Mathematics

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Contents

Chapter 1

Sets and Functions

1.1 Primitive Terms

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B, and call A the *domain* of f and B the *codomain*.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $g \circ f : A \to C$, the *composite* of f and g.

1.2 The Axioms

Axiom 1.1 (Associativity)**.** *Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h(gf) = (hg)f \ .$$

Axiom 1.2 (Identity)**.** *For any set A, there exists a function $i : A \to A$ such that:*

- *for any set B and function $f : A \to B$, we have $fi = f$*
- *for any set B and function $f : B \to A$, we have $if = f$.*

Proposition 1.3. *For any set A, there exists a unique function $i : A \to A$ such that:*

- *for any set B and function $f : A \to B$, we have $fi = f$*
- *for any set B and function $f : B \to A$, we have $if = f$.*

PROOF: If i and j both satisfy these conditions then $i = ij = j$. □

Definition 1.4 (Identity Function)**.** For any set A, the *identity function* on A, id_A, is the unique function $A \to A$ such that:

- for any set B and function $f : A \to B$, we have $f\mathrm{id}_A = f$

- for any set B and function $f : B \to A$, we have $\mathrm{id}_B f = f$.

Definition 1.5 (Isomorphism)**.** A function $f : A \to B$ is an *isomorphism*, $f : A \cong B$, iff there exists a function $g : B \to A$ such that $fg = \mathrm{id}_B$ and $gf = \mathrm{id}_A$.

Axiom 1.6 (Terminal Set)**.** *There exists an* empty *set* $\emptyset$ *such that, for any set* A*, there exists exists exactly one function* $\emptyset \to A$*.*

Proposition 1.7. *If* S *and* T *are empty sets then there exists a unique isomorphism* $S \cong T$*.*

Proof:
⟨1⟩1. Let: f be the unique function $S \to T$
⟨1⟩2. Let: f^{-1} be the unique function $T \to S$
⟨1⟩3. $ff^{-1} = \mathrm{id}_T$
Proof: Each is the unique function $T \to T$.
⟨1⟩4. $f^{-1}f = \mathrm{id}_S$
Proof: Each is the unique function $S \to S$.
□

Definition 1.8 (Empty Set)**.** Let $\emptyset$ be the set such that, for any set A, there exists exactly one function $\emptyset \to A$.

Axiom 1.9 (Terminal Set)**.** *There exists a* terminal *set* 1 *such that, for any set* A*, there exists exists exactly one function* $A \to 1$*.1*

Proposition 1.10. *If* S *and* T *are terminal sets then there exists a unique isomorphism* $S \cong T$*.*

Proof:
⟨1⟩1. Let: f be the unique function $S \to T$
⟨1⟩2. Let: f^{-1} be the unique function $T \to S$
⟨1⟩3. $ff^{-1} = \mathrm{id}_T$
Proof: Each is the unique function $T \to T$.
⟨1⟩4. $f^{-1}f = \mathrm{id}_S$
Proof: Each is the unique function $S \to S$.
□

Definition 1.11 (Terminal Set)**.** Let 1 be the set such that, for any set A, there exists exactly one function $!_A : A \to 1$.

Definition 1.12 (Element)**.** An *element* of a set A is a function $1 \to A$. We write $a \in A$ for $a : 1 \to A$.

Given $f : A \to B$ and $a \in A$, we write $f(a)$ for fa.

Axiom 1.13 (Extensionality)**.** *Let* A *and* B *be sets. Let* $f, g : A \to B$*. If* $\forall x \in A.f(x) = g(x)$ *then* $f = g$*.*

Axiom 1.14 (Non-degeneracy)**.** *The empty set* $\emptyset$ *has no elements.*

Axiom 1.15 (Disjoint Unions)**.** *For any sets A and B, there exists a set $A+B$, the* disjoint union *of A and B, and functions $\kappa_1 : A \to A+B$, $\kappa_2 : B \to A+B$, the* injections*, such that, for any set X and functions $f : A \to X$ and $g : B \to X$, there exists a unique function $[f,g] : A+B \to X$ such that*

$$[f,g]\kappa_1 = f, \qquad [f,g]\kappa_2 = g \ .$$

Definition 1.16 (Surjective)**.** A function $f : A \to B$ is *surjective*, $f : A \twoheadrightarrow B$, iff, for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Proposition 1.17. *If $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$ are surjective then $gf : A \twoheadrightarrow C$ is surjective.*

PROOF:
⟨1⟩1. LET: $c \in C$
⟨1⟩2. PICK $b \in B$ such that $g(b) = c$.
⟨1⟩3. PICK $a \in A$ such that $f(a) = b$.
⟨1⟩4. $gf(a) = c$
□

Definition 1.18 (Injective)**.** A function $f : A \to B$ is *injective*, $f : A \rightarrowtail B$, iff, for all $x, x' \in A$, if $f(x) = f(x')$ then $x = x'$.

