Encyclopaedia of Mathematics and Physics

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Set Theory

Proposition 1.1. Every infinite subset of a countably infinite set is countable.

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Proof:
\langle 1 \rangle 1. Let: i: A \hookrightarrow \mathbb{N} be an infinite subset of \mathbb{N}.
\langle 1 \rangle 2. Define j : \mathbb{N} \to A by: j(k) is the element such that i(j(k)) is least such
        that i(j(k)) \notin \{i(j(0)), \dots, i(j(k-1))\}.
\langle 1 \rangle 3. j is a bijection.
Proposition 1.2. A countable union of countable sets is countable.
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Proof:

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\langle 1 \rangle 1. Let: (A_n) be a sequence of countable sets.
\langle 1 \rangle 2. For n \in \mathbb{N}, PICK an enumeration (e_{nm})_m of A_n.
\langle 1 \rangle 3. Let: (p_k) be the following enumeration of \mathbb{N} \times \mathbb{N}:
```

 $(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots$ $\langle 1 \rangle 4$. $(e_{\pi_1(p_k)\pi_2(p_k)})_k$ is an enumeration of $\bigcup_n A_n$.

Theorem 1.3. $2^{\mathbb{N}}$ is uncountable.

Proof:

- $\langle 1 \rangle 1$. Assume: for a contradiction $f : \mathbb{N} \approx 2^{\mathbb{N}}$
- $\langle 1 \rangle 2$. Let: $S = \{ n \in \mathbb{N} : n \notin f(n) \}$
- $\langle 1 \rangle 3$. For all n, we have $n \in S \Leftrightarrow n \notin f(n)$
- $\langle 1 \rangle 4$. For all n we have $S \neq f(n)$.
- $\langle 1 \rangle$ 5. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 1$.

Relations

Definition 2.1 (Antisymmetric). A relation R on a set A is antisymmetric iff, whenever xRy and yRx, then x = y.

Definition 2.2 (Transitive). A relation R on a type A is *transitive* iff, whenever xRy and yRz, then xRz.

Order Theory

Definition 3.1 (Linear Order). A *linear order* on a set A is a binary relation \leq on A that is transitive, antisymmetric and:

$$\forall x, y \in A.x \le y \lor y \le x$$
.

A linearly ordered set is a pair (A, \leq) where A is a set and \leq is a binary relation on A.

We write x < y for $x \le y$ and $x \ne y$.

Definition 3.2 (Upper Bound). Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is an *upper bound* in E iff $\forall x \in E.x \leq u$. We say E is *bounded above* iff it has an upper bound.

The *up-set* of E, denoted $E \uparrow$, is the set of upper bounds of E.

Definition 3.3 (Lower Bound). Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then u is an *lower bound* in E iff $\forall x \in E.l \leq x$. We say E is *bounded below* iff it has a lower bound.

The down-set of E, denoted $E \downarrow$, is the set of lower bounds of E.

Definition 3.4 (Supremum). Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is the *least upper bound* or *supremum* of E iff u is an upper bound for E and, for any upper bound u' for E, we have $u \le u'$.

Definition 3.5 (Infimum). Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then l is the *greatest lower bound* or *infimum* of E iff l is a lower bound for E and, for any lower bound l' for E, we have $l' \leq l$.

Definition 3.6 (Least Upper Bound Property). A linearly ordered set S has the *least upper bound property* iff every nonempty subset of S that is bounded above has a least upper bound.

Proposition 3.7. Let S be a linearly ordered set and $E \subseteq S$.

1. If $E \downarrow has$ a supremum l, then l is the infimum of E.

2. If $E \uparrow has$ an infimum u, then U is the supremum of E.

PROOF

- $\langle 1 \rangle 1$. If $E \downarrow$ has a supremum l, then l is the infimum of E.
 - $\langle 2 \rangle 1$. l is a lower bound for E.
 - $\langle 3 \rangle 1$. Let: $x \in E$
 - $\langle 3 \rangle 2$. x is an upper bound for $E \downarrow$.

PROOF: For all $y \in E \downarrow$ we have $y \leq x$.

- $\langle 3 \rangle 3. \ l \leq x$
- $\langle 2 \rangle 2$. For any lower bound l' for E, we have $l' \leq l$.

PROOF: Since l is an upper bound for $E \downarrow$.

 $\langle 1 \rangle$ 2. If $E \uparrow$ has an infimum u, then u is the supremum of E. PROOF: Dual.

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Corollary 3.7.1. A linearly ordered set has the least upper bound property if and only if every nonempty set bounded below has an infimum.

Definition 3.8 (Closed Downwards). Let S be a linearly ordered set and $E \subseteq S$. Then E is closed downwards iff, whenever $x \in E$ and y < x, then $y \in E$.

Definition 3.9 (Closed Upwards). Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed upwards* iff, whenever $x \in E$ and x < y, then $y \in E$.

Definition 3.10 (Greatest). Let S be a linearly ordered set and $u \in S$. Then u is greatest in S iff $\forall x \in S.x \leq u$.

Definition 3.11 (Least). Let S be a linearly ordered set and $l \in S$. Then l is least in S iff $\forall x \in S.l \leq x$.

Proposition 3.12. Let \leq be a linear order on a set S and $E \subseteq S$. Then $\leq \cap E^2$ is a linear order on E.

Proof: Easy. \sqcup

Given a linearly ordered set (S, \leq) and $E \subseteq S$, we write just E for the linearly ordered set $(E, \leq \cap E^2)$.

Definition 3.13 (Lexicographic Order). Let A and B be linearly ordered sets. The *lexicographic order* or *dictionary order* on $A \times B$ is the order defined by

$$(a,b) \le (a',b') \Leftrightarrow a = a' \lor (a < a' \land b \le b')$$
.

Proposition 3.14. The lexicographic order is a linear order.

Field Theory

Definition 4.1 (Field). A *field* F consists of a set F, two operations $+, \cdot : F^2 \to F$ and an element $0 \in F$ such that:

- \bullet + is commutative.
- \bullet + is associative.
- $\bullet \ \forall x \in F.x + 0 = x$
- $\forall x \in F. \exists y \in F. x + y = 0$
- \bullet · is commutative.
- \bullet · is associative.
- There exists $1 \in F$ such that $1 \neq 0$ and $\forall x \in F.x1 = x$ and $\forall x \in F.x \neq 0 \Rightarrow \exists y \in F.xy = 1$
- Distributive Law $\forall x, y, z \in F.x(y+z) = xy + xz$

Proposition 4.2. In any field F, the element 0 is the unique element such that $\forall x \in F.x + 0 = x$.

PROOF: If 0 and 0' both have this property then 0 = 0 + 0' = 0'. \square

Proposition 4.3. In any field F, given $x \in F$, there is a unique $y \in F$ such that x + y = 0.

PROOF: If
$$x + y = x + y' = 0$$
 then
$$y = y + 0$$
$$= y + x + y'$$
$$= 0 + y'$$
$$= y'$$

Definition 4.4. Let F be a field. Let $x \in F$. We denote by -x the unique element of F such that x + (-x) = 0.

Given $x, y \in F$, we write x - y for x + (-y).

Proposition 4.5. In any field F, if x + y = x + z then y = z.

PROOF: If x+y=x+z we have -x+x+y=-x+x+z $\therefore 0+y=0+z$ $\therefore y=z$

Proposition 4.6. In any field F, we have -(-x) = x.

PROOF: Since x + (-x) = 0. \square

Proposition 4.7. In any field F, the element 1 such that $\forall x \in F.x1 = x$ is unique.

PROOF: If 1 and 1' both have this property then $1 = 1 \cdot 1' = 1'$. \square

Proposition 4.8. In any field F, given $x \in F$ with $x \neq 0$, the element y such that xy = 1 is unique.

PROOF: If y and y' both have this property then we have

$$y = y1$$

$$= yxy'$$

$$= 1y'$$

$$= y'$$

Definition 4.9. In any field F, if $x \neq 0$, we write x^{-1} for the unique element such that $xx^{-1} = 1$.

We write x/y for xy^{-1} .

Proposition 4.10. In any field F, if xy = xz and $x \neq 0$ then y = z.

Proof:

$$y = 1y$$

$$= x^{-1}xy$$

$$= x^{-1}xz$$

$$= 1z$$

$$= z$$

Proposition 4.11. In any field F, if $x \neq 0$ then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.

PROOF: Since $xx^{-1} = 1$. \square

Proposition 4.12. In any field F, we have x0 = 0.

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Proof:

$$x0 + 0 = x0$$

$$= x(0 + 0)$$

$$= x0 + x0$$

$$\therefore 0 = x0$$

Proposition 4.13. In any field F, if xy = 0 then x = 0 or y = 0.

PROOF: If xy = 0 and $x \neq 0$ then we have $y = x^{-1}xy = x^{-1}0 = 0$. \square

Proposition 4.14. In any field F, we have (-x)y = -(xy).

Proof:

$$xy + (-x)y = (x + (-x))y$$

$$= 0y$$

$$= 0 (Proposition 4.12) \square$$

Corollary 4.14.1. In any field F, we have (-x)(-y) = xy.

Proof:

$$(-x)(-y) = -(x(-y))$$

$$= -(-(xy))$$

$$= xy (Proposition 4.6) \Box$$

Proposition 4.15. Let K be a field. Let $a, b \in K$. If $a^2 = b^2$ then a = b or a = -b.

Proof:

$$a^2 - b^2 = 0$$
$$\therefore (a - b)(a + b) = 0$$

Hence either a - b = 0 or a + b = 0, and the conclusion follows. \square

4.1 Ordered Fields

Definition 4.16 (Ordered Field). An ordered field F consists of a field F and a linear order \leq on F such that:

- For all $x, y, z \in F$, if y < z then x + y < x + z
- For all $x, y \in F$, if x > 0 and y > 0 then xy > 0.

We call x positive iff x > 0 and negative iff x < 0.

Example 4.17. \mathbb{Q} is an ordered field.

Proposition 4.18. In any ordered field, if x is positive then -x is negative.

PROOF: If
$$x > 0$$
 then $0 = x + (-x) > 0 = (-x) = -x$. \Box

Proposition 4.19. In any ordered field, if y < z and x is positive then xy < xz.

PROOF: If y < z then we have

$$0 < z - y$$

$$0 < x(z - y)$$

$$= xz - xy$$

$$xy < xz$$

Proposition 4.20. In any ordered field, if y < z and x is negative then xy > xz.

Proof:

- $\langle 1 \rangle 1$. -x is positive.
- $\langle 1 \rangle 2$. (-x)y < (-x)z
- $\langle 1 \rangle 3. -(xy) < -(xz)$
- $\langle 1 \rangle 4$. xz < xy

Proposition 4.21. In any ordered field, if $x \neq 0$ then $x^2 > 0$.

 $\langle 1 \rangle 1$. If x > 0 then $x^2 > 0$.

PROOF: Proposition 4.19.

 $\langle 1 \rangle 2$. If x < 0 then $x^2 > 0$.

Proof: Proposition 4.20.

Corollary 4.21.1. In any ordered field, we have 1 > 0.

Proposition 4.22. In any ordered field, if x is positive then x^{-1} is positive.

PROOF: If $x^{-1} < 0$ then we would have $1 = xx^{-1} < x0 = 0$ contradicting Corollary 4.21.1. \square

Proposition 4.23. In any ordered field, if 0 < x < y then $y^{-1} < x^{-1}$.

- $\langle 1 \rangle 1$. Assume: 0 < x < y
- $\langle 1 \rangle 2$. x^{-1} and y^{-1} are positive.

Proof: Proposition 4.22.

- $\langle 1 \rangle 3. \ xy^{-1} < yy^{-1} = 1$ $\langle 1 \rangle 4. \ y^{-1} = x^{-1}xy^{-1} < x^{-1}1 = x^{-1}$

Lemma 4.24. Let K be an ordered field. Let $b \in K$ with b > 1. Let n be a positive integer. Then

$$b^n - 1 \ge n(b - 1)$$

Proof:

$$b^{n} - 1 = (b-1)(b^{n-1} + b^{n-2} + \dots + 1)$$

$$\geq (b-1)(1+1+\dots+1)$$

$$= n(b-1)$$

Real Analysis

5.1 Construction of the Real Numbers

Definition 5.1 (Cut). A *cut* is a subset α of \mathbb{Q} such that:

- $\emptyset \neq \alpha \neq \mathbb{Q}$
- α is closed downwards.
- α has no greatest element.

In this section, we write R for the set of all cuts.

Proposition 5.2. R is linearly ordered by \subseteq .

```
PROOF: The only difficult part is to prove that, for any cuts \alpha and \beta, either \alpha \subseteq \beta or \beta \subseteq \alpha. 
(1)1. Assume: \alpha \nsubseteq \beta Prove: \beta \subseteq \alpha
```

 $\langle 1 \rangle 2$. PICK $q \in \alpha$ such that $q \notin \beta$ $\langle 1 \rangle 3$. Let: $r \in \beta$

 $\langle 1 \rangle 4. \ q \not< r$

 $\langle 1 \rangle 5. \ r < q$

 $\langle 1 \rangle 6. \ r \in \alpha$

Proposition 5.3. R has the least upper bound property.

Proof:

 $\langle 1 \rangle 1$. Let: $E \subseteq R$ be nonempty and bounded above.

 $\langle 1 \rangle 2$. Let: $s = \bigcup E$

Prove: s is a cut.

 $\langle 1 \rangle 3. \ \emptyset \neq s$

PROOF: Since E is nonempty and every element of E is nonempty.

 $\langle 1 \rangle 4. \ s \neq \mathbb{Q}$

- $\langle 2 \rangle 1$. PICK an upper bound u for E.
- $\langle 2 \rangle 2$. Pick $q \notin u$ Prove: $q \notin s$
- $\langle 2 \rangle 3. \ \forall \alpha \in E.\alpha \subseteq u$
- $\langle 2 \rangle 4. \ s \subseteq u$
- $\langle 2 \rangle 5. \ q \notin s$
- $\langle 1 \rangle 5$. s is closed downwards.
 - $\langle 2 \rangle 1$. Let: $q \in s$ and r < q.
 - $\langle 2 \rangle 2$. Pick $\alpha \in E$ such that $q \in \alpha$.
 - $\langle 2 \rangle 3. \ r \in \alpha$
 - $\langle 2 \rangle 4. \ r \in s$
- $\langle 1 \rangle 6$. s has no greatest element.
 - $\langle 2 \rangle 1$. Let: $q \in s$
 - $\langle 2 \rangle 2$. PICK $\alpha \in E$ such that $q \in \alpha$.
 - $\langle 2 \rangle 3$. Pick $r \in \alpha$ such that q < r.
- $\langle 2 \rangle 4. \ r \in s$

Definition 5.4 (Addition). Given cuts α and β , we define

$$\alpha + \beta = \{q + r : q \in \alpha, r \in \beta\} .$$

Proposition 5.5. Given cuts α and β , we have $\alpha + \beta$ is a cut.

Proof:

 $\langle 1 \rangle 1$. $\alpha + \beta$ is nonempty.

PROOF: Since α and β are nonempty.

- $\langle 1 \rangle 2. \ \alpha + \beta \neq \mathbb{Q}$
 - $\langle 2 \rangle 1$. Pick $q \in \mathbb{Q} \alpha$ and $r \in \mathbb{Q} \beta$. Prove: $q + r \notin \alpha + \beta$
 - $\langle 2 \rangle 2$. Assume: for a contradiction $q + r \in \alpha + \beta$.
 - $\langle 2 \rangle 3$. Pick $x \in \alpha$ and $y \in \beta$ such that q + r = x + y
 - $\langle 2 \rangle 4$. x < q
 - $\langle 2 \rangle 5$. y < r
 - $\langle 2 \rangle 6$. x + y < q + r
 - $\langle 2 \rangle$ 7. Q.E.D.

PROOF: This is a contradiction.

- $\langle 1 \rangle 3$. $\alpha + \beta$ is closed downwards.
 - $\langle 2 \rangle 1$. Let: $q \in \alpha$, $r \in \beta$ and x < q + r
 - $\langle 2 \rangle 2$. x q < r
 - $\langle 2 \rangle 3. \ x q \in \beta$
 - $\langle 2 \rangle 4. \ x \in \alpha + \beta$
- $\langle 1 \rangle 4$. $\alpha + \beta$ has no greatest element.
 - $\langle 2 \rangle 1$. Let: $q \in \alpha$ and $r \in \beta$.

PROVE: q + r is not greatest in $\alpha + \beta$.

- $\langle 2 \rangle 2$. Pick $q' \in \alpha$ with q < q' and $r' \in \beta$ with r < r'.
- $\langle 2 \rangle 3. \ q + r < q' + r' \in \alpha + \beta$

Proposition 5.6. Addition is commutative and associative on R.

PROOF: Immediate from definitions and the fact that addition is commutative and associative on \mathbb{Q} . \square

Definition 5.7. For any $q \in \mathbb{Q}$, let $q^* = \{r \in \mathbb{Q} : r < q\}$.

Proposition 5.8. For any $q \in \mathbb{Q}$, we have q^* is a cut.

```
Proof:
```

```
\langle 1 \rangle 1. \ q^* \neq \emptyset
   PROOF: Since q - 1 \in q^*.
\langle 1 \rangle 2. \ q^* \neq \mathbb{Q}
   PROOF: Since q \notin q^*.
\langle 1 \rangle 3. q^* is closed downwards.
   PROOF: Immediate from definition.
```

 $\langle 1 \rangle 4$. q^* has no greatest element.

PROOF: For all $r \in q^*$ we have $r < (q+r)/2 \in q^*$.

Proposition 5.9. For any cut α we have $\alpha + 0^* = \alpha$.

Proof:

$$\begin{array}{l} \langle 1 \rangle 1. \ \alpha + 0^* \subseteq \alpha \\ \langle 2 \rangle 1. \ \text{Let:} \ q \in \alpha \ \text{and} \ r \in 0^* \\ \text{Prove:} \ q + r \in \alpha \\ \langle 2 \rangle 2. \ r < 0 \\ \langle 2 \rangle 3. \ q + r < q \\ \langle 2 \rangle 4. \ q + r \in \alpha \\ \langle 1 \rangle 2. \ \alpha \subseteq \alpha + 0^* \\ \langle 2 \rangle 1. \ \text{Let:} \ q \in \alpha \\ \langle 2 \rangle 2. \ \text{Pick} \ r \in \alpha \ \text{such that} \ q < r \\ \langle 2 \rangle 3. \ q = r + (q - r) \in \alpha + 0^* \end{array}$$

Proposition 5.10. For any cut α , there exists a cut β such that $\alpha + \beta = 0$.

```
\langle 1 \rangle 1. Let: \beta = \{ p \in \mathbb{Q} : \exists r > 0. - p - r \notin \alpha \}
\langle 1 \rangle 2. \beta is a cut.
    \langle 2 \rangle 1. \ \beta \neq \emptyset
         \langle 3 \rangle 1. Pick q \notin \alpha
         \langle 3 \rangle 2. -q - 1 \in \beta
     \langle 2 \rangle 2. \ \beta \neq \mathbb{Q}
         \langle 3 \rangle 1. Pick q \in \alpha
                      Prove: -q \notin \beta
         \langle 3 \rangle 2. Assume: for a contradiction -q \in \beta
```

```
\langle 3 \rangle 3. Pick r > 0 such that q - r \notin \alpha
         \langle 3 \rangle 4. \ q - r < q
         \langle 3 \rangle 5. Q.E.D.
            PROOF: This contradicts the fact that \alpha is closed downwards.
    \langle 2 \rangle 3. \beta is closed downwards.
         \langle 3 \rangle 1. Let: p \in \beta and q < p.
         \langle 3 \rangle 2. Pick r > 0 such that -p - r \notin \alpha
         \langle 3 \rangle 3. -p-r < -q-r
         \langle 3 \rangle 4. -q - r \notin \alpha
         \langle 3 \rangle 5. \ q \in \beta
    \langle 2 \rangle 4. \beta has no greatest element.
         \langle 3 \rangle 1. Let: p \in \beta
         \langle 3 \rangle 2. Pick r > 0 such that -p - r \notin \alpha
         \langle 3 \rangle 3. \ -(p+r/2) - r/2 \notin \alpha
         \langle 3 \rangle 4. \ p + r/2 \in \beta
\langle 1 \rangle 3. \ \alpha + \beta \subseteq 0^*
    \langle 2 \rangle 1. Let: p \in \alpha and q \in \beta.
    \langle 2 \rangle 2. Pick r > 0 such that -q - r \notin \alpha.
    \langle 2 \rangle 3. p < -q - r
    \langle 2 \rangle 4. p+q < -r
    \langle 2 \rangle 5. p+q < 0
    \langle 2 \rangle 6. \ p+q \in 0^*
\langle 1 \rangle 4. \ 0^* \subseteq \alpha + \beta
    \langle 2 \rangle 1. Let: v \in 0^*
    \langle 2 \rangle 2. Let: w = -v/2
    \langle 2 \rangle 3. \ w > 0
    \langle 2 \rangle 4. PICK an integer n such that nw \in \alpha and (n+1)w \notin \alpha.
    \langle 2 \rangle5. Let: p = -(n+2)w
    \langle 2 \rangle 6. \ p \in \beta
    \langle 2 \rangle 7. \ v = nw + p
    \langle 2 \rangle 8. \ v \in \alpha + \beta
```

Proposition 5.11. Given $\alpha, \beta, \gamma \in R$, if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.

```
PROOF:  \begin{array}{l} \langle 1 \rangle 1. \ \alpha + \beta \subseteq \alpha + \gamma \\ \text{PROOF: Immediate from definitions.} \\ \langle 1 \rangle 2. \ \alpha + \beta \neq \alpha + \gamma \\ \text{PROOF: If } \alpha + \beta = \alpha + \gamma \text{ then } \beta = \gamma \text{ by cancellation.} \\ \end{array}
```

Definition 5.12. Given cuts α and β , define $\alpha\beta$ by:

$$\alpha\beta = \begin{cases} \{p \in \mathbb{Q} : \exists r \in \alpha. \exists s \in \beta (p \le rs \land r > 0 \land s > 0\} & \text{if } \alpha > 0^* \text{ and } \beta > 0^* \\ (-\alpha)(-\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^* \\ -((-\alpha)\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^* \\ -(\alpha(-\beta)) & \text{if } \alpha > 0^* \text{ and } \beta < 0^* \\ 0^* & \text{if } \alpha > 0^* \text{ and } \beta < 0^* \end{cases}$$

Proposition 5.13. For any cuts α and β , we have $\alpha\beta$ is a cut.

```
Proof:
```

```
\langle 1 \rangle 1. If \alpha > 0^* and \beta > 0^* then \alpha \beta is a cut.
```

- $\langle 2 \rangle 1. \ \alpha \beta \neq \emptyset$
 - $\langle 3 \rangle 1$. Pick $q \in \alpha$ and $r \in \beta$ such that $q, r \notin 0^*$
 - $\langle 3 \rangle 2$. Assume: w.l.o.g. 0 < q and 0 < r.

PROOF: Since α and β have no greatest element.

- $\langle 3 \rangle 3. \ qr \in \alpha \beta$
- $\langle 2 \rangle 2$. $\alpha \beta \neq \mathbb{Q}$
 - $\langle 3 \rangle 1$. PICK $r \notin \alpha$ and $s \notin \beta$ PROVE: $rs \notin \alpha \beta$
 - $\langle 3 \rangle 2$. Assume: for a contradiction $rs \in \alpha \beta$.
 - $\langle 3 \rangle 3$. Pick $r' \in \alpha$ and $s' \in \beta$ such that $rs \leq r's'$ and r' > 0 and s' > 0.
 - $\langle 3 \rangle 4$. r' < r and s' < s
 - $\langle 3 \rangle 5$. r's' < rs
 - $\langle 3 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

- $\langle 2 \rangle 3$. $\alpha \beta$ is closed downwards.
 - $\langle 3 \rangle 1$. Let: $p \in \alpha \beta$ and p' < p
 - $\langle 3 \rangle 2$. Pick $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, r > 0 and s > 0
 - $\langle 3 \rangle 3. \ p' \leq rs$
 - $\langle 3 \rangle 4. \ p' \in \alpha \beta$

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- $\langle 2 \rangle 4$. $\alpha \beta$ has no greatest element.
 - $\langle 3 \rangle 1$. Let: $p \in \alpha \beta$
 - $\langle 3 \rangle 2$. Pick $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, r > 0 and s > 0.
 - $\langle 3 \rangle 3$. Pick $r' \in \alpha$ and $s' \in \beta$ with r < r' and s < s'.
 - $\langle 3 \rangle 4. \ p < r's' \in \alpha \beta$
- $\langle 1 \rangle 2$. For any cuts α and β , we have $\alpha \beta$ is a cut.

PROOF: Since if α is a cut then $-\alpha$ is a cut.

Proposition 5.14. For any cuts α and β we have $\alpha\beta = \beta\alpha$.

PROOF: Easy from the definitions. \square

Proposition 5.15. For any cuts α , β and γ we have

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma$$
.

 $\langle 1 \rangle 1$. Case: α , β and γ are all positive.

PROOF: In this case $\alpha(\beta\gamma) = (\alpha\beta)\gamma = \{p \in \mathbb{Q} : \exists r \in \alpha. \exists s \in \beta. \exists t \in \gamma. (p \leq rst \land r > 0 \land s > 0 \land t > 0)\}.$

 $\langle 1 \rangle 2$. Case: One of α , β or γ is 0^* .

PROOF: Then $\alpha(\beta\gamma) = (\alpha\beta)\gamma = 0^*$.

 $\langle 1 \rangle 3.$ Case: α and β are positive, γ is negative. Proof:

$$\alpha(\beta\gamma) = \alpha(-(\beta(-\gamma)))$$

$$= -(\alpha(\beta(-\gamma)))$$

$$= -((\alpha\beta)(-\gamma))$$

$$= (\alpha\beta)\gamma$$
(\langle 1\rangle 1)

 $\langle 1 \rangle 4.$ Case: α is positive, β is negative, γ is positive. Proof:

$$\alpha(\beta\gamma) = \alpha(-((-\beta)\gamma))$$

$$= -(\alpha((-\beta)\gamma))$$

$$= -((\alpha(-\beta))\gamma)$$

$$= (-(\alpha(-\beta)))\gamma$$

$$= (\alpha\beta)\gamma$$

$$(\langle 1\rangle 1)$$

 $\langle 1 \rangle 5.$ Case: α is positive, β and γ are negative. Proof:

$$\alpha(\beta\gamma) = \alpha((-\beta)(-\gamma))$$

$$= (\alpha(-\beta))(-\gamma)$$

$$= (-(\alpha\beta))(-\gamma)$$

$$= (\alpha\beta)\gamma$$

$$(\langle 1 \rangle 1)$$

 $\langle 1 \rangle$ 6. Case: α is negative, β and γ are positive. Proof: Similar to $\langle 1 \rangle$ 3.

 $\langle 1 \rangle 7.$ Case: α is negative, β is positive, γ is negative. Proof:

$$\alpha(\beta\gamma) = \alpha(-(\beta(-\gamma)))$$

$$= (-\alpha)(\beta(-\gamma))$$

$$= ((-\alpha)\beta)(-\gamma)$$

$$= (-(\alpha\beta))(-\gamma)$$

$$= (\alpha\beta)\gamma$$

$$(\langle 1 \rangle 1)$$

 $\langle 1 \rangle 8$. Case: α and β are negative, γ is positive. Proof: Similar to $\langle 1 \rangle 5$.

 $\langle 1 \rangle 9$. Case: α , β and γ are all negative.

$$\alpha(\beta\gamma) = \alpha(-(-\beta)(-\gamma))$$

$$= -((-\alpha)((-\beta)(-\gamma)))$$

$$= -(((-\alpha)(-\beta))(-\gamma))$$

$$= -((\alpha\beta)(-\gamma))$$

$$= (\alpha\beta)\gamma$$

$$(\langle 1 \rangle 1)$$

П

Proposition 5.16. For any cut α we have $\alpha 1^* = \alpha$.

Proof:

```
\begin{array}{ll} \langle 1 \rangle 1. \  \, \text{Case:} \  \, \alpha \  \, \text{is positive.} \\ \langle 2 \rangle 1. \  \, \alpha 1^* \subseteq \alpha \\ \langle 2 \rangle 2. \  \, \alpha \subseteq \alpha 1^* \\ \langle 1 \rangle 2. \  \, \text{Case:} \  \, \alpha = 0^* \\ \underline{\langle 1 \rangle} 3. \  \, \text{Case:} \  \, \alpha \  \, \text{is negative.} \end{array}
```

Theorem 5.17. There exists an ordered field with the least upper bound property.

Proposition 5.18. There is no rational p such that $p^2 = 2$.

PROOF:

```
PROOF: \langle 1 \rangle 1. Assume: for a contradiction p^2 = 2. \langle 1 \rangle 2. PICK integers m, n not both even such that p = m/n. \langle 1 \rangle 3. m^2 = 2n^2 \langle 1 \rangle 4. m is even. \langle 1 \rangle 5. PICK an integer k such that m = 2k. \langle 1 \rangle 6. 4k^2 = 2n^2 \langle 1 \rangle 7. 2k^2 = n^2 \langle 1 \rangle 8. n is even. \langle 1 \rangle 9. Q.E.D. PROOF: \langle 1 \rangle 2, \langle 1 \rangle 4 and \langle 1 \rangle 8 form a contradiction.
```

Theorem 5.19. Any two complete ordered fields are isomorphic.

Definition 5.20. Let \mathbb{R} be the complete ordered field. We call its elements *real numbers*.

5.2 Properties of the Real Numbers

Theorem 5.21. \mathbb{Q} is a subfield of \mathbb{R} .

Theorem 5.22 (Archimedean Property). Let $x, y \in \mathbb{R}$ with x > 0. There exists a positive integer n such that nx > y.

- $\langle 1 \rangle 1$. Let: $A = \{ nx : n \in \mathbb{Z}^+ \}$
- $\langle 1 \rangle 2$. Assume: for a contradiction there is no positive integer n such that nx > y.
- $\langle 1 \rangle 3$. y is an upper bound for A.
- $\langle 1 \rangle 4$. Let: $\alpha = \sup A$
- $\langle 1 \rangle 5$. αx is not an upper bound for A.
- $\langle 1 \rangle 6$. Pick a positive integer m such that $\alpha x < mx$
- $\langle 1 \rangle 7$. $\alpha < (m+1)x \in A$
- $\langle 1 \rangle 8$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 4$.

П

Theorem 5.23. \mathbb{Q} is dense in \mathbb{R} .

Proof:

- $\langle 1 \rangle 1$. Let: $x, y \in \mathbb{R}$ with x < y
- $\langle 1 \rangle 2$. PICK a positive integer n such that

$$n(y-x) > 1 .$$

PROOF: Archimedean property.

 $\langle 1 \rangle 3$. PICK a positive integer m_1 such that $m_1 > nx$

Proof: Archimedean property.

- $\langle 1 \rangle 4$. PICK a positive integer m_2 such that $m_2 > -nx$ PROOF: Archimedean property.
- $\langle 1 \rangle 5$. $-m_2 < nx < m_1$
- $\langle 1 \rangle 6$. Let: m be the integer such that

$$m-1 \le nx < m$$
.

- $\langle 1 \rangle 7$. $nx < m \le 1 + nx < ny$
- $\langle 1 \rangle 8. \ x < m/n < y$

Theorem 5.24. For every real number x > 0 and positive integer n, there exists a unique positive real number y such that $y^n = x$.

Proof:

- $\langle 1 \rangle 1$. There exists a real y > 0 such that $y^n = x$.
 - $\langle 2 \rangle 1$. Let: $E = \{ t \in \mathbb{R}^+ : t^n < x \}$
 - $\langle 2 \rangle 2$. Let: $y = \sup E$
 - $\langle 3 \rangle 1. \ E \neq \emptyset$
 - $\langle 4 \rangle 1$. Let: t = x/(x+1)
 - $\langle 4 \rangle 2. \ 0 < t < 1$
 - $\langle 4 \rangle 3. \ t^n < t < x$
 - $\langle 4 \rangle 4. \ t \in E$
 - $\langle 3 \rangle 2$. x+1 is an upper bound for E.
 - $\langle 4 \rangle 1$. Let: t > x + 1
 - $\langle 4 \rangle 2$. $t^n > t > x$
 - $\langle 4 \rangle 3. \ t \notin E$

$$\langle 2 \rangle 3. \ y^n = x$$

 $\langle 3 \rangle 1. \ y^n \not< x$

 $\langle 4 \rangle 1$. Assume: for a contradiction $y^n < x$.

 $\langle 4 \rangle 2$. Pick h such that 0 < h < 1 and

$$h < \frac{x - y^n}{n(y+1)^{n-1}}$$
.

$$\langle 4 \rangle 3. \ (y+h)^n - y^n < x - y^n$$

Proof:

$$(y+h)^n - y^n = ((y+h) - y) \sum_{i=0}^{n-1} (y+h)^{n-1-i} y^i$$

$$= h \sum_{i=0}^{n-1} (y+h)^{n-1-i} y^i$$

$$\leq hn(y+h)^{n-1}$$

$$\leq hn(y+1)^{n-1}$$

$$< x - y^n$$

$$\langle 4 \rangle 4$$
. $(y+h)^n < x$

$$\langle 4 \rangle 5. \ y + h \in E$$

 $\langle 4 \rangle 6$. Q.E.D.

PROOF: This contradicts the fact that y is an upper bound for E.

$$\langle 3 \rangle 2. \ y^n \not> x$$

 $\langle 4 \rangle 1$. Assume: for a contradiction $y^n > x$

 $\langle 4 \rangle 2$. Let:

$$k = \frac{y^n - x}{ny^{n-1}}$$

 $\langle 4 \rangle 3$. 0 < k < y

 $\langle 4 \rangle 4$. y - k is an upper bound for E.

$$\langle 5 \rangle 1$$
. Let: $t \geq y - k$

$$\langle 5 \rangle 2$$
. $y^n - t^n \le y^n - x$

Proof:

$$\begin{split} y^n - t^n &\leq y^n - (y - k)^n \\ &= (y - (y - k)) \sum_{i=0}^{n-1} y^{n-i} (y - k)^i \\ &= k \sum_{i=0}^{n-1} y^{n-i} (y - k)^i \\ &\leq k n y^{n-1} \\ &= y^n - x \end{split}$$

$$\langle 5 \rangle 3. \ t^n \ge x$$

$$\langle 5 \rangle 4. \ t \notin E$$

 $\langle 4 \rangle 5$. Q.E.D.

PROOF: This contradicts the fact that y is the least upper bound of E. $\langle 1 \rangle 2$. If y and y' are positive reals with $y^n = y'^n$ then y = y'.

Proof: Since the function that sends y to y^n is strictly monotone. \square

Definition 5.25 (*n*th Root). Given any real number x > 0 and positive integer n, the nth root of x, denoted $x^{1/n}$, is the unique positive real such that

$$(x^{1/n})^n = x .$$

We write \sqrt{x} for $x^{1/2}$.

Proposition 5.26. Let a and b be positive real numbers and n a positive integer. Then

$$(ab)^{1/n} = a^{1/n}b^{1/n}$$
.

PROOF: Since $(a^{1/n}b^{1/n})^n = ab$. \square

Lemma 5.27. Let b be a real number with b > 1. Let n be a positive integer. Then

$$b-1 \ge n(b^{1/n}-1)$$
.

Proof: From Lemma 4.24. \Box

Lemma 5.28. Let b and t be real numbers with b > 1 and t > 1. For any positive integer n, if $n > \frac{b-1}{t-1}$ then $b^{1/n} < t$.

Proof:

$$b-1 \ge n(b^{1/n}-1)$$

$$\therefore \frac{b-1}{n} \ge b^{1/n}-1$$

$$\therefore t-1 > b^{1/n}-1$$

$$\therefore t > b^{1/n}$$

Lemma 5.29. Let b be a real number with b > 0. Let m, n, p, q be integers with n > 0 and q > 0. Assume m/n = p/q. Then

$$(b^m)^{1/n} = (b^p)^{1/q}$$
.

Proof:

$$\langle 1 \rangle 1. \ (b^m)^{1/n} = (b^{1/n})^m$$

Proof:

$$((b^{1/n})^m)^n = ((b^{1/n})^n)^m$$

= b^m

$$\langle 1 \rangle 2. \ ((b^m)^{1/n})^q = b^p$$

Proof:

$$((b^m)^{1/n})^q = (b^{1/n})^{mq}$$
$$= (b^{1/n})^{np}$$
$$= b^p$$

Definition 5.30. For a a positive real and q a rational number, we may therefore define a^q by

$$a^{m/n} = (a^m)^{1/n}$$

for m and n integers with n > 0.

Proposition 5.31. Let a be a positive real and r, s rational numbers. Then

$$a^{r+s} = a^r a^s$$
.

Proof:

$$a^{m/n+p/q} = a^{(mq+np)/nq}$$

$$= (a^{mq+np})^{1/nq}$$

$$= (a^{mq})^{1/nq} (a^{np})^{1/nq}$$

$$= a^{m/n} a^{p/q}$$

Proposition 5.32. Let b > 1 be a real number and q a rational number. Then

$$b^q = \sup\{b^t : t \in \mathbb{Q}, t \le q\}$$

PROOF: It is the greatest element of this set. \square

Definition 5.33. Let b > 1 be a real number and x a real number. Then

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t \le x\} .$$

Lemma 5.34. Let b, w and y be real numbers with b > 1. Assume $b^w < y$. Then there exists a positive integer n such that $b^{w+1/n} < y$.

Proof:

- $\langle 1 \rangle 1$. Let: $t = yb^{-w}$
- $\langle 1 \rangle 2$. PICK a positive integer n such that $n > \frac{b-1}{t-1}$.
- $\langle 1 \rangle 3. \ b^{1/n} < t$

PROOF: Lemma 5.28.

PROOF: Lemma
$$\langle 1 \rangle 4$$
. $b^{w+1/n} < y$

Lemma 5.35. Let b, w and y be real numbers with b > 1. Assume $b^w > y$. Then there exists a positive integer n such that $b^{w-1/n} < y$.

Proof:

- $\langle 1 \rangle 1$. Let: $t = b^w/y$
- $\langle 1 \rangle 2$. PICK a positive integer n such that $n > \frac{b-1}{t-1}$
- $\langle 1 \rangle 3. \ b^{1/n} < t$

Proof: Lemma 5.28.

$$\langle 1 \rangle 4. \ y < b^{w-1/n}$$

Proposition 5.36. For b and x real numbers with b > 1 we have

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t < x\} .$$

Proof:

- $\langle 1 \rangle 1$. b^x is an upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
- $\langle 1 \rangle 2$. Let: u be any upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$. Prove: $b^x \leq u$
- $\langle 1 \rangle 3.$ Let: q be a rational number with $q \leq x.$ Prove: $b^q \leq u$
- $\langle 1 \rangle 4$. Assume: for a contradiction $b^q > u$.
- $\langle 1 \rangle$ 5. PICK a positive integer n such that $b^{q-1/n} > u$.

PROOF: Lemma 5.35.

 $\langle 1 \rangle 6. \ b^{q-1/n} \le u$ PROOF: $\langle 1 \rangle 2$

PROOF: $\langle 1 \rangle 2$ $\langle 1 \rangle 7$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 4$.

Lemma 5.37. Let A be a set of positive real numbers with supremum a > 0 and B a set of positive real numbers with supremum b > 0. Then ab is the supremum of $\{xy : x \in A, y \in B\}$.

Proof:

- $\langle 1 \rangle 1$. For all $x \in A$ and $y \in B$ we have $xy \leq ab$.
- $\langle 1 \rangle 2$. If u is any upper bound for $\{xy : x \in A, y \in B\}$ then $ab \leq u$.
 - $\langle 2 \rangle 1$. Let: u be an upper bound for $\{xy : x \in A, y \in B\}$.
 - $\langle 2 \rangle 2$. For all $x \in A$ we have u/x is an upper bound for B.
 - $\langle 2 \rangle 3$. For all $x \in A$ we have $b \leq u/x$
 - $\langle 2 \rangle 4$. For all $x \in A$ we have $x \leq u/b$
 - $\langle 2 \rangle 5$. $a \leq u/b$
 - $\langle 2 \rangle 6. \ ab \leq u$

Proposition 5.38. *Let* $b, x, y \in \mathbb{R}$ *with* b > 1. *Then*

$$b^{x+y} = b^x b^y .$$

Proof:

- $\langle 1 \rangle 1$. For any rational number q < x + y, there exist rational numbers r < x and s < y such that q = r + s.
 - $\langle 2 \rangle 1. \ q x < y$
 - $\langle 2 \rangle 2$. Pick a rational t such that q x < t < y
 - $\langle 2 \rangle 3$. q = t + (q t) and t < y, q t < x
- $\langle 1 \rangle 2$. $b^x b^y = b^{x+y}$

$$\begin{split} b^x b^y &= \sup\{b^q b^r : q, r \in \mathbb{Q}, q < x, r < y\} \\ &= \sup\{b^{q+r} : q, r \in \mathbb{Q}, q < x, r < y\} \\ &= \sup\{b^q : q \in \mathbb{Q}, q < x + y\} \\ &= b^{x+y} \end{split}$$

5.2.1 Logarithms

Proposition 5.39. Let b and y be real numbers with b > 1 and y > 0. There exists a unique real x such that $b^x = y$.

```
Proof:
```

```
\langle 1 \rangle 1. Let: x = \sup\{w : b^w < y\}
        PROVE: b^x = y
   \langle 2 \rangle 1. \ \{ w : b^w < y \} \neq \emptyset
      Proof: It contains 0.
   \langle 2 \rangle 2. \{w : b^w < y\} is bounded above.
      \langle 3 \rangle 1. Let: n be the least integer such that
         Proof: Archimedean property.
      \langle 3 \rangle 2. Let: w be a real number with b^w < y
              Prove: w < n
      \langle 3 \rangle 3. \ b^w < n(b-1)+1
      \langle 3 \rangle 4. \ b^w < b^n
      \langle 3 \rangle 5. \ w < n
\langle 1 \rangle 2. \ b^x \leq y
   \langle 2 \rangle 1. Assume: for a contradiction b^x > y
   \langle 2 \rangle 2. PICK a positive integer n such that b^{x-1/n} > y
      Proof: Lemma 5.35.
   \langle 2 \rangle 3. Pick w such that x - 1/n < w and b^w < y
      PROOF: Since x - 1/n is not an upper bound for \{w : b^w < y\}.
   \langle 2 \rangle 4. \ b^{x-1/n} < y
   \langle 2 \rangle 5. Q.E.D.
     PROOF: This contradicts \langle 2 \rangle 2.
\langle 1 \rangle 3. \ b^x \geq y
   \langle 2 \rangle 1. Assume: for a contradiction b^x < y.
   \langle 2 \rangle 2. Pick a positive integer n such that b^{x+1/n} < y.
   \langle 2 \rangle 3. \ x + 1/n \le x
   \langle 2 \rangle 4. Q.E.D.
      PROOF: This is a contradiction.
```

Definition 5.40 (Logarithm). Let b and y be real numbers with b > 1 and y > 0. The *logarithm* of y to *base* b, denoted $\log_b y$, is the unique real number

such that

$$b^{\log_b y} = y .$$

5.2.2Intervals

Definition 5.41 (Intervals). Let $a, b \in \mathbb{R}$.

The open interval (a, b) is $\{x \in \mathbb{R} : a < x < b\}$.

The closed interval [a, b] is $\{x \in \mathbb{R} : a \le x \le b\}$.

The half-open intervals [a, b) and (a, b] are defined by

$$[a,b) := \{x \in \mathbb{R} : a \le x < b\}$$
$$(a,b] := \{x \in \mathbb{R} : a < x \le b\}$$

Proposition 5.42. Let (I_n) be a sequence of closed intervals with $I_0 \supseteq I_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} I_n$ is nonempty.

Proof:

- $\langle 1 \rangle 1$. Let: $I_n = [a_n, b_n]$
- $\langle 1 \rangle 2$. Let: $x = \sup_n a_n$

PROOF: $\{a_n : n \in \mathbb{N}\}$ is bounded above by b_0 .

 $\langle 1 \rangle 3.$ $x \in \bigcap_{n=0}^{\infty} I_n$ PROOF: For all n we have $a_n \leq x \leq b_n$ since b_n is an upper bound for $\{a_n : n \in \mathbb{N}\}.$ П

Definition 5.43 (k-cell). Let k be a positive integer. A k-cell is a subset of \mathbb{R}^k of the form

$$\{\vec{x} \in \mathbb{R}^k : \forall i = 1, \dots, k.a_i \le x_i \le b_i\}$$

for some real numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$ with $a_i \leq b_i$ for each i.

Proposition 5.44. Let (I_n) be a sequence of k-cells such that $I_0 \supseteq I_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$.

Proof:

- $\langle 1 \rangle 1$. Let: $I_n = J_{n1} \times \cdots \times J_{nk}$ where each J_{ni} is a closed interval.
- $\langle 1 \rangle 2$. For $i = 1, \ldots, k$, PICK $a_i \in \bigcap_{n=0}^{\infty} J_{ni}$.
- $\langle 1 \rangle 3. \ (a_1, \ldots, a_k) \in \bigcap_{n=0}^{\infty} I_n$

The Cantor Set 5.2.3

Definition 5.45 (Cantor Set). Define a sequence E_n of unions of intervals as follows:

- $E_0 = [0, 1]$
- E_{n+1} is formed from E_n by replacing every interval [a, b] with [a, (2a+b)/3]and [(a+2b)/3, b].

The Cantor set is $\bigcap_{n=0}^{\infty} E_n$.

5.3 The Extended Real Number System

Definition 5.46 (Extended Real Number System). The *extended real number* system is the set $\mathbb{R} \cup \{+\infty, -\infty\}$.

We extend the ordering \leq to the extended reals by defining

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

We extend +, \cdot and / to partial operations on the extended real by defining:

$$x + (+\infty) = +\infty \qquad (x \in \mathbb{R})$$

$$x + (-\infty) = -\infty \qquad (x \in \mathbb{R})$$

$$(+\infty) + x = +\infty \qquad (x \in \mathbb{R})$$

$$(+\infty) + (+\infty) \text{ is undefined}$$

$$(+\infty) + (-\infty) \text{ is undefined}$$

$$(-\infty) + x = -\infty \qquad (x \in \mathbb{R})$$

$$(-\infty) + (+\infty) \text{ is undefined}$$

$$x \cdot (+\infty) = +\infty \qquad (x \in \mathbb{R})$$

$$x \cdot (-\infty) = -\infty \qquad (x \in \mathbb{R})$$

$$x \cdot (-\infty) = -\infty \qquad (x \in \mathbb{R})$$

$$(+\infty) \cdot x = +\infty \qquad (x \in \mathbb{R})$$

$$(+\infty) \cdot (+\infty) \text{ is undefined}$$

$$(+\infty) \cdot (-\infty) \text{ is undefined}$$

$$(-\infty) \cdot (+\infty) \text{ is undefined}$$

$$(-\infty) \cdot (+\infty) \text{ is undefined}$$

$$x/(+\infty) = 0 \qquad (x \in \mathbb{R})$$

$$(x \in \mathbb{R})$$

 $(-\infty)/x$ is undefined

 $(-\infty)/(+\infty)$ is undefined $(-\infty)/(-\infty)$ is undefined

 $(x \in \mathbb{R})$

Complex Analysis

Definition 6.1 (Complex Numbers). A *complex number* is a pair of real numbers. We write \mathbb{C} for the set of complex numbers.

Define + and \cdot on \mathbb{C} by:

$$(a,b) + (c,d) = (a+c,b+d)$$

 $(a,b)(c,d) = (ac-bd,ad+bc)$

Theorem 6.2. The complex numbers form a field.

Theorem 6.3. The function that maps a to (a,0) is an embedding of \mathbb{R} in \mathbb{C} .

Definition 6.4.

$$i = (0, 1)$$

Lemma 6.5.

$$(a,b) = a + ib$$

PROOF: Since (a, 0) + (0, 1)(b, 0) = (a, b).

Lemma 6.6.

$$i^2 = -1$$

PROOF: Immediate from definitions. \square

Corollary 6.6.1. There is no linear order on $\mathbb C$ that makes $\mathbb C$ into an ordered field.

Definition 6.7 (Complex Conjugate). For any complex number z, the complex conjugate \overline{z} is defined by

$$\overline{a+ib} = a-ib \qquad (a,b \in \mathbb{R}) .$$

Definition 6.8 (Real Part). For any complex number z, the *real part* of z, denoted Re(z), is defined by

$$\operatorname{Re}(a+ib) = a \qquad (a, b \in \mathbb{R}) .$$

Definition 6.9 (Imaginary Part). For any complex number z, the *imaginar* part of z, denoted Im(z), is defined by

$$\operatorname{Im}(a+ib) = b \qquad (a, b \in \mathbb{R}) .$$

Theorem 6.10. For all $z, w \in \mathbb{C}$ we have

$$\overline{z+w} = \overline{z} + \overline{w} .$$

Proof:

$$\overline{(a+ib)+(c+id)} = \overline{(a+c)+i(b+d)}$$

$$= (a+c)-i(b+d)$$

$$= (a-ib)+(c-id)$$

$$= \overline{a+ib}+\overline{c+id}$$

Theorem 6.11. For all $z, w \in \mathbb{C}$ we have

$$\overline{zw} = \overline{z} \cdot \overline{w} \ .$$

Proof:

$$\overline{(a+ib)(c+id)} = \overline{(ac-bd) + i(ad+bc)}$$

$$= (ac-bd) - i(ad+bc)$$

$$= (a-ib)(c-id)$$

$$= \overline{a+ib} \cdot \overline{c+id}$$

Theorem 6.12. For all $z \in \mathbb{C}$ we have

$$\operatorname{Re}(z) = \frac{1}{2}(z + \overline{z})$$
.

Proof:

$$(a+ib) + \overline{a+ib} = (a+ib) + (a-ib)$$

$$= 2a$$

$$= 2\operatorname{Re}(a+ib)$$

Theorem 6.13. For all $z \in \mathbb{C}$ we have

$$\operatorname{Im}(z) = \frac{1}{2i}(z - \overline{z}) .$$

Proof:

$$(a+ib) - \overline{a+ib} = (a+ib) - (a-ib)$$

$$= 2ib$$

$$= 2i\operatorname{Im}(a+ib)$$

Theorem 6.14. For all $z \in \mathbb{C}$ we have $z\overline{z}$ is a non-negative real.

$$(a+ib)(\overline{a+ib}) = (a+ib)(a-ib)$$
$$= a^2 + b^2$$

Theorem 6.15. For any $z \in \mathbb{C}$, if $z\overline{z} = 0$ then z = 0.

PROOF: Let z = a + ib. Then $z\overline{z} = a^2 + b^2 = 0$ iff a = b = 0. \square

Definition 6.16 (Absolute Value). For $z \in \mathbb{C}$, the absolute value of z is

$$|z|=(z\overline{z})^{1/2}$$
.

Proposition 6.17. For x a non-negative real we have |x| = x.

PROOF: Since $|x| = \sqrt{x^2} = x$. \square

Proposition 6.18. For x a negative real we have |x| = -x.

Proof: Since $|x| = \sqrt{x^2} = -x$. \square

Theorem 6.19. For any complex number z we have $|z| \ge 0$.

PROOF: Immediate from definition. \Box

Theorem 6.20. For any complex number z, if |z| = 0 then z = 0.

PROOF: From Theorem 6.15. \square

Theorem 6.21. For any complex number z we have

$$|\overline{z}| = |z|$$
.

PROOF: Immediate from definitions. \square

Theorem 6.22. For any complex numbers z and w we have

$$|zw| = |z||w|$$
.

Proof:

$$|zw| = \sqrt{zw\overline{z}w}$$

 $= \sqrt{z\overline{z}}\sqrt{w\overline{w}}$ (Proposition 5.26)
 $= |z||w|$

Theorem 6.23. For any complex number z we have

$$|\operatorname{Re} z| \le |z|$$

PROOF: Let z = a + ib. Then

$$|\operatorname{Re} z| = \sqrt{a^2} \le \sqrt{a^2 + b^2}$$
.

Theorem 6.24. For any complex numbers z and w we have

$$|z+w| \le |z| + |w| .$$

$$|z+w|^2 = (z+w)(\overline{z}+\overline{w})$$

$$= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w}$$

$$= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 \qquad \text{(Theorem 6.12)}$$

$$\leq |z|^2 + 2|z\overline{w}| + |w|^2 \qquad \text{(Theorem 6.23)}$$

$$= |z|^2 + 2|z||w| + |w|^2 \qquad \text{(Theorem 6.22)}$$

$$= (|z| + |w|)^2 \qquad \Box$$

Theorem 6.25 (Schwarz Inequality). Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be complex numbers. Then

$$\left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

Proof:

 $\langle 1 \rangle 1$. Let: $A = \sum_{j=1}^{n} |a_j|^2$ $\langle 1 \rangle 2$. Let: $B = \sum_{j=1}^{n} |b_j|^2$ $\langle 1 \rangle 3$. Let: $C = \sum_{j=1}^{n} a_j \overline{b_j}$ $\langle 1 \rangle 4$. Assume: w.l.o.g. B > 0

PROOF: If B=0 then $b_1=\cdots=b_n=0$ and both sides of the inequality are

$$\langle 1 \rangle$$
5. $\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = B(AB - |C|^2)$

$$\sum_{j=1}^{n} |Ba_{j} - Cb_{j}|^{2} = \sum_{j=1}^{n} (Ba_{j} - Cb_{j})(B\overline{a_{j}} - \overline{Cb_{j}})$$

$$= B^{2} \sum_{j=1}^{n} |a_{j}|^{2} - B\overline{C} \sum_{j=1}^{n} a_{j}\overline{b_{j}} - BC \sum_{j=1}^{n} \overline{a_{j}}b_{j} + |C|^{2} \sum_{j=1}^{n} |b_{j}|^{2}$$

$$= B^{2}A - 2B|C|^{2} + B|C|^{2}$$

$$= B(AB - |C|^{2})$$

$$\langle 1 \rangle 6. \ B(AB - |C|^{2}) \ge 0$$

$$\langle 1 \rangle 7. \ AB \ge |C|^{2}$$

Proposition 6.26. For any non-zero complex number w, there are exactly two complex numbers z such that $z^2 = w$.

Proof:

- $\langle 1 \rangle 1$. There are at most two complex numbers z such that $z^2 = w$. Proof: Proposition 4.15.
- $\langle 1 \rangle 2$. There are at least two complex numbers z such that $z^2 = w$.

 $\langle 2 \rangle 1$. Let: w = u + iv

 $\langle 2 \rangle 2$. Let: $a = \sqrt{\frac{|w| + u}{2}}$

 $\langle 2 \rangle 3$. Let: $b = \sqrt{\frac{|w|-u}{2}}$

$$\begin{array}{lll} \langle 2 \rangle 4. & {\rm Case:} \ v \geq 0 \\ \langle 3 \rangle 1. & {\rm Let:} \ z = a + ib \\ \langle 3 \rangle 2. & z^2 = w \\ & {\rm Proof:} \end{array}$$

$$z^2 = (a + ib)^2 \\ & = a^2 - b^2 + 2iab \\ & = u + i\sqrt{|w|^2 - u^2} \\ & = u + iv \\ & = w \end{array}$$

$$\langle 3 \rangle 3. & (-z)^2 = w \\ \langle 2 \rangle 5. & {\rm Case:} \ v \leq 0 \\ \langle 3 \rangle 1. & {\rm Let:} \ z = a - ib \\ \langle 3 \rangle 2. & z^2 = w \\ & {\rm Proof:} \end{array}$$

$$z^2 = (a - ib)^2 \\ & = a^2 - b^2 - 2iab \\ & = u - i\sqrt{|w|^2 - u^2} \\ & = u - i|v| \\ & = w \end{array}$$

6.1 Algebraic Numbers

Definition 6.27 (Algebraic). A complex number z is algebraic iff there exist integers a_0, a_1, \ldots, a_n not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0$$
;

otherwise, it is transcendental.

Proposition 6.28. The set of algebraic numbers is countable.

PROOF: There are countably many finite sequences of integers (a_0, a_1, \ldots, a_n) , and for each one, there are only finitely many complex numbers z such that $a_0z^n + a_1z^{n-1} + \cdots + a_n = 0$. \square

Part I Linear Algebra

Vector Spaces

7.1 Convex Sets

Definition 7.1 (Convex). Let $E \subseteq \mathbb{R}^k$. Then E is *convex* iff, for all $\vec{x}, \vec{y} \in E$ and $\lambda \in (0,1)$,

$$\lambda \vec{x} + (1 - \lambda) \vec{y} \in E .$$

Proposition 7.2. Every k-cell is convex.

```
Proof:
```

```
\langle 1 \rangle 1. Let: C = \{ \vec{x} \in \mathbb{R}^k : \forall i.a_i \leq x_i \leq b_i \} be a k-cell.
```

 $\langle 1 \rangle 2$. Let: $\vec{x}, \vec{y} \in C$ and $\lambda \in (0, 1)$.

PROVE: $\lambda \vec{x} + (1 - \lambda) \vec{y} \in C$

 $\langle 1 \rangle 3$. For each i we have $a_i \leq \lambda x_i + (1 - \lambda)y_i \leq b_i$

PROOF: Since $\lambda a_1 + (1 - \lambda)a_i \le \lambda x_i + (1 - \lambda)y_i \le \lambda b_i + (1 - \lambda)b_i$.

Real Inner Product Spaces

Definition 8.1 (Inner Product). Given $\vec{x}, \vec{y} \in \mathbb{R}^k$, define the inner product $\vec{x} \cdot \vec{y}$ by

$$(x_1, \ldots, x_k) \cdot (y_1, \ldots, y_k) = x_1 y_1 + \cdots + x_k y_k$$
.

Definition 8.2 (Norm). Define the *norm* of a vector $\vec{x} \in \mathbb{R}^k$ by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$$
.

Proposition 8.3.

$$\|\vec{x}\| \ge 0$$

PROOF: Immediate from the definition. \Box

Proposition 8.4. *If* $||\vec{x}|| = 0$ *then* $\vec{x} = \vec{0}$.

PROOF: If $\|\vec{x}\| = 0$ then $x_1^2 + \cdots + x_n^2 = 0$ so $x_1 = \cdots = x_n = 0$. \square

Proposition 8.5. For $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^k$,

$$\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| .$$

Proof: Easy. \square

Proposition 8.6. For $\vec{x}, \vec{y} \in \mathbb{R}^k$, we have

$$||\vec{x} \cdot \vec{y}|| \le ||\vec{x}|| ||\vec{y}||$$
.

PROOF: By the Schwarz inequality. \square

Proposition 8.7. For $\vec{x}, \vec{y} \in \mathbb{R}^k$ we have

$$\|\vec{x} + \vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$$
.

Proof:

$$\|\vec{x} + \vec{y}\|^{2} = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y})$$

$$= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}$$

$$\leq \|\vec{x}\|^{2} + 2\|\vec{x}\| \|\vec{y}\| + \|\vec{y}\|^{2} \qquad (Proposition 8.6)$$

$$= (\|\vec{x}\| + \|\vec{y}\|)^{2}$$

Corollary 8.7.1. For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ we have

$$\|\vec{x} - \vec{z}\| \le \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$$
.

8.1 Balls

Definition 8.8 (Closed Ball). Let $\vec{x} \in \mathbb{R}^k$ and r > 0. The *closed ball* with *centre* \vec{x} and *radius* r is

$$\{y \in \mathbb{R}^k : \|y - x\| \le r\} .$$

Proposition 8.9. Every closed ball is convex.

Proof:

 $\langle 1 \rangle 1$. Let: B be the closed ball with center \vec{a} and radius r.

 $\langle 1 \rangle 2$. Let: $\vec{x}, \vec{y} \in B$

 $\langle 1 \rangle 3$. Let: $\lambda \in (0,1)$

 $\langle 1 \rangle 4$. $\lambda \vec{x} + (1 - \lambda) \vec{y} \in B$

Proof:

$$\begin{split} \|\lambda \vec{x} + (1 - \lambda)\vec{y} - \vec{a}\| &= \|\lambda (\vec{x} - \vec{a}) + (1 - \lambda)(\vec{y} - \vec{a})\| \\ &= \lambda \|\vec{x} - \vec{a}\| + (1 - \lambda)\|\vec{y} - \vec{a}\| \\ &\leq \lambda r + (1 - \lambda)r \\ &= r \end{split}$$

Complex Inner Product Spaces

Definition 9.1 (Inner Product). Let V be a complex vector space. An *inner product* on V is a function $\langle \ , \ \rangle : V^2 \to \mathbb{C}$ such that, for all $x,y,z \in V$ and $\alpha \in \mathbb{C}$:

- $\langle y, x \rangle = \overline{\langle x, y \rangle}$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\bullet \ \langle \alpha x, y \rangle = \overline{\alpha} \langle x, y \rangle$
- $\langle x, x \rangle \ge 0$
- If $\langle x, x \rangle = 0$ then x = 0.

An inner product space consists of a complex vector space V and an inner product on V.

Definition 9.2 (Norm). Let V be an inner product space and $x \in V$. The norm of x is

$$||x|| = \sqrt{\langle x, x \rangle}$$
.

Proposition 9.3. An inner product space is a metric space under

$$d(x,y) = ||x - y||.$$

Definition 9.4 (Bounded). Let V_1 and V_2 be inner product spaces and $T:V_1 \to V_2$ a linear transformation. Then T is bounded iff $\{\|T(x)\|: \|x\|=1\}$ is bounded above.

Proposition 9.5. Every linear transformation between finite dimensional inner product spaces is bounded.

Definition 9.6 (Outer Product). Let V be an inner product space and $|\psi\rangle$, $|\phi\rangle \in V$. The *outer product* of $|\psi\rangle$ and $|\phi\rangle$ is

$$|\psi\rangle\langle\phi|:V\to V$$
.

Hilbert Spaces 9.1

Definition 9.7 (Hilbert Space). A Hilbert space is a complete inner product space.

Theorem 9.8 (Completeness Relation). Let \mathcal{H} be a Hilbert space. Let $\{|e_n\rangle\}_{n\in\mathbb{N}}$ be a countable orthonormal basis for H. Then

$$\sum_{n=0}^{\infty} |e_n\rangle \langle e_n| = I .$$

Proof:

 $\begin{array}{l} \text{(1)} & \text{(1)} & \text{(1)} & \text{(1)} & \text{(1)} & \text{(1)} & \text{(2)} & \text{(2)$

$$\sum_{n=0}^{\infty} \langle e_n | \phi \rangle | e_n \rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_m \langle e_n | e_m \rangle | e_n \rangle$$
$$= \sum_{n=0}^{\infty} \alpha_n | e_n \rangle$$
$$= | \psi \rangle$$

Definition 9.9 (Separable). A Hilbert space is *separable* iff it has a countable dense orthonormal basis.

Lie Algebras

Definition 10.1 (Lie Algebra). Let K be a field. A Lie algebra \mathcal{L} over K consists of a vector space \mathcal{L} over K and an operation

$$[\ ,\]:\mathcal{L}^2 \to \mathcal{L}\ ,$$

the *Lie bracket* or *commutator*, such that, for all $x, y, z \in \mathcal{L}$ and $\alpha \in K$:

$$\begin{split} [x+y,z] &= [x,z] + [y,z] \\ [x,y+z] &= [x,y] + [x,z] \\ [\alpha x,y] &= \alpha [x,y] \\ [x,x] &= 0 \\ [x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= 0 \end{split} \tag{Jacobi identity}$$

Lemma 10.2. If K has characteristic 0 then the condition [x, x] = 0 can be replaced with [x, y] = -[y, x].

Proposition 10.3. The commutator is determind by its values on any basis for \mathcal{L} .

Example 10.4. \mathbb{R}^3 with the cross product is a real Lie algebra.

Example 10.5. For any $n \geq 0$, we have GL(n, K) is a Lie algebra over K under

$$[A, B] = AB - BA .$$

Definition 10.6 (Linear Lie Algebra). A *linear Lie algebra* over K is a Lie algebra over K that is a subalgebra of GL(n, K) for some n.

Example 10.7 (Special Linear Algebra). The special Linear algebra $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \text{tr} = 0\}$ is a real linear Lie algebra.

Example 10.8 (Orthogonal Lie Algebra). The *orthogonal Lie algebra* $SO(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A \text{ is skew-symmetric} \}$ is a real linear Lie algebra.

Example 10.9. Let u(n) be the set of all skew-Hermitian $n \times n$ -matrices as a real Lie algebra.

Let $su(n) = u(n) \cap SL(n, \mathbb{R})$.

Proposition 10.10. SU(2) is spanned by the Pauli matrices

$$\sigma_x = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_y = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which satisfy

$$[\sigma_x, \sigma_y] = \sigma_z$$
$$[\sigma_y, \sigma_z] = \sigma_x$$
$$[\sigma_z, \sigma_x] = \sigma_y$$

10.1 Lie Algebar Homomorphisms

Definition 10.11 (Homomorphism). Let L_1 and L_2 be Lie algebras over the same field. A *Lie algebra homomorphism* $\phi: L_1 \to L_2$ is a linear transformation such that

$$\phi([x,y]) = [\phi(x), \phi(y)]$$

for all $x, y \in L_1$.

Lemma 10.12. Every bijective Lie algebra homomorphism is an isomorphism.

Definition 10.13 (Representation). Let L be a real (complex) Lie algebra. A representation of L is a Lie algebra homomorphism $L \to GL(n, \mathbb{R})$ ($GL(n, \mathbb{C})$) for some n.

Example 10.14. The linear transformation $\mathbb{R}^3 \to su(2)$ defined by

$$i \mapsto \sigma_x, j \mapsto \sigma_y, k \mapsto \sigma_z$$

is a representation of \mathbb{R}^3 .

Part II Topology

Metric Spaces

Definition 11.1 (Metric). A *metric* on a set X is a function $d: X^2 \to \mathbb{R}$ such that, for all $x, y, z \in X$:

- $d(x,y) \geq 0$
- d(x,y) = 0 iff x = y
- $\bullet \ d(x,y) = d(y,x)$
- Triangle Inequality $d(x,z) \le d(x,y) + d(y,z)$

A $metric\ space\ X$ consists of a set X and a metric on X.

Example 11.2. \mathbb{R}^k is a metric space under $d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||$. The triangle inequality is Corollary 8.7.1.

Example 11.3. For any set X, the discrete metric on X is defined by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Proposition 11.4. Let (X,d) be a metric space and Y a subset of X. Then $d \upharpoonright Y^2$ is a metric on Y.

Proof: Easy.

11.1 Balls

Definition 11.5 (Open Ball). Let $\vec{x} \in \mathbb{R}^k$ and r > 0. The open ball with centre \vec{x} and radius r is

$$\{y \in \mathbb{R}^k : \|y - x\| < r\} .$$

Proposition 11.6. Every open ball in \mathbb{R}^k is convex.

Proof:

```
\langle 1 \rangle 1. Let: B be the open ball with center \vec{a} and radius r.
```

$$\langle 1 \rangle 2$$
. Let: $\vec{x}, \vec{y} \in B$

$$\langle 1 \rangle 3$$
. Let: $\lambda \in (0,1)$

$$\langle 1 \rangle 4$$
. $\lambda \vec{x} + (1 - \lambda) \vec{y} \in B$

Proof:

$$\begin{split} \|\lambda \vec{x} + (1 - \lambda)\vec{y} - \vec{a}\| &= \|\lambda (\vec{x} - \vec{a}) + (1 - \lambda)(\vec{y} - \vec{a})\| \\ &= \lambda \|\vec{x} - \vec{a}\| + (1 - \lambda)\|\vec{y} - \vec{a}\| \\ &< \lambda r + (1 - \lambda)r \\ &= r \end{split}$$

11.2 Limit Points

Definition 11.7 (Limit Point). Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is a *limit point* of E iff every open ball with centre p contains a point of E other than p.

Proposition 11.8. Let X be a metric space. Let $E \subseteq X$. Let p be a limit point of E. Then every neighbourhood of p contains infinitely many points of E.

Proof:

- $\langle 1 \rangle 1$. Assume: for a contradiction N is a neighbourhood of p that contains only finitely many points q_1, \ldots, q_n of $E \{p\}$.
- $\langle 1 \rangle 2$. Let: $r = \min(q_1, \ldots, q_n)$
- $\langle 1 \rangle 3$. Let: B be the open ball with centre p and radius r.
- $\langle 1 \rangle 4$. B is a neighbourhood of p that contains no points of E other than p.

Corollary 11.8.1. A finite set has no limit points.

Definition 11.9 (Isolated Point). Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *isolated point* of E iff $p \in E$ and p is not a limit point of E.

11.3 Closed Sets

Definition 11.10 (Closed Set). Let X be a metric space. Let $E \subseteq X$. Then E is *closed* iff every limit point of E is a member of E.

11.4 Interior Points

Definition 11.11 (Interior Point). Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *interior point* of E iff there exists an open ball E with centre E such that E is E.

11.5. OPEN SETS

Definition 11.12 (Interior). The *interior* of a set E, denoted E° , is the set of all its interior points.

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Proposition 11.13. The interior of E is the largest open set that is included in E.

```
Proof:
\langle 1 \rangle 1. Let: I be the interior of E.
\langle 1 \rangle 2. I is open.
    \langle 2 \rangle 1. Let: p \in I
    \langle 2 \rangle 2. PICK an open ball B with centre p such that B \subseteq E.
    \langle 2 \rangle 3. \ B \subset I
       \langle 3 \rangle 1. Let: q \in B
       \langle 3 \rangle 2. There exists an open ball B' with centre q such that B' \subseteq B.
       \langle 3 \rangle 3. There exists an open ball B' with centre q such that B' \subseteq E.
       \langle 3 \rangle 4. \ q \in I
\langle 1 \rangle 3. If J is any open set and J \subseteq E then J \subseteq I.
    \langle 2 \rangle 1. Let: J be an open set.
    \langle 2 \rangle 2. Assume: J \subseteq E
    \langle 2 \rangle 3. For all p \in J, there exists an open ball B with centre p such that B \subseteq J.
    \langle 2 \rangle 4. For all p \in J, there exists an open ball B with centre p such that B \subseteq E.
    \langle 2 \rangle 5. \ p \in I
П
```

11.5 Open Sets

Definition 11.14 (Open Sets). Let X be a metric space. Let $E \subseteq X$. Then E is *open* iff every point in E is an interior point of E.

Proposition 11.15. Every open ball is open.

```
Proof:
\langle 1 \rangle 1. Let: B be an open ball with centre c and radius r.
\langle 1 \rangle 2. Let: x \in B
\langle 1 \rangle 3. Let: \epsilon = r - d(x, c)
\langle 1 \rangle 4. Let: B' be the open ball with centre x and radius \epsilon.
        Prove: B' \subseteq B
\langle 1 \rangle 5. Let: y \in B'
\langle 1 \rangle 6. \ d(y,c) < r
   Proof:
                  d(y,c) \le d(y,x) + d(x,c)
                                                                      (Triangle Inequality)
                             < \epsilon + d(x,c)
                                                                                            (\langle 1 \rangle 5)
                                                                                            (\langle 1 \rangle 3)
                             = r
```

Proposition 11.16. A set is open if and only if its complement is closed.

```
Proof:
\langle 1 \rangle 1. Let: E \subseteq X
\langle 1 \rangle 2. If E is open then X - E is closed.
   \langle 2 \rangle 1. Assume: E is open.
   \langle 2 \rangle 2. Let: p be a limit point of X - E.
           PROVE: p \in X - E
   \langle 2 \rangle 3. Assume: for a contradiction p \in E.
   \langle 2 \rangle 4. PICK an open ball B with centre p such that B \subseteq E.
   \langle 2 \rangle5. B contains a point of X - E.
      Proof: \langle 2 \rangle 2
   \langle 2 \rangle 6. Q.E.D.
      PROOF: This contradicts \langle 2 \rangle 4.
\langle 1 \rangle 3. If X - E is closed then E is open.
   \langle 2 \rangle 1. Assume: X - E is closed.
   \langle 2 \rangle 2. Let: p \in E
   \langle 2 \rangle 3. Assume: for a contradiction no open ball with centre p is a subset of
   \langle 2 \rangle 4. Every open ball with centre p intersects X - E.
   \langle 2 \rangle5. p is a limit point of X - E.
   \langle 2 \rangle 6. \ p \in X - E
      Proof: \langle 2 \rangle 1
   \langle 2 \rangle 7. Q.E.D.
      Proof: This contradicts \langle 2 \rangle 2.
Corollary 11.16.1. A set is closed if and only if its complement is open.
Proposition 11.17. The union of a set of open sets is open.
\langle 1 \rangle 1. Let: \mathcal{U} be a set of open sets.
\langle 1 \rangle 2. Let: p \in \bigcup \mathcal{U}
\langle 1 \rangle 3. PICK U \in \mathcal{U} such that p \in U.
\langle 1 \rangle 4. PICK an open ball B with centre p such that B \subseteq U.
\langle 1 \rangle 5. \ B \subseteq \bigcup \mathcal{U}
Corollary 11.17.1. The intersection of a set of closed sets is closed.
Proposition 11.18. The intersection of two open sets is open.
Proof:
\langle 1 \rangle 1. Let: U and V be open.
\langle 1 \rangle 2. Let: p \in U \cap V
\langle 1 \rangle 3. PICK open balls B_1 and B_2 with centre p such that B_1 \subseteq U and B_2 \subseteq V.
\langle 1 \rangle 4. Assume: w.l.o.g. the radius of B_1 is \leq the radius of B_2.
\langle 1 \rangle 5. \ B_1 \subseteq U \cap V
```

Corollary 11.18.1. The union of two closed sets is closed.

Example 11.19. The intersection of a set of open sets is not necessarily open.

For every positive integer n, we have (-1/n, 1/n) is open in \mathbb{R} , but $\bigcap_{n=1}^{\infty} (-1/n, 1/n) =$ $\{0\}$ is not open.

Theorem 11.20. Let X be a metric space. Let $Y \subseteq X$ and $E \subseteq Y$. Then E is open in Y if and only if there exists an open subset G of X such that $E = G \cap Y$.

Proof:

- $\langle 1 \rangle 1$. If E is open in Y then there exists an open subset G of X such that $E = G \cap Y$.
 - $\langle 2 \rangle 1$. Assume: E is open in Y.
 - $\langle 2 \rangle 2$. For $p \in E$, Pick $r_p > 0$ such that the open ball in Y with centre p and radius r_p is included in E.
 - $\langle 2 \rangle 3$. For $p \in E$,

Let: V_p be the open ball in X with centre p and radius r_p .

- $\langle 2 \rangle 4$. Let: $G = \bigcup_{p \in E} V_p$ $\langle 2 \rangle 5$. G is open in Y.

Proof: Proposition 11.17.

- $\langle 2 \rangle 6$. $E = G \cap Y$
 - $\langle 3 \rangle 1. \ E \subseteq G \cap Y$
 - $\langle 4 \rangle 1$. Let: $p \in E$
 - $\langle 4 \rangle 2. \ p \in V_p$
 - $\langle 4 \rangle 3. \ p \in G$
 - $\langle 3 \rangle 2$. $G \cap Y \subseteq E$
 - $\langle 4 \rangle 1$. Let: $x \in G \cap Y$
 - $\langle 4 \rangle 2$. PICK $p \in E$ such that $x \in V_p$
 - $\langle 4 \rangle 3. \ d(x,p) < r_p$
 - $\langle 4 \rangle 4. \ x \in E$
- $\langle 1 \rangle 2$. For any open subset G of X, we have $G \cap Y$ is open in Y.
 - $\langle 2 \rangle 1$. Let: G be an open subset of X.
 - $\langle 2 \rangle 2$. Let: $p \in G \cap Y$
 - $\langle 2 \rangle 3$. PICK r > 0 such that the open ball in X with centre p and radius r is included in G.
- $\langle 2 \rangle 4$. The open ball in Y with centre p and radius r is included in $G \cap Y$.

Perfect Sets 11.6

Definition 11.21 (Perfect Set). Let X be a metric space. Let $E \subseteq X$. Then E is perfect iff E is closed and every point in E is a limit point of E.

11.7 Bounded Sets

Definition 11.22 (Bounded Set). Let X be a metric space. Let $E \subseteq X$. Then E is bounded iff there exists a real number M and $q \in X$ such that, for all $p \in E$, we have d(p,q) < M.

11.8 Dense Sets

Definition 11.23 (Dense Set). Let X be a metric space. Let $E \subseteq X$. Then E is *dense* iff every point of X is either a limit point of E or a point of E, or both.

11.9 Closure

Definition 11.24 (Closure). Let X be a metric space. Let $E \subseteq X$. Then the *closure* of E, denoted \overline{E} , is the union of E and the set of limit points of E.

Proposition 11.25. \overline{E} is the smallest closed set that includes E.

```
Proof:
\langle 1 \rangle 1. \overline{E} is closed.
    \langle 2 \rangle 1. Let: p be a limit point of \overline{E}.
    \langle 2 \rangle 2. Assume: p \notin E
             PROVE: p is a limit point of E.
    \langle 2 \rangle 3. Let: B be the open ball with centre p and radius r.
             Prove: B intersects E.
    \langle 2 \rangle 4. Pick a point q \in B \cap \overline{E}.
    \langle 2 \rangle 5. PICK an open ball B' with centre q such that B' \subseteq B.
    \langle 2 \rangle 6. Pick a point r \in E \cap B'
    \langle 2 \rangle 7. \ r \in E \cap B
\langle 1 \rangle 2. If C is closed and E \subseteq C then \overline{E} \subseteq C.
    \langle 2 \rangle 1. Assume: C is closed.
    \langle 2 \rangle 2. Assume: E \subseteq C
    \langle 2 \rangle 3. Let: p \in \overline{E}
    \langle 2 \rangle 4. Assume: for a contradiction p \notin C
    \langle 2 \rangle 5. p is a limit point of C.
       \langle 3 \rangle 1. Let: B be an open ball with centre p.
       \langle 3 \rangle 2. B intersects E.
       \langle 3 \rangle 3. B intersects C.
       \langle 3 \rangle 4. B intersects C in a point other than p.
           Proof: \langle 2 \rangle 3
    \langle 2 \rangle 6. Q.E.D.
       Proof: This contradicts \langle 2 \rangle 1.
```

Corollary 11.25.1. E is closed if and only if $E = \overline{E}$.

Theorem 11.26. Let E be a nonempty set of real numbers bounded above. Then $\sup E \in \overline{E}$.

Proof:

 $\langle 1 \rangle 1$. Assume: $\sup E \notin E$

PROVE: $\sup E$ is a limit point of E.

- $\langle 1 \rangle 2$. Let: B be an open ball with centre sup E and radius r.
- $\langle 1 \rangle 3$. There exists $x \in E$ such that $x > \sup E r$.
- $\langle 1 \rangle 4$. E intersects B in a point other than p.

Proposition 11.27.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proof:

- $\langle 1 \rangle 1$. $\overline{A} \cup \overline{B}$ is a closed set that includes $A \cup B$.
- $\langle 1 \rangle$ 2. If C is a closed set that includes $A \cup B$ then $\overline{A} \cup \overline{B} \subseteq C$.

Example 11.28. It is not true in general. that $\overline{\bigcup A} = \bigcup_{A \in A} \overline{A}$. In \mathbb{R} , let $A = \{\{1/n\} : n \in \mathbb{Z}^+\}$. Then

$$\overline{\bigcup \mathcal{A}} = \{1/n : n \in \mathbb{Z}^+\} \cup \{0\}$$
$$\bigcup_{A \in \mathcal{A}} \overline{A} = \{1/n : n \in \mathbb{Z}^+\}$$

Proposition 11.29.

$$X - E^{\circ} = \overline{X - E}$$

Proof:

$$p \in X - E^{\circ} \Leftrightarrow p \notin E^{\circ}$$

 $\Leftrightarrow \forall B \text{ an open ball with centre } p.B \not\subseteq E$
 $\Leftrightarrow \forall B \text{ an open ball with centre } p.B \text{ intersects} X - E$
 $\Leftrightarrow p \in \overline{X - E}$

11.10 Compact Sets

Definition 11.30 (Open Cover). Let X be a metric space. Let $E \subseteq X$. An open cover of E is a set \mathcal{U} of open sets such that $E \subseteq \bigcup \mathcal{U}$.

Definition 11.31 (Compact Set). Let X be a metric space. Let $K \subseteq X$. Then K is *compact* iff every open cover of K includes a finite subcover.

Proposition 11.32. Every finite set is compact.

Proof: Easy.

Theorem 11.33. Let X be a metric space. Let $Y \subseteq X$ and $K \subseteq Y$. Then K is compact in Y if and only if K is compact in X.

Proof:

- $\langle 1 \rangle 1$. If K is compact in Y then K is compact in X.
 - $\langle 2 \rangle 1$. Assume: K is compact in Y.
 - $\langle 2 \rangle 2$. Let: \mathcal{U} be an open cover of K in X.
 - $\langle 2 \rangle 3$. $\{ U \cap Y : U \in \mathcal{U} \}$ is an open cover of K in Y.
 - $\langle 2 \rangle 4$. Pick a finite subcover $\{U_1 \cap Y, \dots, U_n \cap Y\}$
 - $\langle 2 \rangle 5$. $\{U_1, \ldots, U_n\}$ is a finite subset of \mathcal{U} that is an open cover of K is X.
- $\langle 1 \rangle 2$. If K is compact in X then K is compact in Y.
 - $\langle 2 \rangle 1$. Assume: K is compact in X.
 - $\langle 2 \rangle 2$. Let: \mathcal{U} be an open cover of K in Y.
 - $\langle 2 \rangle 3$. $\{ U \text{ open in } X : U \cap Y \in \mathcal{U} \}$ is an open cover of K in X.
 - $\langle 2 \rangle 4$. PICK a finite subcover $\{U_1, \ldots, U_n\}$.
- $\langle 2 \rangle$ 5. $\{U_1 \cap Y, \dots, U_n \cap Y\}$ is a subset of \mathcal{U} that is an open cover of E in Y.

Proposition 11.34. Every compact set is closed.

Proof:

- $\langle 1 \rangle 1$. Let: E be compact.
- $\langle 1 \rangle 2$. Let: $p \in X E$

PROVE: There exists an open ball with centre p that is a subset of X-E.

- $\langle 1 \rangle 3$. For all $q \in E$, there exist disjoint open balls B with centre q and B' with centre p.
- $\langle 1 \rangle 4$. The set of open balls B such that there exists a disjoint open ball B' with centre p is an open cover of E.
- $\langle 1 \rangle$ 5. PICK a finite subcover $\{B_1, \ldots, B_n\}$.
- $\langle 1 \rangle 6$. For i = 1, ..., n, PICK an open ball B'_i with centre p such that $B_i \cap B'_i = \emptyset$.
- $\langle 1 \rangle 7$. $B'_1 \cap \cdots \cap B'_n$ is an open ball with centre p that is a subset of X E.

Proposition 11.35. Every closed subset of a compact set is compact.

PROOF

- $\langle 1 \rangle 1$. Let: E be compact and $C \subseteq E$ be closed.
- $\langle 1 \rangle 2$. Let: \mathcal{U} be an open cover of C.
- $\langle 1 \rangle 3$. $\mathcal{U} \cup \{X C\}$ is an open cover of E.
- $\langle 1 \rangle 4$. PICK a finite subcover $\{U_1, \ldots, U_n\}$ or $\{U_1, \ldots, U_n, X C\}$.
- $\langle 1 \rangle 5. \{U_1, \ldots, U_n\} \text{ covers } C.$

Corollary 11.35.1. The intersection of a compact set and a closed set is compact.

Proposition 11.36. Let K be a nonempty set of compact sets. If every nonempty finite subset of K has nonempty intersection, then $\bigcap K$ is nonempty.

Proof:

- $\langle 1 \rangle 1$. Pick $K \in \mathcal{K}$
- $\langle 1 \rangle 2$. Assume: $\bigcap \mathcal{K} = \emptyset$
- $\langle 1 \rangle 3$. $\{X K' : K' \in \mathcal{K}\}$ is an open cover of K.
- $\langle 1 \rangle 4$. PICK a finite subcover $\{X K_1, \dots, X K_n\}$.
- $\langle 1 \rangle$ 5. There exists $p \in K \cap K_1 \cap \cdots \cap K_n$
- $\langle 1 \rangle 6$. Q.E.D.

PROOF: $\langle 1 \rangle 4$ and $\langle 1 \rangle 5$ form a contradiction.

Corollary 11.36.1. Let (K_n) be a sequence of nonempty compact sets such that $K_0 \supseteq K_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.

Theorem 11.37. Let X be a metric space and $E \subseteq X$. Then E is compact if and only if every infinite subset of E has a limit point in E.

- $\langle 1 \rangle 1$. If E is compact then every infinite subset of E has a limit point in E.
 - $\langle 2 \rangle 1$. Assume: E is compact.
 - $\langle 2 \rangle 2$. Let: $A \subseteq E$ be infinite.
 - $\langle 2 \rangle 3$. Assume: for a contradiction E has no limit point in K.
 - $\langle 2 \rangle 4$. For all $p \in K$, there exists an open ball B with centre p such that B does not intersect E outside p.
 - $\langle 2 \rangle 5$. The set of open balls that intersect E in at most one point is an open cover for K.
 - $\langle 2 \rangle 6$. Pick a finite subcover B_1, \ldots, B_n .
 - $\langle 2 \rangle 7$. E has at most n points.
 - $\langle 2 \rangle 8$. Q.E.D.

PROOF: This contradicts the fact that E is finite.

- $\langle 1 \rangle 2$. If every infinite subset of K has a limit point in K then K is compact.
 - $\langle 2 \rangle 1$. Assume: Every infinite subset of K has a limit point in K.
 - $\langle 2 \rangle 2$. Let: \mathcal{U} be an open cover of K.
 - $\langle 2 \rangle 3$. Assume: w.l.o.g. \mathcal{U} is countable.

PROOF: We may replace \mathcal{U} with the set of all open balls B with centres in \mathbb{Q}^2 and rational radius such that there exists $U \in \mathcal{U}$ such that $B \subseteq U$.

- $\langle 2 \rangle 4$. Pick an enumeration $\mathcal{U} = \{G_n : n \in \mathbb{N}\}.$
- $\langle 2 \rangle 5$. For $n \in \mathbb{N}$,

Let:
$$F_n = \bigcup_{i=0}^n G_n$$

Let: $F_n = \bigcup_{i=0}^n G_n$. $\langle 2 \rangle 6$. For all $n \in \mathbb{N}$, we have $K - F_n \neq \emptyset$.

PROOF: Since $\{G_0, \ldots, G_n\}$ does not cover K.

 $\langle 2 \rangle 7. \bigcap_{n=0}^{\infty} F_n = \emptyset$

PROOF: Since $\{G_n : n \in \mathbb{N}\}$ covers K.

- $\langle 2 \rangle 8$. For $n \in \mathbb{N}$, Pick $a_n \in K F_n$
- $\langle 2 \rangle 9$. Let: $E = \{a_n : n \in \mathbb{N}\}$
- $\langle 2 \rangle 10$. E is infinite.
 - $\langle 3 \rangle 1$. Let: $n \in \mathbb{N}$

PROVE: there exists m such that $a_m \notin \{a_0, a_1, \dots, a_n\}$.

```
\langle 3 \rangle 2. For i = 0, \ldots, n, PICK k_i such that a_i \in G_{k_i}.
   \langle 3 \rangle 3. Let: m = \max(k_0, \dots, k_n)
   \langle 3 \rangle 4. Assume: for a contradiction a_m = a_i for some i = 0, \ldots, n
   \langle 3 \rangle 5. \ a_i \in G_{k_i}
   \langle 3 \rangle 6. \ a_i \notin F_m
   \langle 3 \rangle7. Q.E.D.
      PROOF: This is a contradiction since k_i \leq m.
\langle 2 \rangle 11. PICK a limit point l for E in K.
   PROOF: From \langle 2 \rangle 1.
\langle 2 \rangle 12. PICK n such that l \in G_n.
\langle 2 \rangle 13. PICK an open ball B with centre l such that B \subseteq G_n
\langle 2 \rangle 14. B \cap E is infinite.
   Proof: Proposition 11.8.
\langle 2 \rangle 15. Pick m \geq n such that a_m \in B.
\langle 2 \rangle 16. \ a_m \in G_n
\langle 2 \rangle 17. Q.E.D.
   PROOF: This is a contradiction since a_m \notin F_m.
```

Theorem 11.38 (Heine-Borel). Let $E \subseteq \mathbb{R}^k$. Then E is compact if and only if it is closed and bounded.

Proof:

 $\langle 1 \rangle 1$. If E is compact then E is closed.

Proof: Proposition 11.34.

 $\langle 1 \rangle 2$. If E is compact then E is bounded.

PROOF: Otherwise $\{(-N,N)^k : N \in \mathbb{Z}^+\}$ would be an open cover of E with no finite subcover.

- $\langle 1 \rangle 3$. If E is closed and bounded then E is compact.
 - $\langle 2 \rangle 1$. Assume: E is closed and bounded.
 - $\langle 2 \rangle 2$. Pick \vec{c} and M such that $\forall \vec{x} \in E. ||\vec{x} \vec{c}|| < M$.
 - $\langle 2 \rangle 3. \ E \subseteq \prod_{i=1}^{k} [c_i M, c_i + M]$ $\langle 2 \rangle 4. \ E \text{ is compact.}$

Proof: Proposition 11.35.

Corollary 11.38.1 (Weierstrass's Theorem). Every bounded infinite subset of \mathbb{R}^k has a limit point.

PROOF: It is a bounded infinite subset of some k-cell and therefore has a limit point in that k-cell. \square

Example 11.39. It is not true that, in any metric space, a set is compact if and only if it is closed and bounded.

In \mathbb{Q} , the set $\{p \in \mathbb{Q} : 2 < p^2 < 3\}$ is closed and bounded but not compact.

Theorem 11.40. Every nonempty perfect set in \mathbb{R}^k is uncountable.

Proof:

```
\langle 1 \rangle 1. Let: P be a nonempty perfect set in \mathbb{R}^k.
\langle 1 \rangle 2. P is infinite.
   Proof: Corollary 11.8.1.
\langle 1 \rangle 3. Assume: for a contradiction P is countable.
\langle 1 \rangle 4. PICK an enumeration P = \{x_n : n \in \mathbb{N}\}.
\langle 1 \rangle5. Pick a sequence (V_n) of open balls such that, for all n, we have \overline{V_{n+1}} \subseteq V_n
         and x_n \notin \overline{V_{n+1}} and V_n \cap P \neq \emptyset
   \langle 2 \rangle 1. Assume: as induction hypothesis we have picked V_0, \ldots, V_{n-1} that
                            satisfy these conditions.
   \langle 2 \rangle 2. Pick p \in P \cap V_n such that p \neq x_n
      PROOF: We cannot have P \cap V_n = \{x_n\} because then V_n would be a
       neighbourhood of x_n that only intersects P at x_n.
   \langle 2 \rangle 3. PICK an open ball B with centre p such that B \subseteq V_n \cap P - \{x_n\}
   \langle 2 \rangle 4. Let: V_{n+1} be the open ball with centre p and half the radius of B.
   \langle 2 \rangle 5. \ \overline{V_{n+1}} \subseteq V_n
       PROOF: Since \overline{V_{n+1}} \subseteq B \subseteq V_n.
   \langle 2 \rangle 6. \ x_n \notin \overline{V_{n+1}}
      PROOF: Since \overline{V_{n+1}} \subseteq B \subseteq P - \{x_n\}.
   \langle 2 \rangle 7. \ V_{n+1} \cap P \neq \emptyset
       PROOF: Since p \in V_{n+1} \cap P.
\langle 1 \rangle 6. For n \in \mathbb{N},
        Let: K_n = \overline{V_n} \cap P.
\langle 1 \rangle 7. For all n \in \mathbb{N}, K_n is compact.
   PROOF: By the Heine-Borel Theorem.
\langle 1 \rangle 8. \bigcap_{n=0}^{\infty} K_n \cap P = \emptyset
   PROOF: Since for each n we have x_n \notin K_{n+1}.
\langle 1 \rangle 9. \bigcap_{n=0}^{\infty} K_n = \emptyset
PROOF: Since \bigcap_{n=0}^{\infty} K_n \subseteq P.
\langle 1 \rangle 10. Q.E.D.
   Proof: This contradicts Proposition 11.36.
```

Corollary 11.40.1. For any $a, b \in \mathbb{R}$ with a < b, the closed interval [a, b] is uncountable.

Corollary 11.40.2. \mathbb{R} is uncountable.

Corollary 11.40.3. The set of transcendental numbers is uncountable.

PROOF: Since the set of algebraic numbers is countable.

Example 11.41. The Cantor set is a perfect set in \mathbb{R} that does not include any open interval.

Proof:

 $\langle 1 \rangle 1$. Let: (E_n) be the sequence of unions of closed intervals from the definition of the Cantor set, and C be the Cantor set.

 $\langle 1 \rangle 2. \ C \neq \emptyset$

```
PROOF: Since 0 \in C.
```

 $\langle 1 \rangle 3$. C is closed.

PROOF: Each E_n is closed and C is their intersection.

- $\langle 1 \rangle 4$. Every point of C is a limit point of C.
 - $\langle 2 \rangle 1$. Let: $p \in C$
 - $\langle 2 \rangle 2$. Let: B be an open ball with centre p and radius r.
 - $\langle 2 \rangle 3$. Pick n such that each of the intervals that make up E_n has length < r/2.
 - $\langle 2 \rangle 4$. Let: I be the interval in E_n that contains p.
 - $\langle 2 \rangle 5. \ I \subseteq B$
 - $\langle 2 \rangle 6$. The endpoint of I that is not p is in $P \cap B$.
- $\langle 1 \rangle 5$. C does not include any open interval.
 - $\langle 2 \rangle 1$. Let: (α, β) be any open interval.
 - $\langle 2 \rangle 2$. Pick m such that $3^{-m} < (\beta \alpha)/6$
 - $\langle 2 \rangle$ 3. PICK k such that $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq (\alpha, \beta)$

 - $\langle 2 \rangle 4. \quad \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq P$ $\langle 2 \rangle 5. \quad \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \cap E_m = \emptyset$
 - $\langle 2 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

Corollary 11.41.1. The Cantor set is uncountable.

Connected Sets 11.11

Definition 11.42 (Separated). Let X be a metric space. Let $A, B \subseteq X$. Then A and B are separated iff $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Proposition 11.43. Any two disjoint open sets are separated.

Proof:

- $\langle 1 \rangle 1$. Let: A and B be disjoint open sets.
- $\langle 1 \rangle 2$. Assume: for a contradiction $p \in \overline{A} \cap B$.
- $\langle 1 \rangle 3$. B is a neighbourhood of p.
- $\langle 1 \rangle 4$. B intersects A.

Definition 11.44 (Connected). Let X be a metric space. Let $E \subseteq X$. Then E is connected iff E is not the union of two nonempty separated sets.

Theorem 11.45. A subset E of the real line is connected if and only if it is convex.

Proof:

- $\langle 1 \rangle 1$. If E is connected then E is convex.
 - $\langle 2 \rangle 1$. Assume: E is connected.
 - $\langle 2 \rangle 2$. Let: $x, y \in E$

```
\langle 2 \rangle 3. Let: z \in (x,y)
    \langle 2 \rangle 4. \ z \in E
       PROOF: Otherwise E \cap (-\infty, z) and E \cap (z, +\infty) would be a separation of
\langle 1 \rangle 2. If E is convex then E is connected.
    \langle 2 \rangle 1. Assume: E is convex.
    \langle 2 \rangle 2. Assume: for a contradiction E = A \cup B where A and B are nonempty
                             and separated.
    \langle 2 \rangle 3. Pick a \in A and b \in B.
    \langle 2 \rangle 4. Assume: w.l.o.g. a < b
    \langle 2 \rangle 5. Let: z = \sup(A \cap [a, b])
    \langle 2 \rangle 6. \ z \in \overline{A}
    \langle 2 \rangle 7. \ z \notin B
    \langle 2 \rangle 8. \ z < b
    \langle 2 \rangle 9. Case: z \in A
        \langle 3 \rangle 1. \ z \notin \overline{B}
        \langle 3 \rangle 2. Pick z_1 \in (z, b) such that z_1 \notin B
        \langle 3 \rangle 3. a < z_1 < b
        \langle 3 \rangle 4. \ z_1 \notin E
           PROOF: We have z_1 \notin A from \langle 2 \rangle 5 since z_1 \in [a,b] and z_1 > z, and
           z_1 \notin B \text{ from } \langle 3 \rangle 2.
        \langle 3 \rangle 5. Q.E.D.
           PROOF: This contradicts \langle 2 \rangle 1.
    \langle 2 \rangle 10. Case: z \notin A
        PROOF: Then a < z < b and z \notin E contradicting \langle 2 \rangle 1.
```

Proposition 11.46. Every connected metric space with more than one point is uncountable.

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Proof:
```

```
\langle 1 \rangle 1. Let: X be a connected metric space with more than one points.
```

- $\langle 1 \rangle 2$. Pick distinct points $p, q \in X$.
- $\langle 1 \rangle 3$. Let: $\epsilon = d(p,q)$
- $\langle 1 \rangle 4$. For every $r \in (0, \epsilon)$, there exists a point $x \in X$ such that d(p, x) = r. PROOF: Otherwise $\{x \in X : d(p,x) < r\}$ and $\{x \in X : d(p,x) > r\}$ would form a separation of X.

Proposition 11.47. The closure of a connected set is connected.

Proof:

- $\langle 1 \rangle 1$. Let: X be a metric space.
- $\langle 1 \rangle 2$. Let: E be a connected subspace of X.
- $\langle 1 \rangle 3$. Assume: for a contradiction A and B form a separation of \overline{E} PROVE: $A \cap E$ and $B \cap E$ form a separation of E.
- $\langle 1 \rangle 4$. $A \cap E \neq \emptyset$

```
 \begin{array}{l} \langle 2 \rangle 1. \  \, \text{Assume: for a contradiction } A \cap E = \emptyset \\ \langle 2 \rangle 2. \  \, E \subseteq B \\ \langle 2 \rangle 3. \  \, \overline{E} \subseteq \overline{B} \\ \langle 2 \rangle 4. \  \, A \subseteq \overline{B} \\ \langle 2 \rangle 5. \  \, A \cap \overline{B} = A \neq \emptyset \\ \langle 2 \rangle 6. \  \, \text{Q.E.D.} \\ \text{PROOF: This contradicts } \langle 1 \rangle 3. \\ \langle 1 \rangle 5. \  \, B \cap E \neq \emptyset \\ \text{PROOF: Similar.} \\ \langle 1 \rangle 6. \  \, \overline{A \cap E} \cap B \cap E = \emptyset \\ \text{PROOF: Since } \overline{A \cap E} \cap B \cap E \subseteq \overline{A} \cap B. \\ \langle 1 \rangle 7. \  \, A \cap E \cap \overline{B} \cap \overline{E} = \emptyset \\ \text{PROOF: Similar.} \\ \end{array}
```

Example 11.48. The interior of a connected set is not necessarily connected. Two touching discs in \mathbb{R}^2 form a connected set but the interior is disconnected.

Proposition 11.49. Every convex set in \mathbb{R}^k is connected.

```
PROOF: \langle 1 \rangle 1. Let: E be a convex set in \mathbb{R}^k. \langle 1 \rangle 2. Assume: for a contradiction A and B form a separation of E. \langle 1 \rangle 3. Pick \vec{a} \in A and \vec{b} \in B. \langle 1 \rangle 4. Define p:[0,1] \to \mathbb{R}^k by p(t)=(1-t)\vec{a}+t\vec{b}. \langle 1 \rangle 5. p^{-1}(A) and p^{-1}(B) are separated sets in \mathbb{R}. \langle 1 \rangle 6. Pick x \in [0,1] such that x \notin p^{-1}(A) and x \notin p^{-1}(B). Proof: There exists such an x since [0,1] is connected. \langle 1 \rangle 7. p(x) \in E Proof: Since E is convex. \langle 1 \rangle 8. p(x) \notin A \cup B \langle 1 \rangle 9. Q.E.D. Proof: This contradicts \langle 1 \rangle 2.
```

11.12 Separable Spaces

Definition 11.50 (Separable). A metric space is *separable* iff it has a countable dense subset.

Example 11.51. \mathbb{R}^k is separable since \mathbb{Q}^k is dense.

Proposition 11.52. Every compact metric space is separable.

Proof:

 $\langle 1 \rangle 1$. Let: X be a compact metric space.

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\langle 1 \rangle 2. For n \in \mathbb{Z}^+, pick finitely many points a_{n1}, \ldots, a_{nr_n} such that \{B(a_{ni}, 1/n) : a_{nr_n}\}
          1 \le i \le r_n covers X.
    PROOF: Since \{B(x, 1/n) : x \in X\} covers X.
\langle 1 \rangle 3. \{a_{ni} : n \in \mathbb{Z}^+, 1 \leq i \leq r_n\} is dense.
    \langle 2 \rangle 1. Let: U be an open set and p \in U.
    \langle 2 \rangle 2. Pick \epsilon > 0 such that B(p, \epsilon) \subseteq U.
    \langle 2 \rangle 3. PICK n such that 1/n < \epsilon.
    \langle 2 \rangle 4. PICK i such that p \in B(a_{ni}, 1/n)
    \langle 2 \rangle 5. \ a_{ni} \in U
```

11.13 Bases

Definition 11.53 (Basis). A basis for a metric space X is a set \mathcal{B} of open sets such that, for every open set U and point $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.

Proposition 11.54. Every separable metric space has a countable basis.

```
\langle 1 \rangle 1. Let: X be a separable metric space.
\langle 1 \rangle 2. PICK a countable dense set D in X.
\langle 1 \rangle 3. Let: \mathcal{B} = \{ B(p, \epsilon) : p \in D, \epsilon \in \mathbb{Q}^+ \}
         Prove: \mathcal{B} is a basis.
\langle 1 \rangle 4. Let: U be an open set in X and p \in U
\langle 1 \rangle 5. Pick \epsilon > 0 such that B(p, \epsilon) \subseteq U
\langle 1 \rangle 6. Pick q \in B(p, \epsilon) \cap D
\langle 1 \rangle 7. PICK a rational \delta such that d(p,q) < \delta < \epsilon.
\langle 1 \rangle 8. \ B(q, \delta) \in \mathcal{B} \text{ and } B(q, \delta) \subseteq U.
```

11.14**Condensation Points**

Definition 11.55 (Condensation Point). Let X be a metric space, $p \in X$ and $E\subseteq X$. Then p is a condensation point of E iff every neighbourhood of p contains uncountably many points in E.

Proposition 11.56. Let X be a metric space. Let $E \subseteq X$. Let P be the set of condensation points of E. Then P is perfect.

Proof:

- $\langle 1 \rangle 1$. P is closed.
 - $\langle 2 \rangle 1$. Let: $p \in X P$
 - $\langle 2 \rangle 2$. PICK a neighbourhood U of p that contains only countably many points
 - $\langle 2 \rangle 3$. For every $x \in U$, we have that U is a neighbourhood of x that contains only countably many points of E.

```
\langle 2 \rangle 4. p \in U \subseteq X - P
\langle 1 \rangle 2. Every point in P is a limit point of P.
PROOF: Immediate from definitions.
```

Proposition 11.57. Let X be a metric space with a countable basis. Let $E \subseteq X$ be uncountable. Let P be the set of condensation points of E. Then E - P is countable.

Proof:

- $\langle 1 \rangle 1$. PICK a countable basis \mathcal{B} for X.
- $\langle 1 \rangle 2$. Let: $W = \bigcup \{ B \in \mathcal{B} : E \cap B \text{ is countable} \}$
- $\langle 1 \rangle 3. \ P = X W$
 - $\langle 2 \rangle 1. \ P \subseteq X W$
 - $\langle 3 \rangle 1$. Assume: for a contradiction $p \in P \cap W$
 - $\langle 3 \rangle 2$. PICK $B \in \mathcal{B}$ such that $p \in B$ and $E \cap B$ is countable.
 - $\langle 3 \rangle 3$. $E \cap B$ is uncountable.
 - $\langle 3 \rangle 4$. Q.E.D.

PROOF: This is a contradiction.

- $\langle 2 \rangle 2$. $X W \subseteq P$
 - $\langle 3 \rangle 1$. Let: $p \in X W$
 - $\langle 3 \rangle 2$. Let: *U* be a neighbourhood of *p*.
 - $\langle 3 \rangle 3$. Pick $B \in \mathcal{B}$ such that $p \in B \subseteq U$.
 - $\langle 3 \rangle 4$. $E \cap B$ is uncountable.

PROOF: Since $p \notin W$.

 $\langle 3 \rangle 5$. $E \cap W$ is uncountable.

- $\langle 1 \rangle 4$. $E P = E \cap W$
- $\langle 1 \rangle 5$. E P is countable.

Corollary 11.57.1. Every closed subset of a metric space with a countable basis is the union of a perfect set and a countable set.

Proof:

- $\langle 1 \rangle 1$. Let: X be a metric space with a countable basis.
- $\langle 1 \rangle 2$. Let: E be a closed subset of X.
- $\langle 1 \rangle 3$. Let: P be the set of condensation points of E.
- $\langle 1 \rangle 4$. E P is countable.

Proof: Proposition 11.57.

- $\langle 1 \rangle 5$. $P \cap E$ is perfect.
 - $\langle 2 \rangle 1$. $P \cap E$ is closed.

Proof: Proposition 11.56.

- $\langle 2 \rangle 2$. Every point in $P \cap E$ is a limit point of $P \cap E$.
 - $\langle 3 \rangle 1$. Let: $l \in P \cap E$
 - $\langle 3 \rangle 2$. Let: U be a neighbourhood of l.
 - $\langle 3 \rangle 3$. Pick $x \in P \cap U$
 - $\langle 3 \rangle 4$. *U* is a neighbourhood of *x*.

- $\langle 3 \rangle 5$. U contains uncountably many points of E.
- $\langle 3 \rangle 6$. U intersects $P \cap E$

PROOF: It cannot be that every point in U and E is not in P since E-P is countable.

Corollary 11.57.2. Let X be a metric space with a countable basis. Then every countable set in X has an isolated point.

Convergence

Definition 12.1 (Converge). Let X be a metric space. Let (p_n) be a sequence in X and $l \in X$. Then we say (p_n) converges to the *limit* l, and write

$$p_n \to l \text{ as } n \to \infty$$
,

iff for every $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon$.

We say (p_n) diverges iff it does not converge to any limit.

Proposition 12.2. A sequence has at most one limit.

Proof:

- $\langle 1 \rangle 1$. Assume: $p_n \to l$ and $p_n \to m$ as $n \to \infty$.
- $\langle 1 \rangle 2$. Assume: for a contradiction $l \neq m$.
- $\langle 1 \rangle 3$. Let: $\epsilon = d(l,m)/2$
- $\langle 1 \rangle 4$. There exists N such that $\forall n \geq N. d(p_n, l) < \epsilon$ and $d(p_n, m) < \epsilon$
- $\langle 1 \rangle 5. \ d(l,m) < 2\epsilon$
- $\langle 1 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

Proposition 12.3. Every convergent sequence is bounded.

Proof:

- $\langle 1 \rangle 1$. Let: $p_n \to l$ as $n \to \infty$
- $\langle 1 \rangle 2$. PICK N such that $\forall n \geq N.d(p_n, l) < 1$
- $\langle 1 \rangle 3$. Let: $M = \max(d(p_0, l), \dots, d(p_{N-1}, l), 1)$
- $\langle 1 \rangle 4$. For all n, we have $d(p_n, l) \leq M$.

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Proposition 12.4. If l is a limit point of E, then there exists a sequence in E that converges to l.

Proof:

 $\langle 1 \rangle 1$. For $n \in \mathbb{Z}^+$, PICK a point $a_n \in E$ such that $d(a_n, l) < 1/n$. PROOF: Since B(l, 1/n) intersects E.

$$\langle 1 \rangle 2$$
. $a_n \to l$ as $n \to \infty$.

Proposition 12.5. Assume $s_n \to s$ and $t_n \to t$ in \mathbb{C} . Then $s_n + t_n \to s + t$.

Proof:

- $\langle 1 \rangle 1$. Let: $\epsilon > 0$
- $\langle 1 \rangle 2$. PICK N such that, for all $n \geq N$, we have $d(s_n, s) < \epsilon/2$ and $d(t_n, t) < \epsilon/2$.
- $\langle 1 \rangle 3$. For all $n \geq N$ we have $d(s_n + t_n, s + t) < \epsilon$.

Lemma 12.6. If $s_n \to s$ as $n \to \infty$ in \mathbb{C} , and $c \in \mathbb{C}$, then $cs_n \to cs$ as $n \to \infty$.

Proof:

- $\langle 1 \rangle 1$. Let: $\epsilon > 0$
- $\langle 1 \rangle 2$. Assume: w.l.o.g. $c \neq 0$
- $\langle 1 \rangle 3$. PICK N such that $\forall n \geq N . |s_n s| < \epsilon / |c|$.

$$\langle 1 \rangle 4. \ \forall n \ge N. |cs_n - cs| < \epsilon$$

Proposition 12.7. If $s_n \to s$ and $t_n \to t$ in \mathbb{C} then $s_n t_n \to st$.

Proof:

- $\langle 1 \rangle 1$. $(s_n s)(t_n t) \to 0$ as $n \to \infty$
 - $\langle 2 \rangle 1$. Let: $\epsilon > 0$
 - $\langle 2 \rangle 2$. PICK N such that, for all $n \geq N$, we have $|s_n s| < \sqrt{\epsilon}$ and $|t_n t| < \sqrt{\epsilon}$.
 - $\langle 2 \rangle 3$. For all $n \geq N$ we have $|(s_n s)(t_n t)| < \epsilon$
- $\langle 1 \rangle 2$. $s_n t_n st \to 0$ as $n \to \infty$

PROOF:

$$s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$$

$$\to 0$$
 as $n \to \infty$

Part III More Algebra

Lie Groups

Definition 13.1 (Lie Group). A *Lie group* G is a group G that is also an analytic differentiable manifold such that the group operation and inverse operation are analytic.

A $homomorphism\ of\ Lie\ groups$ is a group homomorphism that is an analytic function.

Lemma 13.2. Every bijective Lie group homomorphism is an isomorphism.

Definition 13.3 (Unitary Group). The *unitary group* U(n) is the Lie group of all $n \times n$ unitary matrices.

Definition 13.4 (Special Unitary Group). The *special unitary group* SU(n) is the Lie group of all $n \times n$ unitary matrices with determinant 1.

Definition 13.5 (Lie Subgroup). Let G be a Lie group. A Lie subgroup of G is a subgroup that is also an analytic submanifold of G.

Example 13.6. U(n) and SU(n) are Lie subgroups of $GL(n, \mathbb{C})$.