

# Mathematics

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August 15, 2023



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# Chapter 1

## Sets and Classes

### 1.1 Classes

Our language is the language of first-order logic with equality over one primitive binary predicate  $\in$ . We call all the objects we reason about *sets*. When  $a \in b$ , we say  $a$  is a *member* or *element* of  $b$ , or  $b$  *contains*  $a$ . We write  $b \ni a$  for  $a \in b$ , and  $a \notin b$  for  $\neg(a \in b)$ . We write  $\forall x \in a. \phi$  as an abbreviation for  $\forall x(x \in a \rightarrow \phi)$ , and  $\exists x \in a. \phi$  as an abbreviation for  $\exists x(x \in a \wedge \phi)$ .

We shall speak informally of *classes* as an abbreviation for talking about predicates. A *class* is determined by a unary predicate  $\phi[x]$  (possibly with parameters). We write  $\{x \mid \phi[x]\}$  or  $\{x : \phi[x]\}$  for the class determined by  $\phi[x]$ . We write ' $a$  is an element of  $\{x \mid \phi[x]\}$ ' or ' $a \in \{x \mid \phi[x]\}$ ' for  $\phi[a]$ .

We write  $\{t[x_1, \dots, x_n] \mid P[x_1, \dots, x_n]\}$  for

$$\{y \mid \exists x_1, \dots, x_n (y = t[x_1, \dots, x_n] \wedge P[x_1, \dots, x_n])\} .$$

We say two classes  $\mathbf{A}$  and  $\mathbf{B}$  are *equal*, and write  $\mathbf{A} = \mathbf{B}$ , iff  $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$ .

**Proposition Schema 1.1.1.** *For any class  $\mathbf{A}$ , the following is a theorem.*

$$\mathbf{A} = \mathbf{A}$$

PROOF: We have  $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{A})$ .  $\square$

**Proposition Schema 1.1.2.** *For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the following is a theorem.*

*If  $\mathbf{A} = \mathbf{B}$  then  $\mathbf{B} = \mathbf{A}$ .*

PROOF: If  $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$  then  $\forall x(x \in \mathbf{B} \leftrightarrow x \in \mathbf{A})$ .  $\square$

**Proposition Schema 1.1.3.** *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$ , the following is a theorem.*

*If  $\mathbf{A} = \mathbf{B}$  and  $\mathbf{B} = \mathbf{C}$  then  $\mathbf{A} = \mathbf{C}$ .*

PROOF: If  $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$  and  $\forall x(x \in \mathbf{B} \leftrightarrow x \in \mathbf{C})$  then  $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{C})$ .  $\square$

### 1.1.1 Subclasses

**Definition 1.1.4** (Subclass). We say a class  $\mathbf{A}$  is a *subclass* of  $\mathbf{B}$ , or  $\mathbf{B}$  is a *superclass* of  $\mathbf{A}$ , or  $\mathbf{B}$  *includes*  $\mathbf{A}$ , and write  $\mathbf{A} \subseteq \mathbf{B}$  or  $\mathbf{B} \supseteq \mathbf{A}$ , iff every element of  $\mathbf{A}$  is an element of  $\mathbf{B}$ . Otherwise we write  $\mathbf{A} \not\subseteq \mathbf{B}$  or  $\mathbf{B} \not\supseteq \mathbf{A}$ .

We say  $\mathbf{A}$  is a *proper* subclass of  $\mathbf{B}$ ,  $\mathbf{B}$  is a *proper* superclass of  $\mathbf{A}$ , or  $\mathbf{B}$  *properly* includes  $\mathbf{A}$ , and write  $\mathbf{A} \subsetneq \mathbf{B}$  or  $\mathbf{B} \supsetneq \mathbf{A}$ , iff  $\mathbf{A} \subseteq \mathbf{B}$  and  $\mathbf{A} \neq \mathbf{B}$ .

**Proposition Schema 1.1.5.** *For any class  $\mathbf{A}$ , the following is a theorem.*

$$\mathbf{A} \subseteq \mathbf{A}$$

PROOF: Every element of  $\mathbf{A}$  is an element of  $\mathbf{A}$ .  $\square$

**Proposition Schema 1.1.6.** *For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the following is a theorem.*

*If  $\mathbf{A} \subseteq \mathbf{B}$  and  $\mathbf{B} \subseteq \mathbf{A}$  then  $\mathbf{A} = \mathbf{B}$ .*

PROOF: If every element of  $\mathbf{A}$  is an element of  $\mathbf{B}$ , and every element of  $\mathbf{B}$  is an element of  $\mathbf{A}$ , then  $\mathbf{A}$  and  $\mathbf{B}$  have exactly the same elements.  $\square$

**Proposition Schema 1.1.7.** *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$ , the following is a theorem.*

*If  $\mathbf{A} \subseteq \mathbf{B}$  and  $\mathbf{B} \subseteq \mathbf{C}$  then  $\mathbf{A} \subseteq \mathbf{C}$ .*

PROOF: If every element of  $\mathbf{A}$  is an element of  $\mathbf{B}$  and every element of  $\mathbf{B}$  is an element of  $\mathbf{C}$  then every element of  $\mathbf{A}$  is an element of  $\mathbf{C}$ .  $\square$

### 1.1.2 Constructions of Classes

**Definition 1.1.8** (Empty Class). The *empty class*  $\emptyset$  is  $\{x \mid \perp\}$ . Every other class is *nonempty*.

**Definition 1.1.9** (Universal Class). The *universal class*  $\mathbf{V}$  is  $\{x \mid \top\}$ .

**Definition 1.1.10** (Enumeration). Given objects  $a_1, \dots, a_n$ , we define the class  $\{a_1, \dots, a_n\}$  to be the class  $\{x \mid x = a_1 \vee \dots \vee x = a_n\}$ .

**Definition 1.1.11** (Intersection). For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the *intersection*  $\mathbf{A} \cap \mathbf{B}$  is  $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$ .

**Definition 1.1.12** (Union). For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the *union*  $\mathbf{A} \cup \mathbf{B}$  is  $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$ .

**Definition 1.1.13** (Relative Complement). Let  $\mathbf{A}$  and  $\mathbf{B}$  be classes. The *relative complement* of  $\mathbf{B}$  in  $\mathbf{A}$  is the class  $\mathbf{A} - \mathbf{B} := \{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$ .

**Definition 1.1.14** (Symmetric Difference). For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the *symmetric difference* is the class  $\mathbf{A} + \mathbf{B} := (\mathbf{A} - \mathbf{B}) \cup (\mathbf{B} - \mathbf{A})$ .

**Definition 1.1.15** (Pairwise disjoint). Let  $\mathbf{A}$  be a class. We say the elements of  $\mathbf{A}$  are *pairwise disjoint* iff, for all  $x, y \in \mathbf{A}$ , if  $x \cap y \neq \emptyset$  then  $x = y$ .



## 1.2 Sets and the Axiom of Extensionality

**Definition 1.2.1** (Axiom of Extensionality). The *Axiom of Extensionality* is the statement: if two sets have exactly the same members, then they are equal.

$$\forall x, y (\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y) \quad .$$

When working in a theory with the Axiom of Extensionality, we may identify a set  $a$  with the class  $\{x \mid x \in a\}$ . Our use of the symbols  $\in$  and  $=$  is consistent. We say a class  $\mathbf{A}$  is a *set* iff there exists a set  $a$  such that  $a = \mathbf{A}$ ; that is,  $\{x \mid \phi[x]\}$  is a set iff  $\exists a \forall x (x \in a \leftrightarrow \phi[x])$ . Otherwise,  $\mathbf{A}$  is a *proper class*.

**Definition 1.2.2** (Subset). If  $A$  is a set and  $A \subseteq \mathbf{B}$ , we say  $A$  is a *subset* of  $\mathbf{B}$ .

**Definition 1.2.3** (Union). The *union* of a class  $\mathbf{A}$  is  $\{x \mid \exists X \in \mathbf{A}. x \in X\}$ . We write  $\bigcup_{P(x)} t(x)$  for  $\bigcup \{t(x) \mid P(x)\}$ .

**Definition 1.2.4** (Intersection). The *intersection* of a class  $\mathbf{A}$  is  $\{x \mid \forall X \in \mathbf{A}. x \in X\}$ . We write  $\bigcap_{P(x)} t(x)$  for  $\bigcap \{t(x) \mid P(x)\}$ .

**Definition 1.2.5** (Power Class). For any class  $\mathbf{A}$ , the *power class*  $\mathcal{P}\mathbf{A}$  is  $\{X \mid X \subseteq \mathbf{A}\}$ .

## 1.3 The Other Axioms

**Definition 1.3.1** (Empty Set Axiom). The *Empty Set Axiom* is the statement: The empty class  $\emptyset$  is a set.

$$\exists e \forall x x \notin e$$

**Definition 1.3.2** (Pairing Axiom). The *Pairing Axiom* or *Pair Set Axiom* is the statement: for any sets  $a$  and  $b$ , the class  $\{a, b\}$  is a set.

$$\forall a \forall b \exists c \forall x (x \in c \Leftrightarrow x = a \vee x = b)$$

**Definition 1.3.3** (Union Axiom). The *Union Axiom* is the statement: for any set  $A$ , the class  $\bigcup A$  is a set.

$$\forall A \exists B \forall x (x \in B \Leftrightarrow \exists y (y \in A \wedge x \in y))$$

**Definition 1.3.4** (Comprehension Axiom Scheme). The *Comprehension Axiom Scheme*, *Subset Axiom Scheme* or *Aussonderungsaxiom Scheme* is the set of sentences of the form, for any class  $\mathbf{A}$ : If  $\mathbf{A}$  is a subclass of a set then  $\mathbf{A}$  is a set.

That is, for any property  $P[x, y_1, \dots, y_n]$ :

For any sets  $a_1, \dots, a_n$  and  $B$ , the class  $\{x \in B \mid P[x, a_1, \dots, a_n]\}$  is a set.

$$\forall a_1, \dots, a_n, B. \exists C. \forall x (x \in C \Leftrightarrow x \in B \wedge P[x, a_1, \dots, a_n])$$

**Definition 1.3.5** (Replacement Axiom Scheme). The *Replacement Axiom Scheme* is the set of sentences of the form, for some property  $P[x, y, z_1, \dots, z_n]$ :

For any sets  $a_1, \dots, a_n, B$ , assume for all  $x \in B$  there exists at most one  $y$  such that  $P[x, y, a_1, \dots, a_n]$ . Then  $\{y \mid \exists x \in B. P[x, y, a_1, \dots, a_n]\}$  is a set.

$$\forall a_1, \dots, a_n, B (\forall x \in B. \forall y, y' (P[x, y, a_1, \dots, a_n] \wedge P[x, y', a_1, \dots, a_n] \Rightarrow y = y') \Rightarrow \\ \exists C \forall y (y \in C \Leftrightarrow \exists x \in B. P[x, y, a_1, \dots, a_n]))$$

**Definition 1.3.6** (Power Set Axiom). The *Power Set Axiom* is the statement: the power class of a set is a set.

$$\forall A \exists B \forall x (x \in B \Leftrightarrow \forall y (y \in x \Rightarrow y \in A))$$

**Definition 1.3.7** (Axiom of Infinity). The *Axiom of Infinity* is the statement: there exists a set  $I$  such that  $\emptyset \in I$  and  $\forall x \in I. x \cup \{x\} \in I$ .

$$\exists I (\emptyset \in I. \forall x. x \notin e \wedge \forall x \in I. \exists y \in I. \forall z (z \in y \Leftrightarrow z \in x \vee z = x))$$

**Definition 1.3.8** (Axiom of Choice). The *Axiom of Choice* is the statement: For any set  $A$  of pairwise disjoint, nonempty sets, there exists a set  $C$  such that, for all  $x \in A$ , we have  $x \cap C$  has exactly one element.

$$\forall A (\forall x \in A. \exists y y \in x \wedge \\ \forall x, y \in A. \forall z (z \in x \wedge z \in y \Rightarrow x = y) \Rightarrow \\ \exists C. \forall x \in A. \exists y \forall z (z \in x \wedge z \in C \Leftrightarrow z = y))$$

**Definition 1.3.9** (Axiom of Regularity). The *Axiom of Regularity* or *Axiom of Foundation* is the statement: for any  $A$ , if  $A$  has a member, then there exists  $m \in A$  such that  $m \cap A = \emptyset$ .

$$\forall A (\exists x. x \in A \Rightarrow \exists m \in A. \neg \exists x (x \in m \wedge x \in A))$$

**Definition 1.3.10** (Skolem Set Theory). *Skolem set theory* (S) is the theory whose axioms are:

- Extensionality
- Empty Set
- Pairing
- Union
- Comprehension
- Power Set

- Regularity

Let SC be the extension of S with the Axiom of Choice.

**Definition 1.3.11** (Skolem-Fraenkel Set Theory). *Skolem-Fraenkel set theory* (SF) is the theory whose axioms are:

- Extensionality
- Empty Set
- Union
- Replacement
- Power Set
- Regularity

Let SFC be the extension of SF with the Axiom of Choice.

**Definition 1.3.12** (Zermelo Set Theory). *Zermelo set theory* is the theory whose axioms are:

- Extensionality
- Pairing
- Union
- Comprehension
- Power Set
- Infinity
- Regularity

We label theorems with Z when they are provable in Zermelo set theory.

Let ZC be the extension of Z with the Axiom of Choice.

**Definition 1.3.13** (Fraenkel-Mostowski Set Theory). *Fraenkel-Mostowski set theory* (FM) is the theory whose axioms are:

- The *Axiom of Extensionality with Urelements*: For any sets  $x$  and  $y$ , if  $x$  is nonempty and  $x$  and  $y$  have exactly the same elements, then  $x = y$ .
- Union
- Replacement
- Power Set
- Infinity

- Regularity

We write FMC for the extension of FM with Choice.

**Definition 1.3.14** (Zermelo-Fraenkel Set Theory). *Zermelo-Fraenkel set theory* is the theory whose axioms are:

- Extensionality
- Union
- Replacement
- Power Set
- Infinity
- Regularity

We label theorems with ZF when they are provable in Zermelo-Fraenkel set theory.

Let ZFC be the extension of ZF with the Axiom of Choice.

We label a theorem with FOL if it can be proved in first-order logic, i.e. from no axioms.

## 1.4 ZFC Extends Z

**Proposition 1.4.1.** *The Axiom of Infinity implies the Empty Set Axiom.*

PROOF: Trivial.  $\square$

**Corollary 1.4.1.1.** *Z is an extension of S.*

*ZF is an extension of SF.*

*ZC is an extension of SC.*

*ZFC is an extension of SFC.*

**Proposition 1.4.2.** *The Axiom of Pairing is a theorem of SF.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $a, b$  be sets.

$\langle 1 \rangle 2$ . LET:  $P(x, y)$  be the predicate  $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$ .

$\langle 1 \rangle 3$ . For all  $x \in \mathcal{P}\mathcal{P}\emptyset$ , there exists at most one  $y$  such that  $P(x, y)$ .

$\langle 2 \rangle 1$ . LET:  $x \in \mathcal{P}\mathcal{P}\emptyset$

$\langle 2 \rangle 2$ . LET:  $y$  and  $y'$  be sets.

$\langle 2 \rangle 3$ . ASSUME:  $P(x, y)$  and  $P(x, y')$

$\langle 2 \rangle 4$ .  $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$

PROOF: From  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 5$ .  $(x = \emptyset \wedge y' = a) \vee (x = \mathcal{P}\emptyset \wedge y' = b)$

PROOF: From  $\langle 2 \rangle 3$ .

⟨2⟩6.  $\emptyset \neq \mathcal{P}\emptyset$

PROOF: Since  $\emptyset \in \mathcal{P}\emptyset$  and  $\emptyset \notin \emptyset$ .

⟨2⟩7.  $y = y'$

⟨1⟩4. LET:  $A$  be the set  $\{y \mid \exists x \in \mathcal{P}\mathcal{P}\emptyset. P(x, y)\}$ .

⟨1⟩5.  $A = \{a, b\}$

□

**Proposition Schema 1.4.3.** *Every instance of the Comprehension Axiom Scheme is a theorem of SF.*

PROOF:

⟨1⟩1. LET:  $P(x)$  be a predicate.

⟨1⟩2. LET:  $A$  be a set.

⟨1⟩3. LET:  $Q(x, y)$  be the predicate  $P(x) \wedge y = x$ .

⟨1⟩4. For all  $x \in A$ , there exists at most one  $y$  such that  $Q(x, y)$ .

⟨2⟩1. LET:  $x \in A$

⟨2⟩2. LET:  $y$  and  $y'$  be sets.

⟨2⟩3. ASSUME:  $Q(x, y)$  and  $Q(x, y')$

⟨2⟩4.  $x \in A \wedge P(x) \wedge y = x \wedge y' = x$

PROOF: From ⟨2⟩3.

⟨2⟩5.  $y = y'$

PROOF: From ⟨2⟩4.

⟨1⟩5. LET:  $B$  be the set  $\{y \mid \exists x \in A. Q(x, y)\}$

PROOF: This is a set by an Axiom of Replacement and ⟨1⟩4.

⟨1⟩6.  $B = \{y \in A \mid P(y)\}$

PROOF:

$$y \in B \Leftrightarrow \exists x \in A. Q(x, y) \quad (\langle 1 \rangle 5)$$

$$\Leftrightarrow \exists x \in A. (P(x) \wedge y = x) \quad (\langle 1 \rangle 3)$$

$$\Leftrightarrow P(y)$$

□

**Corollary 1.4.3.1.** *SF is an extension of S.*

*SFC is an extension of SC.*

*ZF is an extension of Z.*

*ZFC is an extension of ZC.*

## 1.5 Consequences of the Axioms

**Proposition 1.5.1 (S).** *The union of two sets is a set.*

PROOF: Because  $A \cup B = \bigcup \{A, B\}$ . □

**Proposition Schema 1.5.2 (S).** *For any number  $n$ , the following is a theorem:*

*For any sets  $a_1, \dots, a_n$ , the class  $\{a_1, \dots, a_n\} = \{x \mid x = a_1 \vee \dots \vee x = a_n\}$  is a set.*

PROOF: The case  $n = 1$  follows from Pairing since  $\{a\} = \{a, a\}$ .  
 If we have proved the theorem for  $n$  we have  $\{a_1, \dots, a_n, a_{n+1}\} = \{a_1, \dots, a_n\} \cup \{a_{n+1}\}$ .  $\square$

**Proposition 1.5.3 (S).** *No set is a member of itself.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $x$  be any set.
  - $\langle 1 \rangle 2$ . PICK  $m \in \{x\}$  such that  $m \cap \{x\} = \emptyset$ .  
 PROOF: Axiom of Regularity.
  - $\langle 1 \rangle 3$ .  $m = x$
  - $\langle 1 \rangle 4$ .  $x \cap \{x\} = \emptyset$
  - $\langle 1 \rangle 5$ .  $x \notin x$
- $\square$

**Corollary 1.5.3.1 (S).** *The universal class  $\mathbf{V}$  is a proper class.*

PROOF: If  $\mathbf{V}$  is a set then  $\mathbf{V} \in \mathbf{V}$ , contradicting the Proposition.  $\square$

**Proposition 1.5.4 (S).** *There are no sets  $a$  and  $b$  such that  $a \in b$  and  $b \in a$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $a$  and  $b$  be any sets.
  - $\langle 1 \rangle 2$ . PICK  $m \in \{a, b\}$  such that  $m \cap \{a, b\} = \emptyset$
  - $\langle 1 \rangle 3$ . CASE:  $m = a$   
 PROOF: Then  $b \notin a$ .
  - $\langle 1 \rangle 4$ . CASE:  $m = b$   
 PROOF: Then  $a \notin b$ .
- $\square$

**Proposition 1.5.5 (S).** *The intersection of a set and a class is a set.*

PROOF: Immediate from Comprehension.  $\square$

**Proposition 1.5.6 (S).** *The relative complement of a class in a set is a set.*

PROOF: Immediate from Comprehension.  $\square$

**Corollary 1.5.6.1 (S).** *The symmetric difference of two sets is a set.*

**Proposition 1.5.7 (S).** *The intersection of a nonempty class is a set.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\mathbf{A}$  be a nonempty class.
  - $\langle 1 \rangle 2$ . PICK  $B \in \mathbf{A}$
  - $\langle 1 \rangle 3$ .  $\bigcap \mathbf{A} \subseteq B$
  - $\langle 1 \rangle 4$ .  $\bigcap \mathbf{A}$  is a set.
- PROOF: By Comprehension.
- $\square$

**Proposition Schema 1.5.8** (FOL). *For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the following is a theorem:*

*If  $\mathbf{A} \subseteq \mathbf{B}$  then  $\mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathbf{B}$ .*

PROOF: Every subset of  $\mathbf{A}$  is a subset of  $\mathbf{B}$ .  $\square$

**Proposition Schema 1.5.9** (FOL). *For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the following is a theorem:*

*If  $\mathbf{A} \subseteq \mathbf{B}$  then  $\bigcup \mathbf{A} \subseteq \bigcup \mathbf{B}$ .*

PROOF: If  $x \in X \in \mathbf{A}$  then  $x \in X \in \mathbf{B}$ .  $\square$

**Proposition Schema 1.5.10** (S). *For any class  $\mathbf{A}$ , the following is a theorem:*

$$\mathbf{A} = \bigcup \mathcal{P}\mathbf{A}$$

PROOF:

$\langle 1 \rangle 1. \mathbf{A} \subseteq \bigcup \mathcal{P}\mathbf{A}$

PROOF: For all  $x \in \mathbf{A}$  we have  $x \in \{x\} \in \mathcal{P}\mathbf{A}$ .

$\langle 1 \rangle 2. \bigcup \mathcal{P}\mathbf{A} \subseteq \mathbf{A}$

$\langle 2 \rangle 1. \text{ LET: } x \in \bigcup \mathcal{P}\mathbf{A}$

$\langle 2 \rangle 2. \text{ PICK } X \in \mathcal{P}\mathbf{A} \text{ such that } x \in X$

$\langle 2 \rangle 3. X \subseteq \mathbf{A}$

$\langle 2 \rangle 4. x \in \mathbf{A}$

$\square$

## 1.6 Transitive Classes

**Definition 1.6.1** (Transitive Class). A class  $\mathbf{A}$  is a *transitive class* iff whenever  $x \in y \in \mathbf{A}$  then  $x \in \mathbf{A}$ .

**Proposition Schema 1.6.2** (FOL). *For any class  $\mathbf{A}$ , the following is a theorem:*

*The following are equivalent.*

1.  $\mathbf{A}$  is a transitive class.
2.  $\bigcup \mathbf{A} \subseteq \mathbf{A}$
3. Every element of  $\mathbf{A}$  is a subset of  $\mathbf{A}$ .
4.  $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$

PROOF: Immediate from definitions.  $\square$

**Proposition Schema 1.6.3** (FOL). *For any class  $\mathbf{A}$ , the following is a theorem:*

*If  $\mathbf{A}$  is a transitive class then  $\bigcup \mathbf{A}$  is a transitive class.*

PROOF:

⟨1⟩1. ASSUME:  $\mathbf{A}$  is a transitive class.  
 ⟨1⟩2. LET:  $x \in y \in \bigcup \mathbf{A}$   
 ⟨1⟩3.  $y \in \mathbf{A}$   
 PROOF: Since  $\bigcup \mathbf{A} \subseteq \mathbf{A}$  by Proposition 1.6.2.  
 ⟨1⟩4.  $x \in \bigcup \mathbf{A}$   
 □

**Proposition Schema 1.6.4 (S).** *For any class  $\mathbf{A}$ , the following is a theorem:  
 We have  $\mathbf{A}$  is a transitive class if and only if  $\mathcal{P}\mathbf{A}$  is a transitive class.*

PROOF:

⟨1⟩1. If  $\mathbf{A}$  is a transitive class then  $\mathcal{P}\mathbf{A}$  is a transitive class.  
 ⟨2⟩1. ASSUME:  $\mathbf{A}$  is a transitive class.  
 ⟨2⟩2.  $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$   
 PROOF: Proposition 1.6.2.  
 ⟨2⟩3.  $\mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathcal{P}\mathbf{A}$   
 PROOF: Proposition 1.5.8.  
 ⟨2⟩4.  $\mathcal{P}\mathbf{A}$  is a transitive class.  
 PROOF: Proposition 1.6.2.  
 ⟨1⟩2. If  $\mathcal{P}\mathbf{A}$  is a transitive class then  $\mathbf{A}$  is a transitive class.  
 ⟨2⟩1. ASSUME:  $\mathcal{P}\mathbf{A}$  is a transitive class.  
 ⟨2⟩2.  $\bigcup \mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$   
 PROOF: Proposition 1.6.2.  
 ⟨2⟩3.  $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$   
 PROOF: Proposition 1.5.10.  
 ⟨2⟩4.  $\mathbf{A}$  is a transitive class.  
 PROOF: Proposition 1.6.2.  
 □

**Proposition Schema 1.6.5 (FOL).** *For any class  $\mathbf{A}$ , the following is a theorem:*

*If every member of  $\mathbf{A}$  is a transitive set then  $\bigcup \mathbf{A}$  is a transitive class.*

PROOF:

⟨1⟩1. ASSUME: Every member of  $\mathbf{A}$  is a transitive set.  
 ⟨1⟩2. LET:  $x \in y \in \bigcup \mathbf{A}$   
 ⟨1⟩3. PICK  $A \in \mathbf{A}$  such that  $y \in A$ .  
 ⟨1⟩4.  $x \in A$   
 PROOF: Since  $A$  is a transitive set.  
 ⟨1⟩5.  $x \in \bigcup \mathbf{A}$   
 □

**Proposition Schema 1.6.6 (FOL).** *For any class  $\mathbf{A}$ , the following is a theorem:*

*If every member of  $\mathbf{A}$  is a transitive set then  $\bigcap \mathbf{A}$  is a transitive class.*

PROOF:

⟨1⟩1. ASSUME: Every member of  $\mathbf{A}$  is a transitive set.



$\langle 1 \rangle 2$ . LET:  $x \in y \in \bigcap \mathbf{A}$

PROVE:  $x \in \bigcap \mathbf{A}$

$\langle 1 \rangle 3$ . LET:  $A \in \mathbf{A}$

$\langle 1 \rangle 4$ .  $y \in A$

$\langle 1 \rangle 5$ .  $x \in A$

PROOF: Since  $A$  is a transitive set.

□



## Chapter 2

# Relations

### 2.1 Ordered Pairs

**Definition 2.1.1** (Ordered Pair). For any sets  $a$  and  $b$ , the *ordered pair*  $(a, b)$  is defined to be  $\{\{a\}, \{a, b\}\}$ .

**Theorem 2.1.2** (S). For any sets  $a, b, c, d$ , we have  $(a, b) = (c, d)$  if and only if  $a = c$  and  $b = d$ .

PROOF:

$\langle 1 \rangle 1$ . If  $(a, b) = (c, d)$  then  $a = c$  and  $b = d$ .

$\langle 2 \rangle 1$ . ASSUME:  $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$

$\langle 2 \rangle 2$ .  $\bigcap \{\{a\}, \{a, b\}\} = \bigcap \{\{c\}, \{c, d\}\}$

$\langle 2 \rangle 3$ .  $\{a\} = \{c\}$

$\langle 2 \rangle 4$ .  $a = c$

$\langle 2 \rangle 5$ .  $\bigcup \{\{a\}, \{a, b\}\} = \bigcup \{\{c\}, \{c, d\}\}$

$\langle 2 \rangle 6$ .  $\{a, b\} = \{c, d\}$

$\langle 2 \rangle 7$ .  $b = c$  or  $b = d$

$\langle 2 \rangle 8$ .  $a = d$  or  $b = d$

$\langle 2 \rangle 9$ . If  $b = c$  and  $a = d$  then  $b = d$

PROOF: By  $\langle 2 \rangle 4$ .

$\langle 2 \rangle 10$ .  $b = d$

PROOF: From  $\langle 2 \rangle 7$ ,  $\langle 2 \rangle 8$ ,  $\langle 2 \rangle 9$ .

$\langle 1 \rangle 2$ . If  $a = c$  and  $b = d$  then  $(a, b) = (c, d)$ .

PROOF: First-order logic.

□

**Definition 2.1.3** (Cartesian Product). The *Cartesian product* of classes  $\mathbf{A}$  and  $\mathbf{B}$  is the class  $\mathbf{A} \times \mathbf{B} := \{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$ .

**Proposition 2.1.4** (S). For any sets  $A$  and  $B$ , the class  $A \times B$  is a set.

PROOF: It is a subset of  $\mathcal{PP}(A \cup B)$ . □

**Proposition Schema 2.1.5 (S).** *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$ , the following is a theorem:*

$$\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$$

PROOF:

$$\begin{aligned} (x, y) \in \mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) &\Leftrightarrow x \in \mathbf{A} \wedge (y \in \mathbf{B} \vee y \in \mathbf{C}) \\ &\Leftrightarrow (x \in \mathbf{A} \wedge y \in \mathbf{B}) \vee (x \in \mathbf{A} \wedge y \in \mathbf{C}) \\ &\Leftrightarrow (x, y) \in (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C}) \quad \square \end{aligned}$$

**Proposition Schema 2.1.6 (S).** *For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the following is a theorem:*

*If  $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$  and  $\mathbf{A}$  is nonempty then  $\mathbf{B} = \mathbf{C}$ .*

PROOF:

- $\langle 1 \rangle 1$ . PICK  $a \in \mathbf{A}$   
 $\langle 1 \rangle 2$ . For all  $x$  we have  $x \in \mathbf{B}$  iff  $x \in \mathbf{C}$ .

PROOF:

$$\begin{aligned} x \in \mathbf{B} &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{B} \\ &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{C} \\ &\Leftrightarrow x \in \mathbf{C} \end{aligned}$$

$\square$

**Proposition Schema 2.1.7 (S).** *For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the following is a theorem:*

$$\mathbf{A} \times \bigcup \mathbf{B} = \{(a, b) \mid \exists Y \in \mathbf{B}. (a \in \mathbf{A} \wedge b \in Y)\}$$

PROOF:

$$\begin{aligned} (x, y) \in \mathbf{A} \times \bigcup \mathbf{B} &\Leftrightarrow x \in \mathbf{A} \wedge \exists Y \in \mathbf{B}. y \in Y \\ &\Leftrightarrow \exists Y \in \mathbf{B} (x \in \mathbf{A} \wedge y \in Y) \quad \square \end{aligned}$$

## 2.2 Relations

**Definition 2.2.1 (Relation).** A *relation* is a class of ordered pairs.

A *relation*  $\mathbf{R}$  between classes  $\mathbf{A}$  and  $\mathbf{B}$  is a subclass of  $\mathbf{A} \times \mathbf{B}$ .

A *(binary) relation on  $\mathbf{A}$*  is a relation between  $\mathbf{A}$  and  $\mathbf{A}$ .

We write  $x\mathbf{R}y$  for  $(x, y) \in \mathbf{R}$ .

**Definition 2.2.2 (Domain).** The *domain* of a class  $\mathbf{R}$  is the class

$$\text{dom } \mathbf{R} := \{x \mid \exists y. x\mathbf{R}y\} .$$

**Proposition 2.2.3 (S).** *The domain of a set is a set.*

**Definition 2.2.4 (Range).** The *range* of a class  $\mathbf{R}$  is the class

$$\text{ran } \mathbf{R} := \{y \mid \exists x. x\mathbf{R}y\}$$

**Proposition 2.2.5 (S).** *The range of a set is a set.*

### 2.2.1 Identity Functions

**Definition 2.2.6** (Identity Function). For any class  $\mathbf{A}$ , the *identity function* or *diagonal relation*  $\text{id}_{\mathbf{A}}$  on  $\mathbf{A}$  is

$$\text{id}_{\mathbf{A}} := \{(x, x) \mid x \in \mathbf{A}\} .$$

### 2.2.2 Inverses

**Definition 2.2.7** (Inverse). The *inverse* of a relation  $\mathbf{R}$  between  $\mathbf{A}$  and  $\mathbf{B}$  is the relation  $\mathbf{R}^{-1}$  between  $\mathbf{B}$  and  $\mathbf{A}$  defined by

$$b\mathbf{R}^{-1}a \Leftrightarrow a\mathbf{R}b .$$

**Proposition Schema 2.2.8** (S). For any classes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{R}$ , the following is a theorem:

If  $\mathbf{R}$  is a relation between  $\mathbf{A}$  and  $\mathbf{B}$  then  $(\mathbf{R}^{-1})^{-1} = \mathbf{R}$ .

PROOF:

$$\begin{aligned} x(\mathbf{R}^{-1})^{-1}y &\Leftrightarrow y\mathbf{R}^{-1}x \\ &\Leftrightarrow x\mathbf{R}y \end{aligned} \quad \square$$

### 2.2.3 Composition

**Definition 2.2.9** (Composition). Let  $\mathbf{R}$  be a relation between  $\mathbf{A}$  and  $\mathbf{B}$ , and  $\mathbf{S}$  be a relation between  $\mathbf{B}$  and  $\mathbf{C}$ . The *composition*  $\mathbf{S} \circ \mathbf{R}$  is the relation between  $\mathbf{A}$  and  $\mathbf{C}$  defined by

$$a(\mathbf{S} \circ \mathbf{R})c \Leftrightarrow \exists b(a\mathbf{R}b \wedge b\mathbf{S}c) .$$

**Proposition Schema 2.2.10** (S). For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{R}$  and  $\mathbf{S}$ , the following is a theorem:

If  $\mathbf{R}$  is a relation between  $\mathbf{A}$  and  $\mathbf{B}$ , and  $\mathbf{S}$  is a relation between  $\mathbf{B}$  and  $\mathbf{C}$ , then

$$(\mathbf{S} \circ \mathbf{R})^{-1} = \mathbf{R}^{-1} \circ \mathbf{S}^{-1} .$$

PROOF:

$$\begin{aligned} z(\mathbf{S} \circ \mathbf{R})^{-1}x &\Leftrightarrow x(\mathbf{S} \circ \mathbf{R})z \\ &\Leftrightarrow \exists y.(x\mathbf{R}y \wedge y\mathbf{S}z) \\ &\Leftrightarrow \exists y.(y\mathbf{R}^{-1}x \wedge z\mathbf{S}^{-1}y) \\ &\Leftrightarrow z(\mathbf{R}^{-1} \circ \mathbf{S}^{-1})x \end{aligned} \quad \square$$

### 2.2.4 Properties of Relations

**Definition 2.2.11** (Reflexive). Let  $\mathbf{R}$  be a binary relation on  $\mathbf{A}$ . Then  $\mathbf{R}$  is *reflexive* on  $\mathbf{A}$  iff  $\forall x \in \mathbf{A}.(x, x) \in \mathbf{R}$ .

**Proposition Schema 2.2.12** (S). *For any classes  $\mathbf{A}$  and  $\mathbf{R}$ , the following is a theorem:*

*If  $\mathbf{R}$  is a reflexive relation on  $\mathbf{A}$  then so is  $\mathbf{R}^{-1}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in \mathbf{A}$

$\langle 1 \rangle 2$ .  $x\mathbf{R}x$

PROOF: Since  $\mathbf{R}$  is reflexive.

$\langle 1 \rangle 3$ .  $x\mathbf{R}^{-1}x$

□

**Definition 2.2.13** (Irreflexive). A relation  $\mathbf{R}$  is *irreflexive* iff there is no  $x$  such that  $(x, x) \in \mathbf{R}$ .

**Definition 2.2.14** (Symmetric). A relation  $\mathbf{R}$  is *symmetric* iff, whenever  $x\mathbf{R}y$ , then  $y\mathbf{R}x$ .

**Definition 2.2.15** (Antisymmetric). A relation  $\mathbf{R}$  is *antisymmetric* iff, whenever  $x\mathbf{R}y$  and  $y\mathbf{R}x$ , then  $x = y$ .

**Proposition Schema 2.2.16** (S). *For any classes  $\mathbf{A}$  and  $\mathbf{R}$ , the following is a theorem:*

*If  $\mathbf{R}$  is an antisymmetric relation on  $\mathbf{A}$  then so is  $\mathbf{R}^{-1}$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $x\mathbf{R}^{-1}y$  and  $y\mathbf{R}^{-1}x$

$\langle 1 \rangle 2$ .  $y\mathbf{R}x$  and  $x\mathbf{R}y$

$\langle 1 \rangle 3$ .  $x = y$

PROOF: Since  $\mathbf{R}$  is antisymmetric.

□

**Definition 2.2.17** (Transitive). A relation  $\mathbf{R}$  is *transitive* iff, whenever  $x\mathbf{R}y$  and  $y\mathbf{R}z$ , then  $x\mathbf{R}z$ .

**Proposition Schema 2.2.18** (S). *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{R}$ , the following is a theorem:*

*If  $\mathbf{R}$  is a transitive relation between  $\mathbf{A}$  and  $\mathbf{B}$  then  $\mathbf{R}^{-1}$  is transitive.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $(x, y), (y, z) \in \mathbf{R}^{-1}$

$\langle 1 \rangle 2$ .  $(y, x), (z, y) \in \mathbf{R}$

$\langle 1 \rangle 3$ .  $(z, x) \in \mathbf{R}$

$\langle 1 \rangle 4$ .  $(x, z) \in \mathbf{R}^{-1}$

□

**Proposition 2.2.19** (S). *For any relation  $R$  on a set  $A$ , there exists a smallest transitive relation on  $A$  that includes  $R$ .*

PROOF: The relation is  $\bigcap \{S \in \mathcal{P}A^2 \mid R \subseteq S, S \text{ is transitive}\}$ . □

**Definition 2.2.20** (Transitive Closure). For any relation  $R$  on a set  $A$ , the *transitive closure* of  $R$  is the smallest transitive relation that includes  $R$ .

**Definition 2.2.21** (Minimal). Let  $\mathbf{R}$  be a relation on  $\mathbf{A}$ . An element  $m \in \mathbf{A}$  is *minimal* iff there is no  $x \in \mathbf{A}$  such that  $x\mathbf{R}m$ .

**Definition 2.2.22** (Maximal). Let  $\mathbf{R}$  be a relation on  $\mathbf{A}$ . An element  $m \in \mathbf{A}$  is *maximal* iff there is no  $x \in \mathbf{A}$  such that  $m\mathbf{R}x$ .

## 2.3 n-ary Relations

**Definition Schema 2.3.1.** For any sets  $a_1, \dots, a_n$ , define the *ordered  $n$ -tuple*  $(a_1, \dots, a_n)$  by

$$(a_1) := a_1$$

$$(a_1, \dots, a_n, a_{n+1}) = ((a_1, \dots, a_n), a_{n+1})$$

**Definition Schema 2.3.2.** An  *$n$ -ary relation on  $\mathbf{A}$*  is a class of ordered  $n$ -tuples all of whose components are in  $\mathbf{A}$ .

## 2.4 Well Founded Relations

**Definition 2.4.1** (Well Founded). A relation  $\mathbf{R}$  is *well founded* iff:

- Every nonempty set has an  $\mathbf{R}$ -minimal element.
- For every set  $x$ , there exists a set  $u$  such that  $x \subseteq u$  and, for all  $w, y$ , if  $y \in u$  and  $w\mathbf{R}y$  then  $w \in u$ .

**Proposition Schema 2.4.2** (S). *For any class  $\mathbf{R}$ , the following is a theorem:*  
*Assume  $\mathbf{R}$  is a well founded relation. For any set  $a$ , the class  $\{x \mid x\mathbf{R}a\}$  is a set.*

PROOF:

$\langle 1 \rangle 1$ . PICK a set  $u$  such that  $\{a\} \subseteq u$  and, for all  $w, y$ , if  $y \in u$  and  $w\mathbf{R}y$  then  $w \in u$ .

$\langle 1 \rangle 2$ .  $\{x \mid x\mathbf{R}a\} \subseteq u$

□

**Proposition Schema 2.4.3** (S). *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{R}$ , the following is a theorem:*

*Assume  $\mathbf{R}$  is a well founded relation on  $\mathbf{A}$  and  $\mathbf{B} \subseteq \mathbf{A}$  is nonempty. Then  $\mathbf{B}$  has an  $\mathbf{R}$ -minimal element.*

PROOF:

$\langle 1 \rangle 1$ . PICK  $b \in \mathbf{B}$

$\langle 1 \rangle 2$ . LET:  $S = \{x \in \mathbf{B} \mid x\mathbf{R}b\}$

$\langle 1 \rangle 3$ . CASE:  $S = \emptyset$

PROOF: In this case  $b$  is an  $\mathbf{R}$ -minimal element of  $\mathbf{B}$ .

$\langle 1 \rangle 4$ . CASE:  $S \neq \emptyset$

PROOF: In this cases  $S$  has an  $\mathbf{R}$ -minimal element, which is an  $\mathbf{R}$ -minimal element of  $\mathbf{B}$ .

□

**Proposition Schema 2.4.4** (FOL). *For any classes  $\mathbf{R}$  and  $\mathbf{S}$ , the following is a theorem:*

*Assume  $\mathbf{R}$  is a relation. If  $\mathbf{S}$  is a well-founded relation and  $\mathbf{R} \subseteq \mathbf{S}$  then  $\mathbf{R}$  is a well founded relation.*

PROOF: Immediate from definitions. □

**Theorem Schema 2.4.5** (Transfinite Induction Principle (S)). *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{R}$ , the following is a theorem:*

*Assume  $\mathbf{R}$  is a well founded relation on  $\mathbf{A}$  and  $\mathbf{B} \subseteq \mathbf{A}$ . Assume that, for all  $t \in \mathbf{A}$ ,*

$$\{x \in \mathbf{A} \mid x\mathbf{R}t\} \subseteq \mathbf{B} \Rightarrow t \in \mathbf{B} .$$

*Then  $\mathbf{B} = \mathbf{A}$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $\mathbf{B} \neq \mathbf{A}$

$\langle 1 \rangle 2$ . PICK an  $\mathbf{R}$ -minimal element  $m$  of  $\mathbf{A} - \mathbf{B}$ .

PROOF: Proposition 2.4.3.

$\langle 1 \rangle 3$ .  $\{x \in \mathbf{A} \mid x\mathbf{R}m\} \subseteq \mathbf{B}$

PROOF: By minimality of  $m$ .

$\langle 1 \rangle 4$ .  $m \in \mathbf{B}$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This is a contradiction.

□

**Theorem 2.4.6** (S). *The transitive closure of a well founded relation on a set is well founded.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $R$  be a well founded relation on the set  $A$ .

$\langle 1 \rangle 2$ . LET:  $R^t$  be the transitive closure of  $R$ .

$\langle 1 \rangle 3$ . For any  $x, y \in A$ , if  $xR^t y$  then there exists  $z \in A$  such that  $zRy$ .

PROOF:  $\{(x, y) \in A^2 \mid \exists z \in A. zRy\}$  is a transitive relation on  $A$  that includes  $R$ .

$\langle 1 \rangle 4$ . LET:  $B$  be a nonempty subset of  $A$ .

$\langle 1 \rangle 5$ . PICK an  $R$ -minimal element  $b$  of  $B$ .

$\langle 1 \rangle 6$ .  $b$  is  $R^t$ -minimal in  $B$ .

PROOF: If there exists  $x$  such that  $xR^t b$  then there exists  $z$  such that  $zRb$  by

$\langle 1 \rangle 3$ .

□



**Definition 2.4.7** (Initial Segment). Let  $\mathbf{R}$  be a relation on  $\mathbf{A}$  and  $a \in \mathbf{A}$ . The *initial segment* up to  $a$  is

$$\text{seg } a := \{x \in \mathbf{A} \mid x\mathbf{R}a\} .$$

**Theorem Schema 2.4.8** (Transfinite Recursion Theorem Schema (ZF)). *For any classes  $\mathbf{A}$ ,  $\mathbf{R}$  and any property  $G[x, y, z]$ , there exists a class  $\mathbf{F}$  such that, for any class  $\mathbf{F}'$  the following is a theorem:*

*Assume that  $\mathbf{R}$  is a well-founded relation on  $\mathbf{A}$ . Assume that, for any  $f$  and  $t$ , there exists a unique  $z$  such that  $G[f, t, z]$ . Then  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{V}$  such that, for all  $t \in \mathbf{A}$ , we have  $\mathbf{F} \upharpoonright \text{seg } t$  is a set and*

$$G[\mathbf{F} \upharpoonright \text{seg } t, t, \mathbf{F}(t)] .$$

*If  $\mathbf{F}' : \mathbf{A} \rightarrow \mathbf{V}$  satisfies that, for all  $t \in \mathbf{A}$ , we have  $\mathbf{F}' \upharpoonright \text{seg } t$  is a set and  $G[\mathbf{F}' \upharpoonright \text{seg } t, t, \mathbf{F}'(t)]$ , then  $\mathbf{F}' = \mathbf{F}$ .*

PROOF:

- $\langle 1 \rangle 1$ . For  $B$  a subset of  $\mathbf{A}$ , let us say a function  $v : B \rightarrow \mathbf{V}$  is *acceptable* iff, for all  $x \in B$ , we have  $\text{seg } x \subseteq B$  and  $G[v \upharpoonright \text{seg } x, x, v(x)]$
- $\langle 1 \rangle 2$ . LET:  $\mathbf{K}$  be the class of all acceptable functions.
- $\langle 1 \rangle 3$ . LET:  $\mathbf{F} = \bigcup \mathbf{K}$
- $\langle 1 \rangle 4$ . For all  $B, C \subseteq \mathbf{A}$ , given  $v_1 : B \rightarrow \mathbf{V}$  and  $v_2 : C \rightarrow \mathbf{V}$  acceptable and  $x \in B \cap C$ , we have  $v_1(x) = v_2(x)$ 
  - $\langle 2 \rangle 1$ . ASSUME: as transfinite induction hypothesis  $\forall y \mathbf{R} x. y \in B \cap C \Rightarrow v_1(y) = v_2(y)$
  - $\langle 2 \rangle 2$ .  $v_1 \upharpoonright \text{seg } x = v_2 \upharpoonright \text{seg } x$
  - $\langle 2 \rangle 3$ .  $G[v_1 \upharpoonright \text{seg } x, x, v_1(x)]$
  - $\langle 2 \rangle 4$ .  $G[v_2 \upharpoonright \text{seg } x, x, v_2(x)]$
  - $\langle 2 \rangle 5$ .  $v_1(x) = v_2(x)$
- $\langle 1 \rangle 5$ .  $\mathbf{F}$  is a function.
  - $\langle 2 \rangle 1$ . ASSUME:  $(x, y), (x, z) \in \mathbf{F}$
  - $\langle 2 \rangle 2$ . PICK acceptable  $v_1 : B \rightarrow \mathbf{V}$  and  $v_2 : C \rightarrow \mathbf{V}$  such that  $v_1(x) = y$  and  $v_2(x) = z$
  - $\langle 2 \rangle 3$ .  $y = z$
- PROOF: By  $\langle 1 \rangle 4$ .
- $\langle 1 \rangle 6$ . For all  $t \in \text{dom } \mathbf{F}$ , we have  $\mathbf{F} \upharpoonright \text{seg } t$  is a set and  $G[\mathbf{F} \upharpoonright \text{seg } t, t, \mathbf{F}(t)]$ 
  - $\langle 2 \rangle 1$ . LET:  $t \in \text{dom } \mathbf{F}$
  - $\langle 2 \rangle 2$ . PICK an acceptable  $v : A \rightarrow \mathbf{V}$  such that  $t \in A$
  - $\langle 2 \rangle 3$ . For all  $y \mathbf{R} x$  we have  $v(y) = \mathbf{F}(y)$
  - $\langle 2 \rangle 4$ .  $\mathbf{F} \upharpoonright \text{seg } x = v \upharpoonright \text{seg } x$
  - $\langle 2 \rangle 5$ .  $G[v \upharpoonright \text{seg } x, x, v(x)]$
  - $\langle 2 \rangle 6$ .  $G[\mathbf{F} \upharpoonright \text{seg } x, x, \mathbf{F}(x)]$
- $\langle 1 \rangle 7$ .  $\text{dom } \mathbf{F} = \mathbf{A}$ 
  - $\langle 2 \rangle 1$ . LET:  $x \in \mathbf{A}$
  - $\langle 2 \rangle 2$ . ASSUME: as transfinite induction hypothesis  $\forall y \mathbf{R} x. y \in \mathbf{A}$
  - $\langle 2 \rangle 3$ . ASSUME: for a contradiction  $x \notin \text{dom } \mathbf{F}$

$\langle 2 \rangle 4.$   $\mathbf{F} \upharpoonright \text{seg } x$  is a set

PROOF: Axiom of Replacement, Proposition 2.4.2.

$\langle 2 \rangle 5.$   $\mathbf{F} \upharpoonright \text{seg } x$  is acceptable

$\langle 2 \rangle 6.$  LET:  $y$  be the unique object such that  $G[\mathbf{F} \upharpoonright \text{seg } x, x, y]$

$\langle 2 \rangle 7.$   $\mathbf{F} \upharpoonright \text{seg } x \cup \{(x, y)\}$  is acceptable

$\langle 2 \rangle 8.$   $x \in \text{dom } \mathbf{F}$

$\langle 2 \rangle 9.$  Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 8.$  If  $\mathbf{F}' : \mathbf{A} \rightarrow \mathbf{V}$  satisfies the theorem, then  $\mathbf{F}' = \mathbf{F}$ .

$\langle 2 \rangle 1.$  LET:  $x \in \mathbf{A}$

PROVE:  $\mathbf{F}'(x) = \mathbf{F}(x)$

$\langle 2 \rangle 2.$  ASSUME: as transfinite induction hypothesis  $\forall y \mathbf{R}x. \mathbf{F}'(y) = \mathbf{F}(y)$

$\langle 2 \rangle 3.$   $\mathbf{F} \upharpoonright x = \mathbf{F}' \upharpoonright x$

$\langle 2 \rangle 4.$   $G[\mathbf{F} \upharpoonright x, x, \mathbf{F}(x)]$

$\langle 2 \rangle 5.$   $G[\mathbf{F}' \upharpoonright x, x, \mathbf{F}'(x)]$

$\langle 2 \rangle 6.$   $\mathbf{F}(x) = \mathbf{F}'(x)$

□

## Chapter 3

# Functions

### 3.1 Functions

**Definition 3.1.1** (Function). A *function* from  $\mathbf{A}$  to  $\mathbf{B}$  is a relation  $\mathbf{F}$  between  $\mathbf{A}$  and  $\mathbf{B}$  such that, for all  $x \in \mathbf{A}$ , there is only one  $y$  such that  $x\mathbf{F}y$ . We denote this  $y$  by  $\mathbf{F}(x)$ .

A *binary operation* on a class  $\mathbf{A}$  is a function  $\mathbf{A}^2 \rightarrow \mathbf{A}$ .

**Definition 3.1.2** (Closed). Let  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{A}$  be a function and  $\mathbf{B} \subseteq \mathbf{A}$ . Then  $\mathbf{B}$  is *closed* under  $\mathbf{F}$  iff  $\forall x \in \mathbf{B}. \mathbf{F}(x) \in \mathbf{B}$ .

**Proposition 3.1.3** (S). *For any class  $\mathbf{A}$ , the following is a theorem:*

$$\text{id}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}$$

PROOF: For all  $x \in \mathbf{A}$ , the only  $y$  such that  $(x, y) \in \text{id}_{\mathbf{A}}$  is  $y = x$ .  $\square$

**Proposition Schema 3.1.4** (S). *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{F}$  and  $\mathbf{G}$ , the following is a theorem:*

*Assume  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{G} : \mathbf{B} \rightarrow \mathbf{C}$ . Then  $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \rightarrow \mathbf{C}$  and, for all  $x \in \mathbf{A}$ , we have*

$$(\mathbf{G} \circ \mathbf{F})(x) = \mathbf{G}(\mathbf{F}(x)) .$$

PROOF:

$\langle 1 \rangle 1. \forall x \in \mathbf{A}. (x, \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G} \circ \mathbf{F}$

PROOF: Because  $(x, \mathbf{F}(x)) \in \mathbf{F}$  and  $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$ .

$\langle 1 \rangle 2. \text{ If } (x, z) \in \mathbf{F} \circ \mathbf{G} \text{ then } z = \mathbf{G}(\mathbf{F}(x))$

$\langle 2 \rangle 1. \text{ PICK } y \in \mathbf{B} \text{ such that } x\mathbf{F}y \text{ and } y\mathbf{G}z$

$\langle 2 \rangle 2. y = \mathbf{F}(x)$

$\langle 2 \rangle 3. z = \mathbf{G}(y)$

$\langle 2 \rangle 4. z = \mathbf{G}(\mathbf{F}(x))$

$\square$

**Proposition 3.1.5 (SC).** *For any set  $A$  there exists a function  $F : \mathcal{P}A - \{\emptyset\} \rightarrow A$  (a choice function for  $A$ ) such that, for every nonempty  $B \subseteq A$ , we have  $F(B) \in B$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be a set.

$\langle 1 \rangle 2$ . LET:  $\mathcal{A} = \{\{B\} \times B \mid B \in \mathcal{P}A - \{\emptyset\}\}$

$\langle 1 \rangle 3$ . Every member of  $\mathcal{A}$  is nonempty.

$\langle 1 \rangle 4$ . Any two distinct members of  $\mathcal{A}$  are disjoint.

$\langle 1 \rangle 5$ . PICK a set  $C$  such that, for all  $X \in \mathcal{A}$ , we have  $C \cap X$  is a singleton.

PROOF: Axiom of Choice.

$\langle 1 \rangle 6$ . LET:  $F = C \cap \bigcup \mathcal{A}$

$\langle 1 \rangle 7$ .  $F : \mathcal{P}A - \{\emptyset\} \rightarrow A$

$\langle 2 \rangle 1$ .  $F$  is a function.

$\langle 3 \rangle 1$ . LET:  $(B, b), (B, b') \in F$

$\langle 3 \rangle 2$ .  $(B, b), (B, b') \in \{B\} \times B$

PROOF: Since  $(B, b), (B, b') \in \bigcup \mathcal{A}$ .

$\langle 3 \rangle 3$ .  $(B, b), (B, b') \in C \cap (\{B\} \times B)$

$\langle 3 \rangle 4$ .  $(B, b) = (B, b')$

PROOF: From  $\langle 1 \rangle 5$ .

$\langle 3 \rangle 5$ .  $b = b'$

$\langle 2 \rangle 2$ .  $\text{dom } F = \mathcal{P}A - \{\emptyset\}$

PROOF:

$$B \in \text{dom } F$$

$$\Leftrightarrow \exists b. (B, b) \in F$$

$$\Leftrightarrow \exists b. ((B, b) \in \bigcup \mathcal{A} \wedge (B, b) \in C)$$

$$\Leftrightarrow \exists b. \exists B' \in \mathcal{P}A - \{\emptyset\}. ((B, b) \in \{B'\} \times B' \wedge (B, b) \in C)$$

$$\Leftrightarrow B \in \mathcal{P}A - \{\emptyset\} \wedge \exists b \in B. (B, b) \in C$$

$$\Leftrightarrow B \in \mathcal{P}A - \{\emptyset\} \quad (\langle 1 \rangle 5)$$

$\langle 2 \rangle 3$ .  $\text{ran } F \subseteq A$

$\langle 1 \rangle 8$ . For every nonempty  $B \subseteq A$  we have  $F(B) \in B$

□

**Proposition 3.1.6 (SC).** *For any relation  $R$  between  $A$  and  $B$ , there exists a function  $H : A \rightarrow B$  such that  $H \subseteq R$  (i.e.  $\forall x \in A. xRH(x)$ ).*

PROOF:

$\langle 1 \rangle 1$ . LET:  $R$  be a relation between  $A$  and  $B$ .

$\langle 1 \rangle 2$ . PICK a choice function  $G$  for  $B$ .

$\langle 1 \rangle 3$ . Define  $H : A \rightarrow B$  by  $H(x) = G(\{y \mid xRy\})$

$\langle 1 \rangle 4$ .  $H \subseteq R$

□

### 3.1.1 Injective Functions

**Definition 3.1.7** (Injective). A function  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  is *one-to-one*, *injective* or an *injection*,  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ , iff, for all  $x, y \in \mathbf{A}$ , if  $\mathbf{F}(x) = \mathbf{F}(y)$ , then  $x = y$ .

**Proposition 3.1.8** (S). For any class  $\mathbf{A}$ , the following is a theorem:

$\text{id}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}$  is injective.

PROOF: If  $\text{id}_{\mathbf{A}}(x) = \text{id}_{\mathbf{A}}(y)$  then immediately  $x = y$ .  $\square$

**Proposition Schema 3.1.9** (S). For any classes  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{F}, \mathbf{G}$ , the following is a theorem:

Assume  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{G} : \mathbf{B} \rightarrow \mathbf{C}$ . Then  $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \rightarrow \mathbf{C}$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $x, y \in \mathbf{A}$

$\langle 1 \rangle 2$ . ASSUME:  $(\mathbf{G} \circ \mathbf{F})(x) = (\mathbf{G} \circ \mathbf{F})(y)$

$\langle 1 \rangle 3$ .  $\mathbf{G}(\mathbf{F}(x)) = \mathbf{G}(\mathbf{F}(y))$

$\langle 1 \rangle 4$ .  $\mathbf{F}(x) = \mathbf{F}(y)$

PROOF: Since  $\mathbf{G}$  is injective.

$\langle 1 \rangle 5$ .  $x = y$

PROOF: Since  $\mathbf{F}$  is injective.

$\square$

**Proposition 3.1.10** (S). Let  $F : A \rightarrow B$  where  $A$  is nonempty. There exists  $G : B \rightarrow A$  (a left inverse) such that  $G \circ F = \text{id}_A$  if and only if  $F$  is one-to-one.

PROOF:

$\langle 1 \rangle 1$ . If there exists  $G : B \rightarrow A$  such that  $G \circ F = \text{id}_A$  then  $F$  is one-to-one.

$\langle 2 \rangle 1$ . ASSUME:  $G : B \rightarrow A$  and  $G \circ F = I_A$

$\langle 2 \rangle 2$ . LET:  $x, y \in A$

$\langle 2 \rangle 3$ . ASSUME:  $F(x) = F(y)$

$\langle 2 \rangle 4$ .  $x = y$

PROOF:  $x = G(F(x)) = G(F(y)) = y$

$\langle 1 \rangle 2$ . If  $F$  is one-to-one then there exists  $G : B \rightarrow A$  such that  $G \circ F = I_A$ .

$\langle 2 \rangle 1$ . ASSUME:  $F$  is one-to-one.

$\langle 2 \rangle 2$ . PICK  $a \in A$

$\langle 2 \rangle 3$ . LET:  $G : B \rightarrow A$  be the function defined by:  $G(b)$  is the (unique)  $x \in A$  such that  $F(x) = b$  if there exists such an  $x$ ,  $G(b) = a$  otherwise.

$\langle 2 \rangle 4$ . For all  $x \in A$  we have  $G(F(x)) = x$ .

$\square$

### 3.1.2 Surjective Functions

**Definition 3.1.11** (Surjective). Let  $F : A \rightarrow B$ . We say that  $F$  is *surjective*, or maps  $A$  *onto*  $B$ , and write  $F : A \rightarrow B$ , iff for all  $y \in B$  there exists  $x \in A$  such that  $F(x) = y$ .

**Proposition Schema 3.1.12** (S). For any class  $\mathbf{A}$ , the following is a theorem:

$\text{id}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}$  is surjective.

PROOF: For any  $y \in \mathbf{A}$  we have  $\text{id}_{\mathbf{A}}(y) = y$ .  $\square$

**Proposition Schema 3.1.13 (S).** *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{F}$  and  $\mathbf{G}$ , the following is a theorem:*

*If  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{G} : \mathbf{B} \rightarrow \mathbf{C}$ , then  $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \rightarrow \mathbf{C}$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $c \in \mathbf{C}$
- $\langle 1 \rangle 2$ . PICK  $b \in \mathbf{B}$  such that  $\mathbf{G}(b) = c$ .
- $\langle 1 \rangle 3$ . PICK  $a \in \mathbf{A}$  such that  $\mathbf{F}(a) = b$ .
- $\langle 1 \rangle 4$ .  $(\mathbf{G} \circ \mathbf{F})(a) = c$

$\square$

**Proposition 3.1.14 (SC).** *Let  $F : A \rightarrow B$ . There exists  $H : B \rightarrow A$  (a right inverse) such that  $F \circ H = \text{id}_B$  if and only if  $F$  maps  $A$  onto  $B$ .*

PROOF:

- $\langle 1 \rangle 1$ . If  $F$  has a right inverse then  $F$  is surjective.
  - $\langle 2 \rangle 1$ . ASSUME:  $F$  has a right inverse  $H : B \rightarrow A$ .
  - $\langle 2 \rangle 2$ . LET:  $y \in B$
  - $\langle 2 \rangle 3$ .  $F(H(y)) = y$
  - $\langle 2 \rangle 4$ . There exists  $x \in A$  such that  $F(x) = y$
- $\langle 1 \rangle 2$ . If  $F$  is surjective then  $F$  has a right inverse.
  - $\langle 2 \rangle 1$ . ASSUME:  $F$  is surjective.
  - $\langle 2 \rangle 2$ . PICK a function  $H : B \rightarrow A$  such that  $H \subseteq F^{-1}$

PROOF: Proposition 3.1.6.

- $\langle 2 \rangle 3$ .  $F \circ H = \text{id}_B$ 
  - $\langle 3 \rangle 1$ . LET:  $y \in B$
  - $\langle 3 \rangle 2$ .  $(y, H(y)) \in F^{-1}$
  - $\langle 3 \rangle 3$ .  $F(H(y)) = y$

$\square$

### 3.1.3 Bijections

**Definition 3.1.15 (Bijection).** Let  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ . Then  $\mathbf{F}$  is *bijective* or a *bijection*,  $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$ , iff it is injective and surjective.

**Proposition Schema 3.1.16 (S).** *For any class  $\mathbf{A}$ , the following is a theorem:*  
*The identity function  $\text{id}_{\mathbf{A}} : \mathbf{A} \approx \mathbf{A}$  is a bijection.*

PROOF: Proposition 3.1.8 and 3.1.12.  $\square$

**Proposition Schema 3.1.17 (S).** *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{F}$ , the following is a theorem:*

*If  $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$  then  $\mathbf{F}^{-1} : \mathbf{B} \approx \mathbf{A}$ .*

PROOF:

- $\langle 1 \rangle 1$ .  $\mathbf{F}^{-1} : \mathbf{B} \rightarrow \mathbf{A}$
- $\langle 2 \rangle 1$ . LET:  $b \in \mathbf{B}$

⟨2⟩2. PICK  $a \in \mathbf{A}$  such that  $\mathbf{F}(a) = b$ .

PROOF: Since  $\mathbf{F}$  is surjective.

⟨2⟩3.  $(b, a) \in \mathbf{F}^{-1}$

⟨2⟩4. If  $(b, a') \in \mathbf{F}^{-1}$  then  $a' = a$ .

⟨3⟩1. LET:  $a' \in \mathbf{A}$  such that  $(b, a') \in \mathbf{F}^{-1}$

⟨3⟩2.  $\mathbf{F}(a') = \mathbf{F}(a)$

⟨3⟩3.  $a' = a$

PROOF: Since  $\mathbf{F}$  is injective.

⟨1⟩2.  $\mathbf{F}^{-1}$  is injective.

⟨2⟩1. LET:  $x, y \in \mathbf{B}$

⟨2⟩2. ASSUME:  $\mathbf{F}^{-1}(x) = \mathbf{F}^{-1}(y)$

⟨2⟩3.  $x = y$

PROOF:  $x = \mathbf{F}(\mathbf{F}^{-1}(x)) = \mathbf{F}(\mathbf{F}^{-1}(y)) = y$ .

⟨1⟩3.  $\mathbf{F}^{-1}$  is surjective.

PROOF: For all  $a \in \mathbf{A}$  we have  $\mathbf{F}^{-1}(\mathbf{F}(a)) = a$ .

□

**Proposition Schema 3.1.18 (S).** *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{F}$  and  $\mathbf{G}$ , the following is a theorem:*

*If  $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$  and  $\mathbf{G} : \mathbf{B} \approx \mathbf{C}$  then  $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \approx \mathbf{C}$ .*

PROOF: Propositions 3.1.9 and 3.1.13. □

### 3.1.4 Restrictions

**Definition 3.1.19 (Restriction).** Let  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ . Let  $\mathbf{C} \subseteq \mathbf{A}$ . The *restriction* of  $\mathbf{F}$  to  $\mathbf{C}$ , denoted  $\mathbf{F} \upharpoonright \mathbf{C}$ , is the function

$$\begin{aligned} \mathbf{F} \upharpoonright \mathbf{C} : \mathbf{C} &\rightarrow \mathbf{B} \\ (\mathbf{F} \upharpoonright \mathbf{C})(x) &= \mathbf{F}(x) \quad (x \in \mathbf{C}) \end{aligned}$$

### 3.1.5 Images

**Definition 3.1.20 (Image).** Let  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{C} \subseteq \mathbf{A}$ . The *image* of  $\mathbf{C}$  under  $\mathbf{F}$  is the class

$$\mathbf{F}(\mathbf{C}) := \{\mathbf{F}(x) \mid x \in \mathbf{C}\}.$$

**Proposition Schema 3.1.21 (S).** *For any classes  $\mathbf{F}$ ,  $\mathbf{A}$  and  $\mathbf{B}$ , the following is a theorem.*

*If  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ , then for any subset  $S \subseteq \mathbf{A}$ , the class  $\mathbf{F}(S)$  is a set.*

PROOF: By an Axiom of Replacement. □

**Proposition Schema 3.1.22 (S).** *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  and  $\mathbf{F}$ , the following is a theorem:*

*Assume  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{C} \subseteq \mathcal{P}\mathbf{A}$ . Then*

$$\mathbf{F}\left(\bigcup \mathbf{C}\right) = \{y \mid \exists X \in \mathbf{C}. y \in \mathbf{F}(X)\}$$

PROOF:

$$\begin{aligned}
 y \in \mathbf{F}\left(\bigcup \mathbf{C}\right) &\Leftrightarrow \exists x \in \bigcup \mathbf{C}. y = \mathbf{F}(x) \\
 &\Leftrightarrow \exists x. \exists X. X \in \mathbf{C} \wedge x \in X \wedge y = \mathbf{F}(x) \\
 &\Leftrightarrow \exists X \in \mathbf{C}. y \in \mathbf{F}(X) \quad \square
 \end{aligned}$$

**Proposition Schema 3.1.23 (S).** *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{D}$  and  $\mathbf{F}$ , the following is a theorem:*

*Assume  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$ . Then*

$$\mathbf{F}(\mathbf{C} \cup \mathbf{D}) = \mathbf{F}(\mathbf{C}) \cup \mathbf{F}(\mathbf{D}) .$$

PROOF:

$$\begin{aligned}
 y \in \mathbf{F}(\mathbf{C} \cup \mathbf{D}) &\Leftrightarrow \exists x \in \mathbf{C} \cup \mathbf{D}. y = \mathbf{F}(x) \\
 &\Leftrightarrow \exists x \in \mathbf{C}. y = \mathbf{F}(x) \vee \exists x \in \mathbf{D}. y = \mathbf{F}(x) \\
 &\Leftrightarrow y \in \mathbf{F}(\mathbf{C}) \cup \mathbf{F}(\mathbf{D}) \quad \square
 \end{aligned}$$

**Proposition 3.1.24 (S).** *For any classes  $\mathbf{F}$ ,  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  and  $\mathbf{D}$ , the following is a theorem:*

*Assume  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$ . Then*

$$\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B}) .$$

*Equality holds if  $\mathbf{F}$  is injective.*

PROOF:

- $\langle 1 \rangle 1. \mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
- $\langle 2 \rangle 1. \text{ LET: } y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$
- $\langle 2 \rangle 2. \text{ PICK } x \in \mathbf{A} \cap \mathbf{B} \text{ such that } y = \mathbf{F}(x)$
- $\langle 2 \rangle 3. y \in \mathbf{F}(\mathbf{A})$
- PROOF: Since  $x \in \mathbf{A}$ .
- $\langle 2 \rangle 4. y \in \mathbf{F}(\mathbf{B})$
- PROOF: Since  $x \in \mathbf{B}$ .
- $\langle 1 \rangle 2. \text{ If } \mathbf{F} \text{ is injective then } \mathbf{F}(\mathbf{A} \cap \mathbf{B}) = \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B}).$
- $\langle 2 \rangle 1. \text{ ASSUME: } \mathbf{F} \text{ is injective.}$
- $\langle 2 \rangle 2. \text{ LET: } y \in \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
- $\langle 2 \rangle 3. \text{ PICK } x \in \mathbf{A} \text{ such that } y = \mathbf{F}(x)$
- $\langle 2 \rangle 4. \text{ PICK } x' \in \mathbf{B} \text{ such that } y = \mathbf{F}(x')$
- $\langle 2 \rangle 5. x = x'$
- PROOF:  $\langle 2 \rangle 1$
- $\langle 2 \rangle 6. x \in \mathbf{A} \cap \mathbf{B}$
- $\langle 2 \rangle 7. y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$

□

**Proposition Schema 3.1.25 (S).** *For any classes  $\mathbf{F}$ ,  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$ , the following is a theorem:*

*Let  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{C} \subseteq \mathcal{P}\mathbf{A}$ . Then*



$$\mathbf{F}\left(\bigcap \mathbf{C}\right) \subseteq \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\} .$$

Equality holds if  $\mathbf{F}$  is injective and  $\mathbf{A}$  is nonempty.

