Mathematics

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Contents

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*. We write $A : \mathrm{Set}$ for: A is a set.

For any set A, let there be *elements* of A. We write $a : \mathrm{El}\,(A)$ for: a is an element of A.

For any sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B.

For any function $f : A \to B$ and element $a : \mathrm{El}\,(A)$, let there be an element $f(a) : \mathrm{El}\,(B)$, the *value* of the function f at the *argument* a.

1.2 Axioms

Axiom Schema 1.1 (Choice)**.** *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x : \mathrm{El}\,(X)$ and $y : \mathrm{El}\,(Y)$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a : \mathrm{El}\,(A)$, there exists $b : \mathrm{El}\,(B)$ such that $P[A, B, a, b]$. Then there exists a function $f : A \to B$ such that $\forall a : \mathrm{El}\,(A)\,.P[A, B, a, f(a)]$.

Axiom 1.2 (Pairing)**.** *For any sets A and B, there exists a set $A \times B$, the* Cartesian product *of A and B, and functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that, for all $a : \mathrm{El}\,(A)$ and $b : \mathrm{El}\,(B)$, there exists a unique $(a, b) : \mathrm{El}\,(A \times B)$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Definition 1.3 (Injective)**.** A function $f : A \to B$ is *injective* or an *injection* iff, for all $x, y : \mathrm{El}\,(A)$, if $f(x) = f(y)$ then $x = y$.

Axiom Schema 1.4 (Separation)**.** *For every property $P[X, x]$ where X is a set variable and $x : \mathrm{El}\,(X)$, the following is an axiom:*

For every set A, there exists a set $S = \{x : \mathrm{El}(A) \mid P[A, x]\}$ and an injection $i : S \to A$ such that, for all $x : \mathrm{El}(A)$, we have

$$(\exists y : S. i(y) = x) \Leftrightarrow P[A, x] \ .$$

Axiom 1.5 (Infinity)**.** *There exists a set $\mathbb{N}$, an element $0 : \mathrm{El}(\mathbb{N})$, and a function $s : \mathbb{N} \to \mathbb{N}$ such that:*

- $\forall n : \mathrm{El}(\mathbb{N}) . s(n) \neq 0$
- $\forall m, n : \mathrm{El}(\mathbb{N}) . s(m) = s(n) \Rightarrow m = n.$

1.3 Consequences of the Axioms

1.3.1 Definitions

Definition 1.6. Let $f, g : A \to B$. We say f and g are *equal*, $f = g$, iff $\forall x : \mathrm{El}(A) . f(x) = g(x)$.

Definition 1.7 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for all $y : \mathrm{El}(B)$, there exists $x : \mathrm{El}(A)$ such that $f(x) = y$.

Definition 1.8 (Bijective)**.** A function $f : A \to B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$".

Definition 1.9 (Composition)**.** Given $f : A \to B$ and $g : B \to C$, let $g \circ f$ be the function such that $\forall a : \mathrm{El}(A) . (g \circ f)(a) = g(f(a))$.

1.3.2 The Empty Set

Theorem 1.10. *There exists a set which has no elements.*

PROOF:
⟨1⟩1. PICK a set A
  PROOF: By the Axiom of Infinity, a set exists.
⟨1⟩2. LET: $S = \{x : \mathrm{El}(A) \mid \bot\}$ with injection $i : S \to A$
  PROOF: Axiom of Separation.
⟨1⟩3. S has no elements.
□

Theorem 1.11. *If E and E' have no elements then $E \approx E'$.*

PROOF:

⟨1⟩1. LET: E and E' have no elements.
⟨1⟩2. PICK a function $F : E \to E'$.
PROOF: Axiom of Choice since vacuously $\forall x : \mathrm{El}(E) . \exists y : \mathrm{El}(E') . \top$.
⟨1⟩3. F is injective.
PROOF: Vacuously, for all $x, y : \mathrm{El}(E)$, if $F(x) = F(y)$ then $x = y$.
⟨1⟩4. F is surjective.
PROOF: Vacuously, for all $y : \mathrm{El}(E)$, there exists $x : \mathrm{El}(E)$ such that $F(x) = y$.
□

Definition 1.12 (Empty Set)**.** The *empty set* $\varnothing$ is the set with no elements.

1.3.3 The Singleton

Theorem 1.13. *There exists a set that has exactly one element.*

PROOF:
⟨1⟩1. PICK a set A that has an element.
PROOF: By the Axiom of Infinity, there exists a set that has an element.
⟨1⟩2. PICK $a : \mathrm{El}(A)$
⟨1⟩3. PICK a set S and injection $i : S \rightarrowtail A$ such that, for all $x : \mathrm{El}(A)$, there exists $s : \mathrm{El}(S)$ such that $s = x$ if and only if $x = a$
⟨1⟩4. S has exactly one element.
□

Theorem 1.14. *If A and B both have exactly one element then $A \approx B$.*

PROOF:
⟨1⟩1. LET: A and B both have exactly one element a and b respectively.
⟨1⟩2. LET: $F : A \to B$ be the function such that, for all $x : \mathrm{El}(A)$, we have $(x = a \wedge F(x) = b)$
⟨1⟩3. F is a bijection.
□

Definition 1.15 (Singleton)**.** Let 1 be the set that has exactly one element. Let $*$ be its element.

