

Mathematics

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Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*.

For any set A , let there be *elements* of A . We write $a \in A$ for: a is an element of A .

For any sets A and B , let there be a set B^A , whose elements are called *functions* from A to B . We write $f : A \rightarrow B$ for $f \in B^A$.

For any function $f : A \rightarrow B$ and element $a \in A$, let there be an element $f(a) \in B$, the *value* of the function f at the *argument* a .

1.2 Injections, Surjections and Bijections

Definition 1.2.1 (Injective). A function $f : A \rightarrow B$ is *injective* or an *injection* iff, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

Definition 1.2.2 (Surjective). A function $f : A \rightarrow B$ is *surjective* or a *surjection* iff, for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Definition 1.2.3 (Bijective). A function $f : A \rightarrow B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$ ".

1.3 Axioms

Axiom Schema 1.3.1 (Choice). *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x : \text{El}(X)$ and $y : \text{El}(Y)$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a : \text{El}(A)$, there exists $b : \text{El}(B)$ such that $P[A, B, a, b]$. Then there exists a function $f : A \rightarrow B$ such that $\forall a : \text{El}(A). P[A, B, a, f(a)]$.

Axiom 1.3.2 (Extensionality). *Let $f, g : A \rightarrow B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Axiom 1.3.3 (Pairing). *For any sets A and B , there exists a set $A \times B$, the Cartesian product of A and B , and functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ such that, for all $a : \text{El}(A)$ and $b : \text{El}(B)$, there exists a unique $(a, b) : \text{El}(A \times B)$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Axiom Schema 1.3.4 (Separation). *For every property $P[X, x]$ where X is a set variable and $x : \text{El}(X)$, the following is an axiom:*

For every set A , there exists a set $S = \{x : \text{El}(A) \mid P[A, x]\}$ and an injection $i : S \rightarrow A$ such that, for all $x : \text{El}(A)$, we have

$$(\exists y : S. i(y) = x) \Leftrightarrow P[A, x] .$$

Axiom 1.3.5 (Infinity). *There exists a set \mathbb{N} , an element $0 : \text{El}(\mathbb{N})$, and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that:*

- $\forall n : \text{El}(\mathbb{N}). s(n) \neq 0$
- $\forall m, n : \text{El}(\mathbb{N}). s(m) = s(n) \Rightarrow m = n$.

Axiom Schema 1.3.6 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A , there exists a set B , a function $p : B \rightarrow A$, a set Y and a relation $M : B \rightarrowtail Y$ such that:

- $\forall b \in B. P[A, \{y \in Y : bMy\}, p(b)]$
- *For all $a \in A$, if $\exists Y. P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 1.3.7 (Composite). *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. The composite $g \circ f : A \rightarrow C$ is the function such that, for all $a \in A$, we have*

$$(g \circ f)(a) = g(f(a)) .$$

Axiom 1.3.8 (Universe). *There exists a set E , a set U and a function $el : E \rightarrow U$ such that the following holds.*

Let us say that a set A is small iff there exists $u \in U$ such that $A \approx \{e \in E : el(e) = u\}$.

- \mathbb{N} is small.

- For any U -small sets A and B , the set B^A is small.
- For any U -small sets A and B , the set $A \times B$ is small.
- Let $f : A \rightarrow B$ be a function. If B is small and $f^{-1}(b)$ is U -small for all $b \in B$, then A is small.
- If $p : B \twoheadrightarrow A$ is a surjective function such that A is small, then there exists a U -small set C , a surjection $q : C \twoheadrightarrow A$, and a function $f : C \rightarrow B$ such that $q = p \circ f$.

Chapter 2

Sets and Functions

2.1 Identity Function

Definition 2.1.1 (Identity). For any set A , the *identity* function $\text{id}_A : A \rightarrow A$ is the function defined by $\text{id}_A(a) = a$.

2.2 Composition

Proposition 2.2.1. *Given functions $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f .$$

PROOF: Each is the function that maps $a \in A$ to $h(g(f(a)))$. \square

Proposition 2.2.2. *The composite of injective functions is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$ be injective.

$\langle 1 \rangle 2$. LET: $x, y \in A$ satisfy $(g \circ f)(x) = (g \circ f)(y)$

$\langle 1 \rangle 3$. $g(f(x)) = g(f(y))$

$\langle 1 \rangle 4$. $f(x) = f(y)$

$\langle 1 \rangle 5$. $x = y$

\square

Proposition 2.2.3. *For functions $f : A \rightarrow B$ and $g : B \rightarrow C$, if $g \circ f$ is injective then f is injective.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: $g \circ f$ is injective.

$\langle 1 \rangle 2$. LET: $x, y \in A$

$\langle 1 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 1 \rangle 4$. $g(f(x)) = g(f(y))$

$\langle 1 \rangle 5$. $x = y$

□

Proposition 2.2.4. *Let $f : A \rightarrow B$. Then f is injective if and only if, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.*

PROOF:

⟨1⟩1. If f is injective then, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

⟨2⟩1. ASSUME: f is injective.

⟨2⟩2. LET: X be a set.

⟨2⟩3. LET: $x, y : X \rightarrow A$

⟨2⟩4. ASSUME: $f \circ x = f \circ y$

⟨2⟩5. $\forall t \in X. x(t) = y(t)$

⟨3⟩1. LET: $t \in X$

⟨3⟩2. $f(x(t)) = f(y(t))$

PROOF: ⟨2⟩4

⟨3⟩3. $x(t) = y(t)$

PROOF: ⟨2⟩1

⟨2⟩6. $x = y$

PROOF: Axiom of Extensionality.

⟨1⟩2. If, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

PROOF: Take $X = 1$.

□

Proposition 2.2.5. *The composite of surjective functions is surjective.*

PROOF:

⟨1⟩1. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$ be injective.

⟨1⟩2. LET: $c \in C$

⟨1⟩3. PICK $b \in B$ such that $g(b) = c$.

⟨1⟩4. PICK $a \in A$ such that $f(a) = b$.

⟨1⟩5. $(g \circ f)(a) = c$

□

Proposition 2.2.6. *Let $f : A \rightarrow B$. Then the following are equivalent.*

1. f is surjective.

2. For any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$.

3. There exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

⟨2⟩1. ASSUME: f is surjective.

⟨2⟩2. LET: X be a set.

⟨2⟩3. LET: $g, h : B \rightarrow X$

⟨2⟩4. ASSUME: $g \circ f = h \circ f$

⟨2⟩5. LET: $b \in B$

PROVE: $g(b) = h(b)$
 $\langle 2 \rangle 6$. PICK $a \in A$ such that $f(a) = b$
 $\langle 2 \rangle 7$. $g(b) = h(b)$
 PROOF: $g(b) = g(f(a)) = h(f(a)) = h(b)$
 $\langle 1 \rangle 2$. $1 \Rightarrow 3$
 $\langle 2 \rangle 1$. ASSUME: f is surjective.
 $\langle 2 \rangle 2$. PICK $g : B \rightarrow A$ such that, for all $b \in B$, we have $f(g(b)) = b$.
 PROOF: Axiom of Choice.
 $\langle 2 \rangle 3$. $f \circ g = \text{id}_B$.
 $\langle 1 \rangle 3$. $3 \Rightarrow 2$
 $\langle 2 \rangle 1$. LET: $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$
 $\langle 2 \rangle 2$. LET: X be a set.
 $\langle 2 \rangle 3$. LET: $h, k : B \rightarrow X$
 $\langle 2 \rangle 4$. ASSUME: $h \circ f = k \circ f$
 $\langle 2 \rangle 5$. $h = k$
 PROOF: $h = h \circ f \circ g = k \circ f \circ g = k$
 $\langle 1 \rangle 4$. $2 \Rightarrow 1$
 $\langle 2 \rangle 1$. ASSUME: 2
 $\langle 2 \rangle 2$. LET: $b \in B$
 $\langle 2 \rangle 3$. LET: $h : B \rightarrow 2$ be the function that maps everything to 1.
 $\langle 2 \rangle 4$. LET: $k : B \rightarrow 2$ be the function that maps b to 0 and everything else to 1.
 $\langle 2 \rangle 5$. $h \neq k$
 $\langle 2 \rangle 6$. $h \circ f \neq k \circ f$
 $\langle 2 \rangle 7$. PICK $a \in A$ such that $h(f(a)) \neq k(f(a))$
 $\langle 2 \rangle 8$. $f(a) = b$
 \square

Proposition 2.2.7. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is surjective then g is surjective.*

PROOF:
 $\langle 1 \rangle 1$. LET: $c \in C$
 $\langle 1 \rangle 2$. There exists $a \in A$ such that $g(f(a)) = c$.
 $\langle 1 \rangle 3$. There exists $b \in B$ such that $g(b) = c$.
 \square

Proposition 2.2.8. *The composite of bijections is a bijection.*

PROOF: Propositions 2.2.2 and 2.2.5. \square

Proposition 2.2.9. *Let $f : A \rightarrow B$. Then f is bijective if and only if there exists a function $f^{-1} : B \rightarrow A$, the inverse of f , such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$, in which case the inverse is unique.*

PROOF:
 $\langle 1 \rangle 1$. If f is bijective then there exists $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$.

- ⟨2⟩1. ASSUME: f is bijective.
 ⟨2⟩2. PICK $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$
 PROOF: Proposition 2.2.6.
 ⟨2⟩3. $f \circ g \circ f = f$
 ⟨2⟩4. $g \circ f = \text{id}_A$
 PROOF: Proposition 2.2.4.
 ⟨1⟩2. If there exists $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$, then f is bijective.
 ⟨2⟩1. LET: $f^{-1} : B \rightarrow A$ satisfy $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$
 ⟨2⟩2. f is injective.
 PROOF: If $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$.
 ⟨2⟩3. f is surjective.
 PROOF: Proposition 2.2.6.
 ⟨1⟩3. If $g, h : B \rightarrow A$ satisfy $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$ and $f \circ h = \text{id}_B$ and $h \circ f = \text{id}_A$ then $g = h$.
 PROOF: We have $g = g \circ f \circ h = h$.
 □

Proposition 2.2.10. *Let $f : A \rightarrow B$. Then $\text{id}_B \circ f = f = f \circ \text{id}_A$.*

PROOF: Each is the function that maps a to $f(a)$. □

Proposition 2.2.11.

$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps f to $(\pi_1 \circ f, \pi_2 \circ f)$ is a bijection. □

Proposition 2.2.12.

$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function Φ such that $\Phi(f)(c)(b) = f(b, c)$ is a bijection. □

2.2.1 The Empty Set

Theorem 2.2.13. *There exists a set which has no elements.*

PROOF:

- ⟨1⟩1. PICK a set A
 PROOF: By the Axiom of Infinity, a set exists.
 ⟨1⟩2. LET: $S = \{x : \text{El}(A) \mid \perp\}$ with injection $i : S \rightarrow A$
 PROOF: Axiom of Separation.
 ⟨1⟩3. S has no elements.
 □

Theorem 2.2.14. *If E and E' have no elements then $E \approx E'$.*

PROOF:

- ⟨1⟩1. LET: E and E' have no elements.
 ⟨1⟩2. PICK a function $F : E \rightarrow E'$.

PROOF: Axiom of Choice since vacuously $\forall x : \text{El}(E) . \exists y : \text{El}(E') . \top$.

$\langle 1 \rangle 3$. F is injective.

PROOF: Vacuously, for all $x, y : \text{El}(E)$, if $F(x) = F(y)$ then $x = y$.

$\langle 1 \rangle 4$. F is surjective.

PROOF: Vacuously, for all $y : \text{El}(E)$, there exists $x : \text{El}(E)$ such that $F(x) = y$.

□

Definition 2.2.15 (Empty Set). The *empty set* \emptyset is the set with no elements.

2.2.2 The Singleton

Theorem 2.2.16. *There exists a set that has exactly one element.*

PROOF:

$\langle 1 \rangle 1$. PICK a set A that has an element.

PROOF: By the Axiom of Infinity, there exists a set that has an element.

$\langle 1 \rangle 2$. PICK $a : \text{El}(A)$

$\langle 1 \rangle 3$. PICK a set S and injection $i : S \rightarrow A$ such that, for all $x : \text{El}(A)$, there exists $s : \text{El}(S)$ such that $s = x$ if and only if $x = a$

$\langle 1 \rangle 4$. S has exactly one element.

□

Theorem 2.2.17. *If A and B both have exactly one element then $A \approx B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B both have exactly one element a and b respectively.

$\langle 1 \rangle 2$. LET: $F : A \rightarrow B$ be the function such that, for all $x : \text{El}(A)$, we have
 $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$. F is a bijection.

□

Definition 2.2.18 (Singleton). Let 1 be the set that has exactly one element. Let $*$ be its element.

2.2.3 Subsets

Definition 2.2.19 (Subset). A *subset* of a set A consists of a set S and an injection $i : S \rightarrow A$. We write $(S, i) \subseteq A$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Proposition 2.2.20. *For any subset (S, i) of A we have $(S, i) = (S, i)$.*

PROOF: We have $\text{id}_S : S \approx S$ and $i \circ \text{id}_S = i$.

Proposition 2.2.21. *If $(S, i) = (T, j)$ then $(T, j) = (S, i)$.*

PROOF: If $\phi : S \approx T$ and $j \circ \phi = i$ then $\phi^{-1} : T \approx S$ and $i \circ \phi^{-1} = j$. □

Proposition 2.2.22. *If $(R, i) = (S, j)$ and $(S, j) = (T, k)$ then $(R, i) = (T, k)$.*

PROOF: If $\phi : R \approx S$ and $j \circ \phi = i$, and $\psi : S \approx T$ and $k \circ \psi = j$, then $\psi \circ \phi : R \approx T$ and $k \circ \psi \circ \phi = i$. \square

Definition 2.2.23 (Membership). Given $(S, i) \subseteq A$ and $a \in A$, we write $a \in (S, i)$ for $\exists s \in S. i(s) = a$.

Proposition 2.2.24. *If $a \in (S, i)$ and $(S, i) = (T, j)$ then $a \in (T, j)$.*

PROOF: If $i(s) = a$ then $j(\phi(s)) = a$. \square

Definition 2.2.25 (Union). Given subsets S and T of A , the *union* is the subset $\{x \in A : x \in S \vee x \in T\}$.

Definition 2.2.26 (Intersection). Given subsets S and T of A , the *intersection* is the subset $\{x \in A : x \in S \wedge x \in T\}$.

Proposition 2.2.27 (Distributive Law).

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

Proposition 2.2.28 (Distributive Law).

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

Definition 2.2.29. Given a set A , we write \emptyset for the subset $(\emptyset, !)$ where $!$ is the unique function $\emptyset \rightarrow A$.

Proposition 2.2.30.

$$S \cup \emptyset = S$$

Proposition 2.2.31.

$$S \cap \emptyset = \emptyset$$

Definition 2.2.32 (Inclusion). Given subsets (S, i) and (T, j) of a set A , we write $(S, i) \subseteq (T, j)$ iff there exists $f : S \rightarrow T$ such that $j \circ f = i$.

Proposition 2.2.33.

$$\emptyset \subseteq S$$

Definition 2.2.34 (Disjoint). Subsets S and T of A are *disjoint* iff $S \cap T = \emptyset$.

Definition 2.2.35 (Difference). Given subsets S and T of A , the *difference* of S and T is $S - T = \{x \in A : x \in S \wedge x \notin T\}$.

Proposition 2.2.36 (De Morgan's Law).

$$R - (S \cup T) = (R - S) \cap (R - T)$$

Proposition 2.2.37 (De Morgan's Law).

$$R - (S \cap T) = (R - S) \cup (R - T)$$

2.2.4 Union

Definition 2.2.38 (Union). Given $\mathcal{A} \in \mathcal{PPX}$, its *union* is

$$\bigcup \mathcal{A} := \{x \in X : \exists S \in \mathcal{A}. x \in S\} \in \mathcal{PX} .$$

2.2.5 Intersection

Definition 2.2.39 (Intersection). Given $\mathcal{A} \in \mathcal{PPX}$, its *intersection* is

$$\bigcap \mathcal{A} := \{x \in X : \forall S \in \mathcal{A}. x \in S\} \in \mathcal{PX} .$$

2.2.6 Direct Image

Definition 2.2.40 (Direct Image). Let $f : A \rightarrow B$. Let S be a subset of A . The (*direct*) *image* of S under f is the subset of B given by

$$f(S) := \{f(a) : a \in S\} .$$

2.2.7 Inverse Image

Definition 2.2.41 (Inverse Image). Let $f : A \rightarrow B$. Let S be a subset of B . The *inverse image* or *preimage* of S under f is the subset of A given by

$$f^{-1}(S) := \{x \in A : f(x) \in S\} .$$

2.3 Relations

Definition 2.3.1 (Relation). Let A and B be sets. A *relation* R between A and B , $R : A \looparrowright B$, is a subset of $A \times B$.

Given $a \in A$ and $b \in B$, we write aRb for $(a, b) \in R$.

2.4 Power Set

Definition 2.4.1 (Power Set). The *power set* of a set A is $\mathcal{P}A := 2^A$.

Given $S \in \mathcal{P}A$ and $a \in A$, we write $a \in A$ for $S(a) = 1$.

2.5 Cartesian Product

Definition 2.5.1 (Cartesian Product). Let A and B be sets. The *Cartesian product* of A and B , $A \times B$, is the tabulation of the relation $A \looparrowright B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

2.6 Quotient Sets

Proposition 2.6.1. *Let \sim be an equivalence relation on X . Then there exists a set X/\sim , the quotient set of X with respect to \sim , and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the canonical projection, such that, for all $x, y : \text{El}(X)$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to \sim , then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

2.7 Partitions

Definition 2.7.1 (Partition). A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X .

2.8 Disjoint Union

Theorem 2.8.1. *For any sets A and B , there exists a set $A + B$, the disjoint union of A and B , and functions $\kappa_1 : A \rightarrow A + B$ and $\kappa_2 : B \rightarrow A + B$, the injections, such that, for every set X and functions $f : A \rightarrow X$ and $g : B \rightarrow X$, there exists a unique function $[f, g] : A + B \rightarrow X$ such that $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$.*

PROOF:

$\langle 1 \rangle 1$. LET: $A + B := \{p \in \mathcal{P}A \times \mathcal{P}B : \exists a \in A. p = (\{a\}, \emptyset) \vee \exists b \in B. p = (\emptyset, \{b\})\}$

Definition 2.8.2 (Restriction). Let $f : A \rightarrow B$ and let (S, i) be a subset of A . The *restriction* of f to S is the function $f \upharpoonright S : S \rightarrow B$ defined by $f \upharpoonright S = f \circ i$.

Chapter 3

Order Theory

3.1 Relations

Definition 3.1.1 (Reflexive). A relation $R \subseteq A \times A$ is *reflexive* iff, for all $a \in A$, we have $(a, a) \in R$.

Definition 3.1.2 (Antisymmetric). A relation $R \subseteq A \times A$ is *antisymmetric* iff, for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Definition 3.1.3 (Transitive). A relation $R \subseteq A \times A$ is *transitive* iff, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Definition 3.1.4 (Partial Order). A *partial order* on a set A is a relation on A that is reflexive, antisymmetric and transitive.

We say (A, \leq) is a *partially ordered set* or *poset* iff \leq is a partial order on A .

Chapter 4

Category Theory

4.1 Categories

Definition 4.1.1. A *category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \text{Ob}(\mathcal{C})$.
- for any objects X and Y , a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y . We write $f : X \rightarrow Y$ for $f \in \mathcal{C}[X, Y]$.
- for any objects X, Y and Z , a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \rightarrow \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \rightarrow Y, g : Y \rightarrow Z$ and $h : Z \rightarrow W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X , there exists a morphism $\text{id}_X : X \rightarrow X$, the *identity morphism* on X , such that:
 - for any object Y and morphism $f : Y \rightarrow X$ we have $\text{id}_X \circ f = f$
 - for any object Y and morphism $f : X \rightarrow Y$ we have $f \circ \text{id}_X = f$

We write the composite of morphism f_1, \dots, f_n as $f_n \circ \dots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 4.1.2. Let **Set** be the category of small sets and functions.

Definition 4.1.3. We identify a poset (A, \leq) with the category with:

- set of objects A
- for $a, b \in A$, the set of homomorphisms is $\{x \in 1 : a \leq b\}$

Proposition 4.1.4. A category is a poset iff, for any two objects, there exists at most one morphism between them.

Proposition 4.1.5. *The identity morphism on an object is unique.*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{C} be a category.

$\langle 1 \rangle 2$. LET: $A \in \mathcal{C}$

$\langle 1 \rangle 3$. LET: $i, j : A \rightarrow A$ be identity morphisms on A .

$\langle 1 \rangle 4$. $i = j$

PROOF:

$$\begin{aligned} i &= i \circ j && (j \text{ is an identity on } A) \\ &= j && (i \text{ is an identity on } A) \end{aligned}$$

□

Definition 4.1.6. Given $f : A \rightarrow B$ and an object C , define the function $f^* : \mathcal{C}[B, C] \rightarrow \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 4.1.7. Given $f : A \rightarrow B$ and an object C , define the function $f_* : \mathcal{C}[C, A] \rightarrow \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

4.1.1 Monomorphisms

Definition 4.1.8 (Monomorphism). Let $f : A \rightarrow B$. Then f is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for any object X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

4.1.2 Epimorphisms

Definition 4.1.9 (Epimorphism). Let $f : A \rightarrow B$. Then f is *epic* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for any object X and functions $x, y : B \rightarrow X$, if $x \circ f = y \circ f$ then $x = y$.

4.1.3 Sections and Retractions

Definition 4.1.10 (Section, Retraction). Let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $rs = \text{id}_B$.

Proposition 4.1.11. *Let $f : A \rightarrow B$ and $r, s : B \rightarrow A$. If r is a retraction of f and s is a section of f then $r = s$.*

PROOF:

$$\begin{aligned} r &= r \text{id}_B && (\text{Unit Law}) \\ &= rfs && (s \text{ is a section of } f) \\ &= \text{id}_A s && (r \text{ is a retraction of } f) \\ &= s && (\text{Unit Law}) \square \end{aligned}$$

Proposition 4.1.12. *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $s : B \rightarrow A$ be a section of $r : A \rightarrow B$.

$\langle 1 \rangle 2$. LET: X be an object and $x, y : X \rightarrow B$

$\langle 1 \rangle 3$. ASSUME: $s \circ x = s \circ y$

$\langle 1 \rangle 4$. $x = y$

PROOF: $x = r \circ s \circ x = r \circ s \circ y = y$.

□

Proposition 4.1.13. *Every retraction is epic.*

PROOF: Dual. □

4.1.4 Isomorphisms

Definition 4.1.14 (Isomorphism). A morphism $f : A \rightarrow B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$ that is both a retraction and section of f .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 4.1.15. *The inverse of an isomorphism is unique.*

PROOF: From Proposition 4.1.11. □

Proposition 4.1.16. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $ff^{-1} = \text{id}_B$ and $f^{-1}f = \text{id}_A$. □

Isomorphism.

Define the opposite category.

Slice categories

Definition 4.1.17. Let \mathcal{C} be a category and $B \in \mathcal{C}$. The category \mathcal{C}_B^B of objects *over and under* B is the category with:

- objects all triples (X, u, p) such that $u : B \rightarrow X$ and $p : X \rightarrow B$
- morphisms $f : (X, u, p) \rightarrow (Y, u', p')$ all morphisms $f : X \rightarrow Y$ such that $fu = u'$ and $p'f = p$.

Proposition 4.1.18.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B) \backslash \text{id}_B \cong (\mathcal{C} \backslash B) / \text{id}_B$$

$(B, \text{id}_B, \text{id}_B)$ is the zero object in \mathcal{C}_B^B .

4.1.5 Initial Objects

Definition 4.1.19 (Initial Object). An object I is *initial* iff, for any object X , there exists exactly one morphism $I \rightarrow X$.

Proposition 4.1.20. *The empty set is initial in **Set**.*

PROOF: For any set A , the nowhere-defined function is the unique function $\emptyset \rightarrow A$. \square

Proposition 4.1.21. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : I \rightarrow I'$ be the unique morphism $I \rightarrow I'$.

$\langle 1 \rangle 2$. LET: $i^{-1} : I' \rightarrow I$ be the unique morphism $I' \rightarrow I$.

$\langle 1 \rangle 3$. $ii^{-1} = \text{id}_{I'}$

PROOF: There is only one morphism $I' \rightarrow I'$.

$\langle 1 \rangle 4$. $i^{-1}i = \text{id}_I$

PROOF: There is only one morphism $I \rightarrow I$.

\square

4.1.6 Terminal Objects

Definition 4.1.22 (Terminal Object). An object T is *terminal* iff, for any object X , there exists exactly one morphism $X \rightarrow T$.

Proposition 4.1.23. *1 is terminal in Set.*

PROOF: For any set A , the constant function to $*$ is the only function $A \rightarrow 1$. \square

Proposition 4.1.24. *If T and T' are terminal objects, then there exists a unique isomorphism $T \cong T'$.*

PROOF: Dual to Proposition 4.1.21. \square

4.1.7 Zero Objects

Definition 4.1.25 (Zero Object). An object Z is a *zero object* iff it is an initial object and a terminal object.

Definition 4.1.26 (Zero Morphism). Let \mathcal{C} be a category with a zero object Z . Let $A, B \in \mathcal{C}$. The *zero morphism* $A \rightarrow B$ is the unique morphism $A \rightarrow Z \rightarrow B$.

Proposition 4.1.27. *There is no zero object in Set.*

PROOF: Since $\emptyset \not\approx 1$. \square

4.1.8 Triads

Definition 4.1.28 (Triad). Let \mathcal{C} be a category. A *triad* consists of objects X, Y, M and morphisms $\alpha : X \rightarrow M, \beta : Y \rightarrow M$. We call M the *codomain* of the triad.

4.1.9 Cotriads

Definition 4.1.29 (Cotriad). Let \mathcal{C} be a category. A *cotriad* consists of objects X, Y, W and morphisms $\xi : W \rightarrow X, \eta : W \rightarrow Y$. We call W the *domain* of the triad.

4.1.10 Pullbacks

Definition 4.1.30 (Pullback). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

is a *pullback* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$.

In this case we also say that η is the *pullback* of β along α .

Proposition 4.1.31. *If $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$ form a pullback of $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$, and $\xi' : W' \rightarrow X$ and $\eta' : W' \rightarrow Y$ also form the pullback of α and β , then there exists a unique isomorphism $\phi : W \cong W'$ such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : W \rightarrow W'$ be the unique morphism such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.

$\langle 1 \rangle 2$. LET: $\phi^{-1} : W' \rightarrow W$ be the unique morphism such that $\eta\phi^{-1} = \eta'$ and $\xi\phi^{-1} = \xi'$.

$\langle 1 \rangle 3$. $\phi\phi^{-1} = \text{id}_{W'}$

PROOF: Each is the unique $x : W' \rightarrow W'$ such that $\eta'x = \eta'$ and $\xi'x = \xi'$.

$\langle 1 \rangle 4$. $\phi^{-1}\phi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\eta x = \eta$ and $\xi x = \xi$.

□

Proposition 4.1.32. *For any morphism $h : A \rightarrow B$, the following diagram is a pullback diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: Z be an object.

$\langle 1 \rangle 2$. LET: $f : Z \rightarrow B$ and $g : Z \rightarrow A$ satisfy $\text{id}_B f = hg$

$\langle 1 \rangle 3$. $g : Z \rightarrow A$ is the unique morphism such that $\text{id}_A g = g$ and $hg = f$.

□

Proposition 4.1.33. *The pullback of an isomorphism is an isomorphism.*

PROOF:

⟨1⟩1. LET:

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback diagram.

⟨1⟩2. ASSUME: β is an isomorphism.

⟨1⟩3. LET: ξ^{-1} be the unique morphism $X \rightarrow W$ such that $\xi\xi^{-1} = \text{id}_X$ and $\eta\xi^{-1} = \beta^{-1}\alpha$.

PROOF: This exists since $\alpha\text{id}_X = \beta\beta^{-1}\alpha = \alpha$.

⟨1⟩4. $\xi^{-1}\xi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\xi x = \xi$ and $\eta x = \eta$.

□

Proposition 4.1.34. *Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w : A \rightarrow W$ be the unique morphism such that $\xi w = x$ and $\eta w = y$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ is the pullback of β and α in $\mathcal{C} \setminus A$.

PROOF:

⟨1⟩1. LET: $(Z, z) \in \mathcal{C} \setminus A$

⟨1⟩2. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

⟨1⟩3. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

⟨1⟩4. $hz = w$

⟨2⟩1. $\xi hz = \xi w$

PROOF:

$$\xi hz = fz \quad (\langle 1 \rangle 3)$$

$$= x \quad (\langle 1 \rangle 2)$$

$$= \xi w$$

⟨2⟩2. $\eta hz = \eta w$

PROOF: Similar.

⟨1⟩5. $h : (Z, z) \rightarrow (W, w)$

□

Proposition 4.1.35. *Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in*

\mathcal{C}/A . Let

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w = x\xi : W \rightarrow A$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ form a pullback of α and β in \mathcal{C}/A .

PROOF:

$\langle 1 \rangle 1$. $\eta : (W, w) \rightarrow (Y, y)$

PROOF:

$$\begin{aligned} y\eta &= m\beta\eta \\ &= m\alpha\xi \\ &= x\xi \\ &= w \end{aligned}$$

$\langle 1 \rangle 2$. LET: $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 3$. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle 4$. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle 5$. $h : (Z, z) \rightarrow (W, w)$

PROOF:

$$\begin{aligned} wh &= x\xi h \\ &= xf && (\langle 1 \rangle 4) \\ &= z && (\langle 1 \rangle 3) \end{aligned}$$

□

Proposition 4.1.36. In **Set**, let $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$. Let $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$ with inclusion $i : W \rightarrow X \times Y$. Let $\xi = \pi_1 i : W \rightarrow X$ and $\eta = \pi_2 i : W \rightarrow Y$. Then ξ and η form the pullback of α and β .

PROOF:

$\langle 1 \rangle 1$. $\alpha\xi = \beta\eta$

PROOF: For $w \in W$, if $i(w) = (x, y)$ then $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$.

$\langle 1 \rangle 2$. For every set Z and functions $f : Z \rightarrow X$, $g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$

PROOF: For $z \in Z$, let $h(z)$ be the unique element of W such that $i(h(z)) = (f(z), g(z))$.

□

Pullback lemma

4.1.11 Pushouts

Definition 4.1.37 (Pushout). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array} \quad (4.1)$$

is a *pushout* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ such that $f\xi = g\eta$, there exists a unique $h : M \rightarrow Z$ such that $h\alpha = f$ and $h\beta = g$.

We also say that β is the *pushout* of ξ along η .

Proposition 4.1.38. *If $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$ form a pushout of $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$, and $\alpha' : X \rightarrow M'$ and $\beta' : Y \rightarrow M'$ also form a pushout of ξ and η , then there exists a unique isomorphism $\phi : M \cong M'$ such that $\phi\alpha = \alpha'$ and $\phi\beta = \beta'$.*

PROOF: Dual to Proposition 4.1.31. \square

Proposition 4.1.39. *For any morphism $h : A \rightarrow B$, the following diagram is a pushout diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF: Dual to Proposition 4.1.32.

Proposition 4.1.40. *The diagram (4.1) is a pushout in \mathcal{C} iff it is a pullback in \mathcal{C}^{op} .*

PROOF: Immediate from definitions. \square

Proposition 4.1.41. *The pushout of an isomorphism is an isomorphism.*

PROOF: Dual to Proposition 4.1.33. \square

Proposition 4.1.42. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m := \alpha x : A \rightarrow M$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in $\mathcal{C} \setminus A$.

PROOF: Dual to Proposition 4.1.35. \square

Proposition 4.1.43. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in \mathcal{C}/A . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m : M \rightarrow A$ be the unique morphism such that $m\alpha = x$ and $m\beta = y$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in \mathcal{C}/A .

PROOF: Dual to Proposition 4.1.34. \square

Proposition 4.1.44. *Set has pushouts.*

PROOF:

$\langle 1 \rangle 1$. LET: $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$.

$\langle 1 \rangle 2$. LET: \sim be the equivalence relation on $X + Y$ generated by $\xi(w) \sim \eta(w)$ for all $w \in W$

$\langle 1 \rangle 3$. LET: $M = (X + Y)/\sim$ with canonical projection $\pi : X + Y \twoheadrightarrow M$.

$\langle 1 \rangle 4$. LET: $\alpha = \pi \circ \kappa_1 : X \rightarrow M$

$\langle 1 \rangle 5$. LET: $\beta = \pi \circ \kappa_2 : Y \rightarrow M$

$\langle 1 \rangle 6$. LET: Z be any set, $f : X \rightarrow Z$ and $g : Y \rightarrow Z$.

$\langle 1 \rangle 7$. ASSUME: $f\xi = g\eta$

$\langle 1 \rangle 8$. LET: $h : X + Y \rightarrow Z$ be the function defined by $h(x) = f(x)$ and $h(y) = g(y)$ for $x \in X$ and $y \in Y$

$\langle 1 \rangle 9$. h respects \sim

PROOF: For $w \in W$ we have

$$\begin{aligned} h(\xi(w)) &= f(\xi(w)) && (\langle 1 \rangle 8) \\ &= g(\eta(w)) && (\langle 1 \rangle 7) \\ &= h(\eta(w)) && (\langle 1 \rangle 8) \end{aligned}$$

$\langle 1 \rangle 10$. LET: $\bar{h} : M \rightarrow Z$ be the induced function.

$\langle 1 \rangle 11$. $\bar{h}\alpha = f$

PROOF:

$$\begin{aligned} \bar{h}(\alpha(x)) &= \bar{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x) \end{aligned}$$

$\langle 1 \rangle 12$. $\bar{h}\beta = g$

PROOF: Similar.

$\langle 1 \rangle 13$. For all $k : M \rightarrow Z$, if $k\alpha = f$ and $k\beta = g$ then $k = \bar{h}$.

PROOF:

$$\begin{aligned}
 k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\
 &= f(x) \\
 k(\pi(\kappa_2(y))) &= k(\beta(y)) \\
 &= g(y) \\
 \therefore k \circ \pi &= h \\
 \therefore k &= \bar{h}
 \end{aligned}$$

□

Definition 4.1.45. Let $u : A \rightarrow X$ be an injection. The *pointed set obtained from X by collapsing (A, u)* , denoted $X/(A, u)$, is the pushout

$$\begin{array}{ccc}
 A & \xrightarrow{\quad} & 1 \\
 \downarrow u & & \downarrow * \\
 X & \longrightarrow & X/(A, u)
 \end{array}$$

Proposition 4.1.46. In \mathbf{Set}_* , any two morphisms $1 \rightarrow X$ and $1 \rightarrow Y$ have a pushout.

PROOF: The pushout of $a : (1, *) \rightarrow (X, x)$ and $b : (1, *) \rightarrow (Y, y)$ is $(X+Y/\sim, x)$ where \sim is the equivalence relation generated by $x \sim y$. □

Definition 4.1.47 (Wedge). The *wedge* of pointed sets X and Y , $X \vee Y$, is the pushout of the unique morphism $1 \rightarrow X$ and $1 \rightarrow Y$.

Definition 4.1.48 (Smash). Let X and Y be pointed sets. Let $\xi : X \vee Y \rightarrow X$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & X \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{\quad} & X \vee Y \\
 & \searrow \xi & \nearrow \\
 & & X
 \end{array}$$

(Note: In the original image, there is an additional arrow labeled 0 from Y to X .)

Let $\eta : X \vee Y \rightarrow Y$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & X \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{\quad} & X \vee Y \\
 & \searrow \eta & \nearrow \\
 & & Y
 \end{array}$$

(Note: In the original image, there is an additional arrow labeled 0 from X to Y .)

Let $\zeta = \langle \xi, \eta \rangle : X \vee Y \rightarrow X \times Y$. The *smash* of X and Y , $X \wedge Y$, is the result of collapsing $X \times Y$ with respect to ζ .

Pushout lemma

4.1.12 Subcategories

Definition 4.1.49 (Subcategory). A *subcategory* \mathcal{C}' of a category \mathcal{C} consists of:

- a subset $\text{Ob}(\mathcal{C}')$ of \mathcal{C}
- for all $A, B \in \text{Ob}(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \text{Ob}(\mathcal{C}')$, we have $\text{id}_A \in \mathcal{C}'[A, A]$
- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \text{Ob}(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

4.1.13 Opposite Category

Definition 4.1.50 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- $\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$
- Given $f \in \mathcal{C}^{\text{op}}[A, B]$ and $g \in \mathcal{C}^{\text{op}}[B, C]$, their composite in \mathcal{C}^{op} is $f \circ g$, where \circ is composition in \mathcal{C} .

Proposition 4.1.51. *An object is initial in \mathcal{C} iff it is terminal in \mathcal{C}^{op} .*

PROOF: Immediate from definitions. \square

Proposition 4.1.52. *An object is terminal in \mathcal{C} iff it is initial in \mathcal{C}^{op} .*

PROOF: Immediate from definitions. \square

Corollary 4.1.52.1. *If T and T' are terminal objects in \mathcal{C} then there exists a unique isomorphism $T \cong T'$.*

4.1.14 Groupoids

Definition 4.1.53 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

4.1.15 Concrete Categories

Definition 4.1.54 (Concrete Category). A *concrete category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*
- for any object $A \in \text{Ob}(\mathcal{C})$, a set $|A|$
- for any objects $A, B \in \text{Ob}(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$
- for any object A we have $\text{id}_{|A|} \in \mathcal{C}[A, A]$.

4.1.16 Power of Categories

Definition 4.1.55. Let \mathcal{C} be a category and J a set. The category \mathcal{C}^J is the category with:

- objects all J -indexed families of objects of \mathcal{C}
- morphisms $\{X_j\}_{j \in J} \rightarrow \{Y_j\}_{j \in J}$ all families $\{f_j\}_{j \in J}$ where $f_j : X_j \rightarrow Y_j$

4.1.17 Arrow Category

Definition 4.1.56 (Arrow Category). Let \mathcal{C} be a category. The *arrow category* \mathcal{C}^\rightarrow is the category with:

- objects all triples (A, B, f) where $f : A \rightarrow B$ in \mathcal{C}
- morphisms $(A, B, f) \rightarrow (C, D, g)$ all pairs $(u : A \rightarrow C, v : B \rightarrow D)$ such that $vf = gu$.

4.1.18 Slice Category

Definition 4.1.57 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category under A*, $\mathcal{C} \backslash A$, is the category with:

- objects all pairs (B, f) where $B \in \mathcal{C}$ and $f : A \rightarrow B$
- morphisms $(B, f) \rightarrow (C, g)$ are morphisms $u : B \rightarrow C$ such that $uf = g$.

We identify this with the subcategory of \mathcal{C}^\rightarrow formed by mapping (B, f) to (A, B, f) and u to (id_A, u) .

Proposition 4.1.58. *If $s : (B, f) \rightarrow (C, g)$ in $\mathcal{C} \backslash A$, then any retraction of s in \mathcal{C} is a retraction of s in $\mathcal{C} \backslash A$.*

PROOF:

$\langle 1 \rangle 1$. LET: $r : C \rightarrow B$ be a retraction of s in \mathcal{C} .

$\langle 1 \rangle 2. rg = f$

PROOF: $rg = rsf = f$.

$\langle 1 \rangle 3. r : (C, g) \rightarrow (B, f)$ in $\mathcal{C} \backslash A$

$\langle 1 \rangle 4. rs = \text{id}_{(B, f)}$

PROOF: Because composition is inherited from \mathcal{C} .

□

Proposition 4.1.59. id_A is the initial object in $\mathcal{C} \backslash A$.

PROOF: For any $(B, f) \in \mathcal{C} \backslash A$, we have f is the only morphism $A \rightarrow B$ such that $f \text{id}_A = f$. □

Proposition 4.1.60. If A is terminal in \mathcal{C} then id_A is the zero object in $\mathcal{C} \backslash A$.

PROOF: For any $(B, f) \in \mathcal{C} \backslash A$, the unique morphism $! : B \rightarrow A$ is the unique morphism such that $!f = \text{id}_A$. □

Definition 4.1.61 (Pointed Sets). The category of pointed sets is **Set** $\backslash 1$.

Definition 4.1.62. Let \mathcal{C} be a category and $A \in \mathcal{C}$. The slice category over A , \mathcal{C}/A , is the category with:

- objects all pairs (B, f) with $f : B \rightarrow A$
- morphisms $u : (B, f) \rightarrow (C, g)$ all morphisms $u : B \rightarrow C$ such that $gu = f$.

Proposition 4.1.63. Let $u : (B, f) \rightarrow (C, g) : \mathcal{C}/A$. Any section of u in \mathcal{C} is a section of u in \mathcal{C}/A .

PROOF: Dual to Proposition 4.1.58. □

Proposition 4.1.64. id_A is terminal in \mathcal{C}/A .

PROOF: Dual to Proposition 4.1.59. □

Proposition 4.1.65. If A is initial in \mathcal{C} then id_A is the zero object in \mathcal{C}/A .

PROOF: Dual to Proposition 4.1.60. □

Definition 4.1.66. Let $A \in \mathcal{C}$. The category of objects over and under A , written \mathcal{C}_A^A , is the category with:

- objects all triples (X, u, p) where $u : A \rightarrow X$, $p : X \rightarrow A$ and $pu = \text{id}_A$
- morphism $f : (X, u, p) \rightarrow (Y, v, q)$ all morphisms $f : X \rightarrow Y$ such that $fu = v$ and $qf = p$

Proposition 4.1.67. $(A, \text{id}_A, \text{id}_A)$ is the zero object in \mathcal{C}_A^A .

PROOF: For any object (X, u, p) , we have p is the unique morphism $(X, u, p) \rightarrow (A, \text{id}_A, \text{id}_A)$, and u is the unique morphism $(A, \text{id}_A, \text{id}_A) \rightarrow (X, u, p)$. □

Definition 4.1.68 (Fibre Collapsing). Let B be a set. Let $u : (A, a) \rightarrow (X, x)$ in \mathbf{Set}/B . Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow u & & \downarrow j \\ X & \xrightarrow{i} & C \end{array}$$

Let $c : C \rightarrow B$ be the unique morphism such that $cj = \text{id}_B$ and $ci = x$. Then $(C, j, c) \in \mathbf{Set}_B^B$ is called the set over and under B obtained from X by *fibre collapsing* with respect to u . If (A, u) is a subset of X , we denote this set over and under B by $X/_B(A, u)$.

Definition 4.1.69 (Fibre Wedge). Let B be a small set. Let $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$. The *fibre wedge* of X and Y is the pushout of u_X and u_Y :

$$\begin{array}{ccc} B & \xrightarrow{u_X} & X \\ \downarrow u_Y & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \end{array}$$

Definition 4.1.70 (Fibre Smash). Let $X, Y \in \mathbf{Set}_B^B$. Let $\xi : X \vee_B Y \rightarrow X$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \xi & \nearrow 0 \\ & & X \end{array}$$

Let $\eta : X \vee_B Y \rightarrow Y$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \eta & \nearrow 0 \\ & & Y \end{array}$$

Let $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \rightarrow X \times Y$. The *fibre smash* of X and Y , $X \wedge_B Y$, is the result of collapsing $X \times Y$ with respect to ζ .

Proposition 4.1.71. *Set has products and coproducts.*

Proposition 4.1.72. *Let \mathcal{C} be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in \mathcal{C} and $Z \in \mathcal{C}$. Let $\coprod_{\alpha \in I} X_\alpha$ be the coproduct of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[\coprod_{\alpha \in I} X_\alpha, Z] \approx \prod_{\alpha \in I} \mathcal{C}[X_\alpha, Z] .$$

Proposition 4.1.73. *Let \mathcal{C} be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in \mathcal{C} and $Z \in \mathcal{C}$. Let $\prod_{\alpha \in I} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[Z, \prod_{\alpha \in I} X_\alpha] \approx \prod_{\alpha \in I} \mathcal{C}[Z, X_\alpha] .$$

Proposition 4.1.74. *A product in \mathcal{C} constitutes a product in \mathcal{C}/A .*

Proposition 4.1.75. *A coproduct in \mathcal{C} constitutes a product in \mathcal{C}/A .*

4.2 Functors

Definition 4.2.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- a function $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism $f : A \rightarrow B$ in \mathcal{C} , a morphism $Ff : FA \rightarrow FB$ in \mathcal{D}

such that:

- for all $A : \text{El}(\text{Ob}(\mathcal{C}))$ we have $F\text{id}_A = \text{id}_{FA}$
- for any morphism $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathcal{C} , we have $F(g \circ f) = Fg \circ Ff$

Proposition 4.2.2. *Functors preserve isomorphisms.*

PROOF:

$\langle 1 \rangle 1$. LET: $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

$\langle 1 \rangle 2$. LET: $f : A \cong B$ in \mathcal{C}

$\langle 1 \rangle 3$. $Ff^{-1} \circ Ff = \text{id}_{FA}$

PROOF:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\text{id}_A \\ &= \text{id}_{FA} \end{aligned}$$

$\langle 1 \rangle 4$. $Ff \circ Ff^{-1} = \text{id}_{FB}$

PROOF:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

Definition 4.2.3 (Identity Functor). For any category \mathcal{C} , the *identity* functor on \mathcal{C} is the functor $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$\begin{aligned} I_{\mathcal{C}} A &:= A & (A \in \mathcal{C}) \\ I_{\mathcal{C}} f &:= f & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

Proposition 4.2.4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $r : A \rightarrow B$ is a retraction of $s : B \rightarrow A$ in \mathcal{C} then Fr is a retraction of Fs .

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned} \quad \square$$

Corollary 4.2.4.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $\phi : A \cong B$ is an isomorphism in \mathcal{C} then $F\phi : FA \cong FB$ is an isomorphism in \mathcal{D} with $(F\phi)^{-1} = F\phi^{-1}$.

Definition 4.2.5 (Composition of Functors). Given functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$, the *composite* functor $GF : \mathcal{C} \rightarrow \mathcal{E}$ is defined by

$$\begin{aligned} (GF)A &= G(FA) & (A \in \mathcal{C}) \\ (GF)f &= G(Ff) & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

Definition 4.2.6 (Category of Categories). Let **Cat** be the category of small categories and functors.

Definition 4.2.7 (Isomorphism of Categories). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F is an *isomorphism of categories* iff there exists a functor $F^{-1} : \mathcal{D} \rightarrow \mathcal{C}$, the *inverse* of F , such that $FF^{-1} = I_{\mathcal{D}}$ and $F^{-1}F = I_{\mathcal{C}}$.

Categories \mathcal{C} and \mathcal{D} are *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, iff there exists an isomorphism between them.

Proposition 4.2.8. If A is initial in \mathcal{C} then $\mathcal{C} \setminus A \cong \mathcal{C}$.

PROOF:

$\langle 1 \rangle 1$. Define $F : \mathcal{C} \setminus A \rightarrow \mathcal{C}$ by

$$F(B, f) = B$$

$$F(u : (B, f) \rightarrow (C, g)) = u$$

$\langle 1 \rangle 2$. Define $G : \mathcal{C} \rightarrow \mathcal{C} \setminus A$ by

$$GB = (B, !_B)$$

where $!_B$ is the unique morphism $A \rightarrow B$

$$G(u : B \rightarrow C) = u : (B, !_B) \rightarrow (C, !_C)$$

$\langle 1 \rangle 3$. $FG = \text{id}_{\mathcal{C}}$

$\langle 1 \rangle 4$. $GF = \text{id}_{\mathcal{C} \setminus A}$

PROOF: Since $GF(B, f) = (B, !_B) = (B, f)$ because the morphism $A \rightarrow B$ is unique.

\square

Proposition 4.2.9. If A is terminal in \mathcal{C} then $\mathcal{C}/A \cong \mathcal{C}$.

PROOF: Dual. \square

Proposition 4.2.10.

$$\mathcal{C}_A^A \cong (\mathcal{C}/A) \backslash (A, \text{id}_A) \cong (\mathcal{C} \backslash A) / (A, \text{id}_A)$$

PROOF:

$\langle 1 \rangle 1$. Define a functor $F : \mathcal{C}_A^A \rightarrow (\mathcal{C}/A) \backslash (A, \text{id}_A)$.

$\langle 2 \rangle 1$. Given $A \xrightarrow{u} X \xrightarrow{p} A$ in \mathcal{C}_A^A , let $F(X, u, p) = ((X, p), u)$

$\langle 2 \rangle 2$. Given $f : (A \xrightarrow{u} X \xrightarrow{p} A) \rightarrow (A \xrightarrow{v} Y \xrightarrow{q} A)$, let $Ff = f$.

$\langle 1 \rangle 2$. Define a functor $G : (\mathcal{C}/A) \backslash (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.

$\langle 1 \rangle 3$. Define a functor $H : \mathcal{C}_A^A \rightarrow (\mathcal{C} \backslash A) / (A, \text{id}_A)$.

$\langle 1 \rangle 4$. Define a functor $K : (\mathcal{C} \backslash A) / (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.

\square

Definition 4.2.11 (Forgetful Functor). For any concrete category \mathcal{C} , define the *forgetful* functor $U : \mathcal{C} \rightarrow \mathbf{Set}$ by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

Definition 4.2.12 (Switching Functor). For any category \mathcal{C} , define the *switching* functor $T : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

Definition 4.2.13 (Reduction). Let $\Phi : \mathbf{Set} \rightarrow \mathbf{Set}$ be a functor. The *reduction* of Φ is the functor $\Phi^* : \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ defined by: $\Phi^*(X, a)$ is the collapse of $\Phi(X)$ with respect to $\Phi(a) : \Phi(1) \rightarrow \Phi(X)$.

Definition 4.2.14. Extend the wedge \vee to a functor $\mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ by defining, given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, there $f \vee g$ is the unique morphism that makes the following diagram commute.

$$\begin{array}{ccccc} 1 & \longrightarrow & X & & \\ \downarrow & & \downarrow & \searrow f & \\ Y & \longrightarrow & X \vee Y & & X' \\ & \searrow g & \downarrow f \vee g & \downarrow & \\ & & Y' & \longrightarrow & X' \vee Y' \end{array}$$

Definition 4.2.15. Extend smash to a functor $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ as follows. Given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, let $f \wedge g : X \wedge Y \rightarrow X' \wedge Y'$ be the

unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc}
 X \vee Y & \xrightarrow{\quad} & 1 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 X \times Y & \xrightarrow{\quad} & X \wedge Y & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & X' \vee Y' & \xrightarrow{\quad} & 1 \\
 f \times g \swarrow & & \downarrow & & \downarrow \\
 & & X' \times Y' & \xrightarrow{\quad} & X' \wedge Y'
 \end{array}$$

Definition 4.2.16 (Reduction). Let B be a small set. Let $\Phi_B : \mathbf{Set}/B \rightarrow \mathbf{Set}/B$ be a functor. The *reduction* of Φ_B is the functor $\Phi_B^B : \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ defined as follows.

For $(X, u : B \rightarrow X, p : X \rightarrow B) \in \mathbf{Set}_B^B$, let $\Phi_B^B(X)$ be the set over and under B obtained from $\Phi_B(X)$ by collapsing with respect to $\Phi_B(u) : \Phi_B(B) \rightarrow \Phi_B(X)$.

Definition 4.2.17. Extend \vee_B to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$.

Definition 4.2.18. Extend \wedge_B to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$.

Definition 4.2.19 (Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *faithful* iff, for any objects $A, B \in \mathcal{C}$ and morphisms $f, g : A \rightarrow B : \mathcal{C}$, if $Ff = Fg$ then $f = g$.

Definition 4.2.20 (Full). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *full* iff, for any objects $A, B \in \mathcal{C}$ and morphism $g : FA \rightarrow FB : \mathcal{D}$, there exists $f : A \rightarrow B : \mathcal{C}$ such that $Ff = g$.

Definition 4.2.21 (Fully Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *fully faithful* iff it is full and faithful.

Definition 4.2.22 (Full Embedding). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *full embedding* iff it is fully faithful and injective on objects.

4.3 Natural Transformations

Definition 4.3.1 (Natural Transformation). Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$. A *natural transformation* $\tau : F \Rightarrow G$ is a family of morphisms $\{\tau_X : FX \rightarrow GX\}_{X \in \mathcal{C}}$ such that, for every morphism $f : X \rightarrow Y : \mathcal{C}$, we have $Gf \circ \tau_X = \tau_Y \circ Ff$.

$$\begin{array}{ccc}
 FX & \xrightarrow{Ff} & FY \\
 \tau_X \downarrow & & \downarrow \tau_Y \\
 GX & \xrightarrow{Gf} & GY
 \end{array}$$

Definition 4.3.2 (Natural Isomorphism). A natural transformation $\tau : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$ is a *natural isomorphism*, $\tau : F \cong G$, iff for all $X \in \mathcal{C}$, τ_X is an isomorphism $FX \cong GX$.

Functors F and G are *naturally isomorphic*, $F \cong G$, iff there exists a natural isomorphism between them.

Definition 4.3.3 (Inverse). Let $\tau : F \cong G$. The *inverse* natural isomorphism $\tau^{-1} : G \cong F$ is defined by $(\tau^{-1})_X = \tau_X^{-1}$.

4.4 Bifunctors

Definition 4.4.1 (Commutative). A bifunctor $\square : \mathcal{C}^2 \rightarrow \mathcal{C}$ is *commutative* iff $\square \cong \square \circ T$, where $T : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ is the swap functor.

Proposition 4.4.2. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is commutative.

PROOF: Since the pushout of f and g is the pushout of g and f . \square

Proposition 4.4.3. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is commutative.

PROOF: In the diagram defining $X \wedge Y$, construct the isomorphism between the version with X and Y and the version with Y with X for every object. \square

Proposition 4.4.4. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is commutative.

Proposition 4.4.5. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is commutative.

Definition 4.4.6 (Associative). A bifunctor \square is *associative* iff $\square \circ (\square \times \text{id}) \cong \square \circ (\text{id} \times \square)$.

Proposition 4.4.7. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is associative.

PROOF: Since $X \vee (Y \vee Z)$ and $(X \vee Y) \vee Z$ are both the pushout of the unique morphisms $1 \rightarrow X$, $1 \rightarrow Y$ and $1 \rightarrow Z$. \square

Proposition 4.4.8. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is associative.

PROOF: Draw isomorphisms between the diagrams for $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$. \square

Product and coproduct are commutative and associative.

Proposition 4.4.9. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is associative.

Proposition 4.4.10. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is associative.

Proposition 4.4.11. Let \mathcal{C} be a category with binary coproducts. Let $\square : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ be a bifunctor. Then \square distributes over $+$ iff the canonical morphism

$$(X \square Z) + (Y \square Z) \rightarrow (X + Y) \square Z$$

is an isomorphism for all X, Y, Z .

Proposition 4.4.12. *In a category with binary products and binary coproducts, then \times distributes over $+$.*

Proposition 4.4.13. *In $\mathbf{Set}/*$, we have \times does not distribute over \vee .*

Proposition 4.4.14. *In $\mathbf{Set}/*$, we have \wedge distributes over \vee .*

Proposition 4.4.15. *In \mathbf{Set}/B , we have \times_B distributes over $+_B$.*

Proposition 4.4.16. *In \mathbf{Set}/B^B , we have \wedge_B distributes over \vee_B .*

4.5 Functor Categories

Definition 4.5.1 (Functor Category). Given categories \mathcal{C} and \mathcal{D} , define the *functor category* $\mathcal{C}^{\mathcal{D}}$ to be the category with objects the functors from \mathcal{D} to \mathcal{C} and morphisms the natural transformations.

Definition 4.5.2 (Yoneda Embedding). Let \mathcal{C} be a category. The *Yoneda embedding* $Y : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ is the functor that maps an object A to $\mathcal{C}[-, A]$ and morphisms similarly.

Theorem 4.5.3 (Yoneda Lemma). *Let \mathcal{C} be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\text{op}}}[\mathcal{C}[-, X], F] \cong FX$$

that maps $\tau : \mathcal{C}[-, X] \Rightarrow F$ to $\tau_X(\text{id}_X)$.

PROOF:

$\langle 1 \rangle 1$. ϕ is natural in X .

PROOF:

$\langle 2 \rangle 1$. LET: $f : X \rightarrow Y : \mathcal{C}$

$\langle 2 \rangle 2$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$. $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$

PROOF:

$$\begin{aligned} \phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\text{id}_Y \circ f) \\ &= \tau_Y(f) \\ &= \tau_Y(f \circ \text{id}_X) \\ &= Ff(\tau_X(\text{id}_X)) && (\tau \text{ natural}) \\ &= Ff(\phi(\tau)) \end{aligned}$$

$\langle 1 \rangle 2$. ϕ is natural in F .

$\langle 2 \rangle 1$. LET: $\alpha : F \Rightarrow G : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$

$\langle 2 \rangle 2$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$. $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$

PROOF: $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\text{id}_X)) = \alpha_X(\phi(\tau))$

$\langle 1 \rangle 3$. Each ϕ_{XF} is injective.

$\langle 2 \rangle 1$. LET: $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 2$. ASSUME: $\phi(\sigma) = \phi(\tau)$

$\langle 2 \rangle 3$. LET: $f : Y \rightarrow X$

$\langle 2 \rangle 4$. $\sigma_Y(f) = \tau_Y(f)$

PROOF:

$$\begin{aligned}
 \sigma_Y(f) &= \sigma_Y(\text{id}_X \circ f) \\
 &= Ff(\sigma_X(\text{id}_X)) && (\sigma \text{ is natural}) \\
 &= Ff(\tau_X(\text{id}_X)) && (\langle 2 \rangle 2) \\
 &= \tau_Y(\text{id}_X \circ f) && (\tau \text{ is natural}) \\
 &= \tau_Y(f)
 \end{aligned}$$

$\langle 1 \rangle 4$. Each ϕ_{XF} is surjective.

$\langle 2 \rangle 1$. LET: $X \in \mathcal{C}$ and $F : \mathcal{C} \rightarrow \mathcal{D}$

$\langle 2 \rangle 2$. LET: $a \in FX$

$\langle 2 \rangle 3$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$ be given by $\tau_Y(g) = Fg(a)$ for $g : Y \rightarrow X$

$\langle 2 \rangle 4$. τ is natural.

$\langle 3 \rangle 1$. LET: $h : Y \rightarrow Z : \mathcal{C}$

PROVE: $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \text{id}_X]$

$\langle 3 \rangle 2$. LET: $g : Z \rightarrow X$

$\langle 3 \rangle 3$. $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$

PROOF:

$$\begin{aligned}
 \tau_Y(g \circ h) &= F(g \circ h)(a) \\
 &= Fh(Fg(a)) \\
 &= Fh(\tau_Z(g))
 \end{aligned}$$

$\langle 2 \rangle 5$. $\phi(\tau) = a$

PROOF:

$$\begin{aligned}
 \phi_X(\tau) &= \tau_X(\text{id}_X) \\
 &= F\text{id}_X(a) \\
 &= a
 \end{aligned}$$

□

Corollary 4.5.3.1. *The Yoneda embedding is fully faithful.*

Corollary 4.5.3.2. *Given objects A and B in \mathcal{C} , we have $A \cong B$ if and only if $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$.*

Chapter 5

Monoid Theory

Definition 5.0.1 (Monoid). A *monoid* is a category with one object.

Definition 5.0.2. Let \mathcal{C} be a category and $X \in \mathcal{C}$. The monoid $\text{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \rightarrow X$ under composition.

Proposition 5.0.3. *For any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and $X \in \mathcal{C}$, we have that $F : \text{End}_{\mathcal{C}}(X) \rightarrow \text{End}_{\mathcal{D}}(FX)$ is a monoid homomorphism.*

PROOF: Since $F\text{id}_X = \text{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$. \square

Chapter 6

Group Theory

Definition 6.0.1. Let **Grp** be the category of small groups and group homomorphisms.

Definition 6.0.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G .

Proposition 6.0.3. *The trivial group is a zero object in **Grp**.*

PROOF: Easy. \square

The zero morphism $G \rightarrow H$ maps every element in G to e .

Definition 6.0.4. Let \mathcal{C} be a category and $X \in \mathcal{C}$. We write $\text{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Proposition 6.0.5. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $X \in \mathcal{C}$. Then $F : \text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(FX)$ is a group homomorphism.*

PROOF: Since $F\text{id}_X = \text{id}_{FX}$, $F(g \circ f) = Fg \circ Ff$, and $Ff^{-1} = (Ff)^{-1}$. \square

Proposition 6.0.6. **Grp** has products.

Definition 6.0.7 (Free Product). The product of a family of groups in **Grp** is called the *free product*.

Proposition 6.0.8. **Ab** has products given by direct sums.

Chapter 7

Ring Theory

Definition 7.0.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Definition 7.0.2 (Spectrum). Let R be a commutative ring. The *spectrum* of R , $\text{spec } R$, is the set of all prime ideals of R .

Definition 7.0.3 (Zariski Topology). Let R be a commutative ring. The *Zariski topology* on $\text{spec } R$ is the topology where the closed sets are the sets of the form

$$VE := \{p \in \text{spec } R : E \subseteq p\}$$

for any $E \in \mathcal{P}R$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$

$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{A} \in \mathcal{C}$

$\langle 2 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{C}$

$\langle 2 \rangle 2$. LET: $E = \bigcup \{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$

PROVE: $VE = \bigcap \mathcal{A}$

$\langle 2 \rangle 3$. For all $p \in \text{spec } R$, if $E \subseteq p$ then $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 1$. LET: $p \in \text{spec } R$

$\langle 3 \rangle 2$. ASSUME: $E \subseteq p$

$\langle 3 \rangle 3$. LET: $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$

$\langle 3 \rangle 4$. $E' \subseteq E$

$\langle 3 \rangle 5$. $E' \subseteq p$

$\langle 3 \rangle 6$. $p \in VE'$

$\langle 2 \rangle 4$. For all $p \in \text{spec } R$, if $p \in \bigcap \mathcal{A}$ then $E \subseteq p$

$\langle 3 \rangle 1$. LET: $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 2$. For all $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$ we have $E' \subseteq p$

$\langle 3 \rangle 3$. $E \subseteq p$

$\langle 1 \rangle 3$. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

PROOF: Since $VE \cup VE' = V(E \cap E')$

$\langle 1 \rangle 4. \emptyset \in \mathcal{C}$

$\langle 2 \rangle 1. VR = \emptyset$

PROOF: If $p \in VR$ then $R \subseteq p$ contradicting the fact that p is a prime ideal.

□

Definition 7.0.4. For any ring R , let $R - \mathbf{Mod}$ be the category of small R -modules and R -module homomorphisms.

Proposition 7.0.5. $R - \mathbf{Mod}$ has products and coproducts.

Chapter 8

Field Theory

Proposition 8.0.1. *Field does not have binary products.*

PROOF: There cannot be a field K with field homomorphisms $K \rightarrow \mathbb{Z}_2$ and $K \rightarrow \mathbb{Z}_3$, because its characteristic would be both 2 and 3. \square

Chapter 9

Linear Algebra

Definition 9.0.1. For any field K , we write \mathbf{Vect}_K for $K\text{-Mod}$.

Dual space functor $\mathbf{Vect}_K^{\text{op}} \rightarrow \mathbf{Vect}_K$.

Chapter 10

Topology

10.1 Topological Spaces

Definition 10.1.1 (Topological Space). Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say (X, \mathcal{O}) is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call \mathcal{O} the *topology* of the topological space, and call its elements *open* sets. We shall often write X for the topological space (X, \mathcal{O}) .

Definition 10.1.2 (Discrete Topology). For any set X , the power set $\mathcal{P}X$ is called the *discrete* topology on X .

Proposition 10.1.3. *For any set X , the discrete topology on X is a topology on X .*

Definition 10.1.4 (Indiscrete Topology). For any set X , the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Proposition 10.1.5. *For any set X , the indiscrete topology on X is a topology on X .*

Definition 10.1.6 (Cofinite Topology). For any set X , the *cofinite* topology is $\{X - U : U \subseteq X \text{ is finite}\}$.

Definition 10.1.7 (Cocountable Topology). For any set X , the *cocountable* topology is $\{X - U : U \subseteq X \text{ is countable}\}$.

Definition 10.1.8 (Sierpiński Two-Point Space). The *Sierpiński two-point space* is $\{0, 1\}$ under the topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Definition 10.1.9 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 10.1.10. *A set B is open if and only if $X - B$ is closed.*

Proposition 10.1.11. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that \mathcal{C} is the set of closed sets if and only if:*

- For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$
- For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.
- $\emptyset \in \mathcal{C}$

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Theorem 10.1.12. *There are infinitely many primes.*

Furstenberg's proof:

PROOF:

$\langle 1 \rangle 1$. For $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$,

LET: $S(a, b) := \{an + b : n \in \mathbb{N}\}$

$\langle 1 \rangle 2$. LET: \mathcal{T} be the topology generated by the basis $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$

$\langle 2 \rangle 1$. For every $n \in \mathbb{Z}$, there exist a, b such that $n \in S(a, b)$.

PROOF: $n \in S(n, 0)$

$\langle 2 \rangle 2$. If $n \in S(a_1, b_1) \cap S(a_2, b_2)$ then there exist a_3, b_3 such that $n \in S(a_3, b_3) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 1$. LET: $d = \text{lcm}(a_1, a_2)$

PROVE: $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 2$. LET: $d = a_1k = a_2l$

$\langle 3 \rangle 3$. LET: $n = a_1c + b_1 = a_2d + b_2$

$\langle 3 \rangle 4$. LET: $z \in \mathbb{Z}$

PROVE: $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 5$. $dz + n \in S(a_1, b_1)$

PROOF:

$$\begin{aligned} dz + n &= a_1kz + a_1c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

$\langle 3 \rangle 6$. $dz + n \in S(a_2, b_2)$

PROOF: Similar.

$\langle 1 \rangle 3$. For all $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$ we have $S(a, b)$ is closed.

$\langle 2 \rangle 1$. LET: $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$

$\langle 2 \rangle 2$. LET: $n \in \mathbb{Z} - S(a, b)$

$\langle 2 \rangle 3$. $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$

$\langle 3 \rangle 1$. LET: $x \in S(a, n)$

$\langle 3 \rangle 2$. ASSUME: for a contradiction $x \in S(a, b)$

$\langle 3 \rangle 3$. PICK m such that $x = am + b$

$\langle 3 \rangle 4$. PICK l such that $x = al + n$

$\langle 3 \rangle 5$. $n = a(m - l) + b$

⟨3⟩6. $n \in S(a, b)$

⟨3⟩7. Q.E.D.

PROOF: This contradicts ⟨2⟩2.

⟨1⟩4.

$$\mathbb{Z} - \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$$

PROOF: Since every integer except 1 and -1 is divisible by a prime.

⟨1⟩5. No nonempty finite set is open.

⟨2⟩1. LET: U be a nonempty open set

⟨2⟩2. PICK $n \in U$

⟨2⟩3. There exist a, b such that $n \in S(a, b) \subseteq U$

⟨2⟩4. U is infinite.

⟨1⟩6. $\mathbb{Z} - \{1, -1\}$ is not closed.

⟨1⟩7. $\bigcup_{p \text{ prime}} S(p, 0)$ is not closed.

⟨1⟩8. The union of finitely many closed sets is closed.

⟨1⟩9. There are infinitely many primes.

□

Definition 10.1.13 (Neighbourhood). Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x , and x is an *interior* point of U , iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 10.1.14. A set B is open if and only if it is a neighbourhood of each of its points.

Proposition 10.1.15. Let X be a set and $\mathcal{N} : X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that, for all $x \in X$, we have \mathcal{N}_x is the set of neighbourhoods of x , if and only if:

- For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$
- For all $x \in X$ we have $X \in \mathcal{N}_x$
- For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$
- For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$
- For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M. M \in \mathcal{N}_y$.

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$.

Definition 10.1.16 (Exterior Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x .

Definition 10.1.17 (Boundary Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B .

Definition 10.1.18 (Interior). Let X be a topological space and $B \subseteq X$. The *interior* of B , B° , is the set of all interior points of B .

Proposition 10.1.19. *The interior of B is the union of all the open sets included in B .*

Definition 10.1.20 (Closure). Let X be a topological space and $B \subseteq X$. The closure of B , \overline{B} , is the set of all points that are not exterior points of B .

Proposition 10.1.21. *The closure of B is the intersection of all the closed sets that include B .*

Proposition 10.1.22. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 10.1.23 (Kuratowski Closure Axioms). *Let X be a set and $- : \mathcal{P}X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} such that, for all $B \subseteq X$, \overline{B} is the closure of B , if and only if:*

- $\overline{\emptyset} = \emptyset$
- For all $A \subseteq X$ we have $A \subseteq \overline{A}$
- For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$
- For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$

In this case, \mathcal{O} is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

Definition 10.1.24 (Finer, Coarser). Let \mathcal{T} and \mathcal{T}' be topologies on the set X . Then \mathcal{T} is *coarser*, *smaller* or *weaker* than \mathcal{T}' , or \mathcal{T}' is *finer*, *larger* or *weaker* than \mathcal{T} , iff $\mathcal{T} \subseteq \mathcal{T}'$.

10.1.1 Subspaces

Definition 10.1.25 (Subspace). Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

Example 10.1.26. The *unit sphere* S^2 is $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a subspace of \mathbb{R}^3 .

Theorem 10.1.27. *Let X be a topological space and (Y, i) a subset of X . Then the subspace topology on Y is the unique topology such that, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f : Z \rightarrow X$ is continuous.*

PROOF:

⟨1⟩1. If we give Y the subspace topology then, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.

⟨2⟩1. Given Y the subspace topology.

⟨2⟩2. LET: Z be a topological space.

⟨2⟩3. LET: $f : Z \rightarrow Y$

⟨2⟩4. If f is continuous then $i \circ f$ is continuous.

PROOF: Since i is continuous.

- ⟨2⟩5. If $i \circ f$ is continuous then f is continuous.
 ⟨3⟩1. ASSUME: $i \circ f$ is continuous.
 ⟨3⟩2. LET: U be open in Y .
 ⟨3⟩3. $f^{-1}(i^{-1}(i(U)))$ is open in Z .
 ⟨3⟩4. $f^{-1}(U)$ is open in Z .
 ⟨1⟩2. If, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.
 ⟨2⟩1. ASSUME: For every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.
 ⟨2⟩2. i is continuous.
 ⟨2⟩3. For every open set U in X , we have $i^{-1}(X)$ is open in Y .
 ⟨2⟩4. LET: Z be the set Y under the subspace topology and $f : Z \rightarrow Y$ the identity function.
 ⟨2⟩5. $i \circ f$ is continuous.
 ⟨2⟩6. f is continuous.
 ⟨2⟩7. Every set open in Y is open in Z .
 □

10.1.2 Topological Disjoint Union

Definition 10.1.28 (Coproduct Topology). Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The *coproduct topology* on $\coprod_{\alpha \in A} X_\alpha$ is

$$\mathcal{T} = \left\{ \coprod_{\alpha \in A} U_\alpha : \{U_\alpha\}_{\alpha \in A} \text{ is a family with } U_\alpha \text{ open in } X_\alpha \text{ for all } \alpha \right\}.$$

We prove this is a topology.

PROOF:

- ⟨1⟩1. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF:

$$\bigcup_{i \in I} \coprod_{\alpha \in A} U_{i\alpha} = \coprod_{\alpha \in A} \bigcup_{i \in I} U_{i\alpha}$$

- ⟨1⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF:

$$\coprod_{\alpha \in A} U_\alpha \cap \coprod_{\alpha \in A} V_\alpha = \coprod_{\alpha \in A} (U_\alpha \cap V_\alpha)$$

- ⟨1⟩3. $\coprod_{\alpha \in A} X_\alpha \in \mathcal{T}$

PROOF: Since every X_α is open in X_α .

□

Proposition 10.1.29. The coproduct topology is the finest topology on $\coprod_{\alpha \in A} X_\alpha$ such that every injection $\kappa_\alpha : X_\alpha \rightarrow \coprod_{\alpha \in A} X_\alpha$ is continuous.

PROOF:

- ⟨1⟩1. LET: $P = \coprod_{\alpha \in A} X_\alpha$

- ⟨1⟩2. LET: \mathcal{T}_c be the coproduct topology.

- ⟨1⟩3. LET: \mathcal{T} be any topology on P
- ⟨1⟩4. For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T}_c)$ is continuous.
 - ⟨2⟩1. LET: $\alpha \in A$
 - ⟨2⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with each U_α open in X_α .
 - ⟨2⟩3. For all $\alpha \in A$, we have $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha)$ is open in X_α .

PROOF: Since $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha) = U_\alpha$.
- ⟨1⟩5. If, for all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$ is continuous, then $\mathcal{T} \subseteq \mathcal{T}_c$.
 - ⟨2⟩1. ASSUME: For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$ is continuous.
 - ⟨2⟩2. LET: $U \in \mathcal{T}$
 - ⟨2⟩3. For all $\alpha \in a$, we have $\kappa_\alpha^{-1}(U)$ is open in X_α .
 - ⟨2⟩4. $U = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(U) \in \mathcal{T}_c$

□

Theorem 10.1.30. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The coproduct topology is the unique topology on $\coprod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : \coprod_{\alpha \in A} X_\alpha \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.*

PROOF:

- ⟨1⟩1. LET: $X = \coprod_{\alpha \in A} X_\alpha$
- ⟨1⟩2. LET: \mathcal{T}_c be the coproduct topology.
- ⟨1⟩3. For every topological space Z and function $f : (X, \mathcal{T}_c) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.
 - ⟨2⟩1. LET: Z be a topological space.
 - ⟨2⟩2. LET: $f : X \rightarrow Z$
 - ⟨2⟩3. If f is continuous then $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.

PROOF: Because the composite of two continuous functions is continuous.
 - ⟨2⟩4. If $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous then f is continuous.
 - ⟨3⟩1. ASSUME: $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.
 - ⟨3⟩2. LET: U be open in Z
 - ⟨3⟩3. For all $\alpha \in A$ we have $\kappa_\alpha^{-1}(f^{-1}(U))$ is open in X_α
 - ⟨3⟩4. $f^{-1}(U) = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(f^{-1}(U))$
 - ⟨3⟩5. $f^{-1}(U)$ is open in X
- ⟨1⟩4. For any topology \mathcal{T} on X , if for every topological space Z and function $f : (X, \mathcal{T}) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous, then $\mathcal{T} = \mathcal{T}_c$.
 - ⟨2⟩1. LET: \mathcal{T} be a topology on X .
 - ⟨2⟩2. ASSUME: For every topological space Z and function $f : (X, \mathcal{T}) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.
 - ⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_c$
 - ⟨3⟩1. For all $\alpha \in A$ we have $\kappa_\alpha : X_\alpha \rightarrow (X, \mathcal{T})$ is continuous.

PROOF: From ⟨2⟩1 since id_X is continuous.
 - ⟨3⟩2. $\mathcal{T} \subseteq \mathcal{T}_c$

PROOF: Proposition 10.1.29.
 - ⟨2⟩4. $\mathcal{T}_c \subseteq \mathcal{T}$

- ⟨3⟩1. LET: $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}_c)$ be the identity function.
- ⟨3⟩2. $f \circ \kappa_\alpha$ is continuous for all α .
- ⟨3⟩3. f is continuous.
- PROOF: ⟨2⟩1
- ⟨3⟩4. $\mathcal{T}_c \subseteq \mathcal{T}$

□

10.1.3 Product Topology

Definition 10.1.31 (Product Topology). Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

Proposition 10.1.32. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The product topology on $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis $\mathcal{B} = \{\prod_{\alpha \in A} U_\alpha : \text{for all } \alpha \in A, U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in A\}$.

PROOF:

- ⟨1⟩1. \mathcal{B} is a basis for a topology.
- ⟨1⟩2. LET: \mathcal{T} be the topology generated by \mathcal{B} .
- ⟨1⟩3. LET: \mathcal{T}_p be the product topology.
- ⟨1⟩4. $\mathcal{T} \subseteq \mathcal{T}_p$
 - ⟨2⟩1. LET: $B \in \mathcal{B}$
 - ⟨2⟩2. LET: $B = \prod_{\alpha \in A} U_\alpha$ with each U_α open in X_α and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$
 - ⟨2⟩3. $B = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$
 - ⟨2⟩4. $B \in \mathcal{T}_p$
- ⟨1⟩5. $\mathcal{T}_p \subseteq \mathcal{T}$
 - ⟨2⟩1. For every $\alpha \in A$ we have π_α is continuous.

PROOF: Since $\pi^{-1}(U)$ is open for every U open in X_α .

□

Theorem 10.1.33. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Then the product topology on $\prod_{\alpha \in A} X_\alpha$ is the unique topology such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f : Z \rightarrow X_\alpha$ is continuous.

PROOF:

- ⟨1⟩1. If we give $\prod_{\alpha \in A} X_\alpha$ the product topology, then for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
- ⟨2⟩1. Give $\prod_{\alpha \in A} X_\alpha$ the product topology.
- ⟨2⟩2. LET: Z be a topological space.
- ⟨2⟩3. LET: $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$
- ⟨2⟩4. If f is continuous then, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
- PROOF: Since the composite of two continuous functions is continuous.
- ⟨2⟩5. If, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then f is continuous.

- ⟨3⟩1. ASSUME: For all $\alpha \in A$ we have $\pi_\alpha \circ f$ is continuous.
 ⟨3⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with U_α open in X_α such that $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \dots, \alpha_n$.
 ⟨3⟩3. For all α we have $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ is open in Z .
 ⟨3⟩4. $f^{-1}(\prod_\alpha U_\alpha)$ is open in Z .
 PROOF: Since $f^{-1}(\prod_\alpha U_\alpha) = f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \dots \cap f^{-1}(\pi_{\alpha_n}^{-1}(U_{\alpha_n}))$.
 ⟨1⟩2. If \mathcal{T} is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then \mathcal{T} is the product topology.
 ⟨2⟩1. ASSUME: \mathcal{T} is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
 ⟨2⟩2. LET: \mathcal{T}_p be the product topology.
 ⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_p$
 ⟨3⟩1. LET: $Z = (\prod_\alpha X_\alpha, \mathcal{T}_p)$
 ⟨3⟩2. LET: $f : Z \rightarrow \prod_\alpha X_\alpha$ be the identity function
 ⟨3⟩3. For all α we have $\pi_\alpha \circ f$ is continuous.
 ⟨3⟩4. f is continuous.
 PROOF: ⟨2⟩1
 ⟨3⟩5. Every set open in \mathcal{T} is open in \mathcal{T}_p
 ⟨2⟩4. $\mathcal{T}_p \subseteq \mathcal{T}$
 ⟨3⟩1. $\text{id}_{\prod_\alpha X_\alpha}$ is continuous.
 ⟨3⟩2. For all α we have π_α is continuous.
 PROOF: ⟨2⟩1
 ⟨3⟩3. $\mathcal{T}_p \subseteq \mathcal{T}$
 PROOF: Since \mathcal{T}_p is the coarsest topology such that every π_α is continuous.

□

Example 10.1.34. It is not true that, for any function $f : \prod_{\alpha \in A} X_\alpha \rightarrow Y$, if f is continuous in every variable separately then f is continuous.

Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

Then f is continuous in x and in y , but is not continuous.

Proposition 10.1.35. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology on $\prod_{i \in I} Y_i$ as a subspace of $\prod_{i \in I} X_i$.

PROOF:

- ⟨1⟩1. Given $\prod_{i \in I} Y_i$ the subspace topology.
 ⟨1⟩2. LET: $\iota : \prod_{i \in I} Y_i$ be the inclusion.
 ⟨1⟩3. LET: Z be any topological space.
 ⟨1⟩4. LET: $f : Z \rightarrow \prod_{i \in I} Y_i$

⟨1⟩5. f is continuous if and only if, for all $i \in I$, we have $\pi_i \circ f$ is continuous.

PROOF:

f is continuous $\Leftrightarrow \iota \circ f : Z \rightarrow \prod_{i \in I} X_i$ is continuous (Theorem 10.1.27)

$\Leftrightarrow \forall i \in I. \pi_i \circ \iota \circ f : Z \rightarrow X_i$ is continuous (Theorem 10.1.33)

$\Leftrightarrow \forall i \in I. \iota_i \circ \pi_i \circ f : Z \rightarrow X_i$ is continuous

$\Leftrightarrow \forall i \in I. \pi_i \circ f : Z \rightarrow Y_i$ is continuous (Theorem 10.1.27)

where ι_i is the inclusion $Y_i \rightarrow X_i$.

□

10.1.4 Bases

Definition 10.1.36 (Basis). Let X be a topological space. A *basis* for the topology on X is a set of open sets \mathcal{B} such that every open set is the union of a subset of \mathcal{B} . The elements of \mathcal{B} are called *basic open neighbourhoods* of their elements.

Definition 10.1.37 (Order Topology). Let X be a linearly ordered set. The *order topology* on X is the topology generated by the open interval (a, b) as well as the open rays $(a, +\infty)$ and $(-\infty, b)$ for $a, b \in X$.

Definition 10.1.38 (Lower Limit Topology). The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology generated by the basis consisting of all half-open intervals $[a, b)$.

Proposition 10.1.39. Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is a basis for a topology on X if and only if:

1. $\bigcup \mathcal{B} = X$
2. For all $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

In this case, the topology is unique and is the set of all unions of subsets of \mathcal{B} . We call it the topology generated by \mathcal{B} .

10.1.5 Subbases

Definition 10.1.40 (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of \mathcal{S} .

Definition 10.1.41 (Space with Basepoint). A *space with basepoint* is a pair (X, x) where X is a topological space and $x \in \text{El}(X)$.

10.1.6 Countability Axioms

Definition 10.1.42 (Neighbourhood Basis). Let X be a topological space and $x_0 \in \text{El}(X)$. A *neighbourhood basis* of x_0 is a set \mathcal{U} of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of \mathcal{U} .

Definition 10.1.43 (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

Definition 10.1.44 (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

\mathbb{R}^n is second countable.

An uncountable discrete space is first countable but not second countable.

Proposition 10.1.45. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.

PROOF:

$\langle 1 \rangle 1$. For all $\lambda : \text{El}(\Lambda)$, PICK U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.

$\langle 1 \rangle 2$. For all $\lambda : \text{El}(\Lambda)$, PICK $x_\lambda \in U_\lambda$.

$\langle 1 \rangle 3$. ASSUME: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda \in \Lambda}$.

$\langle 1 \rangle 4$. PICK $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$

$\langle 1 \rangle 5$. There is no $U \in B$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

□

10.2 Continuous Functions

Definition 10.2.1 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* iff, for every open set V in Y , the inverse image $f^{-1}(V)$ is open in X .

Proposition 10.2.2. 1. id_X is continuous

2. The composite of two continuous functions is continuous.

3. If $f : X \rightarrow Y$ is continuous and $X_0 \subseteq X$ then $f|_{X_0} : X_0 \rightarrow Y$ is continuous.

4. If $f : X + Y \rightarrow Z$, then f is continuous iff $f \circ \kappa_1 : X \rightarrow Z$ and $f \circ \kappa_2 : Y \rightarrow Z$ are continuous.

5. If $f : Z \rightarrow X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Proposition 10.2.3. Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{B} be a basis for Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF: Since every element of \mathcal{B} is open in Y .

$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X , then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

$\langle 2 \rangle 2$. LET: U be open in Y .

$\langle 2 \rangle 3$. LET: $x \in f^{-1}(U)$

$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq U$.

$\langle 2 \rangle 5$. $x \in f^{-1}(B) \subseteq f^{-1}(U)$

□

Definition 10.2.4 (Homeomorphism). Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Definition 10.2.5 (Retraction). Let X be a topological space and A a subspace of X . A continuous function $\rho : X \rightarrow A$ is a *retraction* iff $\rho|_A = \text{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 10.2.6. Let **Top** be the category of small topological spaces and continuous functions.

Proposition 10.2.7. \emptyset is initial in **Top**.

Proposition 10.2.8. 1 is terminal in **Top**.

Forgetful functor **Top** \rightarrow **Set**.

Basepoint preserving continuous functor.

Proposition 10.2.9. Let (X, \mathcal{T}) be a topological space. Let S be the Sierpiński two-point space. Define $\Phi : \mathcal{T} \rightarrow \mathbf{Top}[X, S]$ by $\Phi(U)(x) = 1$ iff $x \in U$. Then Φ is a bijection.

PROOF:

$\langle 1 \rangle 1$. For all $U \in \mathcal{T}$ we have $\Phi(U)$ is continuous.

$\langle 2 \rangle 1$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 2$. $\Phi(U)(\{1\})$ is open.

PROOF: Since $\Phi(U)(\{1\}) = U$.

$\langle 1 \rangle 2$. Φ is injective.

PROOF: If $\Phi(U) = \Phi(V)$ then we have $\forall x (x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$.

$\langle 1 \rangle 3$. Φ is surjective.

PROOF: Given $f : X \rightarrow S$ continuous we have $\Phi(f^{-1}(1)) = f$.

□

10.2.1 Paths

Definition 10.2.10 (Path). A *path* in a topological space X is a continuous function $[0, 1] \rightarrow X$.

10.2.2 Loops

Definition 10.2.11 (Loop). A *loop* in a topological space X is a path $\alpha : [0, 1] \rightarrow X$ such that $\alpha(0) = \alpha(1)$.

10.3 Convergence

Definition 10.3.1 (Convergence). Let X be a topological space. Let (x_n) be a sequence in X . A point $a \in \text{El}(X)$ is a *limit* of the sequence iff, for every neighbourhood U of a , there exists n_0 such that $\forall n \geq n_0, x_n \in U$.

Convergence in a product space is pointwise convergence.

If $f : X \rightarrow Y$ is continuous and $x_n \rightarrow l$ in X then $f(x_n) \rightarrow f(l)$ in Y .

Example 10.3.2. The converse does not hold.

Let X be the set of all continuous functions $[0, 1] \rightarrow [-1, 1]$ under the product topology. Let $i : X \rightarrow L^2([0, 1])$ be the inclusion.

If $f_n \rightarrow f$ then $i(f_n) \rightarrow i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K_\epsilon, \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0, 1]} U_\lambda$ where $U_\lambda = [-1, 1]$ except for $\lambda = \lambda_1, \dots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \dots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 10.3.3. *The converse does hold for first countable spaces. If $f : X \rightarrow Y$ where X is first countable, and Y is a topological space, and whenever $x_n \rightarrow x$ then $f(x_n) \rightarrow f(x)$, then f is continuous.*

10.4 Subspaces

Definition 10.4.1 (Subspace). Let X be a topological space, Y a set, and $f : Y \rightarrow X$. The *subspace topology* on Y induced by f is $\mathcal{T} = \{i^{-1}(U) : U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF: Since $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$.

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: Since $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.

$\langle 1 \rangle 3$. $Y \in \mathcal{T}$

PROOF: Since $Y = f^{-1}(X)$.

□

Proposition 10.4.2. *Let X be a topological space, Y a set and $f : Y \rightarrow X$ a function. Then the subspace topology on Y is the coarsest topology such that f is continuous.*

PROOF: Immediate from definition. □

10.5 Embedding

Definition 10.5.1 (Embedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *embedding* iff f is injective and the topology on X is the subspace induced by f .

10.6 Quotient Spaces

Definition 10.6.1 (Quotient Topology). Let X be a topological space, S a set, and $\pi : X \twoheadrightarrow S$ be a surjection. The *quotient topology* on S induced by π is $\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.

PROOF: Since $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}$.

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

PROOF: Since $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$.

$\langle 1 \rangle 3$. $X \in \mathcal{T}$

PROOF: Since $X = \pi^{-1}(Y)$.

□

Proposition 10.6.2. Let X be a topological space, S a set and $\pi : X \twoheadrightarrow S$ a surjection. Then the quotient topology on S is the finest topology such that π is continuous.

PROOF: Immediate from definitions. □

Definition 10.6.3 (Quotient Map). Let X and S be topological spaces and $\pi : X \rightarrow S$. Then π is a *quotient map* iff π is surjective and the topology on S is the quotient topology induced by π .

Theorem 10.6.4. Let X be a topological space, let S be a set, and let $\pi : X \twoheadrightarrow S$ be surjective. Then the quotient topology on S is the unique topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.

PROOF:

$\langle 1 \rangle 1$. If S is given the quotient topology, then for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.

$\langle 2 \rangle 1$. Give S the quotient topology.

$\langle 2 \rangle 2$. LET: Z be a topological space.

$\langle 2 \rangle 3$. LET: $f : S \rightarrow Z$

$\langle 2 \rangle 4$. If f is continuous then $f \circ \pi$ is continuous.

PROOF: The composite of two continuous functions is continuous.

$\langle 2 \rangle 5$. If $f \circ \pi$ is continuous then f is continuous.

- 1

Proposition 10.6.5. *Let Z be a topological space. Define $\pi : [0, 1] \rightarrow S^1$ by $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$. Given any continuous function $f : S^1 \rightarrow Z$, we have $f \circ \pi$ is a loop in Z . This defines a bijection between $\mathbf{Top}[S^1, Z]$ and the set of loops in Z .*

PROOF: Since π is a quotient map. \square

Definition 10.6.6 (Projective Space). The *projective space* \mathbb{RP}^n is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by \sim where $x \sim \lambda x$ for all $x \in \mathbb{R}^{n+1} - \{0\}$ and $\lambda \in \mathbb{R}$.

Definition 10.6.7 (Torus). The *torus* T is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$.

Definition 10.6.8 (Möbius Band). The *Möbius band* is the quotient of $[0, 1]^2$ by \sim where $(0, y) \sim (1, 1 - y)$.

Definition 10.6.9 (Klein Bottle). The *Klein bottle* is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$.

Proposition 10.6.10. \mathbb{RP}^2 is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (1 - x, 1)$ and $(0, y) \sim (1, 1 - y)$.

PROOF:TODO

Example 10.6.11. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and $\{Y_i\}_{i \in I}$ a family of sets. Let $q_i : X_i \rightarrow Y_i$ be a surjective function for all $i \in I$. Give each Y_i the quotient topology. It is not true in general that the product topology on $\prod_{i \in I} Y_i$ is the same as the quotient topology induced by $\prod_{i \in I} q_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$.

PROOF:

- ⟨1⟩1. LET: $X^* = \mathbb{R} - \mathbb{Z}_+ + \{b\}$ be the quotient space obtained from \mathbb{R} by identifying the subset \mathbb{Z}_+ to the point b .
 ⟨1⟩2. LET: $p : \mathbb{R} \rightarrow X^*$ be the quotient map.
 PROVE: $p \times \text{id}_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$ is not a quotient map.
 ⟨1⟩3. For $n \in \mathbb{Z}_+$,
 LET: $c_n = \sqrt{2}/n$
 ⟨1⟩4. For $n \in \mathbb{Z}_+$,
 LET: $U_n = \{(x, y) \in \mathbb{Q} \times \mathbb{R} : n - 1/4 < x < n + 1/4 \text{ and } ((y > x + c_n - n \text{ and } y > -x + c_n + n) \text{ or } (y < x + c_n - n \text{ and } y < -x + c_n + n))\}$
 ⟨1⟩5. For all $n \in \mathbb{Z}_+$, U_n is open in $\mathbb{R} \times \mathbb{Q}$
 ⟨1⟩6. For all $n \in \mathbb{Z}_+$ we have $\{n\} \times \mathbb{Q} \subseteq U_n$
 ⟨1⟩7. LET: $U = \bigcup_{n \in \mathbb{Z}_+} U_n$
 ⟨1⟩8. U is open in $\mathbb{R} \times \mathbb{Q}$.
 ⟨1⟩9. U is saturated with respect to $p \times \text{id}_{\mathbb{Q}}$.
 ⟨1⟩10. LET: $U' = (p \times \text{id}_{\mathbb{Q}})(U)$
 ⟨1⟩11. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$.

10.7 Connected Spaces

Definition 10.7.1 (Connected). A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 10.7.2. *The continuous image of a connected space is connected.*

Proposition 10.7.3. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are connected, then X is connected.*

Proposition 10.7.4. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 10.7.5 (Path-connected). A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \rightarrow X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 10.7.6. *The continuous image of a path connected space is path connected.*

Proposition 10.7.7. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are path connected, then X is path connected.*

Proposition 10.7.8. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

10.8 Hausdorff Spaces

Definition 10.8.1 (Hausdorff). A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 10.8.2. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 10.8.3. 1. *Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

Proposition 10.8.4. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \rightarrow B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X , then $f = g$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.

$\langle 1 \rangle 2$. PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.

$\langle 1 \rangle 3$. PICK $x \in f^{-1}(U) \cap g^{-1}(V)$

$\langle 1 \rangle 4$. $f(x) = g(x) \in U \cap V$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 10.8.5. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$ be uniformly continuous. Let \hat{X} and \hat{Y} be the completions of X and Y . Then f extends uniquely to a continuous map $\hat{X} \rightarrow \hat{Y}$.*

PROOF: The extension maps $\lim_{n \rightarrow \infty} x_n$ to $\lim_{n \rightarrow \infty} f(x_n)$. □

10.9 Separable Spaces

Definition 10.9.1 (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

10.10 Sequential Compactness

Definition 10.10.1 (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

10.11 Compactness

Definition 10.11.1 (Compact). A topological space is *compact* iff every open cover has a finite subcover.

Proposition 10.11.2. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P . Then $X \in P$.*

PROOF:

- $\langle 1 \rangle 1$. P is an open cover of X
 $\langle 1 \rangle 2$. PICK a finite subcover $U_1, \dots, U_n \in P$
 $\langle 1 \rangle 3$. $X = U_1 \cup \dots \cup U_n \in P$
 \square

Corollary 10.11.2.1. *Let f be a compact space and $f : X \rightarrow \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. \square

Proposition 10.11.3. *The continuous image of a compact space is compact.*

Proposition 10.11.4. *A closed subspace of a compact space is compact.*

Proposition 10.11.5. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. X and Y are compact.
2. $X + Y$ is compact.
3. $X \times Y$ is compact.

Proposition 10.11.6. *A compact subspace of a Hausdorff space is closed.*

Proposition 10.11.7. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 10.11.8. *A first countable compact space is sequentially compact.*

10.12 Quotient Spaces

Definition 10.12.1 (Quotient Space). Let X be a topological space and \sim an equivalence relation on X . The *quotient topology* on X/\sim is defined by: $U : \text{El}(\mathcal{P}X)$ is open in X/\sim if and only if $\pi^{-1}(U)$ is open in X .

Proposition 10.12.2. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $f : X/\sim \rightarrow Y$. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proposition 10.12.3. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $\phi : Y \rightarrow X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \rightarrow X$ such that $\pi \circ \Phi = \phi|U$. Then ϕ is continuous.

Proposition 10.12.4. *A quotient of a connected space is connected.*

Proposition 10.12.5. *A quotient of a path connected space is path connected.*

Proposition 10.12.6. *Let X be a topological space and \sim an equivalence relation on X . If X/\sim is Hausdorff then every equivalence class of \sim is closed in X .*

Definition 10.12.7. Let X be a topological space and $A_1, \dots, A_r \subseteq X$. Then $X/A_1, \dots, A_r$ is the quotient space of X with respect to \sim where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 10.12.8 (Cone). Let X be a topological space. The *cone over X* is the space $(X \times [0, 1])/(X \times \{1\})$.

Definition 10.12.9 (Suspension). Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 10.12.10 (Wedge Product). Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 10.12.11 (Smash Product). Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 10.12.12. $D^n/S^{n-1} \cong S^n$

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : D^n/S^{n-1} \rightarrow S^n$ be the function induced by the map $D^n \rightarrow S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.

$\langle 1 \rangle 2$. ϕ is a bijection.

$\langle 1 \rangle 3$. ϕ is a homeomorphism.

PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.

□

10.13 Gluing

Definition 10.13.1 (Gluing). Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \rightarrow Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where \sim is the equivalence relation generated by $x \sim \phi(x)$ for all $x : \text{El}(X)$.

Proposition 10.13.2. *Y is a subspace of $Y \cup_\phi X$.*

Definition 10.13.3. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0, 1])/\alpha$ is the quotient space of $X \times [0, 1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x : \text{El}(X)$.

Definition 10.13.4 (Möbius Strip). The *Möbius strip* is $([-1, 1] \times [0, 1])/\alpha$ where $\alpha(x) = -x$.

Definition 10.13.5 (Klein Bottle). The *Klein bottle* is $(S^1 \times [0, 1])/\alpha$ where $\alpha(z) = \bar{z}$.

Proposition 10.13.6. Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\text{id}_M} M \cong K$.

PROOF:

$\langle 1 \rangle 1$. LET: $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{\pi i \theta/2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-\pi i \theta/2}, t)$.

$\langle 1 \rangle 2$. f induces a bijection $M \cup_{\text{id}_M} M \approx K$

$\langle 1 \rangle 3$. f is a homeomorphism.

□

10.14 Metric Spaces

Definition 10.14.1 (Metric Space). Let X be a set and $d : X^2 \rightarrow \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geq 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d) . We often write X for the metric space (X, d) .

Definition 10.14.2 (Ball). Let X be a metric space. Let $x \in X$ and $r > 0$. The *ball* with *centre* x and *radius* r is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} .$$

Definition 10.14.3 (Metric Topology). Let (X, d) be a metric space. The *metric topology* on X is the topology generated by the basis consisting of the balls.

Definition 10.14.4 (Metrisable). A topological space is *metrisable* iff there exists a metric that induces its topology.

Proposition 10.14.5. Every metrisable space is Hausdorff.

Every metrisable space is first countable.

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

10.14.1 Products

Definition 10.14.6 (Euclidean Metric). Let X and Y be metric spaces. The *Euclidean metric* on $X \times Y$ is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}.$$

We write $X \times Y$ for the set $X \times Y$ under this metric.

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1.$ $d((x_1, y_1), (x_2, y_2)) \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2.$ $d((x_1, y_1), (x_2, y_2)) = 0$ iff $(x_1, y_1) = (x_2, y_2)$

PROOF: $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = 0$ iff $d(x_1, x_2) = d(y_1, y_2) = 0$ iff $x_1 = x_2$ and $y_1 = y_2$.

$\langle 1 \rangle 3.$ $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$

PROOF: Since $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = \sqrt{d(x_2, x_1)^2 + d(y_2, y_1)^2}$.

$\langle 1 \rangle 4.$ The triangle inequality holds.

PROOF:

$$\begin{aligned} & (d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)))^2 \\ &= d((x_1, y_1), (x_2, y_2))^2 + 2d((x_1, y_1), (x_2, y_2))d((x_2, y_2), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3))^2 \\ &= d(x_1, x_2)^2 + d(y_1, y_2)^2 + 2\sqrt{(d(x_1, x_2)^2 + d(y_1, y_2)^2)(d(x_2, x_3)^2 + d(y_2, y_3)^2)} + d(x_2, x_3)^2 + d(y_2, y_3)^2 \\ &\geq d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(y_1, y_2)^2 + d(y_2, y_3)^2 + 2(d(x_1, x_2)d(x_2, x_3) + d(y_1, y_2)d(y_2, y_3)) \\ &\quad \text{(Cauchy-Schwarz)} \\ &= (d(x_1, x_2) + d(x_2, x_3))^2 + (d(y_1, y_2) + d(y_2, y_3))^2 \\ &\geq d(x_1, x_3)^2 + d(y_1, y_3)^2 \\ &= d((x_1, y_1), (x_3, y_3))^2 \end{aligned}$$

□

Proposition 10.14.7. *Let X and Y be metric spaces. The Euclidean metric on $X \times Y$ induces the product topology on $X \times Y$.*

PROOF:

$\langle 1 \rangle 1.$ Every open ball is open in the product topology.

$\langle 2 \rangle 1.$ LET: $(x, y) \in B((a, b), \epsilon)$

PROVE: $B(x, \sqrt{\epsilon}) \times B(y, \sqrt{\epsilon}) \subseteq B((a, b), \epsilon)$

$\langle 2 \rangle 2.$ LET: $x' \in B(x, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$ and $y' \in B(y, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$

PROVE: $d((x', y'), (a, b)) < \epsilon$

$\langle 2 \rangle 3.$ $d((x', y'), (x, y)) < \epsilon - d((x, y), (a, b))$

PROOF:

$$\begin{aligned} d((x', y'), (x, y)) &= \sqrt{d(x', x)^2 + d(y', y)^2} \\ &< \sqrt{(\epsilon - d((x, y), (a, b)))^2/2 + (\epsilon - d((x, y), (a, b)))^2/2} \\ &= \epsilon - d((x, y), (a, b)) \end{aligned}$$

⟨2⟩4. $d((x', y'), (a, b)) < \epsilon$

PROOF:

$$d((x', y'), (a, b)) \leq d((x', y'), (x, y)) + d((x, y), (a, b)) \quad (\text{Triangle Inequality})$$

$$< \epsilon \quad (\langle 2 \rangle 3)$$

⟨1⟩2. If U is open in X and V is open in Y then $U \times V$ is open under the Euclidean metric.

⟨2⟩1. LET: $(x, y) \in U \times V$

⟨2⟩2. PICK $\delta, \epsilon > 0$ such that $B(x, \delta) \subseteq U$ and $B(y, \epsilon) \subseteq V$

PROVE: $(B((x, y), \min(\delta, \epsilon))) \subseteq U \times V$

⟨2⟩3. LET: $(x', y') \in B((x, y), \min(\delta, \epsilon))$

⟨2⟩4. $d(x', x) < \delta$

⟨3⟩1. $d((x', y'), (x, y)) < \min(\delta, \epsilon)$

⟨3⟩2. $d(x', x)^2 + d(y', y)^2 < \delta^2$

⟨3⟩3. $d(x', x)^2 < \delta^2$

⟨2⟩5. $d(y', y) < \epsilon$

PROOF: Similar.

⟨2⟩6. $(x', y') \in U \times V$

□

10.15 Complete Metric Spaces

Definition 10.15.1 (Complete). A metric space is *complete* iff every Cauchy sequence converges.

Example 10.15.2. \mathbb{R} is complete.

Proposition 10.15.3. *The product of two complete metric spaces is complete.*

Proposition 10.15.4. *Every compact metric space is complete.*

Proposition 10.15.5. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 10.15.6 (Completion). Let X be a metric space. A *completion* of X is a complete metric space \hat{X} and injection $i : X \rightarrow \hat{X}$ such that:

- The metric on X is the restriction of the metric on \hat{X}
- X is dense in \hat{X} .

Proposition 10.15.7. *Let $i_1 : X \rightarrow Y_1$ and $i_2 : X \rightarrow Y_2$ be completions of X . Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

PROOF: Define $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$. □

Theorem 10.15.8. *Every metric space has a completion.*

PROOF: Let \hat{X} be the set of Cauchy sequences in X quotiented by \sim where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \rightarrow 0$. □

10.16 Manifolds

Definition 10.16.1 (Manifold). An n -dimensional manifold is a second countable Hausdorff space locally homeomorphic to \mathbb{R}^n .

Chapter 11

Homotopy Theory

11.1 Homotopies

Definition 11.1.1 (Homotopy). Let X and Y be topological spaces. Let $f, g : X \rightarrow Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0, 1] \rightarrow Y$ such that

- $\forall x : \text{El}(X) . h(x, 0) = f(x)$
- $\forall x : \text{El}(X) . h(x, 1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \rightarrow Y$.

Proposition 11.1.2. Let $f, f' : X \rightarrow Y$ and $g, g' : Y \rightarrow Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.

Definition 11.1.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

A *homotopy functor* is a functor $\mathbf{Top} \rightarrow \mathcal{C}$ that factors through the canonical functor $\mathbf{Top} \rightarrow \mathbf{HTop}$.

Definition 11.1.4. A functor $F : \mathbf{Top} \rightarrow \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \rightarrow Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

11.2 Homotopy Equivalence

Definition 11.2.1 (Homotopy Equivalence). Let X and Y be topological spaces. A *homotopy equivalence* between X and Y , $f : X \simeq Y$, is a continuous function $f : X \rightarrow Y$ such that there exists a continuous function $g : Y \rightarrow X$, the *homotopy inverse* to f , such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.

Definition 11.2.2 (Contractible). A topological space X is *contractible* iff $X \simeq 1$.

Example 11.2.3. \mathbb{R}^n is contractible.

Example 11.2.4. D^n is contractible.

Definition 11.2.5 (Deformation Retract). Let X be a topological space and A a subspace of X . A retraction $\rho : X \rightarrow A$ is a *deformation retraction* iff $i \circ \rho \simeq \text{id}_X$, where i is the inclusion $A \hookrightarrow X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 11.2.6 (Strong Deformation Retract). Let X be a topological space and A a subspace of X . A *strong deformation retraction* $\rho : X \rightarrow A$ is a continuous function such that there exists a homotopy $h : X \times [0, 1] \rightarrow X$ between $i \circ \rho$ and id_X such that, for all $a : \text{El}(X)$ and $t : \text{El}([0, 1])$, we have $h(a, t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 11.2.7. $\{0\}$ is a strong deformation retract of \mathbb{R}^n and of D^n .

Example 11.2.8. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 11.2.9. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 11.2.10. For any topological space X , the singleton consisting of the vertex is a strong deformation retract of the cone over X .

Chapter 12

Simplicial Complexes

Definition 12.0.1 (Simplex). A k -dimensional simplex or k -simplex in \mathbb{R}^n is the convex hull $s(x_0, \dots, x_k)$ of $k + 1$ points in general position.

Definition 12.0.2 (Face). A *sub-simplex* or *face* of $s(x_0, \dots, x_k)$ is the convex hull of a subset of $\{x_0, \dots, x_k\}$.

Definition 12.0.3 (Simplicial Complex). A *simplicial complex* in \mathbb{R}^n is a set K of simplices such that:

- for every simplex s in K , every face of s is in K .
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2 .
- K is locally finite, i.e. every point of \mathbb{R}^n has a neighbourhood that only intersects finitely many elements of K .

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of \mathbb{R}^n .

12.1 Cell Decompositions

Definition 12.1.1 (n -cell). An n -cell is a topological space homeomorphic to \mathbb{R}^n .

Definition 12.1.2 (Cell Decomposition). Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n -cells.

Definition 12.1.3 (n -skeleton). Given a cell decomposition of X , the n -skeleton X^n is the union of all the cells of dimension $\leq n$.

12.2 CW-complexes

Definition 12.2.1 (CW-Complex). A *CW-complex* consists of a topological space X and a cell decomposition \mathcal{E} of X such that:

1. *Characteristic Maps* For every n -cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \rightarrow X$ such that $\Phi_e((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in \mathcal{E} of dimension $< n$.
2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have \bar{e} intersects only finitely many other cells in \mathcal{E} .
3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \bar{e}$ is closed.

Proposition 12.2.2. *If a cell decomposition \mathcal{E} satisfies the Characteristic Maps axiom, then for every n -cell $e \in \mathcal{E}$ we have $\bar{e} = \Phi_e(D^n)$. Therefore \bar{e} is compact and $\bar{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

PROOF:

$\langle 1 \rangle 1.$ $e \subseteq \Phi_e(D^n) \subseteq \bar{e}$

PROOF:

$$\begin{aligned}
 e &= \Phi_e((D^n)^\circ) \\
 &\subseteq \Phi_e(D^n) \\
 &= \Phi_e(\overline{(D^n)^\circ}) \\
 &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
 &= \bar{e}
 \end{aligned}$$

$\langle 1 \rangle 2.$ $\Phi_e(D^n)$ is compact.

PROOF: Because D^n is compact.

$\langle 1 \rangle 3.$ $\Phi_e(D^n)$ is closed.

$\langle 1 \rangle 4.$ $\Phi_e(D^n) = \bar{e}$

□

Chapter 13

Topological Groups

Definition 13.0.1 (Topological Group). A *topological group* is a group G with a topology such that the function $G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.

Example 13.0.2. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 13.0.3. Any subgroup of a topological group is a topological group under the subspace topology.

Definition 13.0.4 (Homogeneous Space). A *homogeneous space* is a topological space of the form G/H , where G is a topological group and H is a normal subgroup of G , under the quotient topology.

Proposition 13.0.5. Let G be a topological group and H a normal subgroup of G . Then G/H is Hausdorff if and only if H is closed.

PROOF: See Bourbaki, N., General Topology. III.12 \square

13.1 Continuous Actions

Definition 13.1.1 (Continuous Action). Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \rightarrow X$ such that:

- $\forall x : \text{El}(X) . ex = x$
- $\forall g, h : \text{El}(G) . \forall x : \text{El}(X) . g(hx) = (gh)x$

A G -space consists of a topological space X and a continuous action of G on X .

Definition 13.1.2 (Orbit). Let X be a G -space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 13.1.3. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:

$\langle 1 \rangle 1$. LET: $f_3 : S^2/SO(2) \rightarrow [-1, 1]$ be the function induced by $\pi_3 : S^2 \rightarrow [-1, 1]$

$\langle 1 \rangle 2$. f_3 is bijective.

$\langle 1 \rangle 3$. $S^2/SO(2)$ is compact.

PROOF: It is the continuous image of S^2 which is compact.

$\langle 1 \rangle 4$. $[-1, 1]$ is Hausdorff.

$\langle 1 \rangle 5$. f_3 is a homeomorphism.

□

Definition 13.1.4 (Stabilizer). Let X be a G -space and $x \in X$. The *stabilizer* of x is $G_x := \{g \in G \mid gx = x\}$.

Proposition 13.1.5. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx .*

PROOF:

$\langle 1 \rangle 1$. If $gG_x = hG_x$ then $gx = hx$.

$\langle 2 \rangle 1$. ASSUME: $gG_x = hG_x$

$\langle 2 \rangle 2$. $g^{-1}h \in G_x$

$\langle 2 \rangle 3$. $g^{-1}hx = x$

$\langle 2 \rangle 4$. $gx = hx$

$\langle 1 \rangle 2$. If $gx = hx$ then $gG_x = hG_x$.

PROOF: Similar.

$\langle 1 \rangle 3$. The function is continuous.

PROOF: Proposition 10.12.2.

□

Chapter 14

Topological Vector Spaces

Definition 14.0.1 (Topological Vector Space). Let K be either \mathbb{R} or \mathbb{C} . A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Subtraction is a continuous function $E^2 \rightarrow E$
- Multiplication is a continuous function $K \times E \rightarrow E$

Proposition 14.0.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. \square

Theorem 14.0.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 \square

Proposition 14.0.4. *Let E be a topological vector space and E_0 a subspace of E . Then $\overline{E_0}$ is a subspace of E .*

Definition 14.0.5. Let E be a topological vector space. The topological space associated with E is $E/\overline{\{0\}}$.

14.1 Cauchy Sequences

Definition 14.1.1 (Cauchy Sequence). Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geq n_0, x_n - x_m \in U$.

Definition 14.1.2 (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

14.2 Seminorms

Definition 14.2.1 (Seminorm). Let E be a vector space over K . A *seminorm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ such that:

1. $\forall x : \text{El}(E) . \|x\| \geq 0$
2. $\forall \alpha : \text{El}(K) . \forall x : \text{El}(E) . \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y : \text{El}(E) . \|x + y\| \leq \|x\| + \|y\|$

Example 14.2.2. The function that maps (x_1, \dots, x_n) to $|x_i|$ is a seminorm on \mathbb{R}^n .

Definition 14.2.3. Let E be a vector space over K . Let Λ be a set of seminorms on E . The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x : \text{El}(E)$.

Proposition 14.2.4. E is a topological vector space under this topology. It is Hausdorff iff, for all $x : \text{El}(E)$, if $\forall \lambda \in \Lambda . \lambda(x) = 0$ then $x = 0$.

14.3 Fréchet Spaces

Definition 14.3.1 (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 14.3.2. Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$. Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 14.3.3 (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

14.4 Normed Spaces

Definition 14.4.1 (Normed Space). Let E be a vector space over K . A *norm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ is a seminorm such that, $\forall x \in E . \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 14.4.2. If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E .

Definition 14.4.3 (p -norm). For any $p \geq 1$, the p -norm on \mathbb{R}^n is defined by

$$\|\vec{x}\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1$. For all $\vec{x} \in \mathbb{R}^n$ we have $\|\vec{x}\|_p \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. For all $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ we have $\|\alpha \vec{x}\|_p = |\alpha| \|\vec{x}\|_p$

PROOF:

$$\begin{aligned} \|\alpha(x_1, \dots, x_n)\| &= \|(\alpha x_1, \dots, \alpha x_n)\| \\ &= \left(\sum_{i=1}^n (\alpha x_i)^p \right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|\vec{x}\|_p \end{aligned}$$

$\langle 1 \rangle 3$. The triangle inequality holds.

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\ &= \sum_{i=1}^n |x_i + y_i| |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^n (|x_i| + |y_i|) |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} \\ &\leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} \quad (\text{Hölder's Inequality}) \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p) \|\vec{x} + \vec{y}\|_p^{p-1} \end{aligned}$$

Assuming w.l.o.g. $\|\vec{x} + \vec{y}\|_p^{p-1} \neq 0$ (using ??) we have $\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$.

$\langle 1 \rangle 4$. For any $\vec{x} \in \mathbb{R}^n$, we have $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.

PROOF: $\sum_{i=1}^n x_i^p = 0$ iff $x_1 = \dots = x_n = 0$.

□

Definition 14.4.4 (Sup-norm). The *sup-norm* on \mathbb{R}^n is defined by

$$\|(x_1, \dots, x_n)\|_\infty := \max(|x_1|, \dots, |x_n|) .$$

Proposition 14.4.5. *The 2-norm on \mathbb{R}^n induces the standard metric.*

PROOF: Immediate from definitions. \square

Definition 14.4.6. For $p \geq 1$, the normed space l_p is the set of all sequences (x_n) in \mathbb{R} such that $\sum_{n=1}^\infty x_n^p$ converges, under

$$\|(x_n)\|_p := \left(\sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} .$$

Proposition 14.4.7. *The spaces l_p for $p \geq 1$ are all homeomorphic.*

PROOF: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. <http://dx.doi.org/10.1007/BF01075865>.

Definition 14.4.8. Let l_∞ be the set of all bounded sequences in \mathbb{R} under

$$\|(x_n)\| := \sup_n |x_n|$$

Proposition 14.4.9. *For all $p \geq 1$ we have l_p is not homeomorphic to l_∞ .*

Proposition 14.4.10. *Let $\| \cdot \|$ be a seminorm on the vector space E . Then $\| \cdot \|$ defines a norm on $E/\{0\}$.*

Proposition 14.4.11. *Let E and F be normed spaces. Any continuous linear map $E \rightarrow F$ is uniformly continuous.*

Definition 14.4.12. For $p \geq 1$. let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\{0\} .$$

14.5 Inner Product Spaces

Proposition 14.5.1. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E .*

14.6 Banach Spaces

Definition 14.6.1 (Banach Space). A *Banach space* is a complete normed space.

Example 14.6.2. For any topological space X , the set $C(X)$ of bounded continuous functions $X \rightarrow \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 14.6.3. *The completion of a normed space is a Banach space.*

Proposition 14.6.4. *Let E and F be normed spaces. Let $f : E \rightarrow F$ be a continuous linear map. Then the extension to the completions $\hat{E} \rightarrow \hat{F}$ is linear.*

Proposition 14.6.5. $L^p(\mathbb{R}^n)$ is a Banach space.

Proposition 14.6.6. $C(\mathbb{R})$ is first countable but not second countable.

PROOF: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n . Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.

It is first countable because it is metrizable. \square

14.7 Hilbert Spaces

Definition 14.7.1 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Example 14.7.2. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \rightarrow \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

Proposition 14.7.3. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

14.8 Locally Convex Spaces

Definition 14.8.1 (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 14.8.2. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Proposition 14.8.3. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Example 14.8.4. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \rightarrow \mathbb{R}$ is continuous as a map $E' \rightarrow \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 14.8.5 (Thom Space). Let E be a vector bundle with a Riemannian metric, $DE = \{x : \text{El}(E) \mid \|x\| \leq 1\}$ its disc bundle and $SE := \{v : \text{El}(E) \mid \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE .