

Mathematics

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Part I

Category Theory

Chapter 1

Foundations

This is a placeholder — I am not sure what foundation I want to use for this project yet. I will try to work in a way which is foundation-independent. What I do could be formalized in ZFC, ETCS, or some other system. I will assume the usual set theoretic constructions as needed. Sets will be defined up to bijection only.

Chapter 2

Categories

Definition 2.1 (Category). A *category* \mathcal{C} consists of:

- A class $|\mathcal{C}|$ of *objects*. We write $A \in \mathcal{C}$ for $A \in |\mathcal{C}|$.
- For any objects A, B , a set $\mathcal{C}[A, B]$ of *morphisms* from A to B . We write $f : A \rightarrow B$ for $f \in \mathcal{C}[A, B]$.
- For any object A , a morphism $\text{id}_A : A \rightarrow A$, the *identity* morphism on A .
- For any morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, a morphism $g \circ f : A \rightarrow C$, the *composite* of f and g .

such that:

Associativity Given $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Left Unit Law For any morphism $f : A \rightarrow B$, we have $\text{id}_B \circ f = f$.

Right Unit Law For any morphism $f : A \rightarrow B$, we have $f \circ \text{id}_A = f$.

Proposition 2.2. *The identity morphism on an object is unique.*

PROOF: If i and j are identity morphisms on A then $i = i \circ j = j$. \square

Example 2.3 (Category of Sets). The *category of sets* **Set** has objects all sets and morphisms all functions.

Definition 2.4 (Endomorphism). In a category \mathcal{C} , an *endomorphism* on an object A is a morphism $A \rightarrow A$. We write $\text{End}_{\mathcal{C}}(A)$ for $\mathcal{C}[A, A]$.

Definition 2.5 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with the same objects as \mathcal{C} and

$$\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$$

2.1 Preorders

Definition 2.6 (Preorder). A *preorder* on a set A is a relation \leq on A that is reflexive and transitive.

A *preordered set* is a pair (A, \leq) such that \leq is a preorder on A . We usually write A for the preordered set (A, \leq) .

We identify any preordered set A with the category whose objects are the elements of A , with one morphism $a \rightarrow b$ iff $a \leq b$, and no morphism $a \rightarrow b$ otherwise.

Example 2.7. For any ordinal α , let α be the preorder $\{\beta : \beta < \alpha\}$ under \leq .

Definition 2.8 (Discrete Preorder). We identify any set A with the *discrete* preorder $(A, =)$.

2.2 Monomorphisms and Epimorphisms

Definition 2.9 (Monomorphism). In a category, let $f : A \rightarrow B$. Then f is a *monomorphism* or *monic* iff, for every object X and morphism $x, y : X \rightarrow A$, if $fx = fy$ then $x = y$.

Definition 2.10 (Epimorphism). In a category, let $f : A \rightarrow B$. Then f is a *epimorphism* or *epi* iff, for every object X and morphism $x, y : B \rightarrow X$, if $xf = yf$ then $x = y$.

Proposition 2.11. *The composite of two monomorphisms is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$ be monic.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$

$\langle 1 \rangle 3$. ASSUME: $g \circ f \circ x = g \circ f \circ y$

$\langle 1 \rangle 4$. $f \circ x = f \circ y$

$\langle 1 \rangle 5$. $x = y$

□

Proposition 2.12. *The composite of two epimorphisms is epi.*

PROOF: Dual. □

Proposition 2.13. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is monic then f is monic.*

PROOF: If $f \circ x = f \circ y$ then $gfx = gfy$ and so $x = y$. □

Proposition 2.14. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is epi then g is epi.*

PROOF: Dual. □

Proposition 2.15. *A function is a monomorphism in **Set** iff it is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is monic then f is injective.

$\langle 2 \rangle 1$. ASSUME: f is monic.

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 4$. LET: $\bar{x}, \bar{y} : 1 \rightarrow A$ be the functions such that $\bar{x}(*) = x$ and $\bar{y}(*) = y$

$\langle 2 \rangle 5$. $f \circ \bar{x} = f \circ \bar{y}$

$\langle 2 \rangle 6$. $\bar{x} = \bar{y}$

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. $x = y$

$\langle 1 \rangle 3$. If f is injective then f is monic.

$\langle 2 \rangle 1$. ASSUME: f is injective.

$\langle 2 \rangle 2$. LET: X be a set and $x, y : X \rightarrow A$.

$\langle 2 \rangle 3$. ASSUME: $f \circ x = f \circ y$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $t \in X$

PROVE: $x(t) = y(t)$

$\langle 2 \rangle 5$. $f(x(t)) = f(y(t))$

$\langle 2 \rangle 6$. $x(t) = y(t)$

PROOF: By $\langle 2 \rangle 1$.

□

Proposition 2.16. *A function is an epimorphism in **Set** iff it is surjective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is an epimorphism then f is surjective.

$\langle 2 \rangle 1$. ASSUME: f is an epimorphism.

$\langle 2 \rangle 2$. LET: $b \in B$

$\langle 2 \rangle 3$. LET: $x, y : B \rightarrow 2$ be defined by $x(b) = 1$ and $x(t) = 0$ for all other $t \in B$, $y(t) = 0$ for all $t \in B$.

$\langle 2 \rangle 4$. $x \neq y$

$\langle 2 \rangle 5$. $x \circ f \neq y \circ f$

$\langle 2 \rangle 6$. There exists $a \in A$ such that $f(a) = b$.

$\langle 1 \rangle 3$. If f is surjective then f is an epimorphism.

$\langle 2 \rangle 1$. ASSUME: f is surjective.

$\langle 2 \rangle 2$. LET: $x, y : B \rightarrow X$

$\langle 2 \rangle 3$. ASSUME: $x \circ f = y \circ f$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $b \in B$

PROVE: $x(b) = y(b)$

$\langle 2 \rangle 5$. PICK $a \in A$ such that $f(a) = b$

$\langle 2 \rangle 6$. $x(f(a)) = y(f(a))$

$\langle 2 \rangle 7$. $x(b) = y(b)$

□

Proposition 2.17. *In a preorder, every morphism is monic and epi.*

PROOF: Immediate from definitions. \square

2.3 Sections and Retractions

Definition 2.18 (Section, Retraction). In a category, let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $r \circ s = \text{id}_B$.

Proposition 2.19. *Every identity morphism is a section and retraction of itself.*

PROOF: Immediate from definitions. \square

Proposition 2.20. *Let $r, r' : A \rightarrow B$ and $s : B \rightarrow A$. If r is a retraction of s and r' is a section of s then $r = r'$.*

PROOF:

$$\begin{aligned} r &= r \circ \text{id}_A \\ &= r \circ s \circ r' \\ &= \text{id}_B \circ r' \\ &= r' \end{aligned} \quad \square$$

Proposition 2.21. *Let $r_1 : A \rightarrow B$, $r_2 : B \rightarrow C$, $s_1 : B \rightarrow A$ and $s_2 : C \rightarrow B$. If r_1 is a retraction of s_1 and r_2 is a retraction of s_2 then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= \text{id}_C \end{aligned} \quad \square$$

Proposition 2.22. *Every section is monic.*

PROOF:

- $\langle 1 \rangle 1.$ LET: $s : A \rightarrow B$ be a section of $r : B \rightarrow A$.
- $\langle 1 \rangle 2.$ LET: $x, y : X \rightarrow A$ satisfy $sx = sy$.
- $\langle 1 \rangle 3.$ $rsx = rsy$
- $\langle 1 \rangle 4.$ $x = y$

\square

Proposition 2.23. *Every retraction is epi.*

PROOF: Dual. \square

Proposition 2.24. *In **Set**, every epimorphism has a retraction.*

PROOF: By the Axiom of Choice. \square

Example 2.25. It is not true in general that every monomorphism in any category has a section. nor that every epimorphism in any category has a retraction.

In the category **2**, the morphism $0 \leq 1$ is monic and epi but has no retraction or section.

2.4 Isomorphisms

Definition 2.26 (Isomorphism). In a category \mathcal{C} , a morphism $f : A \rightarrow B$ is an *isomorphism*, denoted $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$, the *inverse* of f , such that $f^{-1} \circ f = \text{id}_A$ and $f \circ f^{-1} = \text{id}_B$.

An *automorphism* on an object A is an isomorphism between A and itself. We write $\text{Aut}_{\mathcal{C}}(A)$ for the set of all automorphisms on A .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 2.27. *The inverse of an isomorphism is unique.*

PROOF: Proposition 2.20. \square

Proposition 2.28. *For any object A we have $\text{id}_A : A \cong A$ and $\text{id}_A^{-1} = \text{id}_A$.*

PROOF: Since $\text{id}_A \circ \text{id}_A = \text{id}_A$ by the Unit Laws. \square

Proposition 2.29. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Immediate from definitions. \square

Proposition 2.30. *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF: From Proposition 2.21. \square

Definition 2.31 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

2.5 Initial and Terminal Objects

Definition 2.32 (Initial Object). An object I in a category is *initial* iff, for any object X , there is exactly one morphism $I \rightarrow X$.

Example 2.33. The empty set is the initial object in **Set**.

Definition 2.34 (Terminal Object). An object T in a category is *terminal* iff, for any object X , there is exactly one morphism $X \rightarrow T$.

Example 2.35. Every singleton is terminal in **Set**.

Proposition 2.36. *If I and J are initial in a category, then there exists a unique isomorphism $I \cong J$.*

PROOF:

- $\langle 1 \rangle 1$. LET: i be the unique morphism $I \rightarrow J$.
- $\langle 1 \rangle 2$. LET: i^{-1} be the unique morphism $J \rightarrow I$.
- $\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_J$

PROOF: Since there is only one morphism $J \rightarrow I$.

- $\langle 1 \rangle 4$. $i^{-1} \circ i = \text{id}_I$

PROOF: Since there is only one morphism $I \rightarrow I$.
 \square

Proposition 2.37. *If S and T are terminal in a category, then there exists a unique isomorphism $S \cong T$.*

PROOF: Dual. \square

Chapter 3

Functors

Definition 3.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for any morphism $f : A \rightarrow B : \mathcal{C}$, a morphism $Ff : FA \rightarrow FB : \mathcal{D}$

such that:

- $F\text{id}_A = \text{id}_{FA}$
- $F(g \circ f) = Fg \circ Ff$

Definition 3.2 (Identity Functor). For any category \mathcal{C} , the *identity functor* $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is defined by

$$\begin{aligned} 1_{\mathcal{C}}A &= A \\ 1_{\mathcal{C}}f &= f \end{aligned}$$

Definition 3.3 (Constant Functor). Given categories \mathcal{C} , \mathcal{D} and an object $D \in \mathcal{D}$, the *constant functor* $K^{\mathcal{C}}D : \mathcal{C} \rightarrow \mathcal{D}$ is the functor defined by

$$\begin{aligned} K^{\mathcal{C}}DC &= D \\ K^{\mathcal{C}}Df &= \text{id}_D \end{aligned}$$

3.1 Comma Categories

Definition 3.4 (Comma Category). Let $F : \mathcal{C} \rightarrow \mathcal{E}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors. The *comma category* $F \downarrow G$ is the category with:

- objects all pairs (C, D, f) where $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f : FC \rightarrow GD : \mathcal{E}$

- morphisms $(u, v) : (C, D, f) \rightarrow (C', D', g)$ all pairs $u : C \rightarrow C' : \mathcal{C}$ and $v : D \rightarrow D' : \mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ \downarrow Fu & & \downarrow Gv \\ FC' & \xrightarrow{g} & GD' \end{array}$$

Definition 3.5 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category* over A , denoted \mathcal{C}/A , is the comma category $1_{\mathcal{C}} \downarrow K^1 A$.

Definition 3.6 (Coslice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *coslice category* over A , denoted $\mathcal{C} \backslash A$, is the comma category $K^1 A \downarrow 1_{\mathcal{C}}$.

Definition 3.7 (Pointed Sets). The *category of pointed sets* \mathbf{Set}_* is the coslice category $\mathbf{Set} \backslash 1$.

Part II

Group Theory

Chapter 4

Semigroups

Definition 4.1 (Semigroup). A *semigroup* consists of a set S and an associative binary operation \cdot on S .

Chapter 5

Monoids

Definition 5.1 (Monoid). A *monoid* consists of a semigroup M such that there exists $e \in M$, the *unit*, such that, for all $x \in M$, we have $xe = ex = x$.

We identify a monoid M with the category with one object whose morphisms are the elements of M , with composition given by \cdot .

Proposition 5.2. *The identity in a group is unique.*

PROOF: Proposition 2.2.

Chapter 6

Groups

Definition 6.1 (Group). Let \mathcal{C} be a category with finite products. A *group (object)* in \mathcal{C} consists of an object $G \in \mathcal{C}$ and morphisms

$$m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G$$

such that the following diagrams commute.

$$\begin{array}{ccc} G^3 & \xrightarrow{m \times \text{id}_G} & G^2 \\ \downarrow \text{id}_G \times m & & \downarrow m \\ G^2 & \xrightarrow{m} & G \end{array}$$

$$\begin{array}{ccc} 1 \times G & \xrightarrow{e \times \text{id}_G} & G^2 \\ & \searrow \cong & \downarrow m \\ & & G \end{array} \quad \begin{array}{ccc} G \times 1 & \xrightarrow{\text{id}_G \times e} & G^2 \\ & \searrow \cong & \downarrow m \\ & & G \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{\Delta} & G^2 & \xrightarrow{\text{id}_G \times i} & G^2 \\ \downarrow & & & & \downarrow m \\ 1 & \xrightarrow{e} & G & & G \end{array} \quad \begin{array}{ccccc} G & \xrightarrow{\Delta} & G^2 & \xrightarrow{i \times \text{id}_G} & G^2 \\ \downarrow & & & & \downarrow m \\ 1 & \xrightarrow{e} & G & & G \end{array}$$

Definition 6.2 (Group). We write just 'group' for 'group in **Set**'. Thus, a *group* G consists of a set G and a binary operation $\cdot : G^2 \rightarrow G$ such that \cdot is associative, and there exists $e \in G$, the *identity* element of the group, such that:

- For all $x \in G$ we have $xe = ex = x$
- For all $x \in G$, there exists $x^{-1} \in G$, the *inverse* of x , such that $xx^{-1} = x^{-1}x = e$.

The *order* of a group G , denoted $|G|$, is the number of elements in G if G is finite; otherwise we write $|G| = \infty$.

Proposition 6.3. *The inverse of an element is unique.*

PROOF: If i and j are inverses of x then $i = ixj = j$. \square

Example 6.4. • The *trivial* group is $\{e\}$ under $ee = e$.

- \mathbb{Z} is a group under addition
 - \mathbb{Q} is a group under addition
 - $\mathbb{Q} - \{0\}$ is a group under multiplication
 - \mathbb{R} is a group under addition
 - $\mathbb{R} - \{0\}$ is a group under multiplication
 - \mathbb{C} is a group under addition
 - $\mathbb{C} - \{0\}$ is a group under multiplication
 - $\{-1, 1\}$ is a group under multiplication
 - For any category \mathcal{C} and object $A \in \mathcal{C}$, we have $\text{Aut}_{\mathcal{C}}(A)$ is a group under $gf = f \circ g$.
- For A a set, we call $S_A = \text{Aut}_{\text{Set}}(A)$ the *symmetric group* or *group of permutations* of A .
- For $n \geq 3$, the *dihedral group* D_{2n} consists of the set of rigid motions that map the regular n -gon onto itself under composition.
 - Let $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ under matrix multiplication.
 - The quaternionic group Q_8 is the group

$$\{1, -1, i, -i, j, -j, k, -k\}$$

with multiplication table

1	-1	i	-i	j	-j	k	-k
-1	1	-i	i	-j	j	-k	k
i	-i	-1	1	k	-k	-j	j
-i	i	1	-1	-k	k	j	-j
j	-j	-k	k	-1	1	i	-i
-j	j	k	-k	1	-1	-i	i
k	-k	j	-j	-i	i	-1	1
-k	k	-j	j	i	-i	1	-1

Example 6.5. • The only group of order 1 is the trivial group.

- The only group of order 2 is \mathbb{Z}_2 .

- The only group of order 3 is \mathbb{Z}_3 .
- There are exactly two groups of order 4: \mathbb{Z}_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ under $(a, b)(c, d) = (ac, bd)$.

Proposition 6.6 (Cancellation). *Let G be a group. Let $a, g, h \in G$. If $ag = ah$ or $ga = ha$ then $g = h$.*

PROOF: If $ag = ah$ then $g = a^{-1}ag = a^{-1}ah = h$. Similarly if $ga = ha$. \square

Proposition 6.7. *Let G be a group and $g, h \in G$. Then $(gh)^{-1} = h^{-1}g^{-1}$.*

PROOF: Since $ghh^{-1}g^{-1} = e$. \square

Definition 6.8. Let G be a group. Let $g \in G$. We define $g^n \in G$ for all $n \in \mathbb{Z}$ as follows:

$$\begin{aligned} g^0 &= e \\ g^{n+1} &= g^n g & (n \geq 0) \\ g^{-n} &= (g^{-1})^n & (n > 0) \end{aligned}$$

Proposition 6.9. *Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then*

$$g^{m+n} = g^m g^n .$$

PROOF:

$\langle 1 \rangle 1$. For all $k \in \mathbb{Z}$ we have $g^{k+1} = g^k g$

$\langle 2 \rangle 1$. For all $k \geq 0$ we have $g^{k+1} = g^k g$

PROOF: Immediate from definition.

$\langle 2 \rangle 2$. $g^{-1+1} = g^{-1} g$

PROOF: Both are equal to e .

$\langle 2 \rangle 3$. For all $k > 1$ we have $g^{-k+1} = g^{-k} g$

PROOF:

$$\begin{aligned} g^{-k+1} &= (g^{-1})^{k-1} \\ &= (g^{-1})^{k-1} g^{-1} g \\ &= (g^{-1})^k g \\ &= g^{-k} g \end{aligned}$$

$\langle 1 \rangle 2$. For all $k \in \mathbb{Z}$ we have $g^{k-1} = g^k g^{-1}$

PROOF: Substitute $k = k - 1$ above and multiply by g^{-1} .

$\langle 1 \rangle 3$. $g^{m+0} = g^m g^0$

PROOF: Since $g^m g^0 = g^m e = g^m$.

$\langle 1 \rangle 4$. If $g^{m+n} = g^m g^n$ then $g^{m+n+1} = g^m g^{n+1}$

PROOF:

$$\begin{aligned} g^{m+n+1} &= g^{m+n} g & (\langle 1 \rangle 1) \\ &= g^m g^n g \\ &= g^m g^{n+1} & (\langle 1 \rangle 1) \end{aligned}$$

$\langle 1 \rangle 5$. If $g^{m+n} = g^m g^n$ then $g^{m+n-1} = g^m g^{n-1}$

PROOF:

$$g^{m+n-1}g = g^{m+n} \quad (\langle 1 \rangle 1)$$

$$= g^m g^n$$

$$\therefore g^{m+n-1} = g^m g^n g^{-1}$$

$$= g^m g^{n-1} \quad (\langle 1 \rangle 2)$$

□

Proposition 6.10. Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then

$$(g^m)^n = g^{mn} .$$

PROOF:

$\langle 1 \rangle 1$. $(g^m)^0 = g^0$

PROOF: Both sides are equal to e .

$\langle 1 \rangle 2$. If $(g^m)^n = g^{mn}$ then $(g^m)^{n+1} = g^{m(n+1)}$.

PROOF:

$$(g^m)^{n+1} = (g^m)^n g^m \quad (\text{Proposition 6.9})$$

$$= g^{mn} g^m$$

$$= g^{mn+m}$$

(Proposition 6.9)

$\langle 1 \rangle 3$. If $(g^m)^n = g^{mn}$ then $(g^m)^{n-1} = g^{m(n-1)}$.

PROOF:

$$(g^m)^n = g^{mn}$$

$$\therefore (g^m)^{n-1} g^m = g^{mn-m} g^m \quad (\text{Proposition 6.9})$$

$$\therefore (g^m)^{n-1} = g^{mn-m} \quad (\text{Cancellation})$$

□

Definition 6.11 (Commute). Let G be a group and $g, h \in G$. We say g and h *commute* iff $gh = hg$.

Definition 6.12. Let G be a group. Given $g \in G$ and $A \subseteq G$, we define

$$gA = \{ga : a \in A\}, \quad Ag = \{ag : a \in A\} .$$

Given sets $A, B \subseteq G$, we define

$$AB = \{ab : a \in A, b \in B\} .$$

6.1 Order of an Element

Definition 6.13 (Order). Let G be a group. Let $g \in G$. Then g has *finite order* iff there exists a positive integer n such that $g^n = e$. In this case, the *order* of g , denoted $|g|$, is the least positive integer n such that $g^n = e$.

If g does not have finite order, we write $|g| = \infty$.

Proposition 6.14. *Let G be a group. Let $g \in G$ and n be a positive integer. If $g^n = e$ then $|g| \mid n$.*

PROOF:

$\langle 1 \rangle 1$. LET: $n = q|g| + d$ where $0 \leq d < |g|$

PROOF: Division Algorithm.

$\langle 1 \rangle 2$. $g^d = e$

PROOF:

$$\begin{aligned} e &= g^n \\ &= g^{q|g|+d} \\ &= (g^{|g|})^q g^d && \text{(Propositions 6.9, 6.10)} \\ &= e^q g^d \\ &= g^d \end{aligned}$$

$\langle 1 \rangle 3$. $d = 0$

PROOF: By minimality of $|g|$.

$\langle 1 \rangle 4$. $n = q|g|$

□

Corollary 6.14.1. *Let G be a group. Let $g \in G$ have finite order and $n \in \mathbb{Z}$. Then $g^n = e$ if and only if $|g| \mid n$.*

Proposition 6.15. *Let G be a group and $g \in G$. Then $|g| \leq |G|$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: w.l.o.g. G is finite.

$\langle 1 \rangle 2$. PICK i, j with $0 \leq i < j \leq |G|$ such that $g^i = g^j$.

PROOF: Otherwise $g^0, g^1, \dots, g^{|G|}$ would be $|G| + 1$ distinct elements of G .

$\langle 1 \rangle 3$. $g^{j-i} = e$

$\langle 1 \rangle 4$. g has finite order and $|g| \leq |G|$

PROOF: Since $|g| \leq j - i \leq j \leq |G|$.

□

Proposition 6.16. *Let G be a group. Let $g \in G$ have finite order. Let $m \in \mathbb{N}$. Then*

$$|g^m| = \frac{\text{lcm}(m, |g|)}{m} = \frac{|g|}{\text{gcd}(m, |g|)}$$

PROOF: Since for any integer d we have

$$g^{md} = e \Leftrightarrow |g| \mid md \quad (\text{Corollary 6.14.1})$$

$$\Leftrightarrow \text{lcm}(m, |g|) \mid md$$

$$\Leftrightarrow \frac{\text{lcm}(m, |g|)}{m} \mid d$$

□

and so $|g^m| = \frac{\text{lcm}(m, |g|)}{m}$ by Corollary 6.14.1. □

Corollary 6.16.1. *If g has odd order then $|g^2| = |g|$.*

Proposition 6.17. *Let G be a group. Let $g, h \in G$ have finite order. Assume $gh = hg$. Then $|gh|$ has finite order and*

$$|gh| \mid \text{lcm}(|g|, |h|)$$

PROOF: Since $(gh)^{\text{lcm}(|g|, |h|)} = g^{\text{lcm}(|g|, |h|)} h^{\text{lcm}(|g|, |h|)} = e$. \square

Example 6.18. This example shows that we cannot remove the hypothesis that $gh = hg$.

In $\text{GL}_2(\mathbb{R})$, take

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then $|g| = 4$, $|h| = 3$ and $|gh| = \infty$.

Proposition 6.19. *Let G be a group and $g, h \in G$ have finite order. If $gh = hg$ and $\gcd(|g|, |h|) = 1$ then $|gh| = |g||h|$.*

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } N = |gh|$$

$$\langle 1 \rangle 2. g^N = (h^{-1})^N$$

$$\langle 1 \rangle 3. g^{N|g|} = e$$

$$\langle 1 \rangle 4. |g^N| \mid |g|$$

$$\langle 1 \rangle 5. h^{-N|h|} = e$$

$$\langle 1 \rangle 6. |g^N| \mid |h|$$

$$\langle 1 \rangle 7. |g^N| = 1$$

PROOF: Since $\gcd(|g|, |h|) = 1$.

$$\langle 1 \rangle 8. g^N = e$$

$$\langle 1 \rangle 9. |g| \mid N$$

$$\langle 1 \rangle 10. h^{-N} = e$$

$$\langle 1 \rangle 11. |h| \mid N$$

$$\langle 1 \rangle 12. N = |g||h|$$

PROOF: Using Proposition 6.17.

\square

Proposition 6.20. *Let G be a finite group. Assume there is exactly one element $f \in G$ of order 2. Then the product of all the elements of G is f .*

PROOF: Let the elements of G be g_1, g_2, \dots, g_n . Apart from e and f , every element and its inverse are distinct elements of the list. Hence the product of the list is $ef = f$. \square

Proposition 6.21. *Let G be a finite group of order n . Let m be the number of elements of G of order 2. Then $n - m$ is odd.*

PROOF: In the list of all elements that are not of order 2, every element and its inverse are distinct except for e . Hence the list has odd length. \square

Corollary 6.21.1. *If a finite group has even order, then it contains an element of order 2.*

Proposition 6.22. *Let G be a group and $a, g \in G$. Then $|aga^{-1}| = |g|$.*

PROOF: Since

$$\begin{aligned} (aga^{-1})^n = e &\Leftrightarrow ag^na^{-1} = e \\ &\Leftrightarrow g^n = e \end{aligned} \quad \square$$

Proposition 6.23. *Let G be a group and $g, h \in G$. Then $|gh| = |hg|$.*

PROOF: Since $|gh| = |ghgg^{-1}| = |hg|$. \square

Proposition 6.24. *Let G be a group of order n . Let k be relatively prime to n . Then every element in G has the form x^k for some x .*

$\langle 1 \rangle 1$. PICK integers a and b such that $an + bk = 1$.

$\langle 1 \rangle 2$. LET: $g \in G$

$\langle 1 \rangle 3$. $g = (g^b)^k$

PROOF:

$$\begin{aligned} g &= g \cdot (g^n)^{-a} & (g^n = e) \\ &= g^{1-an} \\ &= g^{bk} \end{aligned}$$

\square

6.2 Generators

Definition 6.25 (Generator). Let G be a group and $a \in G$. We say a *generates* the group iff, for all $x \in G$, there exists an integer n such that $x^n = a$.

Example 6.26. $\text{SL}_2(\mathbb{Z})$ is generated by

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

PROOF:

$\langle 1 \rangle 1$. LET: $H = \langle s, t \rangle$

$\langle 1 \rangle 2$. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in H$.

PROOF: It is t^q .

$\langle 1 \rangle 3$. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \in H$.

PROOF:

$$\begin{aligned} st^{-q}s^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \end{aligned}$$

⟨1⟩4.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & qa+b \\ c & qc+d \end{pmatrix}$$

⟨1⟩5.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} a+qb & b \\ c+qd & d \end{pmatrix}$$

⟨1⟩6. For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, if c and d are both nonzero, then there exists $N \in H$ such that the bottom row of MN has one entry the same as M and one entry with smaller absolute value.

PROOF: From ⟨1⟩4 and ⟨1⟩5 taking $q = -1$.

⟨1⟩7. For any $M \in \mathrm{SL}_2(\mathbb{Z})$, there exists $N \in H$ such that MN has a zero on the bottom row.

PROOF: Apply ⟨1⟩6 repeatedly.

⟨1⟩8. Any matrix in $\mathrm{SL}_2(\mathbb{Z})$ with a zero on the bottom row is in H .

⟨2⟩1. $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H$

PROOF: ⟨1⟩2

⟨2⟩2. $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \in H$

PROOF: It is $s^2 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ since $s^2 = -I$.

⟨2⟩3. $\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} \in H$

PROOF: It is $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} s$.

⟨2⟩4. $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \in H$

PROOF: It is $s^2 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s$.

⟨1⟩9. Every matrix in $\mathrm{SL}_2(\mathbb{Z})$ is in H .

□

Chapter 7

Group Homomorphisms

Definition 7.1 (Homomorphism). Let G and H be groups. A (group) homomorphism $\phi : G \rightarrow H$ is a function such that, for all $x, y \in G$,

$$\phi(xy) = \phi(x)\phi(y) \ .$$

Proposition 7.2. Let G and H be groups with identities e_G and e_H . Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\phi(e_G) = e_H$.

PROOF: Since $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$ and so $\phi(e_G) = e_H$ by Cancellation. \square

Proposition 7.3. Let $\phi : G \rightarrow H$ be a group homomorphism. For all $x \in G$ we have $\phi(x^{-1}) = \phi(x)^{-1}$.

PROOF: Since $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_H$. \square

Proposition 7.4. Let G, H and K be groups. If $\phi : G \rightarrow H$ and $\psi : H \rightarrow K$ are homomorphisms then $\psi \circ \phi : G \rightarrow K$ is a homomorphism.

PROOF: For $x, y \in G$ we have

$$\psi(\phi(xy)) = \psi(\phi(x)\phi(y)) = \psi(\phi(x))\psi(\phi(y)) \ .$$

Proposition 7.5. Let G be a group. Then $\text{id}_G : G \rightarrow G$ is a group homomorphism.

PROOF: For $x, y \in G$ we have $\text{id}_G(xy) = xy = \text{id}_G(x)\text{id}_G(y)$. \square

Proposition 7.6. Let $\phi : G \rightarrow H$ be a group homomorphism. Let $g \in G$ have finite order. Then $|\phi(g)|$ divides $|g|$.

PROOF: Since $\phi(g)^{|g|} = \phi(g^{|g|}) = e$. \square

Definition 7.7 (Category of Groups). Let **Grp** be the category of groups and group homomorphisms.

Example 7.8. There are 49487365402 groups of order 1024 up to isomorphism.

Proposition 7.9. *A group homomorphism $\phi : G \rightarrow H$ is an isomorphism in **Grp** if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: ϕ is bijective.

PROVE: ϕ^{-1} is a group homomorphism.

$\langle 1 \rangle 2$. LET: $h, h' \in H$

$\langle 1 \rangle 3$. $\phi(\phi^{-1}(hh')) = \phi(\phi^{-1}(h)\phi^{-1}(h'))$

PROOF: Both are equal to hh' .

$\langle 1 \rangle 4$. $\phi^{-1}(hh') = \phi^{-1}(h)\phi^{-1}(h')$

□

Corollary 7.9.1.

$$D_6 \cong C_3$$

PROOF: The canonical homomorphism $D_6 \rightarrow C_3$ is bijective. □

Corollary 7.9.2.

$$(\mathbb{R}, +) \cong (\{x \in \mathbb{R} : x > 0\}, \cdot)$$

PROOF: The function that maps x to e^x is a bijective homomorphism. □

Proposition 7.10. *The trivial group is the zero object in **Grp**.*

PROOF: For any group G , the unique function $G \rightarrow \{e\}$ is a group homomorphism, and the only group homomorphism $\{e\} \rightarrow G$ maps e to e_G . □

Proposition 7.11. *For any groups G and H , the set $G \times H$ under $(g, h)(g', h') = (gg', hh')$ is the product of G and H in **Grp**.*

PROOF:

$\langle 1 \rangle 1$. $G \times H$ is a group.

$\langle 2 \rangle 1$. The multiplication is associative.

PROOF: Since $(g_1, h_1)((g_2, h_2)(g_3, h_3)) = ((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1g_2g_3, h_1h_2h_3)$.

$\langle 2 \rangle 2$. (e_G, e_H) is the identity.

PROOF: Since $(g, h)(e_G, e_H) = (e_G, e_H)(g, h) = (g, h)$.

$\langle 2 \rangle 3$. The inverse of (g, h) is (g^{-1}, h^{-1}) .

PROOF: Since $(g, h)(g^{-1}, h^{-1}) = (g^{-1}, h^{-1})(g, h) = (e_G, e_H)$.

$\langle 1 \rangle 2$. $\pi_1 : G \times H \rightarrow G$ is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$. $\pi_2 : G \times H \rightarrow H$ is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 4$. For any group homomorphism $\phi : K \rightarrow G$ and $\psi : K \rightarrow H$, the function $\langle \phi, \psi \rangle : K \rightarrow G \times H$ where $\langle \phi, \psi \rangle(k) = (\phi(k), \psi(k))$ is a group homomorphism.

PROOF:

$$\begin{aligned} \langle \phi, \psi \rangle(kk') &= (\phi(kk'), \psi(kk')) \\ &= (\phi(k)\phi(k'), \psi(k)\psi(k')) \\ &= (\phi(k), \psi(k))(\phi(k'), \psi(k')) \\ &= \langle \phi, \psi \rangle(k)\langle \phi, \psi \rangle(k') \end{aligned}$$

□

7.1 Subgroups

Definition 7.12 (Subgroup). Let (G, \cdot) and $(H, *)$ be groups such that H is a subset of G . Then H is a *subgroup* of G iff the inclusion $i : H \hookrightarrow G$ is a group homomorphism.

Proposition 7.13. *If $(H, *)$ is a subgroup of (G, \cdot) then $*$ is the restriction of \cdot to H .*

PROOF: Given $x, y \in H$ we have

$$x * y = i(x * y) = i(x) \cdot i(y) = x \cdot y . \quad \square$$

Example 7.14. For any group G we have $\{e\}$ is a subgroup of G .

Proposition 7.15. *Let G be a group. Let H be a subset of G . Then H is a subgroup of G iff H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$.*

PROOF:

$\langle 1 \rangle 1$. If H is a subgroup of G then H is nonempty.

PROOF: Since every group has an identity element and so is nonempty.

$\langle 1 \rangle 2$. If H is a subgroup of G then, for all $x, y \in H$, we have $xy^{-1} \in H$.

PROOF: Easy.

$\langle 1 \rangle 3$. If H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$, then H is a subgroup of G .

$\langle 2 \rangle 1$. ASSUME: H is nonempty.

$\langle 2 \rangle 2$. ASSUME: $\forall x, y \in H. xy^{-1} \in H$

$\langle 2 \rangle 3$. $e \in H$

PROOF: Pick $x \in H$. We have $e = xx^{-1} \in H$.

$\langle 2 \rangle 4$. $\forall x \in H. x^{-1} \in H$

PROOF: Given $x \in H$ we have $x^{-1} = ex^{-1} \in H$.

$\langle 2 \rangle 5$. H is closed under the restriction of \cdot

PROOF: Given $x, y \in H$ we have $xy = x(y^{-1})^{-1} \in H$.

$\langle 2 \rangle 6$. H is a group under the restriction of \cdot

PROOF: Associativity is inherited from G and the existence of an identity element and inverses follows from $\langle 2 \rangle 3$ and $\langle 2 \rangle 4$.

$\langle 2 \rangle 7$. The inclusion $H \hookrightarrow G$ is a group homomorphism.

PROOF: For $x, y \in H$ we have $i(xy) = i(x)i(y) = xy$.

\square

Corollary 7.15.1. *The intersection of a set of subgroups of G is a subgroup of G .*

Corollary 7.15.2. *Let $\phi : G \rightarrow H$ be a group homomorphism. Let K be a subgroup of H . Then $\phi^{-1}(K)$ is a subgroup of G .*

PROOF:

$\langle 1 \rangle 1$. $\phi^{-1}(K)$ is nonempty.

PROOF: Since $e \in \phi^{-1}(K)$.

$\langle 1 \rangle 2$. LET: $x, y \in \phi^{-1}(K)$

- $\langle 1 \rangle 3. \phi(x), \phi(y) \in K$
- $\langle 1 \rangle 4. \phi(x)\phi(y)^{-1} \in K$
- $\langle 1 \rangle 5. \phi(xy^{-1}) \in K$
- $\langle 1 \rangle 6. xy^{-1} \in \phi^{-1}(K)$

□

Corollary 7.15.3. *Let $\phi : G \rightarrow H$ be a group homomorphism. Let K be a subgroup of G . Then $\phi(K)$ is a subgroup of H .*

PROOF:

- $\langle 1 \rangle 1.$ LET: $x, y \in \phi(K)$
- $\langle 1 \rangle 2.$ PICK $a, b \in K$ such that $x = \phi(a)$ and $y = \phi(b)$
- $\langle 1 \rangle 3. xy^{-1} = \phi(ab^{-1})$
- $\langle 1 \rangle 4. xy^{-1} \in \phi(K)$

□

Proposition 7.16. *Let G be a subgroup of \mathbb{Z} . Then there exists $d \geq 0$ such that $G = d\mathbb{Z}$.*

PROOF:

- $\langle 1 \rangle 1.$ ASSUME: w.l.o.g. $G \neq \{0\}$

PROOF: Since $\{0\} = 0\mathbb{Z}$.

- $\langle 1 \rangle 2.$ LET: d be the least positive element of G .

PROVE: $G = d\mathbb{Z}$

PROOF: If $n \in G$ then $-n \in G$ so G must contain a positive element.

- $\langle 1 \rangle 3. G \subseteq d\mathbb{Z}$

- $\langle 2 \rangle 1.$ LET: $n \in G$

- $\langle 2 \rangle 2.$ LET: q and r be the integers such that $n = qd + r$ and $0 \leq r < d$.

- $\langle 2 \rangle 3. r \in G$

PROOF: Since $r = n - qd$.

- $\langle 2 \rangle 4. r = 0$

PROOF: By minimality of d .

- $\langle 2 \rangle 5. n = qd \in d\mathbb{Z}$

- $\langle 1 \rangle 4. d\mathbb{Z} \subseteq G$

□

7.2 Kernel

Definition 7.17 (Kernel). Let $\phi : G \rightarrow H$ be a group homomorphism. The *kernel* of ϕ is

$$\ker \phi = \{g \in G : \phi(g) = e\} .$$

Proposition 7.18. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\ker \phi$ is a subgroup of G .*

PROOF: Corollary 7.15.2. □

Proposition 7.19. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then the inclusion $i : \ker \phi \hookrightarrow G$ is terminal in the category of pairs $(K, \alpha : K \rightarrow G)$ such that $\phi \circ \alpha = 0$.*

PROOF:

$\langle 1 \rangle 1.$ $\phi \circ i = 0$

$\langle 1 \rangle 2.$ For any group K and homomorphism $\alpha : K \rightarrow G$ such that $\phi \circ \alpha = 0$, there exists a unique homomorphism $\beta : K \rightarrow \ker \phi$ such that $i \circ \beta = \alpha$.

□

Proposition 7.20. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then the following are equivalent:*

1. ϕ is monic.
2. $\ker \phi = \{e\}$
3. ϕ is injective.

PROOF:

$\langle 1 \rangle 1.$ $1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: ϕ is monic.

$\langle 2 \rangle 2.$ LET: $i : \ker \phi \hookrightarrow G$, $j : \{e\} \hookrightarrow \ker \phi \hookrightarrow G$ be the inclusions.

$\langle 2 \rangle 3.$ $\phi \circ i = \phi \circ j$

$\langle 2 \rangle 4.$ $i = j$

$\langle 1 \rangle 2.$ $2 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: $\ker \phi = \{e\}$

$\langle 2 \rangle 2.$ LET: $x, y \in G$

$\langle 2 \rangle 3.$ ASSUME: $\phi(x) = \phi(y)$

$\langle 2 \rangle 4.$ $\phi(xy^{-1}) = e$

$\langle 2 \rangle 5.$ $xy^{-1} \in \ker \phi$

$\langle 2 \rangle 6.$ $xy^{-1} = e$

$\langle 2 \rangle 7.$ $x = y$

$\langle 1 \rangle 3.$ $3 \Rightarrow 1$

PROOF: Easy.

□

Proposition 7.21. *A group homomorphism is an epimorphism if and only if it is surjective.*

7.3 Inner Automorphisms

Proposition 7.22. *Let G be a group and $g \in G$. The function $\gamma_g : G \rightarrow G$ defined by $\gamma_g(a) = gag^{-1}$ is an automorphism on G .*

PROOF:

$\langle 1 \rangle 1.$ γ_g is a homomorphism.

PROOF:

$$\begin{aligned}\gamma_g(ab) &= gabg^{-1} \\ &= gag^{-1}gbg^{-1} \\ &= \gamma_g(a)\gamma_g(b)\end{aligned}$$

<1>2. γ_g is injective.

PROOF: By Cancellation.

<1>3. γ_g is surjective.

PROOF: Given $b \in G$, we have $\gamma_g(g^{-1}bg) = b$.

□

Definition 7.23 (Inner Automorphism). Let G be a group. An *inner automorphism* on G is a function of the form $\gamma_g(a) = gag^{-1}$ for some $g \in G$.

We write $\text{Inn}(G)$ for the set of inner automorphisms of G .

Proposition 7.24. Let G be a group. The function $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$ that maps g to γ_g is a group homomorphism.

PROOF: Since $\gamma_{gh}(a) = ghah^{-1}g^{-1} = \gamma_g(\gamma_h(a))$. □

Corollary 7.24.1. $\text{Inn}(G)$ is a subgroup of $\text{Aut}_{\mathbf{Grp}}(G)$.

7.4 Direct Products

Definition 7.25 (Direct Product). The *direct product* of groups G and H is their product in \mathbf{Grp} .

7.5 Free Groups

Proposition 7.26. Let A be a set. Let \mathcal{F}^A be the category whose objects are pairs (G, j) where G is a group and j is a function $A \rightarrow G$, with morphisms $f : (G, j) \rightarrow (H, k)$ the group homomorphisms $f : G \rightarrow H$ such that $f \circ j = k$. Then \mathcal{F}^A has an initial object.

PROOF:

<1>1. LET: $W(A)$ be the set of words in the alphabet whose elements are the elements of A together with $\{a^{-1} : a \in A\}$.

<1>2. LET: $r : W(A) \rightarrow W(A)$ be the function that, given a word w , removes the first pair of letters of the form aa^{-1} or $a^{-1}a$; if there is no such pair, then $r(w) = w$.

<1>3. Let us say that a word w is a *reduced word* iff $r(w) = w$.

<1>4. For any word w of length n , we have $r^{\lceil \frac{n}{2} \rceil}(w)$ is a reduced word.

PROOF: Since we cannot remove more than $n/2$ pairs of letters from w .

<1>5. LET: $R : W(A) \rightarrow W(A)$ be the function $R(w) = r^{\lceil \frac{n}{2} \rceil}(w)$, where n is the length of w .

<1>6. LET: $F(A)$ be the set of reduced words.

<1>7. Define $\cdot : F(A)^2 \rightarrow F(A)$ by $w \cdot w' = R(ww')$

$\langle 1 \rangle 8.$ \cdot is associative.

PROOF: Both $w_1 \cdot (w_2 \cdot w_3)$ and $(w_1 \cdot w_2) \cdot w_3$ are equal to $R(w_1 w_2 w_3)$.

$\langle 1 \rangle 9.$ The empty word is the identity element in $F(A)$

$\langle 1 \rangle 10.$ The inverse of $a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}$ is $a_n^{\mp 1} \dots a_2^{\mp 1} a_1^{\mp 1}$.

$\langle 1 \rangle 11.$ LET: $j : A \rightarrow F(A)$ be the function that maps a to the word a of length

$\langle 1 \rangle 12.$ LET: G be any group and $k : A \rightarrow G$ any function.

$\langle 1 \rangle 13.$ The only morphism $f : (F(A), j) \rightarrow (G, k)$ in \mathcal{F}^A is $f(a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}) = k(a_1)^{\pm 1} k(a_2)^{\pm 1} \dots k(a_n)^{\pm 1}$.

□

Definition 7.27 (Free Group). For any set A , the *free group* on A is the initial object $(F(A), i)$ in \mathcal{F}^A .

Proposition 7.28. $i : A \rightarrow F(A)$ is injective.

PROOF:

$\langle 1 \rangle 1.$ LET: $x, y \in A$

$\langle 1 \rangle 2.$ ASSUME: $x \neq y$

PROVE: $i(x) \neq i(y)$

$\langle 1 \rangle 3.$ LET: $f : A \rightarrow C_2$ be the function that maps x to 0 and all other elements of A to 1.

$\langle 1 \rangle 4.$ LET: $\phi : F(A) \rightarrow C_2$ be the group homomorphism such that $f = \phi \circ i$.

$\langle 1 \rangle 5.$ $f(x) \neq f(y)$

$\langle 1 \rangle 6.$ $\phi(i(x)) \neq \phi(i(y))$

$\langle 1 \rangle 7.$ $i(x) \neq i(y)$

□

Proposition 7.29.

$$F(0) \cong \{e\}$$

PROOF: For any set A , the unique group homomorphism $\{e\} \rightarrow A$ makes the following diagram commute.

$$\begin{array}{ccc} \{e\} & \longrightarrow & A \\ \uparrow & \nearrow & \\ \emptyset & & \end{array}$$

Proposition 7.30. The free group on 1 is \mathbb{Z} with the injection mapping 0 to 1.

PROOF: Given any group G and function $a : 1 \rightarrow G$, the required unique homomorphism $\phi : \mathbb{Z} \rightarrow G$ is defined by $\phi(n) = a(0)^n$. □

Proposition 7.31. For any sets A and B , we have that $F(A + B)$ is the coproduct of $F(A)$ and $F(B)$ in **Grp**.

$$\begin{array}{ccccc}
& & G & & \\
& f \nearrow & \uparrow k & \nwarrow g & \\
F(A) & \xrightarrow{\kappa_1} & F(A+B) & \xleftarrow{\kappa_2} & F(B) \\
i_A \uparrow & & j \uparrow & & i_B \uparrow \\
A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
\end{array}$$

PROOF:

- (1)1. LET: $i_A : A \rightarrow F(A)$, $i_B : B \rightarrow F(B)$, $j : A+B \rightarrow F(A+B)$ be the canonical injections.
 - (1)2. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
 - (1)3. LET: G be any group and $f : F(A) \rightarrow G$, $g : F(B) \rightarrow G$ any group homomorphisms.
 - (1)4. LET: $h : A+B \rightarrow G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.
 - (1)5. LET: $k : F(A+B) \rightarrow G$ be the unique group homomorphism such that $k \circ j = h$.
 - (1)6. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
 - (1)7. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.
-

Definition 7.32 (Subgroup Generated by a Group). Let G be a group and A a subset of G . Let $\phi : F(A) \rightarrow G$ be the unique group homomorphism such that $\phi(a) = a$ for all $a \in A$. The subgroup *generated* by A is

$$\langle A \rangle := \text{im } \phi$$

$$\begin{array}{ccc}
F(A) & \xrightarrow{\phi} & G \\
\uparrow & \nearrow & \\
A & &
\end{array}$$

Proposition 7.33. Let G be a group and A a subset of G . Then $\langle A \rangle$ is the set of all elements of the form $a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$ (where $n \geq 0$) such that $a_1, \dots, a_n \in A$.

PROOF: Immediate from definitions. □

Corollary 7.33.1. Let G be a group and $g \in G$. Then

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\}.$$

Proposition 7.34. Let G be a group and A a subset of G . Then $\langle A \rangle$ is the intersection of all the subgroups of G that include A .

PROOF: Easy. \square

Definition 7.35 (Finitely Generated). Let G be a group. Then G is *finitely generated* iff there exists a finite subset A of G such that $G = \langle A \rangle$.

Proposition 7.36. *Every subgroup of a finitely generated free group is free.*

PROOF: TODO.

Proposition 7.37. *$F(2)$ includes subgroups isomorphic to the free group on arbitrarily many generators.*

PROOF: TODO

Proposition 7.38.

$$[F(2), F(2)] \cong F(\mathbb{Z})$$

PROOF: TODO

7.6 Normal Subgroups

Definition 7.39 (Normal Subgroup). A subgroup N of G is *normal* iff, for all $g \in G$ and $n \in N$, we have $gng^{-1} \in N$.

Example 7.40. Every subgroup of Q_8 is normal.

Proposition 7.41. *Let G be a group and N a subgroup of G . Then the following are equivalent.*

1. N is normal.
2. $\forall g \in G. gNg^{-1} \subseteq N$
3. $\forall g \in G. gNg^{-1} = N$
4. $\forall g \in G. gN \subseteq Ng$
5. $\forall g \in G. gN = Ng$

PROOF:

$\langle 1 \rangle 1. 1 \Leftrightarrow 2$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: If 2 holds then we have $gNg^{-1} \subseteq N$ and $g^{-1}Ng \subseteq N$ hence $N = gNg^{-1}$.

$\langle 1 \rangle 3. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 4. 2 \Leftrightarrow 4$

PROOF: Easy.

$\langle 1 \rangle 5. 3 \Leftrightarrow 5$

PROOF: Easy.

□

Proposition 7.42. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\ker \phi$ is a normal subgroup of G .*

PROOF: Given $g \in G$ and $n \in \ker \phi$ we have

$$\begin{aligned}\phi(gng^{-1}) &= \phi(g)\phi(n)\phi(g)^{-1} \\ &= \phi(g)\phi(g)^{-1} \\ &= e\end{aligned}$$

and so $gng^{-1} \in \ker \phi$. □

7.7 Quotient Groups

Definition 7.43. Let G be a group. Let \sim be an equivalence relation on G . Then we say that \sim is *compatible* with the group operation on G iff, for all $a, a', g \in G$, if $a \sim a'$ then $ga \sim ga'$ and $ag \sim a'g$.

Proposition 7.44. *Let G be a group. Let \sim be an equivalence relation on G . Then there exists an operation $\cdot : (G/\sim)^2 \rightarrow G/\sim$ such that*

$$\forall a, b \in G. [a][b] = [ab]$$

iff \sim is compatible with the group operation on G . In this case, G/\sim is a group under \cdot and the canonical function $\pi : G \rightarrow G/\sim$ is a group homomorphism, and is universal with respect to group homomorphisms $\phi : G \rightarrow G'$ such that if $a \sim a'$ then $\phi(a) = \phi(a')$.

PROOF: Easy. □

Definition 7.45 (Quotient Group). Let G be a group. Let \sim be an equivalence relation on G that is compatible with the group operation on G . Then G/\sim is the *quotient group* of G by \sim under $[a][b] = [ab]$.

Proposition 7.46. *Let G be a group and H a subgroup of G . Then H is normal if and only if there exists a group K and homomorphism $\phi : G \rightarrow K$ such that $H = \ker \phi$.*

PROOF: One direction is given by Proposition 7.42. For the other direction, take $K = G/H$ and ϕ to be the canonical map $G \rightarrow G/H$. □

Definition 7.47 (Modular Group). The *modular group* $\text{PSL}_2(\mathbb{Z})$ is $\text{SL}_2(\mathbb{Z})/\{I, -I\}$.

Proposition 7.48. $\text{PSL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

PROOF: By Example 6.26.

Proposition 7.49 (Roger Alperin). $\text{PSL}_2(\mathbb{Z})$ is presented by $(x, y | x^2, y^3)$.

PROOF:

$\langle 1 \rangle 1.$ LET: $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\langle 1 \rangle 2.$ LET: $y = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$\langle 1 \rangle 3.$ Define an action of $\text{PSL}_2(\mathbb{Z})$ on $\mathbb{R} - \mathbb{Q}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} r = \frac{ar + b}{cr + d}.$$

$\langle 2 \rangle 1.$ Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z})$ and r irrational we have $\frac{ar+b}{cr+d}$ is irrational.

$\langle 3 \rangle 1.$ ASSUME: for a contradiction $\frac{ar+b}{cr+d} = \frac{p}{q}$ where p and q are integers with $q > 0$.

$\langle 3 \rangle 2.$ $aqr + bq = cpr + dp$

$\langle 3 \rangle 3.$ $(aq - cp)r = dp - bq$

$\langle 3 \rangle 4.$ $aq = cp = dp - bq = 0$

$\langle 3 \rangle 5.$ $adq - cdp = 0$

$\langle 3 \rangle 6.$ $cdp - cbq = 0$

$\langle 3 \rangle 7.$ $(ad - cb)q = 0$

PROOF: Since $ad - cb = 1$.

$\langle 3 \rangle 8.$ $q = 0$

$\langle 3 \rangle 9.$ Q.E.D.

PROOF: This contradicts $\langle 3 \rangle 1$.

$\langle 2 \rangle 2.$ $-Ir = r$

PROOF: Since $-Ir = \frac{-r}{-1} = r$.

$\langle 2 \rangle 3.$ Given $A, B \in \text{PSL}_2(\mathbb{Z})$ we have $A(Br) = (AB)r$.

PROOF:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} r \right] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{er + f}{gr + h} \\ &= \frac{a \frac{er+f}{gr+h} + b}{c \frac{er+f}{gr+h} + d} \\ &= \frac{a(er + f) + b(gr + h)}{c(er + f) + d(gr + h)} \\ &= \frac{(ae + bg)r + (af + bh)}{(ce + dg)r + (cf + dh)} \\ &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} r \\ &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right] r \end{aligned}$$

$\langle 1 \rangle 4.$

$$yr = 1 - \frac{1}{r}$$

$\langle 1 \rangle 5.$

$$y^{-1}r = \frac{1}{1 - r}$$

PROOF: Since $y^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

$\langle 1 \rangle 6.$

PROOF: Since $yx = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. $yxr = 1 + r$

$\langle 1 \rangle 7.$

$$y^{-1}xr = \frac{r}{1+r}$$

PROOF: Since $y^{-1}x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

$\langle 1 \rangle 8.$ If $r > -1$ is positive then yxr is positive.

$\langle 1 \rangle 9.$ If r is positive then $y^{-1}xr$ is positive.

$\langle 1 \rangle 10.$ If $r < -1$ then $y^{-1}xr$ is positive.

$\langle 1 \rangle 11.$ If r is negative then yr is positive.

$\langle 1 \rangle 12.$ If r is negative then $y^{-1}r$ is positive.

$\langle 1 \rangle 13.$ No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)$$

with one or more factors can equal the identity.

PROOF: If the last factor is (yx) , then the product maps numbers in $(-1, 0)$ to positive numbers. If the last factor is $(y^{-1}x)$, then the product maps numbers < -1 to positive numbers.

$\langle 1 \rangle 14.$ No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)y^{\pm 1}$$

with one or more factors can equal the identity.

PROOF: The product maps negative numbers to positive numbers.

$\langle 1 \rangle 15.$ $\text{PSL}_2(\mathbb{Z})$ is presented by $(x, y|x^2, y^3)$.

□

Corollary 7.49.1. $\text{PSL}_2(\mathbb{Z})$ is the coproduct of C_2 and C_3 in **Grp**.

Theorem 7.50. Every group homomorphism $\phi : G \rightarrow H$ may be decomposed as

$$G \longrightarrow G/\ker \phi \xrightarrow{\cong} \text{im } \phi \longrightarrow H$$

PROOF: Easy. □

Corollary 7.50.1 (First Isomorphism Theorem). Let $\phi : G \rightarrow H$ be a surjective group homomorphism. Then $H \cong G/\ker \phi$.

Proposition 7.51. Let H_1 be a normal subgroup of G_1 and H_2 a normal subgroup of G_2 . Then $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$, and

$$\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}.$$

PROOF: $\pi \times \pi : G_1 \times G_2 \twoheadrightarrow G_1/H_1 \times G_2/H_2$ is a surjective homomorphism with kernel $H_1 \times H_2$. □

Example 7.52.

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

PROOF: Map a real number r to $(\cos r, \sin r)$. The result is a surjective group homomorphism with kernel \mathbb{Z} . \square

Proposition 7.53. *Let H be a normal subgroup of a group G . For every subgroup K of G that includes H , we have H is a normal subgroup of K , and K/H is a subgroup of G/H . The mapping*

$$u : \{\text{subgroups of } G \text{ including } H\} \rightarrow \{\text{subgroups of } G/H\}$$

with $u(K) = K/H$ is a poset isomorphism.

PROOF:

- $\langle 1 \rangle 1$. If K is a subgroup of G that includes H then H is normal in K .
- $\langle 1 \rangle 2$. If K is a subgroup of G that includes H then K/H is a subgroup of G/H .
- $\langle 1 \rangle 3$. If $H \subseteq K_1 \subseteq K_2$ then $K_1/H \subseteq K_2/H$.
- $\langle 1 \rangle 4$. If $K_1/H = K_2/H$ then $K_1 = K_2$
 - $\langle 2 \rangle 1$. ASSUME: $K_1/H = K_2/H$
 - $\langle 2 \rangle 2$. $K_1 \subseteq K_2$
 - $\langle 3 \rangle 1$. LET: $k \in K_1$
 - $\langle 3 \rangle 2$. $kH \in K_2/H$
 - $\langle 3 \rangle 3$. PICK $k' \in K_2$ such that $kH = k'H$
 - $\langle 3 \rangle 4$. $kk'^{-1} \in H$
 - $\langle 3 \rangle 5$. $kk'^{-1} \in K_2$
 - $\langle 3 \rangle 6$. $k \in K_2$
 - $\langle 2 \rangle 3$. $K_2 \subseteq K_1$
- PROOF: Similar.
- $\langle 1 \rangle 5$. For any subgroup L of G/H , there exists a subgroup K of G that includes H such that $L = K/H$.
 - $\langle 2 \rangle 1$. LET: L be a subgroup of G/H .
 - $\langle 2 \rangle 2$. LET: $K = \{k \in G : kH \in L\}$
 - $\langle 2 \rangle 3$. K is a subgroup of G .
 - PROOF: Given $k, k' \in K$ we have $kH, k'H \in L$ hence $kk'^{-1}H \in L$ and so $kk'^{-1} \in K$.
 - $\langle 2 \rangle 4$. $H \subseteq K$
 - PROOF: For all $h \in H$ we have $hH = H \in L$.
 - $\langle 2 \rangle 5$. $L = K/H$
 - PROOF: By definition.

\square

Proposition 7.54 (Third Isomorphism Theorem). *Let H be a normal subgroup of a group G . Let N be a subgroup of G that includes H . Then N/H is normal in G/H if and only if N is normal in G , in which case*

$$\frac{G/H}{N/H} \cong \frac{G}{N}$$

PROOF:

- ⟨1⟩1. If N/H is normal in G/H then N is normal in G .
 - ⟨2⟩1. ASSUME: N/H is normal in G/H .
 - ⟨2⟩2. LET: $g \in G$ and $n \in N$.
 - ⟨2⟩3. $gng^{-1}H \in N/H$
 - ⟨2⟩4. PICK $n' \in N$ such that $gng^{-1}H = n'H$
 - ⟨2⟩5. $gng^{-1}n'^{-1} \in H$
 - ⟨2⟩6. $gng^{-1}n'^{-1} \in N$
 - ⟨2⟩7. $gng^{-1} \in N$
- ⟨1⟩2. If N is normal in G then N/H is normal in G/H and $(G/H)/(N/H) \cong G/N$.
 - ⟨2⟩1. ASSUME: N is normal in G .
 - ⟨2⟩2. LET: $\phi : G/H \rightarrow G/N$ be the homomorphism $\phi(gH) = gN$
 - ⟨3⟩1. If $gH = g'H$ then $gN = g'N$
 PROOF: If $gg'^{-1} \in H$ then $gg'^{-1} \in N$.
 - ⟨3⟩2. $\phi((gH)(g'H)) = \phi(gH)\phi(g'H)$
 PROOF: Both are $gg'N$.
 - ⟨2⟩3. ϕ is surjective.
 - ⟨2⟩4. $\ker \phi = N/H$
 - ⟨2⟩5. $(G/H)/(N/H) \cong G/N$
 PROOF: By the First Isomorphism Theorem.

□

Proposition 7.55 (Second Isomorphism Theorem). *Let H and K be subgroups of a group G . Assume that H is normal in G . Then:*

- 1. HK is a subgroup of G , and H is normal in HK .
- 2. $H \cap K$ is normal in K , and

$$\frac{HK}{H} \cong \frac{K}{H \cap K}.$$

PROOF:

- ⟨1⟩1. HK is a subgroup of G .
 PROOF: Since $hkh'k' = hh'(h'^{-1}kh')k' \in HK$.
- ⟨1⟩2. H is normal in HK .
- ⟨1⟩3. $H \cap K$ is normal in K and $HK/H \cong K/(H \cap K)$
 PROOF: The function that maps k to kH is a surjective homomorphism $K \twoheadrightarrow HK/H$ with kernel $H \cap K$. Surjectivity follows because $hkh = hkh^{-1}H$.

□

See also Proposition 7.70 for a result that holds even if H is not normal.

7.8 Cosets

Proposition 7.56. *Let G be a group. Let \sim be an equivalence relation on G such that, for all $a, b, g \in G$, if $a \sim b$ then $ga \sim gb$. Let $H = \{h \in G : h \sim e\}$.*

Then H is a subgroup of G and, for all $a, b \in G$, we have

$$a \sim b \Leftrightarrow a^{-1}b \in H \Leftrightarrow aH = bH .$$

PROOF:

$\langle 1 \rangle 1.$ $e \in H$

$\langle 1 \rangle 2.$ For all $x, y \in H$ we have $xy^{-1} \in H$.

$\langle 2 \rangle 1.$ ASSUME: $x \sim e$ and $y \sim e$.

$\langle 2 \rangle 2.$ $e \sim y^{-1}$

PROOF: Since $yy^{-1} \sim ey^{-1}$.

$\langle 2 \rangle 3.$ $xy^{-1} \sim e$

PROOF: Since $xy^{-1} \sim ey^{-1} \sim e$.

$\langle 1 \rangle 3.$ If $a \sim b$ then $a^{-1}b \in H$.

PROOF: If $a \sim b$ then $a^{-1}b \sim a^{-1}a = e$.

$\langle 1 \rangle 4.$ If $a^{-1}b \in H$ then $aH = bH$.

$\langle 2 \rangle 1.$ ASSUME: $a^{-1}b \in H$

$\langle 2 \rangle 2.$ $bH \subseteq aH$

PROOF: For any $h \in H$ we have $bh = aa^{-1}bh \in aH$.

$\langle 2 \rangle 3.$ $aH \subseteq bH$

PROOF: Similar since $b^{-1}a \in H$.

$\langle 1 \rangle 5.$ If $aH = bH$ then $a \sim b$.

$\langle 2 \rangle 1.$ ASSUME: $aH = bH$

$\langle 2 \rangle 2.$ PICK $h \in H$ such that $a = bh$.

$\langle 2 \rangle 3.$ $b^{-1}a = h$

$\langle 2 \rangle 4.$ $b^{-1}a \in H$

$\langle 2 \rangle 5.$ $b^{-1}a \sim e$

$\langle 2 \rangle 6.$ $a \sim b$

PROOF: $a = bb^{-1}a \sim be = b$.

□

Definition 7.57 (Coset). Let G be a group and H a subgroup of G . A *left coset* of H is a set of the form aH for $a \in G$. A *right coset* of H is a set of the form Ha for some $a \in G$.

We write G/H for the set of all left cosets of H , and $G \backslash H$ for the set of all right cosets of H .

Proposition 7.58.

$$G/H \cong G \backslash H$$

PROOF: The function that maps aH to Ha^{-1} is a bijection. □

Proposition 7.59. Let G be a group and H a subgroup of G . Define \sim_H on G by: $a \sim b$ iff $a^{-1}b \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations \sim on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ga \sim gb$. The equivalence class of a is aH .

PROOF:

$\langle 1 \rangle 1.$ For any subgroup H , we have \sim_H is an equivalence relation on G .

⟨2⟩1. \sim is reflexive.

PROOF: For any $a \in G$ we have $a^{-1}a = e \in H$.

⟨2⟩2. \sim is symmetric.

PROOF: If $a^{-1}b \in H$ then $b^{-1}a \in H$.

⟨2⟩3. \sim is transitive.

PROOF: If $a^{-1}b \in H$ and $b^{-1}c \in H$ then $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$.

⟨1⟩2. If $a \sim_H b$ then $ga \sim_H gb$.

PROOF: If $a^{-1}b \in H$ then $(ga)^{-1}(gb) = a^{-1}g^{-1}gb = a^{-1}b \in H$.

⟨1⟩3. For any equivalence relation \sim on G such that, whenever $a \sim b$, then $ga \sim gb$, there exists a subgroup H such that $\sim = \sim_H$.

PROOF: Proposition 7.56.

⟨1⟩4. The \sim_H -equivalence class of a is aH .

PROOF:

$$\begin{aligned} a \sim b &\Leftrightarrow a^{-1}b \in H \\ &\Leftrightarrow \exists h \in H. a^{-1}b = h \\ &\Leftrightarrow \exists h \in H. b = ah \\ &\Leftrightarrow b \in aH \end{aligned}$$

□

Proposition 7.60. Let G be a group and H a subgroup of G . Define \sim_H on G by: $a \sim b$ iff $ab^{-1} \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations \sim on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ag \sim bg$. The equivalence class of a is Ha .

PROOF: Similar. □

Proposition 7.61. Let G be a group and H be a subgroup of G . Define \sim_L and \sim_R on G by:

$$a \sim_L b \Leftrightarrow a^{-1}b \in H, \quad a \sim_R b \Leftrightarrow ab^{-1} \in H.$$

Then $\sim_L = \sim_R$ if and only if H is normal.

PROOF:

⟨1⟩1. If $\sim_L = \sim_R$ then H is normal.

⟨2⟩1. ASSUME: $\sim_L = \sim_R$

⟨2⟩2. LET: $h \in H$ and $g \in G$

⟨2⟩3. $g \sim_L gh^{-1}$

⟨2⟩4. $g \sim_R gh^{-1}h$

⟨2⟩5. $ghg^{-1} \in H$

⟨1⟩2. If H is normal then $\sim_L = \sim_R$.

⟨2⟩1. ASSUME: H is normal.

⟨2⟩2. If $a \sim_L b$ then $a \sim_R b$.

⟨3⟩1. ASSUME: $a \sim_L b$

⟨3⟩2. $a^{-1}b \in H$

⟨3⟩3. $aa^{-1}ba^{-1} \in H$

⟨3⟩4. $ba^{-1} \in H$

- $\langle 3 \rangle 5. a \sim_R b$
 $\langle 2 \rangle 3. \text{ If } a \sim_R b \text{ then } a \sim_L b.$

PROOF: Similar.

□

Corollary 7.61.1. *Let G be a group and H be a normal subgroup of G . Define \sim on G by $a \sim b$ iff $a^{-1}b \in H$. Then G/\sim is a group under $[a][b] = [ab]$.*

Definition 7.62 (Quotient Group). Let G be a group and H be a normal subgroup of G . The *quotient group* G/H is G/\sim where $a \sim b$ iff $a^{-1}b \in H$, under $[a][b] = [ab]$ or $(aH)(bH) = abH$.

Corollary 7.62.1. *Let H be a normal subgroup of a group G . For every group homomorphism $\phi : G \rightarrow G'$ such that $H \subseteq \ker \phi$, there exists a unique group homomorphism $\bar{\phi} : G/H \rightarrow G'$ such that the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ & \searrow \pi & \nearrow \bar{\phi} \\ & G/H & \end{array}$$

Proposition 7.63. $\mathbb{Z}/n\mathbb{Z}$ has exactly n elements.

PROOF: Every integer is congruent to one of $0, 1, \dots, n-1$ by the division algorithm, and no two of them are congruent to one another, since if $0 \leq i < j < n$ then $0 < j - i < n$. □

Proposition 7.64. *Let m and n be integers with $n > 0$. The order of m in $\mathbb{Z}/n\mathbb{Z}$ is $\frac{n}{\gcd(m,n)}$.*

PROOF: By Proposition 6.16 since the order of 1 is n . □

Proposition 7.65. *The integer m generates $\mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(m, n) = 1$.*

PROOF: By Proposition 7.64. □

Corollary 7.65.1. *If p is prime then every non-zero element in $\mathbb{Z}/p\mathbb{Z}$ is a generator.*

Proposition 7.66.

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$$

PROOF: Every permutation of $\{(1, 0), (0, 1), (1, 1)\}$ gives an automorphism of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

Example 7.67. Not all monomorphisms split in \mathbf{Grp} .

Define $\phi : \mathbb{Z}/3\mathbb{Z} \rightarrow S_3$ by

$$\phi(0) = \text{id}_3, \quad \phi(1) = (1 \ 3 \ 2), \quad \phi(2) = (1 \ 2 \ 3) .$$

Then ϕ is monic but has no retraction.

For if $r : S_3 \rightarrow \mathbb{Z}/3\mathbb{Z}$ is a retraction, then we would have

$$r(1\ 2) + r(2\ 3) = 1, \quad r(2\ 3) + r(1\ 2) = 2$$

which is impossible.

Proposition 7.68. *Let G be a group, H a subgroup of G , and $g \in G$. The function that maps h to gh is a bijection $H \cong gH$.*

PROOF: By Cancellation. \square

Proposition 7.69. *Let G be a group, H a subgroup of G , and $g \in G$. The function that maps h to hg is a bijection $H \cong Hg$.*

PROOF: By Cancellation. \square

Proposition 7.70. *Let H and K be finite subgroups of a group G . Then*

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

PROOF:

$\langle 1 \rangle 1$. LET: $f : \{hK : h \in H\} \rightarrow H/(H \cap K)$ be the function $f(hK) = h(H \cap K)$

PROOF: This is well-defined because if $hK = h'K$ then $h^{-1}h' \in H \cap K$ so $h(H \cap K) = h'(H \cap K)$.

$\langle 1 \rangle 2$. f is injective.

PROOF: If $h(H \cap K) = h'(H \cap K)$ then $hK = h'K$.

$\langle 1 \rangle 3$. f is surjective.

PROOF: Clear.

$\langle 1 \rangle 4$.

$$\frac{|HK|}{|K|} = \frac{|H|}{|H \cap K|}$$

\square

7.9 Congruence

Definition 7.71 (Congruence). Given integers a, b, n with n positive, we say a is *congruent to b modulo n* , and write $a \equiv b \pmod{n}$, iff $a + n\mathbb{Z} = b + n\mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$.

Proposition 7.72. *Given integers a, b, n with n positive, we have $a \equiv b \pmod{n}$ iff $n \mid a - b$.*

PROOF: By Proposition 7.56. \square

Proposition 7.73. *If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then $a + b \equiv a' + b' \pmod{n}$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid (a' + b') - (a + b)$. \square

Proposition 7.74. *If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then $ab \equiv a'b' \pmod{n}$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid a'b' - ab = a'(b' - b) + (a' - a)b$. \square

7.10 Cyclic Groups

Definition 7.75 (Cyclic Group). The *cyclic* groups are \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$ for positive integers n .

Proposition 7.76. If m and n are positive integers with $\gcd(m, n) = 1$ then $C_{mn} \cong C_m \times C_n$.

PROOF: The function that maps x to $(x \bmod m, x \bmod n)$ is an isomorphism. \square

Proposition 7.77. Let G be a group and $g \in G$. Then $\langle g \rangle$ is cyclic.

PROOF: If g has finite order then $\langle g \rangle \cong C_{|g|}$, otherwise $\langle g \rangle \cong \mathbb{Z}$. \square

Proposition 7.78. Every finitely generated subgroup of \mathbb{Q} is cyclic.

PROOF:

$\langle 1 \rangle$ 1. LET: $G = \langle a_1/b, \dots, a_n/b \rangle$ where a_1, \dots, a_n, b are integers with $b > 0$

$\langle 1 \rangle$ 2. LET: $a = \gcd(a_1, \dots, a_n)$

$\langle 1 \rangle$ 3. $G = \langle a/b \rangle$

\square

Corollary 7.78.1. \mathbb{Q} is not finitely generated.

7.11 Commutator Subgroup

Definition 7.79 (Commutator Subgroup). Let G be a group. The *commutator subgroup* $[G, G]$ is the subgroup generated by the elements of the form $aba^{-1}b^{-1}$.

Proposition 7.80. The commutator subgroup is normal.

PROOF: Since

$$\begin{aligned} & ga_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_nb_na_n^{-1}b_n^{-1}g^{-1} \\ &= (ga_1g^{-1})(gb_1g^{-1})(ga_1g^{-1})^{-1}(gb_1g^{-1})^{-1} \cdots (ga_ng^{-1})(gb_ng^{-1})(ga_ng^{-1})^{-1}(gb_ng^{-1})^{-1}. \end{aligned} \quad \square$$

7.12 Presentations

Definition 7.81 (Presentation). A *presentation* of a group G is a pair (A, R) where A is a set and $R \subseteq F(A)$ is a set of words such that

$$G \cong F(A)/N(R)$$

where $N(R)$ is the smallest normal subgroup of $F(A)$ that includes R .

Example 7.82. • The free group on a set A is presented by (A, \emptyset) .

- S_3 is presented by $(x, y | x^2, y^3, xyxy)$.
- $(a, b | a^2, b^2, (ab)^n)$ is a presentation of D_{2n} .

- $(x, y \mid x^2y^{-2}, y^4, xyx^{-1}y)$ is a presentation of Q_8 .

Proposition 7.83 (Word Problem). *Let (A, R) be a presentation of the group G . Let $w_1, w_2 \in F(A)$ be two words. Then it is undecidable in general if $w_1N(R) = w_2N(R)$ in G .*

Definition 7.84 (Finitely Presented). A group is *finitely presented* iff it has a presentation (A, R) where both A and R are finite.

Proposition 7.85. *Let $(A|R)$ be a presentation of G and $(A'|R')$ a presentation of H . Assume w.l.o.g. A and A' are disjoint. Then the group $G * G'$ presented by $(A \cup A' | R \cup R')$ is the coproduct of G and G' in **Grp**.*

$$\begin{array}{ccccc}
 A & \longrightarrow & A \cup A' & \longleftarrow & A' \\
 \downarrow & & \downarrow & & \downarrow \\
 F(A) & \longrightarrow & F(A \cup A') & \longleftarrow & F(A') \\
 \downarrow & & \downarrow & & \downarrow \\
 G & \xrightarrow{\kappa_1} & G * G' & \xleftarrow{\kappa_2} & G'
 \end{array}$$

PROOF:

- $\langle 1 \rangle 1$. LET: $\kappa_1 : G \rightarrow G * G'$ and $\kappa_2 : G' \rightarrow G * G'$ be the unique homomorphisms that make the diagram above commute.
 - $\langle 1 \rangle 2$. LET: $\phi : G \rightarrow H$ and $\psi : G' \rightarrow H$ be any homomorphisms.
 - $\langle 1 \rangle 3$. LET: $[\phi, \psi] : F(A \cup A') \rightarrow H$ be the unique homomorphism such that ...
 - $\langle 1 \rangle 4$. $R \cup R' \subseteq \ker[\phi, \psi]$
 - $\langle 1 \rangle 5$. $[\phi, \psi]$ factors uniquely through the morphism $F(A \cup A') \rightarrow G * G'$
-

7.13 Index of a Subgroup

Definition 7.86 (Index). Let G be a group and H a subgroup of G . The *index* of H in G , denoted $|G : H|$, is the number of left cosets of H in G if this is finite, otherwise ∞ .

Theorem 7.87 (Lagrange's Theorem). *Let G be a finite group and H a subgroup of G . Then*

$$|G| = |G : H| |H| .$$

PROOF: G/H is a partition of G into $|G : H|$ subsets, each of size $|H|$. □

Corollary 7.87.1. *For p a prime number, the only group of order p is C_p .*

PROOF: Let G be a group of order p and $g \in G$ with $g \neq e$. Then $|\langle g \rangle|$ divides p and is not 1, hence is p , that is, $G = \langle g \rangle$. □

Theorem 7.88 (Cauchy's Theorem). *Let G be a finite group. If p is prime and $p \mid |G|$ then G has a subgroup of order p .*

Proposition 7.89. *Let G be a group. Let K be a subgroup of G and H a subgroup of K . If $|G : H|$, $|G : K|$ and $|K : H|$ are all finite then*

$$|G : H| = |G : K||K : H| .$$

PROOF:

- $\langle 1 \rangle 1$. LET: $G/K = \{g_1K, g_2K, \dots, g_mK\}$
- $\langle 1 \rangle 2$. LET: $K/H = \{k_1H, k_2H, \dots, k_nH\}$
- $\langle 1 \rangle 3$. $G/H = \{g_ik_jH : 1 \leq i \leq m, 1 \leq j \leq n\}$
- $\langle 2 \rangle 1$. LET: $g \in G$
- $\langle 2 \rangle 2$. PICK i such that $gK = g_iK$
- $\langle 2 \rangle 3$. $g^{-1}g_i \in K$
- $\langle 2 \rangle 4$. PICK j such that $g^{-1}g_iH = k_jH$
- $\langle 2 \rangle 5$. $g^{-1}g_ik_j \in H$
- $\langle 2 \rangle 6$. $gH = g_ik_jH$
- $\langle 1 \rangle 4$. If $g_ik_jH = g_{i'}k_{j'}H$ then $i = i'$ and $j = j'$.
- $\langle 2 \rangle 1$. ASSUME: $g_ik_jH = g_{i'}k_{j'}H$
- $\langle 2 \rangle 2$. $g_iK = g_{i'}K$
- $\langle 2 \rangle 3$. $i = i'$
- $\langle 2 \rangle 4$. $k_jH = k_{j'}H$
- $\langle 2 \rangle 5$. $j = j'$

□

7.14 Cokernels

Proposition 7.90. *Let $\phi : G \rightarrow H$ be a homomorphism between groups. Then there exists a group K and homomorphism $\pi : H \rightarrow K$ that is initial with respect to all homomorphism $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.*

PROOF:

- $\langle 1 \rangle 1$. LET: N be the intersection of all the normal subgroups of H that include $\text{im } \phi$.
- $\langle 1 \rangle 2$. LET: $K = H/N$ and π be the canonical homomorphism.
- $\langle 1 \rangle 3$. LET: $\pi \circ \phi = 0$
- $\langle 1 \rangle 4$. LET: $\alpha : H \rightarrow L$ satisfy $\alpha \circ \phi = 0$
- $\langle 1 \rangle 5$. $\text{im } \phi \subseteq \ker \alpha$
- $\langle 1 \rangle 6$. $N \subseteq \ker \alpha$
- $\langle 1 \rangle 7$. There exists a unique $\bar{\alpha} : H/\text{im } \phi \rightarrow L$ such that $\bar{\alpha} \circ \pi = \alpha$

□

Definition 7.91 (Cokernel). For any homomorphism $\phi : G \rightarrow H$ in **Grp**, the *cokernel* of ϕ is the group $\text{coker } \phi$ and homomorphism $\pi : H \rightarrow \text{coker } \phi$ that is initial among homomorphisms $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.

Example 7.92. It is not true that a homomorphism with trivial cokernel is epi. The inclusion $\langle (1 \ 2) \rangle \hookrightarrow S_3$ has trivial cokernel but is not epi.

7.15 Cayley Graphs

Definition 7.93 (Cayley Graph). Let G be a finitely generated group. Let A be a finite set of generators for G . The *Cayley graph* of G with respect to A is the directed graph whose vertices are the elements of G , with an edge $g_1 \rightarrow g_2$ labelled by $a \in A$ iff $g_2 = g_1 a$.

Proposition 7.94. G is the free group on A iff the Cayley graph with respect to A is a tree.

PROOF: Both are equivalent to saying that the product of two different strings of elements of A and/or their inverses are not equal. \square

Chapter 8

Abelian Groups

Definition 8.1 (Abelian Group). A group is *Abelian* iff any two elements commute.

In an Abelian group G , we often denote the group operation by $+$, the identity element by 0 and the inverse of an element g by $-g$. We write ng for g^n ($g \in G, n \in \mathbb{Z}$).

Example 8.2. Every group of order ≤ 4 is Abelian.

Example 8.3. For any positive integer n , we have $\mathbb{Z}/n\mathbb{Z}$ is an Abelian group under addition.

Example 8.4. S_n is not Abelian for $n \geq 3$. If $x = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ then $xy = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $yx = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$.

Example 8.5. There are 42 Abelian groups of order 1024 up to isomorphism.

Proposition 8.6. Let G be a group. If $g^2 = e$ for all $g \in G$ then G is Abelian.

PROOF: For any $g, h \in G$ we have

$$ghgh = e$$

$$\therefore hgh = g \quad (\text{multiplying on the left by } g)$$

$$\therefore hg = gh \quad (\text{multiplying on the right by } h) \square$$

Proposition 8.7. Let G be a group. Then G is Abelian if and only if the function that maps g to g^{-1} is a group homomorphism.

PROOF:

(1)1. If G is Abelian then the function that maps g to g^{-1} is a group homomorphism.

PROOF: Since $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$.

(1)2. If the function that maps g to g^{-1} is a group homomorphism then G is Abelian.

PROOF: Since $gh = (g^{-1})^{-1}(h^{-1})^{-1} = (g^{-1}h^{-1})^{-1} = hg$.
 \square

Proposition 8.8. *Let G be a group. Then G is Abelian if and only if the function that maps g to g^2 is a group homomorphism.*

PROOF:

$\langle 1 \rangle 1$. If G is Abelian then the function that maps g to g^2 is a group homomorphism.

PROOF: Since $(gh)^2 = g^2h^2$.

$\langle 1 \rangle 2$. If the function that maps g to g^2 is a group homomorphism then G is Abelian.

PROOF: Since we have $(gh)^2 = ghgh = g^2h^2$ and so $hg = gh$.

\square

Proposition 8.9. *Let G be a group. Then G is Abelian if and only if the homomorphism $\gamma : G \rightarrow \text{Aut}_{\text{Grp}}(G)$ is the trivial homomorphism.*

PROOF:

$\langle 1 \rangle 1$. If G is Abelian then γ is trivial.

PROOF: Since $\gamma_g(a) = gag^{-1} = a$.

$\langle 1 \rangle 2$. If γ is trivial then G is Abelian.

PROOF: If $\gamma_g(a) = gag^{-1} = a$ for all g and a then $ga = ag$ for all g, a .

\square

Proposition 8.10. *Let G be an Abelian group. Let $g, h \in G$. If g has maximal finite order in G , and h has finite order, then $|h| \mid |g|$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $|h| \nmid |g|$.

$\langle 1 \rangle 2$. PICK a prime p such that $|g| = p^m r$, $|h| = p^n s$ where $p \nmid r$, $p \nmid s$ and $m < n$.

$\langle 1 \rangle 3$. $|g^{p^m} h^s| = p^n r$

PROOF: Proposition 6.19.

$\langle 1 \rangle 4$. $|g| < |g^{p^m} h^s|$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts the maximality of $|g|$.

\square

Proposition 8.11. *Given a set A and an Abelian group H , the set H^A is an Abelian group under*

$$(\phi + \psi)(a) = \phi(a) + \psi(a) \quad (\phi, \psi \in H^A, a \in A) .$$

PROOF:

$\langle 1 \rangle 1$. $\phi + (\psi + \chi) = (\phi + \psi) + \chi$

$\langle 1 \rangle 2$. $\phi + \psi = \psi + \phi$

$\langle 1 \rangle 3$. LET: $0 : A \rightarrow H$ be the function $0(a) = 0$.

$\langle 1 \rangle 4$. $\phi + 0 = 0 + \phi = \phi$

$\langle 1 \rangle 5$. Given $\phi : A \rightarrow H$, define $-\phi : A \rightarrow H$ by $(-\phi)(a) = -(\phi(a))$.

$\langle 1 \rangle 6$. $\phi + (-\phi) = (-\phi) + \phi = 0$

□

Proposition 8.12. *Given a group G and an Abelian group H , the set $\mathbf{Grp}[G, H]$ is a subgroup of H^G .*

PROOF:

$\langle 1 \rangle 1$. Given $\phi, \psi : G \rightarrow H$ group homomorphisms, we have $\phi - \psi$ is a group homomorphism.

PROOF:

$$\begin{aligned} (\phi - \psi)(g + g') &= \phi(g + g') - \psi(g + g') \\ &= \phi(g) + \phi(g') - \psi(g) - \psi(g') \\ &= \phi(g) - \psi(g) + \phi(g') - \psi(g') \\ &= (\phi - \psi)(g) + (\phi - \psi)(g') \end{aligned}$$

□

Proposition 8.13. *Let G be a group. The following are equivalent.*

1. $\text{Inn}(G)$ is cyclic.
2. $\text{Inn}(G)$ is trivial.
3. G is Abelian.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: $\text{Inn}(G) = \langle \gamma_g \rangle$

$\langle 2 \rangle 2$. g commutes with every element of G

$\langle 3 \rangle 1$. LET: $x \in G$

$\langle 3 \rangle 2$. PICK $n \in \mathbb{Z}$ such that $\gamma_x = \gamma_g^n$

$\langle 3 \rangle 3$. $\forall y \in G. xyx^{-1} = g^n yg^{-n}$

$\langle 3 \rangle 4$. $xgx^{-1} = g$

$\langle 2 \rangle 3$. $\gamma_g = \text{id}_G$

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

$\langle 2 \rangle 1$. ASSUME: $\forall g \in G. \gamma_g = \text{id}_G$

$\langle 2 \rangle 2$. LET: $x, y \in G$

$\langle 2 \rangle 3$. $\gamma_x(y) = y$

$\langle 2 \rangle 4$. $xyx^{-1} = y$

$\langle 2 \rangle 5$. $xy = yx$

$\langle 1 \rangle 3$. $3 \Rightarrow 2$

PROOF: If $xy = yx$ for all x, y then $\gamma_x(y) = y$ for all x, y .

$\langle 1 \rangle 4$. $2 \Rightarrow 1$

PROOF: Easy.

□

Corollary 8.13.1. *If $\text{Aut}_{\mathbf{Grp}}(G)$ is cyclic then G is Abelian.*

Proposition 8.14. *Every subgroup of an Abelian group is normal.*

PROOF: Let G be an Abelian group and N a subgroup of G . Given $g \in G$ and $n \in N$ we have $gng^{-1} = n \in N$. \square

Proposition 8.15. *For any group G , the group $G/[G, G]$ is Abelian.*

PROOF: For any $g, h \in G$ we have

$$gh(hg)^{-1} \in [G, G]$$

$$\therefore gh[G, G] = hg[G, G] \quad \square$$

Proposition 8.16. *Let G be a finite Abelian group. Let p be a prime divisor of $|G|$. Then G has an element of order p .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: as induction hypothesis the result holds for all groups smaller than G .

$\langle 1 \rangle 2$. PICK $g \in G - \{0\}$.

$\langle 1 \rangle 3$. PICK an element $h \in \langle g \rangle$ with prime order q .

$\langle 1 \rangle 4$. CASE: $q = p$

PROOF: h is the required element.

$\langle 1 \rangle 5$. CASE: $q \neq p$

$\langle 2 \rangle 1$. PICK $r \in G$ such that $r + \langle h \rangle$ has order p in $G/\langle h \rangle$.

PROOF: By induction hypothesis since $|G/\langle h \rangle| = |G|/q$.

$\langle 2 \rangle 2$. $pr \in \langle h \rangle$

$\langle 2 \rangle 3$. PICK k such that $pr = kh$

$\langle 2 \rangle 4$. $pqr = e$

$\langle 2 \rangle 5$. qr has order p .

\square

Corollary 8.16.1. *For n an odd integer, any Abelian group of order $2n$ has exactly one element of order 2.*

PROOF: If x and y are distinct elements of order 2 then $\langle x, y \rangle = \{e, x, y, xy\}$ has size 4 and so $4 \mid 2n$ which is a contradiction. \square

Example 8.17. It is not true that, if G is a finite group and $d \mid |G|$, then G has an element of order d . The quaternionic group has no element of order 4.

Proposition 8.18. *If G is a finite Abelian group and $d \mid |G|$ then G has a subgroup of size d .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: as induction hypothesis the result is true for all $d' < d$.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $d \neq 1$.

$\langle 1 \rangle 3$. PICK a prime p such that $p \mid d$.

$\langle 1 \rangle 4$. PICK an element $g \in G$ of order p .

$\langle 1 \rangle 5$. $d/p \mid |G/\langle g \rangle|$

$\langle 1 \rangle 6$. PICK a subgroup H of $G/\langle g \rangle$ of size d/p .

$\langle 1 \rangle 7$. $\pi^{-1}(H)$ is a subgroup of G of size d .

\square

Proposition 8.19. *Let (G, \cdot) be a group. Let $\circ : G^2 \rightarrow G$ be a group homomorphism such that (G, \circ) is a group. Then \circ and \cdot coincide, and G is Abelian.*

PROOF:

$\langle 1 \rangle 1$. For all $g_1, g_2, h_1, h_2 \in G$ we have

$$(g_1 g_2) \circ (h_1 h_2) = (g_1 \circ h_1)(g_2 \circ h_2)$$

$\langle 1 \rangle 2$. $e \circ e = e$

PROOF:

$$\begin{aligned} e \circ e &= (ee) \circ (ee) \\ &= (e \circ e)(e \circ e) \end{aligned}$$

Hence $e \circ e = e$ by Cancellation.

$\langle 1 \rangle 3$. e is the identity of (G, \circ)

$\langle 1 \rangle 4$. For all $g, h \in G$ we have

$$g \circ h = gh$$

PROOF:

$$\begin{aligned} g \circ h &= (ge) \circ (eh) \\ &= (g \circ e)(e \circ h) \\ &= gh \end{aligned}$$

$\langle 1 \rangle 5$. For all $g, h \in G$ we have $gh = hg$.

PROOF:

$$\begin{aligned} gh &= (e \circ g)(h \circ e) \\ &= (eh) \circ (ge) \\ &= h \circ g \\ &= hg \end{aligned}$$

□

Corollary 8.19.1. *If $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$ is a group object in **Grp** then m is the multiplication of G , $e(*)$ is the identity of G , $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Grp** where $e(*) = e$ and $i(g) = g^{-1}$.*

8.1 The Category of Abelian Groups

Definition 8.20 (Category of Abelian Groups). Let **Ab** be the full subcategory of **Grp** whose objects are the Abelian groups.

Proposition 8.21. *If $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$ is a group object in **Ab** then m is the multiplication of G , $e(*)$ is the identity of G , $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Ab** where $e(*) = e$ and $i(g) = g^{-1}$.*

PROOF: Immediate from Corollary 8.19.1. □

Definition 8.22 (Direct Sum). Given Abelian groups G and H , we also call the direct product of G and H the *direct sum* and denote it $G \oplus H$.

Proposition 8.23. *Given Abelian groups G and H , the direct sum $G \oplus H$ is the coproduct of G and H in \mathbf{Ab} .*

PROOF:

- (1)1. LET: $\kappa_1 : G \rightarrow G \oplus H$ be the group homomorphism $\kappa_1(g) = (g, e_H)$.
 (1)2. LET: $\kappa_2 : H \rightarrow G \oplus H$ be the group homomorphism $\kappa_2(h) = (e_G, h)$.
 (1)3. Given group homomorphism $\phi : G \rightarrow K$ and $\psi : H \rightarrow K$, define $[\phi, \psi] : G \oplus H \rightarrow K$ by $[\phi, \psi](g, h) = \phi(g) + \psi(h)$.
 (1)4. $[\phi, \psi]$ is a group homomorphism.

PROOF:

$$\begin{aligned}
 [\phi, \psi]((g, h) + (g', h')) &= [\phi, \psi](g + g', h + h') \\
 &= \phi(g + g') + \psi(h + h') \\
 &= \phi(g) + \phi(g') + \psi(h) + \psi(h') \\
 &= \phi(g) + \psi(h) + \phi(g') + \psi(h') \\
 &= [\phi, \psi](g, h) + [\phi, \psi](g', h')
 \end{aligned}$$

- (1)5. $[\phi, \psi] \circ \kappa_1 = \phi$

PROOF:

$$\begin{aligned}
 [\phi, \psi](\kappa_1(g)) &= [\phi, \psi](g, e_H) \\
 &= \phi(g) + \psi(e_H) \\
 &= \phi(g) + e_K \\
 &= \phi(g)
 \end{aligned}$$

- (1)6. $[\phi, \psi] \circ \kappa_2 = \psi$

PROOF: Similar.

- (1)7. If $f : G \oplus H \rightarrow K$ is a group homomorphism with $f \circ \kappa_1 = \phi$ and $f \circ \kappa_2 = \psi$ then $f = [\phi, \psi]$.

PROOF:

$$\begin{aligned}
 f(g, h) &= f((g, e_H) + (e_G, h)) \\
 &= f(\kappa_1(g)) + f(\kappa_2(h)) \\
 &= \phi(g) + \psi(h)
 \end{aligned}$$

□

Theorem 8.24. *Every finitely generated Abelian group is a direct sum of cyclic groups.*

PROOF: TODO □

8.2 Free Abelian Groups

Proposition 8.25. *Let A be a set. Let \mathcal{F}^A be the category whose objects are pairs (G, j) where G is an Abelian group and j is a function $A \rightarrow G$, with morphisms $f : (G, j) \rightarrow (H, k)$ the group homomorphisms $f : G \rightarrow H$ such that $f \circ j = k$. Then \mathcal{F}^A has an initial object.*

PROOF:

- (1)1. LET: $\mathbb{Z}^{\oplus A}$ be the subgroup of \mathbb{Z}^A consisting of all functions $\alpha : A \rightarrow \mathbb{Z}$ such that $\alpha(a) = 0$ for only finitely many $a \in A$.
- (1)2. LET: $i : A \rightarrow \mathbb{Z}^{\oplus A}$ be the function such that $i(a)(b) = 1$ if $a = b$ and 0 if $a \neq b$.
- (1)3. LET: G be any Abelian group and $j : A \rightarrow G$ any function.
- (1)4. The unique homomorphism $\phi : \mathbb{Z}^{\oplus A} \rightarrow G$ required is defined by $\phi(\alpha) = \sum_{a \in A} \alpha(a)j(a)$

□

Definition 8.26 (Free Abelian Group). For any set A , the *free Abelian group* on A is the initial object $(F^{ab}(A), i)$ in \mathcal{F}^A .

Proposition 8.27. For any sets A and B , we have that $F^{ab}(A + B)$ is the coproduct of $F^{ab}(A)$ and $F^{ab}(B)$ in **Grp**.

$$\begin{array}{ccccc}
 & & G & & \\
 & f \nearrow & \uparrow k & \nwarrow g & \\
 F^{ab}(A) & \xrightarrow{\kappa_1} & F^{ab}(A+B) & \xleftarrow{\kappa_2} & F^{ab}(B) \\
 i_A \uparrow & & j \uparrow & & i_B \uparrow \\
 A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
 \end{array}$$

PROOF:

- (1)1. LET: $i_A : A \rightarrow F^{ab}(A)$, $i_B : B \rightarrow F^{ab}(B)$, $j : A + B \rightarrow F^{ab}(A + B)$ be the canonical injections.
- (1)2. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
- (1)3. LET: G be any group and $f : F^{ab}(A) \rightarrow G$, $g : F^{ab}(B) \rightarrow G$ any group homomorphisms.
- (1)4. LET: $h : A + B \rightarrow G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.
- (1)5. LET: $k : F^{ab}(A + B) \rightarrow G$ be the unique group homomorphism such that $k \circ j = h$.
- (1)6. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
- (1)7. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.

□

Proposition 8.28. For A and B finite sets, if $F^{ab}(A) \cong F^{ab}(B)$ then $A \cong B$.

PROOF:

- (1)1. For any set C , define \sim on $F^{ab}(C)$ by: $f \sim f'$ iff there exists $g \in F^{ab}(C)$ such that $f - f' = 2g$.
- (1)2. For any set C , \sim is an equivalence relation on $F^{ab}(C)$.
- (1)3. For any set C , we have $F^{ab}(C) / \sim$ is finite if and only if C is finite, in which case $|F^{ab}(C) / \sim| = 2^{|C|}$.

PROOF: There is a bijection between $F^{\text{ab}}(C) / \sim$ and the finite subsets of C , which maps f to $\{c \in C : f(c) \text{ is odd}\}$.

$\langle 1 \rangle 4$. If $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$ then $A \cong B$.

PROOF: If $|F^{\text{ab}}(A) / \sim| = |F^{\text{ab}}(B) / \sim|$ then $2^{|A|} = 2^{|B|}$ and so $|A| = |B|$. \square

Proposition 8.29. *Let G be an Abelian group. Then G is finitely generated if and only if there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \rightarrow G$ for some n .*

PROOF:

$\langle 1 \rangle 1$. If G is finitely generated then there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \rightarrow G$ for some n .

PROOF: Let $G = \langle a_1, \dots, a_n \rangle$. Define $\phi : \mathbb{Z}^{\oplus n} \rightarrow G$ by $\phi(i_1, \dots, i_n) = i_1 \cdot a_1 + \dots + i_n \cdot a_n$.

$\langle 1 \rangle 2$. If there exists a surjective homomorphism $\phi : \mathbb{Z}^{\oplus n} \rightarrow G$ for some n then G is finitely generated.

PROOF: G is generated by $\phi(1, 0, \dots, 0), \phi(0, 1, 0, \dots, 0), \dots, \phi(0, \dots, 0, 1)$. \square

Proposition 8.30. *Let A be a set. Let $i : A \hookrightarrow F(A)$ be the free group on A . Then $\pi \circ i : A \rightarrow F(A)/[F(A), F(A)]$ is the free Abelian group on A .*

$$\begin{array}{ccc}
 & F(A)/[F(A), F(A)] & \\
 \pi \uparrow & \searrow h & \\
 F(A) & \xrightarrow{g} & G \\
 i \uparrow & \nearrow f & \\
 A & &
 \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: G be an Abelian group and $f : A \rightarrow G$ a function.

$\langle 1 \rangle 2$. LET: $g : F(A) \rightarrow G$ be the unique group homomorphism such that $g \circ i = f$.

$\langle 1 \rangle 3$. $[F(A), F(A)] \subseteq \ker g$

PROOF: For all $x, y \in F(A)$ we have $g(xy x^{-1} y^{-1}) = g(x) + g(y) - g(x) - g(y) = 0$.

$\langle 1 \rangle 4$. LET: $h : F(A)/[F(A), F(A)] \rightarrow G$ be the unique group homomorphism such that $h \circ \pi = g$.

$\langle 1 \rangle 5$. h is the unique group homomorphism such that $h \circ \pi \circ i = f$. \square

Corollary 8.30.1. *Let A and B be sets. Let $F(A)$ and $F(B)$ be the free groups on A and B respectively. If $F(A) \cong F(B)$ then $A \cong B$.*

PROOF: Proposition 8.28. \square

8.3 Cokernels

Proposition 8.31. *Let $\phi : G \rightarrow H$ be a homomorphism between Abelian groups. Then there exists an Abelian group K and homomorphism $\pi : H \rightarrow K$ that is initial with respect to all homomorphism $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.*

PROOF:

$\langle 1 \rangle 1$. LET: $K = H/\text{im } \phi$ and π be the canonical homomorphism.

$\langle 1 \rangle 2$. LET: $\pi \circ \phi = 0$

$\langle 1 \rangle 3$. LET: $\alpha : H \rightarrow L$ satisfy $\alpha \circ \phi = 0$

$\langle 1 \rangle 4$. $\text{im } \phi \subseteq \ker \alpha$

$\langle 1 \rangle 5$. There exists a unique $\bar{\alpha} : H/\text{im } \phi \rightarrow L$ such that $\bar{\alpha} \circ \pi = \alpha$

□

Definition 8.32 (Cokernel). For any homomorphism $\phi : G \rightarrow H$ in **Ab**, the *cokernel* of ϕ is the Abelian group $\text{coker } \phi$ and homomorphism $\pi : H \rightarrow \text{coker } \phi$ that is initial among homomorphisms $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.

Proposition 8.33. *$\pi : H \rightarrow \text{coker } \phi$ is initial among functions $f : H \rightarrow X$ such that, for all $x, y \in H$, if $x + \text{im } \phi = y + \text{im } \phi$ then $f(x) = f(y)$.*

PROOF: Easy. □

Proposition 8.34. *Let $\phi : G \rightarrow H$ be a homomorphism of Abelian groups. Then the following are equivalent.*

- ϕ is an epimorphism.
- $\text{coker } \phi$ is trivial.
- ϕ is surjective.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: ϕ is epi.

$\langle 2 \rangle 2$. LET: $\pi : H \rightarrow \text{coker } \phi$ be the canonical homomorphism.

$\langle 2 \rangle 3$. $\pi \circ \phi = 0 \circ \phi$

$\langle 2 \rangle 4$. $\pi = 0$

$\langle 2 \rangle 5$. $\text{coker } \phi = \text{im } \pi$ is trivial.

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

PROOF: If $\text{coker } \phi = H/\text{im } \phi$ is trivial then $\text{im } \phi = H$.

$\langle 1 \rangle 3$. $3 \Rightarrow 1$

PROOF: If it is surjective then it is epi in **Set**.

□

Chapter 9

Group Actions

9.1 Group Actions

Definition 9.1 (Action). Let G be a group. Let A be an object of a category \mathcal{C} . A (left) action of G on A is a group homomorphism $G \rightarrow \text{Aut}_{\mathcal{C}}(A)$.

It is *faithful* or *effective* iff it is injective.

Proposition 9.2. Let A be a set. An action of the group G on the set A is given by a function $\cdot : G \times A \rightarrow A$ such that

- $\forall a \in A. ea = a$
- $\forall g, h \in G. \forall a \in A. (gh)a = g(ha)$

PROOF: Just unfolding definitions. \square

Example 9.3. Left multiplication defines a faithful action of any group on its own underlying set.

In fact, for any subgroup H of a group G , left multiplication defines an action of G on G/H .

Corollary 9.3.1 (Cayley's Theorem). Every group G is a subgroup of a symmetric group, namely $\text{Aut}_{\text{Set}}(G)$.

Example 9.4. Conjugation $g * h = ghg^{-1}$ is an action of any group on its own underlying set.

Definition 9.5 (Transitive). An action of a group G on a set A is *transitive* iff, for all $a, b \in A$, there exists $g \in G$ such that $ga = b$.

Example 9.6. Left multiplication of a group G is a transitive action of G on G .

Definition 9.7 (Orbit). Given an action of a group G on a set A and $a \in A$, the *orbit* of a is

$$\text{O}_G(a) := \{ga : g \in G\} .$$

Proposition 9.8. *Given an action of a group G on a set A , the orbits form a partition of A .*

PROOF:

$\langle 1 \rangle 1$. Every element of A is in some orbit.

PROOF: Since $a \in O_G(a)$.

$\langle 1 \rangle 2$. Distinct orbits are disjoint.

$\langle 2 \rangle 1$. LET: $a \in O_G(b) \cap O_G(c)$

$\langle 2 \rangle 2$. PICK $g, h \in G$ such that $a = gb = hc$.

$\langle 2 \rangle 3$. $O_G(b) \subseteq O_G(c)$

PROOF: For all $k \in G$ we have $kb = kg^{-1}hc$.

$\langle 2 \rangle 4$. $O_G(c) \subseteq O_G(b)$

PROOF: Similar.

□

Proposition 9.9. *Given an action of a group G on a set A and $a \in A$, the action is transitive on $O_G(a)$.*

PROOF:

$\langle 1 \rangle 1$. The restriction of the action is an action on $O_G(a)$.

PROOF: Since $g(ha) = (gh)a$, the action maps $O_G(a)$ to itself.

$\langle 1 \rangle 2$. The restricted action is transitive.

PROOF: Given $ga, ha \in O_G(a)$, we have $ha = (hg^{-1})(ga)$.

□

Definition 9.10 (Stabilizer Subgroup). Given an action of a group G on a set A and $a \in A$, the *stabilizer subgroup* of a is

$$\text{Stab}_G(a) := \{g \in G : ga = a\} .$$

Proposition 9.11. *Stabilizer subgroups are subgroups.*

PROOF: If $g, h \in \text{Stab}_G(a)$ then $gh^{-1}a = a$ so $gh^{-1} \in \text{Stab}_G(a)$. □

Proposition 9.12. *Let G act on a set A . Let $a \in A$ and $g \in G$. Then*

$$\text{Stab}_G(ga) = g\text{Stab}_G(a)g^{-1} .$$

PROOF:

$$h \in \text{Stab}_G(ga) \Leftrightarrow hga = ga$$

$$\Leftrightarrow g^{-1}hga = a$$

$$\Leftrightarrow g^{-1}hg \in \text{Stab}_G(a)$$

$$\Leftrightarrow h \in g\text{Stab}_G(a)g^{-1}$$

□

Corollary 9.12.1. *Let G be an action on a set A and $a \in A$. If $\text{Stab}_G(a)$ is normal in G , then for any $b \in O_G(a)$ we have $\text{Stab}_G(a) = \text{Stab}_G(b)$.*

Definition 9.13 (Free). An action of a group G on a set A is *free* iff, whenever $ga = a$, then $g = e$.

Example 9.14. The action of left multiplication is free.

Proposition 9.15. *Let G be a group. Let H be a subgroup of G of finite index n . Then H includes a subgroup K that is normal in G and such that $|G : K|$ divides $\gcd(|G|, n!)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\sigma : G \rightarrow \text{Aut}_{\text{Set}}(G/H)$ be the action of left multiplication.

$\langle 1 \rangle 2$. LET: $K = \ker \sigma$

$\langle 1 \rangle 3$. $K \subseteq H$

$\langle 2 \rangle 1$. LET: $g \in K$

$\langle 2 \rangle 2$. $\sigma(g)(H) = H$

$\langle 2 \rangle 3$. $gH = H$

$\langle 2 \rangle 4$. $g \in H$

$\langle 1 \rangle 4$. K is normal in G .

PROOF: Proposition 7.42.

$\langle 1 \rangle 5$. $|G : K| \mid |G|$

PROOF: Lagrange's Theorem.

$\langle 1 \rangle 6$. $|G : K| \mid n!$

PROOF: Since G/K is a subgroup of $\text{Aut}_{\text{Set}}(G/H)$.

□

Corollary 9.15.1. *Let G be a finite group. Let H be a subgroup of G of index p where p is the smallest prime that divides $|G|$. Then H is normal in G .*

PROOF:

$\langle 1 \rangle 1$. PICK a subgroup K of H normal in G such that $|G : K|$ divides $\gcd(|G|, p!)$.

$\langle 1 \rangle 2$. $|G : K|$ divides p .

$\langle 1 \rangle 3$. $|G : H| |H : K|$ divides p .

$\langle 1 \rangle 4$. $|H : K| = 1$

$\langle 1 \rangle 5$. $H = K$

$\langle 1 \rangle 6$. H is normal.

□

Corollary 9.15.2. *Any subgroup of index 2 is normal.*

Proposition 9.16. *Let G be a group with finite set of generators A . Then left multiplication defines a free action of G on its Cayley graph.*

PROOF: Easy since if $g_2 = g_1 a$ then $hg_2 = hg_1 a$. □

Corollary 9.16.1. *A free group acts freely on a tree.*

Theorem 9.17. *If a group G acts freely on a tree then G is free.*

Corollary 9.17.1. *Every subgroup of the free group on a finite set is free.*

PROOF: If H is a subgroup of $F(A)$ then left multiplication defines a free action of H on the Cayley graph of $F(A)$, which is a tree. □

9.2 Category of G -Sets

Definition 9.18. Given a group G , let $G - \mathbf{Set}$ be the category with:

- objects all pairs (A, ρ) such that A is a set and $\rho : G \times A \rightarrow A$ is an action of G on A ;
- morphisms $f : (A, \rho) \rightarrow (B, \sigma)$ are functions $f : A \rightarrow B$ that are $(G-)$ equivariant, i.e.

$$\forall g \in G. \forall a \in A. f(\rho(g, a)) = \sigma(g, f(a)) .$$

Proposition 9.19. *A G -equivariant function $f : A \rightarrow B$ is an isomorphism in $G - \mathbf{Set}$ if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ be G -equivariant and bijective.

PROVE: f^{-1} is G -equivariant.

$\langle 1 \rangle 2$. LET: $g \in G$ and $b \in B$

$\langle 1 \rangle 3$. $f^{-1}(gb) = gf^{-1}(b)$

PROOF:

$$\begin{aligned} f(f^{-1}(gb)) &= gb \\ &= gf(f^{-1}(b)) \\ &= f(gf^{-1}(b)) \end{aligned}$$

□

Proposition 9.20. *Let G be a group and A a transitive G -set. Let $a \in A$. Then A is isomorphic to $G/\text{Stab}_G(a)$ under left multiplication.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : G/\text{Stab}_G(a) \rightarrow A$ be the function $f(g\text{Stab}_G(a)) = ga$.

$\langle 2 \rangle 1$. ASSUME: $g\text{Stab}_G(a) = h\text{Stab}_G(a)$

PROVE: $ga = ha$

$\langle 2 \rangle 2$. $g^{-1}h \in \text{Stab}_G(a)$

$\langle 2 \rangle 3$. $g^{-1}ha = a$

$\langle 2 \rangle 4$. $ha = ga$

$\langle 1 \rangle 2$. f is G -equivariant.

PROOF: Since $f(gh\text{Stab}_G(a)) = gha = gf(h\text{Stab}_G(a))$.

$\langle 1 \rangle 3$. f is injective.

PROOF: If $ga = ha$ then $g^{-1}h \in \text{Stab}_G(a)$ so $g\text{Stab}_G(a) = h\text{Stab}_G(a)$.

$\langle 1 \rangle 4$. f is surjective.

PROOF: Since for all $b \in A$ there exists $g \in G$ such that $ga = b$.

□

Corollary 9.20.1. *If O is an orbit of the action of a finite group G on a set A , then O is finite and $|O|$ divides $|G|$.*

Corollary 9.20.2. *Let H be a subgroup of G and $g \in G$. Then*

$$G/H \cong G/(gHg^{-1})$$

in $G - \mathbf{Set}$.

PROOF: Taking $A = G/H$ and $a = gH$. \square

Proposition 9.21. *Given a family of G -sets $\{A_i\}_{i \in I}$, we have $\prod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g\{a_i\}_{i \in I} = \{ga_i\}_{i \in I}.$$

PROOF: Easy. \square

Proposition 9.22. *Given a family of G -sets $\{A_i\}_{i \in I}$, we have $\prod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g(i, a_i) = (i, ga_i).$$

PROOF: Easy. \square

Proposition 9.23. *Every finite G -set is a coproduct of G -sets of the form G/H .*

PROOF: If $O(a_1), \dots, O(a_n)$ are the orbits of the G -set A , then G is the coproduct of $G/\text{Stab}_G(a_1), \dots, G/\text{Stab}_G(a_n)$. \square

Proposition 9.24. *For any group G we have $G \cong \text{Aut}_{G-\mathbf{Set}}(G)$ (considering G as a G -set under left multiplication).*

PROOF:

$\langle 1 \rangle 1$. Define $\phi : G \rightarrow \text{Aut}_{G-\mathbf{Set}}(G)$ by $\phi(g)(g') = g'g^{-1}$.

$\langle 2 \rangle 1$. LET: $g \in G$

PROVE: $\lambda g' \in G.g'g^{-1}$ is an automorphism of G in $G - \mathbf{Set}$.

$\langle 2 \rangle 2$. $\phi(g)$ is G -equivariant.

PROOF: Since $\phi(g)(h_1h_2) = h_1h_2g^{-1} = h_1\phi(g)(h_2)$.

$\langle 2 \rangle 3$. $\phi(g)$ is injective.

PROOF: By Cancellation.

$\langle 2 \rangle 4$. $\phi(g)$ is surjective.

PROOF: For any $h \in G$ we have $h = \phi(g)(hg)$.

$\langle 1 \rangle 2$. ϕ is a group homomorphism.

PROOF: $\phi(g_1g_2)(h) = hg_2^{-1}g_1^{-1} = \phi(g_1)(\phi(g_2)(h))$.

$\langle 1 \rangle 3$. ϕ is injective.

PROOF: If $\phi(g) = \phi(g')$ then $g = \phi(g)(e) = \phi(g')(e) = g'$.

$\langle 1 \rangle 4$. ϕ is surjective.

$\langle 2 \rangle 1$. LET: $\sigma \in \text{Aut}_{G-\mathbf{Set}}(G)$

$\langle 2 \rangle 2$. LET: $g = \sigma(e)$

PROVE: $\sigma = \phi(g^{-1})$

$\langle 2 \rangle 3$. $\sigma(h) = hg$

PROOF: $\sigma(h) = \sigma(hg) = h\sigma(e) = hg$.

\square

Part III

Ring Theory

Chapter 10

Rngs

Definition 10.1 (Ring). A *rng* consists of a set R and binary operations $+, \cdot : R^2 \rightarrow R$ such that:

- $(R, +)$ is an Abelian group
- \cdot is associative.
- The *distributive properties* hold: for all $r, s, t \in R$ we have

$$(r + s)t = rt + st, \quad r(s + t) = rs + rt .$$

Example 10.2. • The *zero rng* is $\{0\}$.

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} are rngs.
- $2\mathbb{Z}$ is a rng.
- Given a rng R and natural number n , then the set $\mathfrak{gl}_n(R)$ of all $n \times n$ matrices with entries in R is a rng under matrix addition and matrix multiplication.
- For any set S , the power set $\mathcal{P}S$ is a rng under $A + B = (A \cup B) - (A \cap B)$ and $AB = A \cap B$.
- Given a rng R and a set S , then R^S is a rng under $(f + g)(s) = f(s) + g(s)$ and $(fg)(s) = f(s)g(s)$ for all $f, g \in R^S$ and $s \in S$.
- The set $\mathfrak{sl}_n(\mathbb{R}) = \{M \in \mathfrak{gl}_n(\mathbb{R}) : \text{tr } M = 0\}$ is a rng.
- The set $\mathfrak{sl}_n(\mathbb{C}) = \{M \in \mathfrak{gl}_n(\mathbb{C}) : \text{tr } M = 0\}$ is a rng.
- $\mathbb{Z}/n\mathbb{Z}$ is a rng.

- The ring \mathbb{H} of *quaternions* is \mathbb{R}^4 under the following operations, where we write (a, b, c, d) as $a + bi + cj + dk$:

$$\begin{aligned}
 (a + bi + cj + dk) + (a' + b'i + c'j + d'k) &= (a + a') + (b + b')i \\
 &\quad + (c + c')j + (d + d')k \\
 (a + bi + cj + dk)(a' + b'i + c'j + d'k) &= (aa' - bb' - cc' - dd') \\
 &\quad + (ab' + ba' + cd' - dc')i \\
 &\quad + (ac' - bd' + ca' + db')j \\
 &\quad + (ad' + bc' - cb' + da')k
 \end{aligned}$$

- For any Abelian group G , the set $\text{End}_{\mathbf{Ab}}(G)$ is a ring under pointwise addition and composition.

Proposition 10.3. *In any rng R we have*

$$\forall x \in R. x0 = 0x = 0 \text{ .}$$

PROOF:

$$\begin{aligned}
 x0 &= x(0 + 0) \\
 &= x0 + x0
 \end{aligned}$$

and so $x0 = 0$ by Cancellation. Similarly $0x = 0$. \square

Definition 10.4 (Zero Divisor). Let R be a rng and $a \in R$.

Then a is a *left-zero-divisor* iff there exists $b \in R - \{0\}$ such that $ab = 0$.

The element a is a *right-zero-divisor* iff there exists $b \in R - \{0\}$ such that $ba = 0$.

Example 10.5. 0 is a left- and right-zero-divisor in every non-zero rng.

The zero rng is the only ring with no zero-divisors.

Proposition 10.6. *Let R be a rng and $a \in R$. Then a is not a left-zero-divisor if and only if left multiplication by a is an injective function $R \rightarrow R$.*

PROOF:

$\langle 1 \rangle 1$. If a is not a left-zero-divisor then left multiplication by a is injective.

$\langle 2 \rangle 1$. ASSUME: a is not a left-zero-divisor.

$\langle 2 \rangle 2$. LET: $ab = ac$

$\langle 2 \rangle 3$. $a(b - c) = 0$

$\langle 2 \rangle 4$. $b - c = 0$

$\langle 2 \rangle 5$. $b = c$

$\langle 1 \rangle 2$. If a is a left-zero-divisor then left multiplication by a is not injective.

$\langle 2 \rangle 1$. PICK $b \neq 0$ such that $ab = 0$.

$\langle 2 \rangle 2$. $ab = a0$ but $b \neq 0$

\square

10.1 Commutative Rings

Definition 10.7 (Commutative). A rng R is *commutative* iff $\forall x, y \in R. xy = yx$.

Example 10.8. • The zero rng is commutative.

- \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are commutative.
- $2\mathbb{Z}$ is commutative.
- $\mathfrak{gl}_2(\mathbb{R})$ is not commutative.
- For any set S , the rng $\mathcal{P}S$ is commutative.
- If R is commutative then R^S is commutative.

10.2 Rng Homomorphisms

Definition 10.9. Let R and S be rngs. A *rng homomorphism* $\phi : R \rightarrow S$ is a function such that, for all $x, y \in R$, we have

$$\begin{aligned}\phi(x + y) &= \phi(x) + \phi(y) \\ \phi(xy) &= \phi(x)\phi(y)\end{aligned}$$

Let **Rng** be the category of rngs and rng homomorphisms.

10.3 Quaternions

Definition 10.10 (Norm). The *norm* of a quaternion is defined by

$$N(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2 \ .$$

Chapter 11

Rings

Definition 11.1 (Ring). A *ring* R is a rng such that there exists $1 \in R$, the *multiplicative identity*, such that

$$\forall x \in R. x1 = 1x = x \text{ .}$$

Example 11.2. • The zero rng is a ring with $1 = 0$.

- \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are rngs.
- $2\mathbb{Z}$ is not a ring.
- If R is a ring then $\mathfrak{gl}_n(R)$ is a ring.
- For any set S , the rng $\mathcal{P}S$ is a ring with $1 = S$.
- If R is a ring then R^S is a ring.
- $\mathfrak{sl}_n(\mathbb{R})$ is not a ring for $n > 0$.
- $\mathfrak{sl}_n(\mathbb{C})$ is not a ring for $n > 0$.
- $\mathfrak{so}_n(\mathbb{R}) = \{M \in \mathfrak{sl}_n(\mathbb{R}) : M + M^T = 0\}$ is not a ring.
- $\mathbb{Z}/n\mathbb{Z}$ is a ring.

Proposition 11.3. *In any ring R , if $0 = 1$ then R is the zero ring.*

PROOF: For any $x \in R$ we have $x = 1x = 0x = 0$. \square

Proposition 11.4. *In any ring we have $(-1)x = -x$.*

PROOF: Since

$$\begin{aligned} x + (-1)x &= 1x + (-1)x \\ &= (1 + (-1))x \\ &= 0x \\ &= 0 \end{aligned}$$

\square

11.1 Units

Definition 11.5 (Left-Unit, Right-Unit). Let R be a ring and $a \in R$. Then a is a *left-unit* iff there exists $b \in R$ such that $ab = 1$. The element a is a *right-unit* iff there exists $b \in R$ such that $ba = 1$.

An element is a *unit* iff it is a left-unit and a right-unit.

Proposition 11.6. *Let R be a ring and $a \in R$. Then a is a left-unit iff left multiplication by a is a surjective function $R \rightarrow R$.*

PROOF:

$\langle 1 \rangle 1$. If a is a left-unit then left multiplication by a is surjective.

$\langle 2 \rangle 1$. PICK $b \in R$ such that $ab = 1$.

$\langle 2 \rangle 2$. For all $c \in R$ we have $c = a(bc)$.

$\langle 1 \rangle 2$. If left multiplication by a is surjective then a is a left-unit.

PROOF: Immediate.

□

Proposition 11.7. *Let R be a ring and $a \in R$. Then a is a right-unit iff right multiplication by a is a surjective function $R \rightarrow R$.*

PROOF: Similar. □

Proposition 11.8. *No left-unit is a right-zero-divisor.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $ab = 1$ and $ca = 0$ where $c \neq 0$.

$\langle 1 \rangle 2$. $c = 0$

PROOF:

$$0 = 0b$$

$$= cab$$

$$= c1$$

$$= c$$

$\langle 1 \rangle 3$. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 11.9. *No right-unit is a left-zero-divisor.*

PROOF: Similar. □

Proposition 11.10. *The inverse of a unit is unique.*

PROOF: If $ba = 1$ and $ac = 1$ then $b = bac = c$. □

Proposition 11.11. *The units of a ring form a group under multiplication.*

PROOF:

$\langle 1 \rangle 1$. If a and b are units then ab is a unit.

PROOF: We have $b^{-1}a^{-1}ab = 1$ and $abb^{-1}a^{-1} = 1$.

⟨1⟩2. 1 is a unit.

PROOF: Since $1 \cdot 1 = 1$.

⟨1⟩3. If a is a unit then its inverse is a unit.

PROOF: Immediate from definitions.

□

Definition 11.12 (Group of Units). For any ring R , we write R^* for the group of the units of R under multiplication.

Example 11.13. The quaternionic group is a subgroup of \mathbb{H}^* .

Example 11.14. The norm is a group homomorphism $\mathbb{H}^* \rightarrow \mathbb{R}^+$ where \mathbb{R}^+ is the group of positive real numbers under multiplication with kernel isomorphic to $\text{SU}_2(\mathbb{C})$. The isomorphism maps a quaternion $a + bi + cj + dk$ to

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

Theorem 11.15 (Fermat's Little Theorem). *Let p be a prime number and a any integer. Then $a^p \equiv a \pmod{p}$.*

PROOF: If $p \mid a$ then $a^p \equiv a \equiv 0 \pmod{p}$. Otherwise, we have $a^{p-1} \equiv 1 \pmod{p}$ by applying Lagrange's Theorem to $(\mathbb{Z}/p\mathbb{Z})^*$. □

Example 11.16. It is not true that, if $n \mid |G|$, then G has a subgroup of order n . The group A_4 has order 12 but no subgroup of order 6.

Proposition 11.17. *If p is prime then $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.*

PROOF:

⟨1⟩1. LET: g be an element of maximal order in $(\mathbb{Z}/p\mathbb{Z})^*$.

⟨1⟩2. For all $h \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $h^{|g|} = 1$.

PROOF: Proposition 8.10.

⟨1⟩3. There are at most $|g|$ elements x such that $x^{|g|} = 1$ in $\mathbb{Z}/p\mathbb{Z}$

⟨1⟩4. $p - 1 \leq |g|$

⟨1⟩5. $|g| = p - 1$

⟨1⟩6. g generates $(\mathbb{Z}/p\mathbb{Z})^*$.

□

Example 11.18. $(\mathbb{Z}/12\mathbb{Z})^*$ is not cyclic. Its elements are 1, 5, 7 and 11 with orders 1, 2, 2 and 2.

Theorem 11.19 (Wilson's Theorem). *A positive integer p is prime if and only if $(p - 1)! \equiv 1 \pmod{p}$.*

⟨1⟩1. If p is prime then $(p - 1)! \equiv 1 \pmod{p}$.

⟨2⟩1. ASSUME: p is prime.

⟨2⟩2. $(p - 1)!$ is the product of all the elements of $(\mathbb{Z}/p\mathbb{Z})^*$

⟨2⟩3. The only element of $(\mathbb{Z}/p\mathbb{Z})^*$ with order 2 is -1 .

⟨2⟩4. $(p - 1)! \equiv -1 \pmod{p}$

PROOF: Proposition 6.20.

$\langle 1 \rangle 2$. If $(p-1)! \equiv -1 \pmod{p}$ then p is prime.

$\langle 2 \rangle 1$. ASSUME: $(p-1)! \equiv -1 \pmod{p}$

$\langle 2 \rangle 2$. LET: d be a proper divisor of p .

PROVE: $d = 1$

$\langle 2 \rangle 3$. $d \mid (p-1)!$

$\langle 2 \rangle 4$. $d \mid 1$

PROOF: Since $d \mid p \mid (p-1)! + 1$.

$\langle 2 \rangle 5$. $d = 1$

□

Proposition 11.20. *If p and q are distinct odd primes then $(\mathbb{Z}/pq\mathbb{Z})^*$ is not cyclic.*

PROOF:

$\langle 1 \rangle 1$. $|(\mathbb{Z}/pq\mathbb{Z})^*| = (p-1)(q-1)$

$\langle 1 \rangle 2$. LET: $g \in (\mathbb{Z}/pq\mathbb{Z})^*$

PROVE: g does not have order $(p-1)(q-1)$

$\langle 1 \rangle 3$. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{p}$

$\langle 1 \rangle 4$. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{q}$

$\langle 1 \rangle 5$. $pq \mid g^{(p-1)(q-1)/2} - 1$

$\langle 1 \rangle 6$. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{pq}$

$\langle 1 \rangle 7$. $|g| \mid (p-1)(q-1)/2$

□

Proposition 11.21. *For any prime p , we have $\text{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : \text{Aut}_{\mathbf{Grp}}(C_p) \rightarrow (\mathbb{Z}/p\mathbb{Z})^*$ be the function $\phi(\alpha) = \alpha(1)$.

PROOF: $\alpha(1)$ has order p in C_p and so is coprime with p .

$\langle 1 \rangle 2$. ϕ is a homomorphism.

PROOF: $\phi(\alpha \circ \beta) = \alpha(\beta(1)) = \alpha(\beta(1)1) = \beta(1)\alpha(1) = \phi(\alpha)\phi(\beta)$

$\langle 1 \rangle 3$. ϕ is injective.

PROOF: If $\phi(\alpha) = \phi(\beta)$ then for any n we have $\alpha(n) = n\alpha(1) = n\phi(\alpha) = n\phi(\beta) = n\beta(1) = \beta(n)$.

$\langle 1 \rangle 4$. ϕ is surjective.

PROOF: For any $r \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $r = \phi(\alpha)$ where $\alpha(n) = nr \pmod{p}$.

$\langle 1 \rangle 5$. $(\mathbb{Z}/p\mathbb{Z})^* \cong C_{p-1}$

□

11.2 Euler's ϕ -function

Proposition 11.22. *For n a positive integer, we have $(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$.*

PROOF:

$$\begin{aligned} m \in (\mathbb{Z}/n\mathbb{Z})^* &\Leftrightarrow \exists a. am \equiv 1 \pmod{n} \\ &\Leftrightarrow \exists a, b. am + bn = 1 \\ &\Leftrightarrow \gcd(m, n) = 1 \quad \square \end{aligned}$$

Definition 11.23 (Euler's Totient Function). For n a positive integer, let $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$.

Proposition 11.24. If n is an odd positive integer then $\phi(2n) = \phi(n)$.

PROOF:

(1)1. LET: n be an odd positive integer.

(1)2. For any integer m , if $\gcd(m, n) = 1$ then $\gcd(2m + n, 2n) = 1$

PROOF: For p a prime, if $p \mid 2m + n$ and $p \mid 2n$ then $p \neq 2$ (since $2m + n$ is odd) so $p \mid n$ and hence $p \mid m$, which is a contradiction.

(1)3. For any integer r , if $\gcd(r, 2n) = 1$ then $\gcd(\frac{r+n}{2}, n) = 1$

PROOF: If $p \mid n$ and $p \mid \frac{r+n}{2}$ then $p \mid r + n$ so $p \mid r$ which is a contradiction.

(1)4. The function that maps m to $2m + n$ is a bijection between $(\mathbb{Z}/n\mathbb{Z})^*$ and $(\mathbb{Z}/2n\mathbb{Z})^*$.

□

Theorem 11.25. For any positive integer n we have

$$\sum_{m>0, m|n} \phi(m) = n.$$

PROOF:

(1)1. Define $\chi : \{0, 1, \dots, n-1\} \rightarrow \{(m, d) : m > 0, m \mid n, d \text{ generates } \langle n/m \rangle\}$
by: $\chi(x) = (\gcd(x, n), x)$.

(1)2. χ is injective.

(1)3. χ is surjective.

PROOF: Given (m, d) such that d generates $\langle n/m \rangle$ we have $\chi(d) = (m, d)$.

(1)4. $n = \sum_{m>0, m|n} \phi(m)$

PROOF: Since $\langle n/m \rangle \cong C_m$ and so has $\phi(m)$ generators.

□

Proposition 11.26. For any positive integers a and n , we have $n \mid \phi(a^n - 1)$.

PROOF: Since the order of a is n in $(\mathbb{Z}/(a^n - 1)\mathbb{Z})^*$. □

Theorem 11.27 (Euler's Theorem). For any coprime integers a and n we have $a^{\phi(n)} \equiv a \pmod{n}$.

PROOF: Immediate from Lagrange's Theorem. □

Proposition 11.28.

$$|\text{Aut}_{\mathbf{Grp}}(C_n)| = \phi(n)$$

PROOF: An automorphism α is determined by $\alpha(1)$ which is any element of order n , and g has order n iff $\gcd(g, n) = 1$. □

Example 11.29.

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}) \cong C_2$$

PROOF: The only automorphisms are the identity and multiplication by -1. \square

11.3 Nilpotent Elements

Definition 11.30 (Nilpotent). Let R be a ring and $a \in R$. Then a is *nilpotent* iff there exists n such that $a^n = 0$.

Proposition 11.31. Let R be a ring and $a, b \in R$. If a and b are nilpotent and $ab = ba$ then $a + b$ is nilpotent.

PROOF:

$\langle 1 \rangle 1$. PICK m and n such that $a^m = b^n = 0$.

$\langle 1 \rangle 2$. $(a + b)^{m+n} = 0$

PROOF: Since $(a + b)^{m+n} = \sum_k \binom{m+n}{k} a^k b^{m+n-k}$ and every term in this sum is 0 since, for every k , either $k \geq m$ or $m + n - k \geq n$.

\square

Proposition 11.32. m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$ if and only if m is divisible by all the prime factors of n .

PROOF:

$\langle 1 \rangle 1$. If m is nilpotent then m is divisible by all the prime factors of n .

$\langle 2 \rangle 1$. ASSUME: $m^a \equiv 0 \pmod{n}$

$\langle 2 \rangle 2$. For every prime p , if $p \mid n$ then $p \mid m^a$.

$\langle 2 \rangle 3$. For every prime p , if $p \mid n$ then $p \mid m$.

$\langle 1 \rangle 2$. If m is divisible by all the prime factors of n then m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

$\langle 2 \rangle 1$. ASSUME: m is divisible by all the prime factors of n .

$\langle 2 \rangle 2$. LET: a be the largest number such that $p^a \mid n$ for some prime p .

$\langle 2 \rangle 3$. For every prime p that divides n we have $p^a \mid m^a$

$\langle 2 \rangle 4$. $n \mid m^a$

$\langle 2 \rangle 5$. $m^a \equiv 0 \pmod{n}$

$\langle 2 \rangle 6$. m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

\square

Chapter 12

Ring Homomorphisms

Definition 12.1 (Ring Homomorphism). Let R and S be rings. A *ring homomorphism* $\phi : R \rightarrow S$ is a rng homomorphism such that $\phi(1) = 1$.

Proposition 12.2. *The zero-ring is terminal in **Ring**.*

PROOF: Easy. \square

Proposition 12.3. *The ring \mathbb{Z} is initial in **Ring**.*

PROOF: Easy. \square

Proposition 12.4. *Let R and S be rings and $\phi : R \rightarrow S$ be a rng homomorphism. If ϕ is surjective, then ϕ is a ring homomorphism.*

PROOF:

$\langle 1 \rangle 1$. PICK $a \in R$ such that $\phi(a) = 1$

$\langle 1 \rangle 2$. $\phi(1) = 1$

PROOF:

$$\phi(1) = \phi(1)\phi(a)$$

$$= \phi(1a)$$

$$= \phi(a)$$

$$= 1$$

\square

Example 12.5. For any set S we have $\mathcal{P}S \cong (\mathbb{Z}/2\mathbb{Z})^S$ in **Ring** with the isomorphism

$$\begin{aligned} \phi : \mathcal{P}S &\cong (\mathbb{Z}/2\mathbb{Z})^S \\ \phi(A)(s) &= \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases} \end{aligned}$$

Example 12.6. The function $\mathbb{H} \rightarrow \mathfrak{gl}_4(\mathbb{R})$ that maps $a + bi + cj + dk$ to

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

is a monomorphism in **Ring**, as is the function $\mathbb{H} \rightarrow \mathfrak{sl}_2(\mathbb{C})$ that maps $a + bi + cj + dk$ to

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

Proposition 12.7. *Ring homomorphisms preserve units.*

PROOF: If $uv = 1$ then $\phi(u)\phi(v) = 1$. \square

Proposition 12.8. *Let $\phi : R \rightarrow S$ be a ring homomorphism. Then the following are equivalent.*

1. ϕ is a monomorphism.
2. $\ker \phi = \{0\}$
3. ϕ is injective.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: ϕ is a monomorphism.

$\langle 2 \rangle 2.$ LET: $r \in \ker \phi$

$\langle 2 \rangle 3.$ LET: $\text{ev}_r : \mathbb{Z}[x] \rightarrow R$ be the unique ring homomorphism such that $\text{ev}_r(x) = r$.

$\langle 2 \rangle 4.$ LET: $\text{ev}_0 : \mathbb{Z}[x] \rightarrow R$ be the unique ring homomorphism such that $\text{ev}_0(x) = 0$.

$\langle 2 \rangle 5.$ $\phi \circ \text{ev}_r = \phi \circ \text{ev}_0$

$\langle 2 \rangle 6.$ $\text{ev}_r = \text{ev}_0$

$\langle 2 \rangle 7.$ $r = 0$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: Proposition 7.20.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: Easy.

\square

Example 12.9. It is not true that every epimorphism in **Ring** is surjective. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism but not surjective.

The same example shows that a ring homomorphism may be a monomorphism and an epimorphism but not be an isomorphism.

Example 12.10.

$$\text{End}_{\mathbf{Ab}}(\mathbb{Z}) \cong \mathbb{Z}$$

The isomorphism maps any group endomorphism $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ to $\phi(1)$.

Example 12.11. The group of units of $\text{End}_{\mathbf{Ab}}(G)$ is $\text{Aut}_{\mathbf{Ab}}(G)$.

Example 12.12. Let R be a ring. Then the function $\lambda : R \rightarrow \text{End}_{\mathbf{Ab}}(R)$ defined by

$$\lambda(a)(b) = ab$$

is a ring monomorphism.

PROOF: Easy. \square

12.1 Products

Proposition 12.13. *Let R and S be rings. Then $R \times S$ is a ring under componentwise addition and multiplication, and this ring is the product of R and S in \mathbf{Ring} .*

PROOF: Easy. \square

Chapter 13

Subrings

Definition 13.1 (Subring). Let S be a ring. A *subring* of S is a ring R such that R is a subset of S and the inclusion $R \hookrightarrow S$ is a ring homomorphism.

Proposition 13.2. *Let R and S be rings. Then R is a subring of S if and only if R is a subset of S , the unit 1 of S is an element of R , and the operations of R are the restrictions of the operations of S to R .*

PROOF: Easy. \square

Corollary 13.2.1. *The zero ring is not a subring of any non-zero ring.*

Proposition 13.3. *Let $\phi : R \rightarrow S$ be a ring homomorphism. Then $\phi(R)$ is a subring of S .*

PROOF: Easy. \square

13.1 Centralizer

Definition 13.4 (Centralizer). Let R be a ring and $a \in R$. The *centralizer* of a is $\{r \in R : ar = ra\}$.

Proposition 13.5. *The centralizer of a is a subring of R .*

PROOF: Easy. \square

13.2 Center

Definition 13.6 (Center). The *center* of a ring R is $\{x \in R : \forall y \in R. xy = yx\}$.

Proposition 13.7. *The center of a ring is a subring.*

PROOF: Easy. \square

Proposition 13.8. *Let R be a ring. The center of $\text{End}_{\mathbf{Ab}}(R)$ is isomorphic to the center of R .*

PROOF:

$\langle 1 \rangle 1$. LET: $\lambda : R \rightarrow \text{End}_{\mathbf{Ab}}(R)$ be left multiplication.

$\langle 1 \rangle 2$. λ maps $Z(R)$ to $Z(\text{End}_{\mathbf{Ab}}(R))$.

$\langle 2 \rangle 1$. LET: $a \in Z(R)$

$\langle 2 \rangle 2$. LET: $\phi \in \text{End}_{\mathbf{Ab}}(R)$

PROVE: $\lambda(a) \circ \phi = \phi \circ \lambda(a)$

$\langle 2 \rangle 3$. LET: $x \in R$

$\langle 2 \rangle 4$. $a + \phi(x) = \phi(a + x)$

$\langle 1 \rangle 3$. $\lambda(Z(R)) = Z(\text{End}_{\mathbf{Ab}}(R))$

$\langle 2 \rangle 1$. LET: $\phi \in Z(\text{End}_{\mathbf{Ab}}(R))$

$\langle 2 \rangle 2$. For all $r \in R$,

LET: $\mu_r \in \text{End}_{\mathbf{Ab}}(R)$ be right multiplication by r .

$\langle 2 \rangle 3$. For all $r \in R$ we have $\phi \circ \mu_r = \mu_r \circ \phi$.

$\langle 2 \rangle 4$. For all $r, x \in R$ we have $\phi(xr) = \phi(x)r$

$\langle 2 \rangle 5$. For all $r \in R$ we have $\phi(r) = \phi(1)r$

$\langle 2 \rangle 6$. $\phi = \lambda(\phi(1))$

□

Corollary 13.8.1. *If R is a commutative ring then R is isomorphic to the center of $\text{End}_{\mathbf{Ab}}(R)$.*

Example 13.9. For n a positive integer we have $\mathbb{Z}/n\mathbb{Z} \cong \text{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$.

Since, for any $\phi \in \text{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$ we have $\phi(m) = m\phi(1)$ and so the whole of $\text{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$ is the image of λ .

Chapter 14

Monoid Rings

Definition 14.1 (Monoid Ring). Let R be a ring and M a monoid. Define $R[M]$ to be the ring whose elements are the families $\{a_m\}_{m \in M}$ such that $a_m = 0$ for all but finitely many $m \in M$, written

$$\sum_{m \in M} a_m m \text{ ,}$$

under

$$\begin{aligned} \sum_m a_m m + \sum_m b_m m &= \sum_m (a_m + b_m) m \\ \left(\sum_m a_m m \right) \left(\sum_m b_m m \right) &= \sum_{m \in M} \sum_{m_1 m_2 = m} a_{m_1} b_{m_2} m \end{aligned}$$

Example 14.2. Ring homomorphisms do not necessarily preserve zero-divisors. The canonical homomorphism $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z}$ maps the non-zero-divisor 2 to a zero-divisor.

14.1 Polynomials

Definition 14.3 (Polynomial). Let R be a ring. The ring of *polynomials* $R[x]$ is $R[\mathbb{N}]$. We write

$$\sum_n a_n x^n \text{ for } \sum_n a_n n \text{ .}$$

Concretely, a *polynomial* in R is a sequence (a_n) in R such that there exists N such that $\forall n \geq N. a_n = 0$. We write the polynomial as

$$\sum_{n=0}^{N-1} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1} \text{ .}$$

We write $R[x]$ for the set of all polynomials in R .

Define addition and multiplication on $R[x]$ by

$$\begin{aligned}\sum_n a_n x^n + \sum_n b_n x^n &= \sum_n (a_n + b_n) x^n \\ \left(\sum_n a_n x^n \right) \left(\sum_n b_n x^n \right) &= \sum_n \sum_{i+j=n} a_i b_j x^n\end{aligned}$$

A *constant* is a polynomial of the form $a + 0x + 0x^2 + \cdots$ for some $a \in R$. We write $R[x_1, \dots, x_n]$ for $R[x_1][x_2] \cdots [x_n]$.

Proposition 14.4. *For any ring R , the set of polynomials $R[x]$ is a ring.*

PROOF: Easy. \square

Definition 14.5 (Degree). The *degree* of a polynomial $\sum_n a_n x^n$ is the largest integer d such that $a_d \neq 0$. We take the degree of the zero polynomial to be $-\infty$.

Proposition 14.6. *Let R be a ring and $f, g \in R[x]$ be nonzero polynomials. Then*

$$\deg(f + g) \leq \max(\deg f, \deg g) .$$

PROOF: If $a_n + b_n \neq 0$ then $a_n \neq 0$ or $b_n \neq 0$. \square

Proposition 14.7. *The function $i : n \rightarrow \mathbb{Z}[x_1, \dots, x_n]$ that maps k to x_k is initial in the category with:*

- *objects all pairs $j : n \rightarrow R$ where R is a commutative ring and j a function*
- *morphisms $\phi : (j_1, R_1) \rightarrow (j_2, R_2)$ are ring homomorphisms $\phi : R_1 \rightarrow R_2$ such that $\phi \circ j_1 = j_2$.*

PROOF: The unique morphism $(i, \mathbb{Z}[x_1, \dots, x_n]) \rightarrow (j, R)$ maps a polynomial p to $p(j(0), j(1), \dots, j(n-1))$. \square

Proposition 14.8. *Let $\alpha : R \rightarrow S$ be a ring homomorphism. Let $s \in S$ commute with $\alpha(r)$ for all $r \in R$. Then there exists a unique ring homomorphism $\bar{\alpha} : R[x] \rightarrow S$ such that $\bar{\alpha}(x) = s$ and the following diagram commutes:*

$$\begin{array}{ccc} R[x] & \xrightarrow{\bar{\alpha}} & S \\ \uparrow & \nearrow \alpha & \\ R & & \end{array}$$

PROOF: The map $\bar{\alpha}$ is given by

$$\bar{\alpha}(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = \alpha(a_0) + \alpha(a_1)s + \alpha(a_2)s^2 + \cdots + \alpha(a_n)s^n .$$

\square

Definition 14.9. Let R be a commutative ring. Given a polynomial $p \in R[x]$, the *polynomial function* $p : R \rightarrow R$ is the function given by: $p(r) = \alpha_r(p)$, where $\alpha_r : R[x] \rightarrow R$ is the unique ring homomorphism such that the following diagram commutes.

$$\begin{array}{ccc} R[x] & \xrightarrow{\alpha_r} & R \\ x \uparrow & \nearrow r & \\ 1 & & \end{array}$$

Proposition 14.10. $\mathbb{Z}[x, y]$ is the coproduct of $\mathbb{Z}[x]$ and $\mathbb{Z}[y]$ in the category of commutative rings.

PROOF: Given ring homomorphisms $f : \mathbb{Z}[x] \rightarrow R$ and $g : \mathbb{Z}[y] \rightarrow R$, the required morphism $\mathbb{Z}[x, y] \rightarrow R$ maps $p(x, y)$ to $p(f(x), g(y))$. \square

Example 14.11. $\mathbb{Z}[x, y]$ is not the coproduct of $\mathbb{Z}[x]$ and $\mathbb{Z}[y]$ in **Ring**. Given $f : \mathbb{Z}[x] \rightarrow R$ and $g : \mathbb{Z}[y] \rightarrow R$ with $f(x) \neq g(y)$, the mediating morphism $\mathbb{Z}[x, y] \rightarrow R$ cannot exist since it must map xy to both $f(x)g(y)$ and $g(y)f(x)$. \square

Definition 14.12. A polynomial is *monic* iff its last non-zero coefficient is 1.

Proposition 14.13. A monic polynomial is not a left- or right-zero-divisor.

PROOF: Easy. \square

Proposition 14.14. Let R be a ring. Let $f, g \in R[x]$ with f monic. Then there exist unique polynomials $q, r \in R[x]$ with $\deg r < \deg f$ such that

$$g = qf + r .$$

PROOF:

$\langle 1 \rangle 1$. LET: $d = \deg f$

$\langle 1 \rangle 2$. For all $a \in R$ and $n > d$, there exists $h \in R[x]$ with $\deg h < n$ such that

$$ax^n = ax^{n-d}f + h .$$

PROOF: Take $h = ax^n - ax^{n-d}f$.

$\langle 1 \rangle 3$. For all $a \in R$ and $n > d$, there exists $q, h \in R[x]$ with $\deg h \leq d$ such that

$$ax^n = qf + h .$$

PROOF: Repeating $\langle 1 \rangle 2$ by induction.

$\langle 1 \rangle 4$. LET: $g = \sum_{i=0}^n a_i x^i$

$\langle 1 \rangle 5$. For $i > d$, PICK $q_i h_i \in R[x]$ with $\deg h_i < \deg f$ such that $a_i x^i = q_i f + h_i$

$\langle 1 \rangle 6$. $g = (\sum_{i=d+1}^n q_i) f + \sum_{i=d+1}^n h_i$

$\langle 1 \rangle 7$. q and r are unique.

PROOF: If $q_1 f + r_1 = q_2 f + r_2$ then $r_1 - r_2 = (q_2 - q_1)f$ and so $r_1 - r_2 = (q_2 - q_1)f = 0$ since $\deg(r_1 - r_2) < \deg f$.

\square

14.2 Laurent Polynomials

Definition 14.15 (Laurent Polynomial). Let R be a ring. The ring of *Laurent polynomials* is the group ring $R[\mathbb{Z}]$. We write $\sum_{n \in \mathbb{Z}} a_n x^n$ for $\sum_n a_n n$.

14.3 Power Series

Definition 14.16 (Power Series). Let R be a ring. A *power series* in R is a sequence (a_n) in R . We write the power series as

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

We write $R[[x]]$ for the set of all power series in R .

Define addition and multiplication on $R[[x]]$ by

$$\begin{aligned} \sum_n a_n x^n + \sum_n b_n x^n &= \sum_n (a_n + b_n) x^n \\ \left(\sum_n a_n x^n \right) \left(\sum_n b_n x^n \right) &= \sum_n \sum_{i+j=n} a_i b_j x^n \end{aligned}$$

Proposition 14.17. *For any ring R , the set of power series $R[[x]]$ is a ring.*

PROOF: Easy. \square

Proposition 14.18. *A power series $\sum_n a_n x^n$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R .*

PROOF:

$\langle 1 \rangle 1$. If $\sum_n a_n x^n$ is a unit then a_0 is a unit.

$\langle 2 \rangle 1$. LET: $\sum_n b_n x^n$ be the inverse of $\sum_n a_n x^n$.

$\langle 2 \rangle 2$. $a_0 b_0 = b_0 a_0 = 1$

$\langle 1 \rangle 2$. If a_0 is a unit then $\sum_n a_n x^n$ is a unit.

PROOF: Define the sequence (b_n) in R by

$$b_n = -a_0^{-1} \sum_{i=1}^n a_i b_{n-i}$$

Then $\sum_n b_n x^n$ is the inverse of $\sum_n a_n x^n$.

\square

Chapter 15

Ideals

Definition 15.1 (Left-Ideal). Let R be a ring.

A subgroup I of R is a *left-ideal* iff, for all $r \in R$, we have $rI \subseteq I$.

A subgroup I of R is a *right-ideal* iff, for all $r \in R$, we have $Ir \subseteq I$.

A subgroup I of R is a *(two-sided) ideal* iff it is a left-ideal and a right-ideal.

Example 15.2. Let R be a ring and $a \in R$. Then Ra is a left-ideal and aR is a right-ideal.

In particular, $\{0\}$ is always a two-sided ideal.

Example 15.3. Let S be a set and $T \subseteq S$. Then $\{X \in \mathcal{P}S : X \subseteq T\}$ is an ideal in $\mathcal{P}S$.

Proposition 15.4. Let S be a finite set. Then every ideal in $\mathcal{P}S$ is of the form $\{X \in \mathcal{P}S : X \subseteq T\}$ for some $T \subseteq S$.

PROOF:

$\langle 1 \rangle 1$. LET: I be an ideal in $\mathcal{P}S$.

$\langle 1 \rangle 2$. LET: $T = \bigcup I$

$\langle 1 \rangle 3$. For all $i \in T$ we have $\{i\} \in I$.

$\langle 2 \rangle 1$. LET: $i \in T$

$\langle 2 \rangle 2$. PICK $X \in I$ such that $i \in X$

$\langle 2 \rangle 3$. $\{i\} = \{i\} \cap X \in I$

$\langle 1 \rangle 4$. For all $X \subseteq T$ we have $X \in I$.

PROOF: If $X = \{x_1, \dots, x_n\}$ then $X = \{x_1\} + \dots + \{x_n\} \in I$.

□

Example 15.5. If S is an infinite set, then there is always an ideal in $\mathcal{P}S$ that is not of the form $\{X \in \mathcal{P}S : X \subseteq T\}$ for some $T \subseteq S$, namely the set of all finite subsets of S .

Proposition 15.6. Let $\phi : R \twoheadrightarrow S$ be a surjective ring homomorphism. Let J be an ideal in R . Then $\phi(J)$ is an ideal in S .

PROOF:

- $\langle 1 \rangle 1$. LET: $j \in J$ and $s \in S$
 PROVE: $s\phi(j), \phi(j)s \in \phi(J)$
 $\langle 1 \rangle 2$. PICK $r \in R$ such that $\phi(r) = s$
 $\langle 1 \rangle 3$. $rj, jr \in J$
 $\langle 1 \rangle 4$. $s\phi(j), \phi(j)s \in \phi(J)$
 \square

Example 15.7. We cannot remove the hypothesis that ϕ is surjective.

Let $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion. Then $i(2\mathbb{Z}) = 2\mathbb{Z}$ is not an ideal in \mathbb{Q} .

Proposition 15.8. Let $\phi : R \rightarrow S$ be a ring homomorphism and I a (left-, right-)ideal in S . Then $\phi^{-1}I$ is a (left-, right-)ideal in R .

PROOF: Easy. \square

Corollary 15.8.1. Let $\phi : R \rightarrow S$ be a ring homomorphism. Then $\ker \phi$ is an ideal in R .

Definition 15.9 (Quotient Ring). Let I be an ideal in R . The *quotient ring* R/I is the quotient group R/I under

$$(a + I)(b + I) = ab + I.$$

This is well-defined as, if $a + I = a' + I$ and $b + I = b' + I$ then

$$\begin{aligned}
 a - a' &\in I \\
 b - b' &\in I \\
 \therefore ab - a'b &\in I \\
 a'b - a'b' &\in I \\
 \therefore ab - a'b' &\in I
 \end{aligned}$$

Proposition 15.10. Let I be an ideal in R . Then the canonical group homomorphism $\pi : R \rightarrow R/I$ is a ring homomorphism.

PROOF: By construction. \square

Proposition 15.11. Let I be an ideal in a ring R . For every ring homomorphism $\phi : R \rightarrow S$ such that $I \subseteq \ker \phi$, there exists a unique ring homomorphism $\bar{\phi} : R/I \rightarrow S$ such that the following diagram commutes.

$$\begin{array}{ccc}
 R & \xrightarrow{\phi} & S \\
 \searrow \pi & & \nearrow \bar{\phi} \\
 & R/I &
 \end{array}$$

PROOF: Easy. \square

Corollary 15.11.1. Every ring homomorphism $\phi : R \rightarrow S$ decomposes as follows.

$$\begin{array}{ccccc}
 & & \phi & & \\
 & \searrow & & \nearrow & \\
 R & \twoheadrightarrow & R/\ker \phi & \xrightarrow{\cong} & \text{im } \phi & \hookrightarrow S
 \end{array}$$

Corollary 15.11.2 (First Isomorphism Theorem). *Let $\phi : R \rightarrow S$ be a surjective ring homomorphism. Then*

$$S \cong R/\ker \phi .$$

Theorem 15.12 (Third Isomorphism Theorem). *Let I and J be ideals in R with $I \subseteq J$. Then J/I is an ideal in R/I , and*

$$\frac{R/I}{J/I} \cong R/J$$

PROOF: Since the function $R/I \rightarrow R/J$ that maps $r + I$ to $r + J$ is a surjective ring homomorphism with kernel J/I . \square

Corollary 15.12.1. *Let $\phi : R \rightarrow S$ be a surjective ring homomorphism. Let J be an ideal in R . Then*

$$\frac{S}{\phi(J)} \cong \frac{R}{\ker \phi + J}$$

Proposition 15.13. *Let R be a ring and J an ideal in $\mathfrak{gl}_n(R)$. Let $A \in \mathfrak{gl}_n(R)$. Then $A \in J$ if and only if the matrices obtained by placing any entry of A in any position and zeros elsewhere all belong to J .*

PROOF: Each such matrix can be obtained by pre- and post-multiplying A by matrices which have a single 1 and 0s elsewhere. Conversely, A is a sum of such matrices. \square

Corollary 15.13.1. *Let R be a ring. Let J be an ideal in $\mathfrak{gl}_n(R)$. Let I be the set of all entries of elements of J . Then I is an ideal in R , and J is the set of all matrices whose entries are in I .*

Proposition 15.14. *Let R be a ring. Let $\{I_\alpha\}_{\alpha \in A}$ be a family of ideals in R . Let*

$$\sum_{\alpha \in A} I_\alpha = \left\{ \sum_{\alpha \in A} r_\alpha : \forall \alpha, r_\alpha \in I_\alpha, r_\alpha = 0 \text{ for all but finitely many } \alpha \in A \right\} .$$

Then $\sum_{\alpha \in A} I_\alpha$ is an ideal, and is the smallest ideal that includes every I_α .

PROOF: Easy. \square

Proposition 15.15. *The intersection of a set of ideals is an ideal.*

PROOF: Easy. \square

15.1 Characteristic

Definition 15.16 (Characteristic). The *characteristic* of a ring R is the non-negative integer n such that $n\mathbb{Z}$ is the kernel of the unique ring homomorphism $\mathbb{Z} \rightarrow R$.

Proposition 15.17. *Let R be a ring. If the unit 1 has finite order in R , then its order is the characteristic of R ; otherwise, the characteristic of R is 0.*

PROOF: Easy. \square

Example 15.18. The zero ring is the only ring with characteristic 1.

15.2 Nilradical

Definition 15.19 (Nilradical). Let R be a commutative ring. The *nilradical* of R is the set of all nilpotent elements.

Proposition 15.20. *Let R be a commutative ring. The nilradical of R is an ideal in R .*

PROOF: If $a^n = 0$ then for any b we have $(ba)^n = 0$. \square

Example 15.21. We cannot remove the assumption that R is commutative. In $\mathfrak{gl}_2(\mathbb{R})$ we have that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not.

15.3 Principal Ideals

Definition 15.22 (Principal Ideal). Let R be a commutative ring and $a \in R$. The *principal ideal* generated by a is $(a) = Ra = aR$.

Example 15.23. $\{0\} = (0)$ and $R = \{1\}$ are principal ideals.

Definition 15.24. Let R be a commutative ring and $\{a_\alpha\}_{\alpha \in A}$ be a family of elements of R . The *ideal generated by the elements a_α* is

$$(a_\alpha)_{\alpha \in A} := \sum_{\alpha \in A} (a_\alpha) \ .$$

An ideal is *finitely generated* iff it is generated by a finite family of elements.

Definition 15.25. Let R be a commutative ring and I, J be ideals in R . Then IJ is the ideal generated by $\{ij\}_{i \in I, j \in J}$.

Proposition 15.26.

$$IJ \subseteq I \cap J$$

PROOF: Easy. \square

Proposition 15.27. *Let R be a commutative ring. Let I and J be ideals in R . If $I + J = R$ then $IJ = I \cap J$.*

PROOF:

$\langle 1 \rangle 1$. LET: $r \in I \cap J$

$\langle 1 \rangle 2$. PICK $i \in I$ and $j \in J$ such that $i + j = 1$.

$\langle 1 \rangle 3$. $ri, rj \in IJ$

$\langle 1 \rangle 4$. $r = ri + rj \in IJ$

\square

Proposition 15.28. *Let R be a commutative ring. Let $f \in R[x]$ be a monic polynomial of degree d . Then the function*

$$\phi : R[x] \rightarrow R^{\oplus d}$$

that sends a polynomial g to the remainder of the division of g by f induces an isomorphism of Abelian groups

$$\frac{R[x]}{(f(x))} \cong R^{\oplus d}.$$

PROOF: It is clearly a group homomorphism; it is surjective since it maps any polynomial of degree $< d$ to itself, and its kernel is $(f(x))$ since these are the polynomials with remainder 0. \square

Corollary 15.28.1. *Let R be a commutative ring and $a \in R$. Then we have*

$$\frac{R[x]}{(x - a)} \cong R$$

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : R[x] \rightarrow R$ be evaluation at a .

$\langle 1 \rangle 2$. $\phi(g)$ is the remainder when dividing g by $x - a$.

PROOF: If $g = (x - a)q + r$ then $g(a) = (a - a)q(a) + r = r$.

$\langle 1 \rangle 3$. ϕ induces a group isomorphism $R[x]/(x - a) \cong R$

PROOF: By the theorem.

$\langle 1 \rangle 4$. This isomorphism is a ring isomorphism.

PROOF: Since evaluation at a is a ring homomorphism.

\square

Example 15.29. We have

$$\frac{\mathbb{R}[x]}{(x^2 + 1)} \cong \mathbb{C}$$

as rings.

15.4 Maximal Ideals

Definition 15.30 (Maximal Ideal). Let R be a ring and I an ideal in R . Then I is a *maximal ideal* iff $I \neq R$ and, whenever J is an ideal with $I \subseteq J$, then either $I = J$ or $J = R$.

Chapter 16

Integral Domains

Definition 16.1 (Integral Domain). An *integral domain* is a non-trivial commutative ring with no nonzero zero-divisors.

Example 16.2. \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are integral domains.

Proposition 16.3. $\mathbb{Z}/n\mathbb{Z}$ is an integral domain if and only if n is prime.

PROOF:

$$\begin{aligned} n \text{ is prime} &\Leftrightarrow \forall a, b \in \mathbb{Z} (n \mid ab \Rightarrow n \mid a \vee n \mid b) \\ &\Leftrightarrow \forall a, b \in \mathbb{Z}/n\mathbb{Z} (ab \cong 0(\text{mod } n) \Rightarrow a \cong 0(\text{mod } n) \vee b \cong 0(\text{mod } n)) \\ &\Leftrightarrow \mathbb{Z}/n\mathbb{Z} \text{ is an integral domain} \quad \square \end{aligned}$$

Proposition 16.4. In an integral domain, if $x^2 = 1$ then $x = \pm 1$.

PROOF: We have $x^2 - 1 = (x - 1)(x + 1) = 0$ so $x - 1 = 0$ or $x + 1 = 0$. \square

Proposition 16.5. Let R be an integral domain and $f, g \in R[x]$. Then

$$\deg(fg) = \deg f + \deg g$$

PROOF:

$\langle 1 \rangle 1$. LET: $f = \sum_n a_n x^n$ and $g = \sum_n b_n x^n$.

$\langle 1 \rangle 2$. LET: $d = \deg f$ and $e = \deg g$.

$\langle 1 \rangle 3$. The $d + e$ th term of fg is

$$a_d b_e x^{d+e}$$

which is non-zero.

$\langle 1 \rangle 4$. For $n > d + e$ the n th term of fg is 0.

\square

Corollary 16.5.1. Let R be a ring. Then $R[x]$ is an integral domain if and only if R is an integral domain.

Proposition 16.6. Let R be a ring. Then $R[[x]]$ is an integral domain if and only if R is an integral domain.

PROOF:

$\langle 1 \rangle 1$. If $R[[x]]$ is an integral domain then R is an integral domain.

PROOF: Easy.

$\langle 1 \rangle 2$. If R is an integral domain then $R[[x]]$ is an integral domain.

$\langle 2 \rangle 1$. ASSUME: R is an integral domain.

$\langle 2 \rangle 2$. LET: $(\sum_n a_n x^n)(\sum_n b_n x^n) = 0$

$\langle 2 \rangle 3$. $a_0 b_0 = 0$

$\langle 2 \rangle 4$. $a_0 = 0$ or $b_0 = 0$

$\langle 2 \rangle 5$. ASSUME: w.l.o.g. $b_0 \neq 0$

PROVE: For all n we have $a_n = 0$

$\langle 2 \rangle 6$. ASSUME: as induction hypothesis $a_0 = a_1 = \cdots = a_{n-1} = 0$

$\langle 2 \rangle 7$. $\sum_{i=0}^n a_i b_{n-i} = 0$

$\langle 2 \rangle 8$. $a_n b_0 = 0$

$\langle 2 \rangle 9$. $a_n = 0$

□

Proposition 16.7. *Let R be a ring and S an integral domain. Every ring homomorphism $\phi : R \rightarrow S$ is a ring homomorphism.*

PROOF:

$$\begin{aligned}\phi(1) &= \phi(1 \cdot 1) \\ &= \phi(1)\phi(1)\end{aligned}$$

and so $\phi(1) = 1$ by Cancellation. □

Proposition 16.8. *The characteristic of an integral domain is either 0 or a prime number.*

PROOF:

$\langle 1 \rangle 1$. LET: D be an integral domain.

$\langle 1 \rangle 2$. LET: n be the characteristic of D

$\langle 1 \rangle 3$. ASSUME: $n \neq 0$

$\langle 1 \rangle 4$. ASSUME: $n = ab$

$\langle 1 \rangle 5$. $ab = 0$ in D

$\langle 1 \rangle 6$. $a = 0$ or $b = 0$ in D

$\langle 1 \rangle 7$. $n \mid a$ or $n \mid b$

$\langle 1 \rangle 8$. One of a, b is 1 and the other is n .

□

16.1 Prime Ideals

Definition 16.9 (Prime Ideal). Let I be an ideal in a commutative ring R . Then I is a *prime ideal* iff R/I is an integral domain.

Example 16.10. Let R be a commutative ring and $a \in R$. Then $(x - a)$ is a prime ideal in R iff R is an integral domain.

Proposition 16.11. *Let R be a commutative ring and I a proper ideal in R . Then I is prime iff, whenever $ab \in I$, then $a \in I$ or $b \in I$.*

PROOF: The condition is the same as saying that, if $(a + I)(b + I) = I$, then $a + I = I$ or $b + I = I$. \square

Definition 16.12 (Spectrum). The *spectrum* of a commutative ring R , $\text{Spec } R$, is the set of prime ideals.

Proposition 16.13. Let $\phi : R \rightarrow S$ be a ring homomorphism. If I is a prime ideal in S then $\phi^{-1}(I)$ is a prime ideal in R .

PROOF: If $ab \in \phi^{-1}(I)$ then $\phi(a)\phi(b) \in I$ so either $\phi(a) \in I$ or $\phi(b) \in I$, i.e. either $a \in \phi^{-1}(I)$ or $b \in \phi^{-1}(I)$. \square

Proposition 16.14. Let R be a commutative ring. Suppose there exists a prime ideal P in R such that the only zero-divisor in P is 0. Then R is an integral domain.

PROOF:

$\langle 1 \rangle$ 1. ASSUME: $ab = 0$ in R

$\langle 1 \rangle$ 2. $ab \in P$

$\langle 1 \rangle$ 3. $a \in P$ or $b \in P$

$\langle 1 \rangle$ 4. $a = 0$ or $b = 0$

\square

Proposition 16.15. Let R be a commutative ring. The nilradical of R is included in every prime ideal of R .

PROOF: Let P be a prime ideal. If $a^n = 0$ then $a^n \in P$ hence $a \in P$. \square

Definition 16.16 (Krull Dimension). The (*Krull*) *dimension* of a commutative ring R is the length of the longest chain of prime ideals in R .

Example 16.17. $\mathbb{Z}[x]$ has Krull dimension 2.

Chapter 17

Unique Factorization Domains

Example 17.1. \mathbb{Z} is a UFD.

Chapter 18

Noetherian Rings

Definition 18.1 (Noetherian Ring). A commutative ring is *Noetherian* iff every ideal is finitely generated.

Proposition 18.2. *The homomorphic image of a Noetherian ring is Noetherian.*

PROOF:

⟨1⟩1. LET: R be a Noetherian ring, S be a commutative ring, and $\phi : R \rightarrow S$ a surjective ring homomorphism.

⟨1⟩2. LET: I be an ideal in S .

⟨1⟩3. LET: $\phi^{-1}(I) = (a_1, \dots, a_n)$

⟨1⟩4. $I = (\phi(a_1), \dots, \phi(a_n))$

□

Chapter 19

Principal Ideal Domains

Definition 19.1 (Principal Ideal Domain). A commutative ring is a *principal ideal domain* (PID) iff every ideal is principal.

Example 19.2. \mathbb{Z} is a PID by Proposition 7.16.

Example 19.3. $\mathbb{Z}[x]$ is not a PID. The ideal $(2, x)$ is not principal.

Proposition 19.4. *Every PID is Noetherian.*

PROOF: Trivial. \square

Proposition 19.5. *Every nonzero prime ideal in a PID is maximal.*

PROOF:

$\langle 1 \rangle$ 1. LET: R be a PID.

$\langle 1 \rangle$ 2. LET: I be a nonzero prime ideal in R .

$\langle 1 \rangle$ 3. PICK $a \in R$ such that $I = (a)$.

$\langle 1 \rangle$ 4. LET: J be an ideal such that $I \subseteq J$

$\langle 1 \rangle$ 5. PICK $b \in R$ such that $J = (b)$.

$\langle 1 \rangle$ 6. PICK $t \in R$ such that $a = bt$.

$\langle 1 \rangle$ 7. $b \in I$ or $t \in I$

$\langle 1 \rangle$ 8. CASE: $b \in I$

PROOF: Then $J \subseteq I$ so $I = J$.

$\langle 1 \rangle$ 9. CASE: $t \in I$

$\langle 2 \rangle$ 1. PICK $s \in R$ such that $t = as$.

$\langle 2 \rangle$ 2. $a = ast$

$\langle 2 \rangle$ 3. $st = 1$

PROOF: Since R is an integral domain.

$\langle 2 \rangle$ 4. $1 \in I$

$\langle 2 \rangle$ 5. $I = R$

\square

Corollary 19.5.1. *Any PID has Krull dimension 1.*

Chapter 20

Euclidean Domains

Example 20.1. \mathbb{Z} is a Euclidean domain.

Chapter 21

Division Rings

Definition 21.1 (Division Ring). A *division ring* is a ring in which every nonzero element is a two-sided unit.

Example 21.2. The quaternions form a division ring, with the inverse of a non-zero element $a + bi + cj + dk$ being

$$\frac{1}{a^2 + b^2 + c^2 + d^2}(a - bi - cj - dk) .$$

Example 21.3. For any ring R , the ring of polynomials $R[x]$ is not a division ring, since x has no inverse.

Proposition 21.4. *Every centralizer in a division ring is a division ring.*

PROOF: If $ar = ra$ then $ra^{-1} = a^{-1}r$. \square

Proposition 21.5. *A non-trivial ring R is a division ring if and only if its only left-ideals and right-ideals are $\{0\}$ and R .*

PROOF:

$\langle 1 \rangle 1$. If R is a division ring then the only left-ideals and right-ideals are $\{0\}$ and R .

$\langle 2 \rangle 1$. ASSUME: R is a division ring.

$\langle 2 \rangle 2$. The only left-ideals are $\{0\}$ and R .

$\langle 3 \rangle 1$. LET: I be a left-ideal that is not $\{0\}$.

PROVE: $I = R$

$\langle 3 \rangle 2$. PICK $a \in I - \{0\}$

$\langle 3 \rangle 3$. PICK a left inverse b for a

$\langle 3 \rangle 4$. $1 \in I$

PROOF: Since $1 = ba$.

$\langle 3 \rangle 5$. $I = R$

PROOF: For any $r \in R$ we have $r = r1 \in I$.

$\langle 2 \rangle 3$. The only right-ideals are $\{0\}$ and R .

PROOF: Similar.

$\langle 1 \rangle 2$. If the only left-ideals and right-ideals are $\{0\}$ and R then R is a division ring.

□

Proposition 21.6. *Let K be a division ring and R a non-trivial ring. Every ring homomorphism $K \rightarrow R$ is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : K \rightarrow R$ be a ring homomorphism.

PROVE: $\ker \phi = \{0\}$

$\langle 1 \rangle 2$. LET: $x \in \ker \phi$

$\langle 1 \rangle 3$. ASSUME: for a contradiction $x \neq 0$.

$\langle 1 \rangle 4$. $\phi(xx^{-1}) = 1$

$\langle 1 \rangle 5$. $0 = 1$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts the assumption that R is non-trivial.

□

Chapter 22

Simple Rings

Definition 22.1 (Simple Ring). A non-trivial ring is *R simple* iff its only two-sided ideals are $\{0\}$ and R .

Example 22.2. For any simple ring R we have $\mathfrak{gl}_n(R)$ is simple, by Corollary 15.13.1.

Proposition 22.3. *Let R be a ring and I an ideal in R . Then I is maximal iff R/I is simple.*

PROOF:

$$\begin{aligned} R/I \text{ is simple} &\Leftrightarrow \text{the only ideals in } R/I \text{ are } \{I\} \text{ and } R/I \\ &\Leftrightarrow \text{the only ideals in } R \text{ that include } I \text{ are } I \text{ and } R \\ &\Leftrightarrow I \text{ is maximal} \end{aligned}$$

□

Chapter 23

Reduced Rings

Definition 23.1 (Reduced Ring). A ring is *reduced* iff it has no non-zero nilpotent elements.

Proposition 23.2. *Let R be a commutative ring. Let N be its nilradical. Then R/N is reduced.*

PROOF:

- $\langle 1 \rangle 1$. LET: $r + N$ be nilpotent.
- $\langle 1 \rangle 2$. PICK n such that $(r + N)^n = N$
- $\langle 1 \rangle 3$. $r^n \in N$
- $\langle 1 \rangle 4$. PICK k such that $(r^n)^k = 0$
- $\langle 1 \rangle 5$. $r^{nk} = 0$
- $\langle 1 \rangle 6$. $r \in N$
- $\langle 1 \rangle 7$. $r + N = N$

□

Proposition 23.3. *Let R be a commutative ring. Let I and J be ideals in R . If R/IJ is reduced then $IJ = I \cap J$.*

PROOF:

- $\langle 1 \rangle 1$. LET: $r \in I \cap J$
PROVE: $r \in IJ$
- $\langle 1 \rangle 2$. $r^2 \in IJ$
- $\langle 1 \rangle 3$. $(r + IJ)^2 = IJ$
- $\langle 1 \rangle 4$. $r + IJ = IJ$

PROOF: Since R/IJ is reduced.

- $\langle 1 \rangle 5$. $r \in IJ$

□

Chapter 24

Boolean Rings

Definition 24.1 (Boolean). A ring is *Boolean* iff $a^2 = a$ for every element a .

Example 24.2. For any set S , the ring $\mathcal{P}S$ is Boolean.

Proposition 24.3. *Every non-trivial Boolean ring has characteristic 2.*

PROOF: We have $4 = 2$ and so $2 = 0$. \square

Proposition 24.4. *Every Boolean ring is commutative.*

PROOF:

$$\begin{aligned}(a + b)^2 &= a + b \\ \therefore a^2 + ab + ba + b^2 &= a + b \\ \therefore a + ab + ba + b &= a + b \\ \therefore ab + ba &= 0 \\ \therefore ab &= -ba \\ &= ba \quad (\text{Proposition 24.3})\end{aligned}$$

Example 24.5. The only Boolean integral domain is $\mathbb{Z}/2\mathbb{Z}$. For, if D is a Boolean integral domain and $x \in D$, we have $x^2 = x$, so $x^2 - x = x(x - 1) = 0$ and so $x = 0$ or $x = 1$, i.e. $D = \{0, 1\}$.

Proposition 24.6. *Every Boolean ring has Krull dimension 0.*

PROOF:

$\langle 1 \rangle 1$. LET: R be a Boolean ring.

$\langle 1 \rangle 2$. LET: I be a prime ideal in R .

PROVE: I is maximal.

$\langle 1 \rangle 3$. LET: J be an ideal with $I \subsetneq J$

$\langle 1 \rangle 4$. PICK $a \in J$ with $a \notin I$

$\langle 1 \rangle 5$. $a^2 - a = 0 \in I$

$\langle 1 \rangle 6$. $a(a - 1) \in I$

$$\langle 1 \rangle 7. \ a - 1 \in I$$

$$\langle 1 \rangle 8. \ a - 1 \in J$$

$$\langle 1 \rangle 9. \ 1 \in J$$

$$\langle 1 \rangle 10. \ J = R$$

□

Chapter 25

Modules

Definition 25.1 (Left Module). Let R be a ring and M an Abelian group. A *left-action* of R on M is a ring homomorphism

$$R \rightarrow \text{End}_{\mathbf{Ab}}(M) \quad .$$

A *left R -module* consists of an Abelian group M and a left-action of R on M .

Proposition 25.2. *Let R be a ring and M an Abelian group. Let $\cdot : R \times M \rightarrow M$. Then \cdot defines a left-action of R on M if and only if, for all $r, s \in R$ and $m, n \in M$:*

- $r(m + n) = rm + rn$
- $(r + s)m = rm + sm$
- $(rs)m = r(sm)$
- $1m = m$

PROOF: Immediate from definitions. \square

Proposition 25.3. *In any R -module M we have $0m = 0$ for all $m \in M$.*

PROOF: Since $0m = (0 + 0)m = 0m + 0m$ and so $0m = 0$ by cancellation in M . \square

Proposition 25.4. *In any R -module M we have $(-1)m = -m$ for all $m \in M$.*

PROOF: Since $m + (-1)m = 1m + (-1)m = (1 + (-1))m = 0m = 0$. \square

Proposition 25.5. *Every Abelian group is a \mathbb{Z} -module in exactly one way.*

PROOF: Since \mathbb{Z} is initial in **Ring**. \square

Definition 25.6 (Right Module). Let R be a ring. A *right R -module* consists of an Abelian group M and a function $\cdot : M \times R \rightarrow M$ such that, for all $r, s \in R$ and $m, n \in M$:

- $(m + n)r = mr + nr$
- $m(r + s) = mr + ms$
- $m(rs) = (mr)s$
- $m1 = m$

25.1 Homomorphisms

Definition 25.7 (Homomorphism of Left-Modules). Let R be a ring. Let M and N be left- R -modules. A *homomorphism of left- R -modules* $\phi : M \rightarrow N$ is a group homomorphism such that, for all $r \in R$ and $m \in M$, we have $\phi(rm) = r\phi(m)$.

Let $R - \mathbf{Mod}$ be the category of left- R -modules and left- R -module homomorphisms.

Example 25.8.

$$\mathbb{Z} - \mathbf{Mod} \cong \mathbf{Ab}$$

Example 25.9. The trivial group 0 is the zero object in $R - \mathbf{Mod}$.

Proposition 25.10. *Every bijective R -module homomorphism is an isomorphism.*

PROOF: Easy. \square

Proposition 25.11. *Let R be a ring. Let M be an R -module. Then*

$$M \cong R - \mathbf{Mod}[R, M]$$

as R -modules.

PROOF: The isomorphism maps m to the function $\lambda r.rm$. Its inverse maps an R -module homomorphism α to $\alpha(1)$. \square

Proposition 25.12. *Let R be a commutative ring. Let M be an R -module. Then there is a bijection between the set of $R[x]$ -module structures on M that extend the given R -module structure and $\text{End}_{R - \mathbf{Mod}}(M)$.*

PROOF:

- (1)1. LET: $\alpha : R \rightarrow \text{End}_{\mathbf{Ab}}(M)$ be the given R -module structure on M .
- (1)2. An $R[x]$ -module structure on M that extends α is a ring homomorphism $\beta : R[x] \rightarrow \text{End}_{\mathbf{Ab}}(M)$ such that $\beta \circ i = \alpha$, where i is the inclusion $R \rightarrow R[x]$.
- (1)3. There is a bijection between the $R[x]$ -module structures on M that extend α and the elements $s \in \text{End}_{\mathbf{Ab}}(M)$ that commute with $\alpha(r)$ for all $r \in R$.

PROOF: By the universal property for polynomials.

- (1)4. There is a bijection between the $R[x]$ -module structures on M that extend α and the R -module homomorphisms $(M, \alpha) \rightarrow (M, \alpha)$.

□

Proposition 25.13. *Let R be a commutative ring. Let M and N be R -modules. Then $R - \mathbf{Mod}[M, N]$ is an R -module under*

$$\begin{aligned}(\phi + \psi)(m) &= \phi(m) + \psi(m) \\ (r\phi)(m) &= r\phi(m)\end{aligned}$$

PROOF: Easy. □

Proposition 25.14. *Let R be an integral domain. Let I be a nonzero principal ideal of R . Then $I \cong R$ in $R - \mathbf{Mod}$.*

PROOF:

⟨1⟩1. PICK $a \in R$ such that $I = (a)$.

⟨1⟩2. LET: $\phi : R \rightarrow I$ be the map $\phi(r) = ra$.

⟨1⟩3. ϕ is an R -module homomorphism.

PROOF: Since $(r + s)a = ra + sa$ and $(rs)a = r(sa)$.

⟨1⟩4. ϕ is surjective.

⟨1⟩5. ϕ is injective.

PROOF: If $ra = sa$ then $(r - s)a = 0$ so $r - s = 0$ and $r = s$.

⟨1⟩6. $\phi : R \cong I$

□

25.2 Submodules

Definition 25.15 (Submodule). Let M be a left- R -module and $N \subseteq M$. Then N is a *submodule* of M iff N is a subgroup of M and $\forall r \in R, \forall n \in N, rn \in N$.

Proposition 25.16. *Let R be a ring and $I \subseteq R$. Then I is a left-ideal in R iff I is a submodule of R as an R -module.*

PROOF: Immediate from definitions. □

Proposition 25.17. *Let R be a ring. Let M and N be left- R -modules and $\phi : M \rightarrow N$ an R -module homomorphism. Then $\ker \phi$ is a submodule of M and $\text{im } \phi$ is a submodule of N .*

PROOF: Easy. □

Proposition 25.18. *Let R be a commutative ring. Let M be a left- R -module. Let $r \in R$. Then $rM = \{rm : m \in M\}$ is a submodule of M .*

PROOF: Easy. □

Proposition 25.19. *Let R be a ring. Let M be a left- R -module. Let I be a left-ideal in R . Then $IM = \{rm : r \in I, m \in M\}$ is a submodule of M .*

PROOF:

- $\langle 1 \rangle 1.$ IM is a subgroup of M .
 $\langle 2 \rangle 1.$ LET: $r, s \in I$ and $m, n \in M$.
 PROVE: $rm + sn \in IM$
 $\langle 2 \rangle 2.$ $rm + sn = r(m - n) + (s - r)n$
 $\langle 1 \rangle 2.$ For all $r \in R$ and $x \in IM$ we have $rx \in IM$.
 \square

25.3 Quotient Modules

Definition 25.20 (Quotient Module). Let R be a ring. Let M be a left- R -module. Let N be a submodule of M . Then the *quotient module* M/N is the quotient group M/N under

$$r(m + N) = rm + N \ .$$

Proposition 25.21. Let R be a ring. Let M and P be left- R -modules. Let N be a submodule of M . Let $\phi : M \rightarrow P$ be an R -module homomorphism. If $N \subseteq \ker \phi$, then there exists a unique R -module homomorphism $\bar{\phi} : M/N \rightarrow P$ such that the following diagram commutes.

$$\begin{array}{ccc}
 M & \xrightarrow{\phi} & P \\
 & \searrow & \nearrow \bar{\phi} \\
 & M/N &
 \end{array}$$

PROOF: Easy. \square

Theorem 25.22. Every R -module homomorphism $\phi : M \rightarrow M'$ may be decomposed as:

$$M \longrightarrow M/\ker \phi \xrightarrow{\cong} \text{im } \phi \longrightarrow N$$

PROOF: Easy. \square

Corollary 25.22.1 (First Isomorphism Theorem). Let $\phi : M \rightarrow M'$ be a surjective R -module homomorphism. Then

$$M' \cong \frac{M}{\ker \phi} \ .$$

Proposition 25.23 (Second Isomorphism Theorem). Let R be a ring. Let M be a left- R -module. Let N and P be submodules of M . Then $N + P$ is a submodule of M , $N \cap P$ is a submodule of P , and

$$\frac{N + P}{N} \cong \frac{P}{N \cap P}$$

PROOF: The function that maps P to $p + N$ is a surjective homomorphism $P \rightarrow (N + P)/N$ with kernel $N \cap P$. \square

Proposition 25.24 (Third Isomorphism Theorem). *Let R be a ring. Let M be a left- R -module. Let N be a submodule of M and P a submodule of N . Then N/P is a submodule of M/P and*

$$\frac{M/P}{N/P} \cong \frac{M}{N}$$

PROOF: The canonical map $M \rightarrow M/N$ induces a surjective homomorphism $M/P \rightarrow M/N$ which has kernel N/P . \square

Proposition 25.25. *Let R be a ring. Let M be a left- R -module. The sum and intersection of a family of submodules of M are submodules of M .*

PROOF: Easy. \square

25.4 Products

Proposition 25.26. *$R - \mathbf{Mod}$ has products.*

PROOF: Given a family $\{M_\alpha\}_{\alpha \in A}$ of left- R -modules, we make $\prod_{\alpha \in A} M_\alpha$ into a left- R -module by

$$\begin{aligned} (f + g)(\alpha) &= f(\alpha) + g(\alpha) \\ (rf)(\alpha) &= rf(\alpha) \end{aligned}$$

\square

25.5 Coproducts

Proposition 25.27. *$R - \mathbf{Mod}$ has coproducts.*

PROOF: Given a family $\{M_\alpha\}_{\alpha \in A}$ of left- R -modules, take $\bigoplus_{\alpha \in A} M_\alpha$ to be $\{f \in \prod_{\alpha \in A} M_\alpha : f(\alpha) = 0 \text{ for all but finitely many } \alpha \in A\}$. \square

25.6 Direct Sum

Definition 25.28 (Direct Sum). *Let R be a ring. Let M and N be left- R -modules. Then the direct sum $M \oplus N$ is an R -module under*

$$r(m, n) = (rm, rn) .$$

Proposition 25.29. *$M \oplus N$ is the biproduct of M and N in $R - \mathbf{Mod}$.*

PROOF: Easy. \square

Example 25.30. Infinite products and coproducts are in general different. We have $\mathbb{Z}^{\mathbb{N}} \not\cong \mathbb{Z}^{\oplus \mathbb{N}}$ since $\mathbb{Z}^{\mathbb{N}}$ is uncountable but $\mathbb{Z}^{\oplus \mathbb{N}}$ is countable.

25.7 Kernels and Cokernels

Proposition 25.31. *Let R be a ring. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then $\ker \phi \hookrightarrow M$ is terminal in the category of left- R -module homomorphisms $\alpha : P \rightarrow M$ such that $\phi \circ \alpha = 0$.*

PROOF: Easy. \square

Proposition 25.32. *Let R be a ring. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then $N \twoheadrightarrow \operatorname{coker} \phi$ is initial in the category of left- R -module homomorphisms $\alpha : N \rightarrow P$ such that $\alpha \circ \phi = 0$.*

PROOF: Easy. \square

Proposition 25.33. *Let R be a ring. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then the following are equivalent.*

- ϕ is a monomorphism.
- $\ker \phi$ is trivial.
- ϕ is injective.

PROOF: Easy. \square

Proposition 25.34. *Let R be a ring. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then the following are equivalent.*

- ϕ is an epimorphism.
- $\operatorname{coker} \phi$ is trivial.
- ϕ is surjective.

PROOF: Easy. \square

Proposition 25.35. *Every monomorphism in $R - \mathbf{Mod}$ is the kernel of some homomorphism.*

PROOF: If $\phi : M \rightarrow N$ is a monomorphism then it is the kernel of $N \twoheadrightarrow N/\operatorname{im} \phi$. \square

Proposition 25.36. *Every epimorphism in $R - \mathbf{Mod}$ is the cokernel of some homomorphism.*

PROOF: If $\phi : M \rightarrow N$ is epi then it is the cokernel of $\ker \phi \hookrightarrow M$. \square

Example 25.37. Monomorphisms do not split in $R - \mathbf{Mod}$. Multiplication by 2 is a monomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$ but has no left inverse.

Example 25.38. Epimorphisms do not split in $R - \mathbf{Mod}$. The canonical map $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ is an epimorphism without a right inverse.

25.8 Free Modules

Proposition 25.39. *Let R be a ring and A a set. Then there exists a left- R -module $F^R(A)$ and function $j : A \rightarrow F^R(A)$ such that, for any left- R -module M and function $f : A \rightarrow M$, there exists a unique left- R -module homomorphism $\bar{f} : F^R(A) \rightarrow M$ such that the following diagram commutes.*

$$\begin{array}{ccc} F^R(A) & \xrightarrow{\bar{f}} & M \\ j \uparrow & \nearrow f & \\ A & & \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: $R^{\oplus A} = \{\alpha : A \rightarrow R : \alpha(a) = 0 \text{ for all but finitely many } a \in A\}$
under the operations

$$\begin{aligned} (\alpha + \beta)(a) &= \alpha(a) + \beta(a) \\ (r\alpha)(a) &= r\alpha(a) \end{aligned}$$

$\langle 1 \rangle 2$. $R^{\oplus A}$ is a left- R -module.

$\langle 1 \rangle 3$. LET: $j : A \rightarrow R^{\oplus A}$ be the function

$$j(a)(a') = \begin{cases} 1 & \text{if } a = a' \\ 0 & \text{if } a \neq a' \end{cases}$$

$\langle 1 \rangle 4$. LET: M be any left- R -module.

$\langle 1 \rangle 5$. LET: $f : A \rightarrow M$ be a function.

$\langle 1 \rangle 6$. LET: $\bar{f} : R^{\oplus A} \rightarrow M$ be the function

$$\bar{f}(\alpha) = \sum_{a \in A, \alpha(a) \neq 0} \alpha(a)f(a)$$

$\langle 1 \rangle 7$. \bar{f} is a left- R -module homomorphism.

$\langle 1 \rangle 8$. $\bar{f} \circ j = f$

$\langle 1 \rangle 9$. \bar{f} is unique.

Definition 25.40. We call $j : A \rightarrow F^R(A)$ the *free* left- R -module over A .

Proposition 25.41. j is injective.

PROOF: By the proof of the previous proposition. \square

Proposition 25.42. *Let R be a ring. Let F be a non-zero free left- R -module. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then ϕ is onto if and only if, for every left- R -module homomorphism $\alpha : F \rightarrow N$, there exists a left- R -module homomorphism $\beta : F \rightarrow M$ such that the diagram below commutes.*

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \beta \uparrow & \nearrow \alpha & \\ F & & \end{array}$$

PROOF:

- (1)1. LET: F be the free left- R -module over A with injection $j : A \rightarrow F$.
 (1)2. If ϕ is onto then, for every homomorphism $\alpha : F \rightarrow N$, there exists a homomorphism $\beta : F \rightarrow M$ such that $\phi \circ \beta = \alpha$.
 (2)1. ASSUME: ϕ is onto.
 (2)2. LET: $\alpha : F \rightarrow N$ be a homomorphism.
 (2)3. For $a \in A$, PICK $f(a) \in M$ such that $\phi(f(a)) = \alpha(j(a))$
 (2)4. LET: $\beta : F \rightarrow M$ be the unique homomorphism such that $\beta \circ j = f$
 (2)5. $\phi \circ \beta = \alpha$
 PROOF: Each is the unique homomorphism such that $\alpha \circ j = \phi \circ f$.

□

$$\begin{array}{ccccc}
 & & M & \xrightarrow{\phi} & N \\
 & f \nearrow & \uparrow \beta & \nearrow \alpha & \\
 A & \xrightarrow{j} & F & &
 \end{array}$$

- (1)3. If, for every homomorphism $\alpha : F \rightarrow N$, there exists a homomorphism $\beta : F \rightarrow M$ such that $\phi \circ \beta = \alpha$, then ϕ is onto.
 (2)1. ASSUME: For every homomorphism $\alpha : F \rightarrow N$ there exists a homomorphism $\beta : F \rightarrow M$ such that $\phi \circ \beta = \alpha$.
 (2)2. LET: $n \in N$
 (2)3. LET: $\alpha : F \rightarrow N$ be the unique homomorphism such that, for all $a \in A$, we have $\alpha(j(a)) = n$
 (2)4. PICK a homomorphism $\beta : F \rightarrow M$ such that $\phi \circ \beta = \alpha$
 (2)5. PICK $a \in A$
 (2)6. $\phi(\beta(j(a))) = n$

□

25.9 Generators

Definition 25.43 (Submodule Generated by a Set). Let R be a ring. Let M be a left- R -module. Let A be a subset of M . Let $\phi_A : F^R(A) \rightarrow M$ be the unique left- R -module homomorphism such that the following diagram commutes.

$$\begin{array}{ccc}
 F^R(A) & \xrightarrow{\phi_A} & M \\
 \uparrow & \nearrow & \\
 A & &
 \end{array}$$

The submodule of M generated by A , denoted $\langle A \rangle$, is defined to be $\text{im } \phi_A$.

Definition 25.44 (Finitely Generated). Let R be a ring. Let M be a left- R -module. Then M is *finitely generated* iff there exists a finite set $A \subseteq M$ such that $M = \langle A \rangle$.

Example 25.45. A submodule of a finitely generated module is not necessarily finitely generated.

Let $R = \mathbb{Z}[x_1, x_2, \dots]$. Then R is finitely generated as an R -module, but (x_1, x_2, \dots) is not.

Proposition 25.46. *The homomorphic image of a finitely generated module is finitely generated.*

PROOF: Easy. \square

Proposition 25.47. *Let R be a ring. Let M be a left- R -module. Let N be a submodule of M . If N and M/N are finitely generated then M is finitely generated.*

PROOF:

$\langle 1 \rangle 1$. PICK a_1, \dots, a_n that generate N .

$\langle 1 \rangle 2$. PICK b_1, \dots, b_m such that $b_1 + N, \dots, b_m + N$ generate M/N .

PROVE: $a_1, \dots, a_n, b_1, \dots, b_m$ generate M .

$\langle 1 \rangle 3$. LET: $m \in M$

$\langle 1 \rangle 4$. PICK $r_1, \dots, r_m \in R$ such that $m + N = r_1 b_1 + \dots + r_m b_m + N$

$\langle 1 \rangle 5$. $m - r_1 b_1 - \dots - r_m b_m \in N$

$\langle 1 \rangle 6$. PICK $s_1, \dots, s_n \in R$ such that $m - r_1 b_1 - \dots - r_m b_m = s_1 a_1 + \dots + s_n a_n$

$\langle 1 \rangle 7$. $m = r_1 b_1 + \dots + r_m b_m + s_1 a_1 + \dots + s_n a_n$

\square

25.10 Projections

Definition 25.48 (Projection). Let R be a ring. Let M be a left- R -module. Let $p : M \rightarrow M$ be a left- R -module homomorphism. Then p is a *projection* iff $p^2 = p$.

Proposition 25.49. *Let R be a ring. Let M be a left- R -module. Let $p : M \rightarrow M$ be a projection. Then*

$$M \cong \ker p \oplus \operatorname{im} p .$$

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : M \rightarrow \ker p \oplus \operatorname{im} p$ be the map $\phi(m) = (m - p(m), p(m))$

$\langle 1 \rangle 2$. ϕ is a left- R -module homomorphism.

$\langle 1 \rangle 3$. ϕ is injective.

$\langle 1 \rangle 4$. ϕ is surjective.

\square

25.11 Pullbacks

Proposition 25.50. *$R - \operatorname{Mod}$ has pullbacks.*

PROOF:

$\langle 1 \rangle 1$. LET: $\mu : M \rightarrow Z, \nu : N \rightarrow Z$ be left- R -module homomorphisms.

$\langle 1 \rangle 2$. LET: $M \times_Z N = \{(m, n) \in M \times N : \mu(m) = \nu(n)\}$ under

$$(m, n) + (m', n') = (m + m', n + n')$$

$$r(m, n) = (rm, rn)$$

$\langle 1 \rangle 3$. $M \times_Z N$ is the pullback of M and N .

\square

25.12 Pushouts

Proposition 25.51. $R - \mathbf{Mod}$ has pushouts.

PROOF:

$\langle 1 \rangle 1$. LET: $\mu : A \rightarrow M$ and $\nu : A \rightarrow N$ be left- R -module homomorphisms.

Chapter 26

Cyclic Modules

Definition 26.1 (Cyclic Module). Let R be a ring. Let M be a left- R -module. Then M is *cyclic* iff there exists $m \in M$ such that $M = \langle m \rangle$.

Proposition 26.2. Let R be a ring. Let M be a left- R -module. Then M is cyclic if and only if there exists a left-ideal I in R such that $M \cong R/I$.

PROOF:

$\langle 1 \rangle 1$. If M is cyclic then there exists a left-ideal I in R such that $M \cong R/I$.

$\langle 2 \rangle 1$. ASSUME: M is cyclic.

$\langle 2 \rangle 2$. PICK $m \in M$ such that $M = \langle m \rangle$

$\langle 2 \rangle 3$. LET: $\phi : R \rightarrow M$ be the left- R -module homomorphism $\phi(r) = rm$.

$\langle 2 \rangle 4$. ϕ is surjective.

$\langle 2 \rangle 5$. $M \cong R/\ker \phi$

$\langle 1 \rangle 2$. For every left-ideal I in R , we have that R/I is cyclic.

PROOF: R/I is generated by $1 + I$.

□

Proposition 26.3. A quotient of a cyclic module is cyclic.

PROOF: If M is generated by m then M/N is generated by $m + N$. □

Proposition 26.4. Let R be a ring. For any left-ideal I in R and any left- R -module N , we have

$$R - \mathbf{Mod}[R/I, N] \cong \{n \in N : \forall a \in I. an = 0\} .$$

PROOF:

$\langle 1 \rangle 1$. LET: $\Phi : R - \mathbf{Mod}[R/I, N] \rightarrow \{n \in N : \forall a \in I. an = 0\}$ be the function

$$\Phi(\alpha) = \alpha(1 + I)$$

PROOF: For all $a \in I$ we have $a\alpha(1 + I) = \alpha(a + I) = \alpha(I) = 0$.

$\langle 1 \rangle 2$. Φ is injective.

PROOF: If $\alpha(1 + I) = \beta(1 + I)$ then $\alpha(r + I) = r\alpha(1 + I) = r\beta(1 + I) = \beta(r + I)$ for all $r \in R$, hence $\alpha = \beta$.

$\langle 1 \rangle 3$. Φ is surjective.

PROOF: Given $n \in N$ such that $\forall a \in I. an = 0$, define $\alpha : R/I \rightarrow N$ by $\alpha(r + I) = rn$.

$\langle 1 \rangle 4$. If R is commutative then Φ is an R -module homomorphism.

□

Corollary 26.4.1. *For all $a, b \in \mathbb{Z}$ we have $\mathbf{Ab}[\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}] \cong \mathbb{Z}/\gcd(a, b)\mathbb{Z}$.*

PROOF:

$$\begin{aligned} \mathbf{Ab}[\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}] &\cong \mathbb{Z} - \mathbf{Mod}[\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}] \\ &\cong \{n \in \mathbb{Z}/b\mathbb{Z} : \forall x \in a\mathbb{Z}. xn \cong 0(\text{mod } b)\} \\ &\cong \{n \in \mathbb{Z}/b\mathbb{Z} : \forall x \in \mathbb{Z}. b \mid xan\} \\ &= \{n \in \mathbb{Z}/b\mathbb{Z} : b \mid an\} \end{aligned}$$

Chapter 27

Simple Modules

Definition 27.1 (Simple Module). Let R be a ring. An R -module M is *simple* or *irreducible* iff its only submodules are $\{0\}$ and M .

Proposition 27.2 (Schur's Lemma). *Let R be a ring. Let M and N be simple R -modules. Let $\phi : M \rightarrow N$ be an R -module homomorphism. Then either $\phi = 0$ or ϕ is an isomorphism.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: $\phi \neq 0$

$\langle 1 \rangle 2$. $\ker \phi = 0$

PROOF: Since $\ker \phi$ is a submodule of M that is not M .

$\langle 1 \rangle 3$. $\operatorname{im} \phi = N$

PROOF: Since $\operatorname{im} \phi$ is a submodule of N that is not $\{0\}$.

□

Proposition 27.3. *Every simple module is cyclic.*

PROOF:

$\langle 1 \rangle 1$. LET: M be a simple module.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $M \neq \{0\}$

PROOF: $\{0\} = \langle 0 \rangle$ is cyclic.

$\langle 1 \rangle 3$. PICK $m \in M$ with $m \neq 0$

$\langle 1 \rangle 4$. $\langle m \rangle = M$

PROOF: Since $\langle m \rangle$ is a submodule of M that is not $\{0\}$.

□

Chapter 28

Noetherian Modules

Definition 28.1 (Noetherian Module). Let R be a ring. A left- R -module is *Noetherian* iff every submodule is finitely generated.

Proposition 28.2. *Let R be a ring. Let M be a left- R -module and N a submodule of M . Then M is Noetherian if and only if N and M/N are Noetherian.*

PROOF:

⟨1⟩1. If M is Noetherian then N is Noetherian.

PROOF: Every submodule of N is a submodule of M , hence finitely generated.

⟨1⟩2. If M is Noetherian then M/N is Noetherian.

⟨2⟩1. ASSUME: M is Noetherian.

⟨2⟩2. LET: $\pi : M \rightarrow M/N$ be the canonical epimorphism.

⟨2⟩3. LET: P be a submodule of M/N .

⟨2⟩4. PICK $a_1, \dots, a_n \in M$ that generate $\pi^{-1}(P)$.

⟨2⟩5. $a_1 + N, \dots, a_n + N$ generate P .

⟨1⟩3. If N and M/N are Noetherian then M is Noetherian.

⟨2⟩1. ASSUME: N and M/N are Noetherian.

⟨2⟩2. LET: P be a submodule of M .

⟨2⟩3. PICK $a_1, \dots, a_m \in P$ such that $a_1 + N, \dots, a_m + N$ generate $\pi(P)$.

⟨2⟩4. PICK $b_1, \dots, b_n \in M$ that generated $P \cap N$.

PROVE: $a_1, \dots, a_m, b_1, \dots, b_n$ generate P .

⟨2⟩5. LET: $p \in P$

⟨2⟩6. PICK $r_1, \dots, r_m \in R$ such that $p + N = r_1 a_1 + \dots + r_m a_m + N$

⟨2⟩7. $p - r_1 a_1 - \dots - r_m a_m \in P \cap N$

⟨2⟩8. PICK $s_1, \dots, s_n \in R$ such that $p - r_1 a_1 - \dots - r_m a_m = s_1 b_1 + \dots + s_n b_n$

⟨2⟩9. $p = r_1 a_1 + \dots + r_m a_m + s_1 b_1 + \dots + s_n b_n$

□

Corollary 28.2.1. *If R is a Noetherian ring then $R^{\oplus n}$ is a Noetherian left- R -module.*

PROOF: The proof is by induction on n . The case $n = 1$ is immediate.

The induction step holds since $R^{\oplus(n+1)}/R^{\oplus n} \cong R$. □

Corollary 28.2.2. *If R is a Noetherian ring and M is a finitely generated left- R -module then M is Noetherian.*

PROOF: There is a surjective homomorphism $R^{\oplus n} \twoheadrightarrow M$ for some n , so M is a quotient of $R^{\oplus n}$. \square

Chapter 29

Algebras

Definition 29.1 (Algebra). Let R be a commutative ring. An R -algebra consists of a ring S and a ring homomorphism $\alpha : R \rightarrow S$ such that $\alpha(R)$ is included in the center of S . We write rs for $\alpha(r)s$.

Proposition 29.2. Let R be a commutative ring and S a ring. Let $\cdot : R \times S \rightarrow S$. Then there exists $\alpha : R \rightarrow S$ that makes S into an R -algebra such that

$$rs = \alpha(r)s \quad (r \in R, s \in S)$$

iff S is an R -module under \cdot and, for all $r_1, r_2 \in R$ and $s_1, s_2 \in S$,

$$(r_1 s_1)(r_2 s_2) = (r_1 r_2)(s_1 s_2) .$$

PROOF: Immediate from definitions. \square

Example 29.3. Let R be a commutative ring. Then R is an R -algebra under multiplication.

Example 29.4. Let R be a commutative ring and I an ideal in R . Then R/I is an R -algebra.

Example 29.5. Let R be a commutative ring and M an R -module. Then $\text{End}_{R\text{-Mod}}(M)$ is an R -algebra under composition.

Example 29.6. Let R be a commutative ring. Then $\mathfrak{gl}_n(R)$ is an R -algebra under matrix multiplication.

Definition 29.7 (Algebra Homomorphism). Let R be a commutative ring. Let S and T be R -algebras. An R -algebra homomorphism $\phi : S \rightarrow T$ is a ring homomorphism such that, for all $r \in R$ and $s \in S$, we have $\phi(rs) = r\phi(s)$.

Let $R\text{-Alg}$ be the category of R -algebras and R -algebra homomorphisms.

Example 29.8.

$$\mathbb{Z}\text{-Alg} \cong \mathbf{Ring}$$

Example 29.9. Let R be a commutative ring. Then $R[x_1, \dots, x_n]$, and any quotient ring of $R[x_1, \dots, x_n]$, is a commutative R -algebra.

Example 29.10. R is the initial object in $R\text{-Alg}$.

29.1 Rees Algebra

Definition 29.11 (Rees Algebra). Let R be a commutative ring. Let I be an ideal in R . The *Rees algebra* is the direct sum

$$\text{Rees}_R(I) = \bigoplus_{j \geq 0} I^j$$

under the multiplication

$$\begin{aligned} (r_0, r_1, r_2, r_3, \dots)(s_0, s_1, s_2, \dots) &= (r_0s_0, r_1s_0 + r_0s_1, r_2s_0 + r_1s_1 + r_0s_2, \dots) \\ r(r_0, r_1, r_2, \dots) &= (rr_0, rr_1, rr_2, \dots) \end{aligned}$$

Proposition 29.12. Let R be a commutative ring. Let $a \in R$ be a non-zero-divisor. Then $R[x]$ is the Rees algebra of (a) .

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : R[x] \rightarrow \text{Rees}_R((a))$ be the function $\phi(r_0 + r_1x + r_2x^2 + \dots) = (r_0, r_1a, r_2a^2, \dots)$.

$\langle 1 \rangle 2$. ϕ is an R -algebra homomorphism.

$\langle 1 \rangle 3$. ϕ is injective.

$\langle 2 \rangle 1$. LET: $\phi(r_0 + r_1x + r_2x^2 + \dots) = \phi(s_0 + s_1x + s_2x^2 + \dots)$

$\langle 2 \rangle 2$. For all n we have $r_na^n = s_na^n$

$\langle 2 \rangle 3$. $(r_n - s_n)a^n = 0$

$\langle 2 \rangle 4$. $r_n - s_n = 0$

PROOF: Since a is not a zero-divisor.

$\langle 2 \rangle 5$. $r_n = s_n$

$\langle 1 \rangle 4$. ϕ is surjective.

□

Proposition 29.13. Let R be a commutative ring. Let $a \in R$ be a non-zero-divisor. Let I be an ideal of R . Then $\text{Rees}_R(I) \cong \text{Rees}_R(aI)$.

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : \text{Rees}_R(I) \rightarrow \text{Rees}_R(aI)$ be the function $\phi(r_0, r_1, r_2, \dots) = (r_0, ar_1, a^2r_2, \dots)$.

$\langle 1 \rangle 2$. ϕ is an R -algebra homomorphism.

$\langle 1 \rangle 3$. ϕ is injective.

$\langle 1 \rangle 4$. ϕ is surjective.

□

29.2 Free Algebras

Proposition 29.14. Let R be a ring. Then $R[x_1, \dots, x_n]$ is the free commutative R -algebra on $\{1, \dots, n\}$.

PROOF: Easy. □

Proposition 29.15. Let R be a ring and A a set. Let A^* be the free monoid on A . Then the monoid ring $R[A^*]$ is the free R -algebra on A .

PROOF: Easy. \square

Proposition 29.16. *Let R be a commutative ring and S a commutative R -algebra. Then S is finitely generated as an R -algebra if and only if S is finitely generated as a commutative R -algebra.*

PROOF: Since a subalgebra of a commutative subalgebra is commutative, so the smallest algebra that contains $\{a_1, \dots, a_n\}$ is the smallest commutative subalgebra that contains $\{a_1, \dots, a_n\}$. \square

Chapter 30

Algebras of Finite Type

Definition 30.1 (Algebra of Finite Type). Let R be a ring. Let S be an R -algebra. Then R is of *finite type* iff S is a finitely generated R -algebra.

Proposition 30.2. *Let R be a Noetherian ring. Let S be a finite-type R -algebra. Then S is a Noetherian ring.*

Chapter 31

Finite Algebras

Definition 31.1 (Finite Algebra). Let R be a ring. Let S be an R -algebra. Then S is a *finite* R -algebra iff it is a finitely generated left- R -module.

Proposition 31.2. *Let R be a ring. Every finite R -algebra is of finite type.*

PROOF: If S is generated by a_1, \dots, a_n as an R -module, then it is generated by a_1, \dots, a_n as an R -algebra. \square

Example 31.3. The converse does not hold. $R[x]$ is of finite type but is not finite.

Chapter 32

Division Algebras

Definition 32.1 (Division Algebra). Let R be a commutative ring. A *division R -algebra* is an R -algebra that is a division ring.

Example 32.2. Let R be a commutative ring. Let M be a simple R -algebra. Then $\text{End}_{R\text{-Mod}}(M)$ is a division algebra. For if $\phi \circ \psi = 0$ then ϕ and ψ cannot both be isomorphisms, hence $\phi = 0$ or $\psi = 0$ by Schur's Lemma.

Chapter 33

Chain Complexes

Definition 33.1 (Chain Complex). Let R be a ring. A *chain complex of left- R -modules* $M_\bullet = (M_\bullet, d_\bullet)$ consists of a family of left- R -modules $\{M_i\}_{i \in \mathbb{Z}}$ and a family of left- R -module homomorphisms $\{d_i : M_i \rightarrow M_{i-1}\}_{i \in \mathbb{Z}}$ such that, for all i ,

$$d_i \circ d_{i+1} = 0 \ .$$

We call each d_i a *differential* and the family $\{d_i\}_i$ the *boundary* of the chain complex.

Definition 33.2 (Exact). A chain complex M_\bullet is *exact* at M_i iff $\text{im } d_{i+1} = \ker d_i$.

It is *exact* or an *exact sequence* iff it is exact at M_i for all i .

Proposition 33.3. A complex

$$\cdots \rightarrow 0 \rightarrow L \xrightarrow{\alpha} M \rightarrow \cdots$$

is exact at L iff α is a monomorphism.

PROOF: Since both are equivalent to $\ker \alpha = 0$. \square

Proposition 33.4. A complex

$$\cdots \rightarrow M \xrightarrow{\beta} N \rightarrow 0 \rightarrow \cdots$$

is exact at N iff β is an epimorphism.

PROOF: Since both are equivalent to $\text{im } \beta = N$. \square

Definition 33.5 (Short Exact Sequence). A *short exact sequence* is an exact complex of the form

$$0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0 \ .$$

Proposition 33.6 (Four-Lemma). *If*

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & D_1 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta \\ A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{h_2} & D_2 \end{array}$$

is a commutative diagram of left- R -modules with exact rows, α is an epimorphism, and β and δ are monomorphisms, then γ is an monomorphism.

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in C_1$

$\langle 1 \rangle 2$. ASSUME: $\gamma(x) = \gamma(y)$

$\langle 1 \rangle 3$. $\delta(h_1(x)) = \delta(h_1(y))$

$\langle 1 \rangle 4$. $h_1(x) = h_1(y)$

PROOF: δ is injective.

$\langle 1 \rangle 5$. $x - y \in \ker h_1$

$\langle 1 \rangle 6$. $x - y \in \operatorname{im} g_1$

$\langle 1 \rangle 7$. PICK $b \in B_1$ such that $g_1(b) = x - y$.

$\langle 1 \rangle 8$. $g_2(\beta(b)) = 0$

PROOF: $g_2(\beta(b)) = \gamma(g_1(b)) = \gamma(x - y) = 0$

$\langle 1 \rangle 9$. $\beta(b) \in \ker g_2$

$\langle 1 \rangle 10$. $\beta(b) \in \operatorname{im} f_2$

$\langle 1 \rangle 11$. PICK $a' \in A_2$ such that $f_2(a') = \beta(b)$

$\langle 1 \rangle 12$. PICK $a \in A_1$ such that $\alpha(a) = a'$

PROOF: α is surjective.

$\langle 1 \rangle 13$. $\beta(f_1(a)) = \beta(b)$

$\langle 1 \rangle 14$. $f_1(a) = b$

PROOF: β is injective.

$\langle 1 \rangle 15$. $0 = g_1(b)$

PROOF: Since $g_1(b) = g_1(f_1(a)) = 0$.

$\langle 1 \rangle 16$. $x = y$

PROOF: $\langle 1 \rangle 7$

□

Proposition 33.7 (Four-Lemma). *If*

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & D_1 \\ \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{h_2} & D_2 \end{array}$$

is a commutative diagram of left- R -modules with exact rows, β and δ are epimorphisms, and ϵ is a monomorphism, then γ is an epimorphism.

PROOF:

$\langle 1 \rangle 1$. LET: $b_2 \in B_2$

$\langle 1 \rangle 2$. PICK $c_1 \in C_1$ such that $\delta(c_1) = g_2(b_2)$

PROOF: δ is surjective.

$\langle 1 \rangle 3$. $\epsilon(h_1(c_1)) = 0$

$\langle 1 \rangle 4$. $h_1(c_1) = 0$

PROOF: ϵ is injective.

$\langle 1 \rangle 5$. $c_1 \in \ker h_1$

$\langle 1 \rangle 6$. $c_1 \in \operatorname{im} g_1$

$\langle 1 \rangle 7$. PICK $b_1 \in B_1$ such that $g_1(b_1) = c_1$

$\langle 1 \rangle 8$. $g_2(\gamma(b_1)) = g_2(b_2)$

$\langle 1 \rangle 9$. $\gamma(b_1) - b_2 \in \ker g_2$

$\langle 1 \rangle 10$. $\gamma(b_1) - b_2 \in \operatorname{im} f_2$

$\langle 1 \rangle 11$. PICK $a_2 \in A_2$ such that $f_2(a_2) = \gamma(b_1) - b_2$.

$\langle 1 \rangle 12$. PICK $a_1 \in A_1$ such that $\beta(a_1) = a_2$.

PROOF: β is surjective.

$\langle 1 \rangle 13$. $\gamma(f_1(a_1)) = \gamma(b_1) - b_2$

$\langle 1 \rangle 14$. $b_2 = \gamma(b_1 - f_1(a_1))$

□

Theorem 33.8 (Snake Lemma). *Suppose we have R -modules and homomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 & \longrightarrow & 0 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & & \\ 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 & \longrightarrow & 0 \end{array}$$

such that the diagram commutes and the two rows are short exact sequences. Then there exists a homomorphism $\delta : \ker \nu \rightarrow \operatorname{coker} \lambda$ such that the following is an exact sequence.

$$0 \rightarrow \ker \lambda \xrightarrow{\alpha_1} \ker \mu \xrightarrow{\beta_1} \ker \nu \xrightarrow{\delta} \operatorname{coker} \lambda \xrightarrow{\alpha_0} \operatorname{coker} \mu \xrightarrow{\beta_0} \operatorname{coker} \nu \rightarrow 0 .$$

PROOF:

$\langle 1 \rangle 1$. Define $\delta : \ker \nu \rightarrow \operatorname{coker} \lambda$.

$\langle 2 \rangle 1$. LET: $a \in \ker \nu$

$\langle 2 \rangle 2$. PICK $c \in M_1$ such that $\beta_1(c) = a$.

PROOF: Since β_1 is surjective.

$\langle 2 \rangle 3$. LET: $d = \mu(c)$

$\langle 2 \rangle 4$. $d \in \ker \beta_0 = \operatorname{im} \alpha_0$

PROOF: Since $\beta_0(d) = \beta_0(\mu(c)) = \nu(a) = 0$.

$\langle 2 \rangle 5$. LET: $e \in L_0$ be the element such that $\alpha_0(e) = d$.

$\langle 2 \rangle 6$. LET: $\delta(a) = e + \operatorname{im} \lambda$

$\langle 1 \rangle 2$. δ is a left- R -module homomorphism.

$\langle 2 \rangle 1$. For $a, a' \in \ker \nu$ we have $\delta(a + a') = \delta(a) + \delta(a')$.

$\langle 3 \rangle 1$. LET: $a, a' \in \ker \nu$

$\langle 3 \rangle 2$. LET: $c, c', c'' \in M_1$ and $e, e', e'' \in L_0$ be the elements such that

$$\beta_1(c) = a$$

$$\beta_1(c') = a'$$

$$\beta_1(c'') = a + a'$$

$$\alpha_0(e) = \mu(c)$$

$$\alpha_0(e') = \mu(c')$$

$$\alpha_0(e'') = \mu(c'')$$

$$\delta(a) = e + \text{im } \lambda$$

$$\delta(a') = e' + \text{im } \lambda$$

$$\delta(a + a') = e'' + \text{im } \lambda$$

$\langle 3 \rangle 3$. $c'' - c - c' \in \ker \beta_1 = \text{im } \alpha_1$

$\langle 3 \rangle 4$. PICK $g \in L_1$ such that $\alpha_1(g) = c'' - c - c'$.

$\langle 3 \rangle 5$. $\alpha_0(\lambda(g)) = \alpha_0(e'' - e - e')$

$\langle 3 \rangle 6$. $\lambda(g) = e'' - e - e'$

$\langle 3 \rangle 7$. $e'' - e - e' \in \text{im } \lambda$

$\langle 3 \rangle 8$. $e'' + \text{im } \lambda = e + e' + \text{im } \lambda$

$\langle 3 \rangle 9$. $\delta(a + a') = \delta(a) + \delta(a')$

$\langle 2 \rangle 2$. For $r \in R$ and $a \in \ker \nu$ we have $\delta(ra) = r\delta(a)$.

$\langle 3 \rangle 1$. LET: $r \in R$ and $a \in \ker \nu$

$\langle 3 \rangle 2$. LET: $c, c' \in M_1$ and $e, e' \in L_0$ be the elements such that

$$\beta_1(c) = a$$

$$\beta_1(c') = ra$$

$$\alpha_0(e) = \mu(c)$$

$$\alpha_0(e') = \mu(c')$$

$$\delta(a) = e + \text{im } \lambda$$

$$\delta(ra) = e' + \text{im } \lambda$$

$\langle 3 \rangle 3$. $rc - c' \in \ker \beta_1 = \text{im } \alpha_1$

$\langle 3 \rangle 4$. PICK $g \in L_1$ such that $\alpha_1(g) = rc - c'$.

$\langle 3 \rangle 5$. $\alpha_0(\lambda(g)) = \alpha_0(re - e')$

$\langle 3 \rangle 6$. $\lambda(g) = re - e'$

$\langle 3 \rangle 7$. $re - e' \in \text{im } \lambda$

$\langle 3 \rangle 8$. $re + \text{im } \lambda = e' + \text{im } \lambda$

$\langle 3 \rangle 9$. $r\delta(a) = \delta(ra)$

$\langle 1 \rangle 3$. The sequence is exact at $\ker \lambda$.

PROOF: Since α_1 is injective.

$\langle 1 \rangle 4$. The sequence is exact at $\ker \mu$.

PROOF: Since $\text{im } \alpha_1 = \ker \beta_1$.

$\langle 1 \rangle 5$. The sequence is exact at $\ker \nu$, i.e.

$$\beta a_1(\ker \mu) = \ker \delta.$$

$\langle 2 \rangle 1$. LET: $a \in \ker \nu$

$\langle 2 \rangle 2$. LET: $c \in M_1$ and $e \in L_0$ be the elements such that $\beta_1(c) = a$, $\alpha_0(e) = \mu(c)$, and $\delta(a) = e + \text{im } \lambda$.

- (2)3. If $\delta(a) = \text{im } \lambda$ then $a \in \beta_1(\ker \mu)$
 (3)1. ASSUME: $\delta(a) = \text{im } \lambda$
 (3)2. $e \in \text{im } \lambda$
 (3)3. PICK $g \in L_1$ such that $\lambda(g) = e$
 (3)4. $\mu(\alpha_1(g)) = \mu(c)$
 (3)5. $c - \alpha_1(g) \in \ker \mu$
 (3)6. $a = \beta_1(c - \alpha_1(g))$
 (2)4. If $a \in \beta_1(\ker \mu)$ then $\delta(a) = \text{im } \lambda$
 (3)1. ASSUME: $c' \in \ker \mu$ and $a = \beta_1(c')$
 (3)2. $c - c' \in \ker \beta_1 = \text{im } \alpha_1$
 (3)3. PICK $g \in L_1$ such that $\alpha_1(g) = c - c'$
 (3)4. $\alpha_0(\lambda(g)) = \mu(c) - \mu(c') = \alpha_0(e) - 0 = \alpha_0(e)$
 (3)5. $\lambda(g) = e$
 (3)6. $e \in \text{im } \lambda$
 (3)7. $\delta(a) = \text{im } \lambda$
 (1)6. The sequence is exact at $\text{coker } \lambda$.
 (2)1. LET: $e \in L_0$
 PROVE: $e + \text{im } \lambda \in \text{im } \delta$ iff $\alpha_0(e) \in \text{im } \mu$.
 (2)2. For all $a \in \ker \nu$, if $\delta(a) = e + \text{im } \lambda$ then $\alpha_0(e) \in \text{im } \mu$
 PROOF: From (1)1 and the fact that α_0 is injective hence e is unique given a .
 (2)3. For all $e \in L_0$, if $\alpha_0(e) \in \text{im } \mu$ then $e + \text{im } \lambda \in \text{im } \delta$.
 (3)1. LET: $e \in L_0$
 (3)2. ASSUME: $\alpha_0(e) \in \text{im } \mu$
 (3)3. PICK $c \in M_1$ such that $\mu(c) = \alpha_0(e)$.
 PROVE: $e + \text{im } \lambda = \delta(\beta_1(c))$
 (3)4. PICK $c' \in M_1$ and $e' \in L_0$ such that $\beta_1(c') = \beta_1(c)$, $\alpha_0(e') = \mu(c')$
 and $\delta(\beta_1(c)) = e' + \text{im } \lambda$
 (3)5. $c - c' \in \ker \beta_1 = \text{im } \alpha_1$
 (3)6. PICK $g \in L_1$ such that $\alpha_1(g) = c - c'$.
 (3)7. $\alpha_0(\lambda(g)) = \alpha_0(e - e')$
 (3)8. $\lambda(g) = e - e'$
 (3)9. $e + \text{im } \lambda = e' + \text{im } \lambda = \delta(\beta_1(c))$
 (1)7. The sequence is exact at $\text{coker } \mu$.
 PROOF: Since $\text{im } \alpha_0 = \ker \beta_0$.
 (1)8. The sequence is exact at $\text{coker } \nu$.
 PROOF: Since β_0 is surjective.

□

Corollary 33.8.1. *Suppose we have R -modules and homomorphisms*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 \longrightarrow 0 \\
 & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu \\
 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 \longrightarrow 0
 \end{array}$$

such that the diagram commutes and the two rows are short exact sequences.

Suppose μ is surjective and ν is injective. Then λ is surjective and ν is an isomorphism.

PROOF: We have $\ker \nu = \operatorname{coker} \mu = 0$ and so $0 \xrightarrow{\delta} \operatorname{coker} \lambda \xrightarrow{\alpha_0} 0$ is an exact sequence, hence $\operatorname{coker} \lambda = 0$ and so λ is surjective.

Since $\operatorname{coker} \mu = 0$ we have $0 \rightarrow \operatorname{coker} \nu \rightarrow 0$ is an exact sequence and so $\operatorname{coker} \nu = 0$, hence ν is surjective, hence ν is an isomorphism. \square

Proposition 33.9 (Short Five-Lemma). *Suppose we have R -modules and homomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 & \longrightarrow & 0 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & & \\ 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 & \longrightarrow & 0 \end{array}$$

such that the diagram commutes and the two rows are short exact sequences. If λ and ν are isomorphisms then μ is an isomorphism.

PROOF:

$\langle 1 \rangle 1$. There exists a homomorphism $\delta : 0 \rightarrow L_0$ such that the following is an exact sequence.

$$0 \rightarrow 0 \rightarrow \ker \mu \rightarrow 0 \xrightarrow{\delta} L_0 \xrightarrow{\alpha_0} \operatorname{coker} \mu \xrightarrow{\beta_0} N_0 \rightarrow 0.$$

PROOF: Snake Lemma

$\langle 1 \rangle 2$. $\ker \mu = 0$

$\langle 1 \rangle 3$. $\operatorname{coker} \mu = M_0$

\square

Proposition 33.10. *If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is an exact sequence and L and N are Noetherian then M is Noetherian.*

PROOF:

$\langle 1 \rangle 1$. LET: P be a submodule of M .

$\langle 1 \rangle 2$. PICK a_1, \dots, a_m generate $\alpha^{-1}(P)$.

$\langle 1 \rangle 3$. PICK c_1, \dots, c_n that generate $\beta(P)$.

$\langle 1 \rangle 4$. For $i = 1, \dots, n$, PICK b_i such that $\beta(b_i) = c_i$.

PROVE: $\alpha(a_1), \dots, \alpha(a_m), b_1, \dots, b_n$ generate P .

$\langle 1 \rangle 5$. LET: $p \in P$

$\langle 1 \rangle 6$. PICK $r_1, \dots, r_n \in R$ such that $r_1 c_1 + \dots + r_n c_n = \beta(p)$

$\langle 1 \rangle 7$. $r_1 b_1 + \dots + r_n b_n - p \in \ker \beta = \operatorname{im} \alpha$

$\langle 1 \rangle 8$. PICK $s_1, \dots, s_m \in R$ such that $\alpha(s_1 a_1 + \dots + s_m a_m) = r_1 b_1 + \dots + r_n b_n - p$.

$\langle 1 \rangle 9$. $p = s_1 \alpha(a_1) + \dots + s_m \alpha(a_m) + r_1 b_1 + \dots + r_n b_n$

\square

Proposition 33.11. *Let R be a ring. Let*

$$0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} P \rightarrow 0$$

be a short exact sequence of left- R -modules. Let L be an R -module. Then the following is an exact sequence:

$$0 \rightarrow R\text{-}\mathbf{Mod}[P, L] \xrightarrow{R\text{-}\mathbf{Mod}[\beta, \text{id}_L]} R\text{-}\mathbf{Mod}[N, L] \xrightarrow{R\text{-}\mathbf{Mod}[\alpha, \text{id}_L]} R\text{-}\mathbf{Mod}[M, L] .$$

PROOF:

$\langle 1 \rangle 1.$ $R\text{-}\mathbf{Mod}[\beta, \text{id}_L]$ is injective.

PROOF: Since β is epi.

$\langle 1 \rangle 2.$ $\text{im } R\text{-}\mathbf{Mod}[\beta, \text{id}_L] = \ker R\text{-}\mathbf{Mod}[\alpha, \text{id}_L]$

$\langle 2 \rangle 1.$ $\text{im } R\text{-}\mathbf{Mod}[\beta, \text{id}_L] \subseteq \ker R\text{-}\mathbf{Mod}[\alpha, \text{id}_L]$

PROOF: For any $\gamma \in R\text{-}\mathbf{Mod}[P, L]$ we have $\gamma \circ \beta \circ \alpha = 0$ because $\beta \circ \alpha = 0$.

$\langle 2 \rangle 2.$ $\ker R\text{-}\mathbf{Mod}[\alpha, \text{id}_L] \subseteq \text{im } R\text{-}\mathbf{Mod}[\beta, \text{id}_L]$

$\langle 3 \rangle 1.$ LET: $\gamma \in \ker R\text{-}\mathbf{Mod}[\alpha, \text{id}_L]$

$\langle 3 \rangle 2.$ $\gamma \circ \alpha = 0$

$\langle 3 \rangle 3.$ PICK $\delta : P \rightarrow L$ by: for all $p \in P$, we have $\delta(p) = \gamma(n)$ where $n \in N$ is an element such that $\beta(n) = p$.

PROVE: $\delta \circ \beta = \gamma$

$\langle 3 \rangle 4.$ LET: $n \in N$

PROVE: $\delta(\beta(n)) = \gamma(n)$

$\langle 3 \rangle 5.$ PICK $n' \in N$ such that $\delta(\beta(n)) = \gamma(n')$ and $\beta(n') = \beta(n)$

$\langle 3 \rangle 6.$ $n - n' \in \ker \beta = \text{im } \alpha$

$\langle 3 \rangle 7.$ PICK $m \in M$ such that $\alpha(m) = n - n'$

$\langle 3 \rangle 8.$ $0 = \gamma(\alpha(m)) = \gamma(n) - \gamma(n')$

$\langle 3 \rangle 9.$ $\gamma(n) = \gamma(n') = \delta(\beta(n))$

□

33.1 Split Exact Sequences

Definition 33.12 (Split Sequence). Let $0 \rightarrow M_1 \xrightarrow{\alpha} N \xrightarrow{\beta} M_2 \rightarrow 0$ be a short exact sequence. Then this sequence *splits* iff there exists an isomorphism

$$\phi : N \cong M_1 \oplus M_2$$

such that $\phi \circ \alpha = \kappa_1 : M_1 \rightarrow M_1 \oplus M_2$ and $\beta \circ \phi^{-1} = \pi_2 : M_1 \oplus M_2 \rightarrow M_2$.

Proposition 33.13. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then ϕ has a left-inverse if and only if the sequence

$$0 \rightarrow M \xrightarrow{\phi} N \rightarrow \text{coker } \phi \rightarrow 0$$

splits.

PROOF:

$\langle 1 \rangle 1.$ If ϕ has a left-inverse then the sequence splits.

$\langle 2 \rangle 1.$ ASSUME: ϕ has a left-inverse $\psi : N \rightarrow M$.

$\langle 2 \rangle 2.$ Define $i : N \rightarrow M \oplus \text{coker } \phi$ by $i(n) = (\psi(n), n + \text{im } \phi)$.

⟨2⟩3. Define $i^{-1} : M \oplus \text{coker } \phi$ by $i^{-1}(m, x + \text{im } \phi) = \phi(m) + x - \phi(\psi(x))$.

⟨2⟩4. $i \circ i^{-1} = \text{id}_{M \oplus \text{coker } \phi}$

PROOF:

$$\begin{aligned} \psi(\phi(m) + x - \phi(\psi(x))) &= m + \psi(x) - \psi(x) \\ &= m \end{aligned}$$

⟨2⟩5. $i^{-1} \circ i = \text{id}_N$

PROOF:

$$\begin{aligned} i^{-1}(\psi(n), n + \text{im } \phi) &= \phi(\psi(n)) + n - \phi(\psi(n)) \\ &= n \end{aligned}$$

⟨2⟩6. $i \circ \phi = \kappa_1 : M \rightarrow M \oplus \text{coker } \phi$

PROOF:

$$\begin{aligned} i(\phi(m)) &= (\psi(\phi(m)), \phi(m) + \text{im } \phi) \\ &= (m, \text{im } \phi) \end{aligned}$$

⟨2⟩7. $\pi \circ i^{-1} = \pi_2 : M \oplus \text{coker } \phi \rightarrow \text{coker } \phi$

PROOF:

$$\begin{aligned} i^{-1}(\psi(n), n + \text{im } \phi) + \text{im } \phi &= \phi(\psi(n)) + n - \phi(\psi(n)) + \text{im } \phi \\ &= n + \text{im } \phi \end{aligned}$$

⟨1⟩2. If the sequence splits then ϕ has a left-inverse.

PROOF: Since $\kappa_1 : M \rightarrow M \oplus \text{coker } \phi$ has left inverse π_1 .

□

Proposition 33.14. *Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then ϕ has a right-inverse if and only if the sequence*

$$0 \rightarrow \ker \phi \rightarrow M \xrightarrow{\phi} N \rightarrow 0$$

splits.

PROOF:

⟨1⟩1. If ϕ has a right-inverse then the sequence splits.

⟨2⟩1. LET: $\psi : N \rightarrow M$ be a right inverse to ϕ .

⟨2⟩2. LET: $i : M \rightarrow \ker \phi \oplus N$ be the function $i(m) = (m - \psi(\phi(m)), \phi(m))$.

PROOF: $m - \psi(\phi(m)) \in \ker \phi$ since $\phi(m - \psi(\phi(m))) = \phi(m) - \phi(m) = 0$.

⟨2⟩3. LET: $i^{-1} : \ker \phi \oplus N \rightarrow M$ be the function $i^{-1}(x, n) = x + \psi(n)$.

⟨2⟩4. $i \circ i^{-1} = \text{id}_{\ker \phi \oplus N}$

PROOF:

$$\begin{aligned} i(i^{-1}(x, n)) &= i(x + \psi(n)) \\ &= (x + \psi(n) - \psi(\phi(x)) - \psi(\phi(\psi(n))), \phi(x) + \phi(\psi(n))) \\ &= (x + \psi(n) - \psi(n), n) \\ &= (x, n) \end{aligned}$$

⟨2⟩5. $i^{-1} \circ i = \text{id}_M$

PROOF:

$$\begin{aligned} i^{-1}(i(m)) &= m - \psi(\phi(m)) + \psi(\phi(m)) \\ &= m \end{aligned}$$

⟨2⟩6. $i \circ \iota = \kappa_1$

PROOF: For $m \in \ker \phi$ we have $i(m) = (m - \psi(\phi(m)), \phi(m)) = (m, 0)$.

$\langle 2 \rangle 7$. $\phi \circ i^{-1} = \pi_2$

PROOF:

$$\begin{aligned}\phi(i^{-1}(x, n)) &= \phi(x) + \phi(\psi(n)) \\ &= 0 + n \\ &= n\end{aligned}$$

$\langle 1 \rangle 2$. If the sequence splits then ϕ has a right-inverse.

PROOF: Since $\kappa_2 : N \rightarrow M \oplus N$ is a right-inverse to π_2 .

□

Proposition 33.15. *Let*

$$0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} F \rightarrow 0$$

be a short exact sequence where F is free. Then the sequence splits.

PROOF:

$\langle 1 \rangle 1$. LET: $F = R^{\oplus A}$

$\langle 1 \rangle 2$. PICK $\gamma : F \rightarrow N$ such that $\text{id}_F = \beta \circ \gamma$

$\langle 1 \rangle 3$. LET: $i : M \oplus F \rightarrow N$ be the homomorphism $i(m, f) = \alpha(m) + \gamma(f)$

$\langle 1 \rangle 4$. i is injective.

$\langle 2 \rangle 1$. ASSUME: $i(m, f) = i(m', f')$

$\langle 2 \rangle 2$. $\alpha(m) + \gamma(f) = \alpha(m') + \gamma(f')$

$\langle 2 \rangle 3$. $\alpha(m - m') = \gamma(f - f')$

$\langle 2 \rangle 4$. $f - f' = 0$

PROOF: Applying β to both sides of $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. $f = f'$

$\langle 2 \rangle 6$. $\alpha(m - m') = 0$

$\langle 2 \rangle 7$. $m = m'$

PROOF: Since α is injective.

$\langle 1 \rangle 5$. i is surjective.

$\langle 2 \rangle 1$. LET: $n \in N$

$\langle 2 \rangle 2$. $n - \gamma(\beta(n)) \in \ker \beta = \text{im } \alpha$

$\langle 2 \rangle 3$. PICK $m \in M$ such that $\alpha(m) = n - \gamma(\beta(n))$

$\langle 2 \rangle 4$. $n = i(m, \beta(n))$

$\langle 1 \rangle 6$. $\alpha = i \circ \kappa_1$

$\langle 1 \rangle 7$. $\beta \circ i = \pi_2$

□

Chapter 34

Homology

Definition 34.1 (Homology). Let (M_\bullet, d_\bullet) be a chain complex. The *i*th homology of the complex is the R -module

$$H_i(M_\bullet) := \frac{\ker d_i}{\operatorname{im} d_{i+1}} .$$

Proposition 34.2. *Consider the complex*

$$0 \rightarrow M_1 \xrightarrow{\phi} M_0 \rightarrow 0 .$$

The 1st homology is $\ker \phi$, and the 0th homology is $\operatorname{coker} \phi$.

Part IV

Field Theory

Chapter 35

Fields

Definition 35.1 (Field). A *field* is a non-trivial commutative division ring.

Example 35.2. \mathbb{Q} , \mathbb{R} and \mathbb{C} are fields.

Proposition 35.3. *Every field is an integral domain.*

PROOF: By Propositions 11.8 and 11.9. \square

Example 35.4. The converse does not hold: \mathbb{Z} is an integral domain but not a field.

Proposition 35.5. *Every finite integral domain is a field.*

PROOF: In a finite integral domain, multiplication by any non-zero element is injective, hence surjective. \square

Corollary 35.5.1. *For any positive integer n , the following are equivalent:*

- n is prime.
- $\mathbb{Z}/n\mathbb{Z}$ is an integral domain.
- $\mathbb{Z}/n\mathbb{Z}$ is a field.

Theorem 35.6 (Wedderburn's Little Theorem). *Every finite division ring is a field.*

Proposition 35.7. *Every subring of a field is an integral domain.*

PROOF: Easy. \square

Proposition 35.8. *The center of a division ring is a field.*

PROOF:

$\langle 1 \rangle$ 1. LET: R be a division ring.

$\langle 1 \rangle$ 2. LET: Z be the center of R .

$\langle 1 \rangle$ 3. Z is non-trivial.

PROOF: Since $1 \in Z$.

$\langle 1 \rangle 4$. Z is commutative.

$\langle 1 \rangle 5$. Z is a division ring.

$\langle 2 \rangle 1$. LET: $a \in Z$

$\langle 2 \rangle 2$. $a^{-1} \in Z$

$\langle 3 \rangle 1$. LET: $x \in R$

$\langle 3 \rangle 2$. $ax = xa$

$\langle 3 \rangle 3$. $xa^{-1} = a^{-1}x$

□

Definition 35.9. For any prime p and positive integer r , define a multiplication on $(\mathbb{Z}/p\mathbb{Z})^r$ that makes this group into a field by:

Proposition 35.10. *A commutative ring is a field if and only if it is simple.*

PROOF: Proposition 21.5. □

Corollary 35.10.1. *Every field has Krull dimension 0.*

Proposition 35.11. *Let K be a field. Then $K[x]$ is a PID, and every non-zero ideal in $K[x]$ is generated by a unique monic polynomial.*

PROOF:

$\langle 1 \rangle 1$. LET: I be a non-zero ideal in $K[x]$

$\langle 1 \rangle 2$. PICK a monic polynomial $f \in K[x]$ of minimal degree.

PROVE: $I = (f)$

$\langle 1 \rangle 3$. LET: $g \in I$

$\langle 1 \rangle 4$. PICK polynomials q, r with $\deg r < \deg f$ such that $g = qf + r$

$\langle 1 \rangle 5$. $r \in I$

$\langle 1 \rangle 6$. $r = 0$

$\langle 1 \rangle 7$. $g \in (f)$

□

Proposition 35.12. *Let R be a commutative ring and I an ideal in R . Then I is maximal iff R/I is a field.*

PROOF: From Proposition 22.3. □

Example 35.13. Let R be a commutative ring and $a \in R$. Then $(x - a)$ is a maximal ideal in $R[x]$ iff R is a field, since $R[x]/(x - a) \cong R$.

Example 35.14. The ideal $(2, x)$ is a maximal ideal in $\mathbb{Z}[x]$, since $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}/2\mathbb{Z}$.

Proposition 35.15. *Every maximal ideal in a commutative ring is a prime ideal.*

PROOF: Since every field is an integral domain. □

Proposition 35.16. *Let R be a commutative ring and I an ideal in R . If I is a prime ideal and R/I is finite then I is a maximal ideal.*

PROOF: Since every finite integral domain is a field. \square

Proposition 35.17. *Let R be a commutative ring and I a proper ideal in R . Then I is maximal iff, whenever J is an ideal and $I \subseteq J$, then $I = J$ or $J = R$.*

Example 35.18. The inverse image of a maximal ideal under a homomorphism is not necessarily maximal.

Let $i : \mathbb{Z}[x] \rightarrow \mathbb{Q}[x]$ be the inclusion. Then (x) is maximal in $\mathbb{Q}[x]$ but its inverse image (x) is not maximal in $\mathbb{Z}[x]$.

Definition 35.19 (Maximal Spectrum). Let R be a commutative ring. The *maximal spectrum* of R is the set of all maximal ideals in R .

Proposition 35.20. *Let K be a field. The Krull dimension of $K[x_1, \dots, x_n]$ is n .*

Theorem 35.21 (Hilbert's Nullstellensatz). *Let K be a field and L a subfield of K . If K is an L -algebra of finite type, then K is a finite L -algebra.*

Proposition 35.22. *Let K be a subfield of L . Then L is a K -algebra under multiplication.*

PROOF: Easy. \square

Chapter 36

Algebraically Closed Fields

Definition 36.1 (Algebraically Closed). A field K is *algebraically closed* iff, for every $f \in K[x]$ that is not constant, there exists $r \in K$ such that $f(r) = 0$.

Theorem 36.2. \mathbb{C} is algebraically closed.

Proposition 36.3. Let K be an algebraically closed field. Let I be an ideal in $K[x]$. Then I is maximal if and only if $I = (x - c)$ for some $c \in K$.

PROOF:

$\langle 1 \rangle 1$. If I is maximal then there exists $c \in K$ such that $I = (x - c)$.

$\langle 2 \rangle 1$. ASSUME: I is maximal.

$\langle 2 \rangle 2$. PICK f monic of minimal degree such that $f \in I$.

$\langle 2 \rangle 3$. f is not constant.

PROOF: Otherwise $f = 1$ and $I = K[x]$.

$\langle 2 \rangle 4$. PICK $c \in K$ such that $f(c) = 0$

$\langle 2 \rangle 5$. $x - c \mid f$

$\langle 2 \rangle 6$. $I \subseteq (x - c)$

$\langle 2 \rangle 7$. $I = (x - c)$

$\langle 1 \rangle 2$. For all $c \in K$ we have $(x - c)$ is maximal.

PROOF: Example 35.13.

□

Part V

Linear Algebra

Chapter 37

Vector Spaces

Definition 37.1 (Vector Space). Let K be a field. A K -vector space is a K -module. A *linear map* is a homomorphism of K -modules. We write $K - \mathbf{Vect}$ for $K - \mathbf{Mod}$.

Definition 37.2. Let $\mathrm{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ real matrices. $\mathrm{GL}_n(\mathbb{R})$ acts on \mathbb{R}^n by matrix multiplication.

Definition 37.3. Let $\mathrm{GL}_n(\mathbb{C})$ be the group of invertible $n \times n$ complex matrices. $\mathrm{GL}_n(\mathbb{C})$ acts on \mathbb{C}^n by matrix multiplication.

Definition 37.4. Let $\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : \det M = 1\}$.

Proposition 37.5. $\mathrm{SL}_n(\mathbb{R})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$.

PROOF: If $\det M = 1$ then $\det(AMA^{-1}) = (\det A)(\det M)(\det A)^{-1} = 1$. \square

Proposition 37.6.

$$\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$$

Definition 37.7. Let $\mathrm{SL}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : \det M = 1\}$.

Definition 37.8. Let $\mathrm{O}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : MM^T = M^T M = I_n\}$.

Proposition 37.9. The action of $\mathrm{O}_n(\mathbb{R})$ on \mathbb{R}^n preserves lengths and angles.

Definition 37.10. Let $\mathrm{SO}_n(\mathbb{R}) = \{M \in \mathrm{O}_n(\mathbb{R}) : \det M = 1\}$.

Definition 37.11. Let $\mathrm{U}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : MM^\dagger = M^\dagger M = I_n\}$.

Definition 37.12. Let $\mathrm{SU}_n(\mathbb{C}) = \{M \in \mathrm{U}_n(\mathbb{C}) : \det M = 1\}$.

Proposition 37.13. Every matrix in $\mathrm{SU}_2(\mathbb{C})$ can be written in the form

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 + c^2 + d^2 = 1$.

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SU}_2(\mathbb{C})$$

$$\langle 1 \rangle 2. M^{-1} = M^\dagger$$

$$\langle 1 \rangle 3. \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

$$\langle 1 \rangle 4. \text{ LET: } \alpha = a + bi \text{ and } \beta = c + di.$$

$$\langle 1 \rangle 5. \delta = \bar{\alpha} = a - bi$$

$$\langle 1 \rangle 6. \gamma = -\bar{\beta} = -c + di$$

$$\langle 1 \rangle 7. \det M = a^2 + b^2 + c^2 + d^2 = 1$$

□

Corollary 37.13.1. $\text{SU}_2(\mathbb{C})$ is simply connected.

Corollary 37.13.2.

$$\text{SO}_3(\mathbb{R}) \cong \text{SU}_2(\mathbb{C}) / \{I, -I\}$$

PROOF: The function that maps $\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$ to $\begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(ad + bc) \\ 2(ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(bd - ac) \\ 2(bd - ac) & 2(ab + cd) & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$

is a surjective homomorphism with kernel $\{I, -I\}$. □

Corollary 37.13.3. The fundamental group of $\text{SO}_3(\mathbb{R})$ is C_2 .