Mathematics

Robin Adams

September 15, 2023

Contents

Chapter 1

Sets and Functions

1.1 Primitive Terms

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B, and call A the *domain* of f and B the *codomain*.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $gf = g \circ f : A \to C$, the *composite* of f and g.

1.2 Axioms

1.2.1 Associativity

Axiom 1.1 (Associativity)**.** *For any functions $f : A \to B$, $g : B \to C$ and $h : C \to D$ we have*

$$h(gf) = (hg)f \enspace .$$

Thanks to this axiom, we shall often omit parentheses when writing the composite of a sequence of functions.

1.2.2 Identity Functions

Definition 1.2 (Identity Function)**.** For any set A, an *identity function* on A is a function $i : A \to A$ such that:

- for every set B and function $f : A \to B$ we have $fi = f$;
- for every set B and function $f : B \to A$ we have $if = f$.

Axiom 1.3 (Identity Functions)**.** *Every set has an identity function.*

Proposition 1.4. *Every set has a unique identity function.*

PROOF:

$\langle 1 \rangle$1. Let: A be a set.
$\langle 1 \rangle$2. A has an identity function.
 Proof: Axiom of Identity Functions
$\langle 1 \rangle$3. For any identity functions i and j on A we have $i = j$.
 $\langle 2 \rangle$1. Let: i and j be identity functions on A.
 $\langle 2 \rangle$2. $i = j$
  Proof: $i = ij = j$
□

Definition 1.5 (Identity Function)**.** For any set A, let id_A be the identity function on A.

Definition 1.6 (Retraction, Section)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $rs = \mathrm{id}_B$.

Proposition 1.7. *Let $f : A \to B$ and $g, h : B \to A$. If g is a retraction of f and h is a section of f then $g = h$.*

Proof:

$$\begin{aligned} g &= g\mathrm{id}_B \\ &= gfh \\ &= \mathrm{id}_A h \\ &= h \end{aligned}$$ □

Definition 1.8 (Bijection)**.** Let $f : A \to B$ be a function. We say f is a *bijection*, and write $f : A \approx B$, iff there exists a function $f^{-1} : B \to A$, an *inverse* to f, such that $f^{-1}f = \mathrm{id}_A$ and $ff^{-1} = \mathrm{id}_B$.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

Proposition 1.9. *The inverse to a bijection is unique.*

Proof: From Proposition 1.7. □

1.2.3 The Terminal Set

Definition 1.10 (Terminal Set)**.** A set T is *terminal* iff, for every set X, there exists exactly one function $X \to T$.

Axiom 1.11 (Terminal Set)**.** *There exists a terminal set.*

Proposition 1.12. *If T and T' are terminal sets then there exists a unique bijection $T \approx T'$.*

Proof:
$\langle 1 \rangle$1. Let: i be the unique function $T \to T'$
$\langle 1 \rangle$2. Let: i^{-1} be the unique function $T' \to T$
$\langle 1 \rangle$3. $ii^{-1} = \mathrm{id}_{T'}$
 Proof: Since there is only one function $T' \to T'$.

$\langle 1 \rangle 4.$ $i^{-1}i = \mathrm{id}_T$
PROOF: Since there is only one function $T \to T$.
□

Definition 1.13. Let 1 be the terminal set. For any set A, let $!_A$ be the function $A \to 1$.

Definition 1.14 (Element)**.** For any set A, an *element* of A is a function $1 \to a$. We write $a \in A$ for $a : 1 \to A$.

Given $f : A \to B$ and $a \in A$, we write $f(a)$ for $fa : 1 \to B$.

Axiom 1.15 (Extensionality)**.** *Let A and B be sets and $f, g : A \to B$. If $\forall a \in A. f(a) = g(a)$ then $f = g$.*

1.2.4 The Empty Set

Axiom 1.16 (Empty Set)**.** *There exists a set that has no elements.*

1.2.5 Products

Definition 1.17 (Product)**.** Let A and B be sets. A *product* of A and B consists of a set $A \times B$ and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, the *projections*, such that, for any set X and functions $f : X \to A$, $g : X \to B$, there exists a unique function $\langle f, g \rangle : X \to A \times B$ such that

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g \enspace .$$

Axiom 1.18 (Products)**.** *Any two sets have a product.*

Proposition 1.19. *If P and Q are products of A and B with projections $p_1 : P \to A$, $p_2 : P \to B$, $q_1 : Q \to A$ and $q_2 : Q \to B$, then there exists a unique isomorphism $i : P \approx Q$ such that $q_1 i = p_1$ and $q_2 i = p_2$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $i : P \to Q$ be the unique function such that $p_1 i = q_1$ and $p_2 i = q_2$.
$\langle 1 \rangle 2.$ LET: $i^{-1} : Q \to P$ be the unique function such that $q_1 i^{-1} = p_1$ and $q_2 i^{-1} = p_2$.
$\langle 1 \rangle 3.$ $i^{-1}i = \mathrm{id}_P$
PROOF: Each is the unique x such that $p_1 x = p_1$ and $p_2 x = p_2$.
$\langle 1 \rangle 4.$ $ii^{-1} = \mathrm{id}_Q$
PROOF: Each is the unique x such that $q_1 x = q_1$ and $q_2 x = q_2$.
□

Definition 1.20. For any sets A and B, we write $A \times B$ for the product of A and B, and $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ for the projections. Given $f : X \to A$ and $g : X \to B$, we write $\langle f, g \rangle$ for the unique function $X \to A \times B$ such that

$$\pi_1 \langle f, g \rangle = f, \qquad \pi_2 \langle f, g \rangle = g \enspace .$$

Definition 1.21. Given functions $f : A \to B$ and $g : C \to D$, let $f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle : A \times C \to B \times D$.

1.2.6 Function Sets

Definition 1.22 (Function Set)**.** Let A and B be sets. A *function set* from A to B consists of a set B^A and function $\epsilon : B^A \times A \to B$, the *evaluation* map, such that, for any set I and function $q : I \times A \to B$, there exists a unique function $\lambda q : I \to B^A$ such that $\epsilon \circ (\lambda q \times \mathrm{id}_A) = q$.

Axiom 1.23 (Function Sets)**.** *Any two sets have a function set.*

Proposition 1.24. *If F and G are function sets of A and B with evaluation maps $e : F \times A \to B$ and $e' : G \times A \to B$, then there exists a unique isomorphism $i : F \cong G$ such that $e'(i \times \mathrm{id}_A) = e$.*

PROOF:
⟨1⟩1. LET: $i : F \to G$ be the unique function such that $e'(i \times \mathrm{id}_A) = e$.
⟨1⟩2. LET: $i^{-1} : G \to F$ be the unique function such that $e(i^{-1} \times \mathrm{id}_A) = e'$
⟨1⟩3. $ii^{-1} = \mathrm{id}_G$
PROOF: Each is the unique x such that $e'(x \times \mathrm{id}_A) = e'$.
⟨1⟩4. $i^{-1}i = \mathrm{id}_F$
PROOF: Each is the unique x such that $e(x \times \mathrm{id}_B) = e$.
□

1.2.7 Inverse Images

Definition 1.25 (Pullback)**.** Let $p : A \to B$, $q : A \to C$, $f : B \to D$ and $g : C \to D$. Then we say that A, p and q form the *pullback* of f and g if and only if:

- $fp = gq$
- For any set X and functions $x : X \to B$, $y : X \to C$ such that $fx = gy$, there exists a unique function $(x, y) : X \to A$ such that $p(x, y) = x$ and $q(x, y) = y$.

We also say p is the pullback of g along f, or q is the pullback of f along g.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Axiom 1.26 (Inverse Images)**.** *Given any function $f : X \to Y$ and element $y \in Y$, then there exists a pullback of f and y.*

1.2.8 The Subset Classifier

Definition 1.27 (Injective)**.** A function $f : A \to B$ is *injective* iff, for every $x, y \in A$, if $fx = fy$ then $x = y$.

