Mathematics

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Contents

Chapter 1

The Foundations

1.1 Primitive Notions and Axioms

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ for 'f is a function from A to B'. We call A the *domain* of A, and B the *codomain*.

Given sets A, B and C, and functions $f : A \to B$ and $g : B \to C$, let there be a function $gf = g \circ f : A \to C$, the *composite* of f and g.

Axiom 1.1 (Associativity). *For any functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \enspace .$$

Axiom 1.2 (Identity). *For any set A, there exists a function $\mathrm{id}_A : A \to A$, called an* identity *function on A, such that:*

- *for every set B and function $f : A \to B$, we have $f \circ \mathrm{id}_A = f$;*
- *for every set B and function $f : B \to A$, we have $\mathrm{id}_A \circ f = f$.*

Proposition 1.3. *The identity function on a set is unique.*

Proof: If $i, j : A \to A$ are identity functions on A then we have $i = i \circ j = j$. □

Definition 1.4 (Isomorphism). A function $i : A \to B$ is an *isomorphism*, $i : A \cong B$, iff there exists a function $i^{-1} : B \to A$, the *inverse* of i, such that $i^{-1} \circ i = \mathrm{id}_A$ and $i \circ i^{-1} = \mathrm{id}_B$.

Axiom 1.5 (Terminal Set). *There exists a set 1 such that, for any set A, there exists a unique function $A \to 1$.*

Proposition 1.6. *The terminal set is unique up to unique isomorphism.*

Proof:

$\langle 1 \rangle$1. LET: A and B be terminal sets.
$\langle 1 \rangle$2. LET: i be the unique function $A \to B$.
$\langle 1 \rangle$3. LET: i^{-1} be the unique function $B \to A$.
$\langle 1 \rangle$4. $i \circ i^{-1} = \mathrm{id}_B$
PROOF: Since there is only one function $B \to B$.
$\langle 1 \rangle$5. $i^{-1} \circ i = \mathrm{id}_A$
PROOF: Since there is only one function $A \to A$.
□

Definition 1.7 (Element)**.** For any set A, an *element* of A is a function $1 \to A$.

We write $a \in A$ for $a : 1 \to A$. Given $f : A \to B$ and $a \in A$, we write $f(a)$ for $f \circ a$.

Axiom 1.8 (Extensionality)**.** *Let A and B be sets. Let $f, g : A \to B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Axiom 1.9 (Empty Set)**.** *There exists a set with no elements.*

Axiom 1.10 (Products)**.** *Let A and B be sets. There exists a set $A \times B$ and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, the* projections*, such that, for every set X and functions $f : X \to A$, $g : X \to B$, there exists a unique function $\langle f, g \rangle : X \to A \times B$ such that*

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g \ .$$

Proposition 1.11. *If $\pi_1 : P \to A$ and $\pi_2 : P \to B$ form a product of A and B, and $p_1 : Q \to A$ and $p_2 : Q \to B$ form a product of A and B, then there exists a unique isomorphism $i : P \cong Q$ such that $p_1 \circ i = \pi_1$ and $p_2 \circ i = \pi_2$.*

PROOF:
$\langle 1 \rangle$1. LET: $i : P \to Q$ be the unique function such that $p_1 \circ i = \pi_1$ and $p_2 \circ i = \pi_2$.
$\langle 1 \rangle$2. LET: $i^{-1} : Q \to P$ be the unique function such that $\pi_1 \circ i^{-1} = p_1$ and $\pi_2 \circ i^{-1} = p_2$
$\langle 1 \rangle$3. $i \circ i^{-1} = \mathrm{id}_Q$
PROOF: Each is the unique $x : Q \to Q$ such that $p_1 \circ x = p_1$ and $p_2 \circ x = p_2$.
$\langle 1 \rangle$4. $i^{-1} \circ i = \mathrm{id}_P$
PROOF: Each is the unique $x : P \to P$ such that $\pi_1 \circ x = \pi_1$ and $\pi_2 \circ x = \pi_2$.
□

Definition 1.12. Given functions $f : A \to B$ and $g : C \to D$, define $f \times g : A \times C \to B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle \ .$$

Axiom 1.13 (Function Sets)**.** *Let A and B be sets. There exists a set A^B and function $\epsilon : A^B \times B \to A$ such that, for any set X and function $f : X \times B \to A$, there exists a unique function $\lambda f : X \to A^B$ such that*

$$f = \epsilon \circ \langle \lambda f \circ \pi_1, \pi_2 \rangle \ .$$

Definition 1.14 (Inverse Image)**.** Let A, X and Y be sets. Let $f : X \to Y$, $a \in Y$ and $j : A \to X$. Then j is the *inverse image* of a under f if and only if:

- $f \circ j = a \circ !_A$
- for every set I and function $q : I \to X$ such that $f \circ q = a \circ !_I$, there exists a unique $\overline{q} : I \to A$ such that $q = j \circ \overline{q}$.

Axiom 1.15 (Inverse Images)**.** *For any sets X and Y, function $f : X \to Y$ and element $a \in Y$, there exists a set $f^{-1}(a)$ and function $j : f^{-1}(a) \to X$ such that j is the inverse image of a under f.*

Definition 1.16 (Injective)**.** A function $f : A \to B$ is *injective*, $f : A \rightarrowtail B$, iff, for every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

Definition 1.17 (Surjective)**.** A function $f : A \to B$ is *surjective*, $f : A \twoheadrightarrow B$, iff, for every set X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

Axiom 1.18 (Subset Classifier)**.** *There exists a set 2 and function $\top : 1 \to 2$ such that, for any sets A and X and any injective function $f : A \to X$, there exists a unique function $\chi : X \to 2$ such that f is the inverse image of $\top$ under χ.*

Axiom 1.19 (Natural Numbers)**.** *There exists a set $\mathbb{N}$, an element $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \to \mathbb{N}$ such that, for every set X, element $a \in X$ and function $r : X \to X$, there exists a unique function $x : \mathbb{N} \to X$ such that $x \circ 0 = a$ and $x \circ s = r \circ x$.*

Axiom 1.20 (Choice)**.** *For every surjective function $r : X \to Y$, there exists $s : Y \to X$ such that $r \circ s$ is an identity function on X.*

1.2 Subsets of a Set

Definition 1.21 (Subset)**.** A *subset* of a set A is a function $A \to 2$.

Proposition 1.22. *Let $i : X \rightarrowtail B$ and $j : Y \rightarrowtail B$ be injective functions. Then i and j have the same characteristic function if and only if there exists an isomorphism $k : X \cong Y$ such that $j \circ k = i$, in which case k is unique.*

$$\begin{tikzcd}
X \arrow[dr, "k"] \arrow[drr, "!"] \arrow[ddr, "i"'] & & \\
& Y \arrow[r, "!"'] \arrow[d, "j"] & 1 \arrow[d, "\top"] \\
& B \arrow[r, "\chi"'] & 2
\end{tikzcd}$$

Proof:

⟨1⟩1. If i and j have the same characteristic function then there exists a unique isomorphism $k : X \to Y$ such that $j \circ k = i$.

⟨2⟩1. LET: $\chi : B \to 2$

⟨2⟩2. ASSUME: χ is the characteristic function of i and j.

⟨2⟩3. LET: $k : X \to Y$ be the unique function such that $j \circ k = i$.

⟨2⟩4. LET: $k^{-1} : Y \to X$ be the unique function such that $i \circ k^{-1} = j$.

⟨2⟩5. $k \circ k^{-1} = \mathrm{id}_Y$

PROOF: Each is the unique function x such that $j \circ x = x$.

⟨2⟩6. $k^{-1} \circ k = \mathrm{id}_X$

PROOF: Each is the unique function x such that $i \circ x = x$.

⟨1⟩2. If there exists an isomorphism $k : X \cong Y$ such that $j \circ k = i$ then i and j have the same characteristic function.

⟨2⟩1. LET: $k : X \cong Y$ satisfy $j \circ k = i$.

⟨2⟩2. LET: $\chi : B \to 2$ be the characteristic function of j.

PROVE: χ is the characteristic function of i.

⟨2⟩3. $\chi \circ i = \top \circ !_X$

PROOF:

$$\begin{aligned}
\chi \circ i &= \chi \circ j \circ k && (\langle 2\rangle 1) \\
&= \top \circ !_Y \circ k && (\langle 2\rangle 2) \\
&= \top \circ !_X && (\text{Uniqueness of } !_X)
\end{aligned}$$

⟨2⟩4. For every set I and function $q : I \to B$ such that $\chi \circ q = \top \circ !_I$, there exists a unique $\overline{q} : I \to X$ such that $q = i \circ \overline{q}$.

⟨3⟩1. LET: I be a set.

⟨3⟩2. LET: $q : I \to B$

⟨3⟩3. ASSUME: $\chi \circ q = \top \circ !_I$

⟨3⟩4. LET: $r : I \to Y$ be the unique function such that $q = j \circ r$

⟨3⟩5. $k^{-1} \circ r$ is unique such that $q = i \circ k^{-1} \circ r$

□