# Mathematics

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September 13, 2023

# Contents

# Chapter 1

# Primitive Terms and Axioms

## 1.1 Primitive Terms

Let there be *sets*. We write $A : \mathrm{Set}$ for: $A$ is a set.

For any set $A$, let there be *elements* of $A$. We write $a : \mathrm{El}\,(A)$ for: $a$ is an element of $A$.

For any sets $A$ and $B$, let there be *relations* between $A$ and $B$. We write $R : A \looparrowright B$ for: $R$ is a relation between $A$ and $B$.

For any set $A$ and elements $a, b : \mathrm{El}\,(A)$, let there be a proposition that $a$ and $b$ are *equal*, $a = b$.

For any relation $R : A \looparrowright B$ and elements $a : \mathrm{El}\,(A)$, $b : \mathrm{El}\,(B)$, let there be a proposition $aRb$, that $R$ *holds* between $a$ and $b$.

## 1.2 Definitions Used in the Axioms

**Definition 1.1** (Function)**.** Let $A$ and $B$ be sets and $F : A \looparrowright B$. Then $F$ is a *function* from $A$ to $B$, $F : A \to B$, if and only if, for all $x \in A$, there exists a unique $y \in B$ such that $xFy$. We denote this unique $y$ by $F(x)$.

**Definition 1.2** (Injective)**.** A function $f : A \to B$ is *injective* iff, for all $x, y : \mathrm{El}\,(A)$, if $f(x) = f(y)$ then $x = y$.

**Definition 1.3** (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for all $y : \mathrm{El}\,(B)$, there exists $x : \mathrm{El}\,(A)$ such that $f(x) = y$.

**Definition 1.4** (Bijective)**.** A function $f : A \to B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets $A$ and $B$ are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set $X$ such that $P(X)$, and that any two subsets that satisfy $P$ are bijective, then we may introduce a constant $C$ and define "Let $C$ be the set such that $P(C)$".

## 1.3 Axioms

**Axiom Schema 1.5** (Comprehension)**.** *For any formula $\phi[X, Y, x, y]$ where $X$ and $Y$ are set variables and $x \in X$ and $y \in Y$, the following is an axiom:*

*For any sets $A$ and $B$, there exists a relation $R$ such that, for all $a \in A$ and $b \in B$, we have $aRb$ if and only if $\phi[A, B, a, b]$.*

**Axiom 1.6** (Tabulations)**.** *For any sets $A$ and $B$ and relation $R : A \nrightarrow B$, there exists a set $|R|$, a* tabulation *of $R$, and functions $p : |R| \to A$ and $q : |R| \to B$ such that:*

- *For all $x : \mathrm{El}\,(A)$ and $y : \mathrm{El}\,(B)$, we have $xRy$ if and only if there exists $r : \mathrm{El}\,(|R|)$ such that $p(r) = x$ and $q(r) = y$*
- *For all $r, s : \mathrm{El}\,(|R|)$, if $p(r) = p(s)$ and $q(r) = q(s)$ then $r = s$.*

**Axiom 1.7** (Infinity)**.** *There exists a set $\mathbb{N}$, an element $0 : \mathrm{El}\,(\mathbb{N})$, and a function $s : \mathbb{N} \to \mathbb{N}$ such that:*

- $\forall n : \mathrm{El}\,(\mathbb{N})\,.s(n) \neq 0$
- $\forall m, n : \mathrm{El}\,(\mathbb{N})\,.s(m) = s(n) \Rightarrow m = n.$

## 1.4 Consequences of the Axioms

### 1.4.1 The Empty Set

**Theorem 1.8.** *There exists a set which has no elements.*

PROOF:
⟨1⟩1. PICK a set $A$
PROOF: By the Axiom of Infinity, a set exists.
⟨1⟩2. LET: $R : A \nrightarrow A$ be the relation such that, for all $x, y \in A$, we have $\neg(xRy)$
PROOF: By the Axiom of Comprehension.
⟨1⟩3. LET: $|R|$ be the tabulation of $R$ with projections $p, q : |R| \to A$.
PROVE: $|R|$ has no elements.
PROOF: By the Axiom of Tabulations.
⟨1⟩4. ASSUME: for a contradiction $r : \mathrm{El}\,(|R|)$
⟨1⟩5. $p(r)Rq(r)$
⟨1⟩6. Q.E.D.
PROOF: This contradicts ⟨1⟩2.
□

**Theorem 1.9.** *If $E$ and $E'$ have no elements then $E \approx E'$.*

PROOF:
$\langle 1 \rangle 1$. LET: $E$ and $E'$ have no elements.
$\langle 1 \rangle 2$. LET: $F : E \nrightarrow E'$ be the relation such that, for all $x : \mathrm{El}(E)$ and $y : \mathrm{El}(E')$, we have $xFy$.
PROOF: Axiom of Comprehension.
$\langle 1 \rangle 3$. $F$ is a function.
PROOF: Vacuously, for all $x : \mathrm{El}(E)$, there exists a unique $y : \mathrm{El}(E')$ such that $xFy$.
$\langle 1 \rangle 4$. $F$ is injective.
PROOF: Vacuously, for all $x, y : \mathrm{El}(E)$, if $F(x) = F(y)$ then $x = y$.
$\langle 1 \rangle 5$. $F$ is surjective.
PROOF: Vacuously, for all $y : \mathrm{El}(E)$, there exists $x : \mathrm{El}(E)$ such that $F(x) = y$.
□

**Definition 1.10** (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

### 1.4.2 The Singleton

**Theorem 1.11.** *There exists a set that has exactly one element.*

PROOF:
$\langle 1 \rangle 1$. PICK a set $A$ that has an element.
PROOF: By the Axiom of Infinity, there exists a set that has an element.
$\langle 1 \rangle 2$. PICK $a : \mathrm{El}(A)$
$\langle 1 \rangle 3$. LET: $R : A \nrightarrow A$ be the relation such that, for all $x, y : \mathrm{El}(A)$, we have $xRy$ if and only if $x = y = a$.
PROOF: By the Axiom of Comprehension.
$\langle 1 \rangle 4$. LET: $|R|$ be the tabulation of $R$ with projections $p, q : |R| \to A$.
PROVE: $|R|$ has exactly one element.
PROOF: By the Axiom of Tabulations.
$\langle 1 \rangle 5$. LET: $r : \mathrm{El}(|R|)$ be the element such that $p(r) = q(r) = a$
PROOF: Since $aRa$ by $\langle 1 \rangle 3$.
$\langle 1 \rangle 6$. LET: $s : \mathrm{El}(|R|)$
PROVE: $s = r$
$\langle 1 \rangle 7$. $p(s)Rq(s)$
PROOF: By the Axiom of Tabulations.
$\langle 1 \rangle 8$. $p(s) = q(s) = a$
PROOF: By $\langle 1 \rangle 3$.
$\langle 1 \rangle 9$. $p(s) = p(r)$ and $q(s) = q(r)$
PROOF: By $\langle 1 \rangle 5$.
$\langle 1 \rangle 10$. $s = r$
PROOF: By the Axiom of Tabulations.
□

**Theorem 1.12.** *If $A$ and $B$ both have exactly one element then $A \approx B$.*

PROOF:
⟨1⟩1. LET: $A$ and $B$ both have exactly one element.
⟨1⟩2. LET: $F : A \looparrowright B$ be the relation such that, for all $x : \mathrm{El}(A)$ and $y : \mathrm{El}(B)$, we have $xFy$.
⟨1⟩3. $F$ is a function.
PROOF: If $xFy$ and $xFy'$ then $y = y'$ because $B$ has only one element.
⟨1⟩4. $F$ is injective.
PROOF: If $F(x) = F(x')$ then $x = x'$ because $A$ has only one element.
⟨1⟩5. $F$ is surjective.
⟨2⟩1. LET: $y : \mathrm{El}(B)$
⟨2⟩2. LET: $x$ be the element of $A$.
⟨2⟩3. $F(x) = y$
□

**Definition 1.13** (Singleton)**.** Let 1 be the set that has exactly one element. Let $*$ be its element.

### 1.4.3 Subsets

**Definition 1.14** (Subset)**.** A *subset* of a set $A$ is a relation $1 \looparrowright S$.

Given $S : 1 \looparrowright S$ and $a : \mathrm{El}(A)$, we write $a \in S$ for $*Sa$.