

Mathematics

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Contents

| | | |
|----------|----------------------------------------------|----------|
| 1 | The Foundations | 5 |
| 1.1 | Primitive Notions and Axioms | 5 |
| 1.2 | Injective and Surjective Functions | 9 |
| 1.3 | Subsets of a Set | 9 |
| 1.4 | Equalizers | 10 |
| 1.5 | Pullbacks | 12 |

Chapter 1

The Foundations

1.1 Primitive Notions and Axioms

Let there be *sets*.

Given sets A and B , let there be *functions* from A to B . We write $f : A \rightarrow B$ for ' f is a function from A to B '. We call A the *domain* of f , and B the *codomain*.

Given sets A , B and C , and functions $f : A \rightarrow B$ and $g : B \rightarrow C$, let there be a function $gf = g \circ f : A \rightarrow C$, the *composite* of f and g .

Axiom 1.1 (Associativity). *For any functions $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f .$$

Axiom 1.2 (Identity). *For any set A , there exists a function $\text{id}_A : A \rightarrow A$, called an identity function on A , such that:*

- *for every set B and function $f : A \rightarrow B$, we have $f \circ \text{id}_A = f$;*
- *for every set B and function $f : B \rightarrow A$, we have $\text{id}_A \circ f = f$.*

Proposition 1.3. *The identity function on a set is unique.*

PROOF: If $i, j : A \rightarrow A$ are identity functions on A then we have $i = i \circ j = j$. \square

Definition 1.4 (Isomorphism). A function $i : A \rightarrow B$ is an *isomorphism*, $i : A \cong B$, iff there exists a function $i^{-1} : B \rightarrow A$, the *inverse* of i , such that $i^{-1} \circ i = \text{id}_A$ and $i \circ i^{-1} = \text{id}_B$.

Proposition 1.5. *The inverse of an isomorphism is unique.*

PROOF: If j and k are inverses of i we have $j = jik = k$. \square

Proposition 1.6. *For any set A we have $\text{id}_A : A \cong A$ and $\text{id}_A^{-1} = \text{id}_A$.*

PROOF: Since $\text{id}_A \circ \text{id}_A = \text{id}_A$. \square

Proposition 1.7. *If $i : A \cong B$ then $i^{-1} : B \cong A$ and $(i^{-1})^{-1} = i$.*

PROOF: Since $i \circ i^{-1} = \text{id}_B$ and $i^{-1} \circ i = \text{id}_A$. \square

Proposition 1.8. *If $i : A \cong B$ and $j : B \cong C$ then $j \circ i : A \cong C$ and $(j \circ i)^{-1} = i^{-1} \circ j^{-1}$.*

PROOF: Since $j \circ i \circ i^{-1} \circ j^{-1} = \text{id}_C$ and $i^{-1} \circ j^{-1} \circ j \circ i = \text{id}_A$. \square

Axiom 1.9 (Terminal Set). *There exists a set 1 such that, for any set A , there exists a unique function $A \rightarrow 1$.*

Proposition 1.10. *The terminal set is unique up to unique isomorphism.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B be terminal sets.

$\langle 1 \rangle 2$. LET: i be the unique function $A \rightarrow B$.

$\langle 1 \rangle 3$. LET: i^{-1} be the unique function $B \rightarrow A$.

$\langle 1 \rangle 4$. $i \circ i^{-1} = \text{id}_B$

PROOF: Since there is only one function $B \rightarrow B$.

$\langle 1 \rangle 5$. $i^{-1} \circ i = \text{id}_A$

PROOF: Since there is only one function $A \rightarrow A$.

\square

Definition 1.11 (Element). For any set A , an *element* of A is a function $1 \rightarrow A$.

We write $a \in A$ for $a : 1 \rightarrow A$. Given $f : A \rightarrow B$ and $a \in A$, we write $f(a)$ for $f \circ a$.

Axiom 1.12 (Extensionality). *Let A and B be sets. Let $f, g : A \rightarrow B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Axiom 1.13 (Empty Set). *There exists a set with no elements.*

Axiom 1.14 (Products). *Let A and B be sets. There exists a set $A \times B$ and functions $\pi_1 : A \times B \rightarrow A$, $\pi_2 : A \times B \rightarrow B$, the projections, such that, for every set X and functions $f : X \rightarrow A$, $g : X \rightarrow B$, there exists a unique function $\langle f, g \rangle : X \rightarrow A \times B$ such that*

$$\pi_1 \circ \langle f, g \rangle = f, \quad \pi_2 \circ \langle f, g \rangle = g .$$

Proposition 1.15. *If $\pi_1 : P \rightarrow A$ and $\pi_2 : P \rightarrow B$ form a product of A and B , and $p_1 : Q \rightarrow A$ and $p_2 : Q \rightarrow B$ form a product of A and B , then there exists a unique isomorphism $i : P \cong Q$ such that $p_1 \circ i = \pi_1$ and $p_2 \circ i = \pi_2$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : P \rightarrow Q$ be the unique function such that $p_1 \circ i = \pi_1$ and $p_2 \circ i = \pi_2$.

$\langle 1 \rangle 2$. LET: $i^{-1} : Q \rightarrow P$ be the unique function such that $\pi_1 \circ i^{-1} = p_1$ and $\pi_2 \circ i^{-1} = p_2$

$\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_Q$

PROOF: Each is the unique $x : Q \rightarrow Q$ such that $p_1 \circ x = p_1$ and $p_2 \circ x = p_2$.

⟨1⟩4. $i^{-1} \circ i = \text{id}_P$

PROOF: Each is the unique $x : P \rightarrow P$ such that $\pi_1 \circ x = \pi_1$ and $\pi_2 \circ x = \pi_2$.
□

Proposition 1.16. *Let $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : B \rightarrow D$. Then*

$$\langle g, h \rangle \circ f = \langle g \circ f, h \circ f \rangle$$

PROOF: Each is the unique x such that $\pi_1 \circ x = g \circ f$ and $\pi_2 \circ x = h \circ f$. □

Definition 1.17. Given functions $f : A \rightarrow B$ and $g : C \rightarrow D$, define $f \times g : A \times C \rightarrow B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle .$$

Proposition 1.18. *Let $f : A \rightarrow B$, $g : C \rightarrow D$, $h : B \rightarrow E$ and $k : D \rightarrow F$. Then*

$$(h \times k) \circ (f \times g) = (h \circ f) \times (k \circ g) .$$

PROOF:

$$\begin{aligned} (h \times k) \circ (f \times g) &= \langle h \circ \pi_1, k \circ \pi_2 \rangle \circ (f \times g) \\ &= \langle h \circ \pi_1 \circ (f \times g), k \circ \pi_2 \circ (f \times g) \rangle \quad (\text{Proposition 1.16}) \\ &= \langle h \circ f \circ \pi_1, k \circ g \circ \pi_2 \rangle \\ &= (h \circ f) \times (k \circ g) \end{aligned} \quad \square$$

Axiom 1.19 (Function Sets). *Let A and B be sets. There exists a set A^B , called the function set of A and B , and function $\epsilon : A^B \times B \rightarrow A$, the evaluation map, such that, for any set X and function $f : X \times B \rightarrow A$, there exists a unique function $\lambda f : X \rightarrow A^B$ such that*

$$f = \epsilon \circ (\lambda f \times \text{id}_B) .$$

Proposition 1.20. *For any sets A and B , if F and G are function sets with evaluation maps $e : F \times B \rightarrow A$ and $e' : G \times B \rightarrow A$, then there exists a unique isomorphism $i : F \cong G$ such that $e' \circ (i \times \text{id}_B) = e$.*

PROOF:

⟨1⟩1. LET: $i : F \rightarrow G$ be the unique function such that $e = e' \circ (i \times \text{id}_B)$.
 ⟨1⟩2. LET: $i^{-1} : G \rightarrow F$ be the unique function such that $e' = e \circ (i^{-1} \times \text{id}_B)$
 ⟨1⟩3. $i \circ i^{-1} = \text{id}_G$

PROOF: Each is the unique x such that $e' = e' \circ (x \times \text{id}_B)$.

⟨1⟩4. $i^{-1} \circ i = \text{id}_F$

PROOF: Each is the unique x such that $e = e \circ (x \times \text{id}_B)$.

□

Definition 1.21 (Inverse Image). Let A , X and Y be sets. Let $f : X \rightarrow Y$, $a \in Y$ and $j : A \rightarrow X$. Then j is the *inverse image* of a under f if and only if:

$$\bullet f \circ j = a \circ !_A$$

- for every set I and function $q : I \rightarrow X$ such that $f \circ q = a \circ !_I$, there exists a unique $\bar{q} : I \rightarrow A$ such that $q = j \circ \bar{q}$.

Axiom 1.22 (Inverse Images). *For any sets X and Y , function $f : X \rightarrow Y$ and element $a \in Y$, there exists a set $f^{-1}(a)$ and function $j : f^{-1}(a) \rightarrow X$ such that j is the inverse image of a under f .*

Proposition 1.23. *If $j : A \rightarrow X$ and $k : B \rightarrow X$ are inverse images of $a : 1 \rightarrow Y$ under $f : X \rightarrow Y$, then there exists a unique isomorphism $i : A \cong B$ such that $k \circ i = j$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : A \rightarrow B$ be the unique function such that $j = k \circ i$.

$\langle 1 \rangle 2$. LET: $i^{-1} : B \rightarrow A$ be the unique function such that $k = j \circ i^{-1}$

$\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_B$

PROOF: Each is the unique x such that $k = k \circ x$.

$\langle 1 \rangle 4$. $i^{-1} \circ i = \text{id}_A$

PROOF: Each is the unique x such that $j = j \circ x$.

□

Definition 1.24 (Injective). A function $f : A \rightarrow B$ is *injective*, $f : A \rightarrowtail B$, iff, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

Proposition 1.25. *Any function $a : 1 \rightarrow A$ is injective.*

PROOF: If $x, y : X \rightarrow 1$ satisfy $a \circ x = a \circ y$, then $x = y$ because there is only one function $X \rightarrow 1$. □

Definition 1.26 (Surjective). A function $f : A \rightarrow B$ is *surjective*, $f : A \twoheadrightarrow B$, iff, for every set X and functions $x, y : B \rightarrow X$, if $x \circ f = y \circ f$ then $x = y$.

Axiom 1.27 (Subset Classifier). *There exists a set 2 , a subset classifier, and element $\top \in 2$, truth, such that, for any sets A and X and any injective function $f : A \rightarrow X$, there exists a unique function $\chi : X \rightarrow 2$ such that f is the inverse image of \top under χ .*

Proposition 1.28. *If S and S' are subobject classifiers with truth elements t and t' , then there exists a unique isomorphism $i : S \cong S'$ such that $i(t) = t'$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : S \rightarrow S'$ be the characteristic function of t .

$\langle 1 \rangle 2$. LET: $i^{-1} : S' \rightarrow S$ be the characteristic function of t' .

$\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_{S'}$

PROOF: Each is the characteristic function of t' .

$\langle 1 \rangle 4$. $i^{-1} \circ i = \text{id}_S$

PROOF: Each is the characteristic function of t .

□

Axiom 1.29 (Natural Numbers). *There exists a set \mathbb{N} of natural numbers, an element $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that, for every set X , element $a \in X$ and function $r : X \rightarrow X$, there exists a unique function $x : \mathbb{N} \rightarrow X$ such that $x(0) = a$ and $x \circ s = r \circ x$.*

Proposition 1.30. *If $N, 0 \in N, s : N \rightarrow N$ and $N', 0' \in N', s' : N' \rightarrow N'$ are natural number sets, then there exists a unique isomorphism $i : N \cong N'$ such that $i(0) = 0'$ and $s' \circ i = i \circ s$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : N \rightarrow N'$ be the unique function such that $i(0) = 0'$ and $i \circ s = s' \circ i$.

$\langle 1 \rangle 2$. LET: $i^{-1} : N' \rightarrow N$ be the unique function such that $i^{-1}(0') = 0$ and $i^{-1} \circ s' = s \circ i^{-1}$.

$\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_{N'}$

PROOF: Each is the unique x such that $x(0') = 0'$ and $s' \circ x = x \circ s'$.

$\langle 1 \rangle 4$. $i^{-1} \circ i = \text{id}_N$

PROOF: Each is the unique x such that $x(0) = 0$ and $s \circ x = x \circ s$.

□

Definition 1.31 (Retraction, Section). Let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $r \circ s = \text{id}_B$.

Axiom 1.32 (Choice). *Every surjective function has a section.*

1.2 Injective and Surjective Functions

Proposition 1.33. *Every section is injective.*

PROOF: Let $r \circ s = \text{id}$. If $s \circ x = s \circ y$ then $x = r \circ s \circ x = r \circ s \circ y = y$. □

Proposition 1.34. *Every retraction is surjective.*

PROOF: Similar. □

Corollary 1.34.1. *A function is surjective if and only if it has a section.*

1.3 Subsets of a Set

Definition 1.35 (Subset). Let $i : X \rightarrow A$. We write ' (X, i) is a subset of A ' for ' i is injective'.

Given subsets $i : X \rightarrow A$ and $j : Y \rightarrow A$, we write ' $(X, i) = (Y, j)$ ' for 'there exists an isomorphism $k : X \cong Y$ such that $j \circ k = i$ '.

Proposition 1.36. *Given subsets $(X, i), (Y, j)$ of A , if $(X, i) = (Y, j)$ then the isomorphism $k : X \cong Y$ such that $i \circ k = j$ is unique.*

PROOF: Since i is injective. □

Proposition 1.37. *If (X, i) is a subset of A then $(X, i) = (X, i)$.*

PROOF: Since $\text{id}_X : X \cong X$ and $i \circ \text{id}_X = i$. □

Proposition 1.38. *Given subsets $(X, i), (Y, j)$ of A , if $(X, i) = (Y, j)$ then $(Y, j) = (X, i)$.*

PROOF: If $k : X \cong Y$ and $j \circ k = i$ then $k^{-1} : Y \cong X$ and $i \circ k^{-1} = j$. \square

Proposition 1.39. *Given subsets (X, i) , (Y, j) , (Z, k) of A , if $(X, i) = (Y, j)$ and $(Y, j) = (Z, k)$ then $(X, i) = (Z, k)$.*

PROOF: If $f : X \cong Y$ satisfies $j \circ f = i$ and $g : Y \cong Z$ satisfies $k \circ g = j$, then $g \circ f : X \cong Z$ and $k \circ g \circ f = i$. \square

Definition 1.40 (Inclusion). Let (X, i) and (Y, j) be subsets of A . We say (X, i) is *included* in (Y, j) , and write $(X, i) \subseteq (Y, j)$, iff there exists $k : X \rightarrow Y$ such that $j \circ k = i$.

Proposition 1.41. *For any subsets (X, i) , (Y, j) of A , if $(X, i) = (Y, j)$ then $(X, i) \subseteq (Y, j)$.*

PROOF: Immediate from definitions. \square

Corollary 1.41.1. *For any subset (X, i) of A we have $(X, i) \subseteq (X, i)$.*

Proposition 1.42. *For any subsets (X, i) , (Y, j) , (Z, k) of A , if $(X, i) \subseteq (Y, j)$ and $(Y, j) \subseteq (Z, k)$, then $(X, i) \subseteq (Z, k)$.*

PROOF: If $f : X \rightarrow Y$ satisfies $j \circ f = i$ and $g : Y \rightarrow Z$ satisfies $k \circ g = j$, then $g \circ f : X \rightarrow Z$ and $k \circ g \circ f = i$. \square

Corollary 1.42.1. *Inclusion is well defined. That is, if $(X, i) = (X', i')$, $(Y, j) = (Y', j')$ and $(X, i) \subseteq (Y, j)$ then $(X', i') \subseteq (Y', j')$.*

Proposition 1.43. *For any subsets (X, i) and (Y, j) of A , if $(X, i) \subseteq (Y, j)$ and $(Y, j) \subseteq (X, i)$ then $(X, i) = (Y, j)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : X \rightarrow Y$ satisfy $j \circ f = i$.

$\langle 1 \rangle 2$. LET: $g : Y \rightarrow X$ satisfy $i \circ g = j$.

$\langle 1 \rangle 3$. $g \circ f = \text{id}_X$

PROOF: Since $i \circ g \circ f = i$ and i is injective.

$\langle 1 \rangle 4$. $f \circ g = \text{id}_Y$

PROOF: Since $j \circ f \circ g = j$ and j is injective.

$\langle 1 \rangle 5$. $f : X \cong Y$ and $j \circ f = i$.

$\langle 1 \rangle 6$. $(X, i) = (Y, j)$

\square

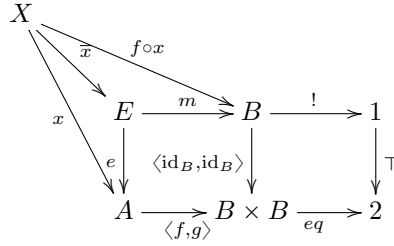
1.4 Equalizers

Proposition 1.44. *For any set A , the function $\langle \text{id}_A, \text{id}_A \rangle : A \rightarrow A \times A$ is injective.*

PROOF: Since $\pi_1 \circ \langle \text{id}_A, \text{id}_A \rangle = \text{id}_A$. \square

Proposition 1.45. *Given sets A and B and functions $f, g : A \rightarrow B$, there exists a set E and function $e : E \rightarrow A$, called the equalizer of f and g , such that:*

- $f \circ e = g \circ e$
- for any set X and function $x : X \rightarrow A$, if $f \circ x = g \circ x$ then there exists a unique $\bar{x} : X \rightarrow E$ such that $x = e \circ \bar{x}$.



PROOF:

⟨1⟩1. LET: $eq : B \times B \rightarrow 2$ be the characteristic function of $\langle id_B, id_B \rangle : B \rightarrow B \times B$

PROOF: By the Axiom of the Subset Classifier.

⟨1⟩2. LET: $e : E \rightarrow A$ be the inverse image of \top under $eq \circ \langle f, g \rangle$

PROOF: By the Axiom of Inverse Images.

⟨1⟩3. $f \circ e = g \circ e$

⟨2⟩1. $eq \circ \langle f, g \rangle \circ e = \top$

⟨2⟩2. LET: $m : E \rightarrow B$ be the unique function such that $\langle id_B, id_B \rangle \circ m = \langle f, g \rangle \circ e$

⟨2⟩3. $\langle m, m \rangle = \langle f \circ e, g \circ e \rangle$

⟨2⟩4. $f \circ e = g \circ e = m$

⟨1⟩4. For any set X and function $x : X \rightarrow A$, if $f \circ x = g \circ x$ then there exists a unique $\bar{x} : X \rightarrow E$ such that $x = e \circ \bar{x}$.

⟨2⟩1. LET: X be a set.

⟨2⟩2. LET: $x : X \rightarrow A$

⟨2⟩3. ASSUME: $f \circ x = g \circ x$

⟨2⟩4. $\langle f, g \rangle \circ x = \langle id_B, id_B \rangle \circ f \circ x$

⟨2⟩5. $eq \circ \langle f, g \rangle \circ x = \top \circ !_X$

PROOF:

$$\begin{aligned} eq \circ \langle f, g \rangle \circ x &= eq \circ \langle id_B, id_B \rangle \circ f \circ x \\ &= \top \circ !_B \circ f \circ x \\ &= \top \circ !_X \end{aligned}$$

⟨2⟩6. There exists a unique $\bar{x} : X \rightarrow E$ such that $e \circ \bar{x} = x$

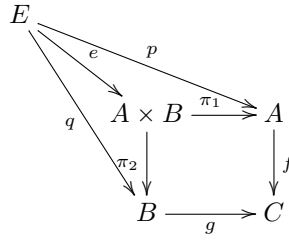
PROOF: From ⟨1⟩2.

□

1.5 Pullbacks

Proposition 1.46. *Let $f : A \rightarrow C$ and $g : B \rightarrow C$. Then there exists a set P and functions $p : P \rightarrow A$, $q : P \rightarrow B$ such that:*

- $f \circ p = g \circ q$
- For any set X and functions $x : X \rightarrow A$, $y : X \rightarrow B$ such that $f \circ x = g \circ y$, there exists a unique function $(x, y) : X \rightarrow P$ such that $p \circ (x, y) = x$ and $q \circ (x, y) = y$.



PROOF:

- <1>1. LET: $e : P \rightarrow A \times B$ be the equalizer of $f \circ \pi_1, g \circ \pi_2 : A \times B \rightarrow C$.
- <1>2. LET: $p = \pi_1 \circ e : E \rightarrow A$ and $q = \pi_2 \circ e : E \rightarrow B$.
- <1>3. $f \circ p = g \circ q$
- <1>4. For any set X and functions $x : X \rightarrow A$, $y : X \rightarrow B$ such that $f \circ x = g \circ y$, there exists a unique function $(x, y) : X \rightarrow P$ such that $p \circ (x, y) = x$ and $q \circ (x, y) = y$.
- <2>1. LET: X be a set.
- <2>2. LET: $x : X \rightarrow A$ and $y : X \rightarrow B$
- <2>3. ASSUME: $f \circ x = g \circ y$
- <2>4. $f \circ \pi_1 \circ \langle x, y \rangle = g \circ \pi_2 \circ \langle x, y \rangle$
- <2>5. LET: $(x, y) : X \rightarrow E$ be the unique morphism such that $e \circ (x, y) = \langle x, y \rangle$
- <2>6. (x, y) is unique such that $\pi_1 \circ e \circ (x, y) = x$ and $\pi_2 \circ e \circ (x, y) = y$
- <2>7. (x, y) is unique such that $p \circ (x, y) = x$ and $q \circ (x, y) = y$.

□