Mathematics

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Contents

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*.

For any set A, let there be *elements* of A. We write $a \in A$ for: a is an element of A.

For any sets A and B, let there be a set B^A, whose elements are called *functions* from A to B. We write $f : A \to B$ for $f \in B^A$.

For any function $f : A \to B$ and element $a \in A$, let there be an element $f(a) \in B$, the *value* of the function f at the *argument* a.

1.2 Axioms

Axiom Schema 1.2.1 (Choice)**.** *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x : \mathrm{El}(X)$ and $y : \mathrm{El}(Y)$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a : \mathrm{El}(A)$, there exists $b : \mathrm{El}(B)$ such that $P[A, B, a, b]$. Then there exists a function $f : A \to B$ such that $\forall a : \mathrm{El}(A) . P[A, B, a, f(a)]$.

Axiom 1.2.2 (Extensionality)**.** *Let $f, g : A \to B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Axiom 1.2.3 (Pairing)**.** *For any sets A and B, there exists a set $A \times B$, the* Cartesian product *of A and B, and functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that, for all $a : \mathrm{El}(A)$ and $b : \mathrm{El}(B)$, there exists a unique $(a, b) : \mathrm{El}(A \times B)$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Definition 1.2.4 (Injective)**.** A function $f : A \to B$ is *injective* or an *injection* iff, for all $x, y : \mathrm{El}(A)$, if $f(x) = f(y)$ then $x = y$.

Axiom Schema 1.2.5 (Separation)**.** *For every property $P[X, x]$ where X is a set variable and $x : \mathrm{El}\,(X)$, the following is an axiom:*

For every set A, there exists a set $S = \{x : \mathrm{El}\,(A) \mid P[A, x]\}$ and an injection $i : S \to A$ such that, for all $x : \mathrm{El}\,(A)$, we have

$$(\exists y : S.i(y) = x) \Leftrightarrow P[A, x] \ .$$

Axiom 1.2.6 (Infinity)**.** *There exists a set $\mathbb{N}$, an element $0 : \mathrm{El}\,(\mathbb{N})$, and a function $s : \mathbb{N} \to \mathbb{N}$ such that:*

- $\forall n : \mathrm{El}\,(\mathbb{N})\,.s(n) \neq 0$
- $\forall m, n : \mathrm{El}\,(\mathbb{N})\,.s(m) = s(n) \Rightarrow m = n.$

Axiom Schema 1.2.7 (Collection)**.** *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A, there exists a set B, a function $p : B \to A$, a set Y and a relation $M : B \looparrowright Y$ such that:

- $\forall b \in B.P[A, \{y \in Y : bMy\}, p(b)]$
- *For all $a \in A$, if $\exists Y.P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 1.2.8 (Composite)**.** Let $f : A \to B$ and $g : B \to C$. The *composite* $g \circ f : A \to C$ is the function such that, for all $a \in A$, we have

$$(g \circ f)(a) = g(f(a)) \ .$$

Axiom 1.2.9 (Universe)**.** *There exists a set E, a set U and a function $el : E \to U$ such that the following holds.*

Let us say that a set A is small *iff there exists $u \in U$ such that $A \approx \{e \in E : el(e) = u\}$.*

- *$\mathbb{N}$ is small.*
- *For any U-small sets A and B, the set B^A is small.*
- *For any U-small sets A and B, the set $A \times B$ is small.*
- *Let $f : A \to B$ be a function. If B is small and $f^{-1}(b)$ is U-small for all $b \in B$, then A is small.*
- *If $p : B \twoheadrightarrow A$ is a surjective function such that A is small, then there exists a U-small set C, a surjection $q : C \twoheadrightarrow A$, and a function $f : C \to B$ such that $q = p \circ f$.*

Chapter 2

Sets and Functions

2.1 Surjective Functions

Definition 2.1.1 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for all $y : \mathrm{El}\,(B)$, there exists $x : \mathrm{El}\,(A)$ such that $f(x) = y$.

2.2 Bijections

Definition 2.2.1 (Bijective)**.** A function $f : A \to B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$".

2.3 Identity Function

Definition 2.3.1 (Identity)**.** For any set A, the *identity* function $\mathrm{id}_A : A \to A$ is the function defined by $\mathrm{id}_A(a) = a$.

2.4 Composition

Proposition 2.4.1. *Given functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \quad .$$

Proof: Each is the function that maps $a \in A$ to $h(g(f(a)))$. $\square$

Proposition 2.4.2. *The composite of injective functions is injective.*

PROOF:
⟨1⟩1. LET: $f : A \rightarrowtail B$ and $g : B \rightarrowtail C$ be injective.
⟨1⟩2. LET: $x, y \in A$ satisfy $(g \circ f)(x) = (g \circ f)(y)$
⟨1⟩3. $g(f(x)) = g(f(y))$
⟨1⟩4. $f(x) = f(y)$
⟨1⟩5. $x = y$
□

Proposition 2.4.3. *For functions $f : A \to B$ and $g : B \to C$, if $g \circ f$ is injective then f is injective.*

PROOF:
⟨1⟩1. ASSUME: $g \circ f$ is injective.
⟨1⟩2. LET: $x, y \in A$
⟨1⟩3. ASSUME: $f(x) = f(y)$
⟨1⟩4. $g(f(x)) = g(f(y))$
⟨1⟩5. $x = y$
□

Proposition 2.4.4. *Let $f : A \to B$. Then f is injective if and only if, for every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.*

PROOF:
⟨1⟩1. If f is injective then, for every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.
  ⟨2⟩1. ASSUME: f is injective.
  ⟨2⟩2. LET: X be a set.
  ⟨2⟩3. LET: $x, y : X \to A$
  ⟨2⟩4. ASSUME: $f \circ x = f \circ y$
  ⟨2⟩5. $\forall t \in X. x(t) = y(t)$
    ⟨3⟩1. LET: $t \in X$
    ⟨3⟩2. $f(x(t)) = f(y(t))$
      PROOF: ⟨2⟩4
    ⟨3⟩3. $x(t) = y(t)$
      PROOF: ⟨2⟩1
  ⟨2⟩6. $x = y$
    PROOF: Axiom of Extensionality.
⟨1⟩2. If, for every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.
  PROOF: Take $X = 1$.
□

Proposition 2.4.5. *The composite of surjective functions is surjective.*

PROOF:
⟨1⟩1. LET: $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$ be injective.
⟨1⟩2. LET: $c \in C$
⟨1⟩3. PICK $b \in B$ such that $g(b) = c$.
⟨1⟩4. PICK $a \in A$ such that $f(a) = b$.

$\langle 1\rangle 5.$ $(g \circ f)(a) = c$
□

Proposition 2.4.6. *Let $f : A \to B$. Then the following are equivalent.*

1. *f is surjective.*
2. *For any set X and functions $g, h : B \to X$, if $g \circ f = h \circ f$ then $g = h$.*
3. *There exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$*

PROOF:
$\langle 1\rangle 1.$ $1 \Rightarrow 2$
  $\langle 2\rangle 1.$ ASSUME: f is surjective.
  $\langle 2\rangle 2.$ LET: X be a set.
  $\langle 2\rangle 3.$ LET: $g, h : B \to X$
  $\langle 2\rangle 4.$ ASSUME: $g \circ f = h \circ f$
  $\langle 2\rangle 5.$ LET: $b \in B$
    PROVE: $g(b) = h(b)$
  $\langle 2\rangle 6.$ PICK $a \in A$ such that $f(a) = b$
  $\langle 2\rangle 7.$ $g(b) = h(b)$
    PROOF: $g(b) = g(f(a)) = h(f(a)) = h(b)$
$\langle 1\rangle 2.$ $1 \Rightarrow 3$
  $\langle 2\rangle 1.$ ASSUME: f is surjective.
  $\langle 2\rangle 2.$ PICK $g : B \to A$ such that, for all $b \in B$, we have $f(g(b)) = b$.
    PROOF: Axiom of Choice.
  $\langle 2\rangle 3.$ $f \circ g = \mathrm{id}_B$.
$\langle 1\rangle 3.$ $3 \Rightarrow 2$
  $\langle 2\rangle 1.$ LET: $g : B \to A$ such that $f \circ g = \mathrm{id}_B$
  $\langle 2\rangle 2.$ LET: X be a set.
  $\langle 2\rangle 3.$ LET: $h, k : B \to X$
  $\langle 2\rangle 4.$ ASSUME: $h \circ f = k \circ f$
  $\langle 2\rangle 5.$ $h = k$
    PROOF: $h = h \circ f \circ g = k \circ f \circ g = k$
$\langle 1\rangle 4.$ $2 \Rightarrow 1$
  $\langle 2\rangle 1.$ ASSUME: 2
  $\langle 2\rangle 2.$ LET: $b \in B$
  $\langle 2\rangle 3.$ LET: $h : B \to 2$ be the function that maps everything to 1.
  $\langle 2\rangle 4.$ LET: $k : B \to 2$ be the function that maps b to 0 and everything else to 1.
  $\langle 2\rangle 5.$ $h \neq k$
  $\langle 2\rangle 6.$ $h \circ f \neq k \circ f$
  $\langle 2\rangle 7.$ PICK $a \in A$ such that $h(f(a)) \neq k(f(a))$
  $\langle 2\rangle 8.$ $f(a) = b$
□

Proposition 2.4.7. *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is surjective then g is surjective.*

PROOF:
⟨1⟩1. LET: $c \in C$
⟨1⟩2. There exists $a \in A$ such that $g(f(a)) = c$.
⟨1⟩3. There exists $b \in B$ such that $g(b) = c$.
$\square$

Proposition 2.4.8. *The composite of bijections is a bijection.*

PROOF: Propositions 2.4.2 and 2.4.5. $\square$

Proposition 2.4.9. *Let $f : A \to B$. Then f is bijective if and only if there exists a function $f^{-1} : B \to A$, the* inverse *of f, such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$, in which case the inverse is unique.*

PROOF:
⟨1⟩1. If f is bijective then there exists $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.
  ⟨2⟩1. ASSUME: f is bijective.
  ⟨2⟩2. PICK $g : B \to A$ such that $f \circ g = \mathrm{id}_B$
    PROOF: Proposition 2.4.6.
  ⟨2⟩3. $f \circ g \circ f = f$
  ⟨2⟩4. $g \circ f = \mathrm{id}_A$
    PROOF: Proposition 2.4.4.
⟨1⟩2. If there exists $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$, then f is bijective.
  ⟨2⟩1. LET: $f^{-1} : B \to A$ satisfy $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$
  ⟨2⟩2. f is injective.
    PROOF: If $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$.
  ⟨2⟩3. f is surjective.
    PROOF: Proposition 2.4.6.
⟨1⟩3. If $g, h : B \to A$ satisfy $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ and $h \circ f = \mathrm{id}_A$ then $g = h$.
  PROOF: We have $g = g \circ f \circ h = h$.
$\square$

Proposition 2.4.10. *Let $f : A \to B$. Then $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.*

PROOF: Each is the function that maps a to $f(a)$. $\square$

Proposition 2.4.11.
$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps f to $(\pi_1 \circ f, \pi_2 \circ f)$ is a bijection. $\square$

Proposition 2.4.12.
$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function Φ such that $\Phi(f)(c)(b) = f(b, c)$ is a bijection. $\square$

2.4.1 The Empty Set

Theorem 2.4.13. *There exists a set which has no elements.*

PROOF:
⟨1⟩1. PICK a set A
PROOF: By the Axiom of Infinity, a set exists.
⟨1⟩2. LET: $S = \{x : \mathrm{El}(A) \mid \bot\}$ with injection $i : S \to A$
PROOF: Axiom of Separation.
⟨1⟩3. S has no elements.
□

Theorem 2.4.14. *If E and E' have no elements then $E \approx E'$.*

PROOF:
⟨1⟩1. LET: E and E' have no elements.
⟨1⟩2. PICK a function $F : E \to E'$.
PROOF: Axiom of Choice since vacuously $\forall x : \mathrm{El}(E) . \exists y : \mathrm{El}(E') . \top$.
⟨1⟩3. F is injective.
PROOF: Vacuously, for all $x, y : \mathrm{El}(E)$, if $F(x) = F(y)$ then $x = y$.
⟨1⟩4. F is surjective.
PROOF: Vacuously, for all $y : \mathrm{El}(E)$, there exists $x : \mathrm{El}(E)$ such that $F(x) = y$.
□

Definition 2.4.15 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

2.4.2 The Singleton

Theorem 2.4.16. *There exists a set that has exactly one element.*

PROOF:
⟨1⟩1. PICK a set A that has an element.
PROOF: By the Axiom of Infinity, there exists a set that has an element.
⟨1⟩2. PICK $a : \mathrm{El}(A)$
⟨1⟩3. PICK a set S and injection $i : S \rightarrowtail A$ such that, for all $x : \mathrm{El}(A)$, there exists $s : \mathrm{El}(S)$ such that $s = x$ if and only if $x = a$
⟨1⟩4. S has exactly one element.
□

Theorem 2.4.17. *If A and B both have exactly one element then $A \approx B$.*

PROOF:
⟨1⟩1. LET: A and B both have exactly one element a and b respectively.
⟨1⟩2. LET: $F : A \to B$ be the function such that, for all $x : \mathrm{El}(A)$, we have $(x = a \wedge F(x) = b)$
⟨1⟩3. F is a bijection.
□

Definition 2.4.18 (Singleton)**.** Let 1 be the set that has exactly one element. Let $*$ be its element.

2.4.3 Subsets

Definition 2.4.19 (Subset)**.** A *subset* of a set A consists of a set S and an injection $i : S \rightarrowtail A$. We write $(S, i) : \mathrm{Sub}(A)$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Proposition 2.4.20. *For any subset (S, i) of A we have $(S, i) = (S, i)$.*

Proof: We have $\mathrm{id}_S : S \approx S$ and $i \circ \mathrm{id}_S = i$.

Proposition 2.4.21. *If $(S, i) = (T, j)$ then $(T, j) = (S, i)$.*

Proof: If $\phi : S \approx T$ and $j \circ \phi = i$ then $\phi^{-1} : T \approx S$ and $i \circ \phi^{-1} = j$. □

Proposition 2.4.22. *If $(R, i) = (S, j)$ and $(S, j) = (T, k)$ then $(R, i) = (T, k)$.*

Proof: If $\phi : R \approx S$ and $j \circ \phi = i$, and $\psi : S \approx T$ and $k \circ \psi = j$, then $\psi \circ \phi : R \approx T$ and $k \circ \psi \circ \phi = i$. □

Definition 2.4.23 (Membership)**.** Given $(S, i) : \mathrm{Sub}(A)$ and $a \in A$, we write $a \in (S, i)$ for $\exists s : \mathrm{El}\,(S)\,.i(s) = a$.

Proposition 2.4.24. *If $a \in (S, i)$ and $(S, i) = (T, j)$ then $a \in (T, j)$.*

Proof: If $i(s) = a$ then $j(\phi(s)) = a$. □

2.5 Relations

Definition 2.5.1 (Relation)**.** Let A and B be sets. A *relation* R between A and B, $R : A \looparrowright B$, is a subset of $A \times B$.

Given $a \in A$ and $b \in B$, we write aRb for $(a, b) \in R$.

2.6 Power Set

Definition 2.6.1 (Power Set)**.** The *power set* of a set A is $\mathcal{P}A := 2^A$.

Given $S \in \mathcal{P}A$ and $a \in A$, we write $a \in A$ for $S(a) = 1$.

2.7 Cartesian Product

Definition 2.7.1 (Cartesian Product)**.** Let A and B be sets. The *Cartesian product* of A and B, $A \times B$, is the tabulation of the relation $A \looparrowright B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

2.8 Quotient Sets

Proposition 2.8.1. *Let $\sim$ be an equivalence relation on X. Then there exists a set $X/\sim$, the* quotient set *of X with respect to $\sim$, and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the* canonical projection*, such that, for all $x, y : \mathrm{El}(X)$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to $\sim$, then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

2.9 Partitions

Definition 2.9.1 (Partition)**.** A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X.

Chapter 3

Order Theory

3.1 Relations

Definition 3.1.1 (Reflexive)**.** A relation $R \subseteq A \times A$ is *reflexive* iff, for all $a \in A$, we have $(a, a) \in R$.

Definition 3.1.2 (Antisymmetric)**.** A relation $R \subseteq A \times A$ is *antisymmetric* iff, for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Definition 3.1.3 (Transitive)**.** A relation $R \subseteq A \times A$ is *transitive* iff, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Definition 3.1.4 (Partial Order)**.** A *partial order* on a set A is a relation on A that is reflexive, antisymmetric and transitive.

We say $(A, \leqslant)$ is a *partially ordered set* or *poset* iff $\leqslant$ is a partial order on A.

Chapter 4

Category Theory

4.1 Categories

Definition 4.1.1. A *category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \mathrm{Ob}(\mathcal{C})$.
- for any objects X and Y, a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y. We write $f : X \to Y$ for $f \in \mathcal{C}[X, Y]$.
- for any objects X, Y and Z, a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \to \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X, there exists a morphism $\mathrm{id}_X : X \to X$, the *identity morphism* on X, such that:
  - for any object Y and morphism $f : Y \to X$ we have $\mathrm{id}_X \circ f = f$
  - for any object Y and morphism $f : X \to Y$ we have $f \circ \mathrm{id}_X = f$

We write the composite of morphism $f_1, \ldots, f_n$ as $f_n \circ \cdots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 4.1.2. Let **Set** be the category of small sets and functions.

Definition 4.1.3. We identify a poset $(A, \leqslant)$ with the category with:

- set of objects A
- for $a, b \in A$, the set of homomorphisms is $\{x \in 1 : a \leqslant b\}$

Proposition 4.1.4. *A category is a poset iff, for any two objects, there exists at most one morphism between them.*

Proposition 4.1.5. *The identity morphism on an object is unique.*

PROOF:
⟨1⟩1. LET: $\mathcal{C}$ be a category.
⟨1⟩2. LET: $A \in \mathcal{C}$
⟨1⟩3. LET: $i, j : A \to A$ be identity morphisms on A.
⟨1⟩4. $i = j$
PROOF:

$$\begin{aligned} i &= i \circ j && (j \text{ is an identity on } A) \\ &= j && (i \text{ is an identity on } A) \end{aligned}$$

□

Definition 4.1.6. Given $f : A \to B$ and an object C, define the function $f^* : \mathcal{C}[B, C] \to \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 4.1.7. Given $f : A \to B$ and an object C, define the function $f_* : \mathcal{C}[C, A] \to \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

4.1.1 Monomorphisms

Definition 4.1.8 (Monomorphism)**.** Let $f : A \to B$. Then f is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for any object X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

4.1.2 Epimorphisms

Definition 4.1.9 (Epimorphism)**.** Let $f : A \to B$. Then f is *epic* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for any object X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

4.1.3 Sections and Retractions

Definition 4.1.10 (Section, Retraction)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $rs = \mathrm{id}_B$.

Proposition 4.1.11. *Let $f : A \to B$ and $r, s : B \to A$. If r is a retraction of f and s is a section of f then $r = s$.*

PROOF:

$$\begin{aligned} r &= r\mathrm{id}_B && \text{(Unit Law)} \\ &= rfs && (s \text{ is a section of } f) \\ &= \mathrm{id}_A s && (r \text{ is a retraction of } f) \\ &= s && \text{(Unit Law)} \end{aligned}$$

□

Proposition 4.1.12. *Every section is monic.*

PROOF:

⟨1⟩1. Let: $s : B \to A$ be a section of $r : A \to B$.
⟨1⟩2. Let: X be an object and $x, y : X \to B$
⟨1⟩3. Assume: $s \circ x = s \circ y$
⟨1⟩4. $x = y$
Proof: $x = r \circ s \circ x = r \circ s \circ y = y$.
□

Proposition 4.1.13. *Every retraction is epic.*

Proof: Dual. □

4.1.4 Isomorphisms

Definition 4.1.14 (Isomorphism)**.** A morphism $f : A \to B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \to A$ that is both a retraction and section of f.

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 4.1.15. *The inverse of an isomorphism is unique.*

Proof: From Proposition 4.1.11. □

Proposition 4.1.16. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

Proof: Since $ff^{-1} = \mathrm{id}_B$ and $f^{-1}f = \mathrm{id}_A$. □

Isomorphism.
Define the opposite category.
Slice categories

Definition 4.1.17. Let $\mathcal{C}$ be a category and $B \in \mathcal{C}$. The category $\mathcal{C}_B^B$ of objects *over and under* B is the category with:

- objects all triples (X, u, p) such that $u : B \to X$ and $p : X \to B$
- morphisms $f : (X, u, p) \to (Y, u', p')$ all morphisms $f : X \to Y$ such that $fu = u'$ and $p'f = p$.

Proposition 4.1.18.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B)\backslash\mathrm{id}_B \cong (\mathcal{C}\backslash B)/\mathrm{id}_B$$

$(B, \mathrm{id}_B, \mathrm{id}_B)$ is the zero object in $\mathcal{C}_B^B$.

4.1.5 Initial Objects

Definition 4.1.19 (Initial Object)**.** An object I is *initial* iff, for any object X, there exists exactly one morphism $I \to X$.

Proposition 4.1.20. *The empty set is initial in* **Set**.

PROOF: For any set A, the nowhere-defined function is the unique function $\varnothing \to A$. ∎

Proposition 4.1.21. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:
⟨1⟩1. LET: $i : I \to I'$ be the unique morphism $I \to I'$.
⟨1⟩2. LET: $i^{-1} : I' \to I$ be the unique morphism $I' \to I$.
⟨1⟩3. $ii^{-1} = \mathrm{id}_{I'}$
PROOF: There is only one morphism $I' \to I'$.
⟨1⟩4. $i^{-1}i = \mathrm{id}_I$
PROOF: There is only one morphism $I \to I$.
∎

4.1.6 Terminal Objects

Definition 4.1.22 (Terminal Object)**.** An object T is *terminal* iff, for any object X, there exists exactly one morphism $X \to T$.

Proposition 4.1.23. *1 is terminal in* **Set**.

PROOF: For any set A, the constant function to $*$ is the only function $A \to 1$. ∎

Proposition 4.1.24. *If T and T' are terminal objects, then there exists a unique isomorphism $T \cong T'$.*

PROOF: Dual to Proposition 4.1.21. ∎

4.1.7 Zero Objects

Definition 4.1.25 (Zero Object)**.** An object Z is a *zero object* iff it is an initial object and a terminal object.

Definition 4.1.26 (Zero Morphism)**.** Let $\mathcal{C}$ be a category with a zero object Z. Let $A, B \in \mathcal{C}$. The *zero morphism* $A \to B$ is the unique morphism $A \to Z \to B$.

Proposition 4.1.27. *There is no zero object in* **Set**.

PROOF: Since $\varnothing \not\approx 1$. ∎

4.1.8 Triads

Definition 4.1.28 (Triad)**.** Let $\mathcal{C}$ be a category. A *triad* consists of objects X, Y, M and morphisms $\alpha : X \to M$, $\beta : Y \to M$. We call M the *codomain* of the triad.

4.1.9 Cotriads

Definition 4.1.29 (Cotriad)**.** Let $\mathcal{C}$ be a category. A *cotriad* consists of objects X, Y, W and morphisms $\xi : W \to X$, $\eta : W \to Y$. We call W the *domain* of the triad.

4.1.10 Pullbacks

Definition 4.1.30 (Pullback)**.** A diagram

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

is a *pullback* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : Z \to X$ and $g : Z \to Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \to W$ such that $\xi h = f$ and $\eta h = g$.

In this case we also say that η is the *pullback* of β along α.

Proposition 4.1.31. *If $\xi : W \to X$ and $\eta : W \to Y$ form a pullback of $\alpha : X \to M$ and $\beta : Y \to M$, and $\xi' : W' \to X$ and $\eta' : W' \to Y$ also form the pullback of α and β, then there exists a unique isomorphism $\phi : W \cong W'$ such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.*

PROOF:
⟨1⟩1. LET: $\phi : W \to W'$ be the unique morphism such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.
⟨1⟩2. LET: $\phi^{-1} : W' \to W$ be the unique morphism such that $\eta\phi^{-1} = \eta'$ and $\xi\phi^{-1} = \xi'$.
⟨1⟩3. $\phi\phi^{-1} = \mathrm{id}_{W'}$
PROOF: Each is the unique $x : W' \to W'$ such that $\eta' x = \eta'$ and $\xi' x = \xi'$.
⟨1⟩4. $\phi^{-1}\phi = \mathrm{id}_W$
PROOF: Each is the unique $x : W \to W$ such that $\eta x = \eta$ and $\xi x = \xi$.
□

Proposition 4.1.32. *For any morphism $h : A \to B$, the following diagram is a pullback diagram.*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

PROOF:
⟨1⟩1. LET: Z be an object.
⟨1⟩2. LET: $f : Z \to B$ and $g : Z \to A$ satisfy $\mathrm{id}_B f = hg$
⟨1⟩3. $g : Z \to B$ is the unique morphism such that $\mathrm{id}_A g = g$ and $hg = f$.
□

Proposition 4.1.33. *The pullback of an isomorphism is an isomorphism.*

PROOF:

⟨1⟩1. Let:

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pullback diagram.
⟨1⟩2. Assume: β is an isomorphism.
⟨1⟩3. Let: ξ^{-1} be the unique morphism $X \to W$ such that $\xi\xi^{-1} = \mathrm{id}_X$ and $\eta\xi^{-1} = \beta^{-1}\alpha$.
Proof: This exists since $\alpha\mathrm{id}_X = \beta\beta^{-1}\alpha = \alpha$.
⟨1⟩4. $\xi^{-1}\xi = \mathrm{id}_W$
Proof: Each is the unique $x : W \to W$ such that $\xi x = \xi$ and $\eta x = \eta$.
□

Proposition 4.1.34. *Let $\beta : (Y, y) \to (M, m)$ and $\alpha : (X, x) \to (M, m)$ in $\mathcal{C}\backslash A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pullback in $\mathcal{C}$. Let $w : A \to W$ be the unique morphism such that $\xi w = x$ and $\eta w = y$. Then $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ is the pullback of β and α in $\mathcal{C}\backslash A$.

Proof:
⟨1⟩1. Let: $(Z, z) \in \mathcal{C}\backslash A$
⟨1⟩2. Let: $f : (Z, z) \to (X, x)$ and $g : (Z, z) \to (Y, y)$ satisfy $\alpha f = \beta g$.
⟨1⟩3. Let: $h : Z \to W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.
⟨1⟩4. $hz = w$
⟨2⟩1. $\xi hz = \xi w$
Proof:

$$\begin{aligned} \xi hz &= fz && (\langle 1\rangle 3) \\ &= x && (\langle 1\rangle 2) \\ &= \xi w \end{aligned}$$

⟨2⟩2. $\eta hz = \eta w$
Proof: Similar.
⟨1⟩5. $h : (Z, z) \to (W, w)$
□

Proposition 4.1.35. *Let $\beta : (Y, y) \to (M, m)$ and $\alpha : (X, x) \to (M, m)$ in*

$\mathcal{C}/A$. Let

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pullback in $\mathcal{C}$. Let $w = x\xi : W \to A$. Then $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ form a pullback of α and β in $\mathcal{C}/A$.

PROOF:
⟨1⟩1. $\eta : (W, w) \to (Y, y)$
PROOF:

$$\begin{aligned} y\eta &= m\beta\eta \\ &= m\alpha\xi \\ &= x\xi \\ &= w \end{aligned}$$

⟨1⟩2. LET: $(Z, z) \in \mathcal{C}/A$
⟨1⟩3. LET: $f : (Z, z) \to (X, x)$ and $g : (Z, z) \to (Y, y)$ satisfy $\alpha f = \beta g$.
⟨1⟩4. LET: $h : Z \to W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.
⟨1⟩5. $h : (Z, z) \to (W, w)$
PROOF:

$$\begin{aligned} wh &= x\xi h \\ &= xf && (\langle 1\rangle 4) \\ &= z && (\langle 1\rangle 3) \end{aligned}$$

□

Proposition 4.1.36. *In* **Set***, let $\alpha : X \to M$ and $\beta : Y \to M$. Let $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$ with inclusion $i : W \to X \times Y$. Let $\xi = \pi_1 i : W \to X$ and $\eta : \pi_2 i : W \to Y$. Then ξ and η form the pullback of α and β.*

PROOF:
⟨1⟩1. $\alpha\xi = \beta\eta$
PROOF: For $w \in W$, if $i(w) = (x, y)$ then then $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$.
⟨1⟩2. For every set Z and functions $f : Z \to X$, $g : Z \to Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \to W$ such that $\xi h = f$ and $\eta h = g$
PROOF: For $z \in Z$, let $h(z)$ be the unique element of W such that $i(h(z)) = (f(z), g(z))$.
□

Pullback lemma

4.1.11 Pushouts

Definition 4.1.37 (Pushout)**.** A diagram

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd} \tag{4.1}$$

is a *pushout* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : X \to Z$ and $g : Y \to Z$ such that $f\xi = g\eta$, there exists a unique $h : M \to Z$ such that $h\alpha = f$ and $h\beta = g$.

We also say that β is the *pushout* of ξ along η.

Proposition 4.1.38. *If $\alpha : X \to M$ and $\beta : Y \to M$ form a pushout of $\xi : W \to X$ and $\eta : W \to Y$, and $\alpha' : X \to M'$ and $\beta' : Y \to M'$ also form a pushout of ξ and η, then there exists a unique isomorphism $\phi : M \cong M'$ such that $\phi\alpha = \alpha'$ and $\phi\beta = \beta'$.*

Proof: Dual to Proposition 4.1.31. □

Proposition 4.1.39. *For any morphism $h : A \to B$, the following diagram is a pushout diagram.*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

Proof: Dual to Proposition 4.1.32.

Proposition 4.1.40. *The diagram (4.1) is a pushout in $\mathcal{C}$ iff it is a pullback in $\mathcal{C}^{\mathrm{op}}$.*

Proof: Immediate from definitions. □

Proposition 4.1.41. *The pushout of an isomorphism is an isomorphism.*

Proof: Dual to Proposition 4.1.33. □

Proposition 4.1.42. *Let $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ in $\mathcal{C}\backslash A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pushout in $\mathcal{C}$. Let $m := \alpha x : A \to M$. Then $\alpha : (X, x) \to (M, m)$ and $\beta : (Y, y) \to (M, m)$ is the pushout of ξ and η in $\mathcal{C}\backslash A$.

PROOF: Dual to Proposition 4.1.35. ☐

Proposition 4.1.43. *Let $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ in $\mathcal{C}/A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pushout in $\mathcal{C}$. Let $m : M \to A$ be the unique morphism such that $m\alpha = x$ and $m\beta = y$. Then $\alpha : (X, x) \to (M, m)$ and $\beta : (Y, y) \to (M, m)$ is the pushout of ξ and η in $\mathcal{C}\backslash A$.

PROOF: Dual to Proposition 4.1.34. ☐

Proposition 4.1.44. Set *has pushouts.*

PROOF:
$\langle 1 \rangle 1$. LET: $\xi : W \to X$ and $\eta : W \to Y$.
$\langle 1 \rangle 2$. LET: $\sim$ be the equivalence relation on $X + Y$ generated by $\xi(w) \sim \eta(w)$ for all $w \in W$
$\langle 1 \rangle 3$. LET: $M = (X + Y)/\sim$ with canonical projection $\pi : X + Y \twoheadrightarrow M$.
$\langle 1 \rangle 4$. LET: $\alpha = \pi \circ \kappa_1 : X \to M$
$\langle 1 \rangle 5$. LET: $\beta = \pi \circ \kappa_2 : Y \to M$
$\langle 1 \rangle 6$. LET: Z be any set, $f : X \to Z$ and $g : Y \to Z$.
$\langle 1 \rangle 7$. ASSUME: $f\xi = g\eta$
$\langle 1 \rangle 8$. LET: $h : X + Y \to Z$ be the function defined by $h(x) = f(x)$ and $h(y) = g(y)$ for $x \in X$ and $y \in Y$
$\langle 1 \rangle 9$. h respects $\sim$
PROOF: For $w \in W$ we have

$$\begin{aligned} h(\xi(w)) &= f(\xi(w)) && (\langle 1 \rangle 8) \\ &= g(\eta(w)) && (\langle 1 \rangle 7) \\ &= h(\eta(w)) && (\langle 1 \rangle 8) \end{aligned}$$

$\langle 1 \rangle 10$. LET: $\overline{h} : M \to Z$ be the induced function.
$\langle 1 \rangle 11$. $\overline{h}\alpha = f$
PROOF:

$$\begin{aligned} \overline{h}(\alpha(x)) &= \overline{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x) \end{aligned}$$

$\langle 1 \rangle 12$. $\overline{h}\beta = g$
PROOF: Similar.
$\langle 1 \rangle 13$. For all $k : M \to Z$, if $k\alpha = f$ and $k\beta = g$ then $k = \overline{h}$.

Proof:

$$
\begin{aligned}
k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\
&= f(x) \\
k(\pi(\kappa_2(y))) &= k(\beta(y)) \\
&= g(y) \\
\therefore k \circ \pi &= h \\
\therefore k &= \overline{h}
\end{aligned}
$$

□

Definition 4.1.45. Let $u : A \rightarrowtail X$ be an injection. The *pointed set obtained from X by collapsing* (A, u), denoted $X/(A, u)$, is the pushout

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "u"] & 1 \arrow[d, "*"'] \\
X \arrow[r] & X/(A,u)
\end{tikzcd}
$$

Proposition 4.1.46. *In* $\mathbf{Set}_*$*, any two morphisms* $1 \to X$ *and* $1 \to Y$ *have a pushout.*

Proof: The pushout of $a : (1, *) \to (X, x)$ and $b : (1, *) \to (Y, y)$ is $(X+Y/\sim, x)$ where $\sim$ is the equivalence relation generated by $x \sim y$. □

Definition 4.1.47 (Wedge)**.** The *wedge* of pointed sets X and Y, $X \vee Y$, is the pushout of the unique morphism $1 \to X$ and $1 \to Y$.

Definition 4.1.48 (Smash)**.** Let X and Y be pointed sets. Let $\xi : X \vee Y \to X$ be the unique morphism such that the following diagram commutes.

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, equal, bend left] & \\
Y \arrow[r] \arrow[drr, bend right, "0"'] & X \vee Y \arrow[dr, "\xi"] & \\
& & X
\end{tikzcd}
$$

Let $\eta : X \vee Y \to Y$ be the unique morphism such that the following diagram commutes.

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "0"] & \\
Y \arrow[r] \arrow[drr, equal, bend right] & X \vee Y \arrow[dr, "\eta"] & \\
& & Y
\end{tikzcd}
$$

Let $\zeta = \langle \xi, \eta \rangle : X \vee Y \to X \times Y$. The *smash* of X and Y, $X \wedge Y$, is the result of collapsing $X \times Y$ with respect to ζ.

Pushout lemma

4.1.12 Subcategories

Definition 4.1.49 (Subcategory)**.** A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ consists of:

- a subset $\mathrm{Ob}(\mathcal{C}')$ of $\mathcal{C}$
- for all $A, B \in \mathrm{Ob}(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \mathrm{Ob}(\mathcal{C}')$, we have $\mathrm{id}_A \in \mathcal{C}'[A, A]$
- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \mathrm{Ob}(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

4.1.13 Opposite Category

Definition 4.1.50 (Opposite Category)**.** For any category $\mathcal{C}$, the *opposite* category $\mathcal{C}^{\mathrm{op}}$ is the category with

- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$
- $\mathcal{C}^{\mathrm{op}}[A, B] = \mathcal{C}[B, A]$
- Given $f \in \mathcal{C}^{\mathrm{op}}[A, B]$ and $g \in \mathcal{C}^{\mathrm{op}}[B, C]$, their composite in $\mathcal{C}^{\mathrm{op}}$ is $f \circ g$, where $\circ$ is composition in $\mathcal{C}$.

Proposition 4.1.51. *An object is initial in $\mathcal{C}$ iff it is terminal in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. □

Proposition 4.1.52. *An object is terminal in $\mathcal{C}$ iff it is initial in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. □

Corollary 4.1.52.1. *If T and T' are terminal objects in $\mathcal{C}$ then there exists a unique isomorphism $T \cong T'$.*

4.1.14 Groupoids

Definition 4.1.53 (Groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

4.1.15 Concrete Categories

Definition 4.1.54 (Concrete Category)**.** A *concrete category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}(\mathcal{C})$ of *objects*
- for any object $A \in \mathrm{Ob}(\mathcal{C})$, a set $|A|$
- for any objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$
- for any object A we have $\mathrm{id}_{|A|} \in \mathcal{C}[A, A]$.

4.1.16 Power of Categories

Definition 4.1.55. Let $\mathcal{C}$ be a category and J a set. The category $\mathcal{C}^J$ is the category with:

- objects all J-indexed families of objects of $\mathcal{C}$
- morphisms $\{X_j\}_{j \in J} \to \{Y_j\}_{j \in J}$ all families $\{f_j\}_{j \in J}$ where $f_j : X_j \to Y_j$

4.1.17 Arrow Category

Definition 4.1.56 (Arrow Category)**.** Let $\mathcal{C}$ be a category. The *arrow category* $\mathcal{C}^{\to}$ is the category with:

- objects all triples (A, B, f) where $f : A \to B$ in $\mathcal{C}$
- morphisms $(A, B, f) \to (C, D, g)$ all pairs $(u : A \to C, v : B \to D)$ such that $vf = gu$.

4.1.18 Slice Category

Definition 4.1.57 (Slice Category)**.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category under* A, $\mathcal{C} \backslash A$, is the category with:

- objects all pairs (B, f) where $B \in \mathcal{C}$ and $f : A \to B$
- morphisms $(B, f) \to (C, g)$ are morphisms $u : B \to C$ such that $uf = g$.

We identify this with the subcategory of $\mathcal{C}^{\to}$ formed by mapping (B, f) to (A, B, f) and u to (id_A, u).

Proposition 4.1.58. *If* $s : (B, f) \to (C, g)$ *in* $\mathcal{C} \backslash A$*, then any retraction of* s *in* $\mathcal{C}$ *is a retraction of* s *in* $\mathcal{C} \backslash A$*.*

PROOF:
$\langle 1 \rangle 1$. LET: $r : C \to B$ be a retraction of s in $\mathcal{C}$.

$\langle 1\rangle 2.$ $rg = f$
PROOF: $rg = rsf = f$.
$\langle 1\rangle 3.$ $r : (C, g) \to (B, f)$ in $\mathcal{C}\backslash A$
$\langle 1\rangle 4.$ $rs = \mathrm{id}_{(B,f)}$
PROOF: Because composition is inherited from $\mathcal{C}$.
□

Proposition 4.1.59. id_A *is the initial object in* $\mathcal{C}\backslash A$*.*

PROOF: For any $(B, f) \in \mathcal{C}\backslash A$, we have f is the only morphism $A \to B$ such that $f\mathrm{id}_A = f$. □

Proposition 4.1.60. *If* A *is terminal in* $\mathcal{C}$ *then* id_A *is the zero object in* $\mathcal{C}\backslash A$*.*

PROOF: For any $(B, f) \in \mathcal{C}\backslash A$, the unique morphism $! : B \to A$ is the unique morphism such that $!f = \mathrm{id}_A$. □

Definition 4.1.61 (Pointed Sets)**.** The *category of pointed sets* is $\mathbf{Set}\backslash 1$.

Definition 4.1.62. Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category over* A, $\mathcal{C}/A$, is the category with:

- objects all pairs (B, f) with $f : B \to A$
- morphisms $u : (B, f) \to (C, g)$ all morphisms $u : B \to C$ such that $gu = f$.

Proposition 4.1.63. *Let* $u : (B, f) \to (C, g) : \mathcal{C}/A$*. Any section of* u *in* $\mathcal{C}$ *is a section of* u *in* $\mathcal{C}/A$*.*

PROOF: Dual to Proposition 4.1.58. □

Proposition 4.1.64. id_A *is terminal in* $\mathcal{C}/A$*.*

PROOF: Dual to Proposition 4.1.59. □

Proposition 4.1.65. *If* A *is initial in* $\mathcal{C}$ *then* id_A *is the zero object in* $\mathcal{C}/A$*.*

PROOF: Dual to Proposition 4.1.60. □

Definition 4.1.66. Let $A \in \mathcal{C}$. The category of objects *over and under* A, written $\mathcal{C}^A_A$, is the category with:

- objects all triples (X, u, p) where $u : A \to X$, $p : X \to A$ and $pu = \mathrm{id}_A$
- morphism $f : (X, u, p) \to (Y, v, q)$ all morphisms $f : X \to Y$ such that $fu = v$ and $qf = p$

Proposition 4.1.67. $(A, \mathrm{id}_A, \mathrm{id}_A)$ *is the zero object in* $\mathcal{C}^A_A$*.*

PROOF: For any object (X, u, p), we have p is the unique morphism $(X, u, p) \to (A, \mathrm{id}_A, \mathrm{id}_A)$, and u is the unique morphism $(A, \mathrm{id}_A, \mathrm{id}_A) \to (X, u, p)$. □

Definition 4.1.68 (Fibre Collapsing)**.** Let B be a set. Let $u : (A, a) \to (X, x)$ in $\mathbf{Set}/B$. Form the pushout

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"] & B \arrow[d, "j"'] \\ X \arrow[r, "i"'] & C \end{tikzcd}$$

Let $c : C \to B$ be the unique morphism such that $cj = \mathrm{id}_B$ and $ci = x$. Then $(C, j, c) \in \mathbf{Set}_B^B$ is called the set over and under B obtained from X by *fibre collapsing* with respect to u. If (A, u) is a subset of X, we denote this set over and under B by $X/_B(A, u)$.

Definition 4.1.69 (Fibre Wedge)**.** Let B be a small set. Let $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$. The *fibre wedge* of X and Y is the pushout of u_X and u_Y:

$$\begin{tikzcd} B \arrow[r, "u_X"] \arrow[d, "u_Y"] & X \arrow[d] \\ Y \arrow[r] & X \vee_B Y \end{tikzcd}$$

Definition 4.1.70 (Fibre Smash)**.** Let $X, Y \in \mathbf{Set}_B^B$. Let $\xi : X \vee_B Y \to X$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, equal, bend left] & \\ Y \arrow[r] \arrow[drr, bend right, "0"'] & X \vee_B Y \arrow[dr, "\xi"] & \\ & & X \end{tikzcd}$$

Let $\eta : X \vee_B Y \to Y$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "0"] & \\ Y \arrow[r] \arrow[drr, equal, bend right] & X \vee_B Y \arrow[dr, "\eta"] & \\ & & Y \end{tikzcd}$$

Let $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \to X \times Y$. The *fibre smash* of X and Y, $X \wedge_B Y$, is the result of collapsing $X \times Y$ with respect to ζ.

Proposition 4.1.71. **Set** *has products and coproducts.*

Proposition 4.1.72. *Let $\mathcal{C}$ be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in $\mathcal{C}$ and $Z \in \mathcal{C}$. Let $\coprod_{\alpha \in I} X_\alpha$ be the coproduct of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[\coprod_{\alpha \in I} X_\alpha, Z] \approx \prod_{\alpha \in I} \mathcal{C}[X_\alpha, Z] \ .$$

Proposition 4.1.73. *Let $\mathcal{C}$ be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in $\mathcal{C}$ and $Z \in \mathcal{C}$. Let $\prod_{\alpha \in I} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[Z, \prod_{\alpha \in I} X_\alpha] \approx \prod_{\alpha \in I} \mathcal{C}[Z, X_\alpha] \ .$$

Proposition 4.1.74. *A product in $\mathcal{C}$ constitutes a product in $\mathcal{C} \backslash A$.*

Proposition 4.1.75. *A coproduct in $\mathcal{C}$ constitutes a product in $\mathcal{C}/A$.*

4.2 Functors

Definition 4.2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- a function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$
- for every morphism $f : A \to B$ in $\mathcal{C}$, a morphism $Ff : FA \to FB$ in $\mathcal{D}$

such that:

- for all $A : \mathrm{El}\,(\mathrm{Ob}(C))$ we have $F\mathrm{id}_A = \mathrm{id}_{FA}$
- for any morphism $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, we have $F(g \circ f) = Fg \circ Ff$

Proposition 4.2.2. *Functors preserve isomorphisms.*

PROOF:
$\langle 1 \rangle 1$. LET: $F : \mathcal{C} \to \mathcal{D}$ be a functor.
$\langle 1 \rangle 2$. LET: $f : A \cong B$ in $\mathcal{C}$
$\langle 1 \rangle 3$. $Ff^{-1} \circ Ff = \mathrm{id}_{FA}$
PROOF:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\mathrm{id}_A \\ &= \mathrm{id}_{FA} \end{aligned}$$

$\langle 1 \rangle 4$. $Ff \circ Ff^{-1} = \mathrm{id}_{FB}$
PROOF:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\mathrm{id}_B \\ &= \mathrm{id}_{FB} \end{aligned}$$

□

Definition 4.2.3 (Identity Functor)**.** For any category $\mathcal{C}$, the *identity* functor on $\mathcal{C}$ is the functor $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ defined by

$$\begin{aligned} I_{\mathcal{C}} A &:= A && (A \in \mathcal{C}) \\ I_{\mathcal{C}} f &:= f && (f : A \to B \text{ in } \mathcal{C}) \end{aligned}$$

Proposition 4.2.4. *Let $F : \mathcal{C} \to \mathcal{D}$. If $r : A \to B$ is a retraction of $s : B \to A$ in $\mathcal{C}$ then Fr is a retraction of Fs.*

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\mathrm{id}_B \\ &= \mathrm{id}_{FB} \end{aligned}$$

∎

Corollary 4.2.4.1. *Let $F : \mathcal{C} \to \mathcal{D}$. If $\phi : A \cong B$ is an isomorphism in $\mathcal{C}$ then $F\phi : FA \cong FB$ is an isomorphism in $\mathcal{D}$ with $(F\phi)^{-1} = F\phi^{-1}$.*

Definition 4.2.5 (Composition of Functors)**.** Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, the *composite* functor $GF : \mathcal{C} \to \mathcal{E}$ is defined by

$$\begin{aligned} (GF)A &= G(FA) && (A \in \mathcal{C}) \\ (GF)f &= G(Ff) && (f : A \to B : \mathcal{C}) \end{aligned}$$

Definition 4.2.6 (Category of Categories)**.** Let **Cat** be the category of small categories and functors.

Definition 4.2.7 (Isomorphism of Categories)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an *isomorphism of categories* iff there exists a functor $F^{-1} : \mathcal{D} \to \mathcal{C}$, the *inverse* of F, such that $FF^{-1} = I_{\mathcal{D}}$ and $F^{-1}F = I_{\mathcal{C}}$.

Categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, iff there exists an isomorphism between them.

Proposition 4.2.8. *If A is initial in $\mathcal{C}$ then $\mathcal{C} \backslash A \cong \mathcal{C}$.*

PROOF:

⟨1⟩1. Define $F : \mathcal{C}\backslash A \to \mathcal{C}$ by

$$\begin{aligned} F(B, f) &= B \\ F(u : (B, f) \to (C, g)) &= u \end{aligned}$$

⟨1⟩2. Define $G : \mathcal{C} \to \mathcal{C}\backslash A$ by

$$GB = (B, !_B) \qquad \text{where } !_B \text{ is the unique morphism } A \to B$$

$$G(u : B \to C) = u : (B, !_B) \to (C, !_C)$$

⟨1⟩3. $FG = \mathrm{id}_{\mathcal{C}}$

⟨1⟩4. $GF = \mathrm{id}_{\mathcal{C}\backslash A}$

PROOF: Since $GF(B, f) = (B, !_B) = (B, f)$ because the morphism $A \to B$ is unique.

∎

Proposition 4.2.9. *If A is terminal in $\mathcal{C}$ then $\mathcal{C}/A \cong \mathcal{C}$.*

Proof: Dual. $\square$

Proposition 4.2.10.

$$\mathcal{C}_A^A \cong (\mathcal{C}/A)\backslash(A, \mathrm{id}_A) \cong (\mathcal{C}\backslash A)/(A, \mathrm{id}_A)$$

Proof:
$\langle 1\rangle 1$. Define a functor $F : \mathcal{C}_A^A \to (\mathcal{C}/A)\backslash(A, \mathrm{id}_A)$.
  $\langle 2\rangle 1$. Given $A \xrightarrow{u} X \xrightarrow{p} A$ in $\mathcal{C}_A^A$, let $F(X, u, p) = ((X, p), u)$
  $\langle 2\rangle 2$. Given $f : (A \xrightarrow{u} X \xrightarrow{p} A) \to (A \xrightarrow{v} Y \xrightarrow{q} A)$, let $Ff = f$.
$\langle 1\rangle 2$. Define a functor $G : (\mathcal{C}/A)\backslash(A, \mathrm{id}_A) \to \mathcal{C}_A^A$.
$\langle 1\rangle 3$. Define a functor $H : \mathcal{C}_A^A \to (\mathcal{C}\backslash A)/(A, \mathrm{id}_A)$.
$\langle 1\rangle 4$. Define a functor $K : (\mathcal{C}\backslash A)/(A, \mathrm{id}_A) \to \mathcal{C}_A^A$.
$\square$

Definition 4.2.11 (Forgetful Functor)**.** For any concrete category $\mathcal{C}$, define the *forgetful* functor $U : \mathcal{C} \to \mathbf{Set}$ by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

Definition 4.2.12 (Switching Functor)**.** For any category $\mathcal{C}$, define the *switching functor* $T : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

Definition 4.2.13 (Reduction)**.** Let $\Phi : \mathbf{Set} \to \mathbf{Set}$ be a functor. The *reduction* of Φ is the functor $\Phi^* : \mathbf{Set}_* \to \mathbf{Set}_*$ defined by: $\Phi^*(X, a)$ is the collapse of $\Phi(X)$ with respect to $\Phi(a) : \Phi(1) \rightarrowtail \Phi(X)$.

Definition 4.2.14. Extend the wedge $\vee$ to a functor $\mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ by defining, given $f : X \to X'$ and $g : Y \to Y'$, thene $f \vee g$ is the unique morphism that makes the following diagram commute.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[dr, "f"] & \\
Y \arrow[r] \arrow[dr, "g"'] & X \vee Y \arrow[dr, "f \vee g"] & X' \arrow[d] \\
& Y' \arrow[r] & X' \vee Y'
\end{tikzcd}$$

Definition 4.2.15. Extend smash to a functor $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ as follows. Given $f : X \to X'$ and $g : Y \to Y'$, let $f \wedge g : X \wedge Y \to X' \wedge Y'$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd}
X \vee Y \arrow[r] \arrow[d] \arrow[ddr] & 1 \arrow[d] \arrow[ddr, equals] & \\
X \times Y \arrow[r] \arrow[ddr, "f \times g"'] & X \wedge Y \arrow[ddr] & \\
& X' \vee Y' \arrow[r] \arrow[d] & 1 \arrow[d] \\
& X' \times Y' \arrow[r] & X' \wedge Y'
\end{tikzcd}$$

Definition 4.2.16 (Reduction)**.** Let B be a small set. Let $\Phi_B : \mathbf{Set}/B \to \mathbf{Set}/B$ be a functor. The *reduction* of Φ_B is the functor $\Phi_B^B : \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ defined as follows.

For $(X, u : B \to X, p : X \to B) \in \mathbf{Set}_B^B$, let $\Phi_B^B(X)$ be the set over and under B obtained from $\Phi_B(X)$ by collapsing with respect to $\Phi_B(u) : \Phi_B(B) \to \Phi_B(X)$.

Definition 4.2.17. Extend $\vee_B$ to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$.

Definition 4.2.18. Extend $\wedge_B$ to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$.

Definition 4.2.19 (Faithful)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* iff, for any objects $A, B \in \mathcal{C}$ and morphisms $f, g : A \to B : \mathcal{C}$, if $Ff = Fg$ then $f = g$.

Definition 4.2.20 (Full)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* iff, for any objects $A, B \in \mathcal{C}$ and morphism $g : FA \to FB : \mathcal{D}$, there exists $f : A \to B : \mathcal{C}$ such that $Ff = g$.

Definition 4.2.21 (Fully Faithful)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* iff it is full and faithful.

Definition 4.2.22 (Full Embedding)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is a *full embedding* iff it is fully faithful and injective on objects.

4.3 Natural Transformations

Definition 4.3.1 (Natural Transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\tau : F \Rightarrow G$ is a family of morphisms $\{\tau_X : FX \to GX\}_{X \in \mathcal{C}}$ such that, for every morphism $f : X \to Y : \mathcal{C}$, we have $Gf \circ \tau_X = \tau_Y \circ Ff$.

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\tau_X"'] & FY \arrow[d, "\tau_Y"] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}$$

Definition 4.3.2 (Natural Isomorphism)**.** A natural transformation $\tau : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ is a *natural isomorphism*, $\tau : F \cong G$, iff for all $X \in \mathcal{C}$, τ_X is an isomorphism $FX \cong GX$.

Functors F and G are *naturally isomorphic*, $F \cong G$, iff there exists a natural isomorphism between them.

Definition 4.3.3 (Inverse)**.** Let $\tau : F \cong G$. The *inverse* natural isomorphism $\tau^{-1} : G \cong F$ is defined by $(\tau^{-1})_X = {\tau_X}^{-1}$.

4.4 Bifunctors

Definition 4.4.1 (Commutative)**.** A bifunctor $\Box : \mathcal{C}^2 \to \mathcal{C}$ is *commutative* iff $\Box \cong \Box \circ T$, where $T : \mathcal{C}^2 \to \mathcal{C}^2$ is the swap functor.

Proposition 4.4.2. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is commutative.*

Proof: Since the pushout of f and g is the pushout of g and f. $\Box$

Proposition 4.4.3. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is commutative.*

Proof: In the diagram defining $X \wedge Y$, construct the isomorphism between the version with X and Y and the version with Y with X for every object. $\Box$

Proposition 4.4.4. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is commutative.*

Proposition 4.4.5. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is commutative.*

Definition 4.4.6 (Associative)**.** A bifunctor $\Box$ is *associative* iff $\Box \circ (\Box \times \mathrm{id}) \cong \Box \circ (\mathrm{id} \times \Box)$.

Proposition 4.4.7. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is associative.*

Proof: Since $X \vee (Y \vee Z)$ and $(X \vee Y) \vee Z$ are both the pushout of the unique morphisms $1 \to X$, $1 \to Y$ and $1 \to Z$. $\Box$

Proposition 4.4.8. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is associative.*

Proof: Draw isomorphisms between the diagrams for $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$. $\Box$

Product and coproduct are commutative and associative.

Proposition 4.4.9. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is associative.*

Proposition 4.4.10. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is associative.*

Proposition 4.4.11. *Let $\mathcal{C}$ be a category with binary coproducts. Let $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a bifunctor. Then $\Box$* distributes *over $+$ iff the canonical morphism*

$$(X \Box Z) + (Y \Box Z) \to (X + Y) \Box Z$$

is an isomorphism for all X, Y, Z.

Proposition 4.4.12. *In a category with binary products and binary coproducts, then $\times$ distributes over $+$.*

Proposition 4.4.13. *In* $\mathbf{Set}/*$*, we have $\times$ does not distribute over $\vee$.*

Proposition 4.4.14. *In* $\mathbf{Set}/*$*, we have $\wedge$ distributes over $\vee$.*

Proposition 4.4.15. *In* $\mathbf{Set}/B$*, we have $\times_B$ distributes over $+_B$.*

Proposition 4.4.16. *In* $\mathbf{Set}/B^B$*, we have $\wedge_B$ distributes over $\vee_B$.*

4.5 Functor Categories

Definition 4.5.1 (Functor Category)**.** Given categories $\mathcal{C}$ and $\mathcal{D}$, define the *functor category* $\mathcal{C}^{\mathcal{D}}$ to be the category with objects the functors from $\mathcal{D}$ to $\mathcal{C}$ and morphisms the natural transformations.

Definition 4.5.2 (Yoneda Embedding)**.** Let $\mathcal{C}$ be a category. The *Yoneda embedding* $Y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the functor that maps an object A to $\mathcal{C}[-, A]$ and morphisms similarly.

Theorem 4.5.3 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}[\mathcal{C}[-, X], F] \cong FX$$

that maps $\tau : \mathcal{C}[-, X] \Rightarrow F$ to $\tau_X(\mathrm{id}_X)$.

Proof:
⟨1⟩1. ϕ is natural in X.
  Proof:
  ⟨2⟩1. Let: $f : X \to Y : \mathcal{C}$
  ⟨2⟩2. Let: $\tau : \mathcal{C}[-, X] \Rightarrow F$
  ⟨2⟩3. $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$
    Proof:

$$\begin{aligned} \phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\mathrm{id}_Y \circ f) \\ &= \tau_Y(f) \\ &= \tau_Y(f \circ \mathrm{id}_X) \\ &= Ff(\tau_X(\mathrm{id}_X)) && (\tau \text{ natural}) \\ &= Ff(\phi(\tau)) \end{aligned}$$

⟨1⟩2. ϕ is natural in F.
  ⟨2⟩1. Let: $\alpha : F \Rightarrow G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$
  ⟨2⟩2. Let: $\tau : \mathcal{C}[-, X] \Rightarrow F$
  ⟨2⟩3. $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$
    Proof: $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\mathrm{id}_X)) = \alpha_X(\phi(\tau))$
⟨1⟩3. Each ϕ_{XF} is injective.
  ⟨2⟩1. Let: $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$
  ⟨2⟩2. Assume: $\phi(\sigma) = \phi(\tau)$

⟨2⟩3. LET: $f : Y \to X$

⟨2⟩4. $\sigma_Y(f) = \tau_Y(f)$

PROOF:

$$
\begin{aligned}
\sigma_Y(f) &= \sigma_Y(\mathrm{id}_X \circ f) \\
&= Ff(\sigma_X(\mathrm{id}_X)) && (\sigma \text{ is natural}) \\
&= Ff(\tau_X(\mathrm{id}_X)) && (\langle 2\rangle 2) \\
&= \tau_Y(\mathrm{id}_X \circ f) && (\tau \text{ is natural}) \\
&= \tau_Y(f)
\end{aligned}
$$

⟨1⟩4. Each ϕ_{XF} is surjective.

⟨2⟩1. LET: $X \in \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$

⟨2⟩2. LET: $a \in FX$

⟨2⟩3. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$ be given by $\tau_Y(g) = Fg(a)$ for $g : Y \to X$

⟨2⟩4. τ is natural.

⟨3⟩1. LET: $h : Y \to Z : \mathcal{C}$

PROVE: $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \mathrm{id}_X]$

⟨3⟩2. LET: $g : Z \to X$

⟨3⟩3. $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$

PROOF:

$$
\begin{aligned}
\tau_Y(g \circ h) &= F(g \circ h)(a) \\
&= Fh(Fg(a)) \\
&= Fh(\tau_Z(g))
\end{aligned}
$$

⟨2⟩5. $\phi(\tau) = a$

PROOF:

$$
\begin{aligned}
\phi_X(\tau) &= \tau_X(\mathrm{id}_X) \\
&= F\mathrm{id}_X(a) \\
&= a
\end{aligned}
$$

∎

Corollary 4.5.3.1. *The Yoneda embedding is fully faithful.*

Corollary 4.5.3.2. *Given objects A and B in $\mathcal{C}$, we have $A \cong B$ if and only if $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$.*

Chapter 5

Monoid Theory

Definition 5.0.1 (Monoid)**.** A *monoid* is a category with one object.

Definition 5.0.2. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. The monoid $\mathrm{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \to X$ under composition.

Proposition 5.0.3. *For any functor $F : \mathcal{C} \to \mathcal{D}$ and $X \in \mathcal{C}$, we have that $F : \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{D}}(FX)$ is a monoid homomorphism.*

PROOF: Since $F\mathrm{id}_X = \mathrm{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$. $\square$

Chapter 6

Group Theory

Definition 6.0.1. Let **Grp** be the category of small groups and group homomorphisms.

Definition 6.0.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G.

Proposition 6.0.3. *The trivial group is a zero object in* **Grp**.

PROOF: Easy. $\square$

The zero morphism $G \to H$ maps every element in G to e.

Definition 6.0.4. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. We write $\mathrm{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Proposition 6.0.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $X \in \mathcal{C}$. Then $F : \mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{D}}(FX)$ is a group homomorphism.*

PROOF: Since $F\mathrm{id}_X = \mathrm{id}_{FX}$, $F(g \circ f) = Fg \circ Ff$, and $Ff^{-1} = (Ff)^{-1}$. $\square$

Proposition 6.0.6. **Grp** *has products.*

Definition 6.0.7 (Free Product)**.** The product of a family of groups in **Grp** is called the *free product*.

Proposition 6.0.8. **Ab** *has products given by direct sums.*

Chapter 7

Ring Theory

Definition 7.0.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Definition 7.0.2 (Spectrum)**.** Let R be a commutative ring. The *spectrum* of R, $\operatorname{spec} R$, is the set of all prime ideals of R.

Definition 7.0.3 (Zariski Topology)**.** Let R be a commutative ring. The *Zariski topology* on $\operatorname{spec} R$ is the topology where the closed sets are the sets of the form

$$VE := \{p \in \operatorname{spec} R : E \subseteq p\}$$

for any $E \in \mathcal{P}R$.

We prove this is a topology.

PROOF:
⟨1⟩1. LET: $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$
⟨1⟩2. For all $\mathcal{A} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{A} \in \mathcal{C}$
  ⟨2⟩1. LET: $\mathcal{A} \subseteq \mathcal{C}$
  ⟨2⟩2. LET: $E = \bigcup\{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$
    PROVE: $VE = \bigcap \mathcal{A}$
  ⟨2⟩3. For all $p \in \operatorname{spec} R$, if $E \subseteq p$ then $p \in \bigcap \mathcal{A}$
    ⟨3⟩1. LET: $p \in \operatorname{spec} R$
    ⟨3⟩2. ASSUME: $E \subseteq p$
    ⟨3⟩3. LET: $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$
    ⟨3⟩4. $E' \subseteq E$
    ⟨3⟩5. $E' \subseteq p$
    ⟨3⟩6. $p \in VE'$
  ⟨2⟩4. For all $p \in \operatorname{spec} R$, if $p \in \bigcap \mathcal{A}$ then $E \subseteq p$
    ⟨3⟩1. LET: $p \in \bigcap \mathcal{A}$
    ⟨3⟩2. For all $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$ we have $E' \subseteq p$
    ⟨3⟩3. $E \subseteq p$
⟨1⟩3. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.
  PROOF: Since $VE \cup VE' = V(E \cap E')$

$\langle 1 \rangle 4.$ $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 1.$ $VR = \emptyset$

PROOF: If $p \in VR$ then $R \subseteq p$ contradicting the fact that p is a prime ideal.

□

Definition 7.0.4. For any ring R, let $R - \mathbf{Mod}$ be the category of small R-modules and R-module homomorphisms.

Proposition 7.0.5. $R - \mathbf{Mod}$ *has products and coproducts.*

Chapter 8

Field Theory

Proposition 8.0.1. **Field** *does not have binary products.*

PROOF: There cannot be a field K with field homomorphisms $K \to \mathbb{Z}_2$ and $K \to \mathbb{Z}_3$, because its characteristic would be both 2 and 3. □

Chapter 9

Linear Algebra

Definition 9.0.1. For any field K, we write $\mathbf{Vect}_K$ for $K - \mathbf{Mod}$.

Dual space functor $\mathbf{Vect}_K^{\mathrm{op}} \to \mathbf{Vect}_K$.

Chapter 10

Topology

10.1 Topological Spaces

Definition 10.1.1 (Topological Space)**.** Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say $(X, \mathcal{O})$ is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call $\mathcal{O}$ the *topology* of the toplogical space, and call its elements *open* sets. We shall often write X for the topological space $(X, \mathcal{O})$.

Definition 10.1.2 (Discrete Topology)**.** For any set X, the power set $\mathcal{P}X$ is called the *discrete* topology on X.

Proposition 10.1.3. *For any set X, the discrete topology on X is a topology on X.*

Definition 10.1.4 (Indiscrete Topology)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Proposition 10.1.5. *For any set X, the indiscrete topology on X is a topology on X.*

Definition 10.1.6 (Cofinite Topology)**.** For any set X, the *cofinite* topology is $\{X - U : U \subseteq X \text{ is finite}\}$.

Definition 10.1.7 (Cocountable Topology)**.** For any set X, the *cocountable* topology is $\{X - U : U \subseteq X \text{ is countable}\}$.

Definition 10.1.8 (Sierpiński Two-Point Space)**.** The *Sierpiński two-point space* is $\{0, 1\}$ under the topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Definition 10.1.9 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 10.1.10. *A set B is open if and only if $X - B$ is closed.*

Proposition 10.1.11. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that $\mathcal{C}$ is the set of closed sets if and only if:*

- *For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$*
- *For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*
- $\emptyset \in \mathcal{C}$

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Theorem 10.1.12. *There are infinitely many primes.*

Furstenberg's proof:
PROOF:
$\langle 1 \rangle 1.$ For $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$,
LET: $S(a, b) := \{an + b : n \in \mathbb{N}\}$
$\langle 1 \rangle 2.$ LET: $\mathcal{T}$ be the topology generated by the basis $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$
$\langle 2 \rangle 1.$ For every $n \in \mathbb{Z}$, there exist a, b such that $n \in S(a, b)$.
PROOF: $n \in S(n, 0)$
$\langle 2 \rangle 2.$ If $n \in S(a_1, b_1) \cap S(a_2, b_2)$ then there exist a_3, b_3 such that $n \in S(a_3, b_3) \subseteq S_(a_1, b_1) \cap S(a_2, b_2)$
$\langle 3 \rangle 1.$ LET: $d = \mathrm{lcm}(a_1, a_2)$
PROVE: $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$
$\langle 3 \rangle 2.$ LET: $d = a_1 k = a_2 l$
$\langle 3 \rangle 3.$ LET: $n = a_1 c + b_1 = a_2 d + b_2$
$\langle 3 \rangle 4.$ LET: $z \in \mathbb{Z}$
PROVE: $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$
$\langle 3 \rangle 5.$ $dz + n \in S(a_1, b_1)$
PROOF:

$$\begin{aligned} dz + n &= a_1 kz + a_1 c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

$\langle 3 \rangle 6.$ $dz + n \in S(a_2, b_2)$
PROOF: Similar.
$\langle 1 \rangle 3.$ For all $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$ we have $S(a, b)$ is closed.
$\langle 2 \rangle 1.$ LET: $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$
$\langle 2 \rangle 2.$ LET: $n \in \mathbb{Z} - S(a, b)$
$\langle 2 \rangle 3.$ $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$
$\langle 3 \rangle 1.$ LET: $x \in S(a, n)$
$\langle 3 \rangle 2.$ ASSUME: for a contradiction $x \in S(a, b)$
$\langle 3 \rangle 3.$ PICK m such that $x = am + b$
$\langle 3 \rangle 4.$ PICK l such that $x = al + n$
$\langle 3 \rangle 5.$ $n = a(m - l) + b$

$\langle 3\rangle 6.$ $n \in S(a,b)$
$\langle 3\rangle 7.$ Q.E.D.
PROOF: This contradicts $\langle 2\rangle 2$.
$\langle 1\rangle 4.$

$$\mathbb{Z} - \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$$

PROOF: Since every integer except 1 and -1 is divisible by a prime.
$\langle 1\rangle 5.$ No nonempty finite set is open.
$\langle 2\rangle 1.$ LET: U be a nonempty open set
$\langle 2\rangle 2.$ PICK $n \in U$
$\langle 2\rangle 3.$ There exist a, b such that $n \in S(a,b) \subseteq U$
$\langle 2\rangle 4.$ U is infinite.
$\langle 1\rangle 6.$ $\mathbb{Z} - \{1, -1\}$ is not closed.
$\langle 1\rangle 7.$ $\bigcup_{p \text{ prime}} S(p, 0)$ is not closed.
$\langle 1\rangle 8.$ The union of finitely many closed sets is closed.
$\langle 1\rangle 9.$ There are infinitely many primes.
$\square$

Definition 10.1.13 (Neighbourhood)**.** Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x, and x is an *interior* point of U, iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 10.1.14. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 10.1.15. *Let X be a set and $\mathcal{N} : X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that, for all $x \in X$, we have $\mathcal{N}_x$ is the set of neighbourhoods of x, if and only if:*

- *For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$*
- *For all $x \in X$ we have $X \in \mathcal{N}_x$*
- *For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$*
- *For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$*
- *For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M. M \in \mathcal{N}_y$.*

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$.

Definition 10.1.16 (Exterior Point)**.** Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x.

Definition 10.1.17 (Boundary Point)**.** Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B.

Definition 10.1.18 (Interior)**.** Let X be a topological space and $B \subseteq X$. The *interior* of B, B°, is the set of all interior points of B.

Proposition 10.1.19. *The interior of B is the union of all the open sets included in B.*

Definition 10.1.20 (Closure)**.** Let X be a topological space and $B \subseteq X$. The *closure* of B, $\overline{B}$, is the set of all points that are not exterior points of B.

Proposition 10.1.21. *The closure of B is the intersection of all the closed sets that include B.*

Proposition 10.1.22. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 10.1.23 (Kuratowski Closure Axioms)**.** *Let X be a set and $\overline{} : \mathcal{P}X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ such that, for all $B \subseteq X$, $\overline{B}$ is the closure of B, if and only if:*

- $\overline{\emptyset} = \emptyset$
- *For all $A \subseteq X$ we have $A \subseteq \overline{A}$*
- *For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$*
- *For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

In this case, $\mathcal{O}$ is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

Definition 10.1.24 (Finer, Coarser)**.** Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the set X. Then $\mathcal{T}$ is *coarser*, *smaller* or *weaker* than $\mathcal{T}'$, or $\mathcal{T}'$ is *finer*, *larger* or *weaker* than $\mathcal{T}$, iff $\mathcal{T} \subseteq \mathcal{T}'$.

10.1.1 Subspaces

Definition 10.1.25 (Subspace)**.** Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

Example 10.1.26. The *unit sphere* S^2 is $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a subspace of $\mathbb{R}^3$.

Theorem 10.1.27. *Let X be a topological space and (Y, i) a subset of X. Then the subspace topology on Y is the unique topology such that, for every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f : Z \to X$ is continuous.*

Proof:
⟨1⟩1. If we give Y the subspace topology then, for every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f$ is continuous.
  ⟨2⟩1. Given Y the subspace topology.
  ⟨2⟩2. Let: Z be a topological space.
  ⟨2⟩3. Let: $f : Z \to Y$
  ⟨2⟩4. If f is continuous then $i \circ f$ is continuous.
   Proof: Since i is continuous.

⟨2⟩5. If $i \circ f$ is continuous then f is continuous.
⟨3⟩1. Assume: $i \circ f$ is continuous.
⟨3⟩2. Let: U be open in Y.
⟨3⟩3. $f^{-1}(i^{-1}(i(U))$ is open in Z.
⟨3⟩4. $f^{-1}(U)$ is open in Z.
⟨1⟩2. If, for every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f$ is continuous.
⟨2⟩1. Assume: For every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f$ is continuous.
⟨2⟩2. i is continuous.
⟨2⟩3. For every open set U in X, we have $i^{-1}(X)$ is open in Y
⟨2⟩4. Let: Z be the set Y under the subspace topology and $f : Z \to Y$ the identity function.
⟨2⟩5. $i \circ f$ is continuous.
⟨2⟩6. f is continuous.
⟨2⟩7. Every set open in Y is open in Z.
□

10.1.2 Topological Disjoint Union

Definition 10.1.28 (Coproduct Topology)**.** Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The *coproduct topology* on $\coprod_{\alpha \in A} X_\alpha$ is

$$\mathcal{T} = \left\{ \coprod_{\alpha \in A} U_\alpha : \{U_\alpha\}_{\alpha \in A} \text{ is a family with } U_\alpha \text{ open in } X_\alpha \text{ for all } \alpha \right\} .$$

We prove this is a topology.

Proof:
⟨1⟩1. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
Proof:
$$\bigcup_{i \in I} \coprod_{\alpha \in A} U_{i\alpha} = \coprod_{\alpha \in A} \bigcup_{i \in I} U_{i\alpha}$$
⟨1⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
Proof:
$$\coprod_{\alpha \in A} U_\alpha \cap \coprod_{\alpha \in A} V_\alpha = \coprod_{\alpha \in A} (U_\alpha \cap V_\alpha)$$
⟨1⟩3. $\coprod_{\alpha \in A} X_\alpha \in \mathcal{T}$
Proof: Since every X_α is open in X_α.
□

Proposition 10.1.29. *The coproduct topology is the finest topology on $\coprod_{\alpha \in A} X_\alpha$ such that every injection $\kappa_\alpha : X_\alpha \to \coprod_{\alpha \in A} X_\alpha$ is continuous.*

Proof:
⟨1⟩1. Let: $P = \coprod_{\alpha \in A} X_\alpha$
⟨1⟩2. Let: $\mathcal{T}_c$ be the coproduct topology.

⟨1⟩3. LET: $\mathcal{T}$ be any topology on P
⟨1⟩4. For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \to (P, \mathcal{T}_c)$ is continuous.
⟨2⟩1. LET: $\alpha \in A$
⟨2⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with each U_α open in X_α.
⟨2⟩3. For all $\alpha \in A$, we have ${\kappa_\alpha}^{-1}(\coprod_{\alpha \in A} U_\alpha)$ is open in X_α.
PROOF: Since ${\kappa_\alpha}^{-1}(\coprod_{\alpha \in A} U_\alpha) = U_\alpha$.
⟨1⟩5. If, for all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \to (P, \mathcal{T})$ is continuous, then $\mathcal{T} \subseteq \mathcal{T}_c$.
⟨2⟩1. ASSUME: For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \to (P, \mathcal{T})$ is continuous.
⟨2⟩2. LET: $U \in \mathcal{T}$
⟨2⟩3. For all $\alpha \in a$, we have ${\kappa_\alpha}^{-1}(U)$ is open in X_α.
⟨2⟩4. $U = \coprod_{\alpha \in A} {\kappa_\alpha}^{-1}(U) \in \mathcal{T}_c$
□

Theorem 10.1.30. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The coproduct topology is the unique topology on $\coprod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : \coprod_{\alpha \in A} X_\alpha \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.*

PROOF:
⟨1⟩1. LET: $X = \coprod_{\alpha \in A} X_\alpha$
⟨1⟩2. LET: $\mathcal{T}_c$ be the coproduct topology.
⟨1⟩3. For every topological space Z and function $f : (X, \mathcal{T}_c) \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
⟨2⟩1. LET: Z be a topological space.
⟨2⟩2. LET: $f : X \to Z$
⟨2⟩3. If f is continuous then $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
PROOF: Because the composite of two continuous functions is continuous.
⟨2⟩4. If $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous then f is continuous.
⟨3⟩1. ASSUME: $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
⟨3⟩2. LET: U be open in Z
⟨3⟩3. For all $\alpha \in A$ we have ${\kappa_\alpha}^{-1}(f^{-1}(U))$ is open in X_α
⟨3⟩4. $f^{-1}(U) = \coprod_{\alpha \in A} {\kappa_\alpha}^{-1}(f^{-1}(U))$
⟨3⟩5. $f^{-1}(U)$ is open in X
⟨1⟩4. For any topology $\mathcal{T}$ on X, if for every topological space Z and function $f : (X, \mathcal{T}) \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous, then $\mathcal{T} = \mathcal{T}_c$.
⟨2⟩1. LET: $\mathcal{T}$ be a topology on X.
⟨2⟩2. ASSUME: For every topological space Z and function $f : (X, \mathcal{T}) \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_c$
⟨3⟩1. For all $\alpha \in A$ we have $\kappa_\alpha : X_\alpha \to (X, \mathcal{T})$ is continuous.
PROOF: From ⟨2⟩1 since id_X is continuous.
⟨3⟩2. $\mathcal{T} \subseteq \mathcal{T}_c$
PROOF: Proposition 10.1.29.
⟨2⟩4. $\mathcal{T}_c \subseteq \mathcal{T}$

⟨3⟩1. LET: $f : (X, \mathcal{T}) \to (X, \mathcal{T}_c)$ be the identity function.
⟨3⟩2. $f \circ \kappa_\alpha$ is continuous for all α.
⟨3⟩3. f is continuous.
PROOF: ⟨2⟩1
⟨3⟩4. $\mathcal{T}_c \subseteq \mathcal{T}$
□

10.1.3 Product Topology

Definition 10.1.31 (Product Topology)**.** Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

Proposition 10.1.32. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The product topology on $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis $\mathcal{B} = \{\prod_{\alpha \in A} U_\alpha :$ for all $\alpha \in A, U_\alpha$ is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A\}$.*

PROOF:
⟨1⟩1. $\mathcal{B}$ is a basis for a topology.
⟨1⟩2. LET: $\mathcal{T}$ be the topology generated by $\mathcal{B}$.
⟨1⟩3. LET: $\mathcal{T}_p$ be the product topology.
⟨1⟩4. $\mathcal{T} \subseteq \mathcal{T}_p$
⟨2⟩1. LET: $B \in \mathcal{B}$
⟨2⟩2. LET: $B = \prod_{\alpha \in A} U_\alpha$ with each U_α open in X_α and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$
⟨2⟩3. $B = {\pi_{\alpha_1}}^{-1}(U_{\alpha_1}) \cap \cdots \cap {\pi_{\alpha_n}}^{-1}(U_{\alpha_n})$
⟨2⟩4. $B \in \mathcal{T}_p$
⟨1⟩5. $\mathcal{T}_p \subseteq \mathcal{T}$
⟨2⟩1. For every $\alpha \in A$ we have π_α is continuous.
PROOF: Since $\pi^{-1}(U)$ is open for every U open in X_α.
□

Theorem 10.1.33. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Then the product topology on $\prod_{\alpha \in A} X_\alpha$ is the unique topology such that, for every topological space Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f : Z \to X_\alpha$ is continuous.*

PROOF:
⟨1⟩1. If we give $\prod_{\alpha \in A} X_\alpha$ the product topology, then for every topological space Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
⟨2⟩1. Give $\prod_{\alpha \in A} X_\alpha$ the product topology.
⟨2⟩2. LET: Z be a topological space.
⟨2⟩3. LET: $f : Z \to \prod_{\alpha \in A} X_\alpha$
⟨2⟩4. If f is continuous then, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
PROOF: Since the composite of two continuous functions is continuous.
⟨2⟩5. If, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then f is continuous.

⟨3⟩1. ASSUME: For all $\alpha \in A$ we have $\pi_\alpha \circ f$ is continuous.
⟨3⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with U_α open in X_α such that $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \ldots, \alpha_n$.
⟨3⟩3. For all α we have $f^{-1}({\pi_\alpha}^{-1}(U_\alpha))$ is open in Z.
⟨3⟩4. $f^{-1}(\prod_\alpha U_\alpha)$ is open in Z
PROOF: Since $f^{-1}(\prod_\alpha U_\alpha) = f^{-1}({\pi_{\alpha_1}}^{-1}(U_{\alpha_1})) \cap \cdots \cap f^{-1}({\pi_{\alpha_n}}^{-1}(U_{\alpha_n}))$.
⟨1⟩2. If $\mathcal{T}$ is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological pace Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then $\mathcal{T}$ is the product topology.
⟨2⟩1. ASSUME: $\mathcal{T}$ is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological pace Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
⟨2⟩2. LET: $\mathcal{T}_p$ be the product topology.
⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_p$
⟨3⟩1. LET: $Z = (\prod_\alpha X_\alpha, \mathcal{T}_p)$
⟨3⟩2. LET: $f : Z \to \prod_\alpha X_\alpha$ be the identity function
⟨3⟩3. For all α we have $\pi_\alpha \circ f$ is continuous.
⟨3⟩4. f is continuous.
PROOF: ⟨2⟩1
⟨3⟩5. Every set open in $\mathcal{T}$ is open in $\mathcal{T}_p$
⟨2⟩4. $\mathcal{T}_p \subseteq \mathcal{T}$
⟨3⟩1. $\mathrm{id}_{\prod_\alpha X_\alpha}$ is continuous.
⟨3⟩2. For all α we have π_α is continuous.
PROOF: ⟨2⟩1
⟨3⟩3. $\mathcal{T}_p \subseteq \mathcal{T}$
PROOF: Since $\mathcal{T}_p$ is the coarsest topology such that every π_α is continuous.

□

Example 10.1.34. It is not true that, for any function $f : \prod_{\alpha \in A} X_\alpha \to Y$, if f is continuous in every variable separately then f is continuous.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

Then f is continuous in x and in y, but is not continuous.

Proposition 10.1.35. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology on $\prod_{i \in I} Y_i$ as a subspace of $\prod_{i \in I} X_i$.*

PROOF:
⟨1⟩1. Given $\prod_{i \in I} Y_i$ the subspace topology.
⟨1⟩2. LET: $\iota : \prod_{i \in I} Y_i$ be the inclusion.
⟨1⟩3. LET: Z be any topological space.
⟨1⟩4. LET: $f : Z \to \prod_{i \in I} Y_i$

⟨1⟩5. f is continuous if and only if, for all $i \in I$, we have $\pi_i \circ f$ is continuous.

PROOF:

$$
\begin{aligned}
f \text{ is continuous} &\Leftrightarrow \iota \circ f : Z \to \prod_{i \in I} X_i \text{ is continuous} && \text{(Theorem 10.1.27)} \\
&\Leftrightarrow \forall i \in I.\pi_i \circ \iota \circ f : Z \to X_i \text{ is continuous} && \text{(Theorem 10.1.33)} \\
&\Leftrightarrow \forall i \in I.\iota_i \circ \pi_i \circ f : Z \to X_i \text{ is continuous} \\
&\Leftrightarrow \forall i \in I.\pi_i \circ f : Z \to Y_i \text{ is continuous} && \text{(Theorem 10.1.27)}
\end{aligned}
$$

where ι_i is the inclusion $Y_i \to X_i$.

□

10.1.4 Bases

Definition 10.1.36 (Basis)**.** Let X be a topological space. A *basis* for the topology on X is a set of open sets $\mathcal{B}$ such that every open set is the union of a subset of $\mathcal{B}$. The elements of $\mathcal{B}$ are called *basic open neighbourhoods* of their elements.

Definition 10.1.37 (Order Topology)**.** Let X be a linearly ordered set. The *order topology* on X is the topology generated by the open interval (a, b) as well as the open rays $(a, +\infty)$ and $(-\infty, b)$ for $a, b \in X$.

Definition 10.1.38 (Lower Limit Topology)**.** The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology generated by the basis consisting of all half-open intervals $[a, b)$.

Proposition 10.1.39. *Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a basis for a topology on X if and only if:*

1. $\bigcup \mathcal{B} = X$
2. *For all $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.*

In this case, the topology is unique and is the set of all unions of subsets of $\mathcal{B}$. We call it the topology generated *by $\mathcal{B}$.*

10.1.5 Subbases

Definition 10.1.40 (Subbasis)**.** Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of $\mathcal{S}$.

Definition 10.1.41 (Space with Basepoint)**.** A *space with basepoint* is a pair (X, x) where X is a topological space and $x : \mathrm{El}(X)$.

10.1.6 Countability Axioms

Definition 10.1.42 (Neighbourhood Basis)**.** Let X be a topological space and $x_0 : \mathrm{El}(X)$. A *neighbourhood basis* of x_0 is a set $\mathcal{U}$ of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of $\mathcal{U}$.

Definition 10.1.43 (First Countable)**.** A topological space is *first countable* iff every point has a countable neighbourhood basis.

Definition 10.1.44 (Second Countable)**.** A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.
A subspace of a first countable space is first countable.
A subspace of a second countable space is second countable.
$\mathbb{R}^n$ is second countable.
An uncountable discrete space is first countable but not second countable.

Proposition 10.1.45. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.*

Proof:
⟨1⟩1. For all $\lambda : \mathrm{El}(\Lambda)$, Pick U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.
⟨1⟩2. For all $\lambda : \mathrm{El}(\lambda)$, Pick $x_\lambda \in U_\lambda$.
⟨1⟩3. Assume: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda \in \Lambda}$.
⟨1⟩4. Pick $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$
⟨1⟩5. There is no $U \in \lambda$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$
⟨1⟩6. Q.E.D.
Proof: This is a contradiction.
□

10.2 Continuous Functions

Definition 10.2.1 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* iff, for every open set V in Y, the inverse image $f^{-1}(V)$ is open in X.

Proposition 10.2.2.

1. id_X *is continuous*
2. *The composite of two continuous functions is continuous.*
3. *If $f : X \to Y$ is continuous and $X_0 \subseteq X$ then $f \restriction X_0 : X_0 \to Y$ is continuous.*
4. *If $f : X + Y \to Z$, then f is continuous iff $f \circ \kappa_1 : X \to Z$ and $f \circ \kappa_2 : Y \to Z$ are continuous.*
5. *If $f : Z \to X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.*

Proposition 10.2.3. *Let X and Y be topological spaces. Let $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

Proof:

⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
  PROOF: Since every element of $\mathcal{B}$ is open in Y.
⟨1⟩2. If, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X, then f is continuous.
  ⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
  ⟨2⟩2. LET: U be open in Y.
  ⟨2⟩3. LET: $x \in f^{-1}(U)$
  ⟨2⟩4. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq U$.
  ⟨2⟩5. $x \in f^{-1}(B) \subseteq f^{-1}(U)$
□

Definition 10.2.4 (Homeomorphism)**.** Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Definition 10.2.5 (Retraction)**.** Let X be a topological space and A a subspace of X. A continuous function $\rho : X \to A$ is a *retraction* iff $\rho \restriction A = \mathrm{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 10.2.6. Let **Top** be the category of small topological spaces and continuous functions.

Proposition 10.2.7. $\emptyset$ *is initial in* **Top**.

Proposition 10.2.8. 1 *is terminal in* **Top**.

Forgetful functor **Top** → **Set**.
Basepoint preserving continuous functor.

Proposition 10.2.9. *Let* $(X, \mathcal{T})$ *be a topological space. Let* S *be the Sierpiński two-point space. Define* $\Phi : \mathcal{T} \to \mathbf{Top}[X, S]$ *by* $\Phi(U)(x) = 1$ *iff* $x \in U$*. Then* Φ *is a bijection.*

PROOF:
⟨1⟩1. For all $U \in \mathcal{T}$ we have $\Phi(U)$ is continuous.
  ⟨2⟩1. LET: $U \in \mathcal{T}$
  ⟨2⟩2. $\Phi(U)(\{1\})$ is open.
    PROOF: Since $\Phi(U)(\{1\}) = U$.
⟨1⟩2. Φ is injective.
  PROOF: If $\Phi(U) = \Phi(V)$ then we have $\forall x (x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$.
⟨1⟩3. Φ is surjective.
  PROOF: Given $f : X \to S$ continuous we have $\Phi(f^{-1}(1)) = f$.
□

10.2.1 Paths

Definition 10.2.10 (Path)**.** A *path* in a topological space X is a continuous function $[0, 1] \to X$.

10.2.2 Loops

Definition 10.2.11 (Loop)**.** A *loop* in a topological space X is a path $\alpha : [0,1] \to X$ such that $\alpha(0) = \alpha(1)$.

10.3 Convergence

Definition 10.3.1 (Convergence)**.** Let X be a topological space. Let (x_n) be a sequence in X. A point $a : \mathrm{El}(X)$ is a *limit* of the sequence iff, for every neighbourhood U of a, there exists n_0 such that $\forall n \geqslant n_0 . x_n \in U$.

Convergence in a product space is pointwise convergence.

If $f : X \to Y$ is continuous and $x_n \to l$ in X then $f(x_n) \to f(l)$ in Y.

Example 10.3.2. The converse does not hold.

Let X be the set of all continuous functions $[0,1] \to [-1,1]$ under the product topology. Let $i : X \to L^2([0,1])$ be the inclusion.

If $f_n \to f$ then $i(f_n) \to i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K_\epsilon . \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0,1]} U_\lambda$ where $U_\lambda = [-1,1]$ except for $\lambda = \lambda_1, \ldots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \ldots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 10.3.3. *The converse does hold for first countable spaces. If $f : X \to Y$ where X is first countable, and Y is a topological space, and whenever $x_n \to x$ then $f(x_n) \to f(x)$, then f is continuous.*

10.4 Subspaces

Definition 10.4.1 (Subspace)**.** Let X be a topological space, Y a set, and $f : Y \to X$. The *subspace topology* on Y induced by f is $\mathcal{T} = \{i^{-1}(U) : U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
  PROOF: Since $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$.
$\langle 1 \rangle 2.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  PROOF: Since $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.
$\langle 1 \rangle 3.$ $Y \in \mathcal{T}$
  PROOF: Since $Y = f^{-1}(X)$.
□

Proposition 10.4.2. *Let X be a topological space, Y a set and $f : Y \to X$ a function. Then the subspace topology on Y is the coarsest topology such that f is continuous.*

PROOF: Immediate from definition. □

10.5 Embedding

Definition 10.5.1 (Embedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *embedding* iff f is injective and the topology on X is the subspace induced by f.

10.6 Quotient Spaces

Definition 10.6.1 (Quotient Topology)**.** Let X be a topological space, S a set, and $\pi : X \twoheadrightarrow S$ be a surjection. The *quotient topology* on S induced by π is $\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
PROOF: Since $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}$.
$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.
PROOF: Since $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$.
$\langle 1 \rangle 3$. $X \in \mathcal{T}$
PROOF: Since $X = \pi^{-1}(Y)$.
□

Proposition 10.6.2. *Let X be a topological space, S a set and $\pi : X \twoheadrightarrow S$ a surjection. Then the quotient topology on S is the finest topology such that π is continuous.*

PROOF: Immediate from definitions. □

Definition 10.6.3 (Quotient Map)**.** Let X and S be topological spaces and $\pi : X \to S$. Then π is a *quotient map* iff π is surjective and the topology on S is the quotient topology induced by π.

Theorem 10.6.4. *Let X be a topological space, let S be a set, and let $\pi : X \twoheadrightarrow S$ be surjective. Then the quotient topology on S is the unique topology such that, for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.*

PROOF:
$\langle 1 \rangle 1$. If S is given the quotient topology, then for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.
$\langle 2 \rangle 1$. Give S the quotient topology.
$\langle 2 \rangle 2$. LET: Z be a topological space.
$\langle 2 \rangle 3$. LET: $f : S \to Z$
$\langle 2 \rangle 4$. If f is continuous then $f \circ \pi$ is continuous.
PROOF: The composite of two continuous functions is continuous.
$\langle 2 \rangle 5$. If $f \circ \pi$ is continuous then f is continuous.

⟨3⟩1. ASSUME: $f \circ \pi$ is continuous.
⟨3⟩2. LET: U be open in Z.
⟨3⟩3. $\pi^{-1}(f^{-1}(U))$ is open in X.
⟨3⟩4. $f^{-1}(U)$ is open in S.

⟨1⟩2. If S is given a topology such that, for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous, then that topology is the quotient topology.
⟨2⟩1. Give S a topology such that, for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.
⟨2⟩2. LET: $U \subseteq S$
⟨2⟩3. If $\pi^{-1}(U)$ is open in X then U is open in S.
⟨3⟩1. LET: Z be S under the quotient topology induced by π.
⟨3⟩2. LET: $f : S \to Z$ be the identity function.
⟨3⟩3. $f \circ \pi$ is continuous.
⟨3⟩4. f is continuous.
PROOF: ⟨2⟩1
⟨3⟩5. U is open in Z.
⟨3⟩6. U is open in X.
⟨2⟩4. If U is open in S then $\pi^{-1}(U)$ is open in X.
PROOF: Since π is continuous (taking $Z = S$ and $f = \mathrm{id}_S$ in ⟨2⟩1).

□

Corollary 10.6.4.1. *Let $\pi : X \twoheadrightarrow S$ be a quotient map. Let Z be a topological space. Let $f : X \to Z$ be continuous. Then there exists a continuous map $g : S \to Z$ such that $f = g \circ \pi$ if and only if, for all $s \in S$, we have f is constant on $\pi^{-1}(s)$.*

Proposition 10.6.5. *Let Z be a topological space. Define $\pi : [0,1] \to S^1$ by $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$. Given any continuous function $f : S^1 \to Z$, we have $f \circ \pi$ is a loop in Z. This defines a bijection between* $\mathbf{Top}[S^1, Z]$ *and the set of loops in Z.*

PROOF: Since π is a quotient map. □

Definition 10.6.6 (Projective Space)**.** The *projective space* $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by $\sim$ where $x \sim \lambda x$ for all $x \in \mathbb{R}^{n+1} - \{0\}$ and $\lambda \in \mathbb{R}$.

Definition 10.6.7 (Torus)**.** The *torus* T is the quotient of $[0,1]^2$ by $\sim$ where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$.

Definition 10.6.8 (Möbius Band)**.** The *Möbius band* is the quotient of $[0,1]^2$ by $\sim$ where $(0,y) \sim (1,1-y)$.

Definition 10.6.9 (Klein Bottle)**.** The *Klein bottle* is the quotient of $[0,1]^2$ by $\sim$ where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,1-y)$.

Proposition 10.6.10. $\mathbb{RP}^2$ *is the quotient of* $[0,1]^2$ *by* $\sim$ *where* $(x,0) \sim (1-x,1)$ *and* $(0,y) \sim (1,1-y)$.

PROOF:TODO

10.7 Connected Spaces

Definition 10.7.1 (Connected)**.** A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 10.7.2. *The continuous image of a connected space is connected.*

Proposition 10.7.3. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are connected, then X is connected.*

Proposition 10.7.4. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 10.7.5 (Path-connected)**.** A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \to X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 10.7.6. *The continuous image of a path connected space is path connected.*

Proposition 10.7.7. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are path connected, then X is path connected.*

Proposition 10.7.8. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

10.8 Hausdorff Spaces

Definition 10.8.1 (Hausdorff)**.** A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 10.8.2. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 10.8.3. 1. *Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

Proposition 10.8.4. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \to B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X, then $f = g$.*

Proof:
$\langle 1 \rangle 1$. Assume: for a contradiction $a \in A$ and $f(a) \neq g(a)$.
$\langle 1 \rangle 2$. Pick disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.
$\langle 1 \rangle 3$. Pick $x \in f^{-1}(U) \cap g^{-1}(V)$
$\langle 1 \rangle 4$. $f(x) = g(x) \in U \cap V$
$\langle 1 \rangle 5$. Q.E.D.
Proof: This is a contradiction.

∎

Proposition 10.8.5. *Let X and Y be metric spaces. Let $f : X \to Y$ be uniformly continuous. Let $\hat{X}$ and $\hat{Y}$ be the completions of X and Y. Then f extends uniquely to a continuous map $\hat{X} \to \hat{Y}$.*

PROOF: The extension maps $\lim_{n\to\infty} x_n$ to $\lim_{n\to\infty} f(x_n)$. ∎

10.9 Separable Spaces

Definition 10.9.1 (Separable)**.** A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

10.10 Sequential Compactness

Definition 10.10.1 (Sequentially Compact)**.** A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

10.11 Compactness

Definition 10.11.1 (Compact)**.** A topological space is *compact* iff every open cover has a finite subcover.

Proposition 10.11.2. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P. Then $X \in P$.*

PROOF:
⟨1⟩1. P is an open cover of X
⟨1⟩2. PICK a finite subcover $U_1, \ldots, U_n \in P$
⟨1⟩3. $X = U_1 \cup \cdots \cup U_n \in P$
∎

Corollary 10.11.2.1. *Let f be a compact space and $f : X \to \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. ∎

Proposition 10.11.3. *The continuous image of a compact space is compact.*

Proposition 10.11.4. *A closed subspace of a compact space is compact.*

Proposition 10.11.5. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. *X and Y are compact.*

2. *$X + Y$ is compact.*

3. *$X \times Y$ is compact.*

Proposition 10.11.6. *A compact subspace of a Hausdorff space is closed.*

Proposition 10.11.7. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 10.11.8. *A first countable compact space is sequentially compact.*

10.12 Quotient Spaces

Definition 10.12.1 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. The *quotient topology* on $X/\sim$ is defined by: $U : \mathrm{El}\,(\mathcal{P}X)$ is open in $X/\sim$ if and only if $\pi^{-1}(U)$ is open in X.

Proposition 10.12.2. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $f : X/\sim \to Y$. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proposition 10.12.3. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $\phi : Y \to X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \to X$ such that $\pi \circ \Phi = \phi \restriction U$. Then ϕ is continuous.

Proposition 10.12.4. *A quotient of a connected space is connected.*

Proposition 10.12.5. *A quotient of a path connected space is path connected.*

Proposition 10.12.6. *Let X be a topological space and $\sim$ an equivalence relation on X. If $X/\sim$ is Hausdorff then every equivalence class of $\sim$ is closed in X.*

Definition 10.12.7. Let X be a topological space and $A_1, \ldots, A_r \subseteq X$. Then $X/A_1, \ldots, A_r$ is the quotient space of X with respect to $\sim$ where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 10.12.8 (Cone)**.** Let X be a topological space. The *cone over X* is the space $(X \times [0,1])/(X \times \{1\})$.

Definition 10.12.9 (Suspension)**.** Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1,1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 10.12.10 (Wedge Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 10.12.11 (Smash Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 10.12.12. $D^n/S^{n-1} \cong S^n$

PROOF:
⟨1⟩1. LET: $\phi : D^n/S^{n-1} \to S^n$ be the function induced by the map $D^n \to S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.
⟨1⟩2. ϕ is a bijection.
⟨1⟩3. ϕ is a homeomorphism.
PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.
□

10.13 Gluing

Definition 10.13.1 (Gluing)**.** Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \to Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where $\sim$ is the equivalence relation generated by $x \sim \phi(x)$ for all $x : \mathrm{El}(X)$.

Proposition 10.13.2. *Y is a subspace of $Y \cup_\phi X$.*

Definition 10.13.3. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0,1])/\alpha$ is the quotient space of $X \times [0,1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x : \mathrm{El}(X)$.

Definition 10.13.4 (Möbius Strip)**.** The *Möbius strip* is $([-1,1] \times [0,1])/\alpha$ where $\alpha(x) = -x$.

Definition 10.13.5 (Klein Bottle)**.** The *Klein bottle* is $(S^1 \times [0,1])/\alpha$ where $\alpha(z) = \overline{z}$.

Proposition 10.13.6. *Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\mathrm{id}_{\partial M}} M \cong K$.*

PROOF:
⟨1⟩1. LET: $f : ([-1,1] \times [0,1]) + ([-1,1] \times [0,1]) \to S^1 \times [0,1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{\pi i\theta/2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-\pi i\theta/2}, t)$.
⟨1⟩2. f induces a bijection $M \cup_{\mathrm{id}_{\partial M}} M \approx K$
⟨1⟩3. f is a homeomorphism.
□

10.14 Metric Spaces

Definition 10.14.1 (Metric Space)**.** Let X be a set and $d : X^2 \to \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geqslant 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$

- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leqslant d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d). We often write X for the metric space (X, d).

Definition 10.14.2 (Ball)**.** Let X be a metric space. Let $x \in X$ and $r > 0$. The *ball* with *centre* x and *radius* r is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} \ .$$

Definition 10.14.3 (Metric Topology)**.** Let (X, d) be a metric space. The *metric topology* on X is the topology generated by the basis consisting of the balls.

Definition 10.14.4 (Metrizable)**.** A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 10.14.5. *Every metrizable space is Hausdorff.*

Every metrizable space is first countable.

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

10.14.1 Products

Definition 10.14.6 (Euclidean Metric)**.** Let X and Y be metric spaces. The *Euclidean metric* on $X \times Y$ is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} \ .$$

We write $X \times Y$ for the set $X \times Y$ under this metric.

We prove this is a metric.

PROOF:
⟨1⟩1. $d((x_1, y_1), (x_2, y_2)) \geqslant 0$
PROOF: Immediate from definition.
⟨1⟩2. $d((x_1, y_1), (x_2, y_2)) = 0$ iff $(x_1, y_1) = (x_2, y_2)$
PROOF: $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = 0$ iff $d(x_1, x_2) = d(y_1, y_2) = 0$ iff $x_1 = x_2$ and $y_1 = y_2$.
⟨1⟩3. $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$
PROOF: Since $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = \sqrt{d(x_2, x_1)^2 + d(y_2, y_1)^2}$.
⟨1⟩4. The triangle inequality holds.

PROOF:

$(d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)))^2$

$= d((x_1, y_1), (x_2, y_2))^2 + 2d((x_1, y_1), (x_2, y_2))d((x_2, y_2), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3))^2$

$= d(x_1, x_2)^2 + d(y_1, y_2)^2 + 2\sqrt{(d(x_1, x_2)^2 + d(y_1, y_2)^2)(d(x_2, x_3)^2 + d(y_2, y_3)^2)} + d(x_2, x_3)^2 + d(y_2, y_3)^2$

$\geqslant d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(y_1, y_2)^2 + d(y_2, y_3)^{+}2(d(x_1, x_2)d(x_2, x_3) + d(y_1, y_2)d(y_2, y_3))$

(Cauchy-Schwarz)

$= (d(x_1, x_2) + d(x_2, x_3))^2 + (d(y_1, y_2) + d(y_2, y_3))^2$

$\geqslant d(x_1, x_3)^2 + d(y_1, y_3)^2$

$= d((x_1, y_1), (x_3, y_3))^2$

□

Proposition 10.14.7. *Let X and Y be metric spaces. The Euclidean metric on $X \times Y$ induces the product topology on $X \times Y$.*

PROOF:

⟨1⟩1. Every open ball is open in the product topology.

⟨2⟩1. LET: $(x, y) \in B((a, b), \epsilon)$

PROVE: $B(x, \sqrt{\epsilon}) \times B(y, \sqrt{\epsilon}) \subseteq B((a, b), \epsilon)$

⟨2⟩2. LET: $x' \in B(x, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$ and $y' \in B(y, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$

PROVE: $d((x', y'), (a, b)) < \epsilon$

⟨2⟩3. $d((x', y'), (x, y)) < \epsilon - d((x, y), (a, b))$

PROOF:

$$\begin{aligned} d((x', y'), (x, y)) &= \sqrt{d(x', x)^2 + d(y', y)^2} \\ &< \sqrt{(\epsilon - d((x, y), (a, b)))^2/2 + (\epsilon - d((x, y), (a, b))^2/2} \\ &= \epsilon - d((x, y), (a, b)) \end{aligned}$$

⟨2⟩4. $d((x', y'), (a, b)) < \epsilon$

PROOF:

$$\begin{aligned} d((x', y'), (a, b)) &\leqslant d((x', y'), (x, y)) + d((x, y), (a, b)) && \text{(Triangle Inequality)} \\ &< \epsilon && (\langle 2\rangle 3) \end{aligned}$$

⟨1⟩2. If U is open in X and V is open in Y then $U \times V$ is open under the Euclidean metric.

⟨2⟩1. LET: $(x, y) \in U \times V$

⟨2⟩2. PICK $\delta, \epsilon > 0$ such that $B(x, \delta) \subseteq U$ and $B(y, \epsilon) \subseteq V$

PROVE: $(B((x, y), \min(\delta, \epsilon)) \subseteq U \times V$

⟨2⟩3. LET: $(x', y') \in B((x, y), \min(\delta, \epsilon))$

⟨2⟩4. $d(x', x) < \delta$

⟨3⟩1. $d((x', y'), (x, y)) < \min(\delta, \epsilon)$

⟨3⟩2. $d(x', x)^2 + d(y', y)^2 < \delta^2$

⟨3⟩3. $d(x', x)^2 < \delta^2$

⟨2⟩5. $d(y', y) < \epsilon$

PROOF: Similar.

⟨2⟩6. $(x', y') \in U \times V$

□

10.15 Complete Metric Spaces

Definition 10.15.1 (Complete)**.** A metric space is *complete* iff every Cauchy sequence converges.

Example 10.15.2. $\mathbb{R}$ is complete.

Proposition 10.15.3. *The product of two complete metric spaces is complete.*

Proposition 10.15.4. *Every compact metric space is complete.*

Proposition 10.15.5. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 10.15.6 (Completion)**.** Let X be a metric space. A *completion* of X is a complete metric space $\hat{X}$ and injection $i : X \rightarrowtail \hat{X}$ such that:

- The metric on X is the restriction of the metric on $\hat{X}$
- X is dense in $\hat{X}$.

Proposition 10.15.7. *Let $i_1 : X \to Y_1$ and $i_2 : X \to Y_2$ be completions of X. Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

Proof: Define $\phi(\lim_{n\to\infty} i_1(x_n)) = \lim_{n\to\infty} i_2(x_n)$. □

Theorem 10.15.8. *Every metric space has a completion.*

Proof: Let $\hat{X}$ be the set of Cauchy sequences in X quotiented by $\sim$ where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \to 0$. □

10.16 Manifolds

Definition 10.16.1 (Manifold)**.** An *n-dimensional manifold* is a second countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$.

Chapter 11

Homotopy Theory

11.1 Homotopies

Definition 11.1.1 (Homotopy)**.** Let X and Y be topological spaces. Let $f, g : X \to Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0,1] \to Y$ such that

- $\forall x : \mathrm{El}\,(X)\,.h(x, 0) = f(x)$
- $\forall x : \mathrm{El}\,(X)\,.h(x, 1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \to Y$.

Proposition 11.1.2. *Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.*

Definition 11.1.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

A *homotopy functor* is a functor $\mathbf{Top} \to \mathcal{C}$ that factors through the canonical functor $\mathbf{Top} \to \mathbf{HTop}$.

Definition 11.1.4. A functor $F : \mathbf{Top} \to \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \to Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

11.2 Homotopy Equivalence

Definition 11.2.1 (Homotopy Equivalence)**.** Let X and Y be topological spaces. A *homotopy equivalence* between X and Y, $f : X \simeq Y$, is a continuous function $f : X \to Y$ such that there exists a continuous function $g : Y \to X$, the *homotopy inverse* to f, such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Definition 11.2.2 (Contractible)**.** A topological space X is *contractible* iff $X \simeq 1$.

Example 11.2.3. $\mathbb{R}^n$ is contractible.

Example 11.2.4. D^n is contractible.

Definition 11.2.5 (Deformation Retract)**.** Let X be a topological space and A a subspace of X. A retraction $\rho : X \to A$ is a *deformation retraction* iff $i \circ \rho \simeq \mathrm{id}_X$, where i is the inclusion $A \rightarrowtail X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 11.2.6 (Strong Deformation Retract)**.** Let X be a topological space and A a subspace of X. A *strong deformation retraction* $\rho : X \to A$ is a continuous function such that there exists a homotopy $h : X \times [0,1] \to X$ between $i \circ \rho$ and id_X such that, for all $a : \mathrm{El}(X)$ and $t : \mathrm{El}([0,1])$, we have $h(a,t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 11.2.7. $\{0\}$ is a strong deformation retract of $\mathbb{R}^n$ and of D^n.

Example 11.2.8. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 11.2.9. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 11.2.10. For any topological space X, the singleton consisting of the vertex is a strong deformation retract of the cone over X.

Chapter 12

Simplicial Complexes

Definition 12.0.1 (Simplex)**.** A *k-dimensional simplex* or *k-simplex* in $\mathbb{R}^n$ is the convex hull $s(x_0, \ldots, x_k)$ of $k+1$ points in general position.

Definition 12.0.2 (Face)**.** A *sub-simplex* or *face* of $s(x_0, \ldots, x_k)$ is the convex hull of a subset of $\{x_0, \ldots, x_k\}$.

Definition 12.0.3 (Simplicial Complex)**.** A *simplicial complex* in $\mathbb{R}^n$ is a set K of simplices such that:

- for every simplex s in K, every face of s is in K.
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2.
- K is locally finite, i.e. every point of $\mathbb{R}^n$ has a neighbourhood that only intersects finitely many elements of K.

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of $\mathbb{R}^n$.

12.1 Cell Decompositions

Definition 12.1.1 (n-cell)**.** An *n-cell* is a topological space homeomorphic to $\mathbb{R}^n$.

Definition 12.1.2 (Cell Decomposition)**.** Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n-cells.

Definition 12.1.3 (n-skeleton)**.** Given a cell decomposition of X, the *n-skeleton* X^n is the union of all the cells of dimension $\leqslant n$.

12.2 CW-complexes

Definition 12.2.1 (CW-Complex)**.** A *CW-complex* consists of a topological space X and a cell decomposition $\mathcal{E}$ of X such that:

1. *Characteristic Maps* For every n-cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \to X$ such that $\Phi((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in $\mathcal{E}$ of dimension $< n$.

2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have $\overline{e}$ intersects only finitely many other cells in $\mathcal{E}$.

3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \overline{e}$ is closed.

Proposition 12.2.2. *If a cell decomposition $\mathcal{E}$ satisfies the Characteristic Maps axiom, then for every n-cell $e \in \mathcal{E}$ we have $\overline{e} = \Phi_e(D^n)$. Therefore $\overline{e}$ is compact and $\overline{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

PROOF:
⟨1⟩1. $e \subseteq \Phi_e(D^n) \subseteq \overline{e}$
PROOF:

$$\begin{aligned} e &= \Phi_e((D^n)^\circ) \\ &\subseteq \Phi_e(D^n) \\ &= \Phi_e(\overline{(D^n)^\circ}) \\ &\subseteq \overline{\Phi_e((D^n)^\circ)} \\ &= \overline{e} \end{aligned}$$

⟨1⟩2. $\Phi_e(D^n)$ is compact.
PROOF: Because D^n is compact.
⟨1⟩3. $\Phi_e(D^n)$ is closed.
⟨1⟩4. $\Phi_e(D^n) = \overline{e}$
□

Chapter 13

Topological Groups

Definition 13.0.1 (Topological Group)**.** A *topological group* is a group G with a topology such that the function $G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.

Example 13.0.2. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 13.0.3. *Any subgroup of a topological group is a topological group under the subspace topology.*

Definition 13.0.4 (Homogeneous Space)**.** A *homogeneous space* is a topological space of the form G/H, where G is a topological group and H is a normal subgroup of G, under the quotient topology.

Proposition 13.0.5. *Let G be a topological group and H a normal subgroup of G. Then G/H is Hausdorff if and only if H is closed.*

Proof: See Bourbaki, N., General Topology. III.12 ∎

13.1 Continuous Actions

Definition 13.1.1 (Continuous Action)**.** Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \to X$ such that:

- $\forall x : \mathrm{El}(X) . ex = x$
- $\forall g, h : \mathrm{El}(G) . \forall x : \mathrm{El}(X) . g(hx) = (gh)x$

A *G-space* consists of a topological space X and a continuous action of G on X.

Definition 13.1.2 (Orbit)**.** Let X be a G-space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 13.1.3. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:
⟨1⟩1. LET: $f_3 : S^2/SO(2) \to [-1, 1]$ be the function induced by $\pi_3 : S^2 \to [-1, 1]$
⟨1⟩2. f_3 is bijective.
⟨1⟩3. $S^2/SO(2)$ is compact.
PROOF: It is the continuous image of S^2 which is compact.
⟨1⟩4. $[-1, 1]$ is Hausdorff.
⟨1⟩5. f_3 is a homeomorphism.
□

Definition 13.1.4 (Stabilizer)**.** Let X be a G-space and $x \in X$. The *stabilizer* of x is $G_x := \{g : \mathrm{El}\,(G) \mid gx = x\}$.

Proposition 13.1.5. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx.*

PROOF:
⟨1⟩1. If $gG_x = hG_x$ then $gx = hx$.
⟨2⟩1. ASSUME: $gG_x = hG_x$
⟨2⟩2. $g^{-1}h \in G_x$
⟨2⟩3. $g^{-1}hx = x$
⟨2⟩4. $gx = hx$
⟨1⟩2. If $gx = hx$ then $gG_x = hG_x$.
PROOF: Similar.
⟨1⟩3. The function is continuous.
PROOF: Proposition 10.12.2.
□

Chapter 14

Topological Vector Spaces

Definition 14.0.1 (Topological Vector Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Substraction is a continuous function $E^2 \to E$
- Multiplication is a continuous function $K \times E \to E$

Proposition 14.0.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. □

Theorem 14.0.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 □

Proposition 14.0.4. *Let E be a topological vector space and E_0 a subspace of E. Then $\overline{E_0}$ is a subspace of E.*

Definition 14.0.5. Let E be a topological vector space. The topological space *associated* with E is $E/\overline{\{0\}}$.

14.1 Cauchy Sequences

Definition 14.1.1 (Cauchy Sequence)**.** Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geqslant n_0 . x_n - x_m \in U$.

Definition 14.1.2 (Complete Topological Vector Space)**.** A topological vector space is *complete* iff every Cauchy sequence converges.

14.2 Seminorms

Definition 14.2.1 (Seminorm)**.** Let E be a vector space over K. A *seminorm* on E is a function $\| \ \| : E \to \mathbb{R}$ such that:

1. $\forall x : \mathrm{El}\,(E)\,.\|x\| \geqslant 0$
2. $\forall \alpha : \mathrm{El}\,(K)\,.\forall x : \mathrm{El}\,(E)\,.\|\alpha x\| = |\alpha|\|x\|$
3. *Triangle Inequality* $\forall x, y : \mathrm{El}\,(E)\,.\|x + y\| \leqslant \|x\| + \|y\|$

Example 14.2.2. The function that maps $(x_1, \ldots, x_n)$ to $|x_i|$ is a seminorm on $\mathbb{R}^n$.

Definition 14.2.3. Let E be a vector space over K. Let Λ be a set of seminorms on E. The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B^\lambda_\epsilon(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x : \mathrm{El}\,(E)$.

Proposition 14.2.4. *E is a topological vector space under this topology. It is Hausdorff iff, for all $x : \mathrm{El}\,(E)$, if $\forall \lambda \in \Lambda.\lambda(x) = 0$ then $x = 0$.*

14.3 Fréchet Spaces

Definition 14.3.1 (Pre-Fréchet Space)**.** A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 14.3.2. *Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \ \|_n : n \in \mathbb{Z}^+\}$. Then*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 14.3.3 (Fréchet Space)**.** A *Fréchet space* is a complete pre-Fréchet space.

14.4 Normed Spaces

Definition 14.4.1 (Normed Space)**.** Let E be a vector space over K. A *norm* on E is a function $\| \ \| : E \to \mathbb{R}$ is a seminorm such that, $\forall x \in E.\|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 14.4.2. *If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E.*

Definition 14.4.3 (p-norm)**.** For any $p \geqslant 1$, the *p-norm* on $\mathbb{R}^n$ is defined by

$$\|\vec{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} .$$

We prove this is a norm.

PROOF:
⟨1⟩1. For all $\vec{x} \in \mathbb{R}^n$ we have $\|\vec{x}\|_p \geqslant 0$
PROOF: Immediate from definition.
⟨1⟩2. For all $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ we have $\|\alpha\vec{x}\|_p = |\alpha| \|\vec{x}\|_p$
PROOF:

$$\begin{aligned} \|\alpha(x_1, \dots, x_n)\| &= \|(\alpha x_1, \dots, \alpha x_n)\| \\ &= \left(\sum_{i=1}^{n} (\alpha x_i)^p \right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|\vec{x}\|_p \end{aligned}$$

⟨1⟩3. The triangle inequality holds.
PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|_p^p &= \sum_{i=1}^{n} |x_i + y_i|^p \\ &= \sum_{i=1}^{n} |x_i + y_i| |x_i + y_i|^{p-1} \\ &\leqslant \sum_{i=1}^{n} (|x_i| + |y_i|) |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^{n} |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| |x_i + y_i|^{p-1} \\ &\leqslant \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{p-1}{p}} && \text{(Hölder's Inequality)} \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p) \|\vec{x} + \vec{y}\|^{p-1} \end{aligned}$$

Assuming w.l.o.g. $\|\vec{x} + \vec{y}\|^{p-1} \neq 0$ (using **??**) we have $\|\vec{x} + \vec{y}\|_p \leqslant \|\vec{x}\|_p + \|\vec{y}\|_p$.
⟨1⟩4. For any $\vec{x} \in \mathbb{R}^n$, we have $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.
PROOF: $\sum_{i=1}^{n} x_i^p = 0$ iff $x_1 = \cdots = x_n = 0$.
□

Definition 14.4.4 (Sup-norm)**.** The *sup-norm* on $\mathbb{R}^n$ is defined by

$$\|(x_1, \ldots, x_n)\|_\infty := \max(|x_1|, \ldots, |x_n|) \ .$$

Proposition 14.4.5. *The 2-norm on $\mathbb{R}^n$ induces the standard metric.*

Proof: Immediate from definitions. □

Definition 14.4.6. For $p \geqslant 1$, the normed space l_p is the set of all sequences (x_n) in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} x_n^p$ converges, under

$$\|(x_n)\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \ .$$

Proposition 14.4.7. *The spaces l_p for $p \geqslant 1$ are all homeomorphic.*

Proof: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. http://dx.doi.org/10.1007/BF01075865.

Definition 14.4.8. Let l_∞ be the set of all bounded sequences in $\mathbb{R}$ under

$$\|(x_n)\| := \sup_n |x_n|$$

Proposition 14.4.9. *For all $p \geqslant 1$ we have l_p is not homeomorphic to l_∞.*

Proposition 14.4.10. *Let $\| \ \|$ be a seminorm on the vector space E. Then $\| \ \|$ defines a norm on $E/\overline{\{0\}}$.*

Proposition 14.4.11. *Let E and F be normed spaces. Any continuous linear map $E \to F$ is uniformly continuous.*

Definition 14.4.12. For $p \geqslant 1$. let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt{p} \int_{\mathbb{R}^n} |f(x)|^p dx$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\overline{\{0\}} \ .$$

14.5 Inner Product Spaces

Proposition 14.5.1. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E.*

14.6 Banach Spaces

Definition 14.6.1 (Banach Space)**.** A *Banach space* is a complete normed space.

Example 14.6.2. For any topological space X, the set $C(X)$ of bounded continuous functions $X \to \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 14.6.3. *The completion of a normed space is a Banach space.*

Proposition 14.6.4. *Let E and F be normed spaces. Let $f : E \to F$ be a continuous linear map. Then the extension to the completions $\hat{E} \to \hat{F}$ is linear.*

Proposition 14.6.5. $L^p(\mathbb{R}^n)$ *is a Banach space.*

Proposition 14.6.6. $C(\mathbb{R})$ *is first countable but not second countable.*

PROOF: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n. Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.
It is first countable because it is metrizable. □

14.7 Hilbert Spaces

Definition 14.7.1 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Example 14.7.2. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \to \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi} \pi f(x) g(x) dx \ .$$

Proposition 14.7.3. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

14.8 Locally Convex Spaces

Definition 14.8.1 (Locally Convex Space)**.** A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 14.8.2. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Proposition 14.8.3. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Example 14.8.4. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \to \mathbb{R}$ is continuous as a map $E' \to \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 14.8.5 (Thom Space)**.** Let E be a vector bundle with a Riemannian metric, $DE = \{x : \mathrm{El}\,(E) \mid \|x\| \leqslant 1\}$ its disc bundle and $SE := \{v : \mathrm{El}\,(E) \mid \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE.