

Mathematics

Robin Adams

September 18, 2023

Contents

1	Primitive Terms and Axioms	5
1.1	Primitive Terms	5
1.2	Axioms	5
1.3	Consequences of the Axioms	6
1.3.1	Definitions	6
1.3.2	The Empty Set	6
1.3.3	The Singleton	7
1.3.4	Subsets	8
1.4	Composition	9
1.5	Axioms Part Two	9
1.6	Cartesian Product	10
1.7	Quotient Sets	10
2	Topology	11
2.1	Topological Spaces	11
2.1.1	Subspaces	13
2.1.2	Topological Disjoint Union	13
2.1.3	Product Topology	13
2.1.4	Bases	13
2.1.5	Subbases	13
2.2	Continuous Functions	13
2.3	Convergence	14
2.4	Connected Spaces	14
2.5	Hausdorff Spaces	14
2.6	Compactness	15
2.7	Quotient Spaces	15
2.8	Metric Spaces	16
3	Topological Groups	17
3.1	Continuous Actions	17
4	Topological Vector Spaces	19
4.1	Cauchy Sequences	19
4.2	Seminorms	20

4.3	Fréchet Spaces	20
4.4	Normed Spaces	20
4.5	Inner Product Spaces	21
4.6	Banach Spaces	21
4.7	Hilbert Spaces	21
4.8	Locally Convex Spaces	21

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*. We write $A : \text{Set}$ for: A is a set.

For any set A , let there be *elements* of A . We write $a : \text{El}(A)$ for: a is an element of A .

For any sets A and B , let there be *functions* from A to B . We write $f : A \rightarrow B$ iff f is a function from A to B .

For any function $f : A \rightarrow B$ and element $a : \text{El}(A)$, let there be an element $f(a) : \text{El}(B)$, the *value* of the function f at the *argument* a .

For any sets A and B , let there be a set $A \times B$, the *Cartesian product* of A and B , and functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$, the *projections*.

For any elements $a : \text{El}(A)$ and $b : \text{El}(B)$, let there be an element $(a, b) : \text{El}(A \times B)$, the *ordered pair* of a and b .

1.2 Axioms

Axiom 1.1 (Strong Extensionality). *Let $i : A \rightarrow B$. Suppose that, for every $y : \text{El}(B)$, there exists a unique $x : \text{El}(A)$ such that $i(x) = y$. Then there exists a function $i^{-1} : B \rightarrow A$ such that $\forall x : \text{El}(A) . i^{-1}(i(x)) = x$ and $\forall y : \text{El}(B) . i(i^{-1}(y)) = y$.*

Axiom 1.2 (Pairing).

- $\forall x : \text{El}(A) . \forall y : \text{El}(B) . \pi_1(x, y) = x$
- $\forall x : \text{El}(A) . \forall y : \text{El}(B) . \pi_2(x, y) = y$
- $\forall p : \text{El}(A \times B) . p = (\pi_1(p), \pi_2(p))$

Definition 1.3 (Injective). A function $f : A \rightarrow B$ is *injective* or an *injection* iff, for all $x, y : \text{El}(A)$, if $f(x) = f(y)$ then $x = y$.

Axiom 1.4 (Separation). For every property $P[X, x]$ where X is a set variable and $x : \text{El}(X)$, the following is an axiom:

For every set A , there exists a set $S = \{x : \text{El}(A) \mid P[A, x]\}$ and an injection $i : S \rightarrow A$ such that, for all $x : \text{El}(A)$, we have

$$(\exists y : S. i(y) = x) \Leftrightarrow P[A, x] .$$

Axiom 1.5 (Infinity). There exists a set \mathbb{N} , an element $0 : \text{El}(\mathbb{N})$, and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that:

- $\forall n : \text{El}(\mathbb{N}). s(n) \neq 0$
- $\forall m, n : \text{El}(\mathbb{N}). s(m) = s(n) \Rightarrow m = n$.

Axiom 1.6 (Choice). Let R be a set and $i : R \rightarrow A \times B$ an injection. Assume $\forall a : \text{El}(A). \exists r : \text{El}(R). \pi_1(i(r)) = a$. Then there exists a function $f : A \rightarrow B$ such that $\forall a : \text{El}(A). \exists r : \text{El}(R). i(r) = (a, f(a))$.

1.3 Consequences of the Axioms

1.3.1 Definitions

Definition 1.7. Let $f, g : A \rightarrow B$. We say f and g are *equal*, $f = g$, iff $\forall x : \text{El}(A). f(x) = g(x)$.

Definition 1.8 (Surjective). A function $f : A \rightarrow B$ is *surjective* iff, for all $y : \text{El}(B)$, there exists $x : \text{El}(A)$ such that $f(x) = y$.

Definition 1.9 (Bijective). A function $f : A \rightarrow B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$ ".

1.3.2 The Empty Set

Theorem 1.10. There exists a set which has no elements.

PROOF:

$\langle 1 \rangle$ 1. PICK a set A

PROOF: By the Axiom of Infinity, a set exists.

$\langle 1 \rangle$ 2. LET: $R : A \looparrowright A$ be the relation such that, for all $x, y \in A$, we have $\neg(xRy)$

PROOF: By the Axiom of Comprehension.

$\langle 1 \rangle 3$. LET: $|R|$ be the tabulation of R with projections $p, q : |R| \rightarrow A$.

PROVE: $|R|$ has no elements.

PROOF: By the Axiom of Tabulations.

$\langle 1 \rangle 4$. ASSUME: for a contradiction $r : \text{El}(|R|)$

$\langle 1 \rangle 5$. $p(r)Rq(r)$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 2$.

□

Theorem 1.11. *If E and E' have no elements then $E \approx E'$.*

PROOF:

$\langle 1 \rangle 1$. LET: E and E' have no elements.

$\langle 1 \rangle 2$. LET: $F : E \rightarrowtail E'$ be the relation such that, for all $x : \text{El}(E)$ and $y : \text{El}(E')$, we have xFy .

PROOF: Axiom of Comprehension.

$\langle 1 \rangle 3$. F is a function.

PROOF: Vacuously, for all $x : \text{El}(E)$, there exists a unique $y : \text{El}(E')$ such that xFy .

$\langle 1 \rangle 4$. F is injective.

PROOF: Vacuously, for all $x, y : \text{El}(E)$, if $F(x) = F(y)$ then $x = y$.

$\langle 1 \rangle 5$. F is surjective.

PROOF: Vacuously, for all $y : \text{El}(E')$, there exists $x : \text{El}(E)$ such that $F(x) = y$.

□

Definition 1.12 (Empty Set). The *empty set* \emptyset is the set with no elements.

1.3.3 The Singleton

Theorem 1.13. *There exists a set that has exactly one element.*

PROOF:

$\langle 1 \rangle 1$. PICK a set A that has an element.

PROOF: By the Axiom of Infinity, there exists a set that has an element.

$\langle 1 \rangle 2$. PICK $a : \text{El}(A)$

$\langle 1 \rangle 3$. LET: $R : A \rightarrowtail A$ be the relation such that, for all $x, y : \text{El}(A)$, we have xRy if and only if $x = y = a$.

PROOF: By the Axiom of Comprehension.

$\langle 1 \rangle 4$. LET: $|R|$ be the tabulation of R with projections $p, q : |R| \rightarrow A$.

PROVE: $|R|$ has exactly one element.

PROOF: By the Axiom of Tabulations.

$\langle 1 \rangle 5$. LET: $r : \text{El}(|R|)$ be the element such that $p(r) = q(r) = a$

PROOF: Since aRa by $\langle 1 \rangle 3$.

$\langle 1 \rangle 6$. LET: $s : \text{El}(|R|)$

PROVE: $s = r$

$\langle 1 \rangle 7. p(s)Rq(s)$

PROOF: By the Axiom of Tabulations.

$\langle 1 \rangle 8. p(s) = q(s) = a$

PROOF: By $\langle 1 \rangle 3$.

$\langle 1 \rangle 9. p(s) = p(r)$ and $q(s) = q(r)$

PROOF: By $\langle 1 \rangle 5$.

$\langle 1 \rangle 10. s = r$

PROOF: By the Axiom of Tabulations.

□

Theorem 1.14. *If A and B both have exactly one element then $A \approx B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B both have exactly one element.

$\langle 1 \rangle 2$. LET: $F : A \rightarrowtail B$ be the relation such that, for all $x : \text{El}(A)$ and $y : \text{El}(B)$, we have xFy .

$\langle 1 \rangle 3$. F is a function.

PROOF: If xFy and xFy' then $y = y'$ because B has only one element.

$\langle 1 \rangle 4$. F is injective.

PROOF: If $F(x) = F(x')$ then $x = x'$ because A has only one element.

$\langle 1 \rangle 5$. F is surjective.

$\langle 2 \rangle 1$. LET: $y : \text{El}(B)$

$\langle 2 \rangle 2$. LET: x be the element of A .

$\langle 2 \rangle 3$. $F(x) = y$

□

Definition 1.15 (Singleton). Let 1 be the set that has exactly one element. Let $*$ be its element.

1.3.4 Subsets

Definition 1.16 (Subset). A *subset* of a set A is a relation $1 \rightarrowtail S$.

Given $S : 1 \rightarrowtail S$ and $a : \text{El}(A)$, we write $a \in S$ for $*Sa$.

Theorem Schema 1.17. *For any property $P[X, x]$ where X is a set variable and $x : \text{El}(X)$, the following is a theorem:*

For any set A , there exists a set B and injection $i : B \rightarrow A$ such that, for all $x : \text{El}(A)$, we have $P[A, x]$ if and only if there exists $b : \text{El}(B)$ such that $i(b) = x$.

PROOF:

$\langle 1 \rangle 1$. LET: $S : 1 \rightarrowtail A$ be the relation such that, for all $e : \text{El}(1)$ and $a : \text{El}(A)$, we have eSa if and only if $P[A, a]$.

PROOF: Axiom of Comprehension.

$\langle 1 \rangle 2$. LET: B be the tabulation of S with projections $p : B \rightarrow 1$ and $i : B \rightarrow A$.

PROOF: Axiom of Tabulations.

$\langle 1 \rangle 3$. i is injective.

$\langle 2 \rangle 1$. LET: $r, s : \text{El}(B)$

$\langle 2 \rangle 2$. ASSUME: $i(r) = i(s)$

$\langle 2 \rangle 3$. $p(r) = p(s)$

PROOF: Since 1 has only one element.

$\langle 2 \rangle 4$. $r = s$

PROOF: Axiom of Tabulations.

$\langle 1 \rangle 4$. For all $x : \text{El}(A)$, we have $P[A, x]$ if and only if there exists $b : \text{El}(B)$ such that $i(b) = x$.

$\langle 2 \rangle 1$. LET: $x : \text{El}(A)$

$\langle 2 \rangle 2$. If $P[A, x]$ then there exists $b : \text{El}(B)$ such that $i(b) = x$

$\langle 3 \rangle 1$. ASSUME: $P[A, x]$

$\langle 3 \rangle 2$. $*Sx$

PROOF: $\langle 1 \rangle 1$

$\langle 3 \rangle 3$. There exists $b : \text{El}(B)$ such that $p(b) = *$ and $i(b) = x$

PROOF: Axiom of Tabulations.

$\langle 2 \rangle 3$. For all $b : \text{El}(B)$ we have $P[A, i(b)]$

$\langle 3 \rangle 1$. LET: $b : \text{El}(B)$

$\langle 3 \rangle 2$. $p(b)Si(b)$

PROOF: Axiom of Tabulations.

$\langle 3 \rangle 3$. $P[A, i(b)]$

PROOF: $\langle 1 \rangle 1$

□

1.4 Composition

Definition 1.18 (Composite). Let $\phi : A \rightharpoonup B$ and $\psi : B \rightharpoonup C$. The *composite* $\psi \circ \phi : A \rightharpoonup C$ is the relation such that $a(\psi \circ \phi)c$ iff there exists b such that $a\phi b$ and $b\psi c$.

Definition 1.19 (Identity). For any set A , the *identity* function $\text{id}_A : A \rightarrow A$ is the function defined by $\text{id}_A(a) = a$.

Theorem 1.20. *Composition of relations is associative, and the identity function is an identity for composition. The composite of functions is a function. The composite of injective functions is injective. The composite of surjective functions is surjective. The composite of bijections is a bijection. A function $f : A \rightarrow B$ is a bijection iff there exists a function $f^{-1} : B \rightarrow A$ such that $f^{-1}f = \text{id}_A$ and $ff^{-1} = \text{id}_B$, in which case f^{-1} is unique.*

1.5 Axioms Part Two

Axiom 1.21 (Power Set). For any set A , there exists a set $\mathcal{P}A$, the power set of A , and a relation $\in : A \rightharpoonup \mathcal{P}A$, called membership, such that, for any subset S of A , there exists a unique $\overline{S} \in \mathcal{P}A$ such that, for all $x \in A$, we have $x \in \overline{S}$ if and only if $x \in S$.

We usually write just S for \overline{S} .

Axiom Schema 1.22 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A , there exists a set B , a function $p : B \rightarrow A$, a set Y and a relation $M : B \rightarrowtail Y$ such that:

- $\forall b \in B. P[A, \{y \in Y : bMy\}, p(b)]$
- *For all $a \in A$, if $\exists Y. P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 1.23 (Universe). Let $E : U \rightarrowtail X$ be a relation. Let us say that a set A is *small* iff there exists $u \in U$ such that $A \approx \{x \in X : uEx\}$.

Then (U, X, E) form a *universe* if and only if:

- \mathbb{N} is U -small.
- For any U -small sets A and B and relation $R : A \rightarrowtail B$, the tabulation of R is U -small.
- If A is U -small then so is $\mathcal{P}A$
- Let $f : A \rightarrow B$ be a function. If B is U -small and $f^{-1}(b)$ is U -small for all $b \in B$, then A is U -small.
- If $p : B \twoheadrightarrow A$ is a surjective function such that A is U -small, then there exists a U -small set C , a surjection $q : C \twoheadrightarrow A$, and a function $f : C \rightarrow B$ such that $q = pf$.

Axiom 1.24 (Universe). *There exists a universe.*

Let $E : U \rightarrowtail X$ be a universe. We shall say a set is *small* iff it is U -small, and *large* otherwise.

1.6 Cartesian Product

Definition 1.25 (Cartesian Product). Let A and B be sets. The *Cartesian product* of A and B , $A \times B$, is the tabulation of the relation $A \rightarrowtail B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

1.7 Quotient Sets

Proposition 1.26. *Let \sim be an equivalence relation on X . Then there exists a set X/\sim , the quotient set of X with respect to \sim , and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the canonical projection, such that, for all $x, y \in \text{El}(X)$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to \sim , then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

Chapter 2

Topology

2.1 Topological Spaces

Definition 2.1 (Topological Space). Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say (X, \mathcal{O}) is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call \mathcal{O} the *topology* of the topological space, and call its elements *open* sets. We shall often write X for the topological space (X, \mathcal{O}) .

Definition 2.2 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 2.3. *A set B is open if and only if $X - B$ is closed.*

Proposition 2.4. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that \mathcal{C} is the set of closed sets if and only if:*

- For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$
- For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.
- $\emptyset \in \mathcal{C}$

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Definition 2.5 (Neighbourhood). Let X be a topological space, $x \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x , and x is an *interior* point of U , iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 2.6. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 2.7. *Let X be a set and $\mathcal{N} : X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that, for all $x \in X$, we have \mathcal{N}_x is the set of neighbourhoods of x , if and only if:*

- *For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$*
- *For all $x \in X$ we have $X \in \mathcal{N}_x$*
- *For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$*
- *For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$*
- *For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M. M \in \mathcal{N}_y$.*

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$.

Definition 2.8 (Exterior Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x .

Definition 2.9 (Boundary Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B .

Definition 2.10 (Interior). Let X be a topological space and $B \subseteq X$. The *interior* of B , B° , is the set of all interior points of B .

Proposition 2.11. *The interior of B is the union of all the open sets included in B .*

Definition 2.12 (Closure). Let X be a topological space and $B \subseteq X$. The *closure* of B , \overline{B} , is the set of all points that are not exterior points of B .

Proposition 2.13. *The closure of B is the intersection of all the closed sets that include B .*

Proposition 2.14. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 2.15 (Kuratowski Closure Axioms). *Let X be a set and $- : \mathcal{P}X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} such that, for all $B \subseteq X$, \overline{B} is the closure of B , if and only if:*

- $\overline{\emptyset} = \emptyset$
- *For all $A \subseteq X$ we have $A \subseteq \overline{A}$*
- *For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$*
- *For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

In this case, \mathcal{O} is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

2.1.1 Subspaces

Definition 2.16 (Subspace). Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

2.1.2 Topological Disjoint Union

Definition 2.17. Let X and Y be topological spaces. The *disjoint union* is $X + Y$ where $U \subseteq X + Y$ is open if and only if $\kappa_1^{-1}(U)$ is open in X and $\kappa_2^{-1}(U)$ is open in Y .

2.1.3 Product Topology

Definition 2.18. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the set of all subsets $W \subseteq X \times Y$ such that, for all $(x, y) \in W$, there exist neighbourhoods U of x in X and V of y in Y such that $U \times V \subseteq W$.

2.1.4 Bases

Definition 2.19 (Basis). Let X be a topological space. A *basis* for the topology on X is a set of open sets \mathcal{B} such that every open set is the union of a subset of \mathcal{B} .

2.1.5 Subbases

Definition 2.20 (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of \mathcal{S} .

2.2 Continuous Functions

Definition 2.21 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* iff, for every open set V in Y , the inverse image $f^{-1}(V)$ is open in X .

- Proposition 2.22.**
1. id_X is continuous
 2. The composite of two continuous functions is continuous.
 3. If $f : X \rightarrow Y$ is continuous and $X_0 \subseteq X$ then $f|_{X_0} : X_0 \rightarrow Y$ is continuous.
 4. If $f : X + Y \rightarrow Z$, then f is continuous iff $f \circ \kappa_1 : X \rightarrow Z$ and $f \circ \kappa_2 : Y \rightarrow Z$ are continuous.
 5. If $f : Z \rightarrow X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Definition 2.23 (Homeomorphism). Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

2.3 Convergence

Definition 2.24 (Convergence). Let X be a topological space. Let (x_n) be a sequence in X . A point $a \in \text{El}(X)$ is a *limit* of the sequence iff, for every neighbourhood U of a , there exists n_0 such that $\forall n \geq n_0. x_n \in U$.

2.4 Connected Spaces

Definition 2.25 (Connected). A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 2.26. *The continuous image of a connected space is connected.*

Proposition 2.27. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are connected, then X is connected.*

Proposition 2.28. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 2.29 (Path-connected). A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \rightarrow X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 2.30. *The continuous image of a path connected space is path connected.*

Proposition 2.31. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are path connected, then X is path connected.*

Proposition 2.32. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

2.5 Hausdorff Spaces

Definition 2.33 (Hausdorff). A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 2.34. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 2.35. 1. *Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

2.6 Compactness

Definition 2.36 (Compact). A topological space is *compact* iff every open cover has a finite subcover.

Proposition 2.37. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P . Then $X \in P$.*

PROOF:

- $\langle 1 \rangle 1$. P is an open cover of X
 - $\langle 1 \rangle 2$. PICK a finite subcover $U_1, \dots, U_n \in P$
 - $\langle 1 \rangle 3$. $X = U_1 \cup \dots \cup U_n \in P$
-

Corollary 2.37.1. *Let f be a compact space and $f : X \rightarrow \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. □

Proposition 2.38. *The continuous image of a compact space is compact.*

Proposition 2.39. *A closed subspace of a compact space is compact.*

Proposition 2.40. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. X and Y are compact.
2. $X + Y$ is compact.
3. $X \times Y$ is compact.

Proposition 2.41. *A compact subspace of a Hausdorff space is closed.*

Proposition 2.42. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

2.7 Quotient Spaces

Definition 2.43 (Quotient Space). Let X be a topological space and \sim an equivalence relation on X . The *quotient topology* on X/\sim is defined by: $U : \text{El}(\mathcal{P}X)$ is open in X/\sim if and only if $\pi^{-1}(U)$ is open in X .

Proposition 2.44. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $f : X/\sim \rightarrow Y$. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proposition 2.45. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $\phi : Y \rightarrow X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \rightarrow X$ such that $\pi \circ \Phi = \phi|_U$. Then ϕ is continuous.

Proposition 2.46. *A quotient of a connected space is connected.*

Proposition 2.47. *A quotient of a path connected space is path connected.*

Proposition 2.48. *Let X be a topological space and \sim an equivalence relation on X . If X/\sim is Hausdorff then every equivalence class of \sim is closed in X .*

2.8 Metric Spaces

Definition 2.49 (Metric Space). Let X be a set and $d : X^2 \rightarrow \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geq 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d) . We often write X for the metric space (X, d) .

Definition 2.50 (Topology of a Metric Space). Let (X, d) be a metric space. The topology *induced* by the metric d is defined by: for $V \subseteq X$, we have V is open if and only if, for all $x \in V$, there exists $\epsilon > 0$ such that $\{y \in X : d(x, y) < \epsilon\} \subseteq V$.

Definition 2.51 (Metrizable). A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 2.52. *Every metrizable space is Hausdorff.*

Chapter 3

Topological Groups

Definition 3.1 (Topological Group). A *topological group* is a group G with a topology such that the function $G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.

Example 3.2. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 3.3. *Any subgroup of a topological group is a topological group under the subspace topology.*

Definition 3.4 (Homogeneous Space). A *homogeneous space* is a topological space of the form G/H , where G is a topological group and H is a normal subgroup of G , under the quotient topology.

Proposition 3.5. *Let G be a topological group and H a normal subgroup of G . Then G/H is Hausdorff if and only if H is closed.*

PROOF: See Bourbaki, N., General Topology. III.12 \square

3.1 Continuous Actions

Definition 3.6 (Continuous Action). Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \rightarrow X$ such that:

- $\forall x : \text{El}(X) . ex = x$
- $\forall g, h : \text{El}(G) . \forall x : \text{El}(X) . g(hx) = (gh)x$

A *G -space* consists of a topological space X and a continuous action of G on X .

Definition 3.7 (Orbit). Let X be a G -space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Chapter 4

Topological Vector Spaces

Definition 4.1 (Topological Vector Space). Let K be either \mathbb{R} or \mathbb{C} . A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Subtraction is a continuous function $E^2 \rightarrow E$
- Multiplication is a continuous function $K \times E \rightarrow E$

Proposition 4.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. \square

Theorem 4.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 \square

Proposition 4.4. *Let E be a topological vector space and E_0 a subspace of E . Then $\overline{E_0}$ is a subspace of E .*

Definition 4.5. Let E be a topological vector space. The topological space associated with E is $E/\overline{\{0\}}$.

4.1 Cauchy Sequences

Definition 4.6 (Cauchy Sequence). Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geq n_0. x_n - x_m \in U$.

Definition 4.7 (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

4.2 Seminorms

Definition 4.8 (Seminorm). Let E be a vector space over K . A *seminorm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ such that:

1. $\forall x : \text{El}(E) . \|x\| \geq 0$
2. $\forall \alpha : \text{El}(K) . \forall x : \text{El}(E) . \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y : \text{El}(E) . \|x + y\| \leq \|x\| + \|y\|$

Example 4.9. The function that maps (x_1, \dots, x_n) to $|x_i|$ is a seminorm on \mathbb{R}^n .

Definition 4.10. Let E be a vector space over K . Let Λ be a set of seminorms on E . The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x : \text{El}(E)$.

Proposition 4.11. E is a topological vector space under this topology. It is Hausdorff iff, for all $x : \text{El}(E)$, if $\forall \lambda \in \Lambda . \lambda(x) = 0$ then $x = 0$.

4.3 Fréchet Spaces

Definition 4.12 (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 4.13. Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$. Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 4.14 (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

4.4 Normed Spaces

Definition 4.15 (Normed Space). Let E be a vector space over K . A *norm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ is a seminorm such that, $\forall x \in E . \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 4.16. If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E .

Proposition 4.17. Let $\| \cdot \|$ be a seminorm on the vector space E . Then $\| \cdot \|$ defines a norm on $E/\overline{\{0\}}$.

4.5 Inner Product Spaces

Proposition 4.18. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E .*

4.6 Banach Spaces

Definition 4.19 (Banach Space). A *Banach space* is a complete normed space.

Example 4.20. For any topological space X , the set $C(X)$ of bounded continuous functions $X \rightarrow \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

4.7 Hilbert Spaces

Definition 4.21 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Example 4.22. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \rightarrow \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \pi f(x)g(x)dx .$$

4.8 Locally Convex Spaces

Definition 4.23 (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 4.24. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Proposition 4.25. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Example 4.26. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \rightarrow \mathbb{R}$ is continuous as a map $E' \rightarrow \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonné, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.