PROOF:

- $\langle 1 \rangle 1.$   $\mathbf{F}(\bigcap \mathbf{A}) \subseteq \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
- $\langle 2 \rangle 1.$  LET:  $y \in \mathbf{F}(\bigcap \mathbf{A})$
- $\langle 2 \rangle 2.$  PICK  $x \in \bigcap \mathbf{A}$  such that  $y = \mathbf{F}(x)$
- $\langle 2 \rangle 3.$  LET:  $X \in \mathbf{A}$
- $\langle 2 \rangle 4.$   $x \in X$
- $\langle 2 \rangle 5.$   $y \in \mathbf{F}(X)$
- $\langle 1 \rangle 2.$  If  $\mathbf{F}$  is injective then  $\mathbf{F}(\bigcap \mathbf{A}) = \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
- $\langle 2 \rangle 1.$  ASSUME:  $\mathbf{F}$  is injective.
- $\langle 2 \rangle 2.$  ASSUME:  $\mathbf{A}$  is nonempty.
- $\langle 2 \rangle 3.$  LET:  $y \in \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
- $\langle 2 \rangle 4.$  PICK  $X_0 \in \mathbf{A}$
- $\langle 2 \rangle 5.$  PICK  $x \in X_0$  such that  $(x, y) \in \mathbf{F}$
- $\langle 2 \rangle 6.$   $x \in \bigcap \mathbf{A}$
- $\langle 3 \rangle 1.$  LET:  $X \in \mathbf{A}$
- $\langle 3 \rangle 2.$  PICK  $x' \in X$  such that  $(x', y) \in \mathbf{F}$ .
- $\langle 3 \rangle 3.$   $x = x'$
- PROOF:  $\langle 2 \rangle 1$
- $\langle 3 \rangle 4.$   $x \in X$
- $\langle 2 \rangle 7.$   $y \in \mathbf{F}(\bigcap \mathbf{A})$

□

**Proposition 3.1.26 (S).** For any classes  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  and  $\mathbf{F}$ , the following is a theorem:

Assume  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$ . Then

$$\mathbf{F}(\mathbf{C}) - \mathbf{F}(\mathbf{D}) \subseteq \mathbf{F}(\mathbf{C} - \mathbf{D}) .$$

Equality holds if  $\mathbf{F}$  is injective.

PROOF:

- $\langle 1 \rangle 1.$   $\mathbf{F}(\mathbf{C}) - \mathbf{F}(\mathbf{D}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
- $\langle 2 \rangle 1.$  LET:  $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
- $\langle 2 \rangle 2.$  PICK  $x \in \mathbf{A}$  such that  $y = \mathbf{F}(x)$
- $\langle 2 \rangle 3.$   $x \notin \mathbf{B}$
- $\langle 2 \rangle 4.$   $x \in \mathbf{A} - \mathbf{B}$
- $\langle 2 \rangle 5.$   $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
- $\langle 1 \rangle 2.$  If  $\mathbf{F}$  is injective then  $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) = \mathbf{F}(\mathbf{A} - \mathbf{B})$
- $\langle 2 \rangle 1.$  ASSUME:  $\mathbf{F}$  is injective.
- $\langle 2 \rangle 2.$  LET:  $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
- $\langle 2 \rangle 3.$  PICK  $x \in \mathbf{A} - \mathbf{B}$  such that  $y = \mathbf{F}(x)$
- $\langle 2 \rangle 4.$   $y \in \mathbf{F}(\mathbf{A})$
- $\langle 2 \rangle 5.$   $y \notin \mathbf{F}(\mathbf{B})$

- ⟨3⟩1. ASSUME: for a contradiction  $y \in \mathbf{F}(\mathbf{B})$   
 ⟨3⟩2. PICK  $x' \in \mathbf{B}$  such that  $y = \mathbf{F}(x')$   
 ⟨3⟩3.  $x = x'$   
 PROOF: ⟨2⟩1  
 ⟨3⟩4.  $x \in \mathbf{B}$   
 ⟨3⟩5. Q.E.D.  
 PROOF: This contradicts ⟨2⟩3.

□

### 3.1.6 Inverse Images

**Definition 3.1.27** (Inverse Image). Let  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{C} \subseteq \mathbf{B}$ . Then the *inverse image* of  $\mathbf{C}$  under  $\mathbf{F}$  is

$$\mathbf{F}^{-1}(\mathbf{C}) = \{x \in \mathbf{A} \mid \mathbf{F}(x) \in \mathbf{C}\} .$$

**Proposition Schema 3.1.28** (S). *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  and  $\mathbf{F}$ , the following is a theorem:*

*Assume  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{C} \subseteq \mathcal{PB}$ . Then*

$$\mathbf{F}^{-1}\left(\bigcap \mathbf{C}\right) = \bigcap \{\mathbf{F}^{-1}(X) \mid X \in \mathbf{C}\} .$$

PROOF:

$$\begin{aligned}
 x \in \mathbf{F}^{-1}\left(\bigcap \mathbf{C}\right) &\Leftrightarrow \mathbf{F}(x) \in \bigcap \mathbf{C} \\
 &\Leftrightarrow \forall X \in \mathbf{C}. \mathbf{F}(x) \in X \\
 &\Leftrightarrow \forall X \in \mathbf{C}. x \in \mathbf{F}^{-1}(X) \quad \square
 \end{aligned}$$

**Proposition Schema 3.1.29** (S). *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{D}$  and  $\mathbf{F}$ , the following is a theorem:*

*Assume  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{C}, \mathbf{D} \subseteq \mathbf{B}$ . Then*

$$\mathbf{F}^{-1}(\mathbf{C} - \mathbf{D}) = \mathbf{F}^{-1}(\mathbf{C}) - \mathbf{F}^{-1}(\mathbf{D}) .$$

PROOF:

$$\begin{aligned}
 x \in \mathbf{F}^{-1}(\mathbf{C} - \mathbf{D}) &\Leftrightarrow \mathbf{F}(x) \in \mathbf{C} - \mathbf{D} \\
 &\Leftrightarrow \mathbf{F}(x) \in \mathbf{C} \wedge \mathbf{F}(x) \notin \mathbf{D} \\
 &\Leftrightarrow x \in \mathbf{F}^{-1}(\mathbf{C}) \wedge x \notin \mathbf{F}^{-1}(\mathbf{D}) \\
 &\Leftrightarrow x \in \mathbf{F}^{-1}(\mathbf{C}) - \mathbf{F}^{-1}(\mathbf{D}) \quad \square
 \end{aligned}$$

### 3.1.7 Function Sets

**Proposition 3.1.30** (SF). *For any classes  $\mathbf{B}$  and  $\mathbf{F}$ , the following is a theorem:*

*Let  $A$  be a set. If  $\mathbf{F} : A \rightarrow \mathbf{B}$  then  $\mathbf{F}$  is a set.*

PROOF: By an Axiom of Replacement, we have  $R = \{\mathbf{F}(x) \mid x \in A\}$  is a set. Hence  $\mathbf{F}$  is a set since  $\mathbf{F} \subseteq A \times R$ . □

**Definition 3.1.31** (Dependent Product Class). Let  $I$  be a set and let  $\mathbf{H}(i)$  be a class for all  $i \in I$ . We write  $\prod_{i \in I} \mathbf{H}(i)$  for the class of all functions  $f : I \rightarrow \bigcup_{i \in I} \mathbf{H}(i)$  such that  $\forall i \in I. f(i) \in \mathbf{H}(i)$ .

We write  $\mathbf{B}^I$  for  $\prod_{i \in I} \mathbf{B}$  where  $\mathbf{B}$  does not depend on  $I$ .

**Proposition Schema 3.1.32** (SF). *Let  $I$  be a set. Let  $H(i)$  be a set for every  $i \in I$ . Then  $\prod_{i \in I} H(i)$  is a set.*

PROOF:

$\langle 1 \rangle 1$ .  $\{\mathbf{H}(i) \mid i \in I\}$  is a set.

PROOF: By an Axiom of Replacement.

$\langle 1 \rangle 2$ .  $\bigcup_{i \in I} \mathbf{H}(i)$  is a set.

$\langle 1 \rangle 3$ .  $\prod_{i \in I} \mathbf{H}(i)$  is a set.

PROOF: It is a subset of  $\mathcal{P}(I \times \bigcup_{i \in I} \mathbf{H}(i))$ .

□

**Proposition 3.1.33** (SFC). *Let  $I$  be a set. Let  $H(i)$  be a set for all  $i \in I$ . If  $\forall i \in I. H(i) \neq \emptyset$  then  $\prod_{i \in I} H(i) \neq \emptyset$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\forall i \in I. H(i) \neq \emptyset$

$\langle 1 \rangle 2$ . LET:  $R = \{(i, x) \mid i \in I, x \in H(i)\}$

$\langle 1 \rangle 3$ . PICK a function  $f : I \rightarrow \bigcup_{i \in I} H(i)$  such that  $f \subseteq R$

PROOF: Proposition 3.1.6.

$\langle 1 \rangle 4$ .  $f \in \prod_{i \in I} H(i)$

□

## 3.2 Equinumerosity

**Definition 3.2.1** (Equinumerous). Sets  $A$  and  $B$  are *equinumerous*,  $A \approx B$ , iff there exists a bijection between  $A$  and  $B$ .

## 3.3 Domination

**Definition 3.3.1** (Dominate). A set  $A$  is *dominated* by a set  $B$ ,  $A \preceq B$ , iff there exists an injection  $A \rightarrow B$ .

**Proposition 3.3.2** (SC). *Given sets  $A$  and  $B$ , if  $A \neq \emptyset$  or  $B = \emptyset$ , then we have  $A \preceq B$  iff there exists a surjective function  $B \rightarrow A$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $A \preceq B$  and  $A \neq \emptyset$  then there exists a surjective function  $B \rightarrow A$ .

$\langle 2 \rangle 1$ . ASSUME:  $f : A \rightarrow B$  be injective.

$\langle 2 \rangle 2$ . PICK  $a \in A$

$\langle 2 \rangle 3$ . LET:  $g : B \rightarrow A$  be the function defined by  $g(b) = f^{-1}(b)$  if  $b \in \text{ran } f$ ,  
and  $g(b) = a$  otherwise.

- $\langle 2 \rangle 4$ .  $g$  is surjective.
- $\langle 1 \rangle 2$ . If there exists a surjective function  $B \rightarrow A$  then  $A \preceq B$ .
- $\langle 2 \rangle 1$ . ASSUME: there exists a surjective function  $g : B \rightarrow A$
- $\langle 2 \rangle 2$ .  $\forall a \in A. \exists b \in B. g(b) = a$
- $\langle 2 \rangle 3$ . Choose a function  $f : A \rightarrow B$  such that  $\forall a \in A. g(f(a)) = a$
- $\langle 2 \rangle 4$ .  $f$  is injective.

□

## Chapter 4

# Category Theory

### 4.1 Categories

**Definition 4.1.1** (Category). A *category*  $\mathbf{C}$  consists of:

- a class of *objects*;
- for any objects  $X, Y$ , a set  $\mathbf{C}(X, Y)$  whose elements are called *morphisms*. We write  $f : X \rightarrow Y$  for  $f \in \mathbf{C}(X, Y)$ .
- for any morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , a morphism  $gf : X \rightarrow Z$

such that:

- For all  $f : X \rightarrow Y, g : Y \rightarrow Z, h : Z \rightarrow W$ , we have  $h(gf) = (hg)f$ .
- For every object  $X$ , there exists a morphism  $\text{id}_X : X \rightarrow X$  such that:
  - for any object  $Y$  and morphism  $f : X \rightarrow Y$  we have  $f\text{id}_X = f$ .
  - for any object  $Y$  and morphism  $f : Y \rightarrow X$  we have  $\text{id}_X f = f$ .

**Definition 4.1.2** (Category of Sets). Let  $\text{Set}$  be the category of sets and functions.

**Definition 4.1.3** (Category of Pointed Sets). The *category of pointed sets*  $\text{Set}_*$  has:

- objects all pairs  $(A, a)$  where  $A$  is a set and  $a \in A$ ;
- morphisms  $f : (A, a) \rightarrow (B, b)$  all functions  $f : A \rightarrow B$  such that  $f(a) = b$ .

**Definition 4.1.4** (Opposite Category). For any category  $\mathbf{C}$ , the *opposite category*  $\mathbf{C}^{\text{op}}$  is the category with the same objects as  $\mathbf{C}$  and  $\mathbf{C}^{\text{op}}(X, Y) = \mathbf{C}(Y, X)$ .

## 4.2 Invertible Morphisms

**Definition 4.2.1** (Left Inverse, Right Inverse). In any category, let  $f : A \rightarrow B$  and  $g : B \rightarrow A$ . Then  $g$  is a *left inverse* of  $f$ , and  $f$  is a *right inverse* of  $f$ , iff  $gf = \text{id}_A$ .

**Proposition 4.2.2** (FOL). Let  $f : A \rightarrow B$  and  $g, h : B \rightarrow A$ . If  $g$  is a left inverse to  $f$  and  $h$  is a right inverse to  $f$  then  $g = h$ .

PROOF: Since  $g = g\text{id}_B = gfh = \text{id}_Ah = h$ .  $\square$

**Definition 4.2.3** (Isomorphism). A morphism  $f : A \rightarrow B$  is an *isomorphism*,  $f : A \cong B$ , iff it has both a left and a right inverse. In this case, its unique inverse is denoted  $f^{-1}$ .

Two objects  $A$  and  $B$  are *isomorphic*,  $A \cong B$ , iff there exists an isomorphism between them.

**Proposition 4.2.4** (S). A function is an isomorphism in  $\text{Set}$  iff it is a bijection.

**Proposition 4.2.5** (S). Let  $f : A \rightarrow B$  in the category  $\mathbf{C}$ . Then the following are equivalent.

1.  $f$  is an isomorphism.
2. For every object  $X$ , the function  $\mathbf{C}(X, f) : \mathbf{C}(X, A) \rightarrow \mathbf{C}(X, B)$  is a bijection.
3. For every object  $X$ , the function  $\mathbf{C}(f, X) : \mathbf{C}(B, X) \rightarrow \mathbf{C}(A, X)$  is a bijection.

## Chapter 5

# Equivalence Relations

**Definition 5.0.1** (Equivalence Relation). An *equivalence relation* on a class  $\mathbf{A}$  is a binary relation on  $\mathbf{A}$  that is reflexive, symmetric and transitive.

**Proposition 5.0.2** (Z). *Equinumerosity is an equivalence relation on the class of all sets.*

PROOF: Propositions 3.1.16, 3.1.17, 3.1.18.  $\square$

**Definition 5.0.3** (Respects). Let  $\mathbf{R}$  be an equivalence relation on  $\mathbf{A}$  and  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ . Then  $\mathbf{F}$  *respects*  $\mathbf{A}$  iff, whenever  $(x, y) \in \mathbf{R}$ , then  $\mathbf{F}(x) = \mathbf{F}(y)$ .

**Definition 5.0.4** (Equivalence Class). Let  $\mathbf{R}$  be an equivalence relation on  $\mathbf{A}$  and  $a \in \mathbf{A}$ . The *equivalence class* of  $a$  modulo  $\mathbf{R}$  is

$$[a]_{\mathbf{R}} := \{x \mid a\mathbf{R}x\} .$$

**Proposition Schema 5.0.5** (Z). *For any classes  $\mathbf{A}$  and  $\mathbf{R}$ , the following is a theorem.*

*Assume  $\mathbf{R}$  be an equivalence relation on  $\mathbf{A}$ . Let  $a, b \in \mathbf{A}$ . Then  $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$  if and only if  $a\mathbf{R}b$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$  then  $a\mathbf{R}b$ .

$\langle 2 \rangle 1$ . ASSUME:  $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$

$\langle 2 \rangle 2$ .  $b\mathbf{R}b$

PROOF: Reflexivity

$\langle 2 \rangle 3$ .  $b \in [b]_{\mathbf{R}}$

$\langle 2 \rangle 4$ .  $b \in [a]_{\mathbf{R}}$

$\langle 2 \rangle 5$ .  $a\mathbf{R}b$

$\langle 1 \rangle 2$ . If  $a\mathbf{R}b$  then  $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ .

$\langle 2 \rangle 1$ . For all  $x, y \in \mathbf{A}$ , if  $x\mathbf{R}y$  then  $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$

$\langle 3 \rangle 1$ . LET:  $x, y \in \mathbf{A}$

$\langle 3 \rangle 2$ . ASSUME:  $x\mathbf{R}y$

$\langle 3 \rangle 3$ . LET:  $t \in [y]_{\mathbf{R}}$   
 $\langle 3 \rangle 4$ .  $y \mathbf{R} t$   
 $\langle 3 \rangle 5$ .  $x \mathbf{R} t$   
 PROOF: Transitivity,  $\langle 3 \rangle 2$ ,  $\langle 3 \rangle 4$ .  
 $\langle 3 \rangle 6$ .  $t \in [x]_{\mathbf{R}}$   
 $\langle 2 \rangle 2$ . ASSUME:  $a \mathbf{R} b$   
 $\langle 2 \rangle 3$ .  $[b]_{\mathbf{R}} \subseteq [a]_{\mathbf{R}}$   
 PROOF:  $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 2$ .  
 $\langle 2 \rangle 4$ .  $b \mathbf{R} a$   
 PROOF: Symmetry,  $\langle 2 \rangle 2$ .  
 $\langle 2 \rangle 5$ .  $[a]_{\mathbf{R}} \subseteq [b]_{\mathbf{R}}$   
 PROOF:  $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 4$ .  
 $\langle 2 \rangle 6$ .  $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$   
 PROOF:  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 5$ .

□

**Definition 5.0.6** (Partition). A *partition*  $\Pi$  of a set  $A$  is a set of nonempty subsets of  $A$  that is disjoint and exhaustive, i.e.

1. no two different sets in  $\Pi$  have any common elements, and
2. each element of  $A$  is in some set in  $\Pi$ .

**Definition 5.0.7**. Let  $R$  be an equivalence relation on a set  $A$ . The *quotient set*  $A/R$  is the set of all equivalence classes.

**Theorem 5.0.8** (Z). Let  $A$  be a set and  $\mathbf{B}$  a class. Let  $R$  be an equivalence relation on  $A$  and  $F : A \rightarrow \mathbf{B}$ . Then  $F$  respects  $R$  if and only if there exists  $\hat{F} : A/R \rightarrow \mathbf{B}$  such that

$$\forall a \in A. \hat{F}([a]_R) = F(a) \text{ .}$$

In this case,  $\hat{F}$  is unique.

PROOF:

$\langle 1 \rangle 1$ . If  $F$  respects  $R$  then there exists  $\hat{F} : A/R \rightarrow \mathbf{B}$  such that  $\forall a \in A. \hat{F}([a]_R) = F(a)$ .  
 $\langle 2 \rangle 1$ . ASSUME:  $F$  respects  $R$ .  
 $\langle 2 \rangle 2$ . LET:  $\hat{F} = \{([a]_R, F(a)) \mid a \in A\}$   
 $\langle 2 \rangle 3$ .  $\hat{F}$  is a function.  
 $\langle 3 \rangle 1$ . ASSUME:  $a, a' \in A$  and  $[a]_R = [a']_R$   
 PROVE:  $F(a) = F(a')$   
 $\langle 3 \rangle 2$ .  $(a, a') \in R$   
 PROOF: Proposition 5.0.5.  
 $\langle 3 \rangle 3$ .  $F(a) = F(a')$   
 PROOF:  $\langle 2 \rangle 1$   
 $\langle 2 \rangle 4$ .  $\text{dom } \hat{F} = A/R$   
 $\langle 2 \rangle 5$ .  $\text{ran } \hat{F} \subseteq \mathbf{B}$



- $\langle 2 \rangle 6. \forall a \in A. \hat{F}([a]_R) = F(a)$   
 $\langle 1 \rangle 2.$  If there exists  $\hat{F} : A/R \rightarrow \mathbf{B}$  such that  $\forall a \in A. \hat{F}([a]_R) = F(a)$  then  $F$  respects  $R$ .  
 $\langle 2 \rangle 1.$  ASSUME:  $\hat{F} : A/R \rightarrow \mathbf{B}$  and  $\forall a \in A. \hat{F}([a]_R) = F(a)$   
 $\langle 2 \rangle 2.$  LET:  $a, a' \in A$   
 $\langle 2 \rangle 3.$  ASSUME:  $(a, a') \in R$   
 $\langle 2 \rangle 4.$   $[a]_R = [a']_R$   
 PROOF: Proposition 5.0.5.  
 $\langle 2 \rangle 5.$   $F(a) = F(a')$   
 PROOF:  $\langle 2 \rangle 1$   
 $\langle 1 \rangle 3.$  If  $G, H : A/R \rightarrow \mathbf{B}$  and  $\forall a \in A. G([a]_R) = H([a]_R)$  then  $G = H$ .  
 $\square$

**Proposition 5.0.9 (Z).** *Let  $R$  be an equivalence relation on a set  $A$ . Then  $A/R$  is a partition of  $A$ .*

PROOF:

- $\langle 1 \rangle 1.$  Every member of  $A/R$  is nonempty.  
 PROOF: Since  $a \in [a]_R$  by reflexivity.  
 $\langle 1 \rangle 2.$  No two different sets in  $A/R$  have any common elements.  
 $\langle 2 \rangle 1.$  LET:  $[a]_R, [b]_R \in A/R$   
 $\langle 2 \rangle 2.$  LET:  $c \in [a]_R \cap [b]_R$   
 PROVE:  $[a]_R = [b]_R$   
 $\langle 2 \rangle 3.$   $aRc$   
 PROOF:  $\langle 2 \rangle 2$   
 $\langle 2 \rangle 4.$   $bRc$   
 PROOF:  $\langle 2 \rangle 2$   
 $\langle 2 \rangle 5.$   $cRb$   
 PROOF: Symmetry,  $\langle 2 \rangle 4$   
 $\langle 2 \rangle 6.$   $aRb$   
 PROOF: Transitivity,  $\langle 2 \rangle 3, \langle 2 \rangle 5$   
 $\langle 2 \rangle 7.$   $[a]_R = [b]_R$   
 PROOF: Proposition 5.0.5,  $\langle 2 \rangle 6$   
 $\langle 1 \rangle 3.$  Each element of  $A$  is in some set in  $A/R$ .  
 PROOF: Since  $a \in [a]_R$  by reflexivity.  
 $\square$

**Proposition 5.0.10 (Z).** *For any partition  $P$  of a set  $A$ , there exists a unique equivalence relation  $R$  on  $A$  such that  $A/R = P$ , namely  $xRy$  iff  $\exists X \in P(x \in X \wedge y \in X)$ .*

PROOF: Easy.  $\square$

**Definition 5.0.11 (Natural Map).** Let  $A$  be a set and  $R$  an equivalence relation on  $A$ . The *natural map*  $A \rightarrow A/R$  is the function that maps  $a \in A$  to  $[a]_R$ .



## Chapter 6

# Ordering Relations

### 6.1 Partial Orders

**Definition 6.1.1** (Partial Ordering). Let  $\mathbf{A}$  be a class. A *partial ordering* on  $\mathbf{A}$  is a relation  $\mathbf{R}$  on  $\mathbf{A}$  that is reflexive, antisymmetric and transitive.

We often write  $\leq$  for a partial ordering, and then write  $x < y$  for  $x \leq y \wedge x \neq y$ .

**Proposition Schema 6.1.2** (Z). *For any classes  $\mathbf{A}$  and  $\mathbf{R}$ , the following is a theorem:*

*If  $\mathbf{R}$  is a partial order on  $\mathbf{A}$  then so is  $\mathbf{R}^{-1}$ .*

PROOF:

$\langle 1 \rangle 1.$   $\mathbf{R}^{-1}$  is reflexive.

PROOF: Proposition 2.2.12.

$\langle 1 \rangle 2.$   $\mathbf{R}^{-1}$  is antisymmetric.

PROOF: Proposition 2.2.16.

$\langle 1 \rangle 3.$   $\mathbf{R}^{-1}$  is transitive.

$\langle 2 \rangle 1.$  ASSUME:  $x\mathbf{R}^{-1}y$  and  $y\mathbf{R}^{-1}z$

$\langle 2 \rangle 2.$   $y\mathbf{R}x$  and  $z\mathbf{R}y$

$\langle 2 \rangle 3.$   $z\mathbf{R}x$

PROOF: Since  $\mathbf{R}$  is transitive.

$\langle 2 \rangle 4.$   $x\mathbf{R}^{-1}z$

□

**Proposition Schema 6.1.3** (Z). *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{F}$  and  $\mathbf{R}$ , the following is a theorem:*

*Assume  $\mathbf{R}$  is a partial order on  $\mathbf{B}$  and  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  is injective. Define  $\mathbf{S}$  on  $\mathbf{A}$  by  $x\mathbf{S}y$  iff  $\mathbf{F}(x)\mathbf{R}\mathbf{F}(y)$ . Then  $\mathbf{S}$  is a partial order on  $\mathbf{A}$ .*

PROOF:

$\langle 1 \rangle 1.$   $\mathbf{S}$  is reflexive.

PROOF: For any  $x \in \mathbf{A}$  we have  $\mathbf{F}(x)\mathbf{R}\mathbf{F}(x)$ .

- ⟨1⟩2. **S** is antisymmetric.  
 ⟨2⟩1. LET:  $x, y \in \mathbf{A}$   
 ⟨2⟩2. ASSUME:  $x\mathbf{S}y$  and  $y\mathbf{S}x$   
 ⟨2⟩3.  $\mathbf{F}(x)\mathbf{R}\mathbf{F}(y)$  and  $\mathbf{F}(y)\mathbf{R}\mathbf{F}(x)$   
 ⟨2⟩4.  $\mathbf{F}(x) = \mathbf{F}(y)$   
 PROOF: **R** is antisymmetric.  
 ⟨2⟩5.  $x = y$   
 ⟨1⟩3. **S** is transitive.  
 □

**Corollary Schema 6.1.3.1 (Z).** *For any classes **A**, **B** and **R**, the following is a theorem:*

*Assume **R** be a partial order on **A** and  $\mathbf{B} \subseteq \mathbf{A}$ . Then  $\mathbf{R} \cap \mathbf{B}^2$  is a partial order on **B**.*

**Definition 6.1.4** (Partially Ordered Set). A *partially ordered set* or *poset* is a pair  $(A, \leq)$  where  $A$  is a set and  $\leq$  is a partial ordering on  $A$ . We often write just  $A$  for  $(A, \leq)$ .

If  $(A, \leq)$  is a poset and  $B \subseteq A$  we write just  $B$  for the poset  $(B, \leq \cap B^2)$ .

**Definition 6.1.5** (Strictly Monotone). Let  $(A, <_A)$  and  $(B, <_B)$  be posets. A function  $f : A \rightarrow B$  is *strictly monotone* iff, whenever  $x <_A y$ , then  $f(x) <_B f(y)$ .

**Definition 6.1.6** (Least). Let  $\leq$  be a partial order on **A**. An element  $m \in \mathbf{A}$  is *least* iff for all  $x \in \mathbf{A}$  we have  $m \leq x$ .

**Proposition 6.1.7 (Z).** *A partial order has at most one least element.*

PROOF: If  $m$  and  $m'$  are least then  $m \leq m'$  and  $m' \leq m$ , so  $m = m'$ . □

**Definition 6.1.8** (Greatest). Let  $\leq$  be a partial order on **A**. An element  $m \in \mathbf{A}$  is *greatest* iff for all  $x \in \mathbf{A}$  we have  $x \leq m$ .

**Proposition 6.1.9 (Z).** *A poset has at most one greatest element.*

PROOF: If  $m$  and  $m'$  are greatest then  $m \leq m'$  and  $m' \leq m$ , so  $m = m'$ . □

**Definition 6.1.10** (Upper Bound). Let  $\leq$  be a partial ordering on **A** and  $\mathbf{B} \subseteq \mathbf{A}$ . Let  $u \in \mathbf{A}$ . Then  $u$  is an *upper bound* for **B** iff  $\forall x \in \mathbf{B}. x \leq u$ .

**Definition 6.1.11** (Lower Bound). Let  $\leq$  be a partial ordering on **A** and  $\mathbf{B} \subseteq \mathbf{A}$ . Let  $l \in \mathbf{A}$ . Then  $l$  is a *lower bound* for **B** iff  $\forall x \in \mathbf{B}. l \leq x$ .

**Definition 6.1.12** (Bounded Above). Let  $\leq$  be a partial ordering on **A** and  $\mathbf{B} \subseteq \mathbf{A}$ . Then **B** is *bounded above* iff it has an upper bound.

**Definition 6.1.13** (Bounded Below). Let  $\leq$  be a partial ordering on **A** and  $\mathbf{B} \subseteq \mathbf{A}$ . Then **B** is *bounded below* iff it has a lower bound.

**Definition 6.1.14** (Least Upper Bound). Let  $\leq$  be a partial ordering on  $\mathbf{A}$  and  $\mathbf{B} \subseteq \mathbf{A}$ . Let  $s \in \mathbf{A}$ . Then  $s$  is the *least upper bound* or *supremum* of  $\mathbf{B}$  iff  $s$  is an upper bound for  $\mathbf{B}$  and, for every upper bound  $u$  for  $\mathbf{B}$ , we have  $s \leq u$ .

**Definition 6.1.15** (Greatest Lower Bound). Let  $\leq$  be a partial ordering on  $\mathbf{A}$  and  $\mathbf{B} \subseteq \mathbf{A}$ . Let  $i \in \mathbf{A}$ . Then  $i$  is the *greatest lower bound* or *infimum* of  $\mathbf{B}$  iff  $i$  is a lower bound for  $\mathbf{B}$  and, for every lower bound  $l$  for  $\mathbf{B}$ , we have  $i \leq l$ .

**Definition 6.1.16** (Complete). A partial order is *complete* iff every nonempty subset bounded above has a supremum, and every nonempty subset bounded below has an infimum.

**Definition 6.1.17** (Order Isomorphism). Let  $A$  and  $B$  be posets. An *order isomorphism* between  $A$  and  $B$ ,  $f : A \cong B$ , is a bijection  $f : A \approx B$  such that  $f$  and  $f^{-1}$  are monotone.

**Theorem 6.1.18** (Knaster Fixed-Point Theorem (Z)). *Let  $A$  be a complete poset with a greatest and least element. Let  $\phi : A \rightarrow A$  be monotone. Then there exists  $a \in A$  such that  $\phi(a) = a$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $B = \{x \in A \mid x \leq \phi(x)\}$

$\langle 1 \rangle 2$ . LET:  $a = \sup B$

PROOF:  $B$  is nonempty because the least element of  $A$  is in  $B$ , and it is bounded above by the greatest element of  $A$ .

$\langle 1 \rangle 3$ . For all  $b \in B$  we have  $b \leq \phi(a)$

$\langle 2 \rangle 1$ . LET:  $b \in B$

$\langle 2 \rangle 2$ .  $b \leq \phi(b)$

$\langle 2 \rangle 3$ .  $b \leq a$

$\langle 2 \rangle 4$ .  $\phi(b) \leq \phi(a)$

$\langle 2 \rangle 5$ .  $b \leq \phi(a)$

$\langle 1 \rangle 4$ .  $a \leq \phi(a)$

$\langle 1 \rangle 5$ .  $\phi(a) \leq \phi(\phi(a))$

$\langle 1 \rangle 6$ .  $\phi(a) \in B$

$\langle 1 \rangle 7$ .  $\phi(a) \leq a$

$\langle 1 \rangle 8$ .  $\phi(a) = a$

□

**Definition 6.1.19** (Dense). Let  $\leq$  be a partial order on  $\mathbf{A}$  and  $\mathbf{B} \subseteq \mathbf{A}$ . Then  $\mathbf{B}$  is *dense* iff, for all  $x, y \in \mathbf{A}$ , if  $x < y$  then there exists  $z \in \mathbf{B}$  such that  $x < z < y$ .

**Proposition 6.1.20** (Z). *Let  $A$  be a complete poset with no least element. Let  $B \subseteq A$  be dense. Let  $\theta : A \rightarrow A$  be a monotone map that is the identity on  $B$ . Then  $\theta = \text{id}_A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $a \in A$

PROVE:  $\theta(a) = a$

- ⟨1⟩2. LET:  $S(a) = \{b \in B \mid b < a\}$
- ⟨1⟩3.  $S(a)$  is nonempty and bounded above.
  - ⟨2⟩1.  $S(a)$  is nonempty.
    - ⟨3⟩1. PICK  $a_1 < a$ 
      - PROOF: Since  $a$  is not least.
    - ⟨3⟩2. There exists  $b \in B$  such that  $a_1 < b < a$ .
  - ⟨2⟩2.  $S(a)$  is bounded above by  $a$ .
- ⟨1⟩4.  $\sup S(a) \leq a$
- ⟨1⟩5.  $\sup S(a) = a$ 
  - ⟨2⟩1. ASSUME: for a contradiction  $\sup S(a) < a$
  - ⟨2⟩2. PICK  $b \in B$  such that  $\sup S(a) < b < a$
  - ⟨2⟩3.  $b \in S(a)$
  - ⟨2⟩4. Q.E.D.
  - PROOF: This contradicts the fact that  $\sup S(a) < b$ .
- ⟨1⟩6. For all  $b \in S(a)$  we have  $b \leq \theta(a)$ 
  - ⟨2⟩1. LET:  $b \in S(a)$
  - ⟨2⟩2.  $b < a$
  - ⟨2⟩3.  $\theta(b) \leq \theta(a)$
  - ⟨2⟩4.  $b \leq \theta(a)$ 
    - PROOF:  $\theta(b) = b$
- ⟨1⟩7.  $a \leq \theta(a)$ 
  - PROOF: Since  $a = \sup S(a)$  and  $\theta(a)$  is an upper bound for  $S(a)$ .
- ⟨1⟩8.  $a \not\leq \theta(a)$ 
  - ⟨2⟩1. ASSUME: for a contradiction  $a < \theta(a)$ .
  - ⟨2⟩2. PICK  $b \in B$  such that  $a < b < \theta(a)$
  - ⟨2⟩3.  $\theta(a) \leq \theta(b) = b$
  - ⟨2⟩4. Q.E.D.
  - PROOF: This contradicts the fact that  $b < \theta(a)$ .
- ⟨1⟩9.  $\theta(a) = a$

□

**Theorem 6.1.21 (Z).** *Let  $A$  and  $P$  be complete posets with no least or greatest element. Let  $B$  be dense in  $A$  and  $Q$  be dense in  $P$ . Every order isomorphism  $\phi : B \cong Q$  extends uniquely to an order isomorphism  $A \cong P$ .*

PROOF:

- ⟨1⟩1. For  $a \in A$ , let  $S(a) = \{b \in B \mid b < a\}$ .
- ⟨1⟩2. Define  $\bar{\phi} : A \rightarrow P$  by  $\bar{\phi}(a) = \sup \phi(S(a))$ .
  - ⟨2⟩1.  $\phi(S(a))$  is nonempty.
    - ⟨3⟩1. PICK  $a_1 < a$ 
      - PROOF: Since  $a$  is not least.
    - ⟨3⟩2. PICK  $b \in B$  such that  $a_1 < b < a$ .
    - ⟨3⟩3.  $\phi(b) \in \phi(S(a))$
  - ⟨2⟩2.  $\phi(S(a))$  is bounded above.
    - ⟨3⟩1. PICK  $a_2 > a$ 
      - PROOF: Since  $a$  is not greatest.
    - ⟨3⟩2. PICK  $b \in B$  such that  $a < b < a_2$

- (3)3.  $\phi(b)$  is an upper bound for  $\phi(S(a))$ .  
 (1)3.  $\bar{\phi}$  is monotone.  
 PROOF: If  $a \leq a'$  then  $S(a) \subseteq S(a')$  and so  $\bar{\phi}(a) \leq \bar{\phi}(a')$ .  
 (1)4.  $\bar{\phi}$  extends  $\phi$ .  
 (2)1. LET:  $b \in B$   
 PROVE:  $\phi(b) = \sup \phi(S(b))$   
 (2)2.  $\phi(b)$  is an upper bound for  $\phi(S(b))$   
 (2)3. LET:  $u$  be any upper bound for  $\phi(S(b))$   
 PROVE:  $\phi(b) \leq u$   
 (2)4. ASSUME: for a contradiction  $u < \phi(b)$   
 (2)5. PICK  $q \in Q$  such that  $u < q < \phi(b)$   
 (2)6. PICK  $b' \in B$  such that  $\phi(b') = q$   
 (2)7.  $b' < b$   
 (2)8.  $b' \in S(b)$   
 (2)9.  $q = \phi(b') \leq u$   
 (2)10. Q.E.D.  
 PROOF: This is a contradiction.  
 (1)5. LET:  $\bar{\psi} = \phi^{-1}$   
 (1)6. LET:  $\bar{\psi} : P \rightarrow A$  be the function  $\bar{\psi}(p) = \sup\{\psi(q) \mid q \in Q, q < p\}$   
 (1)7.  $\bar{\psi}$  is monotone and extends  $\psi$   
 PROOF: Similar.  
 (1)8.  $\bar{\psi} \circ \bar{\phi} : A \rightarrow A$  is monotone and the identity on  $B$ .  
 (1)9.  $\bar{\psi} \circ \bar{\phi} = \text{id}_A$   
 PROOF: Proposition 6.1.20.  
 (1)10.  $\bar{\phi} \circ \bar{\psi} = \text{id}_B$   
 PROOF: Proposition 6.1.20.  
 (1)11. If  $\phi^* : A \cong P$  is any order isomorphism that extends  $\phi$  then  $\phi^* = \bar{\phi}$ .  
 (2)1. LET:  $a \in A$   
 PROVE:  $\phi^*(a) = \sup \phi(S(a))$   
 (2)2.  $\phi^*(a)$  is an upper bound for  $\phi(S(a))$   
 (2)3. LET:  $u$  be any upper bound for  $\phi(S(a))$   
 PROVE:  $\phi^*(a) \leq u$   
 (2)4. ASSUME: for a contradiction  $u < \phi^*(a)$   
 (2)5. PICK  $q \in Q$  such that  $u < q < \phi^*(a)$   
 (2)6. PICK  $b \in B$  such that  $q = \phi(b)$   
 (2)7.  $b < a$   
 (2)8.  $b \in S(a)$   
 (2)9.  $q = \phi(b) \leq u$   
 (2)10. Q.E.D.  
 PROOF: This is a contradiction.

□

**Definition 6.1.22** (Initial Segment). Let  $\leq$  be a partial order on  $\mathbf{A}$  and  $t \in A$ . The *initial segment* up to  $t$  is the class

$$\text{seg } t := \{x \in \mathbf{A} \mid x < t\} .$$

**Definition 6.1.23** (Lexicographic Ordering). Let  $\mathbf{R}$  be a partial order on  $\mathbf{A}$  and  $\mathbf{S}$  a partial order on  $\mathbf{B}$ . The *lexicographic ordering*  $\leq$  on  $\mathbf{A} \times \mathbf{B}$  is defined by:

$$(a, b) \leq (a', b') \Leftrightarrow (a\mathbf{R}a' \wedge a \neq a') \vee (a = a' \wedge b\mathbf{S}b') .$$

**Proposition Schema 6.1.24** (Z). *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{R}$  and  $\mathbf{S}$ , the following is a theorem:*

*If  $\mathbf{R}$  is a partial order on  $\mathbf{A}$  and  $\mathbf{S}$  is a partial order on  $\mathbf{B}$  then the lexicographic ordering on  $\mathbf{A} \times \mathbf{B}$  is a partial order.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\leq$  be the lexicographic ordering on  $\mathbf{A} \times \mathbf{B}$

$\langle 1 \rangle 2$ .  $\leq$  is reflexive.

PROOF: For any  $a \in \mathbf{A}$  and  $b \in \mathbf{B}$  we have  $a = a$  and  $b\mathbf{S}b$ , so  $(a, b) \leq (a, b)$ .

$\langle 1 \rangle 3$ .  $\leq$  is antisymmetric.

$\langle 2 \rangle 1$ . ASSUME:  $(a, b) \leq (a', b')$  and  $(a', b') \leq (a, b)$

$\langle 2 \rangle 2$ .  $(a\mathbf{R}a' \wedge a \neq a') \vee (a = a' \wedge b\mathbf{S}b')$

$\langle 2 \rangle 3$ .  $(a'\mathbf{R}a \wedge a' \neq a) \vee (a' = a \wedge b\mathbf{S}b')$

$\langle 2 \rangle 4$ . CASE:  $a = a'$

PROOF: Then  $b\mathbf{S}b'$  and  $b'\mathbf{S}b$  hence  $b = b'$  and  $(a, b) = (a', b')$ .

$\langle 2 \rangle 5$ . CASE:  $a \neq a'$

PROOF: Then  $a\mathbf{R}a'$  and  $a'\mathbf{R}a$  hence  $a = a'$  which is a contradiction.

$\langle 1 \rangle 4$ .  $\leq$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $(a_1, b_1) \leq (a_2, b_2) \leq (a_3, b_3)$

$\langle 2 \rangle 2$ .  $(a_1\mathbf{R}a_2 \wedge a_1 \neq a_2) \vee (a_1 = a_2 \wedge b_1\mathbf{S}b_2)$

$\langle 2 \rangle 3$ .  $(a_2\mathbf{R}a_3 \wedge a_2 \neq a_3) \vee (a_2 = a_3 \wedge b_2\mathbf{S}b_3)$

$\langle 2 \rangle 4$ . CASE:  $a_1\mathbf{R}a_2, a_1 \neq a_2, a_2\mathbf{R}a_3, a_2 \neq a_3$

$\langle 3 \rangle 1$ .  $a_1\mathbf{R}a_3$

PROOF: Since  $\mathbf{R}$  is transitive.

$\langle 3 \rangle 2$ .  $a_1 \neq a_3$

PROOF: If  $a_1 = a_3$  then  $a_1\mathbf{R}a_2$  and  $a_2\mathbf{R}a_1$  so  $a_1 = a_2$  which is a contradiction.

$\langle 2 \rangle 5$ . CASE:  $a_1\mathbf{R}a_2, a_1 \neq a_2, a_2 = a_3, b_2\mathbf{S}b_3$

PROOF: Then  $a_1\mathbf{R}a_3$  and  $a_1 \neq a_3$ .

$\langle 2 \rangle 6$ . CASE:  $a_1 = a_2, b_1\mathbf{S}b_2, a_2\mathbf{R}a_3, a_2 \neq a_3$

PROOF: Then  $a_1\mathbf{R}a_3$  and  $a_1 \neq a_3$ .

$\langle 2 \rangle 7$ . CASE:  $a_1 = a_2, b_1\mathbf{S}b_2, a_2 = a_3, b_2\mathbf{S}b_3$

PROOF: Then  $a_1 = a_3$  and  $b_1\mathbf{S}b_3$ .

□

## 6.2 Linear Orders

**Definition 6.2.1** (Linear Ordering). Let  $\mathbf{A}$  be a class. A *linear ordering* or *total ordering* on  $\mathbf{A}$  is a partial ordering  $\leq$  on  $\mathbf{A}$  that is *total*, i.e.

$$\forall x, y \in \mathbf{A}. x \leq y \vee y \leq x$$



We often use the symbol  $<$  for a linear ordering, and then write  $x < y$  for  $(x, y) \in <$ .

**Proposition Schema 6.2.2** (Trichotomy (Z)). *For any classes  $\mathbf{A}$  and  $\leq$ , the following is a theorem:*

*Assume  $\leq$  be a linear ordering on  $\mathbf{A}$ . For any  $x, y \in \mathbf{A}$ , exactly one of  $x < y$ ,  $x = y$ ,  $y < x$  holds.*

PROOF: Immediate from definitions.  $\square$

**Proposition Schema 6.2.3** (Z). *For any classes  $\mathbf{A}$  and  $<$ , the following is a theorem:*

*Let  $<$  be a transitive relation on  $\mathbf{A}$  that satisfies trichotomy. Define  $\leq$  on  $\mathbf{A}$  by  $x \leq y$  iff  $x < y$  or  $x = y$ . Then  $\leq$  is a linear ordering on  $\mathbf{A}$  and  $x < y$  iff  $x \leq y$  and  $x \neq y$ .*

PROOF:

$\langle 1 \rangle 1.$   $\leq$  is reflexive.

PROOF: By definition we have  $\forall x \in \mathbf{A}. x \leq x$ .

$\langle 1 \rangle 2.$   $\leq$  is antisymmetric.

$\langle 2 \rangle 1.$  ASSUME:  $x \leq y$  and  $y \leq x$

$\langle 2 \rangle 2.$   $x < y$  or  $x = y$

$\langle 2 \rangle 3.$   $y < x$  or  $y = x$

$\langle 2 \rangle 4.$  We cannot have  $x < y$  and  $y < x$

PROOF: Trichotomy.

$\langle 2 \rangle 5.$   $x = y$

$\langle 1 \rangle 3.$   $\leq$  is transitive.

$\langle 2 \rangle 1.$  ASSUME:  $x \leq y$  and  $y \leq z$

$\langle 2 \rangle 2.$   $x < y$  or  $x = y$

$\langle 2 \rangle 3.$   $y < z$  or  $y = z$

$\langle 2 \rangle 4.$  CASE:  $x < y$  and  $y < z$

PROOF: Then  $x < z$  by transitivity, so  $x \leq z$ .

$\langle 2 \rangle 5.$  CASE:  $x = y$

PROOF: Then we have  $y \leq z$  and so  $x \leq z$ .

$\langle 2 \rangle 6.$  CASE:  $y = z$

PROOF: Then we have  $x \leq y$  and so  $x \leq z$ .

$\langle 1 \rangle 4.$   $\leq$  is total.

PROOF: Immediate from trichotomy.

$\square$

**Proposition Schema 6.2.4** (Z). *For any classes  $\mathbf{A}$  and  $\mathbf{R}$ , the following is a theorem:*

*If  $\mathbf{R}$  is a linear ordering on  $\mathbf{A}$  then  $\mathbf{R}^{-1}$  is also a linear ordering on  $\mathbf{A}$ .*

PROOF:

$\langle 1 \rangle 1.$   $\mathbf{R}^{-1}$  is a partial order on  $\mathbf{A}$ .

PROOF: Proposition 6.1.2.

$\langle 1 \rangle 2.$   $\mathbf{R}^{-1}$  is total.

- $\langle 2 \rangle 1.$  LET:  $x, y \in \mathbf{A}$
- $\langle 2 \rangle 2.$   $x\mathbf{R}y$  or  $y\mathbf{R}x$ .
- $\langle 2 \rangle 3.$   $y\mathbf{R}^{-1}x$  or  $x\mathbf{R}^{-1}y$ .

□

**Proposition Schema 6.2.5 (Z).** *For any classes  $\mathbf{A}, \mathbf{B}, \mathbf{F}, \mathbf{R}, \mathbf{S}$ , the following is a theorem:*

*Assume  $\mathbf{R}$  is a linear order on  $\mathbf{A}$ ,  $\mathbf{S}$  is a partial order on  $\mathbf{B}$ , and  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ . If  $\mathbf{F}$  is strictly monotone then it is injective.*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $x, y \in \mathbf{A}$
- $\langle 1 \rangle 2.$  ASSUME:  $x \neq y$   
PROVE:  $\mathbf{F}(x) \neq \mathbf{F}(y)$
- $\langle 1 \rangle 3.$  ASSUME: w.l.o.g.  $x\mathbf{R}y$   
PROOF:  $\mathbf{R}$  is total.
- $\langle 1 \rangle 4.$   $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$  and  $\mathbf{F}(x) \neq \mathbf{F}(y)$   
PROOF:  $\mathbf{F}$  is strictly monotone.

□

**Proposition Schema 6.2.6 (Z).** *For any classes  $\mathbf{A}, \mathbf{B}$ ,  $\leq$ ,  $\preceq$  and  $\mathbf{F}$ , the following is a theorem:*

*Assume  $\leq$  is a linear order on  $\mathbf{A}$  and  $\preceq$  is a linear order on  $\mathbf{B}$ . Assume  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{F}$  is strictly monotone. For all  $x, y \in \mathbf{A}$ , if  $\mathbf{F}(x) \prec \mathbf{F}(y)$  then  $x < y$ .*

PROOF:

- $\langle 1 \rangle 1.$   $\mathbf{F}(x) \neq \mathbf{F}(y)$  and  $\mathbf{F}(y) \not\prec \mathbf{F}(x)$   
PROOF: Trichotomy.
- $\langle 1 \rangle 2.$   $x \neq y$  and  $y \not\prec x$   
PROOF:  $\mathbf{F}$  is strictly monotone.
- $\langle 1 \rangle 3.$   $x < y$   
PROOF: Trichotomy.

□

**Corollary Schema 6.2.6.1 (Z).** *For any classes  $\mathbf{A}, \mathbf{B}$ ,  $\leq$ ,  $\preceq$  and  $\mathbf{F}$ , the following is a theorem:*

*Assume  $\leq$  is a linear order on  $\mathbf{A}$  and  $\preceq$  is a linear order on  $\mathbf{B}$ . Assume  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  and  $\mathbf{F}$  is strictly monotone. Then  $\mathbf{F}$  is an order isomorphism.*

**Proposition Schema 6.2.7 (Z).** *For any classes  $\mathbf{A}, \mathbf{B}, \mathbf{F}$  and  $\mathbf{S}$ , the following is a theorem:*

*Assume  $\mathbf{S}$  is a linear order on  $\mathbf{B}$  and  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ . Define  $\mathbf{R}$  on  $\mathbf{A}$  by  $x\mathbf{R}y$  if and only if  $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$ . Then  $\mathbf{R}$  is a linear order on  $\mathbf{A}$ .*

PROOF:

- $\langle 1 \rangle 1.$   $\mathbf{R}$  is a partial order on  $\mathbf{A}$ .  
PROOF: Proposition 6.1.3.

$\langle 1 \rangle 2$ .  $\mathbf{R}$  is total.

PROOF: For all  $x, y \in \mathbf{A}$  we have  $\mathbf{F}(x)\mathbf{SF}(y)$  or  $\mathbf{F}(y)\mathbf{SF}(x)$ .

□

**Corollary Schema 6.2.7.1** (Z). *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{R}$ , the following is a theorem:*

*Assume  $\mathbf{R}$  be a linear order on  $\mathbf{A}$  and  $\mathbf{B} \subseteq \mathbf{A}$ . Then  $\mathbf{R} \cap \mathbf{B}^2$  is a linear order on  $\mathbf{B}$ .*

**Proposition Schema 6.2.8** (Z). *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{R}$  and  $\mathbf{S}$ , the following is a theorem:*

*Assume  $\mathbf{R}$  is a linear order on  $\mathbf{A}$  and  $\mathbf{S}$  is a linear order on  $\mathbf{B}$ . Then the lexicographic ordering is a linear order on  $\mathbf{A} \times \mathbf{B}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\leq$  be the lexicographic order on  $\mathbf{A} \times \mathbf{B}$

$\langle 1 \rangle 2$ .  $\leq$  is a partial order.

PROOF: Proposition 6.1.24.

$\langle 1 \rangle 3$ .  $\leq$  is total.

$\langle 2 \rangle 1$ . LET:  $a, a' \in \mathbf{A}$  and  $b, b' \in \mathbf{B}$

$\langle 2 \rangle 2$ . CASE:  $a\mathbf{R}a'$  and  $a \neq a'$

PROOF: Then  $(a, b) \leq (a', b')$ .

$\langle 2 \rangle 3$ . CASE:  $a = a'$

PROOF: We have  $b\mathbf{S}b'$  or  $b'\mathbf{S}b$ , so  $(a, b) \leq (a', b')$  or  $(a', b') \leq (a, b)$ .

$\langle 2 \rangle 4$ . CASE:  $a'\mathbf{R}a$  and  $a \neq a'$

PROOF: Then  $(a', b') \leq (a, b)$ .

□

## 6.3 Well Orderings

**Definition 6.3.1** (Well Ordering). A *well ordering* on a class  $\mathbf{A}$  is a well-founded linear ordering on  $\mathbf{A}$ .

**Proposition 6.3.2** (Z). *Let  $S$  be a well ordering of the set  $B$  and  $f : A \rightarrow B$  a function. Define  $R$  on  $A$  by  $xRy$  if and only if  $F(x)SF(y)$ . Then  $R$  well orders  $A$ .*

PROOF:

$\langle 1 \rangle 1$ .  $R$  linearly orders  $A$ .

PROOF: Proposition 6.2.7.

$\langle 1 \rangle 2$ . Every nonempty subset of  $A$  has a least element.

$\langle 2 \rangle 1$ . LET:  $C$  be a nonempty subset of  $A$ .

$\langle 2 \rangle 2$ . LET:  $y$  be the least element of  $f(C)$ .

$\langle 2 \rangle 3$ . PICK  $x \in C$  such that  $f(x) = y$ .

$\langle 2 \rangle 4$ .  $x$  is least in  $C$ .

□

**Proposition Schema 6.3.3 (Z).** *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{R}$ , the following is a theorem:*

*Assume  $\mathbf{R}$  well orders  $\mathbf{B}$  and  $\mathbf{A} \subseteq \mathbf{B}$ . Then  $\mathbf{R} \cap \mathbf{A}^2$  well orders  $\mathbf{A}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathbf{R}' = \mathbf{R} \cap \mathbf{A}^2$

$\langle 1 \rangle 2$ .  $\mathbf{R}'$  linearly orders  $\mathbf{A}$ .

PROOF: Corollary 6.2.7.1.

$\langle 1 \rangle 3$ .  $\mathbf{R}'$  is well founded.

PROOF: Proposition 2.4.4.

□

**Proposition Schema 6.3.4 (ZFC).** *For any classes  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{F}$  and  $\mathbf{S}$ , the following is a theorem:*

*Assume  $\mathbf{S}$  well orders  $\mathbf{B}$  and  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ . Define  $\mathbf{R}$  on  $\mathbf{A}$  by  $x\mathbf{R}y$  if and only if  $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$ . Then  $\mathbf{R}$  well orders  $\mathbf{A}$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\mathbf{R}$  linearly orders  $\mathbf{A}$ .

PROOF: Proposition 6.2.7.

$\langle 1 \rangle 2$ . For all  $t \in \mathbf{A}$  we have  $\{x \in \mathbf{A} \mid x\mathbf{R}t \wedge x \neq t\}$  is a set.

$\langle 2 \rangle 1$ . LET:  $t \in \mathbf{A}$

$\langle 2 \rangle 2$ . LET:  $S = \{y \in \mathbf{B} \mid y\mathbf{S}\mathbf{F}(t) \wedge y \neq \mathbf{F}(t)\}$

$\langle 2 \rangle 3$ . LET:  $P(x, y)$  be the property  $\mathbf{F}(y) = x$

$\langle 2 \rangle 4$ . For all  $x \in S$  there exists at most one  $y$  such that  $P(x, y)$

PROOF:  $\mathbf{F}$  is injective.

$\langle 2 \rangle 5$ . LET:  $T = \{y \mid \exists x \in S. P(x, y)\}$

PROOF: Axiom of Replacement.

$\langle 2 \rangle 6$ .  $T = \{x \in \mathbf{A} \mid x\mathbf{R}t \wedge x \neq t\}$

$\langle 1 \rangle 3$ . Every nonempty subset of  $\mathbf{A}$  has a least element.

$\langle 2 \rangle 1$ . LET:  $S$  be a nonempty subset of  $\mathbf{A}$ .

$\langle 2 \rangle 2$ .  $\mathbf{F}(S)$  is a nonempty subset of  $\mathbf{B}$

PROOF: Axiom of Replacement.

$\langle 2 \rangle 3$ . LET:  $y$  be the least element of  $\mathbf{F}(S)$ .

$\langle 2 \rangle 4$ . PICK  $x \in S$  such that  $\mathbf{F}(x) = y$ .

$\langle 2 \rangle 5$ .  $x$  is least in  $S$ .

□

**Proposition 6.3.5 (Z).** *For any well ordered sets  $A$  and  $B$ , the lexicographic order well orders  $A \times B$ .*

PROOF:

$\langle 1 \rangle 1$ .  $A \times B$  is linearly ordered.

PROOF: Proposition 6.2.8.

$\langle 1 \rangle 2$ . Every nonempty subset of  $A \times B$  has a least element.

$\langle 2 \rangle 1$ . LET:  $S$  be a nonempty subset of  $A \times B$ .

$\langle 2 \rangle 2$ . LET:  $a$  be the least element of  $\{x \in A \mid \exists y \in B. (x, y) \in S\}$ .

$\langle 2 \rangle 3$ . LET:  $b$  be the least element of  $\{y \in B \mid (a, y) \in S\}$ .

⟨2⟩4.  $(a, b)$  is least in  $S$ .  
 $\square$

**Definition 6.3.6** (End Extension). Let  $A$  and  $B$  be well ordered sets. Then  $B$  is an *end extension* of  $A$  iff  $A \subseteq B$  and:

- Whenever  $x, y \in A$  then  $x \leq_A y$  iff  $x \leq_B y$ .
- Whenever  $x \in A$  and  $y \in B - A$  then  $x < y$ .

**Theorem 6.3.7** (Z). Let  $\leq$  be a linear ordering on  $A$ . Assume that, for any  $B \subseteq A$  such that  $\forall t \in A. \text{seg } t \subseteq B \Rightarrow t \in B$ , we have  $B = A$ . Then  $\leq$  is a well ordering on  $A$ .

PROOF:

- ⟨1⟩1. LET:  $C \subseteq A$  be nonempty.  
 ⟨1⟩2. LET:  $B = \{t \in A \mid \forall x \in C. t < x\}$   
 ⟨1⟩3.  $B \cap C = \emptyset$   
 ⟨1⟩4.  $B \neq A$   
 ⟨1⟩5. PICK  $t \in A$  such that  $\text{seg } t \subseteq B$  and  $t \notin B$   
 ⟨1⟩6.  $t$  is least in  $C$ .  
 $\square$

**Proposition Schema 6.3.8** (Z). For any classes  $\mathbf{A}, \mathbf{B}, \mathbf{F}, \mathbf{G}, \leq$  and  $\preceq$ , the following is a theorem:

Assume  $\leq$  well orders  $\mathbf{A}$  and  $\preceq$  well orders  $\mathbf{B}$ . Assume  $\mathbf{F}$  and  $\mathbf{G}$  are order isomorphisms between  $\mathbf{A}$  and  $\mathbf{B}$ . Then  $\mathbf{F} = \mathbf{G}$ .

PROOF:

- ⟨1⟩1. For all  $x \in \mathbf{A}$ , if  $\forall t < x. \mathbf{F}(t) = \mathbf{G}(t)$ , then  $\mathbf{F}(x) = \mathbf{G}(x)$   
 ⟨2⟩1. LET:  $x \in \mathbf{A}$   
 ⟨2⟩2. ASSUME:  $\forall t < x. \mathbf{F}(t) = \mathbf{G}(t)$   
 ⟨2⟩3.  $\mathbf{F}(\text{seg } x) = \mathbf{G}(\text{seg } x)$   
 ⟨2⟩4.  $\mathbf{F}(x)$  is the least element of  $\mathbf{B} - \mathbf{F}(\text{seg } x)$   
 ⟨2⟩5.  $\mathbf{G}(x)$  is the least element of  $\mathbf{B} - \mathbf{G}(\text{seg } x)$   
 ⟨2⟩6.  $\mathbf{F}(x) = \mathbf{G}(x)$   
 ⟨1⟩2.  $\forall x \in \mathbf{A}. \mathbf{F}(x) = \mathbf{G}(x)$

PROOF: Transfinite induction.

$\square$

**Theorem 6.3.9** (ZFC). Let  $A$  and  $B$  be well ordered sets. Then one of the following holds:  $A \cong B$ ; there exists  $b \in B$  such that  $A \cong \text{seg } b$ ; there exists  $a \in A$  such that  $\text{seg } a \cong B$ .

PROOF:

- ⟨1⟩1. PICK  $e$  that is not in  $A$  or  $B$ .  
 ⟨1⟩2. LET:  $F : A \rightarrow B \cup \{e\}$  be the function defined by transfinite recursion thus:

$$F(t) = \begin{cases} \text{the least element of } B - F(\text{seg } t) & \text{if } B - F(\text{seg } t) \neq \emptyset \\ e & \text{if } B - F(\text{seg } t) = \emptyset \end{cases}$$

$\langle 2 \rangle 1$ . LET:  $t$  be least such that  $F(t) = e$

$$\langle 2 \rangle 2. \quad F \upharpoonright \text{seg } t : \text{seg } t \cong B$$

PROOF: We have  $F : A \cong B$

$\langle 2 \rangle 1$ . LET:  $b$  be the least element of  $B - \text{ran } F$

$\langle 2 \rangle 2.$   $F : A \cong \text{seg } b$

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# Chapter 7

## Ordinal Numbers

### 7.1 Ordinals

**Definition 7.1.1** (Ordinal). A set  $\alpha$  is an *ordinal (number)* iff:

- $\alpha$  is a transitive set
- $\{(x, y) \in \alpha^2 \mid x \in y\}$  is a strict well ordering of  $\alpha$ .

Given ordinals  $\alpha, \beta$ , we write  $\alpha < \beta$  iff  $\alpha \in \beta$ , and  $\alpha \leq \beta$  iff  $\alpha \in \beta$  or  $\alpha = \beta$ .

Let **On** be the class of ordinals.

**Proposition 7.1.2** (S). *Let  $\alpha$  be a set. Then  $\alpha$  is an ordinal if and only if  $\alpha$  is a transitive set and, for all  $x, y \in \alpha$ , we have  $x \in y$  or  $x = y$  or  $y \in x$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\alpha$  be a set.

$\langle 1 \rangle 2$ . If  $\alpha$  is an ordinal then  $\alpha$  is a transitive set and, for all  $x, y \in \alpha$ , we have  $x \in y$  or  $x = y$  or  $y \in x$ .

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$ . If  $\alpha$  is a transitive set and  $\forall x, y \in \alpha (x \in y \vee x = y \vee y \in x)$ , then  $\alpha$  is an ordinal.

$\langle 2 \rangle 1$ . ASSUME:  $\alpha$  is a transitive set.

$\langle 2 \rangle 2$ . ASSUME:  $\forall x, y \in \alpha (x \in y \vee x = y \vee y \in x)$

$\langle 2 \rangle 3$ . LET:  $E = \{(x, y) \in \alpha^2 \mid x \in y\}$

$\langle 2 \rangle 4$ .  $\forall x, y, z \in \alpha. xEy \wedge yEz \Rightarrow xEz$

$\langle 3 \rangle 1$ . LET:  $x, y, z \in \alpha$

$\langle 3 \rangle 2$ . ASSUME:  $xEy$  and  $yEz$

$\langle 3 \rangle 3$ .  $x \in z$  or  $x = z$  or  $z \in x$

$\langle 3 \rangle 4$ .  $x \neq z$

PROOF: We cannot have  $x \in y \in x$  by the Axiom of Foundation.

$\langle 3 \rangle 5$ .  $z \notin x$

PROOF: We cannot have  $x \in y \in z \in x$  by the Axiom of Foundation.

- $\langle 3 \rangle 6. x \in z$
- $\langle 2 \rangle 5.$  There is no  $x \in \alpha$  such that  $xEx$ .  
PROOF: Axiom of Foundation.
- $\langle 2 \rangle 6.$  Every nonempty subset of  $\alpha$  has an  $E$ -least element.  
PROOF: Axiom of Foundation.

□

**Proposition 7.1.3** (S without Foundation). **On** is a transitive class. That is, if  $\beta$  is an ordinal and  $\alpha \in \beta$  then  $\alpha$  is an ordinal.

PROOF:

- $\langle 1 \rangle 1.$  LET:  $\beta$  be an ordinal.
- $\langle 1 \rangle 2.$  LET:  $\alpha \in \beta$
- $\langle 1 \rangle 3.$   $\alpha$  is a transitive set.
  - $\langle 2 \rangle 1.$  LET:  $x \in y \in \alpha$
  - $\langle 2 \rangle 2.$   $y \in \beta$   
PROOF:  $\beta$  is a transitive set.
  - $\langle 2 \rangle 3.$   $x \in \beta$   
PROOF:  $\beta$  is a transitive set.
  - $\langle 2 \rangle 4.$   $x \in \alpha$  or  $x = \alpha$  or  $\alpha \in x$
  - $\langle 2 \rangle 5.$   $x \neq \alpha$   
PROOF: We cannot have  $x \in y \in x$  since  $\beta$  is well-ordered by  $\in$ .
  - $\langle 2 \rangle 6.$   $\alpha \notin x$   
PROOF: We cannot have  $x \in y \in \alpha \in x$  since  $\beta$  is well-ordered by  $\in$ .
  - $\langle 2 \rangle 7.$   $x \in \alpha$
- $\langle 1 \rangle 4.$   $\alpha$  is well-ordered by  $\in$   
PROOF: This holds because  $\alpha \subseteq \beta$  since  $\beta$  is a transitive set.