1.3.4 Subsets

Definition 1.16 (Subset)**.** A *subset* of a set A consists of a set S and an injection $i : S \rightarrowtail A$. We write $(S, i) : \mathrm{Sub}(A)$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Definition 1.17 (Membership)**.** Given $(S, i) : \mathrm{Sub}(A)$ and $a : \mathrm{El}(A)$, we write $a \in S$ for $\exists s : \mathrm{El}(S) . i(s) = a$.

1.4 Composition

Definition 1.18 (Composite)**.** Let $\phi : A \nrightarrow B$ and $\psi : B \nrightarrow C$. The *composite* $\psi \circ \phi : A \nrightarrow C$ is the relation such that $a(\psi \circ \phi)c$ iff there exists b such that $a\phi b$ and $b\psi c$.

Definition 1.19 (Identity)**.** For any set A, the *identity* function $\mathrm{id}_A : A \to A$ is the function defined by $\mathrm{id}_A(a) = a$.

Theorem 1.20. *Composition of relations is associative, and the identity function is an identity for composition. The composite of functions is a function. The composite of injective functions is injective. The composite of surjective functions is surjective. The composite of bijections is a bijection. A function $f : A \to B$ is a bijection iff there exists a function $f^{-1} : B \to A$ such that $f^{-1}f = \mathrm{id}_A$ and $ff^{-1} = \mathrm{id}_B$, in which case f^{-1} is unique.*

1.5 Axioms Part Two

Axiom 1.21 (Power Set)**.** *For any set A, there exists a set $\mathcal{P}A$, the* power set *of A, and a relation $\in: A \nrightarrow \mathcal{P}A$, called* membership*, such that, for any subset S of A, there exists a unique $\overline{S} \in \mathcal{P}A$ such that, for all $x \in A$, we have $x \in \overline{S}$ if and only if $x \in S$.*

We usually write just S for $\overline{S}$.

Axiom Schema 1.22 (Collection)**.** *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A, there exists a set B, a function $p : B \to A$, a set Y and a relation $M : B \nrightarrow Y$ such that:

- $\forall b \in B.P[A, \{y \in Y : bMy\}, p(b)]$
- *For all $a \in A$, if $\exists Y.P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 1.23 (Universe)**.** Let $E : U \nrightarrow X$ be a relation. Let us say that a set A is *small* iff there exists $u \in U$ such that $A \approx \{x \in X : uEx\}$.

Then (U, X, E) form a *universe* if and only if:

- $\mathbb{N}$ is U-small.
- For any U-small sets A and B and relation $R : A \nrightarrow B$, the tabulation of R is U-small.
- If A is U-small then so is $\mathcal{P}A$
- Let $f : A \to B$ be a function. If B is U-small and $f^{-1}(b)$ is U-small for all $b \in B$, then A is U-small.
- If $p : B \twoheadrightarrow A$ is a surjective function such that A is U-small, then there exists a U-small set C, a surjection $q : C \twoheadrightarrow A$, and a function $f : C \to B$ such that $q = pf$.

Axiom 1.24 (Universe)**.** *There exists a universe.*

Let $E : U \looparrowright X$ be a universe. We shall say a set is *small* iff it is U-small, and *large* otherwise.

1.6 Cartesian Product

Definition 1.25 (Cartesian Product)**.** Let A and B be sets. The *Cartesian product* of A and B, $A \times B$, is the tabulation of the relation $A \looparrowright B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

1.7 Quotient Sets

Proposition 1.26. *Let $\sim$ be an equivalence relation on X. Then there exists a set $X/\sim$, the* quotient set *of X with respect to $\sim$, and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the* canonical projection*, such that, for all $x, y : \mathrm{El}(X)$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to $\sim$, then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

1.8 Partitions

Definition 1.27 (Partition)**.** A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X.

Chapter 2

Category Theory

2.1 Categories

Definition 2.1. A *category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \mathrm{Ob}(\mathcal{C})$.
- for any objects X and Y, a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y. We write $f : X \to Y$ for $f \in \mathcal{C}[X, Y]$.
- for any objects X, Y and Z, a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \to \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X, there exists a morphism $\mathrm{id}_X : X \to X$, the *identity morphism* on X, such that:
  - for any object Y and morphism $f : Y \to X$ we have $\mathrm{id}_X \circ f = f$
  - for any object Y and morphism $f : X \to Y$ we have $f \circ \mathrm{id}_X = f$

We write the composite of morphism $f_1, \ldots, f_n$ as $f_n \circ \cdots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 2.2. Let **Set** be the category of small sets and functions.

Proposition 2.3. *The identity morphism on an object is unique.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{C}$ be a category.
$\langle 1 \rangle 2$. LET: $A \in \mathcal{C}$
$\langle 1 \rangle 3$. LET: $i, j : A \to A$ be identity morphisms on A.
$\langle 1 \rangle 4$. $i = j$

PROOF:

$$\begin{aligned} i &= i \circ j && (j \text{ is an identity on } A) \\ &= j && (i \text{ is an identity on } A) \end{aligned}$$

□

Definition 2.4. Given $f : A \to B$ and an object C, define the function $f^* : \mathcal{C}[B, C] \to \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 2.5. Given $f : A \to B$ and an object C, define the function $f_* : \mathcal{C}[C, A] \to \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

2.1.1 Sections and Retractions

Definition 2.6 (Section, Retraction)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $rs = \mathrm{id}_B$.

Proposition 2.7. *Let $f : A \to B$ and $r, s : B \to A$. If r is a retraction of f and s is a section of f then $r = s$.*

PROOF:

$$\begin{aligned} r &= r\mathrm{id}_B && \text{(Unit Law)} \\ &= rfs && (s \text{ is a section of } f) \\ &= \mathrm{id}_A s && (r \text{ is a retraction of } f) \\ &= s && \text{(Unit Law)} \end{aligned}$$ □

2.1.2 Isomorphisms

Definition 2.8 (Isomorphism)**.** A morphism $f : A \to B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \to A$ that is both a retraction and section of f.

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 2.9. *The inverse of an isomorphism is unique.*

PROOF: From Proposition 2.7. □

Proposition 2.10. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $ff^{-1} = \mathrm{id}_B$ and $f^{-1}f = \mathrm{id}_A$. □

Isomorphism.
Define the opposite category.
Initial object.
Terminal object.
Zero object.

Definition 2.11 (Zero Morphism)**.** In a category with a zero object O, for any objects A and B, the composite $A \to O \to B$ is called the *zero morphism* from A to B.

Proposition 2.12. *The empty set is initial in* **Set**.

PROOF: For any set A, the nowhere-defined function is the unique function $\varnothing \to A$. ∎

Proposition 2.13. *1 is terminal in* **Set**.

PROOF: For any set A, the constant function to $*$ is the only function $A \to 1$. ∎

Proposition 2.14. *There is no zero object in* **Set**.

PROOF: Since $\varnothing \not\approx 1$. ∎

Slice categories

Definition 2.15. Let $\mathcal{C}$ be a category and $B \in \mathcal{C}$. The category $\mathcal{C}_B^B$ of objects *over and under* B is the category with:

- objects all triples (X, u, p) such that $u : B \to X$ and $p : X \to B$
- morphisms $f : (X, u, p) \to (Y, u', p')$ all morphisms $f : X \to Y$ such that $fu = u'$ and $p'f = p$.

Proposition 2.16.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B)\backslash \mathrm{id}_B \cong (\mathcal{C}\backslash B)/\mathrm{id}_B$$

$(B, \mathrm{id}_B, \mathrm{id}_B)$ is the zero object in $\mathcal{C}_B^B$.

2.1.3 Initial Objects

Definition 2.17 (Initial Object)**.** An object I is *initial* iff, for any object X, there exists exactly one morphism $I \to X$.

2.1.4 Terminal Objects

Definition 2.18 (Terminal Object)**.** An object T is *terminal* iff, for any object X, there exists exactly one morphism $X \to T$.

2.1.5 Opposite Category

Definition 2.19 (Opposite Category)**.** For any category $\mathcal{C}$, the *opposite* category $\mathcal{C}^{\mathrm{op}}$ is the category with

- $\mathrm{Ob}\,(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}\,(\mathcal{C})$
- $\mathcal{C}^{\mathrm{op}}[A, B] = \mathcal{C}[B, A]$
- Given $f \in \mathcal{C}^{\mathrm{op}}[A, B]$ and $g \in \mathcal{C}^{\mathrm{op}}[B, C]$, their composite in $\mathcal{C}^{\mathrm{op}}$ is $f \circ g$, where $\circ$ is composition in $\mathcal{C}$.

Proposition 2.20. *An object is initial in $\mathcal{C}$ iff it is terminal in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. ∎

Proposition 2.21. *An object is terminal in $\mathcal{C}$ iff it is initial in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. ∎

2.1.6 Subcategories

Definition 2.22 (Subcategory)**.** A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ consists of:

- a subset $\mathrm{Ob}\,(\mathcal{C}')$ of $\mathcal{C}$
- for all $A, B \in \mathrm{Ob}\,(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \mathrm{Ob}\,(\mathcal{C}')$, we have $\mathrm{id}_A \in \mathcal{C}'[A, A]$
- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \mathrm{Ob}\,(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

2.1.7 Groupoids

Definition 2.23 (Groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

2.1.8 Concrete Categories

Definition 2.24 (Concrete Category)**.** A *concrete category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}\,(\mathcal{C})$ of *objects*
- for any object $A \in \mathrm{Ob}\,(\mathcal{C})$, a set $|A|$
- for any objects $A, B \in \mathrm{Ob}\,(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$
- for any object A we have $\mathrm{id}_{|A|} \in \mathcal{C}[A, A]$.

2.1.9 Power of Categories

Definition 2.25. Let $\mathcal{C}$ be a category and J a set. The category $\mathcal{C}^J$ is the category with:

- objects all J-indexed families of objects of $\mathcal{C}$
- morphisms $\{X_j\}_{j\in J} \to \{Y_j\}_{j\in J}$ all families $\{f_j\}_{j\in J}$ where $f_j : X_j \to Y_j$

2.1.10 Arrow Category

Definition 2.26 (Arrow Category)**.** Let $\mathcal{C}$ be a category. The *arrow category* $\mathcal{C}^{\to}$ is the category with:

- objects all triples (A, B, f) where $f : A \to B$ in $\mathcal{C}$
- morphisms $(A, B, f) \to (C, D, g)$ all pairs $(u : A \to C, v : B \to D)$ such that $vf = gu$.

