

# Mathematics

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**Part I**

**Set Theory**



# Chapter 1

## Primitive Terms and Axioms

### 1.1 Primitive Terms

Let there be *sets*.

For any set  $A$ , let there be *elements* of  $A$ . We write  $a \in A$  for:  $a$  is an element of  $A$ .

For any sets  $A$  and  $B$ , let there be a set  $B^A$ , whose elements are called *functions* from  $A$  to  $B$ . We write  $f : A \rightarrow B$  for  $f \in B^A$ .

For any function  $f : A \rightarrow B$  and element  $a \in A$ , let there be an element  $f(a) \in B$ , the *value* of the function  $f$  at the *argument*  $a$ .

### 1.2 Injections, Surjections and Bijections

**Definition 1.2.1** (Injective). A function  $f : A \rightarrow B$  is *injective* or an *injection* iff, for all  $x, y \in A$ , if  $f(x) = f(y)$  then  $x = y$ .

**Definition 1.2.2** (Surjective). A function  $f : A \rightarrow B$  is *surjective* or a *surjection* iff, for all  $y \in B$ , there exists  $x \in A$  such that  $f(x) = y$ .

**Definition 1.2.3** (Bijective). A function  $f : A \rightarrow B$  is *bijective* or a *bijection* iff it is injective and surjective.

Sets  $A$  and  $B$  are *equinumerous*,  $A \approx B$ , iff there exists a bijection between them.

If we prove there exists a set  $X$  such that  $P(X)$ , and that any two sets that satisfy  $P$  are bijective, then we may introduce a constant  $C$  and define "Let  $C$  be the set such that  $P(C)$ ".

### 1.3 Axioms

**Axiom Schema 1.3.1** (Choice). *Let  $P[X, Y, x, y]$  be a formula where  $X$  and  $Y$  are set variables,  $x \in X$  and  $y \in Y$ . Then the following is an axiom.*

*Let  $A$  and  $B$  be sets. Assume that, for all  $a \in A$ , there exists  $b \in B$  such that  $P[A, B, a, b]$ . Then there exists a function  $f : A \rightarrow B$  such that  $\forall a \in A. P[A, B, a, f(a)]$ .*

**Axiom 1.3.2** (Extensionality). *Let  $f, g : A \rightarrow B$ . If, for all  $x \in A$ , we have  $f(x) = g(x)$ , then  $f = g$ .*

**Definition 1.3.3** (Composition). *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . The composite  $g \circ f : A \rightarrow C$  is the function such that, for all  $a \in A$ , we have*

$$(g \circ f)(a) = g(f(a)) .$$

**Axiom 1.3.4** (Pairing). *For any sets  $A$  and  $B$ , there exists a set  $A \times B$ , the Cartesian product of  $A$  and  $B$ , and functions  $\pi_1 : A \times B \rightarrow A$  and  $\pi_2 : A \times B \rightarrow B$  such that, for all  $a \in A$  and  $b \in B$ , there exists a unique  $(a, b) \in A \times B$  such that  $\pi_1(a, b) = a$  and  $\pi_2(a, b) = b$ .*

**Axiom Schema 1.3.5** (Separation). *For every property  $P[X, x]$  where  $X$  is a set variable and  $x \in X$ , the following is an axiom:*

*For every set  $A$ , there exists a set  $S = \{x \in A : P[A, x]\}$  and an injection  $i : S \rightarrow A$  such that, for all  $x \in A$ , we have*

$$(\exists y \in S. i(y) = x) \Leftrightarrow P[A, x] .$$

**Axiom 1.3.6** (Infinity). *There exists a set  $\mathbb{N}$ , an element  $0 \in \mathbb{N}$ , and a function  $s : \mathbb{N} \rightarrow \mathbb{N}$  such that:*

- $\forall n \in \mathbb{N}. s(n) \neq 0$
- $\forall m, n \in \mathbb{N}. s(m) = s(n) \Rightarrow m = n$ .

**Axiom Schema 1.3.7** (Collection). *Let  $P[X, Y, x]$  be a formula with set variables  $X$  and  $Y$  and an element variable  $x \in X$ . Then the following is an axiom.*

*For any set  $A$ , there exist sets  $B$  and  $Y$  and functions  $p : B \rightarrow A$ , and  $m : B \times Y \Rightarrow \mathbb{N}$  such that:*

- $m$  is injective.
- $\forall b \in B. P[A, \{y \in Y : m(b, y) = 0\}, p(b)]$
- For all  $a \in A$ , if  $\exists Y. P[A, Y, a]$ , then there exists  $b \in B$  such that  $a = p(b)$ .

**Axiom 1.3.8** (Universe). *There exists a set  $E$ , a set  $U$  and a function  $el : E \rightarrow U$  such that the following holds.*

*Let us say that a set  $A$  is small iff there exists  $u \in U$  such that  $A \approx \{e \in E : el(e) = u\}$ .*

- $\mathbb{N}$  is small.
- For any  $U$ -small sets  $A$  and  $B$ , the set  $B^A$  is small.
- For any  $U$ -small sets  $A$  and  $B$ , the set  $A \times B$  is small.
- Let  $f : A \rightarrow B$  be a function. If  $B$  is small and  $\{a \in A : f(a) = b\}$  is small for all  $b \in B$ , then  $A$  is small.
- If  $p : B \rightarrow A$  is a surjective function such that  $A$  is small, then there exists a  $U$ -small set  $C$ , a surjection  $q : C \rightarrow A$ , and a function  $f : C \rightarrow B$  such that  $q = p \circ f$ .



## Chapter 2

# Sets and Functions

### 2.1 Composition

**Proposition 2.1.1.** *Given functions  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h : C \rightarrow D$ , we have*

$$h \circ (g \circ f) = (h \circ g) \circ f .$$

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in A$  we have  $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$ .

$\langle 2 \rangle 1$ . LET:  $x \in A$

$\langle 2 \rangle 2$ .  $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$

PROOF:

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) \quad (\text{Definition of composition})$$

$$= h(g(f(x))) \quad (\text{Definition of composition})$$

$$= (h \circ g)(f(x)) \quad (\text{Definition of composition})$$

$$= ((h \circ g) \circ f)(x) \quad (\text{Definition of composition})$$

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: By the Axiom of Extensionality.

□

### 2.2 Injections

**Proposition 2.2.1.** *The composite of injective functions is injective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$ ,  $B$  and  $C$  be sets.

$\langle 1 \rangle 2$ . LET:  $f : A \rightarrow B$

$\langle 1 \rangle 3$ . LET:  $g : B \rightarrow C$

$\langle 1 \rangle 4$ . ASSUME:  $g$  is injective.

$\langle 1 \rangle 5$ . ASSUME:  $f$  is injective.

$\langle 1 \rangle 6$ . LET:  $x, y \in A$

$\langle 1 \rangle 7$ . ASSUME:  $(g \circ f)(x) = (g \circ f)(y)$

PROVE:  $x = y$

$\langle 1 \rangle 8$ .  $g(f(x)) = g(f(y))$

PROOF:

$$g(f(x)) = (g \circ f)(x) \quad (\text{definition of composition})$$

$$= (g \circ f)(y) \quad (\langle 1 \rangle 7)$$

$$= g(f(y)) \quad (\text{definition of composition})$$

$\langle 1 \rangle 9$ .  $f(x) = f(y)$

PROOF:  $\langle 1 \rangle 4, \langle 1 \rangle 8$

$\langle 1 \rangle 10$ .  $x = y$

PROOF:  $\langle 1 \rangle 5, \langle 1 \rangle 9$

□

**Proposition 2.2.2.** *For functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , if  $g \circ f$  is injective then  $f$  is injective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A, B$  and  $C$  be sets.

$\langle 1 \rangle 2$ . LET:  $f : A \rightarrow B$

$\langle 1 \rangle 3$ . LET:  $g : B \rightarrow C$

$\langle 1 \rangle 4$ . ASSUME:  $g \circ f$  is injective.

$\langle 1 \rangle 5$ . LET:  $x, y \in A$

$\langle 1 \rangle 6$ . ASSUME:  $f(x) = f(y)$

$\langle 1 \rangle 7$ .  $(g \circ f)(x) = (g \circ f)(y)$

PROOF:

$$(g \circ f)(x) = g(f(x)) \quad (\text{definition of composition})$$

$$= g(f(y)) \quad (\langle 1 \rangle 6)$$

$$= (g \circ f)(y) \quad (\text{definition of composition})$$

$\langle 1 \rangle 8$ .  $x = y$

PROOF:  $\langle 1 \rangle 4, \langle 1 \rangle 7$

□

**Proposition 2.2.3.** *Let  $f : A \rightarrow B$  be injective. For every set  $X$  and functions  $x, y : X \rightarrow A$ , if  $f \circ x = f \circ y$  then  $x = y$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $f$  is injective.

$\langle 1 \rangle 2$ . LET:  $X$  be a set.

$\langle 1 \rangle 3$ . LET:  $x, y : X \rightarrow A$

$\langle 1 \rangle 4$ . ASSUME:  $f \circ x = f \circ y$

$\langle 1 \rangle 5$ .  $\forall t \in X. x(t) = y(t)$

$\langle 2 \rangle 1$ . LET:  $t \in X$

$\langle 2 \rangle 2$ .  $f(x(t)) = f(y(t))$



PROOF:

$$\begin{aligned} f(x(t)) &= (f \circ x)(t) && \text{(definition of composition)} \\ &= (f \circ y)(t) && (\langle 1 \rangle 4) \\ &= f(y(t)) && \text{(definition of composition)} \end{aligned}$$

$\langle 2 \rangle 3$ .  $x(t) = y(t)$

PROOF:  $\langle 1 \rangle 1$ ,  $\langle 2 \rangle 2$

$\langle 1 \rangle 6$ .  $x = y$

PROOF: Axiom of Extensionality,  $\langle 1 \rangle 5$

□

We will prove the converse as Proposition 2.8.4.

## 2.3 Surjections

**Proposition 2.3.1.** *The composite of surjective functions is surjective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$ ,  $B$  and  $C$  be sets.

$\langle 1 \rangle 2$ . LET:  $f : A \rightarrow B$  and  $g : B \rightarrow C$

$\langle 1 \rangle 3$ . ASSUME:  $g$  is surjective.

$\langle 1 \rangle 4$ . ASSUME:  $f$  is surjective.

$\langle 1 \rangle 5$ . LET:  $c \in C$

$\langle 1 \rangle 6$ . PICK  $b \in B$  such that  $g(b) = c$ .

PROOF:  $\langle 1 \rangle 3$

$\langle 1 \rangle 7$ . PICK  $a \in A$  such that  $f(a) = b$ .

PROOF:  $\langle 1 \rangle 4$

$\langle 1 \rangle 8$ .  $(g \circ f)(a) = c$

PROOF:

$$\begin{aligned} (g \circ f)(a) &= g(f(a)) && \text{(definition of composition)} \\ &= g(b) && (\langle 1 \rangle 7) \\ &= c && (\langle 1 \rangle 6) \end{aligned}$$

□

**Proposition 2.3.2.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $g \circ f$  is surjective then  $g$  is surjective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$ ,  $B$  and  $C$  be sets.

$\langle 1 \rangle 2$ . LET:  $f : A \rightarrow B$  and  $g : B \rightarrow C$ .

$\langle 1 \rangle 3$ . ASSUME:  $g \circ f$  is surjective.

$\langle 1 \rangle 4$ . LET:  $c \in C$

$\langle 1 \rangle 5$ . PICK  $a \in A$  such that  $(g \circ f)(a) = c$

PROOF:  $\langle 1 \rangle 3$

$\langle 1 \rangle 6$ .  $g(f(a)) = c$

PROOF: From  $\langle 1 \rangle 5$  and the definition of composition.

$\langle 1 \rangle 7$ . Q.E.D.

PROOF: There exists  $b \in B$  such that  $g(b) = c$ , namely  $b = f(a)$ .  
 $\square$

**Proposition 2.3.3.** *Let  $f : A \rightarrow B$  be a surjection. For any set  $X$  and functions  $x, y : B \rightarrow X$ , if  $x \circ f = y \circ f$  then  $x = y$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $b \in B$
- $\langle 1 \rangle 2$ . PICK  $a \in A$  such that  $f(a) = b$
- $\langle 1 \rangle 3$ .  $x(f(a)) = y(f(a))$
- $\langle 1 \rangle 4$ .  $x(b) = y(b)$
- $\langle 1 \rangle 5$ . Q.E.D.

PROOF: Axiom of Extensionality.  
 $\square$

We will prove the converse as Proposition 2.9.2.

## 2.4 Bijections

**Proposition 2.4.1.** *The composite of bijections is a bijection.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $A, B$  and  $C$  be sets.
- $\langle 1 \rangle 2$ . LET:  $f : A \rightarrow B$  and  $g : B \rightarrow C$
- $\langle 1 \rangle 3$ . ASSUME:  $g$  is bijective.
- $\langle 1 \rangle 4$ . ASSUME:  $f$  is bijective.
- $\langle 1 \rangle 5$ .  $g$  is injective.  
 PROOF: From  $\langle 1 \rangle 3$ .
- $\langle 1 \rangle 6$ .  $g$  is surjective.  
 PROOF: From  $\langle 1 \rangle 3$ .
- $\langle 1 \rangle 7$ .  $f$  is injective.  
 PROOF: From  $\langle 1 \rangle 4$ .
- $\langle 1 \rangle 8$ .  $f$  is surjective.  
 PROOF: From  $\langle 1 \rangle 4$ .
- $\langle 1 \rangle 9$ .  $g \circ f$  is injective.  
 PROOF: Proposition 2.2.1,  $\langle 1 \rangle 5$ ,  $\langle 1 \rangle 7$ .
- $\langle 1 \rangle 10$ .  $g \circ f$  is surjective.  
 PROOF: Proposition 2.3.1,  $\langle 1 \rangle 6$ ,  $\langle 1 \rangle 8$ .
- $\langle 1 \rangle 11$ .  $g \circ f$  is bijective.  
 PROOF:  $\langle 1 \rangle 9$ ,  $\langle 1 \rangle 10$

$\square$

**Proposition 2.4.2.**

$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps  $f$  to  $(\pi_1 \circ f, \pi_2 \circ f)$  is a bijection.  $\square$

**Proposition 2.4.3.**

$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function  $\Phi$  such that  $\Phi(f)(c)(b) = f(b, c)$  is a bijection.  $\square$

## 2.5 Domination

**Definition 2.5.1** (Dominate). Let  $A$  and  $B$  be sets. We say that  $B$  *dominates*  $A$ , and write  $A \leqslant B$ , iff there exists an injective function  $A \rightarrow B$ .

**Theorem 2.5.2** (Schroeder-Bernstein). *Let  $A$  and  $B$  be sets. If  $A \leqslant B$  and  $B \leqslant A$  then  $A \approx B$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$  and  $g : B \rightarrow A$  be injections.

$\langle 1 \rangle 2$ . Define the subsets  $A_n$  of  $A$  by

$$\begin{aligned} A_0 &:= A - g(B) \\ A_{n+1} &:= g(f(A_n)) \end{aligned}$$

$\langle 1 \rangle 3$ . Define  $h : A \rightarrow B$  by

$$h(x) = \begin{cases} f(x) & \text{if } \exists n. x \in A_n \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

$\langle 1 \rangle 4$ .  $h$  is injective.

$\langle 2 \rangle 1$ . LET:  $x, y \in A$

$\langle 2 \rangle 2$ . ASSUME:  $h(x) = h(y)$

$\langle 2 \rangle 3$ . CASE:  $x \in A_m$  and  $y \in A_n$ .

PROOF: Then  $f(x) = f(y)$  so  $x = y$  since  $f$  is injective.

$\langle 2 \rangle 4$ . CASE:  $x \in A_m$  and there is no  $y$  such that  $y \in A_n$ .

$\langle 3 \rangle 1$ .  $f(x) = g^{-1}(y)$

$\langle 3 \rangle 2$ .  $y = g(f(x))$

$\langle 3 \rangle 3$ .  $y \in A_{m+1}$

$\langle 3 \rangle 4$ . Q.E.D.

PROOF: This is a contradiction.

$\langle 2 \rangle 5$ . CASE:  $y \in A_n$  and there is no  $m$  such that  $x \in A_m$ .

PROOF: Similar.

$\langle 2 \rangle 6$ . CASE: There is no  $m$  such that  $x \in A_m$  and there is no  $n$  such that  $y \in A_n$ .

PROOF: Then  $g^{-1}(x) = g^{-1}(y)$  and so  $x = y$ .

$\langle 1 \rangle 5$ .  $h$  is surjective.

$\langle 2 \rangle 1$ . LET:  $y \in B$

$\langle 2 \rangle 2$ . CASE:  $g(y) \in A_n$

$\langle 3 \rangle 1$ .  $n \neq 0$

$\langle 3 \rangle 2$ . PICK  $x \in A_{n-1}$  such that  $g(y) = g(f(x))$

$\langle 3 \rangle 3$ .  $y = f(x)$

$\langle 3 \rangle 4$ .  $y = h(x)$

$\langle 2 \rangle 3$ . CASE: There is no  $n$  such that  $g(y) \in A_n$ .

PROOF: Then  $h(g(y)) = y$ .

$\square$

## 2.6 Identity Function

**Definition 2.6.1** (Identity). For any set  $A$ , the *identity* function  $\text{id}_A : A \rightarrow A$  is the function defined by  $\text{id}_A(a) = a$ .

### 2.6.1 Injections, Surjections, Bijections

**Proposition 2.6.2.** *For any set  $A$ , the identity function  $\text{id}_A$  is a bijection.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be a set.

$\langle 1 \rangle 2$ .  $\text{id}_A$  is injective.

PROOF: If  $\text{id}_A(x) = \text{id}_A(y)$  then  $x = y$ .

$\langle 1 \rangle 3$ .  $\text{id}_A$  is surjective.

PROOF: For any  $y \in A$ , there exists  $x \in A$  such that  $\text{id}_A(x) = y$ , namely  $x = y$ .

□

### 2.6.2 Composition

**Proposition 2.6.3.** *Let  $f : A \rightarrow B$ . Then  $\text{id}_B \circ f = f = f \circ \text{id}_A$ .*

PROOF: Each is the function that maps  $a$  to  $f(a)$ . □

**Proposition 2.6.4.** *Let  $f : A \rightarrow B$ .*

1. *If there exists  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$  then  $f$  is injective.*

2. *If  $f$  is injective and  $A$  is nonempty, then there exists  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$ .*

PROOF:

$\langle 1 \rangle 1$ . If there exists  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$  then  $f$  is injective.

PROOF: If  $f(x) = f(y)$  then  $x = g(f(x)) = g(f(y)) = y$ .

$\langle 1 \rangle 2$ . If  $f$  is injective and  $A$  is nonempty, then there exists  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$ .

$\langle 2 \rangle 1$ . ASSUME:  $f$  is injective and  $A$  is nonempty.

$\langle 2 \rangle 2$ . PICK  $a \in A$

$\langle 2 \rangle 3$ . Choose a function  $g : B \rightarrow A$  such that  $f(g(x)) = x$  if there exists  $y \in A$  such that  $f(y) = x$ , otherwise  $g(x) = a$ .

$\langle 2 \rangle 4$ . LET:  $x \in A$

PROVE:  $g(f(x)) = x$

$\langle 2 \rangle 5$ .  $f(g(f(x))) = f(x)$

$\langle 2 \rangle 6$ .  $g(f(x)) = x$

□

**Proposition 2.6.5.** *Let  $f : A \rightarrow B$ . Then  $f$  is surjective if and only if there exists  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$ .*

PROOF:

⟨1⟩1. If  $f$  is surjective then there exists  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$ .

⟨2⟩1. ASSUME:  $f$  is surjective.

⟨2⟩2. PICK  $g : B \rightarrow A$  such that, for all  $b \in B$ , we have  $f(g(b)) = b$ .

PROOF: Axiom of Choice.

⟨2⟩3.  $f \circ g = \text{id}_B$ .

⟨1⟩2. If there exists  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$  then  $f$  is surjective.

⟨2⟩1. LET:  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$

⟨2⟩2. LET:  $X$  be a set.

⟨2⟩3. LET:  $h, k : B \rightarrow X$

⟨2⟩4. ASSUME:  $h \circ f = k \circ f$

⟨2⟩5.  $h = k$

PROOF:  $h = h \circ f \circ g = k \circ f \circ g = k$

□

**Corollary 2.6.5.1.** *Let  $A$  and  $B$  be sets.*

1. *If there exists a surjective function  $A \rightarrow B$  then there exists an injective function  $B \rightarrow A$ .*
2. *If there exists an injective function  $A \rightarrow B$  and  $A$  is nonempty then there exists a surjective function  $B \rightarrow A$ .*

**Proposition 2.6.6.** *Let  $f : A \rightarrow B$ . Then  $f$  is bijective if and only if there exists a function  $f^{-1} : B \rightarrow A$ , the inverse of  $f$ , such that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ , in which case the inverse is unique.*

PROOF:

⟨1⟩1. If  $f$  is bijective then there exists  $f^{-1} : B \rightarrow A$  such that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ .

⟨2⟩1. ASSUME:  $f$  is bijective.

⟨2⟩2. PICK  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$

PROOF: Proposition 2.9.2.

⟨2⟩3.  $f \circ g \circ f = f$

⟨2⟩4.  $g \circ f = \text{id}_A$

PROOF: Proposition 2.2.3.

⟨1⟩2. If there exists  $f^{-1} : B \rightarrow A$  such that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ , then  $f$  is bijective.

⟨2⟩1. LET:  $f^{-1} : B \rightarrow A$  satisfy  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$

⟨2⟩2.  $f$  is injective.

PROOF: If  $f(x) = f(y)$  then  $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$ .

⟨2⟩3.  $f$  is surjective.

PROOF: Proposition 2.9.2.

⟨1⟩3. If  $g, h : B \rightarrow A$  satisfy  $f \circ g = \text{id}_B$  and  $g \circ f = \text{id}_A$  and  $f \circ h = \text{id}_B$  and  $h \circ f = \text{id}_A$  then  $g = h$ .

PROOF: We have  $g = g \circ f \circ h = h$ .

□

## 2.7 The Empty Set

**Theorem 2.7.1.** *There exists a set which has no elements.*

PROOF: Take  $\{x \in \mathbb{N} : \perp\}$ .  $\square$

**Theorem 2.7.2.** *If  $E$  and  $E'$  have no elements then  $E \approx E'$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $E$  and  $E'$  have no elements.

$\langle 1 \rangle 2$ . PICK a function  $F : E \rightarrow E'$ .

PROOF: Axiom of Choice since vacuously  $\forall x \in E. \exists y \in E'. \top$ .

$\langle 1 \rangle 3$ .  $F$  is injective.

PROOF: Vacuously, for all  $x, y \in E$ , if  $F(x) = F(y)$  then  $x = y$ .

$\langle 1 \rangle 4$ .  $F$  is surjective.

PROOF: Vacuously, for all  $y \in E'$ , there exists  $x \in E$  such that  $F(x) = y$ .

$\square$

**Definition 2.7.3** (Empty Set). The *empty set*  $\emptyset$  is the set with no elements.

## 2.8 The Singleton

**Theorem 2.8.1.** *There exists a set that has exactly one element.*

PROOF: The set  $\{x \in \mathbb{N} : x = 0\}$  has exactly one element.  $\square$

**Theorem 2.8.2.** *If  $A$  and  $B$  both have exactly one element then  $A \approx B$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $B$  both have exactly one element  $a$  and  $b$  respectively.

$\langle 1 \rangle 2$ . LET:  $F : A \rightarrow B$  be the function such that, for all  $x \in A$ , we have  
 $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$ .  $F$  is a bijection.

$\square$

**Definition 2.8.3** (Singleton). Let  $1$  be the set that has exactly one element.  
 Let  $*$  be its element.

### 2.8.1 Injections

**Proposition 2.8.4.** *Let  $f : A \rightarrow B$ . Assume that, for every set  $X$  and functions  $x, y : X \rightarrow A$ , if  $f \circ x = f \circ y$  then  $x = y$ . Then  $f$  is injective.*

PROOF: Take  $X = 1$ .  $\square$

## 2.9 The Set Two

**Definition 2.9.1** (The Set Two). Let  $2 = \{x \in \mathbb{N} : x = 0 \vee x = 1\}$ .

**Proposition 2.9.2.** *Let  $f : A \rightarrow B$ . Assume that, for any set  $X$  and functions  $g, h : B \rightarrow X$ , if  $g \circ f = h \circ f$  then  $g = h$ . Then  $f$  is surjective.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: For any set  $X$  and functions  $g, h : B \rightarrow X$ , if  $g \circ f = h \circ f$  then  $g = h$ .

$\langle 1 \rangle 2$ . LET:  $b \in B$

$\langle 1 \rangle 3$ . LET:  $h : B \rightarrow 2$  be the function that maps everything to 1.

$\langle 1 \rangle 4$ . LET:  $k : B \rightarrow 2$  be the function that maps  $b$  to 0 and everything else to 1.

$\langle 1 \rangle 5$ .  $h \neq k$

$\langle 1 \rangle 6$ .  $h \circ f \neq k \circ f$

$\langle 1 \rangle 7$ . PICK  $a \in A$  such that  $h(f(a)) \neq k(f(a))$

$\langle 1 \rangle 8$ .  $f(a) = b$

□

## 2.10 Subsets

**Definition 2.10.1** (Subset). A *subset* of a set  $A$  consists of a set  $S$  and an injection  $i : S \rightarrow A$ . We write  $(S, i) \subseteq A$ .

We say two subsets  $(S, i)$  and  $(T, j)$  are *equal*,  $(S, i) = (T, j)$ , iff there exists a bijection  $\phi : S \approx T$  such that  $j \circ \phi = i$ .

**Proposition 2.10.2.** *For any subset  $(S, i)$  of  $A$  we have  $(S, i) = (S, i)$ .*

PROOF: We have  $\text{id}_S : S \approx S$  and  $i \circ \text{id}_S = i$ .

**Proposition 2.10.3.** *If  $(S, i) = (T, j)$  then  $(T, j) = (S, i)$ .*

PROOF: If  $\phi : S \approx T$  and  $j \circ \phi = i$  then  $\phi^{-1} : T \approx S$  and  $i \circ \phi^{-1} = j$ . □

**Proposition 2.10.4.** *If  $(R, i) = (S, j)$  and  $(S, j) = (T, k)$  then  $(R, i) = (T, k)$ .*

PROOF: If  $\phi : R \approx S$  and  $j \circ \phi = i$ , and  $\psi : S \approx T$  and  $k \circ \psi = j$ , then  $\psi \circ \phi : R \approx T$  and  $k \circ \psi \circ \phi = i$ . □

**Definition 2.10.5** (Membership). Given  $(S, i) \subseteq A$  and  $a \in A$ , we write  $a \in (S, i)$  for  $\exists s \in S. i(s) = a$ .

**Proposition 2.10.6.** *If  $a \in (S, i)$  and  $(S, i) = (T, j)$  then  $a \in (T, j)$ .*

PROOF: If  $i(s) = a$  then  $j(\phi(s)) = a$ . □

**Definition 2.10.7** (Union). Given subsets  $S$  and  $T$  of  $A$ , the *union* is the subset  $\{x \in A : x \in S \vee x \in T\}$ .

**Definition 2.10.8** (Intersection). Given subsets  $S$  and  $T$  of  $A$ , the *intersection* is the subset  $\{x \in A : x \in S \wedge x \in T\}$ .

**Proposition 2.10.9** (Distributive Law).

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

**Proposition 2.10.10** (Distributive Law).

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

**Definition 2.10.11.** Given a set  $A$ , we write  $\emptyset$  for the subset  $(\emptyset, !)$  where  $!$  is the unique function  $\emptyset \rightarrow A$ .

**Proposition 2.10.12.**

$$S \cup \emptyset = S$$

**Proposition 2.10.13.**

$$S \cap \emptyset = \emptyset$$

**Definition 2.10.14** (Inclusion). Given subsets  $(S, i)$  and  $(T, j)$  of a set  $A$ , we write  $(S, i) \subseteq (T, j)$  iff there exists  $f : S \rightarrow T$  such that  $j \circ f = i$ .

**Proposition 2.10.15.**

$$\emptyset \subseteq S$$

**Definition 2.10.16** (Disjoint). Subsets  $S$  and  $T$  of  $A$  are *disjoint* iff  $S \cap T = \emptyset$ .

**Definition 2.10.17** (Difference). Given subsets  $S$  and  $T$  of  $A$ , the *difference* of  $S$  and  $T$  is  $S - T = \{x \in A : x \in S \wedge x \notin T\}$ .

**Proposition 2.10.18** (De Morgan's Law).

$$R - (S \cup T) = (R - S) \cap (R - T)$$

**Proposition 2.10.19** (De Morgan's Law).

$$R - (S \cap T) = (R - S) \cup (R - T)$$

## 2.11 Power Set

**Definition 2.11.1** (Power Set). The *power set* of a set  $A$  is

$$\mathcal{P}A := 2^A$$

**Definition 2.11.2** (Cover). Let  $X$  be a set and  $\mathcal{A} \subseteq \mathcal{P}X$ . Then  $\mathcal{A}$  is a *cover* of  $X$ , or *covers*  $X$ , iff  $\bigcup \mathcal{A} = X$ .

Given a subset  $Y$  of  $X$  and  $\mathcal{A} \subseteq \mathcal{P}X$ , we say  $\mathcal{A}$  *covers*  $Y$  iff  $Y \subseteq \bigcup \mathcal{A}$ .

## 2.12 Saturated Set

**Definition 2.12.1** (Saturated). Let  $A$  and  $B$  be sets. Let  $f : A \rightarrow B$  be surjective. Let  $C \subseteq A$ . Then  $C$  is *saturated* with respect to  $f$  iff, for all  $x \in C$  and  $y \in A$ , if  $f(x) = f(y)$  then  $y \in C$ .



## 2.13 Union

**Definition 2.13.1** (Union). Given  $\mathcal{A} \in \mathcal{PPX}$ , its *union* is

$$\bigcup \mathcal{A} := \{x \in X : \exists S \in \mathcal{A}. x \in S\} \in \mathcal{PX} .$$

### 2.13.1 Intersection

**Definition 2.13.2** (Intersection). Given  $\mathcal{A} \in \mathcal{PPX}$ , its *intersection* is

$$\bigcap \mathcal{A} := \{x \in X : \forall S \in \mathcal{A}. x \in S\} \in \mathcal{PX} .$$

### 2.13.2 Direct Image

**Definition 2.13.3** (Direct Image). Let  $f : A \rightarrow B$ . Let  $S$  be a subset of  $A$ . The *(direct) image* of  $S$  under  $f$  is the subset of  $B$  given by

$$f(S) := \{f(a) : a \in S\} .$$

**Proposition 2.13.4.**

1. If  $S \subseteq T$  then  $f(S) \subseteq f(T)$
2.  $f(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f(S)$

**Example 2.13.5.** It is not true in general that  $f(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f(S)$ . Take  $f$  to be the only function  $\{0, 1\} \rightarrow \{0\}$ , and  $\mathcal{S} = \{\{0\}, \{1\}\}$ . Then  $f(\bigcap \mathcal{S}) = \emptyset$  but  $\bigcap_{S \in \mathcal{S}} f(S) = \{0\}$ .

**Example 2.13.6.** It is not true in general that  $f(S - T) = f(S) - f(T)$ . Take  $f$  to be the only function  $\{0, 1\} \rightarrow \{0\}$ ,  $S = \{0\}$  and  $T = \{1\}$ . Then  $f(S - T) = \{0\}$  but  $f(S) - f(T) = \emptyset$ .

## 2.14 Inverse Image

**Definition 2.14.1** (Inverse Image). Let  $f : A \rightarrow B$ . Let  $S$  be a subset of  $B$ . The *inverse image* or *preimage* of  $S$  under  $f$  is the subset of  $A$  given by

$$f^{-1}(S) := \{x \in A : f(x) \in S\} .$$

**Proposition 2.14.2.** 1. If  $S \subseteq T$  then  $f^{-1}(S) \subseteq f^{-1}(T)$

2.  $f^{-1}(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f^{-1}(S)$
3.  $f^{-1}(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f^{-1}(S)$
4.  $f^{-1}(S - T) = f^{-1}(S) - f^{-1}(T)$
5.  $S \subseteq f^{-1}(f(S))$ . Equality holds if  $f$  is injective.
6.  $f(f^{-1}(T)) \subseteq T$ . Equality holds if  $f$  is surjective.
7.  $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$

### 2.14.1 Saturated Sets

**Proposition 2.14.3.** *Let  $A$  and  $B$  be sets. Let  $f : A \rightarrow B$  be surjective. Let  $C \subseteq A$ . Then  $C$  is saturated if and only if there exists  $D \subseteq B$  such that  $C = f^{-1}(D)$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $C$  is saturated then there exists  $D \subseteq B$  such that  $C = f^{-1}(D)$ .

$\langle 2 \rangle 1$ . ASSUME:  $C$  is saturated.

$\langle 2 \rangle 2$ . LET:  $D = f(C)$

$\langle 2 \rangle 3$ .  $C \subseteq f^{-1}(D)$

$\langle 3 \rangle 1$ . LET:  $x \in C$

$\langle 3 \rangle 2$ .  $f(x) \in D$

PROOF:  $\langle 2 \rangle 2$

$\langle 3 \rangle 3$ .  $x \in f^{-1}(D)$

$\langle 2 \rangle 4$ .  $f^{-1}(D) \subseteq C$

$\langle 3 \rangle 1$ . LET:  $x \in f^{-1}(D)$

$\langle 3 \rangle 2$ .  $f(x) \in D$

$\langle 3 \rangle 3$ . PICK  $y \in C$  such that  $f(x) = f(y)$

PROOF:  $\langle 2 \rangle 2$

$\langle 3 \rangle 4$ .  $x \in C$

PROOF:  $\langle 2 \rangle 1$

$\langle 1 \rangle 2$ . If there exists  $D \subseteq B$  such that  $C = f^{-1}(D)$  then  $C$  is saturated.

$\langle 2 \rangle 1$ . LET:  $D \subseteq B$  be such that  $C = f^{-1}(D)$ .

$\langle 2 \rangle 2$ . LET:  $x \in C$  and  $y \in A$

$\langle 2 \rangle 3$ . ASSUME:  $f(x) = f(y)$

$\langle 2 \rangle 4$ .  $f(x) \in D$

$\langle 2 \rangle 5$ .  $f(y) \in D$

$\langle 2 \rangle 6$ .  $y \in C$

□

## 2.15 Relations

**Definition 2.15.1** (Relation). Let  $A$  and  $B$  be sets. A *relation*  $R$  between  $A$  and  $B$ ,  $R : A \rightarrow B$ , is a subset of  $A \times B$ .

Given  $a \in A$  and  $b \in B$ , we write  $aRb$  for  $(a, b) \in R$ .

A relation *on* a set  $A$  is a relation between  $A$  and  $A$ .

**Definition 2.15.2** (Reflexive). A relation  $R$  on a set  $A$  is *reflexive* iff  $\forall a \in A. aRa$ .

**Definition 2.15.3** (Symmetric). A relation  $R$  on a set  $A$  is *symmetric* iff, whenever  $xRy$ , then  $yRx$ .

**Definition 2.15.4** (Transitive). A relation  $R$  on a set  $A$  is *transitive* iff, whenever  $xRy$  and  $yRz$ , then  $xRz$ .

### 2.15.1 Equivalence Relations

**Definition 2.15.5** (Equivalence Relation). A relation  $R$  on a set  $A$  is an *equivalence relation* iff it is reflexive, symmetric and transitive.

**Definition 2.15.6** (Equivalence Class). Let  $R$  be an equivalence relation on a set  $A$  and  $a \in A$ . The *equivalence class* of  $a$  with respect to  $R$  is

$$\{x \in A : xRa\} .$$

**Proposition 2.15.7.** *Two equivalence classes are either disjoint or equal.*

## 2.16 Power Set

**Definition 2.16.1** (Power Set). The *power set* of a set  $A$  is  $\mathcal{P}A := 2^A$ .

Given  $S \in \mathcal{P}A$  and  $a \in A$ , we write  $a \in S$  for  $S(a) = 1$ .

**Definition 2.16.2** (Pairwise Disjoint). Let  $P \subseteq \mathcal{P}A$ . We say the members of  $P$  are *pairwise disjoint* iff, for all  $S, T \in P$ , if  $S \neq T$  then  $S \cap T = \emptyset$ .

### 2.16.1 Partitions

**Definition 2.16.3** (Partition). Let  $A$  be a set. A *partition* of  $A$  is a set  $P \in \mathcal{P}\mathcal{P}A$  such that:

- $\bigcup P = A$
- Every member of  $P$  is nonempty.
- The members of  $P$  are pairwise disjoint.

## 2.17 Cartesian Product

**Definition 2.17.1** (Cartesian Product). Let  $A$  and  $B$  be sets. The *Cartesian product* of  $A$  and  $B$ ,  $A \times B$ , is the tabulation of the relation  $A \looparrowright B$  that holds for all  $a \in A$  and  $b \in B$ . The associated functions  $\pi_1 : A \times B \rightarrow A$  and  $\pi_2 : A \times B \rightarrow B$  are called the *projections*.

Given  $a \in A$  and  $b \in B$ , we write  $(a, b)$  for the unique element of  $A \times B$  such that  $\pi_1(a, b) = a$  and  $\pi_2(a, b) = b$ .

## 2.18 Quotient Sets

**Proposition 2.18.1.** *Let  $\sim$  be an equivalence relation on  $X$ . Then there exists a set  $X/\sim$ , the quotient set of  $X$  with respect to  $\sim$ , and a surjective function  $\pi : X \twoheadrightarrow X/\sim$ , the canonical projection, such that, for all  $x, y \in X$ , we have  $x \sim y$  if and only if  $\pi(x) = \pi(y)$ .*

*Further, if  $p : X \twoheadrightarrow Q$  is another quotient with respect to  $\sim$ , then there exists a unique bijection  $\phi : X/\sim \approx Q$  such that  $\phi \circ \pi = p$ .*

## 2.19 Partitions

**Definition 2.19.1** (Partition). A *partition* of a set  $X$  is a set of pairwise disjoint subsets of  $X$  whose union is  $X$ .

## 2.20 Disjoint Union

**Theorem 2.20.1.** For any sets  $A$  and  $B$ , there exists a set  $A + B$ , the disjoint union of  $A$  and  $B$ , and functions  $\kappa_1 : A \rightarrow A + B$  and  $\kappa_2 : B \rightarrow A + B$ , the injections, such that, for every set  $X$  and functions  $f : A \rightarrow X$  and  $g : B \rightarrow X$ , there exists a unique function  $[f, g] : A + B \rightarrow X$  such that  $[f, g] \circ \kappa_1 = f$  and  $[f, g] \circ \kappa_2 = g$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $A + B := \{p \in \mathcal{P}A \times \mathcal{P}B : \exists a \in A. p = (\{a\}, \emptyset) \vee \exists b \in B. p = (\emptyset, \{b\})\}$

**Definition 2.20.2** (Restriction). Let  $f : A \rightarrow B$  and let  $(S, i)$  be a subset of  $A$ . The *restriction* of  $f$  to  $S$  is the function  $f \upharpoonright S : S \rightarrow B$  defined by  $f \upharpoonright S = f \circ i$ .

## 2.21 Natural Numbers

**Theorem 2.21.1** (Principle of Recursive Definition). Let  $A$  be a set. Let  $F$  be the set of all functions  $\{m \in \mathbb{N} : m < n\} \rightarrow A$  for some  $n$ . Let  $\rho : F \rightarrow A$ . Then there exists a unique  $g : \mathbb{N} \rightarrow A$  such that, for all  $n \in \mathbb{N}$ , we have

$$g(n) = \rho(g \upharpoonright \{m \in \mathbb{N} : m < n\}) .$$

PROOF:

$\langle 1 \rangle 1$ . Given a subset  $B \subseteq \mathbb{N}$ , let us say that a function  $g : B \rightarrow A$  is *acceptable* iff, for all  $n \in B$ , we have

$$\forall m < n. m \in B$$

and

$$g(n) = \rho(g \upharpoonright \{m \in \mathbb{N} : m < n\}) .$$

$\langle 1 \rangle 2$ . For all  $n \in \mathbb{N}$ , there exists an acceptable function  $\{m \in \mathbb{N} : m < n\} \rightarrow A$ .

$\langle 2 \rangle 1$ . LET:  $P[n]$  be the property: There exists an acceptable function  $\{m \in \mathbb{N} : m < n\} \rightarrow A$ .

$\langle 2 \rangle 2$ .  $P[0]$

PROOF: The unique function  $\emptyset \rightarrow A$  is acceptable.

$\langle 2 \rangle 3$ . For any natural number  $n$ , if  $P[n]$  then  $P[n + 1]$ .

$\langle 3 \rangle 1$ . ASSUME:  $P[n]$

$\langle 3 \rangle 2$ . PICK an acceptable  $f : \{m \in \mathbb{N} : m < n\} \rightarrow A$ .

$\langle 3 \rangle 3$ . LET:  $g : \{m \in \mathbb{N} : m < n + 1\} \rightarrow A$  be the function

$$g(m) = \begin{cases} f(m) & \text{if } m < n \\ \rho(f) & \text{if } m = n \end{cases}$$

$\langle 3 \rangle 4$ .  $g$  is acceptable.

- ⟨1⟩3. If  $g : B \rightarrow A$  and  $h : C \rightarrow A$  are acceptable, then  $g$  and  $h$  agree on  $B \cap C$ .  
 ⟨1⟩4. Define  $g : \mathbb{N} \rightarrow A$  by:  $g(n) = a$  iff there exists an acceptable  $h : \{m \in \mathbb{N} : m < n + 1\}$  such that  $h(n) = a$ .  
 ⟨1⟩5.  $g$  is acceptable.  
 ⟨1⟩6. If  $g' : \mathbb{N} \rightarrow A$  is acceptable then  $g' = g$ .  
 □

## 2.22 Finite and Infinite Sets

**Definition 2.22.1** (Finite). A set  $A$  is *finite* iff there exists  $n \in \mathbb{N}$  such that  $A \approx \{m \in \mathbb{N} : m < n\}$ . In this case, we say  $A$  has *cardinality*  $n$ .

**Proposition 2.22.2.** Let  $n \in \mathbb{N}$ . Let  $A$  be a set. Let  $a_0 \in A$ . Then  $A \approx \{m \in \mathbb{N} : m < n + 1\}$  if and only if  $A - \{a_0\} \approx \{m \in \mathbb{N} : m < n\}$ .

**Theorem 2.22.3.** Let  $A$  be a set. Suppose that  $A \approx \{m \in \mathbb{N} : m < n\}$ . Let  $B$  be a proper subset of  $A$ . Then  $B \not\approx \{m \in \mathbb{N} : m < n\}$  but there exists  $m < n$  such that  $B \approx \{k \in \mathbb{N} : k < m\}$ .

PROOF:

- ⟨1⟩1. LET:  $P[n]$  be the property: for every set  $A$ , if  $A \approx \{m \in \mathbb{N} : m < n\}$ , then for every proper subset  $B$  of  $A$ , we have  $B \not\approx \{m \in \mathbb{N} : m < n\}$  but there exists  $m < n$  such that  $B \approx \{k \in \mathbb{N} : k < m\}$ .  
 ⟨1⟩2.  $P[0]$   
 PROOF: If  $A \approx \{m \in \mathbb{N} : m < 0\}$  then  $A$  is empty and so has no proper subset.  
 ⟨1⟩3. For every natural number  $n$ , if  $P[n]$  then  $P[n + 1]$ .  
 ⟨2⟩1. LET:  $n$  be a natural number.  
 ⟨2⟩2. ASSUME:  $P[n]$   
 ⟨2⟩3. LET:  $A$  be a set.  
 ⟨2⟩4. ASSUME:  $A \approx \{m \in \mathbb{N} : m < n + 1\}$   
 ⟨2⟩5. LET:  $B$  be a proper subset of  $A$ .  
 ⟨2⟩6. CASE:  $B = \emptyset$   
 PROOF: Then  $B \not\approx \{m \in \mathbb{N} : m < n + 1\}$  but  $B \approx \{k \in \mathbb{N} : k < 0\}$ .  
 ⟨2⟩7. CASE:  $B \neq \emptyset$   
 ⟨3⟩1. PICK  $b_0 \in B$   
 ⟨3⟩2.  $A - \{b_0\} \approx \{m \in \mathbb{N} : m < n\}$   
 ⟨3⟩3.  $B - \{b_0\}$  is a proper subset of  $A - \{b_0\}$   
 ⟨3⟩4.  $B - \{b_0\} \not\approx \{m \in \mathbb{N} : m < n\}$   
 ⟨3⟩5.  $B \approx \{m \in \mathbb{N} : m < n + 1\}$   
 ⟨3⟩6. PICK  $m < n$  such that  $B - \{b_0\} \approx \{k \in \mathbb{N} : k < m\}$   
 ⟨3⟩7.  $m + 1 < n + 1$   
 ⟨3⟩8.  $B \approx \{k \in \mathbb{N} : k < m + 1\}$   
 □

**Corollary 2.22.3.1.** If  $A$  is finite then there is no bijection between  $A$  and a proper subset of  $A$ .

**Corollary 2.22.3.2.**  $\mathbb{N}$  is infinite.

**Corollary 2.22.3.3.** The cardinality of a finite set is unique.

**Corollary 2.22.3.4.** A subset of a finite set is finite.

**Corollary 2.22.3.5.** If  $A$  is finite and  $B$  is a proper subset of  $A$  then  $|B| < |A|$ .

**Corollary 2.22.3.6.** Let  $A$  be a set. Then the following are equivalent:

1.  $A$  is finite.
2. There exists a surjection from an initial segment of  $\mathbb{N}$  onto  $A$ .
3. There exists an injection from  $A$  to an initial segment of  $\mathbb{N}$ .

**Corollary 2.22.3.7.** A finite union of finite sets is finite.

**Corollary 2.22.3.8.** A finite Cartesian product of finite sets is finite.

**Theorem 2.22.4.** Let  $A$  be a set. The following are equivalent:

1. There exists an injective function  $\mathbb{N} \rightarrow A$ .
2. There exists a bijection between  $A$  and a proper subset of  $A$ .
3.  $A$  is infinite.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$  LET:  $f : \mathbb{N} \rightarrow A$  be injective.

$\langle 2 \rangle 2.$  LET:  $s : \mathbb{N} \approx \mathbb{N} - \{0\}$  be the function  $s(n) = n + 1$ .

$\langle 2 \rangle 3.$   $f \circ s \circ f^{-1} : A \approx A - \{f(0)\}$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: Corollary 2.22.3.1.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: Choose a function  $f : \mathbb{N} \rightarrow A$  such that  $f(n) \in A - \{f(m) : m < n\}$  for all  $n$ .

□

## 2.23 Countable Sets

**Definition 2.23.1** (Countable). A set  $A$  is *countably infinite* iff  $A \approx \mathbb{N}$ .

**Proposition 2.23.2.**  $\mathbb{N} \times \mathbb{N}$  is countably infinite.

PROOF: Define  $f : \mathbb{N} \times \mathbb{N} \approx \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x\}$  by

$$f(x, y) = (x + y, y)$$

Define  $g : \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x\} \approx \mathbb{N}$  by

$$g(x, y) = x(x - 1)/2 + y. \quad \square$$

**Proposition 2.23.3.** *Every infinite subset of  $\mathbb{N}$  is countably infinite.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $C$  be an infinite subset of  $\mathbb{N}$

$\langle 1 \rangle 2$ . Define  $h : \mathbb{Z} \rightarrow C$  by recursion thus:  $h(n)$  is the smallest element of  $C - \{h(m) : m < n\}$ .

$\langle 1 \rangle 3$ .  $h$  is injective.

PROOF: If  $m < n$  then  $h(m) \neq h(n)$  because  $h(n) \in C - \{h(m) : m < n\}$ .

$\langle 1 \rangle 4$ .  $h$  is surjective.

$\langle 2 \rangle 1$ . For all  $n \in \mathbb{N}$  we have  $n \leq h(n)$ .

$\langle 2 \rangle 2$ . LET:  $c \in C$

$\langle 2 \rangle 3$ .  $c \leq h(c)$

$\langle 2 \rangle 4$ . LET:  $n$  be least such that  $c \leq h(n)$

$\langle 2 \rangle 5$ .  $c \in C - \{h(m) : m < n\}$

$\langle 2 \rangle 6$ .  $h(n) \leq c$

$\langle 2 \rangle 7$ .  $h(n) = c$

□

**Definition 2.23.4** (Countable). A set is *countable* iff it is either finite or countably infinite; otherwise it is *uncountable*.

**Proposition 2.23.5.** *Let  $B$  be a nonempty set. Then the following are equivalent.*

1.  $B$  is countable.
2. There exists a surjection  $\mathbb{N} \twoheadrightarrow B$ .
3. There exists an injection  $B \hookrightarrow \mathbb{N}$ .

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

$\langle 2 \rangle 1$ . ASSUME:  $B$  is countable.

$\langle 2 \rangle 2$ . CASE:  $B$  is finite.

$\langle 3 \rangle 1$ . PICK a natural number  $n$  and bijection  $f : \{m \in \mathbb{N} : m < n\} \approx B$

$\langle 3 \rangle 2$ . PICK  $b \in B$

$\langle 3 \rangle 3$ . Extend  $f$  to a surjection  $g : \mathbb{N} \twoheadrightarrow B$  by setting  $g(m) = b$  for  $m \geq n$ .

$\langle 2 \rangle 3$ . CASE:  $B$  is countably infinite.

PROOF: Then there exists a bijection  $\mathbb{N} \approx B$ .

$\langle 1 \rangle 2$ .  $2 \Rightarrow 3$

PROOF: Given a surjection  $f : \mathbb{N} \twoheadrightarrow B$ , define  $g : B \hookrightarrow \mathbb{N}$  by  $g(b)$  is the smallest number such that  $f(g(b)) = b$ .

$\langle 1 \rangle 3$ .  $3 \Rightarrow 1$

$\langle 2 \rangle 1$ . LET:  $f : B \hookrightarrow \mathbb{N}$  be injective.

$\langle 2 \rangle 2$ .  $f(B)$  is countable.

$\langle 2 \rangle 3$ .  $B \approx f(B)$

$\langle 2 \rangle 4$ .  $B$  is countable.

□

**Corollary 2.23.5.1.** *A subset of a countable set is countable.*

**Corollary 2.23.5.2.**  $\mathbb{N} \times \mathbb{N}$  *is countably infinite.*

PROOF: The function that maps  $(m, n)$  to  $2^m 3^n$  is injective.  $\square$

**Corollary 2.23.5.3.** *The Cartesian product of two countable sets is countable.*

**Theorem 2.23.6.** *A countable union of countable sets is countable.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be a set.

$\langle 1 \rangle 2$ . LET:  $\mathcal{B} \subseteq \mathcal{P}A$  be a countable set of countable sets such that  $\bigcup \mathcal{B} = A$

$\langle 1 \rangle 3$ . PICK a surjection  $B : \mathbb{N} \rightarrow \mathcal{B}$

$\langle 1 \rangle 4$ . ASSUME: w.l.o.g. each  $B(n)$  is nonempty.

$\langle 1 \rangle 5$ . For  $n \in \mathbb{N}$ , PICK a surjective function  $g_n : \mathbb{N} \rightarrow B(n)$

$\langle 1 \rangle 6$ . LET:  $h : \mathbb{N} \times \mathbb{N} \rightarrow A$  be the function  $h(m, n) = g_m(n)$

$\langle 1 \rangle 7$ .  $h$  is surjective.

$\square$

**Theorem 2.23.7.**  $2^{\mathbb{N}}$  *is uncountable.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : \mathbb{N} \rightarrow 2^{\mathbb{N}}$

PROVE:  $f$  is not surjective.

$\langle 1 \rangle 2$ . Define  $g : \mathbb{N} \rightarrow 2$  by  $g(n) = 1 - f(n)(n)$ .

$\langle 1 \rangle 3$ . For all  $n \in \mathbb{N}$  we have  $g(n) \neq f(n)(n)$ .

$\langle 1 \rangle 4$ . For all  $n \in \mathbb{N}$  we have  $g \neq f(n)$ .

$\square$

**Theorem 2.23.8.** *For any set  $A$ , there is no surjective function  $A \rightarrow \mathcal{P}A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow \mathcal{P}A$

$\langle 1 \rangle 2$ . LET:  $S = \{x \in A : x \notin f(x)\}$

$\langle 1 \rangle 3$ . For all  $a \in A$  we have  $S \neq f(a)$

PROOF: We have  $a \in S$  if and only if  $a \notin f(a)$ .

$\square$

**Corollary 2.23.8.1.** *For any set  $A$ , there is no injective function  $\mathcal{P}A \rightarrow A$ .*

## 2.24 Fixed Points

**Definition 2.24.1** (Fixed Point). Let  $A$  be a set and  $f : A \rightarrow A$ . A *fixed point* of  $f$  is an element  $a \in A$  such that  $f(a) = a$ .

## 2.25 Finite Intersection Property

**Definition 2.25.1** (Finite Intersection Property). Let  $X$  be a set. Let  $\mathcal{C} \subseteq \mathcal{P}X$ . Then  $\mathcal{C}$  has the *finite intersection property* iff every finite nonempty subset of  $\mathcal{C}$  has nonempty intersection.



## Chapter 3

# Relations

**Definition 3.0.1** (Reflexive). A relation  $R \subseteq A \times A$  is *reflexive* iff, for all  $a \in A$ , we have  $(a, a) \in R$ .

**Definition 3.0.2** (Antisymmetric). A relation  $R \subseteq A \times A$  is *antisymmetric* iff, for all  $a, b \in A$ , if  $(a, b) \in R$  and  $(b, a) \in R$  then  $a = b$ .

**Definition 3.0.3** (Transitive). A relation  $R \subseteq A \times A$  is *transitive* iff, for all  $a, b, c \in A$ , if  $(a, b) \in R$  and  $(b, c) \in R$  then  $(a, c) \in R$ .

**Definition 3.0.4** (Partial Order). A *partial order* on a set  $A$  is a relation on  $A$  that is reflexive, antisymmetric and transitive.

We say  $(A, \leq)$  is a *partially ordered set* or *poset* iff  $\leq$  is a partial order on  $A$ .

**Definition 3.0.5** (Greatest). Let  $A$  be a poset and  $a \in A$ . Then  $a$  is the *greatest* element iff  $\forall x \in A. x \leq a$ .

**Definition 3.0.6** (Least). Let  $A$  be a poset and  $a \in A$ . Then  $a$  is the *least* element iff  $\forall x \in A. a \leq x$ .

**Definition 3.0.7** (Upper Bound). Let  $A$  be a poset,  $S \subseteq A$ , and  $u \in A$ . Then  $u$  is an *upper bound* for  $S$  iff  $\forall x \in S. x \leq u$ . We say  $S$  is *bounded above* iff it has an upper bound.

**Definition 3.0.8** (Lower Bound). Let  $A$  be a poset,  $S \subseteq A$ , and  $l \in A$ . Then  $l$  is a *lower bound* for  $S$  iff  $\forall x \in S. l \leq x$ . We say  $S$  is *bounded below* iff it has a lower bound.

**Definition 3.0.9** (Supremum). Let  $A$  be a poset,  $S \subseteq A$  and  $s \in A$ . Then  $s$  is the *supremum* or *least upper bound* for  $S$  iff  $s$  is the least element in the sub-poset of upper bounds for  $A$ .

**Definition 3.0.10** (Infimum). Let  $A$  be a poset,  $S \subseteq A$  and  $i \in A$ . Then  $i$  is the *infimum* or *greatest lower bound* for  $S$  iff  $i$  is the greatest element in the sub-poset of lower bounds for  $A$ .

**Definition 3.0.11** (Least Upper Bound Property). A poset  $A$  has the *least upper bound property* iff every nonempty subset of  $A$  that is bounded above has a least upper bound.

**Proposition 3.0.12.** *Let  $A$  be a poset. Then  $A$  has the least upper bound property if and only if every nonempty subset of  $A$  that is bounded below has a greatest lower bound.*

PROOF:

$\langle 1 \rangle 1$ . If  $A$  has the least upper bound property then every subset of  $A$  that is bounded below has a greatest lower bound.

$\langle 2 \rangle 1$ . ASSUME:  $A$  has the least upper bound property.

$\langle 2 \rangle 2$ . LET:  $S \subseteq A$  be nonempty and bounded below.

$\langle 2 \rangle 3$ . LET:  $L$  be the set of lower bounds of  $S$ .

$\langle 2 \rangle 4$ .  $L$  is nonempty.

PROOF: Because  $S$  is bounded below.

$\langle 2 \rangle 5$ .  $L$  is bounded above.

PROOF: Pick an element  $s \in S$ . Then  $s$  is an upper bound for  $L$ .

$\langle 2 \rangle 6$ . LET:  $s$  be the supremum of  $L$ .

$\langle 2 \rangle 7$ .  $s$  is the greatest lower bound of  $S$ .

$\langle 3 \rangle 1$ .  $s$  is a lower bound of  $S$ .

$\langle 4 \rangle 1$ . LET:  $x \in S$

$\langle 4 \rangle 2$ .  $x$  is an upper bound for  $L$ .

$\langle 4 \rangle 3$ .  $s \leq x$

$\langle 3 \rangle 2$ . For any lower bound  $l$  of  $S$  we have  $l \leq s$ .

PROOF: Immediate from  $\langle 2 \rangle 6$ .

$\langle 1 \rangle 2$ . If every subset of  $A$  that is bounded below has a greatest lower bound, then  $A$  has the least upper bound property.

PROOF: Dual.

□

## Chapter 4

# Order Theory

### 4.1 Strict Partial Orders

**Definition 4.1.1** (Strict Partial Order). A *strict partial order* on a set  $A$  is a relation on  $A$  that is irreflexive and transitive.

**Proposition 4.1.2.** 1. If  $\leq$  is a partial order on  $A$  then  $<$  is a strict partial order on  $A$ , where  $x < y$  iff  $x \leq y \wedge x \neq y$ .

2. If  $<$  is a strict partial order on  $A$  then  $\leq$  is a partial order on  $A$ , where  $x \leq y$  iff  $x < y \vee x = y$ .

3. These two relations are inverses of one another.

#### 4.1.1 Linear Orders

**Definition 4.1.3** (Linear Order). A *linear order* on a set  $A$  is a partial order  $\leq$  on  $A$  such that, for all  $x, y \in A$ , we have  $x \leq y$  or  $y \leq x$ .

A *linearly ordered set* is a pair  $(X, \leq)$  such that  $X$  is a set and  $\leq$  is a linear order on  $X$ .

**Definition 4.1.4** (Open Interval). Let  $X$  be a linearly ordered set and  $a, b \in X$ . The *open interval*  $(a, b)$  is the set

$$\{x \in X : a < x < b\} .$$

**Definition 4.1.5** (Immediate Predecessor, Immediate Successor). Let  $X$  be a linearly ordered set and  $a, b \in X$ . Then  $b$  is the (*immediate*) *successor* of  $a$ , and  $a$  is the (*immediate*) *predecessor* of  $b$ , iff  $a < b$  and there is no  $x$  such that  $a < x < b$ .

**Definition 4.1.6** (Dictionary Order). Let  $A$  and  $B$  be linearly ordered sets. The *dictionary order* on  $A \times B$  is the order defined by

$$(a, b) < (a', b') \Leftrightarrow a < a' \vee (a = a' \wedge b < b') .$$

**Theorem 4.1.7** (Maximum Principle). *Every poset has a maximal linearly ordered subset.*

PROOF:

⟨1⟩1. LET:  $(A, \leq)$  be a poset.

⟨1⟩2. PICK a well ordering  $\preceq$  of  $A$ .

PROOF: Well Ordering Theorem.

⟨1⟩3. LET:  $h : A \rightarrow 2$  be the function defined by  $\preceq$ -recursion thus:

$$h(a) = \begin{cases} 1 & \text{if } a \text{ is } \leq\text{-comparable with every } b < a \text{ such that } h(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

⟨1⟩4. LET:  $B = \{x \in A : h(x) = 1\}$

PROVE:  $B$  is a maximal subset linearly ordered by  $\leq$ .

⟨1⟩5.  $B$  is linearly ordered by  $\leq$ .

⟨2⟩1. LET:  $x, y \in B$

⟨2⟩2. ASSUME: w.l.o.g.  $x \preceq y$

⟨2⟩3.  $y$  is  $\leq$ -comparable with  $x$

⟨1⟩6. For any subset  $C \subseteq A$  linearly ordered by  $\leq$ , if  $B \subseteq C$  then  $B = C$ .

⟨2⟩1. LET:  $x \in C$

⟨2⟩2.  $x$  is comparable with every  $y \preceq x$  such that  $h(y) = 1$

⟨2⟩3.  $x \in B$

□

**Theorem 4.1.8** (Zorn's Lemma). *Let  $A$  be a poset. If every linearly ordered subset of  $A$  is bounded above, then  $A$  has a maximal element.*

PROOF:

⟨1⟩1. PICK a maximal linearly ordered subset  $B$  of  $A$ .

PROOF: Maximal Principle

⟨1⟩2. PICK an upper bound  $c$  for  $B$ .

PROVE:  $c$  is maximal.

⟨1⟩3. LET:  $x \in A$

⟨1⟩4. ASSUME:  $c \leq x$

PROVE:  $x = c$

⟨1⟩5.  $x$  is an upper bound for  $B$ .

⟨1⟩6.  $x \in B$

PROOF: By the maximality of  $B$ , since  $B \cup \{x\}$  is linearly ordered.

⟨1⟩7.  $x \leq c$

PROOF: ⟨1⟩2

⟨1⟩8.  $x = c$

□

**Corollary 4.1.8.1** (Kuratowski's Lemma). *Let  $\mathcal{A} \subseteq \mathcal{P}X$ . Suppose that, for every subset  $\mathcal{B} \subseteq \mathcal{A}$  that is linearly ordered by inclusion, we have  $\bigcup \mathcal{B} \in \mathcal{A}$ . Then  $\mathcal{A}$  has a maximal element.*

**Definition 4.1.9** (Closed Interval). Let  $X$  be a linearly ordered set. Let  $a, b \in X$  with  $a < b$ . The *closed interval*  $[a, b]$  is

$$[a, b] := \{x \in X : a \leq x \leq b\} .$$

**Definition 4.1.10** (Half-Open Interval). Let  $X$  be a linearly ordered set. Let  $a, b \in X$  with  $a < b$ . The *half-open intervals*  $(a, b]$  and  $[a, b)$  are defined by

$$\begin{aligned}(a, b] &:= \{x \in X : a < x \leq b\} \\ [a, b) &:= \{x \in X : a \leq x < b\}\end{aligned}$$

**Definition 4.1.11** (Open Ray). Let  $X$  be a linearly ordered set and  $a \in X$ . The *open rays*  $(a, +\infty)$  and  $(-\infty, a)$  are defined by:

$$\begin{aligned}(a, +\infty) &:= \{x \in X : a < x\} \\ (-\infty, a) &:= \{x \in X : x < a\}\end{aligned}$$

**Definition 4.1.12** (Closed Ray). Let  $X$  be a linearly ordered set and  $a \in X$ . The *closed rays*  $[a, +\infty)$  and  $(-\infty, a]$  are defined by:

$$\begin{aligned}[a, +\infty) &:= \{x \in X : a \leq x\} \\ (-\infty, a] &:= \{x \in X : x \leq a\}\end{aligned}$$

**Definition 4.1.13** (Convex). Let  $X$  be a linearly ordered set and  $Y \subseteq X$ . Then  $Y$  is *convex* iff, for all  $a, b \in Y$  and  $c \in X$ , if  $a < c < b$  then  $c \in Y$ .

### 4.1.2 Sets of Finite Type

**Definition 4.1.14** (Finite Type). Let  $X$  be a set. Let  $\mathcal{A} \subseteq \mathcal{P}X$ . Then  $\mathcal{A}$  is of *finite type* if and only if, for any  $B \subseteq X$ , we have  $B \in \mathcal{A}$  if and only if every finite subset of  $B$  is in  $\mathcal{A}$ .

**Proposition 4.1.15** (Tukey's Lemma). *Let  $X$  be a set. Let  $\mathcal{A} \subseteq \mathcal{P}X$ . If  $\mathcal{A}$  is of finite type, then  $\mathcal{A}$  has a maximal element.*

PROOF:

$\langle 1 \rangle 1$ . For every subset  $\mathcal{B} \subseteq \mathcal{A}$  that is linearly ordered by inclusion, we have  $\bigcup \mathcal{B} \in \mathcal{A}$ .

$\langle 2 \rangle 1$ . LET:  $\mathcal{B} \subseteq \mathcal{A}$

$\langle 2 \rangle 2$ . ASSUME:  $\mathcal{B}$  is linearly ordered by inclusion.

$\langle 2 \rangle 3$ . Every finite subset of  $\bigcup \mathcal{B}$  is in  $\mathcal{A}$

$\langle 2 \rangle 4$ .  $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: Kuratowski's Lemma.

□

## 4.2 Linear Continua

**Definition 4.2.1** (Linear Continuum). A *linear continuum* is a linearly ordered set with more than one element that is dense and has the least upper bound property.

**Proposition 4.2.2.** *Every convex subset of a linear continuum with more than one element is a linear continuum.*

PROOF: Easy.  $\square$

**Corollary 4.2.2.1.** *Every interval and ray in a linear continuum is a linear continuum.*

### 4.3 Well Orders

**Definition 4.3.1** (Well Ordered Set). A *well ordered set* is a linearly ordered set such that every nonempty subset has a least element.

**Proposition 4.3.2.** *Any subset of a well ordered set is well ordered.*

**Proposition 4.3.3.** *The product of two well ordered sets is well ordered under the dictionary order.*

**Theorem 4.3.4** (Well Ordering Theorem). *Every set has a well ordering.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a set.

$\langle 1 \rangle 2$ . PICK a choice function  $c : \mathcal{P}X - \{\emptyset\} \rightarrow X$

$\langle 1 \rangle 3$ . Define a *tower* to be a pair  $(T, <)$  where  $T \subseteq X$ ,  $<$  is a well ordering of  $T$ , and

$$\forall x \in T. x = c(X - \{y \in T : y < x\}) .$$

$\langle 1 \rangle 4$ . Given two towers, either they are equal or one is a section of the other.

$\langle 2 \rangle 1$ . LET:  $(T_1, <_1)$  and  $(T_2, <_2)$  be towers.

$\langle 2 \rangle 2$ . ASSUME: w.l.o.g. there exists a strictly monotone function  $h : T_1 \rightarrow T_2$

$\langle 2 \rangle 3$ .  $h(T_1)$  is either  $T_2$  or a section of  $T_2$

PROOF: Proposition 4.3.11.

$\langle 2 \rangle 4$ .  $\forall x \in T_1. h(x) = x$

$\langle 3 \rangle 1$ . LET:  $x \in T_1$

$\langle 3 \rangle 2$ . ASSUME: as transfinite induction hypothesis  $\forall y < x. h(y) = y$

$\langle 3 \rangle 3$ .  $h(x)$  is the least element of  $T_2 - \{h(y) \in T_1 : y < x\}$

$\langle 3 \rangle 4$ .  $h(x)$  is the least element of  $T_2 - \{y \in T_1 : y < x\}$

PROOF:  $\langle 3 \rangle 2$

$\langle 3 \rangle 5$ .  $h(x) = x$

PROOF:

$$h(x) = c(X - \{y \in T_2 : y < h(x)\}) \quad (\langle 1 \rangle 3)$$

$$= c(X - \{y \in T_2 : y < x\}) \quad (\langle 3 \rangle 4)$$

$$= c(X - \{y \in T_1 : y < x\}) \quad (\langle 3 \rangle 2)$$

$$= x \quad (\langle 1 \rangle 3)$$

$\langle 1 \rangle 5$ . If  $(T, <)$  is a tower and  $T \neq X$ , then there exists a tower of which  $(T, <)$  is a section.

PROOF: Let  $T_1 = T \cup \{c(T)\}$  and  $<_1$  be the extension of  $<$  such that  $x < c(T)$  for all  $x \in T$ .

- $\langle 1 \rangle 6$ . LET:  $\mathbf{T} = \bigcup \{T : \exists R. (T, R) \text{ is a tower}\}$  and  $\mathbf{R} = \bigcup \{R : \exists T. (T, R) \text{ is a tower}\}$   
 $\langle 1 \rangle 7$ .  $(\mathbf{T}, \mathbf{R})$  is a tower.  
 $\langle 2 \rangle 1$ .  $\mathbf{R}$  is irreflexive.  
 PROOF: Since for every tower  $(T, <)$  we have  $<$  is irreflexive.  
 $\langle 2 \rangle 2$ .  $\mathbf{R}$  is transitive.  
 $\langle 3 \rangle 1$ . ASSUME:  $x\mathbf{R}y$  and  $y\mathbf{R}z$   
 $\langle 3 \rangle 2$ . PICK towers  $(T_1, <_1)$  and  $(T_2, <_2)$  such that  $x <_1 y$  and  $y <_2 z$   
 $\langle 3 \rangle 3$ . ASSUME: w.l.o.g.  $(T_1, <_1)$  is either  $(T_2, <_2)$  or a section of  $(T_2, <_2)$   
 $\langle 3 \rangle 4$ .  $x <_2 y <_2 z$   
 $\langle 3 \rangle 5$ .  $x <_2 z$   
 $\langle 3 \rangle 6$ .  $x\mathbf{R}z$   
 $\langle 2 \rangle 3$ . For all  $x, y \in \mathbf{T}$ , either  $x\mathbf{R}y$  or  $x = y$  or  $y\mathbf{R}x$   
 PROOF: There exists a tower that has both  $x$  and  $y$ .  
 $\langle 2 \rangle 4$ . Every nonempty subset of  $\mathbf{T}$  has an  $\mathbf{R}$ -least element.  
 $\langle 3 \rangle 1$ . LET:  $A \subseteq \mathbf{T}$  be nonempty.  
 $\langle 3 \rangle 2$ . PICK  $a \in A$   
 $\langle 3 \rangle 3$ . PICK a tower  $(T, <)$  such that  $a \in T$ .  
 $\langle 3 \rangle 4$ . LET:  $b$  be the  $<$ -least element of  $A \cap T$   
 PROVE:  $b$  is  $\mathbf{R}$ -least in  $A$ .  
 $\langle 3 \rangle 5$ . LET:  $x \in A$   
 $\langle 3 \rangle 6$ . Etc.  
 $\langle 2 \rangle 5$ .  $\forall x \in \mathbf{T}. x = c(X - \{y \in \mathbf{T} : y\mathbf{R}x\})$   
 $\langle 1 \rangle 8$ .  $\mathbf{T} = X$   
 $\langle 1 \rangle 9$ .  $\mathbf{R}$  is a well ordering of  $X$ .  
 $\square$

**Proposition 4.3.5.** *There exists a well-ordered set with a largest element  $\Omega$  such that  $(-\infty, \Omega)$  is uncountable but, for all  $\alpha < \Omega$ , we have  $(-\infty, \alpha)$  is countable.*

PROOF:

- $\langle 1 \rangle 1$ . PICK an uncountable well ordered set  $B$ .  
 $\langle 1 \rangle 2$ . LET:  $C = 2 \times B$  under the dictionary order.  
 $\langle 1 \rangle 3$ . LET:  $\Omega$  be the least element of  $C$  such that  $(-\infty, \Omega)$  is uncountable.  
 $\langle 1 \rangle 4$ . LET:  $A = (-\infty, \Omega]$   
 $\langle 1 \rangle 5$ .  $A$  is a well ordered set with largest element  $\Omega$  such that  $(-\infty, \Omega)$  is uncountable but, for all  $\alpha < \Omega$ , we have  $(-\infty, \alpha)$  is countable.  
 $\square$

**Proposition 4.3.6.** *Every well ordered set has the least upper bound property.*

PROOF: For any subset that is bounded above, the set of upper bounds is nonempty, hence has a least element.  $\square$

**Proposition 4.3.7.** *In a well ordered set, every element that is not greatest has a successor.*

PROOF: If  $a$  is not greatest, then  $\{x : x > a\}$  is nonempty, hence has a least element.  $\square$

**Theorem 4.3.8** (Transfinite Induction). *Let  $J$  be a well ordered set. Let  $S \subseteq J$ . Assume that, for every  $\alpha \in J$ , if  $\forall x < \alpha. x \in S$  then  $\alpha \in S$ . Then  $S = J$ .*

PROOF: Otherwise  $J - S$  would be a nonempty subset of  $J$  with no least element.  $\square$

**Proposition 4.3.9.** *Let  $I$  be a well ordered set. Let  $\{A_i\}_{i \in I}$  be a family of well ordered sets. Define  $<$  on  $\coprod_{i \in I} A_i$  by:  $\kappa_i(a) < \kappa_j(b)$  iff either  $i < j$ , or  $i = j$  and  $a < b$  in  $A_i$ . Then  $<$  well orders  $\coprod_{i \in I} A_i$ .*

PROOF: Easy.  $\square$

**Theorem 4.3.10** (Principle of Transfinite Recursion). *Let  $J$  be a well ordered set. Let  $C$  be a set. Let  $\mathcal{F}$  be the set of all functions from a section of  $J$  into  $C$ . Let  $\rho : \mathcal{F} \rightarrow C$ . Then there exists a unique function  $h : J \rightarrow C$  such that, for all  $\alpha \in J$ , we have*

$$h(\alpha) = \rho(h \upharpoonright (-\infty, \alpha)) .$$

PROOF:

$\langle 1 \rangle 1$ . For a function  $h$  mapping either a section of  $J$  or all of  $J$  into  $C$ , let us say  $h$  is *acceptable* iff, for all  $x \in \text{dom } h$ , we have  $(-\infty, x) \subseteq \text{dom } h$  and  $h(x) = \rho(h \upharpoonright (-\infty, x))$ .

$\langle 1 \rangle 2$ . If  $h$  and  $k$  are acceptable functions then  $h(x) = k(x)$  for all  $x$  in both domains.

$\langle 2 \rangle 1$ . LET:  $x \in J$

$\langle 2 \rangle 2$ . ASSUME: as transfinite induction hypothesis that, for all  $y < x$  and any acceptable functions  $h$  and  $k$  with  $y \in \text{dom } h \cap \text{dom } k$ , we have  $h(y) = k(y)$

$\langle 2 \rangle 3$ . LET:  $h$  and  $k$  be acceptable functions with  $x \in \text{dom } h \cap \text{dom } k$

$\langle 2 \rangle 4$ .  $h \upharpoonright (-\infty, x) = k \upharpoonright (-\infty, x)$

PROOF: By  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 5$ .  $h(x) = k(x)$

PROOF: By  $\langle 2 \rangle 3$ , each is the least element of the set in  $\langle 2 \rangle 4$ .

$\langle 1 \rangle 3$ . For  $\alpha \in J$ , if there exists an acceptable function  $(-\infty, \alpha) \rightarrow C$ , then there exists an acceptable function  $(-\infty, \alpha] \rightarrow C$ .

$\langle 2 \rangle 1$ . LET:  $\alpha \in J$

$\langle 2 \rangle 2$ . LET:  $f : (-\infty, \alpha) \rightarrow C$  be acceptable.

$\langle 2 \rangle 3$ . LET:  $g : (-\infty, \alpha] \rightarrow C$  be the function given by

$$g(x) = \begin{cases} f(x) & \text{if } x < \alpha \\ \rho(f) & \text{if } x = \alpha \end{cases}$$

$\langle 2 \rangle 4$ .  $g$  is acceptable.

$\langle 1 \rangle 4$ . Let  $K \subseteq J$ . Assume that, for all  $\alpha \in K$ , there exists an acceptable function  $(-\infty, \alpha) \rightarrow C$ . Then there exists an acceptable function  $\bigcup_{\alpha \in K} (-\infty, \alpha) \rightarrow C$ .

$\langle 2 \rangle 1$ . Define  $f : \bigcup_{\alpha \in K} (-\infty, \alpha) \rightarrow C$  by:  $f(x) = y$  iff there exists  $\alpha \in K$  and  $g : (-\infty, \alpha) \rightarrow C$  acceptable such that  $g(x) = y$ .

$\langle 1 \rangle 5$ . For every  $\beta \in J$ , there exists an acceptable function  $(-\infty, \beta) \rightarrow C$



- ⟨2⟩1. LET:  $\beta \in J$
- ⟨2⟩2. ASSUME: as transfinite induction hypothesis that, for all  $\alpha < \beta$ , there exists an acceptable function  $(-\infty, \alpha) \rightarrow C$
- ⟨2⟩3. CASE:  $\beta$  has a predecessor
  - ⟨3⟩1. LET:  $\alpha$  be the predecessor of  $\beta$ .
  - ⟨3⟩2. There exists an acceptable function  $(-\infty, \alpha) \rightarrow C$ .
  - ⟨3⟩3. There exists an acceptable function  $(-\infty, \beta) \rightarrow C$ .
 PROOF: By ⟨1⟩3 since  $(-\infty, \beta) = (-\infty, \alpha]$ .
- ⟨2⟩4. CASE:  $\beta$  has no predecessor.
 PROOF: The result follows by ⟨1⟩4 since  $(-\infty, \beta) = \bigcup_{\alpha < \beta} (-\infty, \alpha)$ .
- ⟨1⟩6. There exists an acceptable function  $J \rightarrow C$ .
  - ⟨2⟩1. CASE:  $J$  has a greatest element.
    - ⟨3⟩1. LET:  $g$  be greatest.
    - ⟨3⟩2. There exists an acceptable function  $(-\infty, g) \rightarrow C$ .
 PROOF: ⟨1⟩5
    - ⟨3⟩3. There exists an acceptable function  $J \rightarrow C$ .
 PROOF: By ⟨1⟩3 since  $J = (-\infty, g]$ .
  - ⟨2⟩2. CASE:  $J$  has no greatest element.
 PROOF: By ⟨1⟩4 since  $J = \bigcup_{\alpha \in J} (-\infty, \alpha)$ .

□

**Corollary 4.3.10.1** (Cardinal Comparability). *Let  $A$  and  $B$  be sets. Then either  $A \leq B$  or  $B \leq A$ .*

PROOF: Choose well orderings of  $A$  and  $B$ . Then either there exists a surjection  $A \twoheadrightarrow B$ , or there exists an injective function  $h : A \rightarrow B$  defined by transfinite recursion by  $h(x)$  is the least element of  $B - h((-\infty, x))$ . □

**Proposition 4.3.11.** *Let  $J$  and  $E$  be well ordered sets. Let  $h : J \rightarrow E$ . Then the following are equivalent.*

1.  $h$  is strictly monotone and  $h(J)$  is either  $E$  or a section of  $E$ .
2. For all  $\alpha \in J$ , we have  $h(\alpha)$  is the least element of  $E - h((-\infty, \alpha))$ .

PROOF:

- ⟨1⟩1.  $1 \Rightarrow 2$ 
  - ⟨2⟩1. ASSUME: 1
  - ⟨2⟩2.  $h(J)$  is closed downwards.
  - ⟨2⟩3. LET:  $\alpha \in J$
  - ⟨2⟩4.  $h(\alpha) \in E - h((-\infty, \alpha))$ 
 PROOF: If  $\beta < \alpha$  then  $h(\beta) < h(\alpha)$ .
  - ⟨2⟩5. For all  $y \in E - h((-\infty, \alpha))$  we have  $h(\alpha) \leq y$ 
    - ⟨3⟩1. ASSUME: for a contradiction  $y < h(\alpha)$
    - ⟨3⟩2.  $y \in h(J)$
    - ⟨3⟩3. PICK  $\beta \in J$  such that  $h(\beta) = y$
    - ⟨3⟩4.  $h(\beta) < h(\alpha)$
    - ⟨3⟩5.  $\beta < \alpha$

⟨3⟩6. Q.E.D.

PROOF: This contradicts the fact that  $y \notin h((-\infty, \alpha))$ .

⟨1⟩2.  $2 \Rightarrow 1$

⟨2⟩1. ASSUME: 2

⟨2⟩2.  $h$  is strictly monotone.

⟨3⟩1. LET:  $\alpha, \beta \in J$  with  $\alpha < \beta$

⟨3⟩2.  $h(\alpha) \neq h(\beta)$

PROOF: Because  $h(\beta) \in E - h((-\infty, \beta))$ .

⟨3⟩3.  $h(\alpha) \leq h(\beta)$

PROOF: Because  $h(\alpha)$  is least in  $E - h((-\infty, \alpha))$ .

⟨3⟩4.  $h(\alpha) < h(\beta)$

⟨2⟩3.  $h(J)$  is either  $E$  or a section of  $E$ .

⟨3⟩1. ASSUME:  $h(J) \neq E$

⟨3⟩2. LET:  $e$  be least in  $E - h(J)$

PROVE:  $h(J) = (-\infty, e)$

⟨3⟩3.  $h(J) \subseteq (-\infty, e)$

⟨4⟩1. LET:  $\alpha \in J$

⟨4⟩2.  $h(\alpha) \neq e$

PROOF:  $e \notin h(J)$

⟨4⟩3.  $h(\alpha) \leq e$

PROOF: Since  $h(\alpha)$  is least in  $E - h((-\infty, \alpha))$ .

⟨4⟩4.  $h(\alpha) < e$

⟨3⟩4.  $(-\infty, e) \subseteq h(J)$

PROOF: If  $e' < e$  then  $e' \in h(J)$  by leastness of  $e$ .

□

Part II

Category Theory



## Chapter 5

# Category Theory

### 5.1 Categories

**Definition 5.1.1.** A *category*  $\mathcal{C}$  consists of:

- a set  $\text{Ob}(\mathcal{C})$  of *objects*. We write  $A \in \mathcal{C}$  for  $A \in \text{Ob}(\mathcal{C})$ .
- for any objects  $X$  and  $Y$ , a set  $\mathcal{C}[X, Y]$  of *morphisms* from  $X$  to  $Y$ . We write  $f : X \rightarrow Y$  for  $f \in \mathcal{C}[X, Y]$ .
- for any objects  $X, Y$  and  $Z$ , a function  $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \rightarrow \mathcal{C}[X, Z]$ , called *composition*.

such that:

- Given  $f : X \rightarrow Y, g : Y \rightarrow Z$  and  $h : Z \rightarrow W$ , we have  $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object  $X$ , there exists a morphism  $\text{id}_X : X \rightarrow X$ , the *identity morphism* on  $X$ , such that:
  - for any object  $Y$  and morphism  $f : Y \rightarrow X$  we have  $\text{id}_X \circ f = f$
  - for any object  $Y$  and morphism  $f : X \rightarrow Y$  we have  $f \circ \text{id}_X = f$

We write the composite of morphism  $f_1, \dots, f_n$  as  $f_n \circ \dots \circ f_1$ . This is unambiguous thanks to Associativity.

**Definition 5.1.2.** Let **Set** be the category of small sets and functions.

**Definition 5.1.3.** Let **LPos** be the category of linearly ordered sets and monotone functions.

**Proposition 5.1.4.** Any finite linearly ordered set is isomorphic to  $\{m \in \mathbb{N} : m < n\}$  for some  $n$ .

PROOF:

$\langle 1 \rangle$ 1. Every finite nonempty linearly ordered set has a greatest element.

- $\langle 2 \rangle 1$ . LET:  $P[n]$  be the property: for any linearly ordered set  $A$ , if there exists a bijection  $A \approx \{m \in \mathbb{N} : m < n\}$  and  $A$  is nonempty then  $A$  has a greatest element.
- $\langle 2 \rangle 2$ .  $P[0]$   
 PROOF: Vacuous.
- $\langle 2 \rangle 3$ .  $\forall n \in \mathbb{N}. P[n] \Rightarrow P[n+1]$
- $\langle 3 \rangle 1$ . LET:  $n \in \mathbb{N}$
- $\langle 3 \rangle 2$ . ASSUME:  $P[n]$
- $\langle 3 \rangle 3$ . LET:  $A$  be a nonempty linearly ordered set.
- $\langle 3 \rangle 4$ . LET:  $f : A \approx \{m \in \mathbb{N} : m < n+1\}$
- $\langle 3 \rangle 5$ . LET:  $a = f^{-1}(n)$
- $\langle 3 \rangle 6$ .  $f \upharpoonright (A - \{a\}) : A - \{a\} \approx \{m \in \mathbb{N} : m < n\}$
- $\langle 3 \rangle 7$ . ASSUME: w.l.o.g.  $a$  is not greatest in  $A$ .
- $\langle 3 \rangle 8$ . LET:  $b$  be greatest in  $A - \{a\}$   
 PROOF:  $\langle 3 \rangle 2$
- $\langle 3 \rangle 9$ .  $b$  is greatest in  $A$ .
- $\langle 1 \rangle 2$ . LET:  $P[n]$  be the property: for any linearly ordered set  $A$ , if there exists a bijection  $A \approx \{m \in \mathbb{N} : m < n\}$  then there exists an isomorphism in **LPos**  $A \cong \{m \in \mathbb{N} : m < n\}$ .
- $\langle 1 \rangle 3$ .  $P[0]$   
 PROOF: If there exists a bijection  $A \approx \emptyset$  then  $A$  is empty and so the unique function  $A \rightarrow \emptyset$  is an order isomorphism.
- $\langle 1 \rangle 4$ . For every natural number  $n$ , if  $P[n]$  then  $P[n+1]$ .
- $\langle 2 \rangle 1$ . LET:  $n$  be a natural number.
- $\langle 2 \rangle 2$ . ASSUME:  $P[n]$
- $\langle 2 \rangle 3$ . LET:  $A$  be a linearly ordered set.
- $\langle 2 \rangle 4$ . ASSUME:  $A$  has  $n+1$  elements.
- $\langle 2 \rangle 5$ . LET:  $a$  be the greatest element in  $A$ .
- $\langle 2 \rangle 6$ . LET:  $f : A - \{a\} \cong \{m \in \mathbb{N} : m < n\}$  be an order isomorphism.  
 PROOF:  $\langle 2 \rangle 2$
- $\langle 2 \rangle 7$ . Define  $g : A \rightarrow \{m \in \mathbb{N} : m < n+1\}$  by
 
$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ n & \text{if } x = a \end{cases}$$
- $\langle 2 \rangle 8$ .  $g$  is an order isomorphism.
- $\langle 1 \rangle 5$ .  $\forall n \in \mathbb{N}. P[n]$   
 $\square$

**Corollary 5.1.4.1.** *Any finite linearly ordered set is well ordered.*

**Proposition 5.1.5.** *Let  $J$  and  $E$  be well ordered sets. Suppose there is a strictly monotone map  $J \rightarrow E$ . Then  $J$  is isomorphic either to  $E$  or a section of  $E$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $k : J \rightarrow E$  be strictly monotone.
- $\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $E$  is nonempty.
- $\langle 1 \rangle 3$ . PICK  $e_0 \in E$

⟨1⟩4. LET:  $h : J \rightarrow E$  be the function defined by transfinite recursion thus:

$$h(\alpha) = \begin{cases} \text{the least element in } E - h((-\infty, \alpha)) & \text{if } h((-\infty, \alpha)) \neq E \\ e_0 & \text{if } h((-\infty, \alpha)) = E \end{cases}$$

⟨1⟩5.  $\forall \alpha \in J, h(\alpha) \leq k(\alpha)$

⟨2⟩1. LET:  $\alpha \in J$

⟨2⟩2. ASSUME: as transfinite induction hypothesis  $\forall \beta < \alpha, h(\beta) \leq k(\beta)$ .

⟨2⟩3.  $\forall \beta < \alpha, h(\beta) < k(\alpha)$

⟨2⟩4.  $h((-\infty, \alpha)) \neq E$

⟨2⟩5.  $h(\alpha)$  is the least element in  $E - h((-\infty, \alpha))$ .

⟨2⟩6.  $k(\alpha) \in E - h((-\infty, \alpha))$

⟨2⟩7.  $h(\alpha) \leq k(\alpha)$

⟨1⟩6.  $\forall \alpha \in J, h((-\infty, \alpha)) \neq E$

PROOF: For  $\beta < \alpha$  we have  $h(\beta) \leq k(\beta) < k(\alpha)$  so  $k(\alpha) \notin h((-\infty, \alpha))$ .

⟨1⟩7. For all  $\alpha \in J$ , we have  $h(\alpha)$  is the least element of  $E - h((-\infty, \alpha))$ .

⟨1⟩8.  $h$  is strictly monotone and  $h(J)$  is either  $E$  or a section of  $E$ .

PROOF: Proposition 4.3.11.

□

**Proposition 5.1.6.** *If  $A$  and  $B$  are well ordered sets, then exactly one of the following conditions hold:  $A \cong B$ , or  $A$  is isomorphic to a section of  $B$ , or  $B$  is isomorphic to a section of  $A$ .*

PROOF:

⟨1⟩1. At least one of the conditions holds.

⟨2⟩1.  $B$  is isomorphic to either  $A + B$  or a section of  $A + B$ .

⟨2⟩2. CASE:  $B \cong A + B$

⟨3⟩1. LET:  $\phi$  be the isomorphism  $B \cong A + B$

⟨3⟩2. LET:  $b_0$  be the least element in  $B$ .

⟨3⟩3.  $A$  is isomorphic to the section  $(-\infty, \phi^{-1}(\kappa_2(b_0)))$  of  $B$ .

⟨2⟩3. CASE:  $a \in A$  and  $B \cong (-\infty, \kappa_1(a))$

PROOF: Then  $B$  is isomorphic to the section  $(-\infty, a)$  of  $A$ .

⟨2⟩4. CASE:  $b \in B$  and  $\phi : B \cong (-\infty, \kappa_2(b))$

⟨3⟩1. CASE:  $b$  is least in  $B$ .

PROOF: Then  $A \cong B$ .

⟨3⟩2. CASE:  $b$  is not least in  $B$ .

⟨4⟩1. LET:  $b_0$  be least in  $B$ .

⟨4⟩2.  $A$  is isomorphic to the section  $(-\infty, \phi^{-1}(\kappa_2(b_0)))$  of  $B$ .

⟨1⟩2. At most one of the conditions holds.

PROOF: Since a well ordered set cannot be isomorphic to a section of itself.

□

**Theorem 5.1.7.** *There exists a well ordered set, unique up to order isomorphism, that is uncountable but such that every section is countable.*

PROOF:

⟨1⟩1. There exists a well ordered set that is uncountable but such that every section is countable.

- $\langle 2 \rangle 1$ . PICK a well ordered set  $A$  with an element  $\Omega \in A$  such that  $(-\infty, \Omega)$  is uncountable but  $\forall \alpha < \Omega. (-\infty, \alpha)$  is countable.  
 $\langle 2 \rangle 2$ . LET:  $(-\infty, \Omega)$  is uncountable but every section is countable.  
 $\langle 1 \rangle 2$ . If  $A$  and  $B$  are uncountable well ordered sets such that every section is countable, then  $A \cong B$ .

PROOF: Since it cannot be that one of  $A$  and  $B$  is isomorphic to a section of the other.

□

**Definition 5.1.8** (Minimal Uncountable Well Ordered Set). The *minimal uncountable well ordered set*  $\Omega$  is the well ordered set that is uncountable but such that every section is countable.

We write  $\bar{\Omega}$  for the well ordered set  $\Omega \cup \{\Omega\}$  where  $\Omega$  is greatest.

**Proposition 5.1.9.** *Every countable subset of  $\Omega$  is bounded above.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $A$  be a countable subset of  $\Omega$ .  
 $\langle 1 \rangle 2$ . For all  $a \in A$  we have  $(-\infty, a)$  is countable.  
 $\langle 1 \rangle 3$ .  $\bigcup_{a \in A} (-\infty, a)$  is countable.  
 $\langle 1 \rangle 4$ .  $\bigcup_{a \in A} (-\infty, a) \neq \Omega$   
 $\langle 1 \rangle 5$ . PICK  $x \in \Omega - \bigcup_{a \in A} (-\infty, a)$   
 $\langle 1 \rangle 6$ .  $x$  is an upper bound for  $A$ .

□

**Proposition 5.1.10.**  *$\Omega$  has no greatest element.*

PROOF: For any  $\alpha \in \Omega$  we have  $(-\infty, \alpha]$  is countable and hence not the whole of  $\Omega$ . □

**Proposition 5.1.11.** *There are uncountably many elements of  $\Omega$  that have no predecessor.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $A$  be the set of all elements of  $\Omega$  that have no predecessor.  
 $\langle 1 \rangle 2$ . LET:  $f : A \times \mathbb{N} \rightarrow \Omega$  be the function that maps  $(a, n)$  to the  $n$ th successor of  $a$ .  
 $\langle 1 \rangle 3$ .  $f$  is surjective.  
 $\langle 2 \rangle 1$ . ASSUME: for a contradiction  $x \in \Omega$  and there is no element  $a \in A$  and  $n \in \mathbb{N}$  such that  $x$  is the  $n$ th successor of  $a$ .  
 $\langle 2 \rangle 2$ . LET:  $x_n$  be the  $n$ th predecessor of  $x$  for  $n \in \mathbb{N}$ .  
 $\langle 2 \rangle 3$ .  $\{x_n : n \in \mathbb{N}\}$  is a nonempty subset of  $\Omega$  with no least element.  
 $\langle 1 \rangle 4$ .  $A \times \mathbb{N}$  is uncountable.  
 $\langle 1 \rangle 5$ .  $A$  is uncountable.

□

**Definition 5.1.12.** We identify a poset  $(A, \leq)$  with the category with:

- set of objects  $A$



- for  $a, b \in A$ , the set of homomorphisms is  $\{x \in 1 : a \leq b\}$

**Proposition 5.1.13.** *A category is a poset iff, for any two objects, there exists at most one morphism between them.*

**Proposition 5.1.14.** *The identity morphism on an object is unique.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C}$  be a category.

$\langle 1 \rangle 2$ . LET:  $A \in \mathcal{C}$

$\langle 1 \rangle 3$ . LET:  $i, j : A \rightarrow A$  be identity morphisms on  $A$ .

$\langle 1 \rangle 4$ .  $i = j$

PROOF:

$$\begin{aligned} i &= i \circ j & (j \text{ is an identity on } A) \\ &= j & (i \text{ is an identity on } A) \end{aligned}$$

□

**Proposition 5.1.15.** *Let  $A$  be a linearly ordered set. Then  $A$  is well ordered if and only if it does not contain a subset of order type  $\mathbb{N}^{\text{op}}$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $A$  is well ordered then it does not contain a subset of order type  $\mathbb{N}^{\text{op}}$ .

PROOF: A subset of order type  $\mathbb{N}^{\text{op}}$  would be a subset with no least element.

$\langle 1 \rangle 2$ . If  $A$  is not well ordered then it contains a subset of order type  $\mathbb{N}^{\text{op}}$ .

$\langle 2 \rangle 1$ . ASSUME:  $A$  is not well ordered.

$\langle 2 \rangle 2$ . PICK a nonempty subset  $S$  with no least element.

$\langle 2 \rangle 3$ . PICK  $a_0 \in S$

$\langle 2 \rangle 4$ . Extend to a sequence  $(a_n)$  in  $S$  such that  $a_{n+1} < a_n$  for all  $n$ .

$\langle 2 \rangle 5$ .  $\{a_n : n \in \mathbb{N}\}$  has order type  $\mathbb{N}^{\text{op}}$ .

□

**Corollary 5.1.15.1.** *Let  $A$  be a linearly ordered set. If every countable subset of  $A$  is well ordered, then  $A$  is well ordered.*

**Definition 5.1.16.** Given  $f : A \rightarrow B$  and an object  $C$ , define the function  $f^* : \mathcal{C}[B, C] \rightarrow \mathcal{C}[A, C]$  by  $f^*(g) = g \circ f$ .

**Definition 5.1.17.** Given  $f : A \rightarrow B$  and an object  $C$ , define the function  $f_* : \mathcal{C}[C, A] \rightarrow \mathcal{C}[C, B]$  by  $f_*(g) = f \circ g$ .

### 5.1.1 Monomorphisms

**Definition 5.1.18** (Monomorphism). Let  $f : A \rightarrow B$ . Then  $f$  is *monic* or a *monomorphism*,  $f : A \rightarrowtail B$ , iff, for any object  $X$  and functions  $x, y : X \rightarrow A$ , if  $f \circ x = f \circ y$  then  $x = y$ .

### 5.1.2 Epimorphisms

**Definition 5.1.19** (Epimorphism). Let  $f : A \rightarrow B$ . Then  $f$  is *epic* or an *epimorphism*,  $f : A \twoheadrightarrow B$ , iff, for any object  $X$  and functions  $x, y : B \rightarrow X$ , if  $x \circ f = y \circ f$  then  $x = y$ .

### 5.1.3 Sections and Retractions

**Definition 5.1.20** (Section, Retraction). Let  $r : A \rightarrow B$  and  $s : B \rightarrow A$ . Then  $r$  is a *retraction* of  $s$ , and  $s$  is a *section* of  $r$ , iff  $rs = \text{id}_B$ .

**Proposition 5.1.21.** Let  $f : A \rightarrow B$  and  $r, s : B \rightarrow A$ . If  $r$  is a retraction of  $f$  and  $s$  is a section of  $f$  then  $r = s$ .

PROOF:

$$\begin{aligned}
 r &= r \text{id}_B && \text{(Unit Law)} \\
 &= rfs && (s \text{ is a section of } f) \\
 &= \text{id}_A s && (r \text{ is a retraction of } f) \\
 &= s && \text{(Unit Law)} \square
 \end{aligned}$$

**Proposition 5.1.22.** Every section is monic.

PROOF:

$\langle 1 \rangle 1$ . LET:  $s : B \rightarrow A$  be a section of  $r : A \rightarrow B$ .

$\langle 1 \rangle 2$ . LET:  $X$  be an object and  $x, y : X \rightarrow B$

$\langle 1 \rangle 3$ . ASSUME:  $s \circ x = s \circ y$

$\langle 1 \rangle 4$ .  $x = y$

PROOF:  $x = r \circ s \circ x = r \circ s \circ y = y$ .

$\square$

**Proposition 5.1.23.** Every retraction is epic.

PROOF: Dual.  $\square$

### 5.1.4 Isomorphisms

**Definition 5.1.24** (Isomorphism). A morphism  $f : A \rightarrow B$  is an *isomorphism*,  $f : A \cong B$ , iff there exists a morphism  $f^{-1} : B \rightarrow A$  that is both a retraction and section of  $f$ .

Objects  $A$  and  $B$  are *isomorphic*,  $A \cong B$ , iff there exists an isomorphism between them.

**Proposition 5.1.25.** The inverse of an isomorphism is unique.

PROOF: From Proposition 5.1.21.  $\square$

**Proposition 5.1.26.** If  $f : A \cong B$  then  $f^{-1} : B \cong A$  and  $(f^{-1})^{-1} = f$ .

PROOF: Since  $ff^{-1} = \text{id}_B$  and  $f^{-1}f = \text{id}_A$ .  $\square$

Isomorphism.

Define the opposite category.

Slice categories

**Definition 5.1.27.** Let  $\mathcal{C}$  be a category and  $B \in \mathcal{C}$ . The category  $\mathcal{C}_B^B$  of objects *over and under*  $B$  is the category with:

- objects all triples  $(X, u, p)$  such that  $u : B \rightarrow X$  and  $p : X \rightarrow B$
- morphisms  $f : (X, u, p) \rightarrow (Y, u', p')$  all morphisms  $f : X \rightarrow Y$  such that  $fu = u'$  and  $p'f = p$ .

**Proposition 5.1.28.**

$$\mathcal{C}_B^B \cong (\mathcal{C}/B) \backslash \text{id}_B \cong (\mathcal{C} \backslash B) / \text{id}_B$$

$(B, \text{id}_B, \text{id}_B)$  is the zero object in  $\mathcal{C}_B^B$ .

### 5.1.5 Initial Objects

**Definition 5.1.29** (Initial Object). An object  $I$  is *initial* iff, for any object  $X$ , there exists exactly one morphism  $I \rightarrow X$ .

**Proposition 5.1.30.** *The empty set is initial in Set.*

PROOF: For any set  $A$ , the nowhere-defined function is the unique function  $\emptyset \rightarrow A$ .  $\square$

**Proposition 5.1.31.** *If  $I$  and  $I'$  are initial objects, then there exists a unique isomorphism  $I \cong I'$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $i : I \rightarrow I'$  be the unique morphism  $I \rightarrow I'$ .

$\langle 1 \rangle 2$ . LET:  $i^{-1} : I' \rightarrow I$  be the unique morphism  $I' \rightarrow I$ .

$\langle 1 \rangle 3$ .  $ii^{-1} = \text{id}_{I'}$

PROOF: There is only one morphism  $I' \rightarrow I'$ .

$\langle 1 \rangle 4$ .  $i^{-1}i = \text{id}_I$

PROOF: There is only one morphism  $I \rightarrow I$ .

$\square$

### 5.1.6 Terminal Objects

**Definition 5.1.32** (Terminal Object). An object  $T$  is *terminal* iff, for any object  $X$ , there exists exactly one morphism  $X \rightarrow T$ .

**Proposition 5.1.33.** *1 is terminal in Set.*

PROOF: For any set  $A$ , the constant function to  $*$  is the only function  $A \rightarrow 1$ .  $\square$

**Proposition 5.1.34.** *If  $T$  and  $T'$  are terminal objects, then there exists a unique isomorphism  $T \cong T'$ .*

PROOF: Dual to Proposition 5.1.31.  $\square$

### 5.1.7 Zero Objects

**Definition 5.1.35** (Zero Object). An object  $Z$  is a *zero object* iff it is an initial object and a terminal object.

**Definition 5.1.36** (Zero Morphism). Let  $\mathcal{C}$  be a category with a zero object  $Z$ . Let  $A, B \in \mathcal{C}$ . The *zero morphism*  $A \rightarrow B$  is the unique morphism  $A \rightarrow Z \rightarrow B$ .

**Proposition 5.1.37.** *There is no zero object in **Set**.*

PROOF: Since  $\emptyset \not\approx 1$ .  $\square$

### 5.1.8 Triads

**Definition 5.1.38** (Triad). Let  $\mathcal{C}$  be a category. A *triad* consists of objects  $X, Y, M$  and morphisms  $\alpha : X \rightarrow M, \beta : Y \rightarrow M$ . We call  $M$  the *codomain* of the triad.

### 5.1.9 Cotriads

**Definition 5.1.39** (Cotriad). Let  $\mathcal{C}$  be a category. A *cotriad* consists of objects  $X, Y, W$  and morphisms  $\xi : W \rightarrow X, \eta : W \rightarrow Y$ . We call  $W$  the *domain* of the triad.

### 5.1.10 Pullbacks

**Definition 5.1.40** (Pullback). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

is a *pullback* iff  $\alpha\xi = \beta\eta$  and, for every object  $Z$  and morphism  $f : Z \rightarrow X$  and  $g : Z \rightarrow Y$  such that  $\alpha f = \beta g$ , there exists a unique  $h : Z \rightarrow W$  such that  $\xi h = f$  and  $\eta h = g$ .

In this case we also say that  $\eta$  is the *pullback* of  $\beta$  along  $\alpha$ .

**Proposition 5.1.41.** *If  $\xi : W \rightarrow X$  and  $\eta : W \rightarrow Y$  form a pullback of  $\alpha : X \rightarrow M$  and  $\beta : Y \rightarrow M$ , and  $\xi' : W' \rightarrow X$  and  $\eta' : W' \rightarrow Y$  also form the pullback of  $\alpha$  and  $\beta$ , then there exists a unique isomorphism  $\phi : W \cong W'$  such that  $\eta'\phi = \eta$  and  $\xi'\phi = \xi$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : W \rightarrow W'$  be the unique morphism such that  $\eta'\phi = \eta$  and  $\xi'\phi = \xi$ .

$\langle 1 \rangle 2$ . LET:  $\phi^{-1} : W' \rightarrow W$  be the unique morphism such that  $\eta\phi^{-1} = \eta'$  and  $\xi\phi^{-1} = \xi'$ .

$\langle 1 \rangle 3$ .  $\phi\phi^{-1} = \text{id}_{W'}$

PROOF: Each is the unique  $x : W' \rightarrow W'$  such that  $\eta'x = \eta'$  and  $\xi'x = \xi'$ .

$\langle 1 \rangle 4$ .  $\phi^{-1}\phi = \text{id}_W$

PROOF: Each is the unique  $x : W \rightarrow W$  such that  $\eta x = \eta$  and  $\xi x = \xi$ .

□

**Proposition 5.1.42.** *For any morphism  $h : A \rightarrow B$ , the following diagram is a pullback diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $Z$  be an object.

$\langle 1 \rangle 2$ . LET:  $f : Z \rightarrow B$  and  $g : Z \rightarrow A$  satisfy  $\text{id}_B f = hg$

$\langle 1 \rangle 3$ .  $g : Z \rightarrow A$  is the unique morphism such that  $\text{id}_A g = g$  and  $hg = f$ .

□

**Proposition 5.1.43.** *The pullback of an isomorphism is an isomorphism.*

PROOF:

$\langle 1 \rangle 1$ . LET:

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback diagram.

$\langle 1 \rangle 2$ . ASSUME:  $\beta$  is an isomorphism.

$\langle 1 \rangle 3$ . LET:  $\xi^{-1}$  be the unique morphism  $X \rightarrow W$  such that  $\xi\xi^{-1} = \text{id}_X$  and  $\eta\xi^{-1} = \beta^{-1}\alpha$ .

PROOF: This exists since  $\alpha\text{id}_X = \beta\beta^{-1}\alpha = \alpha$ .

$\langle 1 \rangle 4$ .  $\xi^{-1}\xi = \text{id}_W$

PROOF: Each is the unique  $x : W \rightarrow W$  such that  $\xi x = \xi$  and  $\eta x = \eta$ .

□

**Proposition 5.1.44.** *Let  $\beta : (Y, y) \rightarrow (M, m)$  and  $\alpha : (X, x) \rightarrow (M, m)$  in  $\mathcal{C} \setminus A$ . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

*be a pullback in  $\mathcal{C}$ . Let  $w : A \rightarrow W$  be the unique morphism such that  $\xi w = x$  and  $\eta w = y$ . Then  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  is the pullback of  $\beta$  and  $\alpha$  in  $\mathcal{C} \setminus A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $(Z, z) \in \mathcal{C} \backslash A$

$\langle 1 \rangle 2$ . LET:  $f : (Z, z) \rightarrow (X, x)$  and  $g : (Z, z) \rightarrow (Y, y)$  satisfy  $\alpha f = \beta g$ .

$\langle 1 \rangle 3$ . LET:  $h : Z \rightarrow W$  be the unique morphism such that  $\xi h = f$  and  $\eta h = g$ .

$\langle 1 \rangle 4$ .  $hz = w$

$\langle 2 \rangle 1$ .  $\xi hz = \xi w$

PROOF:

$$\xi hz = fz \quad (\langle 1 \rangle 3)$$

$$= x \quad (\langle 1 \rangle 2)$$

$$= \xi w$$

$\langle 2 \rangle 2$ .  $\eta hz = \eta w$

PROOF: Similar.

$\langle 1 \rangle 5$ .  $h : (Z, z) \rightarrow (W, w)$

□

**Proposition 5.1.45.** Let  $\beta : (Y, y) \rightarrow (M, m)$  and  $\alpha : (X, x) \rightarrow (M, m)$  in  $\mathcal{C}/A$ . Let

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in  $\mathcal{C}$ . Let  $w = x\xi : W \rightarrow A$ . Then  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  form a pullback of  $\alpha$  and  $\beta$  in  $\mathcal{C}/A$ .

PROOF:

$\langle 1 \rangle 1$ .  $\eta : (W, w) \rightarrow (Y, y)$

PROOF:

$$y\eta = m\beta\eta$$

$$= m\alpha\xi$$

$$= x\xi$$

$$= w$$

$\langle 1 \rangle 2$ . LET:  $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 3$ . LET:  $f : (Z, z) \rightarrow (X, x)$  and  $g : (Z, z) \rightarrow (Y, y)$  satisfy  $\alpha f = \beta g$ .

$\langle 1 \rangle 4$ . LET:  $h : Z \rightarrow W$  be the unique morphism such that  $\xi h = f$  and  $\eta h = g$ .

$\langle 1 \rangle 5$ .  $h : (Z, z) \rightarrow (W, w)$

PROOF:

$$wh = x\xi h$$

$$= xf \quad (\langle 1 \rangle 4)$$

$$= z \quad (\langle 1 \rangle 3)$$

□

**Proposition 5.1.46.** In **Set**, let  $\alpha : X \rightarrow M$  and  $\beta : Y \rightarrow M$ . Let  $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$  with inclusion  $i : W \rightarrow X \times Y$ . Let  $\xi = \pi_1 i : W \rightarrow X$  and  $\eta = \pi_2 i : W \rightarrow Y$ . Then  $\xi$  and  $\eta$  form the pullback of  $\alpha$  and  $\beta$ .

PROOF:

$\langle 1 \rangle 1. \alpha\xi = \beta\eta$

PROOF: For  $w \in W$ , if  $i(w) = (x, y)$  then  $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$ .

$\langle 1 \rangle 2.$  For every set  $Z$  and functions  $f : Z \rightarrow X, g : Z \rightarrow Y$  such that  $\alpha f = \beta g$ , there exists a unique  $h : Z \rightarrow W$  such that  $\xi h = f$  and  $\eta h = g$

PROOF: For  $z \in Z$ , let  $h(z)$  be the unique element of  $W$  such that  $i(h(z)) = (f(z), g(z))$ .

□

Pullback lemma

### 5.1.11 Pushouts

**Definition 5.1.47** (Pushout). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array} \quad (5.1)$$

is a *pushout* iff  $\alpha\xi = \beta\eta$  and, for every object  $Z$  and morphism  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  such that  $f\xi = g\eta$ , there exists a unique  $h : M \rightarrow Z$  such that  $h\alpha = f$  and  $h\beta = g$ .

We also say that  $\beta$  is the *pushout* of  $\xi$  along  $\eta$ .

**Proposition 5.1.48.** If  $\alpha : X \rightarrow M$  and  $\beta : Y \rightarrow M$  form a pushout of  $\xi : W \rightarrow X$  and  $\eta : W \rightarrow Y$ , and  $\alpha' : X \rightarrow M'$  and  $\beta' : Y \rightarrow M'$  also form a pushout of  $\xi$  and  $\eta$ , then there exists a unique isomorphism  $\phi : M \cong M'$  such that  $\phi\alpha = \alpha'$  and  $\phi\beta = \beta'$ .

PROOF: Dual to Proposition 5.1.41. □

**Proposition 5.1.49.** For any morphism  $h : A \rightarrow B$ , the following diagram is a pushout diagram.

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF: Dual to Proposition 5.1.42.

**Proposition 5.1.50.** The diagram (5.1) is a pushout in  $\mathcal{C}$  iff it is a pullback in  $\mathcal{C}^{\text{op}}$ .

PROOF: Immediate from definitions. □

**Proposition 5.1.51.** The pushout of an isomorphism is an isomorphism.

PROOF: Dual to Proposition 5.1.43.  $\square$

**Proposition 5.1.52.** *Let  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  in  $\mathcal{C} \setminus A$ . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow[\beta]{} & M \end{array}$$

*be a pushout in  $\mathcal{C}$ . Let  $m := \alpha x : A \rightarrow M$ . Then  $\alpha : (X, x) \rightarrow (M, m)$  and  $\beta : (Y, y) \rightarrow (M, m)$  is the pushout of  $\xi$  and  $\eta$  in  $\mathcal{C} \setminus A$ .*

PROOF: Dual to Proposition 5.1.45.  $\square$

**Proposition 5.1.53.** *Let  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  in  $\mathcal{C}/A$ . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow[\beta]{} & M \end{array}$$

*be a pushout in  $\mathcal{C}$ . Let  $m : M \rightarrow A$  be the unique morphism such that  $m\alpha = x$  and  $m\beta = y$ . Then  $\alpha : (X, x) \rightarrow (M, m)$  and  $\beta : (Y, y) \rightarrow (M, m)$  is the pushout of  $\xi$  and  $\eta$  in  $\mathcal{C} \setminus A$ .*

PROOF: Dual to Proposition 5.1.44.  $\square$

**Proposition 5.1.54.** *Set has pushouts.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\xi : W \rightarrow X$  and  $\eta : W \rightarrow Y$ .

$\langle 1 \rangle 2$ . LET:  $\sim$  be the equivalence relation on  $X + Y$  generated by  $\xi(w) \sim \eta(w)$  for all  $w \in W$

$\langle 1 \rangle 3$ . LET:  $M = (X + Y)/\sim$  with canonical projection  $\pi : X + Y \rightarrow M$ .

$\langle 1 \rangle 4$ . LET:  $\alpha = \pi \circ \kappa_1 : X \rightarrow M$

$\langle 1 \rangle 5$ . LET:  $\beta = \pi \circ \kappa_2 : Y \rightarrow M$

$\langle 1 \rangle 6$ . LET:  $Z$  be any set,  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$ .

$\langle 1 \rangle 7$ . ASSUME:  $f\xi = g\eta$

$\langle 1 \rangle 8$ . LET:  $h : X + Y \rightarrow Z$  be the function defined by  $h(x) = f(x)$  and  $h(y) = g(y)$  for  $x \in X$  and  $y \in Y$

$\langle 1 \rangle 9$ .  $h$  respects  $\sim$

PROOF: For  $w \in W$  we have

$$h(\xi(w)) = f(\xi(w)) \quad (\langle 1 \rangle 8)$$

$$= g(\eta(w)) \quad (\langle 1 \rangle 7)$$

$$= h(\eta(w)) \quad (\langle 1 \rangle 8)$$

$\langle 1 \rangle 10$ . LET:  $\bar{h} : M \rightarrow Z$  be the induced function.

$\langle 1 \rangle 11$ .  $\bar{h}\alpha = f$



PROOF:

$$\begin{aligned}\bar{h}(\alpha(x)) &= \bar{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x)\end{aligned}$$

$\langle 1 \rangle 12.$   $\bar{h}\beta = g$

PROOF: Similar.

$\langle 1 \rangle 13.$  For all  $k : M \rightarrow Z$ , if  $k\alpha = f$  and  $k\beta = g$  then  $k = \bar{h}$ .

PROOF:

$$\begin{aligned}k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\ &= f(x) \\ k(\pi(\kappa_2(y))) &= k(\beta(y)) \\ &= g(y) \\ \therefore k \circ \pi &= h \\ \therefore k &= \bar{h}\end{aligned}$$

□

**Definition 5.1.55.** Let  $u : A \rightarrowtail X$  be an injection. The *pointed set obtained from  $X$  by collapsing  $(A, u)$* , denoted  $X/(A, u)$ , is the pushout

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ \downarrow u & & \downarrow * \\ X & \longrightarrow & X/(A, u) \end{array}$$

**Proposition 5.1.56.** In  $\mathbf{Set}_*$ , any two morphisms  $1 \rightarrow X$  and  $1 \rightarrow Y$  have a pushout.

PROOF: The pushout of  $a : (1, *) \rightarrow (X, x)$  and  $b : (1, *) \rightarrow (Y, y)$  is  $(X+Y/\sim, x)$  where  $\sim$  is the equivalence relation generated by  $x \sim y$ . □

**Definition 5.1.57** (Wedge). The *wedge* of pointed sets  $X$  and  $Y$ ,  $X \vee Y$ , is the pushout of the unique morphism  $1 \rightarrow X$  and  $1 \rightarrow Y$ .

**Definition 5.1.58** (Smash). Let  $X$  and  $Y$  be pointed sets. Let  $\xi : X \vee Y \rightarrow X$  be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc} 1 & \longrightarrow & X & & \\ \downarrow & & \downarrow & \searrow & \\ Y & \longrightarrow & X \vee Y & \xrightarrow{\xi} & X \\ & \searrow 0 & & & \uparrow \end{array}$$

Let  $\eta : X \vee Y \rightarrow Y$  be the unique morphism such that the following diagram

commutes.



Let  $\zeta = \langle \xi, \eta \rangle : X \vee Y \rightarrow X \times Y$ . The *smash* of  $X$  and  $Y$ ,  $X \wedge Y$ , is the result of collapsing  $X \times Y$  with respect to  $\zeta$ .

Pushout lemma

### 5.1.12 Subcategories

**Definition 5.1.59** (Subcategory). A *subcategory*  $\mathcal{C}'$  of a category  $\mathcal{C}$  consists of:

- a subset  $\text{Ob}(\mathcal{C}')$  of  $\mathcal{C}$
- for all  $A, B \in \text{Ob}(\mathcal{C}')$ , a subset  $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all  $A \in \text{Ob}(\mathcal{C}')$ , we have  $\text{id}_A \in \mathcal{C}'[A, A]$
- for all  $f \in \mathcal{C}'[A, B]$  and  $g \in \mathcal{C}'[B, C]$ , we have  $g \circ f \in \mathcal{C}'[A, C]$ .

It is a *full* subcategory iff, for all  $A, B \in \text{Ob}(\mathcal{C}')$ , we have  $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$ .

### 5.1.13 Opposite Category

**Definition 5.1.60** (Opposite Category). For any category  $\mathcal{C}$ , the *opposite* category  $\mathcal{C}^{\text{op}}$  is the category with

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- $\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$
- Given  $f \in \mathcal{C}^{\text{op}}[A, B]$  and  $g \in \mathcal{C}^{\text{op}}[B, C]$ , their composite in  $\mathcal{C}^{\text{op}}$  is  $f \circ g$ , where  $\circ$  is composition in  $\mathcal{C}$ .

**Proposition 5.1.61.** An object is initial in  $\mathcal{C}$  iff it is terminal in  $\mathcal{C}^{\text{op}}$ .

PROOF: Immediate from definitions.  $\square$

**Proposition 5.1.62.** An object is terminal in  $\mathcal{C}$  iff it is initial in  $\mathcal{C}^{\text{op}}$ .

PROOF: Immediate from definitions.  $\square$

**Corollary 5.1.62.1.** If  $T$  and  $T'$  are terminal objects in  $\mathcal{C}$  then there exists a unique isomorphism  $T \cong T'$ .

### 5.1.14 Groupoids

**Definition 5.1.63** (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

### 5.1.15 Concrete Categories

**Definition 5.1.64** (Concrete Category). A *concrete category*  $\mathcal{C}$  consists of:

- a set  $\text{Ob}(\mathcal{C})$  of *objects*
- for any object  $A \in \text{Ob}(\mathcal{C})$ , a set  $|A|$
- for any objects  $A, B \in \text{Ob}(\mathcal{C})$ , a set of functions  $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any  $f \in \mathcal{C}[A, B]$  and  $g \in \mathcal{C}[B, C]$ , we have  $g \circ f \in \mathcal{C}[A, C]$
- for any object  $A$  we have  $\text{id}_{|A|} \in \mathcal{C}[A, A]$ .

### 5.1.16 Power of Categories

**Definition 5.1.65.** Let  $\mathcal{C}$  be a category and  $J$  a set. The category  $\mathcal{C}^J$  is the category with:

- objects all  $J$ -indexed families of objects of  $\mathcal{C}$
- morphisms  $\{X_j\}_{j \in J} \rightarrow \{Y_j\}_{j \in J}$  all families  $\{f_j\}_{j \in J}$  where  $f_j : X_j \rightarrow Y_j$

### 5.1.17 Arrow Category

**Definition 5.1.66** (Arrow Category). Let  $\mathcal{C}$  be a category. The *arrow category*  $\mathcal{C}^\rightarrow$  is the category with:

- objects all triples  $(A, B, f)$  where  $f : A \rightarrow B$  in  $\mathcal{C}$
- morphisms  $(A, B, f) \rightarrow (C, D, g)$  all pairs  $(u : A \rightarrow C, v : B \rightarrow D)$  such that  $vf = gu$ .

### 5.1.18 Slice Category

**Definition 5.1.67** (Slice Category). Let  $\mathcal{C}$  be a category and  $A \in \mathcal{C}$ . The *slice category under A*,  $\mathcal{C}_{\backslash A}$ , is the category with:

- objects all pairs  $(B, f)$  where  $B \in \mathcal{C}$  and  $f : A \rightarrow B$
- morphisms  $(B, f) \rightarrow (C, g)$  are morphisms  $u : B \rightarrow C$  such that  $uf = g$ .

We identify this with the subcategory of  $\mathcal{C}^\rightarrow$  formed by mapping  $(B, f)$  to  $(A, B, f)$  and  $u$  to  $(\text{id}_A, u)$ .

**Proposition 5.1.68.** *If  $s : (B, f) \rightarrow (C, g)$  in  $\mathcal{C} \setminus A$ , then any retraction of  $s$  in  $\mathcal{C}$  is a retraction of  $s$  in  $\mathcal{C} \setminus A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $r : C \rightarrow B$  be a retraction of  $s$  in  $\mathcal{C}$ .

$\langle 1 \rangle 2$ .  $rg = f$

PROOF:  $rg = rsf = f$ .

$\langle 1 \rangle 3$ .  $r : (C, g) \rightarrow (B, f)$  in  $\mathcal{C} \setminus A$

$\langle 1 \rangle 4$ .  $rs = \text{id}_{(B, f)}$

PROOF: Because composition is inherited from  $\mathcal{C}$ .

□

**Proposition 5.1.69.**  $\text{id}_A$  is the initial object in  $\mathcal{C} \setminus A$ .

PROOF: For any  $(B, f) \in \mathcal{C} \setminus A$ , we have  $f$  is the only morphism  $A \rightarrow B$  such that  $f\text{id}_A = f$ . □

**Proposition 5.1.70.** *If  $A$  is terminal in  $\mathcal{C}$  then  $\text{id}_A$  is the zero object in  $\mathcal{C} \setminus A$ .*

PROOF: For any  $(B, f) \in \mathcal{C} \setminus A$ , the unique morphism  $! : B \rightarrow A$  is the unique morphism such that  $!f = \text{id}_A$ . □

**Definition 5.1.71** (Pointed Sets). The *category of pointed sets* is  $\mathbf{Set} \setminus 1$ .

**Definition 5.1.72.** Let  $\mathcal{C}$  be a category and  $A \in \mathcal{C}$ . The *slice category over  $A$* ,  $\mathcal{C}/A$ , is the category with:

- objects all pairs  $(B, f)$  with  $f : B \rightarrow A$
- morphisms  $u : (B, f) \rightarrow (C, g)$  all morphisms  $u : B \rightarrow C$  such that  $gu = f$ .

**Proposition 5.1.73.** *Let  $u : (B, f) \rightarrow (C, g) : \mathcal{C}/A$ . Any section of  $u$  in  $\mathcal{C}$  is a section of  $u$  in  $\mathcal{C}/A$ .*

PROOF: Dual to Proposition 5.1.68. □

**Proposition 5.1.74.**  $\text{id}_A$  is terminal in  $\mathcal{C}/A$ .

PROOF: Dual to Proposition 5.1.69. □

**Proposition 5.1.75.** *If  $A$  is initial in  $\mathcal{C}$  then  $\text{id}_A$  is the zero object in  $\mathcal{C}/A$ .*

PROOF: Dual to Proposition 5.1.70. □

**Definition 5.1.76.** Let  $A \in \mathcal{C}$ . The category of objects *over and under*  $A$ , written  $\mathcal{C}_A^A$ , is the category with:

- objects all triples  $(X, u, p)$  where  $u : A \rightarrow X$ ,  $p : X \rightarrow A$  and  $pu = \text{id}_A$
- morphism  $f : (X, u, p) \rightarrow (Y, v, q)$  all morphisms  $f : X \rightarrow Y$  such that  $fu = v$  and  $qf = p$

**Proposition 5.1.77.**  $(A, \text{id}_A, \text{id}_A)$  is the zero object in  $\mathcal{C}_A^A$ .

PROOF: For any object  $(X, u, p)$ , we have  $p$  is the unique morphism  $(X, u, p) \rightarrow (A, \text{id}_A, \text{id}_A)$ , and  $u$  is the unique morphism  $(A, \text{id}_A, \text{id}_A) \rightarrow (X, u, p)$ .  $\square$

**Definition 5.1.78** (Fibre Collapsing). Let  $B$  be a set. Let  $u : (A, a) \rightarrow (X, x)$  in  $\mathbf{Set}/B$ . Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow u & & \downarrow j \\ X & \xrightarrow{i} & C \end{array}$$

Let  $c : C \rightarrow B$  be the unique morphism such that  $cj = \text{id}_B$  and  $ci = x$ . Then  $(C, j, c) \in \mathbf{Set}_B^B$  is called the set over and under  $B$  obtained from  $X$  by *fibre collapsing* with respect to  $u$ . If  $(A, u)$  is a subset of  $X$ , we denote this set over and under  $B$  by  $X/_B(A, u)$ .

**Definition 5.1.79** (Fibre Wedge). Let  $B$  be a small set. Let  $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$ . The *fibre wedge* of  $X$  and  $Y$  is the pushout of  $u_X$  and  $u_Y$ :

$$\begin{array}{ccc} B & \xrightarrow{u_X} & X \\ \downarrow u_Y & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \end{array}$$

**Definition 5.1.80** (Fibre Smash). Let  $X, Y \in \mathbf{Set}_B^B$ . Let  $\xi : X \vee_B Y \rightarrow X$  be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \xi & \\ & & X \end{array}$$

$0$

Let  $\eta : X \vee_B Y \rightarrow Y$  be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \eta & \\ & & Y \end{array}$$

$0$

Let  $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \rightarrow X \times Y$ . The *fibre smash* of  $X$  and  $Y$ ,  $X \wedge_B Y$ , is the result of collapsing  $X \times Y$  with respect to  $\zeta$ .

**Proposition 5.1.81.** *Set has products and coproducts.*

**Proposition 5.1.82.** *Let  $\mathcal{C}$  be a category. Let  $\{X_\alpha\}_{\alpha \in I}$  be a family of objects in  $\mathcal{C}$  and  $Z \in \mathcal{C}$ . Let  $\coprod_{\alpha \in I} X_\alpha$  be the coproduct of  $\{X_\alpha\}_{\alpha \in I}$ . Then*

$$\mathcal{C}[\coprod_{\alpha \in I} X_\alpha, Z] \approx \prod_{\alpha \in I} \mathcal{C}[X_\alpha, Z] .$$

**Proposition 5.1.83.** *Let  $\mathcal{C}$  be a category. Let  $\{X_\alpha\}_{\alpha \in I}$  be a family of objects in  $\mathcal{C}$  and  $Z \in \mathcal{C}$ . Let  $\prod_{\alpha \in I} X_\alpha$  be the product of  $\{X_\alpha\}_{\alpha \in I}$ . Then*

$$\mathcal{C}[Z, \prod_{\alpha \in I} X_\alpha] \approx \prod_{\alpha \in I} \mathcal{C}[Z, X_\alpha] .$$

**Proposition 5.1.84.** *A product in  $\mathcal{C}$  constitutes a product in  $\mathcal{C}/A$ .*

**Proposition 5.1.85.** *A coproduct in  $\mathcal{C}$  constitutes a product in  $\mathcal{C}/A$ .*

## 5.2 Functors

**Definition 5.2.1** (Functor). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of:

- a function  $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism  $f : A \rightarrow B$  in  $\mathcal{C}$ , a morphism  $Ff : FA \rightarrow FB$  in  $\mathcal{D}$

such that:

- for all  $A \in \text{Ob}(\mathcal{C})$  we have  $F\text{id}_A = \text{id}_{FA}$
- for any morphism  $f : A \rightarrow B$  and  $g : B \rightarrow C$  in  $\mathcal{C}$ , we have  $F(g \circ f) = Fg \circ Ff$

**Proposition 5.2.2.** *Functors preserve isomorphisms.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

$\langle 1 \rangle 2$ . LET:  $f : A \cong B$  in  $\mathcal{C}$

$\langle 1 \rangle 3$ .  $Ff^{-1} \circ Ff = \text{id}_{FA}$

PROOF:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\text{id}_A \\ &= \text{id}_{FA} \end{aligned}$$

$\langle 1 \rangle 4$ .  $Ff \circ Ff^{-1} = \text{id}_{FB}$

PROOF:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

**Definition 5.2.3** (Identity Functor). For any category  $\mathcal{C}$ , the *identity* functor on  $\mathcal{C}$  is the functor  $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  defined by

$$\begin{aligned} I_{\mathcal{C}}A &:= A & (A \in \mathcal{C}) \\ I_{\mathcal{C}}f &:= f & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

**Proposition 5.2.4.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$ . If  $r : A \rightarrow B$  is a retraction of  $s : B \rightarrow A$  in  $\mathcal{C}$  then  $Fr$  is a retraction of  $Fs$ .

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

**Corollary 5.2.4.1.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$ . If  $\phi : A \cong B$  is an isomorphism in  $\mathcal{C}$  then  $F\phi : FA \cong FB$  is an isomorphism in  $\mathcal{D}$  with  $(F\phi)^{-1} = F\phi^{-1}$ .

**Definition 5.2.5** (Composition of Functors). Given functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$ , the *composite* functor  $GF : \mathcal{C} \rightarrow \mathcal{E}$  is defined by

$$\begin{aligned} (GF)A &= G(FA) & (A \in \mathcal{C}) \\ (GF)f &= G(Ff) & (f : A \rightarrow B : \mathcal{C}) \end{aligned}$$

**Definition 5.2.6** (Category of Categories). Let **Cat** be the category of small categories and functors.

**Definition 5.2.7** (Isomorphism of Categories). Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Then  $F$  is an *isomorphism of categories* iff there exists a functor  $F^{-1} : \mathcal{D} \rightarrow \mathcal{C}$ , the *inverse* of  $F$ , such that  $FF^{-1} = I_{\mathcal{D}}$  and  $F^{-1}F = I_{\mathcal{C}}$ .

Categories  $\mathcal{C}$  and  $\mathcal{D}$  are *isomorphic*,  $\mathcal{C} \cong \mathcal{D}$ , iff there exists an isomorphism between them.

**Proposition 5.2.8.** If  $A$  is initial in  $\mathcal{C}$  then  $\mathcal{C} \setminus A \cong \mathcal{C}$ .

PROOF:

⟨1⟩1. Define  $F : \mathcal{C} \setminus A \rightarrow \mathcal{C}$  by

$$F(B, f) = B$$

$$F(u : (B, f) \rightarrow (C, g)) = u$$

⟨1⟩2. Define  $G : \mathcal{C} \rightarrow \mathcal{C} \setminus A$  by

$$GB = (B, !_B)$$

where  $!_B$  is the unique morphism  $A \rightarrow B$

$$G(u : B \rightarrow C) = u : (B, !_B) \rightarrow (C, !_C)$$

⟨1⟩3.  $FG = \text{id}_{\mathcal{C}}$

⟨1⟩4.  $GF = \text{id}_{\mathcal{C} \setminus A}$

PROOF: Since  $GF(B, f) = (B, !_B) = (B, f)$  because the morphism  $A \rightarrow B$  is unique.

□

**Proposition 5.2.9.** *If  $A$  is terminal in  $\mathcal{C}$  then  $\mathcal{C}/A \cong \mathcal{C}$ .*

PROOF: Dual.  $\square$

**Proposition 5.2.10.**

$$\mathcal{C}_A^A \cong (\mathcal{C}/A) \backslash (A, \text{id}_A) \cong (\mathcal{C} \backslash A) / (A, \text{id}_A)$$

PROOF:

- $\langle 1 \rangle 1$ . Define a functor  $F : \mathcal{C}_A^A \rightarrow (\mathcal{C}/A) \backslash (A, \text{id}_A)$ .  
 $\langle 2 \rangle 1$ . Given  $A \xrightarrow{u} X \xrightarrow{p} A$  in  $\mathcal{C}_A^A$ , let  $F(X, u, p) = ((X, p), u)$   
 $\langle 2 \rangle 2$ . Given  $f : (A \xrightarrow{u} X \xrightarrow{p} A) \rightarrow (A \xrightarrow{v} Y \xrightarrow{q} A)$ , let  $Ff = f$ .  
 $\langle 1 \rangle 2$ . Define a functor  $G : (\mathcal{C}/A) \backslash (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$ .  
 $\langle 1 \rangle 3$ . Define a functor  $H : \mathcal{C}_A^A \rightarrow (\mathcal{C} \backslash A) / (A, \text{id}_A)$ .  
 $\langle 1 \rangle 4$ . Define a functor  $K : (\mathcal{C} \backslash A) / (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$ .  
 $\square$

**Definition 5.2.11** (Forgetful Functor). For any concrete category  $\mathcal{C}$ , define the *forgetful* functor  $U : \mathcal{C} \rightarrow \mathbf{Set}$  by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

**Definition 5.2.12** (Switching Functor). For any category  $\mathcal{C}$ , define the *switching* functor  $T : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$  by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

**Definition 5.2.13** (Reduction). Let  $\Phi : \mathbf{Set} \rightarrow \mathbf{Set}$  be a functor. The *reduction* of  $\Phi$  is the functor  $\Phi^* : \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  defined by:  $\Phi^*(X, a)$  is the collapse of  $\Phi(X)$  with respect to  $\Phi(a) : \Phi(1) \rightarrow \Phi(X)$ .

**Definition 5.2.14.** Extend the wedge  $\vee$  to a functor  $\mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  by defining, given  $f : X \rightarrow X'$  and  $g : Y \rightarrow Y'$ , then  $f \vee g$  is the unique morphism that makes the following diagram commute.

$$\begin{array}{ccccc} 1 & \longrightarrow & X & & \\ \downarrow & & \downarrow & \searrow f & \\ Y & \longrightarrow & X \vee Y & & X' \\ & \searrow g & \searrow f \vee g & & \downarrow \\ & & Y' & \longrightarrow & X' \vee Y' \end{array}$$

**Definition 5.2.15.** Extend smash to a functor  $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  as follows. Given  $f : X \rightarrow X'$  and  $g : Y \rightarrow Y'$ , let  $f \wedge g : X \wedge Y \rightarrow X' \wedge Y'$  be the



unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc}
 X \vee Y & \xrightarrow{\quad} & 1 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 X \times Y & \xrightarrow{\quad} & X \wedge Y & & \\
 & \searrow & \downarrow & \searrow & \\
 & & X' \vee Y' & \xrightarrow{\quad} & 1 \\
 f \times g \swarrow & & \downarrow & & \downarrow \\
 & & X' \times Y' & \xrightarrow{\quad} & X' \wedge Y'
 \end{array}$$

**Definition 5.2.16** (Reduction). Let  $B$  be a small set. Let  $\Phi_B : \mathbf{Set}/B \rightarrow \mathbf{Set}/B$  be a functor. The *reduction* of  $\Phi_B$  is the functor  $\Phi_B^B : \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  defined as follows.

For  $(X, u : B \rightarrow X, p : X \rightarrow B) \in \mathbf{Set}_B^B$ , let  $\Phi_B^B(X)$  be the set over and under  $B$  obtained from  $\Phi_B(X)$  by collapsing with respect to  $\Phi_B(u) : \Phi_B(B) \rightarrow \Phi_B(X)$ .

**Definition 5.2.17.** Extend  $\vee_B$  to a functor  $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ .

**Definition 5.2.18.** Extend  $\wedge_B$  to a functor  $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ .

**Definition 5.2.19** (Faithful). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *faithful* iff, for any objects  $A, B \in \mathcal{C}$  and morphisms  $f, g : A \rightarrow B : \mathcal{C}$ , if  $Ff = Fg$  then  $f = g$ .

**Definition 5.2.20** (Full). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *full* iff, for any objects  $A, B \in \mathcal{C}$  and morphism  $g : FA \rightarrow FB : \mathcal{D}$ , there exists  $f : A \rightarrow B : \mathcal{C}$  such that  $Ff = g$ .

**Definition 5.2.21** (Fully Faithful). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *fully faithful* iff it is full and faithful.

**Definition 5.2.22** (Full Embedding). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a *full embedding* iff it is fully faithful and injective on objects.

## 5.3 Natural Transformations

**Definition 5.3.1** (Natural Transformation). Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$ . A *natural transformation*  $\tau : F \Rightarrow G$  is a family of morphisms  $\{\tau_X : FX \rightarrow GX\}_{X \in \mathcal{C}}$  such that, for every morphism  $f : X \rightarrow Y : \mathcal{C}$ , we have  $Gf \circ \tau_X = \tau_Y \circ Ff$ .

$$\begin{array}{ccc}
 FX & \xrightarrow{Ff} & FY \\
 \tau_X \downarrow & & \downarrow \tau_Y \\
 GX & \xrightarrow{Gf} & GY
 \end{array}$$

**Definition 5.3.2** (Natural Isomorphism). A natural transformation  $\tau : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$  is a *natural isomorphism*,  $\tau : F \cong G$ , iff for all  $X \in \mathcal{C}$ ,  $\tau_X$  is an isomorphism  $FX \cong GX$ .

Functors  $F$  and  $G$  are *naturally isomorphic*,  $F \cong G$ , iff there exists a natural isomorphism between them.

**Definition 5.3.3** (Inverse). Let  $\tau : F \cong G$ . The *inverse* natural isomorphism  $\tau^{-1} : G \cong F$  is defined by  $(\tau^{-1})_X = \tau_X^{-1}$ .

## 5.4 Bifunctors

**Definition 5.4.1** (Commutative). A bifunctor  $\square : \mathcal{C}^2 \rightarrow \mathcal{C}$  is *commutative* iff  $\square \cong \square \circ T$ , where  $T : \mathcal{C}^2 \rightarrow \mathcal{C}^2$  is the swap functor.

**Proposition 5.4.2.**  $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is commutative.

PROOF: Since the pushout of  $f$  and  $g$  is the pushout of  $g$  and  $f$ .  $\square$

**Proposition 5.4.3.**  $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is commutative.

PROOF: In the diagram defining  $X \wedge Y$ , construct the isomorphism between the version with  $X$  and  $Y$  and the version with  $Y$  with  $X$  for every object.  $\square$

**Proposition 5.4.4.**  $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is commutative.

**Proposition 5.4.5.**  $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is commutative.

**Definition 5.4.6** (Associative). A bifunctor  $\square$  is *associative* iff  $\square \circ (\square \times \text{id}) \cong \square \circ (\text{id} \times \square)$ .

**Proposition 5.4.7.**  $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is associative.

PROOF: Since  $X \vee (Y \vee Z)$  and  $(X \vee Y) \vee Z$  are both the pushout of the unique morphisms  $1 \rightarrow X$ ,  $1 \rightarrow Y$  and  $1 \rightarrow Z$ .  $\square$

**Proposition 5.4.8.**  $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is associative.

PROOF: Draw isomorphisms between the diagrams for  $X \wedge (Y \wedge Z)$  and  $(X \wedge Y) \wedge Z$ .  $\square$

Product and coproduct are commutative and associative.

**Proposition 5.4.9.**  $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is associative.

**Proposition 5.4.10.**  $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is associative.

**Proposition 5.4.11.** Let  $\mathcal{C}$  be a category with binary coproducts. Let  $\square : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  be a bifunctor. Then  $\square$  distributes over  $+$  iff the canonical morphism

$$(X \square Z) + (Y \square Z) \rightarrow (X + Y) \square Z$$

is an isomorphism for all  $X, Y, Z$ .

**Proposition 5.4.12.** *In a category with binary products and binary coproducts, then  $\times$  distributes over  $+$ .*

**Proposition 5.4.13.** *In  $\mathbf{Set}/*$ , we have  $\times$  does not distribute over  $\vee$ .*

**Proposition 5.4.14.** *In  $\mathbf{Set}/*$ , we have  $\wedge$  distributes over  $\vee$ .*

**Proposition 5.4.15.** *In  $\mathbf{Set}/B$ , we have  $\times_B$  distributes over  $+_B$ .*

**Proposition 5.4.16.** *In  $\mathbf{Set}/B^B$ , we have  $\wedge_B$  distributes over  $\vee_B$ .*

## 5.5 Functor Categories

**Definition 5.5.1** (Functor Category). Given categories  $\mathcal{C}$  and  $\mathcal{D}$ , define the *functor category*  $\mathcal{C}^{\mathcal{D}}$  to be the category with objects the functors from  $\mathcal{D}$  to  $\mathcal{C}$  and morphisms the natural transformations.

**Definition 5.5.2** (Yoneda Embedding). Let  $\mathcal{C}$  be a category. The *Yoneda embedding*  $Y : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$  is the functor that maps an object  $A$  to  $\mathcal{C}[-, A]$  and morphisms similarly.

**Theorem 5.5.3** (Yoneda Lemma). *Let  $\mathcal{C}$  be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\text{op}}}[\mathcal{C}[-, X], F] \cong FX$$

*that maps  $\tau : \mathcal{C}[-, X] \Rightarrow F$  to  $\tau_X(\text{id}_X)$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\phi$  is natural in  $X$ .

PROOF:

$\langle 2 \rangle 1$ . LET:  $f : X \rightarrow Y : \mathcal{C}$

$\langle 2 \rangle 2$ . LET:  $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$ .  $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$

PROOF:

$$\begin{aligned} \phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\text{id}_Y \circ f) \\ &= \tau_Y(f) \\ &= \tau_Y(f \circ \text{id}_X) \\ &= Ff(\tau_X(\text{id}_X)) && (\tau \text{ natural}) \\ &= Ff(\phi(\tau)) \end{aligned}$$

$\langle 1 \rangle 2$ .  $\phi$  is natural in  $F$ .

$\langle 2 \rangle 1$ . LET:  $\alpha : F \Rightarrow G : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$

$\langle 2 \rangle 2$ . LET:  $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$ .  $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$

PROOF:  $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\text{id}_X)) = \alpha_X(\phi(\tau))$

$\langle 1 \rangle 3$ . Each  $\phi_{XF}$  is injective.

$\langle 2 \rangle 1$ . LET:  $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 2$ . ASSUME:  $\phi(\sigma) = \phi(\tau)$

⟨2⟩3. LET:  $f : Y \rightarrow X$

⟨2⟩4.  $\sigma_Y(f) = \tau_Y(f)$

PROOF:

$$\begin{aligned}
 \sigma_Y(f) &= \sigma_Y(\text{id}_X \circ f) \\
 &= Ff(\sigma_X(\text{id}_X)) && (\sigma \text{ is natural}) \\
 &= Ff(\tau_X(\text{id}_X)) && (\langle 2 \rangle 2) \\
 &= \tau_Y(\text{id}_X \circ f) && (\tau \text{ is natural}) \\
 &= \tau_Y(f)
 \end{aligned}$$

⟨1⟩4. Each  $\phi_{XF}$  is surjective.

⟨2⟩1. LET:  $X \in \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$

⟨2⟩2. LET:  $a \in FX$

⟨2⟩3. LET:  $\tau : \mathcal{C}[-, X] \Rightarrow F$  be given by  $\tau_Y(g) = Fg(a)$  for  $g : Y \rightarrow X$

⟨2⟩4.  $\tau$  is natural.

⟨3⟩1. LET:  $h : Y \rightarrow Z : \mathcal{C}$

PROVE:  $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \text{id}_X]$

⟨3⟩2. LET:  $g : Z \rightarrow X$

⟨3⟩3.  $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$

PROOF:

$$\begin{aligned}
 \tau_Y(g \circ h) &= F(g \circ h)(a) \\
 &= Fh(Fg(a)) \\
 &= Fh(\tau_Z(g))
 \end{aligned}$$

⟨2⟩5.  $\phi(\tau) = a$

PROOF:

$$\begin{aligned}
 \phi_X(\tau) &= \tau_X(\text{id}_X) \\
 &= F\text{id}_X(a) \\
 &= a
 \end{aligned}$$

□

**Corollary 5.5.3.1.** *The Yoneda embedding is fully faithful.*

**Corollary 5.5.3.2.** *Given objects  $A$  and  $B$  in  $\mathcal{C}$ , we have  $A \cong B$  if and only if  $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$ .*

# Part III

## Number Systems



## Chapter 6

# The Real Numbers

**Theorem 6.0.1.** *The following hold in the real numbers:*

1.  $x + (y + z) = (x + y) + z$
2.  $x(yz) = (xy)z$
3.  $x + y = y + x$
4.  $xy = yx$
5.  $x + 0 = x$
6.  $x1 = x$
7.  $x + (-x) = 0$
8. If  $x \neq 0$  then  $x \cdot (1/x) = 1$
9.  $x(y + z) = xy + xz$
10. If  $x > y$  then  $x + z > y + z$ .
11. If  $x > y$  and  $z > 0$  then  $xz > yz$ .
12.  $\mathbb{R}$  has the least upper bound property.
13. If  $x < y$  then there exists  $z$  such that  $x < z < y$ .

**Definition 6.0.2.** Given real numbers  $x$  and  $y$  with  $y \neq 0$ , we write  $x/y$  for  $xy^{-1}$ .

**Theorem 6.0.3.** *For any real numbers  $x$  and  $y$ , if  $x + y = x$  then  $y = 0$ .*

PROOF:

$\langle 1 \rangle 1.$  LET:  $x, y \in \mathbb{R}$

$\langle 1 \rangle 2.$  ASSUME:  $x + y = x$

$\langle 1 \rangle 3.$   $y = 0$

PROOF:

$$\begin{aligned}
y &= y + 0 && \text{(Definition of zero)} \\
&= y + (x + (-x)) && \text{(Definition of } -x) \\
&= (y + x) + (-x) && \text{(Associativity of Addition)} \\
&= (x + y) + (-x) && \text{(Commutativity of Addition)} \\
&= x + (-x) && (\langle 1 \rangle 2) \\
&= 0 && \text{(Definition of } -x)
\end{aligned}$$

□

**Theorem 6.0.4.**

$$\forall x \in \mathbb{R}. 0x = 0$$

PROOF:

 $\langle 1 \rangle 1$ . LET:  $x \in \mathbb{R}$  $\langle 1 \rangle 2$ .  $xx + 0x = xx$ 

PROOF:

$$\begin{aligned}
xx + 0x &= (x + 0)x && \text{(Distributive Law)} \\
&= xx && \text{(Definition of 0)}
\end{aligned}$$

 $\langle 1 \rangle 3$ .  $0x = 0$ PROOF: Theorem 6.0.3,  $\langle 1 \rangle 2$ .

□

**Theorem 6.0.5.**

$$-0 = 0$$

PROOF: Since  $0 + 0 = 0$ . □**Theorem 6.0.6.**

$$\forall x \in \mathbb{R}. -(-x) = x$$

PROOF: Since  $-x + x = 0$ . □**Theorem 6.0.7.**

$$\forall x, y \in \mathbb{R}. x(-y) = -(xy)$$

PROOF:

$$\begin{aligned}
x(-y) + xy &= x((-y) + y) && \text{(Distributive Law)} \\
&= x0 && \text{(Definition of } -y) \\
&= 0 && \text{(Theorem 6.0.4) } \square
\end{aligned}$$

**Theorem 6.0.8.**

$$\forall x \in \mathbb{R}. (-1)x = -x$$

PROOF:

$$\begin{aligned}
(-1)x &= -(1 \cdot x) && \text{(Theorem 6.0.7)} \\
&= -x && \text{(Definition of 1) } \square
\end{aligned}$$



**Proposition 6.0.9.** *Let  $X$  be a linearly ordered set. Let  $a, b, c \in X$  with  $a < b < c$ . Then  $[a, c] \cong [0, 1]$  if and only if  $[a, b] \cong [0, 1]$  and  $[b, c] \cong [0, 1]$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in (0, 1)$  we have  $[0, x] \cong [0, 1]$ .

PROOF: The function that maps  $t$  to  $t/x$  is an order isomorphism.

$\langle 1 \rangle 2$ . For all  $x \in (0, 1)$  we have  $[x, 1] \cong [0, 1]$ .

PROOF: The function that maps  $t$  to  $(t - x)/(1 - x)$  is an order isomorphism.

$\langle 1 \rangle 3$ . We have  $[0, 2] \cong [0, 1]$ .

PROOF: The function that maps  $t$  to  $t/2$  is an order isomorphism.

□

**Proposition 6.0.10.** *Let  $X$  be a linearly ordered set. Let  $(a_n)$  be a strictly increasing sequence in  $X$ . Let  $b$  be its supremum. Then  $[a_0, b] \cong [0, 1]$  if and only if, for all  $n$ , we have  $[a_n, a_{n+1}] \cong [0, 1]$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $x, y \in [0, 1]$  with  $x < y$  we have  $[x, y] \cong [0, 1]$ .

PROOF: The function that maps  $t$  to  $(t - x)/(y - x)$  is an order isomorphism.

$\langle 1 \rangle 2$ . We have  $[0, 1] \cong [0, +\infty)$ .

PROOF: The function that maps  $t$  to  $1/(1 - t) - 1$  is an order isomorphism.

□

## 6.1 Subtraction

**Definition 6.1.1** (Subtraction). We write  $x - y$  for  $x + (-y)$ .

**Theorem 6.1.2.**

$$\forall x, y, z \in \mathbb{R}. x(y - z) = xy - xz$$

PROOF:

$$\begin{aligned} x(y - z) &= x(y + (-z)) && \text{(Definition of subtraction)} \\ &= xy + x(-z) && \text{(Distributive Law)} \\ &= xy + (-(xz)) && \text{(Theorem 6.0.7)} \\ &= xy - xz && \text{(Definition of subtraction)} \quad \square \end{aligned}$$

**Theorem 6.1.3.**

$$\forall x, y \in \mathbb{R}. -(x + y) = -x - y$$

PROOF:

$$\begin{aligned} -(x + y) &= (-1)(x + y) && \text{(Theorem 6.0.8)} \\ &= (-1)x + (-1)y && \text{(Distributive Law)} \\ &= -x + (-y) && \text{(Theorem 6.0.8)} \\ &= -x - y && \text{(Definition of subtraction)} \quad \square \end{aligned}$$

**Theorem 6.1.4.**

$$\forall x, y \in \mathbb{R}. -(x - y) = -x + y$$

PROOF:

$$\begin{aligned}
 -(x - y) &= -(x + (-y)) && \text{(Definition of subtraction)} \\
 &= -x - (-y) && \text{(Theorem 6.1.3)} \\
 &= -x + (-(-y)) && \text{(Definition of subtraction)} \\
 &= -x + y && \text{(Theorem 6.0.6)} \quad \square
 \end{aligned}$$

**Definition 6.1.5** (Reciprocal). Given  $x \in \mathbb{R}$  with  $x \neq 0$ , the *reciprocal* of  $x$ ,  $1/x$ , is the unique real number such that  $x \cdot 1/x = 1$ .

**Theorem 6.1.6.** For any real numbers  $x$  and  $y$ , if  $x \neq 0$  and  $xy = x$  then  $y = 1$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $x, y \in \mathbb{R}$   
 $\langle 1 \rangle 2$ . ASSUME:  $x \neq 0$   
 $\langle 1 \rangle 3$ . ASSUME:  $xy = x$   
 $\langle 1 \rangle 4$ .  $y = 1$

PROOF:

$$\begin{aligned}
 y &= y1 && \text{(Definition of 1)} \\
 &= y(x \cdot 1/x) && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\
 &= (yx)1/x && \text{(Associativity of Multiplication)} \\
 &= (xy)1/x && \text{(Commutativity of Multiplication)} \\
 &= x \cdot 1/x && (\langle 1 \rangle 3) \\
 &= 1 && \text{(Definition of } 1/x, \langle 1 \rangle 2)
 \end{aligned}$$

$\square$

**Definition 6.1.7** (Quotient). Given real numbers  $x$  and  $y$  with  $y \neq 0$ , the *quotient*  $x/y$  is defined by

$$x/y = x \cdot 1/y .$$

**Theorem 6.1.8.** For any real number  $x$ , if  $x \neq 0$  then  $x/x = 1$ .

PROOF: Immediate from definitions.  $\square$

**Theorem 6.1.9.**

$$\forall x \in \mathbb{R}. x/1 = x$$

PROOF:

- $\langle 1 \rangle 1$ . LET:  $x \in \mathbb{R}$   
 $\langle 1 \rangle 2$ .  $1/1 = 1$   
 PROOF: Since  $1 \cdot 1 = 1$ .  
 $\langle 1 \rangle 3$ .  $x/1 = x$   
 PROOF: Since  $x/1 = x \cdot 1/1 = x \cdot 1 = x$ .

$\square$

**Theorem 6.1.10.** For any real numbers  $x$  and  $y$ , if  $x \neq 0$  and  $y \neq 0$  then  $xy \neq 0$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $x, y \in \mathbb{R}$

$\langle 1 \rangle 2$ . ASSUME:  $xy = 0$  and  $x \neq 0$

PROVE:  $y = 0$

$\langle 1 \rangle 3$ .  $y = 0$

PROOF:

$$\begin{aligned}
 y &= 1y && \text{(Definition of 1)} \\
 &= (1/x)xy && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\
 &= (1/x)0 && (\langle 1 \rangle 2) \\
 &= 0 && \text{(Theorem 6.0.4)}
 \end{aligned}$$

□

**Theorem 6.1.11.** For any real numbers  $y$  and  $z$ , if  $y \neq 0$  and  $z \neq 0$  then  $(1/y)(1/z) = 1/(yz)$ .

PROOF: Since  $yz(1/y)(1/z) = 1 \cdot 1 = 1$ . □

**Corollary 6.1.11.1.** For any real numbers  $x, y, z, w$  with  $y \neq 0 \neq w$ , we have  $(x/y)(z/w) = (xz)/(yw)$ .

**Theorem 6.1.12.** For any real numbers  $x, y, z, w$  with  $y \neq 0 \neq w$ , we have

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + yz}{yw}$$

PROOF:

$$\begin{aligned}
 yw \left( \frac{x}{y} + \frac{z}{w} \right) &= yw \frac{x}{y} + yw \frac{z}{w} \\
 &= wx + yz
 \end{aligned}$$

□

**Theorem 6.1.13.** For any real number  $x$ , if  $x \neq 0$  then  $1/x \neq 0$ .

PROOF: Since  $x \cdot 1/x = 1 \neq 0$ . □

**Theorem 6.1.14.** For any real numbers  $w, z$ , if  $w \neq 0 \neq z$  then  $1/(w/z) = z/w$ .

PROOF: Since  $(z/w)(w/z) = (wz)/(wz) = 1$ . □

**Theorem 6.1.15.** For any real numbers  $a, x$  and  $y$ , if  $y \neq 0$  then  $(ax)/y = a(x/y)$

PROOF: Since  $ya(x/y) = ax$ . □

**Theorem 6.1.16.** For any real numbers  $x$  and  $y$ , if  $y \neq 0$  then  $(-x)/y = x/(-y) = -(x/y)$ .

PROOF:

$\langle 1 \rangle 1$ .  $(-x)/y = -(x/y)$

PROOF: Take  $a = -1$  in Theorem 6.1.15.

$\langle 1 \rangle 2$ .  $x/(-y) = -(x/y)$

PROOF: Since  $(-y)(-(x/y)) = y(x/y) = x$ .  
 $\square$

**Theorem 6.1.17.** *For any real numbers  $x, y, z$  and  $w$ , if  $x > y$  and  $w > z$  then  $x + w > y + z$ .*

PROOF: We have  $y + z < x + z < x + w$  by Monotonicity of Addition twice.  $\square$

**Corollary 6.1.17.1.** *For any real numbers  $x$  and  $y$ , if  $x > 0$  and  $y > 0$  then  $x + y > 0$ .*

**Theorem 6.1.18.** *For any real numbers  $x$  and  $y$ , if  $x > 0$  and  $y > 0$  then  $xy > 0$ .*

PROOF:

$$\begin{aligned} xy &> 0y && \text{(Monotonicity of Multiplication)} \\ &= 0 && \text{(Theorem 6.0.4)} \quad \square \end{aligned}$$

**Theorem 6.1.19.** *For any real number  $x$ , we have  $x > 0$  iff  $-x < 0$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $0 < x$  then  $-x < 0$

PROOF: By Monotonicity of Addition adding  $-x$  to both sides.

$\langle 1 \rangle 2$ . If  $-x < 0$  then  $0 < x$

PROOF: By Monotonicity of Addition adding  $x$  to both sides.

$\square$

**Theorem 6.1.20.** *For any real numbers  $x$  and  $y$ , we have  $x > y$  iff  $-x < -y$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $y < x$  then  $-x < -y$ .

PROOF: By Monotonicity of Addition adding  $-x - y$  to both sides.

$\langle 1 \rangle 2$ . If  $-x < -y$  then  $y < x$ .

PROOF: By Monotonicity of Addition adding  $x + y$  to both sides.

$\square$

**Theorem 6.1.21.** *For any real numbers  $x, y$  and  $z$ , if  $x > y$  and  $z < 0$  then  $xz < yz$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $x, y$  and  $z$  be real numbers.

$\langle 1 \rangle 2$ . ASSUME:  $x > y$

$\langle 1 \rangle 3$ . ASSUME:  $z < 0$

$\langle 1 \rangle 4$ .  $-z > 0$

PROOF: Theorem 6.1.19,  $\langle 1 \rangle 3$ .

$\langle 1 \rangle 5$ .  $x(-z) > y(-z)$

PROOF:  $\langle 1 \rangle 2$ ,  $\langle 1 \rangle 4$ , Monotonicity of Multiplication.

$\langle 1 \rangle 6$ .  $-(xz) > -(yz)$

PROOF: Theorem 6.0.7,  $\langle 1 \rangle 5$ .

$\langle 1 \rangle 7. \quad xz < yz$

PROOF: Theorem 6.1.19,  $\langle 1 \rangle 6$ .

□

**Theorem 6.1.22.** *For any real number  $x$ , if  $x \neq 0$  then  $xx > 0$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $x > 0$  then  $xx > 0$

PROOF: By Monotonicity of Multiplication.

$\langle 1 \rangle 2$ . If  $x < 0$  then  $xx > 0$

PROOF: Theorem 6.1.21.

□

**Theorem 6.1.23.**

$$0 < 1$$

PROOF: By Theorem 6.1.22 since  $1 = 1 \cdot 1$ . □

**Definition 6.1.24** (Positive). A real number  $x$  is *positive* iff  $x > 0$ .

We write  $\mathbb{R}_+$  for the set of positive reals.

**Theorem 6.1.25.** *For any real numbers  $x$  and  $y$ , we have  $xy$  is positive if and only if  $x$  and  $y$  are both positive or both negative.*

PROOF: By the Monotonicity of Multiplication and Theorem 6.1.21. □

**Corollary 6.1.25.1.** *For any real number  $x$ , if  $x > 0$  then  $1/x > 0$ .*

PROOF: Since  $x \cdot 1/x = 1$  is positive. □

**Theorem 6.1.26.** *For any real numbers  $x$  and  $y$ , if  $x > y > 0$  then  $1/x < 1/y$ .*

PROOF: If  $1/y \leq 1/x$  then  $1 < 1$  by Monotonicity of Multiplication. □

**Theorem 6.1.27.** *For any real numbers  $x$  and  $y$ , if  $x < y$  then  $x < (x+y)/2 < y$ .*

PROOF: We have  $2x < x+y$  and  $x+y < 2y$  by Monotonicity of Addition, hence  $x < (x+y)/2 < y$  by Monotonicity of Multiplication since  $1/2 > 0$ . □

**Corollary 6.1.27.1.**  $\mathbb{R}$  is a linear continuum.

**Definition 6.1.28** (Negative). A real number  $x$  is *negative* iff  $x < 0$ .

We write  $\overline{\mathbb{R}_+}$  for the set of nonnegative reals.

**Theorem 6.1.29.** *For every positive real number  $a$ , there exists a unique positive real  $\sqrt{a}$  such that  $\sqrt{a}^2 = a$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $a$  be a positive real.

$\langle 1 \rangle 2$ . For any real numbers  $x$  and  $h$ , if  $0 \leq h < 1$ , then

$$(x+h)^2 < x^2 + h(2x+1) .$$

⟨2⟩1. LET:  $x$  and  $h$  be real numbers.

⟨2⟩2. ASSUME:  $0 \leq h < 1$

⟨2⟩3.  $(x + h)^2 < x^2 + h(2x + 1)$

PROOF:

$$\begin{aligned} (x + h)^2 &= x^2 + 2hx + h^2 \\ &< x^2 + 2hx + h & (\langle 2 \rangle 2) \\ &= x^2 + h(2x + 1) \end{aligned}$$

⟨1⟩3. For any real numbers  $x$  and  $h$ , if  $h > 0$  then

$$(x - h)^2 > x^2 - 2hx .$$

⟨2⟩1. LET:  $x$  and  $h$  be real numbers.

⟨2⟩2. ASSUME:  $h > 0$

⟨2⟩3.  $(x - h)^2 > x^2 - 2hx$

PROOF:

$$\begin{aligned} (x - h)^2 &= x^2 - 2hx + h^2 \\ &> x^2 - 2hx & (\langle 2 \rangle 2) \end{aligned}$$

⟨1⟩4. For any positive real  $x$ , if  $x^2 < a$  then there exists  $h > 0$  such that  
 $(x + h)^2 < a$ .

⟨2⟩1. LET:  $x$  be a positive real.

⟨2⟩2. ASSUME:  $x^2 < a$

⟨2⟩3. LET:  $h = \min((a - x^2)/(2x + 1), 1/2)$

⟨2⟩4.  $0 < h < 1$

⟨2⟩5.  $(x + h)^2 < a$

PROOF:

$$\begin{aligned} (x + h)^2 &< x^2 + h(2x + 1) & (\langle 1 \rangle 2) \\ &\leq a \end{aligned}$$

⟨1⟩5. For any positive real  $x$ , if  $x^2 > a$  then there exists  $h > 0$  such that  
 $(x - h)^2 > a$ .

⟨2⟩1. LET:  $x$  be a positive real.

⟨2⟩2. ASSUME:  $x^2 > a$

⟨2⟩3. LET:  $h = (x^2 - a)/2x$

⟨2⟩4.  $h > 0$

⟨2⟩5.  $(x - h)^2 > a$

PROOF:

$$\begin{aligned} (x - h)^2 &> x^2 - 2hx \\ &= a & (\langle 2 \rangle 3) \end{aligned}$$

⟨1⟩6. LET:  $B = \{x \in \mathbb{R} : x^2 < a\}$

⟨1⟩7.  $B$  is bounded above.

PROOF: If  $a \geq 1$  then  $a$  is an upper bound. If  $a < 1$  then 1 is an upper bound.

⟨1⟩8.  $B$  contains at least one positive real.

PROOF: If  $a \geq 1$  then  $1 \in B$ . If  $a < 1$  then  $a \in B$ .

⟨1⟩9. LET:  $b = \sup B$

⟨1⟩10.  $b^2 = a$

⟨2⟩1.  $b^2 \geq a$

⟨3⟩1. ASSUME: for a contradiction  $b^2 < a$

- $\langle 3 \rangle 2$ . PICK  $h > 0$  such that  $(b + h)^2 < a$   
 PROOF:  $\langle 1 \rangle 4$   
 $\langle 3 \rangle 3$ .  $b + h \in B$   
 $\langle 3 \rangle 4$ . Q.E.D.  
 PROOF: This contradicts  $\langle 1 \rangle 9$ .  
 $\langle 2 \rangle 2$ .  $b^2 \leq a$   
 $\langle 3 \rangle 1$ . ASSUME: for a contradiction  $b^2 > a$   
 $\langle 3 \rangle 2$ . PICK  $h > 0$  such that  $(b - h)^2 > a$   
 PROOF:  $\langle 1 \rangle 5$   
 $\langle 3 \rangle 3$ . PICK  $x \in B$  such that  $b - h < x$   
 PROOF:  $\langle 1 \rangle 9$   
 $\langle 3 \rangle 4$ .  $(b - h)^2 < x^2 < a$   
 $\langle 3 \rangle 5$ . Q.E.D.  
 PROOF: This contradicts  $\langle 3 \rangle 2$   
 $\langle 1 \rangle 11$ . For any positive reals  $b$  and  $c$ , if  $b^2 = c^2$  then  $b = c$ .  
 $\langle 2 \rangle 1$ . LET:  $b$  and  $c$  be positive reals.  
 $\langle 2 \rangle 2$ . ASSUME:  $b^2 = c^2$   
 $\langle 2 \rangle 3$ .  $b^2 - c^2 = 0$   
 $\langle 2 \rangle 4$ .  $(b - c)(b + c) = 0$   
 $\langle 2 \rangle 5$ .  $b - c = 0$  or  $b + c = 0$   
 $\langle 2 \rangle 6$ .  $b + c \neq 0$   
 PROOF: Since  $b + c > 0$   
 $\langle 2 \rangle 7$ .  $b - c = 0$   
 $\langle 2 \rangle 8$ .  $b = c$   
 $\square$

**Theorem 6.1.30.** *The set of real numbers is uncountable.*

**Definition 6.1.31.** We write  $\mathbb{R}^\omega$  for the set of sequences in  $\mathbb{R}^\omega$  that are eventually zero.

**Definition 6.1.32** (Hilbert Cube). The *Hilbert cube* is  $\prod_{n=0}^\infty [0, 1/(n+1)]$ .

## 6.2 The Ordered Square

**Definition 6.2.1** (Ordered Square). The *ordered square*  $I_o^2$  is the set  $[0, 1]^2$  under the dictionary order.

**Proposition 6.2.2.** *The ordered square is a linear continuum.*

PROOF:

- $\langle 1 \rangle 1$ .  $I_o^2$  has the least upper bound property.  
 $\langle 2 \rangle 1$ . LET:  $S$  be a nonempty subset of  $I_o^2$ .  
 $\langle 2 \rangle 2$ . LET:  $a$  be the supremum of  $\pi_1(S)$   
 $\langle 2 \rangle 3$ . CASE:  $a \in \pi_1(S)$   
 $\langle 3 \rangle 1$ . LET:  $b$  be the supremum of  $\{y \in [0, 1] : (a, y) \in S\}$   
 $\langle 3 \rangle 2$ .  $(a, b)$  is the supremum of  $S$ .

$\langle 2 \rangle 4$ . CASE:  $a \notin \pi_1(S)$

PROOF:  $(a, 0)$  is the supremum of  $S$ .

$\langle 1 \rangle 2$ .  $I_o^2$  is dense.

$\langle 2 \rangle 1$ . LET:  $(x_1, y_1), (x_2, y_2) \in I_o^2$  with  $(x_1, y_1) < (x_2, y_2)$

PROVE: There exists  $(x_3, y_3) \in I_o^2$  such that  $(x_1, y_1) < (x_3, y_3) < (x_2, y_2)$

$\langle 2 \rangle 2$ . CASE:  $x_1 < x_2$

$\langle 3 \rangle 1$ . PICK  $x_3$  such that  $x_1 < x_3 < x_2$

$\langle 3 \rangle 2$ .  $(x_1, y_1) < (x_3, 0) < (x_2, y_2)$

$\langle 2 \rangle 3$ . CASE:  $x_1 = x_2$  and  $y_1 < y_2$

$\langle 3 \rangle 1$ . PICK  $y_3$  such that  $y_1 < y_3 < y_2$

$\langle 3 \rangle 2$ .  $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

### 6.3 Punctured Euclidean Space

**Definition 6.3.1** (Punctured Euclidean Space). Let  $n$  be a positive integer. The *punctured Euclidean space* is  $\mathbb{R}^n - \{\vec{0}\}$ .

### 6.4 Topologist's Sine Curve

**Definition 6.4.1** (Topologist's Sine Curve). The *topologist's sine curve* is

$$(\{0\} \times [-1, 1]) \cup \{(x, \sin 1/x) : 0 < x \leq 1\}.$$

### 6.5 The Long Line

**Definition 6.5.1** (Long Line). The *long line* is  $S_\Omega \times [0, 1)$  in the dictionary order.

**Proposition 6.5.2.** For any  $a \in S_\Omega$  with  $a \neq 0$  we have  $[(0, 0), (a, 0)) \cong [0, 1)$ .

PROOF: By transfinite induction on  $a$  using Propositions 6.0.9 and 6.0.10. □



## Chapter 7

# Integers and Rationals

### 7.1 Positive Integers

**Definition 7.1.1** (Inductive). A set of real numbers  $A$  is *inductive* iff  $1 \in A$  and  $\forall x \in A. x + 1 \in A$ .

**Definition 7.1.2** (Positive Integer). The set  $\mathbb{Z}_+$  of *positive integers* is the intersection of the set of inductive sets.

**Proposition 7.1.3.** *Every positive integer is positive.*

PROOF: The set of positive reals is inductive.  $\square$

**Proposition 7.1.4.** *1 is the least element of  $\mathbb{Z}_+$ .*

PROOF: Since  $\{x \in \mathbb{R} : x \geq 1\}$  is inductive.  $\square$

**Proposition 7.1.5.**  *$\mathbb{Z}_+$  is inductive.*

PROOF: 1 is an element of every inductive set, and for all  $x \in \mathbb{R}$ , if  $x$  is an element of every inductive set then so is  $x + 1$ .  $\square$

**Theorem 7.1.6** (Principle of Induction). *If  $A$  is an inductive set of positive integers then  $A = \mathbb{Z}_+$ .*

PROOF: Immediate from definitions.  $\square$

**Theorem 7.1.7** (Well-Ordering Property).  *$\mathbb{Z}_+$  is well ordered.*

PROOF: Construct the obvious order isomorphism  $\omega \cong \mathbb{Z}_+$ .  $\square$

**Theorem 7.1.8** (Archimedean Ordering Property). *The set  $\mathbb{Z}_+$  is unbounded above.*

PROOF:

$\langle 1 \rangle$ 1. ASSUME: for a contradiction  $\mathbb{Z}_+$  is bounded above.

⟨1⟩2. LET:

$$s = \sup \mathbb{Z}_+$$

⟨1⟩3. PICK  $n \in \mathbb{Z}_+$  such that  $s - 1 < n$

⟨1⟩4.  $s < n + 1$

⟨1⟩5. Q.E.D.

PROOF: ⟨1⟩2 and ⟨1⟩4 form a contradiction.

□

### 7.1.1 Exponentiation

**Definition 7.1.9.** For  $a$  a real number and  $n$  a positive integer, define the real number  $a^n$  recursively as follows:

$$\begin{aligned} a^1 &= a \\ a^{n+1} &= a^n a \end{aligned}$$

**Theorem 7.1.10.** For all  $a \in \mathbb{R}$  and  $m, n \in \mathbb{Z}_+$ , we have

$$a^n a^m = a^{n+m}$$

PROOF:

⟨1⟩1. LET:  $P(m)$  be the property  $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. a^n a^m = a^{n+m}$

⟨1⟩2.  $P(1)$

PROOF:  $a^n a^1 = a^n a = a^{n+1}$ .

⟨1⟩3.  $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$

⟨2⟩1. LET:  $m$  be a positive integer.

⟨2⟩2. ASSUME:  $P(m)$

⟨2⟩3. LET:  $a \in \mathbb{R}$

⟨2⟩4. LET:  $n \in \mathbb{Z}_+$

⟨2⟩5.  $a^n a^{m+1} = a^{n+m+1}$

PROOF:

$$\begin{aligned} a^n a^{m+1} &= a^n a^m a \\ &= a^{n+m} a && (\langle 2 \rangle 2) \\ &= a^{n+m+1} \end{aligned}$$

⟨1⟩4. Q.E.D.

PROOF: By induction.

□

**Theorem 7.1.11.** For all  $a \in \mathbb{R}$  and  $m, n \in \mathbb{Z}_+$ ,

$$(a^n)^m = a^{nm}.$$

PROOF:

⟨1⟩1. LET:  $P(m)$  be the property  $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. (a^n)^m = a^{nm}$ .

⟨1⟩2.  $P(1)$

PROOF:  $(a^n)^1 = a^n = a^{n \cdot 1}$

⟨1⟩3.  $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$

PROOF:

$$\begin{aligned} (a^n)^{m+1} &= (a^n)^m a^n \\ &= a^{nm} a^n \\ &= a^{nm+n} && (\text{Theorem 7.1.10}) \\ &= a^{n(m+1)} \end{aligned}$$

□

**Theorem 7.1.12.** *For any real numbers  $a$  and  $b$  and positive integer  $m$ ,*

$$a^m b^m = (ab)^m .$$

PROOF: Induction on  $m$ . □

## 7.2 Integers

**Definition 7.2.1** (Integer). The set  $\mathbb{Z}$  of *integers* is

$$\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \{-x : x \in \mathbb{Z}_+\} .$$

**Proposition 7.2.2.** *The sum, difference and product of two integers is an integer.*

PROOF: Easy. □

**Example 7.2.3.**  $1/2$  is not an integer.

**Proposition 7.2.4.** *For any integer  $n$ , there is no integer  $a$  such that  $n < a < n+1$ .*

PROOF:

⟨1⟩1. For any positive integer  $n$ , there is no integer  $a$  such that  $n < a < n+1$ .

⟨2⟩1. There is no integer  $a$  such that  $1 < a < 2$ .

⟨3⟩1. There is no positive integer  $a$  such that  $1 < a < 2$ .

⟨4⟩1. We do not have  $1 < 1 < 2$ .

⟨4⟩2. For any positive integer  $n$ , we do not have  $1 < n+1 < 2$ .

PROOF: Since  $n \geq 1$  so  $n+1 \geq 2$ .

⟨3⟩2. We do not have  $1 < 0 < 2$ .

⟨3⟩3. For any positive integer  $a$ , we do not have  $1 < -a < 2$ .

PROOF: Since  $-a < 0 < 1$ .

⟨2⟩2. For any positive integer  $n$ , if there is no integer  $a$  such that  $n < a < n+1$ , then there is no integer  $a$  such that  $n+1 < a < n+2$ .

PROOF: If  $n+1 < a < n+2$  then  $n < a-1 < n+1$ .

⟨1⟩2. There is no integer  $a$  such that  $0 < a < 1$ .

PROOF: If  $0 < a < 1$  then  $1 < a+1 < 2$ .

⟨1⟩3. For any positive integer  $n$ , there is no integer  $a$  such that  $-n < a < -n+1$ .

PROOF: If  $-n < a < -n+1$  then  $n-1 < -a < n$ .

□

**Theorem 7.2.5.** *Every nonempty subset of  $\mathbb{Z}$  bounded above has a largest element.*

PROOF:

⟨1⟩1. LET:  $S$  be a nonempty subset of  $\mathbb{Z}$  bounded above.

⟨1⟩2. LET:  $u$  be an upper bound for  $S$ .

⟨1⟩3. PICK an integer  $n > u$

PROOF: Archimedean property.

⟨1⟩4. LET:  $k$  be the least positive integer such that  $n - k \in S$ .

⟨2⟩1. PICK  $m \in S$

⟨2⟩2.  $n - m$  is a positive integer.

⟨2⟩3. There exists a positive integer  $k$  such that  $n - k \in S$ .

⟨1⟩5.  $n - k$  is the greatest element in  $S$ .

⟨2⟩1. LET:  $m \in S$

⟨2⟩2.  $n - m \geq k$

⟨2⟩3.  $m \leq n - k$

□

**Theorem 7.2.6.** *For any real number  $x$ , if  $x$  is not an integer then there exists a unique integer  $n$  such that  $n < x < n + 1$ .*

PROOF:

⟨1⟩1.  $\{n \in \mathbb{Z} : n < x\}$  is a nonempty set of integers bounded above.

⟨2⟩1. PICK  $m > -x$

PROOF: Archimedean property.

⟨2⟩2.  $-m < x$

⟨2⟩3.  $\{n \in \mathbb{Z} : n < x\}$  is nonempty.

⟨1⟩2. LET:  $n$  be the greatest integer such that  $n < x$

⟨1⟩3.  $x < n + 1$

⟨1⟩4. If  $n'$  is an integer with  $n' < x < n' + 1$  then  $n' = n$ .

PROOF: We have  $n' < n + 1$  so  $n' \leq n$ , and  $n < n' + 1$  so  $n \leq n'$ .

□

**Definition 7.2.7** (Even). An integer  $n$  is *even* iff  $n/2$  is an integer; otherwise,  $n$  is *odd*.

**Theorem 7.2.8.** *If the integer  $m$  is odd then there exists an integer  $n$  such that  $m = 2n + 1$ .*

PROOF:

⟨1⟩1. LET:  $n$  be the integer such that  $n < m/2 < n + 1$

PROOF: Theorem 7.2.6.

⟨1⟩2.  $2n < m < 2n + 2$

⟨1⟩3.  $m = 2n + 1$

□

**Theorem 7.2.9.** *The product of two odd integers is odd.*

PROOF:  $(2m + 1)(2n + 1) = 2(2mn + m + n) + 1$ .  $\square$

**Corollary 7.2.9.1.** *If  $p$  is an odd integer and  $n$  is a positive integer then  $p^n$  is an odd integer.*

**Definition 7.2.10** (Exponentiation). Extend the definition of exponentiation so  $a^n$  is defined for:

- all real numbers  $a$  and non-negative integers  $n$
- all non-zero real numbers  $a$  and integers  $n$

as follows:

$$\begin{aligned} a^0 &= 1 \\ a^{-n} &= 1/a^n \end{aligned} \quad (n \text{ a positive integer})$$

**Theorem 7.2.11** (Laws of Exponents). *For all non-zero reals  $a$  and  $b$  and integers  $m$  and  $n$ ,*

$$\begin{aligned} a^n a^m &= a^{n+m} \\ (a^n)^m &= a^{nm} \\ a^m b^m &= (ab)^m \end{aligned}$$

PROOF: Easy.  $\square$

**Theorem 7.2.12.**  $\mathbb{Z}$  is countable.

PROOF: The function that maps an integer  $n$  to  $2n$  if  $n \geq 0$  and  $-1 - 2n$  if  $n < 0$  is a bijection  $\mathbb{Z} \approx \mathbb{N}$ .  $\square$

## 7.3 Rational Numbers

**Definition 7.3.1** (Rational Number). The set  $\mathbb{Q}$  of *rational numbers* is the set of all real numbers that are the quotient of two integers. A real that is not rational is *irrational*.

**Theorem 7.3.2.**  $\sqrt{2}$  is irrational.

PROOF:

- $\langle 1 \rangle$ 1. For any positive rational  $a$ , there exist positive integers  $m$  and  $n$  not both even such that  $a = m/n$ .
- $\langle 2 \rangle$ 1. LET:  $a$  be a positive rational.
- $\langle 2 \rangle$ 2. LET:  $n$  be the least positive integer such that  $na$  is a positive integer.
- $\langle 2 \rangle$ 3. LET:  $m = na$
- $\langle 2 \rangle$ 4. ASSUME: for a contradiction  $m$  and  $n$  are both even.
- $\langle 2 \rangle$ 5.  $m/2 = (n/2)a$
- $\langle 2 \rangle$ 6. Q.E.D.

PROOF: This contradicts the leastness of  $n$  ( $\langle 2 \rangle 2$ ).

$\langle 1 \rangle 2$ . ASSUME: for a contradiction  $\sqrt{2}$  is rational.

$\langle 1 \rangle 3$ . PICK positive integers  $m$  and  $n$  not both even such that  $\sqrt{2} = m/n$ .

$\langle 1 \rangle 4$ .  $m^2 = 2n^2$

$\langle 1 \rangle 5$ .  $m^2$  is even.

$\langle 1 \rangle 6$ .  $m$  is even.

PROOF: Theorem 7.2.9.

$\langle 1 \rangle 7$ . LET:  $k = m/2$

$\langle 1 \rangle 8$ .  $4k^2 = 2n^2$

$\langle 1 \rangle 9$ .  $n^2 = 2k^2$

$\langle 1 \rangle 10$ .  $n^2$  is even.

$\langle 1 \rangle 11$ .  $n$  is even.

PROOF: Theorem 7.2.9.

$\langle 1 \rangle 12$ . Q.E.D.

PROOF:  $\langle 1 \rangle 3$ ,  $\langle 1 \rangle 6$  and  $\langle 1 \rangle 11$  form a contradiction.

□

**Theorem 7.3.3.**  $\mathbb{Q}$  is countably infinite.

PROOF: The function  $\mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$  that maps  $(m, n)$  to  $m/(n+1)$  is a surjection.

□

## 7.4 Algebraic Numbers

**Definition 7.4.1** (Algebraic Number). A real number  $r$  is *algebraic* iff there exists a natural number  $n$  and rational numbers  $a_0, a_1, \dots, a_{n-1}$  such that

$$r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0$$

Otherwise,  $r$  is *transcendental*.

**Proposition 7.4.2.** The set of algebraic numbers is countably infinite.

PROOF: There are countably many finite sequences of rational numbers, and each corresponding polynomial has only finitely many roots. □

**Corollary 7.4.2.1.** The set of transcendental numbers is uncountable.

**Part IV**

**Algebra**





## Chapter 8

# Monoid Theory

**Definition 8.0.1** (Monoid). A *monoid* is a category with one object.

**Definition 8.0.2.** Let  $\mathcal{C}$  be a category and  $X \in \mathcal{C}$ . The monoid  $\text{End}_{\mathcal{C}}(X)$  is the set of all morphisms  $X \rightarrow X$  under composition.

**Proposition 8.0.3.** *For any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $X \in \mathcal{C}$ , we have that  $F : \text{End}_{\mathcal{C}}(X) \rightarrow \text{End}_{\mathcal{D}}(FX)$  is a monoid homomorphism.*

PROOF: Since  $F\text{id}_X = \text{id}_{FX}$  and  $F(g \circ f) = Fg \circ Ff$ .  $\square$



## Chapter 9

# Group Theory

### 9.1 Category of Small Groups

**Definition 9.1.1.** Let **Grp** be the category of small groups and group homomorphisms.

**Definition 9.1.2.** We identify any group  $G$  with the category with one object whose morphisms are the elements of  $G$  with composition given by the multiplication in  $G$ .

**Proposition 9.1.3.** *The trivial group is a zero object in **Grp**.*

PROOF: Easy.  $\square$

The zero morphism  $G \rightarrow H$  maps every element in  $G$  to  $e$ .

**Definition 9.1.4.** Let  $\mathcal{C}$  be a category and  $X \in \mathcal{C}$ . We write  $\text{Aut}_{\mathcal{C}}(X)$  for the set of all isomorphisms  $X \cong X$  under composition.

**Proposition 9.1.5.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor and  $X \in \mathcal{C}$ . Then  $F : \text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(FX)$  is a group homomorphism.*

PROOF: Since  $F \text{id}_X = \text{id}_{FX}$ ,  $F(g \circ f) = Fg \circ Ff$ , and  $Ff^{-1} = (Ff)^{-1}$ .  $\square$

**Proposition 9.1.6.** **Grp** has products.

**Definition 9.1.7** (Free Product). The product of a family of groups in **Grp** is called the *free product*.

**Proposition 9.1.8.** **Ab** has products given by direct sums.

**Definition 9.1.9** (Left Coset). Let  $G$  be a group and  $H$  a subgroup of  $G$ . The *left cosets* of  $H$  are the sets of the form

$$xH := \{xh : h \in H\}$$

We write  $G/H$  for the set of left cosets of  $H$  in  $G$ .

**Proposition 9.1.10.** *Let  $G$  be a group and  $H$  a subgroup of  $G$ . Then  $G/H$  is a partition of  $G$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\bigcup (G/H) = G$

PROOF: Since  $x = xe$  and so  $x \in xH$ .

$\langle 1 \rangle 2$ . Any two distinct left cosets of  $H$  are disjoint.

PROOF: Since if  $z \in xH$  and  $z \in yH$  then  $xH = yH = zH$ .

□

**Definition 9.1.11.** Let  $G$  be a group. Let  $A$  and  $B$  be subsets of  $G$ . Then

$$AB := \{ab : a \in A, b \in B\} .$$

**Definition 9.1.12.** Let  $G$  be a group. Let  $A$  be a subset of  $G$ . Then

$$A^{-1} := \{a^{-1} : a \in A\} .$$

## Chapter 10

# Ring Theory

**Definition 10.0.1.** Let **Ring** be the concrete category of rings and ring homomorphisms.

**Definition 10.0.2** (Spectrum). Let  $R$  be a commutative ring. The *spectrum* of  $R$ ,  $\text{spec } R$ , is the set of all prime ideals of  $R$ .

**Definition 10.0.3** (Zariski Topology). Let  $R$  be a commutative ring. The *Zariski topology* on  $\text{spec } R$  is the topology where the closed sets are the sets of the form

$$VE := \{p \in \text{spec } R : E \subseteq p\}$$

for any  $E \in \mathcal{P}R$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$

$\langle 1 \rangle 2$ . For all  $\mathcal{A} \subseteq \mathcal{C}$  we have  $\bigcap \mathcal{A} \in \mathcal{C}$

$\langle 2 \rangle 1$ . LET:  $\mathcal{A} \subseteq \mathcal{C}$

$\langle 2 \rangle 2$ . LET:  $E = \bigcup \{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$

PROVE:  $VE = \bigcap \mathcal{A}$

$\langle 2 \rangle 3$ . For all  $p \in \text{spec } R$ , if  $E \subseteq p$  then  $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 1$ . LET:  $p \in \text{spec } R$

$\langle 3 \rangle 2$ . ASSUME:  $E \subseteq p$

$\langle 3 \rangle 3$ . LET:  $E' \in \mathcal{P}R$  with  $VE' \in \mathcal{A}$

$\langle 3 \rangle 4$ .  $E' \subseteq E$

$\langle 3 \rangle 5$ .  $E' \subseteq p$

$\langle 3 \rangle 6$ .  $p \in VE'$

$\langle 2 \rangle 4$ . For all  $p \in \text{spec } R$ , if  $p \in \bigcap \mathcal{A}$  then  $E \subseteq p$

$\langle 3 \rangle 1$ . LET:  $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 2$ . For all  $E' \in \mathcal{P}R$  with  $VE' \in \mathcal{A}$  we have  $E' \subseteq p$

$\langle 3 \rangle 3$ .  $E \subseteq p$

$\langle 1 \rangle 3$ . For all  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .

PROOF: Since  $VE \cup VE' = V(E \cap E')$

$\langle 1 \rangle 4. \emptyset \in \mathcal{C}$

$\langle 2 \rangle 1. VR = \emptyset$

PROOF: If  $p \in VR$  then  $R \subseteq p$  contradicting the fact that  $p$  is a prime ideal.

□

**Definition 10.0.4.** For any ring  $R$ , let  $R - \mathbf{Mod}$  be the category of small  $R$ -modules and  $R$ -module homomorphisms.

**Proposition 10.0.5.**  $R - \mathbf{Mod}$  has products and coproducts.

## Chapter 11

# Field Theory

**Proposition 11.0.1.** *Field does not have binary products.*

PROOF: There cannot be a field  $K$  with field homomorphisms  $K \rightarrow \mathbb{Z}_2$  and  $K \rightarrow \mathbb{Z}_3$ , because its characteristic would be both 2 and 3.  $\square$





## Chapter 12

# Linear Algebra

**Definition 12.0.1** (Span). Let  $V$  be a vector space and  $A \subseteq V$ . The *span* of  $A$  is the set of all linear combinations of elements of  $A$ .

**Definition 12.0.2** (Independent). Let  $V$  be a vector space and  $A \subseteq V$ . Then  $A$  is *linearly independent* iff, whenever

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

where  $v_1, \dots, v_n \in A$ , then

$$\alpha_1 = \cdots = \alpha_n = 0 \text{ .}$$

**Proposition 12.0.3.** *Let  $V$  be a vector space,  $A \subseteq V$  and  $v \in V$ . If  $A$  is linearly independent and  $v \notin \text{span } A$ , then  $A \cup \{v\}$  is independent.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\alpha_1 v_1 + \cdots + \alpha_n v_n + \beta v = 0$  where  $v_1, \dots, v_n \in A$

$\langle 1 \rangle 2$ .  $\beta = 0$

PROOF: Otherwise  $v = (\alpha_1/\beta)v_1 + \cdots + (\alpha_n/\beta)v_n \in \text{span } A$ .

$\langle 1 \rangle 3$ .  $\alpha_1 = \cdots = \alpha_n = 0$

PROOF: Since  $A$  is linearly independent.

□

**Theorem 12.0.4.** *Every vector space has a basis.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $V$  be a vector space.

$\langle 1 \rangle 2$ . PICK a maximal linearly independent set  $\mathcal{B}$ .

PROOF: By Tukey's Lemma.

$\langle 1 \rangle 3$ .  $\text{span } \mathcal{B} = V$

PROOF: Proposition 12.0.3.

□

**Definition 12.0.5.** For any field  $K$ , we write  $\mathbf{Vect}_K$  for  $K - \mathbf{Mod}$ .

Dual space functor  $\mathbf{Vect}_K^{\text{op}} \rightarrow \mathbf{Vect}_K$ .



**Part V**

**Topology**



## Chapter 13

# Topological Spaces

### 13.1 Topological Spaces

**Definition 13.1.1** (Topological Space). Let  $X$  be a set and  $\mathcal{O} \subseteq \mathcal{P}X$ . Then we say  $(X, \mathcal{O})$  is a *topological space* iff:

- For any  $\mathcal{U} \subseteq \mathcal{O}$  we have  $\bigcup \mathcal{U} \in \mathcal{O}$ .
- For any  $U, V \in \mathcal{O}$  we have  $U \cap V \in \mathcal{O}$ .
- $X \in \mathcal{O}$

We call  $\mathcal{O}$  the *topology* of the topological space, and call its elements *open* sets. We shall often write  $X$  for the topological space  $(X, \mathcal{O})$ .

**Example 13.1.2** (Discrete Topology). For any set  $X$ , the power set  $\mathcal{P}X$  is called the *discrete* topology on  $X$ .

**Example 13.1.3** (Indiscrete Topology). For any set  $X$ , the *indiscrete* or *trivial* topology on  $X$  is  $\{\emptyset, X\}$ .

**Example 13.1.4** (Cofinite Topology). For any set  $X$ , the *cofinite* topology is  $\mathcal{T} = \{\emptyset\} \cup \{X - U : U \subseteq X \text{ is finite}\}$ .

We prove this is a topology.

**Example 13.1.5** (Cocountable Topology). For any set  $X$ , the *cocountable* topology is  $\{X - U : U \subseteq X \text{ is countable}\}$ .

**Example 13.1.6** (Sierpiński Two-Point Space). The *Sierpiński two-point space* is  $\{0, 1\}$  under the topology  $\{\emptyset, \{1\}, \{0, 1\}\}$ .

**Proposition 13.1.7.** *Let  $X$  be a topological space and  $U \subseteq X$ . Then  $U$  is open if and only if, for all  $x \in U$ , there exists an open set  $V$  such that  $x \in V \subseteq U$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $U$  is open then, for all  $x \in U$ , there exists an open set  $V$  such that  $x \in V \subseteq U$ .

PROOF: Take  $V = U$ .

$\langle 1 \rangle 2$ . If, for all  $x \in U$ , there exists an open set  $V$  such that  $x \in V \subseteq U$ , then  $U$  is open.

PROOF: Since then  $U$  is the union of all the open subsets of  $U$ .

□

**Proposition 13.1.8.** *The intersection of a set of topologies on a set  $X$  is a topology on  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{T}$  be a set of topologies on  $X$ .

$\langle 1 \rangle 2$ . For all  $\mathcal{U} \subseteq \bigcap \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$ .

$\langle 2 \rangle 1$ . LET:  $\mathcal{U} \subseteq \bigcap \mathcal{T}$

$\langle 2 \rangle 2$ . LET:  $T \in \mathcal{T}$

$\langle 2 \rangle 3$ .  $\mathcal{U} \subseteq T$

$\langle 2 \rangle 4$ .  $\bigcup \mathcal{U} \in T$

$\langle 1 \rangle 3$ . For all  $U, V \in \bigcap \mathcal{T}$  we have  $U \cap V \in \bigcap \mathcal{T}$ .

$\langle 2 \rangle 1$ . LET:  $U, V \in \bigcap \mathcal{T}$

$\langle 2 \rangle 2$ . LET:  $T \in \mathcal{T}$

$\langle 2 \rangle 3$ .  $U, V \in T$

$\langle 2 \rangle 4$ .  $U \cap V \in T$

$\langle 1 \rangle 4$ .  $X \in \bigcap \mathcal{T}$ .

□

## 13.2 Closed Sets

**Definition 13.2.1** (Closed Set). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is *closed* iff  $X - A$  is open.

**Proposition 13.2.2.** *A set  $B$  is open if and only if  $X - B$  is closed.*

PROOF: We have  $B$  is open iff  $X - (X - B)$  is open iff  $X - B$  is closed. □

**Theorem 13.2.3.** *Let  $X$  be a set. Let  $\mathcal{C} \subseteq \mathcal{P}X$ . Then there exists a topology on  $X$  such that  $\mathcal{C}$  is the set of closed sets if and only if:*

$$1. \emptyset \in \mathcal{C}$$

$$2. \forall \mathcal{A} \subseteq \mathcal{C}. \bigcap \mathcal{A} \in \mathcal{C}$$

$$3. \forall C, D \in \mathcal{C}. C \cup D \in \mathcal{C}$$

*In this case, the topology is unique, and is  $\{X - C : C \in \mathcal{C}\}$ .*

PROOF:

$\langle 1 \rangle 1$ . In any topology on  $X$  we have  $\emptyset$  is closed.

PROOF: Since  $X - \emptyset = X$  is open.

⟨1⟩2. In any topology on  $X$ , the intersection of a set  $\mathcal{A}$  of closed sets is closed.

PROOF: Since  $X - \bigcap \mathcal{A} = \bigcup_{A \in \mathcal{A}} (X - A)$  is open.

⟨1⟩3. In any topology on  $X$ , the union of two closed sets is closed.

PROOF: For any closed sets  $C$  and  $D$ , we have  $X - (C \cup D) = (X - C) \cap (X - D)$  is open.

⟨1⟩4. If  $\mathcal{C}$  is a set satisfying 1–3, then  $\{X - C : C \in \mathcal{C}\}$  is a topology on  $X$  with respect to which  $\mathcal{C}$  is the set of closed sets.

⟨2⟩1. LET:  $\mathcal{C}$  be a set satisfying 1–3.

⟨2⟩2. LET:  $\mathcal{T} = \{X - C : C \in \mathcal{C}\}$

⟨2⟩3. For all  $U \in \mathcal{T}$  we have  $X - U \in \mathcal{C}$ .

⟨2⟩4.  $\mathcal{T}$  is a topology on  $X$ .

⟨3⟩1. For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ .

⟨4⟩1. LET:  $\mathcal{U} \subseteq \mathcal{T}$

⟨4⟩2. For all  $U \in \mathcal{U}$  we have  $X - U \in \mathcal{C}$ .

⟨4⟩3.  $X - \bigcup \mathcal{U} \in \mathcal{C}$

PROOF:

$$\begin{aligned} X - \bigcup \mathcal{U} &= \bigcap_{U \in \mathcal{U}} (X - U) \\ &\in \mathcal{C} \end{aligned}$$

⟨4⟩4.  $\bigcup \mathcal{U} \in \mathcal{T}$

⟨3⟩2. For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .

⟨3⟩3.  $X \in \mathcal{T}$

⟨2⟩5. For any set  $C$  we have  $C \in \mathcal{C}$  iff  $C$  is closed with respect to  $\mathcal{T}$ .

⟨1⟩5. If  $\mathcal{T}$  is any topology on  $X$  then  $\mathcal{T} = \{X - C : C \text{ is closed in } \mathcal{T}\}$ .

PROOF: Proposition 13.2.2.

□

## 13.3 Neighbourhoods

**Definition 13.3.1** (Neighbourhood). Let  $X$  be a topological space,  $Sx \in X$  and  $U \subseteq X$ . Then  $U$  is a *neighbourhood* of  $x$ , and  $x$  is an *interior* point of  $U$ , iff there exists an open set  $V$  such that  $x \in V \subseteq U$ .

**Proposition 13.3.2.** *A set  $B$  is open if and only if it is a neighbourhood of each of its points.*

PROOF: This is Proposition 13.1.7. □

**Proposition 13.3.3.** *Let  $X$  be a set and  $\mathcal{N} : X \rightarrow \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  on  $X$  such that, for all  $x \in X$ , we have  $\mathcal{N}_x$  is the set of neighbourhoods of  $x$ , if and only if:*

- For all  $x \in X$  and  $N \in \mathcal{N}_x$  we have  $x \in N$
- For all  $x \in X$  we have  $X \in \mathcal{N}_x$
- For all  $x \in X$ ,  $N \in \mathcal{N}_x$  and  $V \subseteq \mathcal{P}X$ , if  $N \subseteq V$  then  $V \in \mathcal{N}_x$

- For all  $x \in X$  and  $M, N \in \mathcal{N}_x$  we have  $M \cap N \in \mathcal{N}_x$
- For all  $x \in X$  and  $N \in \mathcal{N}_x$ , there exists  $M \in \mathcal{N}_x$  such that  $M \subseteq N$  and  $\forall y \in M. M \in \mathcal{N}_y$ .

In this case,  $\mathcal{O}$  is unique and is given by  $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$ .

PROOF: Straightforward.  $\square$

## 13.4 Interior

**Definition 13.4.1** (Interior). The interior of  $B$  is the union of all the open sets included in  $B$ .

## 13.5 Closure

**Definition 13.5.1** (Closure). Let  $X$  be a topological space and  $B \subseteq X$ . The closure of  $B$ ,  $\overline{B}$ , is the intersection of all the closed sets that include  $B$ .

**Proposition 13.5.2.** A set  $B$  is open iff  $X - B = \overline{X - B}$ .

**Proposition 13.5.3** (Kuratowski Closure Axioms). Let  $X$  be a set and  $- : \mathcal{P}X \rightarrow \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  such that, for all  $B \subseteq X$ ,  $\overline{B}$  is the closure of  $B$ , if and only if:

- $\overline{\emptyset} = \emptyset$
- For all  $A \subseteq X$  we have  $A \subseteq \overline{A}$
- For all  $A \subseteq X$  we have  $\overline{\overline{A}} = \overline{A}$
- For all  $A, B \subseteq X$  we have  $\overline{A \cup B} = \overline{A} \cup \overline{B}$

In this case,  $\mathcal{O}$  is unique and is defined by  $\mathcal{O} = \{U : X - U = \overline{X - U}\}$ .

**Definition 13.5.4** (Finer, Coarser). Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the set  $X$ . Then  $\mathcal{T}$  is *coarser*, *smaller* or *weaker* than  $\mathcal{T}'$ , or  $\mathcal{T}'$  is *finer*, *larger* or *weaker* than  $\mathcal{T}$ , iff  $\mathcal{T} \subseteq \mathcal{T}'$ .

## 13.6 Bases

**Definition 13.6.1** (Basis). Let  $X$  be a topological space. A *basis* for the topology on  $X$  is a set of open sets  $\mathcal{B}$  such that every open set is the union of a subset of  $\mathcal{B}$ . The elements of  $\mathcal{B}$  are called *basic open neighbourhoods* of their elements.

**Proposition 13.6.2.** Let  $X$  be a set. The set of all one-element subsets of  $X$  is a basis for the discrete topology on  $X$ .



**Proposition 13.6.3.** *Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Then the topology on  $X$  is the coarsest topology that includes  $\mathcal{B}$ .*

**Proposition 13.6.4.** *Let  $X$  and  $Y$  be topological spaces. Let  $\mathcal{B}$  be a basis for the topology on  $X$  and  $\mathcal{C}$  a basis for the topology on  $Y$ . Then*

$$\{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$$

*is a basis for the product topology on  $X \times Y$ .*

**Theorem 13.6.5.** *There are infinitely many primes.*

Furstenberg's proof:

PROOF:

$\langle 1 \rangle 1$ . For  $a \in \mathbb{Z} - \{0\}$  and  $b \in \mathbb{Z}$ ,

LET:  $S(a, b) := \{an + b : n \in \mathbb{N}\}$

$\langle 1 \rangle 2$ . LET:  $\mathcal{T}$  be the topology generated by the basis  $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$

$\langle 2 \rangle 1$ . For every  $n \in \mathbb{Z}$ , there exist  $a, b$  such that  $n \in S(a, b)$ .

PROOF:  $n \in S(n, 0)$

$\langle 2 \rangle 2$ . If  $n \in S(a_1, b_1) \cap S(a_2, b_2)$  then there exist  $a_3, b_3$  such that  $n \in S(a_3, b_3) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 1$ . LET:  $d = \text{lcm}(a_1, a_2)$

PROVE:  $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 2$ . LET:  $d = a_1k = a_2l$

$\langle 3 \rangle 3$ . LET:  $n = a_1c + b_1 = a_2d + b_2$

$\langle 3 \rangle 4$ . LET:  $z \in \mathbb{Z}$

PROVE:  $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 5$ .  $dz + n \in S(a_1, b_1)$

PROOF:

$$\begin{aligned} dz + n &= a_1kz + a_1c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

$\langle 3 \rangle 6$ .  $dz + n \in S(a_2, b_2)$

PROOF: Similar.

$\langle 1 \rangle 3$ . For all  $a \in \mathbb{Z} - \{0\}$  and  $b \in \mathbb{Z}$  we have  $S(a, b)$  is closed.

$\langle 2 \rangle 1$ . LET:  $a \in \mathbb{Z} - \{0\}$  and  $b \in \mathbb{Z}$

$\langle 2 \rangle 2$ . LET:  $n \in \mathbb{Z} - S(a, b)$

$\langle 2 \rangle 3$ .  $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$

$\langle 3 \rangle 1$ . LET:  $x \in S(a, n)$

$\langle 3 \rangle 2$ . ASSUME: for a contradiction  $x \in S(a, b)$

$\langle 3 \rangle 3$ . PICK  $m$  such that  $x = am + b$

$\langle 3 \rangle 4$ . PICK  $l$  such that  $x = al + n$

$\langle 3 \rangle 5$ .  $n = a(m - l) + b$

$\langle 3 \rangle 6$ .  $n \in S(a, b)$

$\langle 3 \rangle 7$ . Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 2$ .

$\langle 1 \rangle 4$ .

$$\mathbb{Z} - \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$$

PROOF: Since every integer except 1 and  $-1$  is divisible by a prime.

- $\langle 1 \rangle 5$ . No nonempty finite set is open.
  - $\langle 2 \rangle 1$ . LET:  $U$  be a nonempty open set
  - $\langle 2 \rangle 2$ . PICK  $n \in U$
  - $\langle 2 \rangle 3$ . There exist  $a, b$  such that  $n \in S(a, b) \subseteq U$
  - $\langle 2 \rangle 4$ .  $U$  is infinite.
  - $\langle 1 \rangle 6$ .  $\mathbb{Z} - \{1, -1\}$  is not closed.
  - $\langle 1 \rangle 7$ .  $\bigcup_{p \text{ prime}} S(p, 0)$  is not closed.
  - $\langle 1 \rangle 8$ . The union of finitely many closed sets is closed.
  - $\langle 1 \rangle 9$ . There are infinitely many primes.
- 

## 13.7 Order Topology

**Definition 13.7.1** (Order Topology). Let  $X$  be a linearly ordered set. The *order topology* on  $X$  is the topology generated by the open interval  $(a, b)$  as well as the open rays  $(a, +\infty)$  and  $(-\infty, b)$  for  $a, b \in X$ .

The *standard topology* on  $\mathbb{R}$  is the order topology.

**Proposition 13.7.2.** *Let  $X$  be a linearly ordered set. Then the order topology is generated by the basis consisting of:*

- all open intervals  $(a, b)$
- all intervals of the form  $[\perp, b)$  where  $\perp$  is the least element of  $X$ , if any
- all intervals of the form  $(a, \top]$  where  $\top$  is the greatest element of  $X$ , if any.

**Proposition 13.7.3.** *Let  $X$  be a linearly ordered set. The open rays in  $X$  form a subbasis for the order topology.*

**Definition 13.7.4** (Lower Limit Topology). The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology on  $\mathbb{R}$  generated by the basis consisting of all half-open intervals  $[a, b)$ .

We write  $\mathbb{R}_l$  for  $\mathbb{R}$  under the lower limit topology.

**Definition 13.7.5** ( $K$ -topology). Let  $K = \{1/n : n \in \mathbb{Z}_+\}$ . The  $K$ -topology on  $\mathbb{R}$  is the topology generated by the basis consisting of all open intervals  $(a, b)$  and all sets of the form  $(a, b) - K$ .

We write  $\mathbb{R}_K$  for  $\mathbb{R}$  under the  $K$ -topology.

**Proposition 13.7.6.** *Let  $X$  be a linearly ordered set under the order topology. Let  $Y \subseteq X$  be convex. Then the order topology on  $Y$  is the same as the subspace topology.*

PROOF:

- $\langle 1 \rangle 1$ . The order topology is coarser than the subspace topology.

$\langle 2 \rangle 1$ . For all  $a \in Y$ , the open ray  $\{y \in Y : a < y\}$  is open in the subspace topology.

PROOF: It is  $(a, +\infty) \cap Y$ .

$\langle 2 \rangle 2$ . For all  $a \in Y$ , the open ray  $\{y \in Y : y < a\}$  is open in the subspace topology.

PROOF: It is  $(-\infty, a) \cap Y$ .

$\langle 1 \rangle 2$ . The subspace topology is coarser than the order topology.

$\langle 2 \rangle 1$ . For all  $a \in X$ , the set  $(-\infty, a) \cap Y$  is open in the order topology.

$\langle 3 \rangle 1$ . CASE:  $a \in Y$

PROOF: Then  $(-\infty, a) \cap Y = \{y \in Y : y < a\}$  is an open ray in  $Y$ .

$\langle 3 \rangle 2$ . CASE:  $a$  is an upper bound for  $Y$

PROOF: Then  $(-\infty, a) \cap Y = Y$ .

$\langle 3 \rangle 3$ . CASE:  $a$  is a lower bound for  $Y$

PROOF: Then  $(-\infty, a) \cap Y = \emptyset$ .

$\langle 3 \rangle 4$ . Q.E.D.

PROOF: These are the only three cases because  $Y$  is convex.

$\langle 2 \rangle 2$ . For all  $a \in X$ , the set  $(a, +\infty) \cap Y$  is open in the order topology.

PROOF: Similar.

□

**Example 13.7.7.** We cannot remove the hypothesis that the set  $Y$  is convex.

Let  $X = \mathbb{R}$  and  $Y = [0, 1) \cup \{2\}$ . Then  $\{2\}$  is open in the subspace topology but not in the order topology on  $Y$ .

**Proposition 13.7.8.** *Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology on  $X$  and  $U \subseteq X$ . Then  $U$  is open if and only if, for all  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .*

**Proposition 13.7.9.** *Let  $X$  be a topological space and  $\mathcal{B} \subseteq \mathcal{P}X$ . Assume that, for every open set  $U$  and element  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ . Then  $\mathcal{B}$  is a basis for the topology on  $X$ .*

**Proposition 13.7.10.** *Let  $X$  be a topological space and  $\mathcal{B} \subseteq \mathcal{P}X$ . Then  $\mathcal{B}$  is a basis for a topology on  $X$  if and only if:*

1.  $\bigcup \mathcal{B} = X$

2. For all  $A, B \in \mathcal{B}$  and  $x \in A \cap B$ , there exists  $C \in \mathcal{B}$  such that  $x \in C \subseteq A \cap B$ .

*In this case, the topology is unique and is the set of all unions of subsets of  $\mathcal{B}$ . We call it the topology generated by  $\mathcal{B}$ .*

**Proposition 13.7.11.** *Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively, on  $X$ . Then  $\mathcal{T}'$  is finer than  $\mathcal{T}$  if and only if, for every  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .*

**Corollary 13.7.11.1.** *The topologies of  $\mathbb{R}_l$  and  $\mathbb{R}_K$  are strictly finer than the standard topology on  $\mathbb{R}$  but are not comparable to one another.*

**Proposition 13.7.12.** *In a linearly ordered set under the order topology, every closed interval and closed ray is closed.*

PROOF:

⟨1⟩1. LET:  $X$  be a linearly ordered set under the order topology.

⟨1⟩2. Every closed interval in  $X$  is closed.

PROOF: Since  $X - [a, b] = (-\infty, a) \cup (b, +\infty)$ .

⟨1⟩3. Every closed ray in  $X$  is closed.

PROOF: Since  $X - [a, +\infty) = (-\infty, a)$  and  $X - (-\infty, a] = (a, +\infty)$ .

□

### 13.7.1 Subspaces

**Proposition 13.7.13.** *Let  $X$  be a topological space. Let  $Y$  be a subspace of  $X$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Then  $\{B \cap Y : B \in \mathcal{B}\}$  is a basis for the topology on  $Y$ .*

PROOF:

⟨1⟩1. For all  $B \in \mathcal{B}$  we have  $B \cap Y$  is open in  $Y$ .

PROOF: Since  $B$  is open in  $X$ .

⟨1⟩2. For any open set  $V$  in  $Y$  and  $y \in V$ , there exists  $B \in \mathcal{B}$  such that  $y \in B \cap Y \subseteq V$ .

⟨2⟩1. LET:  $V$  be open in  $Y$ .

⟨2⟩2. LET:  $y \in V$

⟨2⟩3. PICK  $U$  open in  $X$  such that  $V = U \cap Y$ .

⟨2⟩4. PICK  $B \in \mathcal{B}$  such that  $y \in B \subseteq U$ .

⟨2⟩5.  $y \in B \cap Y \subseteq V$

□

**Proposition 13.7.14.** *Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Let  $A \subseteq Y$ . Then  $A$  is closed in  $Y$  if and only if there exists a closed set  $B$  in  $X$  such that  $A = B \cap Y$ .*

PROOF:

$A$  is closed in  $Y \Leftrightarrow Y - A$  is open in  $Y$

$\Leftrightarrow \exists U$  open in  $X. Y - A = U \cap Y$

$\Leftrightarrow \exists C$  closed in  $X. Y - A = Y - C$

$\Leftrightarrow \exists C$  closed in  $X. A = Y \cap C$  □

### 13.7.2 Product Topology

**Proposition 13.7.15.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. For all  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $X_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i : \text{for finitely many } i \in I \text{ we have } B_i \in \mathcal{B}_i, \text{ and } B_i = X_i \text{ otherwise}\}$  is a basis for the product topology on  $\prod_{i \in I} X_i$ .*

PROOF:

⟨1⟩1. Every  $B \in \mathcal{B}$  is open in the product topology.

PROOF: Since every element of  $\mathcal{B}_i$  is open in  $X_i$ .

$\langle 1 \rangle 2$ . For any open set  $U$  in the product topology and  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .

$\langle 2 \rangle 1$ . LET:  $U$  be a set open in the box topology.

$\langle 2 \rangle 2$ . LET:  $x \in U$

$\langle 2 \rangle 3$ . PICK a family  $\{U_i\}_{i \in I}$  where  $U_i$  is open in  $X_i$  for  $i = i_1, \dots, i_n$ , and  $U_i = X_i$  for all other  $i$ , such that  $x \in \prod_{i \in I} U_i \subseteq U$

$\langle 2 \rangle 4$ . For  $i = i_1, \dots, i_n$ , choose  $B_i \in \mathcal{B}_i$  such that  $x_i \in B_i \subseteq U_i$ . Let  $B_i = X_i$  for all other  $i$ .

$\langle 2 \rangle 5$ .  $\prod_{i \in I} B_i \in \mathcal{B}$

$\langle 2 \rangle 6$ .  $x \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} U_i \subseteq U$

□

## 13.8 Subbases

**Definition 13.8.1** (Subbasis). Let  $X$  be a topological space. A *subbasis* for the topology on  $X$  is a set  $\mathcal{S}$  of open sets such that every open set is a union of finite intersections of  $\mathcal{S}$ .

**Proposition 13.8.2.** Let  $X$  be a set and  $\mathcal{S} \subseteq X$ . Then  $\mathcal{S}$  is a subbasis for a topology on  $X$  if and only if  $\bigcup \mathcal{S} = X$ , in which case the topology is unique and is the set of all unions of finite intersections of elements of  $\mathcal{S}$ .

**Proposition 13.8.3.** Let  $X$  be a topological space. Let  $\mathcal{S}$  be a subbasis for the topology on  $X$ . Then the topology on  $X$  is the coarsest topology that includes  $\mathcal{S}$ .

**Proposition 13.8.4.** Let  $X$  and  $Y$  be topological spaces. Then

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$$

is a subbasis for the product topology on  $X \times Y$ .

PROOF:

$\langle 1 \rangle 1$ . Every element of  $\mathcal{S}$  is open.

PROOF: Since  $\pi_1^{-1}(U) = U \times Y$  and  $\pi_2^{-1}(V) = X \times V$ .

$\langle 1 \rangle 2$ . Every open set is a union of finite intersections of elements of  $\mathcal{S}$ .

PROOF: Since, for  $U$  open in  $X$  and  $V$  open in  $Y$ , we have  $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ .

□

**Definition 13.8.5** (Space with Basepoint). A *space with basepoint* is a pair  $(X, x)$  where  $X$  is a topological space and  $x \in X$ .

## 13.9 Neighbourhood Bases

**Definition 13.9.1** (Neighbourhood Basis). Let  $X$  be a topological space and  $x_0 \in X$ . A *neighbourhood basis* of  $x_0$  is a set  $\mathcal{U}$  of neighbourhoods of  $x_0$  such that every neighbourhood of  $x_0$  includes an element of  $\mathcal{U}$ .

### 13.10 First Countable Spaces

**Definition 13.10.1** (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

**Proposition 13.10.2.**  $\mathbb{R}_l$  is first countable.

PROOF: For any  $x \in \mathbb{R}$  we have  $\{[x, x + 1/n) : n \in \mathbb{Z}_+\}$  is a countable local basis.  
□

**Proposition 13.10.3.** The ordered square is first countable.

PROOF:

⟨1⟩1. Every point  $(a, b)$  with  $0 < b < 1$  has a countable local basis.

PROOF: The set of all intervals  $((a, q), (a, r))$  where  $q$  and  $r$  are rational and  $0 \leq q < b < r \leq 1$  is a countable local basis.

⟨1⟩2. Every point  $(a, 0)$  has a countable local basis with  $a > 0$ .

PROOF: The set of all intervals  $((q, 0), (a, r))$  where  $q$  and  $r$  are rational with  $0 \leq q < a$  and  $0 < r \leq 1$  is a countable local basis.

⟨1⟩3. Every point  $(a, 1)$  has a countable local basis with  $a < 1$ .

PROOF: The set of all intervals  $((a, q), (r, 1))$  with  $q$  and  $r$  rational and  $0 \leq q < 1, a < r \leq 1$  is a countable local basis.

⟨1⟩4.  $(0, 0)$  has a countable local basis.

PROOF: The set of all intervals  $[(0, 0), (0, r))$  with  $r$  rational and  $0 < r \leq 1$  is a countable local basis.

⟨1⟩5.  $(1, 1)$  has a countable local basis.

PROOF: The set of all intervals  $((1, q), (1, 1])$  with  $q$  rational and  $0 \leq q < 1$  is a countable local basis.  
□

### 13.11 Second Countable Spaces

**Definition 13.11.1** (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

$\mathbb{R}^n$  is second countable.

An uncountable discrete space is first countable but not second countable.

**Proposition 13.11.2.** Let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be a family of topological spaces such that no  $X_\lambda$  is indiscrete. If  $\Lambda$  is uncountable, then  $\prod_{\lambda \in \Lambda} X_\lambda$  is not first countable.

PROOF:

⟨1⟩1. For all  $\lambda \in \Lambda$ , PICK  $U_\lambda$  open in  $X_\lambda$  such that  $\emptyset \neq U_\lambda \neq X_\lambda$ .

⟨1⟩2. For all  $\lambda \in \Lambda$ , PICK  $x_\lambda \in U_\lambda$ .

- <1>3. ASSUME: for a contradiction  $B$  is a countable neighbourhood basis for  $(x_\lambda)_{\lambda \in \Lambda}$ .  
 <1>4. PICK  $\lambda \in \Lambda$  such that, for all  $U \in B$ , we have  $\pi_\lambda(U) = X_\lambda$   
 <1>5. There is no  $U \in \lambda$  such that  $U \subseteq \pi_\lambda^{-1}(U_\lambda)$   
 <1>6. Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 13.11.3.** *The long line cannot be embedded in  $\mathbb{R}^n$  for any  $n$ .*

PROOF: Since the long line is not second countable but  $\mathbb{R}^n$  is. □

## 13.12 Interior

**Definition 13.12.1** (Interior). Let  $X$  be a topological space. Let  $A \subseteq X$ . The *interior* of  $A$ ,  $A^\circ$ , is the union of all the open sets included in  $A$ .

## 13.13 Closure

**Definition 13.13.1** (Closure). Let  $X$  be a topological space. Let  $A \subseteq X$ . The *closure* of  $A$ ,  $\overline{A}$ , is the intersection of all the closed sets that include  $A$ .

**Proposition 13.13.2.** *Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Then  $x \in \overline{A}$  if and only if every open set that contains  $x$  intersects  $A$ .*

PROOF:

$$\begin{aligned}
 x \in \overline{A} &\Leftrightarrow \text{for every closed set } C, \text{ if } A \subseteq C \text{ then } x \in C \\
 &\Leftrightarrow \text{for every open set } U, \text{ if } A \subseteq X - U \text{ then } x \in X - U \\
 &\Leftrightarrow \text{for every open set } U, \text{ if } A \cap U = \emptyset \text{ then } x \notin U \\
 &\Leftrightarrow \text{for every open set } U, \text{ if } x \in U \text{ then } A \text{ intersects } U \quad \square
 \end{aligned}$$

**Proposition 13.13.3.** *Let  $X$  be a topological space. Let  $A \subseteq B \subseteq X$ . Then  $\overline{A} \subseteq \overline{B}$ .*

PROOF: Since every closed set that includes  $B$  is a closed set that includes  $A$ . □

**Proposition 13.13.4.** *Let  $X$  be a topological space. Let  $A, B \subseteq X$ . Then  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ .*

PROOF:

$$<1>1. \overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$$

PROOF: Since  $\overline{A \cup B}$  is a closed set that includes  $A \cup B$ .

$$<1>2. \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$$

PROOF: Since  $\overline{A} \subseteq \overline{A \cup B}$  and  $\overline{B} \subseteq \overline{A \cup B}$  by Proposition 13.13.3.

□

**Proposition 13.13.5.** *Let  $X$  be a topological space. Let  $\mathcal{A} \subseteq \mathcal{P}X$ . Then*

$$\bigcup \{\bar{A} : A \in \mathcal{A}\} \subseteq \overline{\bigcup \mathcal{A}}.$$

PROOF: For all  $A \in \mathcal{A}$  we have  $\bar{A} \subseteq \overline{\bigcup \mathcal{A}}$  by Proposition 13.13.3.  $\square$

**Example 13.13.6.** The converse does not always hold. In  $\mathbb{R}$ , let  $\mathcal{A} = \{\{x\} : 0 < x < 1\}$ . Then  $\bigcup \{\bar{A} : A \in \mathcal{A}\} = (0, 1)$  but  $\overline{\bigcup \mathcal{A}} = [0, 1]$ .

**Proposition 13.13.7.** *Let  $X$  be a topological space. Let  $\mathcal{A} \subseteq \mathcal{P}X$ . Then  $\bigcap \bar{A} \subseteq \overline{\bigcap \{A : A \in \mathcal{A}\}}$ .*

PROOF: Since  $\bigcap \bar{A} \subseteq \bar{A}$  for all  $A \in \mathcal{A}$  by Proposition 13.13.3.  $\square$

**Example 13.13.8.** The converse does not always hold. In  $\mathbb{R}$ , if  $A$  is the set of all rational numbers and  $B$  is the set of all irrational numbers then  $\bigcap A \cap B = \emptyset$  but  $\bigcap A \cap \bigcap B = \mathbb{R}$ .

### 13.13.1 Bases

**Proposition 13.13.9.** *Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Then  $x \in \bar{A}$  if and only if, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $x \in \bar{A}$  then, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

PROOF: Proposition 13.13.2 since every element of  $\mathcal{B}$  is open.

$\langle 1 \rangle 2$ . If, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ , then  $x \in \bar{A}$ .

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

$\langle 2 \rangle 2$ . LET:  $U$  be an open set that contains  $x$ .

$\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .

$\langle 2 \rangle 4$ .  $B$  intersects  $A$ .

PROOF:  $\langle 2 \rangle 1$

$\langle 2 \rangle 5$ .  $U$  intersects  $A$ .

$\square$

### 13.13.2 Subspaces

**Proposition 13.13.10.** *Let  $X$  be a topological space. Let  $Y$  be a subspace of  $X$ . Let  $A \subseteq Y$ . Let  $\bar{A}$  be the closure of  $A$  in  $X$ . Then the closure of  $A$  in  $Y$  is  $\bar{A} \cap Y$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\bar{A} \cap Y$  is the closed in  $Y$ .

PROOF: Since  $\bar{A}$  is closed in  $X$ .

$\langle 1 \rangle 2$ . For any closed set  $B$  in  $Y$ , if  $A \subseteq B$  then  $\bar{A} \cap Y \subseteq B$ .

$\langle 2 \rangle 1$ . LET:  $B$  be closed in  $Y$ .

$\langle 2 \rangle 2$ . ASSUME:  $A \subseteq B$



- ⟨2⟩3. PICK  $C$  closed in  $X$  such that  $B = C \cap Y$ .
- ⟨2⟩4.  $A \subseteq C$
- ⟨2⟩5.  $\bar{A} \subseteq C$
- ⟨2⟩6.  $\bar{A} \cap Y \subseteq B$

□

### 13.13.3 Product Topology

**Proposition 13.13.11.** *Let  $X$  and  $Y$  be topological spaces. Let  $A \subseteq X$  and  $B \subseteq Y$ . Then  $\overline{A \times B} = \bar{A} \times \bar{B}$ .*

PROOF:

$$\langle 1 \rangle 1. \overline{A \times B} \subseteq \bar{A} \times \bar{B}$$

PROOF: Since  $\bar{A} \times \bar{B}$  is a closed set that includes  $A \times B$  by Proposition 13.24.2.

$$\langle 1 \rangle 2. \bar{A} \times \bar{B} \subseteq \overline{A \times B}$$

⟨2⟩1. LET:  $x \in \bar{A}$  and  $y \in \bar{B}$ .

⟨2⟩2. LET:  $U$  be an open set that contains  $(x, y)$ .

⟨2⟩3. PICK open sets  $V$  in  $X$  and  $W$  in  $Y$  such that  $(x, y) \in V \times W \subseteq U$ .

⟨2⟩4.  $V$  intersects  $A$  and  $W$  intersects  $B$ .

⟨2⟩5.  $U$  intersects  $A \times B$ .

□

### 13.13.4 Interior

**Proposition 13.13.12.** *Let  $X$  be a topological space and  $A \subseteq X$ . Then*

$$X - A^\circ = \overline{X - A}$$

PROOF:

$$\begin{aligned} X - A^\circ &= X - \bigcup \{U \text{ open in } X : U \subseteq A\} \\ &= \bigcap \{X - U : U \text{ open in } X, U \subseteq A\} \quad (\text{De Morgan's Law}) \\ &= \bigcap \{C : C \text{ closed in } X, X - A \subseteq C\} \\ &= \overline{X - A} \end{aligned}$$

□

**Proposition 13.13.13.** *Let  $X$  be a topological space and  $A \subseteq X$ . Then*

$$X - \bar{A} = (X - A)^\circ$$

PROOF: Dual. □

## 13.14 Boundary

**Definition 13.14.1** (Boundary). Let  $X$  be a topological space. Let  $A \subseteq X$ . The *boundary* of  $A$  is

$$\partial A := \bar{A} \cap \overline{X - A}.$$

**Proposition 13.14.2.** *Let  $X$  be a topological space. Let  $A \subseteq X$ . Then*

$$A^\circ \cap \partial A = \emptyset .$$

PROOF:

- $\langle 1 \rangle 1.$   $A^\circ \subseteq A$
- $\langle 1 \rangle 2.$   $X - A \subseteq X - A^\circ$
- $\langle 1 \rangle 3.$   $\overline{X - A} \subseteq X - A^\circ$
- $\langle 1 \rangle 4.$   $\partial A \subseteq X - A^\circ$

□

**Proposition 13.14.3.** *Let  $X$  be a topological space. Let  $A \subseteq X$ . Then*

$$\overline{A} = A^\circ \cup \partial A$$

- $\langle 1 \rangle 1.$   $A^\circ \subseteq \overline{A}$   
PROOF: Since  $A^\circ \subseteq A \subseteq \overline{A}$ .
- $\langle 1 \rangle 2.$   $\partial A \subseteq \overline{A}$   
PROOF: Definition of  $\partial A$ .
- $\langle 1 \rangle 3.$   $\overline{A} \subseteq A^\circ \cup \partial A$ 
  - $\langle 2 \rangle 1.$  LET:  $x \in \overline{A}$
  - $\langle 2 \rangle 2.$  ASSUME:  $x \notin A^\circ$   
PROVE:  $x \in \partial A$
  - $\langle 2 \rangle 3.$   $x \in \overline{X - A}$   
PROOF: Since  $\overline{X - A} = X - A^\circ$ .
  - $\langle 2 \rangle 4.$   $x \in \partial A$   
PROOF: Since  $\partial A = \overline{A} \cap \overline{X - A}$ .

□

**Proposition 13.14.4.** *Let  $X$  be a topological space. Let  $A \subseteq X$ . Then  $\partial A = \emptyset$  if and only if  $A$  is both open and closed.*

PROOF:

- $\langle 1 \rangle 1.$  If  $\partial A = \emptyset$  then  $A$  is open and closed.
- $\langle 2 \rangle 1.$  ASSUME:  $\partial A = \emptyset$
- $\langle 2 \rangle 2.$   $\overline{A} = A^\circ$   
PROOF: Proposition 13.14.3.
- $\langle 2 \rangle 3.$   $\overline{A} = A = A^\circ$
- $\langle 1 \rangle 2.$  If  $A$  is open and closed then  $\partial A = \emptyset$ .

PROOF: If  $A$  is open and closed then

$$\begin{aligned} \partial A &= \overline{A} \cap \overline{X - A} \\ &= \overline{A} \cap (X - A^\circ) \\ &= A \cap (X - A) \\ &= \emptyset \end{aligned}$$

□

**Proposition 13.14.5.** *Let  $X$  be a topological space. Let  $U \subseteq X$ . Then  $U$  is open if and only if  $\partial U = \overline{U} - U$ .*

PROOF:

⟨1⟩1. If  $U$  is open then  $\partial U = \overline{U} - U$

PROOF: If  $U$  is open then

$$\begin{aligned}\partial U &= \overline{U} \cap \overline{X - U} \\ &= \overline{U} \cap (X - U^\circ) \\ &= \overline{U} - U^\circ \\ &= \overline{U} - U\end{aligned}$$

⟨1⟩2. If  $\partial U = \overline{U} - U$  then  $U$  is open.

⟨2⟩1. ASSUME:  $\partial U = \overline{U} - U$

⟨2⟩2.  $\overline{U} - U^\circ = \overline{U} - U$

⟨2⟩3.  $U \subseteq U^\circ$

⟨2⟩4.  $U = U^\circ$

□

## 13.15 Limit Points

**Definition 13.15.1** (Limit Point). Let  $X$  be a topological space,  $x \in X$  and  $A \subseteq X$ . Then  $x$  is a *limit point*, *cluster point* or *point of accumulation* of  $A$  iff every neighbourhood of  $x$  intersects  $A - \{x\}$ .

**Proposition 13.15.2.** Let  $X$  be a topological space. Let  $A \subseteq X$ . Let  $A'$  be the set of limit points of  $A$ . Then

$$\overline{A} = A \cup A'$$

PROOF:

⟨1⟩1.  $\overline{A} \subseteq A \cup A'$

⟨2⟩1. LET:  $x \in \overline{A}$

⟨2⟩2. ASSUME:  $x \notin A$

PROVE:  $x \in A'$

⟨2⟩3. LET:  $U$  be a neighbourhood of  $x$ .

⟨2⟩4. PICK  $y \in U \cap A$

PROOF: Proposition 13.13.2.

⟨2⟩5.  $y \neq x$

⟨1⟩2.  $A \subseteq \overline{A}$

PROOF: Immediate from the definition of  $\overline{A}$ .

⟨1⟩3.  $A' \subseteq \overline{A}$

PROOF: From Proposition 13.13.2.

□

**Corollary 13.15.2.1.** A set is closed if and only if it contains all its limit points.

## 13.16 Continuous Functions

**Definition 13.16.1** (Continuous). Let  $X$  and  $Y$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous* iff, for every open set  $V$  in  $Y$ , the inverse image  $f^{-1}(V)$  is open in  $X$ .

**Proposition 13.16.2.** *The composite of two continuous functions is continuous.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous.

$\langle 1 \rangle 2$ . LET:  $U$  be open in  $Z$ .

$\langle 1 \rangle 3$ .  $g^{-1}(U)$  is open in  $Y$ .

$\langle 1 \rangle 4$ .  $\inf f(g^{-1}(U))$  is open in  $X$ .

□

**Proposition 13.16.3.** 1.  $\text{id}_X$  is continuous

2. If  $f : X \rightarrow Y$  is continuous and  $X_0 \subseteq X$  then  $f \upharpoonright X_0 : X_0 \rightarrow Y$  is continuous.

3. If  $f : X + Y \rightarrow Z$ , then  $f$  is continuous iff  $f \circ \kappa_1 : X \rightarrow Z$  and  $f \circ \kappa_2 : Y \rightarrow Z$  are continuous.

4. If  $f : Z \rightarrow X \times Y$ , then  $f$  is continuous iff  $\pi_1 \circ f$  and  $\pi_2 \circ f$  are continuous.

**Proposition 13.16.4.** Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Then the following are equivalent.

1.  $f$  is continuous.

2. For all  $A \subseteq X$  we have  $f(\overline{A}) \subseteq \overline{f(A)}$ .

3. For every closed  $B$  in  $Y$ , we have  $f^{-1}(B)$  is closed in  $X$ .

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

$\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous.

$\langle 2 \rangle 2$ . LET:  $A \subseteq X$

$\langle 2 \rangle 3$ . LET:  $x \in \overline{A}$

PROVE:  $f(x) \in \overline{f(A)}$

$\langle 2 \rangle 4$ . LET:  $V$  be a neighbourhood of  $f(x)$ .

PROVE:  $V$  intersects  $f(A)$ .

$\langle 2 \rangle 5$ .  $f^{-1}(V)$  is a neighbourhood of  $x$ .

$\langle 2 \rangle 6$ . PICK  $y \in f^{-1}(V) \cap A$

$\langle 2 \rangle 7$ .  $f(y) \in V \cap f(A)$

$\langle 1 \rangle 2$ .  $2 \Rightarrow 3$

$\langle 2 \rangle 1$ . ASSUME: 2

$\langle 2 \rangle 2$ . LET:  $B$  be closed in  $Y$

$\langle 2 \rangle 3$ . LET:  $A = f^{-1}(B)$

PROVE:  $\overline{A} = A$

$\langle 2 \rangle 4$ .  $f(A) \subseteq B$

$\langle 2 \rangle 5$ .  $\overline{A} \subseteq A$

$\langle 3 \rangle 1$ . LET:  $x \in \overline{A}$

$\langle 3 \rangle 2$ .  $f(x) \in B$

PROOF:

$$\begin{aligned} f(x) &\in f(\overline{A}) \\ &\subseteq \overline{f(A)} && (\langle 2 \rangle 1) \\ &\subseteq \overline{B} && (\langle 2 \rangle 4) \\ &= B && (\langle 2 \rangle 2) \end{aligned}$$

$\langle 1 \rangle 3$ .  $3 \Rightarrow 1$

$\langle 2 \rangle 1$ . ASSUME: 3

$\langle 2 \rangle 2$ . LET:  $V$  be open in  $Y$ .

$\langle 2 \rangle 3$ .  $f^{-1}(Y - V)$  is closed in  $X$ .

$\langle 2 \rangle 4$ .  $X - f^{-1}(V)$  is closed in  $X$ .

$\langle 2 \rangle 5$ .  $f^{-1}(V)$  is open in  $X$ .

□

**Proposition 13.16.5.** *Let  $X$  and  $Y$  be topological spaces. Any constant function  $X \rightarrow Y$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $b \in Y$

$\langle 1 \rangle 2$ . LET:  $f : X \rightarrow Y$  be the constant function with value  $b$ .

$\langle 1 \rangle 3$ . LET:  $V \subseteq Y$  be open.

$\langle 1 \rangle 4$ .  $f^{-1}(V)$  is either  $\emptyset$  or  $X$ .

$\langle 1 \rangle 5$ .  $f^{-1}(V)$  is open.

□

**Proposition 13.16.6.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous then, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{B}$  is open in  $Y$ .

$\langle 1 \rangle 2$ . If, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ , then  $f$  is continuous.

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

$\langle 2 \rangle 2$ . LET:  $U$  be open in  $Y$ .

$\langle 2 \rangle 3$ . LET:  $x \in f^{-1}(U)$

$\langle 2 \rangle 4$ . PICK  $B \in \mathcal{B}$  such that  $f(x) \in B \subseteq U$ .

$\langle 2 \rangle 5$ .  $x \in f^{-1}(B) \subseteq f^{-1}(U)$

□

**Proposition 13.16.7.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a subbasis for the topology on  $Y$ . Then  $f$  is continuous if and only if, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .*

PROOF:

⟨1⟩1. If  $f$  is continuous then, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .

PROOF: Immediate from definitions.

⟨1⟩2. If, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ , then  $f$  is continuous.

⟨2⟩1. ASSUME: For all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .

⟨2⟩2. For all  $V_1, \dots, V_n \in \mathcal{S}$  we have  $f^{-1}(V_1 \cap \dots \cap V_n)$  is open in  $X$ .

PROOF: Since  $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$ .

⟨2⟩3. Q.E.D.

PROOF: By Proposition 13.16.6 since the set of all finite intersections of elements of  $\mathcal{S}$  forms a basis for the topology on  $Y$ .

□

**Proposition 13.16.8.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Then  $f$  is continuous if and only if, for all  $x \in \mathbb{R}$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in \mathbb{R}$ , if  $|y - x| < \delta$  then  $|f(y) - f(x)| < \epsilon$ .*

PROOF:

⟨1⟩1. If  $f$  is continuous then, for all  $x \in \mathbb{R}$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in \mathbb{R}$ , if  $|y - x| < \delta$  then  $|f(y) - f(x)| < \epsilon$ .

⟨2⟩1. ASSUME:  $f$  is continuous.

⟨2⟩2. LET:  $x \in \mathbb{R}$

⟨2⟩3. LET:  $\epsilon > 0$

⟨2⟩4.  $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$  is open in  $X$ .

⟨2⟩5. PICK  $a, b$  such that  $x \in (a, b) \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ .

⟨2⟩6. LET:  $\delta = \min(x - a, b - x)$

⟨2⟩7. LET:  $y \in \mathbb{R}$

⟨2⟩8. ASSUME:  $|y - x| < \delta$

⟨2⟩9.  $y \in (a, b)$

⟨2⟩10.  $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$

⟨2⟩11.  $|f(y) - f(x)| < \epsilon$

⟨1⟩2. If, for all  $x \in \mathbb{R}$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in \mathbb{R}$ , if  $|y - x| < \delta$  then  $|f(y) - f(x)| < \epsilon$ , then  $f$  is continuous.

⟨2⟩1. ASSUME: For all  $x \in \mathbb{R}$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in \mathbb{R}$ , if  $|y - x| < \delta$  then  $|f(y) - f(x)| < \epsilon$ .

⟨2⟩2. For all  $a \in \mathbb{R}$  we have  $f^{-1}((a, +\infty))$  is open.

⟨3⟩1. LET:  $a \in \mathbb{R}$

⟨3⟩2. LET:  $x \in f^{-1}((a, +\infty))$

⟨3⟩3. LET:  $\epsilon = f(x) - a$

⟨3⟩4. PICK  $\delta > 0$  such that, for all  $y \in \mathbb{R}$ , if  $|y - x| < \delta$  then  $|f(y) - f(x)| < \epsilon$

⟨3⟩5.  $x \in (x - \delta, x + \delta) \subseteq f^{-1}((a, +\infty))$

⟨2⟩3. For all  $a \in \mathbb{R}$  we have  $f^{-1}((-\infty, a))$  is open.

PROOF: Similar.

⟨2⟩4. Q.E.D.

PROOF: Proposition 13.16.8.

□

**Definition 13.16.9** (Continuity at a Point). Let  $X$  and  $Y$  be topological spaces.

Let  $f : X \rightarrow Y$ . Let  $a \in X$ . Then  $f$  is *continuous at  $a$*  iff, for every neighbourhood  $V$  of  $f(a)$ , there exists a neighbourhood  $U$  of  $a$  such that  $f(U) \subseteq V$ .

**Proposition 13.16.10.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if  $f$  is continuous at every point in  $X$ .*

$\langle 1 \rangle 1$ . If  $f$  is continuous then  $f$  is continuous at every point in  $X$ .

$\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous.

$\langle 2 \rangle 2$ . LET:  $a \in X$

$\langle 2 \rangle 3$ . LET:  $V$  be a neighbourhood of  $f(a)$

$\langle 2 \rangle 4$ . LET:  $U = f^{-1}(V)$

$\langle 2 \rangle 5$ .  $U$  is a neighbourhood of  $a$ .

$\langle 2 \rangle 6$ .  $f(U) \subseteq V$

$\langle 1 \rangle 2$ . If  $f$  is continuous at every point in  $X$  then  $f$  is continuous.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous at every point in  $X$ .

$\langle 2 \rangle 2$ . LET:  $V$  be open in  $Y$ .

$\langle 2 \rangle 3$ . LET:  $x \in f^{-1}(V)$

$\langle 2 \rangle 4$ .  $V$  is a neighbourhood of  $f(x)$

$\langle 2 \rangle 5$ . PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$

$\langle 2 \rangle 6$ .  $x \in U \subseteq f^{-1}(V)$

□

**Definition 13.16.11** (Homeomorphism). Let  $X$  and  $Y$  be topological spaces. A *homeomorphism* between  $X$  and  $Y$  is a bijection  $f : X \approx Y$  such that  $f$  and  $f^{-1}$  are continuous.

**Proposition 13.16.12.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is a homeomorphism iff  $f$  is bijective and, for all  $U \subseteq X$ , we have  $f(U)$  is open if and only if  $U$  is open.*

PROOF: Immediate from definitions. □

**Definition 13.16.13** (Topological Property). A property  $P$  of topological spaces is a *topological* property iff, for any topological spaces  $X$  and  $Y$ , if  $P[X]$  and  $X \cong Y$  then  $P[Y]$ .

**Definition 13.16.14** (Retraction). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A continuous function  $\rho : X \rightarrow A$  is a *retraction* iff  $\rho \upharpoonright A = \text{id}_A$ . We say  $A$  is a *retract* of  $X$  iff there exists a retraction.

**Definition 13.16.15.** Let **Top** be the category of small topological spaces and continuous functions.

**Proposition 13.16.16.**  $\emptyset$  is initial in **Top**.

**Proposition 13.16.17.**  $1$  is terminal in **Top**.

Forgetful functor **Top**  $\rightarrow$  **Set**.

Basepoint preserving continuous functor.

**Proposition 13.16.18.** *Let  $(X, \mathcal{T})$  be a topological space. Let  $S$  be the Sierpiński two-point space. Define  $\Phi : \mathcal{T} \rightarrow \mathbf{Top}[X, S]$  by  $\Phi(U)(x) = 1$  iff  $x \in U$ . Then  $\Phi$  is a bijection.*

PROOF:

$\langle 1 \rangle 1$ . For all  $U \in \mathcal{T}$  we have  $\Phi(U)$  is continuous.

$\langle 2 \rangle 1$ . LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 2$ .  $\Phi(U)(\{1\})$  is open.

PROOF: Since  $\Phi(U)(\{1\}) = U$ .

$\langle 1 \rangle 2$ .  $\Phi$  is injective.

PROOF: If  $\Phi(U) = \Phi(V)$  then we have  $\forall x (x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$ .

$\langle 1 \rangle 3$ .  $\Phi$  is surjective.

PROOF: Given  $f : X \rightarrow S$  continuous we have  $\Phi(f^{-1}(1)) = f$ .

□

### 13.16.1 Order Topology

**Proposition 13.16.19.** *Let  $X$  and  $Y$  be linearly ordered sets under the order topology. Let  $f : X \rightarrow Y$  be strictly monotone and surjective. Then  $f$  is a homeomorphism.*

PROOF:

$\langle 1 \rangle 1$ .  $f$  is continuous.

$\langle 2 \rangle 1$ . For all  $b \in Y$  we have  $f^{-1}((b, +\infty))$  is open in  $X$ .

$\langle 3 \rangle 1$ . LET:  $b \in Y$

$\langle 3 \rangle 2$ . LET:  $a$  be the element of  $X$  such that  $f(a) = b$ .

$\langle 3 \rangle 3$ .  $f^{-1}((b, +\infty)) = (a, +\infty)$

$\langle 2 \rangle 2$ . For all  $b \in Y$  we have  $f^{-1}((-\infty, b))$  is open in  $X$ .

PROOF: Similar.

$\langle 1 \rangle 2$ .  $f^{-1}$  is continuous.

PROOF: Similar.

□

**Corollary 13.16.19.1.** *For  $n$  a positive integer, the  $n$ th root function  $\overline{\mathbb{R}_+} \rightarrow \overline{\mathbb{R}_+}$  is continuous.*

### 13.16.2 Paths

**Definition 13.16.20** (Path). A *path* in a topological space  $X$  is a continuous function  $[0, 1] \rightarrow X$ .

**Definition 13.16.21** (Constant Path). Let  $X$  be a topological space and  $a \in X$ . The *constant* path at  $a$  is the path  $p : [0, 1] \rightarrow X$  with  $p(t) = a$  for all  $t \in [0, 1]$ .

**Definition 13.16.22** (Reverse Path). Let  $X$  be a topological space and  $p : [0, 1] \rightarrow X$ . The *reverse* of  $p$  is the path  $q : [0, 1] \rightarrow X$  with  $q(t) = p(1 - t)$  for all  $t \in [0, 1]$ .



**Definition 13.16.23** (Concatenation). Let  $X$  be a topological space and  $p, q : [0, 1] \rightarrow X$  be paths in  $X$  with  $p(1) = q(0)$ . The *concatenation* of  $p$  and  $q$  is the path  $r : [0, 1] \rightarrow X$  with

$$r(t) = \begin{cases} p(2t) & \text{if } 0 \leq t \leq 1/2 \\ q(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

### 13.16.3 Loops

**Definition 13.16.24** (Loop). A *loop* in a topological space  $X$  is a path  $\alpha : [0, 1] \rightarrow X$  such that  $\alpha(0) = \alpha(1)$ .

## 13.17 Convergence

**Definition 13.17.1** (Convergence). Let  $X$  be a topological space. Let  $(x_n)$  be a sequence in  $X$ . A point  $a \in X$  is a *limit* of the sequence iff, for every neighbourhood  $U$  of  $a$ , there exists  $n_0$  such that  $\forall n \geq n_0, x_n \in U$ .

**Proposition 13.17.2.** *If  $f : X \rightarrow Y$  is continuous and  $x_n \rightarrow l$  in  $X$  then  $f(x_n) \rightarrow f(l)$  in  $Y$ .*

**Example 13.17.3.** The converse does not hold.

Let  $X$  be the set of all continuous functions  $[0, 1] \rightarrow [-1, 1]$  under the product topology. Let  $i : X \rightarrow L^2([0, 1])$  be the inclusion.

If  $f_n \rightarrow f$  then  $i(f_n) \rightarrow i(f)$  — Lebesgue convergence theorem.

We prove that  $i$  is not continuous.

Assume for a contradiction  $i$  is continuous. Choose a neighbourhood  $K$  of 0 in  $X$  such that  $\forall \phi \in K, \int \phi^2 < 1/2$ . Let  $K = \prod_{\lambda \in [0, 1]} U_\lambda$  where  $U_\lambda = [-1, 1]$  except for  $\lambda = \lambda_1, \dots, \lambda_n$ . Let  $\phi$  be the function that is 0 at  $\lambda_1, \dots, \lambda_n$  and 1 everywhere else. Then  $\phi \in K$  but  $\int \phi^2 = 1$ .

**Proposition 13.17.4.** *The converse does hold for first countable spaces. If  $f : X \rightarrow Y$  where  $X$  is first countable, and  $Y$  is a topological space, and whenever  $x_n \rightarrow x$  then  $f(x_n) \rightarrow f(x)$ , then  $f$  is continuous.*

**Proposition 13.17.5.** *If  $(s_n)$  is an increasing sequence of real numbers bounded above, then  $(s_n)$  converges.*

PROOF:

<1>1. LET:  $s$  be the supremum of  $\{s_n : n \in \mathbb{N}\}$ .

PROVE:  $s_n \rightarrow s$  as  $n \rightarrow \infty$ .

<1>2. LET:  $\epsilon > 0$

<1>3. PICK  $N$  such that  $s_N > s - \epsilon$ .

<1>4.  $\forall n \geq N, s - \epsilon \leq s_n \leq s$

<1>5.  $\forall n \geq N, |s_n - s| < \epsilon$

□

### 13.17.1 Closure

**Proposition 13.17.6.** *Let  $X$  be a topological space. Let  $A \subseteq X$ . Let  $(a_n)$  be a sequence in  $A$  and  $l \in X$ . If  $a_n \rightarrow l$  as  $n \rightarrow \infty$ , then  $l \in \bar{A}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $U$  be a neighbourhood of  $l$ .

$\langle 1 \rangle 2$ . PICK  $N$  such that  $\forall n \in N. a_n \in U$

$\langle 1 \rangle 3$ .  $a_N \in A \cap U$

□

### 13.17.2 Continuous Functions

**Proposition 13.17.7.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$  be continuous. Let  $x_n \rightarrow x$  as  $n \rightarrow \infty$  in  $X$ . Then  $f(x_n) \rightarrow f(x)$  as  $n \rightarrow \infty$  in  $Y$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $V$  be a neighbourhood of  $f(x)$ .

$\langle 1 \rangle 2$ . PICK  $N$  such that  $\forall n \geq N. x_n \in f^{-1}(V)$

$\langle 1 \rangle 3$ .  $\forall n \geq N. f(x_n) \in V$

□

### 13.17.3 Infinite Series

**Definition 13.17.8** (Series). Let  $(a_n)$  be a sequence of real numbers. We say that the infinite series  $\sum_{n=0}^{\infty} a_n$  *converges* to  $s$ , and write

$$\sum_{n=0}^{\infty} a_n = s$$

iff  $\sum_{n=0}^N a_n \rightarrow s$  as  $N \rightarrow \infty$ .

## 13.18 Strong Continuity

**Definition 13.18.1** (Strong Continuity). Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is *strongly continuous* iff, for every  $V \subseteq Y$ , we have  $V$  is open in  $Y$  if and only if  $f^{-1}(V)$  is open in  $X$ .

**Proposition 13.18.2.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is strongly continuous if and only if, for all  $C \subseteq Y$ , we have  $C$  is closed in  $Y$  if and only if  $f^{-1}(C)$  is closed in  $X$ .*

PROOF:

$f$  is continuous  $\Leftrightarrow \forall V \subseteq Y (V \text{ is open in } Y \Leftrightarrow f^{-1}(V) \text{ is open in } X)$

$\Leftrightarrow \forall C \subseteq Y (Y - C \text{ is open in } Y \Leftrightarrow f^{-1}(Y - C) \text{ is open in } X)$

$\Leftrightarrow \forall C \subseteq Y (C \text{ is closed in } Y \Leftrightarrow f^{-1}(C) \text{ is closed in } X)$  □

## 13.19 Subspaces

**Definition 13.19.1** (Subspace). Let  $X$  be a topological space,  $Y$  a set, and  $f : Y \rightarrow X$ . The *subspace topology* on  $Y$  induced by  $f$  is  $\mathcal{T} = \{i^{-1}(U) : U \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF: Since  $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$ .

$\langle 1 \rangle 2$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF: Since  $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$ .

$\langle 1 \rangle 3$ .  $Y \in \mathcal{T}$

PROOF: Since  $Y = f^{-1}(X)$ .

□

**Proposition 13.19.2.** Let  $X$  be a topological space,  $Y$  a set and  $f : Y \rightarrow X$  a function. Then the subspace topology on  $Y$  is the coarsest topology such that  $f$  is continuous.

PROOF: Immediate from definition. □

**Proposition 13.19.3** (Local Formulation of Continuity). Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Let  $\mathcal{U}$  be a set of open subspaces of  $X$  such that  $X = \bigcup \mathcal{U}$ . If  $f|_U : U \rightarrow Y$  is continuous for all  $U \in \mathcal{U}$ , then  $f$  is continuous.

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in X$

PROVE:  $f$  is continuous at  $x$ .

$\langle 1 \rangle 2$ . LET:  $V$  be a neighbourhood of  $f(x)$ .

$\langle 1 \rangle 3$ . PICK  $U \in \mathcal{U}$  such that  $x \in U$ .

$\langle 1 \rangle 4$ . PICK  $W$  open in  $U$  such that  $x \in W$  and  $f(W) \subseteq V$ .

$\langle 1 \rangle 5$ .  $W$  is open in  $X$ .

□

**Theorem 13.19.4.** Let  $X$  be a topological space and  $(Y, i)$  a subset of  $X$ . Then the subspace topology on  $Y$  is the unique topology such that, for every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f : Z \rightarrow X$  is continuous.

PROOF:

$\langle 1 \rangle 1$ . If we give  $Y$  the subspace topology then, for every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f$  is continuous.

$\langle 2 \rangle 1$ . Given  $Y$  the subspace topology.

$\langle 2 \rangle 2$ . LET:  $Z$  be a topological space.

$\langle 2 \rangle 3$ . LET:  $f : Z \rightarrow Y$

$\langle 2 \rangle 4$ . If  $f$  is continuous then  $i \circ f$  is continuous.

PROOF: Since  $i$  is continuous.

- ⟨2⟩5. If  $i \circ f$  is continuous then  $f$  is continuous.
  - ⟨3⟩1. ASSUME:  $i \circ f$  is continuous.
  - ⟨3⟩2. LET:  $U$  be open in  $Y$ .
  - ⟨3⟩3.  $f^{-1}(i^{-1}(i(U)))$  is open in  $Z$ .
  - ⟨3⟩4.  $f^{-1}(U)$  is open in  $Z$ .
  - ⟨1⟩2. If, for every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f$  is continuous.
  - ⟨2⟩1. ASSUME: For every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f$  is continuous.
  - ⟨2⟩2.  $i$  is continuous.
  - ⟨2⟩3. For every open set  $U$  in  $X$ , we have  $i^{-1}(X)$  is open in  $Y$ .
  - ⟨2⟩4. LET:  $Z$  be the set  $Y$  under the subspace topology and  $f : Z \rightarrow Y$  the identity function.
  - ⟨2⟩5.  $i \circ f$  is continuous.
  - ⟨2⟩6.  $f$  is continuous.
  - ⟨2⟩7. Every set open in  $Y$  is open in  $Z$ .
- 

**Proposition 13.19.5.** *Let  $X$  be a topological space,  $Y$  a subspace of  $X$  and  $U \subseteq Y$ . If  $Y$  is open in  $X$  and  $U$  is open in  $Y$  then  $U$  is open in  $X$ .*

PROOF:

- ⟨1⟩1. PICK  $V$  open in  $X$  such that  $U = V \cap Y$
- ⟨1⟩2.  $U$  is open in  $X$ .

PROOF: It is the intersection of two open sets in  $X$ .

□

**Proposition 13.19.6.** *Let  $X$  be a topological space. Let  $Y$  be a subspace of  $X$ . Let  $C \subseteq Y$ . If  $Y$  is closed in  $X$  and  $C$  is closed in  $Y$  then  $C$  is closed in  $X$ .*

PROOF: Similar. □

**Proposition 13.19.7.** *Let  $Y$  be a subspace of  $X$  and  $A \subseteq Y$ . Then the subspace topology on  $A$  as a subspace of  $Y$  is the same as the subspace topology on  $A$  as a subspace of  $X$ .*

PROOF:

- ⟨1⟩1. LET:  $\mathcal{T}_Y$  be the subspace topology on  $A$  as a subspace of  $Y$ .
- ⟨1⟩2. LET:  $\mathcal{T}_X$  be the subspace topology on  $A$  as a subspace of  $X$ .
- ⟨1⟩3. LET:  $U \subseteq A$
- ⟨1⟩4.  $U \in \mathcal{T}_Y \Leftrightarrow U \in \mathcal{T}_X$

PROOF:

$$\begin{aligned}
 U \in \mathcal{T}_Y &\Leftrightarrow \exists V \text{ open in } Y. U = V \cap A \\
 &\Leftrightarrow \exists V. \exists W \text{ open in } X. (V = Y \cap W \wedge U = V \cap A) \\
 &\Leftrightarrow \exists W \text{ open in } X. U = Y \cap W \cap A \\
 &\Leftrightarrow \exists W \text{ open in } X. U = W \cap A \\
 &\Leftrightarrow U \in \mathcal{T}_X
 \end{aligned}$$

□

**Proposition 13.19.8.** *Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Let  $Y \subseteq X$ . Then  $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}\}$  is a basis for the topology on  $Y$ .*

PROOF:

⟨1⟩1. Every element of  $\mathcal{B}'$  is open.

PROOF: For all  $B \in \mathcal{B}$ , we have  $B$  is open in  $X$ , so  $B \cap Y$  is open in  $Y$ .

⟨1⟩2. For any open set  $V$  in  $Y$  and  $y \in V$ , there exists  $B' \in \mathcal{B}'$  such that  $y \in B' \subseteq V$

⟨2⟩1. LET:  $V$  be open in  $Y$ .

⟨2⟩2. LET:  $y \in V$

⟨2⟩3. PICK  $U$  open in  $X$  such that  $V = U \cap Y$ .

⟨2⟩4. PICK  $B \in \mathcal{B}$  such that  $y \in B \subseteq U$

⟨2⟩5.  $B \cap Y \in \mathcal{B}'$  and  $y \in B \cap Y \subseteq V$

□

**Proposition 13.19.9.** *Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Let  $A \subseteq Y$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$  then  $A$  is closed in  $X$ .*

PROOF:

⟨1⟩1. PICK  $C$  closed in  $X$  such that  $A = C \cap Y$ .

⟨1⟩2.  $A$  is closed in  $X$ .

PROOF: It is the intersection of two closed sets in  $X$ .

□

### 13.19.1 Product Topology

**Proposition 13.19.10.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $Y_i$  be a subspace of  $X_i$  for all  $i \in I$ . Then the product topology on  $\prod_{i \in I} Y_i$  is the same as the subspace topology on  $\prod_{i \in I} Y_i$  as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF:

⟨1⟩1. Given  $\prod_{i \in I} Y_i$  the subspace topology.

⟨1⟩2. LET:  $\iota : \prod_{i \in I} Y_i$  be the inclusion.

⟨1⟩3. LET:  $Z$  be any topological space.

⟨1⟩4. LET:  $f : Z \rightarrow \prod_{i \in I} Y_i$

⟨1⟩5.  $f$  is continuous if and only if, for all  $i \in I$ , we have  $\pi_i \circ f$  is continuous.

PROOF:

$$f \text{ is continuous} \Leftrightarrow \iota \circ f : Z \rightarrow \prod_{i \in I} X_i \text{ is continuous} \quad (\text{Theorem 13.19.4})$$

$$\Leftrightarrow \forall i \in I. \pi_i \circ \iota \circ f : Z \rightarrow X_i \text{ is continuous} \quad (\text{Theorem 13.24.4})$$

$$\Leftrightarrow \forall i \in I. \iota_i \circ \pi_i \circ f : Z \rightarrow X_i \text{ is continuous}$$

$$\Leftrightarrow \forall i \in I. \pi_i \circ f : Z \rightarrow Y_i \text{ is continuous} \quad (\text{Theorem 13.19.4})$$

where  $\iota_i$  is the inclusion  $Y_i \rightarrow X_i$ .

□

## 13.20 Embedding

**Definition 13.20.1** (Embedding). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *embedding* iff  $f$  is injective and the topology on  $X$  is the subspace induced by  $f$ .

**Proposition 13.20.2.** *Every embedding is continuous.*

PROOF: Theorem 13.19.4.  $\square$

**Proposition 13.20.3.** *Let  $X$  and  $Y$  be topological spaces. Let  $b \in Y$ . The function  $\kappa : X \rightarrow X \times Y$  that maps  $x$  to  $(x, b)$  is an embedding.*

PROOF:

$\langle 1 \rangle 1$ . For all  $U$  open in  $X$ , we have  $U = \kappa^{-1}(V)$  for some  $V$  open in  $X \times Y$ .

PROOF: Take  $V = U \times Y$ .

$\langle 1 \rangle 2$ . For all  $V$  open in  $X \times Y$  we have  $\kappa^{-1}(V)$  is open in  $X$ .

PROOF: Since  $\pi_1 \circ \kappa = \text{id}_X$  and  $\pi_2 \circ \kappa$  (which is the constant function with value  $b$ ) are both continuous, hence  $\kappa$  is continuous.

$\square$

## 13.21 Open Maps

**Definition 13.21.1** (Open Map). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *open map* iff, for all  $U$  open in  $X$ , we have  $f(U)$  is open in  $Y$ .

**Proposition 13.21.2.** *Let  $X$  and  $Y$  be topological spaces. The projections  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  are open maps.*

PROOF:

$\langle 1 \rangle 1$ .  $\pi_1$  is an open map.

$\langle 2 \rangle 1$ . LET:  $U$  be open in  $X \times Y$ .

$\langle 2 \rangle 2$ . LET:  $x \in \pi_1(U)$

$\langle 2 \rangle 3$ . PICK  $y$  such that  $(x, y) \in U$

$\langle 2 \rangle 4$ . PICK  $V$  and  $W$  open in  $X$  and  $Y$  respectively such that  $(x, y) \in V \times W \subseteq U$

$\langle 2 \rangle 5$ .  $x \in V \subseteq \pi_1(U)$

$\langle 1 \rangle 2$ .  $\pi_2$  is an open map.

PROOF: Similar.

$\square$

### 13.21.1 Subspaces

**Proposition 13.21.3.** *Let  $X$  and  $Y$  be topological spaces. Let  $p : X \rightarrow Y$  be an open map. Let  $A$  be an open set in  $X$ . Then  $p|_A : A \rightarrow p(A)$  is an open map.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $U$  be open in  $A$ .

$\langle 1 \rangle 2$ .  $U$  is open in  $X$ .

PROOF: Proposition 13.19.5.

$\langle 1 \rangle 3$ .  $p(U)$  is open in  $Y$ .

$\langle 1 \rangle 4$ .  $p(U)$  is open in  $p(A)$ .

PROOF: Since  $p(U) = p(U) \cap p(A)$ .

□

## 13.22 Locally Finite

**Definition 13.22.1** (Locally Finite). Let  $X$  be a topological space. Let  $\{A_i\}_{i \in I}$  be a family of subsets of  $X$ . Then  $\{A_i\}_{i \in I}$  is *locally finite* iff, for every  $x \in X$ , there exist only finitely many  $i \in I$  such that  $x \in A_i$ .

**Theorem 13.22.2** (Pasting Lemma). Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Let  $\{A_i\}_{i \in I}$  be a locally finite family of closed subspaces of  $X$  such that  $X = \bigcup_{i \in I} A_i$ . If  $f|_{A_i} : A_i \rightarrow Y$  is continuous for all  $i \in I$ , then  $f$  is continuous.

PROOF:

$\langle 1 \rangle 1$ . LET:  $B$  be closed in  $Y$ .

$\langle 1 \rangle 2$ . LET:  $A = f^{-1}(B)$

PROVE:  $A$  is closed in  $X$ .

$\langle 1 \rangle 3$ .  $A = \bigcup_{i \in I} f|_{A_i}^{-1}(B)$

$\langle 1 \rangle 4$ . LET:  $x \in X - A$

PROVE: There exists a neighbourhood  $U'$  of  $x$  such that  $U' \subseteq X - A$ .

$\langle 1 \rangle 5$ . PICK a neighbourhood  $U$  of  $x$  such that  $U$  intersects  $A_i$  for only finitely many  $i \in I$ .

$\langle 1 \rangle 6$ . LET:  $i_1, \dots, i_n$  be the elements of  $I$  such that  $U$  intersects  $A_{i_1}, \dots, A_{i_n}$ .

$\langle 1 \rangle 7$ . For  $j = 1, \dots, n$ ,

LET:  $S_j = f|_{A_{i_j}}^{-1}(B)$

$\langle 1 \rangle 8$ . For  $j = 1, \dots, n$ , we have  $S_j$  is closed in  $X$ .

$\langle 1 \rangle 9$ . For  $j = 1, \dots, n$ , we have  $x \notin S_j$ .

$\langle 1 \rangle 10$ . LET:  $U' = U \cap \bigcap_{j=1}^n (X - S_j)$

$\langle 1 \rangle 11$ .  $U'$  is a neighbourhood of  $x$ .

$\langle 1 \rangle 12$ .  $U' \subseteq X - A$

□

## 13.23 Closed Maps

**Definition 13.23.1** (Closed Map). Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is a *closed map* iff, for every closed set  $C$  in  $X$ , we have  $f(C)$  is closed in  $Y$ .

## 13.24 Product Topology

**Definition 13.24.1** (Product Topology). Let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be a family of topological spaces. The *product topology* on  $\prod_{\lambda \in \Lambda} X_\lambda$  is the coarsest topology such that every projection onto  $X_\lambda$  is continuous.

### 13.24.1 Closed Sets

**Proposition 13.24.2.** *Let  $X$  and  $Y$  be topological spaces. Let  $A$  be a closed set in  $X$  and  $B$  a closed set in  $Y$ . Then  $A \times B$  is closed in  $X \times Y$ .*

PROOF: Since  $(X \times Y) - (A \times B) = ((X - A) \times Y) \cup (X \times (Y - B))$ .  $\square$

**Proposition 13.24.3.** *Let  $\{X_\alpha\}_{\alpha \in A}$  be a family of topological spaces. The product topology on  $\prod_{\alpha \in A} X_\alpha$  is the topology generated by the basis  $\mathcal{B} = \{\prod_{\alpha \in A} U_\alpha : \text{for all } \alpha \in A, U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in A\}$ .*

PROOF:

- $\langle 1 \rangle 1$ .  $\mathcal{B}$  is a basis for a topology.
- $\langle 1 \rangle 2$ . LET:  $\mathcal{T}$  be the topology generated by  $\mathcal{B}$ .
- $\langle 1 \rangle 3$ . LET:  $\mathcal{T}_p$  be the product topology.
- $\langle 1 \rangle 4$ .  $\mathcal{T} \subseteq \mathcal{T}_p$ 
  - $\langle 2 \rangle 1$ . LET:  $B \in \mathcal{B}$
  - $\langle 2 \rangle 2$ . LET:  $B = \prod_{\alpha \in A} U_\alpha$  with each  $U_\alpha$  open in  $X_\alpha$  and  $U_\alpha = X_\alpha$  except for  $\alpha = \alpha_1, \dots, \alpha_n$
  - $\langle 2 \rangle 3$ .  $B = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$
  - $\langle 2 \rangle 4$ .  $B \in \mathcal{T}_p$
- $\langle 1 \rangle 5$ .  $\mathcal{T}_p \subseteq \mathcal{T}$ 
  - $\langle 2 \rangle 1$ . For every  $\alpha \in A$  we have  $\pi_\alpha$  is continuous.

PROOF: Since  $\pi^{-1}(U)$  is open for every  $U$  open in  $X_\alpha$ .

$\square$

**Theorem 13.24.4.** *Let  $\{X_\alpha\}_{\alpha \in A}$  be a family of topological spaces. Then the product topology on  $\prod_{\alpha \in A} X_\alpha$  is the unique topology such that, for every topological space  $Z$  and function  $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$ , we have  $f$  is continuous if and only if, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f : Z \rightarrow X_\alpha$  is continuous.*

PROOF:

- $\langle 1 \rangle 1$ . If we give  $\prod_{\alpha \in A} X_\alpha$  the product topology, then for every topological space  $Z$  and function  $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$ , we have  $f$  is continuous if and only if, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f$  is continuous.
- $\langle 2 \rangle 1$ . Give  $\prod_{\alpha \in A} X_\alpha$  the product topology.
- $\langle 2 \rangle 2$ . LET:  $Z$  be a topological space.
- $\langle 2 \rangle 3$ . LET:  $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$
- $\langle 2 \rangle 4$ . If  $f$  is continuous then, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f$  is continuous.
- PROOF: Since the composite of two continuous functions is continuous.
- $\langle 2 \rangle 5$ . If, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f$  is continuous, then  $f$  is continuous.
- $\langle 3 \rangle 1$ . ASSUME: For all  $\alpha \in A$  we have  $\pi_\alpha \circ f$  is continuous.



- $\langle 3 \rangle 2$ . LET:  $\{U_\alpha\}_{\alpha \in A}$  be a family with  $U_\alpha$  open in  $X_\alpha$  such that  $U_\alpha = X_\alpha$  for all  $\alpha$  except  $\alpha = \alpha_1, \dots, \alpha_n$ .  
 $\langle 3 \rangle 3$ . For all  $\alpha$  we have  $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$  is open in  $Z$ .  
 $\langle 3 \rangle 4$ .  $f^{-1}(\prod_\alpha U_\alpha)$  is open in  $Z$ .  
 PROOF: Since  $f^{-1}(\prod_\alpha U_\alpha) = f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \dots \cap f^{-1}(\pi_{\alpha_n}^{-1}(U_{\alpha_n}))$ .  
 $\langle 1 \rangle 2$ . If  $\mathcal{T}$  is a topology on  $\prod_{\alpha \in A} X_\alpha$  such that, for every topological space  $Z$  and function  $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$ , we have  $f$  is continuous if and only if, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f$  is continuous, then  $\mathcal{T}$  is the product topology.  
 $\langle 2 \rangle 1$ . ASSUME:  $\mathcal{T}$  is a topology on  $\prod_{\alpha \in A} X_\alpha$  such that, for every topological space  $Z$  and function  $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$ , we have  $f$  is continuous if and only if, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f$  is continuous.  
 $\langle 2 \rangle 2$ . LET:  $\mathcal{T}_p$  be the product topology.  
 $\langle 2 \rangle 3$ .  $\mathcal{T} \subseteq \mathcal{T}_p$   
 $\langle 3 \rangle 1$ . LET:  $Z = (\prod_\alpha X_\alpha, \mathcal{T}_p)$   
 $\langle 3 \rangle 2$ . LET:  $f : Z \rightarrow \prod_\alpha X_\alpha$  be the identity function  
 $\langle 3 \rangle 3$ . For all  $\alpha$  we have  $\pi_\alpha \circ f$  is continuous.  
 $\langle 3 \rangle 4$ .  $f$  is continuous.  
 PROOF:  $\langle 2 \rangle 1$   
 $\langle 3 \rangle 5$ . Every set open in  $\mathcal{T}$  is open in  $\mathcal{T}_p$   
 $\langle 2 \rangle 4$ .  $\mathcal{T}_p \subseteq \mathcal{T}$   
 $\langle 3 \rangle 1$ .  $\text{id}_{\prod_\alpha X_\alpha}$  is continuous.  
 $\langle 3 \rangle 2$ . For all  $\alpha$  we have  $\pi_\alpha$  is continuous.  
 PROOF:  $\langle 2 \rangle 1$   
 $\langle 3 \rangle 3$ .  $\mathcal{T}_p \subseteq \mathcal{T}$   
 PROOF: Since  $\mathcal{T}_p$  is the coarsest topology such that every  $\pi_\alpha$  is continuous.

□

**Example 13.24.5.** It is not true that, for any function  $f : \prod_{\alpha \in A} X_\alpha \rightarrow Y$ , if  $f$  is continuous in every variable separately then  $f$  is continuous.

Define  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

Then  $f$  is continuous in  $x$  and in  $y$ , but is not continuous.

**Proposition 13.24.6.** Let  $\{X_i\}_{i \in I}$  be a nonempty family of topological spaces. The product topology on  $\prod_{i \in I} X_i$  is the topology generated by the subbasis  $\{\pi_i^{-1}(U) : i \in I, U \text{ is open in } X_i\}$ .

PROOF:

- $\langle 1 \rangle 1$ .  $\{\pi_i^{-1}(U) : i \in I, U \text{ is open in } X_i\}$  is a subbasis for a topology on  $\prod_{i \in I} X_i$ .  
 $\langle 2 \rangle 1$ . PICK  $i_0 \in I$   
 $\langle 2 \rangle 2$ .  $\prod_{i \in I} X_i = \pi_{i_0}^{-1}(X_{i_0})$   
 $\langle 1 \rangle 2$ . The topology generated by this subbasis is the product topology.  
 PROOF: Since the basis in Proposition 13.24.3 is the set of all finite intersections of elements of this subbasis.

□

### 13.24.2 Closure

**Proposition 13.24.7.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

⟨1⟩1.  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET:  $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. For any family  $\{U_i\}_{i \in I}$  where each  $U_i$  is open in  $X_i$ , and  $U_i = X_i$  for all but finitely many  $i \in I$ , if  $x \in \prod_{i \in I} U_i$  then  $\prod_{i \in I} U_i$  intersects  $\prod_{i \in I} A_i$ .

⟨3⟩1. LET:  $\{U_i\}_{i \in I}$  be a family where each  $U_i$  is open in  $X_i$ , and  $U_i = X_i$  for all but finitely many  $i$ .

⟨3⟩2. ASSUME:  $x \in \prod_{i \in I} \overline{A_i}$

⟨3⟩3. For all  $i \in I$  we have  $\overline{U_i}$  intersects  $A_i$

PROOF: Since  $\pi_i(x) \in \overline{A_i}$  and  $U_i$  is a neighbourhood of  $\pi_i(x)$ .

⟨3⟩4.  $\prod_{i \in I} U_i$  intersects  $\prod_{i \in I} A_i$

⟨2⟩3.  $x \in \overline{\prod_{i \in I} A_i}$

PROOF: Proposition 13.13.9.

⟨1⟩2.  $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. LET:  $x \in \overline{\prod_{i \in I} A_i}$

⟨2⟩2. LET:  $i \in I$

PROVE:  $\pi_i(x) \in \overline{A_i}$

⟨2⟩3. LET:  $U$  be a neighbourhood of  $\pi_i(x)$  in  $X_i$

⟨2⟩4.  $\pi_i^{-1}(U)$  is a neighbourhood of  $x$  in  $\prod_{i \in I} X_i$

⟨2⟩5. PICK  $y \in \pi_i^{-1}(U) \cap \prod_{i \in I} A_i$

⟨2⟩6.  $\pi_i(y) \in U \cap A_i$

□

### 13.24.3 Convergence

**Proposition 13.24.8.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $(x_n)$  be a sequence of points in  $\prod_{i \in I} X_i$  and  $l \in \prod_{i \in I} X_i$ . Then  $x_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for all  $i \in I$ , we have  $\pi_i(x_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ .*

PROOF:

⟨1⟩1. If  $x_n \rightarrow l$  as  $n \rightarrow \infty$  then, for all  $i \in I$ , we have  $\pi_i(x_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ .

PROOF: Proposition 13.17.2.

⟨1⟩2. If, for all  $i \in I$ , we have  $\pi_i(x_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ , then  $x_n \rightarrow l$  as  $n \rightarrow \infty$ .

⟨2⟩1. ASSUME: For all  $i \in I$  we have  $\pi_i(x_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ .

⟨2⟩2. LET:  $U$  be a neighbourhood of  $l$ .

⟨2⟩3. PICK  $i_1, \dots, i_n \in I$  and open sets  $U_j$  in  $X_{i_j}$  for  $j = 1, \dots, n$  such that  $l \in \pi_{i_1}^{-1}(U_1) \cap \dots \cap \pi_{i_n}^{-1}(U_n) \subseteq U$

- $\langle 2 \rangle 4$ . For  $j = 1, \dots, n$  we have  $\pi_{i_j}(l) \in U_j$   
 $\langle 2 \rangle 5$ . PICK  $N$  such that, for all  $m \geq N$ , we have  $\pi_{i_j}(x_m) \in U_j$   
 $\langle 2 \rangle 6$ .  $\forall m \geq N. x_m \in U$

□

## 13.25 Topological Disjoint Union

**Definition 13.25.1** (Coproduct Topology). Let  $\{X_\alpha\}_{\alpha \in A}$  be a family of topological spaces. The *coproduct topology* on  $\coprod_{\alpha \in A} X_\alpha$  is

$$\mathcal{T} = \left\{ \coprod_{\alpha \in A} U_\alpha : \{U_\alpha\}_{\alpha \in A} \text{ is a family with } U_\alpha \text{ open in } X_\alpha \text{ for all } \alpha \right\}.$$

We prove this is a topology.

PROOF:

- $\langle 1 \rangle 1$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF:

$$\bigcup_{i \in I} \coprod_{\alpha \in A} U_{i\alpha} = \coprod_{\alpha \in A} \bigcup_{i \in I} U_{i\alpha}$$

- $\langle 1 \rangle 2$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF:

$$\coprod_{\alpha \in A} U_\alpha \cap \coprod_{\alpha \in A} V_\alpha = \coprod_{\alpha \in A} (U_\alpha \cap V_\alpha)$$

- $\langle 1 \rangle 3$ .  $\coprod_{\alpha \in A} X_\alpha \in \mathcal{T}$

PROOF: Since every  $X_\alpha$  is open in  $X_\alpha$ .

□

**Proposition 13.25.2.** *The coproduct topology is the finest topology on  $\coprod_{\alpha \in A} X_\alpha$  such that every injection  $\kappa_\alpha : X_\alpha \rightarrow \coprod_{\alpha \in A} X_\alpha$  is continuous.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $P = \coprod_{\alpha \in A} X_\alpha$

- $\langle 1 \rangle 2$ . LET:  $\mathcal{T}_c$  be the coproduct topology.

- $\langle 1 \rangle 3$ . LET:  $\mathcal{T}$  be any topology on  $P$

- $\langle 1 \rangle 4$ . For all  $\alpha \in A$ , the injection  $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T}_c)$  is continuous.

- $\langle 2 \rangle 1$ . LET:  $\alpha \in A$

- $\langle 2 \rangle 2$ . LET:  $\{U_\alpha\}_{\alpha \in A}$  be a family with each  $U_\alpha$  open in  $X_\alpha$ .

- $\langle 2 \rangle 3$ . For all  $\alpha \in A$ , we have  $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha)$  is open in  $X_\alpha$ .

PROOF: Since  $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha) = U_\alpha$ .

- $\langle 1 \rangle 5$ . If, for all  $\alpha \in A$ , the injection  $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$  is continuous, then  $\mathcal{T} \subseteq \mathcal{T}_c$ .

- $\langle 2 \rangle 1$ . ASSUME: For all  $\alpha \in A$ , the injection  $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$  is continuous.

- $\langle 2 \rangle 2$ . LET:  $U \in \mathcal{T}$

- $\langle 2 \rangle 3$ . For all  $\alpha \in A$ , we have  $\kappa_\alpha^{-1}(U)$  is open in  $X_\alpha$ .

- $\langle 2 \rangle 4$ .  $U = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(U) \in \mathcal{T}_c$

□

**Theorem 13.25.3.** *Let  $\{X_\alpha\}_{\alpha \in A}$  be a family of topological spaces. The coproduct topology is the unique topology on  $\coprod_{\alpha \in A} X_\alpha$  such that, for every topological space  $Z$  and function  $f : \coprod_{\alpha \in A} X_\alpha \rightarrow Z$ , we have  $f$  is continuous if and only if  $\forall \alpha \in A. f \circ \kappa_\alpha$  is continuous.*

PROOF:

- ⟨1⟩1. LET:  $X = \coprod_{\alpha \in A} X_\alpha$
- ⟨1⟩2. LET:  $\mathcal{T}_c$  be the coproduct topology.
- ⟨1⟩3. For every topological space  $Z$  and function  $f : (X, \mathcal{T}_c) \rightarrow Z$ , we have  $f$  is continuous if and only if  $\forall \alpha \in A. f \circ \kappa_\alpha$  is continuous.
  - ⟨2⟩1. LET:  $Z$  be a topological space.
  - ⟨2⟩2. LET:  $f : X \rightarrow Z$
  - ⟨2⟩3. If  $f$  is continuous then  $\forall \alpha \in A. f \circ \kappa_\alpha$  is continuous.
 

PROOF: Because the composite of two continuous functions is continuous.
  - ⟨2⟩4. If  $\forall \alpha \in A. f \circ \kappa_\alpha$  is continuous then  $f$  is continuous.
    - ⟨3⟩1. ASSUME:  $\forall \alpha \in A. f \circ \kappa_\alpha$  is continuous.
    - ⟨3⟩2. LET:  $U$  be open in  $Z$
    - ⟨3⟩3. For all  $\alpha \in A$  we have  $\kappa_\alpha^{-1}(f^{-1}(U))$  is open in  $X_\alpha$
    - ⟨3⟩4.  $f^{-1}(U) = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(f^{-1}(U))$
    - ⟨3⟩5.  $f^{-1}(U)$  is open in  $X$
- ⟨1⟩4. For any topology  $\mathcal{T}$  on  $X$ , if for every topological space  $Z$  and function  $f : (X, \mathcal{T}) \rightarrow Z$ , we have  $f$  is continuous if and only if  $\forall \alpha \in A. f \circ \kappa_\alpha$  is continuous, then  $\mathcal{T} = \mathcal{T}_c$ .
  - ⟨2⟩1. LET:  $\mathcal{T}$  be a topology on  $X$ .
  - ⟨2⟩2. ASSUME: For every topological space  $Z$  and function  $f : (X, \mathcal{T}) \rightarrow Z$ , we have  $f$  is continuous if and only if  $\forall \alpha \in A. f \circ \kappa_\alpha$  is continuous.
    - ⟨2⟩3.  $\mathcal{T} \subseteq \mathcal{T}_c$ 
      - ⟨3⟩1. For all  $\alpha \in A$  we have  $\kappa_\alpha : X_\alpha \rightarrow (X, \mathcal{T})$  is continuous.
 

PROOF: From ⟨2⟩1 since  $\text{id}_X$  is continuous.
      - ⟨3⟩2.  $\mathcal{T} \subseteq \mathcal{T}_c$ 

PROOF: Proposition 13.25.2.
    - ⟨2⟩4.  $\mathcal{T}_c \subseteq \mathcal{T}$ 
      - ⟨3⟩1. LET:  $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}_c)$  be the identity function.
      - ⟨3⟩2.  $f \circ \kappa_\alpha$  is continuous for all  $\alpha$ .
      - ⟨3⟩3.  $f$  is continuous.
 

PROOF: ⟨2⟩1
      - ⟨3⟩4.  $\mathcal{T}_c \subseteq \mathcal{T}$

□

## 13.26 Quotient Spaces

**Definition 13.26.1** (Quotient Topology). Let  $X$  be a topological space,  $S$  a set, and  $\pi : X \twoheadrightarrow S$  be a surjection. The *quotient topology* on  $S$  induced by  $\pi$  is

$\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}.$

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1.$  For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}.$

PROOF: Since  $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}.$

$\langle 1 \rangle 2.$  For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}.$

PROOF: Since  $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V).$

$\langle 1 \rangle 3.$   $X \in \mathcal{T}$

PROOF: Since  $X = \pi^{-1}(Y).$

□

**Proposition 13.26.2.** *Let  $X$  be a topological space,  $S$  a set and  $\pi : X \twoheadrightarrow S$  a surjection. Then the quotient topology on  $S$  is the finest topology such that  $\pi$  is continuous.*

PROOF: Immediate from definitions. □

**Theorem 13.26.3.** *Let  $X$  be a topological space, let  $S$  be a set, and let  $\pi : X \twoheadrightarrow S$  be surjective. Then the quotient topology on  $S$  is the unique topology such that, for every topological space  $Z$  and function  $f : S \rightarrow Z$ , we have  $f$  is continuous if and only if  $f \circ \pi$  is continuous.*

PROOF:

$\langle 1 \rangle 1.$  If  $S$  is given the quotient topology, then for every topological space  $Z$  and function  $f : S \rightarrow Z$ , we have  $f$  is continuous if and only if  $f \circ \pi$  is continuous.

$\langle 2 \rangle 1.$  Give  $S$  the quotient topology.

$\langle 2 \rangle 2.$  LET:  $Z$  be a topological space.

$\langle 2 \rangle 3.$  LET:  $f : S \rightarrow Z$

$\langle 2 \rangle 4.$  If  $f$  is continuous then  $f \circ \pi$  is continuous.

PROOF: The composite of two continuous functions is continuous.

$\langle 2 \rangle 5.$  If  $f \circ \pi$  is continuous then  $f$  is continuous.

$\langle 3 \rangle 1.$  ASSUME:  $f \circ \pi$  is continuous.

$\langle 3 \rangle 2.$  LET:  $U$  be open in  $Z$ .

$\langle 3 \rangle 3.$   $\pi^{-1}(f^{-1}(U))$  is open in  $X$ .

$\langle 3 \rangle 4.$   $f^{-1}(U)$  is open in  $S$ .

$\langle 1 \rangle 2.$  If  $S$  is given a topology such that, for every topological space  $Z$  and function  $f : S \rightarrow Z$ , we have  $f$  is continuous if and only if  $f \circ \pi$  is continuous, then that topology is the quotient topology.

$\langle 2 \rangle 1.$  Give  $S$  a topology such that, for every topological space  $Z$  and function  $f : S \rightarrow Z$ , we have  $f$  is continuous if and only if  $f \circ \pi$  is continuous.

$\langle 2 \rangle 2.$  LET:  $U \subseteq S$

$\langle 2 \rangle 3.$  If  $\pi^{-1}(U)$  is open in  $X$  then  $U$  is open in  $S$ .

$\langle 3 \rangle 1.$  LET:  $Z$  be  $S$  under the quotient topology induced by  $\pi$ .

$\langle 3 \rangle 2.$  LET:  $f : S \rightarrow Z$  be the identity function.

$\langle 3 \rangle 3.$   $f \circ \pi$  is continuous.

$\langle 3 \rangle 4.$   $f$  is continuous.

PROOF:  $\langle 2 \rangle 1$

$\langle 3 \rangle 5$ .  $U$  is open in  $Z$ .

$\langle 3 \rangle 6$ .  $U$  is open in  $X$ .

$\langle 2 \rangle 4$ . If  $U$  is open in  $S$  then  $\pi^{-1}(U)$  is open in  $X$ .

PROOF: Since  $\pi$  is continuous (taking  $Z = S$  and  $f = \text{id}_S$  in  $\langle 2 \rangle 1$ ).

□

### 13.26.1 Quotient Maps

**Definition 13.26.4** (Quotient Map). Let  $X$  and  $S$  be topological spaces and  $\pi : X \rightarrow S$ . Then  $\pi$  is a *quotient map* iff  $\pi$  is surjective and the topology on  $S$  is the quotient topology induced by  $\pi$ .

**Proposition 13.26.5.** Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is a quotient map if and only if  $f$  is surjective and strongly continuous.

PROOF: Immediate from definition. □

**Proposition 13.26.6.** Let  $X$  and  $Y$  be topological spaces. Let  $p : X \twoheadrightarrow Y$  be surjective. Then the following are equivalent.

1.  $p$  is a quotient map.
2.  $p$  is continuous and maps saturated open sets to open sets.
3.  $p$  is continuous and maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

$\langle 2 \rangle 1$ . ASSUME:  $p$  is a quotient map.

$\langle 2 \rangle 2$ .  $p$  is continuous.

$\langle 2 \rangle 3$ .  $p$  maps saturated open sets to open sets.

$\langle 3 \rangle 1$ . LET:  $U \subseteq X$  be a saturated open set.

$\langle 3 \rangle 2$ .  $p^{-1}(p(U)) = U$

$\langle 3 \rangle 3$ .  $p^{-1}(p(U))$  is open in  $X$ .

$\langle 3 \rangle 4$ .  $p(U)$  is open in  $Y$ .

$\langle 1 \rangle 2$ .  $2 \Rightarrow 3$

$\langle 2 \rangle 1$ . ASSUME:  $p$  is continuous and maps saturated open sets to open sets.

$\langle 2 \rangle 2$ . LET:  $C$  be a saturated closed set in  $X$ .

$\langle 2 \rangle 3$ .  $X - C$  is a saturated open set.

$\langle 2 \rangle 4$ .  $Y - p(C)$  is open.

$\langle 2 \rangle 5$ .  $p(C)$  is closed.

$\langle 1 \rangle 3$ .  $3 \Rightarrow 1$

$\langle 2 \rangle 1$ . ASSUME:  $p$  is continuous and maps closed sets to closed sets.

$\langle 2 \rangle 2$ . LET:  $C \subseteq Y$

$\langle 2 \rangle 3$ . ASSUME:  $p^{-1}(C)$  is closed in  $X$ .

PROVE:  $C$  is closed in  $Y$ .

$\langle 2 \rangle 4$ .  $p^{-1}(C)$  is saturated.

$\langle 2 \rangle 5$ .  $p(p^{-1}(C))$  is closed.

$\langle 2 \rangle 6$ .  $C$  is closed.

□

**Corollary 13.26.6.1.** *Let  $X$  and  $Y$  be topological spaces. Let  $p : X \rightarrow Y$  be continuous and surjective. If  $p$  is either an open map or a closed map, then  $p$  is a quotient map.*

**Example 13.26.7.** The converse does not hold.

Let  $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \vee y = 0\}$ . Then the first projection  $\pi_1 : A \rightarrow \mathbb{R}$  is a quotient map that is neither an open map nor a closed map.

PROOF:

$\langle 1 \rangle 1$ .  $\pi_1$  is a quotient map.

$\langle 2 \rangle 1$ . LET:  $U \subseteq \mathbb{R}$

$\langle 2 \rangle 2$ . If  $U$  is open then  $\pi_1^{-1}(U)$  is open.

PROOF: Since  $\pi_1^{-1}(U) = (U \times \mathbb{R}) \cap A$ .

$\langle 2 \rangle 3$ . If  $\pi_1^{-1}(U)$  is open then  $U$  is open.

$\langle 3 \rangle 1$ . ASSUME:  $\pi_1^{-1}(U)$  is open.

$\langle 3 \rangle 2$ . LET:  $x \in U$

$\langle 3 \rangle 3$ .  $(x, 0) \in \pi_1^{-1}(U)$

$\langle 3 \rangle 4$ . PICK open neighbourhoods  $V$  of  $x$  and  $W$  of  $0$  such that  $V \times W \subseteq \pi_1^{-1}(U)$

$\langle 3 \rangle 5$ .  $V \subseteq U$

PROOF: For all  $x' \in V$  we have  $(x', 0) \in V \times W \subseteq \pi_1^{-1}(U)$ .

$\langle 1 \rangle 2$ .  $\pi_1$  is not an open map.

PROOF:  $\pi_1((( -1, 1) \times (1, 2)) \cap A) = [0, 1)$  which is not open in  $\mathbb{R}$ .

$\langle 1 \rangle 3$ .  $\pi_1$  is not a closed map.

PROOF:  $\pi_1(\{(x, 1/x) \in \mathbb{R}^2 : x > 0\}) = (0, +\infty)$  is not closed in  $\mathbb{R}$ .

□

**Corollary 13.26.7.1.** *Let  $\{X_i\}_{i \in I}$  and  $\{Y_i\}_{i \in I}$  be families of topological spaces and  $p_i : X_i \rightarrow Y_i$  for all  $i \in I$ .*

1. *If every  $p_i$  is an open quotient map, then  $\prod_{i \in I} p_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$  is an open quotient map.*

2. *If every  $p_i$  is a closed quotient map, then  $\prod_{i \in I} p_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$  is a closed quotient map.*

**Example 13.26.8.** The product of two quotient maps is not necessarily a quotient map.

Let  $Y$  be the quotient space of  $\mathbb{R}_K$  obtained by collapsing the set  $K$  to a point. Let  $p : \mathbb{R}_K \twoheadrightarrow Y$  be the quotient map. Then  $q \times q : \mathbb{R}_K^2 \rightarrow Y^2$  is not a quotient map.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\Delta = \{(y, y) : y \in Y\}$

$\langle 1 \rangle 2$ .  $Y$  is not Hausdorff.

- $\langle 2 \rangle 1$ . LET:  $*_K \in Y$  be the point such that  $q(K) = \{*_K\}$   
 $\langle 2 \rangle 2$ . ASSUME: for a contradiction  $U$  and  $V$  are disjoint neighbourhoods of 0  
 and  $*_K$   
 $\langle 2 \rangle 3$ .  $q^{-1}(U)$  and  $q^{-1}(V)$  are disjoint open sets with  $0 \in q^{-1}(U)$  and  $K \subseteq q^{-1}(V)$   
 $\langle 2 \rangle 4$ . Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 3$ .  $\Delta$  is not closed in  $Y^2$ .

$\langle 1 \rangle 4$ .  $(q \times q)^{-1}(\Delta)$  is closed in  $\mathbb{R}_K^2$

PROOF: It is  $\{(x, x) : x \in \mathbb{R}\} \cup K^2$ .

□

**Proposition 13.26.9.** Let  $\pi : X \twoheadrightarrow S$  be a quotient map. Let  $Z$  be a topological space. Let  $f : X \rightarrow Z$  be continuous. Then there exists a continuous map  $g : S \rightarrow Z$  such that  $f = g \circ \pi$  if and only if, for all  $s \in S$ , we have  $f$  is constant on  $\pi^{-1}(s)$ .

PROOF: From Theorem 13.26.3. □

**Proposition 13.26.10.** Let  $Z$  be a topological space. Define  $\pi : [0, 1] \rightarrow S^1$  by  $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$ . Given any continuous function  $f : S^1 \rightarrow Z$ , we have  $f \circ \pi$  is a loop in  $Z$ . This defines a bijection between  $\mathbf{Top}[S^1, Z]$  and the set of loops in  $Z$ .

PROOF: Since  $\pi$  is a quotient map. □

**Definition 13.26.11** (Projective Space). The *projective space*  $\mathbb{RP}^n$  is the quotient of  $\mathbb{R}^{n+1} - \{0\}$  by  $\sim$  where  $x \sim \lambda x$  for all  $x \in \mathbb{R}^{n+1} - \{0\}$  and  $\lambda \in \mathbb{R}$ .

**Definition 13.26.12** (Torus). The *torus*  $T$  is the quotient of  $[0, 1]^2$  by  $\sim$  where  $(x, 0) \sim (x, 1)$  and  $(0, y) \sim (1, y)$ .

**Definition 13.26.13** (Möbius Band). The *Möbius band* is the quotient of  $[0, 1]^2$  by  $\sim$  where  $(0, y) \sim (1, 1 - y)$ .

**Definition 13.26.14** (Klein Bottle). The *Klein bottle* is the quotient of  $[0, 1]^2$  by  $\sim$  where  $(x, 0) \sim (x, 1)$  and  $(0, y) \sim (1, 1 - y)$ .

**Proposition 13.26.15.**  $\mathbb{RP}^2$  is the quotient of  $[0, 1]^2$  by  $\sim$  where  $(x, 0) \sim (1 - x, 1)$  and  $(0, y) \sim (1, 1 - y)$ .

PROOF: TODO

**Example 13.26.16.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces and  $\{Y_i\}_{i \in I}$  a family of sets. Let  $q_i : X_i \twoheadrightarrow Y_i$  be a surjective function for all  $i \in I$ . Give each  $Y_i$  the quotient topology. It is not true in general that the product topology on  $\prod_{i \in I} Y_i$  is the same as the quotient topology induced by  $\prod_{i \in I} q_i : \prod_{i \in I} X_i \twoheadrightarrow \prod_{i \in I} Y_i$ .

PROOF:



- <1>1. LET:  $X^* = \mathbb{R} - \mathbb{Z}_+ + \{b\}$  be the quotient space obtained from  $\mathbb{R}$  by identifying the subset  $\mathbb{Z}_+$  to the point  $b$ .  
 <1>2. LET:  $p : \mathbb{R} \rightarrow X^*$  be the quotient map.  
 PROVE:  $p \times \text{id}_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$  is not a quotient map.  
 <1>3. For  $n \in \mathbb{Z}_+$ ,  
 LET:  $c_n = \sqrt{2}/n$   
 <1>4. For  $n \in \mathbb{Z}_+$ ,  
 LET:  $U_n = \{(x, y) \in \mathbb{Q} \times \mathbb{R} : n - 1/4 < x < n + 1/4 \text{ and } ((y > x + c_n - n \text{ and } y > -x + c_n + n) \text{ or } (y < x + c_n - n \text{ and } y < -x + c_n + n))\}$   
 <1>5. For all  $n \in \mathbb{Z}_+$ ,  $U_n$  is open in  $\mathbb{R} \times \mathbb{Q}$   
 <1>6. For all  $n \in \mathbb{Z}_+$  we have  $\{n\} \times \mathbb{Q} \subseteq U_n$   
 <1>7. LET:  $U = \bigcup_{n \in \mathbb{Z}_+} U_n$   
 <1>8.  $U$  is open in  $\mathbb{R} \times \mathbb{Q}$ .  
 <1>9.  $U$  is saturated with respect to  $p \times \text{id}_{\mathbb{Q}}$ .  
 <1>10. LET:  $U' = (p \times \text{id}_{\mathbb{Q}})(U)$   
 <1>11. ASSUME: for a contradiction  $U'$  is open in  $X^* \times \mathbb{Q}$ .

**Proposition 13.26.17.** Let  $X$  and  $Y$  be topological spaces. Let  $\sim$  be an equivalence relation on  $X$ . Let  $\phi : Y \rightarrow X/\sim$ .

Assume that, for all  $y \in Y$ , there exists a neighbourhood  $U$  of  $y$  and a continuous function  $\Phi : U \rightarrow X$  such that  $\pi \circ \Phi = \phi|_U$ . Then  $\phi$  is continuous.

**Proposition 13.26.18.** Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . If  $X/\sim$  is Hausdorff then every equivalence class of  $\sim$  is closed in  $X$ .

**Definition 13.26.19.** Let  $X$  be a topological space and  $A_1, \dots, A_r \subseteq X$ . Then  $X/A_1, \dots, A_r$  is the quotient space of  $X$  with respect to  $\sim$  where  $x \sim y$  iff  $x = y$  or  $\exists i(x \in A_i \wedge y \in A_i)$ .

**Definition 13.26.20** (Cone). Let  $X$  be a topological space. The *cone over  $X$*  is the space  $(X \times [0, 1])/(X \times \{1\})$ .

**Definition 13.26.21** (Suspension). Let  $X$  be a topological space. The *suspension* of  $X$  is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

**Definition 13.26.22** (Wedge Product). Let  $x_0 \in X$  and  $y_0 \in Y$ . The *wedge product*  $X \vee Y$  is  $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$  as a subspace of  $X \times Y$ .

**Definition 13.26.23** (Smash Product). Let  $x_0 \in X$  and  $y_0 \in Y$ . The *smash product*  $X \wedge Y$  is  $(X \times Y)/(X \vee Y)$ .

**Example 13.26.24.**  $D^n/S^{n-1} \cong S^n$

PROOF:

- <1>1. LET:  $\phi : D^n/S^{n-1} \rightarrow S^n$  be the function induced by the map  $D^n \rightarrow S^n$  that maps the radii of  $D^n$  onto the meridians of  $S^n$  from the north to the south pole.

$\langle 1 \rangle 2$ .  $\phi$  is a bijection.

$\langle 1 \rangle 3$ .  $\phi$  is a homeomorphism.

PROOF: Since  $D^n/S^{n-1}$  is compact and  $S^n$  is Hausdorff.

□

## 13.27 Box Topology

**Definition 13.27.1** (Box Topology). Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. The *box topology* on  $X = \prod_{i \in I} X_i$  is the topology generated by the basis  $\mathcal{B} = \{\prod_{i \in I} U_i : \{U_i\}_{i \in I} \text{ is a family with each } U_i \text{ an open set in } X_i\}$ .

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ .  $\bigcup \mathcal{B} = X$

PROOF: Since  $\prod_{i \in I} X_i \in \mathcal{B}$ .

$\langle 1 \rangle 2$ . For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

$\langle 2 \rangle 1$ . LET:  $B_1, B_2 \in \mathcal{B}$

$\langle 2 \rangle 2$ . LET:  $x \in B_1 \cap B_2$

$\langle 2 \rangle 3$ . PICK a family  $\{U_i\}_{i \in I}$  such that  $B_1 = \prod_{i \in I} U_i$ .

$\langle 2 \rangle 4$ . PICK a family  $\{V_i\}_{i \in I}$  such that  $B_2 = \prod_{i \in I} V_i$ .

$\langle 2 \rangle 5$ . LET:  $B_3 = \prod_{i \in I} (U_i \cap V_i)$

$\langle 2 \rangle 6$ .  $x \in B_3 \subseteq B_1 \cap B_2$

□

**Proposition 13.27.2.** *The box topology is finer than the product topology.*

PROOF: Immediate from definitions. □

**Proposition 13.27.3.** *On a finite family of topological spaces, the box topology and the product topology are the same.*

PROOF: Immediate from definitions. □

**Proposition 13.27.4.** *The box topology is strictly finer than the product topology on the Hilbert cube.*

PROOF: The set  $\prod_{n=0}^{\infty} (0, 1/(n+1)^2)$  is open in the box topology but not in the product topology. □

### 13.27.1 Bases

**Proposition 13.27.5.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. For all  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $X_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i : \forall i \in I, B_i \in \mathcal{B}_i\}$  is a basis for the box topology on  $\prod_{i \in I} X_i$ .*

PROOF:

$\langle 1 \rangle 1$ . For every family  $\{B_i\}_{i \in I}$  where  $\forall i \in I, B_i \in \mathcal{B}_i$ , we have  $\prod_{i \in I} B_i$  is open in the box topology.

PROOF: Since each  $B_i$  is open in  $X_i$ .

$\langle 1 \rangle 2$ . For any open set  $U$  in the box topology and  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .

$\langle 2 \rangle 1$ . LET:  $U$  be a set open in the box topology.

$\langle 2 \rangle 2$ . LET:  $x \in U$

$\langle 2 \rangle 3$ . PICK a family  $\{U_i\}_{i \in I}$  where each  $U_i$  is open in  $X_i$  such that  $x \in \prod_{i \in I} U_i \subseteq U$

$\langle 2 \rangle 4$ . For  $i \in I$ , choose  $B_i \in \mathcal{B}_i$  such that  $x_i \in B_i \subseteq U_i$ .

$\langle 2 \rangle 5$ .  $\prod_{i \in I} B_i \in \mathcal{B}$

$\langle 2 \rangle 6$ .  $x \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} U_i \subseteq U$

□

### 13.27.2 Subspaces

**Proposition 13.27.6.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $Y_i$  be a subspace of  $X_i$  for all  $i \in I$ . Then the box topology on  $\prod_{i \in I} Y_i$  is the same as the subspace topology that  $\prod_{i \in I} Y_i$  inherits as a subspace of  $\prod_{i \in I} X_i$  under the box topology.*

PROOF: A basis for the box topology is

$$\begin{aligned} & \left\{ \prod_{i \in I} V_i : V_i \text{ open in } Y_i \right\} \\ &= \left\{ \prod_{i \in I} (U_i \cap Y_i) : U_i \text{ open in } X_i \right\} \\ &= \left\{ \prod_{i \in I} U_i \cap \prod_{i \in I} Y_i : U_i \text{ open in } X_i \right\} \end{aligned}$$

which is a basis for the subspace topology by Proposition 13.7.13. □

### 13.27.3 Closure

**Proposition 13.27.7.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Give  $\prod_{i \in I} X_i$  the box topology. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

$\langle 1 \rangle 1$ .  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

$\langle 2 \rangle 1$ . LET:  $x \in \prod_{i \in I} \overline{A_i}$

$\langle 2 \rangle 2$ . For any family  $\{U_i\}_{i \in I}$  where each  $U_i$  is open in  $X_i$ , if  $x \in \prod_{i \in I} U_i$  then  $\prod_{i \in I} U_i$  intersects  $\prod_{i \in I} A_i$ .

$\langle 3 \rangle 1$ . LET:  $\{U_i\}_{i \in I}$  be a family where each  $U_i$  is open in  $X_i$ .

$\langle 3 \rangle 2$ . ASSUME:  $x \in \prod_{i \in I} \overline{A_i}$

$\langle 3 \rangle 3$ . For all  $i \in I$  we have  $U_i$  intersects  $A_i$

PROOF: Since  $\pi_i(x) \in \overline{A_i}$  and  $U_i$  is a neighbourhood of  $\pi_i(x)$ .

$\langle 3 \rangle 4$ .  $\prod_{i \in I} U_i$  intersects  $\prod_{i \in I} A_i$

$\langle 2 \rangle 3.$   $x \in \overline{\prod_{i \in I} A_i}$   
 PROOF: Proposition 13.13.9.  
 $\langle 1 \rangle 2.$   $\prod_{i \in I} A_i \subseteq \overline{\prod_{i \in I} A_i}$   
 $\langle 2 \rangle 1.$  LET:  $x \in \overline{\prod_{i \in I} A_i}$   
 $\langle 2 \rangle 2.$  LET:  $i \in I$   
 PROVE:  $\pi_i(x) \in \overline{A_i}$   
 $\langle 2 \rangle 3.$  LET:  $U$  be a neighbourhood of  $\pi_i(x)$  in  $X_i$   
 $\langle 2 \rangle 4.$   $\pi_i^{-1}(U)$  is a neighbourhood of  $x$  in  $\prod_{i \in I} X_i$   
 $\langle 2 \rangle 5.$  PICK  $y \in \pi_i^{-1}(U) \cap \prod_{i \in I} A_i$   
 $\langle 2 \rangle 6.$   $\pi_i(y) \in U \cap A_i$   
 $\square$

## 13.28 Separations

**Definition 13.28.1** (Separation). Let  $X$  be a topological space. A *separation* of  $X$  is a pair  $(U, V)$  of disjoint nonempty oped subsets in  $X$  such that  $U \cup V = X$ .

### Subspaces

**Proposition 13.28.2.** Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Then a separation of  $Y$  is a pair  $(A, B)$  of disjoint nonempty subsets of  $Y$ , neither of which contains a limit point of the other, such that  $A \cup B = Y$ .

PROOF: Since the following are equivalent:

- Neither of  $A$  and  $B$  contains a limit point of the other.
- $A$  contains all its own limit points in  $Y$ , and  $B$  contains all its own limit points in  $Y$ .
- $A$  and  $B$  are closed in  $Y$ .

$\square$

## 13.29 Connected Spaces

**Definition 13.29.1** (Connected). A topological space is *connected* iff it has no separation.

### 13.29.1 The Real Numbers

**Example 13.29.2.** The space  $\mathbb{R}_l$  is disconnected. The sets  $(-\infty, 0)$  and  $[0, +\infty)$  form a separation.

### 13.29.2 The Indiscrete Topology

**Example 13.29.3.** Any indiscrete space is connected.

### 13.29.3 The Cofinite Topology

**Example 13.29.4.** Any infinite set under the cofinite topology is connected.

PROOF:

⟨1⟩1. LET:  $X$  be an infinite set under the cofinite topology.

⟨1⟩2. ASSUME: for a contradiction  $(C, D)$  is a separation of  $X$ .

⟨1⟩3.  $X = (X - C) \cup (X - D) \cup (C \cap D)$

⟨1⟩4. Q.E.D.

PROOF: This is a contradiction since  $X$  is infinite,  $X - C$  and  $X - D$  are finite, and  $C \cap D = \emptyset$ .

□

**Example 13.29.5.** The rationals are disconnected. For any irrational  $a$ , we have  $(-\infty, a) \cap \mathbb{Q}$  and  $(a, +\infty) \cap \mathbb{Q}$  form a separation of  $\mathbb{Q}$ .

**Example 13.29.6.**  $\mathbb{R}^\omega$  under the box topology is not connected. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 13.29.7.** *A topological space  $X$  is connected if and only if the only sets that are both open and closed are  $\emptyset$  and  $X$ .*

PROOF: Since  $(U, V)$  is a separation of  $X$  iff  $U$  is both open and closed and  $V = X - U$ . □

### 13.29.4 Finer and Coarser

**Proposition 13.29.8.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the same set  $X$ . Assume  $\mathcal{T} \subseteq \mathcal{T}'$ . If  $\mathcal{T}'$  is connected then  $\mathcal{T}$  is connected.*

PROOF: If  $(C, D)$  is a separation of  $(X, \mathcal{T})$  then it is a separation of  $(X, \mathcal{T}')$ . □

### 13.29.5 Boundary

**Proposition 13.29.9.** *Let  $X$  be a topological space. Let  $A \subseteq X$ . Let  $C$  be a connected subspace of  $X$ . If  $C$  intersects  $A$  and  $X - A$  then  $C$  intersects  $\partial A$ .*

PROOF: Otherwise  $(C \cap \overline{A}, C \cap \overline{X - A})$  would be a separation of  $C$ . □

### 13.29.6 Continuous Functions

**Proposition 13.29.10.** *The continuous image of a connected space is connected.*

PROOF:

⟨1⟩1. LET:  $X$  and  $Y$  be topological spaces.

⟨1⟩2. LET:  $f : X \rightarrow Y$  be a surjective continuous function.

⟨1⟩3. LET:  $(C, D)$  be a separation of  $Y$ .

⟨1⟩4.  $(f^{-1}(C), f^{-1}(D))$  is a separation of  $X$ .

□

### 13.29.7 Subspaces

**Proposition 13.29.11.** *Let  $X$  be a topological space. Let  $(C, D)$  be a separation of  $X$ . Let  $Y$  be a connected subspace of  $X$ . Then either  $Y \subseteq C$  or  $Y \subseteq D$ .*

PROOF: Otherwise  $(Y \cap C, Y \cap D)$  would be a separation of  $Y$ .  $\square$

**Proposition 13.29.12.** *Let  $X$  be a topological space. Let  $\mathcal{A}$  be a set of connected subspaces of  $X$  and  $B$  a connected subspace of  $X$ . Assume that, for all  $A \in \mathcal{A}$ , we have  $A \cap B \neq \emptyset$ . Then  $\bigcup \mathcal{A} \cup B$  is connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $(C, D)$  is a separation of  $\bigcup \mathcal{A} \cup B$ .

$\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $B \subseteq C$

PROOF: Proposition 13.29.11.

$\langle 1 \rangle 3$ . For all  $A \in \mathcal{A}$  we have  $A \subseteq C$

PROOF: Proposition 13.29.11.

$\langle 1 \rangle 4$ .  $D = \emptyset$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This is a contradiction.

$\square$

**Proposition 13.29.13.** *Let  $X$  be a topological space. Let  $A$  be a connected subspace of  $X$ . Let  $B$  be a subspace of  $X$ . If  $A \subseteq B \subseteq \overline{A}$  then  $B$  is connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $(C, D)$  is a separation of  $B$ .

$\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $A \subseteq C$

PROOF: Proposition 13.29.11.

$\langle 1 \rangle 3$ .  $\overline{A} \subseteq \overline{C}$

$\langle 1 \rangle 4$ .  $\overline{C} \cap D = \emptyset$

$\langle 1 \rangle 5$ .  $B \cap D = \emptyset$

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This is a contradiction.

$\square$

**Corollary 13.29.13.1.** *The topologist's sine curve is connected.*

PROOF: The set  $\{(x, \sin 1/x) : 0 < x \leq 1\}$  is connected, since it is the continuous image of the connected set  $(0, 1]$ . The topologist's sine curve is its closure, hence connected by Proposition 13.29.13.  $\square$

**Proposition 13.29.14.** *Let  $X$  be a topological space. Let  $(A_n)$  be a sequence of connected subspaces of  $X$  such that, for all  $n$ , we have  $A_n \cap A_{n+1} \neq \emptyset$ . Then  $\bigcup_n A_n$  is connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $(C, D)$  is a separation of  $\bigcup_n A_n$

$\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $A_0 \subseteq C$

PROOF: Proposition 13.29.11.

- ⟨1⟩3.  $\forall n. A_n \subseteq C$
- ⟨2⟩1. ASSUME: as induction hypothesis  $A_n \subseteq C$
- ⟨2⟩2. PICK  $x \in A_n \cap A_{n+1}$
- ⟨2⟩3.  $x \in C$
- ⟨2⟩4.  $A_{n+1} \subseteq C$

PROOF: Proposition 13.29.11.

- ⟨1⟩4.  $\bigcup_n A_n \subseteq C$
- ⟨1⟩5. Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 13.29.15.** *Let  $X$  be a connected topological space. Let  $Y \subseteq X$  be connected. Let  $(A, B)$  be a separation of  $X - Y$ . Then  $Y \cup A$  and  $Y \cup B$  are connected.*

PROOF:

- ⟨1⟩1.  $Y \cup A$  is connected.
- ⟨2⟩1. ASSUME: for a contradiction  $(C, D)$  is a separation of  $Y \cup A$
- ⟨2⟩2. ASSUME: w.l.o.g.  $Y \subseteq C$
- ⟨2⟩3. PICK  $C'$  and  $D'$  open in  $X$  such that  $C = C' \cap (Y \cup A)$  and  $D = D' \cap (Y \cup A)$
- ⟨2⟩4.  $D = D' \cap A$
- ⟨2⟩5.  $C' \cap D' \cap A = \emptyset$
- ⟨2⟩6.  $A \subseteq C' \cup D'$
- ⟨2⟩7. PICK  $A'$  and  $B'$  open in  $X$  such that  $A = A' - Y$  and  $B = B' - Y$
- ⟨2⟩8.  $A' \cap B' \subseteq Y$
- ⟨2⟩9.  $X - Y \subseteq A' \cup B'$
- ⟨2⟩10.  $A' \subseteq C' \cup D'$
- ⟨2⟩11.  $(D' \cap A', B' \cup C')$  is a separation of  $X$ .
- ⟨1⟩2.  $Y \cup B$  is connected.

PROOF: Similar.

□

### 13.29.8 Order Topology

**Proposition 13.29.16.** *Let  $L$  be a linearly ordered set under the order topology. Then  $L$  is connected if and only if  $X$  is a linear continuum.*

PROOF:

- ⟨1⟩1. If  $L$  is a linear continuum then  $L$  is connected.
- ⟨2⟩1. LET:  $L$  be a linear continuum.
- ⟨2⟩2. ASSUME: for a contradiction  $(A, B)$  is a separation of  $L$ .
- ⟨2⟩3. PICK  $a \in A$  and  $b \in B$ .
- ⟨2⟩4. ASSUME: w.l.o.g.  $a < b$
- ⟨2⟩5. LET:  $c = \sup\{x \in A : x < b\}$
- ⟨2⟩6.  $c \notin A$

- ⟨3⟩1. ASSUME: for a contradiction  $c \in A$ .
- ⟨3⟩2. PICK  $e > c$  such that  $[c, e] \subseteq A$ .
- ⟨3⟩3. PICK  $z$  such that  $c < z < e$ .
- ⟨3⟩4.  $z \in A$
- ⟨3⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩5.

- ⟨2⟩7.  $c \notin B$

- ⟨3⟩1. ASSUME: for a contradiction  $c \in B$ .
- ⟨3⟩2. PICK  $d < c$  such that  $(d, c] \subseteq B$ .
- ⟨3⟩3. PICK  $z$  such that  $d < z < c$
- ⟨3⟩4.  $z$  is an upper bound for  $\{x \in A : x < b\}$
- ⟨3⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩5.

- ⟨2⟩8. Q.E.D.

PROOF: This is a contradiction.

- ⟨1⟩2. If  $L$  is connected then  $L$  is a linear continuum.

- ⟨2⟩1. ASSUME:  $L$  is connected.

- ⟨2⟩2.  $L$  is dense.

- ⟨3⟩1. LET:  $a, b \in L$  with  $a < b$ .
- ⟨3⟩2. ASSUME: for a contradiction there is no  $c$  such that  $a < c < b$ .
- ⟨3⟩3.  $((-\infty, b), (a, +\infty))$  is a separation of  $L$ .

- ⟨2⟩3.  $L$  has the least upper bound property.

- ⟨3⟩1. ASSUME: for a contradiction  $S \subseteq L$  is a nonempty set bounded above with no least upper bound.

- ⟨3⟩2. LET:  $S \uparrow$  be the set of upper bounds for  $S$ .

- ⟨3⟩3. LET:  $S \uparrow \downarrow$  be the set of lower bounds for  $S \uparrow$ .

PROVE:  $(S \uparrow \downarrow, S \uparrow)$  is a separation of  $L$ .

- ⟨3⟩4.  $S \uparrow \neq \emptyset$

PROOF: Since  $S$  is bounded above.

- ⟨3⟩5.  $S \uparrow \downarrow \neq \emptyset$

PROOF: Since  $\emptyset \neq S \subseteq S \uparrow \downarrow$ .

- ⟨3⟩6.  $S \uparrow$  is open.

- ⟨4⟩1. LET:  $u \in S \uparrow$

- ⟨4⟩2. PICK  $v \in S \uparrow$  such that  $v < u$

PROOF: Since  $u$  is not the least upper bound for  $S$ .

- ⟨4⟩3.  $u \in (v, +\infty) \subseteq S \uparrow$

- ⟨3⟩7.  $S \uparrow \downarrow$  is open.

- ⟨4⟩1. LET:  $l \in S \uparrow \downarrow$

- ⟨4⟩2.  $l \notin S \uparrow$

PROOF: Since  $l$  is not the least upper bound for  $S$ .

- ⟨4⟩3. PICK  $s \in S$  such that  $l < s$

- ⟨4⟩4.  $l \in (-\infty, s) \subseteq S \uparrow \downarrow$

- ⟨3⟩8.  $S \uparrow \cap S \uparrow \downarrow \neq \emptyset$

PROOF: An element of both would be a least upper bound for  $S$ .

- ⟨3⟩9.  $S \uparrow \cup S \uparrow \downarrow = L$

- ⟨4⟩1. LET:  $x \in L$



- $\langle 4 \rangle 2$ . ASSUME:  $x \notin S \uparrow$
- $\langle 4 \rangle 3$ . There exists  $s \in S$  such that  $x < s$ .
- $\langle 4 \rangle 4$ .  $\forall u \in S \uparrow . x < u$
- $\langle 4 \rangle 5$ .  $x \in S \uparrow \downarrow$

□

**Theorem 13.29.17** (Intermediate Value Theorem). *Let  $X$  be a connected space. Let  $Y$  be a linearly ordered set under the order topology. Let  $f : X \rightarrow Y$  be continuous. Let  $a, b \in X$  and  $r \in Y$ . If  $f(a) < r < f(b)$ , then there exists  $c \in X$  such that  $f(c) = r$ .*

PROOF: Otherwise  $\{x \in X : f(x) < r\}$  and  $\{x \in X : f(x) > r\}$  would form a separation of  $X$ . □

**Corollary 13.29.17.1.** *Every continuous function  $[0, 1] \rightarrow [0, 1]$  has a fixed point.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $f : [0, 1] \rightarrow [0, 1]$  be continuous.
- $\langle 1 \rangle 2$ . LET:  $g : [0, 1] \rightarrow [-1, 1]$  be the function  $g(x) = f(x) - x$ .
- $\langle 1 \rangle 3$ .  $g(0) \geq 0$
- $\langle 1 \rangle 4$ .  $g(1) \leq 0$
- $\langle 1 \rangle 5$ . There exists  $x \in [0, 1]$  such that  $g(x) = 0$ .

PROOF: Intermediate Value Theorem.

- $\langle 1 \rangle 6$ . There exists  $x \in [0, 1]$  such that  $f(x) = x$ .

□

### 13.29.9 Product Topology

**Proposition 13.29.18.** *The product of a family of connected spaces is connected.*

PROOF:

- $\langle 1 \rangle 1$ . The product of two connected spaces is connected.

PROOF:

- $\langle 2 \rangle 1$ . LET:  $X$  and  $Y$  be connected topological spaces.
- $\langle 2 \rangle 2$ . ASSUME: w.l.o.g.  $X$  and  $Y$  are nonempty.
- $\langle 2 \rangle 3$ . PICK  $(a, b) \in X \times Y$
- $\langle 2 \rangle 4$ .  $X \times \{b\}$  is connected.

PROOF: It is homeomorphic to  $X$ .

- $\langle 2 \rangle 5$ . For all  $x \in X$  we have  $\{x\} \times Y$  is connected.

PROOF: It is homeomorphic to  $Y$ .

- $\langle 2 \rangle 6$ . For all  $x \in X$  we have  $(X \times \{b\}) \cup (\{x\} \times Y)$  is connected.

PROOF: Proposition 13.29.12.

- $\langle 2 \rangle 7$ .  $X \cup Y$  is connected.

PROOF: Proposition 13.29.12 since  $X \cup Y = \bigcup_{x \in X} ((X \times \{b\}) \cup (\{x\} \times Y))$  and the subspaces all have the point  $(a, b)$  in common.

- $\langle 1 \rangle 2$ . LET:  $\{X_i\}_{i \in I}$  be a family of connected spaces.

- <1>3. LET:  $X = \prod_{i \in I} X_i$   
 <1>4. ASSUME: w.l.o.g. each  $X_i$  is nonempty.  
 <1>5. PICK  $a \in X$   
 <1>6. For every finite  $K \subseteq I$ ,  
     LET:  $X_K = \{x \in X : \forall i \notin K. \pi_i(x) = \pi_i(a)\}$   
 <1>7. For every finite  $K \subseteq I$ , we have  $X_K$  is connected.  
     PROOF: It is homeomorphic to  $\prod_{i \in K} X_i$  which is connected by <1>1.  
 <1>8. LET:  $Y = \bigcup_K$  a finite subset of  $I$   $X_K$   
 <1>9.  $Y$  is connected.  
     PROOF: Proposition 13.29.12 since  $a \in X_K$  for all  $K$ .  
 <1>10.  $X = \overline{Y}$   
     <2>1. LET:  $x \in X$   
     <2>2. LET:  $U$  be a neighbourhood of  $x$ .  
         PROVE:  $U$  intersects  $Y$ .  
     <2>3. PICK a finite subset  $K$  of  $I$  and  $U_i$  open in each  $X_i$  such that  $U_i = X_i$   
         for all  $i \notin K$ , and  $x \in \prod_i U_i \subseteq U$   
     <2>4. LET:  $y \in X$  be the point with  $\pi_i(y) = \pi_i(x)$  for  $i \in K$  and  $\pi_i(y) = \pi_i(a)$   
         for  $i \notin K$   
     <2>5.  $y \in U \cap Y$   
 <1>11.  $X$  is connected.  
     PROOF: Proposition 13.29.13.  
 □

**Proposition 13.29.19.** *Let  $X$  and  $Y$  be topological spaces. Let  $A$  be a proper subset of  $X$  and  $B$  a proper subset of  $Y$ . Then  $(X \times Y) - (A \times B)$  is connected.*

PROOF:

- <1>1. PICK  $x_0 \in X - A$   
 <1>2. PICK  $y_0 \in Y - B$   
 <1>3. LET:  $C = ((X - A) \times Y) \cup (X \times \{y_0\})$   
 <1>4. LET:  $D = (\{x_0\} \times Y) \cup (X \times (Y - B))$   
 <1>5.  $C$  is connected.  
     <2>1.  $C = \bigcup_{x \in X - A} (\{x\} \times Y) \cup (X \times \{y_0\})$   
     <2>2. For all  $x \in X - A$  we have  $\{x\} \times Y$  is connected.  
         PROOF: It is homeomorphic to  $Y$ .  
     <2>3.  $X \times \{y_0\}$  is connected.  
         PROOF: It is homeomorphic to  $X$ .  
     <2>4. For all  $x \in X - A$  we have  $(x, y_0) \in (\{x\} \times Y) \cap (X \times \{y_0\})$   
     <2>5.  $C$  is connected.  
     PROOF: Proposition 13.29.12.  
 <1>6.  $D$  is connected.  
     PROOF: Similar.  
 <1>7.  $(X \times Y) - (A \times B) = C \cup D$   
 <1>8.  $(X \times Y) - (A \times B)$  is connected.  
     PROOF: Proposition 13.29.12 since  $(x_0, y_0) \in C \cap D$ .  
 □

### 13.29.10 Quotient Spaces

**Proposition 13.29.20.** *A quotient of a connected space is connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $p : X \rightarrow Y$  be a quotient map.

$\langle 1 \rangle 2$ . If  $(C, D)$  is a separation of  $Y$  then  $(p^{-1}(C), p^{-1}(D))$  is a separation of  $X$ .

□

**Proposition 13.29.21.** *Let  $p : X \rightarrow Y$  be a quotient map. Assume that  $Y$  is connected, for all  $y \in Y$ , we have  $p^{-1}(y)$  is connected. Then  $X$  is connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $(A, B)$  is a separation of  $X$ .

$\langle 1 \rangle 2$ . For all  $y \in Y$ , either  $p^{-1}(y) \subseteq A$  or  $p^{-1}(y) \subseteq B$ .

$\langle 1 \rangle 3$ .  $(\{y \in Y : p^{-1}(y) \subseteq A\}, \{y \in Y : p^{-1}(y) \subseteq B\})$  form a separation of  $Y$ .

$\langle 1 \rangle 4$ . Q.E.D.

PROOF: This is a contradiction.

□

## 13.30 $T_1$ Spaces

**Definition 13.30.1** ( $T_1$ ). A topological space is  $T_1$  iff every one-point set is closed.

**Proposition 13.30.2.** *A topological space is  $T_1$  iff every finite set is closed.*

PROOF: Since the union of finitely many closed sets is closed. □

**Proposition 13.30.3.** *Let  $X$  be a topological space. Then  $X$  is  $T_1$  if and only if, for all  $x, y \in X$ , if  $x \neq y$  then there exists a neighbourhood of  $x$  that does not contain  $y$ , and there exists a neighbourhood of  $y$  that does not contain  $x$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $X$  is  $T_1$  then, for all  $x, y \in X$ , if  $x \neq y$  then there exists a neighbourhood of  $x$  that does not contain  $y$ , and there exists a neighbourhood of  $y$  that does not contain  $x$ .

$\langle 2 \rangle 1$ . ASSUME:  $X$  is  $T_1$ .

$\langle 2 \rangle 2$ . LET:  $x, y \in X$

$\langle 2 \rangle 3$ . ASSUME:  $x \neq y$

$\langle 2 \rangle 4$ .  $X - \{y\}$  is a neighbourhood of  $x$  that does not contain  $y$ .

$\langle 2 \rangle 5$ .  $X - \{x\}$  is a neighbourhood of  $y$  that does not contain  $x$ .

$\langle 1 \rangle 2$ . If, for all  $x, y \in X$ , if  $x \neq y$  then there exists a neighbourhood of  $x$  that does not contain  $y$ , and there exists a neighbourhood of  $y$  that does not contain  $x$ , then  $X$  is  $T_1$ .

$\langle 2 \rangle 1$ . ASSUME: For all  $x, y \in X$ , if  $x \neq y$  then there exists a neighbourhood of  $x$  that does not contain  $y$ , and there exists a neighbourhood of  $y$  that does not contain  $x$ .

- ⟨2⟩2. LET:  $x \in X$   
PROVE:  $\{x\}$  is closed.
- ⟨2⟩3. LET:  $y \in X - \{x\}$
- ⟨2⟩4. PICK a neighbourhood  $U$  of  $y$  that does not contain  $x$ .
- ⟨2⟩5.  $y \in U \subseteq X - \{x\}$

□

### 13.30.1 Limit Points

**Proposition 13.30.4.** *Let  $X$  be a  $T_1$  space. Let  $A \subseteq X$  and  $l \in X$ . Then  $l$  is a limit point of  $A$  if and only if every neighbourhood of  $l$  contains infinitely many points of  $A$ .*

PROOF:

- ⟨1⟩1. If  $l$  is a limit point of  $A$  then every neighbourhood of  $l$  contains infinitely many points of  $A$ .
- ⟨2⟩1. ASSUME:  $l$  is a limit point of  $A$ .
- ⟨2⟩2. LET:  $U$  be a neighbourhood of  $l$ .
- ⟨2⟩3. ASSUME: for a contradiction  $U \cap A - \{l\}$  is finite.
- ⟨2⟩4.  $U \cap A - \{l\}$  is closed.
- PROOF: Since  $X$  is  $T_1$ .
- ⟨2⟩5.  $U - (A - \{l\})$  is a neighbourhood of  $l$ .
- ⟨2⟩6.  $U - (A - \{l\})$  intersects  $A$ .
- ⟨2⟩7. Q.E.D.
- ⟨1⟩2. If every neighbourhood of  $l$  contains infinitely many points of  $A$  then  $l$  is a limit point of  $A$ .

PROOF: Immediate from definitions.

□

### 13.31 Hausdorff Spaces

**Definition 13.31.1** (Hausdorff). A topological space is a *Hausdorff* space or a  $T_2$  space iff any two distinct points have disjoint neighbourhoods.

**Proposition 13.31.2.** *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a Hausdorff space.
- ⟨1⟩2. LET:  $(a_n)$  be a sequence in  $X$  and  $l, m \in X$
- ⟨1⟩3. ASSUME:  $a_n \rightarrow l$  and  $a_n \rightarrow m$
- ⟨1⟩4. ASSUME: for a contradiction  $l \neq m$
- ⟨1⟩5. PICK disjoint open sets  $U$  and  $V$  with  $l \in U$  and  $m \in V$
- ⟨1⟩6. PICK  $M, N$  such that  $\forall n \geq M. a_n \in U$  and  $\forall n \geq N. a_n \in V$
- ⟨1⟩7.  $a_{\max(M, N)} \in U \cap V$
- ⟨1⟩8. Q.E.D.

PROOF: This contradicts the fact that  $U \cap V = \emptyset$ .

□

**Example 13.31.3.** We cannot weaken the hypothesis from being Hausdorff to being  $T_1$ .

In the cofinite topology on any infinite set, every sequence converges to every point.

**Proposition 13.31.4.** *Any linearly ordered set is Hausdorff under the order topology.*

PROOF:

⟨1⟩1. LET:  $X$  be a linearly ordered set under the order topology.

⟨1⟩2. LET:  $a, b \in X$  with  $a \neq b$ .

⟨1⟩3. ASSUME: w.l.o.g.  $a < b$ .

⟨1⟩4. CASE: There exists  $c \in X$  such that  $a < c < b$ .

⟨2⟩1. LET:  $U = (-\infty, c)$

⟨2⟩2. LET:  $V = (c, +\infty)$

⟨2⟩3.  $U$  and  $V$  are disjoint open sets with  $a \in U$  and  $b \in V$

⟨1⟩5. CASE: There is no  $c \in X$  such that  $a < c < b$ .

⟨2⟩1. LET:  $U = (-\infty, b)$

⟨2⟩2. LET:  $V = (a, +\infty)$

⟨2⟩3.  $U$  and  $V$  are disjoint open sets with  $a \in U$  and  $b \in V$

□

**Proposition 13.31.5.** *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $X$  be a Hausdorff space.

⟨1⟩2. LET:  $Y$  be a subspace of  $X$ .

⟨1⟩3. LET:  $a, b \in Y$  with  $a \neq b$ .

⟨1⟩4. PICK disjoint open sets  $U$  and  $V$  in  $X$  with  $a \in U$  and  $b \in V$ .

⟨1⟩5.  $U \cap Y$  and  $V \cap Y$  are disjoint open sets in  $Y$  with  $a \in U \cap Y$  and  $b \in V \cap Y$ .

□

**Proposition 13.31.6.** *The disjoint union of two Hausdorff spaces is Hausdorff.*

**Proposition 13.31.7.** *Let  $A$  be a topological space and  $B$  a Hausdorff space. Let  $f, g : A \rightarrow B$  be continuous. Let  $X \subseteq A$  be dense. If  $f$  and  $g$  agree on  $X$ , then  $f = g$ .*

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $a \in A$  and  $f(a) \neq g(a)$ .

⟨1⟩2. PICK disjoint neighbourhoods  $U$  and  $V$  of  $f(a)$  and  $g(a)$  respectively.

⟨1⟩3. PICK  $x \in f^{-1}(U) \cap g^{-1}(V)$

⟨1⟩4.  $f(x) = g(x) \in U \cap V$

⟨1⟩5. Q.E.D.

PROOF: This is a contradiction.

□

### 13.31.1 Product Topology

**Proposition 13.31.8.** *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.
  - $\langle 1 \rangle 2$ . LET:  $x, y \in \prod_{i \in I} X_i$  with  $x \neq y$ .
  - $\langle 1 \rangle 3$ . PICK  $i \in I$  such that  $\pi_i(x) \neq \pi_i(y)$
  - $\langle 1 \rangle 4$ . PICK disjoint open sets  $U$  and  $V$  in  $X_i$  such that  $\pi_i(x) \in U$  and  $\pi_i(y) \in V$ .
  - $\langle 1 \rangle 5$ .  $x \in \pi_i^{-1}(U)$  and  $y \in \pi_i^{-1}(V)$ .
- 

### 13.31.2 Box Topology

**Proposition 13.31.9.** *The box product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.
  - $\langle 1 \rangle 2$ . LET:  $x, y \in \prod_{i \in I} X_i$  with  $x \neq y$ .
  - $\langle 1 \rangle 3$ . PICK  $i \in I$  such that  $\pi_i(x) \neq \pi_i(y)$
  - $\langle 1 \rangle 4$ . PICK disjoint open sets  $U$  and  $V$  in  $X_i$  such that  $\pi_i(x) \in U$  and  $\pi_i(y) \in V$ .
  - $\langle 1 \rangle 5$ .  $x \in \pi_i^{-1}(U)$  and  $y \in \pi_i^{-1}(V)$ .
- 

### 13.31.3 $T_1$ Spaces

**Proposition 13.31.10.** *Every Hausdorff space is  $T_1$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X$  be a Hausdorff space.
  - $\langle 1 \rangle 2$ . LET:  $a \in X$
  - PROVE:  $X - \{a\}$  is open.
  - $\langle 1 \rangle 3$ . LET:  $x \in X - \{a\}$
  - $\langle 1 \rangle 4$ . PICK disjoint open sets  $U$  and  $V$  with  $a \in U$  and  $x \in V$
  - $\langle 1 \rangle 5$ .  $x \in V \subseteq X - U \subseteq X - \{a\}$
- 

**Example 13.31.11.** The converse does not hold. If  $X$  is an infinite set under the cofinite topology, then  $X$  is  $T_1$  but not Hausdorff.

**Proposition 13.31.12.** *Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$  be uniformly continuous. Let  $\hat{X}$  and  $\hat{Y}$  be the completions of  $X$  and  $Y$ . Then  $f$  extends uniquely to a continuous map  $\hat{X} \rightarrow \hat{Y}$ .*

PROOF: The extension maps  $\lim_{n \rightarrow \infty} x_n$  to  $\lim_{n \rightarrow \infty} f(x_n)$ . □

**Proposition 13.31.13.** *Let  $X$  be a topological space. Then  $X$  is Hausdorff if and only if the diagonal  $\Delta = \{(x, x) : x \in X\}$  is closed in  $X^2$ .*

PROOF:

$\Delta$  is closed

$\Leftrightarrow X^2 - \Delta$  is open

$\Leftrightarrow \forall x, y \in X ((x, y) \notin \Delta \Rightarrow \exists V, W \text{ open in } X (x \in V \wedge y \in W \wedge V \times W \subseteq X^2 - \Delta))$

$\Leftrightarrow \forall x, y \in X (x \neq y \Rightarrow \exists V, W \text{ open in } X (x \in V \wedge y \in W \wedge V \cap W = \emptyset))$

$\Leftrightarrow X$  is Hausdorff □

## 13.32 Separable Spaces

**Definition 13.32.1** (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

## 13.33 Sequential Compactness

**Definition 13.33.1** (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

## 13.34 Compactness

**Definition 13.34.1** (Compact). A topological space is *compact* iff every open cover has a finite subcover.

**Proposition 13.34.2.** *Let  $X$  be a compact topological space. Let  $P$  be a set of open sets such that, for all  $U, V \in P$ , we have  $U \cup V \in P$ . Assume that every point has an open neighbourhood in  $P$ . Then  $X \in P$ .*

PROOF:

$\langle 1 \rangle 1$ .  $P$  is an open cover of  $X$

$\langle 1 \rangle 2$ . PICK a finite subcover  $U_1, \dots, U_n \in P$

$\langle 1 \rangle 3$ .  $X = U_1 \cup \dots \cup U_n \in P$

□

**Corollary 13.34.2.1.** *Let  $f$  be a compact space and  $f : X \rightarrow \mathbb{R}$  be locally bounded. Then  $f$  is bounded.*

PROOF: Take  $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$ . □

**Proposition 13.34.3.** *The continuous image of a compact space is compact.*

**Proposition 13.34.4.** *A closed subspace of a compact space is compact.*

**Proposition 13.34.5.** *Let  $X$  and  $Y$  be nonempty spaces. Then the following are equivalent.*

1.  $X$  and  $Y$  are compact.
2.  $X + Y$  is compact.
3.  $X \times Y$  is compact.

**Proposition 13.34.6.** *A compact subspace of a Hausdorff space is closed.*

**Proposition 13.34.7.** *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

**Proposition 13.34.8.** *A first countable compact space is sequentially compact.*

### 13.35 Gluing

**Definition 13.35.1** (Gluing). Let  $X$  and  $Y$  be topological spaces,  $X_0 \subseteq X$  and  $\phi : X_0 \rightarrow Y$  a continuous map. Then  $Y \cup_\phi X$  is the quotient space  $(X + Y)/\sim$ , where  $\sim$  is the equivalence relation generated by  $x \sim \phi(x)$  for all  $x \in X_0$ .

**Proposition 13.35.2.**  *$Y$  is a subspace of  $Y \cup_\phi X$ .*

**Definition 13.35.3.** Let  $X$  be a topological space and  $\alpha : X \cong X$  a homeomorphism. Then  $(X \times [0, 1])/\alpha$  is the quotient space of  $X \times [0, 1]$  by the equivalence relation generated by  $(x, 0) \sim (\alpha(x), 1)$  for all  $x \in X$ .

**Definition 13.35.4** (Möbius Strip). The *Möbius strip* is  $([-1, 1] \times [0, 1])/\alpha$  where  $\alpha(x) = -x$ .

**Definition 13.35.5** (Klein Bottle). The *Klein bottle* is  $(S^1 \times [0, 1])/\alpha$  where  $\alpha(z) = \bar{z}$ .

**Proposition 13.35.6.** *Let  $M$  be the Möbius strip and  $K$  the Klein bottle. Then  $M \cup_{\text{id}_{\partial M}} M \cong K$ .*

PROOF:

$\langle 1 \rangle$ 1. LET:  $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$  be the function that maps  $\kappa_1(\theta, t)$  to  $(e^{\pi i \theta/2}, t)$  and  $\kappa_2(\theta, t)$  to  $(-e^{-\pi i \theta/2}, t)$ .

$\langle 1 \rangle$ 2.  $f$  induces a bijection  $M \cup_{\text{id}_{\partial M}} M \approx K$

$\langle 1 \rangle$ 3.  $f$  is a homeomorphism.

□

### 13.36 Homogeneous Spaces

**Definition 13.36.1** (Homogeneous). A topological space  $X$  is *homogeneous* iff, for any  $x, y \in X$ , there exists a homeomorphism  $f : X \cong X$  such that  $f(x) = y$ .

### 13.37 Regular Spaces

**Definition 13.37.1** (Regular). A topological space  $X$  is *regular* iff it is  $T_1$  and, for every closed set  $A$  and point  $x \notin A$ , there exist disjoint open sets  $U$  and  $V$  with  $A \subseteq U$  and  $x \in V$ .



## 13.38 Totally Disconnected Spaces

**Definition 13.38.1** (Totally Disconnected). A topological space  $X$  is *totally disconnected* iff the only connected subspaces are the one-point subspaces.

**Example 13.38.2.** Every discrete space is totally disconnected.

**Example 13.38.3.** The rationals are totally disconnected.

## 13.39 Path Connected Spaces

**Definition 13.39.1** (Path-connected). A topological space  $X$  is *path-connected* iff, for any points  $a, b \in X$ , there exists a continuous function  $\alpha : [0, 1] \rightarrow X$ , called a *path*, such that  $\alpha(0) = a$  and  $\alpha(1) = b$ .

### 13.39.1 The Ordered Square

**Proposition 13.39.2.** *The ordered square is not path connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $p : [a, b] \rightarrow I_o^2$  is a path from  $(0, 0)$  to  $(1, 1)$ .

$\langle 1 \rangle 2$ .  $p$  is surjective.

PROOF: Intermediate Value Theorem.

$\langle 1 \rangle 3$ . For all  $x \in [0, 1]$ , the set  $p^{-1}(\{x\} \times (0, 1))$  is a nonempty open set in  $[0, 1]$ .

$\langle 1 \rangle 4$ . For all  $x \in [0, 1]$  choose a rational  $q_x \in p^{-1}(\{x\} \times (0, 1))$ .

$\langle 1 \rangle 5$ . The mapping that maps  $x$  to  $q_x$  is an injective function  $[0, 1] \rightarrow \mathbb{Q}$

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts the fact that  $[0, 1]$  is uncountable and  $\mathbb{Q}$  is countable.

□

### 13.39.2 Punctured Euclidean Space

**Proposition 13.39.3.** *For  $n > 1$ , the punctured Euclidean space  $\mathbb{R}^n - \{0\}$  is path connected.*

PROOF: Given points  $x$  and  $y$ , take the straight line from  $x$  to  $y$  if this does not pass through 0. Otherwise pick a point  $z$  not on this line, and take the two straight lines from  $x$  to  $z$  then from  $z$  to  $y$ . □

### 13.39.3 The Topologist's Sine Curve

**Proposition 13.39.4.** *The topologist's sine curve is not path connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $S = \{(x, \sin 1/x) : 0 < x \leq 1\}$

$\langle 1 \rangle 2$ . ASSUME: for a contradiction  $p : [0, 1] \rightarrow \bar{S}$  is a path from  $(0, 0)$  to  $(1, \sin 1)$ .

- <1>3. LET:  $b$  be the largest element of  $p^{-1}(\{0\} \times [-1, 1])$   
 <1>4. For  $n$  a positive integer, choose  $t_n$  such that  $b < t_n < ((n-1)b + 1)/n$   
 and  $\pi_2(p(t_i)) = (-1)^n$   
 <1>5.  $t_n \rightarrow b$  as  $n \rightarrow \infty$   
 <1>6.  $(p(t_n))$  does not converge.  
 <1>7. Q.E.D.

PROOF: This is a contradiction.

□

### 13.39.4 The Long Line

**Proposition 13.39.5.** *The long line is path connected.*

PROOF:

- <1>1. LET:  $L = S_\Omega \times [0, 1)$  be the long line.  
 <1>2. LET:  $(a, b), (c, d) \in L$   
 <1>3. PICK  $e$  such that  $a < e$  and  $c < e$   
 <1>4.  $(a, b), (c, d) \in [(0, 0), (e, 0)) \cong [0, 1)$

PROOF: Using Proposition 6.5.2.

- <1>5. There is a path from  $(a, b)$  to  $(c, d)$ .

□

### 13.39.5 Continuous Functions

**Proposition 13.39.6.** *The continuous image of a path connected space is path connected.*

PROOF:

- <1>1. LET:  $X$  be a path connected space and  $Y$  a topological space.  
 <1>2. LET:  $f : X \rightarrow Y$  be a surjective continuous function.  
 PROVE:  $Y$  is path connected.  
 <1>3. LET:  $a, b \in Y$   
 <1>4. PICK  $x, y \in X$  with  $f(x) = a$  and  $f(y) = b$ .  
 <1>5. PICK a path  $p : [0, 1] \rightarrow X$  from  $x$  to  $y$ .  
 <1>6.  $f \circ p$  is a path from  $a$  to  $b$ .

□

### 13.39.6 Subspaces

**Proposition 13.39.7.** *Let  $\{X\}$  be a topological space. Let  $\mathcal{A}$  be a set of connected subspaces of  $X$ . If  $\bigcap \mathcal{A} \neq \emptyset$  then  $\bigcup \mathcal{A}$  is connected.*

PROOF:

- <1>1. PICK  $a \in \bigcap \mathcal{A}$   
 <1>2. PICK  $x, y \in \bigcup \mathcal{A}$   
 <1>3. PICK  $A, B \in \mathcal{A}$  with  $x \in A$  and  $y \in B$ .  
 <1>4. PICK a path  $p$  from  $x$  to  $a$  in  $A$ , and a path  $q$  from  $a$  to  $y$  in  $B$ .

⟨1⟩5. The concatenation of  $p$  and  $q$  is a path from  $x$  to  $y$  in  $\bigcup \mathcal{A}$ .

□

**Proposition 13.39.8.** *A quotient of a path connected space is path connected.*

### 13.39.7 Product Topology

**Proposition 13.39.9.** *The product of a family of path connected spaces is path connected.*

PROOF:

⟨1⟩1. LET:  $\{X_i\}_{i \in I}$  be a family of path connected spaces.

⟨1⟩2. LET:  $x, y \in \prod_{i \in I} X_i$

⟨1⟩3. For  $i \in I$ , PICK a path  $p_i : [0, 1] \rightarrow X_i$  from  $\pi_i(x)$  to  $\pi_i(y)$

⟨1⟩4.  $\lambda t \in [0, 1]. \lambda i \in I. p_i(t)$  is a path from  $x$  to  $y$  in  $\prod_{i \in I} X_i$ .

□

**Proposition 13.39.10.** *Let  $A \subseteq \mathbb{R}^2$ . If  $A$  is countable then  $\mathbb{R}^2 - A$  is path connected.*

PROOF:

⟨1⟩1. LET:  $x, y \in \mathbb{R}^2 - A$

⟨1⟩2. PICK two non-parallel lines  $L$  through  $x$  and  $L'$  through  $y$  that do not pass through any points in  $A$ .

PROOF: These exist since uncountably many lines pass through any point.

⟨1⟩3. There exists a path from  $x$  to  $y$  that follows  $L$  from  $x$  to the point of intersection of  $L$  and  $L'$ , and then follows  $L'$  to  $y$ .

□

### 13.39.8 Connected Spaces

**Proposition 13.39.11.** *Every path connected space is connected.*

PROOF:

⟨1⟩1. LET:  $X$  be a path connected space.

⟨1⟩2. ASSUME: for a contradiction  $(A, B)$  is a separation of  $X$ .

⟨1⟩3. PICK  $a \in A$  and  $b \in B$

⟨1⟩4. PICK a path  $p : [0, 1] \rightarrow X$  from  $a$  to  $b$ .

⟨1⟩5.  $(p^{-1}(A), p^{-1}(B))$  is a separation of  $[0, 1]$ .

⟨1⟩6. Q.E.D.

PROOF: This contradicts Proposition 13.29.16.

□

**Corollary 13.39.11.1.** *For  $n > 1$ , we have  $\mathbb{R}^n$  and  $\mathbb{R}$  are not homeomorphic.*

PROOF: Removing a point from  $\mathbb{R}$  gives a disconnected space. □

**Proposition 13.39.12.** *Every open connected subspace of  $\mathbb{R}^2$  is path connected.*

PROOF:

- <1>1. LET:  $U$  be an open connected subspace of  $\mathbb{R}^2$ .  
 <1>2. ASSUME: w.l.o.g.  $U \neq \emptyset$   
 <1>3. PICK  $x_0 \in U$   
 <1>4. LET:  $V = \{x \in U : \text{there exists a path from } x_0 \text{ to } x\}$   
 <1>5.  $V$  is open in  $U$ .  
     <2>1. LET:  $x \in V$   
     <2>2. PICK  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$   
     <2>3.  $B(x, \epsilon) \subseteq V$   
         PROOF: For all  $y \in B(x, \epsilon)$ , take a path from  $x_0$  to  $x$  and then a straight line from  $x$  to  $y$ .  
 <1>6.  $V$  is closed in  $U$ .  
     <2>1. LET:  $x \in U - V$   
     <2>2. PICK  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$   
     <2>3.  $B(x, \epsilon) \subseteq U - V$   
         <3>1. LET:  $y \in B(x, \epsilon)$   
         <3>2. There is a path from  $y$  to  $x$ .  
         <3>3. There is no path from  $x_0$  to  $y$ .  
 <1>7.  $V = U$   
     PROOF:  $U$  is connected.  
 □

## 13.40 Locally Homeomorphic

**Definition 13.40.1.** Let  $X$  and  $Y$  be topological spaces. Then  $X$  is *locally homeomorphic* to  $Y$  if and only if every point in  $X$  has a neighbourhood that is homeomorphic to an open set in  $Y$ .

### 13.40.1 The Long Line

**Proposition 13.40.2.** *The long line is locally homeomorphic to  $[0, 1)$ .*

PROOF: By Proposition 6.5.2. □

## 13.41 Components

**Definition 13.41.1** ((Connected) Component). Let  $X$  be a topological space. Define the equivalence relation  $\sim$  on  $X$  by:  $x \sim y$  iff there exists a connected  $C \subseteq X$  such that  $x \in C$  and  $y \in C$ . The *components* of  $X$  are the equivalence classes with respect to  $\sim$ .

We prove this is an equivalence relation.

PROOF:

<1>1.  $\sim$  is reflexive.

PROOF: For any  $x \in X$ , we have  $\{x\}$  is connected and  $x \in \{x\}$ , hence  $x \sim x$ .

<1>2.  $\sim$  is symmetric.

PROOF: Immediate from definition.

$\langle 1 \rangle 3$ .  $\sim$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $x \sim y$  and  $y \sim z$

$\langle 2 \rangle 2$ . PICK connected subspaces  $C$  and  $D$  of  $X$  with  $x \in C$ ,  $y \in C$ ,  $y \in D$  and  $z \in D$ .

$\langle 2 \rangle 3$ .  $C \cup D$  is connected.

PROOF: Proposition 13.29.12.

$\langle 2 \rangle 4$ .  $x \in C \cup D$  and  $z \in C \cup D$ .

$\langle 2 \rangle 5$ .  $x \sim z$

□

**Example 13.41.2.** The components of  $\mathbb{Q}$  are the singleton subsets.

**Example 13.41.3.** The components of  $\mathbb{R}_l$  are the singleton subsets.

**Proposition 13.41.4.** *Every component of a topological space is connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $C$  be a component of the topological space  $X$ .

$\langle 1 \rangle 2$ . ASSUME: for a contradiction  $(A, B)$  is a separation of  $C$ .

$\langle 1 \rangle 3$ . PICK  $a \in A$  and  $b \in B$ .

$\langle 1 \rangle 4$ .  $a \sim b$

$\langle 1 \rangle 5$ . PICK a connected subspace  $D$  of  $X$  such that  $a \in D$  and  $b \in D$ .

$\langle 1 \rangle 6$ .  $D \subseteq C$

$\langle 1 \rangle 7$ .  $(A \cap D, B \cap D)$  is a separation of  $D$ .

$\langle 1 \rangle 8$ . Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 13.41.5.** *Let  $X$  be a topological space. Let  $A$  be a nonempty connected subspace of  $X$ . Then there exists a unique component  $C$  of  $X$  such that  $A \subseteq C$ .*

PROOF:

$\langle 1 \rangle 1$ . PICK  $a \in A$

$\langle 1 \rangle 2$ . LET:  $C$  be the  $\sim$ -equivalence class of  $a$ .

$\langle 1 \rangle 3$ .  $A \subseteq C$

PROOF: For all  $x \in A$  we have  $a \sim x$  hence  $x \in C$ .

$\langle 1 \rangle 4$ . For any component  $C'$ , if  $A \subseteq C'$  then  $C' = C$ .

PROOF: Since the components are pairwise disjoint.

□

**Proposition 13.41.6.** *Every component of a topological space is closed.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a topological space.

$\langle 1 \rangle 2$ . LET:  $C$  be a component of  $X$ .

$\langle 1 \rangle 3$ .  $\overline{C}$  is connected.

PROOF: Proposition 13.29.13.

⟨1⟩4.  $\overline{C} \subseteq C$

PROOF: Proposition 13.41.5.

⟨1⟩5.  $C = \overline{C}$

□

**Corollary 13.41.6.1.** *If a topological space has only finitely many components, then its components are open.*

## 13.42 Path Components

**Definition 13.42.1** (Path Component). Let  $X$  be a topological space. Define the equivalence relation  $\sim$  on  $X$  by:  $x \sim y$  iff there exists a path from  $x$  to  $y$ . The *path components* of  $X$  are the equivalence classes with respect to  $\sim$ .

We prove  $\sim$  is an equivalence relation.

PROOF:

⟨1⟩1.  $\sim$  is reflexive.

PROOF: For any  $a \in X$  the constant path at  $a$  is a path from  $a$  to  $a$ .

⟨1⟩2.  $\sim$  is symmetric.

PROOF: If  $p$  is a path from  $a$  to  $b$  then the reverse of  $p$  is a path from  $b$  to  $a$ .

⟨1⟩3.  $\sim$  is transitive.

PROOF: If  $p$  is a path from  $a$  to  $b$  and  $q$  is a path from  $b$  to  $c$  then the concatenation of  $p$  and  $q$  is a path from  $a$  to  $c$ .

□

**Example 13.42.2.** The topologist's sine curve has two path components, namely  $\{0\} \times [0, 1]$  (which is closed and not open) and  $\{(x, \sin 1/x) : 0 < x \leq 1\}$  (which is open and not closed).

**Proposition 13.42.3.** *Every path component is path connected.*

PROOF: If  $x$  and  $y$  are in the same path component then  $x \sim y$  so there is a path from  $x$  to  $y$ . □

**Corollary 13.42.3.1.** *Every path component is a subset of a component.*

**Proposition 13.42.4.** *Let  $X$  be a topological space. Let  $A$  be a nonempty path connected subspace of  $X$ . Then there exists a unique path component  $C$  of  $X$  such that  $A \subseteq C$ .*

PROOF:

⟨1⟩1. PICK  $a \in A$

⟨1⟩2. LET:  $C$  be the path component of  $a$ .

⟨1⟩3.  $A \subseteq C$

PROOF: For all  $x \in A$  we have  $a \sim x$  (because  $A$  is path connected) hence  $x \in C$ .

⟨1⟩4. For any path component  $C'$ , if  $A \subseteq C'$  then  $C = C'$ .

PROOF: This holds because the path components are pairwise disjoint.

□

**Example 13.42.5.** In  $\mathbb{R}^\omega$  under the box topology,  $\vec{x}$  and  $\vec{y}$  are in the same component if and only if  $\vec{x} - \vec{y}$  is eventually zero.

PROOF:

⟨1⟩1. LET:  $B$  be the set of sequences that are eventually zero.

⟨1⟩2.  $B$  is connected.

PROOF: For  $\vec{x} \in B$ , the straight line path from 0 to  $\vec{x}$  is continuous.

⟨1⟩3.  $B$  is maximally connected.

PROOF: Since  $(B, \mathbb{R}^\omega - B)$  form a separation of  $\mathbb{R}^\omega$ .

⟨1⟩4. For all  $\vec{y} \in \mathbb{R}^\omega$ , the component that contains  $\vec{y}$  is  $\{\vec{x} \in \mathbb{R}^\omega : \vec{x} - \vec{y} \text{ is eventually zero}\}$ .

PROOF: Since the function that maps  $\vec{x}$  to  $\vec{x} + \vec{y}$  is a homeomorphism of  $\mathbb{R}^\omega$  with itself.

□

**Example 13.42.6.** The path components of  $I_o^2$  are  $\{\{x\} \times [0, 1] : 0 \leq x \leq 1\}$ .

PROOF:

⟨1⟩1. For all  $x \in [0, 1]$  we have  $\{x\} \times [0, 1]$  is path connected.

PROOF: It is homeomorphic to  $[0, 1]$ .

⟨1⟩2. Given  $x, y, s, t \in [0, 1]$  with  $x \neq y$ , there is no path from  $(x, s)$  to  $(y, t)$ .

⟨2⟩1. ASSUME: for a contradiction  $p : [0, 1] \rightarrow I_o^2$  is a path from  $(x, s)$  to  $(y, t)$ .

⟨2⟩2. For  $z$  between  $x$  and  $y$ , PICK a rational  $q_z \in [0, 1]$  such that  $p(q_z) \in \{z\} \times [0, 1]$ .

⟨2⟩3.  $\{q_z : z \text{ is between } x \text{ and } y\}$  is an uncountable set of rationals.

⟨2⟩4. Q.E.D.

PROOF: This is a contradiction.

□

## 13.43 Weak Local Connectedness

**Definition 13.43.1** (Weakly Locally Connected). Let  $X$  be a topological space and  $x \in X$ . Then  $X$  is *weakly locally connected* at  $x$  iff, for every neighbourhood  $U$  of  $x$ , there exists a connected  $Y \subseteq X$  and a neighbourhood  $V$  of  $x$  such that  $V \subseteq Y \subseteq U$ .

## 13.44 Local Connectedness

**Definition 13.44.1** (Locally Connected). Let  $X$  be a topological space and  $x \in X$ . Then  $X$  is *locally connected* at  $x$  iff, for every neighbourhood  $U$  of  $x$ , there exists a connected neighbourhood  $V$  of  $x$  such that  $V \subseteq U$ .

The space  $X$  is *locally connected* iff it is locally connected at every point.

**Example 13.44.2.** Every interval and ray in the real line is connected and locally connected.

**Example 13.44.3.** The space  $[-1, 0) \cup (0, 1]$  is locally connected but not connected.

**Example 13.44.4.** The topologist's sine curve is connected but not locally connected.

**Example 13.44.5.** The rationals  $\mathbb{Q}$  are neither connected nor locally connected.

**Example 13.44.6.** For  $n$  a positive integer, let  $a_n = (1/n, 0)$ . Let  $p = (0, 0)$ . Let the infinite broom  $X$  be the union of all the line segments joining  $(a_{n+1}, q)$  to  $(a_n, 0)$  for  $n$  any positive integer and  $q$  any rational in  $[0, 1/n]$ . Then  $X$  is weakly locally connected at  $p$  but not locally connected at  $p$ .

PROOF:

- ⟨1⟩1.  $X$  is weakly locally connected at  $p$ .
  - ⟨2⟩1. LET:  $U$  be any neighbourhood of  $p$ .
  - ⟨2⟩2. PICK  $N$  such that, for all  $n \geq N$  and every rational  $q \in [0, 1/n]$ , the line segment joining  $(a_{n+1}, q)$  to  $(a_n, 0)$  is included in  $U$ .
  - ⟨2⟩3. LET:  $Y$  be the union of all these line segments.
  - ⟨2⟩4.  $Y$  is connected.
  - ⟨2⟩5. LET:  $V = B(p, 1/n) \cap X$
  - ⟨2⟩6.  $V \subseteq Y \subseteq U$
- ⟨1⟩2.  $X$  is not locally connected at  $p$ .
  - ⟨2⟩1. LET:  $U = B(p, 1/2) \cap X$
  - ⟨2⟩2. LET:  $V$  be a neighbourhood of  $p$  with  $V \subseteq U$   
PROVE:  $V$  is disconnected.
  - ⟨2⟩3. LET:  $n$  be least such that  $(a_n, 0) \in V$
  - ⟨2⟩4.  $(a_{n-1}, 0) \notin V$
  - ⟨2⟩5. Some part of a line segment joining some  $(a_n, q)$  to  $(a_{n-1}, 0)$  is in  $V$
  - ⟨2⟩6.  $V$  is disconnected.

□

**Theorem 13.44.7.** Let  $X$  be a topological space. Then  $X$  is locally connected if and only if, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .

PROOF:

- ⟨1⟩1. If  $X$  is locally connected then, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .
  - ⟨2⟩1. ASSUME:  $X$  is locally connected.
  - ⟨2⟩2. LET:  $U$  be an open set in  $X$ .
  - ⟨2⟩3. LET:  $C$  be a component of  $U$ .
  - ⟨2⟩4. LET:  $x \in C$
  - ⟨2⟩5. PICK a connected neighbourhood  $V$  of  $x$  in  $X$  such that  $V \subseteq U$
  - ⟨2⟩6.  $x \in V \subseteq C$
- ⟨1⟩2. If, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ , then  $X$  is locally connected.
  - ⟨2⟩1. ASSUME: For every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .



- ⟨2⟩2. LET:  $x \in X$
- ⟨2⟩3. LET:  $U$  be a neighbourhood of  $x$ .
- ⟨2⟩4. LET:  $V$  be the component of  $U$  that contains  $x$ .
- ⟨2⟩5.  $V$  is a connected neighbourhood of  $x$  and  $V \subseteq U$ .

□

**Proposition 13.44.8.** *The ordered square is locally connected.*

PROOF: Since every basic open set is connected because it is a linear continuum.

□

**Example 13.44.9.** Let  $T$  be the union of all line segments connecting a point  $(q, 0)$  to  $(0, 1)$  where  $q \in [0, 1]$  is rational, and all line segments connecting a point  $(q, 1)$  to  $(1, 0)$  where  $q \in [0, 1]$  is rational. Then  $T$  is path connected but is locally connected at no point.

**Proposition 13.44.10.** *If a topological space is weakly locally connected at every point then it is locally connected.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a topological space that is weakly locally connected at every point.
- ⟨1⟩2. For every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .
  - ⟨2⟩1. LET:  $U$  be an open set in  $X$ .
  - ⟨2⟩2. LET:  $C$  be a component of  $U$ .
  - ⟨2⟩3. For all  $x \in C$ , there exists a neighbourhood  $V$  of  $x$  such that  $V \subseteq C$ .
    - ⟨3⟩1. LET:  $x \in C$
    - ⟨3⟩2. PICK a connected  $Y \subseteq X$  and a neighbourhood  $V$  of  $x$  such that  $V \subseteq Y \subseteq U$
  - PROOF: ⟨1⟩1
  - ⟨3⟩3.  $Y \subseteq C$
  - PROOF: Proposition 13.41.5.
  - ⟨3⟩4.  $V \subseteq C$
  - ⟨2⟩4.  $C$  is open.
  - PROOF: Proposition 13.1.7.
- ⟨1⟩3.  $X$  is locally connected.
- PROOF: Theorem 13.44.7.

□

**Proposition 13.44.11.** *A quotient of a locally connected space is locally connected.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a locally connected space.
- ⟨1⟩2. LET:  $p : X \rightarrow Y$  be a quotient map.
- ⟨1⟩3. For every open set  $V$  in  $Y$ , every component of  $V$  is open in  $Y$ .
  - ⟨2⟩1. LET:  $V$  be an open set in  $Y$ .
  - ⟨2⟩2. LET:  $C$  be a component of  $V$ .

- ⟨2⟩3.  $p^{-1}(C)$  is a union of components of  $p^{-1}(V)$
- ⟨3⟩1. LET:  $x \in p^{-1}(C)$
- ⟨3⟩2. LET:  $D$  be the component of  $p^{-1}(V)$  that contains  $x$ .  
PROVE:  $D \subseteq p^{-1}(C)$
- ⟨3⟩3.  $p(D)$  is connected.  
PROOF: Proposition 13.29.10.
- ⟨3⟩4.  $p(D) \subseteq C$
- ⟨3⟩5.  $D \subseteq p^{-1}(C)$
- ⟨2⟩4. Every component of  $p^{-1}(V)$  is open in  $X$ .  
PROOF: Theorem 13.44.7.
- ⟨2⟩5.  $p^{-1}(C)$  is open in  $X$ .
- ⟨2⟩6.  $C$  is open in  $Y$ .
- ⟨1⟩4.  $Y$  is locally connected.  
PROOF: Theorem 13.44.7.

□

### 13.45 Local Path Connectedness

**Definition 13.45.1** (Locally Path Connected). Let  $X$  be a topological space and  $x \in X$ . Then  $X$  is *locally path connected* at  $x$  iff, for every neighbourhood  $U$  of  $x$ , there exists a path connected neighbourhood  $V$  of  $x$  such that  $V \subseteq U$ .

The space  $X$  is *locally path connected* iff it is locally connected at every point.

**Theorem 13.45.2.** *Let  $X$  be a topological space. Then  $X$  is locally path connected if and only if, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .*

PROOF:

- ⟨1⟩1. If  $X$  is locally path connected then, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .
- ⟨2⟩1. ASSUME:  $X$  is locally path connected.
- ⟨2⟩2. LET:  $U$  be an open set in  $X$ .
- ⟨2⟩3. LET:  $C$  be a path component of  $U$ .
- ⟨2⟩4. LET:  $x \in C$
- ⟨2⟩5. PICK a path connected neighbourhood  $V$  of  $x$  in  $X$  such that  $V \subseteq U$
- ⟨2⟩6.  $x \in V \subseteq C$
- ⟨1⟩2. If, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ , then  $X$  is locally path connected.
- ⟨2⟩1. ASSUME: For every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .
- ⟨2⟩2. LET:  $x \in X$
- ⟨2⟩3. LET:  $U$  be a neighbourhood of  $x$ .
- ⟨2⟩4. LET:  $V$  be the path component of  $U$  that contains  $x$ .
- ⟨2⟩5.  $V$  is a path connected neighbourhood of  $x$  and  $V \subseteq U$ .

□

**Theorem 13.45.3.** *In a locally path connected space, the components are the same as the path components.*

PROOF:

⟨1⟩1. LET:  $X$  be a locally path connected space.

⟨1⟩2. LET:  $P$  be a path component of  $X$ .

⟨1⟩3. LET:  $C$  be the component that includes  $P$ .

PROVE:  $P = C$

⟨1⟩4. LET:  $Q$  be the union of all the path components of  $C$  other than  $P$ .

⟨1⟩5.  $P$  and  $Q$  are open in  $C$ .

PROOF: Theorem 13.45.2.

⟨1⟩6.  $P \cup Q = C$  and  $P \cap Q = \emptyset$

⟨1⟩7.  $Q = \emptyset$

PROOF: Otherwise  $(P, Q)$  would be a separation of  $C$ .

⟨1⟩8.  $P = C$

□

**Example 13.45.4.** The converse does not hold. In  $\mathbb{Q}$ , the components are the same as the path components, namely the one-point sets, but  $\mathbb{Q}$  is not locally path connected.

**Example 13.45.5.** The ordered square is not locally path connected.

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $I_o^2$  is locally path connected at  $(0, 1)$ .

⟨1⟩2. PICK a path connected neighbourhood  $U$  of  $(0, 1)$ .

⟨1⟩3. PICK  $a > 0$  such that  $[(0, 1), (a, 0)] \subseteq U$

⟨1⟩4. PICK a path  $p : [0, 1] \rightarrow I_o^2$  from  $(0, 1)$  to  $(a, 0)$ .

⟨1⟩5. For every  $x \in (0, a)$ , PICK a rational  $q_x \in [0, 1]$  such that  $q_x \in ((x, 0), (x, 1))$

⟨1⟩6.  $\{q_x : x \in (0, a)\}$  is an uncountable set of rationals.

⟨1⟩7. Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 13.45.6.** *Every connected open subspace of a locally path connected space is path connected.*

PROOF:

⟨1⟩1. LET:  $X$  be a locally path connected space.

⟨1⟩2. LET:  $U$  be a connected open subspace.

⟨1⟩3. LET:  $P$  be a path component of  $U$ .

PROVE:  $P = U$

⟨1⟩4. LET:  $Q$  be the union of the path components of  $U$  that are not  $P$ .

⟨1⟩5.  $P$  and  $Q$  are open.

PROOF: Theorem 13.45.2.

⟨1⟩6.  $Q = \emptyset$

PROOF: Otherwise  $(P, Q)$  would be a separation of  $U$ .

⟨1⟩7.  $P = U$

□

### 13.46 Quasicomponents

**Definition 13.46.1** (Quasicomponent). Let  $X$  be a topological space. Define the equivalence relation  $\sim$  on  $X$  by:  $x \sim y$  iff there is no separation  $(U, V)$  of  $X$  with  $x \in U$  and  $y \in V$ . The *quasicomponents* of  $X$  are the equivalence classes with respect to  $\sim$ .

We prove this is an equivalence relation.

PROOF:

$\langle 1 \rangle 1.$   $\sim$  is reflexive.

PROOF: For any  $x \in X$ , there cannot exist a separation  $(U, V)$  of  $X$  with  $x \in U$  and  $x \in V$ .

$\langle 1 \rangle 2.$   $\sim$  is symmetric.

PROOF: Immediate from definition.

$\langle 1 \rangle 3.$   $\sim$  is transitive.

$\langle 2 \rangle 1.$  ASSUME:  $x \sim y$  and  $y \sim z$

$\langle 2 \rangle 2.$  ASSUME: for a contradiction  $(U, V)$  is a separation of  $X$  with  $x \in U$  and  $z \in V$ .

$\langle 2 \rangle 3.$   $y \in U$  or  $y \in V$

$\langle 2 \rangle 4.$   $y \notin U$

PROOF:  $y \in U$  would contradict the fact that  $y \sim z$ .

$\langle 2 \rangle 5.$   $y \notin V$

PROOF:  $y \in V$  would contradict the fact that  $x \sim y$ .

$\langle 2 \rangle 6.$  Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 13.46.2.** *Every component of a topological space is a subset of a quasicomponent.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a topological space.

$\langle 1 \rangle 2.$  LET:  $C$  be a component of  $X$ .

PROVE:  $\forall x, y \in C. x \sim y$

$\langle 1 \rangle 3.$  LET:  $x, y \in C$

$\langle 1 \rangle 4.$  ASSUME: for a contradiction  $(U, V)$  is a separation of  $X$  with  $x \in U$  and  $y \in V$

$\langle 1 \rangle 5.$   $(U \cap C, V \cap C)$  is a separation of  $C$ .

$\langle 1 \rangle 6.$  Q.E.D.

PROOF: This contradicts the fact that  $C$  is connected (Proposition 13.41.4).

□

**Proposition 13.46.3.** *In a locally connected topological space, the components are the same as the quasicomponents.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a locally connected topological space.

$\langle 1 \rangle 2.$  LET:  $C$  be a component of  $X$ .

- <1>3. LET:  $Q$  be the quasicomponent that includes  $C$ .  
 PROVE:  $Q = C$   
 <1>4. ASSUME: for a contradiction  $C \neq Q$   
 <1>5. PICK  $c \in C$  and  $d \in Q - C$   
 <1>6.  $(C, X - C)$  is a separation of  $X$  with  $c \in C$  and  $d \in X - C$ .  
 PROOF: Since the components of  $X$  are open (Theorem 13.44.7).  
 <1>7. Q.E.D.  
 PROOF: This contradicts the fact that  $c \sim d$ .  
 □

## 13.47 Compact Spaces

**Definition 13.47.1** (Open Cover). Let  $X$  be a topological space. An *open cover* of  $X$  is a cover of  $X$  whose elements are open sets.

**Definition 13.47.2** (Compact). A topological space is *compact* iff every open cover includes a finite subcover.

**Example 13.47.3.** The space  $\mathbb{R}$  is not compact, because the open cover  $\{(n, n+2) : n \in \mathbb{Z}\}$  has no finite subcover.

**Example 13.47.4.** Every finite topological space is compact.

**Example 13.47.5.** Any set under the cofinite topology is compact.

**Lemma 13.47.6.** Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Then  $Y$  is compact if and only if every covering of  $Y$  by sets open in  $X$  contains a finite subcollection that covers  $Y$ .

PROOF:

- <1>1. If  $Y$  is compact then every covering of  $Y$  by sets open in  $X$  contains a finite subcollection that covers  $Y$ .  
 <2>1. ASSUME:  $Y$  is compact.  
 <2>2. LET:  $\mathcal{A}$  be a covering of  $Y$  by sets open in  $X$ .  
 <2>3.  $\{U \cap Y : U \in \mathcal{A}\}$  is an open covering of  $Y$ .  
 <2>4. PICK a finite subcovering  $\{U_1 \cap Y, \dots, U_n \cap Y\}$ .  
 <2>5.  $\{U_1, \dots, U_n\}$  is a finite subcollection of  $\mathcal{A}$  that covers  $Y$ .  
 <1>2. If every covering of  $Y$  by sets open in  $X$  contains a finite subcollection that covers  $Y$  then  $Y$  is compact.  
 <2>1. ASSUME: Every covering of  $Y$  by sets open in  $X$  contains a finite subcollection that covers  $Y$ .  
 <2>2. LET:  $\mathcal{A}$  be an open cover of  $Y$ .  
 <2>3.  $\{U \text{ open in } X : U \cap Y \in \mathcal{A}\}$  covers  $Y$ .  
 <2>4. PICK a finite subcollection  $\{U_1, \dots, U_n\}$  that covers  $Y$ .  
 <2>5.  $\{U_1 \cap Y, \dots, U_n \cap Y\}$  is a finite subcollection of  $\mathcal{A}$  that covers  $Y$ .  
 □

**Theorem 13.47.7.** Every closed subspace of a compact space is compact.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X$  be a compact space.
  - $\langle 1 \rangle 2$ . LET:  $C$  be a closed subspace of  $X$ .
  - $\langle 1 \rangle 3$ . Every covering of  $C$  by sets open in  $X$  contains a finite subcollection that covers  $C$ .
    - $\langle 2 \rangle 1$ . LET:  $\mathcal{A}$  be a covering of  $C$  by sets open in  $X$ .
    - $\langle 2 \rangle 2$ .  $\mathcal{A} \cup \{X - C\}$  is an open covering of  $X$ .
    - $\langle 2 \rangle 3$ . PICK a finite subcover  $\mathcal{B}$
    - $\langle 2 \rangle 4$ .  $\mathcal{B} - \{X - C\}$  is a finite subcollection of  $\mathcal{A}$  that covers  $C$ .
  - $\langle 1 \rangle 4$ .  $C$  is compact.
- PROOF: Lemma 13.47.6.

□

**Lemma 13.47.8.** *Let  $X$  be a Hausdorff space. Let  $Y$  be a compact subspace. Let  $x \in X - Y$ . Then there exist disjoint open sets  $U$  and  $V$  such that  $x \in U$  and  $Y \subseteq V$ .*

PROOF:

- $\langle 1 \rangle 1$ . For all  $y \in Y$ , there exist disjoint open sets  $U'$  and  $V'$  with  $x \in U'$  and  $y \in V'$ .
- $\langle 1 \rangle 2$ .  $\{V' \text{ open in } X : \exists U' \text{ open in } X. U' \cap V' = \emptyset \wedge x \in U'\}$  is an cover of  $Y$  by sets open in  $X$ .
- $\langle 1 \rangle 3$ . PICK a finite subcollection  $\{V_1, \dots, V_n\}$  that covers  $Y$ .
- $\langle 1 \rangle 4$ . For  $i = 1, \dots, n$ , PICK an open set  $U_i$  with  $U_i \cap V_i = \emptyset$  and  $x \in U_i$ .
- $\langle 1 \rangle 5$ . LET:  $U = U_1 \cap \dots \cap U_n$  and  $V = V_1 \cup \dots \cup V_n$
- $\langle 1 \rangle 6$ .  $U$  and  $V$  are disjoint open sets with  $x \in U$  and  $Y \subseteq V$ .

□

**Proposition 13.47.9.** *Let  $X$  be a Hausdorff space. Let  $A$  and  $B$  be disjoint compact subspaces of  $X$ . Then there exist disjoint open sets  $U$  and  $V$  with  $A \subseteq U$  and  $B \subseteq V$ .*

PROOF:

- $\langle 1 \rangle 1$ . For all  $x \in A$ , there exist disjoint open sets  $U'$  and  $V'$  with  $x \in U'$  and  $B \subseteq V'$ .
- PROOF: Lemma 13.47.8.
- $\langle 1 \rangle 2$ .  $\{U' \text{ open in } X : \exists V' \text{ open in } X. U' \cap V' = \emptyset \wedge B \subseteq V'\}$  is a set of open sets that covers  $A$ .
  - $\langle 1 \rangle 3$ . PICK a finite subset  $\{U_1, \dots, U_n\}$  that covers  $A$ .

PROOF: Lemma 13.47.6.

- $\langle 1 \rangle 4$ . For  $i = 1, \dots, n$ , PICK  $V_i$  open in  $X$  such that  $U_i \cap V_i = \emptyset$  and  $B \subseteq V_i$ .
- $\langle 1 \rangle 5$ . LET:  $U = U_1 \cup \dots \cup U_n$
- $\langle 1 \rangle 6$ . LET:  $V = V_1 \cap \dots \cap V_n$
- $\langle 1 \rangle 7$ .  $U$  and  $V$  are disjoint and open.
- $\langle 1 \rangle 8$ .  $A \subseteq U$
- $\langle 1 \rangle 9$ .  $B \subseteq V$

□

**Theorem 13.47.10.** *Every compact subspace of a Hausdorff space is closed.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a Hausdorff space.

$\langle 1 \rangle 2$ . LET:  $Y$  be a compact subspace of  $X$ .

$\langle 1 \rangle 3$ . For any  $x \in X - Y$  there exists an open set  $U$  such that  $x \in U \subseteq X - Y$ .

PROOF: Lemma 13.47.8.

$\langle 1 \rangle 4$ .  $X - Y$  is open.

PROOF: Proposition 13.1.7.

$\langle 1 \rangle 5$ .  $Y$  is closed.

□

**Example 13.47.11.** We cannot weaken the hypothesis from the space being Hausdorff to the space being  $T_1$ .

Let  $X$  be any infinite set under the cofinite topology. Then  $X$  is  $T_1$ . The closed sets are the finite sets and  $X$ , but every subspace is compact.

**Theorem 13.47.12.** *The continuous image of a compact space is compact.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a compact space and  $f : X \rightarrow Y$  be a surjective continuous function.

$\langle 1 \rangle 2$ . LET:  $\mathcal{V}$  be an open cover of  $Y$ .

$\langle 1 \rangle 3$ .  $\{f^{-1}(V) : V \in \mathcal{V}\}$  is an open cover of  $U$ .

$\langle 1 \rangle 4$ . PICK a finite subcover  $\{f^{-1}(V_1), \dots, f^{-1}(V_n)\}$ .

$\langle 1 \rangle 5$ .  $\{V_1, \dots, V_n\}$  covers  $Y$ .

□

**Proposition 13.47.13.** *Let  $X$  be a compact space and  $Y$  a Hausdorff space. Every continuous function from  $X$  to  $Y$  is a closed map.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : X \rightarrow Y$  be continuous.

$\langle 1 \rangle 2$ . LET:  $C$  be a closed set in  $X$ .

$\langle 1 \rangle 3$ .  $C$  is compact.

PROOF: Theorem 13.47.7.

$\langle 1 \rangle 4$ .  $f(C)$  is compact.

PROOF: Theorem 13.47.12.

$\langle 1 \rangle 5$ .  $f(C)$  is closed in  $Y$ .

PROOF: Theorem 13.47.10.

□

**Corollary 13.47.13.1.** *Let  $X$  be a compact space and  $Y$  a Hausdorff space. Let  $f : X \rightarrow Y$  be a continuous bijection. Then  $f$  is a homeomorphism.*

**Corollary 13.47.13.2.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two compact Hausdorff topologies on the same set  $X$ . Then either  $\mathcal{T} = \mathcal{T}'$  or they are incompatible.*

**Theorem 13.47.14.** *Let  $X$  and  $Y$  be topological spaces. Let  $A$  be a compact subspace of  $X$  and  $B$  a compact subspace of  $Y$ . Let  $N$  be an open set that includes  $A \times B$ . Then there exists open sets  $U$  in  $X$  and  $V$  in  $Y$  such that*

$$A \times B \subseteq U \times V \subseteq N .$$

PROOF:

- ⟨1⟩1. For all  $x \in A$ , there exist open sets  $U$  in  $X$  and  $V$  in  $Y$  such that  $x \in U$ ,  $B \subseteq V$ , and  $U \times V \subseteq N$ .
- ⟨2⟩1. LET:  $x \in A$
- ⟨2⟩2. For all  $y \in B$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $U \times V \subseteq N$
- ⟨2⟩3. PICK open sets  $V_1, \dots, V_n$  that cover  $B$  such that, for  $i = 1, \dots, n$ , there exists a neighbourhood  $U_i$  of  $x$  such that  $U_i \times V_i \subseteq N$
- ⟨2⟩4. LET:  $U = U_1 \cap \dots \cap U_n$
- ⟨2⟩5. LET:  $V = V_1 \cup \dots \cup V_n$
- ⟨2⟩6.  $U$  is open in  $X$ .
- ⟨2⟩7.  $V$  is open in  $Y$ .
- ⟨2⟩8.  $x \in U$
- ⟨2⟩9.  $B \subseteq V$
- ⟨2⟩10.  $U \times V \subseteq N$
- ⟨1⟩2. PICK open sets  $U_1, \dots, U_n$  in  $X$  that cover  $A$  such that, for  $i = 1, \dots, n$ , there exists  $V_i$  open in  $Y$  such that  $B \subseteq V_i$  and  $U_i \times V_i \subseteq N$ .
- ⟨1⟩3. LET:  $U = U_1 \cup \dots \cup U_n$
- ⟨1⟩4. LET:  $V = V_1 \cap \dots \cap V_n$
- ⟨1⟩5.  $U$  is open in  $X$ .
- ⟨1⟩6.  $V$  is open in  $Y$ .
- ⟨1⟩7.  $A \times B \subseteq U \times V \subseteq N$

□

**Corollary 13.47.14.1** (Tube Lemma). *Let  $X$  be a topological space and  $Y$  a compact space. Let  $x_0 \in X$ . Let  $N$  be an open set in  $X \times Y$  that includes  $\{x_0\} \times Y$ . Then there exists a neighbourhood  $W$  of  $x_0$  such that  $W \times Y \subseteq N$ .*

**Theorem 13.47.15.** *The product of two compact spaces is compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  and  $Y$  be compact spaces.
- ⟨1⟩2. LET:  $\mathcal{A}$  be an open covering of  $X \times Y$ .
- ⟨1⟩3. For all  $x \in X$ , there exists a neighbourhood  $W$  of  $x$  such that  $W \times Y$  can be covered by finitely many elements of  $\mathcal{A}$ .
- ⟨2⟩1. LET:  $x \in X$
- ⟨2⟩2.  $\{x\} \times Y$  is compact.  
PROOF: It is homeomorphic to  $Y$ .
- ⟨2⟩3. PICK a finite subcollection  $\{A_1, \dots, A_n\}$  of  $\mathcal{A}$  that covers  $\{x\} \times Y$ .
- ⟨2⟩4. There exists a neighbourhood  $W$  of  $x$  such that  $W \times Y \subseteq A_1 \cup \dots \cup A_n$ .

PROOF: Tube Lemma



- <1>4. PICK finitely many open sets  $W_1, \dots, W_n$  that cover  $X$  such that each  $W_i \times Y$  can be covered by finitely many elements of  $\mathcal{A}$ .  
 <1>5. For  $i = 1, \dots, n$ , PICK a finite subset  $\mathcal{A}_i$  of  $\mathcal{A}$  that covers  $W_i \times Y$ .  
 <1>6.  $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_n$  covers  $X \times Y$ .  
 $\square$

**Theorem 13.47.16.** *Let  $X$  be a topological space. Then  $X$  is compact if and only if, for every set  $\mathcal{C}$  of compact sets, if  $\mathcal{C}$  has the finite intersection property then  $\bigcap \mathcal{C} \neq \emptyset$ .*

PROOF: The following are equivalent:

- $X$  is compact.
- For every set  $\mathcal{A}$  of open sets, if  $\bigcup \mathcal{A} = X$  then there is a finite subset of  $\mathcal{A}$  that covers  $X$ .
- For every set  $\mathcal{C}$  of closed sets, if  $\bigcup_{C \in \mathcal{C}} (X - C) = X$  then there is a finite subset  $\mathcal{C}_0 \subseteq \mathcal{C}$  such that  $\bigcup_{C \in \mathcal{C}_0} (X - C) = X$ .
- For every set  $\mathcal{C}$  of closed sets, if  $\bigcap \mathcal{C} = \emptyset$  then there is a finite subset  $\mathcal{C}_0 \subseteq \mathcal{C}$  such that  $\bigcap \mathcal{C}_0 = \emptyset$ .
- For every set  $\mathcal{C}$  of closed sets, if  $\mathcal{C}$  has the finite intersection property then  $\bigcap \mathcal{C} \neq \emptyset$ .

$\square$

**Corollary 13.47.16.1.** *Let  $X$  be a compact set. Let  $(C_n)$  be a sequence of nonempty closed sets such that  $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$ . Then  $\bigcap_{n=0}^{\infty} C_n \neq \emptyset$ .*

**Proposition 13.47.17.** *Let  $X$  be a topological space. Let  $Y$  and  $Z$  be compact subspaces of  $X$ . The  $Y \cup Z$  is compact.*

PROOF:

- <1>1. LET:  $\mathcal{U}$  be a set of open sets that covers  $Y \cup Z$ .  
 <1>2. PICK finite subsets  $\mathcal{U}_1$  that covers  $Y$  and  $\mathcal{U}_2$  that covers  $Z$ .  
 <1>3.  $\mathcal{U}_1 \cup \mathcal{U}_2$  is a finite subset of  $\mathcal{U}$  that covers  $Y \cup Z$ .  
 $\square$

**Proposition 13.47.18.** *Let  $X$  be a topological space and  $Y$  a compact space. Then the projection  $\pi_1 : X \times Y \rightarrow X$  is a closed map.*

PROOF:

- <1>1. LET:  $C$  be closed in  $X \times Y$ .  
 <1>2. LET:  $x \in X - \pi_1(C)$   
 <1>3.  $\{x\} \times Y \subseteq (X \times Y) - C$   
 <1>4. PICK a neighbourhood  $U$  of  $x$  such that  $U \times Y \subseteq (X \times Y) - C$   
 PROOF: Tube Lemma.  
 <1>5.  $x \in U \subseteq X - \pi_1(C)$

□

**Theorem 13.47.19.** *Let  $X$  be a topological space and  $Y$  a compact Hausdorff space. Let  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if the graph of  $f$ ,*

$$G_f = \{(x, f(x)) : x \in X\} ,$$

*is closed in  $X \times Y$ .*

PROOF:

⟨1⟩1. If  $f$  is continuous then  $G_f$  is closed in  $X \times Y$ .

⟨2⟩1. ASSUME:  $f$  is continuous.

⟨2⟩2. LET:  $(x, y) \in (X \times Y) - G_f$

⟨2⟩3. PICK disjoint open neighbourhoods  $U$  of  $y$  and  $V$  of  $f(x)$ .

⟨2⟩4.  $(x, y) \in (f^{-1}(V) \times U) \subseteq (X \times Y) - G_f$

⟨1⟩2. If  $G_f$  is closed in  $X \times Y$  then  $f$  is continuous.

⟨2⟩1. ASSUME:  $G_f$  is closed in  $X \times Y$ .

⟨2⟩2. LET:  $x_0 \in X$

⟨2⟩3. LET:  $V$  be a neighbourhood of  $f(x_0)$ .

⟨2⟩4.  $G_f \cap (X \times (Y - V))$  is closed.

⟨2⟩5.  $\pi_1(G_f \cap (X \times (Y - V)))$  is closed.

PROOF: Proposition 13.47.18.

⟨2⟩6. LET:  $U = X - \pi_1(G_f \cap (X \times (Y - V)))$

⟨2⟩7.  $x_0 \in U$

⟨2⟩8.  $f(U) \subseteq V$

□

**Theorem 13.47.20.** *Let  $X$  be a compact Hausdorff space. Let  $\mathcal{A}$  be a set of closed connected subspaces of  $X$  that is linearly ordered under inclusion. Then  $\bigcap \mathcal{A}$  is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $(C, D)$  is a separation of  $\bigcap \mathcal{A}$ .

⟨1⟩2. PICK disjoint open sets  $U$  and  $V$  such that  $C \subseteq U$  and  $D \subseteq V$ .

⟨2⟩1.  $C$  and  $D$  are closed in  $X$ .

PROOF: Proposition 13.19.6.

⟨2⟩2.  $C$  and  $D$  are compact.

PROOF: Theorem 13.47.7.

⟨2⟩3. Q.E.D.

PROOF: Proposition 13.47.9.

⟨1⟩3.  $\bigcap_{A \in \mathcal{A}} (A - (U \cup V))$  is nonempty.

⟨2⟩1.  $\{A - (U \cup V) : A \in \mathcal{A}\}$  has the finite intersection property.

⟨3⟩1. LET:  $A_1, \dots, A_n \in \mathcal{A}$

⟨3⟩2. ASSUME: w.l.o.g.  $A_1 \supseteq \dots \supseteq A_n$

PROOF:  $\mathcal{A}$  is linearly ordered by inclusion.

⟨3⟩3.  $\bigcap_{i=1}^n (A_i - (U \cup V)) = A_n - (U \cup V)$

⟨3⟩4.  $A_n - (U \cup V) \neq \emptyset$

PROOF: Otherwise  $(A_n \cap U, A_n \cap V)$  would be a separation of  $A_n$ .

⟨2⟩2. Q.E.D.

PROOF: Theorem 13.47.16.

⟨1⟩4. Q.E.D.

PROOF: This is a contradiction.

□



## Chapter 14

# Metric Spaces

**Definition 14.0.1** (Metric Space). Let  $X$  be a set and  $d : X^2 \rightarrow \mathbb{R}$ . We say  $(X, d)$  is a *metric space* iff:

- For all  $x, y \in X$  we have  $d(x, y) \geq 0$
- For all  $x, y \in X$  we have  $d(x, y) = 0$  iff  $x = y$
- For all  $x, y \in X$  we have  $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all  $x, y, z \in X$  we have  $d(x, z) \leq d(x, y) + d(y, z)$

We call  $d$  the *metric* of the metric space  $(X, d)$ . We often write  $X$  for the metric space  $(X, d)$ .

**Definition 14.0.2** (Discrete Metric). On any set  $X$ , define the *discrete* metric by  $d(x, y) = 0$  if  $x = y$ , 1 if  $x \neq y$ .

**Definition 14.0.3** (Standard Metric). The *standard metric* on  $\mathbb{R}$  is defined by  $d(x, y) = |x - y|$ .

**Definition 14.0.4** (Square Metric). The *square metric*  $\rho$  on  $\mathbb{R}^n$  is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1$ . For all  $\vec{x}, \vec{y} \in \mathbb{R}^n$  we have  $\rho(\vec{x}, \vec{y}) \geq 0$ .

PROOF: Immediate from definition.

$\langle 1 \rangle 2$ . For all  $\vec{x}, \vec{y} \in \mathbb{R}^n$  we have  $\rho(\vec{x}, \vec{y}) = 0$  iff  $\vec{x} = \vec{y}$ .

PROOF:

$$\begin{aligned} \rho(\vec{x}, \vec{y}) = 0 &\Leftrightarrow \max(|x_1 - y_1|, \dots, |x_n - y_n|) = 0 \\ &\Leftrightarrow |x_1 - y_1| = \dots = |x_n - y_n| = 0 \\ &\Leftrightarrow x_1 = y_1 \wedge \dots \wedge x_n = y_n \\ &\Leftrightarrow \vec{x} = \vec{y} \end{aligned}$$

⟨1⟩3. For all  $\vec{x}, \vec{y} \in \mathbb{R}^n$  we have  $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$ .

PROOF: Immediate from definition.

⟨1⟩4. For all  $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$  we have  $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$ .

PROOF:

$$\begin{aligned} & \max(|x_1 - z_1|, \dots, |x_n - z_n|) \\ & \leq \max(|x_1 - y_1| + |y_1 - z_1|, \dots, |x_n - y_n| + |y_n - z_n|) \\ & \leq \max(|x_1 - y_1|, \dots, |x_n - y_n|) + \max(|y_1 - z_1|, \dots, |y_n - z_n|) \\ & = \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z}) \end{aligned}$$

□

### 14.0.1 Balls

**Definition 14.0.5** (Ball). Let  $X$  be a metric space. Let  $x \in X$  and  $r > 0$ . The *ball* with *centre*  $x$  and *radius*  $r$  is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} .$$

**Definition 14.0.6** (Closed Ball). Let  $X$  be a metric space. Let  $x \in X$  and  $r > 0$ . The *closed ball* with *centre*  $x$  and *radius*  $r$  is

$$\overline{B(x, r)} = \{y \in X \mid d(x, y) \leq r\} .$$

**Definition 14.0.7** (Metric Topology). Let  $(X, d)$  be a metric space. The *metric topology* on  $X$  is the topology generated by the basis consisting of the balls.

We prove this is a basis for a topology.

PROOF:

⟨1⟩1. Every point is a member of some ball.

PROOF: Since  $x \in B(x, 1)$ .

⟨1⟩2. If  $B_1$  and  $B_2$  are balls and  $x \in B_1 \cap B_2$ , then there exists a ball  $B_3$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

⟨2⟩1. LET:  $x \in B(a, \epsilon_1) \cap B(b, \epsilon_2)$

⟨2⟩2. LET:  $\epsilon = \min(\epsilon_1 - d(x, a), \epsilon_2 - d(x, b))$

PROVE:  $x \in B(x, \epsilon) \subseteq B(a, \epsilon_1) \cap B(b, \epsilon_2)$

⟨2⟩3.  $B(x, \epsilon) \subseteq B(a, \epsilon_1)$

⟨3⟩1. LET:  $y \in B(x, \epsilon)$

⟨3⟩2.  $d(y, a) < \epsilon_1$

PROOF:

$$d(y, a) \leq d(y, x) + d(x, a) \quad (\text{Triangle Inequality})$$

$$< \epsilon + d(x, a) \quad (\langle 3 \rangle 1)$$

$$\leq \epsilon_1 \quad (\langle 2 \rangle 2)$$

⟨2⟩4.  $B(x, \epsilon) \subseteq B(b, \epsilon_2)$

PROOF: Similar.

□

**Proposition 14.0.8.** *The discrete metric on a set  $X$  induces the discrete topology.*

PROOF: Since  $B(x, 1/2) = \{x\}$  for all  $x \in X$ .  $\square$

**Proposition 14.0.9.** *The standard metric on  $\mathbb{R}$  induces the standard topology.*

PROOF:

$\langle 1 \rangle 1$ . Every ball is open in the standard topology.

PROOF: Since  $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$ .

$\langle 1 \rangle 2$ . Every open ray is open in the metric topology.

PROOF: If  $x \in (a, +\infty)$  then  $x \in B(x, x - a) \subseteq (a, +\infty)$ . Similarly for  $(-\infty, a)$ .

$\square$

**Proposition 14.0.10.** *The square metric on  $\mathbb{R}^n$  induces the product topology.*

PROOF:

$\langle 1 \rangle 1$ . For any real numbers  $a_1, \dots, a_n, b_1, \dots, b_n$  with  $a_1 < b_1, \dots, a_n < b_n$ , we have  $(a_1, b_1) \times \dots \times (a_n, b_n)$  is open in the metric topology.

$\langle 2 \rangle 1$ . LET:  $\vec{x} \in (a_1, b_1) \times \dots \times (a_n, b_n)$

$\langle 2 \rangle 2$ . LET:  $\epsilon = \min(x_1 - a_1, b_1 - x_1, \dots, x_n - a_n, b_n - x_n)$

$\langle 2 \rangle 3$ .  $B(\vec{x}, \epsilon) \subseteq (a_1, b_1) \times \dots \times (a_n, b_n)$

$\langle 1 \rangle 2$ . For any  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B(\vec{a}, \epsilon)$  is open in the product topology.

PROOF: Since  $B(\vec{a}, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \dots \times (a_n - \epsilon, a_n + \epsilon)$ .

$\square$

**Proposition 14.0.11.** *Addition is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $(x, y) \in \mathbb{R}^2$  and  $\epsilon > 0$

$\langle 1 \rangle 2$ . LET:  $\delta = \epsilon/2$

$\langle 1 \rangle 3$ . LET:  $(x', y') \in \mathbb{R}^2$  with  $\rho((x, y), (x', y')) < \delta$

$\langle 1 \rangle 4$ .  $|x - x'|, |y - y'| < \delta$

$\langle 1 \rangle 5$ .  $|(x + y) - (x' + y')| < \epsilon$

PROOF:

$$\begin{aligned} |(x + y) - (x' + y')| &\leq |x - x'| + |y - y'| \\ &< \delta + \delta && (\langle 1 \rangle 4) \\ &= \epsilon && (\langle 1 \rangle 2) \end{aligned}$$

$\square$

**Proposition 14.0.12.** *Multiplication is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $(x, y) \in \mathbb{R}^2$  and  $\epsilon > 0$

$\langle 1 \rangle 2$ . LET:  $\delta = \min(\epsilon/(|x| + |y| + 1), 1)$

$\langle 1 \rangle 3$ . LET:  $(x', y') \in \mathbb{R}^2$  with  $\rho((x, y), (x', y')) < \delta$

$\langle 1 \rangle 4$ .  $|x - x'|, |y - y'| < \delta$

$\langle 1 \rangle 5$ .  $|xy - x'y'| < \epsilon$

PROOF:

$$\begin{aligned}
 |xy - x'y'| &= |xy - xy' + xy - x'y - xy + x'y + xy' - x'y'| \\
 &\leq |xy - xy'| + |xy - x'y| + |xy - x'y - xy' + xy'y| = |x||y - y'| + |x - x'||y| + |x - x'||y - y'| \\
 &< |x|\delta + |y|\delta + \delta^2 \\
 &\leq |x|\delta + |y|\delta + \delta \\
 &= (|x| + |y| + 1)\delta \\
 &\leq \epsilon
 \end{aligned}$$

□

**Corollary 14.0.12.1.** *The unit circle  $S^1$  is a closed subset of  $\mathbb{R}^2$ .*

PROOF: The function  $f$  that maps  $(x, y)$  to  $x^2 + y^2$  is continuous, and  $S^1 = f^{-1}(\{1\})$ . □

**Corollary 14.0.12.2.** *The unit ball  $B^2$  is a closed subset of  $\mathbb{R}^2$ .*

PROOF: The function  $f$  that maps  $(x, y)$  to  $x^2 + y^2$  is continuous, and  $B^2 = f^{-1}([0, 1])$ . □

**Proposition 14.0.13.** *Let  $(a_n)$  and  $(b_n)$  be sequences of real numbers. Let  $c, s, t \in \mathbb{R}$ . Assume*

$$\sum_{n=0}^{\infty} a_n = s \text{ and } \sum_{n=0}^{\infty} b_n = t .$$

*Then*

$$\sum_{n=0}^{\infty} (ca_n + b_n) = cs + t .$$

PROOF:

$$\sum_{n=0}^N (ca_n + b_n) = c \sum_{n=0}^N a_n + \sum_{n=0}^N b_n \rightarrow cs + t \text{ as } n \rightarrow \infty \quad \square$$

**Proposition 14.0.14** (Comparison Test). *Let  $(a_n)$  and  $(b_n)$  be sequences of real numbers. Assume  $|a_n| \leq b_n$  for all  $n$ . Assume  $\sum_{n=0}^{\infty} b_n$  converges. Then  $\sum_{n=0}^{\infty} a_n$  converges.*

PROOF:

⟨1⟩1. For all  $n$ ,

$$\text{LET: } c_n = |a_n| + a_n$$

⟨1⟩2.  $\sum_{n=0}^{\infty} |a_n|$  converges.

PROOF: Since  $(\sum_{n=0}^N |a_n|)_N$  is an increasing sequence of real numbers bounded above by  $\sum_{n=0}^{\infty} b_n$ .

⟨1⟩3.  $\sum_{n=0}^{\infty} c_n$  converges.

PROOF: Since  $(\sum_{n=0}^N c_n)_N$  is an increasing sequence of real numbers bounded above by  $2 \sum_{n=0}^{\infty} a_n$ .

⟨1⟩4.  $\sum_{n=0}^{\infty} a_n$  converges.

PROOF: Since  $a_n = c_n - |a_n|$ .



□

**Proposition 14.0.15.** *Let  $X$  be a metric space. Let  $U \subseteq X$ . Then  $U$  is open if and only if, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .*

PROOF:

⟨1⟩1. If  $U$  is open then, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

⟨2⟩1. ASSUME:  $U$  is open.

⟨2⟩2. LET:  $x \in U$

⟨2⟩3. PICK a ball  $B(a, \delta)$  such that  $x \in B(a, \delta) \subseteq U$

⟨2⟩4. LET:  $\epsilon = \delta - d(a, x)$

PROVE:  $B(x, \epsilon) \subseteq U$

⟨2⟩5. LET:  $y \in B(x, \epsilon)$

⟨2⟩6.  $y \in B(a, \delta)$

PROOF:

$$\begin{aligned} d(a, y) &\leq d(a, x) + d(x, y) && \text{(Triangle Inequality)} \\ &< d(a, x) + \epsilon && (\langle 2 \rangle 5) \\ &= \delta \end{aligned}$$

⟨2⟩7.  $y \in U$

PROOF: ⟨2⟩3

⟨1⟩2. If, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ , then  $U$  is open.

PROOF: Immediate from definition of the metric topology.

□

**Proposition 14.0.16.** *Let  $X$  be a metric space. Let  $a, b, c \in X$ . Then*

$$|d(a, b) - d(a, c)| \leq d(b, c) .$$

PROOF:

⟨1⟩1.  $d(a, b) - d(a, c) \leq d(b, c)$

PROOF: Triangle Inequality.

⟨1⟩2.  $d(a, c) - d(a, b) \leq d(b, c)$

PROOF: Triangle Inequality.

□

**Proposition 14.0.17.** *Let  $(X, d)$  be a metric space. Then the metric topology on  $X$  is the coarsest topology such that  $d : X^2 \rightarrow \mathbb{R}$  is continuous.*

PROOF:

⟨1⟩1.  $d$  is continuous with respect to the metric topology.

⟨2⟩1. LET:  $(a, b) \in X^2$

⟨2⟩2. LET:  $V$  be a neighbourhood of  $d(a, b)$ .

⟨2⟩3. PICK  $\epsilon > 0$  such that  $(d(a, b) - \epsilon, d(a, b) + \epsilon) \subseteq V$ .

⟨2⟩4. LET:  $U = B(a, \epsilon/2) \times B(b, \epsilon/2)$

⟨2⟩5. LET:  $(x, y) \in U$

⟨2⟩6.  $|d(x, y) - d(a, b)| < \epsilon$

PROOF:

$$\begin{aligned} |d(x, y) - d(a, b)| &\leq |d(x, y) - d(a, y)| + |d(a, y) - d(a, b)| \\ &\leq d(a, x) + d(b, y) && \text{(Proposition 14.0.16)} \\ &< \epsilon \end{aligned}$$

$\langle 2 \rangle 7. d(x, y) \in V$

$\langle 1 \rangle 2.$  If  $\mathcal{T}$  is a topology on  $X$  with respect to which  $d$  is continuous then  $\mathcal{T}$  is finer than the metric topology.

$\langle 2 \rangle 1.$  LET:  $\mathcal{T}$  be a topology on  $X$  with respect to which  $d$  is continuous.

$\langle 2 \rangle 2.$  LET:  $a \in X$  and  $\epsilon > 0$ .

PROVE:  $B(a, \epsilon) \in \mathcal{T}$

$\langle 2 \rangle 3.$  LET:  $x \in B(a, \epsilon)$

$\langle 2 \rangle 4.$   $(a, x) \in d^{-1}((0, \epsilon))$

$\langle 2 \rangle 5.$  PICK  $U, V \in \mathcal{T}$  such that  $(a, x) \in U \times V \subseteq d^{-1}((0, \epsilon))$

$\langle 2 \rangle 6.$   $x \in V \subseteq B(a, \epsilon)$

□

**Proposition 14.0.18.** Let  $d$  and  $d'$  be two metrics on the same set  $X$ . Let  $\mathcal{T}$  and  $\mathcal{T}'$  be the topologies they induce. Then  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1.$  If  $\mathcal{T} \subseteq \mathcal{T}'$  then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ .

$\langle 2 \rangle 1.$  ASSUME:  $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 2.$  LET:  $x \in X$  and  $\epsilon > 0$

$\langle 2 \rangle 3.$   $x \in B_d(x, \epsilon) \in \mathcal{T}'$

$\langle 2 \rangle 4.$  There exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: Proposition 14.0.15.

$\langle 1 \rangle 2.$  If, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ , then  $\mathcal{T} \subseteq \mathcal{T}'$ .

$\langle 2 \rangle 1.$  ASSUME: For all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ .

$\langle 2 \rangle 2.$  LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 3.$  For all  $x \in U$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq U$

$\langle 3 \rangle 1.$  LET:  $x \in U$

$\langle 3 \rangle 2.$  PICK  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 14.0.15.

$\langle 3 \rangle 3.$  PICK  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ .

PROOF:  $\langle 2 \rangle 1$

$\langle 3 \rangle 4.$   $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4.$   $U \in \mathcal{T}'$

PROOF: Proposition 14.0.15.

□

**Definition 14.0.19** (Metrizability). A topological space is *metrizable* iff there exists a metric that induces its topology.

**Proposition 14.0.20.**  $\mathbb{R}^2$  under the dictionary order is metrizable.

PROOF:

$\langle 1 \rangle 1$ . LET:  $d : (\mathbb{R}^2)^2 \rightarrow \mathbb{R}$  be defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} \min(|y_2 - y_1|, 1) & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

$\langle 1 \rangle 2$ .  $d$  is a metric.

$\langle 2 \rangle 1$ . For all  $x, y \in \mathbb{R}^2$  we have  $d(x, y) \geq 0$ .

PROOF: Immediate from definition.

$\langle 2 \rangle 2$ . For all  $x, y \in \mathbb{R}^2$  we have  $d(x, y) = 0$  iff  $x = y$ .

PROOF: Immediate from definition.

$\langle 2 \rangle 3$ . For all  $x, y \in \mathbb{R}^2$  we have  $d(x, y) = d(y, x)$ .

PROOF: Immediate from definition.

$\langle 2 \rangle 4$ . For all  $x, y, z \in \mathbb{R}^2$  we have  $d(x, z) \leq d(x, y) + d(y, z)$ .

PROOF: Easy.

$\langle 1 \rangle 3$ . The metric topology induced by  $d$  is finer than the order topology.

$\langle 2 \rangle 1$ . LET:  $a, b \in \mathbb{R}^2$

$\langle 2 \rangle 2$ . LET:  $x \in (a, b)$

$\langle 2 \rangle 3$ . CASE:  $\pi_1(x) = \pi_1(a) = \pi_1(b)$

$\langle 3 \rangle 1$ . LET:  $\epsilon = \min(\pi_2(x) - \pi_2(a), \pi_2(b) - \pi_2(x))$

$\langle 3 \rangle 2$ .  $B(x, \epsilon) \subseteq (a, b)$

$\langle 2 \rangle 4$ . CASE:  $\pi_1(a) = \pi_1(x) < \pi_1(b)$

$\langle 3 \rangle 1$ . LET:  $\epsilon = \pi_2(x) - \pi_2(a)$

$\langle 3 \rangle 2$ .  $B(x, \epsilon) \subseteq (a, b)$

$\langle 2 \rangle 5$ . CASE:  $\pi_1(a) < \pi_1(x) = \pi_1(b)$

PROOF: Similar.

$\langle 2 \rangle 6$ . CASE:  $\pi_1(a) < \pi_1(x) < \pi_1(b)$

PROOF: Then  $B(x, \epsilon) \subseteq (a, b)$ .

$\langle 1 \rangle 4$ . The order topology is finer than the metric topology.

PROOF: Since  $B((a, b), \epsilon) = ((a, b - \epsilon), (a, b + \epsilon))$  if  $\epsilon \leq 1$ , and  $\mathbb{R}^2$  if  $\epsilon > 1$ .

□

**Proposition 14.0.21.** Every metrizable space is first countable.

PROOF: For any point  $a$ , the set  $\{B(a, 1/n) : n \in \mathbb{Z}_+\}$  is a local basis at  $a$ . □

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

**Proposition 14.0.22.** Every compact metric space is bounded.

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a compact metric space.

$\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $X$  is nonempty.

$\langle 1 \rangle 3$ . PICK  $a \in X$

$\langle 1 \rangle 4$ .  $\{B(a, n) : n \in \mathbb{Z}_+\}$  is an open cover of  $X$ .

$\langle 1 \rangle 5$ . PICK a finite subcover  $\{B(a, n_1), \dots, B(a, n_k)\}$ .

$\langle 1 \rangle 6$ . LET:  $N = \max(n_1, \dots, n_k)$

⟨1⟩7.  $X = B(a, N)$

□

**Example 14.0.23.** The converse does not hold. An infinite discrete space is bounded but not compact.

### 14.0.2 Subspaces

**Proposition 14.0.24.** *Let  $(X, d)$  be a metric space and  $Y \subseteq X$ . Then  $d|_{Y^2}$  is a metric on  $Y$  that induces the subspace topology.*

PROOF:

⟨1⟩1. LET:  $d' = d|_{Y^2} : Y^2 \rightarrow \mathbb{R}$

⟨1⟩2.  $d'$  is a metric.

PROOF: Each of the axioms follows from the axiom in  $X$ .

⟨1⟩3. The metric topology induced by  $d'$  is finer than the subspace topology.

⟨2⟩1. LET:  $U$  be open in  $X$

PROVE:  $U \cap Y$  is open in the  $d'$ -topology.

⟨2⟩2. LET:  $y \in U \cap Y$

⟨2⟩3. PICK  $\epsilon > 0$  such that  $B_d(y, \epsilon) \subseteq U$

⟨2⟩4.  $B_{d'}(y, \epsilon) \subseteq U \cap Y$

⟨1⟩4. The subspace topology is finer than the metric topology induced by  $d'$ .

⟨2⟩1. LET:  $y \in Y$  and  $\epsilon > 0$

PROVE:  $B_{d'}(y, \epsilon)$  is open in the subspace topology.

⟨2⟩2.  $B_{d'}(y, \epsilon) = B_d(y, \epsilon) \cap Y$

□

### 14.0.3 Convergence

**Proposition 14.0.25** (Sequence Lemma). *Let  $X$  be a metric space. Let  $A \subseteq X$ . Let  $l \in \overline{A}$ . Then there exists a sequence in  $A$  that converges to  $l$ .*

PROOF:

⟨1⟩1. For  $n \in \mathbb{N}$ , PICK  $a_n \in B(l, 1/(n+1)) \cap A$ .

⟨1⟩2.  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .

□

**Corollary 14.0.25.1.**  $\mathbb{R}^\omega$  under the box topology is not first countable.

PROOF:

⟨1⟩1. LET:  $A$  be the set of all sequences of positive reals.

⟨1⟩2.  $0 \in \overline{A}$

⟨1⟩3. LET:  $(a_n)$  be a sequence in  $A$

PROVE:  $(a_n)$  does not converge to 0.

⟨1⟩4. For all  $n \in \mathbb{N}$ ,

LET:  $a_n = (x_{nm})$

⟨1⟩5. LET:  $B' = \prod_{n=0}^{\infty} (-x_{nn}, x_{nn})$

⟨1⟩6.  $B'$  is open in the box topology.

- $\langle 1 \rangle 7. 0 \in B'$   
 $\langle 1 \rangle 8. \text{ For all } n \text{ we have } a_n \notin B'$   
 $\square$

**Corollary 14.0.25.2.** *If  $J$  is an uncountable set then  $\mathbb{R}^J$  under the product topology is not first countable.*

PROOF:

- $\langle 1 \rangle 1. \text{ LET: } A = \{x \in \mathbb{R}^J : \pi_j(x) = 1 \text{ for all but finitely many } j \in J\}$   
 $\langle 1 \rangle 2. 0 \in \overline{A}$   
 $\langle 1 \rangle 3. \text{ LET: } (a_n) \text{ be a sequence in } A.$   
 $\text{PROVE: } (a_n) \text{ does not converge to } 0.$   
 $\langle 1 \rangle 4. \text{ For } n \in \mathbb{N},$   
 $\text{LET: } J_n = \{j \in J : \pi_j(a_n) \neq 1\}$   
 $\langle 1 \rangle 5. \bigcup_{n \in \mathbb{N}} J_n \text{ is countable.}$   
 $\langle 1 \rangle 6. \text{ PICK } \beta \in J - \bigcup_{n \in \mathbb{N}} J_n$   
 $\langle 1 \rangle 7. \forall n \in \mathbb{N}. \pi_\beta(a_n) = 1$   
 $\langle 1 \rangle 8. \text{ LET: } U = \pi_\beta^{-1}((-1, 1))$   
 $\langle 1 \rangle 9. 0 \in U$   
 $\langle 1 \rangle 10. \forall n \in \mathbb{N}. a_n \notin U$   
 $\langle 1 \rangle 11. (a_n) \text{ does not converge to } 0.$   
 $\square$

#### 14.0.4 Continuous Functions

**Proposition 14.0.26.** *Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .*

PROOF:

- $\langle 1 \rangle 1. \text{ If } f \text{ is continuous then, for all } x \in X \text{ and } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such}$   
 $\text{that, for all } y \in X, \text{ if } d(x, y) < \delta \text{ then } d(f(x), f(y)) < \epsilon.$   
 $\langle 2 \rangle 1. \text{ ASSUME: } f \text{ is continuous.}$   
 $\langle 2 \rangle 2. \text{ LET: } x \in X$   
 $\langle 2 \rangle 3. \text{ LET: } \epsilon > 0$   
 $\langle 2 \rangle 4. x \in f^{-1}(B(f(x), \epsilon))$   
 $\langle 2 \rangle 5. \text{ There exists } \delta > 0 \text{ such that } B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon)).$   
 $\langle 1 \rangle 2. \text{ If, for all } x \in X \text{ and } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that, for all } y \in X, \text{ if}$   
 $d(x, y) < \delta \text{ then } d(f(x), f(y)) < \epsilon, \text{ then } f \text{ is continuous.}$   
 $\langle 2 \rangle 1. \text{ ASSUME: For all } x \in X \text{ and } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that, for all}$   
 $y \in X, \text{ if } d(x, y) < \delta \text{ then } d(f(x), f(y)) < \epsilon.$   
 $\langle 2 \rangle 2. \text{ LET: } V \text{ be open in } Y$   
 $\langle 2 \rangle 3. \text{ LET: } x \in f^{-1}(V)$   
 $\langle 2 \rangle 4. \text{ PICK } \epsilon > 0 \text{ such that } B(f(x), \epsilon) \subseteq V$   
 $\langle 2 \rangle 5. \text{ PICK } \delta > 0 \text{ such that, for all } y \in X, \text{ if } d(x, y) < \delta \text{ then } d(f(x), f(y)) < \epsilon.$   
 $\langle 2 \rangle 6. B(x, \delta) \subseteq f^{-1}(V)$   
 $\square$

**Proposition 14.0.27.** *Let  $X$  be a metrizable space and  $Y$  a topological space. Let  $f : X \rightarrow Y$ . Assume that, for every sequence  $(x_n)$  in  $X$  and  $l \in X$ , if  $x_n \rightarrow l$  as  $n \rightarrow \infty$  then  $f(x_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ . Then  $f$  is continuous.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $A \subseteq X$   
 PROVE:  $f(\overline{A}) \subseteq \overline{f(A)}$   
 $\langle 1 \rangle 2$ . LET:  $l \in \overline{A}$   
 PROVE:  $f(l) \in \overline{f(A)}$   
 $\langle 1 \rangle 3$ . PICK a sequence  $(x_n)$  in  $A$  such that  $x_n \rightarrow l$  as  $n \rightarrow \infty$ .  
 $\langle 1 \rangle 4$ .  $f(x_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ .  
 $\langle 1 \rangle 5$ .  $f(l) \in \overline{f(A)}$   
 $\square$

**Proposition 14.0.28.** *The function  $i : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$  that maps  $x$  to  $x^{-1}$  is continuous.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $a, b \in \mathbb{R}$  with  $a < b$   
 PROVE:  $i^{-1}((a, b))$  is open.  
 $\langle 1 \rangle 2$ . CASE:  $0 < a$   
 PROOF:  $i^{-1}((a, b)) = (b^{-1}, a^{-1})$   
 $\langle 1 \rangle 3$ . CASE:  $a = 0$   
 PROOF:  $i^{-1}((a, b)) = (b^{-1}, +\infty)$   
 $\langle 1 \rangle 4$ . CASE:  $a < 0 < b$   
 PROOF:  $i^{-1}((a, b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$   
 $\langle 1 \rangle 5$ . CASE:  $b = 0$   
 PROOF:  $i^{-1}((a, b)) = (-\infty, a^{-1})$   
 $\langle 1 \rangle 6$ . CASE:  $b < 0$   
 PROOF:  $i^{-1}((a, b)) = (b^{-1}, a^{-1})$   
 $\square$

**Proposition 14.0.29.** *Subtraction is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .*

PROOF: Since  $a - b = a + (-1)b$  and both addition and multiplication are continuous.  $\square$

**Proposition 14.0.30.** *Division is a continuous function  $\mathbb{R} \times (\mathbb{R} - \{0\}) \rightarrow \mathbb{R}$ .*

PROOF: Since both multiplication and the function that maps  $x$  to  $x^{-1}$  are continuous.  $\square$

### 14.0.5 First Countable Spaces

**Proposition 14.0.31.** *Every metrizable space is first countable.*

PROOF: For any point  $x$ , the set  $\{B(x, 1/n) : n \in \mathbb{Z}_+\}$  is a countable basis at  $x$ .  
 $\square$

**Corollary 14.0.31.1.**  $\mathbb{R}^\omega$  under the box topology is not metrizable.

**Corollary 14.0.31.2.** *If  $J$  is an uncountable set then  $\mathbb{R}^J$  under the product topology is not metrizable.*

### 14.0.6 Hausdorff Spaces

**Proposition 14.0.32.** *Every metric space is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a metric space.

$\langle 1 \rangle 2$ . LET:  $x, y \in X$  with  $x \neq y$ .

$\langle 1 \rangle 3$ . LET:  $\epsilon = d(x, y)$

$\langle 1 \rangle 4$ .  $B(x, \epsilon/2)$  and  $B(y, \epsilon/2)$  are disjoint neighbourhoods of  $x$  and  $y$ .

□

### 14.0.7 Bounded Sets

**Definition 14.0.33** (Bounded). Let  $X$  be a metric space. Let  $A \subseteq X$ . Then  $A$  is *bounded* iff there exists  $M$  such that  $\forall x, y \in A. d(x, y) \leq M$ . Its *diameter* is then defined to be

$$\text{diam } A := \sup\{d(x, y) : x, y \in A\} .$$

### 14.0.8 Uniform Convergence

**Definition 14.0.34** (Uniform Convergence). Let  $X$  be a set and  $Y$  a metric space. Let  $(f_n)$  be a sequence of functions  $X \rightarrow Y$ , and  $f : X \rightarrow Y$ . Then  $(f_n)$  *converges uniformly* to  $f$  iff, for all  $\epsilon > 0$ , there exists  $N$  such that

$$\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon .$$

**Example 14.0.35.** For  $n \in \mathbb{N}$  define  $f_n : [0, 1] \rightarrow \mathbb{R}$  by  $f_n(x) = x^n$ . Define  $f : [0, 1] \rightarrow \mathbb{R}$  by  $f(x) = 0$  for  $x < 1$ ,  $f(1) = 1$ . Then  $f_n$  converges pointwise to  $f$ , but does not converge uniformly to  $f$ .

We prove that, for all  $N$ , there exists  $n \geq N$  and  $x \in [0, 1]$  such that  $|x^n - f(x)| \geq 1/2$ . Take  $n = N$  and  $x$  to be the  $N$ th root of  $3/4$ .

**Example 14.0.36.** For  $n \in \mathbb{N}$ , define  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} .$$

Then for all  $x \in \mathbb{R}$  we have  $f_n(x) \rightarrow 0$  as  $n \rightarrow \infty$ , but  $(f_n)$  does not converge uniformly to 0.

We prove that, for all  $N$ , there exists  $n \geq N$  and  $x \in \mathbb{R}$  such that  $|f_n(x)| \geq 1/2$ . Take  $n = N$  and  $x = 1/N$ . We have  $f_N(1/N) = 1$ .

**Theorem 14.0.37** (Uniform Limit Theorem). *Let  $X$  be a topological space and  $Y$  a metric space. Let  $(f_n)$  be a sequence of functions  $X \rightarrow Y$ , and  $f : X \rightarrow Y$ . If every  $f_n$  is continuous and  $(f_n)$  converges uniformly to  $f$ , then  $f$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $V$  be open in  $Y$ .

$\langle 1 \rangle 2$ . LET:  $x_0 \in f^{-1}(V)$

PROVE: There exists a neighbourhood  $U$  of  $x_0$  such that  $f(U) \subseteq V$ .

$\langle 1 \rangle 3$ . LET:  $y_0 = f(x_0)$

$\langle 1 \rangle 4$ . PICK  $\epsilon > 0$  such that  $B(y_0, \epsilon) \subseteq V$ .

$\langle 1 \rangle 5$ . PICK  $N$  such that  $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/3$ .

$\langle 1 \rangle 6$ . PICK a neighbourhood  $U$  of  $x_0$  such that  $f_N(U) \subseteq B(f_N(x_0), \epsilon/3)$ .

PROVE:  $f(U) \subseteq V$

$\langle 1 \rangle 7$ . LET:  $y \in U$

$\langle 1 \rangle 8$ .  $d(f(y), y_0) < \epsilon$

PROOF:

$$\begin{aligned} d(f(y), y_0) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x_0)) + d(f_N(x_0), y_0) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1 \rangle 5, \langle 1 \rangle 6) \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 9$ .  $f(y) \in V$

PROOF:  $\langle 1 \rangle 4$

□

**Proposition 14.0.38.** *Let  $X$  be a topological space. Let  $Y$  be a metric space. Let  $f_n$  be a sequence of functions  $X \rightarrow Y$  and  $f : X \rightarrow Y$ . Let  $x_n$  be a sequence of points in  $X$  and  $l \in X$ . If  $f_n$  converges uniformly to  $f$ ,  $x_n$  converges to  $l$ , and  $f$  is continuous, then  $f_n(x_n)$  converges to  $f(l)$ .*

PROOF:

$\langle 1 \rangle 1$ .  $f$  is continuous.

$\langle 1 \rangle 2$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 3$ . PICK  $\delta > 0$  such that  $\forall y \in X. d(y, l) < \delta \Rightarrow d(f(y), f(l)) < \epsilon/2$

$\langle 1 \rangle 4$ . PICK  $N$  such that  $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/2$  and  $\forall n \geq N. d(x_n, l) < \delta$

$\langle 1 \rangle 5$ . For all  $n \geq N$  we have  $d(f_n(x_n), f(l)) < \epsilon$

PROOF:

$$\begin{aligned} d(f_n(x_n), f(l)) &\leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(l)) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

**Theorem 14.0.39** (Weierstrass M-Test). *Let  $X$  be a set. Let  $(f_n)$  be a sequence of functions  $X \rightarrow \mathbb{R}$ . Let  $(M_n)$  be a sequence of real numbers. For  $n \in \mathbb{N}$ , let*

$$s_n(x) = \sum_{i=0}^n f_i(x) \quad .$$

*Assume that  $\forall n \in \mathbb{N}. \forall x \in X. |f_n(x)| \leq M_n$ . Assume that  $\sum_{n=0}^{\infty} M_n$  converges. Then  $(s_n)$  uniformly converges to  $s$  where  $s(x) = \sum_{n=0}^{\infty} f_n(x)$ .*

PROOF:



⟨1⟩1. For all  $x \in X$  we have  $\sum_{n=0}^{\infty} f_n(x)$  converges.

PROOF: By the Comparison Test.

⟨1⟩2. For  $n \in \mathbb{N}$ ,

LET:  $r_n = \sum_{i=n+1}^{\infty} M_i$ .

⟨1⟩3. For all  $k, n \in \mathbb{N}$  and  $x \in X$ , if  $k > n$  then  $|s_k(x) - s_n(x)| \leq r_n$ .

PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\ &\leq \sum_{i=n+1}^k |f_i(x)| \\ &\leq \sum_{i=n+1}^k M_i \\ &\leq \sum_{i=n+1}^{\infty} M_i \\ &= r_n \end{aligned}$$

⟨1⟩4. For all  $n \in \mathbb{N}$  we have  $|s(x) - s_n(x)| \leq r_n$ .

PROOF: Taking the limit  $k \rightarrow \infty$  in ⟨1⟩3.

⟨1⟩5.  $(s_n)$  converges uniformly to  $s$ .

PROOF: We have  $\bar{\rho}(s_n, s) \leq r_n$  and so  $\bar{\rho}(s_n, s) \rightarrow 0$  as  $n \rightarrow \infty$  by the Sandwich Theorem.

□

**Theorem 14.0.40.** *Let  $X$  be a compact space. Let  $f_n : X \rightarrow \mathbb{R}$  be a monotone increasing sequence of continuous functions, and  $f : X \rightarrow \mathbb{R}$  be continuous. If  $f_n$  converges to  $f$  pointwise, then  $f_n$  converges to  $f$  uniformly.*

PROOF:

⟨1⟩1. LET:  $\epsilon > 0$

⟨1⟩2. For all  $x \in X$ , there exists a neighbourhood  $U$  of  $x$  and an integer  $N$  such that, for all  $y \in U$ , we have  $|f_N(y) - f(y)| < \epsilon$

⟨2⟩1. LET:  $x \in X$

⟨2⟩2. PICK  $N$  such that  $\forall n \geq N, |f_n(x) - f(x)| < \epsilon/3$ .

⟨2⟩3. PICK neighbourhoods  $U_1, U_2$  of  $x$  such that  $\forall y \in U_1, |f_N(y) - f_N(x)| < \epsilon/3$  and  $\forall y \in U_2, |f(y) - f(x)| < \epsilon/3$ .

⟨2⟩4. LET:  $U = U_1 \cap U_2$

⟨2⟩5. LET:  $y \in U$

⟨2⟩6.  $|f_N(y) - f(y)| < \epsilon$

PROOF:

$$\begin{aligned} |f_N(y) - f(y)| &\leq |f_N(y) - f_N(x)| + |f_N(x) - f(x)| + |f(x) - f(y)| \quad (\text{Triangle Inequality}) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \quad (\langle 2 \rangle 2, \langle 2 \rangle 3) \\ &= \epsilon \end{aligned}$$

⟨1⟩3. PICK open sets  $U_1, \dots, U_k$  that cover  $X$  such that, for all  $i$ , there exists  $N_i$  such that  $\forall y \in U_i, |f_{N_i}(y) - f(y)| < \epsilon$

- <1>4. LET:  $N = \max(N_1, \dots, N_k)$   
 <1>5. For all  $x \in X$  we have  $|f_N(x) - f(x)| < \epsilon$   
 <1>6. For all  $n \geq N$  and  $x \in X$  we have  $|f_n(x) - f(x)| < \epsilon$   
 $\square$

**Example 14.0.41.** We cannot remove the requirement that  $(f_n)$  is monotone increasing.

For all  $n \in \mathbb{Z}_+$ , define  $f_n : [0, 1] \rightarrow \mathbb{R}$  by

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} .$$

Then  $f_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x$ , but the convergence is not uniform.

**Example 14.0.42.** We cannot remove the requirement that  $X$  is compact.

For all  $n \in \mathbb{Z}_+$ , define  $f_n : [0, 1) \rightarrow \mathbb{R}$  by  $f_n(x) = -x^n$ . Then  $(f_n)$  is monotone increasing and  $f_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x$ , but the convergence is not uniform.

### 14.0.9 Standard Bounded Metric

**Definition 14.0.43** (Standard Bounded Metric). Let  $(X, d)$  be a metric space. The *standard bounded metric* corresponding to  $d$  is

$$\bar{d}(x, y) := \min(d(x, y), 1) .$$

**Proposition 14.0.44.** The standard bounded metric associated with  $d$  induces the same topology as  $d$ .

PROOF:

- <1>1. LET:  $(X, d)$  be a metric space.  
 <1>2. Every  $d$ -ball is open under the topology induced by  $\bar{d}$ .  
     <2>1. LET:  $a \in X$  and  $\epsilon > 0$   
     <2>2. LET:  $x \in B_d(a, \epsilon)$   
     <2>3. LET:  $\delta = \min(\epsilon - d(a, x), 1/2)$   
     <2>4.  $B_{\bar{d}}(x, \delta) \subseteq B_d(a, \epsilon)$   
 <1>3. Every  $\bar{d}$ -ball is open under the topology induced by  $d$ .  
 PROOF: Since  $B_{\bar{d}}(a, \epsilon) = B_d(a, \epsilon)$  if  $\epsilon \leq 1$ , and  $X$  if  $\epsilon > 1$ .  
 $\square$

### 14.0.10 Product Spaces

**Proposition 14.0.45.** The product of a countable family of metrizable spaces is metrizable.

PROOF:

- <1>1. LET:  $(X_n, d_n)$  be a sequence of metric spaces.  
 <1>2. For  $n \in \mathbb{N}$ ,

LET:  $\overline{d}_n$  be the standard bounded metric associated with  $d_n$ .

- ⟨1⟩3. LET:  $X = \prod_{n \in \mathbb{N}} X_n$
- ⟨1⟩4. Define  $D : X^2 \rightarrow \mathbb{R}$  by  $D(x, y) = \sup_{n \in \mathbb{N}} \overline{d}_n(\pi_n(x), \pi_n(y)) / (n + 1)$ .
- ⟨1⟩5.  $D$  is a metric on  $X$ .
  - ⟨2⟩1. For all  $x, y \in X$  we have  $D(x, y) \geq 0$ .
  - ⟨2⟩2. For all  $x, y \in X$  we have  $D(x, y) = 0$  iff  $x = y$ .
  - ⟨2⟩3. For all  $x, y \in X$  we have  $D(x, y) = D(y, x)$ .
  - ⟨2⟩4. For all  $x, y, z \in X$  we have  $D(x, z) \leq D(x, y) + D(y, z)$ .
- ⟨1⟩6. The product topology is finer than the metric topology induced by  $D$ .
  - ⟨2⟩1. LET:  $a \in X$  and  $\epsilon > 0$ .
  - ⟨2⟩2. LET:  $x \in B(a, \epsilon)$
  - ⟨2⟩3. LET:  $\delta = \epsilon - D(a, x)$
  - ⟨2⟩4. PICK  $N \in \mathbb{N}$  such that  $1/(N + 1) < \delta$
  - ⟨2⟩5.  $x \in \prod_{n=0}^N B_{\overline{d}_n}(\pi_n(a), n\delta) \times \prod_{n=N+1}^{\infty} B(a, \epsilon)$
- ⟨1⟩7. The metric topology induced by  $D$  is finer than the product topology.
  - ⟨2⟩1. LET:  $n \in \mathbb{N}$  and  $U$  be an open set in  $X_n$ .  
PROVE:  $\pi_n^{-1}(U)$  is open in the metric topology.
  - ⟨2⟩2. LET:  $x \in \pi_n^{-1}(U)$
  - ⟨2⟩3. PICK  $\epsilon > 0$  such that  $B_{\overline{d}_n}(\pi_n(x), \epsilon) \subseteq U$
  - ⟨2⟩4.  $B(x, \epsilon/(n + 1)) \subseteq \pi_n^{-1}(U)$

□

**Definition 14.0.46.** For  $n \geq 1$ , the *unit ball*  $B^n$  is the closed ball  $\overline{B(0, 1)}$  in  $\mathbb{R}^n$  under the Euclidean metric.

## 14.1 Uniform Metric

**Definition 14.1.1** (Uniform Metric). Let  $J$  be a nonempty set. The *uniform metric*  $\bar{\rho}$  on  $\mathbb{R}^J$  is defined by

$$\bar{\rho}(x, y) = \sup_{j \in J} \bar{d}(x_j, y_j)$$

where  $\bar{d}$  is the standard bounded metric associated with the standard metric on  $\mathbb{R}$ .

The topology it induces is called the *uniform topology*.

We prove this is a metric.

PROOF:

- ⟨1⟩1. For all  $x, y \in \mathbb{R}^\omega$  we have  $\bar{\rho}(x, y) \geq 0$ .

PROOF: Pick  $j_0 \in J$ . Then

$$\begin{aligned} \bar{\rho}(x, y) &= \sup_j \bar{d}(x_j, y_j) \\ &\geq \bar{d}(x_{j_0}, y_{j_0}) \\ &\geq 0 \end{aligned}$$

- ⟨1⟩2. For all  $x, y \in \mathbb{R}^\omega$  we have  $\bar{\rho}(x, y) = 0$  iff  $x = y$ .

PROOF:

$$\begin{aligned}\bar{\rho}(x, y) = 0 &\Leftrightarrow \sup_j \bar{d}(x_j, y_j) = 0 \\ &\Leftrightarrow \forall j. \bar{d}(x_j, y_j) = 0 \\ &\Leftrightarrow \forall j. x_j = y_j \\ &\Leftrightarrow x = y\end{aligned}$$

⟨1⟩3. For all  $x, y \in \mathbb{R}^\omega$  we have  $\bar{\rho}(x, y) = \bar{\rho}(y, x)$ .

PROOF:

$$\begin{aligned}\bar{\rho}(x, y) &= \sup_j \bar{d}(x_j, y_j) \\ &= \sup_j \bar{d}(y_j, x_j) \\ &= \bar{\rho}(y, x)\end{aligned}$$

⟨1⟩4. For all  $x, y, z \in \mathbb{R}^\omega$  we have  $\bar{\rho}(x, z) \leq \bar{\rho}(x, y) + \bar{\rho}(y, z)$ .

PROOF:

$$\begin{aligned}\bar{\rho}(x, z) &= \sup_j \bar{d}(x_j, z_j) \\ &\leq \sup_j (\bar{d}(x_j, y_j) + \bar{d}(y_j, z_j)) \\ &\leq \sup_j \bar{d}(x_j, y_j) + \sup_j \bar{d}(y_j, z_j) \\ &= \bar{\rho}(x, y) + \bar{\rho}(y, z)\end{aligned}$$

□

**Proposition 14.1.2.** *The uniform topology is finer than the product topology. It is strictly finer iff  $J$  is infinite.*

PROOF:

⟨1⟩1. The uniform topology is finer than the product topology.

⟨2⟩1. LET:  $U$  be open in  $\mathbb{R}$  and  $j \in J$

PROVE:  $\pi_j^{-1}(U)$  is open in the uniform topology.

⟨2⟩2. LET:  $x \in \pi_j^{-1}(U)$

⟨2⟩3.  $\pi_j(x) \in U$

⟨2⟩4. PICK  $\epsilon > 0$  such that  $B_{\bar{d}}(\pi_j(x), \epsilon) \subseteq U$

⟨2⟩5.  $B_{\bar{\rho}}(x, \epsilon) \subseteq \pi_j^{-1}(U)$

⟨1⟩2. If  $J$  is finite then the uniform topology is equal to the product topology.

PROOF: In  $\mathbb{R}^n$ , the uniform topology is the square topology.

⟨1⟩3. If  $J$  is infinite then the uniform topology is not equal to the product topology.

PROOF: If  $J$  is infinite then  $B(0, 1)$  is not open in the product topology.

□

**Proposition 14.1.3.** *The uniform topology is coarser than the box topology. It is strictly coarser iff  $J$  is infinite.*

PROOF:

⟨1⟩1. The uniform topology is coarser than the box topology.

- $\langle 2 \rangle 1$ . LET:  $U$  be open in the uniform topology.  
 PROVE:  $U$  is open in the box topology.  
 $\langle 2 \rangle 2$ . LET:  $x \in U$   
 $\langle 2 \rangle 3$ . PICK  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$   
 $\langle 2 \rangle 4$ .  $\prod_{j \in J} (x_j - \epsilon, x_j + \epsilon) \subseteq U$   
 $\langle 1 \rangle 2$ . If  $J$  is finite then the uniform topology is equal to the box topology.  
 PROOF: On  $\mathbb{R}^n$ , the uniform metric is the square metric.  
 $\langle 1 \rangle 3$ . If  $J$  is infinite then the uniform topology is not equal to the box topology.  
 $\langle 2 \rangle 1$ . ASSUME:  $J$  is infinite.  
 $\langle 2 \rangle 2$ . PICK a sequence  $(j_n)$  of distinct elements in  $J$ .  
 $\langle 2 \rangle 3$ . LET:  $U = \prod_j U_j$  where  $J_{j_n} = (-1/(n+1), 1/(n+1))$  for  $n \in \mathbb{N}$  and  $J_j = (-1, 1)$  for all other  $j$ .  
 $\langle 2 \rangle 4$ .  $U$  is not open in the uniform topology.  
 $\square$

**Proposition 14.1.4.** *The uniform topology on  $\mathbb{R}^\infty$  is strictly finer than the product topology.*

PROOF: The set of all sequences  $(x_n) \in \mathbb{R}^\infty$  such that  $\forall n, |x_n| < 1$  is open in the uniform topology but not in the product topology.  $\square$

**Proposition 14.1.5.** *The uniform topology on  $\mathbb{R}^\infty$  is strictly coarser than the box topology.*

PROOF: The set of sequences  $(x_n) \in \mathbb{R}^\infty$  such that  $\forall n, |x_n| < 1/n$  is open in the box topology but not in the uniform topology.  $\square$

**Proposition 14.1.6.** *The uniform topology on the Hilbert cube is the same as the product topology.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $(x_n)$  be in the Hilbert cube  $H$  and  $\epsilon > 0$ .  
 PROVE:  $B((x_n), \epsilon) \cap H$  is open in the product topology.  
 $\langle 1 \rangle 2$ . PICK  $N$  such that  $1/N < \epsilon$   
 $\langle 1 \rangle 3$ .  $B((x_n), \epsilon) = (\prod_{n=0}^N (x_n - \epsilon, x_n + \epsilon) \times \prod_{n=N+1}^\infty [0, 1/(n+1)]) \cap H$   
 $\square$

**Corollary 14.1.6.1.** *The uniform topology on the Hilbert cube is strictly finer than the box topology.*

**Proposition 14.1.7.** *Let  $X$  be a set and  $Y$  a metric space. Let  $(f_n)$  be a sequence of functions  $X \rightarrow Y$ , and  $f : X \rightarrow Y$ . Then  $(f_n)$  converges uniformly to  $f$  iff  $(f_n)$  converges to  $f$  in  $Y^X$  under the uniform topology.*

PROOF:

- $\langle 1 \rangle 1$ . If  $(f_n)$  converges uniformly to  $f$  then  $(f_n)$  converges to  $f$  in  $Y^X$  under the uniform topology.  
 $\langle 2 \rangle 1$ . ASSUME:  $(f_n)$  converges uniformly to  $f$ .  
 $\langle 2 \rangle 2$ . LET:  $\epsilon > 0$

- ⟨2⟩3. PICK  $N$  such that  $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/2$
- ⟨2⟩4.  $\forall n \geq N. \bar{\rho}(f_n, f) \leq \epsilon/2$
- ⟨2⟩5.  $\forall n \geq N. \bar{\rho}(f_n, f) < \epsilon$
- ⟨1⟩2. If  $(f_n)$  converges to  $f$  in  $Y^X$  under the uniform topology then  $(f_n)$  converges uniformly to  $f$ .
- ⟨2⟩1. ASSUME:  $(f_n)$  converges to  $f$  in  $Y^X$  under the uniform topology.
- ⟨2⟩2. LET:  $\epsilon > 0$
- ⟨2⟩3. PICK  $N$  such that  $\forall n \geq N. \bar{\rho}(f_n, f) < \epsilon$
- ⟨2⟩4.  $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon$

□

**Proposition 14.1.8.** *In  $\mathbb{R}^\omega$  under the uniform topology,  $\vec{x}$  and  $\vec{y}$  lie in the same component if and only if  $\vec{x} - \vec{y}$  is bounded.*

PROOF:

- ⟨1⟩1. The set of bounded sequences form a component of  $\mathbb{R}^\omega$ .
- ⟨2⟩1. LET:  $B$  be the set of bounded sequences.
- ⟨2⟩2.  $B$  is connected.
- ⟨3⟩1. LET:  $\vec{x} \in B$
- PROVE: The straight line path  $p : [0, 1] \rightarrow \mathbb{R}^\omega$  from 0 to  $\vec{x}$  is continuous.
- ⟨3⟩2. LET:  $t \in [0, 1]$  and  $\epsilon > 0$
- ⟨3⟩3. PICK  $B > 0$  such that  $\forall n. |x_n| < B$
- ⟨3⟩4. LET:  $\delta = \epsilon/B$
- ⟨3⟩5. LET:  $s \in [0, 1]$  with  $|s - t| < \delta$
- ⟨3⟩6. For all  $n$  we have  $|p(s)_n - p(t)_n| < \epsilon/2$
- PROOF:
 
$$\begin{aligned}
 |p(s)_n - p(t)_n| &= |s - t| |x_n| \\
 &< \delta B \\
 &= \epsilon
 \end{aligned}$$
- ⟨3⟩7.  $\bar{\rho}(p(s), p(t)) \leq \epsilon/2$
- ⟨3⟩8.  $\bar{\rho}(p(s), p(t)) < \epsilon$
- ⟨2⟩3.  $B$  is maximally connected.
- PROOF: Since  $(B, \mathbb{R}^\omega - B)$  form a separation of  $\mathbb{R}^\omega$ .
- ⟨1⟩2. For any  $\vec{y} \in \mathbb{R}^\omega$ , the component containing  $\vec{y}$  is  $\{\vec{x} \in \mathbb{R}^\omega : \vec{x} - \vec{y} \text{ is bounded}\}$ .
- PROOF: Since the function that maps  $\vec{x}$  to  $\vec{x} + \vec{y}$  is a homeomorphism between  $\mathbb{R}^\omega$  and itself.

□

### 14.1.1 Products

**Definition 14.1.9** (Euclidean Metric). Let  $X$  and  $Y$  be metric spaces. The *Euclidean metric* on  $X \times Y$  is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}.$$

We write  $X \times Y$  for the set  $X \times Y$  under this metric.

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. d((x_1, y_1), (x_2, y_2)) \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2. d((x_1, y_1), (x_2, y_2)) = 0$  iff  $(x_1, y_1) = (x_2, y_2)$

PROOF:  $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = 0$  iff  $d(x_1, x_2) = d(y_1, y_2) = 0$  iff  $x_1 = x_2$  and  $y_1 = y_2$ .

$\langle 1 \rangle 3. d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$

PROOF: Since  $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = \sqrt{d(x_2, x_1)^2 + d(y_2, y_1)^2}$ .

$\langle 1 \rangle 4. \text{The triangle inequality holds.}$

PROOF:

$$\begin{aligned}
 & (d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)))^2 \\
 &= d((x_1, y_1), (x_2, y_2))^2 + 2d((x_1, y_1), (x_2, y_2))d((x_2, y_2), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3))^2 \\
 &= d(x_1, x_2)^2 + d(y_1, y_2)^2 + 2\sqrt{(d(x_1, x_2)^2 + d(y_1, y_2)^2)(d(x_2, x_3)^2 + d(y_2, y_3)^2)} + d(x_2, x_3)^2 + d(y_2, y_3)^2 \\
 &\geq d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(y_1, y_2)^2 + d(y_2, y_3)^2 + 2(d(x_1, x_2)d(x_2, x_3) + d(y_1, y_2)d(y_2, y_3)) \\
 &\quad (\text{Cauchy-Schwarz}) \\
 &= (d(x_1, x_2) + d(x_2, x_3))^2 + (d(y_1, y_2) + d(y_2, y_3))^2 \\
 &\geq d(x_1, x_3)^2 + d(y_1, y_3)^2 \\
 &= d((x_1, y_1), (x_3, y_3))^2
 \end{aligned}$$

□

**Proposition 14.1.10.** *Let  $X$  and  $Y$  be metric spaces. The Euclidean metric on  $X \times Y$  induces the product topology on  $X \times Y$ .*

PROOF:

$\langle 1 \rangle 1. \text{Every open ball is open in the product topology.}$

$\langle 2 \rangle 1. \text{LET: } (x, y) \in B((a, b), \epsilon)$

PROVE:  $B(x, \sqrt{\epsilon}) \times B(y, \sqrt{\epsilon}) \subseteq B((a, b), \epsilon)$

$\langle 2 \rangle 2. \text{LET: } x' \in B(x, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2}) \text{ and } y' \in B(y, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$

PROVE:  $d((x', y'), (a, b)) < \epsilon$

$\langle 2 \rangle 3. d((x', y'), (x, y)) < \epsilon - d((x, y), (a, b))$

PROOF:

$$\begin{aligned}
 d((x', y'), (x, y)) &= \sqrt{d(x', x)^2 + d(y', y)^2} \\
 &< \sqrt{(\epsilon - d((x, y), (a, b)))^2/2 + (\epsilon - d((x, y), (a, b)))^2/2} \\
 &= \epsilon - d((x, y), (a, b))
 \end{aligned}$$

$\langle 2 \rangle 4. d((x', y'), (a, b)) < \epsilon$

PROOF:

$$d((x', y'), (a, b)) \leq d((x', y'), (x, y)) + d((x, y), (a, b)) \quad (\text{Triangle Inequality})$$

$$< \epsilon$$

( $\langle 2 \rangle 3$ )

$\langle 1 \rangle 2. \text{If } U \text{ is open in } X \text{ and } V \text{ is open in } Y \text{ then } U \times V \text{ is open under the Euclidean metric.}$

$\langle 2 \rangle 1. \text{LET: } (x, y) \in U \times V$

⟨2⟩2. PICK  $\delta, \epsilon > 0$  such that  $B(x, \delta) \subseteq U$  and  $B(y, \epsilon) \subseteq V$

PROVE:  $(B((x, y), \min(\delta, \epsilon))) \subseteq U \times V$

⟨2⟩3. LET:  $(x', y') \in B((x, y), \min(\delta, \epsilon))$

⟨2⟩4.  $d(x', x) < \delta$

⟨3⟩1.  $d((x', y'), (x, y)) < \min(\delta, \epsilon)$

⟨3⟩2.  $d(x', x)^2 + d(y', y)^2 < \delta^2$

⟨3⟩3.  $d(x', x)^2 < \delta^2$

⟨2⟩5.  $d(y', y) < \epsilon$

PROOF: Similar.

⟨2⟩6.  $(x', y') \in U \times V$

□

**Proposition 14.1.11.** *The square metric on  $\mathbb{R}^n$  induces the product topology.*

PROOF:

⟨1⟩1. LET:  $d$  be the Euclidean metric on  $\mathbb{R}^n$  and  $\rho$  the square metric.

⟨1⟩2. For all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_d(x, \delta) \subseteq B_\rho(x, \epsilon)$

PROOF: If  $d(x, y) < \epsilon$  then  $\rho(x, y) < \epsilon$ .

⟨1⟩3. For all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_\rho(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: If  $\rho(x, y) < \epsilon/\sqrt{n}$  then  $d(x, y) < \epsilon$ .

⟨1⟩4.  $d$  and  $\rho$  induce the same topology.

PROOF: Proposition 14.0.18.

□

### 14.1.2 Connected Spaces

**Example 14.1.12.** The space  $\mathbb{R}^\omega$  under the uniform topology is disconnected. The set of bounded sequences and the set of unbounded sequences form a separation.

## 14.2 Isometric Embeddings

**Definition 14.2.1** (Isometric Embedding). Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is an *isometric embedding* of  $X$  in  $Y$  iff, for all  $x, y \in X$ , we have  $d(f(x), f(y)) = d(x, y)$ .

**Proposition 14.2.2.** *Every isometric embedding is an embedding.*

PROOF:

⟨1⟩1. LET:  $X$  and  $Y$  be metric spaces.

⟨1⟩2. LET:  $f : X \rightarrow Y$  be an isometric embedding.

⟨1⟩3.  $f$  is injective.

⟨1⟩4. The subspace topology induced by  $f$  is finer than the metric topology.

⟨2⟩1. LET:  $x \in X$  and  $\epsilon > 0$

PROVE:  $B(x, \epsilon)$  is open in the subspace topology.

⟨2⟩2.  $B(x, \epsilon) = f^{-1}(B(f(x), \epsilon))$

⟨1⟩5. The metric topology is finer than the subspace topology induced by  $f$ .



- $\langle 2 \rangle 1$ . LET:  $V$  be open in  $Y$   
 PROVE:  $f^{-1}(V)$  is open in  $X$   
 $\langle 2 \rangle 2$ . LET:  $x \in f^{-1}(V)$   
 $\langle 2 \rangle 3$ . PICK  $\epsilon > 0$  such that  $B(f(x), \epsilon) \subseteq V$   
 $\langle 2 \rangle 4$ .  $B(x, \epsilon) \subseteq f^{-1}(V)$

□

## 14.3 Complete Metric Spaces

**Definition 14.3.1** (Complete). A metric space is *complete* iff every Cauchy sequence converges.

**Example 14.3.2.**  $\mathbb{R}$  is complete.

**Proposition 14.3.3.** *The product of two complete metric spaces is complete.*

**Proposition 14.3.4.** *Every compact metric space is complete.*

**Proposition 14.3.5.** *Let  $X$  be a complete metric space and  $A \subseteq X$ . Then  $A$  is complete if and only if  $A$  is closed.*

**Definition 14.3.6** (Completion). Let  $X$  be a metric space. A *completion* of  $X$  is a complete metric space  $\hat{X}$  and injection  $i : X \rightarrow \hat{X}$  such that:

- The metric on  $X$  is the restriction of the metric on  $\hat{X}$
- $X$  is dense in  $\hat{X}$ .

**Proposition 14.3.7.** *Let  $i_1 : X \rightarrow Y_1$  and  $i_2 : X \rightarrow Y_2$  be completions of  $X$ . Then there exists a unique isometry  $\phi : Y_1 \cong Y_2$  such that  $\phi \circ i_1 = i_2$ .*

PROOF: Define  $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$ . □

**Theorem 14.3.8.** *Every metric space has a completion.*

PROOF: Let  $\hat{X}$  be the set of Cauchy sequences in  $X$  quotiented by  $\sim$  where  $(x_n) \sim (y_n)$  if and only if  $d(x_n, y_n) \rightarrow 0$ . □

## 14.4 Manifolds

**Definition 14.4.1** (Manifold). An  *$n$ -dimensional manifold* is a second countable Hausdorff space locally homeomorphic to  $\mathbb{R}^n$ .



## Chapter 15

# Homotopy Theory

### 15.1 Homotopies

**Definition 15.1.1** (Homotopy). Let  $X$  and  $Y$  be topological spaces. Let  $f, g : X \rightarrow Y$  be continuous. A *homotopy* between  $f$  and  $g$  is a continuous function  $h : X \times [0, 1] \rightarrow Y$  such that

- $\forall x \in X. h(x, 0) = f(x)$
- $\forall x \in X. h(x, 1) = g(x)$

We say  $f$  and  $g$  are *homotopic*,  $f \simeq g$ , iff there exists a homotopy between them.

Let  $[X, Y]$  be the set of all homotopy classes of functions  $X \rightarrow Y$ .

**Proposition 15.1.2.** Let  $f, f' : X \rightarrow Y$  and  $g, g' : Y \rightarrow Z$  be continuous. If  $f \simeq f'$  and  $g \simeq g'$  then  $g \circ f \simeq g' \circ f'$ .

**Definition 15.1.3.** Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

A *homotopy functor* is a functor  $\mathbf{Top} \rightarrow \mathcal{C}$  that factors through the canonical functor  $\mathbf{Top} \rightarrow \mathbf{HTop}$ .

**Definition 15.1.4.** A functor  $F : \mathbf{Top} \rightarrow \mathcal{C}$  is *homotopy invariant* iff, for any topological spaces  $X, Y$  and continuous functions  $f, g : X \rightarrow Y$ , if  $f \simeq g$  then  $Hf = Hg$ .

Basepoint-preserving homotopy.

### 15.2 Homotopy Equivalence

**Definition 15.2.1** (Homotopy Equivalence). Let  $X$  and  $Y$  be topological spaces. A *homotopy equivalence* between  $X$  and  $Y$ ,  $f : X \simeq Y$ , is a continuous function  $f : X \rightarrow Y$  such that there exists a continuous function  $g : Y \rightarrow X$ , the *homotopy inverse* to  $f$ , such that  $g \circ f \simeq \text{id}_X$  and  $f \circ g \simeq \text{id}_Y$ .

**Definition 15.2.2** (Contractible). A topological space  $X$  is *contractible* iff  $X \simeq 1$ .

**Example 15.2.3.**  $\mathbb{R}^n$  is contractible.

**Example 15.2.4.**  $D^n$  is contractible.

**Definition 15.2.5** (Deformation Retract). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A retraction  $\rho : X \rightarrow A$  is a *deformation retraction* iff  $i \circ \rho \simeq \text{id}_X$ , where  $i$  is the inclusion  $A \hookrightarrow X$ . We say  $A$  is a *deformation retract* of  $X$  iff there exists a deformation retraction.

**Definition 15.2.6** (Strong Deformation Retract). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A *strong deformation retraction*  $\rho : X \rightarrow A$  is a continuous function such that there exists a homotopy  $h : X \times [0, 1] \rightarrow X$  between  $i \circ \rho$  and  $\text{id}_X$  such that, for all  $a \in X$  and  $t \in [0, 1]$ , we have  $h(a, t) = a$ .

We say  $A$  is a *strong deformation retract* of  $X$  iff a strong deformation retraction exists.

**Example 15.2.7.**  $\{0\}$  is a strong deformation retract of  $\mathbb{R}^n$  and of  $D^n$ .

**Example 15.2.8.**  $S^1$  is a strong deformation retract of the torus  $S^1 \times D^2$ .

**Example 15.2.9.**  $S^{n-1}$  is a strong deformation retract of  $D^n - \{0\}$ .

**Example 15.2.10.** For any topological space  $X$ , the singleton consisting of the vertex is a strong deformation retract of the cone over  $X$ .

## Chapter 16

# Simplicial Complexes

**Definition 16.0.1** (Simplex). A  $k$ -dimensional simplex or  $k$ -simplex in  $\mathbb{R}^n$  is the convex hull  $s(x_0, \dots, x_k)$  of  $k + 1$  points in general position.

**Definition 16.0.2** (Face). A *sub-simplex* or *face* of  $s(x_0, \dots, x_k)$  is the convex hull of a subset of  $\{x_0, \dots, x_k\}$ .

**Definition 16.0.3** (Simplicial Complex). A *simplicial complex* in  $\mathbb{R}^n$  is a set  $K$  of simplices such that:

- for every simplex  $s$  in  $K$ , every face of  $s$  is in  $K$ .
- The intersection of two simplices  $s_1, s_2 \in K$  is either empty or is a face of both  $s_1$  and  $s_2$ .
- $K$  is locally finite, i.e. every point of  $\mathbb{R}^n$  has a neighbourhood that only intersects finitely many elements of  $K$ .

The topological space *underlying*  $K$  is  $|K| = \bigcup K$  as a subspace of  $\mathbb{R}^n$ .

### 16.1 Cell Decompositions

**Definition 16.1.1** ( $n$ -cell). An  $n$ -cell is a topological space homeomorphic to  $\mathbb{R}^n$ .

**Definition 16.1.2** (Cell Decomposition). Let  $X$  be a topological space. A *cell decomposition* of  $X$  is a partition of  $X$  into subspaces that are  $n$ -cells.

**Definition 16.1.3** ( $n$ -skeleton). Given a cell decomposition of  $X$ , the  $n$ -skeleton  $X^n$  is the union of all the cells of dimension  $\leq n$ .

### 16.2 CW-complexes

**Definition 16.2.1** (CW-Complex). A *CW-complex* consists of a topological space  $X$  and a cell decomposition  $\mathcal{E}$  of  $X$  such that:

1. *Characteristic Maps* For every  $n$ -cell  $e \in \mathcal{E}$ , there exists a continuous map  $\Phi_e : D^n \rightarrow X$  such that  $\Phi_e((D^n)^\circ) = e$ , the corestriction  $\Phi_e : (D^n)^\circ \approx e$  is a homeomorphism, and  $\Phi_e(S^n)$  is the union of all the cells in  $\mathcal{E}$  of dimension  $< n$ .
2. *Closure Finiteness* For all  $e \in \mathcal{E}$ , we have  $\bar{e}$  intersects only finitely many other cells in  $\mathcal{E}$ .
3. *Weak Topology* Given  $A \subseteq X$ , we have  $A$  is closed iff for all  $e \in \mathcal{E}$ ,  $A \cap \bar{e}$  is closed.

**Proposition 16.2.2.** *If a cell decomposition  $\mathcal{E}$  satisfies the Characteristic Maps axiom, then for every  $n$ -cell  $e \in \mathcal{E}$  we have  $\bar{e} = \Phi_e(D^n)$ . Therefore  $\bar{e}$  is compact and  $\bar{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$ .*

PROOF:

$\langle 1 \rangle 1.$   $e \subseteq \Phi_e(D^n) \subseteq \bar{e}$

PROOF:

$$\begin{aligned}
 e &= \Phi_e((D^n)^\circ) \\
 &\subseteq \Phi_e(D^n) \\
 &= \Phi_e(\overline{(D^n)^\circ}) \\
 &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
 &= \bar{e}
 \end{aligned}$$

$\langle 1 \rangle 2.$   $\Phi_e(D^n)$  is compact.

PROOF: Because  $D^n$  is compact.

$\langle 1 \rangle 3.$   $\Phi_e(D^n)$  is closed.

$\langle 1 \rangle 4.$   $\Phi_e(D^n) = \bar{e}$

□

## Chapter 17

# Topological Groups

### 17.1 Topological Groups

**Definition 17.1.1** (Topological Group). A *topological group* is a group  $G$  with a topology such that the function  $G^2 \rightarrow G$  that maps  $(x, y)$  to  $xy^{-1}$  is continuous.

**Example 17.1.2.**  $\mathbb{Z}$  is a topological group under addition.

PROOF: The function that sends  $(x, y)$  to  $xy^{-1}$  is continuous because the topology on  $\mathbb{Z}$  is discrete.  $\square$

**Example 17.1.3.**  $\mathbb{R}$  is a topological group under addition.

PROOF: From Propositions 14.0.11 and 14.0.12.  $\square$

**Example 17.1.4.**  $\mathbb{R}_+$  is a topological group under multiplication.

PROOF: From Propositions 14.0.12 and 14.0.28.  $\square$

**Example 17.1.5.**  $S^1$  as a subspace of  $\mathbb{C}$  is a topological group under multiplication.

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : S^1 \rightarrow S^1$  be the function  $f(x, y) = xy^{-1}$

$\langle 1 \rangle 2$ . LET:  $U$  be an open set in  $S^1$

PROVE:  $f^{-1}(U)$  is open in  $(S^1)^2$

$\langle 1 \rangle 3$ . LET:  $(x, y) \in f^{-1}(U)$

$\langle 1 \rangle 4$ .  $xy^{-1} \in U$

$\langle 1 \rangle 5$ . LET:  $x = e^{i\phi}$  and  $y = e^{i\psi}$

$\langle 1 \rangle 6$ .  $xy^{-1} = e^{i(\phi-\psi)} \in U$

$\langle 1 \rangle 7$ . PICK  $\epsilon > 0$  such that, for all  $t$ , if  $|\phi - \psi - t| < \epsilon$  then  $e^{it} \in U$

$\langle 1 \rangle 8$ .  $(x, y) \in \{e^{it} : |\phi - t| < \epsilon/2\} \times \{e^{it} : |\psi - t| < \epsilon/2\} \subseteq f^{-1}(U)$

$\square$

**Example 17.1.6.**  $GL(n, \mathbb{R})$  is a topological group considered as a subspace of  $\mathbb{R}^{n^2}$ .

PROOF: Since the calculations for matrix multiplication and inverse are compositions of continuous functions.  $\square$

**Example 17.1.7.**  $GL(n, \mathbb{R})$  and  $GL(n, \mathbb{C})$  are topological groups.

**Proposition 17.1.8.** *Let  $G$  be a group with a topology. Then  $G$  is a topological group if and only if the functions  $m : G^2 \rightarrow G$  that sends  $(x, y)$  to  $xy$  and the function  $i : G \rightarrow G$  that sends  $x$  to  $x^{-1}$  are continuous.*

PROOF:

$\langle 1 \rangle 1$ . If  $G$  is a topological group then  $i$  is continuous.

PROOF: Since  $x^{-1} = ex^{-1}$ .

$\langle 1 \rangle 2$ . If  $G$  is a topological group then  $m$  is continuous.

PROOF: Since  $xy = x(y^{-1})^{-1}$ .

$\langle 1 \rangle 3$ . If  $m$  and  $i$  are continuous then  $G$  is a topological group.

PROOF: Since  $xy^{-1} = m(x, i(y))$ .

$\square$

**Proposition 17.1.9.** *Let  $G$  be a topological group. Let  $\alpha \in G$ . The function that maps  $x$  to  $\alpha x$  is a homeomorphism between  $G$  and itself.*

PROOF:

$\langle 1 \rangle 1$ . For any  $\alpha \in G$ , the function that maps  $x$  to  $\alpha x$  is continuous.

PROOF: From the definition of topological group.

$\langle 1 \rangle 2$ . For any  $\alpha \in G$ , the function that maps  $x$  to  $\alpha x$  is a homeomorphism between  $G$  and itself.

PROOF: Its inverse is the function that maps  $x$  to  $\alpha^{-1}x$ .

$\square$

**Corollary 17.1.9.1.** *Every topological group is homogeneous.*

**Proposition 17.1.10.** *Let  $G$  be a topological group. Let  $\alpha \in G$ . The function that maps  $x$  to  $x\alpha$  is a homeomorphism between  $G$  and itself.*

PROOF: Similar.  $\square$

### 17.1.1 Subgroups

**Proposition 17.1.11.** *Any subgroup of a topological group is a topological group under the subspace topology.*

PROOF: Since the restriction of continuous functions is continuous.  $\square$

**Proposition 17.1.12.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Then  $\overline{H}$  is a topological group under the subspace topology.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $x, y \in \overline{H}$

PROVE:  $xy^{-1} \in \overline{H}$

$\langle 1 \rangle 2$ . LET:  $U$  be a neighbourhood of  $xy^{-1}$ .



PROVE:  $U$  intersects  $H$ .

- ⟨1⟩3. LET:  $f : G^2 \rightarrow G$  be the function that maps  $(x, y)$  to  $xy^{-1}$ .
- ⟨1⟩4.  $f^{-1}(U)$  is a neighbourhood of  $(x, y)$
- ⟨1⟩5. PICK neighbourhoods  $V$  of  $x$  and  $W$  of  $y$  such that  $V \times W \subseteq f^{-1}(U)$ .
- ⟨1⟩6. PICK elements  $x' \in V \cap H$  and  $y' \in W \cap H$
- ⟨1⟩7.  $x'y'^{-1} \in U \cap H$

□

**Proposition 17.1.13.** *Let  $G$  be a topological group. The component of  $G$  that contains  $e$  is a normal subgroup of  $G$ .*

PROOF:

- ⟨1⟩1. LET:  $C$  be the component that contains  $e$ .
- ⟨1⟩2. For all  $x \in G$ , we have  $Cx$  is the component of  $G$  that contains  $x$ .  
 PROOF: Since right multiplication by  $x$  is a homeomorphism between  $G$  and itself.
- ⟨1⟩3.  $C$  is a subgroup of  $G$ .  
 ⟨2⟩1. LET:  $g, h \in C$   
 ⟨2⟩2.  $C = Ch$   
 PROOF: ⟨1⟩2  
 ⟨2⟩3. PICK  $x \in C$  such that  $xh = g$   
 ⟨2⟩4.  $x = gh^{-1}$   
 ⟨2⟩5.  $gh^{-1} \in C$
- ⟨1⟩4.  $C$  is a normal subgroup of  $G$ .  
 ⟨2⟩1. LET:  $g \in G$  and  $h \in C$ .  
 PROVE:  $ghg^{-1} \in C$   
 ⟨2⟩2.  $C = Ch^{-1}$   
 ⟨2⟩3.  $Cg = Ch^{-1}g$   
 ⟨2⟩4.  $g \in Ch^{-1}g$   
 ⟨2⟩5. PICK  $x \in C$  such that  $g = xh^{-1}g$   
 ⟨2⟩6.  $x = ghg^{-1}$   
 ⟨2⟩7.  $ghg^{-1} \in C$

□

### 17.1.2 Left Cosets

**Proposition 17.1.14.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. Let  $\alpha \in G$ . Define  $f_\alpha : G/H \rightarrow G/H$  by*

$$f_\alpha(xH) = \alpha xH .$$

*Then  $f_\alpha$  is a homeomorphism.*

PROOF:

- ⟨1⟩1. For all  $\alpha \in G$  we have  $f_\alpha$  is well defined.  
 ⟨2⟩1. LET:  $x, y \in G$   
 ⟨2⟩2. ASSUME:  $xH = yH$

PROVE:  $\alpha xH = \alpha yH$

$\langle 2 \rangle 3$ .  $x^{-1}y \in H$

$\langle 2 \rangle 4$ .  $x^{-1}\alpha^{-1}\alpha y \in H$

$\langle 2 \rangle 5$ .  $\alpha xH = \alpha yH$

$\langle 1 \rangle 2$ . For all  $\alpha \in G$  we have  $f_\alpha$  is injective.

$\langle 2 \rangle 1$ . LET:  $x, y \in G$

$\langle 2 \rangle 2$ . ASSUME:  $\alpha xH = \alpha yH$

PROVE:  $xH = yH$

$\langle 2 \rangle 3$ .  $\alpha x^{-1}\alpha y \in H$

$\langle 2 \rangle 4$ .  $x^{-1}y \in H$

$\langle 2 \rangle 5$ .  $xH = yH$

$\langle 1 \rangle 3$ . For all  $\alpha \in G$  we have  $f_\alpha$  is surjective.

PROOF: For all  $x \in G$  we have  $xH = f_\alpha(\alpha^{-1}xH)$ .

$\langle 1 \rangle 4$ . For all  $\alpha \in G$  we have  $f_\alpha$  is continuous.

$\langle 2 \rangle 1$ . LET:  $V$  be open in  $G/H$

$\langle 2 \rangle 2$ .  $\pi^{-1}(f_\alpha^{-1}(V))$  is open in  $G$ .

PROOF: It is  $g_\alpha^{-1}(\pi^{-1}(V))$  where  $g_\alpha : V \rightarrow V$  is the homeomorphism  $g_\alpha(x) = \alpha x$ .

$\langle 2 \rangle 3$ .  $f_\alpha^{-1}(V)$  is open in  $G/H$ .

$\langle 1 \rangle 5$ . For all  $\alpha \in G$  we have  $f_\alpha^{-1}$  is continuous.

PROOF: It is  $f_{\alpha^{-1}}$ .

□

**Corollary 17.1.14.1.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Then  $G/H$  is a homogeneous space.*

**Proposition 17.1.15.** *Let  $G$  be a  $T_1$  topological group and  $H$  a closed subgroup of  $G$ . Then  $G/H$  is  $T_1$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in G$

PROVE:  $xH$  is closed.

$\langle 1 \rangle 2$ .  $\pi^{-1}(xH)$  is closed in  $G$ .

PROOF: It is  $f_x(H)$  and  $f_x$  is a homeomorphism.

$\langle 1 \rangle 3$ .  $xH$  is closed in  $G/H$ .

□

**Proposition 17.1.16.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Then the canonical map  $\pi : G \rightarrow G/H$  is an open map.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $U$  be open in  $G$ .

$\langle 1 \rangle 2$ .  $\forall h \in H. Uh$  is open in  $G$ .

PROOF: Since the function that maps  $g$  to  $gh$  is an automorphism of  $G$ .

$\langle 1 \rangle 3$ .  $UH$  is open in  $G$

PROOF: It is  $\bigcup_{h \in H} Uh$ .

$\langle 1 \rangle 4$ .  $UH = \pi^{-1}(\pi(U))$

PROOF:

$$\begin{aligned}
 \pi^{-1}(\pi(U)) &= \{x \in G : \exists y \in U. xH = yH\} \\
 &= \{x \in G : \exists y \in U. x^{-1}y \in H\} \\
 &= \{x \in G : \exists y \in U. \exists h \in H. y^{-1}x = h\} \\
 &= \{x \in G : \exists y \in U. \exists h \in H. x = yh\} \\
 &= UH
 \end{aligned}$$

$\langle 1 \rangle 5$ .  $\pi^{-1}(\pi(U))$  is open in  $G$ .

$\langle 1 \rangle 6$ .  $\pi(U)$  is open in  $G/H$ .

□

**Proposition 17.1.17.** *Let  $G$  be a topological group. Let  $H$  be a normal subgroup of  $G$ . Then  $G/H$  is a topological group.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : G^2 \rightarrow G$  be the map  $f(x, y) = xy^{-1}$

$\langle 1 \rangle 2$ . LET:  $g : (G/H)^2 \rightarrow G/H$  be the map  $g(xH, yH) = xy^{-1}H$

$\langle 1 \rangle 3$ .  $g \circ (\pi \times \pi) = \pi \circ f : G^2 \rightarrow G/H$

$\langle 1 \rangle 4$ .  $g \circ (\pi \times \pi)$  is continuous.

PROOF: Since  $\pi$  and  $f$  are continuous.

$\langle 1 \rangle 5$ .  $\pi$  is an open quotient map.

PROOF: Proposition 17.1.16.

$\langle 1 \rangle 6$ .  $\pi \times \pi$  is an open quotient map.

PROOF: Corollary 13.26.7.1.

$\langle 1 \rangle 7$ .  $g$  is continuous.

PROOF: Theorem 13.26.3.

□

### 17.1.3 Homogeneous Spaces

**Definition 17.1.18** (Homogeneous Space). A *homogeneous space* is a topological space of the form  $G/H$ , where  $G$  is a topological group and  $H$  is a normal subgroup of  $G$ , under the quotient topology.

**Proposition 17.1.19.** *Let  $G$  be a topological group and  $H$  a normal subgroup of  $G$ . Then  $G/H$  is Hausdorff if and only if  $H$  is closed.*

PROOF: See Bourbaki, N., General Topology. III.12 □

## 17.2 Symmetric Neighbourhoods

**Definition 17.2.1** (Symmetric Neighbourhood). Let  $G$  be a topological group. Let  $V$  be a neighbourhood of  $e$ . Then  $V$  is *symmetric* iff  $V = V^{-1}$ .

**Proposition 17.2.2.** *Let  $G$  be a topological group. Let  $U$  be a neighbourhood of  $e$ . Then there exists a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$ .*

PROOF:

- ⟨1⟩1. PICK a neighbourhood  $V'$  of  $e$  such that  $V'V' \subseteq U$ .
- ⟨2⟩1. LET:  $m : G^2 \rightarrow G$  be the function  $m(x, y) = xy$
- ⟨2⟩2.  $m^{-1}(U)$  is open in  $G^2$
- ⟨2⟩3.  $(e, e) \in m^{-1}(U)$
- ⟨2⟩4. PICK neighbourhoods  $V_1, V_2$  of  $e$  such that  $V_1 \times V_2 \subseteq m^{-1}(U)$
- ⟨2⟩5. LET:  $V' = V_1 \cap V_2$
- ⟨1⟩2. PICK a neighbourhood  $W$  of  $e$  such that  $WW^{-1} \subseteq V'$
- ⟨2⟩1. LET:  $f : G^2 \rightarrow G$  be the function  $m(x, y) = xy^{-1}$
- ⟨2⟩2.  $f^{-1}(V')$  is open in  $G^2$
- ⟨2⟩3.  $(e, e) \in m^{-1}(V')$
- ⟨2⟩4. PICK neighbourhoods  $W_1, W_2$  of  $e$  such that  $W_1 \times W_2 \subseteq f^{-1}(V')$
- ⟨2⟩5. LET:  $W = W_1 \cap W_2$
- ⟨1⟩3. LET:  $V = WW^{-1}$
- ⟨1⟩4.  $V$  is a neighbourhood of  $e$ .
- ⟨1⟩5.  $V$  is symmetric.
- ⟨1⟩6.  $VV \subseteq U$

□

**Proposition 17.2.3.** *Every  $T_1$  topological group is regular.*

PROOF:

- ⟨1⟩1. LET:  $G$  be a  $T_1$  topological group.
- ⟨1⟩2. LET:  $A$  be a closed set in  $G$  and  $x \in G - A$ .
- ⟨1⟩3.  $G - Ax^{-1}$  is a neighbourhood of  $e$ .
- ⟨1⟩4. PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq G - Ax^{-1}$ .
- ⟨1⟩5. LET:  $U = VA$  and  $U' = Vx$
- ⟨1⟩6.  $U$  and  $U'$  are disjoint open sets with  $A \subseteq U$  and  $x \in U'$ .

□

**Proposition 17.2.4.** *Let  $G$  be a  $T_1$  topological group. Let  $H$  be a closed subgroup of  $G$ . Then  $G/H$  is regular.*

PROOF:

- ⟨1⟩1. LET:  $A$  be a closed set in  $G/H$  and  $xH \in G/H - A$ .
- ⟨1⟩2.  $G - \pi^{-1}(A)x^{-1}$  is a neighbourhood of  $e$ .
- ⟨1⟩3. PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq G - \pi^{-1}(A)x^{-1}$ .
- ⟨1⟩4. LET:  $U = \pi(V)A$  and  $U' = \pi(V)(xH)$ .
- ⟨1⟩5.  $U$  and  $U'$  are disjoint open sets with  $A \subseteq U$  and  $xH \in U'$
- ⟨2⟩1. ASSUME: for a contradiction  $U \cap U' \neq \emptyset$ .
- ⟨2⟩2. PICK  $v_1, v_2 \in V$  and  $a \in G$  such that  $aH \in A$  and  $v_1aH = v_2xH$ .
- ⟨2⟩3.  $a^{-1}v_1^{-1}v_2x \in H$
- ⟨2⟩4.  $v_1^{-1}v_2 \in \pi^{-1}(A)x^{-1}$
- ⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩3.

□

## 17.3 Continuous Actions

**Definition 17.3.1** (Continuous Action). Let  $G$  be a topological group and  $X$  a topological space. A *continuous action* of  $G$  on  $X$  is a continuous function  $\cdot : G \times X \rightarrow X$  such that:

- $\forall x \in X. ex = x$
- $\forall g, h \in G. \forall x \in X. g(hx) = (gh)x$

A  $G$ -space consists of a topological space  $X$  and a continuous action of  $G$  on  $X$ .

**Definition 17.3.2** (Orbit). Let  $X$  be a  $G$ -space and  $x \in X$ . The *orbit* of  $x$  is  $\{gx : g \in G\}$ .

The *orbit space*  $X/G$  is the set of all orbits under the quotient topology.

**Proposition 17.3.3.** Define an action of  $SO(2)$  on  $S^2$  by

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \text{ .}$$

Then  $S^2/SO(2) \cong [-1, 1]$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $f_3 : S^2/SO(2) \rightarrow [-1, 1]$  be the function induced by  $\pi_3 : S^2 \rightarrow [-1, 1]$

$\langle 1 \rangle 2$ .  $f_3$  is bijective.

$\langle 1 \rangle 3$ .  $S^2/SO(2)$  is compact.

PROOF: It is the continuous image of  $S^2$  which is compact.

$\langle 1 \rangle 4$ .  $[-1, 1]$  is Hausdorff.

$\langle 1 \rangle 5$ .  $f_3$  is a homeomorphism.

□

**Definition 17.3.4** (Stabilizer). Let  $X$  be a  $G$ -space and  $x \in X$ . The *stabilizer* of  $x$  is  $G_x := \{g \in G : gx = x\}$ .

**Proposition 17.3.5.** The function that maps  $gG_x$  to  $gx$  is a continuous bijection from  $G/G_x$  to  $Gx$ .

PROOF:

$\langle 1 \rangle 1$ . If  $gG_x = hG_x$  then  $gx = hx$ .

$\langle 2 \rangle 1$ . ASSUME:  $gG_x = hG_x$

$\langle 2 \rangle 2$ .  $g^{-1}h \in G_x$

$\langle 2 \rangle 3$ .  $g^{-1}hx = x$

$\langle 2 \rangle 4$ .  $gx = hx$

$\langle 1 \rangle 2$ . If  $gx = hx$  then  $gG_x = hG_x$ .

PROOF: Similar.

$\langle 1 \rangle 3$ . The function is continuous.

PROOF: Theorem 13.26.3.

□



## Chapter 18

# Topological Vector Spaces

**Definition 18.0.1** (Topological Vector Space). Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . A *topological vector space* over  $K$  consists of a vector space  $E$  over  $K$  and a topology on  $E$  such that:

- Subtraction is a continuous function  $E^2 \rightarrow E$
- Multiplication is a continuous function  $K \times E \rightarrow E$

**Proposition 18.0.2.** *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition.  $\square$

**Theorem 18.0.3.** *The usual topology on a finite dimensional vector space over  $K$  is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18  $\square$

**Proposition 18.0.4.** *Let  $E$  be a topological vector space and  $E_0$  a subspace of  $E$ . Then  $\overline{E_0}$  is a subspace of  $E$ .*

**Definition 18.0.5.** Let  $E$  be a topological vector space. The topological space associated with  $E$  is  $E/\overline{\{0\}}$ .

### 18.1 Cauchy Sequences

**Definition 18.1.1** (Cauchy Sequence). Let  $E$  be a topological vector space. A sequence  $(x_n)$  in  $E$  is a *Cauchy sequence* iff, for every neighbourhood  $U$  of 0, there exists  $n_0$  such that  $\forall m, n \geq n_0, x_n - x_m \in U$ .

**Definition 18.1.2** (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

## 18.2 Seminorms

**Definition 18.2.1** (Seminorm). Let  $E$  be a vector space over  $K$ . A *seminorm* on  $E$  is a function  $\| \cdot \| : E \rightarrow \mathbb{R}$  such that:

1.  $\forall x \in E, \|x\| \geq 0$
2.  $\forall \alpha \in K, \forall x \in E, \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality*  $\forall x, y \in E, \|x + y\| \leq \|x\| + \|y\|$

**Example 18.2.2.** The function that maps  $(x_1, \dots, x_n)$  to  $|x_i|$  is a seminorm on  $\mathbb{R}^n$ .

**Definition 18.2.3.** Let  $E$  be a vector space over  $K$ . Let  $\Lambda$  be a set of seminorms on  $E$ . The topology *generated* by  $\Lambda$  is the topology generated by the subbasis consisting of all sets of the form  $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$  for  $\epsilon > 0$ ,  $\lambda \in \Lambda$  and  $x \in E$ .

**Proposition 18.2.4.**  $E$  is a topological vector space under this topology. It is Hausdorff iff, for all  $x \in E$ , if  $\forall \lambda \in \Lambda, \lambda(x) = 0$  then  $x = 0$ .

## 18.3 Fréchet Spaces

**Definition 18.3.1** (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

**Proposition 18.3.2.** Let  $E$  be a pre-Fréchet space whose topology is generated by the family of seminorms  $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$ . Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

**Definition 18.3.3** (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

## 18.4 Normed Spaces

**Definition 18.4.1** (Normed Space). Let  $E$  be a vector space over  $K$ . A *norm* on  $E$  is a function  $\| \cdot \| : E \rightarrow \mathbb{R}$  is a seminorm such that,  $\forall x \in E, \|x\| = 0 \Leftrightarrow x = 0$ .

A *normed space* consists of a vector space with a norm.

**Proposition 18.4.2.** If  $E$  is a normed space then  $d(x, y) = \|x - y\|$  is a metric on  $E$  that makes  $E$  into a topological vector space. The two definitions of Cauchy sequence agree on  $E$ .



**Definition 18.4.3** ( $p$ -norm). For any  $p \geq 1$ , the  $p$ -norm on  $\mathbb{R}^n$  is defined by

$$\|\vec{x}\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1$ . For all  $\vec{x} \in \mathbb{R}^n$  we have  $\|\vec{x}\|_p \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$ . For all  $\alpha \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^n$  we have  $\|\alpha \vec{x}\|_p = |\alpha| \|\vec{x}\|_p$

PROOF:

$$\begin{aligned} \|\alpha(x_1, \dots, x_n)\| &= \|(\alpha x_1, \dots, \alpha x_n)\| \\ &= \left( \sum_{i=1}^n (\alpha x_i)^p \right)^{\frac{1}{p}} \\ &= \left( |\alpha|^p \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|\vec{x}\|_p \end{aligned}$$

$\langle 1 \rangle 3$ . The triangle inequality holds.

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\ &= \sum_{i=1}^n |x_i + y_i| |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^n (|x_i| + |y_i|) |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} \\ &\leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} \quad (\text{Hölder's Inequality}) \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p) \|\vec{x} + \vec{y}\|_p^{p-1} \end{aligned}$$

Assuming w.l.o.g.  $\|\vec{x} + \vec{y}\|_p^{p-1} \neq 0$  (using ??) we have  $\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$ .

$\langle 1 \rangle 4$ . For any  $\vec{x} \in \mathbb{R}^n$ , we have  $\|\vec{x}\| = 0$  iff  $\vec{x} = \vec{0}$ .

PROOF:  $\sum_{i=1}^n x_i^p = 0$  iff  $x_1 = \dots = x_n = 0$ .

□

**Proposition 18.4.4.** The  $p$ -norm on  $\mathbb{R}^n$  induces the product topology.

PROOF:

⟨1⟩1. LET:  $d$  be the metric induced by the  $p$ -norm and  $\rho$  the square metric on  $\mathbb{R}^n$ .

⟨1⟩2. The metric topology is finer than the product topology.

⟨2⟩1. LET:  $\vec{x} \in \mathbb{R}^n$  and  $\epsilon > 0$

⟨2⟩2. LET:  $\delta = \epsilon/n^{\frac{1}{p}}$

PROVE:  $B_\rho(\vec{x}, \delta) \subseteq B_d(\vec{x}, \epsilon)$

⟨2⟩3. LET:  $\vec{y} \in B_\rho(\vec{x}, \delta)$

⟨2⟩4.  $\forall i. |x_i - y_i| < \delta$

⟨2⟩5.  $d(\vec{x}, \vec{y}) < \epsilon$

PROOF:

$$\begin{aligned} d(\vec{x}, \vec{y}) &= \left( \sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}} \\ &< \left( \sum_{i=1}^n \delta^p \right)^{\frac{1}{p}} && (\langle 2 \rangle 4) \\ &= n^{\frac{1}{p}} \delta \\ &= \epsilon && (\langle 2 \rangle 2) \end{aligned}$$

⟨1⟩3. The product topology is finer than the metric topology.

⟨2⟩1. LET:  $\vec{x} \in \mathbb{R}^n$  and  $\epsilon > 0$

⟨2⟩2. LET:  $\vec{y} \in B_d(\vec{x}, \epsilon)$

⟨2⟩3.  $d(\vec{x}, \vec{y}) < \epsilon$

⟨2⟩4.  $\sum_{i=1}^n |x_i - y_i|^p < \epsilon^p$

⟨2⟩5.  $\forall i. |x_i - y_i|^p < \epsilon^p$

⟨2⟩6.  $\forall i. |x_i - y_i| < \epsilon$

⟨2⟩7.  $\rho(\vec{x}, \vec{y}) < \epsilon$

□

**Definition 18.4.5** (Sup-norm). The *sup-norm* on  $\mathbb{R}^n$  is defined by

$$\|(x_1, \dots, x_n)\|_\infty := \max(|x_1|, \dots, |x_n|) .$$

**Proposition 18.4.6.** The 2-norm on  $\mathbb{R}^n$  induces the standard metric.

PROOF: Immediate from definitions. □

**Definition 18.4.7.** For  $p \geq 1$ , the normed space  $l_p$  is the set of all sequences  $(x_n)$  in  $\mathbb{R}$  such that  $\sum_{n=1}^\infty x_n^p$  converges, under

$$\|(x_n)\|_p := \left( \sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} .$$

**Proposition 18.4.8.** The spaces  $l_p$  for  $p \geq 1$  are all homeomorphic.

PROOF: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. <http://dx.doi.org/10.1007/BF01075865>.

**Proposition 18.4.9.** *The metric topology on  $l_2$  is strictly finer than the uniform topology.*

PROOF:

⟨1⟩1. LET:  $d$  be the metric induced by the  $l^2$ -norm and  $\bar{\rho}$  the uniform topology.

⟨1⟩2. The metric topology is finer than the uniform topology.

⟨2⟩1. LET:  $x \in l_2$

⟨2⟩2. LET:  $\epsilon > 0$

⟨2⟩3. LET:  $\delta = \epsilon/2$

⟨2⟩4. LET:  $y \in B_d(x, \delta)$

⟨2⟩5.  $\sum_{n=0}^{\infty} (x_n - y_n)^2 < \delta^2$

⟨2⟩6.  $\forall n. (x_n - y_n)^2 < \delta^2$

⟨2⟩7.  $\forall n. |x_n - y_n| < \delta$

⟨2⟩8.  $\forall n. \bar{d}(x_n, y_n) < \delta$

⟨2⟩9.  $\bar{\rho}(x, y) \leq \delta$

⟨2⟩10.  $\bar{\rho}(x, y) < \epsilon$

⟨2⟩11.  $y \in B_{\bar{\rho}}(x, \epsilon)$

⟨1⟩3. The metric topology is not the same as the uniform topology.

⟨2⟩1. ASSUME: for a contradiction  $B_d(0, 1)$  is open in the uniform topology.

⟨2⟩2. PICK  $\epsilon > 0$  such that  $B_{\bar{\rho}}(0, \epsilon) \subseteq B_d(0, 1)$

⟨2⟩3. PICK an integer  $N$  such that  $1/N < \epsilon^2/4$

⟨2⟩4. LET:  $(x_n)$  be the sequence with  $x_n = \epsilon/2$  for  $n < N$  and  $x_n = 0$  for  $n \geq N$

⟨2⟩5.  $(x_n) \in l_2$

⟨2⟩6.  $(x_n) \in B_{\bar{\rho}}(0, \epsilon)$

PROOF: Since  $\bar{\rho}((x_n), 0) = \epsilon/2$ .

⟨2⟩7.  $d((x_n), 0) > 1$

PROOF:

$$\begin{aligned} d((x_n), 0)^2 &= \sum_{n=0}^{\infty} x_n^2 \\ &= N\epsilon^2/4 \\ &> 1 \end{aligned}$$

□

**Proposition 18.4.10.** *The metric topology on  $l_2$  is strictly coarser than the box topology.*

PROOF:

⟨1⟩1. The box topology is finer than the metric topology.

⟨2⟩1. LET:  $(x_n) \in l_2$  and  $\epsilon > 0$ .

⟨2⟩2. LET:  $(y_n) \in B((x_n), \epsilon)$

⟨2⟩3. PICK a sequence of real numbers  $(\delta_n)$  such that  $\sum_{n=0}^{\infty} \delta_n^2 < (\epsilon - d((x_n), (y_n)))^2$

⟨2⟩4. LET:  $U = \prod_n (y_n - \delta_n, y_n + \delta_n)$

PROVE:  $U \subseteq B((x_n), \epsilon)$

⟨2⟩5. LET:  $(z_n) \in U$

⟨2⟩6.  $d((z_n), (y_n)) < \epsilon - d((x_n), (y_n))$

PROOF:

$$\begin{aligned} d((z_n), (y_n))^2 &= \sum_{n=0}^{\infty} (z_n - y_n)^2 \\ &< \sum_{n=0}^{\infty} \delta_n^2 \\ &< (\epsilon - d((x_n), (y_n)))^2 \end{aligned}$$

$$\langle 2 \rangle 7. d((z_n), (x_n)) < \epsilon$$

$\langle 1 \rangle 2.$  The box topology is not equal to the metric topology.

$$\langle 2 \rangle 1. \text{ LET: } U = \prod_n (-1/n, 1/n)$$

$\langle 2 \rangle 2.$  ASSUME: for a contradiction  $U$  is open in the metric topology.

$\langle 2 \rangle 3.$  PICK  $\epsilon > 0$  such that  $B(0, \epsilon) \subseteq U$

$\langle 2 \rangle 4.$  PICK  $N$  such that  $1/N < \epsilon/2$ .

$\langle 2 \rangle 5.$  LET:  $(x_n)$  be the sequence with  $x_N = \epsilon/2$  and  $x_n = 0$  for all other  $n$ .

$\langle 2 \rangle 6.$   $d((x_n), 0) = \epsilon/2$

$\langle 2 \rangle 7.$   $(x_n) \notin U$

□

**Proposition 18.4.11.** *The  $l^2$ -topology on  $\mathbb{R}^\infty$  is strictly finer than the uniform topology.*

PROOF:

$\langle 1 \rangle 1.$  ASSUME: for a contradiction  $B_d(0, 1) \cap \mathbb{R}^\infty$  is open in the uniform topology.

$\langle 1 \rangle 2.$  PICK  $\epsilon > 0$  such that  $B_{\bar{\rho}}(0, \epsilon) \cap \mathbb{R}^\infty \subseteq B_d(0, 1) \cap \mathbb{R}^\infty$

$\langle 1 \rangle 3.$  PICK an integer  $N$  such that  $1/N < \epsilon^2/4$

$\langle 1 \rangle 4.$  LET:  $(x_n)$  be the sequence with  $x_n = \epsilon/2$  for  $n < N$  and  $x_n = 0$  for  $n \geq N$

$\langle 1 \rangle 5.$   $(x_n) \in \mathbb{R}^\infty$

$\langle 1 \rangle 6.$   $(x_n) \in B_{\bar{\rho}}(0, \epsilon)$

PROOF: Since  $\bar{\rho}((x_n), 0) = \epsilon/2$ .

$\langle 1 \rangle 7.$   $d((x_n), 0) > 1$

PROOF:

$$\begin{aligned} d((x_n), 0)^2 &= \sum_{n=0}^{\infty} x_n^2 \\ &= N\epsilon^2/4 \\ &> 1 \end{aligned}$$

□

**Proposition 18.4.12.** *The  $l^2$ -topology on  $\mathbb{R}^\infty$  is strictly coarser than the box topology.*

$\langle 1 \rangle 1.$  LET:  $U = \prod_n (-1/n, 1/n) \cap \mathbb{R}^\infty$

$\langle 1 \rangle 2.$  ASSUME: for a contradiction  $U$  is open in the metric topology.

$\langle 1 \rangle 3.$  PICK  $\epsilon > 0$  such that  $B(0, \epsilon) \cap \mathbb{R}^\infty \subseteq U \cap \mathbb{R}^\infty$

$\langle 1 \rangle 4.$  PICK  $N$  such that  $1/N < \epsilon/2$ .

⟨1⟩5. LET:  $(x_n)$  be the sequence with  $x_N = \epsilon/2$  and  $x_n = 0$  for all other  $n$ .

⟨1⟩6.  $d((x_n), 0) = \epsilon/2$

⟨1⟩7.  $(x_n) \notin U$

□

**Proposition 18.4.13.** *The  $l^2$ -topology on the Hilbert cube the same as the product topology.*

PROOF:

⟨1⟩1. For every  $(x_n) \in H$  and  $\epsilon > 0$ , there exists a neighbourhood  $U$  of  $(x_n)$  in the product topology such that  $U \subseteq B((x_n), \epsilon)$ .

⟨2⟩1. LET:  $(x_n) \in H$

⟨2⟩2. LET:  $\epsilon > 0$

⟨2⟩3. PICK  $N$  such that  $\sum_{i=N+1}^{\infty} 1/i^2 < \epsilon^2/2$

⟨2⟩4. LET:  $B' = (\prod_{i=0}^N (x_i - \epsilon/\sqrt{2N}, x_i + \epsilon/\sqrt{2N}) \times \prod_{i=N+1}^{\infty} [0, 1/(i+1)]) \cap H$

PROVE:  $B' \subseteq B((x_n), \epsilon)$

⟨2⟩5. LET:  $(y_n) \in B'$

⟨2⟩6.  $d((x_n), (y_n)) < \epsilon$

PROOF:

$$\begin{aligned} d((x_n), (y_n))^2 &= \sum_{i=0}^{\infty} |x_n - y_n|^2 \\ &< \sum_{i=0}^N \epsilon^2/2N + \sum_{i=N+1}^{\infty} 1/(i+1)1/(i+1)^2 \\ &< \epsilon^2/2 + \epsilon^2/2 \\ &= \epsilon^2 \end{aligned}$$

⟨1⟩2. The product topology is finer than the  $l^2$ -topology.

⟨2⟩1. LET:  $(x_n) \in H$  and  $\epsilon > 0$

PROVE:  $B((x_n), \epsilon) \cap H$  is open in the product topology.

⟨2⟩2. LET:  $(y_n) \in B((x_n), \epsilon)$

⟨2⟩3. PICK a neighbourhood  $U$  of  $(y_n)$  in the product topology such that

$U \subseteq B((y_n), \epsilon - d((x_n), (y_n)))$

⟨2⟩4.  $U \subseteq B((x_n), \epsilon)$

□

**Definition 18.4.14.** Let  $l_{\infty}$  be the set of all bounded sequences in  $\mathbb{R}$  under

$$\|(x_n)\| := \sup_n |x_n|$$

**Proposition 18.4.15.** *For all  $p \geq 1$  we have  $l_p$  is not homeomorphic to  $l_{\infty}$ .*

**Proposition 18.4.16.** *Let  $\| \cdot \|$  be a seminorm on the vector space  $E$ . Then  $\| \cdot \|$  defines a norm on  $E/\{0\}$ .*

**Proposition 18.4.17.** *Let  $E$  and  $F$  be normed spaces. Any continuous linear map  $E \rightarrow F$  is uniformly continuous.*

**Definition 18.4.18.** For  $p \geq 1$ , let  $\mathcal{L}^p(\mathbb{R}^n)$  be the vector space of all Lebesgue-measurable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $|f|^p$  is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on  $\mathcal{L}^p(\mathbb{R}^n)$ . Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n) / \{0\}.$$

## 18.5 Unit Ball

**Proposition 18.5.1.** *Let  $n$  be a positive integer. Every open ball  $B(\vec{x}, \epsilon)$  in  $\mathbb{R}^n$  is path connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\vec{y}, \vec{z} \in B(\vec{x}, \epsilon)$

$\langle 1 \rangle 2$ . LET:  $\vec{p} : [0, 1] \rightarrow B(\vec{x}, \epsilon)$  be the path  $\vec{p}(t) = (1-t)\vec{y} + t\vec{z}$ .

$\langle 2 \rangle 1$ . LET:  $t \in [0, 1]$

PROVE:  $\vec{p}(t) \in B(\vec{x}, \epsilon)$

$\langle 2 \rangle 2$ .  $d(\vec{p}(t), \vec{x}) < \epsilon$

PROOF:

$$\begin{aligned} d(\vec{p}(t), \vec{x}) &= \|(1-t)\vec{y} + t\vec{z} - \vec{x}\| \\ &= \|(1-t)(\vec{y} - \vec{x}) + t(\vec{z} - \vec{x})\| \\ &\leq (1-t)\|\vec{y} - \vec{x}\| + t\|\vec{z} - \vec{x}\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$ .  $\vec{p}$  is a path from  $\vec{x}$  to  $\vec{y}$ .

□

**Proposition 18.5.2.** *Let  $n$  be a positive integer. Every closed ball  $B(\vec{x}, \epsilon)$  in  $\mathbb{R}^n$  is path connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\vec{y}, \vec{z} \in \overline{B(\vec{x}, \epsilon)}$

$\langle 1 \rangle 2$ . LET:  $\vec{p} : [0, 1] \rightarrow \overline{B(\vec{x}, \epsilon)}$  be the path  $\vec{p}(t) = (1-t)\vec{y} + t\vec{z}$ .

$\langle 2 \rangle 1$ . LET:  $t \in [0, 1]$

PROVE:  $\vec{p}(t) \in \overline{B(\vec{x}, \epsilon)}$

$\langle 2 \rangle 2$ .  $d(\vec{p}(t), \vec{x}) \leq \epsilon$

PROOF:

$$\begin{aligned} d(\vec{p}(t), \vec{x}) &= \|(1-t)\vec{y} + t\vec{z} - \vec{x}\| \\ &= \|(1-t)(\vec{y} - \vec{x}) + t(\vec{z} - \vec{x})\| \\ &\leq (1-t)\|\vec{y} - \vec{x}\| + t\|\vec{z} - \vec{x}\| \\ &\leq (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$ .  $\vec{p}$  is a path from  $\vec{x}$  to  $\vec{y}$ .

□

## 18.6 Unit Sphere

**Definition 18.6.1** (Unit Sphere). Let  $n$  be a positive integer. The *unit sphere*  $S^{n-1}$  is

$$S^{n-1} := \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = 1\} .$$

**Proposition 18.6.2.** For  $n > 1$ , the unit sphere  $S^{n-1}$  is path connected.

PROOF: The map  $g : \mathbb{R}^n - \{\vec{0}\} \rightarrow S^{n-1}$  defined by  $g(\vec{x}) = \vec{x}/\|\vec{x}\|$  is continuous and surjective. Hence  $S^{n-1}$  is the continuous image of a path connected space.  $\square$

## 18.7 Inner Product Spaces

**Definition 18.7.1** (Inner Product). Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n .$$

**Proposition 18.7.2.**

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF:

$$\begin{aligned} \vec{x} \cdot (\vec{y} + \vec{z}) &= x_1(y_1 + z_1) + \cdots + x_n(y_n + z_n) \\ &= x_1 y_1 + x_1 z_1 + \cdots + x_n y_n + x_n z_n \\ &= \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z} \end{aligned} \quad \square$$

**Proposition 18.7.3.** For all  $\vec{x}, \vec{y} \in \mathbb{R}^n$  we have

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\| .$$

PROOF:

- $\langle 1 \rangle 1$ . ASSUME: w.l.o.g.  $\vec{x} \neq \vec{0} \neq \vec{y}$
- $\langle 1 \rangle 2$ . LET:  $a = 1/\|\vec{x}\|$
- $\langle 1 \rangle 3$ . LET:  $b = 1/\|\vec{y}\|$
- $\langle 1 \rangle 4$ .  $\|a\vec{x} + b\vec{y}\| \geq 0$
- $\langle 1 \rangle 5$ .  $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
- $\langle 1 \rangle 6$ .  $ab\vec{x} \cdot \vec{y} \geq -1$
- $\langle 1 \rangle 7$ .  $\|a\vec{x} - b\vec{y}\| \geq 0$
- $\langle 1 \rangle 8$ .  $ab\vec{x} \cdot \vec{y} \leq 1$
- $\langle 1 \rangle 9$ .  $|\vec{x} \cdot \vec{y}| \leq 1/ab$

$\square$

**Proposition 18.7.4.** Let  $(x_n), (y_n)$  be sequences of real numbers. If  $\sum_{n=0}^{\infty} x_n^2$  and  $\sum_{n=0}^{\infty} y_n^2$  converge then  $\sum_{n=0}^{\infty} |x_n y_n|$  converges.

PROOF:

$$\begin{aligned} \sum_{n=0}^N |x_n y_n| &\leq \sqrt{\sum_{n=0}^N x_n^2 \sum_{n=0}^N y_n^2} && \text{(Proposition 18.7.3)} \\ &\leq \sqrt{\sum_{n=0}^{\infty} x_n^2 \sum_{n=0}^{\infty} y_n^2} && \square \end{aligned}$$

**Proposition 18.7.5.** *If  $E$  is an inner product space then  $\|x\| = \sqrt{\langle x, x \rangle}$  is a norm on  $E$ .*

## 18.8 Banach Spaces

**Definition 18.8.1** (Banach Space). A *Banach space* is a complete normed space.

**Example 18.8.2.** For any topological space  $X$ , the set  $C(X)$  of bounded continuous functions  $X \rightarrow \mathbb{R}$  is a Banach space under  $\|f\| = \sup_{x \in X} |f(x)|$ .

**Proposition 18.8.3.** *The completion of a normed space is a Banach space.*

**Proposition 18.8.4.** *Let  $E$  and  $F$  be normed spaces. Let  $f : E \rightarrow F$  be a continuous linear map. Then the extension to the completions  $\hat{E} \rightarrow \hat{F}$  is linear.*

**Proposition 18.8.5.**  *$L^p(\mathbb{R}^n)$  is a Banach space.*

**Proposition 18.8.6.**  *$C(\mathbb{R})$  is first countable but not second countable.*

PROOF: For every sequence of 0s and 1s  $s = (s_n)$ , let  $f_s$  be a continuous bounded function whose value at  $n$  is  $s_n$ . Then the set of all  $f_s$  is an uncountable discrete set in  $C(\mathbb{R})$ . Hence  $C(\mathbb{R})$  is not second countable.

It is first countable because it is metrizable.  $\square$

## 18.9 Hilbert Spaces

**Definition 18.9.1** (Hilbert Space). A *Hilbert space* is a complete inner product space.

**Example 18.9.2.** The set of *square-integrable functions* is the set of Lebesgue integrable functions  $[-\pi, \pi] \rightarrow \mathbb{R}$  quotiented by:  $f \sim g$  iff  $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$  has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

**Proposition 18.9.3.** *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.



## 18.10 Locally Convex Spaces

**Definition 18.10.1** (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

**Proposition 18.10.2.** *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18.  $\square$

**Proposition 18.10.3.** *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18.  $\square$

**Example 18.10.4.** Let  $E$  be an infinite dimensional Hilbert space. Let  $E'$  be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map  $E \rightarrow \mathbb{R}$  is continuous as a map  $E' \rightarrow \mathbb{R}$ . Then  $E$  is locally convex Hausdorff but not metrizable.

Proof: See Dieudonné, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

**Definition 18.10.5** (Thom Space). Let  $E$  be a vector bundle with a Riemannian metric,  $DE = \{x \in E : \|x\| \leq 1\}$  its disc bundle and  $SE := \{v \in E : \|v\| = 1\}$  its sphere bundle. The *Thom space* of  $E$  is the quotient space  $DE/SE$ .