□

**Proposition 7.1.4** (S). For ordinals  $\alpha$  and  $\beta$ , we have  $\alpha \subseteq \beta$  if and only if  $\alpha \leq \beta$ .

PROOF:

- $\langle 1 \rangle 1.$  If  $\alpha \leq \beta$  then  $\alpha \subseteq \beta$ .  
PROOF: Since  $\beta$  is a transitive set.
- $\langle 1 \rangle 2.$  If  $\alpha \subseteq \beta$  then  $\alpha \leq \beta$ .
  - $\langle 2 \rangle 1.$  ASSUME:  $\alpha \subsetneq \beta$   
PROVE:  $\alpha \in \beta$
  - $\langle 2 \rangle 2.$  PICK  $u \in \beta - \alpha$  such that  $u \cap (\beta - \alpha) = \emptyset$   
PROOF: Axiom of Foundation.
  - $\langle 2 \rangle 3.$   $u \in \beta$
  - $\langle 2 \rangle 4.$   $u \notin \alpha$
  - $\langle 2 \rangle 5.$  For all  $v \in \beta$ , if  $v \notin \alpha$  then  $v \notin u$   
PROOF: From  $\langle 2 \rangle 2$
  - $\langle 2 \rangle 6.$  For all  $v \in u$  we have  $v \in \alpha$
  - $\langle 2 \rangle 7.$  For all  $w \in \alpha$  we have  $w \in u$
  - $\langle 3 \rangle 1.$  LET:  $w \in \alpha$



- $\langle 3 \rangle 2. w \in \beta$   
PROOF:  $\langle 2 \rangle 1$
- $\langle 3 \rangle 3. w \in u \text{ or } w = u \text{ or } u \in w$   
PROOF:  $\beta$  is an ordinal,  $\langle 2 \rangle 3, \langle 3 \rangle 2$
- $\langle 3 \rangle 4. w \neq u$   
PROOF:  $\langle 2 \rangle 4, \langle 3 \rangle 1$
- $\langle 3 \rangle 5. u \notin w$   
PROOF:  $\langle 2 \rangle 4, \langle 3 \rangle 1, \alpha$  is a transitive set.
- $\langle 3 \rangle 6. w \in u$   
PROOF:  $\langle 3 \rangle 3, \langle 3 \rangle 4, \langle 3 \rangle 5$
- $\langle 2 \rangle 8. u = \alpha$   
PROOF:  $\langle 2 \rangle 6, \langle 2 \rangle 7$ , Axiom of Extensionality.
- $\langle 2 \rangle 9. \alpha \in \beta$   
PROOF:  $\langle 2 \rangle 3, \langle 2 \rangle 8$

□

**Proposition 7.1.5 (S).** *The intersection of any nonempty set of ordinals is an ordinal.*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $x$  be a nonempty set of ordinals.
- $\langle 1 \rangle 2. \bigcap x$  is a transitive set.  
PROOF: Proposition 1.6.6.
- $\langle 1 \rangle 3. \forall y, z \in \bigcap x. y \in z \vee y = z \vee z \in y$ 
  - $\langle 2 \rangle 1.$  PICK  $\alpha \in x$
  - $\langle 2 \rangle 2. y, z \in \alpha$
  - $\langle 2 \rangle 3. y \in z \text{ or } y = z \text{ or } z \in y$

□

**Theorem 7.1.6 (S).** *For any ordinals  $\alpha$  and  $\beta$ , we have  $\alpha < \beta$  or  $\alpha = \beta$  or  $\beta < \alpha$ .*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $\alpha$  and  $\beta$  be ordinals.
- $\langle 1 \rangle 2.$  CASE:  $\alpha \cap \beta = \alpha$  or  $\alpha \cap \beta = \beta$ 
  - $\langle 2 \rangle 1.$  ASSUME: for a contradiction  $\alpha \cap \beta \subsetneq \alpha$  and  $\alpha \cap \beta \subsetneq \beta$
  - $\langle 2 \rangle 2. \alpha \cap \beta \in \alpha$  and  $\alpha \cap \beta \in \beta$   
PROOF: Propositions 7.1.4, 7.1.5.
  - $\langle 2 \rangle 3. \alpha \cap \beta \in \alpha \cap \beta$
  - $\langle 2 \rangle 4.$  Q.E.D.  
PROOF: This contradicts the Axiom of Foundation.
- $\langle 1 \rangle 3. \alpha \subseteq \beta$  or  $\beta \subseteq \alpha$
- $\langle 1 \rangle 4. \alpha \in \beta$  or  $\alpha = \beta$  or  $\beta \in \alpha$   
PROOF: Proposition 7.1.4.

□

**Corollary 7.1.6.1 (Burali-Forti Paradox (S)).** *The class **On** is a proper class.*

PROOF: If  $\mathbf{On}$  is a set then it is an ordinal by Propositions 7.1.3 and 7.3.8, hence  $\mathbf{On} \in \mathbf{On}$  contradicting the Axiom of Foundation.  $\square$

**Theorem 7.1.7 (S).** *The class  $\mathbf{On}$  is well ordered by  $\leq$ .*

PROOF:

$\langle 1 \rangle 1$ . For every ordinal  $\alpha$  we have  $\alpha \leq \alpha$ .

PROOF: Immediate from definition.

$\langle 1 \rangle 2$ . For any ordinals  $\alpha, \beta$  and  $\gamma$ , if  $\alpha \leq \beta \leq \gamma$  then  $\alpha \leq \gamma$ .

PROOF: Since  $\gamma$  is a transitive set.

$\langle 1 \rangle 3$ . For any ordinals  $\alpha$  and  $\beta$ , if  $\alpha \leq \beta \leq \alpha$  then  $\alpha = \beta$ .

PROOF: If  $\alpha \neq \beta$  then  $\alpha \in \beta \in \alpha$  contradicting the Axiom of Foundation.

$\langle 1 \rangle 4$ . For any ordinals  $\alpha$  and  $\beta$ , either  $\alpha \leq \beta$  or  $\beta \leq \alpha$ .

PROOF: Theorem 7.1.6.

$\langle 1 \rangle 5$ . Any set of ordinals  $x$  has a least element.

PROOF: Axiom of Foundation.

$\square$

**Proposition 7.1.8 (S).** *For any nonempty set of ordinals  $x$ , we have  $\bigcap x$  is the infimum of  $x$  in  $\mathbf{On}$ .*

PROOF: It is an ordinal by Proposition 7.1.5. The fact it is the infimum is immediate from definitions.  $\square$

**Proposition 7.1.9 (S).** *For any nonempty set of ordinals  $x$ , we have  $\bigcup x$  is the supremum of  $x$  in  $\mathbf{On}$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\bigcup x$  is an ordinal.

$\langle 2 \rangle 1$ .  $\bigcup x$  is a transitive set.

PROOF: Proposition 1.6.3.

$\langle 2 \rangle 2$ . For all  $y, z \in \bigcup x$  we have  $y \in z$  or  $y = z$  or  $z \in y$

$\langle 3 \rangle 1$ . PICK  $\alpha, \beta \in x$  such that  $y \in \alpha$  and  $z \in \beta$ .

$\langle 3 \rangle 2$ . ASSUME: w.l.o.g.  $\alpha \leq \beta$ .

$\langle 3 \rangle 3$ .  $y, z \in \beta$

$\langle 3 \rangle 4$ .  $y \in z$  or  $y = z$  or  $z \in y$ .

$\square$

**Proposition 7.1.10 (S).** *For any set  $\alpha$ , we have  $\alpha$  is an ordinal if and only if  $\alpha$  is a transitive set of transitive sets.*

PROOF:

$\langle 1 \rangle 1$ . If  $\alpha$  is an ordinal then  $\alpha$  is a transitive set of transitive sets.

PROOF: Proposition 7.1.3.

$\langle 1 \rangle 2$ . If  $\alpha$  is a transitive set of transitive sets then  $\alpha$  is an ordinal.

$\langle 2 \rangle 1$ . ASSUME:  $\alpha$  is a transitive set of transitive sets.

$\langle 2 \rangle 2$ . ASSUME: for a contradiction there exist  $x, y \in \alpha$  such that  $x \notin y$  and  $y \notin x$

$\langle 2 \rangle 3$ .  $S = \{x \in \alpha \mid \exists y \in \alpha. x \notin y \wedge y \neq x\}$  is nonempty

$\langle 2 \rangle 4$ . PICK  $x \in S$  such that  $x \cap S = \emptyset$

PROOF: Axiom of Foundation.

$\langle 2 \rangle 5$ .  $T = \{y \in \alpha \mid x \notin y \wedge x \neq y \wedge y \notin x\}$  is nonempty.

$\langle 2 \rangle 6$ . PICK  $y \in T$  such that  $y \cap T = \emptyset$

PROOF: Axiom of Foundation.

$\langle 2 \rangle 7$ .  $x \subseteq y$

$\langle 3 \rangle 1$ . LET:  $z \in x$

$\langle 3 \rangle 2$ .  $z \notin S$

PROOF:  $\langle 2 \rangle 4$

$\langle 3 \rangle 3$ .  $z \in y$  or  $z = y$  or  $y \in z$

$\langle 3 \rangle 4$ .  $z \neq y$

PROOF: Since  $y \notin x$  ( $\langle 2 \rangle 6$ ).

$\langle 3 \rangle 5$ .  $y \notin z$

PROOF: Since  $x$  is transitive and  $y \notin x$ .

$\langle 3 \rangle 6$ .  $z \in y$

$\langle 2 \rangle 8$ .  $y \subseteq x$

$\langle 3 \rangle 1$ . LET:  $z \in y$

$\langle 3 \rangle 2$ .  $z \notin T$

PROOF:  $\langle 2 \rangle 6$

$\langle 3 \rangle 3$ .  $z \in x$  or  $z = x$  or  $x \in z$

$\langle 3 \rangle 4$ .  $z \neq x$

PROOF: Since  $x \notin y$

$\langle 3 \rangle 5$ .  $x \notin z$

PROOF: Since  $y$  is a transitive set and  $x \notin y$ .

$\langle 3 \rangle 6$ .  $z \in x$

$\langle 2 \rangle 9$ .  $x = y$

PROOF: Axiom of Extensionality.

$\langle 2 \rangle 10$ . Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 6$ .

□

**Proposition 7.1.11** (S).  $\emptyset$  is an ordinal.

PROOF: Vacuous. □

**Definition 7.1.12.**

$$0 := \emptyset$$

**Definition 7.1.13** (Successor). For any ordinal  $\alpha$ , the *successor* of  $\alpha$  is

$$\alpha^+ := \alpha \cup \{\alpha\} .$$

**Proposition 7.1.14** (S). For any ordinal  $\alpha$ , the successor  $\alpha^+$  is an ordinal.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\alpha$  be an ordinal.

$\langle 1 \rangle 2$ .  $\alpha^+$  is a transitive set.

$\langle 2 \rangle 1$ . LET:  $x \in y \in \alpha^+$

- ⟨2⟩2.  $y \in \alpha$  or  $y = \alpha$
- ⟨2⟩3.  $x \in \alpha$   
 PROOF: If  $y \in \alpha$  then this follows because  $\alpha$  is a transitive set. If  $y = \alpha$  this follows immediately.
- ⟨2⟩4.  $x \in \alpha^+$
- ⟨1⟩3. For any  $x, y \in \alpha^+$ , we have  $x \in y$  or  $x = y$  or  $y \in X$ .  
 ⟨2⟩1. LET:  $x, y \in \alpha^+$   
 ⟨2⟩2. CASE:  $x \in \alpha$  and  $y \in \alpha$   
 PROOF: Then  $x \in y$  or  $x = y$  or  $y \in x$  because  $\alpha$  is an ordinal.  
 ⟨2⟩3. CASE:  $x \in \alpha$  and  $y = \alpha$   
 PROOF: Then  $x \in y$ .  
 ⟨2⟩4. CASE:  $x = \alpha$  and  $y \in \alpha$   
 PROOF: Then  $y \in x$ .  
 ⟨2⟩5. CASE:  $x = \alpha$  and  $y = \alpha$   
 PROOF: Then  $x = y$ .

□

**Proposition 7.1.15 (S).** *There are no ordinals  $\alpha$  and  $\beta$  such that  $\alpha < \beta < \alpha^+$ .*

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction  $\alpha \in \beta \in \alpha^+$
- ⟨1⟩2.  $\beta \in \alpha$  or  $\beta = \alpha$ .
- ⟨1⟩3. CASE:  $\beta \in \alpha$   
 PROOF: We have  $\alpha \in \beta \in \alpha$  contradicting the Axiom of Foundation.
- ⟨1⟩4. CASE:  $\beta = \alpha$   
 PROOF: We have  $\alpha \in \alpha$  contradicting the Axiom of Foundation.

□

**Definition 7.1.16 (Successor Ordinal).** An ordinal  $\alpha$  is a *successor ordinal* iff there exists an ordinal  $\beta$  such that  $\alpha = \beta^+$ .

**Definition 7.1.17 (Limit Ordinal).** An ordinal is a *limit ordinal* iff it is neither 0 nor a successor ordinal.

**Proposition 7.1.18 (S).** *The relation  $\{(\alpha, \beta) \in \mathbf{On}^2 \mid \alpha < \beta\}$  is well founded.*

PROOF:

- ⟨1⟩1. Any nonempty set has an  $\in$ -minimal element.  
 PROOF: Axiom of Foundation.
- ⟨1⟩2. For any set  $x$ , there exists a set  $u$  such that  $x \in u$  and for all ordinals  $\alpha, \beta$ , if  $\beta \in u$  and  $\alpha < \beta$  then  $\alpha \in u$ .  
 ⟨2⟩1. LET:  $x$  be a set.  
 ⟨2⟩2. LET:  $u = \{x\}$  if  $x$  is not an ordinal,  $x^+$  if  $x$  is an ordinal.  
 ⟨2⟩3.  $x \in u$   
 ⟨2⟩4. LET:  $\alpha, \beta$  be ordinals.  
 ⟨2⟩5. ASSUME:  $\beta \in u$   
 ⟨2⟩6. ASSUME:  $\alpha < \beta$   
 ⟨2⟩7.  $x$  is an ordinal

- $\langle 2 \rangle 8. \alpha < \beta \leq x$
- $\langle 2 \rangle 9. \alpha \in x^+ = u$

□

## 7.2 Natural Numbers

**Definition 7.2.1** (Natural Number). An ordinal  $n$  is a *natural number* iff either it is 0, or it is a successor ordinal and every element of  $n$  is either 0 or a successor ordinal.

We write  $\omega$  or  $\mathbb{N}$  for the class of natural numbers.

**Theorem Schema 7.2.2** (Mathematical Induction (S)). *For any predicate  $P(x)$ , the following is a theorem:*

*Assume  $P(0)$ . Assume that, for any ordinal  $\alpha$ , if  $P(\alpha)$  then  $P(\alpha^+)$ . Then, for every natural number  $n$ , we have  $P(n)$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $n$  is a natural number such that  $\neg P(n)$

$\langle 1 \rangle 2$ . LET:  $y = \{\gamma \in n^+ \mid \neg P(\gamma)\}$

$\langle 1 \rangle 3$ .  $y \neq \emptyset$

$\langle 1 \rangle 4$ . PICK  $\gamma_0 \in y$  such that  $y \cap \gamma_0 = \emptyset$

$\langle 1 \rangle 5$ . CASE:  $\gamma_0 = 0$

PROOF: Then  $\neg P(0)$

$\langle 1 \rangle 6$ . CASE:  $\gamma_0 \neq 0$

$\langle 2 \rangle 1$ . PICK  $\gamma_1$  such that  $\gamma_0 = \gamma_1^+$

PROOF: Either  $\gamma_0 = n$  or  $\gamma_0 \in n$ , and in either case  $\gamma_0$  is a successor because  $n$  is a natural number.

$\langle 2 \rangle 2$ .  $\gamma_1 \notin y$

PROOF: From  $\langle 1 \rangle 4$  since  $\gamma_1 \in \gamma_0$ .

$\langle 2 \rangle 3$ .  $\gamma_1 \in n^+$

PROOF: Since  $\gamma_1 \in \gamma_0 \in n^+$  and  $n^+$  is a transitive set.

$\langle 2 \rangle 4$ .  $P(\gamma_1)$  and  $\neg P(\gamma_1^+)$

PROOF:  $\langle 1 \rangle 2$ ,  $\langle 2 \rangle 2$ ,  $\langle 2 \rangle 3$

□

**Theorem 7.2.3** (Z). *The class of natural numbers  $\omega$  is a set.*

PROOF:

$\langle 1 \rangle 1$ . PICK a set  $I$  such that  $\emptyset \in I$  and  $\forall x \in I. x \cup \{x\} \in I$ .

PROOF: Axiom of Infinity.

$\langle 1 \rangle 2$ . Every natural number is in  $I$ .

PROOF: By induction.

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: By an Axiom of Comprehension.

□

**Theorem 7.2.4** (Z).  *$\omega$  is a limit ordinal.*

PROOF:

$\langle 1 \rangle 1.$   $\omega$  is a transitive set.

$\langle 2 \rangle 1.$  LET:  $m \in n \in \omega$

PROVE:  $m$  is a natural number.

$\langle 2 \rangle 2.$  ASSUME:  $m \neq 0$

$\langle 2 \rangle 3.$   $m$  is a successor ordinal.

PROOF: Since  $m \in n$  and  $n$  is a natural number.

$\langle 2 \rangle 4.$  Every element of  $m$  is either 0 or a successor ordinal.

$\langle 3 \rangle 1.$  LET:  $x \in m$

$\langle 3 \rangle 2.$   $x \in n$

PROOF: Since  $n$  is a transitive set.

$\langle 3 \rangle 3.$   $x = 0$  or  $x$  is a successor ordinal.

$\langle 1 \rangle 2.$  For all  $m, n \in \omega$ , either  $m \in n$  or  $m = n$  or  $n \in m$ .

PROOF: Theorem 7.1.6.

$\langle 1 \rangle 3.$   $\omega \neq 0$

PROOF: Since  $0 \in \omega$ .

$\langle 1 \rangle 4.$   $\omega$  is not a successor ordinal.

$\langle 2 \rangle 1.$  ASSUME: for a contradiction  $\omega = n^+$

$\langle 2 \rangle 2.$   $n$  is a natural number.

$\langle 2 \rangle 3.$   $\omega$  is a natural number.

$\langle 2 \rangle 4.$   $\omega \in \omega$

$\langle 2 \rangle 5.$  Q.E.D.

PROOF: This contradicts the Axiom of Foundation.

□

### 7.3 Transitive Closure of a Set

**Definition 7.3.1** (Transitive Closure). Let  $x$  be a set. The *transitive closure* of  $x$  is the class

$$\text{TC}(x) = \bigcap \{y \mid y \text{ is a transitive set} \wedge x \in y\} .$$

**Theorem 7.3.2** (ZF). For any set  $x$ , the transitive closure  $\text{TC}(x)$  is a set.

PROOF:

$\langle 1 \rangle 1.$  LET:  $E(n, y, x)$  be the predicate: There exists a function  $f : n^+ \rightarrow \mathbf{V}$  such that  $f(0) = y$  and  $f(n) = x$  and  $\forall m \in n. f(m) \in f(m^+)$

$\langle 1 \rangle 2.$   $\forall x, y. E(0, y, x) \Leftrightarrow y = x$

$\langle 1 \rangle 3.$  LET:  $x$  be a set.

$\langle 1 \rangle 4.$  LET:  $\mathbf{W} = \{y \mid \exists n \in \mathbb{N}. E(n, y, x)\}$

$\langle 1 \rangle 5.$   $\mathbf{W}$  is a set.

$\langle 2 \rangle 1.$  For  $n \in \mathbb{N}$ ,

LET:  $\mathbf{W}_n = \{y \mid E(n, y, x)\}$

$\langle 2 \rangle 2.$  For all  $n \in \mathbb{N}$ ,  $\mathbf{W}_n$  is a set.

$\langle 3 \rangle 1.$   $\mathbf{W}_0$  is a set.

PROOF:  $\mathbf{W}_0 = \{x\}$ .

- ⟨3⟩2. For all  $n \in \mathbb{N}$ , if  $\mathbf{W}_n$  is a set then  $\mathbf{W}_{n^+}$  is a set.
- ⟨4⟩1.  $\mathbf{W}_{n^+} = \bigcup \mathbf{W}_n$
- ⟨5⟩1.  $\mathbf{W}_{n^+} \subseteq \bigcup \mathbf{W}_n$ 
  - ⟨6⟩1. LET:  $y \in \mathbf{W}_{n^+}$
  - ⟨6⟩2.  $E(n^+, y, x)$
  - ⟨6⟩3. PICK  $f : n^{++} \rightarrow \mathbf{V}$  such that  $f(0) = y$  and  $f(n^+) = x$  and  $\forall m \in n^+. f(m) \in f(m^+)$
  - ⟨6⟩4.  $y \in f(1)$  and  $E(n, f(1), x)$   
PROOF: Define  $g : n^+ \rightarrow \mathbf{V}$  by  $g(m) = f(m^+)$ . Then  $g(0) = f(1)$  and  $g(n) = x$  and  $\forall m \in n. g(m) \in g(m^+)$ .
  - ⟨6⟩5.  $y \in f(1) \in \mathbf{W}_n$
- ⟨5⟩2.  $\bigcup \mathbf{W}_n \subseteq \mathbf{W}_{n^+}$ 
  - ⟨6⟩1. LET:  $y \in z \in \mathbf{W}_n$
  - ⟨6⟩2. PICK  $f : n^+ \rightarrow \mathbf{V}$  such that  $f(0) = z$  and  $f(n) = x$  and  $\forall m \in n. f(m) \in f(m^+)$
  - ⟨6⟩3. LET:  $g : n^{++} \rightarrow \mathbf{V}$  be the function defined by  $g(0) = y$ ,  $g(m^+) = f(m)$
  - ⟨6⟩4.  $g(0) = y$  and  $g(n^+) = x$  and  $\forall m \in n^+. f(m) \in f(m^+)$
  - ⟨6⟩5.  $E(n^+, y, x)$
- ⟨2⟩3.  $\mathbf{W} = \bigcup \{\mathbf{W}_n \mid n \in \mathbb{N}\}$
- ⟨2⟩4.  $\{\mathbf{W}_n \mid n \in \mathbb{N}\}$  is a set.  
PROOF: Axiom of Replacement.
- ⟨2⟩5.  $\mathbf{W}$  is a set.  
PROOF: Axiom of Union.
- ⟨1⟩6.  $\mathbf{W}$  is a transitive set.
  - ⟨2⟩1. LET:  $y \in z \in \mathbf{W}$
  - ⟨2⟩2. PICK  $n \in \mathbb{N}$  and  $f : n^+ \rightarrow \mathbf{V}$  such that  $f(0) = z$ ,  $f(n) = x$  and  $\forall m \in n. f(m^+) \in f(m)$
  - ⟨2⟩3. LET:  $g : n^{++} \rightarrow \mathbf{V}$  be the function with  $g(0) = y$  and  $g(m^+) = f(m)$  for  $m \in n$
  - ⟨2⟩4.  $g(0) = y$  and  $g(n^+) = x$  and  $\forall m \in n^{++}. g(m^+) \in g(m)$
  - ⟨2⟩5.  $y \in \mathbf{W}$
- ⟨1⟩7.  $x \in \mathbf{W}$ 
  - ⟨2⟩1. LET:  $f : 0^+ \rightarrow \mathbf{V}$  be the function  $f = \{(0, x)\}$
  - ⟨2⟩2.  $f(0) = x$  and  $f(0) = x$  and vacuously  $\forall m \in 0. f(m) \in f(m^+)$
  - ⟨2⟩3.  $E(0, x, x)$
- ⟨1⟩8. For any set  $w'$ , if  $w'$  is a transitive set that contains  $x$  then  $\mathbf{W} \subseteq w'$ 
  - ⟨2⟩1. LET:  $w'$  be a transitive set that contains  $x$
  - ⟨2⟩2.  $\forall n \in \mathbb{N}. \forall y. E(n, y, x) \Rightarrow y \in w'$
  - ⟨3⟩1.  $\forall y. E(0, y, x) \Rightarrow y \in w'$ 
    - ⟨4⟩1. LET:  $y$  be a set.
    - ⟨4⟩2. ASSUME:  $E(0, y, x)$
    - ⟨4⟩3. PICK a function  $f : 0^+ \rightarrow \mathbf{V}$  such that  $f(0) = y$  and  $f(0) = x$  and  $\forall m \in 0. f(m) \in f(m^+)$
    - ⟨4⟩4.  $x = y$   
PROOF:  $x = f(0) = y$

- $\langle 4 \rangle 5.$   $y \in w'$   
 PROOF:  $\langle 2 \rangle 1$   
 $\langle 3 \rangle 2.$  For all  $n \in \mathbb{N}$ , if  $\forall y. E(n, y, x) \Rightarrow y \in w'$ , then  $\forall y. E(n^+, y, x) \Rightarrow y \in w'$   
 $\langle 4 \rangle 1.$  LET:  $n \in \mathbb{N}$   
 $\langle 4 \rangle 2.$  ASSUME:  $\forall y. E(n, y, x) \Rightarrow y \in w'$   
 $\langle 4 \rangle 3.$  LET:  $y$  be a set.  
 $\langle 4 \rangle 4.$  ASSUME:  $E(n^+, y, x)$   
 $\langle 4 \rangle 5.$  PICK  $f : n^{++} \rightarrow \mathbf{V}$  such that  $f(0) = y$  and  $f(n^+) = x$  and  
 $\forall m \in n^+. f(m) \in f(m^+)$   
 $\langle 4 \rangle 6.$  LET:  $g : n^+ \rightarrow \mathbf{V}$  be the function  $g(m) = f(m^+)$   
 $\langle 4 \rangle 7.$   $g(0) = f(0^+)$  and  $g(n) = x$  and  $\forall m \in n. g(m) \in g(m^+)$   
 $\langle 4 \rangle 8.$   $E(n, f(0^+), x)$   
 $\langle 4 \rangle 9.$   $f(0^+) \in w'$   
 PROOF:  $\langle 4 \rangle 5$   
 $\langle 4 \rangle 10.$   $y \in f(0^+)$   
 PROOF:  $y = f(0) \in f(0^+)$  by  $\langle 4 \rangle 5$ .  
 $\langle 4 \rangle 11.$   $y \in w'$   
 PROOF: From  $\langle 3 \rangle 2$  and  $\langle 4 \rangle 10$  since  $w'$  is transitive ( $\langle 2 \rangle 1$ ).  
 $\langle 2 \rangle 3.$   $\mathbf{W} \subseteq w'$

□

**Proposition 7.3.3.** *The transitive closure of an ordinal  $\alpha$  is  $\alpha^+$ .*

PROOF:

- $\langle 1 \rangle 1.$   $\alpha^+$  is a transitive set that contains  $\alpha$ .  
 $\langle 1 \rangle 2.$  For any transitive set  $x$ , if  $\alpha \in x$  then  $\alpha^+ \subseteq x$ .

□

**Proposition Schema 7.3.4 (ZF).** *For any class  $\mathbf{A}$ , the following is a theorem:*  
*The relation  $\{(x, y) \in \mathbf{A}^2 \mid x \in y\}$  is well founded.*

PROOF:

- $\langle 1 \rangle 1.$  Every nonempty set has an  $\in$ -minimal element.  
 PROOF: This is the Axiom of Foundation.  
 $\langle 1 \rangle 2.$  For any set  $x$ , there exists  $u$  such that  $x \subseteq u$  and for all  $w, y$ , if  $y \in u$  and  $w, y \in \mathbf{A}$  and  $w \in y$  then  $w \in u$   
 PROOF: Take  $u = \text{TC}(x)$ .

□

**Proposition 7.3.5 (Z).** *For any ordinal numbers  $\alpha$  and  $\beta$ , if  $\alpha \cong \beta$  then  $\alpha = \beta$ .*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $f : \alpha \cong \beta$   
 $\langle 1 \rangle 2.$  For all  $x \in \alpha$ , if  $\forall t < x. f(t) = t$  then  $f(x) = x$   
 $\langle 2 \rangle 1.$   $f(x) \subseteq x$   
 $\langle 3 \rangle 1.$  LET:  $y \in f(x)$   
 $\langle 3 \rangle 2.$   $y \in \beta$   
 $\langle 3 \rangle 3.$  PICK  $t \in \alpha$  such that  $f(t) = y$



PROOF:  $f$  is surjective.

$\langle 3 \rangle 4. f(t) \in f(x)$

$\langle 3 \rangle 5. t \in x$

PROOF: Since  $f$  is an order isomorphism.

$\langle 3 \rangle 6. f(t) = t$

PROOF: Induction hypothesis.

$\langle 3 \rangle 7. y = t$

$\langle 3 \rangle 8. y \in x$

$\langle 2 \rangle 2. x \subseteq f(x)$

$\langle 3 \rangle 1. \text{LET: } t \in x$

$\langle 3 \rangle 2. f(t) \in f(x)$

$\langle 3 \rangle 3. f(t) = t$

$\langle 3 \rangle 4. t \in f(x)$

$\langle 1 \rangle 3. \forall x \in \alpha. f(x) = x$

PROOF: Transfinite induction.

$\langle 1 \rangle 4. \alpha = \beta$

PROOF: Since  $\beta = \{f(t) \mid t \in \alpha\} = \{t \mid t \in \alpha\} = \alpha$ .

□

**Theorem 7.3.6 (ZFC).** *Every well-ordered set is isomorphic to a unique ordinal.*

PROOF:

$\langle 1 \rangle 1.$  For any well-ordered set  $A$ , there exists an ordinal  $\alpha$  such that  $A \cong \alpha$ .

$\langle 2 \rangle 1. \text{LET: } A \text{ be a well-ordered set.}$

$\langle 2 \rangle 2.$  Define the function  $E$  on  $A$  by transfinite recursion thus:

$$E(t) = \{E(x) \mid x < t\} \quad (t \in A) .$$

$\langle 2 \rangle 3. \text{LET: } \alpha = \{E(x) \mid x \in A\}$

$\langle 2 \rangle 4. \alpha \text{ is an ordinal.}$

$\langle 3 \rangle 1. \alpha \text{ is a transitive set.}$

$\langle 4 \rangle 1. \text{LET: } x \in y \in \alpha$

$\langle 4 \rangle 2. \text{PICK } t \in A \text{ such that } y = E(t)$

$\langle 4 \rangle 3. x \in E(t) = \{E(s) \mid s < t\}$

$\langle 4 \rangle 4. \text{PICK } s < t \text{ such that } x = E(s)$

$\langle 4 \rangle 5. x \in \alpha$

$\langle 3 \rangle 2. \alpha \text{ is well-ordered by } \in.$

$\langle 4 \rangle 1. \text{LET: } < = \{(x, y) \in \alpha \mid x \in y\}$

$\langle 4 \rangle 2. < \text{ is transitive.}$

$\langle 5 \rangle 1. \text{LET: } x, y, z \in \alpha \text{ with } x \in y \in z$

$\langle 5 \rangle 2. \text{PICK } t \in A \text{ such that } z = E(t)$

$\langle 5 \rangle 3. \text{PICK } s \in A \text{ such that } s < t \text{ and } y = E(s)$

$\langle 5 \rangle 4. \text{PICK } r \in A \text{ such that } r < s \text{ and } x = E(r)$

$\langle 5 \rangle 5. r < t$

$\langle 5 \rangle 6. x \in z$

$\langle 4 \rangle 3. < \text{ satisfies trichotomy.}$

$\langle 5 \rangle 1. \text{LET: } x, y \in \alpha$

$\langle 5 \rangle 2. \text{PICK } s, t \in A \text{ such that } E(s) = x \text{ and } E(t) = y$

- ⟨5⟩3. Exactly one of  $s < t$ ,  $s = t$ ,  $t < s$  holds.
- ⟨5⟩4. CASE:  $s < t$ 
  - ⟨6⟩1.  $x \in y$
  - ⟨6⟩2.  $x \neq y$  and  $y \notin x$
  - PROOF: Axiom of Regularity.
- ⟨5⟩5. CASE:  $s = t$ 
  - ⟨6⟩1.  $x = y$
  - ⟨6⟩2.  $x \notin y$  and  $y \notin x$
  - PROOF: Axiom of Regularity.
- ⟨5⟩6. CASE:  $t < s$ 
  - PROOF: Similar to ⟨5⟩4.
- ⟨4⟩4.  $\leq$  is a linear order on  $\alpha$ .
  - PROOF: Proposition 6.2.3.
- ⟨4⟩5. Every nonempty subset of  $\alpha$  has a least element.
  - ⟨5⟩1. LET:  $S$  be a nonempty subset of  $\alpha$
  - ⟨5⟩2. LET:  $T = \{x \in A \mid E(x) \in S\}$
  - ⟨5⟩3. LET:  $t$  be the least element of  $T$ .
    - PROVE:  $E(t)$  is least in  $S$
  - ⟨5⟩4. LET:  $y \in S$
  - ⟨5⟩5. PICK  $s \in T$  such that  $E(s) = y$
  - ⟨5⟩6.  $t \leq s$
  - ⟨5⟩7.  $x \leq y$
- ⟨2⟩5.  $E$  is surjective.
  - PROOF: By definition of  $\alpha$ .
- ⟨2⟩6.  $E$  is strictly monotone.
  - PROOF: If  $s < t$  then  $E(s) \in E(t)$  by definition of  $E(t)$ .
- ⟨2⟩7. Q.E.D.
  - PROOF: Corollary 6.2.6.1.
- ⟨1⟩2. For any ordinals  $\alpha$  and  $\beta$ , if  $\alpha \cong \beta$  then  $\alpha = \beta$ .
  - PROOF: Proposition 7.3.5.

□

**Proposition 7.3.7 (Z).** *The class **On** is a transitive class. That is, every element of an ordinal is an ordinal.*

PROOF:

- ⟨1⟩1. LET:  $\alpha$  be an ordinal.
- ⟨1⟩2. LET:  $\beta \in \alpha$
- ⟨1⟩3.  $\beta$  is a transitive set.
  - ⟨2⟩1. LET:  $x \in y \in \beta$
  - ⟨2⟩2.  $y \in \alpha$ 
    - PROOF:  $\alpha$  is transitive.
  - ⟨2⟩3.  $x \in \alpha$ 
    - PROOF:  $\alpha$  is transitive.
  - ⟨2⟩4.  $x \in \beta$ 
    - PROOF: Since  $\{(x, y) \in \alpha^2 \mid x \in y\}$  is transitive.
- ⟨1⟩4.  $\beta$  is well ordered by  $\in$ .

PROOF: By Proposition 6.3.3.

□

**Proposition 7.3.8** (ZFC). *Given two ordinal numbers  $\alpha, \beta$ , exactly one of  $\alpha \in \beta$ ,  $\alpha = \beta$ ,  $\beta \in \alpha$  holds.*

PROOF:

$\langle 1 \rangle 1$ . At most one holds.

PROOF: Since every ordinal is a transitive set and we never have  $\alpha \in \alpha$ .

$\langle 1 \rangle 2$ . At least one holds.

$\langle 2 \rangle 1$ . Either  $\alpha \cong \beta$  or  $\exists t \in \beta. \alpha \cong \text{seg } t$  or  $\exists t \in \alpha. \text{seg } t \cong \beta$ .

$\langle 2 \rangle 2$ . CASE:  $\alpha \cong \beta$

PROOF: Then  $\alpha = \beta$  by Proposition 7.3.5.

$\langle 2 \rangle 3$ . CASE: There exists  $t \in \beta$  such that  $\alpha \cong \text{seg } t$

$\langle 3 \rangle 1$ .  $t$  is an ordinal number.

PROOF: Proposition 7.3.7.

$\langle 3 \rangle 2$ .  $t = \text{seg } t$

$\langle 4 \rangle 1$ .  $t \subseteq \text{seg } t$

$\langle 5 \rangle 1$ . LET:  $s \in t$

$\langle 5 \rangle 2$ .  $s \in \beta$

PROOF:  $\beta$  is a transitive set.

$\langle 5 \rangle 3$ .  $s \in \text{seg } t$

$\langle 4 \rangle 2$ .  $\text{seg } t \subseteq t$

PROOF: Immediate from definitions.

$\langle 3 \rangle 3$ .  $\alpha = t$

PROOF: Proposition 7.3.5.

$\langle 3 \rangle 4$ .  $\alpha \in \beta$

$\langle 2 \rangle 4$ . CASE: There exists  $t \in \alpha$  such that  $\text{seg } t \cong \beta$

PROOF:  $\beta \in \alpha$  similarly.

□

**Proposition 7.3.9** (Z). *Any nonempty set  $S$  of ordinal numbers has a least element.*

PROOF:

$\langle 1 \rangle 1$ . PICK  $\beta \in S$

$\langle 1 \rangle 2$ . CASE:  $\beta \cap S = \emptyset$

PROOF: Then  $\beta$  is least in  $S$ .

$\langle 1 \rangle 3$ . CASE:  $\beta \cap S \neq \emptyset$

PROOF: The least element of  $\beta \cap S$  is least in  $S$ .

□

**Theorem 7.3.10** (ZFC). *The class **On** is well ordered by  $\in$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathbf{E} = \{(x, y) \in \mathbf{On}^2 \mid x \in y\}$

$\langle 1 \rangle 2$ .  $\mathbf{E}$  is transitive.

PROOF: If  $\alpha \in \beta \in \gamma$  then  $\alpha \in \gamma$  because every ordinal is a transitive set.

$\langle 1 \rangle 3$ . **E** satisfies trichotomy.

PROOF: Proposition 7.3.8.

$\langle 1 \rangle 4$ . **E** linearly orders **On**.

PROOF: Proposition 6.2.3.

$\langle 1 \rangle 5$ . **E** is well founded.

PROOF: Proposition 9.6.2.

□

**Corollary 7.3.10.1** (Burali-Forti Paradox (ZFC)). *The class **On** is a proper class.*

PROOF: If it were a set, it would be a transitive set well-ordered by  $\in$ , and hence a member of itself, contradicting Proposition 1.5.3.

**Proposition 7.3.11** (ZFC). *Any transitive set of ordinal numbers is an ordinal number.*

PROOF: It is well-ordered by  $\in$  by Proposition 6.3.3 and Theorem 7.3.10. □

**Proposition 7.3.12** (Z).  *$\emptyset$  is an ordinal number.*

PROOF: Vacuously, it is a transitive set well-ordered by  $\in$ .

**Definition 7.3.13.** We define  $0 = \emptyset$ .

**Proposition 7.3.14** (ZFC). *If  $A$  is a set of ordinal numbers then  $\bigcup A$  is an ordinal number.*

PROOF:

$\langle 1 \rangle 1$ .  $\bigcup A$  is a transitive set.

PROOF: Proposition 1.6.3.

$\langle 1 \rangle 2$ .  $\bigcup A$  is a set of ordinals.

PROOF: Proposition 7.3.7.

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: Proposition 7.3.11.

□

**Corollary 7.3.14.1** (ZFC). *The poset **On** is complete.*

PROOF: For any nonempty set  $A$  of ordinals,  $\bigcup A$  is its supremum. □

**Proposition 7.3.15** (ZFC). *Let  $\alpha$  be an ordinal and  $S \subseteq \alpha$ . Then  $S$  is well-ordered by  $\in$  and the ordinal of  $(S, \in)$  is  $\leq \alpha$ .*

PROOF:

$\langle 1 \rangle 1$ .  $S$  is well ordered by  $\in$ .

$\langle 1 \rangle 2$ . LET:  $\beta$  be the ordinal of  $(S, \in)$

$\langle 1 \rangle 3$ . LET:  $E : S \approx \beta$  be the unique isomorphism.

$\langle 1 \rangle 4$ .  $\forall \gamma \in S. E(\gamma) \leq \gamma$

$\langle 2 \rangle 1$ . LET:  $\gamma \in S$

$\langle 2 \rangle 2$ . ASSUME: as transfinite induction hypothesis  $\forall \delta < \gamma. E(\delta) \leq \delta$

- ⟨2⟩3.  $E(\gamma)$  is the least element of  $\beta$  that is greater than  $E(\delta)$  for all  $\delta < \gamma$
- ⟨2⟩4.  $\gamma$  is greater than  $E(\delta)$  for all  $\delta < \gamma$
- ⟨2⟩5.  $E(\gamma) \leq \gamma$
- ⟨1⟩5.  $\beta \leq \alpha$
- ⟨2⟩1.  $\forall \gamma < \beta. \gamma < \alpha$
- ⟨3⟩1. LET:  $\gamma < \beta$
- ⟨3⟩2. PICK  $\delta \in S$  such that  $E(\delta) = \gamma$
- ⟨3⟩3.  $\gamma = E(\delta) \leq \delta < \alpha$

□

**Proposition 7.3.16 (ZFC).** *Let  $\alpha$  be a set. Then the following are equivalent.*

1.  $\alpha$  is an ordinal.
2.  $\alpha$  is a transitive set and, for all  $x, y \in \alpha$ , either  $x = y$  or  $x \in y$  or  $y \in x$ .
3.  $\alpha$  is a transitive set of transitive sets.

PROOF:

- ⟨1⟩1.  $1 \Rightarrow 2$

PROOF: Immediate from definitions.

- ⟨1⟩2.  $2 \Rightarrow 3$

- ⟨2⟩1. ASSUME:  $\alpha$  is a transitive set and, for all  $x, y \in \alpha$ , either  $x = y$  or  $x \in y$  or  $y \in x$

- ⟨2⟩2. LET:  $z \in \alpha$

PROVE:  $z$  is transitive.

- ⟨2⟩3. LET:  $x \in y \in z$

- ⟨2⟩4.  $y \in \alpha$

- ⟨2⟩5.  $x \in \alpha$

- ⟨2⟩6. Either  $x = z$  or  $x \in z$  or  $z \in x$

- ⟨2⟩7.  $x \neq z$

PROOF: We cannot have  $x \in y \in x$  by the Axiom of Regularity.

- ⟨2⟩8.  $z \notin x$

PROOF: We cannot have  $x \in y \in z \in x$  by the Axiom of Regularity.

- ⟨1⟩3.  $3 \Rightarrow 1$

- ⟨2⟩1. LET:  $x$  be a transitive set of transitive sets.

- ⟨2⟩2. ASSUME: as  $\in$ -induction hypothesis that, for all  $y \in x$ , if  $y$  is a transitive set of transitive sets then  $y$  is a transitive set of ordinals.

- ⟨2⟩3. Every element of  $x$  is an ordinal.

- ⟨3⟩1. LET:  $y \in x$

- ⟨3⟩2.  $y$  is transitive.

- ⟨3⟩3. Every element of  $y$  is transitive.

PROOF: Since every element of  $y$  is an element of  $x$ , because  $x$  is transitive.

- ⟨3⟩4.  $y$  is an ordinal.

PROOF: ⟨2⟩2

- ⟨2⟩4. Q.E.D.

PROOF: Proposition 7.3.11.

□

**Lemma 7.3.17 (Z).** *Let  $A$  and  $B$  be well-ordered sets. If  $B$  is an end extension of  $A$  then the ordinal of  $A$  is  $\leq$  the ordinal of  $B$ .*

PROOF:

- ⟨1⟩1. LET:  $\alpha$  be the ordinal of  $A$  and  $\beta$  the ordinal of  $B$ .
- ⟨1⟩2. LET:  $E_A : A \cong \alpha$  and  $E_B : B \cong \beta$  be the canonical isomorphisms.
- ⟨1⟩3.  $\forall a \in A. E_A(a) = E_B(a)$ 
  - ⟨2⟩1. LET:  $a \in A$
  - ⟨2⟩2. ASSUME: as transfinite induction hypothesis  $\forall x < a. E_A(x) = E_B(x)$
  - ⟨2⟩3.  $E_A(a)$  is the least ordinal that is greater than  $E_A(x)$  for all  $x < a$
  - ⟨2⟩4.  $E_B(a)$  is the least ordinal that is greater than  $E_B(x)$  for all  $x < b$
  - ⟨2⟩5.  $\{x \in A \mid x <_A a\} = \{x \in B \mid x <_B a\}$
  - ⟨2⟩6.  $E_A(a) = E_B(a)$
- ⟨1⟩4.  $\alpha \subseteq \beta$
- ⟨1⟩5.  $\alpha \leq \beta$

□

**Lemma 7.3.18.** *Let  $\mathcal{C}$  be a set of well ordered sets such that, for any  $A, B \in \mathcal{C}$ , we have that one of  $A$  and  $B$  is an end extension of the other. Let  $W = \bigcup \mathcal{C}$  under  $x \leq y$  iff there exists  $A \in W$  such that  $x, y \in A$  and  $x \leq y$ . Then  $W$  is a well ordered set whose ordinal is the supremum of the ordinals of the members of  $\mathcal{C}$ .*

PROOF:

- ⟨1⟩1.  $\leq$  is reflexive on  $W$ .
  - ⟨2⟩1. LET:  $x \in W$
  - ⟨2⟩2. PICK  $A \in W$  such that  $x \in A$ .
  - ⟨2⟩3.  $x \leq x$
- ⟨1⟩2.  $\leq$  is antisymmetric on  $W$ .
  - ⟨2⟩1. LET:  $x, y \in W$
  - ⟨2⟩2. ASSUME:  $x \leq y$  and  $y \leq x$
  - ⟨2⟩3. PICK  $A \in W$  such that  $x, y \in A$  and  $x \leq_A y$ , and  $B \in W$  such that  $x, y \in B$  and  $y \leq_B x$
  - ⟨2⟩4. ASSUME: w.l.o.g.  $B$  is an end extension of  $A$
  - ⟨2⟩5.  $x \leq_B y$  and  $y \leq_B x$
  - ⟨2⟩6.  $x = y$
- ⟨1⟩3.  $\leq$  is transitive on  $W$ .
  - ⟨2⟩1. ASSUME:  $x \leq y \leq z$
  - ⟨2⟩2. PICK  $A, B \in W$  such that  $x \leq_A y$  and  $y \leq_B z$
  - ⟨2⟩3. CASE:  $A$  is an end extension of  $B$ .
    - ⟨3⟩1.  $x \leq_A y$  and  $y \leq_A z$
    - ⟨3⟩2.  $x \leq_A z$
    - ⟨3⟩3.  $x \leq z$
  - ⟨2⟩4. CASE:  $B$  is an end extension of  $A$ .

PROOF: Similar.

- ⟨1⟩4.  $\leq$  is total on  $W$ .
  - ⟨2⟩1. LET:  $x, y \in W$
  - ⟨2⟩2. PICK  $A, B \in \mathcal{C}$  such that  $x \in A$  and  $y \in B$
  - ⟨2⟩3. ASSUME: w.l.o.g.  $B$  is an end extension of  $A$
  - ⟨2⟩4.  $x \leq_B y$  or  $y \leq_B x$
  - ⟨2⟩5.  $x \leq_W y$  or  $y \leq_W x$
- ⟨1⟩5. Every nonempty subset of  $W$  has a least element.
  - ⟨2⟩1. LET:  $S$  be a nonempty subset of  $W$
  - ⟨2⟩2. PICK  $s \in S$
  - ⟨2⟩3. PICK  $A \in \mathcal{C}$  such that  $s \in A$
  - ⟨2⟩4. LET:  $a$  be the  $\leq_A$ -least element of  $S \cap A$   
PROVE:  $a$  is least in  $S$
  - ⟨2⟩5. LET:  $x \in S$   
PROVE:  $a \leq x$
  - ⟨2⟩6. PICK  $B \in \mathcal{C}$  such that  $x \in B$
  - ⟨2⟩7. CASE:  $A$  is an end extension of  $B$ 
    - ⟨3⟩1.  $a \leq_A x$
    - ⟨3⟩2.  $a \leq x$
  - ⟨2⟩8. CASE:  $B$  is an end extension of  $A$ 
    - ⟨3⟩1. CASE:  $x \in A$ 
      - ⟨4⟩1.  $a \leq_A x$
      - ⟨4⟩2.  $a \leq x$
    - ⟨3⟩2. CASE:  $x \in B - A$ 
      - ⟨4⟩1.  $a \leq_B x$
      - ⟨4⟩2.  $a \leq x$
- ⟨1⟩6. For all  $A \in \mathcal{C}$ ,  $W$  is an end extension of  $A$ .
  - ⟨2⟩1. For all  $x, y \in A$ , we have  $x \leq_A y$  if and only if  $x \leq_W y$ 
    - ⟨3⟩1. LET:  $x, y \in A$
    - ⟨3⟩2. If  $x \leq_A y$  then  $x \leq_W y$   
PROOF: Immediate from definitions.
    - ⟨3⟩3. If  $x \leq_W y$  then  $x \leq_A y$ 
      - ⟨4⟩1. ASSUME:  $x \leq_W y$
      - ⟨4⟩2. PICK  $B \in \mathcal{C}$  such that  $x \leq_B y$
      - ⟨4⟩3. CASE:  $A$  is an end extension of  $B$   
PROOF: Then  $x \leq_A y$ .
      - ⟨4⟩4. CASE:  $B$  is an end extension of  $A$   
PROOF: Then  $x \leq_A y$ .
  - ⟨2⟩2. For all  $x \in A$  and  $y \in W - A$  we have  $x < y$ 
    - ⟨3⟩1. LET:  $x \in A$  and  $y \in W - A$
    - ⟨3⟩2. PICK  $B \in \mathcal{C}$  such that  $y \in B$
    - ⟨3⟩3.  $B$  is an end extension of  $A$
    - ⟨3⟩4.  $x <_B y$
    - ⟨3⟩5.  $x <_W y$
- ⟨1⟩7. For all  $A \in \mathcal{C}$ , the ordinal of  $A$  is  $\leq$  the ordinal of  $W$ .  
PROOF: Lemma 7.3.17.
- ⟨1⟩8. For any ordinal  $\alpha$ , if for all  $A \in \mathcal{C}$  the ordinal of  $A$  is  $\leq \alpha$ , then the ordinal

of  $W$  is  $\leq \alpha$ .

- $\langle 2 \rangle 1$ . LET:  $\alpha$  be an ordinal.
- $\langle 2 \rangle 2$ . ASSUME: for all  $A \in \mathcal{C}$ , the ordinal of  $A$  is  $\leq \alpha$
- $\langle 2 \rangle 3$ . LET:  $\beta$  be the ordinal of  $W$
- $\langle 2 \rangle 4$ . LET:  $E : W \approx \beta$  be the canonical isomorphism.
- $\langle 2 \rangle 5$ . ASSUME: for a contradiction  $\alpha < \beta$
- $\langle 2 \rangle 6$ . LET:  $a \in W$  be the element with  $E(a) = \alpha$
- $\langle 2 \rangle 7$ . PICK  $A \in \mathcal{C}$  such that  $a \in A$
- $\langle 2 \rangle 8$ . LET:  $\gamma$  be the ordinal of  $A$  and  $E_A : A \cong \gamma$  be the canonical isomorphism.
- $\langle 2 \rangle 9$ . For all  $x \in A$  we have  $E_A(x) = E(x)$
- PROOF: Transfinite induction on  $x$ .
- $\langle 2 \rangle 10$ .  $E_A(a) = \alpha$
- $\langle 2 \rangle 11$ .  $\alpha < \gamma$
- $\langle 2 \rangle 12$ . Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 2$ .

□

## 7.4 Successors

**Definition 7.4.1** (Successor). The *successor* of a set  $a$  is the set  $a^+ := a \cup \{a\}$ .

**Proposition 7.4.2** (Z). A set  $a$  is a transitive set if and only if

$$\bigcup (a^+) = a .$$

PROOF:

- $\langle 1 \rangle 1$ . If  $a$  is a transitive set then  $\bigcup (a^+) = a$ .
- $\langle 2 \rangle 1$ . ASSUME:  $a$  is a transitive set.
- $\langle 2 \rangle 2$ .  $\bigcup (a^+) \subseteq a$
- $\langle 3 \rangle 1$ . LET:  $x \in \bigcup (a^+)$
- PROVE:  $x \in a$
- $\langle 3 \rangle 2$ . PICK  $y \in a^+$  such that  $x \in y$ .
- $\langle 3 \rangle 3$ .  $y \in a$  or  $y = a$ .
- $\langle 3 \rangle 4$ . CASE:  $y \in a$
- PROOF: Then  $x \in a$  because  $a$  is a transitive set.
- $\langle 3 \rangle 5$ . CASE:  $y = a$
- PROOF: Then  $x \in a$  immediately.
- $\langle 2 \rangle 3$ .  $a \subseteq \bigcup (a^+)$
- PROOF: Since  $a \in a^+$ .
- $\langle 1 \rangle 2$ . If  $\bigcup (a^+) = a$  then  $a$  is a transitive set.
- $\langle 2 \rangle 1$ . ASSUME:  $\bigcup (a^+) = a$
- $\langle 2 \rangle 2$ .  $\bigcup a \subseteq a$
- PROOF:

$$\begin{aligned} \bigcup a &\subseteq \bigcup (a^+) && \text{(Proposition 1.5.9)} \\ &= a && (\langle 2 \rangle 1) \end{aligned}$$



$\langle 2 \rangle 3$ .  $a$  is a transitive set.

PROOF: Proposition 1.6.2.

□

**Proposition 7.4.3.** *For any set  $a$ , we have  $a$  is a transitive set if and only if  $a^+$  is a transitive set.*

PROOF:

$\langle 1 \rangle 1$ . If  $a$  is a transitive set then  $a^+$  is a transitive set.

PROOF: If  $a$  is a transitive set then  $\bigcup(a^+) = a \subseteq a^+$  by Proposition 7.4.2 and so  $a^+$  is a transitive set.

$\langle 1 \rangle 2$ . If  $a^+$  is a transitive set then  $a$  is a transitive set.

$\langle 2 \rangle 1$ . ASSUME:  $a^+$  is a transitive set.

$\langle 2 \rangle 2$ . LET:  $x \in y \in a$

$\langle 2 \rangle 3$ .  $x \in y \in a^+$

$\langle 2 \rangle 4$ .  $x \in a^+$

PROOF:  $\langle 2 \rangle 1$

$\langle 2 \rangle 5$ .  $x \neq a$

PROOF: From  $\langle 2 \rangle 2$  and the Axiom of Regularity.

$\langle 2 \rangle 6$ .  $x \in a$

□

**Definition 7.4.4.** We write 0 for  $\emptyset$ , 1 for  $\emptyset^+$ , 2 for  $\emptyset^{++}$ , etc.

**Proposition 7.4.5.** *For any set  $A$  we have  $\mathcal{P}A \approx 2^A$ .*

PROOF: The function  $H : \mathcal{P}A \rightarrow 2^A$  defined by  $H(S)(a) = \{\emptyset\}$  if  $a \in S$  and  $\emptyset$  if  $a \notin S$  is a bijection. □

**Proposition 7.4.6.** *For any ordinal number  $\alpha$  we have  $\alpha^+$  is an ordinal number.*

PROOF:

$\langle 1 \rangle 1$ .  $\alpha^+$  is a transitive set.

PROOF: Proposition 7.4.3.

$\langle 1 \rangle 2$ .  $\alpha^+$  is well-ordered by  $\in$ .

$\langle 2 \rangle 1$ . For all  $x, y, z \in \alpha^+$ , if  $x \in y \in z$  then  $x \in z$

$\langle 3 \rangle 1$ . CASE:  $z = \alpha$

PROOF: Then  $x \in \alpha$  since  $\alpha$  is a transitive set.

$\langle 3 \rangle 2$ . CASE:  $z \in \alpha$

PROOF: Then  $x \in z$  since  $\alpha$  is well-ordered by  $\in$ .

$\langle 2 \rangle 2$ . For all  $x, y \in \alpha^+$  we have  $x \in y$  or  $x = y$  or  $y \in x$

$\langle 3 \rangle 1$ . CASE:  $x, y \in \alpha$

PROOF: The result follows because  $\alpha$  is well-ordered by  $\in$ .

$\langle 3 \rangle 2$ . CASE:  $x \in \alpha, y = \alpha$

PROOF: Then  $x \in y$ .

$\langle 3 \rangle 3$ . CASE:  $x = \alpha, y \in \alpha$

PROOF: Then  $y \in x$ .

- ⟨3⟩4. CASE:  $x = \alpha, y = \alpha$   
 PROOF: Then  $x = y$ .
- ⟨2⟩3. Every nonempty subset of  $\alpha^+$  has an  $\in$ -least element.
  - ⟨3⟩1. LET:  $S \subseteq \alpha^+$  be nonempty
  - ⟨3⟩2. CASE:  $S = \{\alpha\}$   
 PROOF:  $\alpha$  is least in  $S$ .
  - ⟨3⟩3. CASE:  $S \neq \{\alpha\}$ 
    - ⟨4⟩1.  $S - \{\alpha\}$  is a nonempty subset of  $\alpha$
    - ⟨4⟩2. LET:  $\beta$  be least in  $S - \{\alpha\}$
    - ⟨4⟩3.  $\beta$  is least in  $S$ .

□

**Proposition 7.4.7.** *For ordinals  $\alpha$  and  $\beta$ , if  $\alpha^+ = \beta^+$  then  $\alpha = \beta$ .*

PROOF: If  $\alpha^+ = \beta^+$  then

$$\begin{aligned}
 \alpha &= \bigcup (\alpha^+) && \text{(Proposition 7.4.2)} \\
 &= \bigcup (\beta^+) \\
 &= \beta && \text{(Proposition 7.4.2)}
 \end{aligned}$$

**Proposition 7.4.8.** *For ordinals  $\alpha$  and  $\beta$ , we have  $\alpha < \beta$  if and only if  $\alpha^+ < \beta^+$ .*

PROOF:

$$\begin{aligned}
 \alpha < \beta &\Leftrightarrow \alpha^+ \leq \beta \\
 &\Leftrightarrow \alpha^+ < \beta^+ && \square
 \end{aligned}$$

**Definition 7.4.9** (Successor Ordinal). An ordinal  $\alpha$  is a *successor ordinal* iff  $\alpha = \beta^+$  for some  $\beta$ .

**Definition 7.4.10** (Limit Ordinal). A *limit ordinal* is an ordinal that is neither 0 nor a successor ordinal.

**Proposition 7.4.11.** *If  $\lambda$  is a limit ordinal and  $\beta < \lambda$  then  $\beta^+ < \lambda$ .*

PROOF: Since  $\beta^+ \leq \lambda$  and  $\beta^+ \neq \lambda$ . □

## 7.5 The Well-Ordering Theorem and Zorn's Lemma

**Theorem 7.5.1** (Hartogs). *For any set  $A$ , there exists an ordinal not dominated by  $A$ .*

PROOF:

- ⟨1⟩1. LET:  $\alpha$  be the class of all ordinals  $\beta$  such that  $\beta \preceq A$   
 PROVE:  $\alpha$  is a set.
- ⟨1⟩2. LET:  $W = \{(B, R) \mid B \subseteq A, R \text{ is a well ordering on } B\}$
- ⟨1⟩3.  $\alpha$  is the class of the ordinals of the elements of  $W$ .
  - ⟨2⟩1. For all  $(B, R) \in W$ , the ordinal of  $(B, R)$  is in  $\alpha$ .

- ⟨3⟩1. LET:  $(B, R) \in W$
- ⟨3⟩2. LET:  $\beta$  be the ordinal of  $(B, R)$
- ⟨3⟩3. LET:  $E : B \cong \beta$  be the canonical isomorphism.
- ⟨3⟩4. LET:  $i : B \hookrightarrow A$  be the inclusion
- ⟨3⟩5.  $i \circ E^{-1}$  is an injection  $\beta \rightarrow A$
- ⟨3⟩6.  $\beta \in \alpha$
- ⟨2⟩2. For all  $\beta \in \alpha$ , there exists  $(B, R) \in W$  such that  $\beta$  is the ordinal number of  $(B, R)$ .
  - ⟨3⟩1. LET:  $\beta \in \alpha$
  - ⟨3⟩2. PICK an injection  $f : \beta \rightarrow A$
  - ⟨3⟩3. Define  $\leq$  on  $\text{ran } f$  by  $f(x) \leq f(y)$  iff  $x \leq y$
  - ⟨3⟩4.  $(\text{ran } f, \leq) \in W$
  - ⟨3⟩5.  $\beta$  is the ordinal number of  $(\text{ran } f, \leq)$
- ⟨1⟩4.  $\alpha$  is a set.  
PROOF: By an Axiom of Replacement.
- ⟨1⟩5.  $\alpha$  is an ordinal.  
PROOF: It is a transitive set of ordinals.
- ⟨1⟩6.  $\alpha \not\approx A$   
PROOF: Since  $\alpha \notin \alpha$ .

□

**Theorem 7.5.2** (Numeration Theorem). *Every set is equinumerous with some ordinal.*

PROOF:

- ⟨1⟩1. LET:  $A$  be any set.
- ⟨1⟩2. PICK an ordinal  $\alpha$  not dominated by  $A$ .
- ⟨1⟩3. PICK a choice function  $G$  for  $A$ .
- ⟨1⟩4. PICK  $e \notin A$
- ⟨1⟩5. LET:  $F : \alpha \rightarrow A \cup \{e\}$  by transfinite recursion:
 
$$F(\gamma) = \begin{cases} G(A - F(\{\delta \mid \delta < \gamma\})) & \text{if } A - F(\{\delta \mid \delta < \gamma\}) \neq \emptyset \\ e & \text{if } A - F(\{\delta \mid \delta < \gamma\}) = \emptyset \end{cases}$$
- ⟨1⟩6.  $e \in \text{ran } F$ 
  - ⟨2⟩1. ASSUME: for a contradiction  $e \notin \text{ran } F$
  - ⟨2⟩2.  $F$  is an injection  $\alpha \rightarrow A$ .
    - ⟨3⟩1. LET:  $\beta, \gamma \in \alpha$  with  $\beta \neq \gamma$   
PROVE:  $F(\beta) \neq F(\gamma)$
    - ⟨3⟩2. ASSUME: w.l.o.g.  $\beta < \gamma$
    - ⟨3⟩3.  $F(\gamma) \in A - F(\{\delta \mid \delta < \gamma\})$
    - ⟨3⟩4.  $F(\gamma) \notin F(\{\delta \mid \delta < \gamma\})$
    - ⟨3⟩5.  $F(\gamma) \neq F(\beta)$
  - ⟨2⟩3. Q.E.D.  
PROOF: This contradicts ⟨1⟩2.
- ⟨1⟩7. LET:  $\delta$  be least such that  $F(\delta) = e$
- ⟨1⟩8.  $F \upharpoonright \delta : \delta \approx A$

**Theorem 7.5.3** (Well-Ordering Theorem). *Any set can be well ordered.*

PROOF:

- $\langle 1 \rangle 1$ . PICK an ordinal  $\delta$  and a bijection  $F : A \approx \delta$
- $\langle 1 \rangle 2$ . Define  $\leq$  on  $A$  by  $F(x) \leq F(y)$  iff  $x \leq y$  for  $x, y \in \delta$
- $\langle 1 \rangle 3$ .  $\leq$  is a well ordering on  $A$ .

□

**Theorem 7.5.4** (Zorn's Lemma). *Let  $\mathcal{A}$  be a set such that, for every chain  $\mathcal{B} \subseteq \mathcal{A}$ , we have  $\bigcup \mathcal{B} \in \mathcal{A}$ . Then  $\mathcal{A}$  has a maximal element.*

PROOF:

- $\langle 1 \rangle 1$ . PICK a well ordering  $<$  on  $\mathcal{A}$ .
- $\langle 1 \rangle 2$ . LET:  $F : \mathcal{A} \rightarrow 2$  be the function defined by transfinite recursion by:
 
$$F(A) = \begin{cases} 1 & \text{if } A \text{ includes every set } B < A \text{ for which } F(B) = 1 \\ 0 & \text{otherwise} \end{cases}$$
- $\langle 1 \rangle 3$ . LET:  $\mathcal{C} = \{A \in \mathcal{A} \mid F(A) = 1\}$ 
  - PROVE:  $\bigcup \mathcal{C}$  is a maximal element of  $\mathcal{A}$
- $\langle 1 \rangle 4$ . For all  $A \in \mathcal{A}$ , we have  $A \in \mathcal{C}$  iff  $\forall B < A. B \in \mathcal{C} \Rightarrow B \subseteq A$
- $\langle 1 \rangle 5$ .  $\mathcal{C}$  is a chain.
  - $\langle 2 \rangle 1$ . LET:  $A, A' \in \mathcal{C}$
  - $\langle 2 \rangle 2$ . ASSUME: w.l.o.g.  $A \leq A'$
  - $\langle 2 \rangle 3$ .  $A \subseteq A'$
  - PROOF: By  $\langle 1 \rangle 4$
- $\langle 1 \rangle 6$ .  $\bigcup \mathcal{C} \in \mathcal{A}$
- $\langle 1 \rangle 7$ .  $\bigcup \mathcal{C}$  is maximal in  $\mathcal{A}$ .
  - $\langle 2 \rangle 1$ . LET:  $A \in \mathcal{A}$  and  $\bigcup \mathcal{C} \subseteq A$
  - $\langle 2 \rangle 2$ .  $A \in \mathcal{C}$
  - PROOF: By  $\langle 1 \rangle 4$  since  $\forall B \in \mathcal{C}. B \subseteq A$ .
  - $\langle 2 \rangle 3$ .  $A \subseteq \bigcup \mathcal{C}$
  - $\langle 2 \rangle 4$ .  $A = \bigcup \mathcal{C}$

□

**Proposition 7.5.5** (Teichmüller-Tukey Lemma). *Let  $\mathcal{A}$  be a nonempty set such that, for every  $B$ , we have  $B \in \mathcal{A}$  if and only if every finite subset of  $B$  is a member of  $\mathcal{A}$ . Then  $\mathcal{A}$  has a maximal element.*

PROOF:

- $\langle 1 \rangle 1$ . For every chain  $\mathcal{B} \subseteq \mathcal{A}$ , we have  $\bigcup \mathcal{B} \in \mathcal{A}$ 
  - $\langle 2 \rangle 1$ . LET:  $\mathcal{B} \subseteq \mathcal{A}$  be a chain.
  - $\langle 2 \rangle 2$ . Every finite subset of  $\bigcup \mathcal{B}$  is a member of  $\mathcal{A}$ .
    - $\langle 3 \rangle 1$ . LET:  $C$  be a finite subset of  $\bigcup \mathcal{B}$ .
    - $\langle 3 \rangle 2$ . PICK  $B \in \mathcal{B}$  such that  $C \subseteq B$ .
    - $\langle 3 \rangle 3$ .  $B \in \mathcal{A}$
    - $\langle 3 \rangle 4$ . Every finite subset of  $B$  is in  $\mathcal{A}$ .
    - $\langle 3 \rangle 5$ .  $C \in \mathcal{A}$
  - $\langle 2 \rangle 3$ .  $\bigcup \mathcal{B} \in \mathcal{A}$ .
- $\langle 1 \rangle 2$ . Q.E.D.

PROOF: Zorn's lemma.

□

**Theorem Schema 7.5.6.** *For any class  $\mathbf{A}$ , there exists a class  $\mathbf{F}$  such that the following is a theorem:*

*If  $\mathbf{A}$  is a proper class of ordinals, then  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{A}$  is an order isomorphism.*

PROOF:

⟨1⟩1. Define  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{A}$  by transfinite recursion as follows:  $\mathbf{F}(\alpha)$  is the least element of  $\mathbf{A}$  that is different from  $\mathbf{F}(\beta)$  for all  $\beta < \alpha$ .

⟨1⟩2. For all  $\alpha, \beta \in \mathbf{On}$ , if  $\alpha < \beta$  then  $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$

PROOF: We have  $\mathbf{F}(\alpha) \neq \mathbf{F}(\beta)$  by the definition of  $\mathbf{F}(\beta)$ , and  $\mathbf{F}(\beta) \not< \mathbf{F}(\alpha)$  by the leastness of  $\mathbf{F}(\alpha)$ .

⟨1⟩3.  $\mathbf{F}$  is surjective.

⟨2⟩1. LET:  $\alpha \in \mathbf{A}$

⟨2⟩2. ASSUME: as transfinite induction hypothesis  $\forall \beta \in \mathbf{A}$ , if  $\beta < \alpha$  then there exists  $\gamma$  such that  $\beta = \mathbf{F}(\gamma)$ .

⟨2⟩3. LET:  $\gamma = \{\delta \in \mathbf{On} \mid \mathbf{F}(\delta) < \alpha\}$

⟨2⟩4.  $\gamma$  is a set.

PROOF: Axiom of Replacement applied to  $\alpha$ .

⟨2⟩5.  $\gamma$  is a transitive set.

PROOF: If  $\mathbf{F}(\delta) < \alpha$  and  $\epsilon < \delta$  then  $\mathbf{F}(\epsilon) < \alpha$  by ⟨1⟩2.

⟨2⟩6.  $\gamma$  is an ordinal.

PROOF: Proposition 7.3.11.

⟨2⟩7.  $\mathbf{F}(\gamma) = \alpha$

⟨3⟩1.  $\mathbf{F}(\gamma)$  is the least element of  $\mathbf{A}$  different from  $\mathbf{F}(\delta)$  for all  $\delta < \gamma$

⟨3⟩2.  $\mathbf{F}(\gamma)$  is the least element of  $\mathbf{A}$  different from  $x$  for all  $x \in \mathbf{A}$  with  $x < \alpha$

⟨3⟩3.  $\mathbf{F}(\gamma) = \alpha$

□

## 7.6 Ordinal Operations

**Definition 7.6.1** (Ordinal Operation). An *ordinal operation* is a function  $\mathbf{On} \rightarrow \mathbf{On}$ .

**Definition 7.6.2** (Continuous). An ordinal operation  $\mathbf{T} : \mathbf{On} \rightarrow \mathbf{On}$  is *continuous* iff, for every limit ordinal  $\lambda$ , we have  $\mathbf{T}(\lambda) = \bigcup_{\alpha < \lambda} \mathbf{T}(\alpha)$ .

**Definition 7.6.3** (Normal). An ordinal operation is *normal* iff it is continuous and strictly monotone.

**Proposition Schema 7.6.4.** *For any class  $\mathbf{T}$ , the following is a theorem.*

*If  $\mathbf{T}$  is a continuous ordinal operation and  $\forall \gamma. \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$ , then  $\mathbf{T}$  is normal.*

PROOF:

⟨1⟩1. LET:  $P[\beta]$  be the property  $\forall \gamma < \beta. \mathbf{T}(\gamma) < \mathbf{T}(\beta)$

$\langle 1 \rangle 2. P[0]$

PROOF: Vacuous.

$\langle 1 \rangle 3. \text{ For any ordinal } \gamma, \text{ if } P[\gamma] \text{ then } P[\gamma^+]$

$\langle 2 \rangle 1. \text{ ASSUME: } P[\gamma]$

$\langle 2 \rangle 2. \text{ LET: } \delta < \gamma^+$

$\langle 2 \rangle 3. \text{ CASE: } \delta < \gamma$

PROOF: Then  $\mathbf{T}(\delta) < \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$ .

$\langle 2 \rangle 4. \text{ CASE: } \delta = \gamma$

PROOF: Then  $\mathbf{T}(\delta) = \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$ .

$\langle 1 \rangle 4. \text{ For any limit ordinal } \lambda, \text{ if } \forall \gamma < \lambda. P[\gamma] \text{ then } P[\lambda].$

$\langle 2 \rangle 1. \text{ ASSUME: } \forall \gamma < \lambda. P[\gamma]$

$\langle 2 \rangle 2. \text{ LET: } \delta < \lambda$

$\langle 2 \rangle 3. \mathbf{T}(\delta) < \mathbf{T}(\lambda)$

PROOF:

$$\begin{aligned} \mathbf{T}(\delta) &< \mathbf{T}(\delta^+) \\ &\leq \bigcup_{\epsilon < \lambda} \mathbf{T}(\epsilon) \\ &= \mathbf{T}(\lambda) \end{aligned}$$

□

**Proposition Schema 7.6.5.** *For any class  $\mathbf{T}$ , the following is a theorem:*

*Assume  $\mathbf{T}$  is a normal ordinal operation. For every ordinal  $\alpha$ , we have  $\alpha \leq \mathbf{T}(\alpha)$ .*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } \gamma \text{ be an ordinal.}$

$\langle 1 \rangle 2. \text{ ASSUME: as induction hypothesis } \forall \delta < \gamma. \mathbf{T}(\delta) \geq \delta$

$\langle 1 \rangle 3. \text{ For all } \delta < \gamma \text{ we have } \delta < \mathbf{T}(\gamma)$

PROOF:  $\mathbf{T}$  is strictly monotone.

$\langle 1 \rangle 4. \gamma \leq \mathbf{T}(\gamma)$

□

**Proposition Schema 7.6.6.** *For any class  $\mathbf{T}$ , the following is a theorem:*

*Assume  $\mathbf{T}$  is a normal ordinal operation. For any ordinal  $\beta \geq \mathbf{T}(0)$ , there exists a greatest ordinal  $\gamma$  such that  $\mathbf{T}(\gamma) \leq \beta$ .*

PROOF:

$\langle 1 \rangle 1. \text{ There exists } \gamma \text{ such that } \mathbf{T}(\gamma) > \beta$

$\langle 2 \rangle 1. \text{ For all } \gamma \text{ we have } \mathbf{T}(\gamma) \geq \gamma$

PROOF: Proposition 7.6.5.

$\langle 2 \rangle 2. \mathbf{T}(\beta^+) > \beta$

$\langle 1 \rangle 2. \text{ LET: } \delta \text{ be least such that } \mathbf{T}(\delta) > \beta$

$\langle 1 \rangle 3. \delta \text{ is a successor ordinal.}$

$\langle 2 \rangle 1. \delta \neq 0$

PROOF: Since  $\mathbf{T}(0) \leq \beta$ .

$\langle 2 \rangle 2. \delta \text{ is not a limit ordinal.}$

⟨3⟩1. ASSUME: for a contradiction  $\delta$  is a limit ordinal.

⟨3⟩2.  $\beta < \bigcup_{\epsilon < \delta} \mathbf{T}(\epsilon)$

PROOF:  $\mathbf{T}$  is continuous.

⟨3⟩3. There exists  $\epsilon < \delta$  such that  $\beta < \mathbf{T}(\epsilon)$

⟨3⟩4. Q.E.D.

PROOF: This contradicts the minimality of  $\delta$ .

⟨1⟩4. LET:  $\delta = \gamma^+$

⟨1⟩5.  $\gamma$  is greatest such that  $\mathbf{T}(\gamma) \leq \beta$

□

**Theorem Schema 7.6.7.** *For any class  $\mathbf{T}$ , the following is a theorem:*

*Assume that  $\mathbf{T}$  is a normal ordinal operation. For any nonempty set of ordinals  $S$ , we have*

$$\mathbf{T}(\sup S) = \sup_{\alpha \in S} \mathbf{T}(\alpha) .$$

PROOF:

⟨1⟩1.  $\forall \alpha \in S. \mathbf{T}(\alpha) \leq \mathbf{T}(\sup S)$

PROOF: Since  $\mathbf{T}$  is monotone.

⟨1⟩2. For any ordinal  $\beta$ , if  $\forall \alpha \in S. \mathbf{T}(\alpha) \leq \beta$ , then  $\mathbf{T}(\sup S) \leq \beta$

⟨2⟩1. LET:  $\beta$  be an ordinal.

⟨2⟩2. LET:  $\gamma = \sup S$

⟨2⟩3. ASSUME:  $\forall \alpha \in S. \mathbf{T}(\alpha) \leq \beta$

⟨2⟩4. CASE:  $\gamma$  is 0 or a successor ordinal

PROOF: Then we must have  $\gamma \in S$  so  $\mathbf{T}(\gamma) \leq \beta$  from ⟨2⟩3.

⟨2⟩5. CASE:  $\gamma$  is a limit ordinal

⟨3⟩1.  $\mathbf{T}(\gamma) = \sup_{\alpha < \gamma} \mathbf{T}(\alpha)$

PROOF:  $\mathbf{T}$  is continuous.

⟨3⟩2. ASSUME: for a contradiction  $\beta < \mathbf{T}(\gamma)$

⟨3⟩3. PICK  $\alpha < \gamma$  such that  $\beta < \mathbf{T}(\alpha)$

PROOF: ⟨3⟩1, ⟨3⟩2

⟨3⟩4. PICK  $\alpha' \in S$  such that  $\alpha < \alpha'$

PROOF: ⟨2⟩2, ⟨3⟩3

⟨3⟩5.  $\beta < \mathbf{T}(\alpha') \leq \beta$

PROOF:  $\mathbf{T}$  is strictly monotone, ⟨3⟩3, ⟨3⟩4, ⟨2⟩3.

⟨3⟩6. Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 7.6.8.** *For any classes  $\mathbf{A}$  and  $\mathbf{T}$ , the following is a theorem:*

*Assume  $\mathbf{A}$  is a proper class of ordinals such that, for every set  $S \subseteq \mathbf{A}$ , we have  $\bigcup S \in \mathbf{A}$ . Assume  $\mathbf{T}$  is the unique order isomorphism  $\mathbf{On} \cong \mathbf{A}$ . Then  $\mathbf{T}$  is normal.*

PROOF:

⟨1⟩1.  $\mathbf{T}$  is strictly monotone.

PROOF: Since it is an order isomorphism.

⟨1⟩2.  $\mathbf{T}$  is continuous.

- ⟨2⟩1. LET:  $\lambda$  be a limit ordinal.
- ⟨2⟩2.  $\mathbf{T}'(\lambda)$  is the least member of  $\mathbf{A}$  that is greater than  $\mathbf{T}'(\alpha)$  for all  $\alpha < \lambda$
- ⟨2⟩3.  $\mathbf{T}'(\lambda) = \sup_{\alpha < \lambda} \mathbf{T}'(\alpha)$

□

**Proposition Schema 7.6.9.** *For any class  $\mathbf{T}$ , the following is a theorem:*

*If  $\mathbf{T}$  is a normal ordinal operation, then for any limit ordinal  $\lambda$ , we have  $\mathbf{T}(\lambda)$  is a limit ordinal.*

PROOF:

- ⟨1⟩1.  $\mathbf{T}(\lambda) \neq 0$

PROOF: Since  $0 \leq \mathbf{T}(0) < \mathbf{T}(\lambda)$ .

- ⟨1⟩2.  $\mathbf{T}(\lambda)$  is not a successor ordinal.

- ⟨2⟩1. ASSUME: for a contradiction  $\mathbf{T}(\lambda) = \alpha^+$

- ⟨2⟩2.  $\alpha < \mathbf{T}(\lambda) = \sup_{\beta < \lambda} \mathbf{T}(\beta)$

- ⟨2⟩3. PICK  $\beta < \lambda$  such that  $\alpha < \mathbf{T}(\beta)$

- ⟨2⟩4.  $\alpha^+ \leq \mathbf{T}(\beta) < \mathbf{T}(\lambda)$

- ⟨2⟩5. Q.E.D.

PROOF: This is a contradiction.

□

## 7.7 Ordinal Arithmetic

### 7.7.1 Addition

**Definition 7.7.1.** Let  $A$  and  $B$  be disjoint well-ordered sets. The *concatenation* of  $A$  and  $B$  is the set  $A \cup B$  under the relation:

- if  $a, a' \in A$  then  $a \leq a'$  iff  $a \leq a'$  in  $A$
- if  $b, b' \in B$  then  $b \leq b'$  iff  $b \leq b'$  in  $B$
- if  $a \in A$  and  $b \in B$  then  $a \leq b$  and  $b \not\leq a$ .

**Proposition 7.7.2.** *If  $A$  and  $B$  are disjoint well-ordered sets, then their concatenation is well-ordered.*

PROOF:

- ⟨1⟩1.  $\leq$  is reflexive.

PROOF: For all  $a \in A$  we have  $a \leq a$ , and for all  $b \in B$  we have  $b \leq b$ .

- ⟨1⟩2.  $\leq$  is antisymmetric.

- ⟨2⟩1. ASSUME:  $x \leq y \leq x$

- ⟨2⟩2. CASE:  $x, y \in A$

PROOF: Then  $x = y$  since the order on  $A$  is antisymmetric.

- ⟨2⟩3. CASE:  $x \in A$  and  $y \in B$

PROOF: This is impossible as it would imply  $y \not\leq x$ .

- ⟨2⟩4. CASE:  $x \in B$  and  $y \in A$

PROOF: This is impossible as it would imply  $x \not\leq y$ .



- (2)5. CASE:  $x, y \in B$   
 PROOF: Then  $x = y$  since the order on  $B$  is antisymmetric.
- (1)3.  $\leq$  is transitive.
- (2)1. ASSUME:  $x \leq y \leq z$   
 (2)2. CASE:  $x, z \in A$   
 PROOF: In this case  $y \in A$  since  $y \leq z$ , and so  $x \leq z$  since the order on  $A$  is transitive.
- (2)3. CASE:  $x \in A$  and  $z \in B$   
 PROOF: Then  $x \leq z$  immediately.
- (2)4. CASE:  $x \in B$  and  $z \in A$   
 PROOF: This is impossible because we have  $y \notin A$  since  $x \leq y$  and  $y \notin B$  since  $y \leq z$ .
- (2)5. CASE:  $x, z \in B$   
 PROOF: In this case  $y \in B$  since  $x \leq y$ , and so  $x \leq z$  since the order on  $B$  is transitive.
- (1)4.  $\leq$  is total.
- (2)1. LET:  $x, y \in A \cup B$   
 (2)2. CASE:  $x, y \in A$   
 PROOF: Then  $x \leq y$  or  $y \leq x$  because the order on  $A$  is total.
- (2)3. CASE:  $x \in A$  and  $y \in B$   
 PROOF: Then  $x \leq y$ .
- (2)4. CASE:  $x \in B$  and  $y \in A$   
 PROOF: Then  $y \leq x$ .
- (2)5. CASE:  $x, y \in B$   
 PROOF: Then  $x \leq y$  or  $y \leq x$  because the order on  $B$  is total.
- (1)5. Every nonempty subset of  $A \cup B$  has a least element.
- (2)1. LET:  $S$  be a nonempty subset of  $A \cup B$   
 (2)2. CASE:  $S \cap A = \emptyset$   
 PROOF: Then  $S \subseteq B$  and so  $S$  has a least element.
- (2)3. CASE:  $S \cap A \neq \emptyset$   
 PROOF: The least element of  $S \cap A$  is the least element of  $S$ .

□

**Definition 7.7.3** (Ordinal Addition). Let  $\alpha$  and  $\beta$  be ordinal numbers. Then  $\alpha + \beta$  is the ordinal number of the concatenation of  $A$  and  $B$ , where  $A$  is any well ordered set with ordinal  $\alpha$  and  $B$  is any well ordered set with ordinal  $\beta$ .

**Theorem 7.7.4** (Associative Law for Addition). For any ordinals  $\rho$ ,  $\sigma$  and  $\tau$ , we have

$$\rho + (\sigma + \tau) = (\rho + \sigma) + \tau .$$

PROOF: Given disjoint well ordered sets  $A$ ,  $B$  and  $C$ , the concatenation of  $A$  with (the concatenation of  $B$  and  $C$ ) is the same as the concatenation of (the concatenation of  $A$  and  $B$ ) and  $C$ . □

**Theorem 7.7.5.** For any ordinal  $\rho$  we have

$$\rho + 0 = 0 + \rho = \rho .$$

PROOF: For any well ordered set  $A$ , the concatenation of  $A$  with  $\emptyset$  is  $A$ , and the concatenation of  $\emptyset$  with  $A$  is  $A$ .  $\square$

**Theorem 7.7.6.** *For any ordinal  $\alpha$  we have  $\alpha + 1 = \alpha^+$ .*

PROOF: Since  $\alpha^+$  is the concatenation of  $\alpha$  and  $\{\alpha\}$ .  $\square$

**Theorem 7.7.7.** *For any ordinal  $\alpha$ , the operation that maps  $\beta$  to  $\alpha + \beta$  is normal.*

PROOF:

$\langle 1 \rangle 1$ . For any limit ordinal  $\lambda$ , we have  $\alpha + \lambda = \sup_{\beta < \lambda} (\alpha + \beta)$ .

$\langle 2 \rangle 1$ . LET:  $\lambda$  be a limit ordinal.

$\langle 2 \rangle 2$ .  $(\{0\} \times \alpha) \cup (\{1\} \times \lambda) = \bigcup_{\beta \in \lambda} ((\{0\} \times \alpha) \cup (\{1\} \times \beta))$ , where the order on the right hand side is as in Lemma 7.3.18.

PROOF:

$$\begin{aligned} (\{0\} \times \alpha) \cup (\{1\} \times \lambda) &= (\{0\} \times \alpha) \cup (\{1\} \times \bigcup_{\beta < \lambda} \beta) \\ &= (\{0\} \times \alpha) \cup \bigcup_{\beta < \lambda} (\{1\} \times \beta) \\ &= \bigcup_{\beta < \lambda} ((\{0\} \times \alpha) \cup (\{1\} \times \beta)) \end{aligned}$$

$\langle 1 \rangle 2$ . For any ordinal  $\beta$  we have  $\alpha + \beta < \alpha + \beta^+$

PROOF: Since  $\alpha + \beta^+ = \alpha + \beta + 1 = (\alpha + \beta)^+$ .

$\square$

**Corollary 7.7.7.1.** *For any ordinals  $\alpha, \beta, \gamma$ , we have  $\beta < \gamma$  if and only if  $\alpha + \beta < \alpha + \gamma$ .*

**Corollary 7.7.7.2** (Left Cancellation for Addition). *For any ordinals  $\alpha, \beta$  and  $\gamma$ , if  $\alpha + \beta = \alpha + \gamma$  then  $\beta = \gamma$ .*

**Theorem 7.7.8.** *For any ordinals  $\alpha, \beta, \gamma$ , if  $\beta \leq \gamma$  then  $\beta + \alpha \leq \gamma + \alpha$ .*

PROOF: Transfinite induction on  $\alpha$ .  $\square$

**Theorem 7.7.9** (Subtraction Theorem). *Let  $\alpha$  and  $\beta$  be ordinals with  $\alpha \leq \beta$ . Then there exists a unique ordinal  $\delta$  such that  $\alpha + \delta = \beta$ .*

PROOF:

$\langle 1 \rangle 1$ . For all ordinals  $\alpha$  and  $\beta$  with  $\alpha \leq \beta$ , there exists  $\delta$  such that  $\alpha + \delta = \beta$

$\langle 2 \rangle 1$ . LET:  $\alpha$  and  $\beta$  be ordinals with  $\alpha \leq \beta$

$\langle 2 \rangle 2$ . LET:  $\delta$  be the greatest ordinal such that  $\alpha + \delta \leq \beta$

PROOF: Proposition 7.6.6.

$\langle 2 \rangle 3$ .  $\alpha + \delta = \beta$

PROOF: If  $\alpha + \delta < \beta$  then  $\alpha + \delta + 1 \leq \beta$  contradicting the greatestness of  $\delta$ .

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: Uniqueness follows from the Left Cancellation Law.

$\square$

### 7.7.2 Multiplication

**Definition 7.7.10** (Ordinal Multiplication). Let  $\alpha$  and  $\beta$  be ordinal numbers. Then  $\alpha\beta$  is the ordinal number of  $A \times B$  under the lexicographic order, where  $A$  is any well ordered set with ordinal  $\alpha$  and  $B$  is any well ordered set with ordinal  $\beta$ .

This is well defined by Proposition 6.3.5.

**Theorem 7.7.11** (Associative Law). For any ordinals  $\rho$ ,  $\sigma$  and  $\tau$ , we have

$$\rho(\sigma\tau) = (\rho\sigma)\tau .$$

PROOF: Let  $A$ ,  $B$  and  $C$  be well ordered sets with ordinals  $\rho$ ,  $\sigma$  and  $\tau$ . Then both  $\rho(\sigma\tau)$  and  $(\rho\sigma)\tau$  are the ordinal of  $A \times B \times C$  under  $(a, b, c) \leq (a', b', c') \Leftrightarrow a \leq a' \vee (a = a' \wedge b \leq b') \vee (a = a' \wedge b = b' \wedge c \leq c')$  .  $\square$

**Theorem 7.7.12** (Left Distributive Law). For any ordinals  $\rho$ ,  $\sigma$  and  $\tau$ , we have

$$\rho(\sigma + \tau) = \rho\sigma + \rho\tau$$

PROOF: Let  $A$ ,  $B$  and  $C$  be well ordered sets with ordinals  $\rho$ ,  $\sigma$  and  $\tau$  and with  $B \cap C = \emptyset$ . Then both  $\rho(\sigma + \tau)$  and  $\rho\sigma + \rho\tau$  are the ordinal of  $A \times (B \cup C)$  under the lexicographic ordering.  $\square$

**Theorem 7.7.13.** For any ordinal  $\rho$  we have  $\rho 0 = 0\rho = 0$ .

PROOF: For any well ordered set  $A$  we have  $A \times \emptyset = \emptyset \times A = \emptyset$ .  $\square$

**Theorem 7.7.14.** For any ordinal  $\rho$  we have  $\rho 1 = 1\rho = \rho$ .

PROOF: Easy.  $\square$

**Theorem 7.7.15.** For any ordinals  $\rho$  and  $\sigma$ , if  $\rho\sigma = 0$  then  $\rho = 0$  or  $\sigma = 0$ .

PROOF: If  $A \times B = \emptyset$  then  $A = \emptyset$  or  $B = \emptyset$ .  $\square$

**Theorem 7.7.16.** For any non-zero ordinal  $\alpha$ , the operation that maps  $\beta$  to  $\alpha\beta$  is normal.

PROOF:

$\langle 1 \rangle 1$ . For any limit ordinal  $\lambda$ , we have  $\alpha\lambda = \bigcup_{\beta < \lambda} \alpha\beta$

$\langle 2 \rangle 1$ . LET:  $\lambda$  be a limit ordinal

$\langle 2 \rangle 2$ .  $\alpha \times \lambda = \bigcup_{\beta < \lambda} (\alpha \times \beta)$  as well-ordered sets

$\langle 1 \rangle 2$ . For any ordinal  $\beta$  we have  $\alpha\beta < \alpha\beta^+$

PROOF:  $\alpha\beta^+ = \alpha\beta + \alpha > \alpha\beta$

$\square$

**Corollary 7.7.16.1.** For any ordinals  $\alpha$ ,  $\beta$ ,  $\gamma$ , if  $\alpha \neq 0$  then  $\beta < \gamma$  if and only if  $\alpha\beta < \alpha\gamma$ .

**Corollary 7.7.16.2** (Left Cancellation for Multiplication). *For any ordinals  $\alpha, \beta, \gamma$ , if  $\alpha \neq 0$  and  $\alpha\beta = \alpha\gamma$  then  $\beta = \gamma$ .*

**Theorem 7.7.17.** *For any ordinals  $\alpha, \beta$  and  $\gamma$ , if  $\beta \leq \gamma$  then  $\beta\alpha \leq \gamma\alpha$ .*

PROOF: Transfinite induction on  $\alpha$ .  $\square$

**Theorem 7.7.18** (Division Theorem). *Let  $\alpha$  and  $\delta$  be ordinal numbers with  $\delta \neq 0$ . Then there exist unique ordinals  $\beta$  and  $\gamma$  with  $\gamma < \delta$  and*

$$\alpha = \delta\beta + \gamma.$$

PROOF:

$\langle 1 \rangle 1$ . For any ordinal numbers  $\alpha$  and  $\delta$  with  $\delta \neq 0$ , there exist ordinals  $\beta$  and  $\gamma$  such that  $\gamma < \delta$  and  $\alpha = \delta\beta + \gamma$

$\langle 2 \rangle 1$ . LET:  $\alpha$  and  $\delta$  be ordinals with  $\delta \neq 0$

$\langle 2 \rangle 2$ . LET:  $\beta$  be the greatest ordinal such that  $\delta\beta \leq \alpha$

PROOF: Proposition 7.6.6.

$\langle 2 \rangle 3$ . There exists an ordinal  $\gamma$  such that  $\alpha = \delta\beta + \gamma$

PROOF: Subtraction Theorem

$\langle 1 \rangle 2$ . For any ordinals  $\delta, \beta, \beta', \gamma, \gamma'$ , if  $\delta\beta + \gamma = \delta\beta' + \gamma'$  and  $\delta \neq 0$  and  $\gamma, \gamma' < \delta$  then  $\beta = \beta'$  and  $\gamma = \gamma'$

$\langle 2 \rangle 1$ . LET:  $\delta, \beta, \beta', \gamma, \gamma'$  be ordinals.

$\langle 2 \rangle 2$ . ASSUME:  $\delta \neq 0$  and  $\delta\beta + \gamma = \delta\beta' + \gamma'$

$\langle 2 \rangle 3$ .  $\beta = \beta'$

$\langle 3 \rangle 1$ .  $\beta \not< \beta'$

PROOF: If  $\beta < \beta'$  then

$$\begin{aligned} \delta\beta' + \gamma' &\geq \delta\beta' \\ &\geq \delta(\beta + 1) \\ &= \delta\beta + \delta \\ &> \delta\beta + \gamma \end{aligned}$$

$\langle 3 \rangle 2$ .  $\beta' \not< \beta$

PROOF: Similar.

$\langle 2 \rangle 4$ .  $\gamma = \gamma'$

PROOF: By Cancellation.

$\square$

### 7.7.3 Exponentiation

**Definition 7.7.19.** Given ordinals  $\alpha$  and  $\beta$ , define the ordinal  $\alpha^\beta$  as follows:

$$0^\alpha := 0 \quad (\alpha > 0)$$

$$\alpha^0 := 1$$

$$\alpha^{\beta+} := \alpha^\beta \alpha \quad (\alpha > 0)$$

$$\alpha^\lambda := \sup_{\beta < \lambda} \alpha^\beta \quad (\alpha > 0, \lambda \text{ a limit ordinal})$$

**Theorem 7.7.20.** *Let  $\alpha$  be an ordinal  $\geq 2$ . The operation that maps  $\beta$  to  $\alpha^\beta$  is normal.*

PROOF:

$\langle 1 \rangle 1$ . For  $\lambda$  a limit ordinal we have  $\alpha^\lambda = \sup_{\beta < \lambda} \alpha^\beta$

PROOF: By definition.

$\langle 1 \rangle 2$ . For any ordinal  $\beta$  we have  $\alpha^\beta < \alpha^{\beta+}$

PROOF: We have  $\alpha^{\beta+} = \alpha^\beta \alpha > \alpha^\beta$  by Theorem 7.7.16 since  $\alpha > 1$  and  $\alpha^\beta \neq 0$ .

□

**Corollary 7.7.20.1.** *For any ordinals  $\alpha, \beta, \gamma$ , if  $\alpha \geq 2$  then  $\beta < \gamma$  if and only if  $\alpha^\beta < \alpha^\gamma$ .*

**Corollary 7.7.20.2** (Cancellation for Exponentiation). *For any ordinals  $\alpha, \beta, \gamma$ , if  $\alpha \geq 2$  and  $\alpha^\beta = \alpha^\gamma$  then  $\beta = \gamma$ .*

**Theorem 7.7.21.** *For any ordinals  $\alpha, \beta$  and  $\gamma$ , if  $\beta \leq \gamma$  then  $\beta^\alpha \leq \gamma^\alpha$ .*

PROOF: Transfinite induction on  $\alpha$ .

**Theorem 7.7.22** (Logarithm Theorem). *Let  $\alpha$  and  $\beta$  be ordinal numbers with  $\alpha \neq 0$  and  $\beta > 1$ . Then there exist unique ordinals  $\gamma, \delta$  and  $\rho$  such that*

$$\alpha = \beta^\gamma \delta + \rho, \quad 0 \neq \delta < \beta, \quad \rho < \beta^\gamma.$$

PROOF:

$\langle 1 \rangle 1$ . For any ordinals  $\alpha$  and  $\beta$  with  $\alpha \neq 0$  and  $\beta > 1$ , there exist ordinals  $\gamma, \delta, \rho$  such that

$$\alpha = \beta^\gamma \delta + \rho, \quad 0 \neq \delta < \beta, \quad \rho < \beta^\gamma.$$

$\langle 2 \rangle 1$ . LET:  $\alpha$  and  $\beta$  be ordinals with  $\alpha \neq 0$  and  $\beta > 1$ .

$\langle 2 \rangle 2$ . LET:  $\gamma$  be the greatest ordinal such that  $\beta^\gamma \leq \alpha$ .

PROOF: Proposition 7.6.6.

$\langle 2 \rangle 3$ . LET:  $\delta$  and  $\rho$  be the unique ordinals with  $\rho < \beta^\gamma$  such that  $\alpha = \beta^\gamma \delta + \rho$ .

PROOF: By the Division Theorem.

$\langle 2 \rangle 4$ .  $\delta \neq 0$

PROOF: If  $\delta = 0$  then  $\alpha = \beta^\gamma 0 + \rho = \rho < \beta^\gamma \leq \alpha$  which is a contradiction.

$\langle 2 \rangle 5$ .  $\delta < \beta$

PROOF: If  $\beta \leq \delta$  then  $\alpha \geq \beta^\gamma \delta \geq \beta^\gamma \beta = \beta^{\gamma+1}$ , contradicting the greatestness of  $\gamma$ .

$\langle 1 \rangle 2$ . If  $\beta^\gamma \delta + \rho = \beta^{\gamma'} \delta' + \rho'$  with  $\beta > 1$ ,  $0 \neq \delta < \beta$ ,  $0 \neq \delta' < \beta$ ,  $\rho < \beta^\gamma$  and  $\rho' < \beta^{\gamma'}$ , then  $\gamma = \gamma'$ ,  $\delta = \delta'$  and  $\rho = \rho'$ .

$\langle 2 \rangle 1$ . LET:  $\alpha = \beta^\gamma \delta + \rho = \beta^{\gamma'} \delta' + \rho'$

$\langle 2 \rangle 2$ .  $\beta^\gamma \leq \alpha < \beta^{\gamma+1}$

$\langle 2 \rangle 3$ .  $\beta^{\gamma'} \leq \alpha < \beta^{\gamma'+1}$

$\langle 2 \rangle 4$ .  $\beta^\gamma < \beta^{\gamma'+1}$  and  $\beta^{\gamma'} < \beta^{\gamma+1}$

$\langle 2 \rangle 5$ .  $\gamma < \gamma' + 1$  and  $\gamma' < \gamma + 1$

$\langle 2 \rangle 6$ .  $\gamma = \gamma'$

⟨2⟩7.  $\delta = \delta'$  and  $\rho = \rho'$

PROOF: By the Division Theorem.

□

**Theorem 7.7.23.** *For any ordinal numbers  $\alpha, \beta, \gamma$ , we have*

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma .$$

PROOF:

⟨1⟩1. LET:  $P[\gamma]$  be the property: for any ordinals  $\alpha$  and  $\beta$  we have  $\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$

⟨1⟩2.  $P[0]$

PROOF:

$$\begin{aligned} \alpha^{\beta+0} &= \alpha^\beta \\ &= \alpha^\beta 1 \\ &= \alpha^\beta \alpha^0 \end{aligned}$$

⟨1⟩3. For all  $\gamma$ , if  $P[\gamma]$  then  $P[\gamma+1]$

PROOF:

$$\begin{aligned} \alpha^{\beta+\gamma+1} &= \alpha^{\beta+\gamma} \alpha \\ &= \alpha^\beta \alpha^\gamma \alpha && \text{(induction hypothesis)} \\ &= \alpha^\beta \alpha^{\gamma+1} \end{aligned}$$

⟨1⟩4. For any limit ordinal  $\lambda$ , if  $\forall \gamma < \lambda. P[\gamma]$  then  $P[\lambda]$ .

⟨2⟩1. LET:  $\lambda$  be a limit ordinal

⟨2⟩2. ASSUME:  $\forall \gamma < \lambda. P[\gamma]$

⟨2⟩3. LET:  $\alpha$  and  $\beta$  be any ordinals.

⟨2⟩4. CASE:  $\alpha = 0$

PROOF: We have  $\alpha^{\beta+\lambda} = \alpha^\beta \alpha^\lambda = 0$ .

⟨2⟩5. CASE:  $\alpha = 1$

PROOF: We have  $\alpha^{\beta+\lambda} = \alpha^\beta \alpha^\lambda = 1$ .

⟨2⟩6. CASE:  $\alpha > 1$

PROOF:

$$\begin{aligned} \alpha^{\beta+\lambda} &= \alpha^{\sup_{\gamma < \lambda} (\beta+\gamma)} \\ &= \sup_{\gamma < \lambda} \alpha^{\beta+\gamma} && \text{(Theorem 7.6.7)} \\ &= \sup_{\gamma < \lambda} \alpha^\beta \alpha^\gamma && (\langle 2 \rangle 2) \\ &= \alpha^\beta \sup_{\gamma < \lambda} \alpha^\gamma && \text{(Theorem 7.6.7)} \\ &= \alpha^\beta \alpha^\lambda \end{aligned}$$

□

**Theorem 7.7.24.** *For any ordinal numbers  $\alpha, \beta$  and  $\gamma$ , we have*

$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma} .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $P[\gamma]$  be the property: For any ordinals  $\alpha$  and  $\beta$ , we have  $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$

$\langle 1 \rangle 2$ .  $P[0]$

PROOF:

$$\begin{aligned} (\alpha^\beta)^0 &= 1 \\ &= \alpha^{\beta 0} \end{aligned}$$

$\langle 1 \rangle 3$ .  $\forall \gamma \in \mathbf{On}. P[\gamma] \Rightarrow P[\gamma + 1]$

PROOF:

$$\begin{aligned} (\alpha^\beta)^{\gamma+1} &= (\alpha^\beta)^\gamma \alpha^\beta \\ &= \alpha^{\beta\gamma} \alpha^\beta \\ &= \alpha^{\beta\gamma+\beta} \\ &= \alpha^{\beta(\gamma+1)} \end{aligned}$$

$\langle 1 \rangle 4$ . For any limit ordinal  $\lambda$ , if  $\forall \gamma < \lambda. P[\gamma]$  then  $P[\lambda]$ .

$\langle 2 \rangle 1$ . LET:  $\lambda$  be a limit ordinal.

$\langle 2 \rangle 2$ . ASSUME:  $\forall \gamma < \lambda. P[\gamma]$

$\langle 2 \rangle 3$ . LET:  $\alpha$  and  $\beta$  be any ordinals.

$\langle 2 \rangle 4$ . CASE:  $\alpha = 0$  and  $\beta = 0$

PROOF:

$$\begin{aligned} (0^\beta)^\lambda &= 1^\lambda \\ &= 1 \\ &= 0^0 \\ &= 0^{0\lambda} \end{aligned}$$

$\langle 2 \rangle 5$ . CASE:  $\alpha = 0$  and  $\beta \neq 0$

PROOF:  $(0^\beta)^\lambda = 0^{\beta\lambda} = 0$ .

$\langle 2 \rangle 6$ . CASE:  $\alpha = 1$

PROOF:  $(1^\beta)^\lambda = 1^{\beta\lambda} = 1$

$\langle 2 \rangle 7$ . CASE:  $\alpha > 1$

PROOF:

$$\begin{aligned} (\alpha^\beta)^\lambda &= \sup_{\gamma < \lambda} (\alpha^\beta)^\gamma \\ &= \sup_{\gamma < \lambda} \alpha^{\beta\gamma} \\ &= \alpha^{\sup_{\gamma < \lambda} \beta\gamma} \\ &= \alpha^{\beta\lambda} \end{aligned}$$

□

## 7.8 Sequences

i

**Definition 7.8.1** (Sequence). Given an ordinal  $\alpha$  and class  $\mathbf{A}$ , an  $\alpha$ -sequence in  $\mathbf{A}$  is a function  $a : \alpha \rightarrow \mathbf{A}$ . We write  $a_\beta$  for  $a(\beta)$ , and  $(a_\beta)_{\beta < \alpha}$  for  $a$ .

**Definition 7.8.2** (Strictly Increasing). A sequence  $(a_\beta)$  of ordinals is *strictly increasing* iff, whenever  $\beta < \gamma$ , then  $a_\beta < a_\gamma$ .

**Definition 7.8.3** (Subsequence). Let  $(a_\beta)_{\beta < \gamma}$  be a sequence in  $\mathbf{A}$ . A *subsequence* of  $(a_\beta)$  is a sequence of the form  $(a_{\beta_\xi})_{\xi < \delta}$  where  $(\beta_\xi)_{\xi < \delta}$  is a strictly increasing sequence in  $\gamma$ .

**Definition 7.8.4** (Convergence). Let  $(a_\beta)_{\beta < \gamma}$  be a sequence of ordinals and  $\lambda$  an ordinal. Then  $(a_\beta)$  *converges* to the *limit*  $\lambda$  iff  $\lambda = \sup_{\beta < \gamma} a_\beta$ .

**Lemma 7.8.5.** *Let  $(a_\beta)_{\beta < \gamma}$  be a sequence of ordinals. Then there is a strictly increasing subsequence  $(a_{\beta_\xi})_{\xi < \delta}$  such that  $\sup_{\xi < \delta} a_{\beta_\xi} = \sup_{\beta < \gamma} a_\beta$ .*

PROOF: Define  $\beta_\xi$  by transfinite recursion as follows.  $\beta_\xi$  is the least  $\beta$  such that  $a_\beta > a_{\beta_\zeta}$  for all  $\zeta < \xi$  if there is such an  $a_\beta$ ; if not, the sequence ends.  $\square$

## 7.9 Strict Supremum

**Definition 7.9.1** (Strict Supremum). For any set  $S$  of ordinals, define the *strict supremum* of  $S$ ,  $\text{ssup } S$ , to be the least ordinal greater than every member of  $S$ .



## Chapter 8

# Cardinal Numbers

### 8.1 Cardinal Numbers

**Definition 8.1.1** (Cardinality). For any set  $A$ , the *cardinality* or *cardinal number*  $|A|$  of  $A$  is the least ordinal equinumerous with  $A$ .

Let **Card** be the class of all cardinal numbers.

**Proposition 8.1.2.** For any sets  $A$  and  $B$ , we have  $A \approx B$  iff  $|A| = |B|$ .

PROOF: Easy.  $\square$

**Definition 8.1.3** (Addition). Given cardinal numbers  $\kappa$  and  $\lambda$ , we define  $\kappa + \lambda$  to be  $|A \cup B|$  where  $A$  and  $B$  are disjoint sets of cardinality  $\kappa$  and  $\lambda$  respectively.

We prove this is well-defined.

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $A \approx A'$ ,  $B \approx B'$ , and  $A \cap B = A' \cap B' = \emptyset$

$\langle 1 \rangle 2$ . PICK bijections  $f : A \approx A'$  and  $g : B \approx B'$

$\langle 1 \rangle 3$ . The function  $A \cup B \rightarrow A' \cup B'$  that maps  $a \in A$  to  $f(a)$  and  $b \in B$  to  $g(b)$  is a bijection.

$\square$

**Proposition 8.1.4.** For any cardinal number  $\kappa$ , we have  $\kappa + 0 = \kappa$ .

PROOF: Let  $A$  and  $B$  be disjoint sets of cardinality  $\kappa$  and  $0$ . Then  $B = \emptyset$  so  $A \cup B = A$  and so  $|A \cup B| = \kappa$ .  $\square$

**Theorem 8.1.5** (Associative Law for Addition). For any cardinal numbers  $\kappa$ ,  $\lambda$ ,  $\mu$  we have  $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$ .

PROOF: Since  $A \cup (B \cup C) = (A \cup B) \cup C$ .  $\square$

**Proposition 8.1.6.** For any cardinal numbers  $\kappa$  and  $\lambda$  we have  $\kappa + \lambda = \lambda + \kappa$ .

PROOF: Since  $A \cup B = B \cup A$ .  $\square$

**Definition 8.1.7** (Multiplication). For  $\kappa$  and  $\lambda$  cardinal numbers, we define  $\kappa\lambda$  to be the cardinal number of  $A \times B$ , where  $|A| = \kappa$  and  $|B| = \lambda$ .

We prove this is well-defined.

PROOF: If  $f : A \approx A'$  and  $g : B \approx B'$  then the function that maps  $(a, b)$  to  $(f(a), g(b))$  is a bijection  $A \times B \approx A' \times B'$ .  $\square$

**Proposition 8.1.8.** For any cardinal number  $\kappa$  we have  $\kappa \cdot 0 = 0$ .

PROOF: Since  $A \times \emptyset = \emptyset$ .  $\square$

**Proposition 8.1.9.** For any cardinal number  $\kappa$  we have  $\kappa \cdot 1 = \kappa$ .

PROOF: The function that maps  $(a, e)$  to  $a$  is a bijection  $A \times \{e\} \approx A$ .  $\square$

**Theorem 8.1.10** (Distributive Law). For any cardinal numbers  $\kappa$ ,  $\lambda$  and  $\mu$ , we have  $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$ .

PROOF: Since  $A \times (B \cup C) = (A \times B) \cup (A \times C)$ .  $\square$

**Theorem 8.1.11** (Associative Law for Multiplication). For any cardinal numbers  $\kappa$ ,  $\lambda$  and  $\mu$ , we have  $\kappa(\lambda\mu) = (\kappa\lambda)\mu$ .

PROOF: Since  $A \times (B \times C) \approx (A \times B) \times C$ .  $\square$

**Theorem 8.1.12** (Commutative Law for Multiplication). For any cardinal numbers  $\kappa$  and  $\lambda$ , we have  $\kappa\lambda = \lambda\kappa$ .

PROOF: Since  $A \times B \approx B \times A$ .  $\square$

**Theorem 8.1.13.** For any cardinal numbers  $\kappa$  and  $\lambda$ , if  $\kappa\lambda = 0$  then  $\kappa = 0$  or  $\lambda = 0$ .

PROOF: if  $A \times B = \emptyset$  then  $A = \emptyset$  or  $B = \emptyset$ .  $\square$

**Definition 8.1.14** (Exponentiation). Given cardinal numbers  $\kappa$  and  $\lambda$ , we define  $\kappa^\lambda$  to be  $|A^B|$ , where  $|A| = \kappa$  and  $|B| = \lambda$ .

We prove this is well-defined.

PROOF: If  $f : A \approx A'$  and  $g : B \approx B'$ , then the function that maps  $h : B \rightarrow A$  to  $f \circ h \circ g^{-1}$  is a bijection  $A^B \approx A'^{B'}$ .  $\square$

**Proposition 8.1.15.** For any cardinal numbers  $\kappa$ ,  $\lambda$  and  $\mu$ ,

$$\kappa^{\lambda+\mu} = (\kappa^\lambda)^\mu$$

PROOF: The function that maps  $f : A \times B \rightarrow C$  to  $\lambda a \in A. \lambda b \in B. f(a, b)$  is a bijection  $A^{B \times C} \approx (A^B)^C$ .  $\square$

**Proposition 8.1.16.** For any cardinal numbers  $\kappa$ ,  $\lambda$  and  $\mu$ ,

$$(\kappa\lambda)^\mu = \kappa^\mu \lambda^\mu.$$

PROOF: The function  $f : A^C \times B^C \rightarrow (A \times B)^C$  with  $f(g, h)(c) = (g(c), h(c))$  is a bijection.  $\square$

**Proposition 8.1.17.** *For any cardinal numbers  $\kappa$ ,  $\lambda$  and  $\mu$ , we have*

$$\kappa^{\lambda+\mu} = \kappa^\lambda \kappa^\mu .$$

PROOF: If  $B \cap C = \emptyset$ , then  $f : A^B \times A^C \rightarrow A^{B \cup C}$  given by  $f(g, h)(b) = g(b)$  and  $f(g, h)(c) = h(c)$  is a bijection.  $\square$

**Proposition 8.1.18.** *For any cardinal number  $\kappa$ , we have  $\kappa^0 = 1$ .*

PROOF: For any set  $A$ , we have  $A^\emptyset = \{\emptyset\}$ .  $\square$

**Proposition 8.1.19.** *For any cardinal number  $\kappa$ , we have  $\kappa^1 = \kappa$ .*

PROOF: For any sets  $A$  and  $B$ , if  $B = \{b\}$  then the function  $f : A \rightarrow A^B$  with  $f(a)(b) = a$  is a bijection.  $\square$

**Proposition 8.1.20.** *For any non-zero cardinal number  $\kappa$  we have  $0^\kappa = 0$ .*

PROOF: If  $A$  is nonempty then there is no function  $A \rightarrow \emptyset$ .  $\square$

**Proposition 8.1.21.** *For any set  $A$  we have  $|\mathcal{P}A| = 2^{|A|}$ .*

PROOF: The function  $f : \mathcal{P}A \rightarrow 2^A$  where  $f(X)(a) = 0$  if  $a \notin X$  and  $f(X)(a) = 1$  if  $a \in X$ .  $\square$

**Theorem 8.1.22 (König).** *Let  $I$  be a set. Let  $\{A_i\}_{i \in I}$  and  $\{B_i\}_{i \in I}$  be families of sets. Assume that  $\forall i \in I. |A_i| < |B_i|$ . Then  $|\bigcup_{i \in I} A_i| < |\prod_{i \in I} B_i|$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $i \in I$ , choose an injection  $f_i : A_i \rightarrow B_i$

$\langle 1 \rangle 2$ . For all  $i \in I$ , choose  $b_i \in B_i - f_i(A_i)$

$\langle 1 \rangle 3$ .  $|\bigcup_{i \in I} A_i| \leq |\prod_{i \in I} B_i|$

$\langle 2 \rangle 1$ . Define  $g : \bigcup_{i \in I} A_i \rightarrow \prod_{i \in I} B_i$  by

$$g(i, a)(j) = \begin{cases} f_i(a) & \text{if } i = j \\ b_j & \text{otherwise} \end{cases}$$

$\langle 2 \rangle 2$ .  $g$  is injective.

$\langle 1 \rangle 4$ .  $|\bigcup_{i \in I} A_i| \neq |\prod_{i \in I} B_i|$

$\langle 2 \rangle 1$ . LET:  $h : \bigcup_{i \in I} A_i \rightarrow \prod_{i \in I} B_i$

PROVE:  $h$  is not surjective.

$\langle 2 \rangle 2$ . For  $i \in I$ , PICK  $c_i \in B_i - \{h(i, a)(i) \mid i \in I\}$

$\langle 2 \rangle 3$ .  $c \in \prod_{i \in I} B_i$

$\langle 2 \rangle 4$ .  $c \notin \text{ran } h$

$\square$

**Corollary 8.1.22.1.** *For any cardinal number  $\kappa$  we have  $\kappa < 2^\kappa$ .*

## 8.2 Ordering on Cardinal Numbers

**Definition 8.2.1.** Given cardinal numbers  $\kappa$  and  $\lambda$ , we have  $\kappa \leq \lambda$  iff  $A \preccurlyeq B$ , where  $|A| = \kappa$  and  $|B| = \lambda$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $|A| = \kappa$  and  $|B| = \lambda$

$\langle 1 \rangle 2$ . PICK bijections  $f : A \approx \kappa$  and  $g : B \approx \lambda$

$\langle 1 \rangle 3$ . If  $\kappa \leq \lambda$  then  $A \preccurlyeq B$

PROOF: Let  $i : \kappa \hookrightarrow \lambda$  be the inclusion. Then  $g^{-1} \circ i \circ f$  is an injection  $A \rightarrow B$ .

$\langle 1 \rangle 4$ . If  $A \preccurlyeq B$  then  $\kappa \leq \lambda$

$\langle 2 \rangle 1$ . ASSUME:  $A \preccurlyeq B$

$\langle 2 \rangle 2$ . PICK an injection  $h : A \hookrightarrow B$

$\langle 2 \rangle 3$ .  $g(h(A)) \subseteq B$  is well-ordered by  $\in$

$\langle 2 \rangle 4$ . LET:  $\gamma$  be the ordinal number of  $(g(h(A)), \in)$

$\langle 2 \rangle 5$ .  $\gamma \leq \lambda$

PROOF: Proposition 7.3.15.

$\langle 2 \rangle 6$ .  $\kappa \leq \gamma$

PROOF: By the leastness of  $\kappa$ , since  $A$  is equinumerous with  $\gamma$ .

$\langle 2 \rangle 7$ .  $\kappa \leq \lambda$

□

**Corollary 8.2.1.1.** *There is no largest cardinal number.*

**Proposition 8.2.2.** *For any cardinal numbers  $\kappa$ ,  $\lambda$ ,  $\mu$ , if  $\kappa \leq \lambda$  then  $\kappa + \mu \leq \lambda + \mu$ .*

PROOF: If  $f : A \rightarrow B$  is injective, and  $A \cap C = B \cap C = \emptyset$ , then the function  $A \cup C \rightarrow B \cup C$  that maps  $a$  to  $f(a)$  and maps  $c$  to  $c$  is an injection. □

**Proposition 8.2.3.** *For any cardinal numbers  $\kappa$ ,  $\lambda$ ,  $\mu$ , if  $\kappa \leq \lambda$  then  $\kappa\mu \leq \lambda\mu$ .*

PROOF: If  $f : A \rightarrow B$  is injective, then the function  $A \times C \rightarrow B \times C$  that maps  $(a, c)$  to  $(f(a), c)$  is injective. □

**Proposition 8.2.4.** *For any cardinal numbers  $\kappa$ ,  $\lambda$ ,  $\mu$ , if  $\kappa \leq \lambda$  then  $\kappa^\mu \leq \lambda^\mu$ .*

PROOF: Given an injection  $f : A \rightarrow B$ , the function that maps  $A^C \rightarrow B^C$  that maps  $g$  to  $f \circ g$  is an injection. □

**Proposition 8.2.5.** *For any cardinal numbers  $\kappa$ ,  $\lambda$ ,  $\mu$ , if  $\kappa \leq \lambda$  and  $\mu$  and  $\kappa$  are not both 0, then  $\mu^\kappa \leq \mu^\lambda$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$ ,  $B$  and  $C$  be sets with  $A$  and  $C$  not both empty.

$\langle 1 \rangle 2$ . LET:  $f : A \rightarrow B$  be an injection.

PROVE:  $C^A \preccurlyeq C^B$

$\langle 1 \rangle 3$ . CASE:  $C = \emptyset$

PROOF: Then  $A \neq \emptyset$  so  $C^A = \emptyset \preccurlyeq C^B$ .

$\langle 1 \rangle 4$ . CASE:  $C \neq \emptyset$

- ⟨2⟩1. PICK  $c \in C$   
 ⟨2⟩2. LET:  $g : C^A \rightarrow C^B$  be the function  $g(h)(f(a)) = h(a)$ ,  $g(h)(b) = c$  if  $b \notin f(A)$   
 ⟨2⟩3.  $g$  is an injection.  
 □

**Proposition 8.2.6.** *Let  $\mathcal{A}$  be a set such that  $\forall X \in \mathcal{A}. |X| \leq \kappa$ . Then*

$$\left| \bigcup \mathcal{A} \right| \leq |\mathcal{A}| \kappa .$$

PROOF:

- ⟨1⟩1. For  $X \in \mathcal{A}$ , choose a surjection  $f_X : \kappa \rightarrow X$ .  
 ⟨1⟩2. Define  $g : \mathcal{A} \times \kappa \rightarrow \bigcup \mathcal{A}$  by  $g(X, \alpha) = f_X(\alpha)$   
 ⟨1⟩3.  $g$  is surjective.  
 □

**Lemma 8.2.7.** *The union of a set of cardinal numbers is a cardinal number.*

PROOF:

- ⟨1⟩1. LET:  $A$  be a set of cardinal numbers.  
 PROVE:  $\bigcup A$  is the smallest ordinal equinumerous with  $\bigcup A$   
 ⟨1⟩2. LET:  $\alpha < \bigcup A$   
 PROVE:  $\alpha \not\approx \bigcup A$   
 ⟨1⟩3. PICK  $\kappa \in A$  such that  $\alpha < \kappa$   
 ⟨1⟩4.  $\alpha \prec \kappa$   
 ⟨1⟩5.  $\alpha \prec \bigcup A$   
 □



## Chapter 9

# Natural Numbers

### 9.1 Inductive Sets

**Definition 9.1.1** (Inductive). A set  $I$  is *inductive* iff  $0 \in I$  and  $\forall x \in I. x^+ \in I$ .

**Definition 9.1.2** (Natural Number). A *natural number* is a set that belongs to every inductive set.

**Theorem 9.1.3.** *The class  $\mathbb{N}$  of natural numbers is a set.*

PROOF:

$\langle 1 \rangle 1$ . PICK an inductive set  $I$ .

PROOF: Axiom of Infinity.

$\langle 1 \rangle 2$ .  $\mathbb{N} \subseteq I$

□

**Theorem 9.1.4.**  *$\mathbb{N}$  is inductive, and is a subset of every other inductive set.*

PROOF:

$\langle 1 \rangle 1$ .  $\mathbb{N}$  is inductive.

$\langle 2 \rangle 1$ .  $0 \in \mathbb{N}$

PROOF: Since 0 is a member of every inductive set.

$\langle 2 \rangle 2$ .  $\forall n \in \mathbb{N}. n^+ \in \mathbb{N}$

$\langle 3 \rangle 1$ . LET:  $n \in \mathbb{N}$

$\langle 3 \rangle 2$ . LET:  $I$  be any inductive set.

PROVE:  $n^+ \in I$

$\langle 3 \rangle 3$ .  $n \in I$

PROOF:  $\langle 3 \rangle 1$ ,  $\langle 3 \rangle 2$

$\langle 3 \rangle 4$ .  $n^+ \in I$

PROOF:  $\langle 3 \rangle 2$ ,  $\langle 3 \rangle 3$

$\langle 1 \rangle 2$ .  $\mathbb{N}$  is a subset of every inductive set.

PROOF: Immediate from definitions.

□

**Corollary 9.1.4.1** (Induction Principle for  $\mathbb{N}$ ). *Any inductive subset of  $\mathbb{N}$  coincides with  $\mathbb{N}$ .*

**Theorem 9.1.5.** *Every natural number except 0 is the successor of some natural number.*

PROOF: Trivially by induction.  $\square$

**Proposition 9.1.6.** *Every natural number is an ordinal.*

PROOF: By induction.  $\square$

**Proposition 9.1.7.**  *$\mathbb{N}$  is a transitive set.*

PROOF:

$\langle 1 \rangle 1. 0 \subseteq \mathbb{N}$

$\langle 1 \rangle 2. \forall n \in \mathbb{N}. n \subseteq \mathbb{N} \Rightarrow n^+ \subseteq \mathbb{N}$

$\langle 1 \rangle 3. \forall n \in \mathbb{N}. n \subseteq \mathbb{N}$

PROOF: From  $\langle 1 \rangle 1$  and  $\langle 1 \rangle 2$  by induction.

$\square$

**Corollary 9.1.7.1.**  *$\mathbb{N}$  is an ordinal.*

**Definition 9.1.8.** We define  $\omega = \mathbb{N}$ .

**Proposition 9.1.9** (Dependent Choice). *Let  $A$  be a nonempty set and  $R$  a relation on  $A$  such that  $\forall x \in A. \exists y \in A. (y, x) \in R$ . Then there exists a function  $f : \mathbb{N} \rightarrow A$  such that  $\forall n \in \mathbb{N}. (f(n+1), f(n)) \in R$ .*

PROOF:

$\langle 1 \rangle 1.$  PICK a choice function  $F$  for  $A$ .

$\langle 1 \rangle 2.$  PICK  $a \in A$

$\langle 1 \rangle 3.$  Define  $f : \mathbb{N} \rightarrow A$  by  $f(0) = a$  and  $f(n+1) = F(\{y \in A \mid (y, f(n)) \in R\})$ .

$\square$

**Theorem Schema 9.1.10.** *For any classes  $\mathbf{A}$  and  $\mathbf{R}$ , the following is a theorem:*

*Assume  $\mathbf{R}$  is a relation on  $\mathbf{A}$  and, for all  $a \in \mathbf{A}$ , the class  $\{x \in \mathbf{A} \mid x\mathbf{R}a\}$  is a set. Then  $\mathbf{R}$  is well founded if and only if there does not exist a function  $f : \mathbb{N} \rightarrow \mathbf{A}$  such that  $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$ .*

PROOF:

$\langle 1 \rangle 1.$  If there exists a function  $f : \mathbb{N} \rightarrow \mathbf{A}$  such that  $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$  then  $\mathbf{R}$  is not well founded.

PROOF:  $f(\mathbb{N})$  is a nonempty subset of  $\mathbf{A}$  with no  $\mathbf{R}$ -minimal element.

$\langle 1 \rangle 2.$  If  $\mathbf{R}$  is not well founded then there exists a function  $f : \mathbb{N} \rightarrow \mathbf{A}$  such that  $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$ .

$\langle 2 \rangle 1.$  ASSUME:  $\mathbf{R}$  is not well founded.

$\langle 2 \rangle 2.$  PICK a nonempty subset  $B \subseteq \mathbf{A}$  that has no  $\mathbf{R}$ -minimal element.

$\langle 2 \rangle 3.$   $\forall x \in B. \exists y \in B. y\mathbf{R}x$



- ⟨2⟩4. Choose a function  $g : B \rightarrow B$  such that  $\forall x \in B. g(x) \mathbf{R} x$
- ⟨2⟩5. PICK  $b \in B$
- ⟨2⟩6. Define  $f : \mathbb{N} \rightarrow \mathbf{A}$  recursively by  $f(0) = b$  and  $\forall n \in \mathbb{N}. f(n+1) = g(f(n))$
- ⟨2⟩7.  $\forall n \in \mathbb{N}. f(n+1) \mathbf{R} f(n)$

□

## 9.2 Cardinality

**Definition 9.2.1** (Finite). A set is *finite* iff it is equinumerous to some natural number; otherwise it is *infinite*.

**Theorem 9.2.2** (Pigeonhole Principle). *No natural number is equinumerous to a proper subset of itself.*

PROOF:

⟨1⟩1. LET:  $P(n)$  be the property: any one-to-one function  $n \rightarrow n$  is surjective.

⟨1⟩2.  $P(0)$

PROOF: The only function  $0 \rightarrow 0$  is injective.

⟨1⟩3. For every natural number  $n$ , if  $P(n)$  then  $P(n+1)$ .

⟨2⟩1. ASSUME:  $P(n)$

⟨2⟩2. LET:  $f$  be a one-to-one function  $n+1 \rightarrow n+1$

⟨2⟩3.  $f \upharpoonright n$  is a one-to-one function  $n \rightarrow n+1$

⟨2⟩4. CASE:  $n \notin \text{ran } f$

⟨3⟩1.  $f \upharpoonright n : n \rightarrow n$

⟨3⟩2.  $\text{ran}(f \upharpoonright n) = n$

⟨3⟩3.  $f(n) = n$

PROOF: ⟨2⟩1.

⟨3⟩4.  $\text{ran } f = n+1$

⟨2⟩5. CASE:  $n \in \text{ran } f$

⟨3⟩1. PICK  $p \in n$  such that  $f(p) = n$

⟨3⟩2. LET:  $\hat{f} : n \rightarrow n$  be the function

$$\hat{f}(p) = f(n)$$

$$\hat{f}(x) = f(x) \quad (x \neq p)$$

⟨3⟩3.  $\hat{f}$  is one-to-one

⟨3⟩4.  $\text{ran } \hat{f} = n$

PROOF: ⟨2⟩1

⟨3⟩5.  $\text{ran } f = n+1$

⟨1⟩4. For every natural number  $n$ ,  $P(n)$ .

□

**Corollary 9.2.2.1.** *No finite set is equinumerous to a proper subset of itself.*

**Corollary 9.2.2.2.** *Every natural number is a cardinal number.*

PROOF: For any natural number  $n$ , we have that  $n$  is the least ordinal such that  $n \approx n$ . □

**Corollary 9.2.2.3.**  $\mathbb{N}$  is a cardinal number.

**Corollary 9.2.2.4.**  $\mathbb{N}$  is infinite.

PROOF: The function that maps  $n$  to  $n+1$  is a bijection between  $\mathbb{N}$  and  $\mathbb{N}-\{0\}$ .  $\square$

**Corollary 9.2.2.5.** If  $C$  is a proper subset of a natural number  $n$ , then there exists  $m < n$  such that  $C \approx m$ .

PROOF: By Proposition 7.3.15.  $\square$

**Corollary 9.2.2.6.** Any subset of a finite set is finite.

**Proposition 9.2.3.** For any natural numbers  $m$  and  $n$  we have  $m+n$  (cardinal addition) is a natural number.

PROOF: Induction on  $n$ .  $\square$

**Corollary 9.2.3.1.** The union of two finite sets is finite.

**Corollary 9.2.3.2.** The union of a finite set of finite sets is finite.

PROOF: By induction on the number of elements.  $\square$

**Proposition 9.2.4.** For natural numbers  $m$  and  $n$ , the cardinal sum  $m+n$  is equal to the ordinal sum  $m+n$ .

PROOF: Induction on  $n$ .  $\square$

**Proposition 9.2.5.** For any natural numbers  $m$  and  $n$ , we have  $mn$  (cardinal multiplication) is a natural number.

**Corollary 9.2.5.1.** If  $A$  and  $B$  are finite sets then  $A \times B$  is finite.

**Proposition 9.2.6.** For natural numbers  $m$  and  $n$ , the cardinal product  $mn$  is equal to the ordinal product  $mn$ .

PROOF: Induction on  $n$ .  $\square$

**Proposition 9.2.7.** For any natural numbers  $m$  and  $n$  we have  $m^n$  (cardinal exponentiation) is a natural number.

PROOF: Induction on  $n$ .

**Corollary 9.2.7.1.** If  $A$  and  $B$  are finite sets then  $A^B$  are finite.

**Proposition 9.2.8.** For natural numbers  $m$  and  $n$ , the cardinal exponentiation  $m^n$  and the ordinal exponentiation  $m^n$  agree.

PROOF: Induction on  $n$ .  $\square$

**Proposition 9.2.9.**  $\mathbb{N}^2 \approx \mathbb{N}$

PROOF: The function  $J : \mathbb{N}^2 \rightarrow \mathbb{N}$  defined by  $J(m, n) = ((m + n)^2 + 3m + n)/2$  is a bijection.  $\square$

**Proposition 9.2.10.** *For any infinite cardinal  $\kappa$  we have  $\aleph_0 \leq \kappa$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be an infinite set.

PROVE:  $\aleph_0 \leq A$

$\langle 1 \rangle 2$ . PICK a choice function  $F$  for  $A$ .

$\langle 1 \rangle 3$ . Define  $h : \mathbb{N} \rightarrow \{X \in \mathcal{P}A \mid X \text{ is finite}\}$  by

$$h(0) = \emptyset$$

$$h(n+1) = h(n) \cup \{F(A - \{h(m) \mid m < n\})\}$$

$\langle 1 \rangle 4$ . Define  $g : \mathbb{N} \rightarrow A$  by  $g(n) = F(A - \{h(m) \mid m < n\})$

$\langle 1 \rangle 5$ .  $g$  is injective.

PROOF: If  $m < n$  then  $g(m) \neq g(n)$ .

$\square$

**Theorem Schema 9.2.11** (König's Lemma). *For any classes  $\mathbf{A}$  and  $\mathbf{R}$ , the following is a theorem:*

*Assume  $\mathbf{R}$  is a well founded relation on  $\mathbf{A}$  such that, for all  $y \in \mathbf{A}$ , the class  $\{x \in \mathbf{A} \mid x\mathbf{R}y\}$  is a finite set. Let  $\mathbf{R}^t$  be the transitive closure of  $\mathbf{R}$ . Then, for all  $y \in \mathbf{A}$ , the class  $\{x \in \mathbf{A} \mid x\mathbf{R}^ty\}$  is a finite set.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $y \in \mathbf{A}$

$\langle 1 \rangle 2$ . ASSUME: as transfinite induction hypothesis  $\forall x\mathbf{R}y. \{z \in \mathbf{A} \mid z\mathbf{R}^tx\}$  is a finite set.

$\langle 1 \rangle 3$ .  $\{x \mid x\mathbf{R}^ty\} = \bigcup_{x\mathbf{R}y} (\{x\} \cup \{z \mid z\mathbf{R}^tx\})$

$\langle 1 \rangle 4$ .  $\{x \mid x\mathbf{R}^ty\}$  is finite.

PROOF: Corollary 9.2.3.2.

$\square$

## 9.3 Countable Sets

**Definition 9.3.1** (Countable). A set  $A$  is *countable* iff  $|A| \leq \aleph_0$ .

**Theorem 9.3.2.** *The union of a countable set of countable sets is countable.*

PROOF: Proposition 8.2.6.  $\square$

## 9.4 Arithmetic

**Definition 9.4.1** (Even). A natural number  $n$  is *even* iff there exists  $m \in \mathbb{N}$  such that  $n = 2m$ .

**Definition 9.4.2** (Odd). A natural number  $n$  is *odd* iff there exists  $p \in \mathbb{N}$  such that  $n = 2p + 1$ .

**Proposition 9.4.3.** *Every natural number is either even or odd.*

PROOF:

$\langle 1 \rangle 1$ . 0 is even.

PROOF:  $0 = 2 \times 0$ .

$\langle 1 \rangle 2$ . For any natural number  $n$ , if  $n$  is either even or odd then  $n^+$  is either even or odd.

PROOF:

$\langle 2 \rangle 1$ . LET:  $n \in \mathbb{N}$

$\langle 2 \rangle 2$ . If  $n$  is even then  $n^+$  is odd.

PROOF: If  $n = 2p$  then  $n^+ = 2p + 1$ .

$\langle 2 \rangle 3$ . If  $n$  is odd then  $n^+$  is even.

PROOF: If  $n = 2p + 1$  then  $n^+ = 2(p + 1)$ .

□

**Proposition 9.4.4.** *No natural number is both even and odd.*

PROOF:

$\langle 1 \rangle 1$ . 0 is not odd.

PROOF: For any  $p$  we have  $2p + 1 = (2p)^+ \neq 0$ .

$\langle 1 \rangle 2$ . For any natural number  $n$ , if  $n$  is not both even and odd, then  $n^+$  is not both even and odd.

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . If  $n^+$  is even then  $n$  is odd.

$\langle 3 \rangle 1$ . ASSUME:  $n^+$  is even.

$\langle 3 \rangle 2$ . PICK  $p$  such that  $n^+ = 2p$

$\langle 3 \rangle 3$ .  $p \neq 0$

PROOF: Since  $n^+ \neq 0$ .

$\langle 3 \rangle 4$ . PICK  $q$  such that  $p = q^+$

PROOF: Theorem 9.1.5.

$\langle 3 \rangle 5$ .  $n^+ = 2q + 2$

PROOF:  $\langle 3 \rangle 2$ ,  $\langle 3 \rangle 4$ .

$\langle 3 \rangle 6$ .  $n = 2q + 1$

PROOF: Proposition 7.4.7,  $\langle 3 \rangle 5$

$\langle 3 \rangle 7$ .  $n$  is odd.

$\langle 2 \rangle 3$ . If  $n^+$  is odd then  $n$  is even.

$\langle 3 \rangle 1$ . ASSUME:  $n^+$  is odd.

$\langle 3 \rangle 2$ . PICK  $p$  such that  $n^+ = 2p + 1$

$\langle 3 \rangle 3$ .  $n = 2p$

PROOF: Proposition 7.4.7,  $\langle 3 \rangle 2$

$\langle 3 \rangle 4$ .  $n$  is even.

□

**Proposition 9.4.5.** *Let  $m, n, p, q$  be natural numbers. Assume  $m + n = p + q$ . Then  $m < p$  if and only if  $q < n$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $m < p$  then  $q < n$ .

PROOF: If  $m < p$  and  $n \leq q$  then  $m + n < p + q$ .

(1)2. If  $q < n$  then  $m < p$ .

PROOF: Similar.

□

**Proposition 9.4.6.** *Let  $m, n, p$  and  $q$  be natural numbers. Assume  $n < m$  and  $q < p$ . Then*

$$mq + np < mp + nq .$$

PROOF:

(1)1. PICK positive natural numbers  $a$  and  $b$  such that  $m = n + a$  and  $p = q + b$ .

(1)2.  $mp + nq > mq + np$

PROOF:

$$\begin{aligned} mp + nq &= (n + a)(q + b) + nq \\ &= 2nq + nb + aq + ab \\ mq + np &= (n + a)q + n(q + b) \\ &= 2nq + aq + nb \\ \therefore mp + nq &= mq + np + ab \\ &> mq + np \end{aligned}$$

□

## 9.5 Sequences

**Definition 9.5.1** (Sequence). Let  $A$  be a set. A *finite sequence* in  $A$  is a function  $a : n \rightarrow A$  for some natural number  $n$ ; we write it as  $(a(0), a(1), \dots, a(n - 1))$ . An *(infinite) sequence* in  $A$  is a function  $\mathbb{N} \rightarrow A$ .

We write  $A^*$  for the set of all finite sequences in  $A$ .

**Proposition 9.5.2.** *If  $A$  is countable then  $A^*$  is countable.*

PROOF: For any  $n$ , the set  $A^n$  is countable, and  $A^*$  is equinumerous with  $\bigcup_n A^n$ .

□

## 9.6 Transitive Closure of a Set

**Proposition 9.6.1.** *For any set  $A$ , there exists a unique transitive set  $C$  such that:*

- $A \subseteq C$
- For any transitive set  $X$ , if  $A \subseteq X$  then  $C \subseteq X$

PROOF:

(1)1. Define a function  $F : \mathbb{N} \rightarrow \mathbf{V}$  by

$$F(0) = A$$

$$F(n + 1) = A \cup \bigcup (F(0) \cup \dots \cup F(n))$$

- $\langle 1 \rangle 2$ . For all  $n \in \mathbb{N}$  and  $a \in F(n)$  we have  $a \subseteq F(n+1)$   
 PROOF:  $a \in F(0) \cup \dots \cup F(n)$  so  $a \subseteq \bigcup(F(0) \cup \dots \cup F(n)) \subseteq F(n+1)$ .  
 $\langle 1 \rangle 3$ . LET:  $C = \bigcup_{n \in \mathbb{N}} F(n)$   
 $\langle 1 \rangle 4$ .  $C$  is transitive.  
 $\langle 2 \rangle 1$ . LET:  $x \in y \in C$   
 $\langle 2 \rangle 2$ . PICK  $n \in \mathbb{N}$  such that  $y \in F(n)$   
 $\langle 2 \rangle 3$ .  $y \subseteq F(n+1)$   
 PROOF:  $\langle 1 \rangle 2$   
 $\langle 2 \rangle 4$ .  $x \in F(n+1)$   
 $\langle 2 \rangle 5$ .  $x \in C$   
 $\langle 1 \rangle 5$ .  $A \subseteq C$   
 PROOF: Since  $F(0) = A$ .  
 $\langle 1 \rangle 6$ . For any transitive set  $X$ , if  $A \subseteq X$  then  $C \subseteq X$   
 $\langle 2 \rangle 1$ . LET:  $X$  be a transitive set  
 $\langle 2 \rangle 2$ . ASSUME:  $A \subseteq X$   
 $\langle 2 \rangle 3$ . For all  $n \in \mathbb{N}$  we have  $F(n) \subseteq X$ .  
 $\langle 3 \rangle 1$ .  $F(0) \subseteq X$   
 PROOF:  $\langle 2 \rangle 2$   
 $\langle 3 \rangle 2$ . For all  $n \in \mathbb{N}$ , if  $F(n) \subseteq X$ , then  $F(n+1) \subseteq X$ .  
 $\langle 4 \rangle 1$ . LET:  $n \in \mathbb{N}$   
 $\langle 4 \rangle 2$ . ASSUME:  $\forall m < n. F(m) \subseteq X$   
 $\langle 4 \rangle 3$ .  $F(0) \cup \dots \cup F(n) \subseteq X$   
 $\langle 4 \rangle 4$ .  $\bigcup(F(0) \cup \dots \cup F(n)) \subseteq X$   
 PROOF: Since  $X$  is transitive.  
 $\langle 4 \rangle 5$ .  $F(n+1) \subseteq X$   
 $\langle 2 \rangle 4$ .  $C \subseteq X$   
 $\langle 1 \rangle 7$ . Let  $D$  be a transitive set such that  $A \subseteq D$  and, for any transitive set  $X$ , if  $A \subseteq X$  then  $D \subseteq X$ . Then  $D = C$ .  
 PROOF: We have  $C \subseteq D$  and  $D \subseteq C$ .  
 $\square$

**Proposition 9.6.2 (ZF).** *For any class  $\mathbf{A}$ , the relation  $\mathbf{E} = \{(x, y) \in \mathbf{A}^2 \mid x \in y\}$  is well founded.*

PROOF:

- $\langle 1 \rangle 1$ . Every nonempty set has an  $\mathbf{E}$ -minimal element.  
 PROOF: Axiom of Regularity.  
 $\langle 1 \rangle 2$ . For every set  $x$ , there exists a set  $u$  such that  $x \subseteq u$  and, for all  $w, y$ , if  $y \in u$  and  $w \mathbf{E} y$  then  $w \in u$ .  
 PROOF: Take  $u$  to be the transitive closure of  $x$ .  
 $\square$

## 9.7 The Veblen Fixed Point Theorem

**Theorem Schema 9.7.1** (Veblen Fixed Point Theorem). *For any class  $\mathbf{T}$ , the following is a theorem:*

Assume  $\mathbf{T}$  is a normal ordinal operation. For every ordinal  $\beta$ , there exists  $\gamma \geq \beta$  such that  $\mathbf{T}(\gamma) = \gamma$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $\beta$  be an ordinal.

$\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $\beta < \mathbf{T}(\beta)$

PROOF: We have  $\beta \leq \mathbf{T}(\beta)$  by Proposition 7.6.5, and if  $\beta = \mathbf{T}(\beta)$  we take  $\gamma := \beta$ .

$\langle 1 \rangle 3$ . Define  $f : \mathbb{N} \rightarrow \mathbf{On}$  by recursion thus:

$$f(0) = \beta$$

$$f(n^+) = \mathbf{T}(f(n))$$

$\langle 1 \rangle 4$ . LET:  $\gamma = \sup_{n \in \mathbb{N}} f(n)$

$\langle 1 \rangle 5$ .  $\beta \leq \gamma$

PROOF: Since  $\beta = f(0)$ .

$\langle 1 \rangle 6$ .  $\mathbf{T}(\gamma) = \gamma$

$\langle 2 \rangle 1$ .  $\mathbf{T}(\gamma) \leq \gamma$

PROOF:

$$\mathbf{T}(\gamma) = \sup_{n \in \mathbb{N}} \mathbf{T}(f(n)) \quad (\text{Theorem 7.6.7})$$

$$= \sup_{n \in \mathbb{N}} f(n^+) \quad (\langle 1 \rangle 3)$$

$$\leq \sup_{n \in \mathbb{N}} f(n)$$

$$= \gamma$$

$\langle 2 \rangle 2$ .  $\gamma \leq \mathbf{T}(\gamma)$

PROOF: Proposition 7.6.5.

□

**Definition 9.7.2** (Derived Operation). Let  $\mathbf{T}$  be a normal ordinal operation. The *derived* operation  $\mathbf{T}' : \mathbf{On} \rightarrow \mathbf{V}$  is the unique order isomorphism between  $\mathbf{On}$  and the fixed points of  $\mathbf{T}$ .

**Proposition Schema 9.7.3.** For any class  $\mathbf{T}$ , the following is a theorem:

If  $\mathbf{T}$  is a normal ordinal operation, then the derived operation is normal.

PROOF:

$\langle 1 \rangle 1$ . For any set  $S$  of fixed points of  $\mathbf{T}$ , we have  $\bigcup S$  is a fixed point of  $\mathbf{T}$

$\langle 2 \rangle 1$ . LET:  $S$  be a set of fixed points of  $\mathbf{T}$ .

$\langle 2 \rangle 2$ .  $\mathbf{T}(\sup S) = \sup S$

PROOF:

$$\mathbf{T}(\sup S) = \sup_{\alpha \in S} \mathbf{T}(\alpha) \quad (\text{Theorem 7.6.7})$$

$$= \sup_{\alpha \in S} \alpha \quad (\langle 2 \rangle 1)$$

$$= \sup S$$

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: Proposition 7.6.8.

□

## 9.8 Cantor Normal Form

**Theorem 9.8.1.** *For any ordinal  $\alpha$ , there exist a unique sequence of nonzero natural numbers  $(n_1, \dots, n_k)$  and sequence of ordinals  $(\gamma_1, \dots, \gamma_k)$  such that*

$$\gamma_k < \gamma_{k-1} < \dots < \gamma_1$$

and

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \dots + \omega^{\gamma_k} n_k .$$

PROOF:

$\langle 1 \rangle 1$ . For any ordinal  $\alpha$ , there exist a sequence of nonzero natural numbers  $(n_1, \dots, n_k)$  and sequence of ordinals  $(\gamma_1, \dots, \gamma_k)$  such that

$$\gamma_k < \gamma_{k-1} < \dots < \gamma_1$$

and

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \dots + \omega^{\gamma_k} n_k .$$

$\langle 2 \rangle 1$ . LET:  $\alpha$  be an ordinal

$\langle 2 \rangle 2$ . ASSUME: as an induction hypothesis that, for all  $\beta < \alpha$ , the theorem holds.

$\langle 2 \rangle 3$ . ASSUME: w.l.o.g.  $\alpha \neq 0$

$\langle 2 \rangle 4$ . LET:  $\gamma_1, n_1, \rho_1$  be the unique ordinals such that  $0 \neq n_1 < \omega$ ,  $\rho_1 < \omega^{\gamma_1}$ , and  $\alpha = \omega^{\gamma_1} n_1 + \rho_1$

$\langle 2 \rangle 5$ . LET:  $(\gamma_2, \dots, \gamma_k)$  and  $(n_2, \dots, n_k)$  be sequences such that  $\gamma_k < \gamma_{k-1} < \dots < \gamma_2$  and  $\rho_1 = \omega^{\gamma_2} n_2 + \dots + \omega^{\gamma_k} n_k$

$\langle 2 \rangle 6$ .  $\gamma_2 < \gamma_1$

PROOF: Since  $\omega^{\gamma_2} \leq \rho_1 < \omega^{\gamma_1}$

$\langle 1 \rangle 2$ . If

$$\begin{aligned} \gamma_k &< \gamma_{k-1} < \dots < \gamma_1 \\ \gamma'_k &< \gamma'_{k-1} < \dots < \gamma'_1 \end{aligned}$$

and

$$\omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \dots + \omega^{\gamma_k} n_k = \omega^{\gamma'_1} n'_1 + \omega^{\gamma'_2} n'_2 + \dots + \omega^{\gamma'_k} n'_k$$

then  $\gamma_i = \gamma'_i$  for all  $i$  and  $n_i = n'_i$  for all  $i$

PROOF: Prove by induction on  $i$  using the Logarithm Theorem.

□

**Definition 9.8.2** (Cantor Normal Form). For any ordinal  $\alpha$ , the *Cantor normal form* of  $\alpha$  is the expression  $\alpha = \omega^{\gamma_1} n_1 + \dots + \omega^{\gamma_k} n_k$  such that  $n_1, \dots, n_k$  are nonzero natural numbers and  $\gamma_k < \gamma_{k-1} < \dots < \gamma_1$ .



## Chapter 10

# The Cumulative Hierarchy

**Definition 10.0.1** (Cumulative Hierarchy (ZF)). Define the function  $V : \mathbf{On} \rightarrow \mathbf{V}$  by transfinite recursion thus:

$$V_\alpha = \bigcup_{\beta < \alpha} \mathcal{P}V_\beta$$

**Proposition 10.0.2** (ZF). *For all  $\alpha \in \mathbf{On}$ ,  $V_\alpha$  is a transitive set.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\alpha \in \mathbf{On}$

$\langle 1 \rangle 2$ . ASSUME: as transfinite induction hypothesis  $\forall \beta < \alpha. V_\beta$  is a transitive set.

$\langle 1 \rangle 3$ . For all  $\beta < \alpha$ ,  $\mathcal{P}V_\beta$  is a transitive set.

PROOF: Proposition 1.6.4.

$\langle 1 \rangle 4$ .  $V_\alpha$  is a transitive set.

PROOF: Proposition 1.6.3.

□

**Proposition 10.0.3** (ZF). *For any ordinals  $\alpha$  and  $\beta$ , if  $\beta < \alpha$  then  $V_\beta \subseteq V_\alpha$ .*

PROOF: Since  $V_\beta \in \mathcal{P}V_\beta \subseteq V_\alpha$  and  $V_\alpha$  is a transitive set. □

**Theorem 10.0.4** (ZF).

1.  $V_0 = \emptyset$

2.  $\forall \alpha \in \mathbf{On}. V_{\alpha+} = \mathcal{P}V_\alpha$

3. For any limit ordinal  $\lambda$ ,  $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$ .

PROOF:

$\langle 1 \rangle 1$ .  $V_0 = \emptyset$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$ .  $\forall \alpha \in \mathbf{On}. V_{\alpha+} = \mathcal{P}V_\alpha$

PROOF:

- ⟨2⟩1. LET:  $\alpha \in \mathbf{On}$   
 ⟨2⟩2. For all  $\beta < \alpha$  we have  $\mathcal{P}V_\beta \subseteq \mathcal{P}V_\alpha$   
 PROOF: Propositions 1.5.8 and 10.0.3.  
 ⟨2⟩3.  $V_{\alpha^+} = \mathcal{P}V_\alpha$

$$\begin{aligned}
 V_{\alpha^+} &= \bigcup_{\beta < \alpha^+} \mathcal{P}V_\beta \\
 &= \bigcup_{\beta < \alpha} \mathcal{P}V_\beta \cup \mathcal{P}V_\alpha \\
 &\quad \mathcal{P}V_\alpha
 \end{aligned}$$

□

- ⟨1⟩3. For any limit ordinal  $\lambda$ ,  $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$

PROOF:

- ⟨2⟩1.  $V_\lambda \subseteq \bigcup_{\alpha < \lambda} V_\alpha$

PROOF:

$$\begin{aligned}
 V_\lambda &= \bigcup_{\alpha < \lambda} \mathcal{P}V_\alpha \\
 &= \bigcup_{\alpha < \lambda} V_{\alpha^+} & (\langle 1 \rangle 2) \\
 &\subseteq \bigcup_{\alpha < \lambda} V_\alpha
 \end{aligned}$$

- ⟨2⟩2.  $\bigcup_{\alpha < \lambda} V_\alpha \subseteq V_\lambda$

PROOF: Proposition 10.0.3.

□

**Proposition 10.0.5 (ZF).** *For every set  $A$ , there exists an ordinal  $\alpha$  such that  $A \in V_\alpha$ .*

PROOF:

- ⟨1⟩1. Let us say a set  $A$  is *grounded* iff there exists an ordinal  $\alpha$  such that  $A \in V_\alpha$ .  
 ⟨1⟩2. For any set  $A$ , if every element of  $A$  is grounded, then  $A$  is grounded.  
 ⟨2⟩1. LET:  $A$  be a set.  
 ⟨2⟩2.  $S = \{\alpha \mid \exists a \in A. \alpha \text{ is the least ordinal such that } a \in V_\alpha\}$   
 PROOF:  $S$  is a set by an Axiom of Replacement.  
 ⟨2⟩3. LET:  $\beta = \sup S$   
 ⟨2⟩4.  $A \subseteq V_\beta$   
 ⟨3⟩1. LET:  $a \in A$   
 ⟨3⟩2. LET:  $\alpha$  be the least ordinal such that  $a \in V_\alpha$   
 ⟨3⟩3.  $\alpha \in S$   
 ⟨3⟩4.  $\alpha \leq \beta$   
 ⟨3⟩5.  $a \in V_\beta$   
 ⟨2⟩5.  $A \in V_{\beta^+}$   
 ⟨1⟩3. ASSUME: for a contradiction there exists an ungrounded set.  
 ⟨1⟩4. PICK a transitive set  $B$  that has an ungrounded member.  
 PROOF: Pick a transitive set  $c$ , and take  $B$  to be the transitive closure of  $\{c\}$ .

- ⟨1⟩5. LET:  $A = \{x \in B \mid x \text{ is ungrounded}\}$   
 ⟨1⟩6. PICK  $m \in A$  such that  $m \cap A = \emptyset$   
 PROOF: Axiom of Regularity.  
 ⟨1⟩7. Every member of  $m$  is grounded.  
 ⟨2⟩1. ASSUME: for a contradiction  $x \in m$  is ungrounded.  
 ⟨2⟩2.  $x \in B$   
 PROOF: Since  $B$  is transitive (⟨1⟩4).  
 ⟨2⟩3.  $x \in A$   
 PROOF: ⟨1⟩5  
 ⟨2⟩4. Q.E.D.  
 PROOF: This contradicts ⟨1⟩6.  
 ⟨1⟩8.  $m$  is grounded.  
 PROOF: ⟨1⟩2  
 ⟨1⟩9. Q.E.D.  
 PROOF: This contradicts ⟨1⟩6.

□

**Definition 10.0.6** (Rank (ZF)). The *rank* of a set  $A$  is the least ordinal  $\alpha$  such that  $A \in V_{\alpha^+}$ .

**Proposition 10.0.7** (ZF). For any set  $A$  we have

$$\text{rank } A = \bigcup_{a \in A} (\text{rank } a)^+$$

PROOF:

- ⟨1⟩1. LET:  $\alpha = \bigcup_{a \in A} (\text{rank } a)^+$   
 ⟨1⟩2.  $A \subseteq V_\alpha$   
 ⟨2⟩1. LET:  $a \in A$   
 ⟨2⟩2.  $a \in V_{(\text{rank } a)^+}$   
 ⟨2⟩3.  $a \in V_\alpha$   
 ⟨1⟩3.  $A \in V_{\alpha^+}$   
 ⟨1⟩4. If  $A \subseteq V_\beta$  then  $\alpha \leq \beta$   
 ⟨2⟩1. ASSUME:  $A \subseteq V_\beta$   
 ⟨2⟩2. For all  $a \in A$  we have  $(\text{rank } a)^+ \leq \beta$   
 PROOF: Since  $a \in V_\beta$ .  
 ⟨2⟩3.  $\alpha \leq \beta$

□

**Corollary 10.0.7.1** (ZF). For any sets  $a$  and  $b$ , if  $a \in b$  then  $\text{rank } a < \text{rank } b$ .

**Proposition 10.0.8** (ZF). For any ordinal number  $\alpha$  we have  $\text{rank } \alpha = \alpha$ .

PROOF:

- ⟨1⟩1. LET:  $\alpha$  be an ordinal.  
 ⟨1⟩2. ASSUME: as transfinite induction hypothesis  $\forall \beta < \alpha. \text{rank } \beta = \beta$   
 ⟨1⟩3.  $\text{rank } \alpha = \bigcup_{\beta < \alpha} \beta^+$

PROOF:

$$\begin{aligned}\text{rank } \alpha &= \bigcup_{\beta < \alpha} (\text{rank } \beta)^+ \\ &= \bigcup_{\beta < \alpha} \beta^+\end{aligned}$$

$$\langle 1 \rangle 4. \bigcup_{\beta < \alpha} \beta^+ \leq \alpha$$

PROOF: Since for all  $\beta < \alpha$  we have  $\beta^+ \leq \alpha$ .

$$\langle 1 \rangle 5. \alpha \leq \bigcup_{\beta < \alpha} \beta^+$$

$$\langle 2 \rangle 1. \text{ LET: } \gamma = \bigcup_{\beta < \alpha} \beta^+$$

$$\langle 2 \rangle 2. \text{ ASSUME: for a contradiction } \gamma < \alpha$$

$$\langle 2 \rangle 3. \gamma^+ \leq \bigcup_{\beta < \alpha} \beta^+ = \gamma$$

$$\langle 2 \rangle 4. \text{ Q.E.D.}$$

PROOF: This is a contradiction.

□

**Definition 10.0.9** (Hereditarily Finite (ZF)). A set is *hereditarily finite* iff it is in  $V_\omega$ .

**Proposition 10.0.10** (ZF).  $V_\omega$  is the smallest set  $X$  such that  $\emptyset \in X$  and every finite subset of  $X$  is an element of  $X$ .

## Chapter 11

# Models of Set Theory

**Definition 11.0.1** (Relativization). Let  $\sigma$  be a sentence in the language of set theory and  $\mathbf{M}$  a class. The *relativization* of  $\sigma$  to  $\mathbf{M}$  is the sentence  $\sigma^{\mathbf{M}}$  formed by replacing every quantifier  $\forall x$  with  $\forall x \in \mathbf{M}$ , and  $\exists x$  with  $\exists x \in \mathbf{M}$ .

We write ' $\mathbf{M}$  is a model of  $\sigma$ ' for the sentence  $\sigma^{\mathbf{M}}$ .

**Theorem Schema 11.0.2.** *For any class  $\mathbf{M}$ , the following is a theorem:*

*If  $\mathbf{M}$  is a transitive class, then  $\mathbf{M}$  is a model of the Axiom of Extensionality.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\mathbf{M}$  is a transitive class.

PROVE:  $\forall x, y \in \mathbf{M} (\forall z \in \mathbf{M} (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$

$\langle 1 \rangle 2$ . LET:  $x, y \in \mathbf{M}$

$\langle 1 \rangle 3$ . ASSUME:  $\forall z \in \mathbf{M} (z \in x \Leftrightarrow z \in y)$

$\langle 1 \rangle 4$ .  $\forall z (z \in x \Leftrightarrow z \in y)$

PROOF: Since  $z \in x \Rightarrow z \in \mathbf{M}$  and  $z \in y \Rightarrow z \in \mathbf{M}$  by  $\langle 1 \rangle 1$ .

$\langle 1 \rangle 5$ .  $x = y$

□

**Theorem 11.0.3.** *If  $\alpha$  is a non-zero ordinal then  $V_\alpha$  is a model of the statement: The empty class is a set.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\alpha \neq 0$

PROVE:  $\exists x \in V_\alpha. \forall y \in V_\alpha. y \notin x$

$\langle 1 \rangle 2$ .  $\emptyset \in V_\alpha$

$\langle 1 \rangle 3$ .  $\forall y \in V_\alpha. y \notin \emptyset$

□

**Theorem 11.0.4.** *For any limit ordinal  $\lambda$ , we have  $V_\lambda$  is a model of the statement: for any sets  $a$  and  $b$ , the class  $\{a, b\}$  is a set.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\lambda$  be a limit ordinal.

PROVE:  $\forall a, b \in V_\lambda. \exists c \in V_\lambda. \forall x \in V_\lambda (x \in c \Leftrightarrow x = a \vee x = b)$   
 (1)2. LET:  $a, b \in V_\lambda$   
 (1)3. PICK  $\alpha, \beta < \lambda$  such that  $a \in V_\alpha$  and  $b \in V_\beta$   
 (1)4. ASSUME: w.l.o.g.  $\alpha \leq \beta$   
 (1)5.  $a, b \in V_\beta$   
 (1)6.  $\{a, b\} \in V_{\beta+1}$   
 (1)7.  $\{a, b\} \in V_\lambda$   
 (1)8.  $\forall x \in V_\lambda (x \in \{a, b\} \Leftrightarrow x = a \vee x = b)$   
 $\square$

**Theorem 11.0.5.** *For any ordinal  $\alpha$ , we have  $V_\alpha$  is a model of the Union Axiom.*

PROOF:

(1)1. LET:  $\alpha$  be an ordinal.  
 PROVE:  $\forall a \in V_\alpha. \exists b \in V_\alpha. \forall x \in V_\alpha (x \in b \Leftrightarrow \exists y \in V_\alpha (x \in y \wedge y \in a))$   
 (1)2. LET:  $a \in V_\alpha$   
 (1)3. PICK  $\beta < \alpha$  such that  $a \subseteq V_\beta$   
 (1)4.  $\bigcup a \subseteq V_\beta$   
 PROOF:  $V_\beta$  is a transitive set.  
 (1)5.  $\bigcup a \in V_\alpha$   
 (1)6.  $\forall x \in V_\alpha (x \in \bigcup a \Leftrightarrow \exists y \in V_\alpha (x \in y \wedge y \in a))$   
 PROOF:  $V_\alpha$  is a transitive set.  
 $\square$

**Theorem 11.0.6.** *For any limit ordinal  $\lambda$ , we have  $V_\lambda$  is a model of the Power Set Axiom.*

PROOF:

(1)1. LET:  $\lambda$  be a limit ordinal.  
 PROVE:  $\forall a \in V_\lambda. \exists b \in V_\lambda. \forall x \in V_\lambda (x \in b \Leftrightarrow \forall y \in V_\lambda (y \in x \Rightarrow y \in a))$   
 (1)2. LET:  $a \in V_\lambda$   
 (1)3. PICK  $\alpha < \lambda$  such that  $a \in V_\alpha$   
 (1)4.  $\mathcal{P}a \in V_{\alpha+1}$   
 (1)5.  $\mathcal{P}a \in V_\lambda$   
 (1)6.  $\forall x \in V_\lambda (x \in \mathcal{P}a \Leftrightarrow \forall y \in V_\lambda (y \in x \Rightarrow y \in a))$   
 $\square$

**Theorem Schema 11.0.7.** *For any property  $P[x, y_1, \dots, y_n]$ , the following is a theorem:*

*For any ordinal  $\alpha$ , the set  $V_\alpha$  is a model of the statement: for any sets  $a_1, \dots, a_n, B$ , the class  $\{x \in B \mid P[x, a_1, \dots, a_n]\}$  is a set.*

PROOF:

(1)1. LET:  $\alpha$  be an ordinal.  
 (1)2. LET:  $a_1, \dots, a_n, B \in V_\alpha$   
 (1)3. LET:  $C = \{x \in B \mid P[x, a_1, \dots, a_n]^{V_\alpha}\}$   
 (1)4.  $C \in V_\alpha$

$\langle 1 \rangle 5. \forall x \in V_\alpha (x \in C \Leftrightarrow x \in B \wedge P[x, a_1, \dots, a_n]^{V_\alpha})$

□

**Theorem 11.0.8.** *For any ordinal  $\alpha > \omega$ , we have:  $V_\alpha$  is a model of the Axiom of Infinity.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $\alpha > \omega$

$\langle 1 \rangle 2.$   $\mathbb{N} \in V_\alpha$

$\langle 1 \rangle 3.$   $\exists e \in V_\alpha (e \in \mathbb{N} \wedge \forall x \in V_\alpha. x \notin e)$

$\langle 1 \rangle 4.$   $\forall x \in V_\alpha (x \in \mathbb{N} \Rightarrow \exists y \in V_\alpha \forall z \in V_\alpha (z \in y \Leftrightarrow z \in x \vee z = x))$

□

**Theorem 11.0.9.** *For any ordinal  $\alpha$ , we have  $V_\alpha$  is a model of the Axiom of Choice.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $\alpha$  be an ordinal.

$\langle 1 \rangle 2.$  LET:  $A \in V_\alpha$

$\langle 1 \rangle 3.$  ASSUME:  $\forall x \in V_\alpha (x \in A \Rightarrow \exists y \in V_\alpha. y \in A)$

$\langle 1 \rangle 4.$  ASSUME:  $\forall x, y, z \in V_\alpha (x \in A \wedge y \in A \wedge z \in x \wedge z \in y \Rightarrow x = y)$

$\langle 1 \rangle 5.$   $A$  is a set of pairwise disjoint nonempty sets.

$\langle 1 \rangle 6.$  PICK  $c$  such that, for all  $x \in A$ ,  $x \cap c = \emptyset$

$\langle 1 \rangle 7.$   $c \cap \bigcup A \in V_\alpha$

$\langle 1 \rangle 8.$   $\forall x \in V_\alpha (x \in A \Rightarrow \exists y \in V_\alpha \forall z \in V_\alpha (z \in x \wedge z \in c \cap \bigcup A \Leftrightarrow z = y))$

□

**Theorem 11.0.10.** *For any ordinal  $\alpha$ , we have  $V_\alpha$  is a model of the Axiom of Regularity.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $\alpha$  be an ordinal.

$\langle 1 \rangle 2.$  LET:  $A \in V_\alpha$

$\langle 1 \rangle 3.$  ASSUME:  $\exists x \in V_\alpha. x \in A$

$\langle 1 \rangle 4.$  PICK  $m \in A$  of least rank.

$\langle 1 \rangle 5.$   $m \in V_\alpha$

$\langle 1 \rangle 6.$   $\neg \exists x \in V_\alpha (x \in m \wedge x \in A)$

□

**Theorem Schema 11.0.11.** *For any axiom  $\alpha$  of Zermelo set theory, the following is a theorem:*

*For any limit ordinal  $\lambda > \omega$ , we have  $V_\lambda$  is a model of  $\alpha$ .*

PROOF: Theorems 11.0.2, 11.0.3, 11.0.4, 11.0.5, 11.0.6, 11.0.7, 11.0.8, 11.0.9, 11.0.10. □

**Corollary Schema 11.0.11.1.** *for any axiom  $\alpha$  of Zermelo set theory, the following is a theorem:*

*$V_{\omega_2}$  is a model of  $\alpha$ .*

**Lemma 11.0.12.** *There exists a well-ordered structure in  $V_{\omega^2}$  whose ordinal is not in  $V_{\omega^2}$ .*

PROOF: Take the well-ordered set  $\mathbb{N} \times \{0, 1\}$  whose ordinal is  $\omega^2$ .  $\square$

**Corollary Schema 11.0.12.1.** *There exists an instance  $\alpha$  of the Axiom Schema of Replacement such that the following is a theorem:  
 $V_{\omega^2}$  is not a model of  $\alpha$ .*



## Chapter 12

# Infinite Cardinals

### 12.1 Arithmetic of Infinite Cardinals

**Proposition 12.1.1.** *For any infinite cardinal  $\kappa$  we have  $\kappa\kappa = \kappa$ .*

PROOF:

- $\langle 1 \rangle 1$ . PICK a set  $B$  with  $|B| = \kappa$
- $\langle 1 \rangle 2$ . LET:  $\mathcal{H} = \{f \mid f = \emptyset \vee \exists A \subseteq B. (A \text{ is infinite} \wedge f : A \times A \approx A)\}$
- $\langle 1 \rangle 3$ . For any chain  $\mathcal{C} \subseteq \mathcal{H}$  we have  $\bigcup \mathcal{C} \in \mathcal{H}$ 
  - $\langle 2 \rangle 1$ . LET:  $\mathcal{C} \subseteq \mathcal{H}$  be a chain.
  - $\langle 2 \rangle 2$ . ASSUME: w.l.o.g.  $\mathcal{C}$  has a nonempty element.
  - $\langle 2 \rangle 3$ .  $\bigcup \mathcal{C}$  is a function.
    - $\langle 3 \rangle 1$ . ASSUME:  $(x, y), (x, z) \in \bigcup \mathcal{C}$
    - $\langle 3 \rangle 2$ . PICK  $f, g \in \mathcal{C}$  such that  $f(x) = y$  and  $g(x) = z$
    - $\langle 3 \rangle 3$ . ASSUME: w.l.o.g.  $f \subseteq g$
    - $\langle 3 \rangle 4$ .  $y = z$
  - $\langle 2 \rangle 4$ .  $\bigcup \mathcal{C}$  is injective.
- PROOF: Similar.
- $\langle 2 \rangle 5$ . LET:  $A = \text{ran } \bigcup \mathcal{C}$
- $\langle 2 \rangle 6$ .  $A$  is infinite.
  - $\langle 3 \rangle 1$ . PICK a nonzero  $f \in \mathcal{C}$
  - $\langle 3 \rangle 2$ . LET:  $A'$  be the infinite subset of  $B$  such that  $f : A'^2 \approx A'$
  - $\langle 3 \rangle 3$ .  $A' \subseteq A$
- $\langle 2 \rangle 7$ .  $\text{dom } \bigcup \mathcal{C} = A^2$ 
  - $\langle 3 \rangle 1$ . LET:  $x, y \in A$
  - $\langle 3 \rangle 2$ . PICK  $f, g \in \mathcal{C}$  such that  $x \in \text{ran } f$  and  $y \in \text{ran } g$
  - $\langle 3 \rangle 3$ . ASSUME: w.l.o.g.  $f \subseteq g$
  - $\langle 3 \rangle 4$ . LET:  $A'$  be the infinite subset of  $B$  such that  $g : A'^2 \approx A'$
  - $\langle 3 \rangle 5$ .  $x, y \in A'$
  - $\langle 3 \rangle 6$ .  $(x, y) \in \text{dom } g$
  - $\langle 3 \rangle 7$ .  $(x, y) \in \text{dom } \bigcup \mathcal{C}$
- $\langle 2 \rangle 8$ .  $\bigcup \mathcal{C} \in \mathcal{H}$

- ⟨1⟩4. PICK a maximal  $f_0 \in \mathcal{H}$
- ⟨1⟩5.  $f_0 \neq \emptyset$ 
  - ⟨2⟩1. PICK a countably infinite subset  $A$  of  $B$ .  
PROOF: Proposition 9.2.10.
  - ⟨2⟩2. PICK a bijection  $f : A^2 \approx A$   
PROOF: Proposition 9.2.9.
  - ⟨2⟩3.  $\emptyset \subseteq f \in \mathcal{H}$
  - ⟨2⟩4.  $\emptyset$  is not maximal in  $\mathcal{H}$
- ⟨1⟩6. LET:  $A_0$  be the infinite subset of  $B$  such that  $f_0 : A_0^2 \approx A_0$
- ⟨1⟩7. LET:  $\lambda = |A_0|$
- ⟨1⟩8.  $\lambda$  is infinite.
- ⟨1⟩9.  $\lambda^2 = \lambda$
- ⟨1⟩10.  $\lambda = \kappa$ 
  - ⟨2⟩1. ASSUME: for a contradiction  $\lambda < \kappa$
  - ⟨2⟩2.  $\lambda \leq |B - A_0|$
  - ⟨2⟩3. PICK a subset  $D \subseteq B - A_0$  with  $|D| = \lambda$
  - ⟨2⟩4.  $(A_0 \cup D)^2 = A_0^2 \cup (A_0 \times D) \cup (D \times A_0) \cup D^2$
  - ⟨2⟩5. LET:  $C = (A_0 \times D) \cup (D \times A_0) \cup D^2$
  - ⟨2⟩6.  $|C| = \lambda$   
PROOF:  

$$|(A_0 \times D) \cup (D \times A_0) \cup D^2| = \lambda^2 + \lambda^2 + \lambda^2$$

$$= \lambda + \lambda + \lambda \quad (\langle 1 \rangle 9)$$

$$= 3\lambda$$

$$\leq \lambda \cdot \lambda$$

$$= \lambda \quad (\langle 1 \rangle 9)$$
  - ⟨2⟩7. PICK a bijection  $g : C \approx D$
  - ⟨2⟩8.  $f_0 \cup g : (A_0 \cup D)^2 \approx A_0 \cup D$
  - ⟨2⟩9. Q.E.D.

PROOF: This contradicts the maximality of  $f_0$ .

□

**Theorem 12.1.2** (Absorption Law of Cardinal Arithmetic). *Let  $\kappa$  and  $\lambda$  be nonzero cardinal numbers such that at least one is infinite. Then*

$$\kappa + \lambda = \kappa\lambda = \max(\kappa, \lambda)$$

PROOF:

- ⟨1⟩1. ASSUME: w.l.o.g.  $\lambda \leq \kappa$
- ⟨1⟩2.  $\kappa + \lambda = \kappa\lambda = \kappa$

PROOF:

$$\begin{aligned}
 \kappa &\leq \kappa + \lambda \\
 &\leq \kappa + \kappa \\
 &= 2\kappa \\
 &\leq \kappa\lambda \\
 &\leq \kappa\kappa \\
 &= \kappa
 \end{aligned}$$

(Proposition 12.1.1)

□

## 12.2 Alephs

**Definition 12.2.1** (Aleph). Let  $\aleph$  be the unique order isomorphism between  $\mathbf{On}$  and the class of infinite cardinals.

**Proposition 12.2.2.** *The operation  $\aleph$  is normal.*

PROOF: Proposition 7.6.8 and Lemma 8.2.7. □

**Definition 12.2.3** (Continuum Hypothesis). The *continuum hypothesis* is the statement that  $\aleph_1 = 2^{\aleph_0}$ .

**Definition 12.2.4** (Generalised Continuum Hypothesis). The *generalised continuum hypothesis* is the statement that, for all  $\alpha$ ,  $\aleph_{\alpha+} = 2^{\aleph_\alpha}$ .

## 12.3 Beths

**Definition 12.3.1** (Beth). Define the operation  $\beth : \mathbf{On} \rightarrow \mathbf{Card}$  by transfinite recursion as follows:

$$\begin{aligned}
 \beth_0 &:= \aleph_0 \\
 \beth_{\alpha+} &:= 2^{\beth_\alpha} \\
 \beth_\lambda &:= \bigcup_{\alpha < \lambda} \beth_\alpha \quad (\lambda \text{ a limit ordinal})
 \end{aligned}$$

**Proposition 12.3.2.**  *$\beth$  is a normal operation.*

PROOF: It is continuous by definition, and  $\beth_\alpha < \beth_{\alpha+}$  by Cantor's Theorem. □

**Proposition 12.3.3.** *The continuum hypothesis is equivalent to the statement  $\beth_1 = \aleph_1$ .*

*The generalised continuum hypothesis is equivalent to the statement  $\beth = \aleph$ .*

PROOF: Immediate from definitions. □

**Lemma 12.3.4.** *For any ordinal number  $\alpha$ , we have  $|V_{\omega+\alpha}| = \beth_\alpha$ .*

PROOF:

(1)1.  $|V_\omega| = \beth_0$

PROOF: Since  $V_\omega$  is the union of  $\aleph_0$  finite sets of increasing size.

(1)2. For any ordinal  $\alpha$ , if  $|V_{\omega+\alpha}| = \beth_\alpha$  then  $|V_{\omega+\alpha+1}| = \beth_{\alpha+1}$

PROOF: Since  $V_{\omega+\alpha+1} = \mathcal{P}V_{\omega+\alpha}$ .

(1)3. For any limit ordinal  $\lambda$ , if  $\forall \alpha < \lambda. |V_{\omega+\alpha}| = \beth_\alpha$  then  $|V_{\omega+\lambda}| = \beth_\lambda$ .

PROOF:

$$\begin{aligned} |V_{\omega+\lambda}| &= \left| \bigcup_{\alpha < \lambda} V_{\omega+\alpha} \right| \\ &= \sup_{\alpha < \lambda} |V_{\omega+\alpha}| \\ &= \sup_{\alpha < \lambda} \beth_\alpha \\ &= \beth_\lambda \end{aligned}$$

□

## 12.4 Cofinality

**Definition 12.4.1** (Cofinal). Let  $\lambda$  be a limit ordinal and  $S$  a set of ordinals smaller than  $\lambda$ . Then  $S$  is *cofinal* in  $\lambda$  if and only if  $\lambda = \sup S$ .

**Definition 12.4.2** (Cofinality). For any ordinal  $\alpha$ , define the *cofinality* of  $\alpha$ ,  $\text{cf } \alpha$ , as follows:

- $\text{cf } 0 = 0$
- For any ordinal  $\alpha$ ,  $\text{cf } \alpha^+ = 1$
- For any limit ordinal  $\lambda$ ,  $\text{cf } \lambda$  is the smallest cardinal such that there exists a set  $S$  of ordinals cofinal in  $\lambda$  with  $|S| = \text{cf } \lambda$ .

**Definition 12.4.3** (Regular). A cardinal  $\kappa$  is *regular* iff  $\text{cf } \kappa = \kappa$ ; otherwise it is *singular*.

**Proposition 12.4.4.**  $\aleph_0$  is regular.

PROOF:  $\aleph_0$  is not the supremum of  $< \aleph_0$  smaller ordinals, because a finite union of finite ordinals is finite. □

**Proposition 12.4.5.** For every ordinal  $\alpha$ ,  $\aleph_{\alpha+1}$  is regular.

PROOF: If  $S$  is a set of ordinals with  $|S| < \aleph_{\alpha+1}$  and  $\forall \beta \in S. \beta < \aleph_{\alpha+1}$ , then we have  $|S| \leq \aleph_\alpha$  and  $\forall \beta \in S. \beta \leq \aleph_\alpha$ , hence

$$\begin{aligned} \left| \bigcup S \right| &\leq \aleph_\alpha^2 && \text{(Proposition 8.2.6)} \\ &= \aleph_\alpha && \text{(Proposition 12.1.1)} \end{aligned}$$

**Proposition Schema 12.4.6.** For any class  $\mathbf{T}$ , the following is a theorem.

Assume  $\mathbf{T} : \mathbf{On} \rightarrow \mathbf{On}$  is a normal operation. For any limit ordinal  $\lambda$  we have  $\text{cf } \mathbf{T}(\lambda) = \text{cf } \lambda$ .

PROOF:

- $\langle 1 \rangle 1.$   $\text{cf } \mathbf{T}(\lambda) \leq \text{cf } \lambda$ 
  - $\langle 2 \rangle 1.$  PICK a set  $S$  of ordinals  $< \lambda$  with  $|S| = \text{cf } \lambda$  and  $\sup S = \lambda$
  - $\langle 2 \rangle 2.$   $\mathbf{T}(\lambda) = \sup_{\alpha \in S} \mathbf{T}(\alpha)$   
 PROOF: Theorem 7.6.7.
- $\langle 1 \rangle 2.$   $\text{cf } \lambda \leq \text{cf } \mathbf{T}(\lambda)$ 
  - $\langle 2 \rangle 1.$  PICK a set  $A$  of ordinals  $< \mathbf{T}(\lambda)$  such that  $|A| = \text{cf } \mathbf{T}(\lambda)$  and  $\sup A = \mathbf{T}(\lambda)$
  - $\langle 2 \rangle 2.$  LET:  $B = \{\gamma < \lambda \mid \exists \alpha \in A. |\alpha| = \mathbf{T}(\gamma)\}$
  - $\langle 2 \rangle 3.$   $|B| \leq |A| = \text{cf } \mathbf{T}(\lambda)$   
 PROVE:  $\sup B = \lambda$
  - $\langle 2 \rangle 4.$   $\forall \alpha \in A. |\alpha| \leq \mathbf{T}(\sup B)$
  - $\langle 2 \rangle 5.$   $\forall \alpha \in A. \alpha < \mathbf{T}(\sup B + 1)$
  - $\langle 2 \rangle 6.$   $\aleph_\lambda = \sup A \leq \mathbf{T}(\sup B + 1)$
  - $\langle 2 \rangle 7.$   $\lambda \leq \sup B + 1$
  - $\langle 2 \rangle 8.$   $\lambda \leq \sup B$   
 PROOF:  $\lambda$  is a limit ordinal.
  - $\langle 2 \rangle 9.$   $\sup B = \lambda$

□

**Corollary 12.4.6.1.**  $\aleph_\omega$  is singular.

PROOF:  $\text{cf } \aleph_\omega = \text{cf } \aleph_0 = \aleph_0$ . □

**Corollary 12.4.6.2.** The operation  $\text{cf}$  is not strictly monotone or continuous.

PROOF:  $\text{cf } \aleph_\omega < \text{cf } \aleph_1$  □

**Definition 12.4.7** (Weakly Inaccessible). A cardinal is *weakly inaccessible* iff it is  $\aleph_\lambda$  for some limit ordinal  $\lambda$  and regular.

**Lemma 12.4.8.** Let  $\lambda$  be a limit ordinal. Then there exists a strictly increasing  $\text{cf } \lambda$ -sequence that converges to  $\lambda$ .

PROOF:

- $\langle 1 \rangle 1.$  PICK a set  $S$  of ordinals  $< \lambda$  with  $|S| = \text{cf } \lambda$  and  $\sup S = \lambda$
- $\langle 1 \rangle 2.$  PICK a bijection  $a : \text{cf } \lambda \approx S$
- $\langle 1 \rangle 3.$  PICK a strictly increasing subsequence  $(b_\delta)_{\delta < \beta}$  of  $a$  that converges to  $\lambda$ .  
 PROOF: Lemma 7.8.5.
- $\langle 1 \rangle 4.$   $\beta = \text{cf } \lambda$   
 PROOF: By minimality of  $\text{cf } \lambda$ .

□

**Corollary 12.4.8.1.** Let  $\lambda$  be a limit ordinal. Then  $\text{cf } \lambda$  is the least ordinal such that there exists a strictly increasing  $\text{cf } \lambda$ -sequence that converges to  $\lambda$ .

**Proposition 12.4.9.** For any ordinal  $\lambda$ ,  $\text{cf } \lambda$  is a regular cardinal.

PROOF:

- (1)1. LET:  $\lambda$  be an ordinal.  
 (1)2. ASSUME: w.l.o.g.  $\lambda$  is a limit ordinal.  
 (1)3. PICK a strictly increasing sequence  $(a_\alpha)_{\alpha < \text{cf } \lambda}$  that converges to  $\lambda$ .  
 (1)4. LET:  $S$  be a set of ordinals  $< \text{cf } \lambda$  such that  $|S| = \text{cf } \text{cf } \lambda$  and  $\sup S = \text{cf } \lambda$ .  
 (1)5. LET:  $a(S) = \{a_\alpha \mid \alpha \in S\}$   
 (1)6.  $a(S)$  is cofinal in  $\lambda$ .  
     (2)1. LET:  $\beta < \lambda$   
     (2)2. PICK  $\gamma < \text{cf } \lambda$  such that  $\beta < a_\gamma$   
     (2)3. PICK  $\delta \in S$  such that  $\gamma < \delta$   
     (2)4.  $a_\delta \in a(S)$  and  $\beta < a_\gamma < a_\delta$   
 (1)7.  $\text{cf } \lambda \leq \text{cf } \text{cf } \lambda$   
     PROOF: Since  $a(S)$  is a set of ordinals  $< \lambda$  with  $|a(S)| = \text{cf } \text{cf } \lambda$  and  $\sup a(S) = \lambda$ .  
 (1)8.  $\text{cf } \text{cf } \lambda = \text{cf } \lambda$   
 □

**Theorem 12.4.10.** *Let  $\lambda$  be an infinite cardinal. Then  $\text{cf } \lambda$  is the least cardinal such that  $\lambda$  can be partitioned into  $\text{cf } \lambda$  sets, each of cardinality  $< \lambda$ .*

PROOF:

- (1)1.  $\lambda$  can be partitioned into  $\text{cf } \lambda$  sets, each of cardinality  $< \lambda$   
     (2)1. PICK a strictly increasing sequence of ordinals  $(a_\alpha)_{\alpha < \text{cf } \lambda}$  that converges to  $\lambda$   
     (2)2.  $\{\{\beta \mid a_\alpha \leq \beta < a_{\alpha+1}\} \mid \alpha < \text{cf } \lambda\}$  is a partition of  $\lambda$  into  $\text{cf } \lambda$  sets, each of cardinality  $< \lambda$   
 (1)2. If  $\lambda$  can be partitioned into  $\kappa$  sets, each of cardinality  $< \lambda$ , then  $\text{cf } \lambda \leq \kappa$ .  
     (2)1. LET:  $\mathcal{A}$  be a partition of  $\lambda$  into sets of cardinality  $< \lambda$   
     (2)2. LET:  $\kappa = |P|$   
     (2)3. PICK a bijection  $A : \kappa \approx P$   
     (2)4.  $\lambda = \bigcup_{\xi < \kappa} A(\xi)$   
     (2)5. For all  $\xi < \kappa$  we have  $|A(\xi)| < \lambda$   
     (2)6. LET:  $\mu = \sup_{\xi < \kappa} |A(\xi)|$   
     (2)7.  $\mu \leq \lambda$   
     (2)8. For all  $\xi < \kappa$  we have  $|A(\xi)| \leq \mu$   
     (2)9.  $\lambda \leq \mu\kappa$   
         PROOF: Proposition 8.2.6.  
 (2)10. ASSUME: w.l.o.g.  $\kappa < \lambda$   
     PROOF: If  $\lambda \leq \kappa$  then  $\text{cf } \lambda \leq \kappa$  since  $\text{cf } \lambda \leq \lambda$ .  
 (2)11.  $\lambda = \mu$   
     PROOF:

$$\lambda \leq \mu\kappa \quad (\langle 2 \rangle 9)$$

$$\leq \lambda\lambda \quad (\langle 2 \rangle 7, \langle 2 \rangle 10)$$

$$= \lambda \quad (\text{Proposition 12.1.1})$$

- (2)12.  $\{|A(\xi)| \mid \xi < \kappa\}$  is a set of  $\leq \kappa$  ordinals all  $< \lambda$  whose supremum is  $\lambda$   
 (2)13.  $\text{cf } \lambda \leq \kappa$

□

**Theorem 12.4.11** (König). *For any infinite cardinal  $\kappa$  we have  $\kappa < \text{cf } 2^\kappa$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $\text{cf } 2^\kappa \leq \kappa$

$\langle 1 \rangle 2$ . LET:  $S = 2^\kappa$

$\langle 1 \rangle 3$ . PICK a partition  $\{A_\xi \mid \xi < \kappa\}$  of  $S^\kappa$  with  $\forall \xi < \kappa. |A_\xi| < 2^\kappa$ .

PROOF: Theorem 12.4.10.

$\langle 1 \rangle 4$ .  $\forall \xi < \kappa. \{g(\xi) \mid g \in A_\xi\} \subsetneq S$

PROOF: We do not have equality because  $|\{g(\xi) \mid g \in A_\xi\}| \leq |A_\xi| < 2^\kappa$ .

$\langle 1 \rangle 5$ . For all  $\xi < \kappa$ , choose  $s_\xi \in S - \{g(\xi) \mid g \in A_\xi\}$

$\langle 1 \rangle 6$ .  $s \in S^\kappa$

$\langle 1 \rangle 7$ . For all  $\xi < \kappa$  we have  $s \notin A_\xi$

PROOF: Since for all  $\xi < \kappa$  and  $g \in A_\xi$  we have  $s_\xi(\xi) \neq g(\xi)$ .

$\langle 1 \rangle 8$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 3$ .

□

**Corollary 12.4.11.1.**

$$2^{\aleph_0} \neq \aleph_\omega$$

**Proposition 12.4.12.** *For any ordinal  $\alpha$ , we have  $\text{cf } \alpha$  is the least cardinal such that  $\alpha$  is the strict supremum of  $\text{cf } \alpha$  smaller ordinals.*

PROOF:

$\langle 1 \rangle 1$ . CASE:  $\alpha = 0$

PROOF: Since  $0 = \text{ssup } \emptyset$ .

$\langle 1 \rangle 2$ . CASE:  $\alpha = \beta^+$

PROOF: Since  $\beta^+ = \text{ssup}\{\beta\}$ .

$\langle 1 \rangle 3$ . CASE:  $\alpha$  is a limit ordinal.

$\langle 2 \rangle 1$ . There exists a set  $S$  of ordinals  $< \alpha$  such that  $|S| = \text{cf } \alpha$  and  $\alpha = \text{ssup } S$ .

$\langle 3 \rangle 1$ . PICK a set  $S$  of ordinals  $< \alpha$  such that  $|S| = \text{cf } \alpha$  and  $\sup S = \alpha$

PROVE:  $\alpha = \text{ssup } S$

$\langle 3 \rangle 2$ .  $\forall \beta \in S. \beta < \alpha$

$\langle 3 \rangle 3$ . For any ordinal  $\gamma$ , if  $\forall \beta \in S. \beta < \gamma$  then  $\alpha \leq \gamma$

$\langle 2 \rangle 2$ . If  $T$  is a set of ordinals  $< \alpha$  such that  $\alpha = \text{ssup } T$ , then  $\text{cf } \alpha \leq |T|$ .

$\langle 3 \rangle 1$ . LET:  $T$  be a set of ordinals  $< \alpha$  such that  $\alpha = \text{ssup } T$

$\langle 3 \rangle 2$ .  $\alpha = \sup T$

$\langle 4 \rangle 1$ . For all  $\beta \in T$  we have  $\beta \leq \alpha$

$\langle 4 \rangle 2$ . LET:  $\mu$  be any upper bound for  $T$

PROVE:  $\alpha \leq \mu$

$\langle 4 \rangle 3$ .  $\alpha \leq \mu + 1$

PROOF: Since  $\forall \beta \in T. \beta < \mu + 1$ .

$\langle 4 \rangle 4$ .  $\alpha \neq \mu + 1$

PROOF: Since  $\alpha$  is a limit ordinal.

$\langle 4 \rangle 5$ .  $\alpha < \mu + 1$

$\langle 4 \rangle 6$ .  $\alpha \leq \mu$

$\langle 3 \rangle 3$ .  $\text{cf } \alpha \leq |T|$

□

## 12.5 Inaccessible Cardinals

**Definition 12.5.1** (Inaccessible Cardinal). A cardinal number  $\kappa$  is *inaccessible* iff:

- $\kappa > \aleph_0$
- $\forall \lambda < \kappa. 2^\lambda < \kappa$  (cardinal exponentiation)
- $\kappa$  is regular.

Any inaccessible cardinal is weakly inaccessible.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\kappa = \aleph_\lambda$  be weakly inaccessible.

PROVE:  $\lambda$  is a limit ordinal.

$\langle 1 \rangle 2$ .  $\lambda \neq 0$

$\langle 1 \rangle 3$ . ASSUME: for a contradiction  $\lambda = \beta + 1$

$\langle 1 \rangle 4$ .  $\aleph_\beta < \kappa$

$\langle 1 \rangle 5$ .  $2^{\aleph_\beta} < \kappa$

$\langle 1 \rangle 6$ .  $\aleph_{\beta+1} < \kappa$

PROOF: Since  $\aleph_{\beta+1} \leq 2^{\aleph_\beta}$ .

$\langle 1 \rangle 7$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 3$ .

□

**Proposition 12.5.2.** *If the Generalized Continuum Hypothesis is true, then every weakly inaccessible cardinal is inaccessible.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: The Generalized Continuum Hypothesis.

$\langle 1 \rangle 2$ . LET:  $\kappa = \aleph_\lambda$  be weakly inaccessible.

$\langle 1 \rangle 3$ .  $\kappa > \aleph_0$

PROOF:  $\lambda > 0$  because  $\lambda$  is a limit ordinal.

$\langle 1 \rangle 4$ . For all  $\mu < \kappa$  we have  $2^\mu < \kappa$

$\langle 2 \rangle 1$ . LET:  $\mu < \kappa$

$\langle 2 \rangle 2$ . LET:  $\mu = \aleph_\alpha$

$\langle 2 \rangle 3$ .  $\alpha < \lambda$

$\langle 2 \rangle 4$ .  $\alpha + 1 < \lambda$

PROOF:  $\lambda$  is a limit ordinal.

$\langle 2 \rangle 5$ .  $2^\mu < \kappa$

PROOF:

$$2^\mu = 2^{\aleph_\alpha} \quad (\langle 2 \rangle 2)$$

$$= 2^{\beth_\alpha} \quad (\langle 1 \rangle 1)$$

$$= \beth_{\alpha+1}$$

$$= \aleph_{\alpha+1} \quad (\langle 1 \rangle 1)$$

$$< \aleph_\lambda \quad (\langle 2 \rangle 4)$$

$$= \kappa \quad (\langle 1 \rangle 2)$$



$\langle 1 \rangle 5.$   $\kappa$  is regular.

PROOF:  $\langle 1 \rangle 2$

□

**Lemma 12.5.3.** *Let  $\kappa$  be an inaccessible cardinal. For every ordinal  $\alpha < \kappa$  we have  $\beth_\alpha < \kappa$ .*

PROOF:

$\langle 1 \rangle 1.$   $\beth_0 < \kappa$

PROOF: Since  $\kappa > \aleph_0$ .

$\langle 1 \rangle 2.$  For any ordinal  $\alpha$ , if  $\beth_\alpha < \kappa$  then  $\beth_{\alpha+1} < \kappa$ .

PROOF: Since  $\beth_{\alpha+1} = 2^{\beth_\alpha} < \kappa$ .

$\langle 1 \rangle 3.$  For any limit ordinal  $\lambda$ , if  $\forall \alpha < \lambda. \beth_\alpha < \kappa$  and  $\lambda < \kappa$  then  $\beth_\lambda < \kappa$ .

PROOF: By regularity of  $\kappa$ , since  $\beth_\lambda$  is the union of  $|\lambda|$  cardinals all  $< \kappa$ .

□

**Lemma 12.5.4.** *Let  $\kappa$  be an inaccessible cardinal. For all  $A \in V_\kappa$  we have  $|A| < \kappa$ .*

PROOF:

$\langle 1 \rangle 1.$  LET:  $A \in V_\kappa$

$\langle 1 \rangle 2.$  PICK  $\alpha < \kappa$  such that  $A \in V_\alpha$

$\langle 1 \rangle 3.$   $A \subseteq V_\alpha$

$\langle 1 \rangle 4.$   $|A| \leq |V_\alpha| \leq \beth_\alpha < \kappa$

□

**Theorem Schema 12.5.5.** *For every axiom  $\alpha$  of ZFC, the following is a theorem:*

*For any inaccessible cardinal  $\kappa$ , we have  $V_\kappa$  is a model of  $\alpha$ .*

PROOF: For every axiom except the Replacement Axioms, we have Corollary 11.0.11.1.

For an Axiom of Replacement using the property  $P[x, y, z_1, \dots, z_n]$ , we reason as follows:

$\langle 1 \rangle 1.$  LET:  $\kappa$  be an inaccessible cardinal

PROVE:

$$\begin{aligned} & \forall a_1, \dots, a_n, B \in V_\kappa (\forall x \in B. \forall y, y' \in V_\kappa \\ & (P[x, y, a_1, \dots, a_n]^{V_\kappa} \wedge P[x, y', a_1, \dots, a_n]^{V_\kappa} \Rightarrow y = y') \Rightarrow \\ & \exists C \in V_\kappa \forall y \in V_\kappa (y \in C \Leftrightarrow \exists x \in B. P[x, y, a_1, \dots, a_n]^{V_\kappa})) \end{aligned}$$

$\langle 1 \rangle 2.$  LET:  $a_1, \dots, a_n, B \in V_\kappa$

$\langle 1 \rangle 3.$  ASSUME: for all  $x \in B$ , there exists at most one  $y \in V_\kappa$  such that

$$P[x, y, a_1, \dots, a_n]^{V_\kappa}.$$

$\langle 1 \rangle 4.$  LET:  $F = \{(x, y) \in B \times V_\kappa \mid P[x, y, a_1, \dots, a_n]^{V_\kappa}\}$

$\langle 1 \rangle 5.$  LET:  $C = \text{ran } F$

PROVE:  $C \in V_\kappa$

$\langle 1 \rangle 6.$  LET:  $S = \{\text{rank } F(x) \mid x \in \text{dom } F\}$

$\langle 1 \rangle 7.$   $|S| < \kappa$

PROOF: Since  $|S| \leq |\text{dom } F| \leq |B| < \kappa$ .

$\langle 1 \rangle 8. \forall \alpha \in S. \alpha < \kappa$

PROOF: Since  $F(x) \in V_\kappa$  for all  $x \in \text{dom } F$ .

$\langle 1 \rangle 9. \sup S < \kappa$

PROOF: Since  $\kappa$  is regular.

$\langle 1 \rangle 10. \text{rank } C \leq \sup S + 1$

$\langle 1 \rangle 11. \text{rank } C < \kappa$

$\langle 1 \rangle 12. C \in V_\kappa$

□

# Chapter 13

## Group Theory

### 13.1 Groups

**Definition 13.1.1** (Group). A *group*  $G$  consists of a set  $G$  and a function  $\cdot : G^2 \rightarrow G$  such that:

1.  $\cdot$  is associative
2. There exists  $e \in G$  such that  $\forall x \in G. xe = x$  and  $\forall x \in G. \exists y \in G. xy = e$ .

**Proposition 13.1.2.** *The inverse of an element in a group is unique.*

PROOF:

$\langle 1 \rangle$ 1. ASSUME:  $b$  and  $b'$  are inverses of  $a$ .

$\langle 1 \rangle$ 2.  $b = b'$

PROOF:

$$\begin{aligned} b &= be \\ &= bab' \\ &= eb' \\ &= b' \end{aligned}$$

□

**Definition 13.1.3.** We write  $x^{-1}$  for the inverse of  $x$ .

**Proposition 13.1.4.** *In any group, if  $ab = ac$  then  $b = c$ .*

PROOF:

$$\begin{aligned} b &= eb \\ &= a^{-1}ab \\ &= a^{-1}ac \\ &= ec \\ &= c \end{aligned}$$

□

**Definition 13.1.5.** Let  $\mathbf{Grp}$  be the category of groups and group homomorphisms.

**Definition 13.1.6.** We identify any group  $G$  with the category  $G$  with one object whose morphisms are the elements of  $G$ , with composition given by the multiplication in  $G$ .

## 13.2 Abelian Groups

**Definition 13.2.1** (Abelian group). An *Abelian group* is a group whose multiplication is commutative.

We may say we are writing an Abelian group *additively*, meaning we write  $a + b$  for  $ab$ ,  $0$  for  $e$  and  $-a$  for  $a^{-1}$ . In this case we write  $a - b$  for  $ab^{-1}$ .

# Chapter 14

## Ring Theory

### 14.1 Rings

**Definition 14.1.1** (Commutative Ring). A *commutative ring* consists of a set  $R$  and two binary operations  $+$ ,  $\cdot$  on  $R$  such that:

- $D$  is an Abelian group under  $+$ . Let us write  $0$  for its identity element.
- $\cdot$  is commutative and associative, and distributes over  $+$ .
- $\cdot$  has an identity element  $1$  that is different from  $0$ .

**Proposition 14.1.2.** *In any commutative ring,  $0x = 0$ .*

PROOF:

$$\begin{aligned}(0 + 0)x &= 0x \\ \therefore 0x + 0x &= 0x + 0 \\ \therefore 0x &= 0 && \text{(Proposition 13.1.4)} \square\end{aligned}$$

**Proposition 14.1.3.** *In any commutative ring,  $(-a)b = -(ab)$ .*

PROOF:

$$\begin{aligned}ab + (-a)b &= (a + (-a))b \\ &= 0b \\ &= 0 && \text{(Proposition 14.1.2)} \square\end{aligned}$$

### 14.2 Ordered Rings

**Definition 14.2.1** (Ordered Commutative Ring). An *ordered commutative ring* consists of a commutative ring  $R$  with a linear order  $<$  on  $R$  such that:

- for all  $x, y, z \in R$ , we have  $x < y$  if and only if  $x + z < y + z$ .

- for all  $x, y, z \in R$ , if  $0 < z$  then we have  $x < y$  if and only if  $xz < yz$ .

**Proposition 14.2.2.** *In any ordered commutative ring,  $0 < 1$ .*

PROOF: If  $1 < 0$  then we have  $0 < -1$  and so  $0 < (-1)(-1) = 1$ , which is a contradiction.  $\square$

**Proposition 14.2.3.** *The ordering on an ordered commutative ring is dense; that is, if  $x < y$  then there exists  $z$  such that  $x < z < y$ .*

PROOF: Take  $z = (x + y)/2$ .  $\square$

### 14.3 Integral Domains

**Definition 14.3.1** (Integral Domain). An *integral domain* is a commutative ring such that, for all  $a, b \in D$ , if  $ab = 0$  then  $a = 0$  or  $b = 0$ .

**Proposition 14.3.2.** *In any integral domain, if  $ab = ac$  and  $a \neq 0$  then  $b = c$ .*

PROOF: We have  $a(b - c) = 0$  and  $a \neq 0$  so  $b - c = 0$  hence  $b = c$ .  $\square$

**Definition 14.3.3** (Ordered Integral Domain). An *ordered integral domain* is an ordered commutative ring that is an integral domain.

# Chapter 15

## Field Theory

### 15.1 Fields

**Definition 15.1.1** (Field). A *field*  $F$  is a commutative ring such that  $0 \neq 1$  and, for all  $x \in F$ , if  $x \neq 0$  then there exists  $y \in F$  such that  $xy = 1$ .

**Proposition 15.1.2.** *Every field is an integral domain.*

PROOF: If  $ab = 0$  and  $a \neq 0$  then  $b = a^{-1}ab = 0$ .  $\square$

**Proposition 15.1.3.** *In any field  $F$ , we have  $F - \{0\}$  is an Abelian group under multiplication.*

PROOF: Immediate from the definition.  $\square$

**Definition 15.1.4** (Field of Fractions). Let  $D$  be an integral domain. The *field of fractions* of  $D$  is the quotient set  $F = (D \times (D - \{0\})) / \sim$  where

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

under

$$\begin{aligned} [(a, b)] + [(c, d)] &= [(ad + bc, bd)] \\ [(a, b)][(c, d)] &= [(ac, bd)] \end{aligned}$$

We prove this is a field.

PROOF:

$\langle 1 \rangle 1.$   $\sim$  is an equivalence relation on  $D \times (D - \{0\})$ .

PROOF:

$\langle 2 \rangle 1.$   $\sim$  is reflexive.

PROOF: We always have  $ab = ba$ .

$\langle 2 \rangle 2.$   $\sim$  is symmetric.

PROOF: If  $ad = bc$  then  $cb = da$ .

$\langle 2 \rangle 3$ .  $\sim$  is transitive.

$\langle 3 \rangle 1$ . ASSUME:  $(a, b) \sim (c, d) \sim (e, f)$

$\langle 3 \rangle 2$ .  $ad = bc$  and  $cf = de$

$\langle 3 \rangle 3$ .  $adf = bde$

PROOF:  $adf = bcf = bde$

$\langle 3 \rangle 4$ .  $af = be$

PROOF: Proposition 14.3.2.

□

$\langle 1 \rangle 2$ . Addition is well-defined.

PROOF:

$\langle 2 \rangle 1$ . If  $b \neq 0$  and  $d \neq 0$  then  $bd \neq 0$ .

PROOF: Since  $D$  is an integral domain.

$\langle 2 \rangle 2$ . If  $ab' = a'b$  and  $cd' = c'd$  then  $(ad + bc)b'd' = (a'd' + b'c')bd$ .

PROOF:

$$\begin{aligned} (ad + bc)b'd' &= ab'dd' + bb'cd' \\ &= a'bdd' + bb'c'd \\ &= (a'd' + b'c')bd \end{aligned}$$

□

$\langle 1 \rangle 3$ . Multiplication is well-defined.

PROOF:

$\langle 2 \rangle 1$ . If  $b \neq 0$  and  $d \neq 0$  then  $bd \neq 0$ .

PROOF: Since  $D$  is an integral domain.

$\langle 2 \rangle 2$ . If  $[(a, b)] = [(a', b')]$  and  $[(c, d)] = [(c', d')]$  then  $[(ac, bd)] = [(a'c', b'd')]$ .

PROOF: If  $ab' = a'b$  and  $cd' = c'd$  then  $acb'd' = a'c'bd$ .

□

$\langle 1 \rangle 4$ . Addition is commutative.

PROOF:  $[(a, b)] + [(c, d)] = [(ad + bc, bd)] = [(cb + da, db)] = [(c, d)] + [(a, b)]$  □

$\langle 1 \rangle 5$ . Addition is associative.

PROOF:

$$\begin{aligned} [(a, b)] + ([[(c, d)] + [(e, f)]] &= [(a, b)] + [(cf + de, df)] \\ &= [(adf + bcf + bde, bdf)] \\ &= [(ad + bc, bd)] + [(e, f)] \\ &= ([[(a, b)] + [(c, d)]] + [(e, f)]) \quad \square \end{aligned}$$

$\langle 1 \rangle 6$ . For any  $x \in F$  we have  $x + [(0, 1)] = x$

PROOF:  $[(a, b)] + [(0, 1)] = [(a \cdot 1 + b \cdot 0, b \cdot 1)] = [(a, b)]$  □

$\langle 1 \rangle 7$ . For any  $x \in F$ , there exists  $y \in F$  such that  $x + y = [(0, 1)]$ .

PROOF:  $[(a, b)] + [(-a, b)] = [(ab - ab, b^2)] = [(0, b^2)] = [(0, 1)]$  □

$\langle 1 \rangle 8$ . Multiplication is commutative.

PROOF:  $[(a, b)][(c, d)] = [(c, d)][(a, b)] = [(ac, bd)]$ . □

$\langle 1 \rangle 9$ . Multiplication is associative.

PROOF:  $[(a, b)]([[(c, d)][(e, f)]] = ([[(a, b)][(c, d)]][(e, f)] = [(ace, bdf)]$ . □

$\langle 1 \rangle 10$ . For any  $x \in F$  we have  $x[(1, 1)] = x$

PROOF:  $[(a, b)][(1, 1)] = [(a, b)]$  □

$\langle 1 \rangle 11$ . For any non-zero  $x \in F$ , there exists  $y \in F$  such that  $xy = [(1, 1)]$ .



PROOF:

- $\langle 2 \rangle 1$ . LET:  $[(a, b)] \in \mathbb{Q}$
- $\langle 2 \rangle 2$ . ASSUME:  $[(a, b)] \neq [(0, 1)]$
- $\langle 2 \rangle 3$ .  $a \neq 0$
- $\langle 2 \rangle 4$ .  $[(a, b)][(b, a)] = [(1, 1)]$

□

□

**Definition 15.1.5.** For any field  $F$ , let  $N(F)$  be the intersection of all the subsets  $S \subseteq F$  such that  $1 \in S$  and  $\forall x \in S. x + 1 \in S$ .

**Definition 15.1.6** (Characteristic Zero). A field  $F$  has *characteristic 0* iff  $0 \notin N(F)$ .

**Proposition 15.1.7.** In a field  $F$  with characteristic 0, the function  $n : \mathbb{N} \rightarrow N(F)$  defined by

$$\begin{aligned} n(0) &= 1 \\ n(x + 1) &= n(x) + 1 \end{aligned}$$

is a bijection.

PROOF:

- $\langle 1 \rangle 1$ .  $n$  is injective.
- $\langle 2 \rangle 1$ . ASSUME: for a contradiction  $n(i) = n(j)$  with  $i \neq j$
- $\langle 2 \rangle 2$ . ASSUME: w.l.o.g.  $i < j$
- $\langle 2 \rangle 3$ .  $n(j - i) = 0$
- $\langle 2 \rangle 4$ . Q.E.D.

PROOF: This contradicts the fact that  $F$  has characteristic 0.

- $\langle 1 \rangle 2$ .  $n$  is surjective.

PROOF: Since  $\text{ran } n$  is a subset of  $F$  that includes 1 and is closed under  $+1$ .

□

**Definition 15.1.8.** In any field  $F$ , let

$$I(F) = N(F) \cup \{0\} \cup \{-x \mid x \in N(F)\}$$

**Definition 15.1.9.** In any field  $F$ , let

$$Q(F) = \{x/y \mid x, y \in I(F), y \neq 0\}$$

**Proposition 15.1.10.**  $Q(F)$  is the smallest subfield of  $F$ .

PROOF:  $Q(F)$  is closed under  $+$  and  $\cdot$ , and any subset of  $F$  closed under  $+$  and  $\cdot$  that contains 0 and 1 must include  $Q(F)$ . □

**Theorem 15.1.11.** Let  $F$  and  $G$  be fields of characteristic 0. Then there exists a unique field isomorphism between  $Q(F)$  and  $Q(G)$ .

PROOF:

- (1)1. LET:  $\phi : N(F) \rightarrow N(G)$  be the unique function such that  $\phi(1) = 1$  and  $\forall x \in N(F). \phi(x+1) = \phi(x) + 1$ .
- (1)2.  $\phi$  is a bijection.  
 PROOF: Similar to Proposition 15.1.7.
- (1)3.  $\forall x, y \in N(F). \phi(x+y) = \phi(x) + \phi(y)$   
 PROOF: Induction on  $y$ .
- (1)4.  $\forall x, y \in N(F). \phi(xy) = \phi(x)\phi(y)$   
 PROOF: Induction on  $y$ .
- (1)5. Extend  $\phi$  to a bijection  $I(F) \cong I(G)$  such that  $\forall x, y \in I(F). \phi(x+y) = \phi(x) + \phi(y)$  and  $\forall x, y \in I(F). \phi(xy) = \phi(x)\phi(y)$
- (2)1. Define  $\phi(0) = 0$  and  $\phi(-x) = -\phi(x)$  for  $x \in N(F)$
- (3)1.  $0 \notin N(F)$
- (3)2. For all  $x \in N(F)$  we have  $-x \notin N(F)$   
 PROOF: Then we would have  $x + -x = 0 \in N(F)$ .
- (3)3. For all  $x \in N(F)$  we have  $-x \neq 0$
- (2)2. For all  $x, y \in I(F)$  we have  $\phi(x+y) = \phi(x) + \phi(y)$   
 PROOF: Case analysis on  $x$  and  $y$ .
- (2)3. For all  $x, y \in I(F)$  we have  $\phi(xy) = \phi(x)\phi(y)$   
 PROOF: Case analysis on  $x$  and  $y$ .
- (1)6. Extend  $\phi$  to a bijection  $Q(F) \cong Q(G)$  such that  $\forall x, y \in Q(F). \phi(x+y) = \phi(x) + \phi(y)$  and  $\forall x, y \in Q(F). \phi(xy) = \phi(x)\phi(y)$
- (2)1. Define  $\phi(x/y) = \phi(x)/\phi(y)$
- (1)7.  $\phi$  is unique.
- (2)1. LET:  $\theta$  satisfy the theorem.
- (2)2. For all  $x \in N(F)$  we have  $\theta(x) = \phi(x)$
- (2)3. For all  $x \in I(F)$  we have  $\theta(x) = \phi(x)$
- (2)4. For all  $x \in Q(F)$  we have  $\theta(x) = \phi(x)$

□

## 15.2 Ordered Fields

**Definition 15.2.1** (Ordered Field). An *ordered field* is an ordered commutative ring that is a field.

**Proposition 15.2.2.** Every ordered field  $F$  has characteristic 0.

PROOF: We have  $0 < n$  for all  $n \in N(F)$ . □

**Proposition 15.2.3.** Let  $F$  be a field of characteristic 0. Then there exists a unique relation  $<$  on  $Q(F)$  that makes  $Q(F)$  into an ordered field.

PROOF: Easy. □

**Corollary 15.2.3.1.** Let  $F$  and  $G$  be ordered fields. Let  $\phi$  be the unique field isomorphism between  $Q(F)$  and  $Q(G)$ . Then  $\phi$  is an ordered field isomorphism.

**Definition 15.2.4** (Archimedean). An ordered field  $F$  is *Archimedean* iff

$$\forall x \in F. \exists n \in N(F). n > x .$$

**Proposition 15.2.5.** *Let  $F$  be an Archimedean ordered field. Let  $x, y \in F$  with  $x > 0$ . Then there exists  $n \in N(F)$  such that  $nx > y$ .*

PROOF: Pick  $n > y/x$ .  $\square$

**Proposition 15.2.6.** *Let  $F$  be an Archimedean ordered field. For all  $x, y \in F$ , if  $x < y$ , then there exists  $r \in Q(F)$  such that  $x < r < y$ .*

PROOF:

$\langle 1 \rangle 1$ . CASE:  $x > 0$

$\langle 2 \rangle 1$ . PICK  $n \in N(F)$  such that  $n(y - x) > 1$

PROOF: Proposition 15.2.5.

$\langle 2 \rangle 2$ .  $ny > 1 + nx$

$\langle 2 \rangle 3$ . LET:  $m$  be the least element of  $N(F)$  such that  $m > nx$ .

$\langle 2 \rangle 4$ .  $m - 1 \leq nx$

$\langle 2 \rangle 5$ .  $nx < m < ny$

$\langle 2 \rangle 6$ .  $x < m/n < y$

$\langle 1 \rangle 2$ . CASE:  $x \leq 0$

$\langle 2 \rangle 1$ . PICK  $k \in N(F)$  such that  $k > -x$

$\langle 2 \rangle 2$ .  $0 < x + k < y + k$

$\langle 2 \rangle 3$ . PICK  $r \in Q(F)$  such that  $x + k < r < y + k$

PROOF:  $\langle 1 \rangle 1$

$\langle 2 \rangle 4$ .  $x < r - k < y$

**Definition 15.2.7** (Complete). An ordered field  $F$  is *complete* iff every nonempty subset of  $F$  bounded above has a least upper bound.

**Proposition 15.2.8.** *Every complete ordered field is Archimedean.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $F$  be a complete ordered field.

$\langle 1 \rangle 2$ . LET:  $x \in F$

$\langle 1 \rangle 3$ . ASSUME: for a contradiction there is no member of  $N(F)$  greater than  $x$ .

$\langle 1 \rangle 4$ .  $x$  is an upper bound for  $N(F)$ .

$\langle 1 \rangle 5$ . LET:  $y = \sup N(F)$

$\langle 1 \rangle 6$ . PICK  $n \in N(F)$  such that  $y - 1 < n$

$\langle 1 \rangle 7$ .  $y < n + 1$

$\langle 1 \rangle 8$ . Q.E.D.

PROOF: This is a contradiction.

$\square$

**Proposition 15.2.9.** *Let  $F$  be a complete ordered field and  $a \in F$  be nonnegative. Then there exists  $b \in F$  such that  $b^2 = a$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $B = \{x \in F \mid 0 \leq x \leq 1 + a\}$

$\langle 1 \rangle 2$ . LET:  $\phi : B \rightarrow B$  be the function

$$\phi(x) = x + \frac{1}{2(1+a)}(a - x^2) .$$

- ⟨1⟩3.  $\phi$  is strictly monotone.  
 ⟨2⟩1. LET:  $0 \leq x < y \leq 1 + a$   
 ⟨2⟩2.  $1 - \frac{x+y}{2(1+a)} > 0$   
 ⟨2⟩3.  $\phi(y) - \phi(x) = (y - x)(1 - \frac{x+y}{2(1+a)}) > 0$   
 ⟨2⟩4.  $\phi(x) < \phi(y)$   
 ⟨1⟩4. PICK  $b \in B$  such that  $\phi(b) = b$ .  
 PROOF: Knaster Fixed-Point Theorem.  
 ⟨1⟩5.  $b^2 = a$   
 $\square$

**Theorem 15.2.10** (Uniqueness of the Complete Ordered Field). *If  $F$  and  $G$  are complete ordered fields, then there exists a unique bijection  $\phi : F \cong G$  such that, for all  $x, y \in F$ ,*

$$\begin{aligned}\phi(x + y) &= \phi(x) + \phi(y) \\ \phi(xy) &= \phi(x)\phi(y)\end{aligned}$$

*This bijection also satisfies: for all  $x, y \in F$ ,*

$$x < y \Leftrightarrow \phi(x) < \phi(y) .$$

PROOF:

- ⟨1⟩1. PICK a bijection  $\phi : Q(F) \cong Q(G)$  such that, for all  $x, y \in Q(F)$ ,  
 $\phi(x + y) = \phi(x) + \phi(y)$   
 $\phi(xy) = \phi(x)\phi(y)$   
 $x < y \Leftrightarrow \phi(x) < \phi(y)$

PROOF: Corollary 15.2.3.1.

- ⟨1⟩2.  $Q(F)$  intersects every interval in  $F$ .

PROOF: Proposition 15.2.6.

- ⟨1⟩3.  $Q(G)$  intersects every interval in  $G$ .

PROOF: Proposition 15.2.6.

- ⟨1⟩4. PICK an order isomorphism  $\psi : F \cong G$  that extends  $\phi$ .

PROOF: Theorem 6.1.21.

- ⟨1⟩5.  $\forall x, y \in F. \psi(x + y) = \psi(x) + \psi(y)$   
 ⟨2⟩1. LET:  $x, y \in F$   
 ⟨2⟩2.  $\psi(x) + \psi(y) \not\leq \psi(x + y)$   
 ⟨3⟩1. ASSUME: for a contradiction  $\psi(x) + \psi(y) < \psi(x + y)$   
 ⟨3⟩2. PICK  $r' \in Q(G)$  such that  $\psi(x) < r' < \psi(x + y) - \psi(y)$   
 ⟨3⟩3. PICK  $s' \in Q(G)$  such that  $\psi(y) < s' < \psi(x + y) - r'$   
 ⟨3⟩4.  $r' + s' < \psi(x + y)$   
 ⟨3⟩5. PICK  $r, s \in Q(F)$  such that  $\phi(r) = r'$  and  $\phi(s) = s'$   
 ⟨3⟩6.  $\phi(r + s) = r' + s'$   
 ⟨3⟩7.  $\psi(x) < \psi(r)$   
 ⟨3⟩8.  $\psi(y) < \psi(s)$   
 ⟨3⟩9.  $\psi(x + y) > \psi(r + s)$   
 ⟨3⟩10.  $x < r$

- $\langle 3 \rangle 11. y < s$
- $\langle 3 \rangle 12. x + y > r + s$
- $\langle 3 \rangle 13. \text{Q.E.D.}$

PROOF: This is a contradiction.

- $\langle 2 \rangle 3. \psi(x + y) \not\leq \psi(x) + \psi(y)$

PROOF: Similar.

- $\langle 1 \rangle 6. \forall x, y \in F. \psi(xy) = \psi(x)\psi(y)$

- $\langle 2 \rangle 1. \text{LET: } x, y \in F$

- $\langle 2 \rangle 2. \text{CASE: } x \text{ and } y \text{ are positive.}$

- $\langle 3 \rangle 1. \psi(x)\psi(y) \not\leq \psi(xy)$

- $\langle 4 \rangle 1. \text{ASSUME: for a contradiction } \psi(x)\psi(y) < \psi(xy)$

- $\langle 4 \rangle 2. \text{PICK } r' \in Q(G) \text{ such that } \psi(x) < r' < \psi(xy)/\psi(y)$

- $\langle 4 \rangle 3. \text{PICK } s' \in Q(G) \text{ such that } \psi(y) < s' < \psi(xy)/r'$

- $\langle 4 \rangle 4. r's' < \psi(xy)$

- $\langle 4 \rangle 5. \text{PICK } r, s \in Q(F) \text{ such that } \phi(r) = r' \text{ and } \phi(s) = s'$

- $\langle 4 \rangle 6. \phi(rs) = r's'$

- $\langle 4 \rangle 7. x < r, y < s \text{ and } rs < xy$

- $\langle 4 \rangle 8. \text{Q.E.D.}$

PROOF: This is a contradiction.

- $\langle 3 \rangle 2. \psi(xy) \not\leq \psi(x)\psi(y)$

PROOF: Similar.

- $\langle 2 \rangle 3. \text{CASE: } x \text{ and } y \text{ are not both positive.}$

PROOF: Follows from  $\langle 2 \rangle 2$  since  $\psi(-x) = -\psi(x)$  by  $\langle 1 \rangle 5$ .

- $\langle 1 \rangle 7. \text{For any field isomorphism } \theta : F \cong G, \text{ we have } \theta = \psi.$

- $\langle 2 \rangle 1. \theta \upharpoonright Q(F) = \phi$

PROOF: Theorem 15.1.11.

- $\langle 2 \rangle 2. \theta \text{ is strictly monotone.}$

- $\langle 3 \rangle 1. \text{LET: } x, y \in F \text{ with } x < y$

- $\langle 3 \rangle 2. y - x > 0$

- $\langle 3 \rangle 3. \text{PICK } z \in F \text{ such that } z^2 = y - x$

- $\langle 3 \rangle 4. \theta(z)^2 = \theta(y) - \theta(x)$

- $\langle 3 \rangle 5. \theta(y) - \theta(x) > 0$

- $\langle 3 \rangle 6. \theta(x) < \theta(y)$

- $\langle 2 \rangle 3. \theta = \psi$

PROOF: By the uniqueness of  $\psi$ .

□



## Chapter 16

# Number Systems

### 16.1 The Integers

**Definition 16.1.1.** The set of *integers*  $\mathbb{Z}$  is the quotient set  $\mathbb{N}^2 / \sim$ , where  $(m, n) \sim (p, q)$  iff  $m + q = n + p$ .

We prove  $\sim$  is an equivalence relation on  $\mathbb{N}^2$ .

PROOF:

$\langle 1 \rangle 1.$   $\sim$  is reflexive.

PROOF: For all  $m, n \in \mathbb{N}$  we have  $m + n = n + m$ .

$\langle 1 \rangle 2.$   $\sim$  is symmetric.

PROOF: If  $m + q = n + p$  then  $p + n = q + m$ .

$\langle 1 \rangle 3.$   $\sim$  is transitive.

$\langle 2 \rangle 1.$  ASSUME:  $(m, n) \sim (p, q) \sim (r, s)$

$\langle 2 \rangle 2.$   $m + q = n + p$  and  $p + s = q + r$

$\langle 2 \rangle 3.$   $m + q + s = n + q + r$

$\langle 2 \rangle 4.$   $m + s = n + r$

PROOF: By cancellation.

□

**Definition 16.1.2** (Addition). Define *addition*  $+$  on  $\mathbb{Z}$  by  $[(m, n)] + [(p, q)] = [(m + p, n + q)]$ .

We prove this is well-defined.

PROOF: If  $m + n' = n + m'$  and  $p + q' = q + p'$  then  $m + p + n' + q' = n + q + m' + p'$ .

□

**Proposition 16.1.3.** *Addition on  $\mathbb{Z}$  is commutative.*

PROOF:  $[(m, n)] + [(p, q)] = [(m + p, n + q)] = [(p + m, q + n)] = [(p, q)] + [(m, n)]$ .

□

**Proposition 16.1.4.** *Addition on  $\mathbb{Z}$  is associative.*

PROOF:  $[(m, n)] + [(p, q)] + [(r, s)] = [(m + p + r, n + q + s)] = [(m, n)] + [(p, q)] + [(r, s)]$ .  $\square$

**Proposition 16.1.5.** *Given natural numbers  $m$  and  $n$ , we have  $[(m, 0)] = [(n, 0)]$  iff  $m = n$ .*

PROOF: Immediate from definitions.  $\square$

**Definition 16.1.6.** We identify any natural number  $n$  with the integer  $[(n, 0)]$ .

**Proposition 16.1.7.** *Addition on integers agrees with addition on natural numbers.*

PROOF: Since  $[(m, 0)] + [(n, 0)] = [(m + n, 0)]$ .  $\square$

**Proposition 16.1.8.** *For all  $a \in \mathbb{Z}$  we have  $a + 0 = a$ .*

PROOF:  $[(m, n)] + [(0, 0)] = [(m + 0, n + 0)] = [(m, n)]$ .  $\square$

**Proposition 16.1.9.** *For all  $a \in \mathbb{Z}$ , there exists  $b \in \mathbb{Z}$  such that  $a + b = 0$ .*

PROOF:  $[(m, n)] + [(n, m)] = [(m + n, m + n)] = [(0, 0)]$   $\square$

**Proposition 16.1.10.** *The integers form an Abelian group under addition.*

PROOF: Proposition 16.1.3, 16.1.4, 16.1.8, 16.1.9.  $\square$

**Definition 16.1.11.** Define multiplication  $\cdot$  on  $\mathbb{Z}$  by:  $[(m, n)][(p, q)] = [(mp + nq, mq + np)]$ .

We prove this is well defined.

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $m + n' = n + m'$  and  $p + q' = q + p'$

PROVE:  $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$

$\langle 1 \rangle 2$ .  $mp + n'p = np + m'p$

$\langle 1 \rangle 3$ .  $nq + m'q = mq + n'q$

$\langle 1 \rangle 4$ .  $m'p + m'q' = m'q + m'p'$

$\langle 1 \rangle 5$ .  $n'q + n'p' = n'p + n'q'$

$\langle 1 \rangle 6$ .  $mp + n'p + nq + m'q + m'p + m'q' + n'q + n'p' = np + m'p + mq + n'q + m'q + m'p' + n'p + n'q'$

$\langle 1 \rangle 7$ .  $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$

PROOF: By cancellation.

$\square$

**Proposition 16.1.12.** *Multiplication on integers agrees with multiplication on natural numbers.*

PROOF: Since  $[(m, 0)][(n, 0)] = [(mn + 0, m0 + n0)] = [(mn, 0)]$ .  $\square$

**Proposition 16.1.13.** *Multiplication on  $\mathbb{Z}$  is commutative.*

PROOF:  $[(m, n)][(p, q)] = [(mp + nq, mq + np)] = [(pm + qn, pn + qm)] = [(p, q)][(m, n)]$ .  $\square$



**Proposition 16.1.14.** *Multiplication on  $\mathbb{Z}$  is associative.*

PROOF:

$$\begin{aligned}
 [(m, n)][(p, q)][(r, s)] &= [(m, n)][(pr + qs, ps + qr)] \\
 &= [(mpr + mqs + nps + nqr, mps + mqr + npr + nqs)] \\
 &= [(mp + nq, mq + np)][(r, s)] \\
 &= [(m, n)][(p, q)][(r, s)] \quad \square
 \end{aligned}$$

**Proposition 16.1.15.** *Multiplication distributes over addition.*

PROOF:

$$\begin{aligned}
 [(m, n)][(p, q)] + [(m, n)][(r, s)] &= [(m, n)][(p + r, q + s)] \\
 &= [(mp + mr + nq + ns, mp + nr + mq + ms)] \\
 [(m, n)][(p, q)] + [(m, n)][(r, s)] &= [(mp + nq, mq + np)] + [(mr + ns, ms + nr)] \\
 &= [(mp + nq + mr + ns, mq + np + ms + nr)] \quad \square
 \end{aligned}$$

**Proposition 16.1.16.** *For any integer  $a$  we have  $a1 = a$ .*

PROOF: Since  $[(m, n)][(1, 0)] = [(m1 + n0, m0 + n1)] = [(m, n)]$ .  $\square$

**Proposition 16.1.17.** *For any integers  $a$  and  $b$ , if  $ab = 0$  then  $a = 0$  or  $b = 0$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $[(m, n)][(p, q)] = [(0, 0)]$

$\langle 1 \rangle 2$ .  $mp + nq = mq + np$

$\langle 1 \rangle 3$ . ASSUME:  $[(m, n)] \neq [(0, 0)]$

$\langle 1 \rangle 4$ .  $m \neq n$

PROVE:  $p = q$

$\langle 1 \rangle 5$ . CASE:  $m < n$

$\langle 2 \rangle 1$ .  $p \not\leq q$

PROOF: If  $p < q$  then  $mq + np < mp + nq$  by Proposition 9.4.6.

$\langle 2 \rangle 2$ .  $q \not\leq p$

PROOF: If  $q < p$  then  $mp + nq < mq + np$  by Proposition 9.4.6.

$\langle 2 \rangle 3$ .  $p = q$

PROOF: By trichotomy.

$\langle 1 \rangle 6$ . CASE:  $n < m$

PROOF: Similar.

$\square$

**Proposition 16.1.18.** *The integers  $\mathbb{Z}$  form an integral domain.*

PROOF: Propositions 16.1.13, 16.1.14, 16.1.15, 16.1.16, 16.1.17, 16.1.10.  $\square$

**Definition 16.1.19.** Define  $<$  on  $\mathbb{Z}$  by  $[(m, n)] < [(p, q)]$  if and only if  $m + q < n + p$ .

We prove this is well-defined.

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $m + n' = n + m'$  and  $p + q' = q + p'$ .

PROVE:  $m + q < n + p$  if and only if  $m' + q' < n' + p'$

$\langle 1 \rangle 2$ .  $m + q < n + p$  if and only if  $m' + q' < n' + p'$

PROOF:

$$m + q < n + p \Leftrightarrow m + n' + q < n + n' + p \quad (\text{Corollary 7.7.7.1})$$

$$\Leftrightarrow m' + n + q < n + n' + p$$

$$\Leftrightarrow m' + q < n' + p \quad (\text{Corollary 7.7.7.1})$$

$$\Leftrightarrow m' + q + p' < n' + p + p' \quad (\text{Corollary 7.7.7.1})$$

$$\Leftrightarrow m' + q' + p < n' + p + p'$$

$$\Leftrightarrow m' + q' < n' + p' \quad (\text{Corollary 7.7.7.1}) \square$$

**Proposition 16.1.20.** *The ordering on the integers agrees with the ordering on the natural numbers.*

PROOF: We have  $[(m, 0)] < [(n, 0)]$  iff  $m < n$ .  $\square$

**Proposition 16.1.21.**  *$<$  is a linear order on  $\mathbb{Z}$ .*

PROOF:

$\langle 1 \rangle 1$ .  $<$  is irreflexive.

PROOF: We never have  $m + n < m + n$ .

$\langle 1 \rangle 2$ .  $<$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $[(m, n)] < [(p, q)] < [(r, s)]$

$\langle 2 \rangle 2$ .  $m + q < n + p$  and  $p + s < q + r$

$\langle 2 \rangle 3$ .  $m + q + s < n + q + r$

PROOF:  $m + q + s < n + p + s < n + q + r$

$\langle 2 \rangle 4$ .  $m + s < n + r$

PROOF: Corollary 7.7.7.1.

$\langle 1 \rangle 3$ .  $<$  is total.

PROOF: Given natural numbers  $m, n, p$  and  $q$ , either  $m + q < n + p$ , or  $m + q = n + p$ , or  $n + p < m + q$ .

$\square$

**Definition 16.1.22** (Positive). An integer  $a$  is *positive* iff  $a > 0$ .

**Theorem 16.1.23.** *For any integers  $a, b$  and  $c$ , we have  $a < b$  if and only if  $a + c < b + c$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $a < b$  then  $a + c < b + c$ .

$\langle 2 \rangle 1$ . LET:  $a = [(m, n)]$ ,  $b = [(p, q)]$  and  $c = [(r, s)]$ .

$\langle 2 \rangle 2$ . ASSUME:  $a < b$

$\langle 2 \rangle 3$ .  $m + q < n + p$

$\langle 2 \rangle 4$ .  $m + r + q + s < n + r + p + s$

$\langle 2 \rangle 5$ .  $[(m + r, n + s)] < [(p + r, q + s)]$

$\langle 2 \rangle 6$ .  $a + c < b + c$

$\langle 1 \rangle 2$ . If  $a + c < b + c$  then  $a < b$ .

PROOF: From  $\langle 1 \rangle 1$  and Proposition 6.2.6.

□

**Proposition 16.1.24.** *Let  $a$ ,  $b$  and  $c$  be integers. If  $0 < c$ , then  $a < b$  if and only if  $ac < bc$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $c = [(r, s)]$

$\langle 1 \rangle 2$ . ASSUME:  $0 < c$

$\langle 1 \rangle 3$ .  $s < r$

$\langle 1 \rangle 4$ . For all integers  $a$  and  $b$ , if  $a < b$  then  $ac < bc$

$\langle 2 \rangle 1$ . LET:  $a = [(m, n)]$ ,  $b = [(p, q)]$ .

$\langle 2 \rangle 2$ . ASSUME:  $a < b$

$\langle 2 \rangle 3$ .  $m + q < n + p$

$\langle 2 \rangle 4$ .  $(m + q)r + (p + n)s < (m + q)s + (p + n)r$

PROOF: Proposition 9.4.6,  $\langle 1 \rangle 3$ ,  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 5$ .  $mr + ns + ps + qr < ms + nr + pr + qs$

$\langle 2 \rangle 6$ .  $[(mr + ns, ms + nr)] < [(pr + qs, ps + qr)]$

$\langle 2 \rangle 7$ .  $ac < bc$

$\langle 1 \rangle 5$ . For all integers  $a$  and  $b$ , if  $ac < bc$  then  $a < b$

PROOF: From  $\langle 1 \rangle 4$  and Proposition 6.2.6.

□

**Proposition 16.1.25.** *Let  $a$  be a positive integer. For any integer  $b$ , there exists  $k \in \mathbb{N}$  such that  $b < ak$ .*

PROOF:

$\langle 1 \rangle 1$ . CASE:  $b \leq 0$

PROOF: Take  $k = 1$ .

$\langle 1 \rangle 2$ . CASE:  $b > 0$

PROOF: Take  $k = b + 1$ .

□

## 16.2 The Rationals

**Definition 16.2.1** (Rational Numbers). The set  $\mathbb{Q}$  of *rational numbers* is the field of fractions over the integers.

**Proposition 16.2.2.** *For any integers  $a$  and  $b$ , we have  $[(a, 1)] = [(b, 1)]$  iff  $a = b$ .*

PROOF: Immediate from definitions. □

Henceforth we identify any integer  $a$  with the rational number  $[(a, 1)]$ .

**Proposition 16.2.3.** *Addition on the rationals agrees with addition on the integers.*

PROOF:  $[(a, 1)] + [(b, 1)] = [(a \cdot 1 + b \cdot 1, 1 \cdot 1)] = [(a + b, 1)]$ .  $\square$

**Proposition 16.2.4.** *Multiplication on the rationals agrees with multiplication on the integers.*

PROOF:  $[(a, 1)][(b, 1)] = [(ab, 1)]$   $\square$

**Definition 16.2.5.** Define the ordering  $<$  on the rationals by: if  $b$  and  $d$  are positive, then  $[(a, b)] < [(c, d)]$  iff  $ad < bc$ .

We prove this is well-defined.

PROOF:

$\langle 1 \rangle 1$ . For any rational  $q$ , there exist integers  $a, b$  with  $b$  positive such that  $q = [(a, b)]$ .

PROOF: Since  $[(a, b)] = [(-a, -b)]$ , and if  $b \neq 0$  then one of  $b$  and  $-b$  is positive.

$\langle 1 \rangle 2$ . If  $b, b', d$  and  $d'$  are positive,  $[(a, b)] = [(a', b')]$ , and  $[(c, d)] = [(c', d')]$ , then  $ad < bc$  iff  $a'd' < b'c'$ .

PROOF:

$\langle 2 \rangle 1$ . If  $ad < bc$  then  $a'd' < b'c'$ .

$\langle 3 \rangle 1$ . ASSUME:  $ad < bc$

$\langle 3 \rangle 2$ .  $ab'd < bb'c$

$\langle 3 \rangle 3$ .  $a'bd < bb'c$

$\langle 3 \rangle 4$ .  $a'd < b'c$

$\langle 3 \rangle 5$ .  $a'dd' < b'cd'$

$\langle 3 \rangle 6$ .  $a'dd' < b'c'd$

$\langle 3 \rangle 7$ .  $a'd' < b'c'$

$\langle 2 \rangle 2$ . If  $a'd' < b'c'$  then  $ad < bc$ .

PROOF: Similar.

$\square$

**Proposition 16.2.6.** *The ordering on the rationals agrees with the ordering on the integers.*

PROOF: We have  $[(a, 1)] < [(b, 1)]$  if and only if  $a < b$ .  $\square$

**Proposition 16.2.7.** *The relation  $<$  is a linear ordering on  $\mathbb{Q}$ .*

PROOF:

$\langle 1 \rangle 1$ .  $<$  is irreflexive.

PROOF: We never have  $ab < ab$ .

$\langle 1 \rangle 2$ .  $<$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $[(a, b)] < [(c, d)] < [(e, f)]$  where  $b, d$  and  $f$  are positive.

$\langle 2 \rangle 2$ .  $ad < bc$  and  $cf < de$

$\langle 2 \rangle 3$ .  $adf < bde$

PROOF:  $adf < bcf < bde$

$\langle 2 \rangle 4$ .  $af < be$

$\langle 1 \rangle 3$ .  $<$  is total.

PROOF: For any integers  $a, b, c, d$ , we have  $ad < bc$  or  $ad = bc$  or  $bc < ad$ .

□

**Proposition 16.2.8.** *For any rationals  $r$ ,  $s$  and  $t$ , we have  $r < s$  if and only if  $r + t < s + t$ .*

PROOF:

⟨1⟩1. LET:  $a, b, c, d, e, f$  be integers with  $b, d$  and  $f$  positive.

⟨1⟩2.  $[(a, b)] + [(e, f)] < [(c, d)] + [(e, f)]$  if and only if  $[(a, b)] < [(c, d)]$ .

PROOF:

$$\begin{aligned} [(a, b)] + [(e, f)] < [(c, d)] + [(e, f)] &\Leftrightarrow [(af + be, bf)] < [(cf + de, df)] \\ &\Leftrightarrow (af + be)df < (cf + de)bf \\ &\Leftrightarrow afd f + bedf < cfbf + debf \\ &\Leftrightarrow afd f < cfbf \\ &\Leftrightarrow ad < bc \\ &\Leftrightarrow [(a, b)] < [(c, d)] \end{aligned}$$

□

**Corollary 16.2.8.1.** *For any rational  $r$ , we have  $r < 0$  if and only if  $0 < -r$ .*

**Definition 16.2.9** (Absolute Value). For any rational  $r$ , the *absolute value* of  $r$  is defined by

$$|r| := \begin{cases} -r & \text{if } 0 < -r \\ r & \text{otherwise} \end{cases}$$

**Proposition 16.2.10.** *For any rationals  $r$ ,  $s$  and  $t$ , if  $t$  is positive then  $r < s$  iff  $rt < st$ .*

PROOF:

⟨1⟩1. LET:  $r = [(a, b)]$ ,  $s = [(c, d)]$  and  $t = [(e, f)]$  where  $b, d$  and  $f$  are positive.

⟨1⟩2. ASSUME:  $0 < t$

⟨1⟩3.  $e > 0$

⟨1⟩4.  $rt < st$  iff  $r < s$

PROOF:

$$\begin{aligned} rt < st &\Leftrightarrow [(ae, bf)] < [(ce, df)] \\ &\Leftrightarrow aedf < cebf \\ &\Leftrightarrow ad < bc \\ &\Leftrightarrow r < s \end{aligned}$$

□

**Corollary 16.2.10.1.** *The rationals form an ordered field.*

**Proposition 16.2.11.** *Let  $p$  be a positive rational. For any rational number  $r$ , there exists  $k \in \mathbb{N}$  such that  $r < pk$ .*

PROOF:

⟨1⟩1. LET:  $p = a/b$  and  $r = c/d$  where  $a, b$  and  $d$  are positive.

⟨1⟩2. PICK  $k \in \mathbb{N}$  such that  $bc < adk$

PROOF: Proposition 16.1.25.

⟨1⟩3.  $r < pk$

□

**Proposition 16.2.12.**  $\mathbb{Q} \approx \mathbb{N}$

PROOF: Arrange the rationals in order  $0/1, 1/1, 1/2, 0/2, -1/2, -1/1, -2/1, -2/2, -2/3, -1/3, 0/3, 1/3, 2/3$ , etc. then remove all duplicates. □

## 16.3 The Real Numbers

**Definition 16.3.1** (Cauchy Sequence). A *Cauchy sequence* is a sequence  $(q_n)$  of rationals such that, for every positive rational  $\epsilon$ , there exists  $k \in \mathbb{N}$  such that  $\forall m, n > k. |q_m - q_n| < \epsilon$ .

**Definition 16.3.2** (Dedekind Cut). A *Dedekind cut* is a set  $x \subseteq \mathbb{Q}$  such that:

1.  $\emptyset \neq x \neq \mathbb{Q}$
2.  $x$  is closed downwards.
3.  $x$  has no greatest member.

The set  $\mathbb{R}$  of *real numbers* is the set of Dedekind cuts.

**Proposition 16.3.3.** For any rational  $q$ , we have  $\{r \in \mathbb{Q} \mid r < q\} \in \mathbb{R}$ .

PROOF:

⟨1⟩1. LET:  $q \in \mathbb{Q}$

⟨1⟩2. LET:  $q \downarrow = \{r \mid r < q\}$

⟨1⟩3.  $q \notin q \downarrow$

PROOF: We have  $q - 1 \in q \downarrow$ .

⟨1⟩4.  $q \notin q \downarrow$

PROOF: Since  $q \notin q \downarrow$ .

⟨1⟩5.  $q \downarrow$  is closed downwards.

PROOF: Trivial.

⟨1⟩6.  $q \downarrow$  has no greatest element.

PROOF: For all  $r \in q \downarrow$  we have  $r < (q + r)/2 \in q \downarrow$ .

□

**Proposition 16.3.4.** For rationals  $q$  and  $r$ , we have  $q = r$  if and only if  $\{s \in \mathbb{Q} \mid s < q\} = \{s \in \mathbb{Q} \mid s < r\}$ .

PROOF:

⟨1⟩1. LET:  $q \downarrow = \{s \in \mathbb{Q} \mid s < q\}$

⟨1⟩2. LET:  $r \downarrow = \{s \in \mathbb{Q} \mid s < r\}$

⟨1⟩3. If  $q = r$  then  $q \downarrow = r \downarrow$

PROOF: Trivial.

$\langle 1 \rangle 4$ . If  $q < r$  then  $q \downarrow \neq r \downarrow$

PROOF: We have  $q \in r \downarrow$  and  $q \notin q \downarrow$ .

$\langle 1 \rangle 5$ . If  $r < q$  then  $q \downarrow \neq r \downarrow$

PROOF: We have  $r \in q \downarrow$  and  $q \notin q \downarrow$ .

□

Henceforth we identify a rational  $q$  with the real number  $\{r \in \mathbb{Q} \mid r < q\}$ .

**Definition 16.3.5.** Define the ordering  $<$  on  $\mathbb{R}$  by:  $x < y$  iff  $x \subsetneq y$ .

**Proposition 16.3.6.** *The ordering on the reals agrees with the ordering on the rationals.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $q, r \in \mathbb{Q}$

$\langle 1 \rangle 2$ . LET:  $q \downarrow = \{s \in \mathbb{Q} \mid s < q\}$ .

$\langle 1 \rangle 3$ . LET:  $r \downarrow = \{s \in \mathbb{Q} \mid s < r\}$ .

PROVE:  $q < r$  iff  $q \downarrow \subsetneq r \downarrow$

$\langle 1 \rangle 4$ . If  $q < r$  then  $q \downarrow \subsetneq r \downarrow$

$\langle 2 \rangle 1$ . ASSUME:  $q < r$

$\langle 2 \rangle 2$ .  $q \downarrow \subseteq r \downarrow$

PROOF: If  $s < q$  then  $s < r$ .

$\langle 2 \rangle 3$ .  $q \downarrow \neq r \downarrow$

PROOF: Proposition 16.3.4.

$\langle 1 \rangle 5$ . If  $q \downarrow \subsetneq r \downarrow$  then  $q < r$

$\langle 2 \rangle 1$ . ASSUME:  $q \downarrow \subsetneq r \downarrow$

$\langle 2 \rangle 2$ . PICK  $s \in r \downarrow$  such that  $s \notin q \downarrow$

$\langle 2 \rangle 3$ .  $q \leq s < r$

□

**Proposition 16.3.7.** *The ordering  $<$  is a linear ordering on  $\mathbb{R}$ .*

PROOF:

$\langle 1 \rangle 1$ .  $<$  is irreflexive.

PROOF: No set is a proper subset of itself.

$\langle 1 \rangle 2$ .  $<$  is transitive.

PROOF: Since the relationship  $\subsetneq$  is transitive on the class of all sets.

$\langle 1 \rangle 3$ .  $<$  is total.

$\langle 2 \rangle 1$ . LET:  $x, y$  be Dedekind cuts.

$\langle 2 \rangle 2$ . ASSUME:  $x \not\subseteq y$

PROVE:  $y \subsetneq x$

$\langle 2 \rangle 3$ . PICK  $q \in x$  such that  $q \notin y$

$\langle 2 \rangle 4$ . LET:  $r \in y$

PROVE:  $r \in x$

$\langle 2 \rangle 5$ .  $q \not\leq r$

PROOF: Since  $y$  is closed downwards.

$\langle 2 \rangle 6$ .  $r < q$

$\langle 2 \rangle 7$ .  $r \in x$

PROOF: Since  $x$  is closed downwards.

□

**Proposition 16.3.8.** *Any bounded nonempty subset of  $\mathbb{R}$  has a least upper bound.*

PROOF:

⟨1⟩1. LET:  $A$  be a bounded nonempty subset of  $\mathbb{R}$ .

⟨1⟩2.  $\bigcup A$  is a Dedekind cut.

⟨2⟩1.  $\bigcup A \neq \emptyset$

⟨3⟩1. PICK  $x \in A$

⟨3⟩2. PICK  $q \in x$

⟨3⟩3.  $q \in \bigcup A$

⟨2⟩2.  $\bigcup A \neq \mathbb{Q}$

⟨3⟩1. PICK an upper bound  $u$  for  $A$

⟨3⟩2. PICK  $q \notin u$

PROVE:  $q \notin \bigcup A$

⟨3⟩3. ASSUME: for a contradiction  $q \in \bigcup A$

⟨3⟩4. PICK  $x \in A$  such that  $q \in x$

⟨3⟩5.  $x \leq u$

⟨3⟩6.  $q \in u$

⟨3⟩7. Q.E.D.

PROOF: This is a contradiction.

⟨2⟩3.  $\bigcup A$  is closed downwards.

⟨3⟩1. LET:  $q \in \bigcup A$  and  $r < q$

⟨3⟩2. PICK  $x \in A$  such that  $q \in x$

⟨3⟩3.  $r \in x$

⟨3⟩4.  $r \in \bigcup A$

⟨2⟩4.  $\bigcup A$  has no greatest element.

⟨3⟩1. LET:  $q \in \bigcup A$

⟨3⟩2. PICK  $x \in A$  such that  $q \in x$

⟨3⟩3. PICK  $r \in x$  such that  $q < r$

⟨3⟩4.  $r \in \bigcup A$

⟨1⟩3.  $\bigcup A$  is an upper bound for  $A$ .

PROOF: For all  $x \in A$  we have  $x \subseteq \bigcup A$ .

⟨1⟩4. For any upper bound  $u$  for  $\bigcup A$  we have  $\bigcup A \leq u$ .

PROOF: If  $\forall x \in A. x \subseteq u$  we have  $\bigcup A \subseteq u$ .

□

**Definition 16.3.9** (Addition). Define *addition*  $+$  on the reals by

$$x + y := \{q + r \mid q \in x, r \in y\} .$$

We prove this is well-defined.

PROOF:

⟨1⟩1. LET:  $x, y \in \mathbb{R}$

PROVE:  $X + y$  is a Dedekind cut.



$\langle 1 \rangle 2. x + y \neq \emptyset$

PROOF: Pick  $q \in x$  and  $r \in y$ ; then  $q + r \in x + y$ .

$\langle 1 \rangle 3. x + y \neq \mathbb{Q}$

$\langle 2 \rangle 1. \text{ PICK } q \notin x \text{ and } r \notin y$

PROVE:  $q + r \notin x + y$

$\langle 2 \rangle 2. \text{ ASSUME: for a contradiction } q + r \in x + y$

$\langle 2 \rangle 3. \text{ PICK } q' \in x \text{ and } r' \in y \text{ such that } q + r = q' + r'$

$\langle 2 \rangle 4. q' < q \text{ and } r' < r$

$\langle 2 \rangle 5. q' + r' < q + r$

$\langle 2 \rangle 6. \text{ Q.E.D.}$

PROOF: This is a contradiction.

$\langle 1 \rangle 4. x + y$  is closed downwards.

$\langle 2 \rangle 1. \text{ LET: } q \in x \text{ and } r \in y$

$\langle 2 \rangle 2. \text{ LET: } s < q + r$

PROVE:  $s \in x + y$

$\langle 2 \rangle 3. s - r < q$

$\langle 2 \rangle 4. s - r \in x$

$\langle 2 \rangle 5. s = (s - r) + r \in x + y$

$\langle 1 \rangle 5. x + y$  has no greatest element.

$\langle 2 \rangle 1. \text{ LET: } q \in x \text{ and } r \in y$

PROVE: There exists  $s \in x + y$  such that  $q + r < s$

$\langle 2 \rangle 2. \text{ PICK } q' \in x \text{ and } r' \in y \text{ such that } q < q' \text{ and } r < r'$

$\langle 2 \rangle 3. q + r < q' + r' \in x + y$

□

**Proposition 16.3.10.** *Addition on the reals agrees with addition on the rationals.*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } q, r \in \mathbb{Q}$

$\langle 1 \rangle 2. q \downarrow + r \downarrow \subseteq (q + r) \downarrow$

PROOF: If  $s_1 < q$  and  $s_2 < r$  then  $s_1 + s_2 < q + r$ .

$\langle 1 \rangle 3. (q + r) \downarrow \subseteq q \downarrow + r \downarrow$

$\langle 2 \rangle 1. \text{ LET: } s < q + r$

$\langle 2 \rangle 2. s - r < q$

$\langle 2 \rangle 3. \text{ PICK } t \text{ such that } s - r < t < q$

$\langle 2 \rangle 4. s - t < r$

$\langle 2 \rangle 5. s = t + (s - t) \in q \downarrow + r \downarrow$

□

**Proposition 16.3.11.** *Addition is associative.*

PROOF:

$$\begin{aligned} x + (y + z) &= \{q + r \mid q \in x, r \in y + z\} \\ &= \{q + s_1 + s_2 \mid q \in x, s_1 \in y, s_2 \in z\} \\ &= \{r + s_2 \mid r \in x + y, s_2 \in z\} \\ &= (x + y) + z \end{aligned}$$

□

**Proposition 16.3.12.** *Addition is commutative.*

PROOF:

$$\begin{aligned} x + y &= \{q + r \mid q \in x, r \in y\} \\ &= \{r + q \mid r \in y, q \in x\} \\ &= y + x \end{aligned}$$

□

**Proposition 16.3.13.** *For any  $x \in \mathbb{R}$  we have  $x + 0 = x$ .*

PROOF:

⟨1⟩1.  $x + 0 \subseteq x$

PROOF: If  $q \in x$  and  $r < 0$  then  $q + r < q$  so  $q + r \in x$ .

⟨1⟩2.  $x \subseteq x + 0$

⟨2⟩1. LET:  $q \in x$

⟨2⟩2. PICK  $r \in x$  such that  $q < r$ .

PROOF:  $x$  has no greatest element.

⟨2⟩3.  $q - r < 0$

⟨2⟩4.  $q = r + (q - r) \in x + 0$

□

**Definition 16.3.14.** For  $x \in \mathbb{R}$ , define  $-x := \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$ .

**Proposition 16.3.15.** *For all  $x \in \mathbb{R}$  we have  $-x \in \mathbb{R}$ .*

PROOF:

⟨1⟩1. LET:  $x \in \mathbb{R}$

⟨1⟩2.  $-x \neq \emptyset$

⟨2⟩1. PICK  $s \notin x$

⟨2⟩2.  $-s - 1 \in -x$

⟨1⟩3.  $-x \neq \mathbb{Q}$

⟨2⟩1. PICK  $s \in x$

PROVE:  $-s \notin -x$

⟨2⟩2. ASSUME: for a contradiction  $-s \in -x$

⟨2⟩3. PICK  $r > -s$  such that  $-r \notin x$

⟨2⟩4.  $-r < s$

⟨2⟩5. Q.E.D.

PROOF: This contradicts the fact that  $x$  is closed downwards.

⟨1⟩4.  $-x$  is closed downwards.

PROOF: Immediate from definition.

⟨1⟩5.  $-x$  has no greatest element.

⟨2⟩1. LET:  $q \in -x$

⟨2⟩2. PICK  $r > q$  such that  $-r \notin x$

⟨2⟩3. PICK  $s$  such that  $q < s < r$

⟨2⟩4.  $s \in -x$

□

**Lemma 16.3.16.** *Let  $p$  be a positive rational number. For any real number  $x$ , there exists a rational  $q \in x$  such that  $p + q \notin x$ .*

PROOF:

- $\langle 1 \rangle 1$ . PICK  $q_0 \in x$
- $\langle 1 \rangle 2$ . There exists  $k \in \mathbb{N}$  such that  $q_0 + kp \notin x$ 
  - $\langle 2 \rangle 1$ . PICK  $q_1 \notin x$
  - $\langle 2 \rangle 2$ . PICK  $k \in \mathbb{N}$  such that  $q_1 - q_0 < pk$
  - PROOF: Proposition 16.2.11.
  - $\langle 2 \rangle 3$ .  $q_1 < q_0 + kp$
  - $\langle 2 \rangle 4$ .  $q_0 + kp \notin x$
- $\langle 1 \rangle 3$ . LET:  $k$  be the least natural number such that  $q_0 + kp \notin x$
- $\langle 1 \rangle 4$ .  $k \neq 0$
- PROOF:  $\langle 1 \rangle 1$
- $\langle 1 \rangle 5$ . LET:  $q = q_0 + (k-1)p$
- $\langle 1 \rangle 6$ .  $q \in x$  and  $q + p \notin x$ .

□

**Proposition 16.3.17.** *For every real  $x$  we have  $x + (-x) = 0$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $x$  be a real number.
- $\langle 1 \rangle 2$ .  $x + (-x) \subseteq 0$ 
  - $\langle 2 \rangle 1$ . LET:  $q_1 \in x$  and  $q_2 \in -x$
  - $\langle 2 \rangle 2$ . PICK  $r > q_2$  such that  $-r \notin x$
  - $\langle 2 \rangle 3$ .  $q_1 < -r$
  - $\langle 2 \rangle 4$ .  $r < -q_1$
  - $\langle 2 \rangle 5$ .  $q_2 < -q_1$
  - $\langle 2 \rangle 6$ .  $q_1 + q_2 < 0$
- $\langle 1 \rangle 3$ .  $0 \subseteq x + (-x)$ 
  - $\langle 2 \rangle 1$ . LET:  $p < 0$
  - $\langle 2 \rangle 2$ .  $0 < -p$
  - $\langle 2 \rangle 3$ . PICK  $q \in x$  such that  $q - p/2 \notin x$
  - PROOF: Lemma 16.3.16.
  - $\langle 2 \rangle 4$ . LET:  $s = p/2 - q$
  - $\langle 2 \rangle 5$ .  $-s \notin x$
  - $\langle 2 \rangle 6$ .  $p - q < s$
  - $\langle 2 \rangle 7$ .  $p - q \in -x$
  - $\langle 2 \rangle 8$ .  $p \in x + (-x)$

□

**Corollary 16.3.17.1.** *The reals form an Abelian group under addition.*

**Proposition 16.3.18.** *For any reals  $x, y$  and  $z$ , we have  $x < y$  if and only if  $x + z < y + z$ .*

PROOF:

- $\langle 1 \rangle 1$ .  $\forall x, y, z \in \mathbb{R}. x \leq y \Rightarrow x + z \leq y + z$ 
  - $\langle 2 \rangle 1$ . LET:  $x, y, z \in \mathbb{R}$
  - $\langle 2 \rangle 2$ . ASSUME:  $x \leq y$
  - $\langle 2 \rangle 3$ . For all  $q \in x$  and  $r \in z$  we have  $q + r \in y + z$

⟨1⟩2.  $\forall x, y, z \in \mathbb{R}. x + z = y + z \Leftrightarrow x = y$

PROOF: Proposition 13.1.4.

⟨1⟩3.  $\forall x, y, z \in \mathbb{R}. x < y \Rightarrow x + z < y + z$

⟨1⟩4. Q.E.D.

PROOF: Proposition 6.2.6.

□

**Definition 16.3.19** (Absolute Value). The *absolute value* of a real number  $x$  is defined to be

$$|x| = \begin{cases} x & \text{if } 0 \leq x \\ -x & \text{if } x < 0 \end{cases}$$

**Definition 16.3.20** (Multiplication). Define *multiplication*  $\cdot$  on  $\mathbb{R}$  as follows:

- If  $x$  and  $y$  are non-negative then

$$xy = 0 \cup \{rs \mid 0 \leq r \in x \wedge 0 \leq s \in y\} .$$

- If  $x$  and  $y$  are both negative then

$$xy = (-x)(-y) .$$

- If one of  $x$  and  $y$  is negative and one is non-negative then

$$xy = -(|x||y|) .$$

We prove this is well-defined.

PROOF:

⟨1⟩1. LET:  $x$  and  $y$  be non-negative reals.

PROVE:  $xy$  is real.

⟨1⟩2.  $xy \neq \emptyset$

PROOF: Since  $-1 \in xy$ .

⟨1⟩3.  $xy \neq \mathbb{Q}$

⟨2⟩1. PICK  $r \notin x$  and  $s \notin y$

PROVE:  $rs \notin xy$

⟨2⟩2.  $0 \leq r$  and  $0 \leq s$

PROOF: Since  $0 \subseteq x$  and  $0 \subseteq y$ .

⟨2⟩3. ASSUME: for a contradiction  $rs \in xy$

⟨2⟩4. PICK  $r'$  and  $s'$  such that  $0 \leq r' \in x$ ,  $0 \leq s' \in y$  and  $rs = r's'$

⟨2⟩5.  $r' < r$

⟨2⟩6.  $s' < s$

⟨2⟩7.  $r's' < rs$

⟨2⟩8. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩4.  $xy$  is closed downwards.

⟨2⟩1. LET:  $q \in xy$  and  $r < q$

- ⟨2⟩2. CASE:  $q \in 0$   
 PROOF: Then  $r < q < 0$  so  $r \in xy$
- ⟨2⟩3. CASE:  $q = s_1 s_2$  where  $0 \leq s_1 \in x$  and  $0 \leq s_2 \in y$ 
  - ⟨3⟩1. ASSUME: w.l.o.g.  $0 \leq r$
  - ⟨3⟩2.  $0 < s_1$  and  $0 < s_2$
  - ⟨3⟩3.  $r/s_2 < s_1$
  - ⟨3⟩4.  $r/s_2 \in x$
  - ⟨3⟩5.  $r = (r/s_2)s_2 \in xy$
- ⟨1⟩5.  $xy$  has no greatest element.
  - ⟨2⟩1. LET:  $q \in xy$
  - ⟨2⟩2. CASE:  $q \in 0$   
 PROOF:  $q < q/2 \in 0$
  - ⟨2⟩3. CASE:  $q = rs$  where  $0 \leq r \in x$  and  $0 \leq s \in y$ 
    - ⟨3⟩1. PICK  $r'$  and  $s'$  with  $r < r' \in x$  and  $s < s' \in y$
    - ⟨3⟩2.  $q < r's' \in xy$

□

**Proposition 16.3.21.** *Multiplication is commutative.*

PROOF: Immediate from definition. □

**Proposition 16.3.22.** *Multiplication is associative.*

PROOF:

- ⟨1⟩1. For non-negative reals  $x, y$  and  $z$ , we have  $x(yz) = (xy)z$   
 PROOF: It computes to  $0 \cup \{qrs \mid 0 \leq q \in x, 0 \leq r \in y, 0 \leq s \in z\}$ .
- ⟨1⟩2. For all reals  $x, y$  and  $z$ , we have  $x(yz) = (xy)z$   
 PROOF: It is equal to  $|x||y||z|$  if an even number of them are negative, and  $-(|x||y||z|)$  otherwise.

□

**Proposition 16.3.23.** *Multiplication distributes over addition.*

PROOF:

- ⟨1⟩1. For all non-negative reals  $x, y$  and  $z$ , we have  $x(y + z) = xy + xz$ 
  - ⟨2⟩1. LET:  $x, y$  and  $z$  be non-negative reals.
  - ⟨2⟩2.  $x(y + z) \subseteq xy + xz$ 
    - ⟨3⟩1. LET:  $q \in x(y + z)$
    - ⟨3⟩2. CASE:  $q < 0$   
 PROOF: Then we have  $q/2 \in xy$  and  $q/2 \in xz$  so  $q \in xy + xz$ .
    - ⟨3⟩3. CASE:  $q = rs$  where  $0 \leq r \in x$  and  $0 \leq s \in y + z$ 
      - ⟨4⟩1. PICK  $s_1 \in y$  and  $s_2 \in z$  such that  $s = s_1 + s_2$
      - ⟨4⟩2.  $rs_1 \in xy$   
 PROOF: If  $s_1 < 0$  then  $rs_1 < 0$  so  $rs_1 \in xy$ . If  $0 \leq s_1$  then we also have  $rs_1 \in xy$ .
      - ⟨4⟩3.  $rs_2 \in xz$   
 PROOF: Similar.
      - ⟨4⟩4.  $q \in xy + xz$

PROOF: Since  $q = rs_1 + rs_2$ .

$\langle 2 \rangle 3$ .  $xy + xz \subseteq x(y + z)$

$\langle 3 \rangle 1$ . LET:  $q \in xy$  and  $r \in xz$ .

PROVE:  $q + r \in x(y + z)$

$\langle 3 \rangle 2$ . CASE:  $q < 0$  and  $r < 0$

PROOF: Then  $q + r < 0$  so  $q + r \in x(y + z)$ .

$\langle 3 \rangle 3$ . CASE:  $q < 0$  and  $r = r_1r_2$  where  $0 \leq r_1 \in x$  and  $0 \leq r_2 \in z$

$\langle 4 \rangle 1$ .  $q + r < r$

$\langle 4 \rangle 2$ .  $q + r \in xz$

$\langle 4 \rangle 3$ . ASSUME: w.l.o.g.  $0 \leq q + r$

PROOF: Otherwise  $q + r \in x(y + z)$  immediately.

$\langle 4 \rangle 4$ . PICK  $s_1, s_2$  with  $0 \leq s_1 \in x$ ,  $0 \leq s_2 \in y$  and  $q + r = s_1s_2$

$\langle 4 \rangle 5$ .  $s_2 \in y + z$

PROOF: Since  $0 \in z$  so  $s_2 = s_2 + 0 \in y + z$ .

$\langle 4 \rangle 6$ .  $q + r \in x(y + z)$

$\langle 3 \rangle 4$ . CASE:  $q = q_1q_2$  where  $0 \leq q_1 \in x$  and  $0 \leq q_2 \in y$  and  $r < 0$

PROOF: Similar.

$\langle 3 \rangle 5$ . CASE:  $q = q_1q_2$  where  $0 \leq q_1 \in x$  and  $0 \leq q_2 \in y$  and  $r = r_1r_2$  where  $0 \leq r_1 \in x$  and  $0 \leq r_2 \in z$

$\langle 4 \rangle 1$ . ASSUME: w.l.o.g.  $q_1 \leq r_1$

$\langle 4 \rangle 2$ .  $q + r \leq r_1(q_2 + r_2) \in x(y + z)$

$\langle 1 \rangle 2$ . For any negative real  $x$  and non-negative reals  $y$  and  $z$ , we have  $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned} x(y + z) &= -(-x)(y + z) = -((-x)y + (-x)z) & (\langle 1 \rangle 1) \\ &= -((-x)y) - ((-x)z) \\ &= xy + xz \end{aligned}$$

$\langle 1 \rangle 3$ . For any non-negative real  $x$  and reals  $y$  and  $z$  with one negative and one non-negative, we have  $x(y + z) = xy + xz$

$\langle 2 \rangle 1$ . ASSUME: w.l.o.g.  $y$  is negative and  $z$  is non-negative.

$\langle 2 \rangle 2$ . CASE:  $0 \leq y + z$

PROOF:

$$\begin{aligned} xy + xz &= xy + x(-y + y + z) \\ &= -(x(-y)) + x(-y + y + z) \\ &= -(x(-y)) + x(-y) + x(y + z) & (\langle 1 \rangle 1) \\ &= x(y + z) \end{aligned}$$

$\langle 2 \rangle 3$ . CASE:  $y + z < 0$

$\langle 3 \rangle 1$ .  $-y - z > 0$

$\langle 3 \rangle 2$ .  $-y = z - y - z$

$\langle 3 \rangle 3$ .  $xy + xz = x(y + z)$

PROOF:

$$\begin{aligned}
 xy + xz &= -(x(-y)) + xz \\
 &= -(x(z - y - z)) + xz \\
 &= -(xz + x(-y - z)) + xz & ((1)1) \\
 &= -xy - x(-y - z) + xz \\
 &= -x(-y - z) \\
 &= x(y + z)
 \end{aligned}$$

(1)4. For any non-negative real  $x$  and negative reals  $y$  and  $z$ , we have  $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned}
 x(y + z) &= -x(-y - z) \\
 &= -(x(-y) + x(-z)) & ((1)1) \\
 &= -x(-y) - x(-z) \\
 &= xy + xz
 \end{aligned}$$

(1)5. For any negative real  $x$  and reals  $y$  and  $z$  with one negative and one non-negative, we have  $x(y + z) = xy + xz$

(2)1. ASSUME: w.l.o.g.  $y$  is negative and  $z$  is non-negative.

(2)2. CASE:  $0 \leq y + z$

PROOF:

$$\begin{aligned}
 x(y + z) &= -((-x)(y + z)) \\
 &= -((-x)y + (-x)z) & ((1)3) \\
 &= -((-x)y) - ((-x)z) \\
 &= (-x)(-y) - ((-x)z) \\
 &= xy + xz
 \end{aligned}$$

(2)3. CASE:  $y + z < 0$

PROOF:

$$\begin{aligned}
 x(y + z) &= (-x)(-y - z) \\
 &= (-x)(-y) + (-x)(-z) & ((1)3) \\
 &= xy + xz
 \end{aligned}$$

(1)6. For any negative reals  $x$ ,  $y$  and  $z$ , we have  $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned}
 x(y + z) &= (-x)(-y - z) \\
 &= (-x)(-y) + (-x)(-z) & ((1)1) \\
 &= xy + xz
 \end{aligned}$$

□

**Proposition 16.3.24.** *For any real  $x$  we have  $x1 = x$ .*

PROOF:

(1)1. CASE:  $0 \leq x$

(2)1.  $x1 \subseteq x$

(3)1. LET:  $q \in x1$

- ⟨3⟩2. CASE:  $q < 0$   
 PROOF: Then  $q \in x$  because  $0 \leq x$ .  
 ⟨3⟩3.  $q = rs$  where  $0 \leq r \in x$  and  $0 \leq s < 1$   
 PROOF: Then  $q < r$  so  $q \in x$ .  
 ⟨2⟩2.  $x \subseteq x1$   
 ⟨3⟩1. LET:  $q \in x$   
 ⟨3⟩2. ASSUME: w.l.o.g.  $0 \leq q$   
 ⟨3⟩3. PICK  $r$  such that  $q < r \in x$   
 ⟨3⟩4.  $0 \leq q/r < 1$   
 ⟨3⟩5.  $q = r(q/r) \in x1$   
 ⟨1⟩2. CASE:  $x < 0$   
 PROOF:

$$\begin{aligned}
 x1 &= -((-x)1) \\
 &= -(-x) && (\langle 1 \rangle 1) \\
 &= x
 \end{aligned}$$

□

**Lemma 16.3.25.** *Let  $x \in \mathbb{R}$  and  $c$  be a positive rational. Then there exists  $a \in x$  and a non-least rational upper bound  $b$  for  $x$  such that  $b - a = c$ .*

PROOF:

- ⟨1⟩1. PICK  $a_1 \in x$  such that if  $x$  has a rational supremum  $s$  then  $a_1 > s - c$   
 ⟨1⟩2. There exists a natural number  $n$  such that  $a_1 + nc$  is an upper bound for  $x$ .  
 ⟨2⟩1. PICK a non-least upper bound  $b_1$  for  $x$ .  
 ⟨2⟩2. PICK a natural number  $n$  such that  $nc > b_1 - a_1$   
 PROOF: Proposition 16.2.11.  
 ⟨2⟩3.  $a_1 + nc > b_1$   
 ⟨2⟩4.  $a_1 + nc$  is an upper bound for  $x$ .  
 ⟨1⟩3. LET:  $k$  be the least natural number such that  $a_1 + kc$  is an upper bound for  $x$ .  
 ⟨1⟩4.  $a_1 + (k-1)c \in x$   
 ⟨1⟩5.  $a_1 + kc$  is not the supremum of  $x$ .  
 ⟨2⟩1. ASSUME: for a contradiction  $a_1 + kc$  is the supremum of  $x$ .  
 ⟨2⟩2.  $a_1 > a_1 + (k-1)c$   
 PROOF: ⟨1⟩1  
 ⟨2⟩3. Q.E.D.  
 PROOF: This is a contradiction.  
 ⟨1⟩6. LET:  $a = a_1 + (k-1)c$   
 ⟨1⟩7. LET:  $b = a_1 + kc$   
 ⟨1⟩8.  $b - a = c$

□

**Proposition 16.3.26.** *For any non-zero real  $x$ , there exists a real  $y$  such that  $xy = 1$ .*

PROOF:



- ⟨1⟩1. CASE:  $0 < x$
- ⟨2⟩1. LET:  $y = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{u^{-1} \mid u \text{ is an upper bound for } x \text{ but not the supremum of } x\}$
- ⟨2⟩2.  $y$  is a real number.
  - ⟨3⟩1.  $y \neq \emptyset$   
PROOF: Since  $0 \in y$ .
  - ⟨3⟩2.  $y \neq \mathbb{Q}$ 
    - ⟨4⟩1. PICK  $q \in x$  such that  $0 < q$
    - ⟨4⟩2.  $q^{-1} \notin y$
  - ⟨3⟩3.  $y$  is closed downwards.
    - ⟨4⟩1. LET:  $q \in y$  and  $r < q$   
PROVE:  $r \in y$
    - ⟨4⟩2. ASSUME: w.l.o.g.  $0 < r$
    - ⟨4⟩3.  $q^{-1}$  is a non-least upper bound for  $x$ .
    - ⟨4⟩4.  $q^{-1} < r^{-1}$
    - ⟨4⟩5.  $r^{-1}$  is a non-least upper bound for  $x$ .
    - ⟨4⟩6.  $r \in y$
  - ⟨3⟩4.  $y$  has no greatest element.
    - ⟨4⟩1. LET:  $q \in y$   
PROVE: There exists  $r \in y$  such that  $q < r$
    - ⟨4⟩2. CASE:  $q \leq 0$ 
      - ⟨5⟩1. PICK a non-least upper bound  $u$  for  $x$ .
      - ⟨5⟩2.  $q < u^{-1} \in x$
    - ⟨4⟩3. CASE:  $q = u^{-1}$  where  $u$  is a non-least upper bound for  $x$ .
      - ⟨5⟩1. PICK a non-least upper bound  $v$  with  $v < u$
      - ⟨5⟩2.  $u^{-1} < v^{-1} \in y$
- ⟨2⟩3.  $0 < y$
- ⟨2⟩4.  $xy \subseteq 1$ 
  - ⟨3⟩1. LET:  $q \in xy$
  - ⟨3⟩2. ASSUME: w.l.o.g.  $0 < q$
  - ⟨3⟩3. PICK  $0 < r \in x$  and  $0 < s \in y$  such that  $q = rs$
  - ⟨3⟩4.  $s^{-1}$  is a non-least upper bound for  $x$
  - ⟨3⟩5.  $r < s^{-1}$
  - ⟨3⟩6.  $rs < 1$
- ⟨2⟩5.  $1 \subseteq xy$ 
  - ⟨3⟩1. LET:  $q < 1$   
PROVE:  $q \in xy$
  - ⟨3⟩2. ASSUME: w.l.o.g.  $0 < q$
  - ⟨3⟩3. PICK  $a_1$  with  $0 < a_1 \in x$
  - ⟨3⟩4.  $(1 - q)a_1 > 0$
  - ⟨3⟩5. PICK  $a \in x$  and a non-least upper bound  $w$  of  $x$  such that  $w - a = (1 - q)a_1$   
PROOF: Lemma 16.3.25.
  - ⟨3⟩6.  $w - a < (1 - q)w$
  - ⟨3⟩7.  $qw < a$
  - ⟨3⟩8.  $w < a/q$
  - ⟨3⟩9.  $a/q$  is a non-least upper bound for  $x$

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**Proposition 16.3.30.** *The set of algebraic numbers is countable.*

PROOF: There are countably many integer polynomials, each with finitely many roots.  $\square$

**Corollary 16.3.30.1.** *There are uncountably many transcendental numbers.*

**Proposition 16.3.31.** *Let  $A$  be a set of disks in the plane, no two of which intersect. Then  $A$  is countable.*

PROOF: Every circle includes a point with rational coordinates. Define  $f : \{q \in \mathbb{Q}^2 \mid \exists C \in A. q \in C\} \rightarrow A$  by  $f(q) = C$  iff  $q \in C$ . Then  $f$  is surjective.  $\square$

**Proposition 16.3.32.** *There exists an uncountable set of circles in the plane that do not intersect.*

PROOF: The set of all circles with origin  $O$  is uncountable.  $\square$



## Chapter 17

# Real Analysis

**Theorem 17.0.1** (Weierstrass). *Let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. For every  $\epsilon > 0$ , there exists a polynomial  $p$  such that  $\forall x \in [a, b]. |f(x) - p(x)| < \epsilon$ .*

**Theorem 17.0.2** (Bolzano-Weierstrass). *Every bounded sequence in  $\mathbb{R}^n$  has a convergent subsequence.*

### 17.1 Step Functions

**Definition 17.1.1** (Step Function). A *step function* on  $\mathbb{R}$  is a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , where  $[a_1, b_1), \dots, [a_n, b_n)$  are disjoint half-open intervals, such that  $f$  is constant on each  $[a_i, b_i)$ , and zero outside them.

**Definition 17.1.2** (Basic Representation). Let  $f$  be a step function. The *basic representation* of  $f$  is defined as follows.

Let  $a_0, a_1, \dots, a_n$  be the points of discontinuity of  $f$ . For  $k = 1, \dots, n$ , let  $\alpha_k = f(a_{k-1})$  and  $g_k$  be the characteristic function of  $[a_{k-1}, a_k)$ . Then the basic representation of  $f$  is

$$f = \alpha_1 g_1 + \dots + \alpha_n g_n .$$

**Proposition 17.1.3.** *If  $f$  and  $g$  are step functions then  $\lambda x.f(x) + g(x)$  is a step function.*

**Proposition 17.1.4.** *If  $c \in \mathbb{R}$  and  $f$  is a step function then  $\lambda x.cf(x)$  is a step function.*

**Proposition 17.1.5.** *If  $f$  is a step function then  $\lambda x.|f(x)|$  is a step function.*

**Proposition 17.1.6.** *If  $f$  and  $g$  are step functions then  $\lambda x.\min(f(x), g(x))$  and  $\lambda x.\max(f(x), g(x))$  are step functions.*

**Proposition 17.1.7.** *If  $f$  is a step function and  $c \in \mathbb{R}$  then  $\lambda x.f(x - c)$  is a step function.*

**Definition 17.1.8** (Support). Given  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the *support* of  $f$  is

$$\text{supp } f := \{x \in \mathbb{R} \mid f(x) \neq 0\} .$$

**Definition 17.1.9** (Integral of a Step Function). Given a step function  $f$ , define  $\int f = \int f(x)dx \in \mathbb{R}$  as follows.

Let  $f(x) = \lambda_1 f_1(x) + \cdots + \lambda_n f_n(x)$ , where  $f_i$  is the characteristic function of  $[a_i, b_i)$ . Then

$$\int f = \lambda_1(b_1 - a_1) + \cdots + \lambda_n(b_n - a_n) .$$

We prove this is well defined.

PROOF:

- (1)1. LET:  $f = \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x) = \mu_1 g_1(x) + \cdots + \mu_n g_n(x)$ , where  $f_i$  is the characteristic function of  $[a_i, b_i)$  and  $g_i$  is the characteristic function of  $[c_i, d_i)$ , with  $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$  and  $c_1 < d_1 \leq c_2 < d_2 \leq \cdots \leq c_n < d_n$ .
- (1)2. ASSUME: w.l.o.g. none of the  $\lambda_i$  or  $\mu_i$  is zero.
- (1)3. ASSUME: w.l.o.g. we never have  $\lambda_i = \lambda_{i+1}$  and  $b_i = a_{i+1}$ , and we never have  $\mu_i = \mu_{i+1}$  and  $d_i = c_{i+1}$ .
- (1)4. We have  $m = n$  and for all  $i$ ,  $a_i = b_i$  and  $c_i = d_i$  and  $\lambda_i = \mu_i$ .
  - (2)1. ASSUME: w.l.o.g.  $m \leq n$
  - (2)2. ASSUME: as induction hypothesis  $a_i = b_i$ ,  $c_i = d_i$ ,  $\lambda_i = \mu_i$  for  $i = 1, \dots, k$  with  $k < m$ .
  - (2)3.  $a_{k+1} = b_{k+1}$   
PROOF: It is  $\inf\{x \in (a_k, +\infty) : f(x) \neq 0\}$  (or  $\inf\{x \in \mathbb{R} : f(x) \neq 0\}$  if  $k = 0$ ).
  - (2)4.  $\lambda_{k+1} = \mu_{k+1}$   
PROOF: It is  $f(a_{k+1})$ .
  - (2)5.  $c_{k+1} = d_{k+1}$   
PROOF: It is  $\sup\{x \in (a_{k+1}, +\infty) : f(x) = \lambda_{k+1}\}$ .
  - (2)6.  $m = n$   
PROOF: For all  $x > b_m$  we have  $f(x) = 0$ .

□

**Proposition 17.1.10.** If  $f$  and  $g$  are step functions then  $\int(f(x) + g(x))dx = \int f + \int g$ .

**Proposition 17.1.11.** If  $f$  is a step function then  $\int cf(x)dx = c \int f$ .

**Proposition 17.1.12.** If  $f$  and  $g$  are step functions and  $\forall x. f(x) \leq g(x)$  then  $\int f \leq \int g$ .

PROOF: We have  $g(x) - f(x) \geq 0$  for all  $x$  and so  $\int(g(x) - f(x))dx \geq 0$ . □

**Proposition 17.1.13.** If  $f$  is a step function then  $|\int f| \leq \int |f(x)|dx$ .

**Proposition 17.1.14.** If  $f$  is a step function and  $c \in \mathbb{R}$  then  $\int f(x-c)dx = \int f$ .

**Lemma 17.1.15.** *Let  $f$  be a step function with  $\text{supp } f \subseteq [a_1, b_1] \cup \cdots \cup [a_n, b_n]$ . Let  $M$  be a constant. If  $\forall x. |f(x)| < M$  then*

$$\int |f(x)| dx \leq M \sum_{k=1}^n (b_k - a_k) .$$

**Lemma 17.1.16.** *Let  $\{[a_i, b_i] \mid i \in \mathbb{N}\}$  be a partition of  $[a, b]$ . Then*

$$\sum_{i=0}^{\infty} (b_i - a_i) = b - a .$$

PROOF:

$\langle 1 \rangle 1$ . For all  $c \in (a, b]$  we have  $\{[a_i, b_i] \cap [a, c] \mid i \in \mathbb{N}\}$  is a partition of  $[a, c]$ .

$\langle 1 \rangle 2$ . For  $c \in (a, b]$  and  $n \in \mathbb{N}$ ,

LET:  $b_{cn} := \min(b_n, c)$

$\langle 1 \rangle 3$ . LET:  $S = \{c \in (a, b] \mid \sum_{a_n < b_{cn}} (b_{cn} - a_n) = c - a\}$

$\langle 1 \rangle 4$ .  $S \neq \emptyset$

$\langle 2 \rangle 1$ . PICK  $n$  such that  $a_n = a$

$\langle 2 \rangle 2$ .  $b_n \in S$

$\langle 1 \rangle 5$ . LET:  $s := \sup S$

$\langle 1 \rangle 6$ .  $s \in S$

$\langle 2 \rangle 1$ . PICK an increasing sequence  $(s_n)$  in  $S$  that converges to  $S$ .

$\langle 2 \rangle 2$ . For all  $n$  we have  $s_n - a \leq \sum_{a_m < b_{sm}} (b_{sm} - a_m) \leq s - a$ .

PROOF:

$$\begin{aligned} s_n - a &= \sum_{a_m < b_{s_n m}} (b_{s_n m} - a_m) && (s_n \in S) \\ &\leq \sum_{a_m < b_{sm}} (b_{sm} - a_m) && (s_n \leq s) \\ &\leq s - a && (\langle 1 \rangle 1) \end{aligned}$$

$\langle 2 \rangle 3$ .  $\sum_{a_m < b_{sm}} (b_{sm} - a_m) = s - a$

PROOF: Sandwich Theorem

$\langle 1 \rangle 7$ .  $s = b$

$\langle 2 \rangle 1$ . ASSUME: for a contradiction  $s < b$

$\langle 2 \rangle 2$ . PICK  $k$  such that  $s \in [a_k, b_k)$

$\langle 2 \rangle 3$ .  $b_k \in S$

PROOF: Since  $\sum_{a_m < b_{sm}} (b_{sm} - a_m) = \sum_{a_m < b_{b_k m}} (b_{b_k m} - a_m)$ .

$\langle 2 \rangle 4$ .  $b_k \leq s$

$\langle 2 \rangle 5$ . Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 8$ .  $b \in S$

$\langle 1 \rangle 9$ .  $\sum_{a_n < b_n} (b_n - a_n) = b - a$

□

**Theorem 17.1.17.** *Let  $(f_n)$  be a decreasing sequence of non-negative step functions such that, for all  $x \in \mathbb{R}$ , we have  $f_n(x) \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\int f_n \rightarrow 0$  as  $n \rightarrow \infty$ .*

PROOF:

$\langle 1 \rangle 1.$   $(\int f_n)$  is decreasing and bounded below by 0.

$\langle 1 \rangle 2.$  LET:  $\epsilon := \lim_{n \rightarrow \infty} \int f_n$

$\langle 1 \rangle 3.$  ASSUME: for a contradiction  $\epsilon > 0$ .

$\langle 1 \rangle 4.$  PICK  $a, b \in \mathbb{R}$  such that  $\text{supp } f_0 \subseteq [a, b]$

$\langle 1 \rangle 5.$  LET:  $\alpha := \epsilon/2(b - a)$

$\langle 1 \rangle 6.$  For  $n \in \mathbb{N}$ ,

LET:

$$A_n := \{x \in [a, b) : f_n(x) < \alpha\} .$$

$\langle 1 \rangle 7.$  LET:  $B_0 := A_0$

$\langle 1 \rangle 8.$  For  $n$  a positive integer,

LET:  $B_n = A_n - A_{n-1}$ .

$\langle 1 \rangle 9.$  For all  $n$  we have  $A_n \subseteq A_{n+1}$ .

$\langle 1 \rangle 10.$  For  $m \neq n$  we have  $B_m \cap B_n = \emptyset$ .

$\langle 1 \rangle 11.$   $\bigcup_{n=0}^{\infty} A_n = [a, b)$

PROOF: For all  $x \in [a, b)$ , there exists  $N$  such that  $f_N(x) < \alpha$  because  $f_n(x) \rightarrow 0$  as  $n \rightarrow \infty$ .

$\langle 1 \rangle 12.$   $\bigcup_{n=0}^{\infty} B_n = [a, b)$

$\langle 1 \rangle 13.$  For  $n \in \mathbb{N}$ ,

LET:  $B_n = [a_{n1}, b_{n1}) \cup \dots \cup [a_{nk_n}, b_{nk_n})$ .

$\langle 1 \rangle 14.$   $\sum_{n=0}^{\infty} \sum_{k=1}^{k_n} (b_{nk} - a_{nk}) = b - a$

PROOF: Lemma 17.1.16.

$\langle 1 \rangle 15.$  LET:

$$\delta = \frac{\epsilon}{2 \max_x |f_0(x)|} .$$

$\langle 1 \rangle 16.$  PICK  $n_0$  such that

$$\sum_{n=n_0}^{\infty} \sum_{k=1}^{k_n} (b_{nk} - a_{nk}) < \delta .$$

$\langle 1 \rangle 17.$   $A_{n_0} = B_0 \cup \dots \cup B_{n_0}$

$\langle 1 \rangle 18.$  LET:  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$g(x) = \begin{cases} f_{n_0}(x) & \text{if } x \in A_{n_0} \\ 0 & \text{if } x \notin A_{n_0} \end{cases}$$

$\langle 1 \rangle 19.$  LET:  $h : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$h(x) = \begin{cases} 0 & \text{if } x \in A_{n_0} \\ f_{n_0}(x) & \text{if } x \notin A_{n_0} \end{cases}$$

$\langle 1 \rangle 20.$   $\forall x \in B. f_{n_0}(x) < \alpha$

$\langle 1 \rangle 21.$   $\forall x \in \mathbb{R}. g(x) < \alpha$

$\langle 1 \rangle 22.$

$$\int g < \frac{\epsilon}{2}$$

$\langle 1 \rangle 23.$

$$\int h < \frac{\epsilon}{2}$$



PROOF:

$$\begin{aligned} \int h &< \delta \max_x |f_{n_0}(x)| \\ &\leq \delta \max_x |f_0(x)| \\ &= \epsilon/2 \end{aligned}$$

(1)24.

$$\int f_{n_0} < \epsilon$$

PROOF: Since  $\forall x. f_{n_0}(x) = g(x) + h(x)$ .

(1)25.

$$\lim_{n \rightarrow \infty} \int f_n < \epsilon$$

(1)26. Q.E.D.

PROOF: This contradicts (1)2.

□

**Corollary 17.1.17.1.** *Let  $(f_n)$  be an increasing sequence of step functions. If  $\lim_{n \rightarrow \infty} f_n(x) \geq 0$  for all  $x$ , then  $\lim_{n \rightarrow \infty} \int f_n \geq 0$ .*

PROOF:

(1)1. For  $n \in \mathbb{N}$ ,

LET:  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  be the function  $g_n(x) = \max(0, -f_n(x))$ .

(1)2.  $(g_n)$  is a decreasing sequence of step functions and  $g_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x$ .

(1)3.  $\int g_n \rightarrow 0$  as  $n \rightarrow \infty$

(1)4. For all  $n \in \mathbb{N}$  we have  $f_n(x) = \max(0, f_n(x)) - \max(0, -f_n(x))$ .

(1)5.

$$\int f_n = \int \max(0, f_n(x)) dx - \int \max(0, -f_n(x)) dx$$

(1)6.

$$\lim_{n \rightarrow \infty} \int f_n = \lim_{n \rightarrow \infty} \int \max(0, f_n(x)) dx$$

(1)7.  $\lim_{n \rightarrow \infty} \int f_n \geq 0$

□

## 17.2 Lebesgue Integration

**Definition 17.2.1** (Lebesgue Integration). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Then  $f$  is *Lebesgue integrable* iff there exists a sequence  $(f_n)$  of step functions such that:

•

$$\sum_{n=0}^{\infty} \int |f_n(x)| dx < \infty$$

• For all  $x \in \mathbb{R}$ , if  $\sum_{n=0}^{\infty} |f_n(x)| < \infty$ , then  $f(x) = \sum_{n=0}^{\infty} f_n(x)$ .

We write  $f \simeq \sum_{n=0}^{\infty} f_n$  for these two conditions.

The *integral* of  $f$  is then

$$\int f := \sum_{n=0}^{\infty} \int f_n .$$

## Chapter 18

# Complex Analysis

**Theorem 18.0.1** (Hölder's Inequality). *Let  $p$  and  $q$  be real numbers with  $p > 1$ ,  $q > 1$  and  $1/p + 1/q = 1$ . If  $(x_n) \in l^p$  and  $(y_n) \in l^q$  then*

$$\sum_{n=0}^{\infty} |x_n y_n| \leq \left( \sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} \left( \sum_{n=0}^{\infty} |y_n|^q \right)^{1/q}$$

PROOF:

- ⟨1⟩1. LET:  $p$  and  $q$  be real numbers with  $p > 1$  and  $q > 1$
- ⟨1⟩2. ASSUME:  $1/p + 1/q = 1$
- ⟨1⟩3. LET:  $(x_n) \in l^p$
- ⟨1⟩4. LET:  $(y_n) \in l^q$
- ⟨1⟩5. ASSUME: w.l.o.g.  $x_0 \neq 0$  and  $y_0 \neq 0$
- ⟨1⟩6. For all  $x \in [0, 1]$ , we have

$$x^{1/p} \leq \frac{1}{p}x + \frac{1}{q}.$$

PROOF:

- ⟨2⟩1. LET:  $f : [0, 1] \rightarrow \mathbb{R}$  be the function

$$f(x) = \frac{1}{p}x + \frac{1}{q} - x^{1/p}.$$

- ⟨2⟩2.  $f'(x) = \frac{1}{p} - \frac{1}{p}x^{-1/q}$  for  $x \in (0, 1]$
- ⟨2⟩3.  $f'(x) < 0$  for  $x \in (0, 1]$
- ⟨2⟩4.  $f(1) = 1/p + 1/q - 1 = 0$
- ⟨2⟩5.  $f(x) \geq 0$  for all  $x \in [0, 1]$
- ⟨1⟩7. For all non-negative reals  $a$  and  $b$ , we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

- ⟨2⟩1. LET:  $a$  and  $b$  be non-negative reals.
- ⟨2⟩2. CASE:  $a^p \leq b^q$ 
  - ⟨3⟩1.  $0 \leq a^p/b^q \leq 1$
  - ⟨3⟩2.

$$ab^{-q/p} \leq \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q}$$

PROOF: Taking  $x = a^p/b^q$  in  $\langle 1 \rangle 6$ .

$\langle 3 \rangle 3$ .

$$ab^{1-q} \leq \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q}$$

PROOF:  $-q/p = 1 - q$  from  $\langle 1 \rangle 2$ .

$\langle 3 \rangle 4$ .

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$\langle 2 \rangle 3$ . CASE:  $b^q \leq a^p$

PROOF: Similar.

$\langle 1 \rangle 8$ . For  $j = 1, \dots, n$ , we have

$$\frac{|x_j|}{(\sum_{k=0}^n |x_k|^p)^{1/p}} \frac{|y_j|}{(\sum_{k=0}^n |y_k|^q)^{1/q}} \leq \frac{1}{p} \frac{|x_j|^p}{\sum_{k=0}^n |x_k|^p} + \frac{1}{q} \frac{|y_j|^q}{\sum_{k=0}^n |y_k|^q}$$

PROOF: From  $\langle 1 \rangle 7$  with

$$a = \frac{|x_j|}{(\sum_{k=0}^n |x_k|^p)^{1/p}} \text{ and } b = \frac{|y_j|}{(\sum_{k=0}^n |y_k|^q)^{1/q}}.$$

$\langle 1 \rangle 9$ .

$$\frac{\sum_{j=0}^n |x_j| |y_j|}{(\sum_{k=0}^n |x_k|^p)^{1/p} (\sum_{k=0}^n |y_k|^q)^{1/q}} \leq 1$$

PROOF:

$$\begin{aligned} \frac{\sum_{j=0}^n |x_j| |y_j|}{(\sum_{k=0}^n |x_k|^p)^{1/p} (\sum_{k=0}^n |y_k|^q)^{1/q}} &\leq \frac{1}{p} + \frac{1}{q} \quad (\text{Taking the sum } j = 0 \text{ to } n \text{ in } \langle 1 \rangle 8) \\ &= 1 \end{aligned} \quad (\langle 1 \rangle 2)$$

$\langle 1 \rangle 10$ . Q.E.D.

PROOF: Taking the limit  $n \rightarrow \infty$  in  $\langle 1 \rangle 9$ .

□

**Theorem 18.0.2** (Minkowski's Inequality). *Let  $p$  be a real number,  $p \geq 1$ . Let  $(x_n), (y_n) \in l^p$ . Then*

$$\left( \sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{1/p} \leq \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} + \left( \sum_{n=1}^{\infty} |y_n|^p \right)^{1/p}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $p$  be a real number with  $p \geq 1$

$\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $p > 1$

PROOF: The case  $p = 1$  is just the Triangle Inequality.

$\langle 1 \rangle 3$ . LET:  $q$  be the real such that  $1/p + 1/q = 1$

$\langle 1 \rangle 4$ .

$$\begin{aligned} \sum_{n=0}^{\infty} |x_n + y_n|^p &\leq \left( \sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} \left( \sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \\ &\quad + \left( \sum_{n=0}^{\infty} |y_n|^p \right)^{1/p} \left( \sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \end{aligned}$$

PROOF:

$$\begin{aligned}
\sum_{n=0}^{\infty} |x_n + y_n|^p &= \sum_{n=0}^{\infty} |x_n + y_n| |x_n + y_n|^{p-1} \\
&\leq \sum_{n=0}^{\infty} |x_n| |x_n + y_n|^{p-1} + \sum_{n=0}^{\infty} |y_n| |x_n + y_n|^{p-1} \quad (\text{Triangle Inequality}) \\
&\leq \left( \sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} \left( \sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \\
&\quad + \left( \sum_{n=0}^{\infty} |y_n|^p \right)^{1/p} \left( \sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \quad (\text{Hölder's Inequality})
\end{aligned}$$

$\langle 1 \rangle 5.$

$$\sum_{n=0}^{\infty} |x_n + y_n|^p \leq \left\{ \left( \sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} + \left( \sum_{n=0}^{\infty} |y_n|^p \right)^{1/p} \right\} \left( \sum_{n=0}^{\infty} |x_n + y_n|^p \right)^{1/q}$$

$\langle 1 \rangle 6.$  Q.E.D.

□



# Chapter 19

## Topology

### 19.1 Topological Spaces

**Definition 19.1.1** (Topology). Let  $X$  be a set. A *topology* on  $X$  is a set  $\mathcal{T} \subseteq \mathcal{P}X$ , whose elements are called *open sets*, such that:

- $X \in \mathcal{T}$
- $\forall \mathcal{U} \subseteq \mathcal{T}. \bigcup \mathcal{U} \in \mathcal{T}$
- $\forall U, V \in \mathcal{T}. U \cap V \in \mathcal{T}$

A *topological space* is a pair  $(X, \mathcal{T})$  such that  $X$  is a set and  $\mathcal{T}$  is a topology on  $X$ . We refer to the elements of  $X$  as *points*.

An *open neighbourhood* of a point  $x$  is an open set  $U$  such that  $x \in U$ . We write  $\mathcal{T}_x$  for the set of all open neighbourhoods of  $x$ .

**Definition 19.1.2** (Closed Set). In a topological space  $X$ , a set  $C$  is *closed* iff  $X - C$  is open.

**Definition 19.1.3** (Discrete Topology). The *discrete topology* on a set  $X$  is  $\mathcal{P}X$ .

**Definition 19.1.4** (Indiscrete Topology). The *indiscrete topology* or *trivial topology* on a set  $X$  is  $\{\emptyset, X\}$ .

**Definition 19.1.5** (Finer, Coarser). Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the same set  $X$ . Then  $\mathcal{T}$  is *finer*, *larger* or *stronger* than  $\mathcal{T}'$ , and  $\mathcal{T}'$  is *coarser*, *smaller* or *weaker* than  $\mathcal{T}$ , iff  $\mathcal{T}' \subseteq \mathcal{T}$ .

**Definition 19.1.6** (Basis). Let  $X$  be a set. A *basis* for a topology on  $X$  is a set  $\mathcal{B} \subseteq \mathcal{P}X$ , whose elements we call *basic open neighbourhoods*, such that:

- $\bigcup \mathcal{B} = X$
- $\forall A, B \in \mathcal{B}. \forall x \in A \cap B. \exists C \in \mathcal{B}. x \in C \subseteq A \cap B.$

The topology *generated* by  $\mathcal{B}$  is the coarsest topology that includes  $\mathcal{B}$ .

**Proposition 19.1.7.** *The topology generated by  $\mathcal{B}$  is  $\{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$*

## 19.2 Continuous Functions

**Definition 19.2.1** (Continuous). Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is *continuous* iff, for every open set  $U$  in  $Y$ , the set  $f^{-1}(U)$  is open in  $X$ .

**Proposition 19.2.2.** *For any topological space  $X$ , the identity function  $\text{id}_X : X \rightarrow X$  is continuous.*

**Proposition 19.2.3.** *If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous then  $g \circ f : X \rightarrow Z$  is continuous.*

**Definition 19.2.4.** Let  $\text{Top}$  be the category of topological spaces and continuous functions.

**Proposition 19.2.5.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for every closed set  $C$  in  $Y$ , we have  $f^{-1}(C)$  is closed in  $X$ .*

**Definition 19.2.6** (Continuous at a Point). Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Let  $x \in X$ . Then  $f$  is *continuous at  $x$*  iff, for every open neighbourhood  $V$  of  $f(x)$ , we have  $f^{-1}(V)$  is open.

**Definition 19.2.7.** The category of *pointed* topological spaces,  $\text{Top}_*$ , is the category with:

- objects all pairs  $(A, a)$  where  $A$  is a topological space and  $a \in A$ ;
- morphisms  $f : (A, a) \rightarrow (B, b)$  all continuous functions  $f : A \rightarrow B$  such that  $f(a) = b$ .

**Definition 19.2.8** (Homeomorphism). A *homeomorphism* is an isomorphism in  $\text{Top}$ . Two isomorphic topological spaces are called *homeomorphic*.

**Definition 19.2.9** (Topological Property). A property of topological spaces is a *topological* property iff it is preserved by homeomorphism.

**Proposition 19.2.10.** *Cardinality of a topological space is a topological property.*

## 19.3 Convergence

**Definition 19.3.1** (Convergence). Let  $X$  be a topological space. Let  $(x_n)$  be a sequence in  $X$  and  $l \in X$ . Then  $(x_n)$  *converges* to  $l$ ,  $x_n \rightarrow l$  as  $n \rightarrow \infty$ , if and only if, for every open neighbourhood  $U$  of  $l$ , there exists  $N$  such that  $\forall n \geq N. x_n \in U$ .



**Theorem 19.3.2.** *Let  $X$  and  $Y$  be topological spaces. Let  $Z$  be a closed subspace of  $X$  and  $f : Z \rightarrow Y$  a continuous function. Then the graph of  $f$ ,  $G = \{(x, f(x)) \mid x \in Z\}$ , is closed in  $X \times Y$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $((x_n, f(x_n)))$  be a sequence in  $G$ .

$\langle 1 \rangle 2$ . LET:  $(x_n, T(x_n)) \rightarrow (x, y)$  as  $n \rightarrow \infty$

$\langle 1 \rangle 3$ .  $x \in W$

PROOF: Since  $x_n \rightarrow x$  and  $W$  is closed.

$\langle 1 \rangle 4$ .  $y = f(x)$

PROOF:

$$\begin{aligned} y &= \lim_{n \rightarrow \infty} f(x_n) \\ &= f\left(\lim_{n \rightarrow \infty} x_n\right) \\ &= f(x) \end{aligned}$$

□

## 19.4 Homotopy

**Definition 19.4.1.** Let  $\mathbf{hTop}$  be the category whose objects are topological spaces, and whose morphisms are homotopy classes of continuous functions.

**Definition 19.4.2.** A *homotopy equivalence* is an isomorphism in  $\mathbf{hTop}$ . Isomorphic topological spaces are called *homotopic*.

## 19.5 Metric Spaces

**Definition 19.5.1** (Metric). Let  $X$  be a set. A *metric* on a set  $X$  is a function  $d : X^2 \rightarrow \mathbb{R}$  such that:

- $\forall x, y \in X. d(x, y) \geq 0$
- $\forall x, y \in X. d(x, y) = d(y, x)$
- *Triangle Inequality*  $\forall x, y, z \in X. d(x, y) + d(y, z) \geq d(x, z)$
- $\forall x, y \in X. d(x, y) = 0$  iff  $x = y$ .

A *metric space* is a pair  $(X, d)$  such that  $d$  is a metric on  $X$ .

**Definition 19.5.2** (Open Ball). In a metric space  $X$ , let  $c \in X$  and  $r > 0$ . The *open ball* with *centre*  $c$  and *radius*  $r$  is

$$B(c, r) := \{x \in X \mid d(x, c) < r\} .$$

**Proposition 19.5.3.** *In a metric space, the set of open balls forms a basis for a topology.*

**Definition 19.5.4** (Metric Topology). Given a metric space  $X$ , the *metric topology* on  $X$  is the topology generated by the basis of open balls.

A topological space  $(X, \mathcal{T})$  is *metrizable* iff there exists a metric  $d$  on  $X$  such that  $\mathcal{T}$  is the metric topology induced by  $d$ .

We identify a metric space with this topological space.

**Proposition 19.5.5.** *If  $d$  is a metric on  $X$  and  $Y \subseteq X$  then  $d \upharpoonright Y^2$  is a metric on  $Y$ .*

We write just  $Y$  for the metric space  $(Y, d \upharpoonright Y^2)$ .

**Proposition 19.5.6.** *Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for every sequence  $(x_n)$  in  $X$  and  $l \in X$ , if  $x_n \rightarrow l$  as  $n \rightarrow \infty$  then  $f(x_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ .*

PROOF:

- $\langle 1 \rangle 1$ . If  $f$  is continuous then, for every sequence  $(x_n)$  in  $X$  and  $l \in X$ , if  $x_n \rightarrow l$  as  $n \rightarrow \infty$ , then  $f(x_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ .
- $\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous.
- $\langle 2 \rangle 2$ . LET:  $(x_n)$  be a sequence in  $X$ .
- $\langle 2 \rangle 3$ . LET:  $l \in X$
- $\langle 2 \rangle 4$ . ASSUME:  $x_n \rightarrow l$  as  $n \rightarrow \infty$ .
- $\langle 2 \rangle 5$ . LET:  $V$  be an open neighbourhood of  $f(l)$
- $\langle 2 \rangle 6$ .  $f^{-1}(V)$  is an open neighbourhood of  $l$ .
- $\langle 2 \rangle 7$ . PICK  $N$  such that  $\forall n \geq N. x_n \in f^{-1}(V)$
- $\langle 2 \rangle 8$ .  $\forall n \geq N. f(x_n) \in V$
- $\langle 1 \rangle 2$ . If, for every sequence  $(x_n)$  in  $X$  and  $l \in X$ , if  $x_n \rightarrow l$  as  $n \rightarrow \infty$ , then  $f(x_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ , then  $f$  is continuous.
- $\langle 2 \rangle 1$ . ASSUME:  $f$  is not continuous.
- $\langle 2 \rangle 2$ . PICK an open set  $V$  in  $Y$  such that  $f^{-1}(V)$  is not open in  $X$
- $\langle 2 \rangle 3$ . PICK  $l \in f^{-1}(V)$  such that, for all  $\epsilon > 0$ ,  $B(l, \epsilon) \not\subseteq f^{-1}(V)$ .
- $\langle 2 \rangle 4$ . For  $n \in \mathbb{N}$ , PICK  $x_n \in B(l, 1/(n+1))$  such that  $x_n \notin f^{-1}(V)$ .
- $\langle 2 \rangle 5$ .  $x_n \rightarrow l$  as  $n \rightarrow \infty$ .
- $\langle 2 \rangle 6$ .  $f(x_n) \not\rightarrow f(l)$  as  $n \rightarrow \infty$

□

**Proposition 19.5.7.** *Completeness is not a topological property.*

PROOF: We have  $(-1, 1) \cong \mathbb{R}$ , but  $\mathbb{R}$  is complete and  $(-1, 1)$  is not. □

## Chapter 20

# Ring Theory

**Definition 20.0.1.** Given a ring  $R$ , let  $R\text{-Mod}$  be the category of modules over  $R$  and  $R$ -linear maps.



# Chapter 21

## Linear Algebra

### 21.1 Vector Spaces

**Definition 21.1.1** (Vector Space). Let  $K$  be a field. A *vector space* over  $K$  consists of:

- a set  $V$ , whose elements are called *vectors*;
- an operation  $+$  :  $V^2 \rightarrow V$ , *addition*;
- an operation  $\cdot$  :  $K \times V \rightarrow V$ , *scalar multiplication*

such that:

- $V$  is an Abelian group under  $+$
- $\forall \alpha, \beta \in K. \forall x \in V. \alpha(\beta x) = (\alpha\beta)x$
- $\forall \alpha, \beta \in K. \forall x \in V. (\alpha + \beta)x = \alpha x + \beta x$
- $\forall \alpha \in K. \forall x, y \in V. \alpha(x + y) = \alpha x + \alpha y$
- $\forall x \in V. 1x = x$

We call the elements of  $K$  *scalars*. A *real vector space* is a vector space over  $\mathbb{R}$ , and a *complex vector space* is a vector space over  $\mathbb{C}$ .

**Proposition 21.1.2.** *Let  $K$  be a field. Let  $V$  be a vector space over  $K$ . For any  $\lambda \in K$  we have  $\lambda 0 = 0$ .*

PROOF:

$$\begin{aligned}\lambda 0 &= \lambda(0 + 0) \\ &= \lambda 0 + \lambda 0 \\ \therefore 0 &= \lambda 0\end{aligned}$$

□

**Proposition 21.1.3.** *Let  $K$  be a field. Let  $V$  be a vector space over  $K$ . Let  $\lambda \in K$  and  $x \in V$ . If  $\lambda x = 0$  then either  $\lambda = 0$  or  $x = 0$ .*

PROOF: If  $\lambda \neq 0$  then  $x = 1x = \lambda^{-1}\lambda x = \lambda^{-1}0 = 0$ .  $\square$

**Proposition 21.1.4.** *Let  $K$  be a field. Let  $V$  be a vector space over  $K$ . For any  $x \in V$  we have  $0x = 0$ .*

PROOF:

$$\begin{aligned} 0x &= (0 + 0)x \\ &= 0x + 0x \\ \therefore 0 &= 0x \end{aligned} \quad \square$$

**Proposition 21.1.5.** *Let  $K$  be a field. Let  $V$  be a vector space over  $K$ . For any  $x \in V$ , we have  $(-1)x = -x$ .*

PROOF:

$$\begin{aligned} x + (-1)x &= 1x + (-1)x \\ &= (1 + (-1))x \\ &= 0x \\ &= 0 \\ \therefore (-1)x &= -x \end{aligned} \quad \square$$

**Proposition 21.1.6.** *Let  $K$  be a field. Then  $K$  is a vector space over  $K$  under addition and multiplication in  $K$ .*

PROOF: Easy.  $\square$

**Proposition 21.1.7.**  $\mathbb{C}$  is a vector space over  $\mathbb{R}$ .

PROOF: Easy.  $\square$

**Proposition 21.1.8.** *Let  $K$  be a field. Let  $\{V_i\}_{i \in I}$  be a family of vector spaces over  $K$ . Then  $\prod_{i \in I} V_i$  is a vector space under*

$$\begin{aligned} (f + g)(i) &= f(i) + g(i) & (f, g \in \prod_{i \in I} V_i, x \in X) \\ (\lambda f)(x) &= \lambda f(x) & (\lambda \in K, f \in \prod_{i \in I} V_i, x \in X) \end{aligned}$$

PROOF: Easy.  $\square$

## 21.2 Subspaces

**Definition 21.2.1** (Vector Subspace). Let  $K$  be a field. Let  $V$  be a vector space over  $K$ . A *vector subspace* of  $V$  is a subset  $U \subseteq V$  such that, for all  $\alpha, \beta \in K$  and  $x, y \in U$ , we have  $\alpha x + \beta y \in U$ .

It is a *proper* subspace iff  $U \neq V$ .

**Proposition 21.2.2.** *If  $U$  is a subspace of  $V$  then  $U$  is a vector space under the restrictions of  $+$  and  $\cdot$  to  $U$ .*

PROOF: Easy.  $\square$

**Proposition 21.2.3.**  *$V$  is a subspace of  $V$ .*

PROOF: Easy.  $\square$

**Proposition 21.2.4.** *If  $U$  is a subspace of  $V$  and  $V$  is a subspace of  $W$  then  $U$  is a subspace of  $W$ .*

PROOF: Easy.  $\square$

**Definition 21.2.5.** Let  $\Omega$  be a topological space. Then  $\mathcal{C}(\Omega)$  is the complex vector space of all continuous functions from  $\Omega$  to  $\mathbb{C}$ . This is a subspace of  $\mathbb{C}^\Omega$ .

**Definition 21.2.6.** Let  $n, k \in \mathbb{N}$ . Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . Then  $\mathcal{C}^k(\Omega)$  is the complex vector space of all functions  $\Omega \rightarrow \mathbb{C}$  that have all continuous partial derivatives of order  $k$ . This is a subspace of  $\mathcal{C}(\Omega)$ . If  $l > k$  then  $\mathcal{C}^l(\Omega)$  is a subspace of  $\mathcal{C}^k(\Omega)$ .

**Definition 21.2.7.** Let  $n \in \mathbb{N}$ . Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . Then  $\mathcal{C}^\infty(\Omega)$  is the complex vector space of all infinitely differentiable functions  $\Omega \rightarrow \mathbb{C}$ . This is a subspace of  $\mathcal{C}^k(\Omega)$  for all  $k$ .

**Definition 21.2.8.** Let  $n \in \mathbb{N}$ . Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . Then  $\mathcal{P}(\Omega)$  is the complex vector space of all complex polynomials of  $n$  variables, considered as functions  $\Omega \rightarrow \mathbb{C}$ . This is a subspace of  $\mathcal{C}^\infty(\Omega)$ .

**Proposition 21.2.9.** *The space of all convergent sequences in  $\mathbb{C}$  is a subspace of the space of all bounded sequences in  $\mathbb{C}$ , which is a subspace of  $\mathbb{C}^\mathbb{N}$ .*

PROOF: Easy.  $\square$

**Definition 21.2.10.** Let  $p$  be a real number,  $p \geq 1$ . Let  $l^p$  be the set of all complex sequences  $(z_n)$  such that  $\sum_{n=1}^\infty |z_n|^p < \infty$ .

**Proposition 21.2.11.** *For  $p$  a real number  $\geq 1$ , we have that  $l^p$  is a subspace of  $\mathbb{C}^\mathbb{N}$ .*

PROOF:

(1)1. For all  $(x_n), (y_n) \in l^p$ , we have  $(x_n + y_n) \in l^p$ .

PROOF: From Minkowski's Inequality.

(1)2. For all  $\lambda \in \mathbb{C}$  and  $(x_n) \in l^p$  we have  $(\lambda x_n) \in l^p$

PROOF:

$$\sum_{n=1}^\infty |\lambda x_n|^p = |\lambda|^p \sum_{n=1}^\infty |x_n|^p < \infty$$

$\square$

**Definition 21.2.12** (Linear Combination). Let  $K$  be a field. Let  $V$  be a vector space over  $K$ . Let  $x, x_1, \dots, x_n \in V$ . Then  $x$  is a *linear combination* of  $x_1, \dots, x_n$  iff there exist  $\alpha_1, \dots, \alpha_n \in K$  such that

$$x = \alpha_1 x_1 + \dots + \alpha_n x_n .$$

**Definition 21.2.13** (Linearly Independent). A finite set of vectors  $\{x_1, \dots, x_n\}$  is *linearly independent* iff, whenever  $\alpha_1 x_1 + \dots + \alpha_n x_n = 0$ , then  $\alpha_1 = \dots = \alpha_n = 0$ .

A set of vectors is *linearly independent* iff every finite subset is linearly independent; otherwise, it is *linearly dependent*.

**Definition 21.2.14** (Span). Let  $\mathcal{A}$  be a set of vectors. The *span* of  $\mathcal{A}$ ,  $\text{span } \mathcal{A}$ , is the set of all linear combinations of elements of  $\mathcal{A}$ .

**Proposition 21.2.15.**  $\text{span } \mathcal{A}$  is the smallest subspace of  $V$  that includes  $\mathcal{A}$ .

PROOF: Easy.  $\square$

**Definition 21.2.16** (Basis). A *basis* for  $V$  is a linearly independent set of vectors  $\mathcal{B}$  such that  $\text{span } \mathcal{B} = V$ .

**Definition 21.2.17** (Finite Dimensional). A vector space is *finite dimensional* iff it has a finite basis; otherwise it is *infinite dimensional*.

**Proposition 21.2.18.** In a finite dimensional vector space, any two bases have the same number of elements.

**Definition 21.2.19** (Dimension). The *dimension* of a finite dimensional vector space  $V$ ,  $\dim V$ , is the number of elements in any basis.

**Proposition 21.2.20.**

$$\dim K^n = n$$

PROOF: The standard basis is the set of vectors with one coordinate 1 and all others 0.  $\square$

**Proposition 21.2.21.** The dimension of  $\mathbb{C}^n$  as a real vector space is  $2n$ .

**Proposition 21.2.22.** The set of all step functions is a subspace of  $\mathbb{R}^{\mathbb{R}}$ .

## 21.3 Linear Transformations

**Definition 21.3.1** (Linear Transformation). Let  $K$  be a field. Let  $U$  and  $V$  be vector spaces over  $K$ . Let  $T : U \rightarrow V$ . Then  $T$  is a *linear transformation* iff

$$\forall \alpha, \beta \in K. \forall x, y \in U. T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) .$$

Let  $\text{Vect}_K$  be the category of vector spaces over  $K$  and linear transformations.



**Proposition 21.3.2.** *If  $T : U \rightarrow V$  is a linear transformation then  $T(U)$  is a subspace of  $V$ .*

**Proposition 21.3.3.** *If  $T : U \rightarrow V$  is a linear transformation then the graph of  $T$ ,  $\{(x, T(x)) \mid x \in U\}$ , is a subspace of  $U \times V$ .*

**Definition 21.3.4** (Null Space). Let  $U$  and  $V$  be vector spaces over  $K$  and  $T : U \rightarrow V$ . The *null space* of  $T$  is

$$\mathcal{N}(T) := \{x \in U \mid T(x) = 0\} .$$

**Proposition 21.3.5.** *If  $T : U \rightarrow V$  is a linear transformation then  $\mathcal{N}(T)$  is a subspace of  $U$ .*

**Proposition 21.3.6.** *Let  $U$  and  $V$  be vector spaces over  $K$ . The set of all linear transformations  $U \rightarrow V$  is a vector space over  $K$  under*

$$\begin{aligned} (S + T)(u) &= S(u) + T(u) \\ (\lambda S)(u) &= \lambda S(u) \end{aligned}$$

## 21.4 Normed Spaces

**Definition 21.4.1** (Norm). Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . A *norm* on a vector space  $V$  over  $K$  is a function  $\| \cdot \| : V \rightarrow \mathbb{R}$  such that:

1.  $\forall x \in V. \|x\| = 0 \Rightarrow x = 0$
2.  $\forall \lambda \in K. \forall x \in V. \|\lambda x\| = |\lambda| \|x\|$
3. *Triangle Inequality*  $\forall x, y \in V. \|x + y\| \leq \|x\| + \|y\|$

**Proposition 21.4.2.** *Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a normed space over  $K$ . Define  $d : V^2 \rightarrow \mathbb{R}$  by  $d(x, y) = \|x - y\|$ . Then  $d$  is a metric on  $V$ .*

PROOF: Easy.  $\square$

We identify any normed space  $V$  with this metric space.

**Proposition 21.4.3.** *Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a vector space over  $K$ . Let  $\| \cdot \|$  be a norm on  $V$ . Then*

$$\|0\| = 0 .$$

PROOF:

$$\begin{aligned} \|0\| &= \|0 \cdot 0\| && \text{(Proposition 21.1.4)} \\ &= |0| \|0\| && \text{(Axiom 2 for a norm)} \\ &= 0 && \square \end{aligned}$$

**Proposition 21.4.4.** *Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a vector space over  $K$ . Let  $\| \cdot \|$  be a norm on  $V$ . Let  $x \in V$ . Then*

$$\|x\| \geq 0 .$$

PROOF:

$$\begin{aligned}
 0 &= \|0\| && \text{(Proposition 21.4.3)} \\
 &= \|x - x\| \\
 &\leq \|x\| + \|-x\| && \text{(Triangle Inequality)} \\
 &= \|x\| + \|x\| && \text{(Axiom 2 for a norm)} \\
 &= 2\|x\| && \square
 \end{aligned}$$

**Proposition 21.4.5.** *Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a vector space over  $K$ . Let  $\|\cdot\|$  be a norm on  $V$ . Let  $x, y \in V$ . Then*

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

PROOF:

$$\langle 1 \rangle 1. \quad \|x\| - \|y\| \leq \|x - y\|$$

PROOF:  $\|x\| \leq \|x - y\| + \|y\|$  by the Triangle Inequality.

$$\langle 1 \rangle 2. \quad \|y\| - \|x\| \leq \|x - y\|$$

PROOF:

$$\begin{aligned}
 \|x\| + \|x - y\| &= \|x\| + \|y - x\| && \text{(Axiom 2 for a norm)} \\
 &\leq \|y\| && \text{(Triangle Inequality)}
 \end{aligned}$$

$\square$

**Corollary 21.4.5.1.** *Let  $V$  be a normed space. Then  $\|\cdot\| : V \rightarrow \mathbb{R}$  is continuous.*

**Definition 21.4.6** (Euclidean Norm). The *Euclidean norm* on  $\mathbb{C}^n$  is defined by

$$\|(z_1, \dots, z_n)\| = \sqrt{|z_1|^2 + \dots + |z_n|^2}$$

**Proposition 21.4.7.** *Define  $\|\cdot\| : \mathbb{C}^n \rightarrow \mathbb{R}$  by*

$$\|(z_1, \dots, z_n)\| = |z_1| + \dots + |z_n|$$

*Then this defines a norm on  $\mathbb{C}^n$ .*

PROOF: Easy.  $\square$

**Proposition 21.4.8.** *Define  $\|\cdot\| : \mathbb{C}^n \rightarrow \mathbb{R}$  by*

$$\|(z_1, \dots, z_n)\| = \max(|z_1|, \dots, |z_n|)$$

*Then this defines a norm on  $\mathbb{C}^n$ .*

PROOF: Easy.  $\square$

**Proposition 21.4.9.** *Let  $\Omega$  be a closed bounded subset of  $\mathbb{R}^n$ . Define  $\|\cdot\| : \mathcal{C}(\Omega) \rightarrow \mathbb{R}$  by  $\|f\| = \max_{x \in \Omega} |f(x)|$ . Then  $\|\cdot\|$  defines a norm on  $\mathcal{C}(\Omega)$ .*

PROOF: Easy.  $\square$

**Proposition 21.4.10.** Let  $p$  be a real number,  $p \geq 1$ . Define  $\| \cdot \| : l^p \rightarrow \mathbb{R}$  by

$$\|(z_n)\| = \left( \sum_{n=0}^{\infty} |z_n|^p \right)^{1/p}.$$

Then this defines a norm on  $l^p$ .

PROOF: Easy. The triangle inequality is Minkowski's Inequality.  $\square$

**Definition 21.4.11** (Normed Space). Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . A *normed space* over  $K$  consists of a vector space  $V$  over  $K$  and a norm on  $V$ .

We shall write simply:

- $K^n$  for the normed space  $K^n$  under the Euclidean norm
- $l^p$  for the normed space  $l^p$  under the norm  $\|(z_n)\| = (\sum_{n=0}^{\infty} |z_n|^p)^{1/p}$ .

**Proposition 21.4.12.** Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a normed space over  $K$ . If  $x_n \rightarrow l$  as  $n \rightarrow \infty$  in  $V$  and  $\lambda_n \rightarrow \lambda$  as  $n \rightarrow \infty$  in  $K$ , then  $\lambda_n x_n \rightarrow \lambda l$  as  $n \rightarrow \infty$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 2$ . LET:  $K = |\lambda| + \epsilon/2 \|l\|$

$\langle 1 \rangle 3$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $|\lambda_n - \lambda| < \epsilon/2 \|l\|$  and  $\|x_n - l\| < \epsilon/(2K)$

$\langle 1 \rangle 4$ . For all  $n \geq N$  we have  $|\lambda_n| < K$

$\langle 1 \rangle 5$ .  $\|\lambda_n x_n - \lambda l\| < \epsilon$

PROOF:

$$\begin{aligned} \|\lambda_n x_n - \lambda l\| &\leq \|\lambda_n x_n - \lambda_n l\| + \|\lambda_n l - \lambda l\| \\ &= |\lambda_n| \|x_n - l\| + |\lambda_n - \lambda| \|l\| \\ &< K \frac{\epsilon}{2K} + \frac{\epsilon}{2\|l\|} \|l\| \\ &= \epsilon \end{aligned}$$

$\square$

**Proposition 21.4.13.** In a normed space, if  $x_n \rightarrow l$  and  $y_n \rightarrow m$  then  $x_n + y_n \rightarrow l + m$

PROOF:

$$\begin{aligned} \|(x_n + y_n) - (l + m)\| &\leq \|x_n - l\| + \|y_n - m\| \\ &\rightarrow 0 \end{aligned}$$

$\square$

**Definition 21.4.14** (Uniform Convergence). Let  $\Omega$  be a closed bounded set in  $\mathbb{R}^n$ . Let  $(f_n)$  be a sequence in  $\mathcal{C}(\Omega)$  and  $f \in \mathcal{C}(\Omega)$ . Then  $(f_n)$  *converges uniformly* to  $f$  if and only if, for every  $\epsilon > 0$ , there exists  $N$  such that  $\forall x \in \Omega, \forall n \geq N, |f_n(x) - f(x)| < \epsilon$ .

**Proposition 21.4.15.**  $(f_n)$  converges uniformly to  $f$  iff  $(f_n)$  converges to  $f$  under the uniform convergence norm.

PROOF: Easy.  $\square$

**Proposition 21.4.16.** There is no norm on  $\mathcal{C}([0, 1])$  that induces pointwise convergence.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\| \cdot \|$  be any norm on  $\mathcal{C}([0, 1])$

$\langle 1 \rangle 2$ . For  $n \in \mathbb{N}$ , define  $g_n \in \mathcal{C}([0, 1])$  by

$$g_n(t) = \begin{cases} 2^n t & \text{if } 0 \leq t \leq 2^{-n} \\ 2 - 2^n t & \text{if } 2^{-n} \leq t \leq 2^{1-n} \\ 0 & \text{otherwise} \end{cases}$$

$\langle 1 \rangle 3$ . For all  $n \in \mathbb{N}$  we have  $\|g_n\| \neq 0$

$\langle 1 \rangle 4$ . For  $n \in \mathbb{N}$ ,

LET:  $f_n = g_n / \|g_n\|$

$\langle 1 \rangle 5$ . For all  $n \in \mathbb{N}$ ,  $\|f_n\| = 1$

$\langle 1 \rangle 6$ .  $f_n$  does not converge to 0

$\langle 1 \rangle 7$ .  $f_n \rightarrow 0$  as  $n \rightarrow \infty$  pointwise.

$\square$

**Definition 21.4.17** (Equivalent Norms). Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a vector spaces over  $K$ . Then two norms  $\| \cdot \|_1$  and  $\| \cdot \|_2$  are *equivalent* if and only if, for any sequence  $(x_n)$  in  $V$  and  $l \in V$ , we have  $x_n \rightarrow l$  under  $\| \cdot \|_1$  if and only if  $x_n \rightarrow l$  under  $\| \cdot \|_2$ .

**Proposition 21.4.18.** Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a vector spaces over  $K$ . Let  $\| \cdot \|_1$  and  $\| \cdot \|_2$  be norms on  $V$ . Then  $\| \cdot \|_1$  and  $\| \cdot \|_2$  are equivalent if and only if there exist positive reals  $\alpha$  and  $\beta$  such that, for all  $x \in V$ ,

$$\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1 \quad (21.1)$$

PROOF:

$\langle 1 \rangle 1$ . If  $\| \cdot \|_1$  and  $\| \cdot \|_2$  are equivalent then (21.1) holds.

$\langle 2 \rangle 1$ . ASSUME:  $\| \cdot \|_1$  and  $\| \cdot \|_2$  are equivalent.

$\langle 2 \rangle 2$ . There exists  $\alpha > 0$  such that, for all  $x \in V$ , we have  $\alpha \|x\|_1 \leq \|x\|_2$

$\langle 3 \rangle 1$ . ASSUME: for a contradiction  $\forall \alpha > 0. \exists x \in V. \alpha \|x\|_1 > \|x\|_2$

$\langle 3 \rangle 2$ . For  $n \in \mathbb{Z}^+$ , choose  $x_n \in V$  such that  $1/n \|x_n\|_1 > \|x_n\|_2$

$\langle 3 \rangle 3$ . For  $n \in \mathbb{Z}^+$ ,

LET:

$$y_n = \frac{1}{\sqrt{n}} \frac{x_n}{\|x_n\|_2}$$

$\langle 3 \rangle 4$ .  $\|y_n\|_2 \rightarrow 0$  as  $n \rightarrow \infty$

$\langle 3 \rangle 5$ . For all  $n \in \mathbb{Z}^+$ ,  $\|y_n\|_1 > \sqrt{n}$

$\langle 3 \rangle 6$ .  $\|y_n\| \not\rightarrow 0$  as  $n \rightarrow \infty$

$\langle 2 \rangle 3$ . There exists  $\beta > 0$  such that, for all  $x \in V$ , we have  $\|x\|_2 \leq \beta \|x\|_1$

PROOF: Similar.

(1)2. If (21.1) holds then  $\| \cdot \|_1$  and  $\| \cdot \|_2$  are equivalent.

(2)1. ASSUME: (21.1) holds.

(2)2. LET:  $(x_n)$  be a sequence in  $V$  and  $l \in V$

(2)3. If  $x_n \rightarrow l$  under  $\| \cdot \|_1$  then  $x_n \rightarrow l$  under  $\| \cdot \|_2$ .

(3)1. ASSUME:  $x_n \rightarrow l$  und  $\| \cdot \|_1$ .

(3)2. LET:  $\epsilon > 0$

(3)3. PICK  $N$  such that  $\forall n \geq N. \|x_n - l\| < \epsilon/\beta$

(3)4. LET:  $n \geq N$

(3)5.  $\|x_n - l\|_2 < \epsilon$

PROOF:

$$\begin{aligned} \|x_n - l\|_2 &\leq \beta \|x_n - l\|_1 \\ &< \epsilon \end{aligned}$$

(2)4. If  $x_n \rightarrow l$  under  $\| \cdot \|_2$  then  $x_n \rightarrow l$  under  $\| \cdot \|_1$ .

PROOF: Similar.

□

**Proposition 21.4.19.** *Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a normed space over  $K$ . If  $x_1, \dots, x_n \in V$  are linearly independent, then there exists  $c > 0$  such that, for all  $\alpha_1, \dots, \alpha_n \in K$ ,*

$$\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \geq c(|\alpha_1| + \dots + |\alpha_n|) .$$

PROOF:

(1)1. LET:  $B = \{(\beta_1, \dots, \beta_n) \in K^n \mid |\beta_1| + \dots + |\beta_n| = 1\}$

(1)2. LET:  $f : B \rightarrow \mathbb{R}$  be the function

$$f(\beta_1, \dots, \beta_n) = \|\beta_1 x_1 + \dots + \beta_n x_n\| .$$

(1)3. LET:  $c$  be the minimum value in  $f(B)$

PROOF:  $f$  is continuous and  $B$  is compact.

(1)4.  $c > 0$

PROOF: We never have  $f(\beta_1, \dots, \beta_n) = 0$  by linear independence.

(1)5. LET:  $\alpha_1, \dots, \alpha_n \in K$

(1)6. ASSUME: w.l.o.g.  $\alpha_1, \dots, \alpha_n$  are not all zero.

(1)7. For  $i = 1, \dots, n$ ,

$$\text{LET: } \beta_i = \alpha_i / (|\alpha_1| + \dots + |\alpha_n|)$$

(1)8.  $(\beta_1, \dots, \beta_n) \in B$

(1)9.  $f(\beta_1, \dots, \beta_n) \geq c$

(1)10.  $\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \geq c(|\alpha_1| + \dots + |\alpha_n|)$

□

**Theorem 21.4.20.** *Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a finite dimensional vector space over  $K$ . Then any two norms on  $V$  are equivalent.*

PROOF:

(1)1. PICK a basis  $\{e_1, \dots, e_n\}$  for  $V$

(1)2. LET:  $\| \cdot \|_0 : V \rightarrow \mathbb{R}$  be the function

$$\|\alpha_1 e_1 + \dots + \alpha_n e_n\|_0 = |\alpha_1| + \dots + |\alpha_n| .$$

(1)3.  $\| \cdot \|_0$  is a norm.

⟨2⟩1.  $\forall x \in V. \|x\|_0 = 0 \Rightarrow x = 0$

PROOF: If  $|\alpha_1| + \cdots + |\alpha_n| = 0$  then  $\alpha_1 = \cdots = \alpha_n = 0$ .

⟨2⟩2.  $\forall \lambda \in K. \forall x \in V. \|\lambda x\| = |\lambda| \|x\|$

PROOF:

$$\begin{aligned} \|\lambda(\alpha_1 e_1 + \cdots + \alpha_n e_n)\|_0 &= \|\lambda \alpha_1 e_1 + \cdots + \lambda \alpha_n e_n\|_0 \\ &= |\lambda \alpha_1| + \cdots + |\lambda \alpha_n| \\ &= |\lambda|(|\alpha_1| + \cdots + |\alpha_n|) \\ &= |\lambda| \|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \end{aligned}$$

⟨2⟩3. The triangle inequality holds.

PROOF:

$$\begin{aligned} \|(\alpha_1 e_1 + \cdots + \alpha_n e_n) + (\beta_1 e_1 + \cdots + \beta_n e_n)\| &= \|(\alpha_1 + \beta_1) e_1 + \cdots + (\alpha_n + \beta_n) e_n\| \\ &= |\alpha_1 + \beta_1| + \cdots + |\alpha_n + \beta_n| \\ &\leq (|\alpha_1| + \cdots + |\alpha_n|) + (|\beta_1| + \cdots + |\beta_n|) \\ &= \|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 + \|\beta_1 e_1 + \cdots + \beta_n e_n\|_0 \end{aligned}$$

⟨1⟩4. LET:  $\| \cdot \|$  be any norm on  $V$ .

PROVE:  $\| \cdot \|$  is equivalent to  $\| \cdot \|_0$ .

⟨1⟩5. For all  $\alpha_1, \dots, \alpha_n \in K$ ,

$$\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \leq \max(\|e_1\|, \dots, \|e_n\|)(|\alpha_1| + \cdots + |\alpha_n|)$$

⟨2⟩1. LET:  $\alpha_1, \dots, \alpha_n \in K$

⟨2⟩2.  $\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \leq \max(\|e_1\|, \dots, \|e_n\|)(|\alpha_1| + \cdots + |\alpha_n|)$

PROOF:

$$\begin{aligned} \|\alpha_1 e_1 + \cdots + \alpha_n e_n\| &\leq |\alpha_1| \|e_1\| + \cdots + |\alpha_n| \|e_n\| \\ &\leq (|\alpha_1| + \cdots + |\alpha_n|) \max(\|e_1\|, \dots, \|e_n\|) \quad \square \end{aligned}$$

⟨1⟩6. LET:  $\beta = \max(\|e_1\|, \dots, \|e_n\|)$

⟨1⟩7. For all  $x \in V$ ,

$$\|x\| \leq \beta \|x\|_0 .$$

⟨1⟩8. There exists  $\alpha > 0$  such that, for all  $x \in V$ ,

$$\alpha \|x\|_0 \leq \|x\| .$$

PROOF: Proposition 21.4.19.

□

**Definition 21.4.21** (Closed Ball). Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a vector space over  $K$ . Let  $x \in V$  and let  $r$  be a positive real number. The *closed ball* with *centre*  $x$  and *radius*  $r$  is

$$\overline{B(x, r)} := \{y \in V \mid \|x - y\| \leq r\} .$$

**Definition 21.4.22** (Sphere). Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a vector space over  $K$ . Let  $x \in V$  and let  $r$  be a positive real number. The *sphere* with *centre*  $x$  and *radius*  $r$  is

$$S(x, r) := \{y \in V \mid \|x - y\| = r\} .$$

**Proposition 21.4.23.** *Every closed ball is closed.*

**Proposition 21.4.24.** *Every sphere is closed.*

**Proposition 21.4.25.** *The union of two closed sets is closed.*

**Proposition 21.4.26.** *The intersection of a nonempty set of closed sets is closed.*

**Proposition 21.4.27.** *In a normed space  $V$ , both  $\emptyset$  and  $V$  are closed.*

**Proposition 21.4.28.** *Let  $V$  be a normed space and  $C \subseteq V$ . Then  $C$  is closed iff, for every sequence  $(x_n)$  in  $C$  and  $l \in V$ , if  $x_n \rightarrow l$  then  $l \in C$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $C$  is closed then, for every sequence  $(x_n)$  in  $C$  and  $l \in V$ , if  $x_n \rightarrow l$  then  $l \in C$ .

$\langle 2 \rangle 1$ . ASSUME:  $C$  is closed.

$\langle 2 \rangle 2$ . LET:  $(x_n)$  be a sequence in  $C$ .

$\langle 2 \rangle 3$ . LET:  $l \in V$

$\langle 2 \rangle 4$ . ASSUME:  $x_n \rightarrow l$  as  $n \rightarrow \infty$

$\langle 2 \rangle 5$ . ASSUME: for a contradiction  $l \notin C$

$\langle 2 \rangle 6$ . PICK  $\epsilon > 0$  such that  $B(l, \epsilon) \subseteq V - C$

$\langle 2 \rangle 7$ . PICK  $N$  such that  $\forall n \geq N, \|x_n - l\| < \epsilon$

$\langle 2 \rangle 8$ .  $x_N \in C$

$\langle 2 \rangle 9$ .  $\|x_N - l\| < \epsilon$

$\langle 2 \rangle 10$ . Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 2$ . If, for every sequence  $(x_n)$  in  $C$  and  $l \in V$ , if  $x_n \rightarrow l$  then  $l \in C$ , then  $C$  is closed.

$\langle 2 \rangle 1$ . ASSUME: for every sequence  $(x_n)$  in  $C$  and  $l \in V$ , if  $x_n \rightarrow l$  then  $l \in C$ .

$\langle 2 \rangle 2$ . LET:  $x \in V - C$

$\langle 2 \rangle 3$ . ASSUME: for a contradiction there is no  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq V - C$

$\langle 2 \rangle 4$ . For  $n \in \mathbb{Z}^+$ , PICK  $x_n \in B(x, \epsilon) \cap C$

$\langle 2 \rangle 5$ .  $x_n \rightarrow x$  as  $n \rightarrow \infty$

$\langle 2 \rangle 6$ . Q.E.D.

PROOF: This is a contradiction.

□

**Definition 21.4.29** (Closure). Let  $V$  be a normed space and  $A \subseteq V$ . The *closure* of  $A$ ,  $\text{cl } A$ , is the intersection of all the closed sets that include  $A$ .

**Theorem 21.4.30.** *Let  $V$  be a normed space and  $S \subseteq V$ . Then the closure of  $S$  is the set of all limits of convergent sequences in  $S$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $l \in \text{cl } S$ , there exists a sequence  $(x_n)$  that converges to  $l$ .

$\langle 2 \rangle 1$ . LET:  $l \in \text{cl } S$

$\langle 2 \rangle 2$ . For  $n \in \mathbb{N}$ , PICK  $x_n \in B(l, 1/(n+1)) \cap S$

$\langle 3 \rangle 1$ . ASSUME: for a contradiction  $B(l, 1/(n+1))$  does not intersect  $S$ .

$\langle 3 \rangle 2$ .  $V - B(l, 1/(n+1))$  is a closed set that includes  $S$ .

$\langle 3 \rangle 3$ .  $\text{cl } S \subseteq V - B(l, 1/(n+1))$

$\langle 3 \rangle 4.$   $l \notin B(l, 1/(n+1))$

$\langle 3 \rangle 5.$  Q.E.D.

PROOF: This is a contradiction.

$\langle 2 \rangle 3.$   $x_n \rightarrow l$

$\langle 1 \rangle 2.$  For every sequence  $(x_n)$  in  $S$ , if  $x_n \rightarrow l$  then  $l \in \text{cl } S$ .

PROOF: Proposition 21.4.28.

□

**Definition 21.4.31** (Dense). Let  $V$  be a normed space and  $S \subseteq V$ . Then  $S$  is *dense* iff  $\text{cl } S = V$ .

**Proposition 21.4.32.** In  $\mathcal{C}([a, b])$ , the set of polynomials is dense.

PROOF: By the Weierstrass Theorem. □

**Proposition 21.4.33.** For any real  $p \geq 1$ , the set of all sequences with only finitely many non-zero terms is dense in  $l^p$ .

PROOF:

$\langle 1 \rangle 1.$  LET:  $p \geq 1$

$\langle 1 \rangle 2.$  LET:  $(z_n) \in l^p$

$\langle 1 \rangle 3.$  LET:  $\epsilon > 0$

$\langle 1 \rangle 4.$  PICK  $N$  such that  $\forall n \geq N, |z_n| < \epsilon/2$

$\langle 1 \rangle 5.$  LET:  $(y_n)$  be the sequence with  $y_n = z_n$  for  $n < N$ , and  $y_n = 0$  for  $n \geq N$

$\langle 1 \rangle 6.$   $\|(z_n) - (y_n)\| \leq \epsilon/2$

$\langle 1 \rangle 7.$   $\|(z_n) - (y_n)\| < \epsilon$

□

**Theorem 21.4.34.** Let  $V$  be a normed space. Let  $S \subseteq V$ . Then the following are equivalent:

1.  $S$  is dense.
2. For all  $x \in V$ , there exists a sequence  $(x_n)$  in  $S$  such that  $x_n \rightarrow x$ .
3. Every nonempty open subset of  $V$  intersects  $S$ .

PROOF:

$\langle 1 \rangle 1.$   $1 \Leftrightarrow 2$

PROOF: Theorem 21.4.30.

$\langle 1 \rangle 2.$   $1 \Leftrightarrow 3$

PROOF:

$S$  is dense  $\Leftrightarrow$  the only closed set that includes  $S$  is  $V$

$\Leftrightarrow$  the only open set that does not intersect  $S$  is empty

□

**Definition 21.4.35** (Compact). Let  $V$  be a normed space and  $S \subseteq V$ . Then  $S$  is *compact* if and only if every sequence in  $S$  has a subsequence that converges to a limit in  $S$ .



**Proposition 21.4.36.** *Every compact set is closed.*

PROOF:

- ⟨1⟩1. LET:  $V$  be a normed space.
- ⟨1⟩2. LET:  $C \subseteq V$  be compact.
- ⟨1⟩3. LET:  $(x_n)$  be a sequence in  $C$  that converges to  $l \in V$ .
- ⟨1⟩4. PICK a subsequence  $(y_n)$  of  $(x_n)$  that converges to  $m \in C$ .
- ⟨1⟩5.  $l = m$
- ⟨1⟩6.  $l \in C$
- ⟨1⟩7. Q.E.D.

PROOF: Proposition 21.4.28.

□

**Definition 21.4.37** (Bounded). Let  $V$  be a normed space and  $S \subseteq V$ . Then  $S$  is *bounded* iff there exists  $r > 0$  such that  $S \subseteq B(0, r)$ .

**Proposition 21.4.38.** *In  $\mathbb{R}^n$  and  $\mathbb{C}^n$ , the compact sets are the closed bounded sets.*

PROOF:

- ⟨1⟩1. LET:  $C \subseteq K^n$
- ⟨1⟩2. If  $C$  is compact then  $C$  is closed.
- PROOF: Proposition 21.4.36.
- ⟨1⟩3. If  $C$  is compact then  $C$  is bounded.
- ⟨2⟩1. ASSUME:  $C$  is compact.
- ⟨2⟩2. ASSUME: for a contradiction  $C$  is not bounded.
- ⟨2⟩3. For  $n \in \mathbb{N}$ , PICK  $x_n \in C$  with  $\|x_n\| > n + 1$ .
- ⟨2⟩4. PICK a convergent subsequence  $(x_{n_r})$  that converges to  $l \in C$
- ⟨2⟩5.  $\|x_{n_r}\| \rightarrow \|l\|$
- ⟨2⟩6.  $\|x_{n_r}\| \rightarrow +\infty$
- ⟨2⟩7. Q.E.D.

PROOF: This is a contradiction.

- ⟨1⟩4. If  $C$  is closed and bounded then  $C$  is compact.

PROOF: By the Bolzano-Weierstrass Theorem.

□

**Proposition 21.4.39.** *Let  $V$  be a normed space and  $S \subseteq V$ . Then  $S$  is bounded if and only if, for every sequence  $(x_n)$  in  $S$  and every sequence  $(\lambda_n)$  in  $K$ , if  $\lambda_n \rightarrow 0$  then  $\|\lambda_n x_n\| \rightarrow 0$ .*

PROOF:

- ⟨1⟩1. If  $S$  is bounded then, for every sequence  $(x_n)$  in  $S$  and every sequence  $(\lambda_n)$  in  $K$ , if  $\lambda_n \rightarrow 0$  then  $\|\lambda_n x_n\| \rightarrow 0$ .
- ⟨2⟩1. ASSUME:  $S$  is bounded.
- ⟨2⟩2. PICK  $r > 0$  such that  $S \subseteq B(0, r)$ .
- ⟨2⟩3. LET:  $(x_n)$  be a sequence in  $S$ .
- ⟨2⟩4. LET:  $(\lambda_n)$  be a sequence in  $K$ .
- ⟨2⟩5. ASSUME:  $\lambda_n \rightarrow 0$

- ⟨2⟩6. LET:  $\epsilon > 0$
- ⟨2⟩7. PICK  $N$  such that  $\forall n \geq N, |\lambda_n| < \epsilon/r$
- ⟨2⟩8.  $\forall n \geq N, \|\lambda_n x_n\| < \epsilon$
- ⟨1⟩2. If  $S$  is unbounded then there exists a sequence  $(x_n)$  in  $S$  and  $(\lambda_n)$  in  $K$  such that  $\lambda_n \rightarrow 0$  and  $\|\lambda_n x_n\| \not\rightarrow 0$ .
- ⟨2⟩1.  $S$  is unbounded.
- ⟨2⟩2. For  $n \in \mathbb{N}$ , PICK  $x_n \in S$  such that  $\|x_n\| > n$ .
- ⟨2⟩3. For  $n \in \mathbb{N}$ ,  
LET:  $\lambda_n = 1/n$  if  $n > 0$ , 1 if  $n = 0$
- ⟨2⟩4.  $\lambda_n \rightarrow 0$
- ⟨2⟩5.  $\|\lambda_n x_n\| > 1$  for all  $n > 1$

□

**Proposition 21.4.40.** *In  $\mathcal{C}([0, 1])$ , the unit ball  $\overline{B(0, 1)}$  is closed and bounded but not compact.*

PROOF:

- ⟨1⟩1.  $\overline{B(0, 1)}$  is closed.  
PROOF: Proposition 21.4.23.
- ⟨1⟩2.  $\overline{B(0, 1)}$  is bounded.  
PROOF:  $\overline{B(0, 1)} \subseteq B(0, 2)$ .
- ⟨1⟩3.  $\overline{B(0, 1)}$  is not compact.
- ⟨2⟩1. For  $n \in \mathbb{N}$ ,  
LET:  $x_n : [0, 1] \rightarrow \mathbb{R}$  be the function  $x_n(t) = t^n$ .
- ⟨2⟩2. For  $n \in \mathbb{N}$ , we have  $x_n \in \overline{B(0, 1)}$ .
- ⟨2⟩3. No subsequence of  $(x_n)$  converges.

□

**Theorem 21.4.41** (Riesz's Lemma). *Let  $X$  be a closed proper subspace of a normed space  $V$ . For every  $\epsilon \in (0, 1)$ , there exists  $x_\epsilon \in V$  such that  $\|x_\epsilon\| = 1$  and  $\forall x \in X, \|x_\epsilon - x\| \geq \epsilon$ .*

PROOF:

- ⟨1⟩1. PICK  $z \in E - X$
- ⟨1⟩2. LET:  $d = \inf_{x \in X} \|z - x\|$
- ⟨1⟩3.  $d > 0$
- ⟨2⟩1. PICK-  $\epsilon > 0$  such that  $B(z, \epsilon) \subseteq E - X$
- ⟨2⟩2.  $d \geq \epsilon$
- ⟨1⟩4. For all  $\epsilon \in (0, 1)$ , choose  $y_\epsilon \in X$  such that  
$$d \leq \|z - y_\epsilon\| < d/\epsilon .$$
- ⟨1⟩5. For  $\epsilon \in (0, 1)$ ,  
LET:

$$x_\epsilon = \frac{z - y_\epsilon}{\|z - y_\epsilon\|} .$$

- ⟨1⟩6. For all  $x \in X$  we have  $\|x_\epsilon - x\| > \epsilon$

PROOF:

$$\begin{aligned}
 \|x_\epsilon - x\| &= \left\| \frac{z - y_\epsilon}{\|z - y_\epsilon\|} - x \right\| \\
 &= \frac{1}{\|z - y_\epsilon\|} \|z - y_\epsilon - \|z - y_\epsilon\|x\| \quad (y_\epsilon + \|z - y_\epsilon\|x \in X) \\
 &\geq \frac{1}{\|z - y_\epsilon\|} d \\
 &> \epsilon
 \end{aligned}$$

□

**Theorem 21.4.42.** *Let  $V$  be a normed space. Then  $V$  is finite dimensional if and only if  $\overline{B(0, 1)}$  is compact.*

PROOF:

⟨1⟩1. If  $V$  is finite dimensional then  $\overline{B(0, 1)}$  is compact.

⟨2⟩1. ASSUME:  $V$  is finite dimensional.

⟨2⟩2. PICK a basis  $\{e_1, \dots, e_n\}$ .

⟨2⟩3. ASSUME: w.l.o.g.  $\forall \alpha_1, \dots, \alpha_n \in K, \|\alpha_1 e_1 + \dots + \alpha_n e_n\| = |\alpha_1| + \dots + |\alpha_n|$

⟨2⟩4. LET:  $(x_m)$  be a sequence in  $\overline{B(0, 1)}$

⟨2⟩5. For  $m \in \mathbb{N}$ ,

$$\text{LET: } x_m = \alpha_{m1}e_1 + \dots + \alpha_{mn}e_n$$

⟨2⟩6. For  $m \in \mathbb{N}$  and  $i = 1, \dots, n$ , we have  $|\alpha_{mi}| \leq 1$

⟨2⟩7. For  $i = 1, \dots, n$ , PICK a convergent subsequence  $(\alpha_{m_r i})$  of  $(\alpha_{mi})$  in  $\mathbb{C}$  that converges to  $l_i$

PROOF: Since  $\overline{B(0, 1)}$  is compact in  $K$ .

⟨2⟩8.  $x_m$  converges to  $l_1 e_1 + \dots + l_n e_n$

PROOF:

$$\begin{aligned}
 \|x_m - l_1 e_1 - \dots - l_n e_n\| &= \|(\alpha_{m1} - l_1)e_1 + \dots + (\alpha_{mn} - l_n)e_n\| \\
 &= |\alpha_{m1} - l_1| + \dots + |\alpha_{mn} - l_n| \\
 &\rightarrow 0 \quad \text{as } m \rightarrow \infty
 \end{aligned}$$

⟨1⟩2. If  $V$  is infinite dimensional then  $\overline{B(0, 1)}$  is not compact.

⟨2⟩1. ASSUME:  $V$  is infinite dimensional.

⟨2⟩2. Choose a sequence  $(x_n)$  such that  $\|x_n\| = 1$  and  $\|x_m - x_n\| \geq 1/2$  for all  $m \neq n$ .

⟨3⟩1. ASSUME: as induction hypothesis there exists a sequence  $(x_0, x_1, \dots, x_n)$  such that  $\|x_i\| = 1$  and  $\|x_i - x_j\| \geq 1/2$  for  $i \neq j$

⟨3⟩2. PICK  $x_{n+1}$  such that  $\|x_{n+1}\| = 1$  and  $\|x_{n+1} - x\| \geq 1/2$  for  $x \in \{x_1, \dots, x_n\}$ .

⟨2⟩3. ASSUME: for a contradiction  $(x_{n_r})$  is a subsequence that converges to  $l$

⟨2⟩4. For all  $r$  we have  $1/2 \leq \|x_{n_r} - l\| + \|x_{n_{r+1}} - l\|$

PROOF:

$$\begin{aligned}
 1/2 &\leq \|x_{n_r} - x_{n_{r+1}}\| && (\langle 2 \rangle 2) \\
 &\leq \|x_{n_r} - l\| + \|x_{n_{r+1}} - l\| && (\text{Triangle Inequality})
 \end{aligned}$$

⟨2⟩5. Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 21.4.43.** *Let  $U$  and  $V$  be normed spaces. Let  $f : U \rightarrow V$ . If  $f$  is continuous at one point, then it is continuous.*

PROOF:

- ⟨1⟩1. ASSUME:  $f$  is continuous at  $x_0 \in U$ .
- ⟨1⟩2. LET:  $(x_n)$  be a sequence in  $U$  that converges to  $l \in U$ .
- ⟨1⟩3.  $x_n - l + x_0 \rightarrow x_0$  as  $n \rightarrow \infty$
- ⟨1⟩4.  $f(x_n - l + x_0) \rightarrow f(x_0)$  as  $n \rightarrow \infty$
- ⟨1⟩5.  $f(x_n) - f(l) + f(x_0) \rightarrow f(x_0)$  as  $n \rightarrow \infty$
- ⟨1⟩6.  $f(x_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ .

□

**Definition 21.4.44** (Bounded). Let  $U$  and  $V$  be normed spaces and  $L : U \rightarrow V$  be a linear transformation. Then  $L$  is *bounded* iff there exists  $\alpha > 0$  such that  $\forall x \in U, \|L(x)\| \leq \alpha\|x\|$ .

**Theorem 21.4.45.** *Let  $U$  and  $V$  be normed spaces. Let  $L : U \rightarrow V$  be a linear transformation. Then  $L$  is continuous if and only if it is bounded.*

PROOF:

- ⟨1⟩1. If  $L$  is continuous then  $L$  is bounded.
  - ⟨2⟩1. ASSUME:  $L$  is not bounded.
  - ⟨2⟩2. For  $n \in \mathbb{N}$ , choose  $x_n \in U$  such that  $\|L(x_n)\| > (n+1)\|x_n\|$
  - ⟨2⟩3. For  $n \in \mathbb{N}$ ,  
LET:  $y_n = x_n/(n+1)\|x_n\|$
  - ⟨2⟩4.  $y_n \rightarrow 0$  as  $n \rightarrow \infty$
  - ⟨2⟩5. For  $n \in \mathbb{N}$ , we have  $\|L(y_n)\| > 1$
  - ⟨2⟩6.  $L(y_n) \not\rightarrow 0 = L(0)$  as  $n \rightarrow \infty$
  - ⟨2⟩7.  $L$  is not continuous.
- ⟨1⟩2. If  $L$  is bounded then  $L$  is continuous.
  - ⟨2⟩1. LET:  $\alpha > 0$  be such that  $\forall x \in U, \|L(x)\| \leq \alpha\|x\|$   
PROVE:  $L$  is continuous at 0.
  - ⟨2⟩2. LET:  $(x_n)$  be a sequence in  $U$  that converges to 0.
  - ⟨2⟩3.  $L(x_n) \rightarrow 0$  as  $n \rightarrow \infty$
  - ⟨2⟩4. Q.E.D.

PROOF: Proposition 21.4.43.

□

□

**Corollary 21.4.45.1.** *If  $U$  and  $V$  are finite dimensional normed spaces, then every linear transformation  $U \rightarrow V$  is continuous.*

**Definition 21.4.46.** For  $U$  and  $V$  normed spaces, let  $\mathcal{B}(U, V)$  be the space of all bounded linear transformations  $U \rightarrow V$ . This is a subspace of the space of all linear transformations  $U \rightarrow V$ .

Define the *uniform convergence norm* on  $\mathcal{B}(U, V)$  by

$$\|L\| = \sup\{\|L(x)\| \mid x \in U, \|x\| = 1\} .$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1.$   $\forall L \in \mathcal{B}(U, V). \|L\| = 0 \Rightarrow L = 0$

$\langle 2 \rangle 1.$  LET:  $L \in \mathcal{B}(U, V)$

$\langle 2 \rangle 2.$  ASSUME:  $\|L\| = 0$

$\langle 2 \rangle 3.$  For all  $x \in U$ , if  $\|x\| = 1$  then  $\|L(x)\| = 0$

$\langle 2 \rangle 4.$  LET:  $x \in U$

PROVE:  $L(x) = 0$

$\langle 2 \rangle 5.$  ASSUME: w.l.o.g.  $x \neq 0$

$\langle 2 \rangle 6.$   $\|L(x/\|x\|)\| = 0$

PROOF:  $\langle 2 \rangle 3$

$\langle 2 \rangle 7.$   $L(x/\|x\|) = 0$

$\langle 2 \rangle 8.$   $L(x)/\|x\| = 0$

$\langle 2 \rangle 9.$   $L(x) = 0$

$\langle 1 \rangle 2.$   $\forall \lambda \in K. \forall L \in \mathcal{B}(U, V). \|\lambda L\| = |\lambda| \|L\|$

$\langle 2 \rangle 1.$  LET:  $\lambda \in K$

$\langle 2 \rangle 2.$  LET:  $L \in \mathcal{B}(U, V)$

$\langle 2 \rangle 3.$   $\|\lambda L\| = |\lambda| \|L\|$

PROOF:

$$\begin{aligned} \|\lambda L\| &= \sup_{\|x\|=1} \|\lambda L(x)\| \\ &= \sup_{\|x\|=1} (|\lambda| \|L(x)\|) \\ &= |\lambda| \sup_{\|x\|=1} \|L(x)\| \\ &= |\lambda| \|L\| \end{aligned}$$

$\langle 1 \rangle 3.$  The triangle inequality holds.

$\langle 2 \rangle 1.$  LET:  $L, M \in \mathcal{B}(U, V)$

$\langle 2 \rangle 2.$  For all  $x \in U$ , if  $\|x\| = 1$  then

$$\|L(x) + M(x)\| \leq \|L(x)\| + \|M(x)\| .$$

$\langle 2 \rangle 3.$  For all  $x \in U$ , if  $\|x\| = 1$  then

$$\|L(x) + M(x)\| \leq \|L\| + \|M\| .$$

$\langle 2 \rangle 4.$   $\|L + M\| \leq \|L\| + \|M\|$

□

**Proposition 21.4.47.** Let  $U$  and  $V$  be normed spaces and  $L \in \mathcal{B}(U, V)$ . Then  $\|L\|$  is the least number such that  $\forall x \in U. \|L(x)\| \leq \|L\| \|x\|$ .

**Theorem 21.4.48.** Let  $U$  and  $V$  be normed spaces. Let  $T : U \rightarrow V$  be a continuous linear transformation. Then the null space  $\mathcal{N}(T)$  is closed in  $U$ .

PROOF: If  $(x_n)$  is a sequence in  $\mathcal{N}(T)$  and  $x_n \rightarrow l$  then  $T(l) = \lim_{n \rightarrow \infty} T(x_n) = 0$  so  $l \in \mathcal{N}(T)$ . □

**Theorem 21.4.49** (Diagonal Theorem). Let  $V$  be a normed space. Let  $(x_{ij})_{i,j \in \mathbb{N}}$  be an infinite matrix in  $V$ . If:

1.  $\forall j \in \mathbb{N}. x_{ij} \rightarrow 0$  as  $i \rightarrow \infty$ ;
2. Every strictly increasing sequence of natural numbers  $(p_i)$  has a subsequence  $(q_i)$  such that

$$\sum_{j=0}^{\infty} x_{q_i q_j} \rightarrow 0 \text{ as } i \rightarrow \infty$$

then  $x_{ii} \rightarrow 0$  as  $i \rightarrow \infty$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $V$  be a normed space.
- $\langle 1 \rangle 2$ . LET:  $(x_{ij})_{i,j \in \mathbb{N}}$  be an infinite matrix in  $V$ .
- $\langle 1 \rangle 3$ . ASSUME:  $\forall j \in \mathbb{N}. x_{ij} \rightarrow 0$  as  $i \rightarrow \infty$
- $\langle 1 \rangle 4$ . ASSUME: Every strictly increasing sequence of natural numbers  $(p_i)$  has a subsequence  $(q_i)$  such that
 
$$\sum_{j=0}^{\infty} x_{q_i q_j} \rightarrow 0 \text{ as } i \rightarrow \infty$$
- $\langle 1 \rangle 5$ . ASSUME: for a contradiction  $x_{ii} \not\rightarrow 0$  as  $i \rightarrow \infty$
- $\langle 1 \rangle 6$ . PICK a strictly increasing sequence  $(p_i)$  and  $\epsilon > 0$  such that  $\forall i \in \mathbb{N}. \|x_{p_i p_i}\| \geq \epsilon$ 
  - $\langle 2 \rangle 1$ . PICK  $\epsilon > 0$  such that, for all  $N$ , there exists  $n \geq N$  such that  $\|x_{ii}\| \geq \epsilon$
  - $\langle 2 \rangle 2$ . Choose a sequence  $(p_i)$  such that, for all  $i$ , we have  $p_{i+1} \geq p_i + 1$  and  $\|x_{p_i p_i}\| \geq \epsilon$
- $\langle 1 \rangle 7$ . PICK a subsequence  $(q_i)$  of  $(p_i)$  such that  $\sum_{j=0}^{\infty} x_{q_i q_j} \rightarrow 0$  as  $i \rightarrow \infty$ .
 

PROOF:  $\langle 1 \rangle 4$
- $\langle 1 \rangle 8$ . For all  $i$  we have  $x_{q_i q_j} \rightarrow 0$  as  $j \rightarrow \infty$ .
 

PROOF: Since  $\sum_{j=0}^{\infty} x_{q_i q_j} < \infty$ .
- $\langle 1 \rangle 9$ . For all  $j$  we have  $x_{q_i q_j} \rightarrow 0$  as  $i \rightarrow \infty$ .
 

PROOF:  $\langle 1 \rangle 3$
- $\langle 1 \rangle 10$ . PICK a subsequence  $(r_i)$  of  $(q_i)$  such that, for all  $i, j$  with  $i \neq j$ , we have  $\|x_{r_i r_j}\| < \epsilon/2^{j+2}$ .
  - $\langle 2 \rangle 1$ . LET:  $r_1 = q_1$
  - $\langle 2 \rangle 2$ . ASSUME: as induction hypothesis we have defined  $(r_1, \dots, r_n)$  such that:
    - $r_1 < r_2 < \dots < r_n$
    - Whenever  $q_i > r_j$  then  $\|x_{q_i r_j}\| < \epsilon/2^{j+2}$
    - Whenever  $i < j$  then  $\|x_{r_i r_j}\| < \epsilon/2^{j+2}$
  - $\langle 2 \rangle 3$ . LET:  $r_{n+1}$  be the first element of  $(q_i)$  such that  $r_{n+1} > r_n$ ,  $\|x_{q_i r_{n+1}}\| < \epsilon/2^{n+2}$  whenever  $q_i \geq r_{n+1}$ , and  $\|x_{r_j r_{n+1}}\| < \epsilon/2^{n+3}$  for  $j = 1, \dots, n$ .

PROOF: One exists by  $\langle 1 \rangle 8$ .
- $\langle 2 \rangle 4$ .  $r_1 < \dots < r_n < r_{n+1}$
- $\langle 2 \rangle 5$ . Whenever  $q_i > r_j$  then  $\|x_{q_i r_j}\| < \epsilon/2^{j+2}$
- $\langle 2 \rangle 6$ . Whenever  $i < j$  then  $\|x_{r_i r_j}\| < \epsilon/2^{j+2}$
- $\langle 1 \rangle 11$ . PICK a subsequence  $(s_i)$  of  $(r_i)$  such that  $\sum_{j=0}^{\infty} x_{s_i s_j} \rightarrow 0$  as  $i \rightarrow \infty$

PROOF:  $\langle 1 \rangle 4$

$$\langle 1 \rangle 12. \forall i \in \mathbb{N}. \left\| \sum_{j=0}^{\infty} x_{s_i s_j} \right\| > \epsilon/2$$

PROOF:

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} x_{s_i s_j} \right\| &= \left\| x_{s_i s_i} + \sum_{j \neq i} x_{s_i s_j} \right\| \\ &\geq \left\| x_{s_i s_i} \right\| - \left\| \sum_{j \neq i} x_{s_i s_j} \right\| && \text{(Proposition 21.4.5)} \\ &= \left\| x_{s_i s_i} \right\| - \left\| \sum_{j \neq i} x_{s_i s_j} \right\| && \left( \left\| x_{s_i s_i} \right\| > \epsilon, \left\| \sum_j x_{s_i s_j} \right\| \leq \epsilon/2 \right) \\ &\geq \left\| x_{s_i s_i} \right\| - \sum_{j \neq i} \left\| x_{s_i s_j} \right\| && \text{(Triangle Inequality, )} \\ &> \epsilon - \sum_{j \neq i} \epsilon/2^{j+2} && (\langle 1 \rangle 6, \langle 1 \rangle 10) \\ &> \epsilon/2 \end{aligned}$$

$\langle 1 \rangle 13$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 11$ .

□

### 21.4.1 Functionals

**Definition 21.4.50** (Functional). Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $V$  be a normed space over  $K$ . A *functional* over  $V$  is a bounded linear transformation  $V \rightarrow K$ . The *dual space* of  $V$  is

$$V' := \mathcal{B}(V, K) .$$

### 21.4.2 Contraction Mappings

**Definition 21.4.51** (Contraction). Let  $V$  be a normed space and  $A \subseteq V$ . Let  $f : A \rightarrow V$ . Then  $f$  is a *contraction* iff there exists a real number  $\alpha$  with  $0 < \alpha < 1$  such that, for all  $x, y \in A$ , we have

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| .$$

**Proposition 21.4.52.** *Every contraction mapping is continuous.*

## 21.5 Banach Spaces

**Definition 21.5.1** (Cauchy Sequence). A sequence  $(x_n)$  in a normed space is a *Cauchy sequence* iff, for every  $\epsilon > 0$ , there exists  $N$  such that  $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$ .

**Theorem 21.5.2.** *Let  $V$  be a normed space. Let  $(x_n)$  be a sequence in  $V$ . Then the following are equivalent.*

1.  $(x_n)$  is Cauchy.
2. For every pair of strictly increasing sequences of natural numbers  $(p_n)$  and  $(q_n)$ , we have  $\|x_{p_n} - x_{q_n}\| \rightarrow 0$  as  $n \rightarrow \infty$ .
3. For every strictly increasing sequence of natural numbers  $(p_n)$ , we have  $\|x_{p_{n+1}} - x_{p_n}\| \rightarrow 0$  as  $n \rightarrow \infty$ .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

- $\langle 2 \rangle 1.$  ASSUME:  $(x_n)$  is Cauchy.
- $\langle 2 \rangle 2.$  LET:  $(p_n)$  and  $(q_n)$  be a pair of increasing sequences of natural numbers.
- $\langle 2 \rangle 3.$  LET:  $\epsilon > 0$
- $\langle 2 \rangle 4.$  PICK  $N$  such that  $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$
- $\langle 2 \rangle 5.$   $\forall n \geq N. p_n, q_n \geq N$
- $\langle 2 \rangle 6.$   $\forall n \geq N. \|x_{p_n} - x_{q_n}\| < \epsilon$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: Trivial.

$\langle 1 \rangle 3. 2 \Rightarrow 1$

- $\langle 2 \rangle 1.$  ASSUME:  $(x_n)$  is not Cauchy.
- $\langle 2 \rangle 2.$  PICK  $\epsilon > 0$  such that, for all  $N$ , there exist  $m, n \geq N$  such that  $\|x_m - x_n\| \geq \epsilon$
- $\langle 2 \rangle 3.$  PICK strictly increasing sequences of natural numbers  $(p_n)$  and  $(q_n)$  such that, for all  $n$ ,  $\|x_{p_n} - x_{q_n}\| \geq \epsilon$
- $\langle 3 \rangle 1.$  ASSUME: as induction hypothesis we have chosen  $(p_1, \dots, p_n)$  and  $(q_1, \dots, q_n)$  strictly increasing such that  $\forall i. \|x_{p_i} - x_{q_i}\| \geq \epsilon$
- $\langle 3 \rangle 2.$  PICK  $p_{n+1}, q_{n+1} \geq \max(p_n, q_n)$  such that  $\|x_{p_{n+1}} - x_{q_{n+1}}\| \geq \epsilon$
- $\langle 2 \rangle 4.$  2 is false.

$\langle 1 \rangle 4. 3 \Rightarrow 2$

- $\langle 2 \rangle 1.$  ASSUME:  $(p_n)$  and  $(q_n)$  are strictly increasing sequences such that  $\|x_{p_n} - x_{q_n}\| \not\rightarrow 0$  as  $n \rightarrow \infty$ .
- $\langle 2 \rangle 2.$  PICK  $\epsilon > 0$  such that, for all  $N$ , there exists  $n \geq N$  such that  $\|x_{p_n} - x_{q_n}\| \geq \epsilon$
- $\langle 2 \rangle 3.$  Choose a strictly increasing sequence  $(r_n)$  such that, for all  $n$ , we have  $\|x_{r_{2n}} - x_{r_{2n+1}}\| \geq \epsilon$
- $\langle 3 \rangle 1.$  ASSUME: as induction hypothesis we have chosen  $(r_0, r_1, \dots, r_{2n+1})$  such that, for  $i = 0, 1, \dots, n$ , we have  $\|x_{r_{2i}} - x_{r_{2i+1}}\| \geq \epsilon$
- $\langle 3 \rangle 2.$  PICK  $i, j \geq r_{2n+1}$  such that  $\|x_i - x_j\| \geq \epsilon$
- $\langle 3 \rangle 3.$  Set  $r_{2n+2} = \min(i, j)$  and  $r_{2n+3} = \max(i, j)$
- $\langle 2 \rangle 4.$  3 is false.

□

**Proposition 21.5.3.** *Every convergent sequence is a Cauchy sequence.*

PROOF:



- ⟨1⟩1. LET:  $(x_n)$  be a convergent sequence in a normed space  $V$  with limit  $l$ .  
 ⟨1⟩2. LET:  $\epsilon > 0$   
 ⟨1⟩3. PICK  $N$  such that  $\forall n \geq N. \|x_n - l\| < \epsilon/2$   
 ⟨1⟩4.  $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$   
 $\square$

**Proposition 21.5.4.** *Let  $\mathcal{P}([0, 1])$  be the space of polynomials on  $[0, 1]$  under the norm of uniform convergence. For  $n \in \mathbb{N}$ , let  $P_n = 1 + x + x^2/2! + \cdots + x^n/n!$ . Then  $(P_n)$  is Cauchy but does not converge, since  $e^x$  is not a polynomial.*

PROOF: Easy.  $\square$

**Proposition 21.5.5.** *If  $(x_n)$  is a Cauchy sequence in a normed space  $V$ , then  $(\|x_n\|)$  converges in  $\mathbb{R}$ .*

PROOF:

- ⟨1⟩1.  $(\|x_n\|)$  is Cauchy.  
 ⟨2⟩1. LET:  $\epsilon > 0$   
 ⟨2⟩2. PICK  $N$  such that  $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$   
 ⟨2⟩3.  $\forall m, n \geq N. \|\|x_m\| - \|x_n\|\| < \epsilon$   
 PROOF: Proposition 21.4.5.

⟨1⟩2. Q.E.D.

PROOF: Since  $\mathbb{R}$  is complete.

$\square$

**Corollary 21.5.5.1.** *Every Cauchy sequence is bounded.*

**Definition 21.5.6** (Banach space). A normed space is *complete* or a *Banach space* iff every Cauchy sequence converges.

**Proposition 21.5.7.** *For all  $p \geq 1$ , the space  $l^p$  is complete.*

PROOF:

- ⟨1⟩1. LET:  $(a_n)$  be a Cauchy sequence in  $l^p$ .  
 ⟨1⟩2. For  $n \in \mathbb{N}$ ,  
 LET:  $a_n = (\alpha_{n0}, \alpha_{n1}, \dots)$   
 ⟨1⟩3. For all  $\epsilon > 0$ , there exists  $N$  such that  $\forall m, n \geq N$

$$\sum_{k=0}^{\infty} |\alpha_{mk} - \alpha_{nk}|^p < \epsilon .$$

PROOF: ⟨1⟩1

- ⟨1⟩4. For all  $k \in \mathbb{N}$  and  $\epsilon > 0$ , there exists  $N$  such that  $\forall m, n \geq N$   
 $|\alpha_{mk} - \alpha_{nk}| < \epsilon .$

PROOF: ⟨1⟩3

- ⟨1⟩5. For all  $k \in \mathbb{N}$ , the sequence  $(\alpha_{nk})_n$  converges in  $\mathbb{C}$ .

PROOF:  $\mathbb{C}$  is complete.

- ⟨1⟩6. For  $k \in \mathbb{N}$ ,

LET:

$$\alpha_k = \lim_{n \rightarrow \infty} \alpha_{nk} .$$

⟨1⟩7. LET:  $a = (\alpha_k)_k$

⟨1⟩8. For all  $\epsilon > 0$ , there exists  $N$  such that  $\forall n \geq N$

$$\sum_{k=0}^{\infty} |\alpha_k - \alpha_{nk}|^p < \epsilon .$$

PROOF: Take the limit  $m \rightarrow \infty$  in ⟨1⟩4.

⟨1⟩9.  $a \in l^p$

⟨2⟩1. PICK  $N$  such that  $\forall n \geq N, \sum_{k=0}^{\infty} |\alpha_k - \alpha_{nk}|^p < 1$

⟨2⟩2.  $a - a_N \in l^p$

⟨2⟩3. Q.E.D.

PROOF: Since  $l^p$  is closed under  $+$ .

⟨1⟩10.  $a_n \rightarrow a$  as  $n \rightarrow \infty$ .

PROOF: Immediate from ⟨1⟩8.

□

**Proposition 21.5.8.** *For any real number  $a, b$  with  $a < b$ , the space  $\mathcal{C}([a, b])$  is complete.*

PROOF:

⟨1⟩1. LET:  $(f_n)$  be a Cauchy sequence in  $\mathcal{C}([a, b])$

⟨1⟩2. For all  $\epsilon > 0$ , there exists  $N$  such that, for all  $m, n \geq N$  and  $x \in [a, b]$ ,

$$|f_m(x) - f_n(x)| < \epsilon .$$

⟨1⟩3. For all  $x \in [a, b]$ ,  $(f_n(x))_n$  is Cauchy.

⟨1⟩4. LET:  $f : [a, b] \rightarrow \mathbb{C}$  be the function

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) .$$

⟨1⟩5. For all  $\epsilon > 0$ , there exists  $N$  such that, for all  $n \geq N$  and  $x \in [a, b]$ ,

$$|f_n(x) - f(x)| < \epsilon .$$

PROOF: Take the limit  $m \rightarrow \infty$  in ⟨1⟩2.

⟨1⟩6.  $f$  is continuous.

⟨2⟩1. LET:  $x_0 \in [a, b]$

⟨2⟩2. LET:  $\epsilon > 0$

⟨2⟩3. PICK  $N$  such that, for all  $n \geq N$  and  $y \in [a, b]$ , we have

$$|f_n(y) - f(y)| < \epsilon/3 .$$

PROOF: ⟨1⟩5

⟨2⟩4. PICK  $\delta > 0$  such that, for all  $y \in [a, b]$  with  $|x_0 - y| < \delta$ , we have

$$|f_N(x_0) - f_N(y)| < \epsilon/3$$

⟨2⟩5. For all  $y \in [a, b]$ , if  $|x_0 - y| < \delta$  then  $|f(x_0) - f(y)| < \epsilon$

PROOF:

$$|f(x_0) - f(y)|$$

$$\leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(y)| + |f_N(y) - f(y)| \quad (\text{Triangle Inequality})$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 \quad (\langle 2 \rangle 3, \langle 2 \rangle 4)$$

$$= \epsilon$$

⟨1⟩7.  $f_n \rightarrow f$  as  $n \rightarrow \infty$ .

PROOF: Immediate from ⟨1⟩5.

□

**Definition 21.5.9** (Convergent Series). Let  $(x_n)$  be a sequence in a normed space  $V$ . We say the series  $\sum_{n=0}^{\infty} x_n$  is *convergent* iff the sequence  $(\sum_{n=0}^N x_n)_N$  is convergent, and then we write  $\sum_{n=0}^{\infty} x_n = l$  for  $\lim_{N \rightarrow \infty} \sum_{n=0}^N x_n = l$ .

**Definition 21.5.10** (Absolutely Convergent Series). Let  $(x_n)$  be a sequence in a normed space  $V$ . We say the series  $\sum_{n=0}^{\infty} x_n$  is *absolutely convergent* iff the series  $\sum_{n=0}^{\infty} \|x_n\|$  converges in  $\mathbb{R}$ .

**Theorem 21.5.11.** *A normed space is complete if and only if every absolutely convergent series is convergent.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $V$  be a normed space.

$\langle 1 \rangle 2$ . If  $V$  is complete then every absolutely convergent series is convergent.

$\langle 2 \rangle 1$ . ASSUME:  $V$  is complete.

$\langle 2 \rangle 2$ . LET:  $(x_n)$  be a sequence in  $V$ .

$\langle 2 \rangle 3$ . ASSUME:  $\sum_{n=0}^{\infty} \|x_n\|$  converges.

PROVE:  $(\sum_{n=0}^N x_n)_N$  is Cauchy.

$\langle 2 \rangle 4$ . LET:  $\epsilon > 0$

$\langle 2 \rangle 5$ . PICK  $N$  such that  $\sum_{n=N}^{\infty} \|x_n\| < \epsilon$

$\langle 2 \rangle 6$ . LET:  $m > n \geq N$

$\langle 2 \rangle 7$ .  $\|\sum_{k=0}^m x_k - \sum_{k=0}^n x_k\| < \epsilon$

PROOF:

$$\begin{aligned} \left\| \sum_{k=0}^m x_k - \sum_{k=0}^n x_k \right\| &= \left\| \sum_{k=n+1}^m x_k \right\| \\ &\leq \sum_{k=n+1}^m \|x_k\| \\ &\leq \sum_{k=n+1}^{\infty} \|x_k\| \\ &< \epsilon \end{aligned}$$

$\langle 1 \rangle 3$ . If every absolutely convergent series is convergent then  $V$  is complete.

$\langle 2 \rangle 1$ . ASSUME: Every absolutely convergent series is convergent.

$\langle 2 \rangle 2$ . LET:  $(x_n)$  be a Cauchy sequence in  $V$ .

$\langle 2 \rangle 3$ . Choose an increasing sequence of natural numbers  $(p_k)$  such that, for all  $m, n \geq p_k$ , we have

$$\|x_m - x_n\| < 2^{-k}.$$

$\langle 2 \rangle 4$ .  $\sum_{k=0}^{\infty} \|x_{p_{k+1}} - x_{p_k}\|$  is absolutely convergent.

$\langle 2 \rangle 5$ .  $\sum_{k=0}^{\infty} \|x_{p_{k+1}} - x_{p_k}\|$  is convergent.

$\langle 2 \rangle 6$ .  $(x_{p_k})$  converges.

$\langle 2 \rangle 7$ . LET:  $l = \lim_{k \rightarrow \infty} x_{p_k}$

$\langle 2 \rangle 8$ .  $x_n \rightarrow l$  as  $n \rightarrow \infty$ .

PROOF:

$$\begin{aligned} \|x_n - l\| &\leq \|x_n - x_{p_n}\| + \|x_{p_n} - l\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ (Theorem 21.5.2)} \end{aligned}$$

□

**Proposition 21.5.12.** *A closed subspace of a Banach space is a Banach space.*

PROOF:

- ⟨1⟩1. LET:  $V$  be a Banach space.
- ⟨1⟩2. LET:  $U$  be a closed subspace of  $V$ .
- ⟨1⟩3. LET:  $(x_n)$  be a Cauchy sequence in  $U$ .
- ⟨1⟩4.  $(x_n)$  is a Cauchy sequence in  $V$ .
- ⟨1⟩5. LET:  $l$  be the limit of  $(x_n)$  in  $V$ .
- ⟨1⟩6.  $l \in U$

PROOF: Proposition 21.4.28.

□

**Definition 21.5.13** (Completion). Let  $V$  be a normed space. A *completion* of  $V$  consists of a Banach space  $W$  and a function  $\phi : V \rightarrow W$  such that:

- $\phi$  is injective.
- $\phi$  is a linear transformation.
- $\phi$  preserves the norm.
- $\phi(V)$  is dense in  $W$ .

**Definition 21.5.14** (Equivalent Cauchy Sequences). Let  $V$  be a normed space. Let  $(x_n)$  and  $(y_n)$  be Cauchy sequences in  $V$ . Then  $(x_n)$  and  $(y_n)$  are *equivalent*,  $(x_n) \sim (y_n)$ , iff  $\|x_n - y_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Proposition 21.5.15.** *Equivalence is an equivalence relation on the set of Cauchy sequences.*

**Proposition 21.5.16.** *If  $(x_n) \sim (y_n)$  then  $\lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|y_n\|$ .*

**Theorem 21.5.17.** *Let  $V$  be a normed space. Let  $W$  be the quotient set of all Cauchy sequences modulo  $\sim$ . Define  $+$ ,  $\cdot$  and  $\| \cdot \|$  on  $W$  by*

$$\begin{aligned} [(x_n)] + [(y_n)] &= [(x_n + y_n)] \\ \lambda[(x_n)] &= [(\lambda x_n)] \\ \|[(x_n)]\| &= \lim_{n \rightarrow \infty} \|x_n\| \end{aligned}$$

*Define  $\phi : V \rightarrow W$  by  $\phi(x)$  is the constant sequence  $(x)$ . Then  $\phi : V \rightarrow W$  is the completion of  $V$ .*

PROOF:

- ⟨1⟩1.  $+$ ,  $\cdot$  and  $\| \cdot \|$  are well defined.
- ⟨1⟩2.  $W$  is a normed space.
- ⟨1⟩3.  $\phi(V)$  is dense in  $W$ .

PROOF: For any  $[(x_n)] \in W$  we have  $[(x_n)] = \lim_{n \rightarrow \infty} \phi(x_n)$ .

- ⟨1⟩4.  $W$  is complete.

- (2)1. LET:  $(X_n)$  be a Cauchy sequence in  $W$ .  
 (2)2. For all  $n \in \mathbb{N}$ , PICK  $x_n \in V$  such that  $\|\phi(x_n) - X_n\| < 1/n$   
 PROOF: (1)3  
 (2)3. For all  $m, n$  we have  $\|x_n - x_m\| \leq \|X_n - X_m\| + 1/n + 1/m$   
 PROOF:  

$$\begin{aligned} \|x_n - x_m\| &= \|\phi(x_n) - \phi(x_m)\| \\ &\leq \|\phi(x_n) - X_n\| + \|X_n - X_m\| + \|\phi(x_m) - X_m\| \\ &\leq \|X_n - X_m\| + 1/n + 1/m \end{aligned}$$
  
 (2)4.  $(x_n)$  is a Cauchy sequence in  $V$ .  
 (2)5. LET:  $X = [(x_n)]$   
 PROVE:  $X_n \rightarrow X$  as  $n \rightarrow \infty$   
 (2)6.  $\|X_n - X\| \rightarrow 0$  as  $n \rightarrow \infty$   
 PROOF:  

$$\begin{aligned} \|X_n - X\| &\leq \|X_n - \phi(x_n)\| + \|\phi(x_n) - X\| \\ &< \| \phi(x_n) - X \| + 1/n \\ &\rightarrow 0 \end{aligned}$$

- (1)5.  $\phi$  is injective.  
 (1)6.  $\phi$  is a linear transformation.  
 (1)7.  $\phi$  preserves the norm.  
 $\square$

**Theorem 21.5.18.** *Let  $U$  be a normed space and  $V$  a Banach space. Then  $\mathcal{B}(U, V)$  is a Banach space.*

PROOF:

- (1)1. LET:  $(L_n)$  be a Cauchy sequence in  $\mathcal{B}(U, V)$   
 (1)2. For all  $x \in U$ , we have  $(L_n(x))$  is a Cauchy sequence in  $V$ .  
 (2)1. LET:  $x \in U$   
 (2)2. ASSUME: w.l.o.g.  $x \neq 0$   
 (2)3. LET:  $\epsilon > 0$   
 (2)4. PICK  $N$  such that  $\forall m, n \geq N. \|L_m - L_n\| < \epsilon/\|x\|$   
 (2)5.  $\forall m, n \geq N. \|L_m(x) - L_n(x)\| < \epsilon$   
 PROOF:  $\|L_m(x) - L_n(x)\| \leq \|L_m - L_n\| \|x\| < \epsilon$   
 (1)3. Define  $L : U \rightarrow V$  by  $L(x) = \lim_{n \rightarrow \infty} L_n(x)$   
 (1)4.  $L \in \mathcal{B}(U, V)$   
 (2)1.  $L$  is linear.  
 (3)1. LET:  $\lambda, \mu \in K$  and  $x, y \in U$   
 (3)2.  $L(\lambda x + \mu y) = \lambda L(x) + \mu L(y)$   
 PROOF:  

$$\begin{aligned} L(\lambda x + \mu y) &= \lim_{n \rightarrow \infty} L_n(\lambda x + \mu y) \\ &= \lim_{n \rightarrow \infty} (\lambda L_n(x) + \mu L_n(y)) \\ &= \lambda L(x) + \mu L(y) \end{aligned}$$
  
 (2)2.  $L$  is bounded.  
 (3)1. PICK  $N$  such that  $\forall m, n \geq N. \|L_m - L_n\| < 1$

PROVE:  $\forall x \in U, \|L(x)\| \leq (\|L_N\| + 1)\|x\|$   
 (3)2.  $\forall n \geq N, \|L_n\| \leq \|L_N\| + 1$   
 PROOF: Since  $|\|L_n\| - \|L_N\|| \leq \|L_n - L_N\| < 1$ .  
 (3)3. LET:  $x \in U$   
 (3)4.  $\|L(x)\| \leq (\|L_N\| + 1)\|x\|$   
 PROOF:

$$\begin{aligned} \|L(x)\| &= \left\| \lim_{n \rightarrow \infty} L_n(x) \right\| \\ &= \lim_{n \rightarrow \infty} \|L_n(x)\| \\ &\leq \lim_{n \rightarrow \infty} \|L_n\| \|x\| \\ &\leq (\|L_N\| + 1)\|x\| \end{aligned}$$

(1)5.  $L_n \rightarrow L$  as  $n \rightarrow \infty$   
 (2)1. LET:  $\epsilon > 0$   
 (2)2. PICK  $N$  such that  $\forall m, n \geq N, \|L_m - L_n\| < \epsilon/4$   
 (2)3. LET:  $n \geq N$   
 (2)4. For all  $x \in U$  we have  $\|L_n(x) - L(x)\| < (\epsilon/2)\|x\|$   
 (3)1. LET:  $x \in U$   
 (3)2. For all  $m \geq N$  we have  $\|L_n(x) - L_m(x)\| < (\epsilon/4)\|x\|$   
 (3)3.  $\|L_n(x) - L(x)\| \leq (\epsilon/4)\|x\|$   
 PROOF: Taking the limit as  $m \rightarrow \infty$ .  
 (2)5.  $\|L_n - L\| \leq \epsilon/2$   
 (2)6.  $\|L_n - L\| < \epsilon$

□

**Corollary 21.5.18.1.** *The dual space of a normed space is a Banach space.*

**Theorem 21.5.19.** *Let  $U$  be a normed space and  $V$  a Banach space. Let  $W$  be a subspace of  $U$ . Let  $L : W \rightarrow V$  be a bounded linear transformation. Then  $L$  has a unique extension to a bounded linear transformation  $\bar{W} \rightarrow V$ .*

PROOF:

(1)1. Define  $L' : \bar{W} \rightarrow V$  as follows. Given  $x \in \bar{W}$ , pick a sequence  $(x_n)$  in  $W$  that converges to  $x$ . Then  $L'(x) = \lim_{n \rightarrow \infty} L(x_n)$   
 (2)1. For all  $x \in \bar{W}$ , there exists a sequence  $(x_n)$  in  $W$  that converges to  $x$ .  
 PROOF: Theorem 21.4.30.  
 (2)2. For any sequence  $(x_n)$  in  $W$  that converges in  $\bar{W}$ , we have  $(L(x_n))$  converges in  $V$ .  
 (3)1. LET:  $(x_n)$  be a sequence in  $W$  that converges in  $\bar{W}$   
 (3)2.  $(x_n)$  is Cauchy.  
 (3)3.  $(L(x_n))$  is Cauchy.  
 PROOF: For any strictly increasing sequence of natural numbers  $(p_n)$ , we have  $\|L(x_{p_{n+1}}) - L(x_{p_n})\| \leq \|L\| \|x_{p_{n+1}} - x_{p_n}\| \rightarrow 0$  as  $n \rightarrow \infty$ .  
 (3)4. Q.E.D.  
 PROOF:  $W$  is a Banach space.  
 (2)3. If  $(x_n)$  and  $(y_n)$  are sequences in  $W$  that converge to the same point in  $\bar{W}$ , then  $\lim_{n \rightarrow \infty} L(x_n) = \lim_{n \rightarrow \infty} L(y_n)$

PROOF: Since  $\|L(x_n) - L(y_n)\| \leq \|L\|\|x_n - y_n\| \rightarrow 0$ .

(1)2.  $L'$  extends  $L$

PROOF: For  $x \in W$  we have the constant sequence  $(x)$  converges to  $x$ , and the constant sequence  $(L(x))$  converges to  $L(x)$ , so  $L'(x) = L(x)$ .

(1)3.  $L'$  is a linear transformation.

(2)1. LET:  $\lambda, \mu \in K$  and  $x, y \in \overline{W}$

(2)2. PICK sequences  $(x_n), (y_n)$  in  $W$  that converge to  $x$  and  $y$  respectively.

(2)3.  $\lambda x_n + \mu y_n \rightarrow \lambda x + \mu y$

(2)4. Q.E.D.

PROOF:

$$\begin{aligned} L'(\lambda x + \mu y) &= \lim_{n \rightarrow \infty} L(\lambda x_n + \mu y_n) \\ &= \lim_{n \rightarrow \infty} (\lambda L(x_n) + \mu L(y_n)) \\ &= \lambda L'(x) + \mu L'(y) \end{aligned}$$

(1)4.  $L'$  is bounded.

(2)1. LET:  $x \in \overline{W}$

(2)2. PICK a sequence  $(x_n)$  in  $W$  that converges to  $x$ .

(2)3.  $\|L'(x)\| \leq \|L\|\|x\|$

PROOF:

$$\begin{aligned} \|L'(x)\| &= \left\| \lim_{n \rightarrow \infty} L(x_n) \right\| \\ &= \lim_{n \rightarrow \infty} \|L(x_n)\| \\ &\leq \|L\| \lim_{n \rightarrow \infty} \|x_n\| \\ &= \|L\|\|x\| \end{aligned}$$

(1)5. If  $L'' : \overline{W} \rightarrow V$  is a bounded linear transformation that extends  $L$ , then  $L'' = L'$ .

(2)1. LET:  $x \in \overline{W}$

(2)2. PICK a sequence  $(x_n)$  in  $W$  that converges to  $x$ .

(2)3.  $L''(x) = L'(x)$

PROOF:

$$\begin{aligned} L''(x) &= \lim_{n \rightarrow \infty} L''(x_n) \\ &= \lim_{n \rightarrow \infty} L(x_n) \\ &= L'(x_n) \end{aligned}$$

□

**Theorem 21.5.20** (Banach-Steinhaus). *Let  $X$  be a Banach space and  $Y$  a normed space. Let  $\mathcal{T}$  be a set of bounded linear transformations from  $X$  into  $Y$ . Assume that, for all  $x \in X$ , there exists  $M_x > 0$  such that, for all  $T \in \mathcal{T}$ , we have  $\|T(x)\| \leq M_x$ . Then there exists  $M > 0$  such that, for all  $T \in \mathcal{T}$ , we have  $\|T\| \leq M$ .*

PROOF:

(1)1. LET:  $X$  be a Banach space.

(1)2. LET:  $Y$  be a normed space.

- ⟨1⟩3. LET:  $\mathcal{T}$  be a set of bounded linear transformations.  
 ⟨1⟩4. ASSUME: For all  $x \in X$ , there exists  $M_x > 0$  such that, for all  $T \in \mathcal{T}$ , we have  $\|T(x)\| \leq M_x$ .  
 ⟨1⟩5. ASSUME: for a contradiction there is no  $M > 0$  such that, for all  $T \in \mathcal{T}$ , we have  $\|T\| \leq M$   
 ⟨1⟩6. For every positive integer  $n$ , choose  $T_n \in \mathcal{T}$  such that  $\|T_n\| > n2^n$ .  
 ⟨1⟩7. For every positive integer  $n$ , choose  $x_n \in U$  such that  $\|x_n\| = 1$  and  $\|T_n(x_n)\| > n2^n$ .  
 ⟨1⟩8. For every positive integer  $n$  we have

$$\left\| \frac{1}{n} T_n \left( \frac{x_n}{2^n} \right) \right\| > 1 .$$

- ⟨1⟩9. For positive integers  $i$  and  $j$ ,

LET:

$$y_{ij} = \frac{1}{i} T_i \left( \frac{x_j}{2^j} \right)$$

- ⟨1⟩10. LET:  $z = \sum_{j=1}^{\infty} \frac{x_j}{2^j}$

- ⟨2⟩1.  $\sum_{j=1}^{\infty} \frac{x_j}{2^j}$  is absolutely convergent.

PROOF:

$$\begin{aligned} \sum_{j=1}^{\infty} \left\| \frac{x_j}{2^j} \right\| &= \sum_{j=1}^{\infty} \frac{1}{2^j} \\ &= 1 \end{aligned} \quad (\langle 1 \rangle 7)$$

- ⟨2⟩2. Q.E.D.

PROOF: Theorem 21.5.11.

- ⟨1⟩11. PICK  $C > 0$  such that, for all  $i$ , we have  $\left\| \sum_{j=1}^{\infty} y_{ij} \right\| \leq C/i$ .

- ⟨2⟩1. PICK  $C > 0$  such that, for all  $T \in \mathcal{T}$ , we have  $\|T(z)\| \leq C$ .

PROOF: ⟨1⟩4.

- ⟨2⟩2. For all  $i$  we have  $\left\| \sum_{j=1}^{\infty} y_{ij} \right\| \leq C/i$

PROOF:

$$\left\| \sum_{j=1}^{\infty} y_{ij} \right\| = \left\| \sum_{j=1}^{\infty} \frac{1}{i} T_i \left( \frac{x_j}{2^j} \right) \right\| \quad (\langle 1 \rangle 9)$$

$$= \frac{1}{i} \left\| T_i \left( \sum_{j=1}^{\infty} \frac{x_j}{2^j} \right) \right\| \quad (T_i \text{ continuous by } \langle 1 \rangle 3)$$

$$= \frac{1}{i} \|T_i(z)\| \quad (\langle 1 \rangle 10)$$

$$\leq \frac{C}{i} \quad (\langle 2 \rangle 1)$$

- ⟨1⟩12.  $\sum_{j=1}^{\infty} y_{ij} \rightarrow 0$  as  $i \rightarrow \infty$

- ⟨1⟩13. For any increasing sequence of positive integers  $(q_i)$ , we have  $\sum_{j=0}^{\infty} y_{q_i q_j} \rightarrow 0$  as  $i \rightarrow \infty$

PROOF: Similar.

- ⟨1⟩14. For all  $j$  we have  $y_{ij} \rightarrow 0$  as  $i \rightarrow \infty$

PROOF: From ⟨1⟩9 and the fact that  $T_i$  is continuous.



$\langle 1 \rangle 15.$   $y_{ii} \rightarrow 0$  as  $i \rightarrow \infty$ .

PROOF: By the Diagonal Theorem.

$\langle 1 \rangle 16.$  Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 8$ .

□

**Theorem 21.5.21** (Banach Fixed Point Theorem). *Let  $V$  be a Banach space. Let  $F \subseteq V$  be closed and nonempty. Let  $f : F \rightarrow F$  be a contraction. Then there exists a unique  $z \in F$  such that  $f(z) = z$ .*

PROOF:

$\langle 1 \rangle 1.$  PICK  $\alpha$  such that  $0 < \alpha < 1$  and  $\forall x, y \in V. \|f(x) - f(y)\| \leq \alpha \|x - y\|$ .

$\langle 1 \rangle 2.$  PICK  $x_0 \in F$

$\langle 1 \rangle 3.$  Extend to the sequence  $(x_n)$  in  $F$  by defining  $x_{n+1} := f(x_n)$ .

$\langle 1 \rangle 4.$   $(x_n)$  is Cauchy.

$\langle 2 \rangle 1.$   $\forall n \in \mathbb{N}. \|x_{n+1} - x_n\| \leq \alpha^n \|x_1 - x_0\|$

$\langle 2 \rangle 2.$  LET:  $\epsilon > 0$

$\langle 2 \rangle 3.$  PICK  $N$  such that  $\frac{\|x_1 - x_0\|}{1 - \alpha} \alpha^N \leq \epsilon$

$\langle 2 \rangle 4.$   $\forall m, n \geq N. \|x_n - x_m\| < \epsilon$

$\langle 3 \rangle 1.$  LET:  $m, n \geq N$

$\langle 3 \rangle 2.$  ASSUME: w.l.o.g.  $m < n$

$\langle 3 \rangle 3.$   $\|x_n - x_m\| < \epsilon$

PROOF:

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\| \\ &\leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m) \|x_1 - x_0\| \\ &< \frac{\|x_1 - x_0\|}{1 - \alpha} \alpha^m \\ &\leq \frac{\|x_1 - x_0\|}{1 - \alpha} \alpha^N \\ &\leq \epsilon \end{aligned}$$

$\langle 1 \rangle 5.$  LET:  $z = \lim_{n \rightarrow \infty} x_n$

$\langle 1 \rangle 6.$   $z$  is unique such that  $f(z) = z$ .

$\langle 2 \rangle 1.$   $f(z) = z$

PROOF:

$$\begin{aligned} f(z) &= f\left(\lim_{n \rightarrow \infty} x_n\right) \\ &= \lim_{n \rightarrow \infty} f(x_n) \\ &= \lim_{n \rightarrow \infty} x_{n+1} \\ &= z \end{aligned}$$

$\langle 2 \rangle 2.$  If  $f(w) = w$  then  $w = z$ .

$\langle 3 \rangle 1.$   $\|w - z\| \leq \alpha \|w - z\|$

PROOF:  $\|w - z\| = \|f(w) - f(z)\| \leq \alpha \|w - z\|$

$\langle 3 \rangle 2.$   $\|w - z\| = 0$

PROOF: Otherwise  $\|w - z\| < \|w - z\|$ .

$\langle 3 \rangle 3.$   $w = z$

□

## 21.6 Inner Product Spaces

**Definition 21.6.1** (Inner Product Space). An *inner product* on a complex vector space  $V$  is a function  $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{C}$  such that:

- $\forall x, y \in V. \langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\forall \lambda \in \mathbb{C}. \forall x, y \in V. \langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
- $\forall x, y, z \in V. \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\forall x \in V. \langle x, x \rangle = 0 \Rightarrow x = 0$

An *inner product space* or *pre-Hilbert space* is a complex vector space with an inner product.

**Proposition 21.6.2.**  $\mathbb{C}^n$  is an inner product space under

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = x_1 \overline{y_1} + \dots + x_n \overline{y_n} .$$

**Proposition 21.6.3.**  $l^2$  is an inner product under

$$\langle (x_n), (y_n) \rangle = \sum_{n=0}^{\infty} x_n \overline{y_n} .$$

PROOF:

$\langle 1 \rangle 1$ . For all  $(x_n), (y_n) \in l^2$  we have  $\sum_n x_n \overline{y_n} < \infty$

$\langle 2 \rangle 1$ . LET:  $(x_n), (y_n) \in l^2$

$\langle 2 \rangle 2$ .  $\sum_{n=1}^N |x_n \overline{y_n}| \leq (\sum_{n=1}^{\infty} |x_n|^2)^{1/2} (\sum_{n=1}^{\infty} |y_n|^2)^{1/2}$

PROOF:

$$\begin{aligned} \sum_{n=1}^N |x_n \overline{y_n}| &= \sum_{n=1}^N |x_n| |y_n| \\ &\leq \left( \sum_{n=1}^N |x_n|^2 \right)^{1/2} \left( \sum_{n=1}^N |y_n|^2 \right)^{1/2} \quad (\text{Cauchy-Schwarz}) \\ &\leq \left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} |y_n|^2 \right)^{1/2} \end{aligned}$$

$\langle 2 \rangle 3$ .  $\sum_{n=1}^N x_n \overline{y_n}$  is absolutely convergent.

$\langle 1 \rangle 2$ .  $\forall (x_n), (y_n) \in l^2. \langle (x_n), (y_n) \rangle = \overline{\langle (y_n), (x_n) \rangle}$

$\langle 1 \rangle 3$ .  $\forall \lambda \in \mathbb{C}. \forall (x_n), (y_n) \in l^2. \langle \lambda (x_n), (y_n) \rangle = \lambda \langle (x_n), (y_n) \rangle$

$\langle 1 \rangle 4$ .  $\forall (x_n), (y_n), (z_n) \in l^2. \langle (x_n) + (y_n), (z_n) \rangle = \langle (x_n), (z_n) \rangle + \langle (y_n), (z_n) \rangle$

$\langle 1 \rangle 5$ .  $\forall (x_n) \in l^2. \langle (x_n), (x_n) \rangle = 0 \Rightarrow (x_n) = 0$

□

**Proposition 21.6.4.** *The space  $C[0, 1]$  of all continuous functions  $[0, 1] \rightarrow \mathbb{C}$  is an inner product space under*

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt .$$

**Proposition 21.6.5.** *The space  $\mathbb{C}^{mn}$  of all  $m \times n$  complex matrices is an inner product space under*

$$\langle A, B \rangle = \text{tr}(B^* A)$$

where  $B^*$  is the conjugate transpose of  $B$ .

**Theorem 21.6.6.** *Let  $V$  be an inner product space. Let  $x, y, z \in V$  and  $\lambda \in \mathbb{C}$ . Then:*

1.  $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
2.  $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$
3.  $\langle x, 0 \rangle = \langle 0, x \rangle = 0$
4. If  $\forall w \in V. \langle x, w \rangle = \langle y, w \rangle$  then  $x = y$
5.  $\langle x, x \rangle$  is a non-negative real.

PROOF: For part 4, take  $w = x - y$ .  $\square$

**Theorem 21.6.7** (Cauchy-Schwarz Inequality). *Let  $V$  be an inner product space. Let  $x, y \in V$ . Then*

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} .$$

*Equality holds if and only if  $x$  and  $y$  are linearly dependent.*

PROOF:

$\langle 1 \rangle 1$ . If  $x$  and  $y$  are linearly dependent then  $|\langle x, y \rangle| = \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$ .

PROOF: If  $y = \lambda x$  then both sides are equal to  $|\lambda| \langle x, x \rangle$ .

$\langle 1 \rangle 2$ . If  $x$  and  $y$  are linearly independent then  $|\langle x, y \rangle| < \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$ .

$\langle 2 \rangle 1$ . For any  $\lambda \in \mathbb{C}$  with  $x + \lambda y \neq 0$  we have

$$\langle x, x \rangle + 2\Re(\overline{\lambda} \langle x, y \rangle) + |\lambda|^2 \langle y, y \rangle > 0 .$$

PROOF:

$$\begin{aligned} 0 &< \langle x + \lambda y, x + \lambda y \rangle \\ &= \langle x, x \rangle + \overline{\lambda} \langle x, y \rangle + \lambda \langle y, x \rangle + |\lambda|^2 \langle y, y \rangle \\ &= \langle x, x \rangle + 2\Re(\overline{\lambda} \langle x, y \rangle) + |\lambda|^2 \langle y, y \rangle \end{aligned}$$

$\langle 2 \rangle 2$ . LET:  $u = |\langle x, y \rangle| / \langle x, y \rangle$  or  $u = 1$  if  $\langle x, y \rangle = 0$

$\langle 2 \rangle 3$ . For any  $t \in \mathbb{R}$ ,

$$\langle x, x \rangle + 2|\langle x, y \rangle|t + \langle y, y \rangle t^2 > 0$$

PROOF: Take  $\lambda = tu$  in  $\langle 2 \rangle 1$

$\langle 2 \rangle 4$ .

$$4|\langle x, y \rangle|^2 - 4\langle x, x \rangle \langle y, y \rangle < 0$$

PROOF: The quadratic  $\langle 2 \rangle 3$  must have negative discriminant.  
 $\langle 2 \rangle 5$ .

$$|\langle x, y \rangle| < \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$$

□

**Theorem 21.6.8.** *Every inner product space is a normed space under  $\|x\| = \langle x, x \rangle^{1/2}$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $\|x\| = 0$  then  $x = 0$

$\langle 1 \rangle 2$ .  $\|\lambda x\| = |\lambda| \|x\|$

$\langle 1 \rangle 3$ .  $\|x + y\| \leq \|x\| + \|y\|$

PROOF:

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 2\Re\langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \\ &< \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 && \text{(Cauchy-Schwarz)} \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

□

**Theorem 21.6.9** (Parallelogram Law). *Let  $V$  be an inner product space. Let  $x, y \in V$ . Then*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

PROOF:

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 \\ \|x - y\|^2 &= \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2 \end{aligned}$$

**Theorem 21.6.10** (Polarization Identity). *Let  $V$  be an inner product space. Let  $x, y \in V$ . Then*

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 .$$

PROOF: Straightforward calculation. □