2.2 Functors

Definition 2.27 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- a function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$
- for every morphism $f : A \to B$ in $\mathcal{C}$, a morphism $Ff : FA \to FB$ in $\mathcal{D}$

such that:

- for all $A : \mathrm{El}\,(\mathrm{Ob}(C))$ we have $F\mathrm{id}_A = \mathrm{id}_{FA}$
- for any morphism $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, we have $F(g \circ f) = Fg \circ Ff$

Define the identity functor, constant functors.
Functors preserve isomorphisms.
Isomorphism of categories.
Natural transformation.
Pullback
Pushout
Product
Coproduct
Adjunction

2.3 Bifunctors

Definition 2.28 (Commutative)**.** A bifunctor $\Box : \mathcal{C}^2 \to \mathcal{C}$ is *commutative* iff $\Box \cong \Box \circ T$, where $T : \mathcal{C}^2 \to \mathcal{C}^2$ is the swap functor.

Definition 2.29 (Associative)**.** A bifunctor $\Box$ is *associative* iff $\Box \circ (\Box \times \mathrm{id}) \cong \Box \circ (\mathrm{id} \times \Box)$.

Product and coproduct are commutative and associative.

Chapter 3

Monoid Theory

Definition 3.1 (Monoid)**.** A *monoid* is a category with one object.

Definition 3.2. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. The monoid $\mathrm{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \to X$ under composition.

Chapter 4

Group Theory

Definition 4.1. Let **Grp** be the category of small groups and group homomorphisms.

Definition 4.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G.

Proposition 4.3. *The trivial group is a zero object in* **Grp**.

PROOF: Easy. $\square$

The zero morphism $G \to H$ maps every element in G to e.

Definition 4.4. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. We write $\mathrm{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Chapter 5

Ring Theory

Definition 5.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Chapter 6

Linear Algebra

Definition 6.1. For any field K, let $\mathbf{Vect}_K$ be the concrete category of small vector spaces over K and linear transformations.

Dual space functor $\mathbf{Vect}_K^{\mathrm{op}} \to \mathbf{Vect}_K$.

Chapter 7

Topology

7.1 Topological Spaces

Definition 7.1 (Topological Space)**.** Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say $(X, \mathcal{O})$ is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call $\mathcal{O}$ the *topology* of the toplogical space, and call its elements *open* sets. We shall often write X for the topological space $(X, \mathcal{O})$.

Definition 7.2 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 7.3. *A set B is open if and only if $X - B$ is closed.*

Proposition 7.4. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that $\mathcal{C}$ is the set of closed sets if and only if:*

- *For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$*
- *For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*
- $\emptyset \in \mathcal{C}$

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Definition 7.5 (Neighbourhood)**.** Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x, and x is an *interior* point of U, iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 7.6. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 7.7. *Let X be a set and $\mathcal{N} : X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that, for all $x \in X$, we have $\mathcal{N}_x$ is the set of neighbourhoods of x, if and only if:*

- *For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$*
- *For all $x \in X$ we have $X \in \mathcal{N}_x$*
- *For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$*
- *For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$*
- *For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M. M \in \mathcal{N}_y$.*

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$.

Definition 7.8 (Exterior Point)**.** Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x.

Definition 7.9 (Boundary Point)**.** Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B.

Definition 7.10 (Interior)**.** Let X be a topological space and $B \subseteq X$. The *interior* of B, B°, is the set of all interior points of B.

Proposition 7.11. *The interior of B is the union of all the open sets included in B.*

Definition 7.12 (Closure)**.** Let X be a topological space and $B \subseteq X$. The *closure* of B, $\overline{B}$, is the set of all points that are not exterior points of B.

Proposition 7.13. *The closure of B is the intersection of all the closed sets that include B.*

Proposition 7.14. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 7.15 (Kuratowski Closure Axioms)**.** *Let X be a set and $\overline{} : \mathcal{P}X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ such that, for all $B \subseteq X$, $\overline{B}$ is the closure of B, if and only if:*

- $\overline{\varnothing} = \varnothing$
- *For all $A \subseteq X$ we have $A \subseteq \overline{A}$*
- *For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$*
- *For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

In this case, $\mathcal{O}$ is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

7.1.1 Subspaces

Definition 7.16 (Subspace)**.** Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

Example 7.17. The *unit sphere* S^2 is $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a subspace of $\mathbb{R}^3$.

7.1.2 Topological Disjoint Union

Definition 7.18. Let X and Y be topological spaces. The *disjoint union* is $X + Y$ where $U \subseteq X + Y$ is open if and only if ${\kappa_1}^{-1}(U)$ is open in X and ${\kappa_2}^{-1}(U)$ is open in Y.

7.1.3 Product Topology

Definition 7.19 (Product Topology)**.** Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

7.1.4 Bases

Definition 7.20 (Basis)**.** Let X be a topological space. A *basis* for the topology on X is a set of open sets $\mathcal{B}$ such that every open set is the union of a subset of $\mathcal{B}$.

7.1.5 Subbases

Definition 7.21 (Subbasis)**.** Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of $\mathcal{S}$.

Definition 7.22 (Space with Basepoint)**.** A *space with basepoint* is a pair (X, x) where X is a topological space and $x : \mathrm{El}\,(X)$.

7.1.6 Countability Axioms

Definition 7.23 (Neighbourhood Basis)**.** Let X be a topological space and $x_0 : \mathrm{El}\,(X)$. A *neighbourhood basis* of x_0 is a set $\mathcal{U}$ of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of $\mathcal{U}$.

Definition 7.24 (First Countable)**.** A topological space is *first countable* iff every point has a countable neighbourhood basis.

Definition 7.25 (Second Countable)**.** A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.
A subspace of a first countable space is first countable.
A subspace of a second countable space is second countable.

$\mathbb{R}^n$ is second countable.

An uncountable discrete space is first countable but not second countable.

Proposition 7.26. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.*

Proof:

⟨1⟩1. For all $\lambda : \mathrm{El}(\Lambda)$, Pick U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.

⟨1⟩2. For all $\lambda : \mathrm{El}(\lambda)$, Pick $x_\lambda \in U_\lambda$.

⟨1⟩3. Assume: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda \in \Lambda}$.

⟨1⟩4. Pick $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$

⟨1⟩5. There is no $U \in \lambda$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$

⟨1⟩6. Q.E.D.

Proof: This is a contradiction.

□

7.2 Continuous Functions

Definition 7.27 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* iff, for every open set V in Y, the inverse image $f^{-1}(V)$ is open in X.

Proposition 7.28.

1. *id_X is continuous*
2. *The composite of two continuous functions is continuous.*
3. *If $f : X \to Y$ is continuous and $X_0 \subseteq X$ then $f \restriction X_0 : X_0 \to Y$ is continuous.*
4. *If $f : X + Y \to Z$, then f is continuous iff $f \circ \kappa_1 : X \to Z$ and $f \circ \kappa_2 : Y \to Z$ are continuous.*
5. *If $f : Z \to X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.*

Definition 7.29 (Homeomorphism)**.** Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Definition 7.30 (Retraction)**.** Let X be a topological space and A a subspace of X. A continuous function $\rho : X \to A$ is a *retraction* iff $\rho \restriction A = \mathrm{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 7.31. Let **Top** be the category of small topological spaces and continuous functions.

Forgetful functor **Top** → **Set**.

Basepoint preserving continuous functor.

7.3 Convergence

Definition 7.32 (Convergence)**.** Let X be a topological space. Let (x_n) be a sequence in X. A point $a : \mathrm{El}(X)$ is a *limit* of the sequence iff, for every neighbourhood U of a, there exists n_0 such that $\forall n \geqslant n_0 . x_n \in U$.

Convergence in a product space is pointwise convergence.

If $f : X \to Y$ is continuous and $x_n \to l$ in X then $f(x_n) \to f(l)$ in Y.

Example 7.33. The converse does not hold.

Let X be the set of all continuous functions $[0,1] \to [-1,1]$ under the product topology. Let $i : X \to L^2([0,1])$ be the inclusion.

If $f_n \to f$ then $i(f_n) \to i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K_\epsilon . \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0,1]} U_\lambda$ where $U_\lambda = [-1,1]$ except for $\lambda = \lambda_1, \ldots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \ldots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 7.34. *The converse does hold for first countable spaces. If $f : X \to Y$ where X is first countable, and Y is a topological space, and whenever $x_n \to x$ then $f(x_n) \to f(x)$, then f is continuous.*

7.4 Connected Spaces

Definition 7.35 (Connected)**.** A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 7.36. *The continuous image of a connected space is connected.*

Proposition 7.37. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are connected, then X is connected.*

Proposition 7.38. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 7.39 (Path-connected)**.** A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0,1] \to X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 7.40. *The continuous image of a path connected space is path connected.*

Proposition 7.41. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are path connected, then X is path connected.*

Proposition 7.42. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

7.5 Hausdorff Spaces

Definition 7.43 (Hausdorff)**.** A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 7.44. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 7.45.

1. *Every subspace of a Hausdorff space is Hausdorff.*
2. *The disjoint union of two Hausdorff spaces is Hausdorff.*
3. *The product of two Hausdorff spaces is Hausdorff.*

Proposition 7.46. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \to B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X, then $f = g$.*

Proof:
⟨1⟩1. Assume: for a contradiction $a \in A$ and $f(a) \neq g(a)$.
⟨1⟩2. Pick disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.
⟨1⟩3. Pick $x \in f^{-1}(U) \cap g^{-1}(V)$
⟨1⟩4. $f(x) = g(x) \in U \cap V$
⟨1⟩5. Q.E.D.
Proof: This is a contradiction.
□

Proposition 7.47. *Let X and Y be metric spaces. Let $f : X \to Y$ be uniformly continuous. Let $\hat{X}$ and $\hat{Y}$ be the completions of X and Y. Then f extends uniquely to a continuous map $\hat{X} \to \hat{Y}$.*

Proof: The extension maps $\lim_{n\to\infty} x_n$ to $\lim_{n\to\infty} f(x_n)$. □

7.6 Separable Spaces

Definition 7.48 (Separable)**.** A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

7.7 Sequential Compactness

Definition 7.49 (Sequentially Compact)**.** A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

7.8 Compactness

Definition 7.50 (Compact)**.** A topological space is *compact* iff every open cover has a finite subcover.

Proposition 7.51. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P. Then $X \in P$.*

PROOF:
$\langle 1 \rangle 1$. P is an open cover of X
$\langle 1 \rangle 2$. PICK a finite subcover $U_1, \ldots, U_n \in P$
$\langle 1 \rangle 3$. $X = U_1 \cup \cdots \cup U_n \in P$
□

Corollary 7.51.1. *Let f be a compact space and $f : X \to \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. □

Proposition 7.52. *The continuous image of a compact space is compact.*

Proposition 7.53. *A closed subspace of a compact space is compact.*

Proposition 7.54. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. *X and Y are compact.*
2. *$X + Y$ is compact.*
3. *$X \times Y$ is compact.*

Proposition 7.55. *A compact subspace of a Hausdorff space is closed.*

Proposition 7.56. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 7.57. *A first countable compact space is sequentially compact.*

7.9 Quotient Spaces

Definition 7.58 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. The *quotient topology* on $X/\sim$ is defined by: $U : \mathrm{El}(\mathcal{P}X)$ is open in $X/\sim$ if and only if $\pi^{-1}(U)$ is open in X.

Proposition 7.59. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $f : X/\sim \to Y$. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proposition 7.60. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $\phi : Y \to X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \to X$ such that $\pi \circ \Phi = \phi \restriction U$. Then ϕ is continuous.

Proposition 7.61. *A quotient of a connected space is connected.*

Proposition 7.62. *A quotient of a path connected space is path connected.*

Proposition 7.63. *Let X be a topological space and $\sim$ an equivalence relation on X. If $X/\sim$ is Hausdorff then every equivalence class of $\sim$ is closed in X.*

Definition 7.64. Let X be a topological space and $A_1, \ldots, A_r \subseteq X$. Then $X/A_1, \ldots, A_r$ is the quotient space of X with respect to $\sim$ where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 7.65 (Cone)**.** Let X be a topological space. The *cone over X* is the space $(X \times [0,1])/(X \times \{1\})$.

Definition 7.66 (Suspension)**.** Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1,1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 7.67 (Wedge Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 7.68 (Smash Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 7.69. $D^n/S^{n-1} \cong S^n$

PROOF:
$\langle 1 \rangle$1. LET: $\phi : D^n/S^{n-1} \to S^n$ be the function induced by the map $D^n \to S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.
$\langle 1 \rangle$2. ϕ is a bijection.
$\langle 1 \rangle$3. ϕ is a homeomorphism.
PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.
□

7.10 Gluing

Definition 7.70 (Gluing)**.** Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \to Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where $\sim$ is the equivalence relation generated by $x \sim \phi(x)$ for all $x : \mathrm{El}(X)$.

Proposition 7.71. *Y is a subspace of $Y \cup_\phi X$.*

Definition 7.72. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0,1])/\alpha$ is the quotient space of $X \times [0,1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x : \mathrm{El}(X)$.

Definition 7.73 (Möbius Strip)**.** The *Möbius strip* is $([-1,1] \times [0,1])/\alpha$ where $\alpha(x) = -x$.

Definition 7.74 (Klein Bottle)**.** The *Klein bottle* is $(S^1 \times [0,1])/\alpha$ where $\alpha(z) = \overline{z}$.

Proposition 7.75. *Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\mathrm{id}_{\partial M}} M \cong K$.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : ([-1,1] \times [0,1]) + ([-1,1] \times [0,1]) \to S^1 \times [0,1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{\pi i \theta/2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-\pi i \theta/2}, t)$.
$\langle 1 \rangle 2$. f induces a bijection $M \cup_{\mathrm{id}_{\partial M}} M \approx K$
$\langle 1 \rangle 3$. f is a homeomorphism.
□

7.11 Metric Spaces

Definition 7.76 (Metric Space)**.** Let X be a set and $d : X^2 \to \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geqslant 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leqslant d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d). We often write X for the metric space (X, d).

Definition 7.77 (Topology of a Metric Space)**.** Let (X, d) be a metric space. The topology *induced* by the metric d is defined by: for $V \subseteq X$, we have V is open if and only if, for all $x \in V$, there exists $\epsilon > 0$ such that $\{y \in X : d(x, y) < \epsilon\} \subseteq V$.

Definition 7.78 (Metrizable)**.** A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 7.79. *Every metrizable space is Hausdorff.*

Every metrizable space is first countable.
A metric space is compact if and only if it is sequentially compact.
A metric space is separable if and only if it is second countable.

7.12 Complete Metric Spaces

Definition 7.80 (Complete)**.** A metric space is *complete* iff every Cauchy sequence converges.

Example 7.81. $\mathbb{R}$ is complete.

Proposition 7.82. *The product of two complete metric spaces is complete.*

Proposition 7.83. *Every compact metric space is complete.*

Proposition 7.84. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 7.85 (Completion)**.** Let X be a metric space. A *completion* of X is a complete metric space $\hat{X}$ and injection $i : X \rightarrowtail \hat{X}$ such that:

- The metric on X is the restriction of the metric on $\hat{X}$
- X is dense in $\hat{X}$.

Proposition 7.86. *Let $i_1 : X \to Y_1$ and $i_2 : X \to Y_2$ be completions of X. Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

Proof: Define $\phi(\lim_{n\to\infty} i_1(x_n)) = \lim_{n\to\infty} i_2(x_n)$. □

Theorem 7.87. *Every metric space has a completion.*

Proof: Let $\hat{X}$ be the set of Cauchy sequences in X quotiented by $\sim$ where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \to 0$. □

7.13 Manifolds

Definition 7.88 (Manifold)**.** An *n-dimensional manifold* is a second countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$.

Chapter 8

Homotopy Theory

8.1 Homotopies

Definition 8.1 (Homotopy)**.** Let X and Y be topological spaces. Let $f, g : X \to Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0,1] \to Y$ such that

- $\forall x : \mathrm{El}\,(X)\,.h(x,0) = f(x)$
- $\forall x : \mathrm{El}\,(X)\,.h(x,1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \to Y$.

Proposition 8.2. *Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.*

Definition 8.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

Definition 8.4. A functor $F : \mathbf{Top} \to \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \to Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

8.2 Homotopy Equivalence

Definition 8.5 (Homotopy Equivalence)**.** Let X and Y be topological spaces. A *homotopy equivalence* between X and Y, $f : X \simeq Y$, is a continuous function $f : X \to Y$ such that there exists a continuous function $g : Y \to X$, the *homotopy inverse* to f, such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Definition 8.6 (Contractible)**.** A topological space X is *contractible* iff $X \simeq 1$.

Example 8.7. $\mathbb{R}^n$ is contractible.

Example 8.8. D^n is contractible.

Definition 8.9 (Deformation Retract)**.** Let X be a topological space and A a subspace of X. A retraction $\rho : X \to A$ is a *deformation retraction* iff $i \circ \rho \simeq \mathrm{id}_X$, where i is the inclusion $A \rightarrowtail X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 8.10 (Strong Deformation Retract)**.** Let X be a topological space and A a subspace of X. A *strong deformation retraction* $\rho : X \to A$ is a continuous function such that there exists a homotopy $h : X \times [0,1] \to X$ between $i \circ \rho$ and id_X such that, for all $a : \mathrm{El}\,(X)$ and $t : \mathrm{El}\,([0,1])$, we have $h(a,t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 8.11. $\{0\}$ is a strong deformation retract of $\mathbb{R}^n$ and of D^n.

Example 8.12. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 8.13. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 8.14. For any topological space X, the singleton consisting of the vertex is a strong deformation retract of the cone over X.

Chapter 9

Simplicial Complexes

Definition 9.1 (Simplex)**.** A *k-dimensional simplex* or *k-simplex* in $\mathbb{R}^n$ is the convex hull $s(x_0, \ldots, x_k)$ of $k+1$ points in general position.

Definition 9.2 (Face)**.** A *sub-simplex* or *face* of $s(x_0, \ldots, x_k)$ is the convex hull of a subset of $\{x_0, \ldots, x_k\}$.

Definition 9.3 (Simplicial Complex)**.** A *simplicial complex* in $\mathbb{R}^n$ is a set K of simplices such that:

- for every simplex s in K, every face of s is in K.
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2.
- K is locally finite, i.e. every point of $\mathbb{R}^n$ has a neighbourhood that only intersects finitely many elements of K.

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of $\mathbb{R}^n$.

9.1 Cell Decompositions

Definition 9.4 (n-cell)**.** An *n-cell* is a topological space homeomorphic to $\mathbb{R}^n$.

Definition 9.5 (Cell Decomposition)**.** Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n-cells.

Definition 9.6 (n-skeleton)**.** Given a cell decomposition of X, the *n-skeleton* X^n is the union of all the cells of dimension $\leqslant n$.

9.2 CW-complexes

Definition 9.7 (CW-Complex)**.** A *CW-complex* consists of a topological space X and a cell decomposition $\mathcal{E}$ of X such that:

1. *Characteristic Maps* For every n-cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \to X$ such that $\Phi((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in $\mathcal{E}$ of dimension $< n$.

2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have $\overline{e}$ intersects only finitely many other cells in $\mathcal{E}$.

3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \overline{e}$ is closed.

Proposition 9.8. *If a cell decomposition $\mathcal{E}$ satisfies the Characteristic Maps axiom, then for every n-cell $e \in \mathcal{E}$ we have $\overline{e} = \Phi_e(D^n)$. Therefore $\overline{e}$ is compact and $\overline{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

Proof:
⟨1⟩1. $e \subseteq \Phi_e(D^n) \subseteq \overline{e}$
Proof:

$$\begin{aligned} e &= \Phi_e((D^n)^\circ) \\ &\subseteq \Phi_e(D^n) \\ &= \Phi_e(\overline{(D^n)^\circ}) \\ &\subseteq \overline{\Phi_e((D^n)^\circ)} \\ &= \overline{e} \end{aligned}$$

⟨1⟩2. $\Phi_e(D^n)$ is compact.
Proof: Because D^n is compact.
⟨1⟩3. $\Phi_e(D^n)$ is closed.
⟨1⟩4. $\Phi_e(D^n) = \overline{e}$
□

Chapter 10

Topological Groups

Definition 10.1 (Topological Group)**.** A *topological group* is a group G with a topology such that the function $G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.

Example 10.2. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 10.3. *Any subgroup of a topological group is a topological group under the subspace topology.*

Definition 10.4 (Homogeneous Space)**.** A *homogeneous space* is a topological space of the form G/H, where G is a topological group and H is a normal subgroup of G, under the quotient topology.

Proposition 10.5. *Let G be a topological group and H a normal subgroup of G. Then G/H is Hausdorff if and only if H is closed.*

PROOF: See Bourbaki, N., General Topology. III.12 $\square$

10.1 Continuous Actions

Definition 10.6 (Continuous Action)**.** Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \to X$ such that:

- $\forall x : \mathrm{El}\,(X)\,.ex = x$
- $\forall g, h : \mathrm{El}\,(G)\,.\forall x : \mathrm{El}\,(X)\,.g(hx) = (gh)x$

A *G-space* consists of a topological space X and a continuous action of G on X.