Definition 1.28 (Subset Classifier)**.** A *subset classifier* consists of a set 2 and an element $\top \in 2$ such that, for any sets A and X and injective function $j : A \rightarrowtail X$, there exists a unique function $\chi : X \to 2$, the *classifying function* of j, such that j and $!_A : A \to 1$ form the pullback of $\top$ and χ.

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "j"'] & 1 \arrow[d, "\top"] \\ X \arrow[r, "\chi"'] & 2 \end{tikzcd}$$

Axiom 1.29 (Subset Classifier)**.** *There exists a subset classifier.*

Proposition 1.30. *If $\top \in 2$ and $\top' \in 2'$ are subset classifiers, then there exists a unique isomorphism $i : 2 \approx 2'$ such that $i(\top) = \top'$.*

PROOF:
$\langle 1 \rangle 1$. LET: $i : 2 \to 2'$ be the unique function such that $\top$ and id_1 form the pullback of $\top'$ and i
$\langle 1 \rangle 2$. LET: $i^{-1} : 2' \to 2$ be the unique function such that $\top'$ and id_1 form the pullback of $\top$ and i^{-1}
$\langle 1 \rangle 3$. $ii^{-1} = \mathrm{id}_{2'}$
PROOF: Each is the unique x such that $\top'$ and id_1 form the pullback of $\top'$ and x.
$\langle 1 \rangle 4$. $i^{-1}i = \mathrm{id}_2$
PROOF: Each is the unique x such that $\top$ and id_1 form the pullback of $\top$ and x.
$\square$

Definition 1.31. Let 2 and $\top \in 2$ be the subset classifier.

1.2.9 The Natural Numbers

Definition 1.32 (Natural Numbers Set)**.** A *natural numbers set* consists of a set $\mathbb{N}$, an element $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \to \mathbb{N}$ such that, for any set A, element $a \in A$ and function $f : A \to A$, there exists a unique function $r : \mathbb{N} \to A$ such that $r(0) = a$ and $f \circ r = r \circ s$.

Axiom 1.33 (Infinity)**.** *There exists a natural numbers set.*

Proposition 1.34. *If N, $0 \in N$, $s : N \to N$ and N', $0' \in N'$, $s' : N' \to N'$ are two natural numbers sets, then there exists a unique isomorphism $i : N \approx N'$ such that $i(0) = 0'$ and $s'i = is$.*

PROOF:
$\langle 1 \rangle 1$. LET: $i : N \to N'$ be the unique function such that $i(0) = 0'$ and $s'i = is$.
$\langle 1 \rangle 2$. LET: $i^{-1} : N' \to N$ be the unique function such that $i^{-1}(0') = 0$ and $si^{-1} = i^{-1}s'$.

⟨1⟩3. $ii^{-1} = \mathrm{id}_{N'}$
PROOF: Each is the unique x such that $x(0') = 0'$ and $s'x = xs'$.
⟨1⟩4. $i^{-1}i = \mathrm{id}_N$
PROOF: Each is the unique x such that $x(0) = 0$ and $sx = xs$.
□

Definition 1.35. Let $\mathbb{N}$, $0 \in \mathbb{N}$, $s : \mathbb{N} \to \mathbb{N}$ be the natural numbers set.

1.2.10 The Axiom of Choice

Definition 1.36 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for every element $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Axiom 1.37 (Choice)**.** *Every surjective function has a section.*

1.3 Sections and Retractions

Proposition 1.38. *Let $r : A \to B$, $r' : B \to C$, $s : B \to A$ and $s' : C \to B$. If s is a section of r and s' is a section of r', then ss' is a section of $r'r$.*

PROOF: Since $r'rss' = r'\mathrm{id}_B s' = r's' = \mathrm{id}_C$. □

1.4 Injective Functions

Proposition 1.39. *Let $f : A \to B$ be injective. Let $x, y : X \to A$. If $fx = fy$ then $x = y$.*

PROOF:
⟨1⟩1. $\forall t \in X. x(t) = y(t)$
⟨2⟩1. LET: $t \in X$
⟨2⟩2. $f(x(t)) = f(y(t))$
⟨2⟩3. $x(t) = y(t)$
PROOF: f is injective.
⟨1⟩2. $x = y$
PROOF: Axiom of Extensionality
□

The composite if two injective functions is injective.
If gf is injective then f is injective.
Every section is injective.

1.5 Surjective Functions

The composite of two surjective functions is surjective.
If gf is surjective then g is surjective.
A function is surjective iff it has a section.

1.6 Bijections

Proposition 1.40. *For any set A we have* $\mathrm{id}_A : A \approx A$ *and* ${\mathrm{id}_A}^{-1} = \mathrm{id}_A$.

PROOF: Immediate from the fact that $\mathrm{id}_A \mathrm{id}_A = \mathrm{id}_A$. ∎

Proposition 1.41. *If $f : A \approx B$ then $f^{-1} : B \approx A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $ff^{-1} = \mathrm{id}_B$ and $f^{-1}f = \mathrm{id}_A$. ∎

Proposition 1.42. *If $f : A \approx B$ and $g : B \approx C$ then $gf : A \approx C$ and $(gf)^{-1} = f^{-1}g^{-1}$.*

PROOF: From Proposition 1.38. ∎

A function is bijective iff it is injective and surjective.

1.7 Function Sets

Proposition 1.43. *Let $f : A \times B \to C$. Let $a \in A$ and $b \in B$. Then*

$$\epsilon((\lambda f)(a), b) = f(a, b)$$

PROOF:

$$\begin{aligned} \epsilon((\lambda f)(a), b) &= \epsilon(\lambda f \times \mathrm{id}_B)(a, b) \\ &= f(a, b) \end{aligned} \qquad \blacksquare$$

1.8 Subsets

Definition 1.44 (Subset)**.** Let $i : U \to A$. Then we say that (U, i) is a *subset* of A iff i is injective.

Definition 1.45. Let (U, i) and (V, j) be subsets of A. Then we say (U, i) and (V, j) are *equal*, and write $(U, i) = (V, j)$, iff there exists an bijection $\phi : U \approx V$ such that $j\phi = i$.

Proposition 1.46. *For any subset (U, i) of A we have $(U, i) = (U, i)$.*

PROOF: Since $\mathrm{id}_U : U \approx U$ and $i\mathrm{id}_U = i$. ∎

Proposition 1.47. *For any subsets (U, i) and (V, j) of A, if $(U, i) = (V, j)$ then $(V, j) = (U, i)$.*

PROOF: If $\phi : U \approx V$ and $j\phi = i$ then $\phi^{-1} : V \approx U$ and $i\phi^{-1} = j$. ∎

Proposition 1.48. *For any subsets (U, i), (V, j) and (W, k) of A, if $(U, i) = (V, j) = (W, k)$ then $(U, i) = (W, k)$.*

PROOF: If $\phi : U \approx V$, $j\phi = i$, $\psi : V \approx W$ and $k\psi = j$ then $\psi\phi : U \approx W$ and $k\psi\phi = i$. ∎

Definition 1.49 (Inclusion)**.** Let (U, i) and (V, j) be subsets of A. We say that (U, i) is *included* in (V, j) and write $(U, i) \subseteq (V, j)$ iff there exists $f : U \to V$ such that $if = j$.

Proposition 1.50. *If $(U, i) \subseteq (V, j)$, $(U, i) = (U', i')$ and $(V, j) = (V', j')$ then $(U', i') \subseteq (V', j')$.*

PROOF: If $f : U \to V$ satisfies $jf = i$, $\phi : U \approx U'$ satisfies $i'\phi = i$, and $\psi : V \approx V'$ satisfies $j'\psi = j$, then we have $\psi f \phi^{-1} : U' \to V'$ and $j'\psi f \phi^{-1} = i'$. □

Proposition 1.51. *For any subset (U, i) of A we have $(U, i) \subseteq (U, i)$.*

PROOF: Since $\mathrm{id}_U : U \to U$ and $i\mathrm{id}_U = i$. □

Proposition 1.52. *If $(U, i) \subseteq (V, j) \subseteq (W, k)$ then $(U, i) \subseteq (W, k)$.*

PROOF: If $f : U \to V$ satisfies $jf = i$ and $g : V \to W$ satisfies $kg = j$ then $gf : U \to W$ and $kgf = i$. □

Proposition 1.53. *If $(U, i) \subseteq (V, j)$ and $(V, j) \subseteq (U, i)$ then $(U, i) = (V, j)$.*

PROOF: If $f : U \to V$ satisfies $jf = i$ and $g : V \to U$ satisfies $ig = j$ then we have

$$\begin{aligned} igf &= i \\ \therefore gf &= \mathrm{id}_U && (i \text{ is injective}) \\ jfg &= j \\ \therefore fg &= \mathrm{id}_V && (j \text{ is injective}) \end{aligned}$$

Thus $f : U \approx V$ and $jf = i$. So there exists an isomorphism $\phi : U \approx V$ such that $j\phi = i$ as required. □

1.9 Pullbacks and Equalizers

Proposition 1.54. *Let $f : A \to C$ and $g : B \to D$. Let $p : P \to A$ and $q : P \to B$ form a pullback of f and g, and let $p' : P' \to A$ and $q' : P' \to B$ form another pullback. Then there exists a unique isomorphism $\phi : P \approx P'$ such that $p'\phi = p$ and $q'\phi = q$.*

PROOF: By the now familiar pattern. □

State and prove the Pullback Lemma.

Proposition 1.55. *Let $f : X \to Y$ and $i : V \to Y$. Assume i is injective. Then there exists a pullback of f and i.*

PROOF:
⟨1⟩1. LET: $\chi : Y \to 2$ be the characteristic function of i.
⟨1⟩2. LET: $j : U \to X$ be the pullback of χf and $\top$
  PROOF: Axiom of Inverse Images.

⟨1⟩3. LET: $g : U \to V$ be the unique function such that $ig = fj$ and $!_V g = !_U$
⟨2⟩1. $\chi f j = \top !_U$
PROOF: ⟨1⟩2
⟨2⟩2. Q.E.D.
PROOF: ⟨1⟩1
⟨1⟩4. g and j form the pullback of f and i.
PROOF: By the Pullback Lemma.
□

Theorem 1.56. *Any two functions $f, g : A \to B$ have an equalizer.*

PROOF: Take the inverse image of $\delta_B = \langle \mathrm{id}_B, \mathrm{id}_B \rangle : B \rightarrowtail B^2$ and $\langle f, g \rangle : A \to B^2$. □

Theorem 1.57. *Any two functions $f : A \to C$ and $g : B \to C$ have a pullback.*

PROOF: Take the pullback of $f\pi_1 : A \times B \to C$ and $g\pi_2 : A \times B \to C$. □

1.10 Intersections

Definition 1.58 (Intersection)**.** Let (U, i) and (V, j) be subsets of a set A. Let $p : W \to U$ and $q : W \to V$ form the pullback of i under j. Then the *intersection* of (U, i) and (V, j) is defined to be $(U, i) \cap (V, j) = (W, ip) = (W, jq)$.

$S \cap T \subseteq S$ and $S \cap T \subseteq T$.
If $R \subseteq S$ and $R \subseteq T$ then $R \subseteq S \cap T$.

1.11 The Internal Logic

Proposition 1.59. *Let $i : U \rightarrowtail A$ be injective. Let $\chi : A \to 2$ be its characteristic function. Then, for all $a \in A$, we have $\chi(a) = \top$ if and only if there exists $u \in U$ such that $i(u) = a$.*

PROOF:
⟨1⟩1. If $\chi(a) = \top$ then there exists $u \in U$ such that $i(u) = a$.
PROOF: If $\chi \circ a = \top = \top \circ !_1$ then there exists a unique $u : 1 \to U$ such that $i \circ u = a$ and $!_U \circ u = !_1$.
⟨1⟩2. For all $u \in U$ we have $\chi(i(u)) = \top$.
PROOF: Since $\chi \circ i = \top \circ !_U$.
□

Proposition 1.60. *Subsets of a set A are equal if and only if they have the same characteristic function.*

PROOF: Follows from the fact that pullbacks are unique up to isomorphism and the uniqueness of the characteristic function. □

Proposition 1.61. *There are exactly two subsets of 1.*

PROOF:
⟨1⟩1. PICK a set E with no elements.
  PROOF: Axiom of the Empty Set.
⟨1⟩2. $!_E : E \to 1$ is injective.
  PROOF: Vacuously, $\forall x, y \in E.!_E(x) =!_E(y) \Rightarrow x = y$.
⟨1⟩3. $(E, !_E) \neq (1, \mathrm{id}_1)$
  PROOF: Since there cannot be an isomorphism $1 \cong E$.
⟨1⟩4. For any subsets (U, i) and (V, j) of 1, if $(U, i) \neq (U, i) \cap (V, j)$ then $(U, i) = (1, \mathrm{id}_1)$
  ⟨2⟩1. LET: (U, i) and (V, j) be subsets of 1.
  ⟨2⟩2. LET: $p : W \to U$ and $q : W \to V$ form the intersection of (U, i) and (V, j)
  ⟨2⟩3. ASSUME: $(U, i) \neq (W, k)$
  ⟨2⟩4. LET: $(U, \mathrm{id}_U) \neq (W, p)$ as subsets of U.
  ⟨2⟩5. LET: $\chi_U, \chi_W : U \to 2$ be the characteristic functions of (U, id_U) and (W, p) respectively.
  ⟨2⟩6. $\chi_U \neq \chi_W$
  ⟨2⟩7. PICK $x \in U$
    PROOF: By the Axiom of Extensionality, there exists $x \in U$ such that $\chi_U(x) \neq \chi_W(x)$.
  ⟨2⟩8. $ix = \mathrm{id}_1$
  ⟨2⟩9. $x : 1 \cong U$
  ⟨2⟩10. $(U, i) = (1, \mathrm{id}_1)$
⟨1⟩5. For any subset (U, i) of 1, either $(U, i) = (E, !_E)$ or $(U, i) = (1, \mathrm{id}_1)$.
  ⟨2⟩1. LET: (U, i) be a subset of 1.
  ⟨2⟩2. ASSUME: $(U, i) \neq (E, !_E)$
  ⟨2⟩3. $(U, i) \neq (U, i) \cap (E, !_E)$ or $(E, !_E) \neq (U, i) \cap (E, !_E)$
  ⟨2⟩4. $(U, i) = (1, \mathrm{id}_1)$ or $(E, !_E) = (1, \mathrm{id}_1)$
    PROOF: ⟨1⟩4
  ⟨2⟩5. $(U, i) = (1, \mathrm{id}_1)$
    PROOF: ⟨1⟩3

□

Corollary 1.61.1. *There are exactly two elements of 2.*

Definition 1.62 (Falsehood)**.** Let *falsehood* $\perp$ be the element of 2 that is not $\top$.

Corollary 1.62.1. *2 is the coproduct of 1 and 1 with injections $\top$ and $\perp$.*

Proposition 1.63. *A function $f : A \to B$ is surjective if and only if, for any set X and functions $x, y : B \to X$, if $xf = yf$ then $x = y$.*

PROOF:
⟨1⟩1. If f is surjective then, for any set X and functions $x, y : B \to X$, if $xf = yf$ then $x = y$.
  ⟨2⟩1. LET: $s : B \to A$ be a section of f.
    PROOF: Axiom of Choice.

⟨2⟩2. LET: X be a set and $x, y : B \to X$ satisfy $xf = yf$.
⟨2⟩3. $x = y$
PROOF: $x = xfs = yfs = y$
⟨1⟩2. If, for any set X and functions $x, y : B \to X$, if $xf = yf$ then $x = y$, then f is surjective.
⟨2⟩1. ASSUME: For any set X and functions $x, y : B \to X$, if $xf = yf$ then $x = y$.
⟨2⟩2. LET: $b \in B$
⟨2⟩3. ASSUME: for a contradiction $\forall a \in A. f(a) \neq b$
⟨2⟩4. LET: $\psi_1 : B \to 2$ be the characteristic function of b.
⟨2⟩5. LET: $\psi_2 = \bot \circ !_B : B \to 2$
⟨2⟩6. $\forall x \in A. \psi_1(f(x)) = \psi_2(f(x))$
⟨3⟩1. LET: $x \in A$
⟨3⟩2. $\psi_1(f(x)) \neq \top$
PROOF: Proposition 1.59, ⟨2⟩3, ⟨2⟩4.
⟨3⟩3. $\psi_1(f(x)) = \bot$
⟨3⟩4. $\psi_1(f(x)) = \psi_2(f(x))$
⟨2⟩7. $\psi_1 \circ f = \psi_2 \circ f$
PROOF: Axiom of Extensionality
⟨2⟩8. $\psi_1 = \psi_2$
PROOF: ⟨2⟩1
⟨2⟩9. $\psi_1(b) \neq \psi_2(b)$
PROOF: Since $\psi_1(b) = \top$ and $\psi_2(b) = \bot$.
⟨2⟩10. Q.E.D.
PROOF: This is a contradiction
□

1.12 The Empty Set

Theorem 1.64. *If E is a set with no elements, then E has no proper subsets.*

PROOF: A proper subset of E would give a proper subset of 1 that is different from $(E, !_E)$. □

Theorem 1.65. *If E is a set with no elements, then for any set X there exists exactly one function $E \to X$.*

PROOF:
⟨1⟩1. LET: E be a set with no elements.
⟨1⟩2. LET: X be a set.
⟨1⟩3. There exists a function $E \to X$.
⟨2⟩1. LET: $t : 1 \to 2^X$ be the name of the characteristic function of $\mathrm{id}_X : X \to X$.
⟨2⟩2. LET: $\sigma : X \to 2^X$ be the lambda of the characteristic function of $\delta = \langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \to X \times X$.
⟨2⟩3. LET: $p : P \to E$ and $q : P \to X$ be the pullback of $t \circ !_E$ and σ.

PROOF: $t \circ !_E$ is vacuously injective.
$\langle 2 \rangle 4.$ p is injective.
PROOF: It is the pullback of the injective function σ.
$\langle 2 \rangle 5.$ p is bijective.
$\langle 2 \rangle 6.$ $q \circ p^{-1} : E \to X$
$\langle 1 \rangle 4.$ For any functions $f, g : E \to X$ we have $f = g$.
$\langle 2 \rangle 1.$ LET: $f, g : E \to X$
$\langle 2 \rangle 2.$ LET: $m : M \to E$ be the pullback of f and g.
$\langle 2 \rangle 3.$ $(M, m) = (E, \mathrm{id}_E)$
PROOF: Since E has no proper subsets.
$\langle 2 \rangle 4.$ $m : M \cong E$
$\langle 2 \rangle 5.$ $f = g$
□

Corollary 1.65.1. *If E and E' are sets with no elements then there exists a unique isomorphism $E \cong E'$.*

Definition 1.66 (Empty Set)**.** Let the *empty set* $\emptyset$ be the set with no elements.

Theorem 1.67. *For any set A, if there exists a function $A \to \emptyset$ then $A \cong \emptyset$.*

PROOF: If $f : A \to \emptyset$ then A has no elements, because for any $a \in A$ we have $f(a) \in \emptyset$. □

1.13 Universal Quantification

Definition 1.68. For any set A, let $t_A : 1 \to 2^A$ be the name of the characteristic function of $\top \circ !_A : A \to 2$. Define *universal quantification* $\forall_A : 2^A \to 2$ to be the characteristic function of t_A.

1.14 Intersection

Theorem 1.69. *Let X be a set. There exists a function $\bigcap : 2^{2^X} \to 2^X$ such that, for all $S \in 2^{2^X}$ and $a \in X$, we have*

$$\epsilon(\bigcap S, a) = \top \Leftrightarrow \forall A \in 2^X.(\epsilon(S, A) = \top \Rightarrow \epsilon(A, a) = \top)$$

PROOF:
$\langle 1 \rangle 1.$ LET: X be a set.
$\langle 1 \rangle 2.$ LET: $\phi_2 : X \to 2^{2^X}$ be the lambda of $\epsilon : 2^X \times X \to 2$
$\langle 1 \rangle 3.$ For all $x \in X$ and $S \in 2^X$ we have $\epsilon(\phi_2(x), S) = \epsilon(S, x)$.
$\langle 1 \rangle 4.$ LET: $F_1 = \langle \mathrm{id}_{2^{2^X}}, \phi_2 \rangle : 2^{2^X} \times X \to 2^{2^X} \times 2^{2^X}$
$\langle 1 \rangle 5.$ For all $S \in 2^{2^X}$ and $x \in X$ we have $F_1(S, x) = \langle S, \phi_2(x) \rangle$
$\langle 1 \rangle 6.$ LET: $F_2 : 2^{2^X} \times X \to (2 \times 2)^{2^X}$ be the composition of F_1 with the bijection $2^{2^X} \times 2^{2^X} \approx (2 \times 2)^{2^X}$

⟨1⟩7. For all $S \in 2^{2^X}$, $T \in 2^X$ and $x \in X$ we have $\epsilon(F_2(S,x),T) = \langle \epsilon(S,T), \epsilon(\phi_2(x),T)\rangle = \langle \epsilon(S,T), \epsilon(S,x)\rangle$
⟨1⟩8. LET: $F_3 = (\Rightarrow)^{2^X} \circ F_2$
⟨1⟩9. For all $S \in 2^{2^X}$, $T \in 2^X$ and $x \in X$ we have $\epsilon(F_3(S,x),T) = \epsilon(S,T) \Rightarrow \epsilon(S,x)$
⟨1⟩10. LET: $F_4 = \forall \circ F_3$
⟨1⟩11. For all $S \in 2^{2^X}$ and $x \in X$ we have $F_4(S,x) = \top$ iff, for all $T \in 2^X$, if $\epsilon(S,T) = \top$ then $\epsilon(S,x) = \top$
⟨1⟩12. LET: $\bigcap = \lambda F_4 : 2^{2^X} \to 2^X$
⟨1⟩13. For all $S \in 2^{2^X}$ and $x \in X$, we have $\epsilon(\bigcap S, x) = \top$ iff, for all $T \in 2^X$, if $\epsilon(S,T) = \top$ then $\epsilon(T,x) = \top$
$\square$

1.15 Union

Theorem 1.70. *Any two subsets of a set have a union.*

PROOF:
⟨1⟩1. LET: A and B be subsets of X
⟨1⟩2. LET: $\chi_A \in 2^X$ be the name of the characteristic function of A.
⟨1⟩3. LET: $t_X \in 2^X$ be the name of $\top \circ !_X : X \to 2$
⟨1⟩4. LET: C be the pullback of t_X and $\chi_A \Rightarrow - : 2^X \to 2^X$
⟨1⟩5. LET: D be the pullback of t_X and $\chi_B \Rightarrow -$
⟨1⟩6. $\bigcap(C \cap D)$ is the union of A and B.
$\square$

Theorem 1.71. *Any two sets have a coproduct.*

PROOF:
⟨1⟩1. LET: X and Y be sets.
⟨1⟩2. LET: $\sigma_X : X \to 2^X$ be the lambda of the characteristic function of $\langle \mathrm{id}_X, \mathrm{id}_X\rangle : X \to X \times X$
⟨1⟩3. LET: $\chi_0 : 1 \to Y$ be the characteristic function of the unique function $\varnothing \to Y$
⟨1⟩4. LET: $i_X = \langle \sigma_X, \chi_0 \circ !_X\rangle : X \to 2^X \times 2^Y$
⟨1⟩5. LET: $i_Y : Y \to 2^X \times 2^Y$ be defined similarly.
⟨1⟩6. i_X and i_Y are monic.
⟨1⟩7. $\varnothing$ is the pullback of i_X and i_Y (i.e. $(X, i_X) \cap (Y, i_Y) = \varnothing$).
⟨1⟩8. LET: $j : Z \to 2^X \times 2^Y$ be the union of i_X and i_Y
⟨1⟩9. Z is the coproduct of X and Y.
$\square$