Mathematics

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Contents

Chapter 1

Sets and Functions

1.1 Primitive Terms

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B, and call A the *domain* of f and B the *codomain*.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $g \circ f : A \to C$, the *composite* of f and g.

For any set A, let there be a function $\mathrm{id}_A : A \to A$, the *identity* function on A.

Let there be a set 1, the *terminal* set.

For any sets A and B, let there be a set $A \times B$, the *product* of A and B, and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, the *projections*.

Given functions $f : A \to B$ and $g : A \to C$, let there be a function $\langle f, g \rangle : A \to B, C$.

1.2 Definitions Used in the Axioms

Definition 1.1 (Injective)**.** A function $f : A \to B$ is *injective* iff, for every set X and functions $x, y : X \to A$, if $fx = fy$ then $x = y$.

Definition 1.2 (Element)**.** For any set A, an *element* of A is a function $1 \to a$. We write $a \in A$ for $a : 1 \to A$.

Given $f : A \to B$ and $a \in A$, we write $f(a)$ for $f \circ a : 1 \to B$.

Definition 1.3 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for every element $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Definition 1.4. Given functions $f : A \to B$ and $g : C \to D$, let $f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle$.

Definition 1.5 (Function Set)**.** Let A and B be sets. A *function set* from A to B consists of a set B^A and function $\epsilon : B^A \times A \to B$ such that, for any set

I and function $q : I \times A \to B$, there exists a unique function $\lambda q : I \to B^A$ such that $\epsilon \circ (\lambda q \times \mathrm{id}_A) = q$.

Definition 1.6 (Pullback)**.** Let $p : A \to B$, $q : A \to C$, $f : B \to D$ and $g : C \to D$. Then we say that A, p and q form the *pullback* of f and g if and only if:

- $fp = gq$
- For any set X and functions $x : X \to B$, $y : X \to C$ such that $fx = gy$, there exists a unique function $(x, y) : X \to A$ such that $p(x, y) = x$ and $q(x, y) = y$.

We also say p is the pullback of g along f, or q is the pullback of f along g.

In the case g is injective, we also say A and p form the *inverse image* of g under f.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

1.3 The Axioms

Axiom 1.7 (Associativity)**.** *Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*
$$h(gf) = (hg)f \ .$$

Axiom 1.8 (Unit Laws)**.** *For any function $f : A \to B$, we have $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$.*

Axiom 1.9 (Terminal Set)**.** *For any set X, there is exactly one function $X \to 1$.*

Axiom 1.10 (Empty Set)**.** *There exists a set that has no elements.*

Axiom 1.11 (Extensionality)**.** *Let A and B be sets and $f, g : A \to B$. If $\forall a \in A. f(a) = g(a)$ then $f = g$.*

Axiom 1.12 (Products)**.** *Let $f : A \to B$ and $g : A \to C$. Then $\langle f, g \rangle$ is the unique function $A \to B \times C$ such that*

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g \ .$$

Axiom 1.13 (Function Sets)**.** *Any two sets have a function set.*

Axiom 1.14 (Inverse Images)**.** *Given any function $f : X \to Y$ and element $y \in Y$, then there exists a pullback of f and y.*

Axiom 1.15 (Subset Classifier). *There exists a set* 2 *and element* $\top \in 2$ *such that, for any sets* A *and* X *and injective function* $j : A \rightarrowtail X$*, there exists a unique function* $\chi : X \to 2$ *such that* j *and the unique function* $A \to 1$ *form the pullback of* $\top$ *and* χ*.*

Axiom 1.16 (Natural Numbers Set). *There exists a set* $\mathbb{N}$*, an element* $0 \in \mathbb{N}$ *and a function* $s : \mathbb{N} \to \mathbb{N}$ *such that, for any set* A*, element* $a \in A$ *and function* $f : A \to A$*, there exists a unique function* $r : \mathbb{N} \to A$ *such that* $r(0) = a$ *and* $f \circ r = r \circ s$*.*

Axiom 1.17 (Choice). *For any surjective function* $r : A \to B$*, there exists* $s : B \to A$ *such that* $rs = \mathrm{id}_B$*.*

1.4 Functions

Proposition 1.18. *Let* $f : A \to B$*,* $g : B \to C$ *and* $a \in A$*. Then*

$$(g \circ f)(a) = g(f(a)) \ .$$

PROOF: Immediate from the Axiom of Associativity. □