Mathematics

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Contents

Chapter 1

Sets and Functions

1.1 Primitive Terms

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B, and call A the *domain* of f and B the *codomain*.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $gf = g \circ f : A \to C$, the *composite* of f and g.

1.2 Axioms

1.2.1 Associativity

Axiom 1.1 (Associativity)**.** *For any functions $f : A \to B$, $g : B \to C$ and $h : C \to D$ we have*

$$h(gf) = (hg)f \enspace .$$

Thanks to this axiom, we shall often omit parentheses when writing the composite of a sequence of functions.

1.2.2 Identity Functions

Definition 1.2 (Identity Function)**.** For any set A, an *identity function* on A is a function $i : A \to A$ such that:

- for every set B and function $f : A \to B$ we have $fi = f$;
- for every set B and function $f : B \to A$ we have $if = f$.

Axiom 1.3 (Identity Functions)**.** *Every set has an identity function.*

Proposition 1.4. *Every set has a unique identity function.*

PROOF:

⟨1⟩1. LET: A be a set.
⟨1⟩2. A has an identity function.
  PROOF: Axiom of Identity Functions
⟨1⟩3. For any identity functions i and j on A we have $i = j$.
  ⟨2⟩1. LET: i and j be identity functions on A.
  ⟨2⟩2. $i = j$
    PROOF: $i = ij = j$

□

Definition 1.5 (Identity Function)**.** For any set A, let id_A be the identity function on A.

Definition 1.6 (Retraction, Section)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $rs = \mathrm{id}_B$.

Proposition 1.7. *Let $f : A \to B$ and $g, h : B \to A$. If g is a retraction of f and h is a section of f then $g = h$.*

PROOF:

$$\begin{aligned} g &= g\mathrm{id}_B \\ &= gfh \\ &= \mathrm{id}_A h \\ &= h \end{aligned}$$

□

Definition 1.8 (Bijection)**.** Let $f : A \to B$ be a function. We say f is a *bijection*, and write $f : A \approx B$, iff there exists a function $f^{-1} : B \to A$, an *inverse* to f, such that $f^{-1}f = \mathrm{id}_A$ and $ff^{-1} = \mathrm{id}_B$.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

Proposition 1.9. *The inverse to a bijection is unique.*

PROOF: From Proposition 1.7. □

1.2.3 The Terminal Set

Definition 1.10 (Terminal Set)**.** A set T is *terminal* iff, for every set X, there exists exactly one function $X \to T$.

Axiom 1.11 (Terminal Set)**.** *There exists a terminal set.*

Proposition 1.12. *If T and T' are terminal sets then there exists a unique bijection $T \approx T'$.*

PROOF:
⟨1⟩1. LET: i be the unique function $T \to T'$
⟨1⟩2. LET: i^{-1} be the unique function $T' \to T$
⟨1⟩3. $ii^{-1} = \mathrm{id}_{T'}$
  PROOF: Since there is only one function $T' \to T'$.

⟨1⟩4. $i^{-1}i = \mathrm{id}_T$
  PROOF: Since there is only one function $T \to T$.
□

Definition 1.13. Let 1 be the terminal set. For any set A, let $!_A$ be the function $A \to 1$.

Definition 1.14 (Element)**.** For any set A, an *element* of A is a function $1 \to a$. We write $a \in A$ for $a : 1 \to A$.

Given $f : A \to B$ and $a \in A$, we write $f(a)$ for $fa : 1 \to B$.

Axiom 1.15 (Extensionality)**.** *Let A and B be sets and $f, g : A \to B$. If $\forall a \in A. f(a) = g(a)$ then $f = g$.*

1.2.4 The Empty Set

Axiom 1.16 (Empty Set)**.** *There exists a set that has no elements.*

1.2.5 Products

Definition 1.17 (Product)**.** Let A and B be sets. A *product* of A and B consists of a set $A \times B$ and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, the *projections*, such that, for any set X and functions $f : X \to A$, $g : X \to B$, there exists a unique function $\langle f, g \rangle : X \to A \times B$ such that

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g \ .$$

Axiom 1.18 (Products)**.** *Any two sets have a product.*

Proposition 1.19. *If P and Q are products of A and B with projections $p_1 : P \to A$, $p_2 : P \to B$, $q_1 : Q \to A$ and $q_2 : Q \to B$, then there exists a unique isomorphism $i : P \approx Q$ such that $q_1 i = p_1$ and $q_2 i = p_2$.*

PROOF:
⟨1⟩1. LET: $i : P \to Q$ be the unique function such that $p_1 i = q_1$ and $p_2 i = q_2$.
⟨1⟩2. LET: $i^{-1} : Q \to P$ be the unique function such that $q_1 i^{-1} = p_1$ and $q_2 i^{-1} = p_2$.
⟨1⟩3. $i^{-1}i = \mathrm{id}_P$
  PROOF: Each is the unique x such that $p_1 x = p_1$ and $p_2 x = p_2$.
⟨1⟩4. $ii^{-1} = \mathrm{id}_Q$
  PROOF: Each is the unique x such that $q_1 x = q_1$ and $q_2 x = q_2$.
□

Definition 1.20. For any sets A and B, we write $A \times B$ for the product of A and B, and $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ for the projections. Given $f : X \to A$ and $g : X \to B$, we write $\langle f, g \rangle$ for the unique function $X \to A \times B$ such that

$$\pi_1 \langle f, g \rangle = f, \qquad \pi_2 \langle f, g \rangle = g \ .$$

Definition 1.21. Given functions $f : A \to B$ and $g : C \to D$, let $f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle : A \times C \to B \times D$.

1.2.6 Function Sets

Definition 1.22 (Function Set)**.** Let A and B be sets. A *function set* from A to B consists of a set B^A and function $\epsilon : B^A \times A \to B$, the *evaluation* map, such that, for any set I and function $q : I \times A \to B$, there exists a unique function $\lambda q : I \to B^A$ such that $\epsilon \circ (\lambda q \times \mathrm{id}_A) = q$.

Axiom 1.23 (Function Sets)**.** *Any two sets have a function set.*

Proposition 1.24. *If F and G are function sets of A and B with evaluation maps $e : F \times A \to B$ and $e' : G \times A \to B$, then there exists a unique isomorphism $i : F \cong G$ such that $e'(i \times \mathrm{id}_A) = e$.*

Proof:
$\langle 1 \rangle$1. Let: $i : F \to G$ be the unique function such that $e'(i \times \mathrm{id}_A) = e$.
$\langle 1 \rangle$2. Let: $i^{-1} : G \to F$ be the unique function such that $e(i^{-1} \times \mathrm{id}_A) = e'$
$\langle 1 \rangle$3. $ii^{-1} = \mathrm{id}_G$
Proof: Each is the unique x such that $e'(x \times \mathrm{id}_A) = e'$.
$\langle 1 \rangle$4. $i^{-1}i = \mathrm{id}_F$
Proof: Each is the unique x such that $e(x \times \mathrm{id}_B) = e$.
□

1.2.7 Inverse Images

Definition 1.25 (Pullback)**.** Let $p : A \to B$, $q : A \to C$, $f : B \to D$ and $g : C \to D$. Then we say that A, p and q form the *pullback* of f and g if and only if:

- $fp = gq$
- For any set X and functions $x : X \to B$, $y : X \to C$ such that $fx = gy$, there exists a unique function $(x, y) : X \to A$ such that $p(x, y) = x$ and $q(x, y) = y$.

We also say p is the pullback of g along f, or q is the pullback of f along g.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Axiom 1.26 (Inverse Images)**.** *Given any function $f : X \to Y$ and element $y \in Y$, then there exists a pullback of f and y.*

1.2.8 The Subset Classifier

Definition 1.27 (Injective)**.** A function $f : A \to B$ is *injective* iff, for every set X and functions $x, y : X \to A$, if $fx = fy$ then $x = y$.

Definition 1.28 (Subset Classifier)**.** A *subset classifier* consists of a set 2 and an element $\top \in 2$ such that, for any sets A and X and injective function $j : A \rightarrowtail X$, there exists a unique function $\chi : X \to 2$, the *classifying function* of j, such that j and $!_A : A \to 1$ form the pullback of $\top$ and χ.

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "j"'] & 1 \arrow[d, "\top"] \\ X \arrow[r, "\chi"'] & 2 \end{tikzcd}$$

Axiom 1.29 (Subset Classifier)**.** *There exists a subset classifier.*

Proposition 1.30. *If $\top \in 2$ and $\top' \in 2'$ are subset classifiers, then there exists a unique isomorphism $i : 2 \approx 2'$ such that $i(\top) = \top'$.*

PROOF:
⟨1⟩1. LET: $i : 2 \to 2'$ be the unique function such that $\top$ and id_1 form the pullback of $\top'$ and i
⟨1⟩2. LET: $i^{-1} : 2' \to 2$ be the unique function such that $\top'$ and id_1 form the pullback of $\top$ and i^{-1}
⟨1⟩3. $ii^{-1} = \mathrm{id}_{2'}$
PROOF: Each is the unique x such that $\top'$ and id_1 form the pullback of $\top'$ and x.
⟨1⟩4. $i^{-1}i = \mathrm{id}_2$
PROOF: Each is the unique x such that $\top$ and id_1 form the pullback of $\top$ and x.
□

Definition 1.31. Let 2 and $\top \in 2$ be the subset classifier.

1.2.9 The Natural Numbers

Definition 1.32 (Natural Numbers Set)**.** A *natural numbers set* consists of a set $\mathbb{N}$, an element $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \to \mathbb{N}$ such that, for any set A, element $a \in A$ and function $f : A \to A$, there exists a unique function $r : \mathbb{N} \to A$ such that $r(0) = a$ and $f \circ r = r \circ s$.

Axiom 1.33 (Infinity)**.** *There exists a natural numbers set.*

Proposition 1.34. *If N, $0 \in N$, $s : N \to N$ and N', $0' \in N'$, $s' : N' \to N'$ are two natural numbers sets, then there exists a unique isomorphism $i : N \approx N'$ such that $i(0) = 0'$ and $s'i = is$.*

PROOF:
⟨1⟩1. LET: $i : N \to N'$ be the unique function such that $i(0) = 0'$ and $s'i = is$.
⟨1⟩2. LET: $i^{-1} : N' \to N$ be the unique function such that $i^{-1}(0') = 0$ and $si^{-1} = i^{-1}s'$.

$\langle 1 \rangle 3.$ $ii^{-1} = \mathrm{id}_{N'}$

PROOF: Each is the unique x such that $x(0') = 0'$ and $s'x = xs'$.

$\langle 1 \rangle 4.$ $i^{-1}i = \mathrm{id}_N$

PROOF: Each is the unique x such that $x(0) = 0$ and $sx = xs$.

□

1.2.10 The Axiom of Choice

Definition 1.35 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for every element $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Axiom 1.36 (Choice)**.** *Every surjective function has a section.*

1.3 Sections and Retractions

Proposition 1.37. *Let $r : A \to B$, $r' : B \to C$, $s : B \to A$ and $s' : C \to B$. If s is a section of r and s' is a section of r', then ss' is a section of $r'r$.*

PROOF: Since $r'rss' = r'\mathrm{id}_B s' = r's' = \mathrm{id}_C$. □

1.4 Isomorphisms

Proposition 1.38. *For any set A we have $\mathrm{id}_A : A \approx A$ and ${\mathrm{id}_A}^{-1} = \mathrm{id}_A$.*

PROOF: Immediate from the fact that $\mathrm{id}_A \mathrm{id}_A = \mathrm{id}_A$. □

Proposition 1.39. *If $f : A \approx B$ then $f^{-1} : B \approx A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $ff^{-1} = \mathrm{id}_B$ and $f^{-1}f = \mathrm{id}_A$. □

Proposition 1.40. *If $f : A \approx B$ and $g : B \approx C$ then $gf : A \approx C$ and $(gf)^{-1} = f^{-1}g^{-1}$.*

PROOF: From Proposition 1.37. □

1.5 Subsets

Definition 1.41 (Subset)**.** Let $i : U \to A$. Then we say that (U, i) is a *subset* of A iff i is injective.

Definition 1.42. Let (U, i) and (V, j) be subsets of A. Then we say (U, i) and (V, j) are *equal*, and write $(U, i) = (V, j)$, iff there exists an isomorphism $\phi : U \cong V$ such that $j\phi = i$.

1.6 Intersections

Definition 1.43 (Intersection)**.** Let (U, i) and (V, j) be subsets of a set A. Let $p : W \to U$ and $q : W \to V$ form the pullback of i under j. Then the *intersection* of (U, i) and (V, j) is defined to be $(W, ip) = (W, jq)$.

1.7 Pullbacks

1.8 Functions

Proposition 1.44. *Let $f : A \to B$. Then f is injective if and only if, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.*

PROOF:
$\langle 1 \rangle 1$. If f is injective then, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.
  PROOF: Immediate from the definition of injective.
$\langle 1 \rangle 2$. If $\forall x, y \in A. f(x) = f(y) \Rightarrow x = y$ then f is injective.
  $\langle 2 \rangle 1$. ASSUME: $\forall x, y \in A. f(x) = f(y) \Rightarrow x = y$
  $\langle 2 \rangle 2$. LET: X be a set and $s, t : X \to A$
  $\langle 2 \rangle 3$. ASSUME: $fs = ft$
  $\langle 2 \rangle 4$. $\forall x \in X. s(x) = t(x)$
    $\langle 3 \rangle 1$. LET: $x \in X$
    $\langle 3 \rangle 2$. $f(s(x)) = f(t(x))$
      PROOF: $\langle 2 \rangle 3$
    $\langle 3 \rangle 3$. $s(x) = t(x)$
      PROOF: $\langle 2 \rangle 1$
  $\langle 2 \rangle 5$. $s = t$
    PROOF: Axiom of Extensionality
∎

1.9 The Internal Logic

Proposition 1.45. *Let $i : U \rightarrowtail A$ be injective. Let $\chi : A \to 2$ be its characteristic function. Then, for all $a \in A$, we have $\chi(a) = \top$ if and only if there exists $u \in U$ such that $i(u) = a$.*

PROOF:
$\langle 1 \rangle 1$. If $\chi(a) = \top$ then there exists $u \in U$ such that $i(u) = a$.
  PROOF: If $\chi \circ a = \top = \top \circ !_1$ then there exists a unique $u : 1 \to U$ such that $i \circ u = a$ and $!_U \circ u = !_1$.
$\langle 1 \rangle 2$. For all $u \in U$ we have $\chi(i(u)) = \top$.
  PROOF: Since $\chi \circ i = \top \circ !_U$.
∎

Proposition 1.46. *Subsets of a set A are equal if and only if they have the same characteristic function.*

PROOF: Follows from the fact that pullbacks are unique up to isomorphism and the uniqueness of the characteristic function. ∎

Proposition 1.47. *There are exactly two subsets of 1.*

PROOF:

$\langle 1\rangle 1.$ PICK a set E with no elements.
$\langle 1\rangle 2.$ $!_E : E \to 1$ is injective.
PROOF: Vacuously, $\forall x, y \in E.!_E(x) =!_E(y) \Rightarrow x = y$.
$\langle 1\rangle 3.$ $(E, !_E) \neq (1, \mathrm{id}_1)$
PROOF: Since there cannot be an isomorphism $1 \cong E$.
$\langle 1\rangle 4.$ For any subsets (U, i) and (V, j) of 1, if $(U, i) \neq (U, i) \cap (V, j)$ then $(U, i) = (1, \mathrm{id}_1)$
$\langle 2\rangle 1.$ LET: (U, i) and (V, j) be subsets of 1.
$\langle 2\rangle 2.$ LET: $p : W \to U$ and $q : W \to V$ form the intersection of (U, i) and (V, j)
$\langle 2\rangle 3.$ ASSUME: $(U, i) \neq (W, k)$
$\langle 2\rangle 4.$ LET: $(U, \mathrm{id}_U) \neq (W, p)$ as subsets of U.
$\langle 2\rangle 5.$ LET: $\chi_U, \chi_W : U \to 2$ be the characteristic functions of (U, id_U) and (W, p) respectively.
$\langle 2\rangle 6.$ $\chi_U \neq \chi_W$
$\langle 2\rangle 7.$ PICK $x \in U$
PROOF: By the Axiom of Extensionality, there exists $x \in U$ such that $\chi_U(x) \neq \chi_W(x)$.
$\langle 2\rangle 8.$ $ix = \mathrm{id}_1$
$\langle 2\rangle 9.$ $x : 1 \cong U$
$\langle 2\rangle 10.$ $(U, i) = (1, \mathrm{id}_1)$
$\langle 1\rangle 5.$ For any subset (U, i) of 1, either $(U, i) = (E, !_E)$ or $(U, i) = (1, \mathrm{id}_1)$.
$\langle 2\rangle 1.$ LET: (U, i) be a subset of 1.
$\langle 2\rangle 2.$ ASSUME: $(U, i) \neq (E, !_E)$
$\langle 2\rangle 3.$ $(U, i) \neq (U, i) \cap (E, !_E)$ or $(E, !_E) \neq (U, i) \cap (E, !_E)$
$\langle 2\rangle 4.$ $(U, i) = (1, \mathrm{id}_1)$ or $(E, !_E) = (1, \mathrm{id}_1)$
PROOF: $\langle 1\rangle 4$
$\langle 2\rangle 5.$ $(U, i) = (1, \mathrm{id}_1)$
PROOF: $\langle 1\rangle 3$
□

Corollary 1.47.1. *There are exactly two elements of 2.*

Definition 1.48 (Falsehood)**.** Let *falsehood* $\bot$ be the element of 2 that is not $\top$.

Corollary 1.48.1. *2 is the coproduct of 1 and 1 with injections $\top$ and $\bot$.*

1.10 Functions

Proposition 1.49. *Let $f : A \to B$, $g : B \to C$ and $a \in A$. Then*

$$(g \circ f)(a) = g(f(a)) \ .$$

PROOF: Immediate from the Axiom of Associativity. □

Proposition 1.50. *For any set A, any function $1 \to A$ is injective.*

PROOF: Since there is only one function $X \to 1$ for any set X. ∎

Proposition 1.51. *Let $f : A \to B$. Then the following are equivalent:*

1. *f is surjective.*
2. *f is a retraction (i.e. f has a section).*
3. *For any set X and functions $x, y : B \to X$, if $xf = yf$ then $x = y$.*

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
PROOF: Immediate from the Axiom of Choice.
$\langle 1 \rangle 2.$ $2 \Rightarrow 3$
$\langle 2 \rangle 1.$ LET: $s : B \to A$ be a section of f.
$\langle 2 \rangle 2.$ LET: X be a set and $x, y : B \to X$ satisfy $xf = yf$.
$\langle 2 \rangle 3.$ $x = y$
PROOF: $x = xfs = yfs = y$
$\langle 1 \rangle 3.$ $3 \Rightarrow 1$
$\langle 2 \rangle 1.$ ASSUME: 3
$\langle 2 \rangle 2.$ LET: $b \in B$
$\langle 2 \rangle 3.$ ASSUME: for a contradiction $\forall a \in A. f(a) \neq b$
$\langle 2 \rangle 4.$ LET: $\psi_1 : B \to 2$ be the characteristic function of b.
$\langle 2 \rangle 5.$ LET: $\psi_2 = \bot \circ !_B : B \to 2$
$\langle 2 \rangle 6.$ $\forall x \in A. \psi_1(f(x)) = \psi_2(f(x))$
$\langle 3 \rangle 1.$ LET: $x \in A$
$\langle 3 \rangle 2.$ $\psi_1(f(x)) \neq \top$
PROOF: Proposition 1.45, $\langle 2 \rangle 3$, $\langle 2 \rangle 4$.
$\langle 3 \rangle 3.$ $\psi_1(f(x)) = \bot$
$\langle 3 \rangle 4.$ $\psi_1(f(x)) = \psi_2(f(x))$
$\langle 2 \rangle 7.$ $\psi_1 \circ f = \psi_2 \circ f$
PROOF: Axiom of Extensionality
$\langle 2 \rangle 8.$ $\psi_1 = \psi_2$
PROOF: $\langle 2 \rangle 1$
$\langle 2 \rangle 9.$ $\psi_1(b) \neq \psi_2(b)$
PROOF: Since $\psi_1(b) = \top$ and $\psi_2(b) = \bot$.
$\langle 2 \rangle 10.$ Q.E.D.
PROOF: This is a contradiction
∎

Corollary 1.51.1. *A function is bijective iff it is injective and surjective.*

1.11 Equalizers

Theorem 1.52. *Any two functions $f, g : A \to B$ have an equalizer.*

PROOF: Take the inverse image of $\delta_B = \langle \mathrm{id}_B, \mathrm{id}_B \rangle : B \rightarrowtail B^2$ and $\langle f, g \rangle : A \to B^2$. ∎

1.12 The Empty Set

Theorem 1.53. *If E is a set with no elements, then E has no proper subsets.*

PROOF: A proper subset of E would give a proper subset of 1 that is different from $(E, !_E)$. ∎

Theorem 1.54. *If E is a set with no elements, then for any set X there exists exactly one function $E \to X$.*

PROOF:
⟨1⟩1. LET: E be a set with no elements.
⟨1⟩2. LET: X be a set.
⟨1⟩3. There exists a function $E \to X$.
  ⟨2⟩1. LET: $t : 1 \to 2^X$ be the name of the characteristic function of $\mathrm{id}_X : X \to X$.
  ⟨2⟩2. LET: $\sigma : X \to 2^X$ be the lambda of the characteristic function of $\delta = \langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \to X \times X$.
  ⟨2⟩3. LET: $p : P \to E$ and $q : P \to X$ be the pullback of $t \circ !_E$ and σ.
    PROOF: $t \circ !_E$ is vacuously injective.
  ⟨2⟩4. p is injective.
    PROOF: It is the pullback of the injective function σ.
  ⟨2⟩5. p is bijective.
  ⟨2⟩6. $q \circ p^{-1} : E \to X$
⟨1⟩4. For any functions $f, g : E \to X$ we have $f = g$.
  ⟨2⟩1. LET: $f, g : E \to X$
  ⟨2⟩2. LET: $m : M \to E$ be the pullback of f and g.
  ⟨2⟩3. $(M, m) = (E, \mathrm{id}_E)$
    PROOF: Since E has no proper subsets.
  ⟨2⟩4. $m : M \cong E$
  ⟨2⟩5. $f = g$
∎

Corollary 1.54.1. *If E and E' are sets with no elements then there exists a unique isomorphism $E \cong E'$.*

Definition 1.55 (Empty Set)**.** Let the *empty set* $\varnothing$ be the set with no elements.

Theorem 1.56. *For any set A, if there exists a function $A \to \varnothing$ then $A \cong \varnothing$.*

PROOF: If $f : A \to \varnothing$ then A has no elements, because for any $a \in A$ we have $f(a) \in \varnothing$. ∎

1.13 Universal Quantification

Definition 1.57. For any set A, let $t_A : 1 \to 2^A$ be the name of the characteristic function of $\top \circ !_A : A \to 2$. Define *universal quantification* $\forall_A : 2^A \to 2$ to be the characteristic function of t_A.

1.14 Intersection

Theorem 1.58. *Let X be a set. There exists a function $\bigcap : 2^{2^X} \to 2^X$ such that, for all $S \in 2^{2^X}$ and $a \in X$, we have*

$$\epsilon(\bigcap S, a) = \top \Leftrightarrow \forall A \in 2^X.(\epsilon(S, A) = \top \Rightarrow \epsilon(A, a) = \top)$$

PROOF:
⟨1⟩1. LET: X be a set.
⟨1⟩2. LET: $\phi_2 : X \to 2^{2^X}$ be the lambda of $\epsilon : 2^X \times X \to 2$
⟨1⟩3. LET: F be the function

$$2^{2^X} \times X \xrightarrow{\langle \mathrm{id}_{2^{2^X}}, \phi_2 \rangle} 2^{2^X} \times 2^{2^X} \xrightarrow{\cong} (2 \times 2)^{2^X} \xrightarrow{\Rightarrow} 2^{2^X} \xrightarrow{\forall} 2$$

⟨1⟩4. LET: $\bigcap$ be the lambda
□

1.15 Union

Theorem 1.59. *Any two subsets of a set have a union.*

PROOF:
⟨1⟩1. LET: A and B be subsets of X
⟨1⟩2. LET: $\chi_A \in 2^X$ be the name of the characteristic function of A.
⟨1⟩3. LET: $t_X \in 2^X$ be the name of $\top \circ !_X : X \to 2$
⟨1⟩4. LET: C be the pullback of t_X and $\chi_A \Rightarrow - : 2^X \to 2^X$
⟨1⟩5. LET: D be the pullback of t_X and $\chi_B \Rightarrow -$
⟨1⟩6. $\bigcap(C \cap D)$ is the union of A and B.
□

Theorem 1.60. *Any two sets have a coproduct.*

PROOF:
⟨1⟩1. LET: X and Y be sets.
⟨1⟩2. LET: $\sigma_X : X \to 2^X$ be the lambda of the characteristic function of $\langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \to X \times X$
⟨1⟩3. LET: $\chi_0 : 1 \to Y$ be the characteristic function of the unique function $\varnothing \to Y$
⟨1⟩4. LET: $i_X = \langle \sigma_X, \chi_0 \circ !_X \rangle : X \to 2^X \times 2^Y$
⟨1⟩5. LET: $i_Y : Y \to 2^X \times 2^Y$ be defined similarly.
⟨1⟩6. i_X and i_Y are monic.
⟨1⟩7. $\varnothing$ is the pullback of i_X and i_Y (i.e. $(X, i_X) \cap (Y, i_Y) = \varnothing$).
⟨1⟩8. LET: $j : Z \to 2^X \times 2^Y$ be the union of i_X and i_Y
⟨1⟩9. Z is the coproduct of X and Y.
□