Mathematics

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Chapter 1

Primitive Terms and Axioms

Let there be *sets*.

For any set A, let there be *elements* of A. We write $a \in A$ for: a is an element of A.

For any sets A and B, let there be *relations* between A and B. We write $R : A \looparrowright B$ for: R is a relation between A and B.

For any set A, let there be a relation of *equality* $=$ between the elements of A.

For any sets A and B, relation $R : A \looparrowright B$, and elements $a \in A$ and $b \in B$, let there be a proposition aRb.

Definition 1.1 (Equality of Relations)**.** We say relations $R, S : A \looparrowright B$ are *equal*, $R = S$, iff $\forall x \in A.\forall y \in B.(xRy \Leftrightarrow xSy)$.

Axiom Schema 1.2 (Comprehension)**.** *For any property $\phi[X, Y, x, y]$ where $x \in X$ and $y \in Y$, the following is an axiom:*

For any sets A and B, there exists a relation $R : A \looparrowright B$ such that

$$\forall x \in A.\forall y \in B.xRy \Leftrightarrow \phi[A, B, x, y] \ .$$

Definition 1.3 (Function)**.** Let A and B be sets and $F : A \looparrowright B$. Then we say F is a *function*, and write $F : A \to B$, iff $\forall x \in A.\exists! y \in B.xFy$. In this case we denote this unique y by $F(x)$ and call it the *value* of F at the *argument* x.

Axiom 1.4 (Tabulation)**.** *For any relation $R : A \looparrowright B$, there exists a set $|R|$, the* tabulation *of R, and functions $p : |R| \to A$, $q : |R| \to B$, the* projections*, such that:*

- $\forall x \in A.\forall y \in B.xRy \Leftrightarrow \exists r \in |R|.p(r) = x \wedge q(r) = y$
- *For all $r, s \in |R|$, if $p(r) = p(s)$ and $q(r) = q(s)$ then $r = s$.*

Axiom 1.5 (Infinity)**.** *There exists a set* $\mathbb{N}$*, an element* $0 \in \mathbb{N}$ *and an injective function* $s : \mathbb{N} \to \mathbb{N}$ *such that:*

$$\forall n \in \mathbb{N}.s(n) \neq 0$$

Theorem 1.6. *There exists a set 1, unique up to unique isomorphism, with exactly one element.*

PROOF: By the Axiom of Infinity, a set exists that has an element. Pick a set A and $a \in A$. Define $R : A \looparrowright A$ by xRy iff $x = y = a$. Let 1 be the tabulation of R. □

Definition 1.7 (Subset)**.** A *subset* S of a set A is a relation $S : 1 \looparrowright A$. For $a \in A$, we write $a \in S$ for $*Sa$.

Axiom 1.8 (Power Set)**.** *For any set* A*, there exists a set* $\mathcal{P}A$*, the* power set *of* A*, and a relation* $\in : A \looparrowright \mathcal{P}A$*, called* membership*, such that, for any subset* $S \subseteq A$*, there exists a unique* $\overline{S} \in \mathcal{P}A$ *such that*

$$\forall x \in A.x \in S \Leftrightarrow x \in \overline{S} \ .$$

Axiom Schema 1.9 (Collection)**.** *For any property* $\phi[X, Y, x]$ *where* X *and* Y *are set variables and* $x \in X$*, the following is an axiom:*

For any set A*, there exist sets* B *and* Y*, a function* $p : B \to A$*, and a relation* $M : B \looparrowright Y$ *such that:*

- $\forall b \in B.\phi[A, \{y \in Y : bMy\}, p(b)]$
- *For any* $a \in A$*, if there exists a set* X *such that* $\phi[A, X, a]$*, then there exists* $b \in B$ *such that* $a = p(b)$*.*

Axiom 1.10 (Choice)**.** *Let* $R : A \looparrowright B$*. If* $\forall a \in A.\exists b \in B.aRb$*, then there exists a function* $f : A \to B$ *such that* $\forall a \in A.aRf(a)$*.*