Mathematics

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Part I

Set Theory

Chapter 1

Axioms

1.1 Classes

We speak informally about *classes*. A *class* is determined by a unary predicate. We write $\{x \mid P(x)\}$ for the class determined by P.

We say an object a is a *member* or *element* of the class $\mathbf{A} = \{x \mid P(x)\}$, or $\mathbf{A}$ *contains* a, and write $a \in \mathbf{A}$ or $\mathbf{A} \ni a$, iff $P(a)$ is true.

We say two classes are *equal* iff they have exactly the same elements.

We write $\{x \in \mathbf{A} \mid P(x)\}$ for $\{x \mid x \in \mathbf{A} \wedge P(x)\}$. We write $\{t[x_1, \ldots, x_n] \mid P(x_1, \ldots, x_n)\}$ for $\{y \mid \exists x_1 \cdots \exists x_n (P(x_1, \ldots, x_n) \wedge y = t[x_1, \ldots, x_n])\}$.

Definition 1.1 (Disjoint)**.** Two classes are *disjoint* iff they have no common element.

Definition 1.2 (Subclass)**.** Given classes $\mathbf{A}$ and $\mathbf{B}$, we say $\mathbf{A}$ is a *subclass* of $\mathbf{B}$, $\mathbf{B}$ is a *superclass* of $\mathbf{A}$, or $\mathbf{B}$ *includes* $\mathbf{A}$, and write $\mathbf{A} \subseteq \mathbf{B}$ or $\mathbf{B} \supseteq \mathbf{A}$, iff every element of $\mathbf{A}$ is an element of $\mathbf{B}$.

If, in addition, $\mathbf{A} \neq \mathbf{B}$, then we say $\mathbf{A}$ is a *proper* subclass of $\mathbf{B}$, $\mathbf{B}$ is a *proper* superclass of $\mathbf{A}$, or $\mathbf{B}$ *properly* includes $\mathbf{A}$, and write $\mathbf{A} \subsetneq \mathbf{B}$ or $\mathbf{B} \supsetneq \mathbf{A}$.

Proposition 1.3. *Every class is a subclass of itself.*

PROOF: For any class $\mathbf{A}$, we have that every element of $\mathbf{A}$ is an element of $\mathbf{A}$. □

Definition 1.4 (Empty Class)**.** The *empty class* $\emptyset$ is $\{x \mid \bot\}$. All other classes are *nonempty.*

Proposition 1.5. *The empty class is a subclass of every class.*

PROOF: For any class $\mathbf{A}$, vacuously every element of $\emptyset$ is an element of $\mathbf{A}$. □

Definition 1.6 (Universal Class)**.** The *universal class* $\mathbf{V}$ is $\{x \mid \top\}$.

Definition 1.7. Given objects $a_1, \ldots, a_n$, we write $\{a_1, \ldots, a_n\}$ for the class $\{x \mid x = a_1 \vee \cdots \vee x = a_n\}$.

A class of the form $\{a\}$ is called a *singleton*.

Definition 1.8 (Union)**.** The *union* of classes $\mathbf{A}$ and $\mathbf{B}$ is the class $\mathbf{A} \cup \mathbf{B} = \{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$.

Definition 1.9 (Intersection)**.** The *intersection* of classes $\mathbf{A}$ and $\mathbf{B}$ is the class $\mathbf{A} \cap \mathbf{B} = \{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$.

Definition 1.10 (Relative Complement)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be classes. The *relative complement* of $\mathbf{B}$ in $\mathbf{A}$ is the class $\mathbf{A} - \mathbf{B} = \{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

1.2 Primitive Notions

Let there be *sets*.

Let there be a binary relation $\in$ between sets, called *membership*. When $a \in b$ holds, we say a is a *member* or *element* of b, or a is *in* b, or b *contains* a, and we also write $b \ni a$. When this does not hold, we write $a \notin b$ or $b \not\ni a$.

Definition 1.11 (Pairwise Disjoint)**.** Let A be a set. We say the elements of A are *pairwise disjoint* iff, for all $x, y \in A$, if there exists z such that $z \in x$ and $z \in y$, then $x = y$.

1.3 Axioms

Axiom 1 (Extensionality)**.** *Two sets with exactly the same elements are equal.*

Thanks to this axiom, we may identify a set A with the class $\{x \mid x \in A\}$. Our usage of the symbols $\in$ and $=$ is consistent.

Definition 1.12. We say that a class $\mathbf{A}$ *is a set* iff there exists a set A such that $A = \mathbf{A}$. That is, $\{x \mid P(x)\}$ is a class iff there exists a set A such that $\forall x(x \in A \Leftrightarrow P(x))$. Otherwise, $\mathbf{A}$ is a *proper class*.

Definition 1.13 (Subset)**.** A (proper) *subset* of a class is a (proper) subclass that is a set.

A (proper) *superset* of a class is a (proper) superclass that is a set.

Definition 1.14 (Union)**.** For any class $\mathbf{A}$, the *union* of $\mathbf{A}$ is the class $\{x \mid \exists A \in \mathbf{A}. x \in A\}$.

Axiom 2 (Regularity)**.** *For any nonempty set A, there exists a set $m \in A$ such that m and A are disjoint.*

Axiom 3 (Union)**.** *The union of a set is a set.*

Axiom 4 (Replacement)**.** *For any property $P(x, y)$, the following is an axiom:*

Let A be a set. Assume that, for any $x \in A$, there exists at most one y such that $P(x, y)$. Then $\{y \mid \exists x \in A. P(x, y)\}$ is a set.

Axiom 5 (Infinity)**.** *There exists a set I such that:*

- *I has an element that is empty.*
- *For all $x \in I$, there exists $y \in I$ such that the elements of y are exactly x and the members of x.*

Axiom 6 (Power Set)**.** *For any set A, the class of all subsets of A is a set.*

Axiom 7 (Choice)**.** *Let A be a set whose elements are nonempty and pairwise disjoint. Then there exists a set B that has exactly one member in common with each member of A.*

1.4 Basic Constructions on Sets

Proposition 1.15. *The empty class $\emptyset$ is a set.*

Immediate from the Axiom of Infinity. □

Definition 1.16 (Power Set)**.** For any set A, the *power set* of A, denoted $\mathcal{P}A$, is the set of all subsets of A.

(This is a set by the Power Set Axiom.)

Theorem 1.17. *For any sets a and b, the class $\{a, b\}$ is a set.*

PROOF:
$\langle 1 \rangle 1$. LET: a and b be sets.
$\langle 1 \rangle 2$. LET: $P(x, y)$ be the property $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$.
$\langle 1 \rangle 3$. For any $x \in \mathcal{P}\mathcal{P}\emptyset$, there exists at most one y such that $P(x, y)$.
  $\langle 2 \rangle 1$. LET: $x \in \mathcal{P}\mathcal{P}\emptyset$
  $\langle 2 \rangle 2$. ASSUME: $P(x, y)$ and $P(x, z)$
    PROVE: $y = z$
  $\langle 2 \rangle 3$. CASE: $x = \emptyset$, $y = a$, $x = \emptyset$ and $z = a$
   PROOF: Then $y = z$.
  $\langle 2 \rangle 4$. CASE: $x = \emptyset$, $y = a$, $x = \mathcal{P}\emptyset$ and $z = b$
   PROOF: This case is impossible since we have $\emptyset \in \mathcal{P}\emptyset$ but $\emptyset \notin \emptyset$.
  $\langle 2 \rangle 5$. CASE: $x = \mathcal{P}\emptyset$, $y = b$, $x = \emptyset$ and $z = a$
   PROOF: This case is impossible since we have $\emptyset \in \mathcal{P}\emptyset$ but $\emptyset \notin \emptyset$.
  $\langle 2 \rangle 6$. CASE: $x = \mathcal{P}\emptyset$, $y = b$, $x = \mathcal{P}\emptyset$ and $z = b$
   PROOF: Then $y = z$.
$\langle 1 \rangle 4$. LET: $A = \{y \mid \exists x \in \mathcal{P}\mathcal{P}\emptyset . P(x, y)\}$
  PROOF: By $\langle 1 \rangle 3$ and the Axiom of Replacement.
$\langle 1 \rangle 5$. $A = \{a, b\}$
  $\langle 2 \rangle 1$. $a \in A$
   PROOF: Since $\emptyset \in \mathcal{P}\mathcal{P}\emptyset$ and $P(\emptyset, a)$.
  $\langle 2 \rangle 2$. $b \in A$
   PROOF: Since $\mathcal{P}\emptyset \in \mathcal{P}\mathcal{P}\emptyset$ and $P(\mathcal{P}\emptyset, b)$.
  $\langle 2 \rangle 3$. For all $y \in A$ we have $y = a$ or $y = b$.

PROOF: Immediate from $\langle 1\rangle 4$.
∎

Corollary 1.17.1. *For any set a, the class $\{a\}$ is a set.*

Proposition 1.18. *The union of two sets is a set.*

PROOF: Since for sets A and B we have $A \cup B = \bigcup\{A, B\}$. ∎

Proposition 1.19. *For any sets $a_1, \ldots, a_n$, the class $\{a_1, \ldots, a_n\}$ is a set.*

PROOF: It is $\{a_1\} \cup \cdots \cup \{a_n\}$. ∎

Theorem 1.20 (Comprehension)**.** *Every subclass of a set is a set.*

PROOF:
$\langle 1\rangle 1$. LET: A be a set, $\mathbf{B}$ a class with $\mathbf{B} \subseteq A$.
$\langle 1\rangle 2$. LET: $P(x, y)$ be the property $x \in \mathbf{B} \wedge y = x$
$\langle 1\rangle 3$. For any $x \in A$ there exists at most one y such that $P(x, y)$.
$\langle 1\rangle 4$. $\mathbf{B} = \{y \mid \exists x \in A.P(x, y)\}$.
$\langle 1\rangle 5$. Q.E.D.
PROOF: Hence $\mathbf{B}$ is a set by the Axiom of Replacement.
∎

Corollary 1.20.1. *For any set A and class $\mathbf{B}$, the intersection $A \cap \mathbf{B}$ is a set.*

Corollary 1.20.2. *For any set A and class $\mathbf{B}$, the relative complement $A - \mathbf{B}$ is a set.*

Theorem 1.21 (Russell's Paradox)**.** *The universal class $\mathbf{V}$ is a proper class.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathbf{R} = \{x \mid x \notin x\}$
$\langle 1\rangle 2$. $\mathbf{R}$ is not a set.
PROOF: If it were, we would have $\mathbf{R} \in \mathbf{R}$ if and only if $\mathbf{R} \notin \mathbf{R}$.
$\langle 1\rangle 3$. $\mathbf{V}$ is not a set.
PROOF: By Comprehension.
∎

Definition 1.22 (Intersection)**.** The *intersection* of a class $\mathbf{A}$ is the class

$$\bigcap \mathbf{A} = \{x \mid \forall A \in \mathbf{A}.x \in A\} \enspace .$$

Proposition 1.23. *The intersection of a nonempty class is a set.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathbf{A}$ be a nonempty class.
$\langle 1\rangle 2$. PICK $A \in \mathbf{A}$
$\langle 1\rangle 3$. $\bigcap \mathbf{A} \subseteq A$
$\langle 1\rangle 4$. $\bigcap \mathbf{A}$ is a set.
PROOF: By Comprehension.
∎

Chapter 2

Ordered Pairs and Relations

Definition 2.1 (Ordered Pair)**.** For any sets a and b, the *ordered pair* (a, b) is defined to be $\{\{a\}, \{a, b\}\}$.

Proposition 2.2. *For any sets a, b, c and d, if $(a, b) = (c, d)$ then $a = c$ and $b = d$.*

PROOF:
$\langle 1 \rangle$1. LET: a, b, c, d be sets.
$\langle 1 \rangle$2. ASSUME: $(a, b) = (c, d)$
$\langle 1 \rangle$3. $a = c$
  PROOF: Since $\{a\} = \bigcap(a, b) = \bigcap(c, d) = \{c\}$.
$\langle 1 \rangle$4. $\{a, b\} = \{c, d\}$
  PROOF: Since $\{a, b\} = \bigcup(a, b) = \bigcup(c, d) = \{c, d\}$.
$\langle 1 \rangle$5. $b = d$
  $\langle 2 \rangle$1. CASE: $a = b$
    PROOF: Then the set $\{a, b\} = \{c, d\}$ is a singleton, and so $a = b = c = d$.
  $\langle 2 \rangle$2. CASE: $a \neq b$
    PROOF: Then we have $\{b\} = \{a, b\} - \{a\} = \{c, d\} - \{c\}$ and so $b = d$.
$\square$

Definition 2.3 (Cartesian Product)**.**