \cup A)$
- ⟨2⟩4. $D = D' \cap A$
- ⟨2⟩5. $C' \cap D' \cap A = \emptyset$
- ⟨2⟩6. $A \subseteq C' \cup D'$
- ⟨2⟩7. PICK A' and B' open in X such that $A = A' - Y$ and $B = B' - Y$
- ⟨2⟩8. $A' \cap B' \subseteq Y$
- ⟨2⟩9. $X - Y \subseteq A' \cup B'$
- ⟨2⟩10. $A' \subseteq C' \cup D'$
- ⟨2⟩11. $(D' \cap A', B' \cup C')$ is a separation of X .
- ⟨1⟩2. $Y \cup B$ is connected.

PROOF: Similar.

□

13.25.8 Order Topology

Proposition 13.25.16. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if X is a linear continuum.*

PROOF:

- ⟨1⟩1. If L is a linear continuum then L is connected.
- ⟨2⟩1. LET: L be a linear continuum.
- ⟨2⟩2. ASSUME: for a contradiction (A, B) is a separation of L .
- ⟨2⟩3. PICK $a \in A$ and $b \in B$.
- ⟨2⟩4. ASSUME: w.l.o.g. $a < b$
- ⟨2⟩5. LET: $c = \sup\{x \in A : x < b\}$
- ⟨2⟩6. $c \notin A$

- ⟨3⟩1. ASSUME: for a contradiction $c \in A$.
- ⟨3⟩2. PICK $e > c$ such that $[c, e] \subseteq A$.
- ⟨3⟩3. PICK z such that $c < z < e$.
- ⟨3⟩4. $z \in A$
- ⟨3⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩5.

- ⟨2⟩7. $c \notin B$
- ⟨3⟩1. ASSUME: for a contradiction $c \in B$.
- ⟨3⟩2. PICK $d < c$ such that $(d, c] \subseteq B$.
- ⟨3⟩3. PICK z such that $d < z < c$
- ⟨3⟩4. z is an upper bound for $\{x \in A : x < b\}$
- ⟨3⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩5.

- ⟨2⟩8. Q.E.D.

PROOF: This is a contradiction.

- ⟨1⟩2. If L is connected then L is a linear continuum.

- ⟨2⟩1. ASSUME: L is connected.
- ⟨2⟩2. L is dense.
- ⟨3⟩1. LET: $a, b \in L$ with $a < b$.
- ⟨3⟩2. ASSUME: for a contradiction there is no c such that $a < c < b$.
- ⟨3⟩3. $((-\infty, b), (a, +\infty))$ is a separation of L .

- ⟨2⟩3. L has the least upper bound property.

- ⟨3⟩1. ASSUME: for a contradiction $S \subseteq L$ is a nonempty set bounded above with no least upper bound.

- ⟨3⟩2. LET: $S \uparrow$ be the set of upper bounds for S .
- ⟨3⟩3. LET: $S \uparrow \downarrow$ be the set of lower bounds for $S \uparrow$.

PROVE: $(S \uparrow \downarrow, S \uparrow)$ is a separation of L .

- ⟨3⟩4. $S \uparrow \neq \emptyset$

PROOF: Since S is bounded above.

- ⟨3⟩5. $S \uparrow \downarrow \neq \emptyset$

PROOF: Since $\emptyset \neq S \subseteq S \uparrow \downarrow$.

- ⟨3⟩6. $S \uparrow$ is open.

- ⟨4⟩1. LET: $u \in S \uparrow$

- ⟨4⟩2. PICK $v \in S \uparrow$ such that $v < u$

PROOF: Since u is not the least upper bound for S .

- ⟨4⟩3. $u \in (v, +\infty) \subseteq S \uparrow$

- ⟨3⟩7. $S \uparrow \downarrow$ is open.

- ⟨4⟩1. LET: $l \in S \uparrow \downarrow$

- ⟨4⟩2. $l \notin S \uparrow$

PROOF: Since l is not the least upper bound for S .

- ⟨4⟩3. PICK $s \in S$ such that $l < s$

- ⟨4⟩4. $l \in (-\infty, s) \subseteq S \uparrow \downarrow$

- ⟨3⟩8. $S \uparrow \cap S \uparrow \downarrow \neq \emptyset$

PROOF: An element of both would be a least upper bound for S .

- ⟨3⟩9. $S \uparrow \cup S \uparrow \downarrow = L$

- ⟨4⟩1. LET: $x \in L$

- $\langle 4 \rangle 2$. ASSUME: $x \notin S \uparrow$
- $\langle 4 \rangle 3$. There exists $s \in S$ such that $x < s$.
- $\langle 4 \rangle 4$. $\forall u \in S \uparrow . x < u$
- $\langle 4 \rangle 5$. $x \in S \uparrow \downarrow$

□

Theorem 13.25.17 (Intermediate Value Theorem). *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \rightarrow Y$ be continuous. Let $a, b \in X$ and $r \in Y$. If $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.*

PROOF: Otherwise $\{x \in X : f(x) < r\}$ and $\{x \in X : f(x) > r\}$ would form a separation of X . □

Corollary 13.25.17.1. *Every continuous function $[0, 1] \rightarrow [0, 1]$ has a fixed point.*

PROOF:

- $\langle 1 \rangle 1$. LET: $f : [0, 1] \rightarrow [0, 1]$ be continuous.
- $\langle 1 \rangle 2$. LET: $g : [0, 1] \rightarrow [-1, 1]$ be the function $g(x) = f(x) - x$.
- $\langle 1 \rangle 3$. $g(0) \geq 0$
- $\langle 1 \rangle 4$. $g(1) \leq 0$
- $\langle 1 \rangle 5$. There exists $x \in [0, 1]$ such that $g(x) = 0$.

PROOF: Intermediate Value Theorem.

- $\langle 1 \rangle 6$. There exists $x \in [0, 1]$ such that $f(x) = x$.

□

13.25.9 Product Topology

Proposition 13.25.18. *The product of a family of connected spaces is connected.*

PROOF:

- $\langle 1 \rangle 1$. The product of two connected spaces is connected.

PROOF:

- $\langle 2 \rangle 1$. LET: X and Y be connected topological spaces.
- $\langle 2 \rangle 2$. ASSUME: w.l.o.g. X and Y are nonempty.
- $\langle 2 \rangle 3$. PICK $(a, b) \in X \times Y$
- $\langle 2 \rangle 4$. $X \times \{b\}$ is connected.

PROOF: It is homeomorphic to X .

- $\langle 2 \rangle 5$. For all $x \in X$ we have $\{x\} \times Y$ is connected.

PROOF: It is homeomorphic to Y .

- $\langle 2 \rangle 6$. For all $x \in X$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.

PROOF: Proposition 13.25.12.

- $\langle 2 \rangle 7$. $X \cup Y$ is connected.

PROOF: Proposition 13.25.12 since $X \cup Y = \bigcup_{x \in X} ((X \times \{b\}) \cup (\{x\} \times Y))$ and the subspaces all have the point (a, b) in common.

- $\langle 1 \rangle 2$. LET: $\{X_i\}_{i \in I}$ be a family of connected spaces.

- <1>3. LET: $X = \prod_{i \in I} X_i$
 <1>4. ASSUME: w.l.o.g. each X_i is nonempty.
 <1>5. PICK $a \in X$
 <1>6. For every finite $K \subseteq I$,
 LET: $X_K = \{x \in X : \forall i \notin K, \pi_i(x) = \pi_i(a)\}$
 <1>7. For every finite $K \subseteq I$, we have X_K is connected.
 PROOF: It is homeomorphic to $\prod_{i \in K} X_i$ which is connected by <1>1.
 <1>8. LET: $Y = \bigcup_{K \text{ a finite subset of } I} X_K$
 <1>9. Y is connected.
 PROOF: Proposition 13.25.12 since $a \in X_K$ for all K .
 <1>10. $X = \overline{Y}$
 <2>1. LET: $x \in X$
 <2>2. LET: U be a neighbourhood of x .
 PROVE: U intersects Y .
 <2>3. PICK a finite subset K of I and U_i open in each X_i such that $U_i = X_i$
 for all $i \notin K$, and $x \in \prod_i U_i \subseteq U$
 <2>4. LET: $y \in X$ be the point with $\pi_i(y) = \pi_i(x)$ for $i \in K$ and $\pi_i(y) = \pi_i(a)$
 for $i \notin K$
 <2>5. $y \in U \cap Y$
 <1>11. X is connected.
 PROOF: Proposition 13.25.13.
 □

Proposition 13.25.19. *Let X and Y be topological spaces. Let A be a proper subset of X and B a proper subset of Y . Then $(X \times Y) - (A \times B)$ is connected.*

PROOF:

- <1>1. PICK $x_0 \in X - A$
 <1>2. PICK $y_0 \in Y - B$
 <1>3. LET: $C = ((X - A) \times Y) \cup (X \times \{y_0\})$
 <1>4. LET: $D = (\{x_0\} \times Y) \cup (X \times (Y - B))$
 <1>5. C is connected.
 <2>1. $C = \bigcup_{x \in X - A} (\{x\} \times Y) \cup (X \times \{y_0\})$
 <2>2. For all $x \in X - A$ we have $\{x\} \times Y$ is connected.
 PROOF: It is homeomorphic to Y .
 <2>3. $X \times \{y_0\}$ is connected.
 PROOF: It is homeomorphic to X .
 <2>4. For all $x \in X - A$ we have $(x, y_0) \in (\{x\} \times Y) \cap (X \times \{y_0\})$
 <2>5. C is connected.
 PROOF: Proposition 13.25.12.
 <1>6. D is connected.
 PROOF: Similar.
 <1>7. $(X \times Y) - (A \times B) = C \cup D$
 <1>8. $(X \times Y) - (A \times B)$ is connected.
 PROOF: Proposition 13.25.12 since $(x_0, y_0) \in C \cap D$.
 □

13.25.10 Quotient Spaces

Proposition 13.25.20. *A quotient of a connected space is connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $p : X \rightarrow Y$ be a quotient map.

$\langle 1 \rangle 2$. If (C, D) is a separation of Y then $(p^{-1}(C), p^{-1}(D))$ is a separation of X .
 \square

Proposition 13.25.21. *Let $p : X \rightarrow Y$ be a quotient map. Assume that Y is connected, for all $y \in Y$, we have $p^{-1}(y)$ is connected. Then X is connected.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction (A, B) is a separation of X .

$\langle 1 \rangle 2$. For all $y \in Y$, either $p^{-1}(y) \subseteq A$ or $p^{-1}(y) \subseteq B$.

$\langle 1 \rangle 3$. $(\{y \in Y : p^{-1}(y) \subseteq A\}, \{y \in Y : p^{-1}(y) \subseteq B\})$ form a separation of Y .

$\langle 1 \rangle 4$. Q.E.D.

PROOF: This is a contradiction.

\square

13.26 T_1 Spaces

Definition 13.26.1 (T_1). A topological space is T_1 iff every one-point set is closed.

Proposition 13.26.2. *A topological space is T_1 iff every finite set is closed.*

PROOF: Since the union of finitely many closed sets is closed. \square

Proposition 13.26.3. *Let X be a topological space. Then X is T_1 if and only if, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x .*

PROOF:

$\langle 1 \rangle 1$. If X is T_1 then, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x .

$\langle 2 \rangle 1$. ASSUME: X is T_1 .

$\langle 2 \rangle 2$. LET: $x, y \in X$

$\langle 2 \rangle 3$. ASSUME: $x \neq y$

$\langle 2 \rangle 4$. $X - \{y\}$ is a neighbourhood of x that does not contain y .

$\langle 2 \rangle 5$. $X - \{x\}$ is a neighbourhood of y that does not contain x .

$\langle 1 \rangle 2$. If, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x , then X is T_1 .

$\langle 2 \rangle 1$. ASSUME: For all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x .

- $\langle 2 \rangle 2$. LET: $x \in X$
 PROVE: $\{x\}$ is closed.
 $\langle 2 \rangle 3$. LET: $y \in X - \{x\}$
 $\langle 2 \rangle 4$. PICK a neighbourhood U of y that does not contain x .
 $\langle 2 \rangle 5$. $y \in U \subseteq X - \{x\}$

□

13.26.1 Limit Points

Proposition 13.26.4. *Let X be a T_1 space. Let $A \subseteq X$ and $l \in X$. Then l is a limit point of A if and only if every neighbourhood of l contains infinitely many points of A .*

PROOF:

- $\langle 1 \rangle 1$. If l is a limit point of A then every neighbourhood of l contains infinitely many points of A .
 $\langle 2 \rangle 1$. ASSUME: l is a limit point of A .
 $\langle 2 \rangle 2$. LET: U be a neighbourhood of l .
 $\langle 2 \rangle 3$. ASSUME: for a contradiction $U \cap A - \{l\}$ is finite.
 $\langle 2 \rangle 4$. $U \cap A - \{l\}$ is closed.
 PROOF: Since X is T_1 .
 $\langle 2 \rangle 5$. $U - (A - \{l\})$ is a neighbourhood of l .
 $\langle 2 \rangle 6$. $U - (A - \{l\})$ intersects A .
 $\langle 2 \rangle 7$. Q.E.D.
 $\langle 1 \rangle 2$. If every neighbourhood of l contains infinitely many points of A then l is a limit point of A .

PROOF: Immediate from definitions.

□

13.27 Hausdorff Spaces

Definition 13.27.1 (Hausdorff). A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 13.27.2. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be a Hausdorff space.
 $\langle 1 \rangle 2$. LET: (a_n) be a sequence in X and $l, m \in X$
 $\langle 1 \rangle 3$. ASSUME: $a_n \rightarrow l$ and $a_n \rightarrow m$
 $\langle 1 \rangle 4$. ASSUME: for a contradiction $l \neq m$
 $\langle 1 \rangle 5$. PICK disjoint open sets U and V with $l \in U$ and $m \in V$
 $\langle 1 \rangle 6$. PICK M, N such that $\forall n \geq M. a_n \in U$ and $\forall n \geq N. a_n \in V$
 $\langle 1 \rangle 7$. $a_{\max(M, N)} \in U \cap V$
 $\langle 1 \rangle 8$. Q.E.D.

PROOF: This contradicts the fact that $U \cap V = \emptyset$.

□

Example 13.27.3. We cannot weaken the hypothesis from being Hausdorff to being T_1 .

In the cofinite topology on any infinite set, every sequence converges to every point.

Proposition 13.27.4. *Any linearly ordered set is Hausdorff under the order topology.*

PROOF:

⟨1⟩1. LET: X be a linearly ordered set under the order topology.

⟨1⟩2. LET: $a, b \in X$ with $a \neq b$.

⟨1⟩3. ASSUME: w.l.o.g. $a < b$.

⟨1⟩4. CASE: There exists $c \in X$ such that $a < c < b$.

⟨2⟩1. LET: $U = (-\infty, c)$

⟨2⟩2. LET: $V = (c, +\infty)$

⟨2⟩3. U and V are disjoint open sets with $a \in U$ and $b \in V$

⟨1⟩5. CASE: There is no $c \in X$ such that $a < c < b$.

⟨2⟩1. LET: $U = (-\infty, b)$

⟨2⟩2. LET: $V = (a, +\infty)$

⟨2⟩3. U and V are disjoint open sets with $a \in U$ and $b \in V$

□

Proposition 13.27.5. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a Hausdorff space.

⟨1⟩2. LET: Y be a subspace of X .

⟨1⟩3. LET: $a, b \in Y$ with $a \neq b$.

⟨1⟩4. PICK disjoint open sets U and V in X with $a \in U$ and $b \in V$.

⟨1⟩5. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y with $a \in U \cap Y$ and $b \in V \cap Y$.

□

Proposition 13.27.6. *The disjoint union of two Hausdorff spaces is Hausdorff.*

Proposition 13.27.7. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \rightarrow B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X , then $f = g$.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.

⟨1⟩2. PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.

⟨1⟩3. PICK $x \in f^{-1}(U) \cap g^{-1}(V)$

⟨1⟩4. $f(x) = g(x) \in U \cap V$

⟨1⟩5. Q.E.D.

PROOF: This is a contradiction.

□

13.27.1 Product Topology

Proposition 13.27.8. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

- ⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
- ⟨1⟩2. LET: $x, y \in \prod_{i \in I} X_i$ with $x \neq y$.
- ⟨1⟩3. PICK $i \in I$ such that $\pi_i(x) \neq \pi_i(y)$
- ⟨1⟩4. PICK disjoint open sets U and V in X_i such that $\pi_i(x) \in U$ and $\pi_i(y) \in V$.
- ⟨1⟩5. $x \in \pi_i^{-1}(U)$ and $y \in \pi_i^{-1}(V)$.

□

13.27.2 Box Topology

Proposition 13.27.9. *The box product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

- ⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
- ⟨1⟩2. LET: $x, y \in \prod_{i \in I} X_i$ with $x \neq y$.
- ⟨1⟩3. PICK $i \in I$ such that $\pi_i(x) \neq \pi_i(y)$
- ⟨1⟩4. PICK disjoint open sets U and V in X_i such that $\pi_i(x) \in U$ and $\pi_i(y) \in V$.
- ⟨1⟩5. $x \in \pi_i^{-1}(U)$ and $y \in \pi_i^{-1}(V)$.

□

13.27.3 T_1 Spaces

Proposition 13.27.10. *Every Hausdorff space is T_1 .*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space.
- ⟨1⟩2. LET: $a \in X$
 PROVE: $X - \{a\}$ is open.
- ⟨1⟩3. LET: $x \in X - \{a\}$
- ⟨1⟩4. PICK disjoint open sets U and V with $a \in U$ and $x \in V$
- ⟨1⟩5. $x \in V \subseteq X - U \subseteq X - \{a\}$

□

Example 13.27.11. The converse does not hold. If X is an infinite set under the cofinite topology, then X is T_1 but not Hausdorff.

Proposition 13.27.12. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$ be uniformly continuous. Let \hat{X} and \hat{Y} be the completions of X and Y . Then f extends uniquely to a continuous map $\hat{X} \rightarrow \hat{Y}$.*

PROOF: The extension maps $\lim_{n \rightarrow \infty} x_n$ to $\lim_{n \rightarrow \infty} f(x_n)$. □

Proposition 13.27.13. *Let X be a topological space. Then X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in X^2 .*

PROOF:

Δ is closed

$\Leftrightarrow X^2 - \Delta$ is open

$\Leftrightarrow \forall x, y \in X ((x, y) \notin \Delta \Rightarrow \exists V, W \text{ open in } X (x \in V \wedge y \in W \wedge V \times W \subseteq X^2 - \Delta))$

$\Leftrightarrow \forall x, y \in X (x \neq y \Rightarrow \exists V, W \text{ open in } X (x \in V \wedge y \in W \wedge V \cap W = \emptyset))$

$\Leftrightarrow X$ is Hausdorff □

13.28 Separable Spaces

Definition 13.28.1 (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

13.29 Sequential Compactness

Definition 13.29.1 (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

13.30 Compactness

Definition 13.30.1 (Compact). A topological space is *compact* iff every open cover has a finite subcover.

Proposition 13.30.2. Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P . Then $X \in P$.

PROOF:

$\langle 1 \rangle$ 1. P is an open cover of X

$\langle 1 \rangle$ 2. PICK a finite subcover $U_1, \dots, U_n \in P$

$\langle 1 \rangle$ 3. $X = U_1 \cup \dots \cup U_n \in P$

□

Corollary 13.30.2.1. Let f be a compact space and $f : X \rightarrow \mathbb{R}$ be locally bounded. Then f is bounded.

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. □

Proposition 13.30.3. The continuous image of a compact space is compact.

Proposition 13.30.4. A closed subspace of a compact space is compact.

Proposition 13.30.5. Let X and Y be nonempty spaces. Then the following are equivalent.

1. X and Y are compact.
2. $X + Y$ is compact.
3. $X \times Y$ is compact.

Proposition 13.30.6. *A compact subspace of a Hausdorff space is closed.*

Proposition 13.30.7. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 13.30.8. *A first countable compact space is sequentially compact.*

13.31 Gluing

Definition 13.31.1 (Gluing). Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \rightarrow Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where \sim is the equivalence relation generated by $x \sim \phi(x)$ for all $x \in X_0$.

Proposition 13.31.2. *Y is a subspace of $Y \cup_\phi X$.*

Definition 13.31.3. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0, 1])/\alpha$ is the quotient space of $X \times [0, 1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x \in X$.

Definition 13.31.4 (Möbius Strip). The *Möbius strip* is $([-1, 1] \times [0, 1])/\alpha$ where $\alpha(x) = -x$.

Definition 13.31.5 (Klein Bottle). The *Klein bottle* is $(S^1 \times [0, 1])/\alpha$ where $\alpha(z) = \bar{z}$.

Proposition 13.31.6. *Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\text{id}_M} M \cong K$.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{i\theta/2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-i\theta/2}, t)$.

$\langle 1 \rangle 2$. f induces a bijection $M \cup_{\text{id}_M} M \approx K$

$\langle 1 \rangle 3$. f is a homeomorphism.

□

13.32 Homogeneous Spaces

Definition 13.32.1 (Homogeneous). A topological space X is *homogeneous* iff, for any $x, y \in X$, there exists a homeomorphism $f : X \cong X$ such that $f(x) = y$.

13.33 Regular Spaces

Definition 13.33.1 (Regular). A topological space X is *regular* iff it is T_1 and, for every closed set A and point $x \notin A$, there exist disjoint open sets U and V with $A \subseteq U$ and $x \in V$.

13.34 Totally Disconnected Spaces

Definition 13.34.1 (Totally Disconnected). A topological space X is *totally disconnected* iff the only connected subspaces are the one-point subspaces.

Example 13.34.2. Every discrete space is totally disconnected.

Example 13.34.3. The rationals are totally disconnected.

13.35 Path Connected Spaces

Definition 13.35.1 (Path-connected). A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \rightarrow X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

13.35.1 The Ordered Square

Proposition 13.35.2. *The ordered square is not path connected.*

PROOF:

- $\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [a, b] \rightarrow I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.
 $\langle 1 \rangle 2$. p is surjective.

PROOF: Intermediate Value Theorem.

- $\langle 1 \rangle 3$. For all $x \in [0, 1]$, the set $p^{-1}(\{x\} \times (0, 1))$ is a nonempty open set in $[0, 1]$.
 $\langle 1 \rangle 4$. For all $x \in [0, 1]$ choose a rational $q_x \in p^{-1}(\{x\} \times (0, 1))$.
 $\langle 1 \rangle 5$. The mapping that maps x to q_x is an injective function $[0, 1] \rightarrow \mathbb{Q}$.
 $\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts the fact that $[0, 1]$ is uncountable and \mathbb{Q} is countable.

□

13.35.2 Punctured Euclidean Space

Proposition 13.35.3. *For $n > 1$, the punctured Euclidean space $\mathbb{R}^n - \{0\}$ is path connected.*

PROOF: Given points x and y , take the straight line from x to y if this does not pass through 0. Otherwise pick a point z not on this line, and take the two straight lines from x to z then from z to y . □

13.35.3 The Topologist's Sine Curve

Proposition 13.35.4. *The topologist's sine curve is not path connected.*

PROOF:

- $\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) : 0 < x \leq 1\}$
 $\langle 1 \rangle 2$. ASSUME: for a contradiction $p : [0, 1] \rightarrow \bar{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

- <1>3. LET: b be the largest element of $p^{-1}(\{0\} \times [-1, 1])$
 <1>4. For n a positive integer, choose t_n such that $b < t_n < ((n-1)b + 1)/n$
 and $\pi_2(p(t_n)) = (-1)^n$
 <1>5. $t_n \rightarrow b$ as $n \rightarrow \infty$
 <1>6. $(p(t_n))$ does not converge.
 <1>7. Q.E.D.

PROOF: This is a contradiction.

□

13.35.4 Continuous Functions

Proposition 13.35.5. *The continuous image of a path connected space is path connected.*

PROOF:

- <1>1. LET: X be a path connected space and Y a topological space.
 <1>2. LET: $f : X \rightarrow Y$ be a surjective continuous function.
 PROVE: Y is path connected.
 <1>3. LET: $a, b \in Y$
 <1>4. PICK $x, y \in X$ with $f(x) = a$ and $f(y) = b$.
 <1>5. PICK a path $p : [0, 1] \rightarrow X$ from x to y .
 <1>6. $f \circ p$ is a path from a to b .

□

13.35.5 Subspaces

Proposition 13.35.6. *Let $\{X\}$ be a topological space. Let \mathcal{A} be a set of connected subspaces of X . If $\bigcap \mathcal{A} \neq \emptyset$ then $\bigcup \mathcal{A}$ is connected.*

PROOF:

- <1>1. PICK $a \in \bigcap \mathcal{A}$
 <1>2. PICK $x, y \in \bigcup \mathcal{A}$
 <1>3. PICK $A, B \in \mathcal{A}$ with $x \in A$ and $y \in B$.
 <1>4. PICK a path p from x to a in A , and a path q from a to y in B .
 <1>5. The concatenation of p and q is a path from x to y in $\bigcup \mathcal{A}$.

□

Proposition 13.35.7. *A quotient of a path connected space is path connected.*

13.35.6 Product Topology

Proposition 13.35.8. *The product of a family of path connected spaces is path connected.*

PROOF:

- <1>1. LET: $\{X_i\}_{i \in I}$ be a family of path connected spaces.
 <1>2. LET: $x, y \in \prod_{i \in I} X_i$
 <1>3. For $i \in I$, PICK a path $p_i : [0, 1] \rightarrow X_i$ from $\pi_i(x)$ to $\pi_i(y)$

$\langle 1 \rangle 4.$ $\lambda t \in [0, 1]. \lambda i \in I. p_i(t)$ is a path from x to y in $\prod_{i \in I} X_i$.

□

Proposition 13.35.9. *Let $A \subseteq \mathbb{R}^2$. If A is countable then $\mathbb{R}^2 - A$ is path connected.*

PROOF:

$\langle 1 \rangle 1.$ LET: $x, y \in \mathbb{R}^2 - A$

$\langle 1 \rangle 2.$ PICK two non-parallel lines L through x and L' through y that do not pass through any points in A .

PROOF: These exist since uncountably many lines pass through any point.

$\langle 1 \rangle 3.$ There exists a path from x to y that follows L from x to the point of intersection of L and L' , and then follows L' to y .

□

13.35.7 Connected Spaces

Proposition 13.35.10. *Every path connected space is connected.*

PROOF:

$\langle 1 \rangle 1.$ LET: X be a path connected space.

$\langle 1 \rangle 2.$ ASSUME: for a contradiction (A, B) is a separation of X .

$\langle 1 \rangle 3.$ PICK $a \in A$ and $b \in B$

$\langle 1 \rangle 4.$ PICK a path $p : [0, 1] \rightarrow X$ from a to b .

$\langle 1 \rangle 5.$ $(p^{-1}(A), p^{-1}(B))$ is a separation of $[0, 1]$.

$\langle 1 \rangle 6.$ Q.E.D.

PROOF: This contradicts Proposition 13.25.16.

□

Corollary 13.35.10.1. *For $n > 1$, we have \mathbb{R}^n and \mathbb{R} are not homeomorphic.*

PROOF: Removing a point from \mathbb{R} gives a disconnected space. □

Proposition 13.35.11. *Every open connected subspace of \mathbb{R}^2 is path connected.*

PROOF:

$\langle 1 \rangle 1.$ LET: U be an open connected subspace of \mathbb{R}^2 .

$\langle 1 \rangle 2.$ ASSUME: w.l.o.g. $U \neq \emptyset$

$\langle 1 \rangle 3.$ PICK $x_0 \in U$

$\langle 1 \rangle 4.$ LET: $V = \{x \in U : \text{there exists a path from } x_0 \text{ to } x\}$

$\langle 1 \rangle 5.$ V is open in U .

$\langle 2 \rangle 1.$ LET: $x \in V$

$\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$

$\langle 2 \rangle 3.$ $B(x, \epsilon) \subseteq V$

PROOF: For all $y \in B(x, \epsilon)$, take a path from x_0 to x and then a straight line from x to y .

$\langle 1 \rangle 6.$ V is closed in U .

$\langle 2 \rangle 1.$ LET: $x \in U - V$

- $\langle 2 \rangle 2$. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
 - $\langle 2 \rangle 3$. $B(x, \epsilon) \subseteq U - V$
 - $\langle 3 \rangle 1$. LET: $y \in B(x, \epsilon)$
 - $\langle 3 \rangle 2$. There is a path from y to x .
 - $\langle 3 \rangle 3$. There is no path from x_0 to y .
 - $\langle 1 \rangle 7$. $V = U$
- PROOF: U is connected.
-

Chapter 14

Metric Spaces

Definition 14.0.1 (Metric Space). Let X be a set and $d : X^2 \rightarrow \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geq 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d) . We often write X for the metric space (X, d) .

Definition 14.0.2 (Discrete Metric). On any set X , define the *discrete* metric by $d(x, y) = 0$ if $x = y$, 1 if $x \neq y$.

Definition 14.0.3 (Standard Metric). The *standard metric* on \mathbb{R} is defined by $d(x, y) = |x - y|$.

Definition 14.0.4 (Square Metric). The *square metric* ρ on \mathbb{R}^n is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1$. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have $\rho(\vec{x}, \vec{y}) \geq 0$.

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have $\rho(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$.

PROOF:

$$\begin{aligned} \rho(\vec{x}, \vec{y}) = 0 &\Leftrightarrow \max(|x_1 - y_1|, \dots, |x_n - y_n|) = 0 \\ &\Leftrightarrow |x_1 - y_1| = \dots = |x_n - y_n| = 0 \\ &\Leftrightarrow x_1 = y_1 \wedge \dots \wedge x_n = y_n \\ &\Leftrightarrow \vec{x} = \vec{y} \end{aligned}$$

⟨1⟩3. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$.

PROOF: Immediate from definition.

⟨1⟩4. For all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ we have $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$.

PROOF:

$$\begin{aligned} & \max(|x_1 - z_1|, \dots, |x_n - z_n|) \\ & \leq \max(|x_1 - y_1| + |y_1 - z_1|, \dots, |x_n - y_n| + |y_n - z_n|) \\ & \leq \max(|x_1 - y_1|, \dots, |x_n - y_n|) + \max(|y_1 - z_1|, \dots, |y_n - z_n|) \\ & = \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z}) \end{aligned}$$

□

14.0.1 Balls

Definition 14.0.5 (Ball). Let X be a metric space. Let $x \in X$ and $r > 0$. The *ball* with *centre* x and *radius* r is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} .$$

Definition 14.0.6 (Closed Ball). Let X be a metric space. Let $x \in X$ and $r > 0$. The *closed ball* with *centre* x and *radius* r is

$$\overline{B(x, r)} = \{y \in X \mid d(x, y) \leq r\} .$$

Definition 14.0.7 (Metric Topology). Let (X, d) be a metric space. The *metric topology* on X is the topology generated by the basis consisting of the balls.

We prove this is a basis for a topology.

PROOF:

⟨1⟩1. Every point is a member of some ball.

PROOF: Since $x \in B(x, 1)$.

⟨1⟩2. If B_1 and B_2 are balls and $x \in B_1 \cap B_2$, then there exists a ball B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.

⟨2⟩1. LET: $x \in B(a, \epsilon_1) \cap B(b, \epsilon_2)$

⟨2⟩2. LET: $\epsilon = \min(\epsilon_1 - d(x, a), \epsilon_2 - d(x, b))$

PROVE: $x \in B(x, \epsilon) \subseteq B(a, \epsilon_1) \cap B(b, \epsilon_2)$

⟨2⟩3. $B(x, \epsilon) \subseteq B(a, \epsilon_1)$

⟨3⟩1. LET: $y \in B(x, \epsilon)$

⟨3⟩2. $d(y, a) < \epsilon_1$

PROOF:

$$d(y, a) \leq d(y, x) + d(x, a) \quad (\text{Triangle Inequality})$$

$$< \epsilon + d(x, a) \quad (\langle 3 \rangle 1)$$

$$\leq \epsilon_1 \quad (\langle 2 \rangle 2)$$

⟨2⟩4. $B(x, \epsilon) \subseteq B(b, \epsilon_2)$

PROOF: Similar.

□

Proposition 14.0.8. *The discrete metric on a set X induces the discrete topology.*

PROOF: Since $B(x, 1/2) = \{x\}$ for all $x \in X$. \square

Proposition 14.0.9. *The standard metric on \mathbb{R} induces the standard topology.*

PROOF:

$\langle 1 \rangle 1$. Every ball is open in the standard topology.

PROOF: Since $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$.

$\langle 1 \rangle 2$. Every open ray is open in the metric topology.

PROOF: If $x \in (a, +\infty)$ then $x \in B(x, x - a) \subseteq (a, +\infty)$. Similarly for $(-\infty, a)$.

\square

Proposition 14.0.10. *The square metric on \mathbb{R}^n induces the product topology.*

PROOF:

$\langle 1 \rangle 1$. For any real numbers $a_1, \dots, a_n, b_1, \dots, b_n$ with $a_1 < b_1, \dots, a_n < b_n$, we have $(a_1, b_1) \times \dots \times (a_n, b_n)$ is open in the metric topology.

$\langle 2 \rangle 1$. LET: $\vec{x} \in (a_1, b_1) \times \dots \times (a_n, b_n)$

$\langle 2 \rangle 2$. LET: $\epsilon = \min(x_1 - a_1, b_1 - x_1, \dots, x_n - a_n, b_n - x_n)$

$\langle 2 \rangle 3$. $B(\vec{x}, \epsilon) \subseteq (a_1, b_1) \times \dots \times (a_n, b_n)$

$\langle 1 \rangle 2$. For any $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B(\vec{a}, \epsilon)$ is open in the product topology.

PROOF: Since $B(\vec{a}, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \dots \times (a_n - \epsilon, a_n + \epsilon)$.

\square

Proposition 14.0.11. *Addition is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

$\langle 1 \rangle 2$. LET: $\delta = \epsilon/2$

$\langle 1 \rangle 3$. LET: $(x', y') \in \mathbb{R}^2$ with $\rho((x, y), (x', y')) < \delta$

$\langle 1 \rangle 4$. $|x - x'|, |y - y'| < \delta$

$\langle 1 \rangle 5$. $|(x + y) - (x' + y')| < \epsilon$

PROOF:

$$\begin{aligned} |(x + y) - (x' + y')| &\leq |x - x'| + |y - y'| \\ &< \delta + \delta && (\langle 1 \rangle 4) \\ &= \epsilon && (\langle 1 \rangle 2) \end{aligned}$$

\square

Proposition 14.0.12. *Multiplication is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

$\langle 1 \rangle 2$. LET: $\delta = \min(\epsilon/(|x| + |y| + 1), 1)$

$\langle 1 \rangle 3$. LET: $(x', y') \in \mathbb{R}^2$ with $\rho((x, y), (x', y')) < \delta$

$\langle 1 \rangle 4$. $|x - x'|, |y - y'| < \delta$

$\langle 1 \rangle 5$. $|xy - x'y'| < \epsilon$

PROOF:

$$\begin{aligned}
 |xy - x'y'| &= |xy - xy' + xy' - x'y - xy + x'y + xy' - x'y'| \\
 &\leq |xy - xy'| + |xy' - x'y| + |xy - x'y - xy' + xy'y| = |x||y - y'| + |x - x'||y| + |x - x'||y - y'| \\
 &< |x|\delta + |y|\delta + \delta^2 \\
 &\leq |x|\delta + |y|\delta + \delta \\
 &= (|x| + |y| + 1)\delta \\
 &\leq \epsilon
 \end{aligned}$$

□

Corollary 14.0.12.1. *The unit circle S^1 is a closed subset of \mathbb{R}^2 .*

PROOF: The function f that maps (x, y) to $x^2 + y^2$ is continuous, and $S^1 = f^{-1}(\{1\})$. □

Corollary 14.0.12.2. *The unit ball B^2 is a closed subset of \mathbb{R}^2 .*

PROOF: The function f that maps (x, y) to $x^2 + y^2$ is continuous, and $B^2 = f^{-1}([0, 1])$. □

Proposition 14.0.13. *Let (a_n) and (b_n) be sequences of real numbers. Let $c, s, t \in \mathbb{R}$. Assume*

$$\sum_{n=0}^{\infty} a_n = s \text{ and } \sum_{n=0}^{\infty} b_n = t .$$

Then

$$\sum_{n=0}^{\infty} (ca_n + b_n) = cs + t .$$

PROOF:

$$\sum_{n=0}^N (ca_n + b_n) = c \sum_{n=0}^N a_n + \sum_{n=0}^N b_n \rightarrow cs + t \text{ as } n \rightarrow \infty \quad \square$$

Proposition 14.0.14 (Comparison Test). *Let (a_n) and (b_n) be sequences of real numbers. Assume $|a_n| \leq b_n$ for all n . Assume $\sum_{n=0}^{\infty} b_n$ converges. Then $\sum_{n=0}^{\infty} a_n$ converges.*

PROOF:

⟨1⟩1. For all n ,

$$\text{LET: } c_n = |a_n| + a_n$$

⟨1⟩2. $\sum_{n=0}^{\infty} |a_n|$ converges.

PROOF: Since $(\sum_{n=0}^N |a_n|)_N$ is an increasing sequence of real numbers bounded above by $\sum_{n=0}^{\infty} b_n$.

⟨1⟩3. $\sum_{n=0}^{\infty} c_n$ converges.

PROOF: Since $(\sum_{n=0}^N c_n)_N$ is an increasing sequence of real numbers bounded above by $2 \sum_{n=0}^{\infty} a_n$.

⟨1⟩4. $\sum_{n=0}^{\infty} a_n$ converges.

PROOF: Since $a_n = c_n - |a_n|$.

□

Proposition 14.0.15. *Let X be a metric space. Let $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:

⟨1⟩1. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

⟨2⟩1. ASSUME: U is open.

⟨2⟩2. LET: $x \in U$

⟨2⟩3. PICK a ball $B(a, \delta)$ such that $x \in B(a, \delta) \subseteq U$

⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$

PROVE: $B(x, \epsilon) \subseteq U$

⟨2⟩5. LET: $y \in B(x, \epsilon)$

⟨2⟩6. $y \in B(a, \delta)$

PROOF:

$$\begin{aligned} d(a, y) &\leq d(a, x) + d(x, y) && \text{(Triangle Inequality)} \\ &< d(a, x) + \epsilon && (\langle 2 \rangle 5) \\ &= \delta \end{aligned}$$

⟨2⟩7. $y \in U$

PROOF: ⟨2⟩3

⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.

PROOF: Immediate from definition of the metric topology.

□

Proposition 14.0.16. *Let X be a metric space. Let $a, b, c \in X$. Then*

$$|d(a, b) - d(a, c)| \leq d(b, c) .$$

PROOF:

⟨1⟩1. $d(a, b) - d(a, c) \leq d(b, c)$

PROOF: Triangle Inequality.

⟨1⟩2. $d(a, c) - d(a, b) \leq d(b, c)$

PROOF: Triangle Inequality.

□

Proposition 14.0.17. *Let (X, d) be a metric space. Then the metric topology on X is the coarsest topology such that $d : X^2 \rightarrow \mathbb{R}$ is continuous.*

PROOF:

⟨1⟩1. d is continuous with respect to the metric topology.

⟨2⟩1. LET: $(a, b) \in X^2$

⟨2⟩2. LET: V be a neighbourhood of $d(a, b)$.

⟨2⟩3. PICK $\epsilon > 0$ such that $(d(a, b) - \epsilon, d(a, b) + \epsilon) \subseteq V$.

⟨2⟩4. LET: $U = B(a, \epsilon/2) \times B(b, \epsilon/2)$

⟨2⟩5. LET: $(x, y) \in U$

⟨2⟩6. $|d(x, y) - d(a, b)| < \epsilon$

PROOF:

$$\begin{aligned} |d(x, y) - d(a, b)| &\leq |d(x, y) - d(a, y)| + |d(a, y) - d(a, b)| \\ &\leq d(a, x) + d(b, y) && \text{(Proposition 14.0.16)} \\ &< \epsilon \end{aligned}$$

$\langle 2 \rangle 7. d(x, y) \in V$

$\langle 1 \rangle 2.$ If \mathcal{T} is a topology on X with respect to which d is continuous then \mathcal{T} is finer than the metric topology.

$\langle 2 \rangle 1.$ LET: \mathcal{T} be a topology on X with respect to which d is continuous.

$\langle 2 \rangle 2.$ LET: $a \in X$ and $\epsilon > 0$.

PROVE: $B(a, \epsilon) \in \mathcal{T}$

$\langle 2 \rangle 3.$ LET: $x \in B(a, \epsilon)$

$\langle 2 \rangle 4.$ $(a, x) \in d^{-1}((0, \epsilon))$

$\langle 2 \rangle 5.$ PICK $U, V \in \mathcal{T}$ such that $(a, x) \in U \times V \subseteq d^{-1}((0, \epsilon))$

$\langle 2 \rangle 6.$ $x \in V \subseteq B(a, \epsilon)$

□

Proposition 14.0.18. Let d and d' be two metrics on the same set X . Let \mathcal{T} and \mathcal{T}' be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1.$ If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon).$$

$\langle 2 \rangle 1.$ ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 2.$ LET: $x \in X$ and $\epsilon > 0$

$\langle 2 \rangle 3.$ $x \in B_d(x, \epsilon) \in \mathcal{T}'$

$\langle 2 \rangle 4.$ There exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: Proposition 14.0.15.

$\langle 1 \rangle 2.$ If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$.

$\langle 2 \rangle 1.$ ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

$\langle 2 \rangle 2.$ LET: $U \in \mathcal{T}$

$\langle 2 \rangle 3.$ For all $x \in U$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$

$\langle 3 \rangle 1.$ LET: $x \in U$

$\langle 3 \rangle 2.$ PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 14.0.15.

$\langle 3 \rangle 3.$ PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

PROOF: $\langle 2 \rangle 1$

$\langle 3 \rangle 4.$ $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4.$ $U \in \mathcal{T}'$

PROOF: Proposition 14.0.15.

□

Definition 14.0.19 (Metrizability). A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 14.0.20. \mathbb{R}^2 under the dictionary order is metrizable.

PROOF:

$\langle 1 \rangle 1$. LET: $d : (\mathbb{R}^2)^2 \rightarrow \mathbb{R}$ be defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} \min(|y_2 - y_1|, 1) & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

$\langle 1 \rangle 2$. d is a metric.

$\langle 2 \rangle 1$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) \geq 0$.

PROOF: Immediate from definition.

$\langle 2 \rangle 2$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) = 0$ iff $x = y$.

PROOF: Immediate from definition.

$\langle 2 \rangle 3$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) = d(y, x)$.

PROOF: Immediate from definition.

$\langle 2 \rangle 4$. For all $x, y, z \in \mathbb{R}^2$ we have $d(x, z) \leq d(x, y) + d(y, z)$.

PROOF: Easy.

$\langle 1 \rangle 3$. The metric topology induced by d is finer than the order topology.

$\langle 2 \rangle 1$. LET: $a, b \in \mathbb{R}^2$

$\langle 2 \rangle 2$. LET: $x \in (a, b)$

$\langle 2 \rangle 3$. CASE: $\pi_1(x) = \pi_1(a) = \pi_1(b)$

$\langle 3 \rangle 1$. LET: $\epsilon = \min(\pi_2(x) - \pi_2(a), \pi_2(b) - \pi_2(x))$

$\langle 3 \rangle 2$. $B(x, \epsilon) \subseteq (a, b)$

$\langle 2 \rangle 4$. CASE: $\pi_1(a) = \pi_1(x) < \pi_1(b)$

$\langle 3 \rangle 1$. LET: $\epsilon = \pi_2(x) - \pi_2(a)$

$\langle 3 \rangle 2$. $B(x, \epsilon) \subseteq (a, b)$

$\langle 2 \rangle 5$. CASE: $\pi_1(a) < \pi_1(x) = \pi_1(b)$

PROOF: Similar.

$\langle 2 \rangle 6$. CASE: $\pi_1(a) < \pi_1(x) < \pi_1(b)$

PROOF: Then $B(x, \epsilon) \subseteq (a, b)$.

$\langle 1 \rangle 4$. The order topology is finer than the metric topology.

PROOF: Since $B((a, b), \epsilon) = ((a, b - \epsilon), (a, b + \epsilon))$ if $\epsilon \leq 1$, and \mathbb{R}^2 if $\epsilon > 1$.

□

Every metrizable space is first countable.

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

14.0.2 Subspaces

Proposition 14.0.21. Let (X, d) be a metric space and $Y \subseteq X$. Then $d \upharpoonright Y^2$ is a metric on Y that induces the subspace topology.

PROOF:

$\langle 1 \rangle 1$. LET: $d' = d \upharpoonright Y^2 : Y^2 \rightarrow \mathbb{R}$

$\langle 1 \rangle 2$. d' is a metric.

PROOF: Each of the axioms follows from the axiom in X .

$\langle 1 \rangle 3$. The metric topology induced by d' is finer than the subspace topology.

$\langle 2 \rangle 1$. LET: U be open in X

- PROVE: $U \cap Y$ is open in the d' -topology.
- $\langle 2 \rangle 2$. LET: $y \in U \cap Y$
 - $\langle 2 \rangle 3$. PICK $\epsilon > 0$ such that $B_d(y, \epsilon) \subseteq U$
 - $\langle 2 \rangle 4$. $B_{d'}(y, \epsilon) \subseteq U \cap Y$
 - $\langle 1 \rangle 4$. The subspace topology is finer than the metric topology induced by d' .
 - $\langle 2 \rangle 1$. LET: $y \in Y$ and $\epsilon > 0$
 - PROVE: $B_{d'}(y, \epsilon)$ is open in the subspace topology.
 - $\langle 2 \rangle 2$. $B_{d'}(y, \epsilon) = B_d(y, \epsilon) \cap Y$
-

14.0.3 Convergence

Proposition 14.0.22 (Sequence Lemma). *Let X be a metric space. Let $A \subseteq X$. Let $l \in \overline{A}$. Then there exists a sequence in A that converges to l .*

PROOF:

- $\langle 1 \rangle 1$. For $n \in \mathbb{N}$, PICK $a_n \in B(l, 1/(n+1)) \cap A$.
 - $\langle 1 \rangle 2$. $a_n \rightarrow l$ as $n \rightarrow \infty$.
-

Corollary 14.0.22.1. \mathbb{R}^ω under the box topology is not first countable.

PROOF:

- $\langle 1 \rangle 1$. LET: A be the set of all sequences of positive reals.
 - $\langle 1 \rangle 2$. $0 \in \overline{A}$
 - $\langle 1 \rangle 3$. LET: (a_n) be a sequence in A
 - PROVE: (a_n) does not converge to 0.
 - $\langle 1 \rangle 4$. For all $n \in \mathbb{N}$,
 - LET: $a_n = (x_{nm})$
 - $\langle 1 \rangle 5$. LET: $B' = \prod_{n=0}^{\infty} (-x_{nn}, x_{nn})$
 - $\langle 1 \rangle 6$. B' is open in the box topology.
 - $\langle 1 \rangle 7$. $0 \in B'$
 - $\langle 1 \rangle 8$. For all n we have $a_n \notin B'$
-

Corollary 14.0.22.2. *If J is an uncountable set then \mathbb{R}^J under the product topology is not first countable.*

PROOF:

- $\langle 1 \rangle 1$. LET: $A = \{x \in \mathbb{R}^J : \pi_j(x) = 1 \text{ for all but finitely many } j \in J\}$
- $\langle 1 \rangle 2$. $0 \in \overline{A}$
- $\langle 1 \rangle 3$. LET: (a_n) be a sequence in A .
- PROVE: (a_n) does not converge to 0.
- $\langle 1 \rangle 4$. For $n \in \mathbb{N}$,
- LET: $J_n = \{j \in J : \pi_j(a_n) \neq 1\}$
- $\langle 1 \rangle 5$. $\bigcup_{n \in \mathbb{N}} J_n$ is countable.
- $\langle 1 \rangle 6$. PICK $\beta \in J - \bigcup_{n \in \mathbb{N}} J_n$
- $\langle 1 \rangle 7$. $\forall n \in \mathbb{N}. \pi_\beta(a_n) = 1$

- $\langle 1 \rangle 8.$ LET: $U = \pi_\beta^{-1}((-1, 1))$
 $\langle 1 \rangle 9.$ $0 \in U$
 $\langle 1 \rangle 10.$ $\forall n \in \mathbb{N}. a_n \notin U$
 $\langle 1 \rangle 11.$ (a_n) does not converge to 0.
 \square

14.0.4 Continuous Functions

Proposition 14.0.23. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:

- $\langle 1 \rangle 1.$ If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
 $\langle 2 \rangle 1.$ ASSUME: f is continuous.
 $\langle 2 \rangle 2.$ LET: $x \in X$
 $\langle 2 \rangle 3.$ LET: $\epsilon > 0$
 $\langle 2 \rangle 4.$ $x \in f^{-1}(B(f(x), \epsilon))$
 $\langle 2 \rangle 5.$ There exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$.
 $\langle 1 \rangle 2.$ If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.
 $\langle 2 \rangle 1.$ ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
 $\langle 2 \rangle 2.$ LET: V be open in Y
 $\langle 2 \rangle 3.$ LET: $x \in f^{-1}(V)$
 $\langle 2 \rangle 4.$ PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
 $\langle 2 \rangle 5.$ PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
 $\langle 2 \rangle 6.$ $B(x, \delta) \subseteq f^{-1}(V)$
 \square

Proposition 14.0.24. *Let X be a metrizable space and Y a topological space. Let $f : X \rightarrow Y$. Assume that, for every sequence (x_n) in X and $l \in X$, if $x_n \rightarrow l$ as $n \rightarrow \infty$ then $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$. Then f is continuous.*

PROOF:

- $\langle 1 \rangle 1.$ LET: $A \subseteq X$
PROVE: $f(\overline{A}) \subseteq \overline{f(A)}$
 $\langle 1 \rangle 2.$ LET: $l \in \overline{A}$
PROVE: $f(l) \in \overline{f(A)}$
 $\langle 1 \rangle 3.$ PICK a sequence (x_n) in A such that $x_n \rightarrow l$ as $n \rightarrow \infty$.
 $\langle 1 \rangle 4.$ $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$.
 $\langle 1 \rangle 5.$ $f(l) \in \overline{f(A)}$
 \square

Proposition 14.0.25. *The function $i : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ that maps x to x^{-1} is continuous.*

PROOF:

- $\langle 1 \rangle 1.$ LET: $a, b \in \mathbb{R}$ with $a < b$
 PROVE: $i^{-1}((a, b))$ is open.
 $\langle 1 \rangle 2.$ CASE: $0 < a$
 PROOF: $i^{-1}((a, b)) = (b^{-1}, a^{-1})$
 $\langle 1 \rangle 3.$ CASE: $a = 0$
 PROOF: $i^{-1}((a, b)) = (b^{-1}, +\infty)$
 $\langle 1 \rangle 4.$ CASE: $a < 0 < b$
 PROOF: $i^{-1}((a, b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$
 $\langle 1 \rangle 5.$ CASE: $b = 0$
 PROOF: $i^{-1}((a, b)) = (-\infty, a^{-1})$
 $\langle 1 \rangle 6.$ CASE: $b < 0$
 PROOF: $i^{-1}((a, b)) = (b^{-1}, a^{-1})$
 \square

Proposition 14.0.26. *Subtraction is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF: Since $a - b = a + (-1)b$ and both addition and multiplication are continuous. \square

Proposition 14.0.27. *Division is a continuous function $\mathbb{R} \times (\mathbb{R} - \{0\}) \rightarrow \mathbb{R}$.*

PROOF: Since both multiplication and the function that maps x to x^{-1} are continuous. \square

14.0.5 First Countable Spaces

Proposition 14.0.28. *Every metrizable space is first countable.*

PROOF: For any point x , the set $\{B(x, 1/n) : n \in \mathbb{Z}_+\}$ is a countable basis at x . \square

Corollary 14.0.28.1. \mathbb{R}^ω under the box topology is not metrizable.

Corollary 14.0.28.2. *If J is an uncountable set then \mathbb{R}^J under the product topology is not metrizable.*

14.0.6 Hausdorff Spaces

Proposition 14.0.29. *Every metric space is Hausdorff.*

PROOF:

- $\langle 1 \rangle 1.$ LET: X be a metric space.
 $\langle 1 \rangle 2.$ LET: $x, y \in X$ with $x \neq y$.
 $\langle 1 \rangle 3.$ LET: $\epsilon = d(x, y)$
 $\langle 1 \rangle 4.$ $B(x, \epsilon/2)$ and $B(y, \epsilon/2)$ are disjoint neighbourhoods of x and y .
 \square

14.0.7 Bounded Sets

Definition 14.0.30 (Bounded). Let X be a metric space. Let $A \subseteq X$. Then A is *bounded* iff there exists M such that $\forall x, y \in A. d(x, y) \leq M$. Its *diameter* is then defined to be

$$\text{diam } A := \sup\{d(x, y) : x, y \in A\} .$$

14.0.8 Uniform Convergence

Definition 14.0.31 (Uniform Convergence). Let X be a set and Y a metric space. Let (f_n) be a sequence of functions $X \rightarrow Y$, and $f : X \rightarrow Y$. Then (f_n) *converges uniformly* to f iff, for all $\epsilon > 0$, there exists N such that

$$\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon .$$

Example 14.0.32. For $n \in \mathbb{N}$ define $f_n : [0, 1] \rightarrow \mathbb{R}$ by $f_n(x) = x^n$. Define $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = 0$ for $x < 1$, $f(1) = 1$. Then f_n converges pointwise to f , but does not converge uniformly to f .

We prove that, for all N , there exists $n \geq N$ and $x \in [0, 1]$ such that $|x^n - f(x)| \geq 1/2$. Take $n = N$ and x to be the N th root of $3/4$.

Example 14.0.33. For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} .$$

Then for all $x \in \mathbb{R}$ we have $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$, but (f_n) does not converge uniformly to 0.

We prove that, for all N , there exists $n \geq N$ and $x \in \mathbb{R}$ such that $|f_n(x)| \geq 1/2$. Take $n = N$ and $x = 1/N$. We have $f_N(1/N) = 1$.

Theorem 14.0.34 (Uniform Limit Theorem). *Let X be a topological space and Y a metric space. Let (f_n) be a sequence of functions $X \rightarrow Y$, and $f : X \rightarrow Y$. If every f_n is continuous and (f_n) converges uniformly to f , then f is continuous.*

PROOF:

<1>1. LET: V be open in Y .

<1>2. LET: $x_0 \in f^{-1}(V)$

PROVE: There exists a neighbourhood U of x_0 such that $f(U) \subseteq V$.

<1>3. LET: $y_0 = f(x_0)$

<1>4. PICK $\epsilon > 0$ such that $B(y_0, \epsilon) \subseteq V$.

<1>5. PICK N such that $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/3$.

<1>6. PICK a neighbourhood U of x_0 such that $f_N(U) \subseteq B(f_N(x_0), \epsilon/3)$.

PROVE: $f(U) \subseteq V$

<1>7. LET: $y \in U$

<1>8. $d(f(y), y_0) < \epsilon$

PROOF:

$$\begin{aligned} d(f(y), y_0) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x_0)) + d(f_N(x_0), y_0) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1 \rangle 5, \langle 1 \rangle 6)l \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 9$. $f(y) \text{ in } V$

PROOF: $\langle 1 \rangle 4$

□

Proposition 14.0.35. *Let X be a topological space. Let Y be a metric space. Let f_n be a sequence of functions $X \rightarrow Y$ and $f : X \rightarrow Y$. Let x_n be a sequence of points in X and $l \in X$. If f_n converges uniformly to f , x_n converges to l , and f is continuous, then $f_n(x_n)$ converges to $f(l)$.*

PROOF:

$\langle 1 \rangle 1$. f is continuous.

$\langle 1 \rangle 2$. LET: $\epsilon > 0$

$\langle 1 \rangle 3$. PICK $\delta > 0$ such that $\forall y \in X. d(y, l) < \delta \Rightarrow d(f(y), f(l)) < \epsilon/2$

$\langle 1 \rangle 4$. PICK N such that $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/2$ and $\forall n \geq N. d(x_n, l) < \delta$

$\langle 1 \rangle 5$. For all $n \geq N$ we have $d(f_n(x_n), f(l)) < \epsilon$

PROOF:

$$\begin{aligned} d(f_n(x_n), f(l)) &\leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(l)) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

Theorem 14.0.36 (Weierstrass M-Test). *Let X be a set. Let (f_n) be a sequence of functions $X \rightarrow \mathbb{R}$. Let (M_n) be a sequence of real numbers. For $n \in \mathbb{N}$, let*

$$s_n(x) = \sum_{i=0}^n f_i(x) \ .$$

Assume that $\forall n \in \mathbb{N}. \forall x \in X. |f_n(x)| \leq M_n$. Assume that $\sum_{n=0}^{\infty} M_n$ converges. Then (s_n) uniformly converges to s where $s(x) = \sum_{n=0}^{\infty} f_n(x)$.

PROOF:

$\langle 1 \rangle 1$. For all $x \in X$ we have $\sum_{n=0}^{\infty} f_n(x)$ converges.

PROOF: By the Comparison Test.

$\langle 1 \rangle 2$. For $n \in \mathbb{N}$,

LET: $r_n = \sum_{i=n+1}^{\infty} M_i$.

$\langle 1 \rangle 3$. For all $k, n \in \mathbb{N}$ and $x \in X$, if $k > n$ then $|s_k(x) - s_n(x)| \leq r_n$.

PROOF:

$$\begin{aligned}
 |s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\
 &\leq \sum_{i=n+1}^k |f_i(x)| \\
 &\leq \sum_{i=n+1}^k M_i \\
 &\leq \sum_{i=n+1}^{\infty} M_i \\
 &= r_n
 \end{aligned}$$

$\langle 1 \rangle 4$. For all $n \in \mathbb{N}$ we have $|s(x) - s_n(x)| \leq r_n$.

PROOF: Taking the limit $k \rightarrow \infty$ in $\langle 1 \rangle 3$.

$\langle 1 \rangle 5$. (s_n) converges uniformly to s .

PROOF: We have $\bar{\rho}(s_n, s) \leq r_n$ and so $\bar{\rho}(s_n, s) \rightarrow 0$ as $n \rightarrow \infty$ by the Sandwich Theorem.

□

14.0.9 Standard Bounded Metric

Definition 14.0.37 (Standard Bounded Metric). Let (X, d) be a metric space. The *standard bounded metric* corresponding to d is

$$\bar{d}(x, y) := \min(d(x, y), 1) .$$

Proposition 14.0.38. *The standard bounded metric associated with d induces the same topology as d .*

PROOF:

$\langle 1 \rangle 1$. LET: (X, d) be a metric space.

$\langle 1 \rangle 2$. Every d -ball is open under the topology induced by \bar{d} .

$\langle 2 \rangle 1$. LET: $a \in X$ and $\epsilon > 0$

$\langle 2 \rangle 2$. LET: $x \in B_d(a, \epsilon)$

$\langle 2 \rangle 3$. LET: $\delta = \min(\epsilon - d(a, x), 1/2)$

$\langle 2 \rangle 4$. $B_{\bar{d}}(x, \delta) \subseteq B_d(a, \epsilon)$

$\langle 1 \rangle 3$. Every \bar{d} -ball is open under the topology induced by d .

PROOF: Since $B_{\bar{d}}(a, \epsilon) = B_d(a, \epsilon)$ if $\epsilon \leq 1$, and X if $\epsilon > 1$.

□

14.0.10 Product Spaces

Proposition 14.0.39. *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

- <1>1. LET: (X_n, d_n) be a sequence of metric spaces.
 <1>2. For $n \in \mathbb{N}$,
 LET: \bar{d}_n be the standard bounded metric associated with d_n .
 <1>3. LET: $X = \prod_{n \in \mathbb{N}} X_n$
 <1>4. Define $D : X^2 \rightarrow \mathbb{R}$ by $D(x, y) = \sup_{n \in \mathbb{N}} \bar{d}_n(\pi_n(x), \pi_n(y)) / (n + 1)$.
 <1>5. D is a metric on X .
 <2>1. For all $x, y \in X$ we have $D(x, y) \geq 0$.
 <2>2. For all $x, y \in X$ we have $D(x, y) = 0$ iff $x = y$.
 <2>3. For all $x, y \in X$ we have $D(x, y) = D(y, x)$.
 <2>4. For all $x, y, z \in X$ we have $D(x, z) \leq D(x, y) + D(y, z)$.
 <1>6. The product topology is finer than the metric topology induced by D .
 <2>1. LET: $a \in X$ and $\epsilon > 0$.
 <2>2. LET: $x \in B(a, \epsilon)$
 <2>3. LET: $\delta = \epsilon - D(a, x)$
 <2>4. PICK $N \in \mathbb{N}$ such that $1/(N + 1) < \delta$
 <2>5. $x \in \prod_{n=0}^N B_{\bar{d}_n}(\pi_n(a), n\delta) \times \prod_{n=N+1}^{\infty} B_{\bar{d}_n}(\pi_n(a), n\delta) \subseteq B(a, \epsilon)$
 <1>7. The metric topology induced by D is finer than the product topology.
 <2>1. LET: $n \in \mathbb{N}$ and U be an open set in X_n .
 PROVE: $\pi_n^{-1}(U)$ is open in the metric topology.
 <2>2. LET: $x \in \pi_n^{-1}(U)$
 <2>3. PICK $\epsilon > 0$ such that $B_{\bar{d}_n}(\pi_n(x), \epsilon) \subseteq U$
 <2>4. $B(x, \epsilon/(n + 1)) \subseteq \pi_n^{-1}(U)$

□

Definition 14.0.40. For $n \geq 1$, the *unit ball* B^n is the closed ball $\overline{B(0, 1)}$ in \mathbb{R}^n under the Euclidean metric.

14.1 Uniform Metric

Definition 14.1.1 (Uniform Metric). Let J be a nonempty set. The *uniform metric* $\bar{\rho}$ on \mathbb{R}^J is defined by

$$\bar{\rho}(x, y) = \sup_{j \in J} \bar{d}(x_j, y_j)$$

where \bar{d} is the standard bounded metric associated with the standard metric on \mathbb{R} .

The topology it induces is called the *uniform topology*.

We prove this is a metric.

PROOF:

- <1>1. For all $x, y \in \mathbb{R}^J$ we have $\bar{\rho}(x, y) \geq 0$.

PROOF: Pick $j_0 \in J$. Then

$$\begin{aligned}
 \bar{\rho}(x, y) &= \sup_j \bar{d}(x_j, y_j) \\
 &\geq \bar{d}(x_{j_0}, y_{j_0}) \\
 &\geq 0
 \end{aligned}$$

⟨1⟩2. For all $x, y \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, y) = 0$ iff $x = y$.

PROOF:

$$\begin{aligned}\bar{\rho}(x, y) = 0 &\Leftrightarrow \sup_j \bar{d}(x_j, y_j) = 0 \\ &\Leftrightarrow \forall j. \bar{d}(x_j, y_j) = 0 \\ &\Leftrightarrow \forall j. x_j = y_j \\ &\Leftrightarrow x = y\end{aligned}$$

⟨1⟩3. For all $x, y \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, y) = \bar{\rho}(y, x)$.

PROOF:

$$\begin{aligned}\bar{\rho}(x, y) &= \sup_j \bar{d}(x_j, y_j) \\ &= \sup_j \bar{d}(y_j, x_j) \\ &= \bar{\rho}(y, x)\end{aligned}$$

⟨1⟩4. For all $x, y, z \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, z) \leq \bar{\rho}(x, y) + \bar{\rho}(y, z)$.

PROOF:

$$\begin{aligned}\bar{\rho}(x, z) &= \sup_j \bar{d}(x_j, z_j) \\ &\leq \sup_j (\bar{d}(x_j, y_j) + \bar{d}(y_j, z_j)) \\ &\leq \sup_j \bar{d}(x_j, y_j) + \sup_j \bar{d}(y_j, z_j) \\ &= \bar{\rho}(x, y) + \bar{\rho}(y, z)\end{aligned}$$

□

Proposition 14.1.2. *The uniform topology is finer than the product topology. It is strictly finer iff J is infinite.*

PROOF:

⟨1⟩1. The uniform topology is finer than the product topology.

⟨2⟩1. LET: U be open in \mathbb{R} and $j \in J$

PROVE: $\pi_j^{-1}(U)$ is open in the uniform topology.

⟨2⟩2. LET: $x \in \pi_j^{-1}(U)$

⟨2⟩3. $\pi_j(x) \in U$

⟨2⟩4. PICK $\epsilon > 0$ such that $B_{\bar{d}}(\pi_j(x), \epsilon) \subseteq U$

⟨2⟩5. $B_{\bar{\rho}}(x, \epsilon) \subseteq \pi_j^{-1}(U)$

⟨1⟩2. If J is finite then the uniform topology is equal to the product topology.

PROOF: In \mathbb{R}^n , the uniform topology is the square topology.

⟨1⟩3. If J is infinite then the uniform topology is not equal to the product topology.

PROOF: If J is infinite then $B(0, 1)$ is not open in the product topology.

□

Proposition 14.1.3. *The uniform topology is coarser than the box topology. It is strictly coarser iff J is infinite.*

PROOF:

- ⟨1⟩1. The uniform topology is coarser than the box topology.
- ⟨2⟩1. LET: U be open in the uniform topology.
PROVE: U is open in the box topology.
- ⟨2⟩2. LET: $x \in U$
- ⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
- ⟨2⟩4. $\prod_{j \in J} (x_j - \epsilon, x_j + \epsilon) \subseteq U$
- ⟨1⟩2. If J is finite then the uniform topology is equal to the box topology.
PROOF: On \mathbb{R}^n , the uniform metric is the square metric.
- ⟨1⟩3. If J is infinite then the uniform topology is not equal to the box topology.
- ⟨2⟩1. ASSUME: J is infinite.
- ⟨2⟩2. PICK a sequence (j_n) of distinct elements in J .
- ⟨2⟩3. LET: $U = \prod_j U_j$ where $J_{j_n} = (-1/(n+1), 1/(n+1))$ for $n \in \mathbb{N}$ and $J_j = (-1, 1)$ for all other j .
- ⟨2⟩4. U is not open in the uniform topology.

□

Proposition 14.1.4. *The uniform topology on \mathbb{R}^∞ is strictly finer than the product topology.*

PROOF: The set of all sequences $(x_n) \in \mathbb{R}^\infty$ such that $\forall n. |x_n| < 1$ is open in the uniform topology but not in the product topology. □

Proposition 14.1.5. *The uniform topology on \mathbb{R}^∞ is strictly coarser than the box topology.*

PROOF: The set of sequences $(x_n) \in \mathbb{R}^\infty$ such that $\forall n. |x_n| < 1/n$ is open in the box topology but not in the uniform topology. □

Proposition 14.1.6. *The uniform topology on the Hilbert cube is the same as the product topology.*

PROOF:

- ⟨1⟩1. LET: (x_n) be in the Hilbert cube H and $\epsilon > 0$.
PROVE: $B((x_n), \epsilon) \cap H$ is open in the product topology.
- ⟨1⟩2. PICK N such that $1/N < \epsilon$
- ⟨1⟩3. $B((x_n), \epsilon) = (\prod_{n=0}^N (x_n - \epsilon, x_n + \epsilon) \times \prod_{n=N+1}^\infty [0, 1/(n+1)]) \cap H$

□

Corollary 14.1.6.1. *The uniform topology on the Hilbert cube is strictly finer than the box topology.*

Proposition 14.1.7. *Let X be a set and Y a metric space. Let (f_n) be a sequence of functions $X \rightarrow Y$, and $f : X \rightarrow Y$. Then (f_n) converges uniformly to f iff (f_n) converges to f in Y^X under the uniform topology.*

PROOF:

- ⟨1⟩1. If (f_n) converges uniformly to f then (f_n) converges to f in Y^X under the uniform topology.
- ⟨2⟩1. ASSUME: (f_n) converges uniformly to f .

- $\langle 2 \rangle 2$. LET: $\epsilon > 0$
 $\langle 2 \rangle 3$. PICK N such that $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/2$
 $\langle 2 \rangle 4$. $\forall n \geq N. \bar{\rho}(f_n, f) \leq \epsilon/2$
 $\langle 2 \rangle 5$. $\forall n \geq N. \bar{\rho}(f_n, f) < \epsilon$
 $\langle 1 \rangle 2$. If (f_n) converges to f in Y^X under the uniform topology then (f_n) converges uniformly to f .
 $\langle 2 \rangle 1$. ASSUME: (f_n) converges to f in Y^X under the uniform topology.
 $\langle 2 \rangle 2$. LET: $\epsilon > 0$
 $\langle 2 \rangle 3$. PICK N such that $\forall n \geq N. \bar{\rho}(f_n, f) < \epsilon$
 $\langle 2 \rangle 4$. $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon$
 \square

14.1.1 Products

Definition 14.1.8 (Euclidean Metric). Let X and Y be metric spaces. The *Euclidean metric* on $X \times Y$ is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}.$$

We write $X \times Y$ for the set $X \times Y$ under this metric.

We prove this is a metric.

PROOF:

- $\langle 1 \rangle 1$. $d((x_1, y_1), (x_2, y_2)) \geq 0$
 PROOF: Immediate from definition.
 $\langle 1 \rangle 2$. $d((x_1, y_1), (x_2, y_2)) = 0$ iff $(x_1, y_1) = (x_2, y_2)$
 PROOF: $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = 0$ iff $d(x_1, x_2) = d(y_1, y_2) = 0$ iff $x_1 = x_2$ and $y_1 = y_2$.
 $\langle 1 \rangle 3$. $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$
 PROOF: Since $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = \sqrt{d(x_2, x_1)^2 + d(y_2, y_1)^2}$.
 $\langle 1 \rangle 4$. The triangle inequality holds.
 PROOF:

$$\begin{aligned}
 & (d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)))^2 \\
 &= d((x_1, y_1), (x_2, y_2))^2 + 2d((x_1, y_1), (x_2, y_2))d((x_2, y_2), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3))^2 \\
 &= d(x_1, x_2)^2 + d(y_1, y_2)^2 + 2\sqrt{(d(x_1, x_2)^2 + d(y_1, y_2)^2)(d(x_2, x_3)^2 + d(y_2, y_3)^2)} + d(x_2, x_3)^2 + d(y_2, y_3)^2 \\
 &\geq d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(y_1, y_2)^2 + d(y_2, y_3)^2 + 2(d(x_1, x_2)d(x_2, x_3) + d(y_1, y_2)d(y_2, y_3)) \\
 &\quad \text{(Cauchy-Schwarz)} \\
 &= (d(x_1, x_2) + d(x_2, x_3))^2 + (d(y_1, y_2) + d(y_2, y_3))^2 \\
 &\geq d(x_1, x_3)^2 + d(y_1, y_3)^2 \\
 &= d((x_1, y_1), (x_3, y_3))^2
 \end{aligned}$$

\square

Proposition 14.1.9. Let X and Y be metric spaces. The Euclidean metric on $X \times Y$ induces the product topology on $X \times Y$.

PROOF:

⟨1⟩1. Every open ball is open in the product topology.

⟨2⟩1. LET: $(x, y) \in B((a, b), \epsilon)$

PROVE: $B(x, \sqrt{\epsilon}) \times B(y, \sqrt{\epsilon}) \subseteq B((a, b), \epsilon)$

⟨2⟩2. LET: $x' \in B(x, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$ and $y' \in B(y, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$

PROVE: $d((x', y'), (a, b)) < \epsilon$

⟨2⟩3. $d((x', y'), (x, y)) < \epsilon - d((x, y), (a, b))$

PROOF:

$$\begin{aligned} d((x', y'), (x, y)) &= \sqrt{d(x', x)^2 + d(y', y)^2} \\ &< \sqrt{(\epsilon - d((x, y), (a, b)))^2/2 + (\epsilon - d((x, y), (a, b)))^2/2} \\ &= \epsilon - d((x, y), (a, b)) \end{aligned}$$

⟨2⟩4. $d((x', y'), (a, b)) < \epsilon$

PROOF:

$$\begin{aligned} d((x', y'), (a, b)) &\leq d((x', y'), (x, y)) + d((x, y), (a, b)) \quad (\text{Triangle Inequality}) \\ &< \epsilon \quad (\langle 2 \rangle 3) \end{aligned}$$

⟨1⟩2. If U is open in X and V is open in Y then $U \times V$ is open under the Euclidean metric.

⟨2⟩1. LET: $(x, y) \in U \times V$

⟨2⟩2. PICK $\delta, \epsilon > 0$ such that $B(x, \delta) \subseteq U$ and $B(y, \epsilon) \subseteq V$

PROVE: $(B((x, y), \min(\delta, \epsilon))) \subseteq U \times V$

⟨2⟩3. LET: $(x', y') \in B((x, y), \min(\delta, \epsilon))$

⟨2⟩4. $d(x', x) < \delta$

⟨3⟩1. $d((x', y'), (x, y)) < \min(\delta, \epsilon)$

⟨3⟩2. $d(x', x)^2 + d(y', y)^2 < \delta^2$

⟨3⟩3. $d(x', x)^2 < \delta^2$

⟨2⟩5. $d(y', y) < \epsilon$

PROOF: Similar.

⟨2⟩6. $(x', y') \in U \times V$

□

Proposition 14.1.10. *The square metric on \mathbb{R}^n induces the product topology.*

PROOF:

⟨1⟩1. LET: d be the Euclidean metric on \mathbb{R}^n and ρ the square metric.

⟨1⟩2. For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_d(x, \delta) \subseteq B_\rho(x, \epsilon)$

PROOF: If $d(x, y) < \epsilon$ then $\rho(x, y) < \epsilon$.

⟨1⟩3. For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_\rho(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: If $\rho(x, y) < \epsilon/\sqrt{n}$ then $d(x, y) < \epsilon$.

⟨1⟩4. d and ρ induce the same topology.

PROOF: Proposition 14.0.18.

□

14.1.2 Connected Spaces

Example 14.1.11. The space \mathbb{R}^ω under the uniform topology is disconnected. The set of bounded sequences and the set of unbounded sequences form a sep-

aration.

14.2 Isometric Embeddings

Definition 14.2.1 (Isometric Embedding). Let X and Y be metric spaces. Let $f : X \rightarrow Y$. Then f is an *isometric embedding* of X in Y iff, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 14.2.2. *Every isometric embedding is an embedding.*

PROOF:

$\langle 1 \rangle 1$. LET: X and Y be metric spaces.

$\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$ be an isometric embedding.

$\langle 1 \rangle 3$. f is injective.

$\langle 1 \rangle 4$. The subspace topology induced by f is finer than the metric topology.

$\langle 2 \rangle 1$. LET: $x \in X$ and $\epsilon > 0$

PROVE: $B(x, \epsilon)$ is open in the subspace topology.

$\langle 2 \rangle 2$. $B(x, \epsilon) = f^{-1}(B(f(x), \epsilon))$

$\langle 1 \rangle 5$. The metric topology is finer than the subspace topology induced by f .

$\langle 2 \rangle 1$. LET: V be open in Y

PROVE: $f^{-1}(V)$ is open in X

$\langle 2 \rangle 2$. LET: $x \in f^{-1}(V)$

$\langle 2 \rangle 3$. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$

$\langle 2 \rangle 4$. $B(x, \epsilon) \subseteq f^{-1}(V)$

□

14.3 Complete Metric Spaces

Definition 14.3.1 (Complete). A metric space is *complete* iff every Cauchy sequence converges.

Example 14.3.2. \mathbb{R} is complete.

Proposition 14.3.3. *The product of two complete metric spaces is complete.*

Proposition 14.3.4. *Every compact metric space is complete.*

Proposition 14.3.5. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 14.3.6 (Completion). Let X be a metric space. A *completion* of X is a complete metric space \hat{X} and injection $i : X \rightarrow \hat{X}$ such that:

- The metric on X is the restriction of the metric on \hat{X}
- X is dense in \hat{X} .

Proposition 14.3.7. *Let $i_1 : X \rightarrow Y_1$ and $i_2 : X \rightarrow Y_2$ be completions of X . Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

PROOF: Define $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$. \square

Theorem 14.3.8. *Every metric space has a completion.*

PROOF: Let \hat{X} be the set of Cauchy sequences in X quotiented by \sim where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \rightarrow 0$. \square

14.4 Manifolds

Definition 14.4.1 (Manifold). An n -dimensional manifold is a second countable Hausdorff space locally homeomorphic to \mathbb{R}^n .

Chapter 15

Homotopy Theory

15.1 Homotopies

Definition 15.1.1 (Homotopy). Let X and Y be topological spaces. Let $f, g : X \rightarrow Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0, 1] \rightarrow Y$ such that

- $\forall x \in X. h(x, 0) = f(x)$
- $\forall x \in X. h(x, 1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \rightarrow Y$.

Proposition 15.1.2. Let $f, f' : X \rightarrow Y$ and $g, g' : Y \rightarrow Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.

Definition 15.1.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

A *homotopy functor* is a functor $\mathbf{Top} \rightarrow \mathcal{C}$ that factors through the canonical functor $\mathbf{Top} \rightarrow \mathbf{HTop}$.

Definition 15.1.4. A functor $F : \mathbf{Top} \rightarrow \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \rightarrow Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

15.2 Homotopy Equivalence

Definition 15.2.1 (Homotopy Equivalence). Let X and Y be topological spaces. A *homotopy equivalence* between X and Y , $f : X \simeq Y$, is a continuous function $f : X \rightarrow Y$ such that there exists a continuous function $g : Y \rightarrow X$, the *homotopy inverse* to f , such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.

Definition 15.2.2 (Contractible). A topological space X is *contractible* iff $X \simeq 1$.

Example 15.2.3. \mathbb{R}^n is contractible.

Example 15.2.4. D^n is contractible.

Definition 15.2.5 (Deformation Retract). Let X be a topological space and A a subspace of X . A retraction $\rho : X \rightarrow A$ is a *deformation retraction* iff $i \circ \rho \simeq \text{id}_X$, where i is the inclusion $A \hookrightarrow X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 15.2.6 (Strong Deformation Retract). Let X be a topological space and A a subspace of X . A *strong deformation retraction* $\rho : X \rightarrow A$ is a continuous function such that there exists a homotopy $h : X \times [0, 1] \rightarrow X$ between $i \circ \rho$ and id_X such that, for all $a \in X$ and $t \in [0, 1]$, we have $h(a, t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 15.2.7. $\{0\}$ is a strong deformation retract of \mathbb{R}^n and of D^n .

Example 15.2.8. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 15.2.9. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 15.2.10. For any topological space X , the singleton consisting of the vertex is a strong deformation retract of the cone over X .

Chapter 16

Simplicial Complexes

Definition 16.0.1 (Simplex). A k -dimensional simplex or k -simplex in \mathbb{R}^n is the convex hull $s(x_0, \dots, x_k)$ of $k + 1$ points in general position.

Definition 16.0.2 (Face). A *sub-simplex* or *face* of $s(x_0, \dots, x_k)$ is the convex hull of a subset of $\{x_0, \dots, x_k\}$.

Definition 16.0.3 (Simplicial Complex). A *simplicial complex* in \mathbb{R}^n is a set K of simplices such that:

- for every simplex s in K , every face of s is in K .
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2 .
- K is locally finite, i.e. every point of \mathbb{R}^n has a neighbourhood that only intersects finitely many elements of K .

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of \mathbb{R}^n .

16.1 Cell Decompositions

Definition 16.1.1 (n -cell). An n -cell is a topological space homeomorphic to \mathbb{R}^n .

Definition 16.1.2 (Cell Decomposition). Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n -cells.

Definition 16.1.3 (n -skeleton). Given a cell decomposition of X , the n -skeleton X^n is the union of all the cells of dimension $\leq n$.

16.2 CW-complexes

Definition 16.2.1 (CW-Complex). A *CW-complex* consists of a topological space X and a cell decomposition \mathcal{E} of X such that:

1. *Characteristic Maps* For every n -cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \rightarrow X$ such that $\Phi_e((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in \mathcal{E} of dimension $< n$.
2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have \bar{e} intersects only finitely many other cells in \mathcal{E} .
3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \bar{e}$ is closed.

Proposition 16.2.2. *If a cell decomposition \mathcal{E} satisfies the Characteristic Maps axiom, then for every n -cell $e \in \mathcal{E}$ we have $\bar{e} = \Phi_e(D^n)$. Therefore \bar{e} is compact and $\bar{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

PROOF:

$\langle 1 \rangle 1.$ $e \subseteq \Phi_e(D^n) \subseteq \bar{e}$

PROOF:

$$\begin{aligned}
 e &= \Phi_e((D^n)^\circ) \\
 &\subseteq \Phi_e(D^n) \\
 &= \Phi_e(\overline{(D^n)^\circ}) \\
 &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
 &= \bar{e}
 \end{aligned}$$

$\langle 1 \rangle 2.$ $\Phi_e(D^n)$ is compact.

PROOF: Because D^n is compact.

$\langle 1 \rangle 3.$ $\Phi_e(D^n)$ is closed.

$\langle 1 \rangle 4.$ $\Phi_e(D^n) = \bar{e}$

□

Chapter 17

Topological Groups

17.1 Topological Groups

Definition 17.1.1 (Topological Group). A *topological group* is a group G with a topology such that the function $G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.

Example 17.1.2. \mathbb{Z} is a topological group under addition.

PROOF: The function that sends (x, y) to xy^{-1} is continuous because the topology on \mathbb{Z} is discrete. \square

Example 17.1.3. \mathbb{R} is a topological group under addition.

PROOF: From Propositions 14.0.11 and 14.0.12. \square

Example 17.1.4. \mathbb{R}_+ is a topological group under multiplication.

PROOF: From Propositions 14.0.12 and 14.0.25. \square

Example 17.1.5. S^1 as a subspace of \mathbb{C} is a topological group under multiplication.

PROOF:

$\langle 1 \rangle 1$. LET: $f : S^1 \rightarrow S^1$ be the function $f(x, y) = xy^{-1}$

$\langle 1 \rangle 2$. LET: U be an open set in S^1

PROVE: $f^{-1}(U)$ is open in $(S^1)^2$

$\langle 1 \rangle 3$. LET: $(x, y) \in f^{-1}(U)$

$\langle 1 \rangle 4$. $xy^{-1} \in U$

$\langle 1 \rangle 5$. LET: $x = e^{i\phi}$ and $y = e^{i\psi}$

$\langle 1 \rangle 6$. $xy^{-1} = e^{i(\phi-\psi)} \in U$

$\langle 1 \rangle 7$. PICK $\epsilon > 0$ such that, for all t , if $|\phi - \psi - t| < \epsilon$ then $e^{it} \in U$

$\langle 1 \rangle 8$. $(x, y) \in \{e^{it} : |\phi - t| < \epsilon/2\} \times \{e^{it} : |\psi - t| < \epsilon/2\} \subseteq f^{-1}(U)$

\square

Example 17.1.6. $GL(n, \mathbb{R})$ is a topological group considered as a subspace of \mathbb{R}^{n^2} .

PROOF: Since the calculations for matrix multiplication and inverse are compositions of continuous functions. \square

Example 17.1.7. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 17.1.8. *Let G be a group with a topology. Then G is a topological group if and only if the functions $m : G^2 \rightarrow G$ that sends (x, y) to xy and the function $i : G \rightarrow G$ that sends x to x^{-1} are continuous.*

PROOF:

$\langle 1 \rangle 1$. If G is a topological group then i is continuous.

PROOF: Since $x^{-1} = ex^{-1}$.

$\langle 1 \rangle 2$. If G is a topological group then m is continuous.

PROOF: Since $xy = x(y^{-1})^{-1}$.

$\langle 1 \rangle 3$. If m and i are continuous then G is a topological group.

PROOF: Since $xy^{-1} = m(x, i(y))$.

\square

Proposition 17.1.9. *Let G be a topological group. Let $\alpha \in G$. The function that maps x to αx is a homeomorphism between G and itself.*

PROOF:

$\langle 1 \rangle 1$. For any $\alpha \in G$, the function that maps x to αx is continuous.

PROOF: From the definition of topological group.

$\langle 1 \rangle 2$. For any $\alpha \in G$, the function that maps x to αx is a homeomorphism between G and itself.

PROOF: Its inverse is the function that maps x to $\alpha^{-1}x$.

\square

Corollary 17.1.9.1. *Every topological group is homogeneous.*

Proposition 17.1.10. *Let G be a topological group. Let $\alpha \in G$. The function that maps x to $x\alpha$ is a homeomorphism between G and itself.*

PROOF: Similar. \square

17.1.1 Subgroups

Proposition 17.1.11. *Any subgroup of a topological group is a topological group under the subspace topology.*

PROOF: Since the restriction of continuous functions is continuous. \square

Proposition 17.1.12. *Let G be a topological group and H a subgroup of G . Then \overline{H} is a topological group under the subspace topology.*

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \overline{H}$

PROVE: $xy^{-1} \in \overline{H}$

$\langle 1 \rangle 2$. LET: U be a neighbourhood of xy^{-1} .

PROVE: U intersects H .

- ⟨1⟩3. LET: $f : G^2 \rightarrow G$ be the function that maps (x, y) to xy^{-1} .
- ⟨1⟩4. $f^{-1}(U)$ is a neighbourhood of (x, y)
- ⟨1⟩5. PICK neighbourhoods V of x and W of y such that $V \times W \subseteq f^{-1}(U)$.
- ⟨1⟩6. PICK elements $x' \in V \cap H$ and $y' \in W \cap H$
- ⟨1⟩7. $x'y'^{-1} \in U \cap H$

□

17.1.2 Left Cosets

Proposition 17.1.13. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. Let $\alpha \in G$. Define $f_\alpha : G/H \rightarrow G/H$ by*

$$f_\alpha(xH) = \alpha xH \text{ .}$$

Then f_α is a homeomorphism.

PROOF:

- ⟨1⟩1. For all $\alpha \in G$ we have f_α is well defined.
 - ⟨2⟩1. LET: $x, y \in G$
 - ⟨2⟩2. ASSUME: $xH = yH$
 - PROVE: $\alpha xH = \alpha yH$
 - ⟨2⟩3. $x^{-1}y \in H$
 - ⟨2⟩4. $x^{-1}\alpha^{-1}\alpha y \in H$
 - ⟨2⟩5. $\alpha xH = \alpha yH$
- ⟨1⟩2. For all $\alpha \in G$ we have f_α is injective.
 - ⟨2⟩1. LET: $x, y \in G$
 - ⟨2⟩2. ASSUME: $\alpha xH = \alpha yH$
 - PROVE: $xH = yH$
 - ⟨2⟩3. $\alpha x^{-1}\alpha y \in H$
 - ⟨2⟩4. $x^{-1}y \in H$
 - ⟨2⟩5. $xH = yH$
- ⟨1⟩3. For all $\alpha \in G$ we have f_α is surjective.
 - PROOF: For all $x \in G$ we have $xH = f_\alpha(\alpha^{-1}xH)$.
- ⟨1⟩4. For all $\alpha \in G$ we have f_α is continuous.
 - ⟨2⟩1. LET: V be open in G/H
 - ⟨2⟩2. $\pi^{-1}(f_\alpha^{-1}(V))$ is open in G .
 - PROOF: It is $g_\alpha^{-1}(\pi^{-1}(V))$ where $g_\alpha : V \rightarrow V$ is the homeomorphism $g_\alpha(x) = \alpha x$.
 - ⟨2⟩3. $f_\alpha^{-1}(V)$ is open in G/H .
- ⟨1⟩5. For all $\alpha \in G$ we have f_α^{-1} is continuous.
 - PROOF: It is $f_{\alpha^{-1}}$.

□

Corollary 17.1.13.1. *Let G be a topological group and H a subgroup of G . Then G/H is a homogeneous space.*

Proposition 17.1.14. *Let G be a T_1 topological group and H a closed subgroup of G . Then G/H is T_1 .*

PROOF:

$\langle 1 \rangle 1$. LET: $x \in G$

PROVE: xH is closed.

$\langle 1 \rangle 2$. $\pi^{-1}(xH)$ is closed in G .

PROOF: It is $f_x(H)$ and f_x is a homeomorphism.

$\langle 1 \rangle 3$. xH is closed in G/H .

□

Proposition 17.1.15. *Let G be a topological group and H a subgroup of G . Then the canonical map $\pi : G \rightarrow G/H$ is an open map.*

PROOF:

$\langle 1 \rangle 1$. LET: U be open in G .

$\langle 1 \rangle 2$. $\forall h \in H. Uh$ is open in G .

PROOF: Since the function that maps g to gh is an automorphism of G .

$\langle 1 \rangle 3$. UH is open in G

PROOF: It is $\bigcup_{h \in H} Uh$.

$\langle 1 \rangle 4$. $UH = \pi^{-1}(\pi(U))$

PROOF:

$$\begin{aligned} \pi^{-1}(\pi(U)) &= \{x \in G : \exists y \in U. xH = yH\} \\ &= \{x \in G : \exists y \in U. x^{-1}y \in H\} \\ &= \{x \in G : \exists y \in U. \exists h \in H. y^{-1}x = h\} \\ &= \{x \in G : \exists y \in U. \exists h \in H. x = yh\} \\ &= UH \end{aligned}$$

$\langle 1 \rangle 5$. $\pi^{-1}(\pi(U))$ is open in G .

$\langle 1 \rangle 6$. $\pi(U)$ is open in G/H .

□

Proposition 17.1.16. *Let G be a topological group. Let H be a normal subgroup of G . Then G/H is a topological group.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : G^2 \rightarrow G$ be the map $f(x, y) = xy^{-1}$

$\langle 1 \rangle 2$. LET: $g : (G/H)^2 \rightarrow G/H$ be the map $g(xH, yH) = xy^{-1}H$

$\langle 1 \rangle 3$. $g \circ (\pi \times \pi) = \pi \circ f : G^2 \rightarrow G/H$

$\langle 1 \rangle 4$. $g \circ (\pi \times \pi)$ is continuous.

PROOF: Since π and f are continuous.

$\langle 1 \rangle 5$. π is an open quotient map.

PROOF: Proposition 17.1.15.

$\langle 1 \rangle 6$. $\pi \times \pi$ is an open quotient map.

PROOF: Corollary 13.22.7.1.

$\langle 1 \rangle 7$. g is continuous.

PROOF: Theorem 13.22.3.

□

17.1.3 Homogeneous Spaces

Definition 17.1.17 (Homogeneous Space). A *homogeneous space* is a topological space of the form G/H , where G is a topological group and H is a normal subgroup of G , under the quotient topology.

Proposition 17.1.18. *Let G be a topological group and H a normal subgroup of G . Then G/H is Hausdorff if and only if H is closed.*

PROOF: See Bourbaki, N., General Topology. III.12 \square

17.2 Symmetric Neighbourhoods

Definition 17.2.1 (Symmetric Neighbourhood). Let G be a topological group. Let V be a neighbourhood of e . Then V is *symmetric* iff $V = V^{-1}$.

Proposition 17.2.2. *Let G be a topological group. Let U be a neighbourhood of e . Then there exists a symmetric neighbourhood V of e such that $VV \subseteq U$.*

PROOF:

- $\langle 1 \rangle 1$. PICK a neighbourhood V' of e such that $V'V' \subseteq U$.
 - $\langle 2 \rangle 1$. LET: $m : G^2 \rightarrow G$ be the function $m(x, y) = xy$
 - $\langle 2 \rangle 2$. $m^{-1}(U)$ is open in G^2
 - $\langle 2 \rangle 3$. $(e, e) \in m^{-1}(U)$
 - $\langle 2 \rangle 4$. PICK neighbourhoods V_1, V_2 of e such that $V_1 \times V_2 \subseteq m^{-1}(U)$
 - $\langle 2 \rangle 5$. LET: $V' = V_1 \cap V_2$
 - $\langle 1 \rangle 2$. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
 - $\langle 2 \rangle 1$. LET: $f : G^2 \rightarrow G$ be the function $m(x, y) = xy^{-1}$
 - $\langle 2 \rangle 2$. $f^{-1}(V')$ is open in G^2
 - $\langle 2 \rangle 3$. $(e, e) \in m^{-1}(V')$
 - $\langle 2 \rangle 4$. PICK neighbourhoods W_1, W_2 of e such that $W_1 \times W_2 \subseteq f^{-1}(V')$
 - $\langle 2 \rangle 5$. LET: $W = W_1 \cap W_2$
 - $\langle 1 \rangle 3$. LET: $V = WW^{-1}$
 - $\langle 1 \rangle 4$. V is a neighbourhood of e .
 - $\langle 1 \rangle 5$. V is symmetric.
 - $\langle 1 \rangle 6$. $VV \subseteq U$
- \square

Proposition 17.2.3. *Every T_1 topological group is regular.*

PROOF:

- $\langle 1 \rangle 1$. LET: G be a T_1 topological group.
 - $\langle 1 \rangle 2$. LET: A be a closed set in G and $x \in G - A$.
 - $\langle 1 \rangle 3$. $G - Ax^{-1}$ is a neighbourhood of e .
 - $\langle 1 \rangle 4$. PICK a symmetric neighbourhood V of e such that $VV \subseteq G - Ax^{-1}$.
 - $\langle 1 \rangle 5$. LET: $U = VA$ and $U' = Vx$
 - $\langle 1 \rangle 6$. U and U' are disjoint open sets with $A \subseteq U$ and $x \in U'$.
- \square

Proposition 17.2.4. *Let G be a T_1 topological group. Let H be a closed subgroup of G . Then G/H is regular.*

PROOF:

- ⟨1⟩1. LET: A be a closed set in G/H and $xH \in G/H - A$.
- ⟨1⟩2. $G - \pi^{-1}(A)x^{-1}$ is a neighbourhood of e .
- ⟨1⟩3. PICK a symmetric neighbourhood V of e such that $VV \subseteq G - \pi^{-1}(A)x^{-1}$.
- ⟨1⟩4. LET: $U = \pi(V)A$ and $U' = \pi(V)(xH)$.
- ⟨1⟩5. U and U' are disjoint open sets with $A \subseteq U$ and $xH \in U'$
 - ⟨2⟩1. ASSUME: for a contradiction $U \cap U' \neq \emptyset$.
 - ⟨2⟩2. PICK $v_1, v_2 \in V$ and $a \in G$ such that $aH \in A$ and $v_1aH = v_2xH$.
 - ⟨2⟩3. $a^{-1}v_1^{-1}v_2x \in H$
 - ⟨2⟩4. $v_1^{-1}v_2 \in \pi^{-1}(A)x^{-1}$
 - ⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩3.

□

17.3 Continuous Actions

Definition 17.3.1 (Continuous Action). Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \rightarrow X$ such that:

- $\forall x \in X. ex = x$
- $\forall g, h \in G. \forall x \in X. g(hx) = (gh)x$

A G -space consists of a topological space X and a continuous action of G on X .

Definition 17.3.2 (Orbit). Let X be a G -space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 17.3.3. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:

- ⟨1⟩1. LET: $f_3 : S^2/SO(2) \rightarrow [-1, 1]$ be the function induced by $\pi_3 : S^2 \rightarrow [-1, 1]$
- ⟨1⟩2. f_3 is bijective.
- ⟨1⟩3. $S^2/SO(2)$ is compact.

PROOF: It is the continuous image of S^2 which is compact.

- ⟨1⟩4. $[-1, 1]$ is Hausdorff.
- ⟨1⟩5. f_3 is a homeomorphism.

□

Definition 17.3.4 (Stabilizer). Let X be a G -space and $x \in X$. The *stabilizer* of x is $G_x := \{g \in G : gx = x\}$.

Proposition 17.3.5. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx .*

PROOF:

⟨1⟩1. If $gG_x = hG_x$ then $gx = hx$.

⟨2⟩1. ASSUME: $gG_x = hG_x$

⟨2⟩2. $g^{-1}h \in G_x$

⟨2⟩3. $g^{-1}hx = x$

⟨2⟩4. $gx = hx$

⟨1⟩2. If $gx = hx$ then $gG_x = hG_x$.

PROOF: Similar.

⟨1⟩3. The function is continuous.

PROOF: Theorem 13.22.3.

□

Chapter 18

Topological Vector Spaces

Definition 18.0.1 (Topological Vector Space). Let K be either \mathbb{R} or \mathbb{C} . A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Subtraction is a continuous function $E^2 \rightarrow E$
- Multiplication is a continuous function $K \times E \rightarrow E$

Proposition 18.0.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. \square

Theorem 18.0.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 \square

Proposition 18.0.4. *Let E be a topological vector space and E_0 a subspace of E . Then $\overline{E_0}$ is a subspace of E .*

Definition 18.0.5. Let E be a topological vector space. The topological space associated with E is $E/\overline{\{0\}}$.

18.1 Cauchy Sequences

Definition 18.1.1 (Cauchy Sequence). Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geq n_0, x_n - x_m \in U$.

Definition 18.1.2 (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

18.2 Seminorms

Definition 18.2.1 (Seminorm). Let E be a vector space over K . A *seminorm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ such that:

1. $\forall x \in E, \|x\| \geq 0$
2. $\forall \alpha \in K, \forall x \in E, \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y \in E, \|x + y\| \leq \|x\| + \|y\|$

Example 18.2.2. The function that maps (x_1, \dots, x_n) to $|x_i|$ is a seminorm on \mathbb{R}^n .

Definition 18.2.3. Let E be a vector space over K . Let Λ be a set of seminorms on E . The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x \in E$.

Proposition 18.2.4. E is a topological vector space under this topology. It is Hausdorff iff, for all $x \in E$, if $\forall \lambda \in \Lambda, \lambda(x) = 0$ then $x = 0$.

18.3 Fréchet Spaces

Definition 18.3.1 (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 18.3.2. Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$. Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 18.3.3 (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

18.4 Normed Spaces

Definition 18.4.1 (Normed Space). Let E be a vector space over K . A *norm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ is a seminorm such that, $\forall x \in E, \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 18.4.2. If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E .

Definition 18.4.3 (p -norm). For any $p \geq 1$, the p -norm on \mathbb{R}^n is defined by

$$\|\vec{x}\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1$. For all $\vec{x} \in \mathbb{R}^n$ we have $\|\vec{x}\|_p \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. For all $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ we have $\|\alpha \vec{x}\|_p = |\alpha| \|\vec{x}\|_p$

PROOF:

$$\begin{aligned} \|\alpha(x_1, \dots, x_n)\| &= \|(\alpha x_1, \dots, \alpha x_n)\| \\ &= \left(\sum_{i=1}^n (\alpha x_i)^p \right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|\vec{x}\|_p \end{aligned}$$

$\langle 1 \rangle 3$. The triangle inequality holds.

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\ &= \sum_{i=1}^n |x_i + y_i| |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^n (|x_i| + |y_i|) |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} \\ &\leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} \quad (\text{Hölder's Inequality}) \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p) \|\vec{x} + \vec{y}\|_p^{p-1} \end{aligned}$$

Assuming w.l.o.g. $\|\vec{x} + \vec{y}\|_p^{p-1} \neq 0$ (using ??) we have $\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$.

$\langle 1 \rangle 4$. For any $\vec{x} \in \mathbb{R}^n$, we have $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.

PROOF: $\sum_{i=1}^n x_i^p = 0$ iff $x_1 = \dots = x_n = 0$.

□

Proposition 18.4.4. The p -norm on \mathbb{R}^n induces the product topology.

PROOF:

⟨1⟩1. LET: d be the metric induced by the p -norm and ρ the square metric on \mathbb{R}^n .

⟨1⟩2. The metric topology is finer than the product topology.

⟨2⟩1. LET: $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$

⟨2⟩2. LET: $\delta = \epsilon/n^{\frac{1}{p}}$

PROVE: $B_\rho(\vec{x}, \delta) \subseteq B_d(\vec{x}, \epsilon)$

⟨2⟩3. LET: $\vec{y} \in B_\rho(\vec{x}, \delta)$

⟨2⟩4. $\forall i. |x_i - y_i| < \delta$

⟨2⟩5. $d(\vec{x}, \vec{y}) < \epsilon$

PROOF:

$$\begin{aligned} d(\vec{x}, \vec{y}) &= \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}} \\ &< \left(\sum_{i=1}^n \delta^p \right)^{\frac{1}{p}} && (\langle 2 \rangle 4) \\ &= n^{\frac{1}{p}} \delta \\ &= \epsilon && (\langle 2 \rangle 2) \end{aligned}$$

⟨1⟩3. The product topology is finer than the metric topology.

⟨2⟩1. LET: $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$

⟨2⟩2. LET: $\vec{y} \in B_d(\vec{x}, \epsilon)$

⟨2⟩3. $d(\vec{x}, \vec{y}) < \epsilon$

⟨2⟩4. $\sum_{i=1}^n |x_i - y_i|^p < \epsilon^p$

⟨2⟩5. $\forall i. |x_i - y_i|^p < \epsilon^p$

⟨2⟩6. $\forall i. |x_i - y_i| < \epsilon$

⟨2⟩7. $\rho(\vec{x}, \vec{y}) < \epsilon$

□

Definition 18.4.5 (Sup-norm). The *sup-norm* on \mathbb{R}^n is defined by

$$\|(x_1, \dots, x_n)\|_\infty := \max(|x_1|, \dots, |x_n|) .$$

Proposition 18.4.6. The 2-norm on \mathbb{R}^n induces the standard metric.

PROOF: Immediate from definitions. □

Definition 18.4.7. For $p \geq 1$, the normed space l_p is the set of all sequences (x_n) in \mathbb{R} such that $\sum_{n=1}^\infty x_n^p$ converges, under

$$\|(x_n)\|_p := \left(\sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} .$$

Proposition 18.4.8. The spaces l_p for $p \geq 1$ are all homeomorphic.

PROOF: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. <http://dx.doi.org/10.1007/BF01075865>.

Proposition 18.4.9. *The metric topology on l_2 is strictly finer than the uniform topology.*

PROOF:

⟨1⟩1. LET: d be the metric induced by the l^2 -norm and $\bar{\rho}$ the uniform topology.

⟨1⟩2. The metric topology is finer than the uniform topology.

⟨2⟩1. LET: $x \in l_2$

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. LET: $\delta = \epsilon/2$

⟨2⟩4. LET: $y \in B_d(x, \delta)$

⟨2⟩5. $\sum_{n=0}^{\infty} (x_n - y_n)^2 < \delta^2$

⟨2⟩6. $\forall n. (x_n - y_n)^2 < \delta^2$

⟨2⟩7. $\forall n. |x_n - y_n| < \delta$

⟨2⟩8. $\forall n. \bar{d}(x_n, y_n) < \delta$

⟨2⟩9. $\bar{\rho}(x, y) \leq \delta$

⟨2⟩10. $\bar{\rho}(x, y) < \epsilon$

⟨2⟩11. $y \in B_{\bar{\rho}}(x, \epsilon)$

⟨1⟩3. The metric topology is not the same as the uniform topology.

⟨2⟩1. ASSUME: for a contradiction $B_d(0, 1)$ is open in the uniform topology.

⟨2⟩2. PICK $\epsilon > 0$ such that $B_{\bar{\rho}}(0, \epsilon) \subseteq B_d(0, 1)$

⟨2⟩3. PICK an integer N such that $1/N < \epsilon^2/4$

⟨2⟩4. LET: (x_n) be the sequence with $x_n = \epsilon/2$ for $n < N$ and $x_n = 0$ for $n \geq N$

⟨2⟩5. $(x_n) \in l_2$

⟨2⟩6. $(x_n) \in B_{\bar{\rho}}(0, \epsilon)$

PROOF: Since $\bar{\rho}((x_n), 0) = \epsilon/2$.

⟨2⟩7. $d((x_n), 0) > 1$

PROOF:

$$\begin{aligned} d((x_n), 0)^2 &= \sum_{n=0}^{\infty} x_n^2 \\ &= N\epsilon^2/4 \\ &> 1 \end{aligned}$$

□

Proposition 18.4.10. *The metric topology on l_2 is strictly coarser than the box topology.*

PROOF:

⟨1⟩1. The box topology is finer than the metric topology.

⟨2⟩1. LET: $(x_n) \in l_2$ and $\epsilon > 0$.

⟨2⟩2. LET: $(y_n) \in B((x_n), \epsilon)$

⟨2⟩3. PICK a sequence of real numbers (δ_n) such that $\sum_{n=0}^{\infty} \delta_n^2 < (\epsilon - d((x_n), (y_n)))^2$

⟨2⟩4. LET: $U = \prod_n (y_n - \delta_n, y_n + \delta_n)$

PROVE: $U \subseteq B((x_n), \epsilon)$

⟨2⟩5. LET: $(z_n) \in U$

⟨2⟩6. $d((z_n), (y_n)) < \epsilon - d((x_n), (y_n))$

PROOF:

$$\begin{aligned} d((z_n), (y_n))^2 &= \sum_{n=0}^{\infty} (z_n - y_n)^2 \\ &< \sum_{n=0}^{\infty} \delta_n^2 \\ &< (\epsilon - d((x_n), (y_n)))^2 \end{aligned}$$

$$\langle 2 \rangle 7. d((z_n), (x_n)) < \epsilon$$

$\langle 1 \rangle 2.$ The box topology is not equal to the metric topology.

$\langle 2 \rangle 1.$ LET: $U = \prod_n (-1/n, 1/n)$

$\langle 2 \rangle 2.$ ASSUME: for a contradiction U is open in the metric topology.

$\langle 2 \rangle 3.$ PICK $\epsilon > 0$ such that $B(0, \epsilon) \subseteq U$

$\langle 2 \rangle 4.$ PICK N such that $1/N < \epsilon/2$.

$\langle 2 \rangle 5.$ LET: (x_n) be the sequence with $x_N = \epsilon/2$ and $x_n = 0$ for all other n .

$\langle 2 \rangle 6.$ $d((x_n), 0) = \epsilon/2$

$\langle 2 \rangle 7.$ $(x_n) \notin U$

□

Proposition 18.4.11. *The l^2 -topology on \mathbb{R}^∞ is strictly finer than the uniform topology.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: for a contradiction $B_d(0, 1) \cap \mathbb{R}^\infty$ is open in the uniform topology.

$\langle 1 \rangle 2.$ PICK $\epsilon > 0$ such that $B_{\bar{\rho}}(0, \epsilon) \cap \mathbb{R}^\infty \subseteq B_d(0, 1) \cap \mathbb{R}^\infty$

$\langle 1 \rangle 3.$ PICK an integer N such that $1/N < \epsilon^2/4$

$\langle 1 \rangle 4.$ LET: (x_n) be the sequence with $x_n = \epsilon/2$ for $n < N$ and $x_n = 0$ for $n \geq N$

$\langle 1 \rangle 5.$ $(x_n) \in \mathbb{R}^\infty$

$\langle 1 \rangle 6.$ $(x_n) \in B_{\bar{\rho}}(0, \epsilon)$

PROOF: Since $\bar{\rho}((x_n), 0) = \epsilon/2$.

$\langle 1 \rangle 7.$ $d((x_n), 0) > 1$

PROOF:

$$\begin{aligned} d((x_n), 0)^2 &= \sum_{n=0}^{\infty} x_n^2 \\ &= N\epsilon^2/4 \\ &> 1 \end{aligned}$$

□

Proposition 18.4.12. *The l^2 -topology on \mathbb{R}^∞ is strictly coarser than the box topology.*

$\langle 1 \rangle 1.$ LET: $U = \prod_n (-1/n, 1/n) \cap \mathbb{R}^\infty$

$\langle 1 \rangle 2.$ ASSUME: for a contradiction U is open in the metric topology.

$\langle 1 \rangle 3.$ PICK $\epsilon > 0$ such that $B(0, \epsilon) \cap \mathbb{R}^\infty \subseteq U \cap \mathbb{R}^\infty$

$\langle 1 \rangle 4.$ PICK N such that $1/N < \epsilon/2$.

⟨1⟩5. LET: (x_n) be the sequence with $x_N = \epsilon/2$ and $x_n = 0$ for all other n .

⟨1⟩6. $d((x_n), 0) = \epsilon/2$

⟨1⟩7. $(x_n) \notin U$

□

Proposition 18.4.13. *The l^2 -topology on the Hilbert cube the same as the product topology.*

PROOF:

⟨1⟩1. For every $(x_n) \in H$ and $\epsilon > 0$, there exists a neighbourhood U of (x_n) in the product topology such that $U \subseteq B((x_n), \epsilon)$.

⟨2⟩1. LET: $(x_n) \in H$

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. PICK N such that $\sum_{i=N+1}^{\infty} 1/i^2 < \epsilon^2/2$

⟨2⟩4. LET: $B' = (\prod_{i=0}^N (x_i - \epsilon/\sqrt{2N}, x_i + \epsilon/\sqrt{2N}) \times \prod_{i=N+1}^{\infty} [0, 1/(i+1)]) \cap H$

PROVE: $B' \subseteq B((x_n), \epsilon)$

⟨2⟩5. LET: $(y_n) \in B'$

⟨2⟩6. $d((x_n), (y_n)) < \epsilon$

PROOF:

$$\begin{aligned} d((x_n), (y_n))^2 &= \sum_{i=0}^{\infty} |x_n - y_n|^2 \\ &< \sum_{i=0}^N \epsilon^2/2N + \sum_{i=N+1}^{\infty} 1/(i+1)1/(i+1)^2 \\ &< \epsilon^2/2 + \epsilon^2/2 \\ &= \epsilon^2 \end{aligned}$$

⟨1⟩2. The product topology is finer than the l^2 -topology.

⟨2⟩1. LET: $(x_n) \in H$ and $\epsilon > 0$

PROVE: $B((x_n), \epsilon) \cap H$ is open in the product topology.

⟨2⟩2. LET: $(y_n) \in B((x_n), \epsilon)$

⟨2⟩3. PICK a neighbourhood U of (y_n) in the product topology such that

$U \subseteq B((y_n), \epsilon - d((x_n), (y_n)))$

⟨2⟩4. $U \subseteq B((x_n), \epsilon)$

□

Definition 18.4.14. Let l_{∞} be the set of all bounded sequences in \mathbb{R} under

$$\|(x_n)\| := \sup_n |x_n|$$

Proposition 18.4.15. *For all $p \geq 1$ we have l_p is not homeomorphic to l_{∞} .*

Proposition 18.4.16. *Let $\| \cdot \|$ be a seminorm on the vector space E . Then $\| \cdot \|$ defines a norm on $E/\{0\}$.*

Proposition 18.4.17. *Let E and F be normed spaces. Any continuous linear map $E \rightarrow F$ is uniformly continuous.*

Definition 18.4.18. For $p \geq 1$, let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n) / \{0\}.$$

18.5 Unit Ball

Proposition 18.5.1. *Let n be a positive integer. Every open ball $B(\vec{x}, \epsilon)$ in \mathbb{R}^n is path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $\vec{y}, \vec{z} \in B(\vec{x}, \epsilon)$

$\langle 1 \rangle 2$. LET: $\vec{p} : [0, 1] \rightarrow B(\vec{x}, \epsilon)$ be the path $\vec{p}(t) = (1-t)\vec{y} + t\vec{z}$.

$\langle 2 \rangle 1$. LET: $t \in [0, 1]$

PROVE: $\vec{p}(t) \in B(\vec{x}, \epsilon)$

$\langle 2 \rangle 2$. $d(\vec{p}(t), \vec{x}) < \epsilon$

PROOF:

$$\begin{aligned} d(\vec{p}(t), \vec{x}) &= \|(1-t)\vec{y} + t\vec{z} - \vec{x}\| \\ &= \|(1-t)(\vec{y} - \vec{x}) + t(\vec{z} - \vec{x})\| \\ &\leq (1-t)\|\vec{y} - \vec{x}\| + t\|\vec{z} - \vec{x}\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. \vec{p} is a path from \vec{x} to \vec{y} .

□

Proposition 18.5.2. *Let n be a positive integer. Every closed ball $B(\vec{x}, \epsilon)$ in \mathbb{R}^n is path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $\vec{y}, \vec{z} \in \overline{B(\vec{x}, \epsilon)}$

$\langle 1 \rangle 2$. LET: $\vec{p} : [0, 1] \rightarrow \overline{B(\vec{x}, \epsilon)}$ be the path $\vec{p}(t) = (1-t)\vec{y} + t\vec{z}$.

$\langle 2 \rangle 1$. LET: $t \in [0, 1]$

PROVE: $\vec{p}(t) \in \overline{B(\vec{x}, \epsilon)}$

$\langle 2 \rangle 2$. $d(\vec{p}(t), \vec{x}) \leq \epsilon$

PROOF:

$$\begin{aligned} d(\vec{p}(t), \vec{x}) &= \|(1-t)\vec{y} + t\vec{z} - \vec{x}\| \\ &= \|(1-t)(\vec{y} - \vec{x}) + t(\vec{z} - \vec{x})\| \\ &\leq (1-t)\|\vec{y} - \vec{x}\| + t\|\vec{z} - \vec{x}\| \\ &\leq (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. \vec{p} is a path from \vec{x} to \vec{y} .

□

18.6 Unit Sphere

Definition 18.6.1 (Unit Sphere). Let n be a positive integer. The *unit sphere* S^{n-1} is

$$S^{n-1} := \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = 1\} .$$

Proposition 18.6.2. For $n > 1$, the unit sphere S^{n-1} is path connected.

PROOF: The map $g : \mathbb{R}^n - \{\vec{0}\} \rightarrow S^{n-1}$ defined by $g(\vec{x}) = \vec{x}/\|\vec{x}\|$ is continuous and surjective. Hence S^{n-1} is the continuous image of a path connected space. \square

18.7 Inner Product Spaces

Definition 18.7.1 (Inner Product). Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n .$$

Proposition 18.7.2.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF:

$$\begin{aligned} \vec{x} \cdot (\vec{y} + \vec{z}) &= x_1(y_1 + z_1) + \cdots + x_n(y_n + z_n) \\ &= x_1 y_1 + x_1 z_1 + \cdots + x_n y_n + x_n z_n \\ &= \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z} \end{aligned}$$

\square

Proposition 18.7.3. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\| .$$

PROOF:

- $\langle 1 \rangle 1$. ASSUME: w.l.o.g. $\vec{x} \neq \vec{0} \neq \vec{y}$
- $\langle 1 \rangle 2$. LET: $a = 1/\|\vec{x}\|$
- $\langle 1 \rangle 3$. LET: $b = 1/\|\vec{y}\|$
- $\langle 1 \rangle 4$. $\|a\vec{x} + b\vec{y}\| \geq 0$
- $\langle 1 \rangle 5$. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
- $\langle 1 \rangle 6$. $ab\vec{x} \cdot \vec{y} \geq -1$
- $\langle 1 \rangle 7$. $\|a\vec{x} - b\vec{y}\| \geq 0$
- $\langle 1 \rangle 8$. $ab\vec{x} \cdot \vec{y} \leq 1$
- $\langle 1 \rangle 9$. $|\vec{x} \cdot \vec{y}| \leq 1/ab$

\square

Proposition 18.7.4. Let $(x_n), (y_n)$ be sequences of real numbers. If $\sum_{n=0}^{\infty} x_n^2$ and $\sum_{n=0}^{\infty} y_n^2$ converge then $\sum_{n=0}^{\infty} |x_n y_n|$ converges.

PROOF:

$$\begin{aligned} \sum_{n=0}^N |x_n y_n| &\leq \sqrt{\sum_{n=0}^N x_n^2 \sum_{n=0}^N y_n^2} && \text{(Proposition 18.7.3)} \\ &\leq \sqrt{\sum_{n=0}^{\infty} x_n^2 \sum_{n=0}^{\infty} y_n^2} && \square \end{aligned}$$

Proposition 18.7.5. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E .*

18.8 Banach Spaces

Definition 18.8.1 (Banach Space). A *Banach space* is a complete normed space.

Example 18.8.2. For any topological space X , the set $C(X)$ of bounded continuous functions $X \rightarrow \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 18.8.3. *The completion of a normed space is a Banach space.*

Proposition 18.8.4. *Let E and F be normed spaces. Let $f : E \rightarrow F$ be a continuous linear map. Then the extension to the completions $\hat{E} \rightarrow \hat{F}$ is linear.*

Proposition 18.8.5. *$L^p(\mathbb{R}^n)$ is a Banach space.*

Proposition 18.8.6. *$C(\mathbb{R})$ is first countable but not second countable.*

PROOF: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n . Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.

It is first countable because it is metrizable. \square

18.9 Hilbert Spaces

Definition 18.9.1 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Example 18.9.2. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \rightarrow \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

Proposition 18.9.3. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

18.10 Locally Convex Spaces

Definition 18.10.1 (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 18.10.2. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Proposition 18.10.3. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Example 18.10.4. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \rightarrow \mathbb{R}$ is continuous as a map $E' \rightarrow \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonné, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 18.10.5 (Thom Space). Let E be a vector bundle with a Riemannian metric, $DE = \{x \in E : \|x\| \leq 1\}$ its disc bundle and $SE := \{v \in E : \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE .