Encyclopaedia of Mathematics and Physics

Robin Adams

Contents

Chapter 1

Relations

Definition 1.1 (Antisymmetric)**.** A relation R on a set A is *antisymmetric* iff, whenever xRy and yRx, then $x = y$.

Definition 1.2 (Transitive)**.** A relation R on a type A is *transitive* iff, whenever xRy and yRz, then xRz.

Chapter 2

Order Theory

Definition 2.1 (Linear Order)**.** A *linear order* on a set A is a binary relation $\leq$ on A that is transitive, antisymmetric and:

$$\forall x, y \in A. x \leq y \vee y \leq x \ \ .$$

A *linearly ordered set* is a pair $(A, \leq)$ where A is a set and $\leq$ is a binary relation on A.

We write $x < y$ for $x \leq y$ and $x \neq y$.

Definition 2.2 (Upper Bound)**.** Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is an *upper bound* in E iff $\forall x \in E. x \leq u$. We say E is *bounded above* iff it has an upper bound.

The *up-set* of E, denoted $E\uparrow$, is the set of upper bounds of E.

Definition 2.3 (Lower Bound)**.** Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then u is an *lower bound* in E iff $\forall x \in E. l \leq x$. We say E is *bounded below* iff it has a lower bound.

The *down-set* of E, denoted $E\downarrow$, is the set of lower bounds of E.

Definition 2.4 (Supremum)**.** Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is the *least upper bound* or *supremum* of E iff u is an upper bound for E and, for any upper bound u' for E, we have $u \leq u'$.

Definition 2.5 (Infimum)**.** Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then l is the *greatest lower bound* or *infimum* of E iff l is a lower bound for E and, for any lower bound l' for E, we have $l' \leq l$.

Definition 2.6 (Least Upper Bound Property)**.** A linearly ordered set S has the *least upper bound property* iff every nonempty subset of S that is bounded above has a least upper bound.

Proposition 2.7. *Let S be a linearly ordered set and $E \subseteq S$.*

1. *If $E\downarrow$ has a supremum l, then l is the infimum of E.*

2. If $E\uparrow$ has an infimum u, then U is the supremum of E.

PROOF:
⟨1⟩1. If $E\downarrow$ has a supremum l, then l is the infimum of E.
⟨2⟩1. l is a lower bound for E.
⟨3⟩1. LET: $x \in E$
⟨3⟩2. x is an upper bound for $E\downarrow$.
PROOF: For all $y \in E\downarrow$ we have $y \leq x$.
⟨3⟩3. $l \leq x$
⟨2⟩2. For any lower bound l' for E, we have $l' \leq l$.
PROOF: Since l is an upper bound for $E\downarrow$.
⟨1⟩2. If $E\uparrow$ has an infimum u, then u is the supremum of E.
PROOF: Dual.
□

Corollary 2.7.1. *A linearly ordered set has the least upper bound property if and only if every nonempty set bounded below has an infimum.*

Chapter 3

Real Analysis

Proposition 3.1. *There is no rational p such that $p^2 = 2$.*

PROOF:
$\langle 1\rangle 1$. ASSUME: for a contradiction $p^2 = 2$.
$\langle 1\rangle 2$. PICK integers m, n not both even such that $p = m/n$.
$\langle 1\rangle 3$. $m^2 = 2n^2$
$\langle 1\rangle 4$. m is even.
$\langle 1\rangle 5$. PICK an integer k such that $m = 2k$.
$\langle 1\rangle 6$. $4k^2 = 2n^2$
$\langle 1\rangle 7$. $2k^2 = n^2$
$\langle 1\rangle 8$. n is even.
$\langle 1\rangle 9$. Q.E.D.
PROOF: $\langle 1\rangle 2$, $\langle 1\rangle 4$ and $\langle 1\rangle 8$ form a contradiction.
$\square$