Proposition 1.19. *If $f : A \rightarrowtail B$ and $g : B \rightarrowtail C$ are injective then $gf : A \rightarrowtail C$ is injective.*

PROOF: If $g(f(x)) = g(f(x'))$ then $f(x) = f(x')$ since g is injective, hence $x = x'$ since f is injective. □

Proposition 1.20. *Let $f : A \to B$ and $g : B \to C$. If gf is injective then f is injective.*

PROOF:
⟨1⟩1. LET: $x, x' \in A$
⟨1⟩2. ASSUME: $f(x) = f(x')$
⟨1⟩3. $g(f(x)) = g(f(x'))$
⟨1⟩4. $x = x'$
□

Proposition 1.21. *Let $f : A \to B$ be injective. For any set X and functions $x, y : X \to A$, if $fx = fy$ then $x = y$.*

PROOF:
⟨1⟩1. LET: $f : A \to B$
⟨1⟩2. ASSUME: f is injective.
⟨1⟩3. LET: X be a set.
⟨1⟩4. LET: $x, y : X \to A$
⟨1⟩5. ASSUME: $fx = fy$
⟨1⟩6. LET: $t \in X$

PROVE: $x(t) = y(t)$
$\langle 1\rangle 7.$ $f(x(t)) = f(y(t))$
PROOF: $\langle 1\rangle 5$
$\langle 1\rangle 8.$ $x(t) = y(t)$
PROOF: $\langle 1\rangle 2$
□

Proposition 1.22. *Any function $f : 1 \to A$ is injective.*

PROOF: For any $x, y \in 1$, if $f(x) = f(y)$ then $x = y$ since 1 has only one element.
□

Proposition 1.23. *For any sets A and B, the injections $\kappa_1 : A \to A + B$ and $\kappa_2 : B \to A + B$ are injective.*

PROOF:
$\langle 1\rangle 1.$ κ_1 is injective.
$\langle 2\rangle 1.$ LET: $x, y \in A$
$\langle 2\rangle 2.$ ASSUME: $\kappa_1(x) = \kappa_1(y)$
$\langle 2\rangle 3.$ LET: $f : A + B \to A$ be the function $f = [\mathrm{id}_A, x \circ !_B]$
$\langle 2\rangle 4.$ $x = y$
PROOF: $x = f(\kappa_1(x)) = f(\kappa_1(y)) = y$.
$\langle 1\rangle 2.$ κ_2 is injective.
PROOF: Similar.
□

Definition 1.24 (Bijective)**.** A function is *bijective* iff it is injective and surjective.

Definition 1.25 (Constant)**.** A function $f : A \to B$ is *constant* iff there exists $b \in B$ such that $f = b \circ !_A$.

1.3 Subsets

Definition 1.26 (Subset)**.** A *subset* of a set A is a pair (B, i) such that B is a set and $i : B \rightarrowtail A$ is an injective function.

Definition 1.27 (Equality of Subsets)**.** Given subsets (U, i) and (V, j) of a set A, we say (U, i) and (V, j) are *equal*, $(U, i) = (V, j)$, iff there exists an isomorphism $h : U \cong V$ such that $jh = i$.

Definition 1.28 (Inclusion)**.** Let (B, i) and (C, j) be subsets of A. Then (B, i) is *included* in (C, j), $(B, i) \subseteq (C, j)$, iff there exists $k : B \to C$ such that $jk = i$.

Proposition 1.29. *Let (U, i) and (V, j) be subsets of a set A. Then we have $(U, i) = (V, j)$ iff $(U, i) \subseteq (V, j)$ and $(V, j) \subseteq (U, i)$.*

PROOF:
$\langle 1\rangle 1.$ LET: (U, i) and (V, j) be subsets of a set A.

$\langle 1 \rangle 2.$ If $(U,i) = (V,j)$ then $(U,i) \subseteq (V,j)$ and $(V,j) \subseteq (U,i)$
  $\langle 2 \rangle 1.$ ASSUME: $(U,i) = (V,j)$
  $\langle 2 \rangle 2.$ PICK an isomorphism $h : U \cong V$ such that $jh = i$.
  $\langle 2 \rangle 3.$ $(U,i) \subseteq (V,j)$
    PROOF: Since $jh = i$.
  $\langle 2 \rangle 4.$ $(V,j) \subseteq (U,i)$
    PROOF: Since $ih^{-1} = j$.
$\langle 1 \rangle 3.$ If $(U,i) \subseteq (V,j)$ and $(V,j) \subseteq (U,i)$ then $(U,i) = (V,j)$.
  $\langle 2 \rangle 1.$ ASSUME: $(U,i) \subseteq (V,j)$ and $(V,j) \subseteq (U,i)$
  $\langle 2 \rangle 2.$ PICK $h : U \to V$ such that $jh = i$.
  $\langle 2 \rangle 3.$ PICK $h^{-1} : V \to U$ such that $ih^{-1} = j$.
  $\langle 2 \rangle 4.$ $hh^{-1} = \mathrm{id}_V$
    PROOF: $jhh^{-1} = ih^{-1} = j$
  $\langle 2 \rangle 5.$ $h^{-1}h = \mathrm{id}_U$
    PROOF: $ih^{-1}h = jh = i$
$\square$

Definition 1.30 (Membership)**.** Let (B,i) be a subset of A and $a \in A$. Then a is a *member* of (B,i), $a \in (B,i)$, iff there exists $b \in B$ such that $i(b) = a$.

Proposition 1.31. *Let A be a set. Let $a \in A$, and let S and T be subsets of A. If $a \in S$ and $S \subseteq T$ then $a \in T$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $S = (B,i)$ and $T = (C,j)$
$\langle 1 \rangle 2.$ PICK $k : B \to C$ such that $jk = i$
$\langle 1 \rangle 3.$ PICK $b \in B$ such that $i(b) = a$
$\langle 1 \rangle 4.$ $j(k(b)) = a$
$\langle 1 \rangle 5.$ $a \in T$
$\square$

Corollary 1.31.1. *If $a \in S$ and $S = T$ then $a \in T$.*

1.4 The Subset Classifier

Definition 1.32.

$$2 = 1 + 1$$

Let $\top = \kappa_1 \in 2$.

Definition 1.33 (Characteristic Function)**.** Let $i : X \rightarrowtail A$ be an injective function. Then a function $\chi_{(X,i)} : A \to 2$ is the *characteristic function* of (X,i) if and only if:

- $\chi i = \top !_X$
- for every set T and function $a : T \to A$ such that $\chi_{(X,i)} a = \top !_T$, there exists a unique $\overline{a} : T \to X$ such that $i\overline{a} = a$.

$$\begin{tikzcd} T \arrow[r, "\overline{a}"] \arrow[dr, "a"'] & X \arrow[r, "!"] \arrow[d, "i"] & 1 \arrow[d, "\top"] \\ & A \arrow[r, "\chi(X,i)"'] & 2 \end{tikzcd}$$

Axiom 1.34 (Subset Classifier)**.**

1. *For any set A and function $\phi : A \to 2$, there exists a set X and monomorphism $i : X \rightarrowtail A$ such that ϕ is the characteristic function of (X, i).*

2. *For any set A, every part of A has a characteristic function.*

Proposition 1.35. id_2 *is the characteristic function of* $(1, \top)$

For any set T and function $a : T \to 2$ such that $\mathrm{id}_2 a = \top !_T$, then we have $\top !_T = a$. □

Proposition 1.36. *Let A be a set. Let (U, i) and (V, j) be subsets of A. Then $(U, i) = (V, j)$ if and only if they have the same characteristic function.*

PROOF:
⟨1⟩1. If $(U, i) = (V, j)$ then (U, i) and (V, j) have the same characteristic function.
  ⟨2⟩1. ASSUME: $(U, i) = (V, j)$
  ⟨2⟩2. LET: $h : U \cong V$ be the isomorphism such that $jh = i$.
  ⟨2⟩3. LET: $\chi : A \to 2$ be the characteristic function of (U, i).
    PROVE: χ is the characteristic function of (V, j).
  ⟨2⟩4. $\chi j = \top !_V$
    ⟨3⟩1. $\chi i = \top !_U$
    ⟨3⟩2. $\chi j h = \top !_U$
    ⟨3⟩3. $\chi j = \top !_U h^{-1}$
    ⟨3⟩4. $\chi j = \top !_V$
  ⟨2⟩5. LET: T be a set and $a : T \to A$ satisfy $\chi a = \top !_T$
  ⟨2⟩6. LET: $\overline{a} : T \to U$ be the unique function such that $i\overline{a} = a$.
  ⟨2⟩7. $h\overline{a}$ is the unique function $T \to V$ such that $jh\overline{a} = a$.
⟨1⟩2. If (U, i) and (V, j) have the same characteristic function then $(U, i) = (V, j)$.
  ⟨2⟩1. ASSUME: $\chi : A \to 2$ is the characteristic function of (U, i) and of (V, j)
  ⟨2⟩2. $\chi i = \top !_U$
  ⟨2⟩3. There exists $h : U \to V$ such that $jh = i$.
  ⟨2⟩4. $(U, i) \subseteq (V, j)$
  ⟨2⟩5. $\chi j = \top !_V$
  ⟨2⟩6. There exists $k : V \to U$ such that $ik = j$.
  ⟨2⟩7. $(V, j) \subseteq (U, i)$
  ⟨2⟩8. $(U, i) = (V, j)$
□