Definition 10.7 (Orbit)**.** Let X be a G-space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 10.8. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:
⟨1⟩1. LET: $f_3 : S^2/SO(2) \to [-1, 1]$ be the function induced by $\pi_3 : S^2 \to [-1, 1]$
⟨1⟩2. f_3 is bijective.
⟨1⟩3. $S^2/SO(2)$ is compact.
PROOF: It is the continuous image of S^2 which is compact.
⟨1⟩4. $[-1, 1]$ is Hausdorff.
⟨1⟩5. f_3 is a homeomorphism.
□

Definition 10.9 (Stabilizer)**.** Let X be a G-space and $x \in X$. The *stabilizer* of x is $G_x := \{g : \mathrm{El}\,(G) \mid gx = x\}$.

Proposition 10.10. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx.*

PROOF:
⟨1⟩1. If $gG_x = hG_x$ then $gx = hx$.
⟨2⟩1. ASSUME: $gG_x = hG_x$
⟨2⟩2. $g^{-1}h \in G_x$
⟨2⟩3. $g^{-1}hx = x$
⟨2⟩4. $gx = hx$
⟨1⟩2. If $gx = hx$ then $gG_x = hG_x$.
PROOF: Similar.
⟨1⟩3. The function is continuous.
PROOF: Proposition 7.59.
□

Chapter 11

Topological Vector Spaces

Definition 11.1 (Topological Vector Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Substraction is a continuous function $E^2 \to E$
- Multiplication is a continuous function $K \times E \to E$

Proposition 11.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. □

Theorem 11.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 □

Proposition 11.4. *Let E be a topological vector space and E_0 a subspace of E. Then $\overline{E_0}$ is a subspace of E.*

Definition 11.5. Let E be a topological vector space. The topological space *associated* with E is $E/\overline{\{0\}}$.

11.1 Cauchy Sequences

Definition 11.6 (Cauchy Sequence)**.** Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geqslant n_0 . x_n - x_m \in U$.

Definition 11.7 (Complete Topological Vector Space)**.** A topological vector space is *complete* iff every Cauchy sequence converges.

11.2 Seminorms

Definition 11.8 (Seminorm). Let E be a vector space over K. A *seminorm* on E is a function $\| \, \| : E \to \mathbb{R}$ such that:

1. $\forall x : \mathrm{El}\,(E)\,.\|x\| \geqslant 0$
2. $\forall \alpha : \mathrm{El}\,(K)\,.\forall x : \mathrm{El}\,(E)\,.\|\alpha x\| = |\alpha|\|x\|$
3. *Triangle Inequality* $\forall x, y : \mathrm{El}\,(E)\,.\|x + y\| \leqslant \|x\| + \|y\|$

Example 11.9. The function that maps $(x_1, \dots, x_n)$ to $|x_i|$ is a seminorm on $\mathbb{R}^n$.

Definition 11.10. Let E be a vector space over K. Let Λ be a set of seminorms on E. The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x : \mathrm{El}\,(E)$.

Proposition 11.11. *E is a topological vector space under this topology. It is Hausdorff iff, for all $x : \mathrm{El}\,(E)$, if $\forall \lambda \in \Lambda . \lambda(x) = 0$ then $x = 0$.*

11.3 Fréchet Spaces

Definition 11.12 (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 11.13. *Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \, \|_n : n \in \mathbb{Z}^+\}$. Then*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 11.14 (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

11.4 Normed Spaces

Definition 11.15 (Normed Space). Let E be a vector space over K. A *norm* on E is a function $\| \, \| : E \to \mathbb{R}$ is a seminorm such that, $\forall x \in E . \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 11.16. *If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E.*

Proposition 11.17. *Let $\| \ \|$ be a seminorm on the vector space E. Then $\| \ \|$ defines a norm on $E/\overline{\{0\}}$.*

Proposition 11.18. *Let E and F be normed spaces. Any continuous linear map $E \to F$ is uniformly continuous.*

Definition 11.19. For $p \geqslant 1$. let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt{p}\int_{\mathbb{R}^n} |f(x)|^p dx$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\overline{\{0\}} \ .$$

11.5 Inner Product Spaces

Proposition 11.20. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on E.*

11.6 Banach Spaces

Definition 11.21 (Banach Space)**.** A *Banach space* is a complete normed space.

Example 11.22. For any topological space X, the set $C(X)$ of bounded continuous functions $X \to \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x\in X} |f(x)|$.

Proposition 11.23. *The completion of a normed space is a Banach space.*

Proposition 11.24. *Let E and F be normed spaces. Let $f : E \to F$ be a continuous linear map. Then the extension to the completions $\hat{E} \to \hat{F}$ is linear.*

Proposition 11.25. *$L^p(\mathbb{R}^n)$ is a Banach space.*

Proposition 11.26. *$C(\mathbb{R})$ is first countable but not second countable.*

Proof: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n. Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.
It is first countable because it is metrizable. □

11.7 Hilbert Spaces

Definition 11.27 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Example 11.28. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \to \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi} \pi f(x) g(x) dx \ .$$

Proposition 11.29. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

11.8 Locally Convex Spaces

Definition 11.30 (Locally Convex Space)**.** A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 11.31. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Proposition 11.32. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Example 11.33. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \to \mathbb{R}$ is continuous as a map $E' \to \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 11.34 (Thom Space)**.** Let E be a vector bundle with a Riemannian metric, $DE = \{x : \mathrm{El}\,(E) \mid \|x\| \leqslant 1\}$ its disc bundle and $SE := \{v : \mathrm{El}\,(E) \mid \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE.