Mathematics

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Contents

Part I

Set Theory

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*.

For any set A, let there be *elements* of A. We write $a \in A$ for: a is an element of A.

For any sets A and B, let there be a set B^A, whose elements are called *functions* from A to B. We write $f : A \to B$ for $f \in B^A$.

For any function $f : A \to B$ and element $a \in A$, let there be an element $f(a) \in B$, the *value* of the function f at the *argument* a.

1.2 Injections, Surjections and Bijections

Definition 1.2.1 (Injective)**.** We say a function $f : A \to B$ is *injective* or an *injection*, and write $f : A \rightarrowtail B$, iff, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

We say a set A is *dominated* by a set B, and write $A \preccurlyeq B$, iff there exists an injection from A to B.

Definition 1.2.2 (Surjective)**.** We say a function $f : A \to B$ is *surjective* or a *surjection*, and write $f : A \twoheadrightarrow B$, iff, for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Definition 1.2.3 (Bijective)**.** We say a function $f : A \to B$ is *bijective* or a *bijection*, and write $f : A \approx B$, iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

We say the set A is *strictly* dominated by the set B, and write $A \prec B$, iff $A \preccurlyeq B$ and $A \not\approx B$.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$".

1.3 Axioms

Axiom Schema 1.3.1 (Choice)**.** *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x \in X$ and $y \in Y$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a \in A$, there exists $b \in B$ such that $P[A, B, a, b]$. Then there exists a function $f : A \to B$ such that $\forall a \in A.P[A, B, a, f(a)]$.

Axiom 1.3.2 (Extensionality)**.** *Let $f, g : A \to B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Axiom 1.3.3 (Pairing)**.** *For any sets A and B, there exists a set $A \times B$, the* Cartesian product *of A and B, and functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that, for all $a \in A$ and $b \in B$, there exists a unique $(a, b) \in A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Axiom Schema 1.3.4 (Separation)**.** *For every property $P[X, x]$ where X is a set variable and $x \in X$, the following is an axiom:*

For every set A, there exists a set $S = \{x \in A : P[A, x]\}$ and an injection $i : S \to A$ such that, for all $x \in A$, we have

$$(\exists y \in S.i(y) = x) \Leftrightarrow P[A, x] \ .$$

Axiom 1.3.5 (Infinity)**.** *There exists a set $\mathbb{N}$, an element $0 \in \mathbb{N}$, and a function $s : \mathbb{N} \to \mathbb{N}$ such that:*

- $\forall n \in \mathbb{N}.s(n) \neq 0$
- $\forall m, n \in \mathbb{N}.s(m) = s(n) \Rightarrow m = n.$

Axiom Schema 1.3.6 (Collection)**.** *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A, there exist sets B and Y and functions $p : B \to A$, and $m : B \times Y \Rightarrow \mathbb{N}$ such that:

- *m is injective.*
- $\forall b \in B.P[A, \{y \in Y : m(b, y) = 0\}, p(b)]$
- *For all $a \in A$, if $\exists Y.P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 1.3.7 (Composition)**.** Let $f : A \to B$ and $g : B \to C$. The *composite* $g \circ f : A \to C$ is the function such that, for all $a \in A$, we have

$$(g \circ f)(a) = g(f(a)) \ .$$

Axiom 1.3.8 (Universe)**.** *There exists a set E, a set U and a function $el : E \to U$ such that the following holds.*

Let us say that a set A is small *iff there exists $u \in U$ such that $A \approx \{e \in E : el(e) = u\}$.*

- $\mathbb{N}$ *is small.*
- *For any U-small sets A and B, the set B^A is small.*
- *For any U-small sets A and B, the set $A \times B$ is small.*
- *Let $f : A \to B$ be a function. If B is small and $\{a \in A : f(a) = b\}$ is small for all $b \in B$, then A is small.*
- *If $p : B \twoheadrightarrow A$ is a surjective function such that A is small, then there exists a U-small set C, a surjection $q : C \twoheadrightarrow A$, and a function $f : C \to B$ such that $q = p \circ f$.*

Chapter 2

Sets and Functions

2.1 Composition

Proposition 2.1.1. *Given functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \quad .$$

PROOF:
$\langle 1 \rangle 1$. For all $x \in A$ we have $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$.
  $\langle 2 \rangle 1$. LET: $x \in A$
  $\langle 2 \rangle 2$. $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$
    PROOF:

$$\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) && \text{(Definition of composition)} \\
&= h(g(f(x))) && \text{(Definition of composition)} \\
&= (h \circ g)(f(x)) && \text{(Definition of composition)} \\
&= ((h \circ g) \circ f)(x) && \text{(Definition of composition)}
\end{aligned}$$

$\langle 1 \rangle 2$. Q.E.D.
  PROOF: By the Axiom of Extensionality.
$\square$

2.2 Injections

Proposition 2.2.1. *The composite of injective functions is injective.*

PROOF:
$\langle 1 \rangle 1$. LET: A, B and C be sets.
$\langle 1 \rangle 2$. LET: $f : A \to B$
$\langle 1 \rangle 3$. LET: $g : B \to C$
$\langle 1 \rangle 4$. ASSUME: g is injective.
$\langle 1 \rangle 5$. ASSUME: f is injective.
$\langle 1 \rangle 6$. LET: $x, y \in A$

$\langle 1 \rangle 7$. ASSUME: $(g \circ f)(x) = (g \circ f)(y)$
PROVE: $x = y$
$\langle 1 \rangle 8$. $g(f(x)) = g(f(y))$
PROOF:

$$\begin{aligned} g(f(x)) &= (g \circ f)(x) && \text{(definition of composition)} \\ &= (g \circ f)(y) && (\langle 1 \rangle 7) \\ &= g(f(y)) && \text{(definition of composition)} \end{aligned}$$

$\langle 1 \rangle 9$. $f(x) = f(y)$
PROOF: $\langle 1 \rangle 4$, $\langle 1 \rangle 8$
$\langle 1 \rangle 10$. $x = y$
PROOF: $\langle 1 \rangle 5$, $\langle 1 \rangle 9$
□

Proposition 2.2.2. *For functions $f : A \to B$ and $g : B \to C$, if $g \circ f$ is injective then f is injective.*

PROOF:
$\langle 1 \rangle 1$. LET: A, B and C be sets.
$\langle 1 \rangle 2$. LET: $f : A \to B$
$\langle 1 \rangle 3$. LET: $g : B \to C$
$\langle 1 \rangle 4$. ASSUME: $g \circ f$ is injective.
$\langle 1 \rangle 5$. LET: $x, y \in A$
$\langle 1 \rangle 6$. ASSUME: $f(x) = f(y)$
$\langle 1 \rangle 7$. $(g \circ f)(x) = (g \circ f)(y)$
PROOF:

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{(definition of composition)} \\ &= g(f(y)) && (\langle 1 \rangle 6) \\ &= (g \circ f)(y) && \text{(definition of composition)} \end{aligned}$$

$\langle 1 \rangle 8$. $x = y$
PROOF: $\langle 1 \rangle 4$, $\langle 1 \rangle 7$
□

Proposition 2.2.3. *Let $f : A \to B$ be injective. For every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: f is injective.
$\langle 1 \rangle 2$. LET: X be a set.
$\langle 1 \rangle 3$. LET: $x, y : X \to A$
$\langle 1 \rangle 4$. ASSUME: $f \circ x = f \circ y$
$\langle 1 \rangle 5$. $\forall t \in X. x(t) = y(t)$
$\langle 2 \rangle 1$. LET: $t \in X$
$\langle 2 \rangle 2$. $f(x(t)) = f(y(t))$

PROOF:

$$\begin{aligned} f(x(t)) &= (f \circ x)(t) && \text{(definition of composition)} \\ &= (f \circ y)(t) && (\langle 1 \rangle 4) \\ &= f(y(t)) && \text{(definition of composition)} \end{aligned}$$

$\langle 2 \rangle 3$. $x(t) = y(t)$
PROOF: $\langle 1 \rangle 1$, $\langle 2 \rangle 2$
$\langle 1 \rangle 6$. $x = y$
PROOF: Axiom of Extensionality, $\langle 1 \rangle 5$
□

We will prove the converse as Proposition 2.8.4.

2.3 Surjections

Proposition 2.3.1. *The composite of surjective functions is surjective.*

PROOF:
$\langle 1 \rangle 1$. LET: A, B and C be sets.
$\langle 1 \rangle 2$. LET: $f : A \to B$ and $g : B \to C$
$\langle 1 \rangle 3$. ASSUME: g is surjective.
$\langle 1 \rangle 4$. ASSUME: f is surjective.
$\langle 1 \rangle 5$. LET: $c \in C$
$\langle 1 \rangle 6$. PICK $b \in B$ such that $g(b) = c$.
PROOF: $\langle 1 \rangle 3$
$\langle 1 \rangle 7$. PICK $a \in A$ such that $f(a) = b$.
PROOF: $\langle 1 \rangle 4$
$\langle 1 \rangle 8$. $(g \circ f)(a) = c$
PROOF:

$$\begin{aligned} (g \circ f)(a) &= g(f(a)) && \text{(definition of composition)} \\ &= g(b) && (\langle 1 \rangle 7) \\ &= c && (\langle 1 \rangle 6) \end{aligned}$$

□

Proposition 2.3.2. *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is surjective then g is surjective.*

PROOF:
$\langle 1 \rangle 1$. LET: A, B and C be sets.
$\langle 1 \rangle 2$. LET: $f : A \to B$ and $g : B \to C$.
$\langle 1 \rangle 3$. ASSUME: $g \circ f$ is surjective.
$\langle 1 \rangle 4$. LET: $c \in C$
$\langle 1 \rangle 5$. PICK $a \in A$ such that $(g \circ f)(a) = c$
PROOF: $\langle 1 \rangle 3$
$\langle 1 \rangle 6$. $g(f(a)) = c$
PROOF: From $\langle 1 \rangle 5$ and the definition of composition.
$\langle 1 \rangle 7$. Q.E.D.

PROOF: There exists $b \in B$ such that $g(b) = c$, namely $b = f(a)$.
□

Proposition 2.3.3. *Let $f : A \to B$ be a surjection. For any set X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.*

PROOF:
⟨1⟩1. LET: $b \in B$
⟨1⟩2. PICK $a \in A$ such that $f(a) = b$
⟨1⟩3. $x(f(a)) = y(f(a))$
⟨1⟩4. $x(b) = y(b)$
⟨1⟩5. Q.E.D.
PROOF: Axiom of Extensionality.
□

We will prove the converse as Proposition 2.9.2.

2.4 Bijections

Proposition 2.4.1. *The composite of bijections is a bijection.*

PROOF:
⟨1⟩1. LET: A, B and C be sets.
⟨1⟩2. LET: $f : A \to B$ and $g : B \to C$
⟨1⟩3. ASSUME: g is bijective.
⟨1⟩4. ASSUME: f is bijective.
⟨1⟩5. g is injective.
PROOF: From ⟨1⟩3.
⟨1⟩6. g is surjective.
PROOF: From ⟨1⟩3.
⟨1⟩7. f is injective.
PROOF: From ⟨1⟩4.
⟨1⟩8. f is surjective.
PROOF: From ⟨1⟩4.
⟨1⟩9. $g \circ f$ is injective.
PROOF: Proposition 2.2.1, ⟨1⟩5, ⟨1⟩7.
⟨1⟩10. $g \circ f$ is surjective.
PROOF: Proposition 2.3.1, ⟨1⟩6, ⟨1⟩8.
⟨1⟩11. $g \circ f$ is bijective.
PROOF: ⟨1⟩9, ⟨1⟩10
□

Proposition 2.4.2.
$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps f to $(\pi_1 \circ f, \pi_2 \circ f)$ is a bijection. □

Proposition 2.4.3.
$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function Φ such that $\Phi(f)(c)(b) = f(b, c)$ is a bijection. ∎

2.5 Domination

Definition 2.5.1 (Dominate)**.** Let A and B be sets. We say that B *dominates* A, and write $A \preccurlyeq B$, iff there exists an injective function $A \to B$.

Theorem 2.5.2 (Schroeder-Bernstein)**.** *Let A and B be sets. If $A \preccurlyeq B$ and $B \preccurlyeq A$ then $A \approx B$.*

PROOF:
⟨1⟩1. LET: $f : A \rightarrowtail B$ and $g : B \rightarrowtail A$ be injections.
⟨1⟩2. Define the subsets A_n of A by

$$\begin{aligned} A_0 &:= A - g(B) \\ A_{n+1} &:= g(f(A_n)) \end{aligned}$$

⟨1⟩3. Define $h : A \to B$ by

$$h(x) = \begin{cases} f(x) & \text{if } \exists n. x \in A_n \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

⟨1⟩4. h is injective.
  ⟨2⟩1. LET: $x, y \in A$
  ⟨2⟩2. ASSUME: $h(x) = h(y)$
  ⟨2⟩3. CASE: $x \in A_m$ and $y \in A_n$.
    PROOF: Then $f(x) = f(y)$ so $x = y$ since f is injective.
  ⟨2⟩4. CASE: $x \in A_m$ and there is no y such that $y \in A_n$.
    ⟨3⟩1. $f(x) = g^{-1}(y)$
    ⟨3⟩2. $y = g(f(x))$
    ⟨3⟩3. $y \in A_{m+1}$
    ⟨3⟩4. Q.E.D.
      PROOF: This is a contradiction.
  ⟨2⟩5. CASE: $y \in A_n$ and there is no m such that $x \in A_m$.
    PROOF: Similar.
  ⟨2⟩6. CASE: There is no m such that $x \in A_m$ and there is no n such that $y \in A_n$.
    PROOF: Then $g^{-1}(x) = g^{-1}(y)$ and so $x = y$.
⟨1⟩5. h is surjective.
  ⟨2⟩1. LET: $y \in B$
  ⟨2⟩2. CASE: $g(y) \in A_n$
    ⟨3⟩1. $n \neq 0$
    ⟨3⟩2. PICK $x \in A_{n-1}$ such that $g(y) = g(f(x))$
    ⟨3⟩3. $y = f(x)$
    ⟨3⟩4. $y = h(x)$
  ⟨2⟩3. CASE: There is no n such that $g(y) \in A_n$.
    PROOF: Then $h(g(y)) = y$.
∎

2.6 Identity Function

Definition 2.6.1 (Identity)**.** For any set A, the *identity* function $\mathrm{id}_A : A \to A$ is the function defined by $\mathrm{id}_A(a) = a$.

2.6.1 Injections, Surjections, Bijections

Proposition 2.6.2. *For any set A, the identity function id_A is a bijection.*

PROOF:
⟨1⟩1. LET: A be a set.
⟨1⟩2. id_A is injective.
PROOF: If $\mathrm{id}_A(x) = \mathrm{id}_A(y)$ then $x = y$.
⟨1⟩3. id_A is surjective.
PROOF: For any $y \in A$, there exists $x \in A$ such that $\mathrm{id}_A(x) = y$, namely $x = y$.
□

2.6.2 Composition

Proposition 2.6.3. *Let $f : A \to B$. Then $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.*

PROOF: Each is the function that maps a to $f(a)$. □

Proposition 2.6.4. *Let $f : A \to B$.*

1. *If there exists $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ then f is injective.*
2. *If f is injective and A is nonempty, then there exists $g : B \to A$ such that $g \circ f = \mathrm{id}_A$.*

PROOF:
⟨1⟩1. If there exists $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ then f is injective.
PROOF: If $f(x) = f(y)$ then $x = g(f(x)) = g(f(y)) = y$.
⟨1⟩2. If f is injective and A is nonempty, then there exists $g : B \to A$ such that $g \circ f = \mathrm{id}_A$.
⟨2⟩1. ASSUME: f is injective and A is nonempty.
⟨2⟩2. PICK $a \in A$
⟨2⟩3. Choose a function $g : B \to A$ such that $f(g(x)) = x$ if there exists $y \in A$ such that $f(y) = x$, otherwise $g(x) = a$.
⟨2⟩4. LET: $x \in A$
PROVE: $g(f(x)) = x$
⟨2⟩5. $f(g(f(x))) = f(x)$
⟨2⟩6. $g(f(x)) = x$
□

Proposition 2.6.5. *Let $f : A \to B$. Then f is surjective if and only if there exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$.*

Proof:
⟨1⟩1. If f is surjective then there exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$.
  ⟨2⟩1. Assume: f is surjective.
  ⟨2⟩2. Pick $g : B \to A$ such that, for all $b \in B$, we have $f(g(b)) = b$.
    Proof: Axiom of Choice.
  ⟨2⟩3. $f \circ g = \mathrm{id}_B$.
⟨1⟩2. If there exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ then f is surjective.
  ⟨2⟩1. Let: $g : B \to A$ such that $f \circ g = \mathrm{id}_B$
  ⟨2⟩2. Let: X be a set.
  ⟨2⟩3. Let: $h, k : B \to X$
  ⟨2⟩4. Assume: $h \circ f = k \circ f$
  ⟨2⟩5. $h = k$
    Proof: $h = h \circ f \circ g = k \circ f \circ g = k$
□

Corollary 2.6.5.1. *Let A and B be sets.*

1. *If there exists a surjective function $A \to B$ then there exists an injective function $B \to A$.*

2. *If there exists an injective function $A \to B$ and A is nonempty then there exists a surjective function $B \to A$.*

Proposition 2.6.6. *Let $f : A \to B$. Then f is bijective if and only if there exists a function $f^{-1} : B \to A$, the* inverse *of f, such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$, in which case the inverse is unique.*

Proof:
⟨1⟩1. If f is bijective then there exists $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.
  ⟨2⟩1. Assume: f is bijective.
  ⟨2⟩2. Pick $g : B \to A$ such that $f \circ g = \mathrm{id}_B$
    Proof: Proposition 2.9.2.
  ⟨2⟩3. $f \circ g \circ f = f$
  ⟨2⟩4. $g \circ f = \mathrm{id}_A$
    Proof: Proposition 2.2.3.
⟨1⟩2. If there exists $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$, then f is bijective.
  ⟨2⟩1. Let: $f^{-1} : B \to A$ satisfy $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$
  ⟨2⟩2. f is injective.
    Proof: If $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$.
  ⟨2⟩3. f is surjective.
    Proof: Proposition 2.9.2.
⟨1⟩3. If $g, h : B \to A$ satisfy $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ and $h \circ f = \mathrm{id}_A$ then $g = h$.
  Proof: We have $g = g \circ f \circ h = h$.
□

2.7 The Empty Set

Theorem 2.7.1. *There exists a set which has no elements.*

Proof: Take $\{x \in \mathbb{N} : \bot\}$. □

Theorem 2.7.2. *If E and E' have no elements then $E \approx E'$.*

Proof:
⟨1⟩1. Let: E and E' have no elements.
⟨1⟩2. Pick a function $F : E \to E'$.
  Proof: Axiom of Choice since vacuously $\forall x \in E. \exists y \in E'. \top$.
⟨1⟩3. F is injective.
  Proof: Vacuously, for all $x, y \in E$, if $F(x) = F(y)$ then $x = y$.
⟨1⟩4. F is surjective.
  Proof: Vacuously, for all $y \in E$, there exists $x \in E$ such that $F(x) = y$.
□

Definition 2.7.3 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

2.8 The Singleton

Theorem 2.8.1. *There exists a set that has exactly one element.*

Proof: The set $\{x \in \mathbb{N} : x = 0\}$ has exactly one element. □

Theorem 2.8.2. *If A and B both have exactly one element then $A \approx B$.*

Proof:
⟨1⟩1. Let: A and B both have exactly one element a and b respectively.
⟨1⟩2. Let: $F : A \to B$ be the function such that, for all $x \in A$, we have $(x = a \wedge F(x) = b)$
⟨1⟩3. F is a bijection.
□

Definition 2.8.3 (Singleton)**.** Let 1 be the set that has exactly one element. Let $*$ be its element.

2.8.1 Injections

Proposition 2.8.4. *Let $f : A \to B$. Assume that, for every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$. Then f is injective.*

Proof: Take $X = 1$. □

2.9 The Set Two

Definition 2.9.1 (The Set Two)**.** Let $2 = \{x \in \mathbb{N} : x = 0 \vee x = 1\}$.

Proposition 2.9.2. *Let $f : A \to B$. Assume that, for any set X and functions $g, h : B \to X$, if $g \circ f = h \circ f$ then $g = h$. Then f is surjective.*

PROOF:
⟨1⟩1. ASSUME: For any set X and functions $g, h : B \to X$, if $g \circ f = h \circ f$ then $g = h$.
⟨1⟩2. LET: $b \in B$
⟨1⟩3. LET: $h : B \to 2$ be the function that maps everything to 1.
⟨1⟩4. LET: $k : B \to 2$ be the function that maps b to 0 and everything else to 1.
⟨1⟩5. $h \neq k$
⟨1⟩6. $h \circ f \neq k \circ f$
⟨1⟩7. PICK $a \in A$ such that $h(f(a)) \neq k(f(a))$
⟨1⟩8. $f(a) = b$
□

2.10 Subsets

Definition 2.10.1 (Subset)**.** A *subset* of a set A consists of a set S and an injection $i : S \rightarrowtail A$. We write $(S, i) \subseteq A$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Proposition 2.10.2. *For any subset (S, i) of A we have $(S, i) = (S, i)$.*

PROOF: We have $\mathrm{id}_S : S \approx S$ and $i \circ \mathrm{id}_S = i$.

Proposition 2.10.3. *If $(S, i) = (T, j)$ then $(T, j) = (S, i)$.*

PROOF: If $\phi : S \approx T$ and $j \circ \phi = i$ then $\phi^{-1} : T \approx S$ and $i \circ \phi^{-1} = j$. □

Proposition 2.10.4. *If $(R, i) = (S, j)$ and $(S, j) = (T, k)$ then $(R, i) = (T, k)$.*

PROOF: If $\phi : R \approx S$ and $j \circ \phi = i$, and $\psi : S \approx T$ and $k \circ \psi = j$, then $\psi \circ \phi : R \approx T$ and $k \circ \psi \circ \phi = i$. □

Definition 2.10.5 (Membership)**.** Given $(S, i) \subseteq A$ and $a \in A$, we write $a \in (S, i)$ for $\exists s \in S. i(s) = a$.

Proposition 2.10.6. *If $a \in (S, i)$ and $(S, i) = (T, j)$ then $a \in (T, j)$.*

PROOF: If $i(s) = a$ then $j(\phi(s)) = a$. □

Definition 2.10.7 (Union)**.** Given subsets S and T of A, the *union* is the subset $\{x \in A : x \in S \vee x \in T\}$.

Definition 2.10.8 (Intersection)**.** Given subsets S and T of A, the *intersection* is the subset $\{x \in A : x \in S \wedge x \in T\}$.

Proposition 2.10.9 (Distributive Law)**.**

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

Proposition 2.10.10 (Distributive Law)**.**

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

Definition 2.10.11. Given a set A, we write $\varnothing$ for the subset $(\varnothing, !)$ where $!$ is the unique function $\varnothing \to A$.

Proposition 2.10.12.

$$S \cup \varnothing = S$$

Proposition 2.10.13.

$$S \cap \varnothing = S$$

Definition 2.10.14 (Inclusion)**.** Given subsets (S, i) and (T, j) of a set A, we write $(S, i) \subseteq (T, j)$ iff there exists $f : S \to T$ such that $j \circ f = i$.

Proposition 2.10.15.

$$\varnothing \subseteq S$$

Definition 2.10.16 (Disjoint)**.** Subsets S and T of A are *disjoint* iff $S \cap T = \varnothing$.

Definition 2.10.17 (Difference)**.** Given subsets S and T of A, the *difference* of S and T is $S - T = \{x \in A : x \in S \wedge x \notin T\}$.

Proposition 2.10.18 (De Morgan's Law)**.**

$$R - (S \cup T) = (R - S) \cap (R - T)$$

Proposition 2.10.19 (De Morgan's Law)**.**

$$R - (S \cap T) = (R - S) \cup (R - T)$$

2.11 Power Set

Definition 2.11.1 (Power Set)**.** The *power set* of a set A is

$$\mathcal{P}A := 2^A$$

Definition 2.11.2 (Cover)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ is a *cover* of X, or *covers* X, iff $\bigcup \mathcal{A} = X$.

Given a subset Y of X and $\mathcal{A} \subseteq \mathcal{P}X$, we say $\mathcal{A}$ *covers* Y iff $Y \subseteq \bigcup \mathcal{A}$.

2.12 Saturated Set

Definition 2.12.1 (Saturated)**.** Let A and B be sets. Let $f : A \to B$ be surjective. Let $C \subseteq A$. Then C is *saturated* with respect to f iff, for all $x \in C$ and $y \in A$, if $f(x) = f(y)$ then $y \in C$.

2.13 Union

Definition 2.13.1 (Union)**.** Given $\mathcal{A} \in \mathcal{P}\mathcal{P}X$, its *union* is

$$\bigcup \mathcal{A} := \{x \in X : \exists S \in \mathcal{A}. x \in S\} \in \mathcal{P}X \ .$$

2.13.1 Intersection

Definition 2.13.2 (Intersection)**.** Given $\mathcal{A} \in \mathcal{P}\mathcal{P}X$, its *intersection* is

$$\bigcap \mathcal{A} := \{x \in X : \forall S \in \mathcal{A}. x \in S\} \in \mathcal{P}X \ .$$

2.13.2 Direct Image

Definition 2.13.3 (Direct Image)**.** Let $f : A \to B$. Let S be a subset of A. The *(direct) image* of S under f is the subset of B given by

$$f(S) := \{f(a) : a \in S\} \ .$$

Proposition 2.13.4.

1. *If* $S \subseteq T$ *then* $f(S) \subseteq f(T)$
2. $f(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f(S)$

Example 2.13.5. It is not true in general that $f(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f(S)$. Take f to be the only function $\{0, 1\} \to \{0\}$, and $\mathcal{S} = \{\{0\}, \{1\}\}$. Then $f(\bigcap \mathcal{S}) = \emptyset$ but $\bigcap_{S \in \mathcal{S}} f(S) = \{0\}$.

Example 2.13.6. It is not true in general that $f(S - T) = f(S) - f(T)$. Take f to be the only function $\{0, 1\} \to \{0\}$, $S = \{0\}$ and $T = \{1\}$. Then $f(S - T) = \{0\}$ but $f(S) - f(T) = \emptyset$.

2.14 Inverse Image

Definition 2.14.1 (Inverse Image)**.** Let $f : A \to B$. Let S be a subset of B. The *inverse image* or *preimage* of S under f is the subset of A given by

$$f^{-1}(S) := \{x \in A : f(x) \in S\} \ .$$

Proposition 2.14.2.

1. *If* $S \subseteq T$ *then* $f^{-1}(S) \subseteq f^{-1}(T)$
2. $f^{-1}(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f^{-1}(S)$
3. $f^{-1}(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f^{-1}(S)$
4. $f^{-1}(S - T) = f^{-1}(S) - f^{-1}(T)$
5. $S \subseteq f^{-1}(f(S))$. *Equality holds if* f *is injective.*
6. $f(f^{-1}(T)) \subseteq T$. *Equality holds if* f *is surjective.*
7. $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$

2.14.1 Saturated Sets

Proposition 2.14.3. *Let A and B be sets. Let $f : A \to B$ be surjective. Let $C \subseteq A$. Then C is saturated if and only if there exists $D \subseteq B$ such that $C = f^{-1}(D)$.*

PROOF:
⟨1⟩1. If C is saturated then there exists $D \subseteq B$ such that $C = f^{-1}(D)$.
  ⟨2⟩1. ASSUME: C is saturated.
  ⟨2⟩2. LET: $D = f(C)$
  ⟨2⟩3. $C \subseteq f^{-1}(D)$
    ⟨3⟩1. LET: $x \in C$
    ⟨3⟩2. $f(x) \in D$
      PROOF: ⟨2⟩2
    ⟨3⟩3. $x \in f^{-1}(D)$
  ⟨2⟩4. $f^{-1}(D) \subseteq C$
    ⟨3⟩1. LET: $x \in f^{-1}(D)$
    ⟨3⟩2. $f(x) \in D$
    ⟨3⟩3. PICK $y \in C$ such that $f(x) = f(y)$
      PROOF: ⟨2⟩2
    ⟨3⟩4. $x \in C$
      PROOF: ⟨2⟩1
⟨1⟩2. If there exists $D \subseteq B$ such that $C = f^{-1}(D)$ then C is saturated.
  ⟨2⟩1. LET: $D \subseteq B$ be such that $C = f^{-1}(D)$.
  ⟨2⟩2. LET: $x \in C$ and $y \in A$
  ⟨2⟩3. ASSUME: $f(x) = f(y)$
  ⟨2⟩4. $f(x) \in D$
  ⟨2⟩5. $f(y) \in D$
  ⟨2⟩6. $y \in C$
□

2.15 Relations

Definition 2.15.1 (Relation)**.** Let A and B be sets. A *relation* R between A and B, $R : A \nrightarrow B$, is a subset of $A \times B$.

Given $a \in A$ and $b \in B$, we write aRb for $(a, b) \in R$.

A relation *on* a set A is a relation between A and A.

Definition 2.15.2 (Reflexive)**.** A relation R on a set A is *reflexive* iff $\forall a \in A.aRa$.

Definition 2.15.3 (Symmetric)**.** A relation R on a set A is *symmetric* iff, whenever xRy, then yRx.

Definition 2.15.4 (Transitive)**.** A relation R on a set A is *transitive* iff, whenever xRy and yRz, then xRz.

2.15.1 Equivalence Relations

Definition 2.15.5 (Equivalence Relation)**.** A relation R on a set A is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Definition 2.15.6 (Equivalence Class)**.** Let R be an equivalence relation on a set A and $a \in A$. The *equivalence class* of a with respect to R is

$$\{x \in A : xRa\} \ .$$

Proposition 2.15.7. *Two equivalence classes are either disjoint or equal.*

2.16 Power Set

Definition 2.16.1 (Power Set)**.** The *power set* of a set A is $\mathcal{P}A := 2^A$.

Given $S \in \mathcal{P}A$ and $a \in A$, we write $a \in A$ for $S(a) = 1$.

Definition 2.16.2 (Pairwise Disjoint)**.** Let $P \subseteq \mathcal{P}A$. We say the members of P are *pairwise disjoint* iff, for all $S, T \in P$, if $S \neq T$ then $S \cap T = \emptyset$.

2.16.1 Partitions

Definition 2.16.3 (Partition)**.** Let A be a set. A *partition* of A is a set $P \in \mathcal{P}\mathcal{P}A$ such that:

- $\bigcup P = A$
- Every member of P is nonempty.
- The members of P are pairwise disjoint.

2.17 Cartesian Product

Definition 2.17.1 (Cartesian Product)**.** Let A and B be sets. The *Cartesian product* of A and B, $A \times B$, is the tabulation of the relation $A \looparrowright B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

2.18 Quotient Sets

Proposition 2.18.1. *Let $\sim$ be an equivalence relation on X. Then there exists a set $X/\sim$, the* quotient set *of X with respect to $\sim$, and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the* canonical projection*, such that, for all $x, y \in X$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to $\sim$, then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

2.19 Partitions

Definition 2.19.1 (Partition). A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X.

2.20 Disjoint Union

Theorem 2.20.1. *For any sets A and B, there exists a set $A+B$, the* disjoint union *of A and B, and functions $\kappa_1 : A \to A+B$ and $\kappa_2 : B \to A+B$, the* injections*, such that, for every set X and functions $f : A \to X$ and $g : B \to X$, there exists a unique function $[f, g] : A + B \to X$ such that $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$.*

PROOF:
⟨1⟩1. LET: $A+B := \{p \in \mathcal{P}A \times \mathcal{P}B : \exists a \in A.p = (\{a\}, \emptyset) \vee \exists b \in B.p = (\emptyset, \{b\})\}$

Definition 2.20.2 (Restriction). Let $f : A \to B$ and let (S, i) be a subset of A. The *restriction* of f to S is the function $f \restriction S : S \to B$ defined by $f \restriction S = f \circ i$.

2.21 Natural Numbers

Theorem 2.21.1 (Principle of Recursive Definition). *Let A be a set. Let F be the set of all functions $\{m \in \mathbb{N} : m < n\} \to A$ for some n. Let $\rho : F \to A$. Then there exists a unique $g : \mathbb{N} \to A$ such that, for all $n \in \mathbb{N}$, we have*

$$g(n) = \rho(g \restriction \{m \in \mathbb{N} : m < n\}) \ .$$

PROOF:
⟨1⟩1. Given a subset $B \subseteq \mathbb{N}$, let us say that a function $g : B \to A$ is *acceptable* iff, for all $n \in B$, we have

$$\forall m < n.m \in B$$

and

$$g(n) = \rho(g \restriction \{m \in \mathbb{N} : m < n\}) \ .$$

⟨1⟩2. For all $n \in \mathbb{N}$, there exists an acceptable function $\{m \in \mathbb{N} : m < n\} \to A$.
  ⟨2⟩1. LET: $P[n]$ be the property: There exists an acceptable function $\{m \in \mathbb{N} : m < n\} \to A$.
  ⟨2⟩2. $P[0]$
    PROOF: The unique function $\emptyset \to A$ is acceptable.
  ⟨2⟩3. For any natural number n, if $P[n]$ then $P[n + 1]$.
    ⟨3⟩1. ASSUME: $P[n]$
    ⟨3⟩2. PICK an acceptable $f : \{m \in \mathbb{N} : m < n\} \to A$.
    ⟨3⟩3. LET: $g : \{m \in \mathbb{N} : m < n + 1\} \to A$ be the function

$$g(m) = \begin{cases} f(m) & \text{if } m < n \\ \rho(f) & \text{if } m = n \end{cases}$$

    ⟨3⟩4. g is acceptable.

$\langle 1 \rangle 3$. If $g : B \to A$ and $h : C \to A$ are acceptable, then g and h agree on $B \cap C$.
$\langle 1 \rangle 4$. Define $g : \mathbb{N} \to A$ by: $g(n) = a$ iff there exists an acceptable $h : \{m \in \mathbb{N} : m < n + 1\}$ such that $h(n) = a$.
$\langle 1 \rangle 5$. g is acceptable.
$\langle 1 \rangle 6$. If $g' : \mathbb{N} \to A$ is acceptable then $g' = g$.
□

2.22 Finite and Infinite Sets

Definition 2.22.1 (Finite)**.** A set A is *finite* iff there exists $n \in \mathbb{N}$ such that $A \approx \{m \in \mathbb{N} : m < n\}$. In this case, we say A has *cardinality* n.

Proposition 2.22.2. *Let $n \in \mathbb{N}$. Let A be a set. Let $a_0 \in A$. Then $A \approx \{m \in \mathbb{N} : m < n + 1\}$ if and only if $A - \{a_0\} \approx \{m \in \mathbb{N} : m < n\}$.*

Theorem 2.22.3. *Let A be a set. Suppose that $A \approx \{m \in \mathbb{N} : m < n\}$. Let B be a proper subset of A. Then $B \not\approx \{m \in \mathbb{N} : m < n\}$ but there exists $m < n$ such that $B \approx \{k \in \mathbb{N} : k < m\}$.*

PROOF:
$\langle 1 \rangle 1$. LET: $P[n]$ be the property: for every set A, if $Aapprox\{m \in \mathbb{N} : m < n\}$, then for every proper subset B of A, we have $B \not\approx \{m \in \mathbb{N} : m < n\}$ but there exists $m < n$ such that $B \approx \{k \in \mathbb{N} : k < m\}$.
$\langle 1 \rangle 2$. $P[0]$
PROOF: If $A \approx \{m \in \mathbb{N} : m < 0\}$ then A is empty and so has no proper subset.
$\langle 1 \rangle 3$. For every natural number n, if $P[n]$ then $P[n + 1]$.
$\langle 2 \rangle 1$. LET: n be a natural number.
$\langle 2 \rangle 2$. ASSUME: $P[n]$
$\langle 2 \rangle 3$. LET: A be a set.
$\langle 2 \rangle 4$. ASSUME: $A \approx \{m \in \mathbb{N} : m < n + 1\}$
$\langle 2 \rangle 5$. LET: B be a proper subset of A.
$\langle 2 \rangle 6$. CASE: $B = \emptyset$
PROOF: Then $B \not\approx \{m \in \mathbb{N} : m < n + 1\}$ but $B \approx \{k \in \mathbb{N} : k < 0\}$.
$\langle 2 \rangle 7$. CASE: $B \neq \emptyset$
$\langle 3 \rangle 1$. PICK $b_0 \in B$
$\langle 3 \rangle 2$. $A - \{b_0\} \approx \{m \in \mathbb{N} : m < n\}$
$\langle 3 \rangle 3$. $B - \{b_0\}$ is a proper subset of $A - \{b_0\}$
$\langle 3 \rangle 4$. $B - \{b_0\} \not\approx \{m \in \mathbb{N} : m < n\}$
$\langle 3 \rangle 5$. $B \approx \{m \in \mathbb{N} : m < n + 1\}$
$\langle 3 \rangle 6$. PICK $m < n$ such that $B - \{b_0\} \approx \{k \in \mathbb{N} : k < m\}$
$\langle 3 \rangle 7$. $m + 1 < n + 1$
$\langle 3 \rangle 8$. $B \approx \{k \in \mathbb{N} : k < m + 1\}$
□

Corollary 2.22.3.1. *If A is finite then there is no bijection between A and a proper subset of A.*

Corollary 2.22.3.2. *$\mathbb{N}$ is infinite.*

Corollary 2.22.3.3. *The cardinality of a finite set is unique.*

Corollary 2.22.3.4. *A subset of a finite set is finite.*

Corollary 2.22.3.5. *If A is finite and B is a proper subset of A then $|B| < |A|$.*

Corollary 2.22.3.6. *Let A be a set. Then the following are equivalent:*

1. *A is finite.*
2. *There exists a surjection from an initial segment of $\mathbb{N}$ onto A.*
3. *There exists an injection from A to an initial segment of $\mathbb{N}$.*

Corollary 2.22.3.7. *A finite union of finite sets is finite.*

Corollary 2.22.3.8. *A finite Cartesian product of finite sets is finite.*

Theorem 2.22.4. *Let A be a set. The following are equivalent:*

1. *There exists an injective function $\mathbb{N} \rightarrowtail A$.*
2. *There exists a bijection between A and a proper subset of A.*
3. *A is infinite.*

Proof:
$\langle 1 \rangle$1. $1 \Rightarrow 2$
  $\langle 2 \rangle$1. Let: $f : \mathbb{N} \rightarrowtail A$ be injective.
  $\langle 2 \rangle$2. Let: $s : \mathbb{N} \approx \mathbb{N} - \{0\}$ be the function $s(n) = n + 1$.
  $\langle 2 \rangle$3. $f \circ s \circ f^{-1} : A \approx A - \{f(0)\}$
$\langle 1 \rangle$2. $2 \Rightarrow 3$
  Proof: Corollary 2.22.3.1.
$\langle 1 \rangle$3. $3 \Rightarrow 1$
  Proof: Choose a function $f : \mathbb{N} \to A$ such that $f(n) \in A - \{f(m) : m < n\}$ for all n.
□

2.23 Countable Sets

Definition 2.23.1 (Countable)**.** A set A is *countably infinite* iff $A \approx \mathbb{N}$.

Proposition 2.23.2. *$\mathbb{N} \times \mathbb{N}$ is countably infinite.*

Proof: Define $f : \mathbb{N} \times \mathbb{N} \approx \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leqslant x\}$ by
$$f(x, y) = (x + y, y)$$
Define $g : \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leqslant x\} \approx \mathbb{N}$ by
$$g(x, y) = x(x - 1)/2 + y \ . \square$$

Proposition 2.23.3. *Every infinite subset of* $\mathbb{N}$ *is countably infinite.*

PROOF:
⟨1⟩1. LET: C be an infinite subset of $\mathbb{N}$
⟨1⟩2. Define $h : \mathbb{Z} \to C$ by recursion thus: $h(n)$ is the smallest element of $C - \{h(m) : m < n\}$.
⟨1⟩3. h is injective.
PROOF: If $m < n$ then $h(m) \neq h(n)$ because $h(n) \in C - \{h(m) : m < n\}$.
⟨1⟩4. h is surjective.
⟨2⟩1. For all $n \in \mathbb{N}$ we have $n \leqslant h(n)$.
⟨2⟩2. LET: $c \in C$
⟨2⟩3. $c \leqslant h(c)$
⟨2⟩4. LET: n be least such that $c \leqslant h(n)$
⟨2⟩5. $c \in C - \{h(m) : m < n\}$
⟨2⟩6. $h(n) \leqslant c$
⟨2⟩7. $h(n) = c$
□

Definition 2.23.4 (Countable)**.** A set is *countable* iff it is either finite or countably infinite; otherwise it is *uncountable*.

Proposition 2.23.5. *Let* B *be a nonempty set. Then the following are equivalent.*

1. B *is countable.*
2. *There exists a surjection* $\mathbb{N} \twoheadrightarrow B$*.*
3. *There exists an injection* $B \rightarrowtail \mathbb{N}$*.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
⟨2⟩1. ASSUME: B is countable.
⟨2⟩2. CASE: B is finite.
⟨3⟩1. PICK a natural number n and bijection $f : \{m \in \mathbb{N} : m < n\} \approx B$
⟨3⟩2. PICK $b \in B$
⟨3⟩3. Extend f to a surjection $g : \mathbb{N} \twoheadrightarrow B$ by setting $g(m) = b$ for $m \geqslant n$.
⟨2⟩3. CASE: B is countably infinite.
PROOF: Then there exists a bijection $\mathbb{N} \approx B$.
⟨1⟩2. $2 \Rightarrow 3$
PROOF: Given a surjection $f : \mathbb{N} \twoheadrightarrow B$, define $g : B \rightarrowtail \mathbb{N}$ by $g(b)$ is the smallest number such that $f(g(b)) = b$.
⟨1⟩3. $3 \Rightarrow 1$
⟨2⟩1. LET: $f : B \rightarrowtail \mathbb{N}$ be injective.
⟨2⟩2. $f(B)$ is countable.
⟨2⟩3. $B \approx f(B)$
⟨2⟩4. B is countable.
□

Corollary 2.23.5.1. *A subset of a countable set is countable.*

Corollary 2.23.5.2. $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

PROOF: The function that maps (m, n) to $2^m 3^n$ is injective. □

Corollary 2.23.5.3. *The Cartesian product of two countable sets is countable.*

Theorem 2.23.6. *A countable union of countable sets is countable.*

PROOF:
⟨1⟩1. LET: A be a set.
⟨1⟩2. LET: $\mathcal{B} \subseteq \mathcal{P}A$ be a countable set of countable sets such that $\bigcup \mathcal{B} = A$
⟨1⟩3. PICK a surjection $B : \mathbb{N} \twoheadrightarrow \mathcal{B}$
⟨1⟩4. ASSUME: w.l.o.g. each $B(n)$ is nonempty.
⟨1⟩5. For $n \in \mathbb{N}$, PICK a surjective function $g_n : \mathbb{N} \twoheadrightarrow B(n)$
⟨1⟩6. LET: $h : \mathbb{N} \times \mathbb{N} \to A$ be the function $h(m, n) = g_m(n)$
⟨1⟩7. h is surjective.
□

Theorem 2.23.7. $2^{\mathbb{N}}$ *is uncountable.*

PROOF:
⟨1⟩1. LET: $f : \mathbb{N} \to 2^{\mathbb{N}}$
PROVE: f is not surjective.
⟨1⟩2. Define $g : \mathbb{N} \to 2$ by $g(n) = 1 - f(n)(n)$.
⟨1⟩3. For all $n \in \mathbb{N}$ we have $g(n) \neq f(n)(n)$.
⟨1⟩4. For all $n \in \mathbb{N}$ we have $g \neq f(n)$.
□

Theorem 2.23.8. *For any set A, there is no surjective function $A \to \mathcal{P}A$.*

PROOF:
⟨1⟩1. LET: $f : A \to \mathcal{P}A$
⟨1⟩2. LET: $S = \{x \in A : x \notin f(x)\}$
⟨1⟩3. For all $a \in A$ we have $S \neq f(a)$
PROOF: We have $a \in S$ if and only if $a \notin f(a)$.
□

Corollary 2.23.8.1. *For any set A, there is no injective function $\mathcal{P}A \to A$.*

2.24 Fixed Points

Definition 2.24.1 (Fixed Point)**.** Let A be a set and $f : A \to A$. A *fixed point* of f is an element $a \in A$ such that $f(a) = a$.

2.25 Finite Intersection Property

Definition 2.25.1 (Finite Intersection Property)**.** Let X be a set. Let $\mathcal{C} \subseteq \mathcal{P}X$. Then $\mathcal{C}$ has the *finite intersection property* iff every finite nonempty subset of $\mathcal{C}$ has nonempty intersection.

Chapter 3

Relations

Definition 3.0.1 (Reflexive)**.** A relation $R \subseteq A \times A$ is *reflexive* iff, for all $a \in A$, we have $(a, a) \in R$.

Definition 3.0.2 (Antisymmetric)**.** A relation $R \subseteq A \times A$ is *antisymmetric* iff, for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Definition 3.0.3 (Transitive)**.** A relation $R \subseteq A \times A$ is *transitive* iff, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Definition 3.0.4 (Partial Order)**.** A *partial order* on a set A is a relation on A that is reflexive, antisymmetric and transitive.

We say $(A, \leqslant)$ is a *partially ordered set* or *poset* iff $\leqslant$ is a partial order on A.

Definition 3.0.5 (Greatest)**.** Let A be a poset and $a \in A$. Then a is the *greatest* element iff $\forall x \in A. x \leqslant a$.

Definition 3.0.6 (Least)**.** Let A be a poset and $a \in A$. Then a is the *least* element iff $\forall x \in A. a \leqslant x$.

Definition 3.0.7 (Upper Bound)**.** Let A be a poset, $S \subseteq A$, and $u \in A$. Then u is an *upper bound* for S iff $\forall x \in S. x \leqslant u$. We say S is *bounded above* iff it has an upper bound.

Definition 3.0.8 (Lower Bound)**.** Let A be a poset, $S \subseteq A$, and $l \in A$. Then l is a *lower bound* for S iff $\forall x \in S. l \leqslant x$. We say S is *bounded below* iff it has a lower bound.

Definition 3.0.9 (Supremum)**.** Let A be a poset, $S \subseteq A$ and $s \in A$. Then s is the *supremum* or *least upper bound* for S iff s is the least element in the sub-poset of upper bounds for A.

Definition 3.0.10 (Supremum)**.** Let A be a poset, $S \subseteq A$ and $i \in A$. Then i is the *infimum* or *greatest lower bound* for S iff i is the greatest element in the sub-poset of lower bounds for A.

Definition 3.0.11 (Least Upper Bound Property)**.** A poset A has the *least upper bound property* iff every nonempty subset of A that is bounded above has a least upper bound.

Proposition 3.0.12. *Let A be a poset. Then A has the least upper bound property if and only if every nonempty subset of A that is bounded below has a greatest lower bound.*

PROOF:
$\langle 1\rangle 1$. If A has the least upper bound property then every subset of A that is bounded below has a greatest lower bound.
  $\langle 2\rangle 1$. ASSUME: A has the least upper bound property.
  $\langle 2\rangle 2$. LET: $S \subseteq A$ be nonempty and bounded below.
  $\langle 2\rangle 3$. LET: L be the set of lower bounds of S.
  $\langle 2\rangle 4$. L is nonempty.
   PROOF: Because S is bounded below.
  $\langle 2\rangle 5$. L is bounded above.
   PROOF: Pick an element $s \in S$. Then s is an upper bound for L.
  $\langle 2\rangle 6$. LET: s be the supremum of L.
  $\langle 2\rangle 7$. s is the greatest lower bound of S.
   $\langle 3\rangle 1$. s is a lower bound of S.
    $\langle 4\rangle 1$. LET: $x \in S$
    $\langle 4\rangle 2$. x is an upper bound for L.
    $\langle 4\rangle 3$. $s \leqslant x$
   $\langle 3\rangle 2$. For any lower bound l of S we have $l \leqslant s$.
    PROOF: Immediate from $\langle 2\rangle 6$.
$\langle 1\rangle 2$. If every subset of A that is bounded below has a greatest lower bound, then A has the least upper bound property.
  PROOF: Dual.
□

Chapter 4

Order Theory

4.1 Strict Partial Orders

Definition 4.1.1 (Strict Partial Order)**.** A *strict partial order* on a set A is a relation on A that is irreflexive and transitive.

Proposition 4.1.2.

1. *If $\leqslant$ is a partial order on A then $<$ is a strict partial order on A, where $x < y$ iff $x \leqslant y \wedge x \neq y$.*
2. *If $<$ is a strict partial order on A then $\leqslant$ is a partial order on A, where $x \leqslant y$ iff $x < y \vee x = y$.*
3. *These two relations are inverses of one another.*

4.1.1 Linear Orders

Definition 4.1.3 (Linear Order)**.** A *linear order* on a set A is a partial order $\leqslant$ on A such that, for all $x, y \in A$, we have $x \leqslant y$ or $y \leqslant x$.

A *linearly ordered set* is a pair $(X, \leqslant)$ such that X is a set and $\leqslant$ is a linear order on X.

Definition 4.1.4 (Open Interval)**.** Let X be a linearly ordered set and $a, b \in X$. The *open interval* (a, b) is the set

$$\{x \in X : a < x < b\} \ .$$

Definition 4.1.5 (Immediate Predecessor, Immediate Successor)**.** Let X be a linearly ordered set and $a, b \in X$. Then b is the *(immediate) successor* of a, and a is the *(immediate) predecessor* of b, iff $a < B$ and there is no x such that $a < x < b$.

Definition 4.1.6 (Dictionary Order)**.** Let A and B be linearly ordered sets. The *dictionary order* on $A \times B$ is the order defined by

$$(a, b) < (a', b') \Leftrightarrow a < a' \vee (a = a' \wedge b < b') \ .$$

Theorem 4.1.7 (Maximum Principle)**.** *Every poset has a maximal linearly ordered subset.*

PROOF:
⟨1⟩1. LET: $(A, \leqslant)$ be a poset.
⟨1⟩2. PICK a well ordering $\preccurlyeq$ of A.
 PROOF: Well Ordering Theorem.
⟨1⟩3. LET: $h : A \to 2$ be the function defined by $\preccurlyeq$-recursion thus:

$$h(a) = \begin{cases} 1 & \text{if } a \text{ is } \leqslant\text{-comparable with every } b \prec a \text{ such that } h(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

⟨1⟩4. LET: $B = \{x \in A : h(x) = 1\}$
 PROVE: B is a maximal subset linearly ordered by $\leqslant$.
⟨1⟩5. B is linearly ordered by $\leqslant$.
 ⟨2⟩1. LET: $x, y \in B$
 ⟨2⟩2. ASSUME: w.l.o.g. $x \preccurlyeq y$
 ⟨2⟩3. y is $\leqslant$-comparable with x
⟨1⟩6. For any subset $C \subseteq A$ linearly ordered by $\leqslant$, if $B \subseteq C$ then $B = C$.
 ⟨2⟩1. LET: $x \in C$
 ⟨2⟩2. x is comparable with every $y \preccurlyeq x$ such that $h(x) = 1$
 ⟨2⟩3. $x \in B$
□

Theorem 4.1.8 (Zorn's Lemma)**.** *Let A be a poset. If every linearly ordered subset of A is bounded above, then A has a maximal element.*

PROOF:
⟨1⟩1. PICK a maximal linearly ordered subset B of A.
 PROOF: Maximal Principle
⟨1⟩2. PICK an upper bound c for B.
 PROVE: c is maximal.
⟨1⟩3. LET: $x \in A$
⟨1⟩4. ASSUME: $c \leqslant x$
 PROVE: $x = c$
⟨1⟩5. x is an upper bound for B.
⟨1⟩6. $x \in B$
 PROOF: By the maximality of B, since $B \cup \{x\}$ is linearly ordered.
⟨1⟩7. $x \leqslant c$
 PROOF: ⟨1⟩2
⟨1⟩8. $x = c$
□

Corollary 4.1.8.1 (Kuratowski's Lemma)**.** *Let $\mathcal{A} \subseteq \mathcal{P}X$. Suppose that, for every subset $\mathcal{B} \subseteq \mathcal{A}$ that is linearly ordered by inclusion, we have $\bigcup \mathcal{B} \in \mathcal{A}$. Then $\mathcal{A}$ has a maximal element.*

Definition 4.1.9 (Closed Interval)**.** Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *closed interval* $[a, b]$ is

$$[a, b] := \{x \in X : a \leqslant x \leqslant b\} \ .$$

Definition 4.1.10 (Half-Open Interval)**.** Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *half-open intervals* $(a, b]$ and $[a, b)$ are defined by

$$(a, b] := \{x \in X : a < x \leqslant b\}$$
$$[a, b) := \{x \in X : a \leqslant x < b\}$$

Definition 4.1.11 (Open Ray)**.** Let X be a linearly ordered set and $a \in X$. The *open rays* $(a, +\infty)$ and $(-\infty, a)$ are defined by:

$$(a, +\infty) := \{x \in X : a < x\}$$
$$(-\infty, a) := \{x \in X : x < a\}$$

Definition 4.1.12 (Closed Ray)**.** Let X be a linearly ordered set and $a \in X$. The *closed rays* $[a, +\infty)$ and $(-\infty, a]$ are defined by:

$$[a, +\infty) := \{x \in X : a \leqslant x\}$$
$$(-\infty, a] := \{x \in X : x \leqslant a\}$$

Definition 4.1.13 (Convex)**.** Let X be a linearly ordered set and $Y \subseteq X$. Then Y is *convex* iff, for all $a, b \in Y$ and $c \in X$, if $a < c < b$ then $c \in Y$.

4.1.2 Sets of Finite Type

Definition 4.1.14 (Finite Type)**.** Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ is of *finite type* if and only if, for any $B \subseteq X$, we have $B \in \mathcal{A}$ if and only if every finite subset of B is in $\mathcal{A}$.

Proposition 4.1.15 (Tukey's Lemma)**.** *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$. If $\mathcal{A}$ is of finite type, then $\mathcal{A}$ has a maximal element.*

Proof:
⟨1⟩1. For every subset $\mathcal{B} \subseteq \mathcal{A}$ that is linearly ordered by inclusion, we have $\bigcup \mathcal{B} \in \mathcal{A}$.
  ⟨2⟩1. Let: $\mathcal{B} \subseteq \mathcal{A}$
  ⟨2⟩2. Assume: $\mathcal{B}$ is linearly ordered by inclusion.
  ⟨2⟩3. Every finite subset of $\bigcup \mathcal{B}$ is in $\mathcal{A}$
  ⟨2⟩4. $\bigcup \mathcal{B} \in \mathcal{A}$
⟨1⟩2. Q.E.D.
  Proof: Kuratowski's Lemma.
□

4.2 Linear Continuua

Definition 4.2.1 (Linear Continuum)**.** A *linear continuum* is a linearly ordered set with more than one element that is dense and has the least upper bound property.

Proposition 4.2.2. *Every convex subset of a linear continuum with more than one element is a linear continuum.*

PROOF: Easy. ∎

Corollary 4.2.2.1. *Every interval and ray in a linear continuum is a linear continuum.*

4.3 Well Orders

Definition 4.3.1 (Well Ordered Set)**.** A *well ordered set* is a linearly ordered set such that every nonempty subset has a least element.

Proposition 4.3.2. *Any subset of a well ordered set is well ordered.*

Proposition 4.3.3. *The product of two well ordered sets is well ordered under the dictionary order.*

Theorem 4.3.4 (Well Ordering Theorem)**.** *Every set has a well ordering.*

PROOF:
⟨1⟩1. LET: X be a set.
⟨1⟩2. PICK a choice function $c : \mathcal{P}X - \{\emptyset\} \to X$
⟨1⟩3. Define a *tower* to be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well ordering of T, and

$$\forall x \in T.x = c(X - \{y \in T : y < x\}) \ .$$

⟨1⟩4. Given two towers, either they are equal or one is a section of the other.
⟨2⟩1. LET: $(T_1, <_1)$ and $(T_2, <_2)$ be towers.
⟨2⟩2. ASSUME: w.l.o.g. there exists a strictly monotone function $h : T_1 \to T_2$
⟨2⟩3. $h(T_1)$ is either T_2 or a section of T_2
PROOF: Proposition 4.3.11.
⟨2⟩4. $\forall x \in T_1.h(x) = x$
⟨3⟩1. LET: $x \in T_1$
⟨3⟩2. ASSUME: as transfinite induction hypothesis $\forall y < x.h(y) = y$
⟨3⟩3. $h(x)$ is the least element of $T_2 - \{h(y) \in T_1 : y < x\}$
⟨3⟩4. $h(x)$ is the least element of $T_2 - \{y \in T_1 : y < x\}$
PROOF: ⟨3⟩2
⟨3⟩5. $h(x) = x$
PROOF:

$$\begin{aligned} h(x) &= c(X - \{y \in T_2 : y < h(x)\}) && (\langle 1\rangle 3) \\ &= c(X - \{y \in T_2 : y < x\}) && (\langle 3\rangle 4) \\ &= c(X - \{y \in T_1 : y < x\}) && (\langle 3\rangle 2) \\ &= x && (\langle 1\rangle 3) \end{aligned}$$

⟨1⟩5. If $(T, <)$ is a tower and $T \neq X$, then there exists a tower of which $(T, <)$ is a section.
PROOF: Let $T_1 = T \cup \{c(T)\}$ and $<_1$ be the extension of $<$ such that $x < c(T)$ for all $x \in T$.

$\langle 1 \rangle 6$. LET: $\mathbf{T} = \bigcup \{T : \exists R.(T, R) \text{ is a tower}\}$ and $\mathbf{R} = \bigcup \{R : \exists T.(T, R) \text{ is a tower}\}$
$\langle 1 \rangle 7$. $(\mathbf{T}, \mathbf{R})$ is a tower.
$\langle 2 \rangle 1$. $\mathbf{R}$ is irreflexive.
PROOF: Since for every tower $(T, <)$ we have $<$ is irreflexive.
$\langle 2 \rangle 2$. $\mathbf{R}$ is transitive.
$\langle 3 \rangle 1$. ASSUME: $x\mathbf{R}y$ and $y\mathbf{R}z$
$\langle 3 \rangle 2$. PICK towers $(T_1, <_1)$ and $(T_2, <_2)$ such that $x <_1 y$ and $y <_2 z$
$\langle 3 \rangle 3$. ASSUME: w.l.o.g. $(T_1, <_1)$ is either $(T_2, <_2)$ or a section of $(T_2, <_2)$
$\langle 3 \rangle 4$. $x <_2 y <_2 z$
$\langle 3 \rangle 5$. $x <_2 z$
$\langle 3 \rangle 6$. $x\mathbf{R}z$
$\langle 2 \rangle 3$. For all $x, y \in \mathbf{T}$, either $x\mathbf{R}y$ or $x = y$ or $y\mathbf{R}x$
PROOF: There exists a tower that has both x and y.
$\langle 2 \rangle 4$. Every nonempty subset of $\mathbf{T}$ has an $\mathbf{R}$-least element.
$\langle 3 \rangle 1$. LET: $A \subseteq \mathbf{T}$ be nonempty.
$\langle 3 \rangle 2$. PICK $a \in A$
$\langle 3 \rangle 3$. PICK a tower $(T, <)$ such that $a \in T$.
$\langle 3 \rangle 4$. LET: b be the $<$-least element of $A \cap T$
PROVE: b is $\mathbf{R}$-least in A.
$\langle 3 \rangle 5$. LET: $x \in A$
$\langle 3 \rangle 6$. Etc.
$\langle 2 \rangle 5$. $\forall x \in \mathbf{T}. x = c(X - \{y \in \mathbf{T} : y\mathbf{R}x\})$
$\langle 1 \rangle 8$. $\mathbf{T} = X$
$\langle 1 \rangle 9$. $\mathbf{R}$ is a well ordering of X.
□

Proposition 4.3.5. *There exists a well-ordered set with a largest element Ω such that $(-\infty, \Omega)$ is uncountable but, for all $\alpha < \Omega$, we have $(-\infty, \alpha)$ is countable.*

PROOF:
$\langle 1 \rangle 1$. PICK an uncountable well ordered set B.
$\langle 1 \rangle 2$. LET: $C = 2 \times B$ under the dictionary order.
$\langle 1 \rangle 3$. LET: Ω be the least element of C such that $(-\infty, \Omega)$ is uncountable.
$\langle 1 \rangle 4$. LET: $A = (-\infty, \Omega]$
$\langle 1 \rangle 5$. A is a well ordered set with largest element Ω such that $(-\infty, \Omega)$ is uncountable but, for all $\alpha < \Omega$, we have $(-\infty, \alpha)$ is countable.
□

Proposition 4.3.6. *Every well ordered set has the least upper bound property.*

PROOF: For any subset that is bounded above, the set of upper bounds is nonempty, hence has a least element. □

Proposition 4.3.7. *In a well ordered set, every element that is not greatest has a successor.*

PROOF: If a is not greatest, then $\{x : x > a\}$ is nonempty, hence has a least element. □

Theorem 4.3.8 (Transfinite Induction)**.** *Let J be a well ordered set. Let $S \subseteq J$. Assume that, for every $\alpha \in J$, if $\forall x < \alpha . x \in S$ then $\alpha in S$. Then $S = J$.*

PROOF: Otherwise $J - S$ would be a nonempty subset of J with no least element. □

Proposition 4.3.9. *Let I be a well ordered set. Let $\{A_i\}_{i \in I}$ be a family of well ordered sets. Define $<$ on $\coprod_{i \in I} A_i$ by: $\kappa_i(a) < \kappa_j(b)$ iff either $i < j$, or $i = j$ and $a < b$ in A_i. Then $<$ well orders $\coprod_{i \in I} A_i$.*

PROOF: Easy. □

Theorem 4.3.10 (Principle of Transfinite Recursion)**.** *Let J be a well ordered set. Let C be a set. Let $\mathcal{F}$ be the set of all functions from a section of J into C. Let $\rho : \mathcal{F} \to C$. Then there exists a unique function $h : J \to C$ such that, for all $\alpha \in J$, we have*

$$h(\alpha) = \rho(h \upharpoonright (-\infty, \alpha)) \enspace .$$

PROOF:

⟨1⟩1. For a function h mapping either a section of J or all of J into C, let us say h is *acceptable* iff, for all $x \in \operatorname{dom} h$, we have $(-\infty, x) \subseteq \operatorname{dom} h$ and $h(x) = \rho(h \upharpoonright (-\infty, x))$.

⟨1⟩2. If h and k are acceptable functions then $h(x) = k(x)$ for all x in both domains.

  ⟨2⟩1. LET: $x \in J$

  ⟨2⟩2. ASSUME: as transfinite induction hypothesis that, for all $y < x$ and any acceptable functions h and k with $y \in \operatorname{dom} h \cap \operatorname{dom} k$, we have $h(y) = k(y)$

  ⟨2⟩3. LET: h and k be acceptable functions with $x \in \operatorname{dom} h \cap \operatorname{dom} k$

  ⟨2⟩4. $h \upharpoonright (-\infty, x) = k \upharpoonright (-\infty, x)$

  PROOF: By ⟨2⟩2.

  ⟨2⟩5. $h(x) = k(x)$

  PROOF: By ⟨2⟩3, each is the least element of the set in ⟨2⟩4.

⟨1⟩3. For $\alpha \in J$, if there exists an acceptable function $(-\infty, \alpha) \to C$, then there exists an acceptable function $(-\infty, \alpha] \to C$.

  ⟨2⟩1. LET: $\alpha \in J$

  ⟨2⟩2. LET: $f : (-\infty, \alpha) \to C$ be acceptable.

  ⟨2⟩3. LET: $g : (-\infty, \alpha] \to C$ be the function given by

$$g(x) = \begin{cases} f(x) & \text{if } x < \alpha \\ \rho(f) & \text{if } x = \alpha \end{cases}$$

  ⟨2⟩4. g is acceptable.

⟨1⟩4. Let $K \subseteq J$. Assume that, for all $\alpha \in K$, there exists an acceptable function $(-\infty, \alpha) \to C$. Then there exists an acceptable function $\bigcup_{\alpha \in K} (-\infty, \alpha) \to C$.

  ⟨2⟩1. Define $f : \bigcup_{\alpha \in K} (-\infty, \alpha) \to C$ by: $f(x) = y$ iff there exists $\alpha \in K$ and $g : (-\infty, \alpha) \to C$ acceptable such that $g(x) = y$.

⟨1⟩5. For every $\beta \in J$, there exists an acceptable function $(-\infty, \beta) \to C$

⟨2⟩1. LET: $\beta \in J$
⟨2⟩2. ASSUME: as transfinite induction hypothesis that, for all $\alpha < \beta$, there exists an acceptable function $(-\infty, \alpha) \to C$
⟨2⟩3. CASE: β has a predecessor
⟨3⟩1. LET: α be the predecessor of β.
⟨3⟩2. There exists an acceptable function $(-\infty, \alpha) \to C$.
⟨3⟩3. There exists an acceptable function $(-\infty, \beta) \to C$.
PROOF: By ⟨1⟩3 since $(-\infty, \beta) = (-\infty, \alpha]$.
⟨2⟩4. CASE: β has no predecessor.
PROOF: The result follows by ⟨1⟩4 since $(-\infty, \beta) = \bigcup_{\alpha<\beta}(-\infty, \alpha)$.
⟨1⟩6. There exists an acceptable function $J \to C$.
⟨2⟩1. CASE: J has a greatest element.
⟨3⟩1. LET: g be greatest.
⟨3⟩2. There exists an acceptable function $(-\infty, g) \to C$.
PROOF: ⟨1⟩5
⟨3⟩3. There exists an acceptable function $J \to C$.
PROOF: By ⟨1⟩3 since $J = (-\infty, g]$.
⟨2⟩2. CASE: J has no greatest element.
PROOF: By ⟨1⟩4 since $J = \bigcup_{\alpha \in J}(-\infty, \alpha)$.
□

Corollary 4.3.10.1 (Cardinal Comparability)**.** *Let A and B be sets. Then either $A \preccurlyeq B$ or $B \preccurlyeq A$.*

PROOF: Choose well orderings of A and B. Then either there exists a surjection $A \twoheadrightarrow B$, or there exists an injective function $h : A \rightarrowtail B$ defined by transfinite recursion by $h(x)$ is the least element of $B - h((-\infty, x))$. □

Proposition 4.3.11. *Let J and E be well ordered sets. Let $h : J \to E$. Then the following are equivalent.*

1. *h is strictly monotone and $h(J)$ is either E or a section of E.*
2. *For all $\alpha \in J$, we have $h(\alpha)$ is the least element of $E - h((-\infty, \alpha))$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
⟨2⟩1. ASSUME: 1
⟨2⟩2. $h(J)$ is closed downwards.
⟨2⟩3. LET: $\alpha \in J$
⟨2⟩4. $h(\alpha) \in E - h((-\infty, \alpha))$
PROOF: If $\beta < \alpha$ then $h(\beta) < h(\alpha)$.
⟨2⟩5. For all $y \in E - h((-\infty, \alpha))$ we have $h(\alpha) \leqslant y$
⟨3⟩1. ASSUME: for a contradiction $y < h(\alpha)$
⟨3⟩2. $y \in h(J)$
⟨3⟩3. PICK $\beta \in J$ such that $h(\beta) = y$
⟨3⟩4. $h(\beta) < h(\alpha)$
⟨3⟩5. $\beta < \alpha$

⟨3⟩6. Q.E.D.
PROOF: This contradicts the fact that $y \notin h((-\infty, \alpha))$.
⟨1⟩2. $2 \Rightarrow 1$
⟨2⟩1. ASSUME: 2
⟨2⟩2. h is strictly monotone.
⟨3⟩1. LET: $\alpha, \beta \in J$ with $\alpha < \beta$
⟨3⟩2. $h(\alpha) \neq h(\beta)$
PROOF: Because $h(\beta) \in E - h((-\infty, \beta))$.
⟨3⟩3. $h(\alpha) \leqslant h(\beta)$
PROOF:Because $h(\alpha)$ is least in $E - h((-\infty, \alpha))$.
⟨3⟩4. $h(\alpha) < h(\beta)$
⟨2⟩3. $h(J)$ is either E or a section of E.
⟨3⟩1. ASSUME: $h(J) \neq E$
⟨3⟩2. LET: e be least in $E - h(J)$
PROVE: $h(J) = (-\infty, e)$
⟨3⟩3. $h(J) \subseteq (-\infty, e)$
⟨4⟩1. LET: $\alpha \in J$
⟨4⟩2. $h(\alpha) \neq e$
PROOF: $e \notin h(J)$
⟨4⟩3. $h(\alpha) \leqslant e$
PROOF: Since $h(\alpha)$ is least in $E - h((-\infty, \alpha))$.
⟨4⟩4. $h(\alpha) < e$
⟨3⟩4. $(-\infty, e) \subseteq h(J)$
PROOF: If $e' < e$ then $e' \in h(J)$ by leastness of e.

$\square$

Part II

Category Theory

Chapter 5

Category Theory

5.1 Categories

Definition 5.1.1. A *category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \mathrm{Ob}(\mathcal{C})$.
- for any objects X and Y, a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y. We write $f : X \to Y$ for $f \in \mathcal{C}[X, Y]$.
- for any objects X, Y and Z, a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \to \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X, there exists a morphism $\mathrm{id}_X : X \to X$, the *identity morphism* on X, such that:
  - for any object Y and morphism $f : Y \to X$ we have $\mathrm{id}_X \circ f = f$
  - for any object Y and morphism $f : X \to Y$ we have $f \circ \mathrm{id}_X = f$

We write the composite of morphism $f_1, \ldots, f_n$ as $f_n \circ \cdots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 5.1.2. Let **Set** be the category of small sets and functions.

Definition 5.1.3. Let **LPos** be the category of linearly ordered sets and monotone functions.

Proposition 5.1.4. *Any finite linearly ordered set is isomorphic to* $\{m \in \mathbb{N} : m < n\}$ *for some* n.

PROOF:
$\langle 1 \rangle 1$. Every finite nonempty linearly ordered set has a greatest element.

⟨2⟩1. LET: $P[n]$ be the property: for any linearly ordered set A, if there exists a bijection $A \approx \{m \in \mathbb{N} : m < n\}$ and A is nonempty then A has a greatest element.

⟨2⟩2. $P[0]$

PROOF: Vacuous.

⟨2⟩3. $\forall n \in \mathbb{N}.P[n] \Rightarrow P[n+1]$

⟨3⟩1. LET: $n \in \mathbb{N}$

⟨3⟩2. ASSUME: $P[n]$

⟨3⟩3. LET: A be a nonempty linearly ordered set.

⟨3⟩4. LET: $f : A \approx \{m \in \mathbb{N} : m < n+1\}$

⟨3⟩5. LET: $a = f^{-1}(n)$

⟨3⟩6. $f \restriction (A - \{a\}) : A - \{a\} \approx \{m \in \mathbb{N} : m < n\}$

⟨3⟩7. ASSUME: w.l.o.g. a is not greatest in A.

⟨3⟩8. LET: b be greatest in $A - \{a\}$

PROOF: ⟨3⟩2

⟨3⟩9. b is greatest in A.

⟨1⟩2. LET: $P[n]$ be the property: for any linearly ordered set A, if there exists a bijection $A \approx \{m \in \mathbb{N} : m < n\}$ then there exists an isomorphism in **LPos** $A \cong \{m \in \mathbb{N} : m < n\}$.

⟨1⟩3. $P[0]$

PROOF: If there exists a bijection $A \approx \emptyset$ then A is empty and so the unique function $A \to \emptyset$ is an order isomorphism.

⟨1⟩4. For every natural number n, if $P[n]$ then $P[n+1]$.

⟨2⟩1. LET: n be a natural number.

⟨2⟩2. ASSUME: $P[n]$

⟨2⟩3. LET: A be a linearly ordered set.

⟨2⟩4. ASSUME: A has $n+1$ elements.

⟨2⟩5. LET: a be the greatest element in A.

⟨2⟩6. LET: $f : A - \{a\} \cong \{m \in \mathbb{N} : m < n\}$ be an order isomorphism.

PROOF: ⟨2⟩2

⟨2⟩7. Define $g : A \to \{m \in \mathbb{N} : m < n+1\}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ n & \text{if } x = a \end{cases}$$

⟨2⟩8. g is an order isomorphism.

⟨1⟩5. $\forall n \in \mathbb{N}.P[n]$

□

Corollary 5.1.4.1. *Any finite linearly ordered set is well ordered.*

Proposition 5.1.5. *Let J and E be well ordered sets. Suppose there is a strictly monotone map $J \to E$. Then J is isomorphic either to E or a section of E.*

PROOF:

⟨1⟩1. LET: $k : J \to E$ be strictly monotone.

⟨1⟩2. ASSUME: w.l.o.g. E is nonempty.

⟨1⟩3. PICK $e_0 \in E$

⟨1⟩4. LET: $h : J \to E$ be the function defined by transfinite recursion thus:

$$h(\alpha) = \begin{cases} \text{the least element in } E - h((-\infty, \alpha)) & \text{if } h((-\infty, \alpha)) \neq E \\ e_0 & \text{if } h((-\infty, \alpha)) = E \end{cases}$$

⟨1⟩5. $\forall \alpha \in J. h(\alpha) \leqslant k(\alpha)$
  ⟨2⟩1. LET: $\alpha \in J$
  ⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha. h(\beta) \leqslant k(\beta)$.
  ⟨2⟩3. $\forall \beta < \alpha. h(\beta) < k(\alpha)$
  ⟨2⟩4. $h((-\infty, \alpha)) \neq E$
  ⟨2⟩5. $h(\alpha)$ is the least element in $E - h((-\infty, \alpha))$.
  ⟨2⟩6. $k(\alpha) \in E - h((-\infty, \alpha))$
  ⟨2⟩7. $h(\alpha) \leqslant k(\alpha)$
⟨1⟩6. $\forall \alpha \in J. h((-\infty, \alpha)) \neq E$
  PROOF: For $\beta < \alpha$ we have $h(\beta) \leqslant k(\beta) < k(\alpha)$ so $k(\alpha) \notin h((-\infty, \alpha))$.
⟨1⟩7. For all $\alpha \in J$, we have $h(\alpha)$ is the least element of $E - h((-\infty, \alpha))$.
⟨1⟩8. h is strictly monotone and $h(J)$ is either E or a section of E.
  PROOF: Proposition 4.3.11.
□

Proposition 5.1.6. *If A and B are well ordered sets, then exactly one of the following conditions hold: $A \cong B$, or A is isomorphic to a section of B, or B is isomorphic to a section of A.*

PROOF:
⟨1⟩1. At least one of the conditions holds.
  ⟨2⟩1. B is isomorphic to either $A + B$ or a section of $A + B$.
  ⟨2⟩2. CASE: $B \cong A + B$
    ⟨3⟩1. LET: ϕ be the isomorphism $B \cong A + B$
    ⟨3⟩2. LET: b_0 be the least element in B.
    ⟨3⟩3. A is isomorphic to the section $(-\infty, \phi^{-1}(\kappa_2(b_0)))$ of B.
  ⟨2⟩3. CASE: $a \in A$ and $B \cong (-\infty, \kappa_1(a))$
    PROOF: Then B is isomorphic to the section $(-\infty, a)$ of A.
  ⟨2⟩4. CASE: $b \in B$ and $\phi : B \cong (-\infty, \kappa_2(b))$
    ⟨3⟩1. CASE: b is least in B.
      PROOF: Then $A \cong B$.
    ⟨3⟩2. CASE: b is not least in B.
      ⟨4⟩1. LET: b_0 be least in B.
      ⟨4⟩2. A is isomorphic to the section $(-\infty, \phi^{-1}(\kappa_2(b_0)))$ of B.
⟨1⟩2. At most one of the conditions holds.
  PROOF: Since a well ordered set cannot be isomorphic to a section of itself.
□

Theorem 5.1.7. *There exists a well ordered set, unique up to order isomorphism, that is uncountable but such that every section is countable.*

PROOF:
⟨1⟩1. There exists a well ordered set that is uncountable but such that every section is countable.

⟨2⟩1. PICK a well ordered set A with an element $\Omega \in A$ such that $(-\infty, \Omega)$ is uncountable but $\forall \alpha < \Omega.(-\infty, \alpha)$ is countable.

⟨2⟩2. LET: $(-\infty, Omega)$ is uncountable but every section is countable.

⟨1⟩2. If A and B are uncountable well ordered sets such that every section is countable, then $A \cong B$.

PROOF: Since it cannot be that one of A and B is isomorphic to a section of the other.

□

Definition 5.1.8 (Minimal Uncountable Well Ordered Set)**.** The *minimal uncountable well ordered set* Ω is the well ordered set that is uncountable but such that every section is countable.

We write $\overline{\Omega}$ for the well ordered set $\Omega \cup \{\Omega\}$ where Ω is greatest.

Proposition 5.1.9. *Every countable subset of* Ω *is bounded above.*

PROOF:

⟨1⟩1. LET: A be a countable subset of Ω.

⟨1⟩2. For all $a \in A$ we have $(-\infty, a)$ is countable.

⟨1⟩3. $\bigcup_{a \in A}(-\infty, a)$ is countable.

⟨1⟩4. $\bigcup_{a \in A}(-\infty, a) \neq \Omega$

⟨1⟩5. PICK $x \in \Omega - \bigcup_{a \in A}(-\infty, a)$

⟨1⟩6. x is an upper bound for A.

□

Proposition 5.1.10. Ω *has no greatest element.*

PROOF: For any $\alpha \in \Omega$ we have $(-\infty, \alpha]$ is countable and hence not the whole of Ω. □

Proposition 5.1.11. *There are uncountably many elements of* Ω *that have no predecessor.*

PROOF:

⟨1⟩1. LET: A be the set of all elements of Ω that have no predecessor.

⟨1⟩2. LET: $f : A \times \mathbb{N} \to \Omega$ be the function that maps (a, n) to the nth successor of a.

⟨1⟩3. f is surjective.

⟨2⟩1. ASSUME: for a contradiction $x \in \Omega$ and there is no element $a \in A$ and $n \in \mathbb{N}$ such that x is the nth successor of a.

⟨2⟩2. LET: x_n be the nth predecessor of x for $n \in \mathbb{N}$.

⟨2⟩3. $\{x_n : n \in \mathbb{N}\}$ is a nonempty subset of Ω with no least element.

⟨1⟩4. $A \times \mathbb{N}$ is uncountable.

⟨1⟩5. A is uncountable.

□

Definition 5.1.12. We identify a poset $(A, \leqslant)$ with the category with:

- set of objects A

- for $a, b \in A$, the set of homomorphisms is $\{x \in 1 : a \leqslant b\}$

Proposition 5.1.13. *A category is a poset iff, for any two objects, there exists at most one morphism between them.*

Proposition 5.1.14. *The identity morphism on an object is unique.*

PROOF:
⟨1⟩1. LET: $\mathcal{C}$ be a category.
⟨1⟩2. LET: $A \in \mathcal{C}$
⟨1⟩3. LET: $i, j : A \to A$ be identity morphisms on A.
⟨1⟩4. $i = j$
PROOF:

$$\begin{aligned} i &= i \circ j && (j \text{ is an identity on } A) \\ &= j && (i \text{ is an identity on } A) \end{aligned}$$

□

Proposition 5.1.15. *Let A be a linearly ordered set. Then A is well ordered if and only if it does not contain a subset of order type $\mathbb{N}^{\mathrm{op}}$.*

PROOF:
⟨1⟩1. If A is well ordered then it does not contain a subset of order type $\mathbb{N}^{\mathrm{op}}$.
PROOF: A subset of order type $\mathbb{N}^{\mathrm{op}}$ would be a subset with no least element.
⟨1⟩2. If A is not well ordered then it contains a subset of order type $\mathbb{N}^{\mathrm{op}}$.
⟨2⟩1. ASSUME: A is not well ordered.
⟨2⟩2. PICK a nonempty subset S with no least element.
⟨2⟩3. PICK $a_0 \in S$
⟨2⟩4. Extend to a sequence (a_n) in S such that $a_{n+1} < a_n$ for all n.
⟨2⟩5. $\{a_n : n \in \mathbb{N}\}$ has order type $\mathbb{N}^{\mathrm{op}}$.
□

Corollary 5.1.15.1. *Let A be a linearly ordered set. If every countable subset of A is well ordered, then A is well ordered.*

Definition 5.1.16. Given $f : A \to B$ and an object C, define the function $f^* : \mathcal{C}[B, C] \to \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 5.1.17. Given $f : A \to B$ and an object C, define the function $f_* : \mathcal{C}[C, A] \to \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

5.1.1 Monomorphisms

Definition 5.1.18 (Monomorphism)**.** Let $f : A \to B$. Then f is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for any object X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

5.1.2 Epimorphisms

Definition 5.1.19 (Epimorphism)**.** Let $f : A \to B$. Then f is *epic* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for any object X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

5.1.3 Sections and Retractions

Definition 5.1.20 (Section, Retraction)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $rs = \mathrm{id}_B$.

Proposition 5.1.21. *Let $f : A \to B$ and $r, s : B \to A$. If r is a retraction of f and s is a section of f then $r = s$.*

PROOF:

$$\begin{aligned} r &= r\mathrm{id}_B && \text{(Unit Law)} \\ &= rfs && \text{(s is a section of f)} \\ &= \mathrm{id}_A s && \text{(r is a retraction of f)} \\ &= s && \text{(Unit Law)} \square \end{aligned}$$

Proposition 5.1.22. *Every section is monic.*

PROOF:
⟨1⟩1. LET: $s : B \to A$ be a section of $r : A \to B$.
⟨1⟩2. LET: X be an object and $x, y : X \to B$
⟨1⟩3. ASSUME: $s \circ x = s \circ y$
⟨1⟩4. $x = y$
  PROOF: $x = r \circ s \circ x = r \circ s \circ y = y$.
$\square$

Proposition 5.1.23. *Every retraction is epic.*

PROOF: Dual. $\square$

5.1.4 Isomorphisms

Definition 5.1.24 (Isomorphism)**.** A morphism $f : A \to B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \to A$ that is both a retraction and section of f.

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 5.1.25. *The inverse of an isomorphism is unique.*

PROOF: From Proposition 5.1.21. $\square$

Proposition 5.1.26. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $ff^{-1} = \mathrm{id}_B$ and $f^{-1}f = \mathrm{id}_A$. $\square$

Isomorphism.
Define the opposite category.
Slice categories

Definition 5.1.27. Let $\mathcal{C}$ be a category and $B \in \mathcal{C}$. The category $\mathcal{C}_B^B$ of objects *over and under B* is the category with:

- objects all triples (X, u, p) such that $u : B \to X$ and $p : X \to B$
- morphisms $f : (X, u, p) \to (Y, u', p')$ all morphisms $f : X \to Y$ such that $fu = u'$ and $p'f = p$.

Proposition 5.1.28.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B)\backslash \mathrm{id}_B \cong (\mathcal{C}\backslash B)/\mathrm{id}_B$$

$(B, \mathrm{id}_B, \mathrm{id}_B)$ is the zero object in $\mathcal{C}_B^B$.

5.1.5 Initial Objects

Definition 5.1.29 (Initial Object)**.** An object I is *initial* iff, for any object X, there exists exactly one morphism $I \to X$.

Proposition 5.1.30. *The empty set is initial in* **Set**.

PROOF: For any set A, the nowhere-defined function is the unique function $\varnothing \to A$. □

Proposition 5.1.31. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:
⟨1⟩1. LET: $i : I \to I'$ be the unique morphism $I \to I'$.
⟨1⟩2. LET: $i^{-1} : I' \to I$ be the unique morphism $I' \to I$.
⟨1⟩3. $ii^{-1} = \mathrm{id}_{I'}$
PROOF: There is only one morphism $I' \to I'$.
⟨1⟩4. $i^{-1}i = \mathrm{id}_I$
PROOF: There is only one morphism $I \to I$.
□

5.1.6 Terminal Objects

Definition 5.1.32 (Terminal Object)**.** An object T is *terminal* iff, for any object X, there exists exactly one morphism $X \to T$.

Proposition 5.1.33. *1 is terminal in* **Set**.

PROOF: For any set A, the constant function to $*$ is the only function $A \to 1$. □

Proposition 5.1.34. *If T and T' are terminal objects, then there exists a unique isomorphism $T \cong T'$.*

PROOF: Dual to Proposition 5.1.31. □

5.1.7 Zero Objects

Definition 5.1.35 (Zero Object)**.** An object Z is a *zero object* iff it is an initial object and a terminal object.

Definition 5.1.36 (Zero Morphism)**.** Let $\mathcal{C}$ be a category with a zero object Z. Let $A, B \in \mathcal{C}$. The *zero morphism* $A \to B$ is the unique morphism $A \to Z \to B$.

Proposition 5.1.37. *There is no zero object in* **Set***.*

PROOF: Since $\emptyset \not\cong 1$. □

5.1.8 Triads

Definition 5.1.38 (Triad)**.** Let $\mathcal{C}$ be a category. A *triad* consists of objects X, Y, M and morphisms $\alpha : X \to M$, $\beta : Y \to M$. We call M the *codomain* of the triad.

5.1.9 Cotriads

Definition 5.1.39 (Cotriad)**.** Let $\mathcal{C}$ be a category. A *cotriad* consists of objects X, Y, W and morphisms $\xi : W \to X$, $\eta : W \to Y$. We call W the *domain* of the triad.

5.1.10 Pullbacks

Definition 5.1.40 (Pullback)**.** A diagram

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

is a *pullback* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : Z \to X$ and $g : Z \to Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \to W$ such that $\xi h = f$ and $\eta h = g$.

In this case we also say that η is the *pullback* of β along α.

Proposition 5.1.41. *If $\xi : W \to X$ and $\eta : W \to Y$ form a pullback of $\alpha : X \to M$ and $\beta : Y \to M$, and $\xi' : W' \to X$ and $\eta' : W' \to Y$ also form the pullback of α and β, then there exists a unique isomorphism $\phi : W \cong W'$ such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.*

PROOF:
⟨1⟩1. LET: $\phi : W \to W'$ be the unique morphism such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.
⟨1⟩2. LET: $\phi^{-1} : W' \to W$ be the unique morphism such that $\eta\phi^{-1} = \eta'$ and $\xi\phi^{-1} = \xi'$.
⟨1⟩3. $\phi\phi^{-1} = \mathrm{id}_{W'}$

PROOF: Each is the unique $x : W' \to W'$ such that $\eta' x = \eta'$ and $\xi' x = \xi'$.
⟨1⟩4. $\phi^{-1}\phi = \mathrm{id}_W$
PROOF: Each is the unique $x : W \to W$ such that $\eta x = \eta$ and $\xi x = \xi$.
□

Proposition 5.1.42. *For any morphism $h : A \to B$, the following diagram is a pullback diagram.*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

PROOF:
⟨1⟩1. LET: Z be an object.
⟨1⟩2. LET: $f : Z \to B$ and $g : Z \to A$ satisfy $\mathrm{id}_B f = hg$
⟨1⟩3. $g : Z \to B$ is the unique morphism such that $\mathrm{id}_A g = g$ and $hg = f$.
□

Proposition 5.1.43. *The pullback of an isomorphism is an isomorphism.*

PROOF:
⟨1⟩1. LET:

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pullback diagram.
⟨1⟩2. ASSUME: β is an isomorphism.
⟨1⟩3. LET: ξ^{-1} be the unique morphism $X \to W$ such that $\xi\xi^{-1} = \mathrm{id}_X$ and $\eta\xi^{-1} = \beta^{-1}\alpha$.
PROOF: This exists since $\alpha \mathrm{id}_X = \beta\beta^{-1}\alpha = \alpha$.
⟨1⟩4. $\xi^{-1}\xi = \mathrm{id}_W$
PROOF: Each is the unique $x : W \to W$ such that $\xi x = \xi$ and $\eta x = \eta$.
□

Proposition 5.1.44. *Let $\beta : (Y, y) \to (M, m)$ and $\alpha : (X, x) \to (M, m)$ in $\mathcal{C}\backslash A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pullback in $\mathcal{C}$. Let $w : A \to W$ be the unique morphism such that $\xi w = x$ and $\eta w = y$. Then $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ is the pullback of β and α in $\mathcal{C}\backslash A$.

Proof:
$\langle 1\rangle 1$. Let: $(Z,z) \in \mathcal{C}\backslash A$
$\langle 1\rangle 2$. Let: $f : (Z,z) \to (X,x)$ and $g : (Z,z) \to (Y,y)$ satisfy $\alpha f = \beta g$.
$\langle 1\rangle 3$. Let: $h : Z \to W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.
$\langle 1\rangle 4$. $hz = w$
$\langle 2\rangle 1$. $\xi hz = \xi w$
Proof:

$$\begin{aligned} \xi hz &= fz && (\langle 1\rangle 3) \\ &= x && (\langle 1\rangle 2) \\ &= \xi w \end{aligned}$$

$\langle 2\rangle 2$. $\eta hz = \eta w$
Proof: Similar.
$\langle 1\rangle 5$. $h : (Z,z) \to (W,w)$
□

Proposition 5.1.45. *Let $\beta : (Y,y) \to (M,m)$ and $\alpha : (X,x) \to (M,m)$ in $\mathcal{C}/A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pullback in $\mathcal{C}$. Let $w = x\xi : W \to A$. Then $\xi : (W,w) \to (X,x)$ and $\eta : (W,w) \to (Y,y)$ form a pullback of α and β in $\mathcal{C}/A$.

Proof:
$\langle 1\rangle 1$. $\eta : (W,w) \to (Y,y)$
Proof:

$$\begin{aligned} y\eta &= m\beta\eta \\ &= m\alpha\xi \\ &= x\xi \\ &= w \end{aligned}$$

$\langle 1\rangle 2$. Let: $(Z,z) \in \mathcal{C}/A$
$\langle 1\rangle 3$. Let: $f : (Z,z) \to (X,x)$ and $g : (Z,z) \to (Y,y)$ satisfy $\alpha f = \beta g$.
$\langle 1\rangle 4$. Let: $h : Z \to W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.
$\langle 1\rangle 5$. $h : (Z,z) \to (W,w)$
Proof:

$$\begin{aligned} wh &= x\xi h \\ &= xf && (\langle 1\rangle 4) \\ &= z && (\langle 1\rangle 3) \end{aligned}$$

□

Proposition 5.1.46. *In* **Set**, *let $\alpha : X \to M$ and $\beta : Y \to M$. Let $W = \{(x,y) \in X \times Y : \alpha(x) = \beta(y)\}$ with inclusion $i : W \to X \times Y$. Let $\xi = \pi_1 i : W \to X$ and $\eta : \pi_2 i : W \to Y$. Then ξ and η form the pullback of α and β.*

PROOF:
⟨1⟩1. $\alpha\xi = \beta\eta$
PROOF: For $w \in W$, if $i(w) = (x, y)$ then then $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$.
⟨1⟩2. For every set Z and functions $f : Z \to X$, $g : Z \to Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \to W$ such that $\xi h = f$ and $\eta h = g$
PROOF: For $z \in Z$, let $h(z)$ be the unique element of W such that $i(h(z)) = (f(z), g(z))$.
∎

Pullback lemma

5.1.11 Pushouts

Definition 5.1.47 (Pushout)**.** A diagram

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd} \tag{5.1}$$

is a *pushout* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : X \to Z$ and $g : Y \to Z$ such that $f\xi = g\eta$, there exists a unique $h : M \to Z$ such that $h\alpha = f$ and $h\beta = g$.

We also say that β is the *pushout* of ξ along η.

Proposition 5.1.48. *If $\alpha : X \to M$ and $\beta : Y \to M$ form a pushout of $\xi : W \to X$ and $\eta : W \to Y$, and $\alpha' : X \to M'$ and $\beta' : Y \to M'$ also form a pushout of ξ and η, then there exists a unique isomorphism $\phi : M \cong M'$ such that $\phi\alpha = \alpha'$ and $\phi\beta = \beta'$.*

PROOF: Dual to Proposition 5.1.41. ∎

Proposition 5.1.49. *For any morphism $h : A \to B$, the following diagram is a pushout diagram.*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

PROOF: Dual to Proposition 5.1.42.

Proposition 5.1.50. *The diagram (5.1) is a pushout in $\mathcal{C}$ iff it is a pullback in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. ∎

Proposition 5.1.51. *The pushout of an isomorphism is an isomorphism.*

PROOF: Dual to Proposition 5.1.43. ∎

Proposition 5.1.52. *Let $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ in $\mathcal{C}\backslash A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pushout in $\mathcal{C}$. Let $m := \alpha x : A \to M$. Then $\alpha : (X, x) \to (M, m)$ and $\beta : (Y, y) \to (M, m)$ is the pushout of ξ and η in $\mathcal{C}\backslash A$.

PROOF: Dual to Proposition 5.1.45. ∎

Proposition 5.1.53. *Let $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ in $\mathcal{C}/A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pushout in $\mathcal{C}$. Let $m : M \to A$ be the unique morphism such that $m\alpha = x$ and $m\beta = y$. Then $\alpha : (X, x) \to (M, m)$ and $\beta : (Y, y) \to (M, m)$ is the pushout of ξ and η in $\mathcal{C}\backslash A$.

PROOF: Dual to Proposition 5.1.44. ∎

Proposition 5.1.54. **Set** *has pushouts.*

PROOF:
⟨1⟩1. LET: $\xi : W \to X$ and $\eta : W \to Y$.
⟨1⟩2. LET: $\sim$ be the equivalence relation on $X + Y$ generated by $\xi(w) \sim \eta(w)$ for all $w \in W$
⟨1⟩3. LET: $M = (X + Y)/\sim$ with canonical projection $\pi : X + Y \twoheadrightarrow M$.
⟨1⟩4. LET: $\alpha = \pi \circ \kappa_1 : X \to M$
⟨1⟩5. LET: $\beta = \pi \circ \kappa_2 : Y \to M$
⟨1⟩6. LET: Z be any set, $f : X \to Z$ and $g : Y \to Z$.
⟨1⟩7. ASSUME: $f\xi = g\eta$
⟨1⟩8. LET: $h : X + Y \to Z$ be the function defined by $h(x) = f(x)$ and $h(y) = g(y)$ for $x \in X$ and $y \in Y$
⟨1⟩9. h respects $\sim$
PROOF: For $w \in W$ we have

$$\begin{aligned} h(\xi(w)) &= f(\xi(w)) && (\langle 1\rangle 8) \\ &= g(\eta(w)) && (\langle 1\rangle 7) \\ &= h(\eta(w)) && (\langle 1\rangle 8) \end{aligned}$$

⟨1⟩10. LET: $\overline{h} : M \to Z$ be the induced function.
⟨1⟩11. $\overline{h}\alpha = f$

PROOF:

$$\begin{aligned}\overline{h}(\alpha(x)) &= \overline{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x)\end{aligned}$$

⟨1⟩12. $\overline{h}\beta = g$
PROOF: Similar.
⟨1⟩13. For all $k : M \to Z$, if $k\alpha = f$ and $k\beta = g$ then $k = \overline{h}$.
PROOF:

$$\begin{aligned}k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\ &= f(x) \\ k(\pi(\kappa_2(y))) &= k(\beta(y)) \\ &= g(y) \\ \therefore k \circ \pi &= h \\ \therefore k &= \overline{h}\end{aligned}$$

□

Definition 5.1.55. Let $u : A \rightarrowtail X$ be an injection. The *pointed set obtained from X by collapsing* (A, u), denoted $X/(A, u)$, is the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"] & 1 \arrow[d, "*"'] \\ X \arrow[r] & X/(A,u) \end{tikzcd}$$

Proposition 5.1.56. *In* **Set**$_*$, *any two morphisms* $1 \to X$ *and* $1 \to Y$ *have a pushout.*

PROOF: The pushout of $a : (1, *) \to (X, x)$ and $b : (1, *) \to (Y, y)$ is $(X+Y/\sim, x)$ where $\sim$ is the equivalence relation generated by $x \sim y$. □

Definition 5.1.57 (Wedge)**.** The *wedge* of pointed sets X and Y, $X \vee Y$, is the pushout of the unique morphism $1 \to X$ and $1 \to Y$.

Definition 5.1.58 (Smash)**.** Let X and Y be pointed sets. Let $\xi : X \vee Y \to X$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, equal, bend left] & \\ Y \arrow[r] \arrow[drr, bend right, "0"'] & X \vee Y \arrow[dr, "\xi"] & \\ & & X \end{tikzcd}$$

Let $\eta : X \vee Y \to Y$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "0"] & \\
Y \arrow[r] \arrow[drr, bend right, equal] & X \vee Y \arrow[dr, "\eta"] & \\
& & Y
\end{tikzcd}$$

Let $\zeta = \langle \xi, \eta \rangle : X \vee Y \to X \times Y$. The *smash* of X and Y, $X \wedge Y$, is the result of collapsing $X \times Y$ with respect to ζ.

Pushout lemma

5.1.12 Subcategories

Definition 5.1.59 (Subcategory)**.** A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ consists of:

- a subset $\mathrm{Ob}(\mathcal{C}')$ of $\mathcal{C}$
- for all $A, B \in \mathrm{Ob}(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \mathrm{Ob}(\mathcal{C}')$, we have $\mathrm{id}_A \in \mathcal{C}'[A, A]$
- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \mathrm{Ob}(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

5.1.13 Opposite Category

Definition 5.1.60 (Opposite Category)**.** For any category $\mathcal{C}$, the *opposite* category $\mathcal{C}^{\mathrm{op}}$ is the category with

- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$
- $\mathcal{C}^{\mathrm{op}}[A, B] = \mathcal{C}[B, A]$
- Given $f \in \mathcal{C}^{\mathrm{op}}[A, B]$ and $g \in \mathcal{C}^{\mathrm{op}}[B, C]$, their composite in $\mathcal{C}^{\mathrm{op}}$ is $f \circ g$, where $\circ$ is composition in $\mathcal{C}$.

Proposition 5.1.61. *An object is initial in $\mathcal{C}$ iff it is terminal in $\mathcal{C}^{\mathrm{op}}$.*

Proof: Immediate from definitions. $\square$

Proposition 5.1.62. *An object is terminal in $\mathcal{C}$ iff it is initial in $\mathcal{C}^{\mathrm{op}}$.*

Proof: Immediate from definitions. $\square$

Corollary 5.1.62.1. *If T and T' are terminal objects in $\mathcal{C}$ then there exists a unique isomorphism $T \cong T'$.*

5.1.14 Groupoids

Definition 5.1.63 (Groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

5.1.15 Concrete Categories

Definition 5.1.64 (Concrete Category)**.** A *concrete category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}(\mathcal{C})$ of *objects*
- for any object $A \in \mathrm{Ob}(\mathcal{C})$, a set $|A|$
- for any objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$
- for any object A we have $\mathrm{id}_{|A|} \in \mathcal{C}[A, A]$.

5.1.16 Power of Categories

Definition 5.1.65. Let $\mathcal{C}$ be a category and J a set. The category $\mathcal{C}^J$ is the category with:

- objects all J-indexed families of objects of $\mathcal{C}$
- morphisms $\{X_j\}_{j \in J} \to \{Y_j\}_{j \in J}$ all families $\{f_j\}_{j \in J}$ where $f_j : X_j \to Y_j$

5.1.17 Arrow Category

Definition 5.1.66 (Arrow Category)**.** Let $\mathcal{C}$ be a category. The *arrow category* $\mathcal{C}^{\to}$ is the category with:

- objects all triples (A, B, f) where $f : A \to B$ in $\mathcal{C}$
- morphisms $(A, B, f) \to (C, D, g)$ all pairs $(u : A \to C, v : B \to D)$ such that $vf = gu$.

5.1.18 Slice Category

Definition 5.1.67 (Slice Category)**.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category under* A, $\mathcal{C} \backslash A$, is the category with:

- objects all pairs (B, f) where $B \in \mathcal{C}$ and $f : A \to B$
- morphisms $(B, f) \to (C, g)$ are morphisms $u : B \to C$ such that $uf = g$.

We identify this with the subcategory of $\mathcal{C}^{\to}$ formed by mapping (B, f) to (A, B, f) and u to (id_A, u).

Proposition 5.1.68. *If $s : (B, f) \to (C, g)$ in $\mathcal{C}\backslash A$, then any retraction of s in $\mathcal{C}$ is a retraction of s in $\mathcal{C}\backslash A$.*

PROOF:
⟨1⟩1. LET: $r : C \to B$ be a retraction of s in $\mathcal{C}$.
⟨1⟩2. $rg = f$
  PROOF: $rg = rsf = f$.
⟨1⟩3. $r : (C, g) \to (B, f)$ in $\mathcal{C}\backslash A$
⟨1⟩4. $rs = \mathrm{id}_{(B,f)}$
  PROOF: Because composition is inherited from $\mathcal{C}$.
∎

Proposition 5.1.69. id_A *is the initial object in $\mathcal{C}\backslash A$.*

PROOF: For any $(B, f) \in \mathcal{C}\backslash A$, we have f is the only morphism $A \to B$ such that $f\mathrm{id}_A = f$. ∎

Proposition 5.1.70. *If A is terminal in $\mathcal{C}$ then* id_A *is the zero object in $\mathcal{C}\backslash A$.*

PROOF: For any $(B, f) \in \mathcal{C}\backslash A$, the unique morphism $! : B \to A$ is the unique morphism such that $!f = \mathrm{id}_A$. ∎

Definition 5.1.71 (Pointed Sets)**.** The *category of pointed sets* is $\mathbf{Set}\backslash 1$.

Definition 5.1.72. Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category over A*, $\mathcal{C}/A$, is the category with:

- objects all pairs (B, f) with $f : B \to A$
- morphisms $u : (B, f) \to (C, g)$ all morphisms $u : B \to C$ such that $gu = f$.

Proposition 5.1.73. *Let $u : (B, f) \to (C, g) : \mathcal{C}/A$. Any section of u in $\mathcal{C}$ is a section of u in $\mathcal{C}/A$.*

PROOF: Dual to Proposition 5.1.68. ∎

Proposition 5.1.74. id_A *is terminal in $\mathcal{C}/A$.*

PROOF: Dual to Proposition 5.1.69. ∎

Proposition 5.1.75. *If A is initial in $\mathcal{C}$ then* id_A *is the zero object in $\mathcal{C}/A$.*

PROOF: Dual to Proposition 5.1.70. ∎

Definition 5.1.76. Let $A \in \mathcal{C}$. The category of objects *over and under A*, written $\mathcal{C}^A_A$, is the category with:

- objects all triples (X, u, p) where $u : A \to X$, $p : X \to A$ and $pu = \mathrm{id}_A$
- morphism $f : (X, u, p) \to (Y, v, q)$ all morphisms $f : X \to Y$ such that $fu = v$ and $qf = p$

Proposition 5.1.77. $(A, \mathrm{id}_A, \mathrm{id}_A)$ *is the zero object in $\mathcal{C}^A_A$.*

Proof: For any object (X, u, p), we have p is the unique morphism $(X, u, p) \to (A, \mathrm{id}_A, \mathrm{id}_A)$, and u is the unique morphism $(A, \mathrm{id}_A, \mathrm{id}_A) \to (X, u, p)$. □

Definition 5.1.78 (Fibre Collapsing)**.** Let B be a set. Let $u : (A, a) \to (X, x)$ in $\mathbf{Set}/B$. Form the pushout

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"] & B \arrow[d, "j"'] \\ X \arrow[r, "i"'] & C \end{tikzcd}$$

Let $c : C \to B$ be the unique morphism such that $cj = \mathrm{id}_B$ and $ci = x$. Then $(C, j, c) \in \mathbf{Set}_B^B$ is called the set over and under B obtained from X by *fibre collapsing* with respect to u. If (A, u) is a subset of X, we denote this set over and under B by $X/_B(A, u)$.

Definition 5.1.79 (Fibre Wedge)**.** Let B be a small set. Let $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$. The *fibre wedge* of X and Y is the pushout of u_X and u_Y:

$$\begin{tikzcd} B \arrow[r, "u_X"] \arrow[d, "u_Y"] & X \arrow[d] \\ Y \arrow[r] & X \vee_B Y \end{tikzcd}$$

Definition 5.1.80 (Fibre Smash)**.** Let $X, Y \in \mathbf{Set}_B^B$. Let $\xi : X \vee_B Y \to X$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, equal, bend left] & \\ Y \arrow[r] \arrow[drr, bend right, "0"'] & X \vee_B Y \arrow[dr, "\xi"] & \\ & & X \end{tikzcd}$$

Let $\eta : X \vee_B Y \to Y$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "0"] & \\ Y \arrow[r] \arrow[drr, equal, bend right] & X \vee_B Y \arrow[dr, "\eta"] & \\ & & Y \end{tikzcd}$$

Let $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \to X \times Y$. The *fibre smash* of X and Y, $X \wedge_B Y$, is the result of collapsing $X \times Y$ with respect to ζ.

Proposition 5.1.81. **Set** *has products and coproducts.*

Proposition 5.1.82. *Let $\mathcal{C}$ be a category. Let $\{X_\alpha\}_{\alpha\in I}$ be a family of objects in $\mathcal{C}$ and $Z \in \mathcal{C}$. Let $\coprod_{\alpha\in I} X_\alpha$ be the coproduct of $\{X_\alpha\}_{\alpha\in I}$. Then*

$$\mathcal{C}[\coprod_{\alpha\in I} X_\alpha, Z] \approx \prod_{\alpha\in I} \mathcal{C}[X_\alpha, Z] \ .$$

Proposition 5.1.83. *Let $\mathcal{C}$ be a category. Let $\{X_\alpha\}_{\alpha\in I}$ be a family of objects in $\mathcal{C}$ and $Z \in \mathcal{C}$. Let $\prod_{\alpha\in I} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha\in I}$. Then*

$$\mathcal{C}[Z, \prod_{\alpha\in I} X_\alpha] \approx \prod_{\alpha\in I} \mathcal{C}[Z, X_\alpha] \ .$$

Proposition 5.1.84. *A product in $\mathcal{C}$ constitutes a product in $\mathcal{C}\backslash A$.*

Proposition 5.1.85. *A coproduct in $\mathcal{C}$ constitutes a product in $\mathcal{C}/A$.*

5.2 Functors

Definition 5.2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- a function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$
- for every morphism $f : A \to B$ in $\mathcal{C}$, a morphism $Ff : FA \to FB$ in $\mathcal{D}$

such that:

- for all $A \in \mathrm{Ob}(C)$ we have $F\mathrm{id}_A = \mathrm{id}_{FA}$
- for any morphism $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, we have $F(g \circ f) = Fg \circ Ff$

Proposition 5.2.2. *Functors preserve isomorphisms.*

Proof:
⟨1⟩1. Let: $F : \mathcal{C} \to \mathcal{D}$ be a functor.
⟨1⟩2. Let: $f : A \cong B$ in $\mathcal{C}$
⟨1⟩3. $Ff^{-1} \circ Ff = \mathrm{id}_{FA}$
Proof:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\mathrm{id}_A \\ &= \mathrm{id}_{FA} \end{aligned}$$

⟨1⟩4. $Ff \circ Ff^{-1} = \mathrm{id}_{FB}$
Proof:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\mathrm{id}_B \\ &= \mathrm{id}_{FB} \end{aligned}$$

□

Definition 5.2.3 (Identity Functor)**.** For any category $\mathcal{C}$, the *identity* functor on $\mathcal{C}$ is the functor $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ defined by

$$\begin{aligned} I_{\mathcal{C}} A &:= A && (A \in \mathcal{C}) \\ I_{\mathcal{C}} f &:= f && (f : A \to B \text{ in } \mathcal{C}) \end{aligned}$$

Proposition 5.2.4. *Let $F : \mathcal{C} \to \mathcal{D}$. If $r : A \to B$ is a retraction of $s : B \to A$ in $\mathcal{C}$ then Fr is a retraction of Fs.*

Proof:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\mathrm{id}_B \\ &= \mathrm{id}_{FB} \end{aligned}$$ □

Corollary 5.2.4.1. *Let $F : \mathcal{C} \to \mathcal{D}$. If $\phi : A \cong B$ is an isomorphism in $\mathcal{C}$ then $F\phi : FA \cong FB$ is an isomorphism in $\mathcal{D}$ with $(F\phi)^{-1} = F\phi^{-1}$.*

Definition 5.2.5 (Composition of Functors)**.** Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, the *composite* functor $GF : \mathcal{C} \to \mathcal{E}$ is defined by

$$\begin{aligned} (GF)A &= G(FA) && (A \in \mathcal{C}) \\ (GF)f &= G(Ff) && (f : A \to B : \mathcal{C}) \end{aligned}$$

Definition 5.2.6 (Category of Categories)**.** Let **Cat** be the category of small categories and functors.

Definition 5.2.7 (Isomorphism of Categories)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an *isomorphism of categories* iff there exists a functor $F^{-1} : \mathcal{D} \to \mathcal{C}$, the *inverse* of F, such that $FF^{-1} = I_{\mathcal{D}}$ and $F^{-1}F = I_{\mathcal{C}}$.

Categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, iff there exists an isomorphism between them.

Proposition 5.2.8. *If A is initial in $\mathcal{C}$ then $\mathcal{C} \backslash A \cong \mathcal{C}$.*

Proof:

⟨1⟩1. Define $F : \mathcal{C} \backslash A \to \mathcal{C}$ by

$$\begin{aligned} F(B, f) &= B \\ F(u : (B, f) \to (C, g)) &= u \end{aligned}$$

⟨1⟩2. Define $G : \mathcal{C} \to \mathcal{C} \backslash A$ by

$$\begin{aligned} GB &= (B, !_B) && \text{where } !_B \text{ is the unique morphism } A \to B \\ G(u : B \to C) &= u : (B, !_B) \to (C, !_C) \end{aligned}$$

⟨1⟩3. $FG = \mathrm{id}_{\mathcal{C}}$

⟨1⟩4. $GF = \mathrm{id}_{\mathcal{C} \backslash A}$

Proof: Since $GF(B, f) = (B, !_B) = (B, f)$ because the morphism $A \to B$ is unique.

□

Proposition 5.2.9. *If A is terminal in $\mathcal{C}$ then $\mathcal{C}/A \cong \mathcal{C}$.*

Proof: Dual. □

Proposition 5.2.10.

$$\mathcal{C}_A^A \cong (\mathcal{C}/A)\backslash(A, \mathrm{id}_A) \cong (\mathcal{C}\backslash A)/(A, \mathrm{id}_A)$$

Proof:
⟨1⟩1. Define a functor $F : \mathcal{C}_A^A \to (\mathcal{C}/A)\backslash(A, \mathrm{id}_A)$.
  ⟨2⟩1. Given $A \xrightarrow{u} X \xrightarrow{p} A$ in $\mathcal{C}_A^A$, let $F(X, u, p) = ((X, p), u)$
  ⟨2⟩2. Given $f : (A \xrightarrow{u} X \xrightarrow{p} A) \to (A \xrightarrow{v} Y \xrightarrow{q} A)$, let $Ff = f$.
⟨1⟩2. Define a functor $G : (\mathcal{C}/A)\backslash(A, \mathrm{id}_A) \to \mathcal{C}_A^A$.
⟨1⟩3. Define a functor $H : \mathcal{C}_A^A \to (\mathcal{C}\backslash A)/(A, \mathrm{id}_A)$.
⟨1⟩4. Define a functor $K : (\mathcal{C}\backslash A)/(A, \mathrm{id}_A) \to \mathcal{C}_A^A$.
□

Definition 5.2.11 (Forgetful Functor)**.** For any concrete category $\mathcal{C}$, define the *forgetful* functor $U : \mathcal{C} \to \mathbf{Set}$ by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

Definition 5.2.12 (Switching Functor)**.** For any category $\mathcal{C}$, define the *switching functor* $T : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

Definition 5.2.13 (Reduction)**.** Let $\Phi : \mathbf{Set} \to \mathbf{Set}$ be a functor. The *reduction* of Φ is the functor $\Phi^* : \mathbf{Set}_* \to \mathbf{Set}_*$ defined by: $\Phi^*(X, a)$ is the collapse of $\Phi(X)$ with respect to $\Phi(a) : \Phi(1) \rightarrowtail \Phi(X)$.

Definition 5.2.14. Extend the wedge $\vee$ to a functor $\mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ by defining, given $f : X \to X'$ and $g : Y \to Y'$, thene $f \vee g$ is the unique morphism that makes the following diagram commute.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[dr, "f"] & \\
Y \arrow[r] \arrow[dr, "g"'] & X \vee Y \arrow[dr, "f \vee g"] & X' \arrow[d] \\
& Y' \arrow[r] & X' \vee Y'
\end{tikzcd}$$

Definition 5.2.15. Extend smash to a functor $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ as follows. Given $f : X \to X'$ and $g : Y \to Y'$, let $f \wedge g : X \wedge Y \to X' \wedge Y'$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd}
X \vee Y \arrow[r] \arrow[d] \arrow[ddr] & 1 \arrow[d] \arrow[ddr, equal, bend left] & \\
X \times Y \arrow[r] \arrow[ddr, "f \times g"'] & X \wedge Y \arrow[ddr] & \\
& X' \vee Y' \arrow[r] \arrow[d] & 1 \arrow[d] \\
& X' \times Y' \arrow[r] & X' \wedge Y'
\end{tikzcd}$$

Definition 5.2.16 (Reduction)**.** Let B be a small set. Let $\Phi_B : \mathbf{Set}/B \to \mathbf{Set}/B$ be a functor. The *reduction* of Φ_B is the functor $\Phi_B^B : \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ defined as follows.

For $(X, u : B \to X, p : X \to B) \in \mathbf{Set}_B^B$, let $\Phi_B^B(X)$ be the set over and under B obtained from $\Phi_B(X)$ by collapsing with respect to $\Phi_B(u) : \Phi_B(B) \to \Phi_B(X)$.

Definition 5.2.17. Extend $\vee_B$ to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$.

Definition 5.2.18. Extend $\wedge_B$ to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$.

Definition 5.2.19 (Faithful)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* iff, for any objects $A, B \in \mathcal{C}$ and morphisms $f, g : A \to B : \mathcal{C}$, if $Ff = Fg$ then $f = g$.

Definition 5.2.20 (Full)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* iff, for any objects $A, B \in \mathcal{C}$ and morphism $g : FA \to FB : \mathcal{D}$, there exists $f : A \to B : \mathcal{C}$ such that $Ff = g$.

Definition 5.2.21 (Fully Faithful)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* iff it is full and faithful.

Definition 5.2.22 (Full Embedding)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is a *full embedding* iff it is fully faithful and injective on objects.

5.3 Natural Transformations

Definition 5.3.1 (Natural Transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\tau : F \Rightarrow G$ is a family of morphisms $\{\tau_X : FX \to GX\}_{X \in \mathcal{C}}$ such that, for every morphism $f : X \to Y : \mathcal{C}$, we have $Gf \circ \tau_X = \tau_Y \circ Ff$.

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\tau_X"'] & FY \arrow[d, "\tau_Y"] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}$$

Definition 5.3.2 (Natural Isomorphism)**.** A natural transformation $\tau : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ is a *natural isomorphism*, $\tau : F \cong G$, iff for all $X \in \mathcal{C}$, τ_X is an isomorphism $FX \cong GX$.

Functors F and G are *naturally isomorphic*, $F \cong G$, iff there exists a natural isomorphism between them.

Definition 5.3.3 (Inverse)**.** Let $\tau : F \cong G$. The *inverse* natural isomorphism $\tau^{-1} : G \cong F$ is defined by $(\tau^{-1})_X = {\tau_X}^{-1}$.

5.4 Bifunctors

Definition 5.4.1 (Commutative)**.** A bifunctor $\square : \mathcal{C}^2 \to \mathcal{C}$ is *commutative* iff $\square \cong \square \circ T$, where $T : \mathcal{C}^2 \to \mathcal{C}^2$ is the swap functor.

Proposition 5.4.2. *$\vee : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ is commutative.*

Proof: Since the pushout of f and g is the pushout of g and f. $\square$

Proposition 5.4.3. *$\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ is commutative.*

Proof: In the diagram defining $X \wedge Y$, construct the isomorphism between the version with X and Y and the version with Y with X for every object. $\square$

Proposition 5.4.4. *$\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ is commutative.*

Proposition 5.4.5. *$\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ is commutative.*

Definition 5.4.6 (Associative)**.** A bifunctor $\square$ is *associative* iff $\square \circ (\square \times \mathrm{id}) \cong \square \circ (\mathrm{id} \times \square)$.

Proposition 5.4.7. *$\vee : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ is associative.*

Proof: Since $X \vee (Y \vee Z)$ and $(X \vee Y) \vee Z$ are both the pushout of the unique morphisms $1 \to X$, $1 \to Y$ and $1 \to Z$. $\square$

Proposition 5.4.8. *$\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ is associative.*

Proof: Draw isomorphisms between the diagrams for $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$. $\square$

Product and coproduct are commutative and associative.

Proposition 5.4.9. *$\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ is associative.*

Proposition 5.4.10. *$\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ is associative.*

Proposition 5.4.11. *Let $\mathcal{C}$ be a category with binary coproducts. Let $\square : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a bifunctor. Then $\square$* distributes *over $+$ iff the canonical morphism*

$$(X \square Z) + (Y \square Z) \to (X + Y) \square Z$$

is an isomorphism for all X, Y, Z.

Proposition 5.4.12. *In a category with binary products and binary coproducts, then $\times$ distributes over $+$.*

Proposition 5.4.13. *In* $\mathbf{Set}/*$*, we have $\times$ does not distribute over $\vee$.*

Proposition 5.4.14. *In* $\mathbf{Set}/*$*, we have $\wedge$ distributes over $\vee$.*

Proposition 5.4.15. *In* $\mathbf{Set}/B$*, we have $\times_B$ distributes over $+_B$.*

Proposition 5.4.16. *In* $\mathbf{Set}/B^B$*, we have $\wedge_B$ distributes over $\vee_B$.*

5.5 Functor Categories

Definition 5.5.1 (Functor Category)**.** Given categories $\mathcal{C}$ and $\mathcal{D}$, define the *functor category* $\mathcal{C}^{\mathcal{D}}$ to be the category with objects the functors from $\mathcal{D}$ to $\mathcal{C}$ and morphisms the natural transformations.

Definition 5.5.2 (Yoneda Embedding)**.** Let $\mathcal{C}$ be a category. The *Yoneda embedding* $Y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the functor that maps an object A to $\mathcal{C}[-, A]$ and morphisms similarly.

Theorem 5.5.3 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}[\mathcal{C}[-, X], F] \cong FX$$

that maps $\tau : \mathcal{C}[-, X] \Rightarrow F$ to $\tau_X(\mathrm{id}_X)$.

Proof:
⟨1⟩1. ϕ is natural in X.
Proof:
⟨2⟩1. Let: $f : X \to Y : \mathcal{C}$
⟨2⟩2. Let: $\tau : \mathcal{C}[-, X] \Rightarrow F$
⟨2⟩3. $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$
Proof:

$$\begin{aligned}
\phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\mathrm{id}_Y \circ f) \\
&= \tau_Y(f) \\
&= \tau_Y(f \circ \mathrm{id}_X) \\
&= Ff(\tau_X(\mathrm{id}_X)) && (\tau \text{ natural}) \\
&= Ff(\phi(\tau))
\end{aligned}$$

⟨1⟩2. ϕ is natural in F.
⟨2⟩1. Let: $\alpha : F \Rightarrow G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$
⟨2⟩2. Let: $\tau : \mathcal{C}[-, X] \Rightarrow F$
⟨2⟩3. $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$
Proof: $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\mathrm{id}_X)) = \alpha_X(\phi(\tau))$
⟨1⟩3. Each ϕ_{XF} is injective.
⟨2⟩1. Let: $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$
⟨2⟩2. Assume: $\phi(\sigma) = \phi(\tau)$

⟨2⟩3. Let: $f : Y \to X$
⟨2⟩4. $\sigma_Y(f) = \tau_Y(f)$
Proof:

$$\begin{aligned} \sigma_Y(f) &= \sigma_Y(\mathrm{id}_X \circ f) \\ &= Ff(\sigma_X(\mathrm{id}_X)) && (\sigma \text{ is natural}) \\ &= Ff(\tau_X(\mathrm{id}_X)) && (\langle 2\rangle 2) \\ &= \tau_Y(\mathrm{id}_X \circ f) && (\tau \text{ is natural}) \\ &= \tau_Y(f) \end{aligned}$$

⟨1⟩4. Each ϕ_{XF} is surjective.
⟨2⟩1. Let: $X \in \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$
⟨2⟩2. Let: $a \in FX$
⟨2⟩3. Let: $\tau : \mathcal{C}[-, X] \Rightarrow F$ be given by $\tau_Y(g) = Fg(a)$ for $g : Y \to X$
⟨2⟩4. τ is natural.
⟨3⟩1. Let: $h : Y \to Z : \mathcal{C}$
Prove: $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \mathrm{id}_X]$
⟨3⟩2. Let: $g : Z \to X$
⟨3⟩3. $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$
Proof:

$$\begin{aligned} \tau_Y(g \circ h) &= F(g \circ h)(a) \\ &= Fh(Fg(a)) \\ &= Fh(\tau_Z(g)) \end{aligned}$$

⟨2⟩5. $\phi(\tau) = a$
Proof:

$$\begin{aligned} \phi_X(\tau) &= \tau_X(\mathrm{id}_X) \\ &= F\mathrm{id}_X(a) \\ &= a \end{aligned}$$

□

Corollary 5.5.3.1. *The Yoneda embedding is fully faithful.*

Corollary 5.5.3.2. *Given objects A and B in $\mathcal{C}$, we have $A \cong B$ if and only if $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$.*

Part III

Number Systems

Chapter 6

The Real Numbers

Theorem 6.0.1. *The following hold in the real numbers:*

1. $x + (y + z) = (x + y) + z$
2. $x(yz) = (xy)z$
3. $x + y = y + x$
4. $xy = yx$
5. $x + 0 = x$
6. $x1 = x$
7. $x + (-x) = 0$
8. *If* $x \neq 0$ *then* $x \cdot (1/x) = 1$
9. $x(y + z) = xy + xz$
10. *If* $x > y$ *then* $x + z > y + z$.
11. *If* $x > y$ *and* $z > 0$ *then* $xz > yz$.
12. $\mathbb{R}$ *has the least upper bound property.*
13. *If* $x < y$ *then there exists* z *such that* $x < z < y$.

Definition 6.0.2. Given real numbers x and y with $y \neq 0$, we write x/y for xy^{-1}.

Theorem 6.0.3. *For any real numbers* x *and* y, *if* $x + y = x$ *then* $y = 0$.

PROOF:
$\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$
$\langle 1 \rangle 2$. ASSUME: $x + y = x$
$\langle 1 \rangle 3$. $y = 0$

PROOF:

$$\begin{aligned} y &= y + 0 && \text{(Definition of zero)} \\ &= y + (x + (-x)) && \text{(Definition of } -x\text{)} \\ &= (y + x) + (-x) && \text{(Associativity of Addition)} \\ &= (x + y) + (-x) && \text{(Commutativity of Addition)} \\ &= x + (-x) && (\langle 1\rangle 2) \\ &= 0 && \text{(Definition of } -x\text{)} \end{aligned}$$

□

Theorem 6.0.4.

$$\forall x \in \mathbb{R}. 0x = 0$$

PROOF:
$\langle 1\rangle 1$. LET: $x \in \mathbb{R}$
$\langle 1\rangle 2$. $xx + 0x = xx$
PROOF:

$$\begin{aligned} xx + 0x &= (x + 0)x && \text{(Distributive Law)} \\ &= xx && \text{(Definition of 0)} \end{aligned}$$

$\langle 1\rangle 3$. $0x = 0$
PROOF: Theorem 6.0.3, $\langle 1\rangle 2$.
□

Theorem 6.0.5.

$$-0 = 0$$

PROOF: Since $0 + 0 = 0$. □

Theorem 6.0.6.

$$\forall x \in \mathbb{R}. -(-x) = x$$

PROOF: Since $-x + x = 0$. □

Theorem 6.0.7.

$$\forall x, y \in \mathbb{R}. x(-y) = -(xy)$$

PROOF:

$$\begin{aligned} x(-y) + xy &= x((-y) + y) && \text{(Distributive Law)} \\ &= x0 && \text{(Definition of } -y\text{)} \\ &= 0 && \text{(Theorem 6.0.4)} \end{aligned}$$ □

Theorem 6.0.8.

$$\forall x \in \mathbb{R}. (-1)x = -x$$

PROOF:

$$\begin{aligned} (-1)x &= -(1 \cdot x) && \text{(Theorem 6.0.7)} \\ &= -x && \text{(Definition of 1)} \end{aligned}$$ □

Proposition 6.0.9. *Let X be a linearly ordered set. Let $a, b, c \in X$ with $a < b < c$. Then $[a, c) \cong [0, 1)$ if and only if $[a, b) \cong [0, 1)$ and $[b, c) \cong [0, 1)$.*

PROOF:
$\langle 1 \rangle$1. For all $x \in (0, 1)$ we have $[0, x) \cong [0, 1)$.
  PROOF: The function that maps t to t/x is an order isomorphism.
$\langle 1 \rangle$2. For all $x \in (0, 1)$ we have $[x, 1) \cong [0, 1)$.
  PROOF: The function that maps t to $(t - x)/(1 - x)$ is an order isomorphism.
$\langle 1 \rangle$3. We have $[0, 2) \cong [0, 1)$.
  PROOF: The function that maps t to $t/2$ is an order isomorphism.
□

Proposition 6.0.10. *Let X be a linearly ordered set. Let (a_n) be a strictly increasing sequence in X. Let b be its supremum. Then $[a_0, b) \cong [0, 1)$ if and only if, for all n, we have $[a_n, a_{n+1}) \cong [0, 1)$.*

PROOF:
$\langle 1 \rangle$1. For all $x, y \in [0, 1)$ with $x < y$ we have $[x, y) \cong [0, 1)$.
  PROOF: The function that maps t to $(t - x)/(y - x)$ is an order isomorphism.
$\langle 1 \rangle$2. We have $[0, 1) \cong [0, +\infty)$.
  PROOF: The function that maps t to $1/(1 - t) - 1$ is an order isomorphism.
□

6.1 Subtraction

Definition 6.1.1 (Subtraction)**.** We write $x - y$ for $x + (-y)$.

Theorem 6.1.2.

$$\forall x, y, z \in \mathbb{R}. x(y - z) = xy - xz$$

PROOF:

$$\begin{aligned} x(y - z) &= x(y + (-z)) && \text{(Definition of subtraction)} \\ &= xy + x(-z) && \text{(Distributive Law)} \\ &= xy + (-(xz)) && \text{(Theorem 6.0.7)} \\ &= xy - xz && \text{(Definition of subtraction)} \end{aligned}$$ □

Theorem 6.1.3.

$$\forall x, y \in \mathbb{R}. -(x + y) = -x - y$$

PROOF:

$$\begin{aligned} -(x + y) &= (-1)(x + y) && \text{(Theorem 6.0.8)} \\ &= (-1)x + (-1)y && \text{(Distributive Law)} \\ &= -x + (-y) && \text{(Theorem 6.0.8)} \\ &= -x - y && \text{(Definition of subtraction)} \end{aligned}$$ □

Theorem 6.1.4.

$$\forall x, y \in \mathbb{R}. -(x - y) = -x + y$$

PROOF:

$$\begin{aligned}
-(x-y) &= -(x+(-y)) && \text{(Definition of subtraction)}\\
&= -x-(-y) && \text{(Theorem 6.1.3)}\\
&= -x+(-(-y)) && \text{(Definition of subtraction)}\\
&= -x+y && \text{(Theorem 6.0.6)} \quad \square
\end{aligned}$$

Definition 6.1.5 (Reciprocal)**.** Given $x \in \mathbb{R}$ with $x \neq 0$, the *reciprocal* of x, $1/x$, is the unique real number such that $x \cdot 1/x = 1$.

Theorem 6.1.6. *For any real numbers x and y, if $x \neq 0$ and $xy = x$ then $y = 1$.*

PROOF:
$\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$
$\langle 1 \rangle 2$. ASSUME: $x \neq 0$
$\langle 1 \rangle 3$. ASSUME: $xy = x$
$\langle 1 \rangle 4$. $y = 1$
PROOF:

$$\begin{aligned}
y &= y1 && \text{(Definition of 1)}\\
&= y(x \cdot 1/x) && \text{(Definition of } 1/x, \langle 1 \rangle 2)\\
&= (yx)1/x && \text{(Associativity of Multiplication)}\\
&= (xy)1/x && \text{(Commutativity of Multiplication)}\\
&= x \cdot 1/x && (\langle 1 \rangle 3)\\
&= 1 && \text{(Definition of } 1/x, \langle 1 \rangle 2)
\end{aligned}$$

$\square$

Definition 6.1.7 (Quotient)**.** Given real numbers x and y with $y \neq 0$, the *quotient* x/y is defined by

$$x/y = x \cdot 1/y \ .$$

Theorem 6.1.8. *For any real number x, if $x \neq 0$ then $x/x = 1$.*

PROOF: Immediate from definitions. $\square$

Theorem 6.1.9.

$$\forall x \in \mathbb{R}. x/1 = x$$

PROOF:
$\langle 1 \rangle 1$. LET: $x \in \mathbb{R}$
$\langle 1 \rangle 2$. $1/1 = 1$
PROOF: Since $1 \cdot 1 = 1$.
$\langle 1 \rangle 3$. $x/1 = x$
PROOF: Since $x/1 = x \cdot 1/1 = x \cdot 1 = x$.
$\square$

Theorem 6.1.10. *For any real numbers x and y, if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.*

PROOF:
$\langle 1\rangle 1.$ LET: $x, y \in \mathbb{R}$
$\langle 1\rangle 2.$ ASSUME: $xy = 0$ and $x \neq 0$
PROVE: $y = 0$
$\langle 1\rangle 3.$ $y = 0$
PROOF:

$$\begin{aligned} y &= 1y && \text{(Definition of 1)} \\ &= (1/x)xy && \text{(Definition of } 1/x, \langle 1\rangle 2) \\ &= (1/x)0 && (\langle 1\rangle 2) \\ &= 0 && \text{(Theorem 6.0.4)} \end{aligned}$$

□

Theorem 6.1.11. *For any real numbers y and z, if $y \neq 0$ and $z \neq 0$ then $(1/y)(1/z) = 1/(yz)$.*

PROOF: Since $yz(1/y)(1/z) = 1 \cdot 1 = 1$. □

Corollary 6.1.11.1. *For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have $(x/y)(z/w) = (xz)/(yw)$.*

Theorem 6.1.12. *For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have*

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + yz}{yw}$$

PROOF:

$$\begin{aligned} yw\left(\frac{x}{y} + \frac{z}{w}\right) &= yw\frac{x}{y} + yw\frac{z}{w} \\ &= wx + yz \end{aligned}$$

□

Theorem 6.1.13. *For any real number x, if $x \neq 0$ then $1/x \neq 0$.*

PROOF: Since $x \cdot 1/x = 1 \neq 0$. □

Theorem 6.1.14. *For any real numbers w, z, if $w \neq 0 \neq z$ then $1/(w/z) = z/w$.*

PROOF: Since $(z/w)(w/z) = (wz)/(wz) = 1$. □

Theorem 6.1.15. *For any real numbers a, x and y, if $y \neq 0$ then $(ax)/y = a(x/y)$*

PROOF: Since $ya(x/y) = ax$. □

Theorem 6.1.16. *For any real numbers x and y, if $y \neq 0$ then $(-x)/y = x/(-y) = -(x/y)$.*

PROOF:
$\langle 1\rangle 1.$ $(-x)/y = -(x/y)$
PROOF: Take $a = -1$ in Theorem 6.1.15.
$\langle 1\rangle 2.$ $x/(-y) = -(x/y)$

PROOF: Since $(-y)(-(x/y)) = y(x/y) = x$.
∎

Theorem 6.1.17. *For any real numbers x, y, z and w, if $x > y$ and $w > z$ then $x + w > y + z$.*

PROOF: We have $y + z < x + z < x + w$ by Monotonicity of Addition twice. ∎

Corollary 6.1.17.1. *For any real numbers x and y, if $x > 0$ and $y > 0$ then $x + y > 0$.*

Theorem 6.1.18. *For any real numbers x and y, if $x > 0$ and $y > 0$ then $xy > 0$.*

PROOF:

$$\begin{aligned} xy &> 0y && \text{(Monotonicity of Multiplication)} \\ &= 0 && \text{(Theorem 6.0.4)} \end{aligned}$$ ∎

Theorem 6.1.19. *For any real number x, we have $x > 0$ iff $-x < 0$.*

PROOF:
⟨1⟩1. If $0 < x$ then $-x < 0$
  PROOF: By Monotonicity of Addition adding $-x$ to both sides.
⟨1⟩2. If $-x < 0$ then $0 < x$
  PROOF: By Monotonicity of Addition adding x to both sides.
∎

Theorem 6.1.20. *For any real numbers x and y, we have $x > y$ iff $-x < -y$.*

PROOF:
⟨1⟩1. If $y < x$ then $-x < -y$.
  PROOF: By Monotonicity of Addition adding $-x - y$ to both sides.
⟨1⟩2. If $-x < -y$ then $y < x$.
  PROOF: By Monotonicity of Addition adding $x + y$ to both sides.
∎

Theorem 6.1.21. *For any real numbers x, y and z, if $x > y$ and $z < 0$ then $xz < yz$.*

PROOF:
⟨1⟩1. LET: x, y and z be real numbers.
⟨1⟩2. ASSUME: $x > y$
⟨1⟩3. ASSUME: $z < 0$
⟨1⟩4. $-z > 0$
  PROOF: Theorem 6.1.19, ⟨1⟩3.
⟨1⟩5. $x(-z) > y(-z)$
  PROOF: ⟨1⟩2, ⟨1⟩4, Monotonicity of Multiplication.
⟨1⟩6. $-(xz) > -(yz)$
  PROOF: Theorem 6.0.7, ⟨1⟩5.

$\langle 1 \rangle 7.$ $xz < yz$
PROOF: Theorem 6.1.19, $\langle 1 \rangle 6$.
□

Theorem 6.1.22. *For any real number x, if $x \neq 0$ then $xx > 0$.*

PROOF:
$\langle 1 \rangle 1.$ If $x > 0$ then $xx > 0$
PROOF: By Monotonicity of Multiplication.
$\langle 1 \rangle 2.$ If $x < 0$ then $xx > 0$
PROOF: Theorem 6.1.21.
□

Theorem 6.1.23.

$$0 < 1$$

PROOF: By Theorem 6.1.22 since $1 = 1 \cdot 1$. □

Definition 6.1.24 (Positive)**.** A real number x is *positive* iff $x > 0$.
We write $\mathbb{R}_+$ for the set of positive reals.

Theorem 6.1.25. *For any real numbers x and y, we have xy is positive if and only if x and y are both positive or both negative.*

PROOF: By the Monotonicity of Multiplication and Theorem 6.1.21. □

Corollary 6.1.25.1. *For any real number x, if $x > 0$ then $1/x > 0$.*

PROOF: Since $x \cdot 1/x = 1$ is positive. □

Theorem 6.1.26. *For any real numbers x and y, if $x > y > 0$ then $1/x < 1/y$.*

PROOF: If $1/y \leqslant 1/x$ then $1 < 1$ by Monotonicity of Multiplication. □

Theorem 6.1.27. *For any real numbers x and y, if $x < y$ then $x < (x+y)/2 < y$.*

PROOF: We have $2x < x+y$ and $x+y < 2y$ by Monotonicity of Addition, hence $x < (x+y)/2 < y$ by Monotonicity of Multiplication since $1/2 > 0$. □

Corollary 6.1.27.1. *$\mathbb{R}$ is a linear continuum.*

Definition 6.1.28 (Negative)**.** A real number x is *negative* iff $x < 0$.
We write $\overline{\mathbb{R}_+}$ for the set of nonnegative reals.

Theorem 6.1.29. *For every positive real number a, there exists a unique positive real $\sqrt{a}$ such that $\sqrt{a}^2 = a$.*

PROOF:
$\langle 1 \rangle 1.$ LET: a be a positive real.
$\langle 1 \rangle 2.$ For any real numbers x and h, if $0 \leqslant h < 1$, then

$$(x+h)^2 < x^2 + h(2x+1) \enspace .$$

⟨2⟩1. LET: x and h be real numbers.
⟨2⟩2. ASSUME: $0 \leqslant h < 1$
⟨2⟩3. $(x+h)^2 < x^2 + h(2x+1)$
PROOF:

$$\begin{aligned}(x+h)^2 &= x^2 + 2hx + h^2 \\ &< x^2 + 2hx + h && (\langle 2\rangle 2) \\ &= x^2 + h(2x+1)\end{aligned}$$

⟨1⟩3. For any real numbers x and h, if $h > 0$ then

$$(x-h)^2 > x^2 - 2hx\ .$$

⟨2⟩1. LET: x and h be real numbers.
⟨2⟩2. ASSUME: $h > 0$
⟨2⟩3. $(x-h)^2 > x^2 - 2hx$
PROOF:

$$\begin{aligned}(x-h)^2 &= x^2 - 2hx + h^2 \\ &> x^2 - 2hx && (\langle 2\rangle 2)\end{aligned}$$

⟨1⟩4. For any positive real x, if $x^2 < a$ then there exists $h > 0$ such that $(x+h)^2 < a$.
⟨2⟩1. LET: x be a positive real.
⟨2⟩2. ASSUME: $x^2 < a$
⟨2⟩3. LET: $h = \min((a - x^2)/(2x+1), 1/2)$
⟨2⟩4. $0 < h < 1$
⟨2⟩5. $(x+h)^2 < a$
PROOF:

$$\begin{aligned}(x+h)^2 &< x^2 + h(2x+1) && (\langle 1\rangle 2) \\ &\leqslant a\end{aligned}$$

⟨1⟩5. For any positive real x, if $x^2 > a$ then there exists $h > 0$ such that $(x-h)^2 > a$.
⟨2⟩1. LET: x be a positive real.
⟨2⟩2. ASSUME: $x^2 > a$
⟨2⟩3. LET: $h = (x^2 - a)/2x$
⟨2⟩4. $h > 0$
⟨2⟩5. $(x-h)^2 > a$
PROOF:

$$\begin{aligned}(x-h)^2 &> x^2 - 2hx \\ &= a && (\langle 2\rangle 3)\end{aligned}$$

⟨1⟩6. LET: $B = \{x \in \mathbb{R} : x^2 < a\}$
⟨1⟩7. B is bounded above.
PROOF: If $a \geqslant 1$ then a is an upper bound. If $a < 1$ then 1 is an upper bound.
⟨1⟩8. B contains at least one positive real.
PROOF: If $a \geqslant 1$ then $1 \in B$. If $a < 1$ then $a \in B$.
⟨1⟩9. LET: $b = \sup B$
⟨1⟩10. $b^2 = a$
⟨2⟩1. $b^2 \geqslant a$
⟨3⟩1. ASSUME: for a contradiction $b^2 < a$

⟨3⟩2. PICK $h > 0$ such that $(b + h)^2 < a$
PROOF: ⟨1⟩4
⟨3⟩3. $b + h \in B$
⟨3⟩4. Q.E.D.
PROOF: This contradicts ⟨1⟩9.
⟨2⟩2. $b^2 \leqslant a$
⟨3⟩1. ASSUME: for a contradiction $b^2 > a$
⟨3⟩2. PICK $h > 0$ such that $(b - h)^2 > a$
PROOF: ⟨1⟩5
⟨3⟩3. PICK $x \in B$ such that $b - h < x$
PROOF: ⟨1⟩9
⟨3⟩4. $(b - h)^2 < x^2 < a$
⟨3⟩5. Q.E.D.
PROOF: This contradicts ⟨3⟩2
⟨1⟩11. For any positive reals b and c, if $b^2 = c^2$ then $b = c$.
⟨2⟩1. LET: b and c be positive reals.
⟨2⟩2. ASSUME: $b^2 = c^2$
⟨2⟩3. $b^2 - c^2 = 0$
⟨2⟩4. $(b - c)(b + c) = 0$
⟨2⟩5. $b - c = 0$ or $b + c = 0$
⟨2⟩6. $b + c \neq 0$
PROOF: Since $b + c > 0$
⟨2⟩7. $b - c = 0$
⟨2⟩8. $b = c$
□

Theorem 6.1.30. *The set of real numbers is uncountable.*

Definition 6.1.31. We write $\mathbb{R}^\infty$ for the set of sequences in $\mathbb{R}^\omega$ that are eventually zero.

Definition 6.1.32 (Hilbert Cube)**.** The *Hilbert cube* is $\prod_{n=0}^{\infty}[0, 1/(n+1)]$.

6.2 The Ordered Square

Definition 6.2.1 (Ordered Square)**.** The *ordered square* I_o^2 is the set $[0,1]^2$ under the dictionary order.

Proposition 6.2.2. *The ordered square is a linear continuum.*

PROOF:
⟨1⟩1. I_o^2 has the least upper bound property.
⟨2⟩1. LET: S be a nonempty subset of I_o^2.
⟨2⟩2. LET: a be the supremum of $\pi_1(S)$
⟨2⟩3. CASE: $a \in \pi_1(S)$
⟨3⟩1. LET: b be the supremum of $\{y \in [0,1] : (a, y) \in S\}$
⟨3⟩2. (a, b) is the supremum of S.

⟨2⟩4. CASE: $a \notin \pi_1(S)$
PROOF: $(a, 0)$ is the supremum of S.
⟨1⟩2. I_o^2 is dense.
⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
PROVE: There exists $(x_3, y_3) \in I_o^2$ such that $(x_1, y_1) < (x_3, y_3) < (x_2, y_2)$
⟨2⟩2. CASE: $x_1 < x_2$
⟨3⟩1. PICK x_3 such that $x_1 < x_3 < x_2$
⟨3⟩2. $(x_1, y_1) < (x_3, 0) < (x_2, y_2)$
⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < y_2$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

6.3 Punctured Euclidean Space

Definition 6.3.1 (Punctured Euclidean Space)**.** Let n be a positive integer. The *punctured Euclidean space* is $\mathbb{R}^n - \{\vec{0}\}$.

6.4 Topologist's Sine Curve

Definition 6.4.1 (Topologist's Sine Curve)**.** The *topologist's sine curve* is

$$(\{0\} \times [-1, 1]) \cup \{(x, \sin 1/x) : 0 < x \leqslant 1\} \ .$$

6.5 The Long Line

Definition 6.5.1 (Long Line)**.** The *long line* is $S_\Omega \times [0, 1)$ in the dictionary order.

Proposition 6.5.2. *For any $a \in S_\Omega$ with $a \neq 0$ we have* $[(0, 0), (a, 0)) \cong [0, 1)$.

PROOF: By transfinite induction on a using Propositions 6.0.9 and 6.0.10. □

Chapter 7

Integers and Rationals

7.1 Positive Integers

Definition 7.1.1 (Inductive)**.** A set of real numbers A is *inductive* iff $1 \in A$ and $\forall x \in A . x + 1 \in A$.

Definition 7.1.2 (Positive Integer)**.** The set $\mathbb{Z}_+$ of *positive integers* is the intersection of the set of inductive sets.

Proposition 7.1.3. *Every positive integer is positive.*

PROOF: The set of positive reals is inductive. □

Proposition 7.1.4. *1 is the least element of* $\mathbb{Z}_+$.

PROOF: Since $\{x \in \mathbb{R} : x \geqslant 1\}$ is inductive. □

Proposition 7.1.5. $\mathbb{Z}_+$ *is inductive.*

PROOF: 1 is an element of every inductive set, and for all $x \in \mathbb{R}$, if x is an element of every inductive set then so is $x + 1$. □

Theorem 7.1.6 (Principle of Induction)**.** *If A is an inductive set of positive integers then $A = \mathbb{Z}_+$.*

PROOF: Immediate from definitions. □

Theorem 7.1.7 (Well-Ordering Property)**.** $\mathbb{Z}_+$ *is well ordered.*

PROOF: Construct the obvious order isomorphism $\omega \cong \mathbb{Z}_+$. □

Theorem 7.1.8 (Archimedean Ordering Property)**.** *The set $\mathbb{Z}_+$ is unbounded above.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $\mathbb{Z}_+$ is bounded above.

⟨1⟩2. LET:
$s = \sup \mathbb{Z}_+$
⟨1⟩3. PICK $n \in \mathbb{Z}_+$ such that $s - 1 < n$
⟨1⟩4. $s < n + 1$
⟨1⟩5. Q.E.D.
PROOF: ⟨1⟩2 and ⟨1⟩4 form a contradiction.
□

7.1.1 Exponentiation

Definition 7.1.9. For a a real number and n a positive integer, define the real number a^n recursively as follows:

$$a^1 = a$$
$$a^{n+1} = a^n a$$

Theorem 7.1.10. *For all $a \in \mathbb{R}$ and $m, n \in mathbbZ_+$, we have*

$$a^n a^m = a^{n+m}$$

PROOF:
⟨1⟩1. LET: $P(m)$ be the property $\forall a \in \mathbb{R}.\forall n \in \mathbb{Z}_+.a^n a^m = a^{n+m}$
⟨1⟩2. $P(1)$
PROOF: $a^n a^1 = a^n a = a^{n+1}$.
⟨1⟩3. $\forall m \in \mathbb{Z}_+.P(m) \Rightarrow P(m+1)$
⟨2⟩1. LET: m be a positive integer.
⟨2⟩2. ASSUME: $P(m)$
⟨2⟩3. LET: $a \in \mathbb{R}$
⟨2⟩4. LET: $n \in \mathbb{Z}_+$
⟨2⟩5. $a^n a^{m+1} = a^{n+m+1}$
PROOF:

$$\begin{aligned} a^n a^{m+1} &= a^n a^m a \\ &= a^{n+m} a && (\langle 2\rangle 2) \\ &= a^{n+m+1} \end{aligned}$$

⟨1⟩4. Q.E.D.
PROOF: By induction.
□

Theorem 7.1.11. *For all $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}_+$,*

$$(a^n)^m = a^{nm} \quad .$$

PROOF:
⟨1⟩1. LET: $P(m)$ be the property $\forall a \in \mathbb{R}.\forall n \in \mathbb{Z}_+.(a^n)^m = a^{nm}$.
⟨1⟩2. $P(1)$
PROOF: $(a^n)^1 = a^n = a^{n \cdot 1}$

⟨1⟩3. $\forall m \in \mathbb{Z}_+ . P(m) \Rightarrow P(m+1)$
PROOF:

$$\begin{aligned}(a^n)^{m+1} &= (a^n)^m a^n \\ &= a^{nm} a^n \\ &= a^{nm+n} && \text{(Theorem 7.1.10)} \\ &= a^{n(m+1)}\end{aligned}$$

□

Theorem 7.1.12. *For any real numbers a and b and positive integer m,*

$$a^m b^m = (ab)^m \quad .$$

PROOF: Induction on m. □

7.2 Integers

Definition 7.2.1 (Integer)**.** The set $\mathbb{Z}$ of *integers* is

$$\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \{-x : x \in \mathbb{Z}_+\} \quad .$$

Proposition 7.2.2. *The sum, difference and product of two integers is an integer.*

PROOF: Easy. □

Example 7.2.3. 1/2 is not an integer.

Proposition 7.2.4. *For any integer n, there is no integer a such that $n < a < n+1$.*

PROOF:
⟨1⟩1. For any positive integer n, there is no integer a such that $n < a < n+1$.
⟨2⟩1. There is no integer a such that $1 < a < 2$.
⟨3⟩1. There is no positive integer a such that $1 < a < 2$.
⟨4⟩1. We do not have $1 < 1 < 2$.
⟨4⟩2. For any positive integer n, we do not have $1 < n+1 < 2$.
PROOF: Since $n \geqslant 1$ so $n+1 \geqslant 2$.
⟨3⟩2. We do not have $1 < 0 < 2$.
⟨3⟩3. For any positive integer a, we do not have $1 < -a < 2$.
PROOF: Since $-a < 0 < 1$.
⟨2⟩2. For any positive integer n, if there is no integer a such that $n < a < n+1$, then there is no integer a such that $n+1 < a < n+2$.
PROOF: If $n+1 < a < n+2$ then $n < a-1 < n+1$.
⟨1⟩2. There is no integer a such that $0 < a < 1$.
PROOF: If $0 < a < 1$ then $1 < a+1 < 2$.
⟨1⟩3. For any positive integer n, there is no integer a such that $-n < a < -n+1$.
PROOF: If $-n < a < -n+1$ then $n-1 < -a < n$.

□

Theorem 7.2.5. *Every nonempty subset of $\mathbb{Z}$ bounded above has a largest element.*

PROOF:
⟨1⟩1. LET: S be a nonempty subset of $\mathbb{Z}$ bounded above.
⟨1⟩2. LET: u be an upper bound for S.
⟨1⟩3. PICK an integer $n > u$
  PROOF: Archimedean property.
⟨1⟩4. LET: k be the least positive integer such that $n - k \in S$.
  ⟨2⟩1. PICK $m \in S$
  ⟨2⟩2. $n - m$ is a positive integer.
  ⟨2⟩3. There exists a positive integer k such that $n - k \in S$.
⟨1⟩5. $n - k$ is the greatest element in S.
  ⟨2⟩1. LET: $m \in S$
  ⟨2⟩2. $n - m \geqslant k$
  ⟨2⟩3. $m \leqslant n - k$
□

Theorem 7.2.6. *For any real number x, if x is not an integer then there exists a unique integer n such that $n < x < n + 1$.*

PROOF:
⟨1⟩1. $\{n \in \mathbb{Z} : n < x\}$ is a nonempty set of integers bounded above.
  ⟨2⟩1. PICK $m > -x$
    PROOF: Archimedean property.
  ⟨2⟩2. $-m < x$
  ⟨2⟩3. $\{n \in \mathbb{Z} : n < x\}$ is nonempty.
⟨1⟩2. LET: n be the greatest integer such that $n < x$
⟨1⟩3. $x < n + 1$
⟨1⟩4. If n' is an integer with $n' < x < n' + 1$ then $n' = n$.
  PROOF: We have $n' < n + 1$ so $n' \leqslant n$, and $n < n' + 1$ so $n \leqslant n'$.
□

Definition 7.2.7 (Even)**.** An integer n is *even* iff $n/2$ is an integer; otherwise, n is *odd*.

Theorem 7.2.8. *If the integer m is odd then there exists an integer n such that $m = 2n + 1$.*

PROOF:
⟨1⟩1. LET: n be the integer such that $n < m/2 < n + 1$
  PROOF: Theorem 7.2.6.
⟨1⟩2. $2n < m < 2n + 2$
⟨1⟩3. $m = 2n + 1$
□

Theorem 7.2.9. *The product of two odd integers is odd.*

PROOF: $(2m+1)(2n+1) = 2(2mn+m+n)+1$. □

Corollary 7.2.9.1. *If p is an odd integer and n is a positive integer then p^n is an odd integer.*

Definition 7.2.10 (Exponentiation)**.** Extend the definition of exponentiation so a^n is defined for:

- all real numbers a and non-negative integers n
- all non-zero real numbers a and integers n

as follows:

$$a^0 = 1$$
$$a^{-n} = 1/a^n \qquad (n \text{ a positive integer})$$

Theorem 7.2.11 (Laws of Exponents)**.** *For all non-zero reals a and b and integers m and n,*

$$a^n a^m = a^{n+m}$$
$$(a^n)^m = a^{nm}$$
$$a^m b^m = (ab)^m$$

PROOF: Easy. □

Theorem 7.2.12. *$\mathbb{Z}$ is countable.*

PROOF: The function that maps an integer n to $2n$ if $n \geqslant 0$ and $-1-2n$ if $n < 0$ is a bijection $\mathbb{Z} \approx \mathbb{N}$. □

7.3 Rational Numbers

Definition 7.3.1 (Rational Number)**.** The set $\mathbb{Q}$ of *rational numbers* is the set of all real numbers that are the quotient of two integers. A real that is not rational is *irrational*.

Theorem 7.3.2. *$\sqrt{2}$ is irrational.*

PROOF:
⟨1⟩1. For any positive rational a, there exist positive integers m and n not both even such that $a = m/n$.
  ⟨2⟩1. LET: a be a positive rational.
  ⟨2⟩2. LET: n be the least positive integer such that na is a positive integer.
  ⟨2⟩3. LET: $m = na$
  ⟨2⟩4. ASSUME: for a contradiction m and n are both even.
  ⟨2⟩5. $m/2 = (n/2)a$
  ⟨2⟩6. Q.E.D.

PROOF: This contradicts the leastness of n ($\langle 2\rangle 2$).
$\langle 1\rangle 2$. ASSUME: for a contradiction $\sqrt{2}$ is rational.
$\langle 1\rangle 3$. PICK positive integers m and n not both even such that $\sqrt{2} = m/n$.
$\langle 1\rangle 4$. $m^2 = 2n^2$
$\langle 1\rangle 5$. m^2 is even.
$\langle 1\rangle 6$. m is even.
PROOF: Theorem 7.2.9.
$\langle 1\rangle 7$. LET: $k = m/2$
$\langle 1\rangle 8$. $4k^2 = 2n^2$
$\langle 1\rangle 9$. $n^2 = 2k^2$
$\langle 1\rangle 10$. n^2 is even.
$\langle 1\rangle 11$. n is even.
PROOF: Theorem 7.2.9.
$\langle 1\rangle 12$. Q.E.D.
PROOF: $\langle 1\rangle 3$, $\langle 1\rangle 6$ and $\langle 1\rangle 11$ form a contradiction.
□

Theorem 7.3.3. $\mathbb{Q}$ *is countably infinite.*

PROOF: The function $\mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ that maps (m, n) to $m/(n+1)$ is a surjection.
□

7.4 Algebraic Numbers

Definition 7.4.1 (Algebraic Number)**.** A real number r is *algebraic* iff there exists a natural number n and rational numbers a_0, a_1, ..., a_{n-1} such that

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Otherwise, r is *transcendental*.

Proposition 7.4.2. *The set of algebraic numbers is countably infinite.*

PROOF: There are countably many finite sequences of rational numbers, and each corresponding polynomial has only finitely many roots. □

Corollary 7.4.2.1. *The set of transcendental numbers is uncountable.*

Part IV

Algebra

Chapter 8

Monoid Theory

Definition 8.0.1 (Monoid)**.** A *monoid* is a category with one object.

Definition 8.0.2. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. The monoid $\mathrm{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \to X$ under composition.

Proposition 8.0.3. *For any functor $F : \mathcal{C} \to \mathcal{D}$ and $X \in \mathcal{C}$, we have that $F : \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{D}}(FX)$ is a monoid homomorphism.*

PROOF: Since $F\mathrm{id}_X = \mathrm{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$. □

Chapter 9

Group Theory

9.1 Category of Small Groups

Definition 9.1.1. Let **Grp** be the category of small groups and group homomorphisms.

Definition 9.1.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G.

Proposition 9.1.3. *The trivial group is a zero object in* **Grp**.

PROOF: Easy. $\square$

The zero morphism $G \to H$ maps every element in G to e.

Definition 9.1.4. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. We write $\mathrm{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Proposition 9.1.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $X \in \mathcal{C}$. Then $F : \mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{D}}(FX)$ is a group homomorphism.*

PROOF: Since $F\mathrm{id}_X = \mathrm{id}_{FX}$, $F(g \circ f) = Fg \circ Ff$, and $Ff^{-1} = (Ff)^{-1}$. $\square$

Proposition 9.1.6. **Grp** *has products.*

Definition 9.1.7 (Free Product)**.** The product of a family of groups in **Grp** is called the *free product*.

Proposition 9.1.8. **Ab** *has products given by direct sums.*

Definition 9.1.9 (Left Coset)**.** Let G be a group and H a subgroup of G. The *left cosets* of H are the sets of the form

$$xH := \{xh : h \in H\}$$

We write G/H for the set of left cosets of H in G.

Proposition 9.1.10. *Let G be a group and H a subgroup of G. Then G/H is a partition of G.*

PROOF:
$\langle 1\rangle 1.$ $\bigcup(G/H) = G$
PROOF: Since $x = xe$ and so $x \in xH$.
$\langle 1\rangle 2.$ Any two distinct left cosets of H are disjoint.
PROOF: Since if $z \in xH$ and $z \in yH$ then $xH = yH = zH$.
$\square$

Definition 9.1.11. Let G be a group. Let A and B be subsets of G. Then

$$AB := \{ab : a \in A, b \in B\} \ .$$

Definition 9.1.12. Let G be a group. Let A be a subset of G. Then

$$A^{-1} := \{a^{-1} : a \in A\} \ .$$

Chapter 10

Ring Theory

Definition 10.0.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Definition 10.0.2 (Spectrum)**.** Let R be a commutative ring. The *spectrum* of R, $\operatorname{spec} R$, is the set of all prime ideals of R.

Definition 10.0.3 (Zariski Topology)**.** Let R be a commutative ring. The *Zariski topology* on $\operatorname{spec} R$ is the topology where the closed sets are the sets of the form

$$VE := \{p \in \operatorname{spec} R : E \subseteq p\}$$

for any $E \in \mathcal{P}R$.

We prove this is a topology.

Proof:
⟨1⟩1. Let: $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$
⟨1⟩2. For all $\mathcal{A} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{A} \in \mathcal{C}$
  ⟨2⟩1. Let: $\mathcal{A} \subseteq \mathcal{C}$
  ⟨2⟩2. Let: $E = \bigcup\{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$
     Prove: $VE = \bigcap \mathcal{A}$
  ⟨2⟩3. For all $p \in \operatorname{spec} R$, if $E \subseteq p$ then $p \in \bigcap \mathcal{A}$
    ⟨3⟩1. Let: $p \in \operatorname{spec} R$
    ⟨3⟩2. Assume: $E \subseteq p$
    ⟨3⟩3. Let: $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$
    ⟨3⟩4. $E' \subseteq E$
    ⟨3⟩5. $E' \subseteq p$
    ⟨3⟩6. $p \in VE'$
  ⟨2⟩4. For all $p \in \operatorname{spec} R$, if $p \in \bigcap \mathcal{A}$ then $E \subseteq p$
    ⟨3⟩1. Let: $p \in \bigcap \mathcal{A}$
    ⟨3⟩2. For all $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$ we have $E' \subseteq p$
    ⟨3⟩3. $E \subseteq p$
⟨1⟩3. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.
  Proof: Since $VE \cup VE' = V(E \cap E')$

⟨1⟩4. $\varnothing \in \mathcal{C}$

⟨2⟩1. $VR = \varnothing$

PROOF: If $p \in VR$ then $R \subseteq p$ contradicting the fact that p is a prime ideal.

□

Definition 10.0.4. For any ring R, let $R - \mathbf{Mod}$ be the category of small R-modules and R-module homomorphisms.

Proposition 10.0.5. $R - \mathbf{Mod}$ *has products and coproducts.*

Chapter 11

Field Theory

Proposition 11.0.1. Field *does not have binary products.*

PROOF: There cannot be a field K with field homomorphisms $K \to \mathbb{Z}_2$ and $K \to \mathbb{Z}_3$, because its characteristic would be both 2 and 3. □

Chapter 12

Linear Algebra

Definition 12.0.1 (Span)**.** Let V be a vector space and $A \subseteq V$. The *span* of A is the set of all linear combinations of elements of A.

Definition 12.0.2 (Independent)**.** Let V be a vector space and $A \subseteq V$. Then A is *linearly independent* iff, whenever

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

where $v_1, \ldots, v_n \in A$, then

$$\alpha_1 = \cdots = \alpha_n = 0 \ .$$

Proposition 12.0.3. *Let V be a vector space, $A \subseteq V$ and $v \in V$. If A is linearly independent and $v \notin \operatorname{span} A$, then $A \cup \{v\}$ is independent.*

PROOF:
$\langle 1 \rangle$1. LET: $\alpha_1 v_1 + \cdots + \alpha_n v_n + \beta v = 0$ where $v_1, \ldots, v_n \in A$
$\langle 1 \rangle$2. $\beta = 0$
PROOF: Otherwise $v = (\alpha_1/\beta)v_1 + \cdots + (\alpha_n/\beta)v_n \in \operatorname{span} A$.
$\langle 1 \rangle$3. $\alpha_1 = \cdots = \alpha_n = 0$
PROOF: Since A is linearly independent.
□

Theorem 12.0.4. *Every vector space has a basis.*

PROOF:
$\langle 1 \rangle$1. LET: V be a vector space.
$\langle 1 \rangle$2. PICK a maximal linearly independent set $\mathcal{B}$.
PROOF: By Tukey's Lemma.
$\langle 1 \rangle$3. $\operatorname{span} \mathcal{B} = V$
PROOF: Proposition 12.0.3.
□

Definition 12.0.5. For any field K, we write $\mathbf{Vect}_K$ for $K - \mathbf{Mod}$.

Dual space functor $\mathbf{Vect}_K^{\mathrm{op}} \to \mathbf{Vect}_K$.

Definition 12.0.6 (Invariant). Let $T : V \to V$ be a linear operator and S a subspace of V. Then S is *invariant* under T iff $T(S) \subseteq S$.

12.0.1 Commutator

Definition 12.0.7 (Commutator). Let $S, T : V \to V$ be linear transformations. The *commutator* of S and T is

$$[S, T] = ST - TS$$

12.1 Inner Product Spaces

Definition 12.1.1 (Inner Product). Let V be a complex vector space. An *inner product* on V is a function $\langle\ |\ \rangle : V^2 \to \mathbb{C}$ such that, for all $|\phi\rangle, |\psi\rangle, |\psi_1\rangle, |\psi_2\rangle \in V$ and $c_1, c_2 \in \mathbb{C}$,

1. $\langle\phi|c_1\psi_1 + c_2\psi_2\rangle = c_1\langle\phi|\psi_1\rangle + c_2\langle\phi|\psi_2\rangle$
2. $\langle\psi|\phi\rangle = \overline{\langle\phi|\psi\rangle}$
3. $\langle\psi|\psi\rangle \geqslant 0$
4. If $\langle\psi|\psi\rangle = 0$ then $|\psi\rangle = 0$.

An *inner product space* is a complex vector space with an inner product.

Example 12.1.2. The function $\langle\ |\ \rangle : (\mathbb{C}^n)^2 \to \mathbb{C}$ defined by

$$\langle(a_1, \ldots, a_n)|(b_1, \ldots, b_n)\rangle = \overline{a_1}b_1 + \cdots + \overline{a_n}b_n$$

is an inner product on $\mathbb{C}^n$.

Definition 12.1.3 (Normalised). Let V be an inner product space and $|\psi\rangle \in V$. Then $|\psi\rangle$ is *normalised* iff $\langle\psi|\psi\rangle = 1$.

Definition 12.1.4 (Orthogonal). Let V be an inner product space and $|\phi\rangle, |\psi\rangle \in V$. Then $|\phi\rangle$ and $|\psi\rangle$ ale *orthogonal* iff $\langle\phi|\psi\rangle = 0$.

Definition 12.1.5 (Orthonormal). Let V be an inner product space and $S \subseteq V$. Then S is *orthonormal* iff:

- Every element of S is normalised.
- Any two distinct elements of S are orthogonal.

Definition 12.1.6 (Hermitian). Let V be an inner product space. A linear operator $T : V \to V$ is *Hermitian* iff

$$\langle\psi|T|\phi\rangle = \overline{\langle\phi|T|\psi\rangle}$$

for all $|\phi\rangle, |\psi\rangle \in V$.

Proposition 12.1.7. *Every eigenvalue of a Hermitian operator is real.*

PROOF:
$\langle 1\rangle 1.$ LET: $T : V \to V$ be Hermitian.
$\langle 1\rangle 2.$ LET: $|\phi\rangle$ be an eigenvector with eigenvalue α.
$\langle 1\rangle 3.$ $\langle\phi|T|\phi\rangle = \alpha\langle\phi|\phi\rangle$
$\langle 1\rangle 4.$ $\langle\phi|T|\phi\rangle = \overline{\alpha}\langle\phi|\phi\rangle$
PROOF:

$$\begin{aligned}\langle\phi|T|\phi\rangle &= \overline{\langle\phi|T|\phi\rangle} \\ &= \overline{\alpha}\overline{\langle\phi|\phi\rangle} \\ &= \overline{\alpha}\langle\phi|\phi\rangle\end{aligned}$$

$\langle 1\rangle 5.$ $\alpha = \overline{\alpha}$
∎

Proposition 12.1.8. *If $S, T : V \to V$ are Hermitian then so is $i[S, T]$.*

PROOF:

$$\begin{aligned}\langle\psi|i[S,T]|\phi\rangle &= i\langle\psi|ST|\phi\rangle - i\langle\psi|TS|\phi\rangle \\ &= i\overline{\langle T|\phi\rangle|S|\psi\rangle} - i\overline{\langle S|\phi\rangle|T|\psi\rangle} \\ &= i\langle S|\psi\rangle|T|\phi\rangle - i\langle T|\psi\rangle|S|\phi\rangle \\ &= i\overline{\langle\phi|TS|\psi\rangle} - i\overline{\langle\phi|ST|\psi\rangle} \\ &= -i\overline{\langle\phi|[S,T]|\psi\rangle} \\ &= \overline{\langle\phi|i[S,T]|\psi\rangle}\end{aligned}$$

∎

Definition 12.1.9 (Hermitian Conjugate)**.** Let V be an inner product space and $T : V \to V$ a linear operator. The *Hermitian conjugate* $T^\dagger : V \to V$ is the linear operator such that, for all $|\phi\rangle, |\psi\rangle \in V$, we have

$$\langle\phi|T^\dagger|\psi\rangle = \overline{\langle\psi|T|\phi\rangle} \ .$$

Proposition 12.1.10.

$$(cT)^\dagger = \overline{c}T^\dagger$$

PROOF:

$$\begin{aligned}\langle\phi|\overline{c}T^\dagger|\psi\rangle &= \overline{c}\langle\phi|T^\dagger|\psi\rangle \\ &= \overline{c}\overline{\langle\psi|T|\phi\rangle} \\ &= \overline{\langle\psi|cT|\phi\rangle}\end{aligned}$$

∎

Proposition 12.1.11.

$$(ST)^\dagger = T^\dagger S^\dagger$$

PROOF:

$$\begin{aligned}\langle\phi|T^\dagger S^\dagger|\psi\rangle &= \overline{\langle S^\dagger|\psi\rangle|T|\phi\rangle\rangle} \\ &= \langle T|\phi\rangle|S^\dagger|\psi\rangle \\ &= \overline{\langle\psi|ST|\phi\rangle}\end{aligned}$$

∎

Proposition 12.1.12. *T is Hermitian iff $T = T^\dagger$.*

PROOF: Immediate from definitions. □

Proposition 12.1.13. *If M is the matrix of T then $\overline{M}^{\mathrm{T}}$ is the matrix of $T^\dagger$.*

Definition 12.1.14 (Unitary)**.** Let V be an inner product space and $T : V \to V$ a linear operator. Then T is *unitary* iff $T^\dagger T = I$.

Proposition 12.1.15. *If α is an eigenvalue of a unitary operator then $|\alpha| = 1$.*

PROOF:
⟨1⟩1. LET: $T : V \to V$ be unitary.
⟨1⟩2. LET: $|\phi\rangle$ be an eigenstate with eigenvalue α.
⟨1⟩3. $|\alpha|^2\langle\phi|\phi\rangle = \langle\phi|\phi\rangle$
PROOF:

$$\begin{aligned}\langle\phi|\phi\rangle &= \langle\phi|T^\dagger T|\phi\rangle \\ &= \alpha\langle\phi|T^\dagger|\phi\rangle \\ &= \alpha\overline{\langle\phi|T|\phi\rangle} \\ &= \alpha\overline{\alpha}\overline{\langle\phi|\phi\rangle} \\ &= |\alpha|^2\langle\phi|\phi\rangle\end{aligned}$$

⟨1⟩4. $|\alpha| = 1$
□

Definition 12.1.16 (Bra Vector)**.** Let V be an inner product space and $|\phi\rangle \in V$. The *bra vector* or *Hermitian conjugate* $\langle\phi|$ is the linear functional $\langle\phi| : V \to \mathbb{C}$ that maps $|\psi\rangle$ to $\langle\phi|\psi\rangle$.

Proposition 12.1.17. *The hermitian conjugate of $c\,|\phi\rangle$ is $\overline{c}\langle\phi|$.*

PROOF: Since $\langle c\,|\phi\rangle\,|\psi\rangle = \overline{c}\langle\phi|\psi\rangle$. □

Proposition 12.1.18. *The Hermitian conjugate of $T\,|\phi\rangle$ is $\langle\phi|T^\dagger$, the linear functional that maps $|\psi\rangle$ to $\langle\phi|T^\dagger|\psi\rangle$.*

PROOF:

$$\begin{aligned}\langle T\,|\phi\rangle\,|\psi\rangle &= \overline{\langle\psi|T|\phi\rangle} \\ &= \langle\phi|T^\dagger|\psi\rangle\end{aligned}$$

□

Definition 12.1.19 (Outer Product)**.** Let $|\phi\rangle, |\psi\rangle \in V$. The *outer product* $|\psi\rangle\langle\phi| : V \to \mathbb{C}$ is the linear functional defined by

$$(|\psi\rangle\langle\phi|)\,|\chi\rangle = \langle\phi|\chi\rangle\,|\psi\rangle \quad .$$

Proposition 12.1.20. *Let $\{|\psi_1\rangle, |\psi_2\rangle, \ldots\}$ be a countable orthonormal basis. Then*

$$\sum_{n=1}^{\infty} |\psi_n\rangle\langle\psi_n| = I \quad .$$

Part V

Analysis

Chapter 13

Real Analysis

13.1 Hermite Polynomials

Definition 13.1.1 (Hermite Polynomials)**.** For $n \in \mathbb{N}$, define a sequence of natural numbers $a_n, a_{n-2}, \ldots$ by

$$\begin{aligned} a_n &= 2 \\ a_{k-2} &= -\frac{k(k-1)}{2(n-k+2)} a_k \end{aligned}$$

with the final entry being a_1^n if n is odd and a_0 if n is even.

The nth *Hermite polynomial* H_n is the polynomial

$$H_n(x) = \sum_k a_k x^k \quad .$$

Example 13.1.2.

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \\ H_5(x) &= 32x^5 - 160x^3 + 120x \end{aligned}$$

Proposition 13.1.3.

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Proposition 13.1.4 (Rodrigues Formula)**.**

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Proposition 13.1.5.

$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x)$$

Proposition 13.1.6.

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) = e^{-z^2+2zx}$$

13.2 Fourier Transforms

Definition 13.2.1 (Fourier Transform)**.** The *Fourier transform* of a function f is

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{-ixt}dt \ \ .$$

Definition 13.2.2 (Fourier Transform)**.** The *inverse Fourier transform* of a function F is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(t)e^{ixt}dt \ \ .$$

Theorem 13.2.3 (Plancherel's Theorem)**.** *F is the Fourier transform of f if and only if f is the inverse Fourier transform of F.*

Part VI

Topology

Chapter 14

Topological Spaces

14.1 Topological Spaces

Definition 14.1.1 (Topological Space)**.** Let X be a set and $\mathcal{T} \subseteq \mathcal{P}X$. Then we say $(X, \mathcal{T})$ is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For any $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.
- $X \in \mathcal{T}$

We call $\mathcal{T}$ the *topology* of the toplogical space, and call its elements *open* sets. We shall often write X for the topological space $(X, \mathcal{T})$.

Example 14.1.2 (Discrete Topology)**.** For any set X, the power set $\mathcal{P}X$ is called the *discrete* topology on X.

Example 14.1.3 (Indiscrete Topology)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Example 14.1.4 (Cofinite Topology)**.** For any set X, the *cofinite* topology is $\mathcal{T} = \{\emptyset\} \cup \{X - U : U \subseteq X \text{ is finite}\}$.

We prove this is a topology.

Example 14.1.5 (Cocountable Topology)**.** For any set X, the *cocountable* topology is $\{X - U : U \subseteq X \text{ is countable}\}$.

Example 14.1.6 (Sierpiński Two-Point Space)**.** The *Sierpiński two-point space* is $\{0, 1\}$ under the topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Proposition 14.1.7. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

PROOF:

$\langle 1\rangle 1$. If U is open then, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.

PROOF: Take $V = U$.

$\langle 1\rangle 2$. If, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$, then U is open.

PROOF: Since then U is the union of all the open subsets of U.

□

Proposition 14.1.8. *The intersection of a set of topologies on a set X is a topology on X.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{T}$ be a set of topologies on X.
$\langle 1\rangle 2$. For all $\mathcal{U} \subseteq \bigcap \mathcal{T}$ we have $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$.
  $\langle 2\rangle 1$. LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
  $\langle 2\rangle 2$. LET: $T \in \mathcal{T}$
  $\langle 2\rangle 3$. $\mathcal{U} \subseteq T$
  $\langle 2\rangle 4$. $\bigcup \mathcal{U} \in T$
$\langle 1\rangle 3$. For all $U, V \in \bigcap \mathcal{T}$ we have $U \cap T \in \bigcap \mathcal{T}$.
  $\langle 2\rangle 1$. LET: $U, V \in \bigcap \mathcal{T}$
  $\langle 2\rangle 2$. LET: $T \in \mathcal{T}$
  $\langle 2\rangle 3$. $U, V \in T$
  $\langle 2\rangle 4$. $U \cap V \in T$
$\langle 1\rangle 4$. $X \in \bigcap \mathcal{T}$.

□

Definition 14.1.9 (Finer, Coarser)**.** Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the set X. Then $\mathcal{T}$ is *coarser*, *smaller* or *weaker* than $\mathcal{T}'$, or $\mathcal{T}'$ is *finer*, *larger* or *weaker* than $\mathcal{T}$, iff $\mathcal{T} \subseteq \mathcal{T}'$.

14.2 Closed Sets

Definition 14.2.1 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 14.2.2. *A set B is open if and only if $X - B$ is closed.*

PROOF: We have B is open iff $X - (X - B)$ is open iff $X - B$ is closed. □

Theorem 14.2.3. *Let X be a set. Let $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology on X such that $\mathcal{C}$ is the set of closed sets if and only if:*

1. $\emptyset \in \mathcal{C}$
2. $\forall \mathcal{A} \subseteq \mathcal{C}. \bigcap \mathcal{A} \in \mathcal{C}$
3. $\forall C, D \in \mathcal{C}. C \cup D \in \mathcal{C}$

In this case, the topology is unique, and is $\{X - C : C \in \mathcal{C}\}$.

PROOF:
⟨1⟩1. In any topology on X we have $\emptyset$ is closed.
PROOF: Since $X - \emptyset = X$ is open.
⟨1⟩2. In any topology on X, the intersection of a set $\mathcal{A}$ of closed sets is closed.
PROOF: Since $X - \bigcap \mathcal{A} = \bigcup_{A \in \mathcal{A}} (X - A)$ is open.
⟨1⟩3. In any topology on X, the union of two closed sets is closed.
PROOF: For any closed sets C and D, we have $X - (C \cup D) = (X - C) \cap (X - D)$ is open.
⟨1⟩4. If $\mathcal{C}$ is a set satisfying 1–3, then $\{X - C : C \in \mathcal{C}\}$ is a topology on X with respect to which $\mathcal{C}$ is the set of closed sets.
⟨2⟩1. LET: $\mathcal{C}$ be a set satisfying 1–3.
⟨2⟩2. LET: $\mathcal{T} = \{X - C : C \in \mathcal{C}\}$
⟨2⟩3. For all $U \in \mathcal{T}$ we have $X - U \in \mathcal{C}$.
⟨2⟩4. $\mathcal{T}$ is a topology on X.
⟨3⟩1. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
⟨4⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
⟨4⟩2. For all $U \in \mathcal{U}$ we have $X - U \in \mathcal{C}$.
⟨4⟩3. $X - \bigcup \mathcal{U} \in \mathcal{C}$
PROOF:
$$\begin{aligned} X - \bigcup \mathcal{U} &= \bigcap_{U \in \mathcal{U}} (X - U) \\ &\in \mathcal{C} \end{aligned}$$
⟨4⟩4. $\bigcup \mathcal{U} \in \mathcal{T}$
⟨3⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.
⟨3⟩3. $X \in \mathcal{T}$
⟨2⟩5. For any set C we have $C \in \mathcal{C}$ iff C is closed with respect to $\mathcal{T}$.
⟨1⟩5. If $\mathcal{T}$ is any topology on X then $\mathcal{T} = \{X - C : C \text{ is closed in } \mathcal{T}\}$.
PROOF: Proposition 14.2.2.
□

14.3 Neighbourhoods

Definition 14.3.1 (Neighbourhood)**.** Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x, and x is an *interior* point of U, iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 14.3.2. *A set B is open if and only if it is a neighbourhood of each of its points.*

PROOF: This is Proposition 14.1.7. □

Proposition 14.3.3. *Let X be a set and $\mathcal{N} : X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that, for all $x \in X$, we have $\mathcal{N}_x$ is the set of neighbourhoods of x, if and only if:*

- *For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$*

- *For all* $x \in X$ *we have* $X \in \mathcal{N}_x$
- *For all* $x \in X$, $N \in \mathcal{N}_x$ *and* $V \subseteq \mathcal{P}X$, *if* $N \subseteq V$ *then* $V \in \mathcal{N}_x$
- *For all* $x \in X$ *and* $M, N \in \mathcal{N}_x$ *we have* $M \cap N \in \mathcal{N}_x$
- *For all* $x \in X$ *and* $N \in \mathcal{N}_x$, *there exists* $M \in \mathcal{N}_x$ *such that* $M \subseteq N$ *and* $\forall y \in M.M \in \mathcal{N}_y$.

In this case, $\mathcal{O}$ *is unique and is given by* $\mathcal{O} = \{U : \forall x \in U.U \in \mathcal{N}_x\}$.

PROOF: Straightforward. □

14.4 Interior

Definition 14.4.1 (Interior)**.** The interior of B is the union of all the open sets included in B.

14.5 Closure

Definition 14.5.1 (Closure)**.** Let X be a topological space and $B \subseteq X$. The *closure* of B, $\overline{B}$, is the intersection of all the closed sets that include B.

Proposition 14.5.2. *A set* C *is closed if and only if* $C = \overline{C}$.

PROOF: Easy. □

Corollary 14.5.2.1. *A set* B *is open iff* $X - B = \overline{X - B}$.

Proposition 14.5.3 (Kuratowski Closure Axioms)**.** *Let* X *be a set and* $\overline{} : \mathcal{P}X \to \mathcal{P}X$. *Then there exists a topology* $\mathcal{O}$ *such that, for all* $B \subseteq X$, $\overline{B}$ *is the closure of* B, *if and only if:*

- $\overline{\emptyset} = \emptyset$
- *For all* $A \subseteq X$ *we have* $A \subseteq \overline{A}$
- *For all* $A \subseteq X$ *we have* $\overline{\overline{A}} = \overline{A}$
- *For all* $A, B \subseteq X$ *we have* $\overline{A \cup B} = \overline{A} \cup \overline{B}$

In this case, $\mathcal{O}$ *is unique and is defined by* $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

PROOF: Straightforward. □

14.6 Bases

Definition 14.6.1 (Basis)**.** Let X be a topological space. A *basis* for the topology on X is a set of open sets $\mathcal{B}$ such that every open set is the union of a subset of $\mathcal{B}$. The elements of $\mathcal{B}$ are called *basic open neighbourhoods* of their elements.

Example 14.6.2. Let X be a set. The set of all one-element subsets of X is a basis for the discrete topology on X.

Proposition 14.6.3. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Then the topology on X is the coarsest topology that includes $\mathcal{B}$.*

Proposition 14.6.4. *Let X and Y be topological spaces. Let $\mathcal{B}$ be a basis for the topology on X and $\mathcal{C}$ a basis for the topology on Y. Then*

$$\{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Theorem 14.6.5. *There are infinitely many primes.*

Furstenberg's proof:
PROOF:
$\langle 1 \rangle 1$. For $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$,
LET: $S(a, b) := \{an + b : n \in \mathbb{N}\}$
$\langle 1 \rangle 2$. LET: $\mathcal{T}$ be the topology generated by the basis $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$
$\langle 2 \rangle 1$. For every $n \in \mathbb{Z}$, there exist a, b such that $n \in S(a, b)$.
PROOF: $n \in S(n, 0)$
$\langle 2 \rangle 2$. If $n \in S(a_1, b_1) \cap S(a_2, b_2)$ then there exist a_3, b_3 such that $n \in S(a_3, b_3) \subseteq S_(a_1, b_1) \cap S(a_2, b_2)$
$\langle 3 \rangle 1$. LET: $d = \mathrm{lcm}(a_1, a_2)$
PROVE: $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$
$\langle 3 \rangle 2$. LET: $d = a_1 k = a_2 l$
$\langle 3 \rangle 3$. LET: $n = a_1 c + b_1 = a_2 d + b_2$
$\langle 3 \rangle 4$. LET: $z \in \mathbb{Z}$
PROVE: $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$
$\langle 3 \rangle 5$. $dz + n \in S(a_1, b_1)$
PROOF:

$$\begin{aligned} dz + n &= a_1 kz + a_1 c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

$\langle 3 \rangle 6$. $dz + n \in S(a_2, b_2)$
PROOF: Similar.
$\langle 1 \rangle 3$. For all $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$ we have $S(a, b)$ is closed.
$\langle 2 \rangle 1$. LET: $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$
$\langle 2 \rangle 2$. LET: $n \in \mathbb{Z} - S(a, b)$
$\langle 2 \rangle 3$. $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$
$\langle 3 \rangle 1$. LET: $x \in S(a, n)$

⟨3⟩2. ASSUME: for a contradiction $x \in S(a,b)$
⟨3⟩3. PICK m such that $x = am + b$
⟨3⟩4. PICK l such that $x = al + n$
⟨3⟩5. $n = a(m-l) + b$
⟨3⟩6. $n \in S(a,b)$
⟨3⟩7. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
⟨1⟩4.
$$\mathbb{Z} - \{1,-1\} = \bigcup_{p \text{ prime}} S(p,0)$$
PROOF: Since every integer except 1 and −1 is divisible by a prime.
⟨1⟩5. No nonempty finite set is open.
⟨2⟩1. LET: U be a nonempty open set
⟨2⟩2. PICK $n \in U$
⟨2⟩3. There exist a, b such that $n \in S(a,b) \subseteq U$
⟨2⟩4. U is infinite.
⟨1⟩6. $\mathbb{Z} - \{1,-1\}$ is not closed.
⟨1⟩7. $\bigcup_{p \text{ prime}} S(p,0)$ is not closed.
⟨1⟩8. The union of finitely many closed sets is closed.
⟨1⟩9. There are infinitely many primes.
□

14.7 Order Topology

Definition 14.7.1 (Order Topology)**.** Let X be a linearly ordered set. The *order topology* on X is the topology generated by the open interval (a,b) as well as the open rays $(a,+\infty)$ and $(-\infty,b)$ for $a,b \in X$.

The *standard topology* on $\mathbb{R}$ is the order topology.

Proposition 14.7.2. *Let X be a linearly ordered set. Then the order topology is generated by the basis consisting of:*

- *all open intervals (a,b)*
- *all intervals of the form $[\bot,b)$ where $\bot$ is the least element of X, if any*
- *all intervals of the form $(a,\top]$ where $\top$ is the greatest element of X, if any.*

Proposition 14.7.3. *Let X be a linearly ordered set. The open rays in X form a subbasis for the order topology.*

Definition 14.7.4 (Lower Limit Topology)**.** The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals $[a,b)$.

We write $\mathbb{R}_l$ for $\mathbb{R}$ under the lower limit topology.

Definition 14.7.5 (K-topology)**.** Let $K = \{1/n : n \in \mathbb{Z}_+\}$. The *$K$-topology* on $\mathbb{R}$ is the topology generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) - K$.

We write $\mathbb{R}_K$ for $\mathbb{R}$ under the K -topology.

Proposition 14.7.6. *Let X be a linearly ordered set under the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology.*

PROOF:
⟨1⟩1. The order topology is coarser than the subspace topology.
⟨2⟩1. For all $a \in Y$, the open ray $\{y \in Y : a < y\}$ is open in the subspace topology.
PROOF: It is $(a, +\infty) \cap Y$.
⟨2⟩2. For all $a \in Y$, the open ray $\{y \in Y : y < a\}$ is open in the subspace topology.
PROOF: It is $(-\infty, a) \cap Y$.
⟨1⟩2. The subspace topology is coarser than the order topology.
⟨2⟩1. For all $a \in X$, the set $(-\infty, a) \cap Y$ is open in the order topology.
⟨3⟩1. CASE: $a \in Y$
PROOF: Then $(-\infty, a) \cap Y = \{y \in Y : y < a\}$ is an open ray in Y.
⟨3⟩2. CASE: a is an upper bound for Y
PROOF: Then $(-\infty, a) \cap Y = Y$.
⟨3⟩3. CASE: a is a lower bound for Y
PROOF: Then $(-\infty, a) \cap Y = \emptyset$.
⟨3⟩4. Q.E.D.
PROOF: These are the only three cases because Y is convex.
⟨2⟩2. For all $a \in X$, the set $(a, +\infty) \cap Y$ is open in the order topology.
PROOF: Similar.
□

Example 14.7.7. We cannot remove the hypothesis that the set Y is convex.

Let $X = \mathbb{R}$ and $Y = [0, 1) \cup \{2\}$. Then $\{2\}$ is open in the subspace topology but not in the order topology on Y.

Proposition 14.7.8. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proposition 14.7.9. *Let X be a topological space and $\mathcal{B} \subseteq X$. Assume that, for every open set U and element $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then $\mathcal{B}$ is a basis for the topology on X.*

Proposition 14.7.10. *Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a basis for a topology on X if and only if:*

1. $\bigcup \mathcal{B} = X$
2. *For all $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.*

In this case, the topology is unique and is the set of all unions of subsets of $\mathcal{B}$. We call it the topology generated *by $\mathcal{B}$.*

Proposition 14.7.11. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if, for every $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Corollary 14.7.11.1. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$ but are not comparable to one another.*

Proposition 14.7.12. *In a linearly ordered set under the order topology, every closed interval and closed ray is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a linearly ordered set under the order topology.
$\langle 1 \rangle 2$. Every closed interval in X is closed.
  PROOF: Since $X - [a, b] = (-\infty, a) \cup (b, +\infty)$.
$\langle 1 \rangle 3$. Every closed ray in X is closed.
  PROOF: Since $X - [a, +\infty) = (-\infty, a)$ and $X - (-\infty, a] = (a, +\infty)$.
□

14.7.1 Subspaces

Proposition 14.7.13. *Let X be a topological space. Let Y be a subspace of X. Let $\mathcal{B}$ be a basis for the topology on X. Then $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for the topology on Y.*

PROOF:
$\langle 1 \rangle 1$. For all $B \in \mathcal{B}$ we have $B \cap Y$ is open in Y.
  PROOF: Since B is open in X.
$\langle 1 \rangle 2$. For any open set V in Y and $y \in V$, there exists $B \in \mathcal{B}$ such that $y \in B \cap Y \subseteq V$.
  $\langle 2 \rangle 1$. LET: V be open in Y.
  $\langle 2 \rangle 2$. LET: $y \in V$
  $\langle 2 \rangle 3$. PICK U open in X such that $V = U \cap Y$.
  $\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$.
  $\langle 2 \rangle 5$. $y \in B \cap Y \subseteq V$
□

Proposition 14.7.14. *Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set B in X such that $A = B \cap Y$.*

PROOF:
$$\begin{aligned} A \text{ is closed in } Y &\Leftrightarrow Y - A \text{ is open in } Y \\ &\Leftrightarrow \exists U \text{ open in } X . Y - A = U \cap Y \\ &\Leftrightarrow \exists C \text{ closed in } X . Y - A = Y - C \\ &\Leftrightarrow \exists C \text{ closed in } X . A = Y \cap C \end{aligned}$$
□

14.7.2 Product Topology

Proposition 14.7.15. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. For all $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on X_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i :$ for finitely many $i \in I$ we have $B_i \in \mathcal{B}_i$, and $B_i = X_i$ is a basis for the product topology on $\prod_{i \in I} X_i$.*

PROOF:
⟨1⟩1. Every $B \in \mathcal{B}$ is open in the product topology.
  PROOF: Since every element of $\mathcal{B}_i$ is open in X_i.
⟨1⟩2. For any open set U in the product topology and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
  ⟨2⟩1. LET: U be a set open in the box topology.
  ⟨2⟩2. LET: $x \in U$
  ⟨2⟩3. PICK a family $\{U_i\}_{i \in I}$ where U_i is open in X_i for $i = i_1, \ldots, i_n$, and $U_i = X_i$ for all other i, such that $x \in \prod_{i \in I} U_i \subseteq U$
  ⟨2⟩4. For $i = i_1, \ldots, i_n$, choose $B_i \in \mathcal{B}_i$ such that $x_i \in B_i \subseteq U_i$. Let $B_i = X_i$ for all other i.
  ⟨2⟩5. $\prod_{i \in I} B_i \in \mathcal{B}$
  ⟨2⟩6. $x \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} U_i \subseteq U$
□

14.8 Subbases

Definition 14.8.1 (Subbasis)**.** Let X be a topological space. A *subbasis* for the topology on X is a set $\mathcal{S}$ of open sets such that every open set is a union of finite intersections of $\mathcal{S}$.

Proposition 14.8.2. *Let X be a set and $\mathcal{S} \subseteq X$. Then $\mathcal{S}$ is a subbasis for a topology on X if and only if $\bigcup \mathcal{S} = X$, in which case the topology is unique and is the set of all unions of finite intersections of elements of $\mathcal{S}$.*

Proposition 14.8.3. *Let X be a topological space. Let $\mathcal{S}$ be a subbasis for the topology on X. Then the topology on X is the coarsest topology that includes $\mathcal{S}$.*

Proposition 14.8.4. *Let X and Y be topological spaces. Then*

$$\mathcal{S} = \{{\pi_1}^{-1}(U) : U \text{ is open in } X\} \cup \{{\pi_2}^{-1}(V) : V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

PROOF:
⟨1⟩1. Every element of $\mathcal{S}$ is open.
  PROOF: Since ${\pi_1}^{-1}(U) = U \times Y$ and ${\pi_2}^{-1}(V) = X \times V$.
⟨1⟩2. Every open set is a union of finite intersections of elements of $\mathcal{S}$.
  PROOF: Since, for U open in X and V open in Y, we have $U \times V = {\pi_1}^{-1}(U) \cap {\pi_2}^{-1}(V)$.
□

Definition 14.8.5 (Space with Basepoint)**.** A *space with basepoint* is a pair (X, x) where X is a topological space and $x \in X$.

14.9 Neighbourhood Bases

Definition 14.9.1 (Neighbourhood Basis)**.** Let X be a topological space and $x_0 \in X$. A *neighbourhood basis* of x_0 is a set $\mathcal{U}$ of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of $\mathcal{U}$.

14.10 First Countable Spaces

Definition 14.10.1 (First Countable)**.** A topological space is *first countable* iff every point has a countable neighbourhood basis.

Proposition 14.10.2. $\mathbb{R}_l$ *is first countable.*

PROOF: For any $x \in \mathbb{R}$ we have $\{[x, x+1/n) : n \in \mathbb{Z}_+\}$ is a countable local basis.
□

Proposition 14.10.3. *The ordered square is first countable.*

PROOF:
⟨1⟩1. Every point (a,b) with $0 < b < 1$ has a countable local basis.
  PROOF: The set of all intervals $((a,q),(a,r))$ where q and r are rational and $0 \leqslant q < b < r \leqslant 1$ is a countable local basis.
⟨1⟩2. Every point $(a,0)$ has a countable local basis with $a > 0$.
  PROOF: The set of all intervals $((q,0),(a,r))$ where q and r are rational with $0 \leqslant q < a$ and $0 < r \leqslant 1$ is a countable local basis.
⟨1⟩3. Every point $(a,1)$ has a countable local basis with $a < 1$.
  PROOF: The set of all intervals $((a,q),(r,1))$ with q and r rational and $0 \leqslant q < 1$, $a < r \leqslant 1$ is a countable local basis.
⟨1⟩4. $(0,0)$ has a countable local basis.
  PROOF: The set of all intervals $[(0,0),(0,r))$ with r rational and $0 < r \leqslant 1$ is a countable local basis.
⟨1⟩5. $(1,1)$ has a countable local basis.
  PROOF: The set of all intervals $((1,q),(1,1)]$ with q rational and $0 \leqslant q < 1$ is a countable local basis.
□

14.11 Second Countable Spaces

Definition 14.11.1 (Second Countable)**.** A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.
A subspace of a first countable space is first countable.
A subspace of a second countable space is second countable.
$\mathbb{R}^n$ is second countable.
An uncountable discrete space is first countable but not second countable.

Proposition 14.11.2. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda\in\Lambda} X_\lambda$ is not first countable.*

PROOF:
⟨1⟩1. For all $\lambda \in \Lambda$, PICK U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.
⟨1⟩2. For all $\lambda \in \Lambda$, PICK $x_\lambda \in U_\lambda$.
⟨1⟩3. ASSUME: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda\in\Lambda}$.
⟨1⟩4. PICK $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$
⟨1⟩5. There is no $U \in \lambda$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$
⟨1⟩6. Q.E.D.
PROOF: This is a contradiction.
□

Proposition 14.11.3. *The long line cannot be embedded in $\mathbb{R}^n$ for any n.*

PROOF: Since the long line is not second countable but $\mathbb{R}^n$ is. □

14.12 Interior

Definition 14.12.1 (Interior)**.** Let X be a topological space. Let $A \subseteq X$. The *interior* of A, A°, is the union of all the open sets included in A.

14.13 Closure

Definition 14.13.1 (Closure)**.** Let X be a topological space. Let $A \subseteq X$. The *closure* of A, $\overline{A}$, is the intersection of all the closed sets that include A.

Proposition 14.13.2. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every open set that contains x intersects A.*

PROOF:
$$\begin{aligned} x \in \overline{A} &\Leftrightarrow \text{for every closed set } C \text{, if } A \subseteq C \text{ then } x \in C \\ &\Leftrightarrow \text{for every open set } U \text{, if } A \subseteq X - U \text{ then } x \in X - U \\ &\Leftrightarrow \text{for every open set } U \text{, if } A \cap U = \emptyset \text{ then } x \notin U \\ &\Leftrightarrow \text{for every open set } U \text{, if } x \in U \text{ then } A \text{ intersects } U \end{aligned}$$
□

Proposition 14.13.3. *Let X be a topological space. Let $A \subseteq B \subseteq X$. Then $\overline{A} \subseteq \overline{B}$.*

PROOF: Since every closed set that includes B is a closed set that includes A. □

Proposition 14.13.4. *Let X be a topological space. Let $A, B \subseteq X$. Then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

PROOF:
⟨1⟩1. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$

PROOF: Since $\overline{A} \cup \overline{B}$ is a closed set that includes $A \cup B$.
⟨1⟩2. $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$
PROOF: Since $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ by Proposition 14.13.3.
□

Proposition 14.13.5. *Let X be a topological space. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then*

$$\bigcup\{\overline{A} : A \in \mathcal{A}\} \subseteq \overline{\bigcup \mathcal{A}} \ .$$

PROOF:For all $A \in \mathcal{A}$ we have $\overline{A} \subseteq \overline{\bigcup \mathcal{A}}$ by Proposition 14.13.3. □

Example 14.13.6. The converse does not always hold. In $\mathbb{R}$, let $\mathcal{A} = \{\{x\} : 0 < x < 1\}$. Then $\bigcup\{\overline{A} : A \in \mathcal{A}\} = (0,1)$ but $\overline{\bigcup \mathcal{A}} = [0,1]$.

Proposition 14.13.7. *Let X be a topological space. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then $\overline{\bigcap \mathcal{A}} \subseteq \bigcap\{\overline{A} : A \in \mathcal{A}\}$.*

PROOF: Since $\overline{\bigcap \mathcal{A}} \subseteq \overline{A}$ for all $A \in \mathcal{A}$ by Proposition 14.13.3. □

Example 14.13.8. The converse does not always hold. In $\mathbb{R}$, if A is the set of all rational numbers and B is the set of all irrational numbers then $\overline{\bigcap A \cap B} = \emptyset$ but $\bigcap \overline{A} \cap \bigcap \overline{B} = \mathbb{R}$.

14.13.1 Bases

Proposition 14.13.9. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for the topology on X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

PROOF:
⟨1⟩1. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
PROOF: Proposition 14.13.2 since every element of $\mathcal{B}$ is open.
⟨1⟩2. If, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A, then $x \in \overline{A}$.
⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
⟨2⟩2. LET: U be an open set that contains x.
⟨2⟩3. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
⟨2⟩4. B intersects A.
PROOF: ⟨2⟩1
⟨2⟩5. U intersects A.
□

14.13.2 Subspaces

Proposition 14.13.10. *Let X be a topological space. Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF:

⟨1⟩1. $\overline{A} \cap Y$ is the closed in Y.
  PROOF: Since $\overline{A}$ is closed in X.
⟨1⟩2. For any closed set B in Y, if $A \subseteq B$ then $\overline{A} \cap Y \subseteq B$.
  ⟨2⟩1. LET: B be closed in Y.
  ⟨2⟩2. ASSUME: $A \subseteq B$
  ⟨2⟩3. PICK C closed in X such that $B = C \cap Y$.
  ⟨2⟩4. $A \subseteq C$
  ⟨2⟩5. $\overline{A} \subseteq C$
  ⟨2⟩6. $\overline{A} \cap Y \subseteq B$
∎

14.13.3 Product Topology

Proposition 14.13.11. *Let X and Y be topological spaces. Let $A \subseteq X$ and $B \subseteq Y$. Then $\overline{A \times B} = \overline{A} \times \overline{B}$.*

PROOF:
⟨1⟩1. $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$
  PROOF: Since $\overline{A} \times \overline{B}$ is a closed set that includes $A \times B$ by Proposition 15.8.2.
⟨1⟩2. $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$
  ⟨2⟩1. LET: $x \in \overline{A}$ and $y \in \overline{B}$.
  ⟨2⟩2. LET: U be an open set that contains (x, y).
  ⟨2⟩3. PICK open sets V in X and W in Y such that $(x, y) \in V \times W \subseteq U$.
  ⟨2⟩4. V intersects A and W intersects B.
  ⟨2⟩5. U intersects $A \times B$.
∎

14.13.4 Interior

Proposition 14.13.12. *Let X be a topological space and $A \subseteq X$. Then*

$$X - A^\circ = \overline{X - A}$$

PROOF:

$$\begin{aligned} X - A^\circ &= X - \bigcup\{U \text{ open in } X : U \subseteq A\} \\ &= \bigcap\{X - U : U \text{ open in } X, U \subseteq A\} && \text{(De Morgan's Law)} \\ &= \bigcap\{C : C \text{ closed in } X, X - A \subseteq C\} \\ &= \overline{X - A} \end{aligned}$$

∎

Proposition 14.13.13. *Let X be a topological space and $A \subseteq X$. Then*

$$X - \overline{A} = (X - A)^\circ$$

PROOF: Dual. ∎

14.14 Boundary

Definition 14.14.1 (Boundary)**.** Let X be a topological space. Let $A \subseteq X$. The *boundary* of A is

$$\partial A := \overline{A} \cap \overline{X - A} \ .$$

Proposition 14.14.2. *Let X be a topological space. Let $A \subseteq X$. Then*

$$A^\circ \cap \partial A = \emptyset \ .$$

PROOF:
$\langle 1 \rangle 1.$ $A^\circ \subseteq A$
$\langle 1 \rangle 2.$ $X - A \subseteq X - A^\circ$
$\langle 1 \rangle 3.$ $\overline{X - A} \subseteq X - A^\circ$
$\langle 1 \rangle 4.$ $\partial A \subseteq X - A^\circ$
□

Proposition 14.14.3. *Let X be a topological space. Let $A \subseteq X$. Then*

$$\overline{A} = A^\circ \cup \partial A$$

$\langle 1 \rangle 1.$ $A^\circ \subseteq \overline{A}$
 PROOF: Since $A^\circ \subseteq A \subseteq \overline{A}$.
$\langle 1 \rangle 2.$ $\partial A \subseteq \overline{A}$
 PROOF: Definition of ∂A.
$\langle 1 \rangle 3.$ $\overline{A} \subseteq A^\circ \cup \partial A$
 $\langle 2 \rangle 1.$ LET: $x \in \overline{A}$
 $\langle 2 \rangle 2.$ ASSUME: $x \notin A^\circ$
  PROVE: $x \in \partial A$
 $\langle 2 \rangle 3.$ $x \in \overline{X - A}$
  PROOF: Since $\overline{X - A} = X - A^\circ$.
 $\langle 2 \rangle 4.$ $x \in \partial A$
  PROOF: Since $\partial A = \overline{A} \cap \overline{X - A}$.
□

Proposition 14.14.4. *Let X be a topological space. Let $A \subseteq X$. Then $\partial A = \emptyset$ if and only if A is both open and closed.*

PROOF:
$\langle 1 \rangle 1.$ If $\partial A = \emptyset$ then A is open and closed.
 $\langle 2 \rangle 1.$ ASSUME: $\partial A = \emptyset$
 $\langle 2 \rangle 2.$ $\overline{A} = A^\circ$
  PROOF: Proposition 14.14.3.
 $\langle 2 \rangle 3.$ $\overline{A} = A = A^\circ$
$\langle 1 \rangle 2.$ If A is open and closed then $\partial A = \emptyset$.
 PROOF: If A is open and closed then

$$\begin{aligned}\partial A &= \overline{A} \cap \overline{X - A} \\ &= \overline{A} \cap (X - A^\circ) \\ &= A \cap (X - A) \\ &= \emptyset\end{aligned}$$

∎

Proposition 14.14.5. *Let X be a topological space. Let $U \subseteq X$. Then U is open if and only if $\partial U = \overline{U} - U$.*

PROOF:
⟨1⟩1. If U is open then $\partial U = \overline{U} - U$
  PROOF: If U is open then

$$\begin{aligned}\partial U &= \overline{U} \cap \overline{X - U} \\ &= \overline{U} \cap (X - U^\circ) \\ &= \overline{U} - U^\circ \\ &= \overline{U} - U\end{aligned}$$

⟨1⟩2. If $\partial U = \overline{U} - U$ then U is open.
  ⟨2⟩1. ASSUME: $\partial U = \overline{U} - U$
  ⟨2⟩2. $\overline{U} - U^\circ = \overline{U} - U$
  ⟨2⟩3. $U \subseteq U^\circ$
  ⟨2⟩4. $U = U^\circ$

∎

14.15 Limit Points

Definition 14.15.1 (Limit Point)**.** Let X be a topological space, $x \in X$ and $A \subseteq X$. Then x is a *limit point*, *cluster point* or *point of accumulation* of A iff every neighbourhood of x intersects $A - \{x\}$.

Proposition 14.15.2. *Let X be a topological space. Let $A \subseteq X$. Let A' be the set of limit points of A. Then*

$$\overline{A} = A \cup A'$$

PROOF:
⟨1⟩1. $\overline{A} \subseteq A \cup A'$
  ⟨2⟩1. LET: $x \in \overline{A}$
  ⟨2⟩2. ASSUME: $x \notin A$
    PROVE: $x \in A'$
  ⟨2⟩3. LET: U be a neighbourhood of x.
  ⟨2⟩4. PICK $y \in U \cap A$
    PROOF: Proposition 14.13.2.
  ⟨2⟩5. $y \neq x$
⟨1⟩2. $A \subseteq \overline{A}$
  PROOF: Immediate from the definition of $\overline{A}$.
⟨1⟩3. $A' \subseteq \overline{A}$
  PROOF: From Proposition 14.13.2.

∎

Corollary 14.15.2.1. *A set is closed if and only if it contains all its limit points.*

14.16 Isolated Points

Definition 14.16.1 (Isolated Point)**.** Let X be a topological space. Let $a \in X$. Then a is an *isolated point* iff $\{a\}$ is open.

Chapter 15

Continuous Functions

Definition 15.0.1 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* iff, for every open set V in Y, the inverse image $f^{-1}(V)$ is open in X.

Proposition 15.0.2. *The composite of two continuous functions is continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : X \to Y$ and $g : Y \to Z$ be continuous.
$\langle 1 \rangle 2$. LET: U be open in Z.
$\langle 1 \rangle 3$. $g^{-1}(U)$ is open in Y.
$\langle 1 \rangle 4$. inf $f(g^{-1}(U))$ is open in X.
□

Proposition 15.0.3.

1. *id_X is continuous*
2. *If $f : X \to Y$ is continuous and $X_0 \subseteq X$ then $f \restriction X_0 : X_0 \to Y$ is continuous.*
3. *If $f : X + Y \to Z$, then f is continuous iff $f \circ \kappa_1 : X \to Z$ and $f \circ \kappa_2 : Y \to Z$ are continuous.*
4. *If $f : Z \to X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.*

Proposition 15.0.4. *Let X and Y be topological spaces. Let $f : X \to Y$. Then the following are equivalent.*

1. *f is continuous.*
2. *For all $A \subseteq X$ we have $f(\overline{A}) \subseteq \overline{f(A)}$.*
3. *For every closed B in Y, we have $f^{-1}(B)$ is closed in X.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
  $\langle 2 \rangle 1$. ASSUME: f is continuous.

⟨2⟩2. Let: $A \subseteq X$
⟨2⟩3. Let: $x \in \overline{A}$
Prove: $f(x) \in \overline{f(A)}$
⟨2⟩4. Let: V be a neighbourhood of $f(x)$.
Prove: V intersects $f(A)$.
⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of x.
⟨2⟩6. Pick $y \in f^{-1}(V) \cap A$
⟨2⟩7. $f(y) \in V \cap f(A)$
⟨1⟩2. $2 \Rightarrow 3$
⟨2⟩1. Assume: 2
⟨2⟩2. Let: B be closed in Y
⟨2⟩3. Let: $A = f^{-1}(B)$
Prove: $\overline{A} = A$
⟨2⟩4. $f(A) \subseteq B$
⟨2⟩5. $\overline{A} \subseteq A$
⟨3⟩1. Let: $x \in \overline{A}$
⟨3⟩2. $f(x) \in B$
Proof:

$$\begin{aligned} f(x) &\in f(\overline{A}) \\ &\subseteq \overline{f(A)} && (\langle 2\rangle 1) \\ &\subseteq \overline{B} && (\langle 2\rangle 4) \\ &= B && (\langle 2\rangle 2) \end{aligned}$$

⟨1⟩3. $3 \Rightarrow 1$
⟨2⟩1. Assume: 3
⟨2⟩2. Let: V be open in Y.
⟨2⟩3. $f^{-1}(Y - V)$ is closed in X.
⟨2⟩4. $X - f^{-1}(V)$ is closed in X.
⟨2⟩5. $f^{-1}(V)$ is open in X.
□

Proposition 15.0.5. *Let X and Y be topological spaces. Any constant function $X \to Y$ is continuous.*

Proof:
⟨1⟩1. Let: $b \in Y$
⟨1⟩2. Let: $f : X \to Y$ be the constant function with value b.
⟨1⟩3. Let: $V \subseteq Y$ be open.
⟨1⟩4. $f^{-1}(V)$ is either $\emptyset$ or X.
⟨1⟩5. $f^{-1}(V)$ is open.
□

Proposition 15.0.6. *Let X and Y be topological spaces. Let $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

Proof:

$\langle 1\rangle 1$. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
PROOF: Since every element of $\mathcal{B}$ is open in Y.
$\langle 1\rangle 2$. If, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X, then f is continuous.
$\langle 2\rangle 1$. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
$\langle 2\rangle 2$. LET: U be open in Y.
$\langle 2\rangle 3$. LET: $x \in f^{-1}(U)$
$\langle 2\rangle 4$. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq U$.
$\langle 2\rangle 5$. $x \in f^{-1}(B) \subseteq f^{-1}(U)$
$\square$

Proposition 15.0.7. *Let X and Y be topological spaces. Let $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for the topology on Y. Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.*

PROOF:
$\langle 1\rangle 1$. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
PROOF: Immediate from definitions.
$\langle 1\rangle 2$. If, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X, then f is continuous.
$\langle 2\rangle 1$. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
$\langle 2\rangle 2$. For all $V_1, \ldots, V_n \in \mathcal{S}$ we have $f^{-1}(V_1 \cap \cdots \cap V_n)$ is open in X.
PROOF: Since $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.
$\langle 2\rangle 3$. Q.E.D.
PROOF: By Proposition 15.0.6 since the set of all finite intersections of elements of $\mathcal{S}$ forms a basis for the topology on Y.
$\square$

Proposition 15.0.8. *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is continuous if and only if, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.*

PROOF:
$\langle 1\rangle 1$. If f is continuous then, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.
$\langle 2\rangle 1$. ASSUME: f is continuous.
$\langle 2\rangle 2$. LET: $x \in \mathbb{R}$
$\langle 2\rangle 3$. LET: $\epsilon > 0$
$\langle 2\rangle 4$. $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open in X.
$\langle 2\rangle 5$. PICK a, b such that $x \in (a, b) \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$.
$\langle 2\rangle 6$. LET: $\delta = \min(x - a, b - x)$
$\langle 2\rangle 7$. LET: $y \in \mathbb{R}$
$\langle 2\rangle 8$. ASSUME: $|y - x| < \delta$
$\langle 2\rangle 9$. $y \in (a, b)$
$\langle 2\rangle 10$. $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$
$\langle 2\rangle 11$. $|f(y) - f(x)| < \epsilon$
$\langle 1\rangle 2$. If, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$, then f is continuous.
$\langle 2\rangle 1$. ASSUME: For all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.

⟨2⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.
⟨3⟩1. LET: $a \in \mathbb{R}$
⟨3⟩2. LET: $x \in f^{-1}((a, +\infty))$
⟨3⟩3. LET: $\epsilon = f(x) - a$
⟨3⟩4. PICK $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y-x| < \delta$ then $|f(y)-f(x)| < \epsilon$
⟨3⟩5. $x \in (x - \delta, x + \delta) \subseteq f^{-1}((a, +\infty))$
⟨2⟩3. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.
PROOF: Similar.
⟨2⟩4. Q.E.D.
PROOF: Proposition 15.0.8.
□

Definition 15.0.9 (Continuity at a Point)**.** Let X and Y be topological spaces. Let $f : X \to Y$. Let $a \in X$. Then f is *continuous at a* iff, for every neighbourhood V of $f(a)$, there exists a neighbourhood U of a such that $f(U) \subseteq V$.

Proposition 15.0.10. *Let X and Y be topological spaces. Let $f : X \to Y$. Then f is continuous if and only if f is continuous at every point in X.*

⟨1⟩1. If f is continuous then f is continuous at every point in X.
⟨2⟩1. ASSUME: f is continuous.
⟨2⟩2. LET: $a \in X$
⟨2⟩3. LET: V be a neighbourhood of $f(a)$
⟨2⟩4. LET: $U = f^{-1}(V)$
⟨2⟩5. U is a neighbourhood of a.
⟨2⟩6. $f(U) \subseteq V$
⟨1⟩2. If f is continuous at every point in X then f is continuous.
⟨2⟩1. ASSUME: f is continuous at every point in X.
⟨2⟩2. LET: V be open in Y.
⟨2⟩3. LET: $x \in f^{-1}(V)$
⟨2⟩4. V is a neighbourhood of $f(x)$
⟨2⟩5. PICK a neighbourhood U of x such that $f(U) \subseteq V$
⟨2⟩6. $x \in U \subseteq f^{-1}(V)$
□

Definition 15.0.11 (Homeomorphism)**.** Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Proposition 15.0.12. *Let X and Y be topological spaces. Let $f : X \to Y$. Then f is a homeomorphism iff f is bijective and, for all $U \subseteq X$, we have $f(U)$ is open if and only if U is open.*

PROOF: Immediate from definitions. □

Definition 15.0.13 (Topological Property)**.** A property P of topological spaces is a *topological* property iff, for any topological spaces X and Y, if $P[X]$ and $X \cong Y$ then $P[Y]$.

Definition 15.0.14 (Retraction)**.** Let X be a topological space and A a subspace of X. A continuous function $\rho : X \to A$ is a *retraction* iff $\rho \restriction A = \mathrm{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 15.0.15. Let **Top** be the category of small topological spaces and continuous functions.

Proposition 15.0.16. $\emptyset$ *is initial in* **Top**.

Proposition 15.0.17. 1 *is terminal in* **Top**.

Forgetful functor **Top** $\to$ **Set**.
Basepoint preserving continuous functor.

Proposition 15.0.18. *Let* $(X, \mathcal{T})$ *be a topological space. Let* S *be the Sierpiński two-point space. Define* $\Phi : \mathcal{T} \to \mathbf{Top}[X, S]$ *by* $\Phi(U)(x) = 1$ *iff* $x \in U$. *Then* Φ *is a bijection.*

Proof:
⟨1⟩1. For all $U \in \mathcal{T}$ we have $\Phi(U)$ is continuous.
⟨2⟩1. Let: $U \in \mathcal{T}$
⟨2⟩2. $\Phi(U)(\{1\})$ is open.
Proof: Since $\Phi(U)(\{1\}) = U$.
⟨1⟩2. Φ is injective.
Proof: If $\Phi(U) = \Phi(V)$ then we have $\forall x(x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$.
⟨1⟩3. Φ is surjective.
Proof: Given $f : X \to S$ continuous we have $\Phi(f^{-1}(1)) = f$.
$\square$

15.0.1 Order Topology

Proposition 15.0.19. *Let* X *and* Y *be linearly ordered sets under the order topology. Let* $f : X \to Y$ *be strictly monotone and surjective. Then* f *is a homeomorphism.*

Proof:
⟨1⟩1. f is continuous.
⟨2⟩1. For all $b \in Y$ we have $f^{-1}((b, +\infty))$ is open in X.
⟨3⟩1. Let: $b \in Y$
⟨3⟩2. Let: a be the element of X such that $f(a) = b$.
⟨3⟩3. $f^{-1}((b, +\infty)) = (a, +\infty)$
⟨2⟩2. For all $b \in Y$ we have $f^{-1}((-\infty, b))$ is open in X.
Proof: Similar.
⟨1⟩2. f^{-1} is continuous.
Proof: Similar.
$\square$

Corollary 15.0.19.1. *For* n *a positive integer, the* n*th root function* $\overline{\mathbb{R}_+} \to \overline{\mathbb{R}_+}$ *is continuous.*

15.0.2 Paths

Definition 15.0.20 (Path)**.** A *path* in a topological space X is a continuous function $[0,1] \to X$.

Definition 15.0.21 (Constant Path)**.** Let X be a topological space and $a \in X$. The *constant* path at a is the path $p : [0,1] \to X$ with $p(t) = a$ for all $t \in [0,1]$.

Definition 15.0.22 (Reverse Path)**.** Let X be a topological space and $p : [0,1] \to X$. The *reverse* of p is the path $q : [0,1] \to X$ with $q(t) = p(1-t)$ for all $t \in [0,1]$.

Definition 15.0.23 (Concatenation)**.** Let X be a topological space and $p, q : [0,1] \to X$ be paths in X with $p(1) = q(0)$. The *concatenation* of p and q is the path $r : [0,1] \to X$ with

$$r(t) = \begin{cases} p(2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ q(2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

15.0.3 Loops

Definition 15.0.24 (Loop)**.** A *loop* in a topological space X is a path $\alpha : [0,1] \to X$ such that $\alpha(0) = \alpha(1)$.

15.1 Convergence

Definition 15.1.1 (Convergence)**.** Let X be a topological space. Let (x_n) be a sequence in X. A point $a \in X$ is a *limit* of the sequence iff, for every neighbourhood U of a, there exists n_0 such that $\forall n \geqslant n_0 . x_n \in U$.

Proposition 15.1.2. *If $f : X \to Y$ is continuous and $x_n \to l$ in X then $f(x_n) \to f(l)$ in Y.*

Example 15.1.3. The converse does not hold.

Let X be the set of all continuous functions $[0,1] \to [-1,1]$ under the product topology. Let $i : X \to L^2([0,1])$ be the inclusion.

If $f_n \to f$ then $i(f_n) \to i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K_\epsilon . \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0,1]} U_\lambda$ where $U_\lambda = [-1,1]$ except for $\lambda = \lambda_1, \ldots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \ldots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 15.1.4. *The converse does hold for first countable spaces. If $f : X \to Y$ where X is first countable, and Y is a topological space, and whenever $x_n \to x$ then $f(x_n) \to f(x)$, then f is continuous.*

Proposition 15.1.5. *If (s_n) is an increasing sequence of real numbers bounded above, then (s_n) converges.*

PROOF:
⟨1⟩1. LET: s be the supremum of $\{s_n : n \in \mathbb{N}\}$.
PROVE: $s_n \to s$ as $n \to \infty$.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK N such that $s_N > s - \epsilon$.
⟨1⟩4. $\forall n \geqslant N. s - \epsilon \leqslant s_n \leqslant s$
⟨1⟩5. $\forall n \geqslant N. |s_n - s| < \epsilon$
□

15.1.1 Closure

Proposition 15.1.6. *Let X be a topological space. Let $A \subseteq X$. Let (a_n) be a sequence in A and $l \in X$. If $a_n \to l$ as $n \to \infty$, then $l \in \overline{A}$.*

PROOF:
⟨1⟩1. LET: U be a neighbourhood of l.
⟨1⟩2. PICK N such that $\forall n \in N. a_n \in U$
⟨1⟩3. $a_N \in A \cap U$
□

15.1.2 Continuous Functions

Proposition 15.1.7. *Let X and Y be topological spaces. Let $f : X \to Y$ be continuous. Let $x_n \to x$ as $n \to \infty$ in X. Then $f(x_n) \to f(x)$ as $n \to \infty$ in Y.*

PROOF:
⟨1⟩1. LET: V be a neighbourhood of $f(x)$.
⟨1⟩2. PICK N such that $\forall n \geqslant N. x_n \in f^{-1}(V)$
⟨1⟩3. $\forall n \geqslant N. f(x_n) \in V$
□

15.1.3 Infinite Series

Definition 15.1.8 (Series)**.** Let (a_n) be a sequence of real numbers. We say that the infinite series $\sum_{n=0}^{\infty} a_n$ *converges* to s, and write

$$\sum_{n=0}^{\infty} a_n = s$$

iff $\sum_{n=0}^{N} a_n \to s$ as $N \to \infty$.

15.2 Strong Continuity

Definition 15.2.1 (Strong Continuity)**.** Let X and Y be topological spaces. Let $f : X \to Y$. Then f is *strongly continuous* iff, for every $V \subseteq Y$, we have V is open in Y if and only if $f^{-1}(V)$ is open in X.

Proposition 15.2.2. *Let X and Y be topological spaces. Let $f : X \to Y$. Then f is strongly continuous if and only if, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X.*

Proof:

$$\begin{aligned} f \text{ is continuous} &\Leftrightarrow \forall V \subseteq Y (V \text{ is open in } Y \Leftrightarrow f^{-1}(V) \text{ is open in } X) \\ &\Leftrightarrow \forall C \subseteq Y (Y - C \text{ is open in } Y \Leftrightarrow f^{-1}(Y - C) \text{ is open in } X) \\ &\Leftrightarrow \forall C \subseteq Y (C \text{ is closed in } Y \Leftrightarrow f^{-1}(C) \text{ is closed in } X) \end{aligned}$$ □

15.3 Subspaces

Definition 15.3.1 (Subspace)**.** Let X be a topological space, Y a set, and $f : Y \to X$. The *subspace topology* on Y induced by f is $\mathcal{T} = \{i^{-1}(U) : U \text{ is open in } X\}$.

We prove this is a topology.

Proof:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

Proof: Since $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$.

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

Proof: Since $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.

$\langle 1 \rangle 3$. $Y \in \mathcal{T}$

Proof: Since $Y = f^{-1}(X)$.

□

Proposition 15.3.2. *Let X be a topological space, Y a set and $f : Y \to X$ a function. Then the subspace topology on Y is the coarsest topology such that f is continuous.*

Proof: Immediate from definition. □

Proposition 15.3.3 (Local Formulation of Continuity)**.** *Let X and Y be topological spaces. Let $f : X \to Y$. Let $\mathcal{U}$ be a set of open subspaces of X such that $X = \bigcup \mathcal{U}$. If $f \restriction U : U \to Y$ is continuous for all $U \in \mathcal{U}$, then f is continuous.*

Proof:

$\langle 1 \rangle 1$. Let: $x \in X$

Prove: f is continuous at x.

$\langle 1 \rangle 2$. Let: V be a neighbourhood of $f(x)$.

$\langle 1 \rangle 3$. Pick $U \in \mathcal{U}$ such that $x \in U$.

$\langle 1 \rangle 4$. Pick W open in U such that $x \in W$ and $f(W) \subseteq V$.

$\langle 1 \rangle 5$. W is open in X.

□

Theorem 15.3.4. *Let X be a topological space and (Y, i) a subset of X. Then the subspace topology on Y is the unique topology such that, for every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f : Z \to X$ is continuous.*

PROOF:
⟨1⟩1. If we give Y the subspace topology then, for every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f$ is continuous.
⟨2⟩1. Given Y the subspace topology.
⟨2⟩2. LET: Z be a topological space.
⟨2⟩3. LET: $f : Z \to Y$
⟨2⟩4. If f is continuous then $i \circ f$ is continuous.
PROOF: Since i is continuous.
⟨2⟩5. If $i \circ f$ is continuous then f is continuous.
⟨3⟩1. ASSUME: $i \circ f$ is continuous.
⟨3⟩2. LET: U be open in Y.
⟨3⟩3. $f^{-1}(i^{-1}(i(U))$ is open in Z.
⟨3⟩4. $f^{-1}(U)$ is open in Z.
⟨1⟩2. If, for every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f$ is continuous.
⟨2⟩1. ASSUME: For every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f$ is continuous.
⟨2⟩2. i is continuous.
⟨2⟩3. For every open set U in X, we have $i^{-1}(X)$ is open in Y
⟨2⟩4. LET: Z be the set Y under the subspace topology and $f : Z \to Y$ the identity function.
⟨2⟩5. $i \circ f$ is continuous.
⟨2⟩6. f is continuous.
⟨2⟩7. Every set open in Y is open in Z.
□

Proposition 15.3.5. *Let X be a topological space, Y a subspace of X and $U \subseteq Y$. If Y is open in X and U is open in Y then U is open in X.*

PROOF:
⟨1⟩1. PICK V open in X such that $U = V \cap Y$
⟨1⟩2. U is open in X.
PROOF: It is the intersection of two open sets in X.
□

Proposition 15.3.6. *Let X be a topological space. Let Y be a subspace of X. Let $C \subseteq Y$. If Y is closed in X and C is closed in Y then C is closed in X.*

PROOF: Similar. □

Proposition 15.3.7. *Let Y be a subspace of X and $A \subseteq Y$. Then the subspace topology on A as a subspace of Y is the same as the subspace topology on A as a subspace of X.*

PROOF:
⟨1⟩1. LET: $\mathcal{T}_Y$ be the subspace topology on A as a subspace of Y.
⟨1⟩2. LET: $\mathcal{T}_X$ be the subspace topology on A as a subspace of X.

⟨1⟩3. LET: $U \subseteq A$
⟨1⟩4. $U \in \mathcal{T}_Y \Leftrightarrow U \in \mathcal{T}_X$
PROOF:

$$\begin{aligned} U \in \mathcal{T}_Y &\Leftrightarrow \exists V \text{ open in } Y. U = V \cap A \\ &\Leftrightarrow \exists V. \exists W \text{ open in } X.(V = Y \cap W \wedge U = V \cap A) \\ &\Leftrightarrow \exists W \text{ open in } X. U = Y \cap W \cap A \\ &\Leftrightarrow \exists W \text{ open in } X. U = W \cap A \\ &\Leftrightarrow U \in \mathcal{T}_X \end{aligned}$$

□

Proposition 15.3.8. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $Y \subseteq X$. Then $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the topology on Y.*

PROOF:
⟨1⟩1. Every element of $\mathcal{B}'$ is open.
PROOF: For all $B \in \mathcal{B}$, we have B is open in X, so $B \cap Y$ is open in Y.
⟨1⟩2. For any open set V in Y and $y \in V$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq V$
⟨2⟩1. LET: V be open in Y.
⟨2⟩2. LET: $y \in V$
⟨2⟩3. PICK U open in X such that $V = U \cap Y$.
⟨2⟩4. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$
⟨2⟩5. $B \cap Y \in \mathcal{B}'$ and $y \in B \cap Y \subseteq V$

□

Proposition 15.3.9. *Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X.*

PROOF:
⟨1⟩1. PICK C closed in X such that $A = C \cap Y$.
⟨1⟩2. A is closed in X.
PROOF: It is the intersection of two closed sets in X.

□

15.3.1 Product Topology

Proposition 15.3.10. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology on $\prod_{i \in I} Y_i$ as a subspace of $\prod_{i \in I} X_i$.*

PROOF:
⟨1⟩1. Given $\prod_{i \in I} Y_i$ the subspace topology.
⟨1⟩2. LET: $\iota : \prod_{i \in I} Y_i$ be the inclusion.
⟨1⟩3. LET: Z be any topological space.
⟨1⟩4. LET: $f : Z \to \prod_{i \in I} Y_i$
⟨1⟩5. f is continuous if and only if, for all $i \in I$, we have $\pi_i \circ f$ is continuous.

PROOF:

$$\begin{aligned} f \text{ is continuous} &\Leftrightarrow \iota \circ f : Z \to \prod_{i \in I} X_i \text{ is continuous} && \text{(Theorem 15.3.4)} \\ &\Leftrightarrow \forall i \in I.\pi_i \circ \iota \circ f : Z \to X_i \text{ is continuous} && \text{(Theorem 15.8.4)} \\ &\Leftrightarrow \forall i \in I.\iota_i \circ \pi_i \circ f : Z \to X_i \text{ is continuous} \\ &\Leftrightarrow \forall i \in I.\pi_i \circ f : Z \to Y_i \text{ is continuous} && \text{(Theorem 15.3.4)} \end{aligned}$$

where ι_i is the inclusion $Y_i \to X_i$.

□

15.4 Embedding

Definition 15.4.1 (Embedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *embedding* iff f is injective and the topology on X is the subspace induced by f.

Proposition 15.4.2. *Every embedding is continuous.*

PROOF: Theorem 15.3.4. □

Proposition 15.4.3. *Let X and Y be topological spaces. Let $b \in Y$. The function $\kappa : X \to X \times Y$ that maps x to (x, b) is an embedding.*

PROOF:
⟨1⟩1. For all U open in X, we have $U = \kappa^{-1}(V)$ for some V open in $X \times Y$.
  PROOF: Take $V = U \times Y$.
⟨1⟩2. For all V open in $X \times Y$ we have $\kappa^{-1}(V)$ is open in X.
  PROOF: Since $\pi_1 \circ \kappa = \mathrm{id}_X$ and $\pi_2 \circ \kappa$ (which is the constant function with value b) are both continuous, hence κ is continuous.

□

15.5 Open Maps

Definition 15.5.1 (Open Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* iff, for all U open in X, we have $f(U)$ is open in Y.

Proposition 15.5.2. *Let X and Y be topological spaces. The projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.*

PROOF:
⟨1⟩1. π_1 is an open map.
  ⟨2⟩1. LET: U be open in $X \times Y$.
  ⟨2⟩2. LET: $x \in \pi_1(U)$
  ⟨2⟩3. PICK y such that $(x, y) \in U$
  ⟨2⟩4. PICK V and W open in X and Y respectively such that $(x, y) \in V \times W \subseteq U$

⟨2⟩5. $x \in V \subseteq \pi_1(U)$
⟨1⟩2. π_2 is an open map.
PROOF: Similar.
□

15.5.1 Subspaces

Proposition 15.5.3. *Let X and Y be topological spaces. Let $p : X \to Y$ be an open map. Let A be an open set in X. Then $p \upharpoonright A : A \to p(A)$ is an open map.*

PROOF:
⟨1⟩1. LET: U be open in A.
⟨1⟩2. U is open in X.
PROOF: Proposition 15.3.5.
⟨1⟩3. $p(U)$ is open in Y.
⟨1⟩4. $p(U)$ is open in $p(A)$.
PROOF: Since $p(U) = p(U) \cap p(A)$.
□

15.6 Locally Finite

Definition 15.6.1 (Locally Finite)**.** Let X be a topological space. Let $\{A_i\}_{i \in I}$ be a family of subsets of X. Then $\{A_i\}_{i \in I}$ is *locally finite* iff, for every $x \in X$, there exist only finitely many $i \in I$ such that $x \in A_i$.

Theorem 15.6.2 (Pasting Lemma)**.** *Let X and Y be topological spaces. Let $f : X \to Y$. Let $\{A_i\}_{i \in I}$ be a locally finite family of closed subspaces of X such that $X = \bigcup_{i \in I} A_i$. If $f \upharpoonright A_i : A_i \to Y$ is continuous for all $i \in I$, then f is continuous.*

PROOF:
⟨1⟩1. LET: B be closed in Y.
⟨1⟩2. LET: $A = f^{-1}(B)$
PROVE: A is closed in X.
⟨1⟩3. $A = \bigcup_{i \in I} f \upharpoonright {A_i}^{-1}(B)$
⟨1⟩4. LET: $x \in X - A$
PROVE: There exists a neighbourhood U' of x such that $U' \subseteq X - A$.
⟨1⟩5. PICK a neighbourhood U of x such that U intersects A_i for only finitely many $i \in I$.
⟨1⟩6. LET: $i_1, \ldots, i_n$ be the elements of I such that U intersects $A_{i_1}, \ldots, A_{i_n}$.
⟨1⟩7. For $j = 1, \ldots, n$,
LET: $S_j = f \upharpoonright {A_{i_j}}^{-1}(B)$
⟨1⟩8. For $j = 1, \ldots, n$, we have S_j is closed in X.
⟨1⟩9. For $j = 1, \ldots, n$, we have $x \notin S_j$.
⟨1⟩10. LET: $U' = U \cap \bigcap_{j=1}^{n}(X - S_j)$
⟨1⟩11. U' is a neighbourhood of x.

⟨1⟩12. $U' \subseteq X - A$
□

15.7 Closed Maps

Definition 15.7.1 (Closed Map)**.** Let X and Y be topological spaces. Let $f : X \to Y$. Then f is a *closed map* iff, for every closed set C in X, we have $f(C)$ is closed in Y.

15.8 Product Topology

Definition 15.8.1 (Product Topology)**.** Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

15.8.1 Closed Sets

Proposition 15.8.2. *Let X and Y be topological spaces. Let A be a closed set in X and B a closed set in Y. Then $A \times B$ is closed in $X \times Y$.*

Proof: Since $(X \times Y) - (A \times B) = ((X - A) \times Y) \cup (X \times (Y - B))$. □

Proposition 15.8.3. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The product topology on $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis $\mathcal{B} = \{\prod_{\alpha \in A} U_\alpha :$ for all $\alpha \in A, U_\alpha$ is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A\}$.*

Proof:
⟨1⟩1. $\mathcal{B}$ is a basis for a topology.
⟨1⟩2. Let: $\mathcal{T}$ be the topology generated by $\mathcal{B}$.
⟨1⟩3. Let: $\mathcal{T}_p$ be the product topology.
⟨1⟩4. $\mathcal{T} \subseteq \mathcal{T}_p$
  ⟨2⟩1. Let: $B \in \mathcal{B}$
  ⟨2⟩2. Let: $B = \prod_{\alpha \in A} U_\alpha$ with each U_α open in X_α and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$
  ⟨2⟩3. $B = {\pi_{\alpha_1}}^{-1}(U_{\alpha_1}) \cap \cdots \cap {\pi_{\alpha_n}}^{-1}(U_{\alpha_n})$
  ⟨2⟩4. $B \in \mathcal{T}_p$
⟨1⟩5. $\mathcal{T}_p \subseteq \mathcal{T}$
  ⟨2⟩1. For every $\alpha \in A$ we have π_α is continuous.
    Proof: Since $\pi^{-1}(U)$ is open for every U open in X_α.
□

Theorem 15.8.4. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Then the product topology on $\prod_{\alpha \in A} X_\alpha$ is the unique topology such that, for every topological space Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f : Z \to X_\alpha$ is continuous.*

Proof:

⟨1⟩1. If we give $\prod_{\alpha \in A} X_\alpha$ the product topology, then for every topological space Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
⟨2⟩1. Give $\prod_{\alpha \in A} X_\alpha$ the product topology.
⟨2⟩2. LET: Z be a topological space.
⟨2⟩3. LET: $f : Z \to \prod_{\alpha \in A} X_\alpha$
⟨2⟩4. If f is continuous then, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
PROOF: Since the composite of two continuous functions is continuous.
⟨2⟩5. If, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then f is continuous.
⟨3⟩1. ASSUME: For all $\alpha \in A$ we have $\pi_\alpha \circ f$ is continuous.
⟨3⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with U_α open in X_α such that $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \ldots, \alpha_n$.
⟨3⟩3. For all α we have $f^{-1}({\pi_\alpha}^{-1}(U_\alpha))$ is open in Z.
⟨3⟩4. $f^{-1}(\prod_\alpha U_\alpha)$ is open in Z
PROOF: Since $f^{-1}(\prod_\alpha U_\alpha) = f^{-1}({\pi_{\alpha_1}}^{-1}(U_{\alpha_1})) \cap \cdots \cap f^{-1}({\pi_{\alpha_n}}^{-1}(U_{\alpha_n}))$.
⟨1⟩2. If $\mathcal{T}$ is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological pace Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then $\mathcal{T}$ is the product topology.
⟨2⟩1. ASSUME: $\mathcal{T}$ is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological pace Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
⟨2⟩2. LET: $\mathcal{T}_p$ be the product topology.
⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_p$
⟨3⟩1. LET: $Z = (\prod_\alpha X_\alpha, \mathcal{T}_p)$
⟨3⟩2. LET: $f : Z \to \prod_\alpha X_\alpha$ be the identity function
⟨3⟩3. For all α we have $\pi_\alpha \circ f$ is continuous.
⟨3⟩4. f is continuous.
PROOF: ⟨2⟩1
⟨3⟩5. Every set open in $\mathcal{T}$ is open in $\mathcal{T}_p$
⟨2⟩4. $\mathcal{T}_p \subseteq \mathcal{T}$
⟨3⟩1. $\mathrm{id}_{\prod_\alpha X_\alpha}$ is continuous.
⟨3⟩2. For all α we have π_α is continuous.
PROOF: ⟨2⟩1
⟨3⟩3. $\mathcal{T}_p \subseteq \mathcal{T}$
PROOF: Since $\mathcal{T}_p$ is the coarsest topology such that every π_α is continuous.

□

Example 15.8.5. It is not true that, for any function $f : \prod_{\alpha \in A} X_\alpha \to Y$, if f is continuous in every variable separately then f is continuous.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

Then f is continuous in x and in y, but is not continuous.

Proposition 15.8.6. *Let $\{X_i\}_{i\in I}$ be a nonempty family of topological spaces. The product topology on $\prod_{i\in I}$ is the topology generated by the subbasis $\{{\pi_i}^{-1}(U) : i \in I, U$ is open in $X_i\}$.*

PROOF:
⟨1⟩1. $\{{\pi_i}^{-1}(U) : i \in I, U$ is open in $X_i\}$ is a subbasis for a topology on $\prod_{i\in I} X_i$.
 ⟨2⟩1. PICK $i_0 \in I$
 ⟨2⟩2. $\prod_{i\in I} X_i = {\pi_{i_0}}^{-1}(X_{i_0})$
⟨1⟩2. The topology generated by this subbasis is the product topology.
 PROOF: Since the basis in Proposition 15.8.3 is the set of all finite intersections of elements of this subbasis.
□

15.8.2 Closure

Proposition 15.8.7. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i\in I} \overline{A_i} = \overline{\prod_{i\in I} A_i} \ .$$

PROOF:
⟨1⟩1. $\prod_{i\in I} \overline{A_i} \subseteq \overline{\prod_{i\in I} A_i}$
 ⟨2⟩1. LET: $x \in \prod_{i\in I} \overline{A_i}$
 ⟨2⟩2. For any family $\{U_i\}_{i\in I}$ where each U_i is open in X_i, and $U_i = X_i$ for all but finitely many $i \in I$, if $x \in \prod_{i\in I} U_i$ then $\prod_{i\in I} U_i$ intersects $\prod_{i\in I} A_i$.
  ⟨3⟩1. LET: $\{U_i\}_{i\in I}$ be a family where each U_i is open in X_i, and $U_i = X_i$ for all but finitely many i.
  ⟨3⟩2. ASSUME: $x \in \prod_{i\in I}$
  ⟨3⟩3. For all $i \in I$ we have U_i intersects A_i
   PROOF: Since $\pi_i(x) \in \overline{A_i}$ and U_i is a neighbourhood of $\pi_i(x)$.
  ⟨3⟩4. $\prod_{i\in I} U_i$ intersects $\prod_{i\in I} A_i$
 ⟨2⟩3. $x \in \overline{\prod_{i\in I} A_i}$
  PROOF: Proposition 14.13.9.
⟨1⟩2. $\overline{\prod_{i\in I} A_i} \subseteq \prod_{i\in I} \overline{A_i}$
 ⟨2⟩1. LET: $x \in \overline{\prod_{i\in I} A_i}$
 ⟨2⟩2. LET: $i \in I$
  PROVE: $\pi_i(x) \in \overline{A_i}$
 ⟨2⟩3. LET: U be a neighbourhood of $\pi_i(x)$ in X_i
 ⟨2⟩4. ${\pi_i}^{-1}(U)$ is a neighbourhood of x in $\prod_{i\in I} X_i$
 ⟨2⟩5. PICK $y \in {\pi_i}^{-1}(U) \cap \prod_{i\in I} A_i$
 ⟨2⟩6. $\pi_i(y) \in U \cap A_i$
□

15.8.3 Convergence

Proposition 15.8.8. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Let (x_n) be a sequence of points in $\prod_{i\in I} X_i$ and $l \in \prod_{i\in I} X_i$. Then $x_n \to l$ as $n \to \infty$ if and* *only if, for all $i \in I$, we have $\pi_i(x_n) \to \pi_i(l)$ as $n \to \infty$.*

PROOF:
⟨1⟩1. If $x_n \to l$ as $n \to \infty$ then, for all $i \in I$, we have $\pi_i(x_n) \to \pi_i(l)$ as $n \to \infty$.
PROOF: Proposition 15.1.2.
⟨1⟩2. If, for all $i \in I$, we have $\pi_i(x_n) \to \pi_i(l)$ as $n \to \infty$, then $x_n \to l$ as $n \to \infty$.
⟨2⟩1. ASSUME: For all $i \in I$ we have $\pi_i(x_n) \to \pi_i(l)$ as $n \to \infty$.
⟨2⟩2. LET: U be a neighbourhood of l.
⟨2⟩3. PICK $i_1, \ldots, i_n \in I$ and open sets U_j in X_{i_j} for $j = 1, \ldots, n$ such that $l \in {\pi_{i_1}}^{-1}(U_1) \cap \cdots \cap {\pi_{i_n}}^{-1}(U_n) \subseteq U$
⟨2⟩4. For $j = 1, \ldots, n$ we have $\pi_{i_j}(l) \in U_j$
⟨2⟩5. PICK N such that, for all $m \geqslant N$, we have $\pi_{i_j}(x_m) \in U_j$
⟨2⟩6. $\forall m \geqslant N. x_m \in U$
□

15.9 Topological Disjoint Union

Definition 15.9.1 (Coproduct Topology)**.** Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The *coproduct topology* on $\coprod_{\alpha \in A} X_\alpha$ is

$$\mathcal{T} = \left\{ \coprod_{\alpha \in A} U_\alpha : \{U_\alpha\}_{\alpha \in A} \text{ is a family with } U_\alpha \text{ open in } X_\alpha \text{ for all } \alpha \right\} .$$

We prove this is a topology.

PROOF:
⟨1⟩1. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
PROOF:
$$\bigcup_{i \in I} \coprod_{\alpha \in A} U_{i\alpha} = \coprod_{\alpha \in A} \bigcup_{i \in I} U_{i\alpha}$$
⟨1⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF:
$$\coprod_{\alpha \in A} U_\alpha \cap \coprod_{\alpha \in A} V_\alpha = \coprod_{\alpha \in A} (U_\alpha \cap V_\alpha)$$
⟨1⟩3. $\coprod_{\alpha \in A} X_\alpha \in \mathcal{T}$
PROOF: Since every X_α is open in X_α.
□

Proposition 15.9.2. *The coproduct topology is the finest topology on $\coprod_{\alpha \in A} X_\alpha$ such that every injection $\kappa_\alpha : X_\alpha \to \coprod_{\alpha \in A} X_\alpha$ is continuous.*

PROOF:
⟨1⟩1. LET: $P = \coprod_{\alpha \in A} X_\alpha$
⟨1⟩2. LET: $\mathcal{T}_c$ be the coproduct topology.
⟨1⟩3. LET: $\mathcal{T}$ be any topology on P
⟨1⟩4. For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \to (P, \mathcal{T}_c)$ is continuous.

⟨2⟩1. LET: $\alpha \in A$
⟨2⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with each U_α open in X_α.
⟨2⟩3. For all $\alpha \in A$, we have ${\kappa_\alpha}^{-1}(\coprod_{\alpha \in A} U_\alpha)$ is open in X_α.
PROOF: Since ${\kappa_\alpha}^{-1}(\coprod_{\alpha \in A} U_\alpha) = U_\alpha$.
⟨1⟩5. If, for all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \to (P, \mathcal{T})$ is continuous, then $\mathcal{T} \subseteq \mathcal{T}_c$.
⟨2⟩1. ASSUME: For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \to (P, \mathcal{T})$ is continuous.
⟨2⟩2. LET: $U \in \mathcal{T}$
⟨2⟩3. For all $\alpha \in a$, we have ${\kappa_\alpha}^{-1}(U)$ is open in X_α.
⟨2⟩4. $U = \coprod_{\alpha \in A} {\kappa_\alpha}^{-1}(U) \in \mathcal{T}_c$
□

Theorem 15.9.3. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The coproduct topology is the unique topology on $\coprod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : \coprod_{\alpha \in A} X_\alpha \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.*

PROOF:
⟨1⟩1. LET: $X = \coprod_{\alpha \in A} X_\alpha$
⟨1⟩2. LET: $\mathcal{T}_c$ be the coproduct topology.
⟨1⟩3. For every topological space Z and function $f : (X, \mathcal{T}_c) \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
⟨2⟩1. LET: Z be a topological space.
⟨2⟩2. LET: $f : X \to Z$
⟨2⟩3. If f is continuous then $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
PROOF: Because the composite of two continuous functions is continuous.
⟨2⟩4. If $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous then f is continuous.
⟨3⟩1. ASSUME: $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
⟨3⟩2. LET: U be open in Z
⟨3⟩3. For all $\alpha \in A$ we have ${\kappa_\alpha}^{-1}(f^{-1}(U))$ is open in X_α
⟨3⟩4. $f^{-1}(U) = \coprod_{\alpha \in A} {\kappa_\alpha}^{-1}(f^{-1}(U))$
⟨3⟩5. $f^{-1}(U)$ is open in X
⟨1⟩4. For any topology $\mathcal{T}$ on X, if for every topological space Z and function $f : (X, \mathcal{T}) \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous, then $\mathcal{T} = \mathcal{T}_c$.
⟨2⟩1. LET: $\mathcal{T}$ be a topology on X.
⟨2⟩2. ASSUME: For every topological space Z and function $f : (X, \mathcal{T}) \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_c$
⟨3⟩1. For all $\alpha \in A$ we have $\kappa_\alpha : X_\alpha \to (X, \mathcal{T})$ is continuous.
PROOF: From ⟨2⟩1 since id_X is continuous.
⟨3⟩2. $\mathcal{T} \subseteq \mathcal{T}_c$
PROOF: Proposition 15.9.2.
⟨2⟩4. $\mathcal{T}_c \subseteq \mathcal{T}$
⟨3⟩1. LET: $f : (X, \mathcal{T}) \to (X, \mathcal{T}_c)$ be the identity function.
⟨3⟩2. $f \circ \kappa_\alpha$ is continuous for all α.

⟨3⟩3. f is continuous.
PROOF: ⟨2⟩1
⟨3⟩4. $\mathcal{T}_c \subseteq \mathcal{T}$
□

15.10 Quotient Spaces

Definition 15.10.1 (Quotient Topology)**.** Let X be a topological space, S a set, and $\pi : X \twoheadrightarrow S$ be a surjection. The *quotient topology* on S induced by π is $\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
PROOF: Since $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}$.
⟨1⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.
PROOF: Since $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$.
⟨1⟩3. $X \in \mathcal{T}$
PROOF: Since $X = \pi^{-1}(Y)$.
□

Proposition 15.10.2. *Let X be a topological space, S a set and $\pi : X \twoheadrightarrow S$ a surjection. Then the quotient topology on S is the finest topology such that π is continuous.*

PROOF: Immediate from definitions. □

Theorem 15.10.3. *Let X be a topological space, let S be a set, and let $\pi : X \twoheadrightarrow S$ be surjective. Then the quotient topology on S is the unique topology such that, for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.*

PROOF:
⟨1⟩1. If S is given the quotient topology, then for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.
⟨2⟩1. Give S the quotient topology.
⟨2⟩2. LET: Z be a topological space.
⟨2⟩3. LET: $f : S \to Z$
⟨2⟩4. If f is continuous then $f \circ \pi$ is continuous.
PROOF: The composite of two continuous functions is continuous.
⟨2⟩5. If $f \circ \pi$ is continuous then f is continuous.
⟨3⟩1. ASSUME: $f \circ \pi$ is continuous.
⟨3⟩2. LET: U be open in Z.
⟨3⟩3. $\pi^{-1}(f^{-1}(U))$ is open in X.
⟨3⟩4. $f^{-1}(U)$ is open in S.

$\langle 1 \rangle 2$. If S is given a topology such that, for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous, then that topology is the quotient topology.

$\langle 2 \rangle 1$. Give S a topology such that, for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.

$\langle 2 \rangle 2$. LET: $U \subseteq S$

$\langle 2 \rangle 3$. If $\pi^{-1}(U)$ is open in X then U is open in S.

$\langle 3 \rangle 1$. LET: Z be S under the quotient topology induced by π.

$\langle 3 \rangle 2$. LET: $f : S \to Z$ be the identity function.

$\langle 3 \rangle 3$. $f \circ \pi$ is continuous.

$\langle 3 \rangle 4$. f is continuous.

PROOF: $\langle 2 \rangle 1$

$\langle 3 \rangle 5$. U is open in Z.

$\langle 3 \rangle 6$. U is open in X.

$\langle 2 \rangle 4$. If U is open in S then $\pi^{-1}(U)$ is open in X.

PROOF: Since π is continuous (taking $Z = S$ and $f = \mathrm{id}_S$ in $\langle 2 \rangle 1$).

$\square$

15.10.1 Quotient Maps

Definition 15.10.4 (Quotient Map)**.** Let X and S be topological spaces and $\pi : X \to S$. Then π is a *quotient map* iff π is surjective and the topology on S is the quotient topology induced by π.

Proposition 15.10.5. *Let X and Y be topological spaces. Let $f : X \to Y$. Then f is a quotient map if and only if f is surjective and strongly continuous.*

PROOF: Immediate from definition. $\square$

Proposition 15.10.6. *Let X and Y be topological spaces. Let $p : X \twoheadrightarrow Y$ be surjective. Then the following are equivalent.*

1. *p is a quotient map.*
2. *p is continuous and maps saturated open sets to open sets.*
3. *p is continuous and maps saturated closed sets to closed sets.*

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: p is a quotient map.

$\langle 2 \rangle 2$. p is continuous.

$\langle 2 \rangle 3$. p maps saturated open sets to open sets.

$\langle 3 \rangle 1$. LET: $U \subseteq X$ be a saturated open set.

$\langle 3 \rangle 2$. $p^{-1}(p(U)) = U$

$\langle 3 \rangle 3$. $p^{-1}(p(U))$ is open in X.

$\langle 3 \rangle 4$. $p(U)$ is open in Y.

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

⟨2⟩1. Assume: p is continuous and maps saturated open sets to open sets.
⟨2⟩2. Let: C be a saturated closed set in X.
⟨2⟩3. $X - C$ is a saturated open set.
⟨2⟩4. $Y - p(C)$ is open.
⟨2⟩5. $p(C)$ is closed.
⟨1⟩3. $3 \Rightarrow 1$
⟨2⟩1. Assume: p is continuous and maps closed sets to closed sets.
⟨2⟩2. Let: $C \subseteq Y$
⟨2⟩3. Assume: $p^{-1}(C)$ is closed in X.
Prove: C is closed in Y.
⟨2⟩4. $p^{-1}(C)$ is saturated.
⟨2⟩5. $p(p^{-1}(C))$ is closed.
⟨2⟩6. C is closed.
□

Corollary 15.10.6.1. *Let X and Y be topological spaces. Let $p : X \to Y$ be continuous and surjective. If p is either an open map or a closed map, then p is a quotient map.*

Example 15.10.7. The converse does not hold.

Let $A = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0 \vee y = 0\}$. Then the first projection $\pi_1 : A \to \mathbb{R}$ is a quotient map that is neither an open map nor a closed map.

Proof:
⟨1⟩1. π_1 is a quotient map.
⟨2⟩1. Let: $U \subseteq \mathbb{R}$
⟨2⟩2. If U is open then ${\pi_1}^{-1}(U)$ is open.
Proof: Since ${\pi_1}^{-1}(U) = (U \times \mathbb{R}) \cap A$.
⟨2⟩3. If ${\pi_1}^{-1}(U)$ is open then U is open.
⟨3⟩1. Assume: ${\pi_1}^{-1}(U)$ is open.
⟨3⟩2. Let: $x \in U$
⟨3⟩3. $(x, 0) \in {\pi_1}^{-1}(U)$
⟨3⟩4. Pick open neighbourhoods V of x and W of 0 such that $V \times W \subseteq {\pi_1}^{-1}(U)$
⟨3⟩5. $V \subseteq U$
Proof: For all $x' \in V$ we have $(x', 0) \in V \times W \subseteq {\pi_1}^{-1}(U)$.
⟨1⟩2. π_1 is not an open map.
Proof: $\pi_1(((-1, 1) \times (1, 2)) \cap A) = [0, 1)$ which is not open in $\mathbb{R}$.
⟨1⟩3. π_1 is not a closed map.
Proof: $\pi_1(\{(x, 1/x) \in \mathbb{R}^2 : x > 0\}) = (0, +\infty)$ is not closed in $\mathbb{R}$.
□

Corollary 15.10.7.1. *Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be families of topological spaces and $p_i : X_i \to Y_i$ for all $i \in I$.*

1. *If every p_i is an open quotient map, then $\prod_{i \in I} p_i : \prod_{i \in I} X_i \twoheadrightarrow \prod_{i \in I} Y_i$ is an open quotient map.*

2. *If every p_i is a closed quotient map, then $\prod_{i \in I} p_i : \prod_{i \in I} X_i \twoheadrightarrow \prod_{i \in I} Y_i$ is a closed quotient map.*

Example 15.10.8. The product of two quotient maps is not necessarily a quotient map.

Let Y be the quotient space of $\mathbb{R}_K$ obtained by collapsing the set K to a point. Let $p : \mathbb{R}_K \twoheadrightarrow Y$ be the quotient map. Then $q \times q : \mathbb{R}_K^2 \to Y^2$ is not a quotient map.

PROOF:
⟨1⟩1. LET: $\Delta = \{(y, y) : y \in Y\}$
⟨1⟩2. Y is not Hausdorff.
⟨2⟩1. LET: $*_K \in Y$ be the point such thta $q(K) = \{*_K\}$
⟨2⟩2. ASSUME: for a contradiction U and V are disjoint neighbourhoods of 0 and $*_K$
⟨2⟩3. $q^{-1}(U)$ and $q^{-1}(V)$ are disjoint open sets with $0 \in q^{-1}(U)$ and $K \subseteq q^{-1}(V)$
⟨2⟩4. Q.E.D.
PROOF: This is a contradiction.
⟨1⟩3. Δ is not closed in Y^2.
⟨1⟩4. $(q \times q)^{-1}(\Delta)$ is closed in $\mathbb{R}_K^2$
PROOF: It is $\{(x, x) : x \in \mathbb{R}\} \cup K^2$.
□

Proposition 15.10.9. *Let $\pi : X \twoheadrightarrow S$ be a quotient map. Let Z be a topological space. Let $f : X \to Z$ be continuous. Then there exists a continuous map $g : S \to Z$ such that $f = g \circ \pi$ if and only if, for all $s \in S$, we have f is constant on $\pi^{-1}(s)$.*

PROOF: From Theorem 15.10.3. □

Proposition 15.10.10. *Let Z be a topological space. Define $\pi : [0, 1] \to S^1$ by $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$. Given any continuous function $f : S^1 \to Z$, we have $f \circ \pi$ is a loop in Z. This defines a bijection between* **Top**$[S^1, Z]$ *and the set of loops in Z.*

PROOF: Since π is a quotient map. □

Definition 15.10.11 (Projective Space)**.** The *projective space* $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by $\sim$ where $x \sim \lambda x$ for all $x \in \mathbb{R}^{n+1} - \{0\}$ and $\lambda \in \mathbb{R}$.

Definition 15.10.12 (Torus)**.** The *torus* T is the quotient of $[0, 1]^2$ by $\sim$ where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$.

Definition 15.10.13 (Möbius Band)**.** The *Möbius band* is the quotient of $[0, 1]^2$ by $\sim$ where $(0, y) \sim (1, 1 - y)$.

Definition 15.10.14 (Klein Bottle)**.** The *Klein bottle* is the quotient of $[0, 1]^2$ by $\sim$ where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$.

Proposition 15.10.15. *$\mathbb{RP}^2$ is the quotient of $[0,1]^2$ by $\sim$ where $(x,0) \sim (1-x,1)$ and $(0,y) \sim (1,1-y)$.*

PROOF:TODO

Example 15.10.16. Let $\{X_i\}_{i\in I}$ be a family of topological spaces and $\{Y_i\}_{i\in I}$ a family of sets. Let $q_i : X_i \twoheadrightarrow Y_i$ be a surjective function for all $i \in I$. Give each Y_i the quotient topology. It is not true in general that the product topology on $\prod_{i\in I} Y_i$ is the same as the quotient topology induced by $\prod_{i\in I} q_i : \prod_{i\in I} X_i \twoheadrightarrow \prod_{i\in I} Y_i$.

PROOF:
⟨1⟩1. LET: $X^* = \mathbb{R} - \mathbb{Z}_+ + \{b\}$ be the quotient space obtained from $\mathbb{R}$ by identifying the subset $\mathbb{Z}_+$ to the point b.
⟨1⟩2. LET: $p : \mathbb{R} \to X^*$ be the quotient map.
PROVE: $p \times \mathrm{id}_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \to X^* \times \mathbb{Q}$ is not a quotient map.
⟨1⟩3. For $n \in \mathbb{Z}_+$,
LET: $c_n = \sqrt{2}/n$
⟨1⟩4. For $n \in \mathbb{Z}_+$,
LET: $U_n = \{(x,y) \in \mathbb{Q} \times \mathbb{R} : n - 1/4 < x < n + 1/4$ and $((y > x + c_n - n$ and $y > -x + c_n + n)$ or $(y < x + c_n - n$ and $y < -x + c_n + n))\}$
⟨1⟩5. For all $n \in \mathbb{Z}_+$, U_n is open in $\mathbb{R} \times \mathbb{Q}$
⟨1⟩6. For all $n \in \mathbb{Z}_+$ we have $\{n\} \times \mathbb{Q} \subseteq U_n$
⟨1⟩7. LET: $U = \bigcup_{n\in\mathbb{Z}_+} U_n$
⟨1⟩8. U is open in $\mathbb{R} \times \mathbb{Q}$.
⟨1⟩9. U is saturated with respect to $p \times \mathrm{id}_{\mathbb{Q}}$.
⟨1⟩10. LET: $U' = (p \times \mathrm{id}_{\mathbb{Q}})(U)$
⟨1⟩11. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$.

Proposition 15.10.17. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $\phi : Y \to X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \to X$ such that $\pi \circ \Phi = \phi \restriction U$. Then ϕ is continuous.

Proposition 15.10.18. *Let X be a topological space and $\sim$ an equivalence relation on X. If $X/\sim$ is Hausdorff then every equivalence class of $\sim$ is closed in X.*

Definition 15.10.19. Let X be a topological space and $A_1, \ldots, A_r \subseteq X$. Then $X/A_1, \ldots, A_r$ is the quotient space of X with respect to $\sim$ where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 15.10.20 (Cone)**.** Let X be a topological space. The *cone over X* is the space $(X \times [0,1])/(X \times \{1\})$.

Definition 15.10.21 (Suspension)**.** Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1,1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 15.10.22 (Wedge Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 15.10.23 (Smash Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 15.10.24. $D^n/S^{n-1} \cong S^n$

PROOF:
⟨1⟩1. LET: $\phi : D^n/S^{n-1} \to S^n$ be the function induced by the map $D^n \to S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.
⟨1⟩2. ϕ is a bijection.
⟨1⟩3. ϕ is a homeomorphism.
PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.
□

15.11 Box Topology

Definition 15.11.1 (Box Topology)**.** Let $\{X_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $X = \prod_{i \in I} X_i$ is the topology generated by the basis $\mathcal{B} = \{\prod_{i \in I} U_i : \{U_i\}_{i \in I}$ is a family with each U_i an open set in $X_i\}$.

We prove this is a basis for a topology.

PROOF:
⟨1⟩1. $\bigcup \mathcal{B} = X$
PROOF: Since $\prod_{i \in I} X_i \in \mathcal{B}$.
⟨1⟩2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
⟨2⟩1. LET: $B_1, B_2 \in \mathcal{B}$
⟨2⟩2. LET: $x \in B_1 \cap B_2$
⟨2⟩3. PICK a family $\{U_i\}_{i \in I}$ such that $B_1 = \prod_{i \in I} U_i$.
⟨2⟩4. PICK a family $\{V_i\}_{i \in I}$ such that $B_2 = \prod_{i \in I} V_i$.
⟨2⟩5. LET: $B_3 = \prod_{i \in I}(U_i \cap V_i)$
⟨2⟩6. $x \in B_3 \subseteq B_1 \cap B_2$
□

Proposition 15.11.2. *The box topology is finer than the product topology.*

PROOF: Immediate from definitions. □

Proposition 15.11.3. *On a finite family of topological spaces, the box topology and the product topology are the same.*

PROOF: Immediate from definitions. □

Proposition 15.11.4. *The box topology is strictly finer than the product topology on the Hilbert cube.*

PROOF: The set $\prod_{n=0}^{\infty}(0, 1/(n+1)^2)$ is open in the box topology but not in the product topology. □

15.11.1 Bases

Proposition 15.11.5. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. For all $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on X_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i : \forall i \in I.B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} X_i$.*

PROOF:
⟨1⟩1. For every family $\{B_i\}_{i \in I}$ where $\forall i \in I.B_i \in \mathcal{B}_i$, we have $\prod_{i \in I} B_i$ is open in the box topology.
  PROOF: Since each B_i is open in X_i.
⟨1⟩2. For any open set U in the box topology and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
  ⟨2⟩1. LET: U be a set open in the box topology.
  ⟨2⟩2. LET: $x \in U$
  ⟨2⟩3. PICK a family $\{U_i\}_{i \in I}$ where each U_i is open in X_i such that $x \in \prod_{i \in I} U_i \subseteq U$
  ⟨2⟩4. For $i \in I$, choose $B_i \in \mathcal{B}_i$ such that $x_i \in B_i \subseteq U_i$.
  ⟨2⟩5. $\prod_{i \in I} B_i \in \mathcal{B}$
  ⟨2⟩6. $x \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} U_i \subseteq U$

□

15.11.2 Subspaces

Proposition 15.11.6. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the box topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology that $\prod_{i \in I} Y_i$ inherits as a subspace of $\prod_{i \in I} X_i$ under the box topology.*

PROOF: A basis for the box topology is

$$\begin{aligned} &\{\prod_{i \in I} V_i : V_i \text{ open in } Y_i\} \\ =&\{\prod_{i \in I} (U_i \cap Y_i) : U_i \text{ open in } X_i\} \\ =& \{\prod_{i \in I} U_i \cap \prod_{i \in I} Y_i : U_i \text{ open in } X_i\} \end{aligned}$$

which is a basis for the subspace topology by Proposition 14.7.13. □

15.11.3 Closure

Proposition 15.11.7. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

PROOF:
⟨1⟩1. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$
⟨2⟩2. For any family $\{U_i\}_{i \in I}$ where each U_i is open in X_i, if $x \in \prod_{i \in I} U_i$ then $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$.
⟨3⟩1. LET: $\{U_i\}_{i \in I}$ be a family where each U_i is open in X_i.
⟨3⟩2. ASSUME: $x \in \prod_{i \in I}$
⟨3⟩3. For all $i \in I$ we have U_i intersects A_i
PROOF: Since $\pi_i(x) \in \overline{A_i}$ and U_i is a neighbourhood of $\pi_i(x)$.
⟨3⟩4. $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$
⟨2⟩3. $x \in \overline{\prod_{i \in I} A_i}$
PROOF: Proposition 14.13.9.
⟨1⟩2. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
⟨2⟩1. LET: $x \in \overline{\prod_{i \in I} A_i}$
⟨2⟩2. LET: $i \in I$
PROVE: $\pi_i(x) \in \overline{A_i}$
⟨2⟩3. LET: U be a neighbourhood of $\pi_i(x)$ in X_i
⟨2⟩4. ${\pi_i}^{-1}(U)$ is a neighbourhood of x in $\prod_{i \in I} X_i$
⟨2⟩5. PICK $y \in {\pi_i}^{-1}(U) \cap \prod_{i \in I} A_i$
⟨2⟩6. $\pi_i(y) \in U \cap A_i$
□

15.12 Separations

Definition 15.12.1 (Separation)**.** Let X be a topological space. A *separation* of X is a pair (U, V) of disjoint nonempty oped subsets in X such that $U \cup V = X$.

Subspaces

Proposition 15.12.2. *Let X be a topological space and Y a subspace of X. Then a separation of Y is a pair (A, B) of disjoint nonempty subsets of Y, neither of which contains a limit point of the other, such that $A \cup B = Y$.*

PROOF: Since the following are equivalent:

- Neither of A and B contains a limit point of the other.
- A contains all its own limit points in Y, and B contains all its own limit points in Y.
- A and B are closed in Y.

□

15.13 Connected Spaces

Definition 15.13.1 (Connected)**.** A topological space is *connected* iff it has no separation.

15.13.1 The Real Numbers

Example 15.13.2. The space $\mathbb{R}_l$ is disconnected. The sets $(-\infty, 0)$ and $[0, +\infty)$ form a separation.

15.13.2 The Indiscrete Topology

Example 15.13.3. Any indiscrete space is connected.

15.13.3 The Cofinite Topology

Example 15.13.4. Any infinite set under the cofinite topology is connected.

PROOF:
$\langle 1\rangle 1$. LET: X be an infinite set under the cofinite topology.
$\langle 1\rangle 2$. ASSUME: for a contradiction (C, D) is a separation of X.
$\langle 1\rangle 3$. $X = (X - C) \cup (X - D) \cup (C \cap D)$
$\langle 1\rangle 4$. Q.E.D.
PROOF: This is a contradiction since X is infinite, $X - C$ and $X - D$ are finite, and $C \cap D = \emptyset$.
□

Example 15.13.5. The rationals are disconnected. For any irrational a, we have $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of $\mathbb{Q}$.

Example 15.13.6. $\mathbb{R}^\omega$ under the box topology is not connected. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 15.13.7. *A topological space X is connected if and only if the only sets that are both open and closed are $\emptyset$ and X.*

PROOF: Since (U, V) is a separation of X iff U is both open and closed and $V = X - U$. □

15.13.4 Finer and Coarser

Proposition 15.13.8. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. Assume $\mathcal{T} \subseteq \mathcal{T}'$. If $\mathcal{T}'$ is connected then $\mathcal{T}$ is connected.*

PROOF: If (C, D) is a separation of $(X, \mathcal{T})$ then it is a separation of $(X, \mathcal{T}')$. □

15.13.5 Boundary

Proposition 15.13.9. *Let X be a topological space. Let $A \subseteq X$. Let C be a connected subspace of X. If C intersects A and $X - A$ then C intersects ∂A.*

PROOF: Otherwise $(C \cap \overline{A}, C \cap \overline{X - A})$ would be a separation of C. □

15.13.6 Continuous Functions

Proposition 15.13.10. *The continuous image of a connected space is connected.*

PROOF:
⟨1⟩1. LET: X and Y be topological spaces.
⟨1⟩2. LET: $f : X \twoheadrightarrow Y$ be a surjective continuous function.
⟨1⟩3. LET: (C, D) be a separation of Y.
⟨1⟩4. $(f^{-1}(C), f^{-1}(D))$ is a separation of X.
∎

15.13.7 Subspaces

Proposition 15.13.11. *Let X be a topological space. Let (C, D) be a separation of X. Let Y be a connected subspace of X. Then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $(Y \cap C, Y \cap D)$ would be a separation of Y. ∎

Proposition 15.13.12. *Let X be a topological space. Let $\mathcal{A}$ be a set of connected subspaces of X and B a connected subspace of X. Assume that, for all $A \in \mathcal{A}$, we have $A \cap B \neq \emptyset$. Then $\bigcup \mathcal{A} \cup B$ is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction (C, D) is a separation of $\bigcup \mathcal{A} \cup B$.
⟨1⟩2. ASSUME: w.l.o.g. $B \subseteq C$
PROOF: Proposition 15.13.11.
⟨1⟩3. For all $A \in \mathcal{A}$ we have $A \subseteq C$
PROOF: Proposition 15.13.11.
⟨1⟩4. $D = \emptyset$
⟨1⟩5. Q.E.D.
PROOF: This is a contradiction.
∎

Proposition 15.13.13. *Let X be a topological space. Let A be a connected subspace of X. Let B be a subspace of X. If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction (C, D) is a separation of B.
⟨1⟩2. ASSUME: w.l.o.g. $A \subseteq C$
PROOF: Proposition 15.13.11.
⟨1⟩3. $\overline{A} \subseteq \overline{C}$
⟨1⟩4. $\overline{C} \cap D = \emptyset$
⟨1⟩5. $B \cap D = \emptyset$
⟨1⟩6. Q.E.D.
PROOF: This is a contradiction.
∎

Corollary 15.13.13.1. *The topologist's sine curve is connected.*

PROOF: The set $\{(x, \sin 1/x) : 0 < x \leqslant 1\}$ is connected, since it is the continuous image of the connected set $(0, 1]$. The topologist's sine curve is its closure, hence connected by Proposition 15.13.13. ∎

Proposition 15.13.14. *Let X be a topological space. Let (A_n) be a sequence of connected subspaces of X such that, for all n, we have $A_n \cap A_{n+1} \neq \emptyset$. Then $\bigcup_n A_n$ is connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction (C, D) is a separation of $\bigcup_n A_n$
$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $A_0 \subseteq C$
  PROOF: Proposition 15.13.11.
$\langle 1 \rangle 3$. $\forall n. A_n \subseteq C$
  $\langle 2 \rangle 1$. ASSUME: as induction hypothesis $A_n \subseteq C$
  $\langle 2 \rangle 2$. PICK $x \in A_n \cap A_{n+1}$
  $\langle 2 \rangle 3$. $x \in C$
  $\langle 2 \rangle 4$. $A_{n+1} \subseteq C$
    PROOF: Proposition 15.13.11.
$\langle 1 \rangle 4$. $\bigcup_n A_n \subseteq C$
$\langle 1 \rangle 5$. Q.E.D.
  PROOF: This is a contradiction.
∎

Proposition 15.13.15. *Let X be a connected topological space. Let $Y \subseteq X$ be connected. Let (A, B) be a separation of $X - Y$. Then $Y \cup A$ and $Y \cup B$ are connected.*

PROOF:
$\langle 1 \rangle 1$. $Y \cup A$ is connected.
  $\langle 2 \rangle 1$. ASSUME: for a contradiction (C, D) is a separation of $Y \cup A$
  $\langle 2 \rangle 2$. ASSUME: w.l.o.g. $Y \subseteq C$
  $\langle 2 \rangle 3$. PICK C' and D' open in X such that $C = C' \cap (Y \cup A)$ and $D = D' \cap (Y \cup A)$
  $\langle 2 \rangle 4$. $D = D' \cap A$
  $\langle 2 \rangle 5$. $C' \cap D' \cap A = \emptyset$
  $\langle 2 \rangle 6$. $A \subseteq C' \cup D'$
  $\langle 2 \rangle 7$. PICK A' and B' open in X such that $A = A' - Y$ and $B = B' - Y$
  $\langle 2 \rangle 8$. $A' \cap B' \subseteq Y$
  $\langle 2 \rangle 9$. $X - Y \subseteq A' \cup B'$
  $\langle 2 \rangle 10$. $A' \subseteq C' \cup D'$
  $\langle 2 \rangle 11$. $(D' \cap A', B' \cup C')$ is a separation of X.
$\langle 1 \rangle 2$. $Y \cup B$ is connected.
  PROOF: Similar.
∎

15.13.8 Order Topology

Proposition 15.13.16. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if X is a linear continuum.*

PROOF:
$\langle 1 \rangle 1$. If L is a linear continuum then L is connected.
 $\langle 2 \rangle 1$. LET: L be a linear continuum.
 $\langle 2 \rangle 2$. ASSUME: for a contradiction (A, B) is a separation of L.
 $\langle 2 \rangle 3$. PICK $a \in A$ and $b \in B$.
 $\langle 2 \rangle 4$. ASSUME: w.l.o.g. $a < b$
 $\langle 2 \rangle 5$. LET: $c = \sup\{x \in A : x < b\}$
 $\langle 2 \rangle 6$. $c \notin A$
  $\langle 3 \rangle 1$. ASSUME: for a contradiction $c \in A$.
  $\langle 3 \rangle 2$. PICK $e > c$ such that $[c, e) \subseteq A$.
  $\langle 3 \rangle 3$. PICK z such that $c < z < e$.
  $\langle 3 \rangle 4$. $z \in A$
  $\langle 3 \rangle 5$. Q.E.D.
   PROOF: This contradicts $\langle 2 \rangle 5$.
 $\langle 2 \rangle 7$. $c \notin B$
  $\langle 3 \rangle 1$. ASSUME: for a contradictis $c \in B$.
  $\langle 3 \rangle 2$. PICK $d < c$ such that $(d, c] \subseteq B$.
  $\langle 3 \rangle 3$. PICK z such that $d < z < c$
  $\langle 3 \rangle 4$. z is an upper bound for $\{x \in A : x < b\}$
  $\langle 3 \rangle 5$. Q.E.D.
   PROOF: This contradicts $\langle 2 \rangle 5$.
 $\langle 2 \rangle 8$. Q.E.D.
  PROOF: This is a contradiction.
$\langle 1 \rangle 2$. If L is connected then L is a linear continuum.
 $\langle 2 \rangle 1$. ASSUME: L is connected.
 $\langle 2 \rangle 2$. L is dense.
  $\langle 3 \rangle 1$. LET: $a, b \in L$ with $a < b$.
  $\langle 3 \rangle 2$. ASSUME: for a contradiction there is no c such that $a < c < b$.
  $\langle 3 \rangle 3$. $((-\infty, b), (a, +\infty))$ is a separation of L.
 $\langle 2 \rangle 3$. L has the least upper bound property.
  $\langle 3 \rangle 1$. ASSUME: for a contradiction $S \subseteq L$ is a nonempty set bounded above with no least upper bound.
  $\langle 3 \rangle 2$. LET: $S \uparrow$ be the set of upper bounds for S.
  $\langle 3 \rangle 3$. LET: $S \uparrow\downarrow$ be the set of lower bounds for $S \uparrow$.
   PROVE: $(S \uparrow\downarrow, S \uparrow)$ is a separation of L.
  $\langle 3 \rangle 4$. $S \uparrow \neq \emptyset$
   PROOF: Since S is bounded above.
  $\langle 3 \rangle 5$. $S \uparrow\downarrow \neq \emptyset$
   PROOF: Since $\emptyset \neq S \subseteq S \uparrow\downarrow$.
  $\langle 3 \rangle 6$. $S \uparrow$ is open.
   $\langle 4 \rangle 1$. LET: $u \in S \uparrow$
   $\langle 4 \rangle 2$. PICK $v \in S \uparrow$ such that $v < u$

PROOF: Since u is not the least upper bound for S.
⟨4⟩3. $u \in (v, +\infty) \subseteq S\uparrow$
⟨3⟩7. $S\uparrow\downarrow$ is open.
⟨4⟩1. LET: $l \in S\uparrow\downarrow$
⟨4⟩2. $l \notin S\uparrow$
PROOF: Since l is not the least upper bound for S.
⟨4⟩3. PICK $s \in S$ such that $l < s$
⟨4⟩4. $l \in (-\infty, s) \subseteq S\uparrow\downarrow$
⟨3⟩8. $S\uparrow \cap S\uparrow\downarrow \neq \emptyset$
PROOF: An element of both would be a least upper bound for S.
⟨3⟩9. $S\uparrow \cup S\uparrow\downarrow = L$
⟨4⟩1. LET: $x \in L$
⟨4⟩2. ASSUME: $x \notin S\uparrow$
⟨4⟩3. There exists $s \in S$ such that $x < s$.
⟨4⟩4. $\forall u \in S\uparrow . x < u$
⟨4⟩5. $x \in S\uparrow\downarrow$
□

Theorem 15.13.17 (Intermediate Value Theorem)**.** *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$. If $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.*

PROOF: Otherwise $\{x \in X : f(x) < r\}$ and $\{x \in X : f(x) > r\}$ would form a separation of X. □

Corollary 15.13.17.1. *Every continuous function $[0,1] \to [0,1]$ has a fixed point.*

PROOF:
⟨1⟩1. LET: $f : [0,1] \to [0,1]$ be continuous.
⟨1⟩2. LET: $g : [0,1] \to [-1,1]$ be the function $g(x) = f(x) - x$.
⟨1⟩3. $g(0) \geqslant 0$
⟨1⟩4. $g(1) \leqslant 0$
⟨1⟩5. There exists $x \in [0,1]$ such that $g(x) = 0$.
PROOF: Intermediate Value Theorem.
⟨1⟩6. There exists $x \in [0,1]$ such that $f(x) = x$.
□

15.13.9 Product Topology

Proposition 15.13.18. *The product of a family of connected spaces is connected.*

PROOF:
⟨1⟩1. The product of two connected spaces is connected.
PROOF:
⟨2⟩1. LET: X and Y be connected topological spaces.

$\langle 2\rangle 2$. ASSUME: w.l.o.g. X and Y are nonempty.
$\langle 2\rangle 3$. PICK $(a, b) \in X \times Y$
$\langle 2\rangle 4$. $X \times \{b\}$ is connected.
PROOF: It is homeomorphic to X.
$\langle 2\rangle 5$. For all $x \in X$ we have $\{x\} \times Y$ is connected.
PROOF: It is homeomorphic to Y.
$\langle 2\rangle 6$. For all $x \in X$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.
PROOF: Proposition 15.13.12.
$\langle 2\rangle 7$. $X \cup Y$ is connected.
PROOF: Proposition 15.13.12 since $X \cup Y = \bigcup_{x \in X}((X \times \{b\}) \cup (\{x\} \times Y))$ and the subspaces all have the point (a, b) in common.
$\langle 1\rangle 2$. LET: $\{X_i\}_{i \in I}$ be a family of connected spaces.
$\langle 1\rangle 3$. LET: $X = \prod_{i \in I} X_i$
$\langle 1\rangle 4$. ASSUME: w.l.o.g. each X_i is nonempty.
$\langle 1\rangle 5$. PICK $a \in X$
$\langle 1\rangle 6$. For every finite $K \subseteq I$,
LET: $X_K = \{x \in X : \forall i \notin K . \pi_i(x) = \pi_i(a)\}$
$\langle 1\rangle 7$. For every finite $K \subseteq I$, we have X_K is connected.
PROOF: It is homeomorphic to $\prod_{i \in K} X_i$ which is connected by $\langle 1\rangle 1$.
$\langle 1\rangle 8$. LET: $Y = \bigcup_{K \text{ a finite subset of } I} X_K$
$\langle 1\rangle 9$. Y is connected.
PROOF: Proposition 15.13.12 since $a \in X_K$ for all K.
$\langle 1\rangle 10$. $X = \overline{Y}$
$\langle 2\rangle 1$. LET: $x \in X$
$\langle 2\rangle 2$. LET: U be a neighbourhood of x.
PROVE: U intersects Y.
$\langle 2\rangle 3$. PICK a finite subset K of I and U_i open in each X_i such that $U_i = X_i$ for all $i \notin K$, and $x \in \prod_i U_i \subseteq U$
$\langle 2\rangle 4$. LET: $y \in X$ be the point with $\pi_i(y) = \pi_i(x)$ for $i \in K$ and $\pi_i(y) = \pi_i(a)$ for $i \notin K$
$\langle 2\rangle 5$. $y \in U \cap Y$
$\langle 1\rangle 11$. X is connected.
PROOF: Proposition 15.13.13.
□

Proposition 15.13.19. *Let X and Y be topological spaces. Let A be a proper subset of X and B a proper subset of Y. Then $(X \times Y) - (A \times B)$ is connected.*

PROOF:
$\langle 1\rangle 1$. PICK $x_0 \in X - A$
$\langle 1\rangle 2$. PICK $y_0 \in Y - B$
$\langle 1\rangle 3$. LET: $C = ((X - A) \times Y) \cup (X \times \{y_0\})$
$\langle 1\rangle 4$. LET: $D = (\{x_0\} \times Y) \cup (X \times (Y - B))$
$\langle 1\rangle 5$. C is connected.
$\langle 2\rangle 1$. $C = \bigcup_{x \in X - A}(\{x\} \times Y) \cup (X \times \{y_0\})$
$\langle 2\rangle 2$. For all $x \in X - A$ we have $\{x\} \times Y$ is connected.
PROOF: It is homeomorphic to Y.

⟨2⟩3. $X \times \{y_0\}$ is connected.
PROOF: It is homeomorphic to X.
⟨2⟩4. For all $x \in X - A$ we have $(x, y_0) \in (\{x\} \times Y) \cap (X \times \{y_0\})$
⟨2⟩5. C is connected.
PROOF: Proposition 15.13.12.
⟨1⟩6. D is connected.
PROOF: Similar.
⟨1⟩7. $(X \times Y) - (A \times B) = C \cup D$
⟨1⟩8. $(X \times Y) - (A \times B)$ is connected.
PROOF: Proposition 15.13.12 since $(x_0, y_0) \in C \cap D$.
□

15.13.10 Quotient Spaces

Proposition 15.13.20. *A quotient of a connected space is connected.*

PROOF:
⟨1⟩1. LET: $p : X \twoheadrightarrow Y$ be a quotient map.
⟨1⟩2. If (C, D) is a separation of Y then $(p^{-1}(C), p^{-1}(D))$ is a separation of X.
□

Proposition 15.13.21. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Assume that Y is connected, for all $y \in Y$, we have $p^{-1}(y)$ is connected. Then X is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction (A, B) is a separation of X.
⟨1⟩2. For all $y \in Y$, either $p^{-1}(y) \subseteq A$ or $p^{-1}(y) \subseteq B$.
⟨1⟩3. $(\{y \in Y : p^{-1}(y) \subseteq A\}, \{y \in Y : p^{-1}(y) \subseteq B\})$ form a separation of Y.
⟨1⟩4. Q.E.D.
PROOF: This is a contradiction.
□

15.14 T_1 Spaces

Definition 15.14.1 (T_1). A topological space is T_1 iff every one-point set is closed.

Proposition 15.14.2. *A topological space is T_1 iff every finite set is closed.*

PROOF: Since the union of finitely many closed sets is closed. □

Proposition 15.14.3. *Let X be a topological space. Then X is T_1 if and only if, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y, and there exists a neighbourhood of y that does not contain x.*

PROOF:
⟨1⟩1. If X is T_1 then, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y, and there exists a neighbourhood of y that does not contain x.

⟨2⟩1. Assume: X is T_1.
⟨2⟩2. Let: $x, y \in X$
⟨2⟩3. Assume: $x \neq y$
⟨2⟩4. $X - \{y\}$ is a neighbourhood of x that does not contain y.
⟨2⟩5. $X - \{x\}$ is a neighbourhood of y that does not contain x.
⟨1⟩2. If, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y, and there exists a neighbourhood of y that does not contain x, then X is T_1.
⟨2⟩1. Assume: For all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y, and there exists a neighbourhood of y that does not contain x.
⟨2⟩2. Let: $x \in X$
Prove: $\{x\}$ is closed.
⟨2⟩3. Let: $y \in X - \{x\}$
⟨2⟩4. Pick a neighbourhood U of y that does not contain x.
⟨2⟩5. $y \in U \subseteq X - \{x\}$
□

15.14.1 Limit Points

Proposition 15.14.4. *Let X be a T_1 space. Let $A \subseteq X$ and $l \in X$. Then l is a limit point of A if and only if every neighbourhood of l contains infinitely many points of A.*

Proof:
⟨1⟩1. If l is a limit point of A then every neighbourhood of l contains infinitely many points of A.
⟨2⟩1. Assume: l is a limit point of A.
⟨2⟩2. Let: U be a neighbourhood of l.
⟨2⟩3. Assume: for a contradiction $U \cap A - \{l\}$ is finite.
⟨2⟩4. $U \cap A - \{l\}$ is closed.
Proof: Since X is T_1.
⟨2⟩5. $U - (A - \{l\})$ is a neighbourhood of l.
⟨2⟩6. $U - (A - \{l\})$ intersects A.
⟨2⟩7. Q.E.D.
⟨1⟩2. If every neighbourhood of l contains infinitely many points of A then l is a limit point of A.
Proof: Immediate from definitions.
□

15.15 Hausdorff Spaces

Definition 15.15.1 (Hausdorff)**.** A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 15.15.2. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
$\langle 1\rangle 1$. LET: X be a Hausdorff space.
$\langle 1\rangle 2$. LET: (a_n) be a sequence in X and $l, m \in X$
$\langle 1\rangle 3$. ASSUME: $a_n \to l$ and $a_n \to m$
$\langle 1\rangle 4$. ASSUME: for a contradiction $l \neq m$
$\langle 1\rangle 5$. PICK disjoint open sets U and V with $l \in U$ and $m \in V$
$\langle 1\rangle 6$. PICK M, N such that $\forall n \geqslant M.a_n \in U$ and $\forall n \geqslant N.a_n \in V$
$\langle 1\rangle 7$. $a_{\max(M,N)} \in U \cap V$
$\langle 1\rangle 8$. Q.E.D.
PROOF: This contradicts the fact that $U \cap V = \emptyset$.
□

Example 15.15.3. We cannot weaken the hypothesis from being Hausdorff to being T_1.

In the cofinite topology on any infinite set, every sequence converges to every point.

Proposition 15.15.4. *Any linearly ordered set is Hausdorff under the order topology.*

PROOF:
$\langle 1\rangle 1$. LET: X be a linearly ordered set under the order topology.
$\langle 1\rangle 2$. LET: $a, b \in X$ with $a \neq b$.
$\langle 1\rangle 3$. ASSUME: w.l.o.g. $a < b$.
$\langle 1\rangle 4$. CASE: There exists $c \in X$ such that $a < c < b$.
$\langle 2\rangle 1$. LET: $U = (-\infty, c)$
$\langle 2\rangle 2$. LET: $V = (c, +\infty)$
$\langle 2\rangle 3$. U and V are disjoint open sets with $a \in U$ and $b \in V$
$\langle 1\rangle 5$. CASE: There is no $c \in X$ such that $a < c < b$.
$\langle 2\rangle 1$. LET: $U = (-\infty, b)$
$\langle 2\rangle 2$. LET: $V = (a, +\infty)$
$\langle 2\rangle 3$. U and V are disjoint open sets with $a \in U$ and $b \in V$
□

Proposition 15.15.5. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:
$\langle 1\rangle 1$. LET: X be a Hausdorff space.
$\langle 1\rangle 2$. LET: Y be a subspace of X.
$\langle 1\rangle 3$. LET: $a, b \in Y$ with $a \neq b$.
$\langle 1\rangle 4$. PICK disjoint open sets U and V in X with $a \in U$ and $b \in V$.
$\langle 1\rangle 5$. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y with $a \in U \cap Y$ and $b \in V \cap Y$.
□

Proposition 15.15.6. *The disjoint union of two Hausdorff spaces is Hausdorff.*

Proposition 15.15.7. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \to B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X, then $f = g$.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.
⟨1⟩2. PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.
⟨1⟩3. PICK $x \in f^{-1}(U) \cap g^{-1}(V)$
⟨1⟩4. $f(x) = g(x) \in U \cap V$
⟨1⟩5. Q.E.D.
PROOF: This is a contradiction.
□

15.15.1 Product Topology

Proposition 15.15.8. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:
⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
⟨1⟩2. LET: $x, y \in \prod_{i \in I} X_i$ with $x \neq y$.
⟨1⟩3. PICK $i \in I$ such that $\pi_i(x) \neq \pi_i(y)$
⟨1⟩4. PICK disjoint open sets U and V in X_i such that $\pi_i(x) \in U$ and $\pi_i(y) \in V$.
⟨1⟩5. $x \in {\pi_i}^{-1}(U)$ and $y \in {\pi_i}^{-1}(V)$.
□

15.15.2 Box Topology

Proposition 15.15.9. *The box product of a family of Hausdorff spaces is Hausdorff.*

PROOF:
⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
⟨1⟩2. LET: $x, y \in \prod_{i \in I} X_i$ with $x \neq y$.
⟨1⟩3. PICK $i \in I$ such that $\pi_i(x) \neq \pi_i(y)$
⟨1⟩4. PICK disjoint open sets U and V in X_i such that $\pi_i(x) \in U$ and $\pi_i(y) \in V$.
⟨1⟩5. $x \in {\pi_i}^{-1}(U)$ and $y \in {\pi_i}^{-1}(V)$.
□

15.15.3 T_1 Spaces

Proposition 15.15.10. *Every Hausdorff space is T_1.*

PROOF:
⟨1⟩1. LET: X be a Hausdorff space.
⟨1⟩2. LET: $a \in X$
PROVE: $X - \{a\}$ is open.
⟨1⟩3. LET: $x \in X - \{a\}$
⟨1⟩4. PICK disjoint open sets U and V with $a \in U$ and $x \in V$
⟨1⟩5. $x \in V \subseteq X - U \subseteq X - \{a\}$
□

Example 15.15.11. The converse does not hold. If X is an infinite set under the cofinite topology, then X is T_1 but not Hausdorff.

Proposition 15.15.12. *Let X and Y be metric spaces. Let $f : X \to Y$ be uniformly continuous. Let $\hat{X}$ and $\hat{Y}$ be the completions of X and Y. Then f extends uniquely to a continuous map $\hat{X} \to \hat{Y}$.*

PROOF: The extension maps $\lim_{n\to\infty} x_n$ to $\lim_{n\to\infty} f(x_n)$. ∎

Proposition 15.15.13. *Let X be a topological space. Then X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) : x \in X\}$ is closed in X^2.*

PROOF:

$$
\begin{aligned}
& \Delta \text{ is closed} \\
\Leftrightarrow & X^2 - \Delta \text{ is open} \\
\Leftrightarrow & \forall x, y \in X((x,y) \notin \Delta \Rightarrow \exists V, W \text{ open in } X(x \in V \wedge y \in W \wedge V \times W \subseteq X^2 - \Delta)) \\
\Leftrightarrow & \forall x, y \in X(x \neq y \Rightarrow \exists V, W \text{ open in } X(x \in V \wedge y \in W \wedge V \cap W = \emptyset)) \\
\Leftrightarrow & X \text{ is Hausdorff}
\end{aligned}
$$

∎

15.16 Separable Spaces

Definition 15.16.1 (Separable)**.** A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

15.17 Sequential Compactness

Definition 15.17.1 (Sequentially Compact)**.** A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

15.18 Compactness

Definition 15.18.1 (Compact)**.** A topological space is *compact* iff every open cover has a finite subcover.

Proposition 15.18.2. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P. Then $X \in P$.*

PROOF:
⟨1⟩1. P is an open cover of X
⟨1⟩2. PICK a finite subcover $U_1, \ldots, U_n \in P$
⟨1⟩3. $X = U_1 \cup \cdots \cup U_n \in P$
∎

Corollary 15.18.2.1. *Let f be a compact space and $f : X \to \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. ∎

Proposition 15.18.3. *The continuous image of a compact space is compact.*

Proposition 15.18.4. *A closed subspace of a compact space is compact.*

Proposition 15.18.5. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. *X and Y are compact.*
2. *$X + Y$ is compact.*
3. *$X \times Y$ is compact.*

Proposition 15.18.6. *A compact subspace of a Hausdorff space is closed.*

Proposition 15.18.7. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 15.18.8. *A first countable compact space is sequentially compact.*

15.19 Gluing

Definition 15.19.1 (Gluing)**.** Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \to Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where $\sim$ is the equivalence relation generated by $x \sim \phi(x)$ for all $x \in X$.

Proposition 15.19.2. *Y is a subspace of $Y \cup_\phi X$.*

Definition 15.19.3. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0,1])/\alpha$ is the quotient space of $X \times [0,1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x \in X$.

Definition 15.19.4 (Möbius Strip)**.** The *Möbius strip* is $([-1,1] \times [0,1])/\alpha$ where $\alpha(x) = -x$.

Definition 15.19.5 (Klein Bottle)**.** The *Klein bottle* is $(S^1 \times [0,1])/\alpha$ where $\alpha(z) = \overline{z}$.

Proposition 15.19.6. *Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\mathrm{id}_{\partial M}} M \cong K$.*

PROOF:
⟨1⟩1. LET: $f : ([-1,1] \times [0,1]) + ([-1,1] \times [0,1]) \to S^1 \times [0,1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{\pi i \theta/2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-\pi i \theta/2}, t)$.
⟨1⟩2. f induces a bijection $M \cup_{\mathrm{id}_{\partial M}} M \approx K$
⟨1⟩3. f is a homeomorphism.
∎

15.20 Homogeneous Spaces

Definition 15.20.1 (Homogeneous)**.** A topological space X is *homogeneous* iff, for any $x, y \in X$, there exists a homeomorphism $f : X \cong X$ such that $f(x) = y$.

15.21 Regular Spaces

Definition 15.21.1 (Regular)**.** A topological space X is *regular* iff it is T_1 and, for every closed set A and point $x \notin A$, there exist disjoint open sets U and V with $A \subseteq U$ and $x \in V$.

15.22 Totally Disconnected Spaces

Definition 15.22.1 (Totally Disconnected)**.** A topological space X is *totally disconnected* iff the only connected subspaces are the one-point subspaces.

Example 15.22.2. Every discrete space is totally disconnected.

Example 15.22.3. The rationals are totally disconnected.

15.23 Path Connected Spaces

Definition 15.23.1 (Path-connected)**.** A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \to X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

15.23.1 The Ordered Square

Proposition 15.23.2. *The ordered square is not path connected.*

PROOF:
$\langle 1 \rangle$1. ASSUME: for a contradiction $p : [a, b] \to I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.
$\langle 1 \rangle$2. p is surjective.
PROOF: Intermediate Value Theorem.
$\langle 1 \rangle$3. For all $x \in [0, 1]$, the set $p^{-1}(\{x\} \times (0, 1))$ is a nonempty open set in $[0, 1]$.
$\langle 1 \rangle$4. For all $x \in [0, 1]$ choose a rational $q_x \in p^{-1}(\{x\} \times (0, 1))$.
$\langle 1 \rangle$5. The mapping that maps x to q_x is an injective function $[0, 1] \to \mathbb{Q}$
$\langle 1 \rangle$6. Q.E.D.
PROOF: This contradicts the fact that $[0, 1]$ is uncountable and $\mathbb{Q}$ is countable.
$\square$

15.23.2 Punctured Euclidean Space

Proposition 15.23.3. *For $n > 1$, the punctured Euclidean space $\mathbb{R}^n - \{0\}$ is path connected.*

PROOF: Given points x and y, take the straight line from x to y if this does not pass through 0. Otherwise pick a point z not on this line, and take the two straight lines from x to z then from z to y. □

15.23.3 The Topologist's Sine Curve

Proposition 15.23.4. *The topologist's sine curve is not path connected.*

PROOF:
⟨1⟩1. LET: $S = \{(x, \sin 1/x) : 0 < x \leqslant 1\}$
⟨1⟩2. ASSUME: for a contradiction $p : [0,1] \to \overline{S}$ is a path from $(0,0)$ to $(1, \sin 1)$.
⟨1⟩3. LET: b be the largest element of $p^{-1}(\{0\} \times [-1,1])$
⟨1⟩4. For n a positive integer, choose t_n such that $b < t_n < ((n-1)b+1)/n$ and $\pi_2(p(t_t)) = (-1)^n$
⟨1⟩5. $t_n \to b$ as $n \to \infty$
⟨1⟩6. $(p(t_n))$ does not converge.
⟨1⟩7. Q.E.D.
PROOF: This is a contradiction.
□

15.23.4 The Long Line

Proposition 15.23.5. *The long line is path connected.*

PROOF:
⟨1⟩1. LET: $L = S_\Omega \times [0,1)$ be the long line.
⟨1⟩2. LET: $(a,b), (c,d) \in L$
⟨1⟩3. PICK e such that $a < e$ and $c < e$
⟨1⟩4. $(a,b), (c,d) \in [(0,0), (e,0)) \cong [0,1)$
PROOF: Using Proposition 6.5.2.
⟨1⟩5. There is a path from (a,b) to (c,d).
□

15.23.5 Continuous Functions

Proposition 15.23.6. *The continuous image of a path connected space is path connected.*

PROOF:
⟨1⟩1. LET: X be a path connected space and Y a topological space.
⟨1⟩2. LET: $f : X \twoheadrightarrow Y$ be a surjective continuous function.
PROVE: Y is path connected.
⟨1⟩3. LET: $a, b \in Y$
⟨1⟩4. PICK $x, y \in X$ with $f(x) = a$ and $f(y) = b$.
⟨1⟩5. PICK a path $p : [0,1] \to X$ from x to y.
⟨1⟩6. $f \circ p$ is a path from a to b.
□

15.23.6 Subspaces

Proposition 15.23.7. *Let $\{X\}$ be a topological space. Let $\mathcal{A}$ be a set of connected subspaces of X. If $\bigcap \mathcal{A} \neq \emptyset$ then $\bigcup \mathcal{A}$ is connected.*

PROOF:
⟨1⟩1. PICK $a \in \bigcap \mathcal{A}$
⟨1⟩2. PICK $x, y \in \bigcup \mathcal{A}$
⟨1⟩3. PICK $A, B \in \mathcal{A}$ with $x \in A$ and $y \in B$.
⟨1⟩4. PICK a path p from x to a in A, and a path q from a to y in B.
⟨1⟩5. The concatenation of p and q is a path from x to y in $\bigcup \mathcal{A}$.
□

Proposition 15.23.8. *A quotient of a path connected space is path connected.*

15.23.7 Product Topology

Proposition 15.23.9. *The product of a family of path connected spaces is path connected.*

PROOF:
⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of path connected spaces.
⟨1⟩2. LET: $x, y \in \prod_{i \in I} X_i$
⟨1⟩3. For $i \in I$, PICK a path $p_i : [0, 1] \to X_i$ from $\pi_i(x)$ to $\pi_i(y)$
⟨1⟩4. $\lambda t \in [0, 1].\lambda i \in I.p_i(t)$ is a path from x to y in $\prod_{i \in I} X_i$.
□

Proposition 15.23.10. *Let $A \subseteq \mathbb{R}^2$. If A is countable then $\mathbb{R}^2 - A$ is path connected.*

PROOF:
⟨1⟩1. LET: $x, y \in \mathbb{R}^2 - A$
⟨1⟩2. PICK two non-parallel lines L through x and L' through y that do not pass through any points in A.
PROOF: These exist since uncountably many lines pass through any point.
⟨1⟩3. There exists a path from x to y that follows L from x to the point of intersection of L and L', and then follows L' to y.
□

15.23.8 Connected Spaces

Proposition 15.23.11. *Every path connected space is connected.*

PROOF:
⟨1⟩1. LET: X be a path connected space.
⟨1⟩2. ASSUME: for a contradiction (A, B) is a separation of X.
⟨1⟩3. PICK $a \in A$ and $b \in B$
⟨1⟩4. PICK a path $p : [0, 1] \to X$ from a to b.

$\langle 1 \rangle 5$. $(p^{-1}(A), p^{-1}(B))$ is a separation of $[0,1]$.
$\langle 1 \rangle 6$. Q.E.D.
Proof: This contradicts Proposition 15.13.16.
□

Corollary 15.23.11.1. *For $n > 1$, we have $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic.*

Proof: Removing a point from $\mathbb{R}$ gives a disconnected space. □

Proposition 15.23.12. *Every open connected subspace of $\mathbb{R}^2$ is path connected.*

Proof:
$\langle 1 \rangle 1$. Let: U be an open connected subspace of $\mathbb{R}^2$.
$\langle 1 \rangle 2$. Assume: w.l.o.g. $U \neq \emptyset$
$\langle 1 \rangle 3$. Pick $x_0 \in U$
$\langle 1 \rangle 4$. Let: $V = \{x \in U : \text{there exists a path from } x_0 \text{ to } x\}$
$\langle 1 \rangle 5$. V is open in U.
$\langle 2 \rangle 1$. Let: $x \in V$
$\langle 2 \rangle 2$. Pick $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
$\langle 2 \rangle 3$. $B(x, \epsilon) \subseteq V$
Proof: For all $y \in B(x, \epsilon)$, take a path from x_0 to x and then a straight line from x to y.
$\langle 1 \rangle 6$. V is closed in U.
$\langle 2 \rangle 1$. Let: $x \in U - V$
$\langle 2 \rangle 2$. Pick $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
$\langle 2 \rangle 3$. $B(x, \epsilon) \subseteq U - V$
$\langle 3 \rangle 1$. Let: $y \in B(x, \epsilon)$
$\langle 3 \rangle 2$. There is a path from y to x.
$\langle 3 \rangle 3$. There is no path from x_0 to y.
$\langle 1 \rangle 7$. $V = U$
Proof: U is connected.
□

15.24 Locally Homeomorphic

Definition 15.24.1. Let X and Y be topological spaces. Then X is *locally homeomorphic* to Y if and only if every point in X has a neighbourhood that is homeomorphic to an open set in Y.

15.24.1 The Long Line

Proposition 15.24.2. *The long line is locally homeomorphic to* $[0, 1)$.

Proof: By Proposition 6.5.2. □

15.25 Components

Definition 15.25.1 ((Connected) Component)**.** Let X be a topological space. Define the equivalence relation $\sim$ on X by: $x \sim y$ iff there exists a connected $C \subseteq X$ such that $x \in C$ and $y \in C$. The *components* of X are the equivalence classes with respect to $\sim$.

We prove this is an equivalence relation.

PROOF:
⟨1⟩1. $\sim$ is reflexive.
  PROOF: For any $x \in X$, we have $\{x\}$ is connected and $x \in \{x\}$, hence $x \sim x$.
⟨1⟩2. $\sim$ is symmetric.
  PROOF: Immediate from definition.
⟨1⟩3. $\sim$ is transitive.
  ⟨2⟩1. ASSUME: $x \sim y$ and $y \sim z$
  ⟨2⟩2. PICK connected subspaces C and D of X with $x \in C$, $y \in C$, $y \in D$ and $z \in D$.
  ⟨2⟩3. $C \cup D$ is connected.
    PROOF: Proposition 15.13.12.
  ⟨2⟩4. $x \in C \cup D$ and $z \in C \cup D$.
  ⟨2⟩5. $x \sim z$
□

Example 15.25.2. The components of $\mathbb{Q}$ are the singleton subsets.

Example 15.25.3. The components of $\mathbb{R}_l$ are the singleton subsets.

Proposition 15.25.4. *Every component of a topological space is connected.*

PROOF:
⟨1⟩1. LET: C be a component of the topological space X.
⟨1⟩2. ASSUME: for a contradiction (A, B) is a separation of C.
⟨1⟩3. PICK $a \in A$ and $b \in B$.
⟨1⟩4. $a \sim b$
⟨1⟩5. PICK a connected subspace D of X such that $a \in D$ and $b \in D$.
⟨1⟩6. $D \subseteq C$
⟨1⟩7. $(A \cap D, B \cap D)$ is a separation of D.
⟨1⟩8. Q.E.D.
  PROOF: This is a contradiction.
□

Proposition 15.25.5. *Let X be a topological space. Let A be a nonempty connected subspace of X. Then there exists a unique component C of X such that $A \subseteq C$.*

PROOF:
⟨1⟩1. PICK $a \in A$
⟨1⟩2. LET: C be the $\sim$-equivalence class of a.
⟨1⟩3. $A \subseteq C$

PROOF: For all $x \in A$ we have $a \sim x$ hence $x \in C$.
⟨1⟩4. For any component C', if $A \subseteq C'$ then $C' = C$.
PROOF: Since the components are pairwise disjoint.
□

Proposition 15.25.6. *Every component of a topological space is closed.*

PROOF:
⟨1⟩1. LET: X be a topological space.
⟨1⟩2. LET: C be a component of X.
⟨1⟩3. $\overline{C}$ is connected.
PROOF: Proposition 15.13.13.
⟨1⟩4. $\overline{C} \subseteq C$
PROOF: Proposition 15.25.5.
⟨1⟩5. $C = \overline{C}$
□

Corollary 15.25.6.1. *If a topological space has only finitely many components, then its components are open.*

15.26 Path Components

Definition 15.26.1 (Path Component)**.** Let X be a topological space. Define the equivalence relation $\sim$ on X by: $x \sim y$ iff there exists a path from x to y. The *path components* of X are the equivalence classes with respect to $\sim$.

We prove $\sim$ is an equivalence relation.

PROOF:
⟨1⟩1. $\sim$ is reflexive.
PROOF: For any $a \in X$ the constant path at a is a path from a to a.
⟨1⟩2. $\sim$ is symmetric.
PROOF: If p is a path from a to b then the reverse of p is a path from b to a.
⟨1⟩3. $\sim$ is transitive.
PROOF: If p is a path from a to b and q is a path from b to c then the concatenation of p and q is a path from a to c.
□

Example 15.26.2. The topologist's sine curve has two path components, namely $\{0\} \times [0, 1]$ (which is closed and not open) and $\{(x, \sin 1/x) : 0 < x \leqslant 1\}$ (which is open and not closed).

Proposition 15.26.3. *Every path component is path connected.*

PROOF: If x and y are in the same path component then $x \sim y$ so there is a path from x to y. □

Corollary 15.26.3.1. *Every path component is a subset of a component.*

Proposition 15.26.4. *Let X be a topological space. Let A be a nonempty path connected subspace of X. Then there exists a unique path component C of X such that $A \subseteq C$.*

PROOF:
⟨1⟩1. PICK $a \in A$
⟨1⟩2. LET: C be the path component of a.
⟨1⟩3. $A \subseteq C$
PROOF: For all $x \in A$ we have $a \sim x$ (because A is path connected) hence $x \in C$.
⟨1⟩4. For any path component C', if $A \subseteq C'$ then $C = C'$.
PROOF: This holds because the path components are pairwise disjoint.
□

Example 15.26.5. In $\mathbb{R}^\omega$ under the box topology, $\vec{x}$ and $\vec{y}$ are in the same component if and only if $\vec{x} - \vec{y}$ is eventually zero.

PROOF:
⟨1⟩1. LET: B be the set of sequences that are eventually zero.
⟨1⟩2. B is connected.
PROOF: For $\vec{x} \in B$, the straight line path from 0 to $\vec{x}$ is continuous.
⟨1⟩3. B is maximally connected.
PROOF: Since $(B, \mathbb{R}^\omega - B)$ form a separation of $\mathbb{R}^\omega$.
⟨1⟩4. For all $\vec{y} \in \mathbb{R}^\omega$, the component that contains $\vec{y}$ is $\{\vec{x} \in \mathbb{R}^\omega : \vec{x} - \vec{y} \text{ is eventually zero}\}$.
PROOF: Since the function that maps $\vec{x}$ to $\vec{x} + \vec{y}$ is a homeomorphism of $\mathbb{R}^\omega$ with itself.
□

Example 15.26.6. The path components of I_o^2 are $\{\{x\} \times [0,1] : 0 \leqslant x \leqslant 1\}$.

PROOF:
⟨1⟩1. For all $x \in [0,1]$ we have $\{x\} \times [0,1]$ is path connected.
PROOF: It is homeomorphic to $[0,1]$.
⟨1⟩2. Given $x, y, s, t \in [0,1]$ with $x \neq y$, there is no path from (x,s) to (y,t).
⟨2⟩1. ASSUME: for a contradiction $p : [0,1] \to I_o^2$ is a path from (x,s) to (y,t).
⟨2⟩2. For z between x and y, PICK a rational $q_z \in [0,1]$ such that $p(q_z) \in \{z\} \times [0,1]$.
⟨2⟩3. $\{q_z : z \text{ is between } x \text{ and } y\}$ is an uncountable set of rationals.
⟨2⟩4. Q.E.D.
PROOF: This is a contradiction.
□

15.27 Weak Local Connectedness

Definition 15.27.1 (Weakly Locally Connected)**.** Let X be a topological space and $x \in X$. Then X is *weakly locally connected* at x iff, for every neighbourhood

U of x, there exists a connected $Y \subseteq X$ and a neighbourhood V of x such that $V \subseteq Y \subseteq U$.

15.28 Local Connectedness

Definition 15.28.1 (Locally Connected)**.** Let X be a topological space and $x \in X$. Then X is *locally connected* at x iff, for every neighbourhood U of x, there exists a connected neighbourhood V of x such that $V \subseteq U$.

The space X is *locally connected* iff it is locally connected at every point.

Example 15.28.2. Every interval and ray in the real line is connected and locally connected.

Example 15.28.3. The space $[-1, 0) \cup (0, 1]$ is locally connected but not connected.

Example 15.28.4. The topologist's sine curve is connected but not locally connected.

Example 15.28.5. The rationals $\mathbb{Q}$ are neither connected nor locally connected.

Example 15.28.6. For n a positive integer, let $a_n = (1/n, 0)$. Let $p = (0, 0)$. Let the infinite broom X be the union of all the line segments joining (a_{n+1}, q) to $(a_n, 0)$ for n any positive integer and q any rational in $[0, 1/n]$. Then X is weakly locally connected at p but not locally connected at p.

PROOF:
$\langle 1 \rangle 1$. X is weakly locally connected at p.
  $\langle 2 \rangle 1$. LET: U be any neighbourhood of p.
  $\langle 2 \rangle 2$. PICK N such that, for all $n \geqslant N$ and every rational $q \in [0, 1/n]$, the line segment joining (a_{n+1}, q) to $(a_n, 0)$ is included in U.
  $\langle 2 \rangle 3$. LET: Y be the union of all these line segments.
  $\langle 2 \rangle 4$. Y is connected.
  $\langle 2 \rangle 5$. LET: $V = B(p, 1/n) \cap X$
  $\langle 2 \rangle 6$. $V \subseteq Y \subseteq U$
$\langle 1 \rangle 2$. X is not locally connected at p.
  $\langle 2 \rangle 1$. LET: $U = B(p, 1/2) \cap X$
  $\langle 2 \rangle 2$. LET: V be a neighbourhood of p with $V \subseteq U$
  PROVE: V is disconnected.
  $\langle 2 \rangle 3$. LET: n be least such that $(a_n, 0) \in V$
  $\langle 2 \rangle 4$. $(a_{n-1}, 0) \notin V$
  $\langle 2 \rangle 5$. Some part of a line segment joining some (a_n, q) to $(a_{n-1}, 0)$ is in V
  $\langle 2 \rangle 6$. V is disconnected.
$\square$

Theorem 15.28.7. *Let X be a topological space. Then X is locally connected if and only if, for every open set U in X, every component of U is open in X.*

PROOF:
⟨1⟩1. If X is locally connected then, for every open set U in X, every component of U is open in X.
⟨2⟩1. ASSUME: X is locally connected.
⟨2⟩2. LET: U be an open set in X.
⟨2⟩3. LET: C be a component of U.
⟨2⟩4. LET: $x \in C$
⟨2⟩5. PICK a connected neighbourhood V of x in X such that $V \subseteq U$
⟨2⟩6. $x \in V \subseteq C$
⟨1⟩2. If, for every open set U in X, every component of U is open in X, then X is locally connected.
⟨2⟩1. ASSUME: For every open set U in X, every component of U is open in X.
⟨2⟩2. LET: $x \in X$
⟨2⟩3. LET: U be a neighbourhood of x.
⟨2⟩4. LET: V be the component of U that contains x.
⟨2⟩5. V is a connected neighbourhood of x and $V \subseteq U$.
□

Proposition 15.28.8. *The ordered square is locally connected.*

PROOF: Since every basic open set is connected because it is a linear continuum.
□

Example 15.28.9. Let T be the union of all line segments connecting a point $(q, 0)$ to $(0, 1)$ where $q \in [0, 1]$ is rational, and all line segments connecting a point $(q, 1)$ to $(1, 0)$ where $q \in [0, 1]$ is rational. Then T is path connected but is locally connected at no point.

Proposition 15.28.10. *If a topological space is weakly locally connected at every point then it is locally connected.*

PROOF:
⟨1⟩1. LET: X be a topological space that is weakly locally connected at every point.
⟨1⟩2. For every open set U in X, every component of U is open in X.
⟨2⟩1. LET: U be an open set in X.
⟨2⟩2. LET: C be a component of U.
⟨2⟩3. For all $x \in C$, there exists a neighbourhood V of x such that $V \subseteq C$.
⟨3⟩1. LET: $x \in C$
⟨3⟩2. PICK a connected $Y \subseteq X$ and a neighbourhood V of x such that $V \subseteq Y \subseteq U$
PROOF: ⟨1⟩1
⟨3⟩3. $Y \subseteq C$
PROOF: Proposition 15.25.5.
⟨3⟩4. $V \subseteq C$
⟨2⟩4. C is open.
PROOF: Proposition 14.1.7.

⟨1⟩3. X is locally connected.
  PROOF: Theorem 15.28.7.
□

Proposition 15.28.11. *A quotient of a locally connected space is locally connected.*

PROOF:
⟨1⟩1. LET: X be a locally connected space.
⟨1⟩2. LET: $p : X \twoheadrightarrow Y$ be a quotient map.
⟨1⟩3. For every open set V in Y, every component of V is open in Y.
  ⟨2⟩1. LET: V be an open set in Y.
  ⟨2⟩2. LET: C be a component of V.
  ⟨2⟩3. $p^{-1}(C)$ is a union of components of $p^{-1}(V)$
    ⟨3⟩1. LET: $x \in p^{-1}(C)$
    ⟨3⟩2. LET: D be the component of $p^{-1}(V)$ that contains x.
      PROVE: $D \subseteq p^{-1}(C)$
    ⟨3⟩3. $p(D)$ is connected.
      PROOF: Proposition 15.13.10.
    ⟨3⟩4. $p(D) \subseteq C$
    ⟨3⟩5. $D \subseteq p^{-1}(C)$
  ⟨2⟩4. Every component of $p^{-1}(V)$ is open in X.
    PROOF: Theorem 15.28.7.
  ⟨2⟩5. $p^{-1}(C)$ is open in X.
  ⟨2⟩6. C is open in Y.
⟨1⟩4. Y is locally connected.
  PROOF: Theorem 15.28.7.
□

15.29 Local Path Connectedness

Definition 15.29.1 (Locally Path Connected)**.** Let X be a topological space and $x \in X$. Then X is *locally path connected* at x iff, for every neighbourhood U of x, there exists a path connected neighbourhood V of x such that $V \subseteq U$.

The space X is *locally path connected* iff it is locally connected at every point.

Theorem 15.29.2. *Let X be a topological space. Then X is locally path connected if and only if, for every open set U in X, every path component of U is open in X.*

PROOF:
⟨1⟩1. If X is locally path connected then, for every open set U in X, every path component of U is open in X.
  ⟨2⟩1. ASSUME: X is locally path connected.
  ⟨2⟩2. LET: U be an open set in X.
  ⟨2⟩3. LET: C be a path component of U.

⟨2⟩4. LET: $x \in C$
⟨2⟩5. PICK a path connected neighbourhood V of x in X such that $V \subseteq U$
⟨2⟩6. $x \in V \subseteq C$
⟨1⟩2. If, for every open set U in X, every path component of U is open in X, then X is locally path connected.
⟨2⟩1. ASSUME: For every open set U in X, every path component of U is open in X.
⟨2⟩2. LET: $x \in X$
⟨2⟩3. LET: U be a neighbourhood of x.
⟨2⟩4. LET: V be the path component of U that contains x.
⟨2⟩5. V is a path connected neighbourhood of x and $V \subseteq U$.
□

Theorem 15.29.3. *In a locally path connected space, the components are the same as the path components.*

PROOF:
⟨1⟩1. LET: X be a locally path connected space.
⟨1⟩2. LET: P be a path component of X.
⟨1⟩3. LET: C be the component that includes P.
PROVE: $P = C$
⟨1⟩4. LET: Q be the union of all the path components of C other than P.
⟨1⟩5. P and Q are open in C.
PROOF: Theorem 15.29.2.
⟨1⟩6. $P \cup Q = C$ and $P \cap Q = \emptyset$
⟨1⟩7. $Q = \emptyset$
PROOF: Otherwise (P, Q) would be a separation of C.
⟨1⟩8. $P = C$
□

Example 15.29.4. The converse does not hold. In $\mathbb{Q}$, the components are the same as the path components, namely the one-point sets, but $\mathbb{Q}$ is not locally path connected.

Example 15.29.5. The ordered square is not locally path connected.

PROOF:
⟨1⟩1. ASSUME: for a contradiction I_o^2 is locally path connected at $(0, 1)$.
⟨1⟩2. PICK a path connected neighbourhood U of $(0, 1)$.
⟨1⟩3. PICK $a > 0$ such that $[(0, 1), (a, 0)] \subseteq U$
⟨1⟩4. PICK a path $p : [0, 1] \to I_o^2$ from $(0, 1)$ to $(a, 0)$.
⟨1⟩5. For every $x \in (0, a)$, PICK a rational $q_x \in [0, 1]$ such that $q_x \in ((x, 0), (x, 1))$
⟨1⟩6. $\{q_x : x \in (0, a)\}$ is an uncountable set of rationals.
⟨1⟩7. Q.E.D.
PROOF: This is a contradiction.
□

Proposition 15.29.6. *Every connected open subspace of a locally path connected space is path connected.*

PROOF:
⟨1⟩1. LET: X be a locally path connected space.
⟨1⟩2. LET: U be a connected open subspace.
⟨1⟩3. LET: P be a path component of U.
PROVE: $P = U$
⟨1⟩4. LET: Q be the union of the path components of U that are not P.
⟨1⟩5. P and Q are open.
PROOF: Theorem 15.29.2.
⟨1⟩6. $Q = \emptyset$
PROOF: Otherwise (P, Q) would be a separation of U.
⟨1⟩7. $P = U$
□

15.30 Quasicomponents

Definition 15.30.1 (Quasicomponent)**.** Let X be a topological space. Define the equivalence relation $\sim$ on X by: $x \sim y$ iff there is no separation (U, V) of X with $x \in U$ and $y \in V$. The *quasicomponents* of X are the equivalence classes with respect to $\sim$.

We prove this is an equivalence relation.

PROOF:
⟨1⟩1. $\sim$ is reflexive.
PROOF: For any $x \in X$, there cannot exist a separation (U, V) of X with $x \in U$ and $x \in V$.
⟨1⟩2. $\sim$ is symmetric.
PROOF: Immediate from definition.
⟨1⟩3. $\sim$ is transitive.
⟨2⟩1. ASSUME: $x \sim y$ and $y \sim z$
⟨2⟩2. ASSUME: for a contradiction (U, V) is a separation of X with $x \in U$ and $z \in V$.
⟨2⟩3. $y \in U$ or $y \in V$
⟨2⟩4. $y \notin U$
PROOF: $y \in U$ would contradict the fact that $y \sim z$.
⟨2⟩5. $y \notin V$
PROOF: $y \in V$ would contradict the fact that $x \sim y$.
⟨2⟩6. Q.E.D.
PROOF: This is a contradiction.
□

Proposition 15.30.2. *Every component of a topological space is a subset of a quasicomponent.*

PROOF:
⟨1⟩1. LET: X be a topological space.
⟨1⟩2. LET: C be a component of X.

PROVE: $\forall x, y \in C . x \sim y$
⟨1⟩3. LET: $x, y \in C$
⟨1⟩4. ASSUME: for a contradiction (U, V) is a separation of X with $x \in U$ and $y \in V$
⟨1⟩5. $(U \cap C, V \cap C)$ is a separation of C.
⟨1⟩6. Q.E.D.
PROOF: This contradicts the fact that C is connected (Proposition 15.25.4).
$\square$

Proposition 15.30.3. *In a locally connected topological space, the components are the same as the quasicomponents.*

PROOF:
⟨1⟩1. LET: X be a locally connected topological space.
⟨1⟩2. LET: C be a component of X.
⟨1⟩3. LET: Q be the quasicomponent that includes C.
PROVE: $Q = C$
⟨1⟩4. ASSUME: for a contradiction $C \neq Q$
⟨1⟩5. PICK $c \in C$ and $d \in Q - C$
⟨1⟩6. $(C, X - C)$ is a separation of X with $c \in C$ and $d \in X - C$.
PROOF: Since the components of X are open (Theorem 15.28.7).
⟨1⟩7. Q.E.D.
PROOF: This contradicts the fact that $c \sim d$.
$\square$

15.31 Compact Spaces

Definition 15.31.1 (Open Cover)**.** Let X be a topological space. An *open cover* of X is a cover of X whose elements are open sets.

Definition 15.31.2 (Compact)**.** A topological space is *compact* iff every open cover includes a finite subcover.

Example 15.31.3. The space $\mathbb{R}$ is not compact, because the open cover $\{(n, n+2) : n \in \mathbb{Z}\}$ has no finite subcover.

Example 15.31.4. Every finite topological space is compact.

Example 15.31.5. Any set under the cofinite topology is compact.

Lemma 15.31.6. *Let X be a topological space and Y a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection that covers Y.*

PROOF:
⟨1⟩1. If Y is compact then every covering of Y by sets open in X contains a finite subcollection that covers Y.
⟨2⟩1. ASSUME: Y is compact.

⟨2⟩2. LET: $\mathcal{A}$ be a covering of Y by sets open in X.
⟨2⟩3. $\{U \cap Y : U \in \mathcal{A}\}$ is an open covering of Y.
⟨2⟩4. PICK a finite subcovering $\{U_1 \cap Y, \ldots, U_n \cap Y\}$.
⟨2⟩5. $\{U_1, \ldots, U_n\}$ is a finite subcollection of $\mathcal{A}$ that covers Y.
⟨1⟩2. If every covering of Y by sets open in X contains a finite subcollection that covers Y then Y is compact.
⟨2⟩1. ASSUME: Every covering of Y by sets open in X contains a finite subcollection that covers Y.
⟨2⟩2. LET: $\mathcal{A}$ be an open cover of Y.
⟨2⟩3. $\{U$ open in $X : U \cap Y \in \mathcal{A}\}$ covers Y.
⟨2⟩4. PICK a finite subcollection $\{U_1, \ldots, U_n\}$ that covers Y.
⟨2⟩5. $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ is a finite subcollection of $\mathcal{A}$ that covers Y.
□

Theorem 15.31.7. *Every closed subspace of a compact space is compact.*

PROOF:
⟨1⟩1. LET: X be a compact space.
⟨1⟩2. LET: C be a closed subspace of X.
⟨1⟩3. Every covering of C by sets open in X contains a finite subcollection that covers C.
⟨2⟩1. LET: $\mathcal{A}$ be a covering of C by sets open in X.
⟨2⟩2. $\mathcal{A} \cup \{X - C\}$ is an open covering of X.
⟨2⟩3. PICK a finite subcover $\mathcal{B}$
⟨2⟩4. $\mathcal{B} - \{X - C\}$ is a finite subcollection of $\mathcal{A}$ that covers C.
⟨1⟩4. C is compact.
PROOF: Lemma 15.31.6.
□

Lemma 15.31.8. *Let X be a Hausdorff space. Let Y be a compact subspace. Let $x \in X - Y$. Then there exist disjoint open sets U and V such that $x \in U$ and $Y \subseteq V$.*

PROOF:
⟨1⟩1. For all $y \in Y$, there exist disjoint open sets U' and V' with $x \in U'$ and $y \in V'$.
⟨1⟩2. $\{V'$ open in $X : \exists U'$ open in $X . U' \cap V' = \emptyset \wedge x \in U'\}$ is an cover of Y by sets open in X.
⟨1⟩3. PICK a finite subcollection $\{V_1, \ldots, V_n\}$ that covers Y.
⟨1⟩4. For $i = 1, \ldots, n$, PICK an open set U_i with $U_i \cap V_i = \emptyset$ and $x \in U_i$.
⟨1⟩5. LET: $U = U_1 \cap \cdots \cap U_n$ and $V = V_1 \cup \cdots \cup V_n$
⟨1⟩6. U and V are disjoint open sets with $x \in U$ and $Y \subseteq V$.
□

Proposition 15.31.9. *Let X be a Hausdorff space. Let A and B be disjoint compact subspaces of X. Then there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.*

PROOF:

⟨1⟩1. For all $x \in A$, there exist disjoint open sets U' and V' with $x \in U'$ and $B \subseteq V'$.
PROOF: Lemma 15.31.8.
⟨1⟩2. $\{U'$ open in $X : \exists V'$ open in $X.U' \cap V' = \emptyset \wedge B \subseteq V'\}$ is a set of open sets that covers A.
⟨1⟩3. PICK a finite subset $\{U_1, \ldots, U_n\}$ that covers A.
PROOF: Lemma 15.31.6.
⟨1⟩4. For $i = 1, \ldots, n$, PICK V_i open in X such that $U_i \cap V_i = \emptyset$ and $B \subseteq V_i$.
⟨1⟩5. LET: $U = U_1 \cup \cdots \cup U_n$
⟨1⟩6. LET: $V = V_1 \cap \cdots \cap V_n$
⟨1⟩7. U and V are disjoint and open.
⟨1⟩8. $A \subseteq U$
⟨1⟩9. $B \subseteq V$
□

Theorem 15.31.10. *Every compact subspace of a Hausdorff space is closed.*

PROOF:
⟨1⟩1. LET: X be a Hausdorff space.
⟨1⟩2. LET: Y be a compact subspace of X.
⟨1⟩3. For any $x \in X - Y$ there exists an open set U such that $x \in U \subseteq X - Y$.
PROOF: Lemma 15.31.8.
⟨1⟩4. $X - Y$ is open.
PROOF: Proposition 14.1.7.
⟨1⟩5. Y is closed.
□

Example 15.31.11. We cannot weaken the hypothesis from the space being Hausdorff to the space being T_1.

Let X be any infinite set under the cofinite topology. Then X is T_1. The closed sets are the finite sets and X, but every subspace is compact.

Theorem 15.31.12. *The continuous image of a compact space is compact.*

PROOF:
⟨1⟩1. LET: X be a compact space and $f : X \twoheadrightarrow Y$ be a surjective continuous function.
⟨1⟩2. LET: $\mathcal{V}$ be an open cover of Y.
⟨1⟩3. $\{f^{-1}(V) : V \in \mathcal{V}\}$ is an open cover of U.
⟨1⟩4. PICK a finite subcover $\{f^{-1}(V_1), \ldots, f^{-1}(V_n)\}$.
⟨1⟩5. $\{V_1, \ldots, V_n\}$ covers Y.
□

Proposition 15.31.13. *Let X be a compact space and Y a Hausdorff space. Every continuous function from X to Y is a closed map.*

PROOF:
⟨1⟩1. LET: $f : X \to Y$ be continuous.

⟨1⟩2. LET: C be a closed set in X.
⟨1⟩3. C is compact.
PROOF: Theorem 15.31.7.
⟨1⟩4. $f(C)$ is compact.
PROOF: Theorem 15.31.12.
⟨1⟩5. $f(C)$ is closed in Y.
PROOF: Theorem 15.31.10.
□

Corollary 15.31.13.1. *Let X be a compact space and Y a Hausdorff space. Let $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism.*

Corollary 15.31.13.2. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two compact Hausdorff topologies on the same set X. Then either $\mathcal{T} = \mathcal{T}'$ or they are incompatible.*

Theorem 15.31.14. *Let X and Y be topological spaces. Let A be a compact subspace of X and B a compact subspace of Y. Let N be an open set that includes $A \times B$. Then there exists open sets U in X and V in Y such that*

$$A \times B \subseteq U \times V \subseteq N \enspace .$$

PROOF:
⟨1⟩1. For all $x \in A$, there exist open sets U in X and V in Y such that $x \in U$, $B \subseteq V$, and $U \times V \subseteq N$.
⟨2⟩1. LET: $x \in A$
⟨2⟩2. For all $y \in B$, there exist neighbourhoods U of x and V of y such that $U \times V \subseteq N$
⟨2⟩3. PICK open sets $V_1, \ldots, V_n$ that cover B such that, for $i = 1, \ldots, n$, there exists a neighbourhood U_i of x such that $U_i \times V_i \subseteq N$
⟨2⟩4. LET: $U = U_1 \cap \cdots \cap U_n$
⟨2⟩5. LET: $V = V_1 \cup \cdots \cup V_n$
⟨2⟩6. U is open in X.
⟨2⟩7. V is open in Y.
⟨2⟩8. $x \in U$
⟨2⟩9. $B \subseteq V$
⟨2⟩10. $U \times V \subseteq N$
⟨1⟩2. PICK open sets $U_1, \ldots, U_n$ in X that cover A such that, for $i = 1, \ldots, n$, there exists V_i open in Y such that $B \subseteq V_i$ and $U_i \times V_i \subseteq N$.
⟨1⟩3. LET: $U = U_1 \cup \cdots \cup U_n$
⟨1⟩4. LET: $V = V_1 \cap \cdots \cap V_n$
⟨1⟩5. U is open in X.
⟨1⟩6. V is open in Y.
⟨1⟩7. $A \times B \subseteq U \times V \subseteq N$
□

Corollary 15.31.14.1 (Tube Lemma)**.** *Let X be a topological space and Y a compact space. Let $x_0 \in X$. Let N be an open set in $X \times Y$ that includes $\{x_0\} \times Y$. Then there exists a neighbourhood W of x_0 such that $W \times Y \subseteq N$.*

Theorem 15.31.15. *The product of two compact spaces is compact.*

PROOF:
⟨1⟩1. LET: X and Y be compact spaces.
⟨1⟩2. LET: $\mathcal{A}$ be an open covering of $X \times Y$.
⟨1⟩3. For all $x \in X$, there exists a neighbourhood W of x such that $W \times Y$ can be covered by finitely many elements of $\mathcal{A}$.
⟨2⟩1. LET: $x \in X$
⟨2⟩2. $\{x\} \times Y$ is compact.
PROOF: It is homeomorphic to Y.
⟨2⟩3. PICK a finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ that covers $\{x\} \times Y$.
⟨2⟩4. There exists a neighbourhood W of x such that $W \times Y \subseteq A_1 \cup \cdots \cup A_n$.
PROOF: Tube Lemma
⟨1⟩4. PICK finitely many open sets $W_1, \ldots, W_n$ that cover X such that each $W_i \times Y$ can be covered by finitely many elements of $\mathcal{A}$.
⟨1⟩5. For $i = 1, \ldots, n$, PICK a finite subset $\mathcal{A}_i$ of $\mathcal{A}$ that covers $W_i \times Y$.
⟨1⟩6. $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$ covers $X \times Y$.
□

Theorem 15.31.16. *Let X be a topological space. Then X is compact if and only if, for every set $\mathcal{C}$ of compact sets, if $\mathcal{C}$ has the finite intersection property then $\bigcup \mathcal{C} \neq \emptyset$.*

PROOF: The following are equivalent:

- X is compact.
- For every set $\mathcal{A}$ of open sets, if $\bigcup \mathcal{A} = X$ then there is a finite subset of $\mathcal{A}$ that covers X.
- For every set $\mathcal{C}$ of closed sets, if $\bigcup_{C \in \mathcal{C}} (X - C) = X$ then there is a finite subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\bigcup_{C \in \mathcal{C}_0} (X - C) = X$.
- For every set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\bigcap \mathcal{C}_0 = \emptyset$
- For every set $\mathcal{C}$ of closed sets, if $\mathcal{C}$ has the finite intersection property then $\bigcap \mathcal{C} \neq \emptyset$.

□

Corollary 15.31.16.1. *Let X be a compact set. Let (C_n) be a sequence of nonempty closed sets such that $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} C_n \neq \emptyset$.*

Proposition 15.31.17. *Let X be a topological space. Let Y and Z be compact subspaces of X. The $Y \cup Z$ is compact.*

PROOF:
⟨1⟩1. LET: $\mathcal{U}$ be a set of open sets that covers $Y \cup Z$.
⟨1⟩2. PICK finite subsets $\mathcal{U}_1$ that covers Y and $\mathcal{U}_2$ that covers Z.

⟨1⟩3. $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset of $\mathcal{U}$ that covers $Y \cup Z$.
□

Proposition 15.31.18. *Let X be a topological space and Y a compact space. Then the projection $\pi_1 : X \times Y \to X$ is a closed map.*

PROOF:
⟨1⟩1. LET: C be closed in $X \times Y$.
⟨1⟩2. LET: $x \in X - \pi_1(C)$
⟨1⟩3. $\{x\} \times Y \subseteq (X \times Y) - C$
⟨1⟩4. PICK a neighbourhood U of x such that $U \times Y \subseteq (X \times Y) - C$
PROOF: Tube Lemma.
⟨1⟩5. $x \in U \subseteq X - \pi_1(C)$
□

Theorem 15.31.19. *Let X be a topological space and Y a compact Hausdorff space. Let $f : X \to Y$. Then f is continuous if and only if the graph of f,*

$$G_f = \{(x, f(x)) : x \in X\} \enspace ,$$

is closed in $X \times Y$.

PROOF:
⟨1⟩1. If f is continuous then G_f is closed in $X \times Y$.
⟨2⟩1. ASSUME: f is continuous.
⟨2⟩2. LET: $(x, y) \in (X \times Y) - G_f$
⟨2⟩3. PICK disjoint open neighbourhoods U of y and V of $f(x)$.
⟨2⟩4. $(x, y) \in (f^{-1}(V) \times U) \subseteq (X \times Y) - G_f$
⟨1⟩2. If G_f is closed in $X \times Y$ then f is continuous.
⟨2⟩1. ASSUME: G_f is closed in $X \times Y$.
⟨2⟩2. LET: $x_0 \in X$
⟨2⟩3. LET: V be a neighbourhood of $f(x_0)$.
⟨2⟩4. $G_f \cap (X \times (Y - V))$ is closed.
⟨2⟩5. $\pi_1(G_f \cap (X \times (Y - V)))$ is closed.
PROOF: Proposition 15.31.18.
⟨2⟩6. LET: $U = X - \pi_1(G_f \cap (X \times (Y - V)))$
⟨2⟩7. $x_0 \in U$
⟨2⟩8. $f(U) \subseteq V$
□

Theorem 15.31.20. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a set of closed connected subspaces of X that is linearly ordered under inclusion. Then $\bigcap \mathcal{A}$ is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction (C, D) is a separation of $\bigcap \mathcal{A}$.
⟨1⟩2. PICK disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$.
⟨2⟩1. C and D are closed in X.

PROOF: Proposition 15.3.6.
⟨2⟩2. C and D are compact.
PROOF: Theorem 15.31.7.
⟨2⟩3. Q.E.D.
PROOF: Proposition 15.31.9.
⟨1⟩3. $\bigcap_{A \in \mathcal{A}}(A - (U \cup V))$ is nonempty.
⟨2⟩1. $\{A - (U \cap V) : A \in \mathcal{A}\}$ has the finite intersection property.
⟨3⟩1. LET: $A_1, \ldots, A_n \in \mathcal{A}$
⟨3⟩2. ASSUME: w.l.o.g. $A_1 \supseteq \cdots \supseteq A_n$
PROOF: $\mathcal{A}$ is linearly ordered by inclusion.
⟨3⟩3. $\bigcap_{i=1}^{n}(A_i - (U \cup V)) = A_n - (U \cup V)$
⟨3⟩4. $A_n - (U \cup V) \neq \emptyset$
PROOF: Otherwise $(A_n \cap U, A_n \cap V)$ would be a separation of A_n.
⟨2⟩2. Q.E.D.
PROOF: Theorem 15.31.16.
⟨1⟩4. Q.E.D.
PROOF: This is a contradiction.
□

Theorem 15.31.21. *Let X be a linearly ordered set with the least upper bound property under the order topology. Every closed interval in X is compact.*

PROOF:
⟨1⟩1. LET: $a, b \in X$ with $a < b$.
⟨1⟩2. LET: $\mathcal{A}$ be an open covering of $[a, b]$
⟨1⟩3. For any $x \in [a, b)$, there exists $y \in (x, b]$ such that $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.
⟨2⟩1. LET: $x \in [a, b)$
⟨2⟩2. PICK $U \in \mathcal{A}$ with $x \in U$.
⟨2⟩3. PICK $y > x$ such that $[x, y) \subseteq U$.
⟨2⟩4. PICK $V \in \mathcal{A}$ such that $y \in V$.
⟨2⟩5. $[x, y]$ can be covered by U and V.
⟨1⟩4. LET: $C = \{y \in (a, b] : [a, y]$ can be covered by finitely many elements of $\mathcal{A}\}$
⟨1⟩5. $C \neq \emptyset$
PROOF: By ⟨1⟩3, there exists $y \in (a, b]$ such that $[a, y]$ can be covered by at most two elements of $\mathcal{A}$.
⟨1⟩6. LET: $c = \sup C$
⟨1⟩7. $c \in C$
⟨2⟩1. PICK $U \in \mathcal{A}$ such that $c \in U$.
⟨2⟩2. PICK $y < c$ such that $(y, c] \subseteq U$.
⟨2⟩3. PICK $z \in C$ such that $y < z$
⟨2⟩4. PICK a finite $\mathcal{A}_0 \subseteq \mathcal{A}$ that covers $[a, z]$.
⟨2⟩5. $\mathcal{A}_0 \cup \{U\}$ covers $[a, c]$.
⟨2⟩6. $c \in C$
⟨1⟩8. $c = b$
⟨2⟩1. ASSUME: for a contradiction $c < b$.

⟨2⟩2. PICK $y > c$ such that $[c, y]$ can be covered by at most two elements of $\mathcal{A}$.
⟨2⟩3. $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$.
PROOF: ⟨1⟩7
⟨2⟩4. $[a, y]$ can be covered by finitely many elements of $\mathcal{A}$.
⟨2⟩5. $y \in C$
⟨2⟩6. Q.E.D.
PROOF: This is a contradiction.
⟨1⟩9. $[a, b]$ can be covered by finitely many elements of $\mathcal{A}$.
□

Corollary 15.31.21.1. *Every closed interval in* $\mathbb{R}$ *is compact.*

Proposition 15.31.22. *A linearly ordered set that is compact under the order topology has a greatest and a least element.*

PROOF:
⟨1⟩1. LET: X be a linearly ordered set that is compact under the order topology.
⟨1⟩2. X has a greatest element.
⟨2⟩1. ASSUME: for a contradiction X has no greatest element.
⟨2⟩2. The open rays $(-\infty, a)$ form an open cover of X.
⟨2⟩3. PICK a finite subcover $\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$ where $a_1 \leqslant \cdots \leqslant a_n$
⟨2⟩4. $a_n < a_n$
⟨2⟩5. Q.E.D.
PROOF: This is a contradiction.
⟨1⟩3. X has a least element.
PROOF: Similar.
□

Corollary 15.31.22.1 (Extreme Value Theorem)**.** *Let X be a compact space and Y a linearly ordered set in the order topology. Let $f : X \to Y$ be continuous. Then there exist $c, d \in X$ such that, for all $x \in X$, we have $f(c) \leqslant f(x) \leqslant f(d)$.*

PROOF: Since $f(X)$ is compact, and so has a greatest and least element. □

15.32 Perfect Maps

Definition 15.32.1 (Perfect Map)**.** Let X and Y be topological spaces. A *perfect map* from X to Y is a closed continuous surjective map $p : X \twoheadrightarrow Y$ such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact.

Proposition 15.32.2. *Let X be a topological space and Y a compact space. If there exists a perfect map $X \twoheadrightarrow Y$, then X is compact.*

PROOF:
⟨1⟩1. LET: $p : X \twoheadrightarrow Y$ be a perfect map.
⟨1⟩2. For all $y \in Y$ and every open set U such that $p^{-1}(y) \subseteq U$, there exists a neighbourhood W of y such that $p^{-1}(W) \subseteq U$.

⟨2⟩1. LET: $y \in Y$
⟨2⟩2. LET: U be an open set such that $p^{-1}(y) \subseteq U$.
⟨2⟩3. LET: $W = Y - p(X - U)$
⟨2⟩4. W is open.
⟨3⟩1. $X - U$ is closed.
PROOF: Proposition 14.2.2, ⟨2⟩2.
⟨3⟩2. $p(X - U)$ is closed.
PROOF: Since p is a closed map (⟨1⟩1).
⟨3⟩3. $Y - p(X - U)$ is open.
⟨2⟩5. $y \in W$
⟨3⟩1. ASSUME: for a contradiction $y \in p(X - U)$
⟨3⟩2. PICK $x \in X - U$ such that $p(x) = y$
⟨3⟩3. $x \in p^{-1}(y)$
⟨3⟩4. $x \in U$
PROOF: ⟨2⟩2
⟨3⟩5. Q.E.D.
PROOF: This contradicts ⟨3⟩2.
⟨2⟩6. $p^{-1}(W) \subseteq U$
⟨3⟩1. LET: $x \in p^{-1}(W)$
⟨3⟩2. $p(x) \in W$
⟨3⟩3. $p(x) \notin p(X - U)$
PROOF: ⟨2⟩3
⟨3⟩4. $x \notin X - U$
⟨3⟩5. $x \in U$
⟨1⟩3. LET: $\mathcal{U}$ be an open cover of X.
⟨1⟩4. For every $y \in Y$, there exists a neighbourhood W of y such that $p^{-1}(W)$ can be covered by finitely many elements of $\mathcal{U}$.
⟨2⟩1. LET: $y \in Y$
⟨2⟩2. PICK finitely many sets $U_1, \ldots, U_n \in \mathcal{U}$ that cover $p^{-1}(y)$
PROOF: Since $p^{-1}(y)$ is compact.
⟨2⟩3. There exists a neighbourhood W of y such that $p^{-1}(W) \subseteq U_1 \cup \cdots \cup U_n$.
PROOF: ⟨1⟩2
⟨1⟩5. PICK finitely many open sets $W_1, \ldots, W_n$ in Y such that each $p^{-1}(W_i)$ can be covered by finitely many elements of $\mathcal{U}$.
PROOF: Since Y is compact.
⟨1⟩6. For $i = 1, \ldots, n$, PICK a finite subset $\mathcal{U}_i$ of $\mathcal{U}$ such that $p^{-1}(W_i) \subseteq \bigcup \mathcal{U}_i$.
⟨1⟩7. $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ covers X.
□

Chapter 16

Metric Spaces

16.1 Metric Spaces

Definition 16.1.1 (Metric Space)**.** Let X be a set and $d : X^2 \to \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geqslant 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leqslant d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d). We often write X for the metric space (X, d).

Definition 16.1.2 (Discrete Metric)**.** On any set X, define the *discrete* metric by $d(x, y) = 0$ if $x = y$, 1 if $x \neq y$.

Definition 16.1.3 (Standard Metric)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

Definition 16.1.4 (Distance to a Set)**.** Let X be a metric space. Let $A \subseteq X$ be nonempty and $x \in X$. The *distance from x to A* is

$$d(x, A) = \inf_{a \in A} d(x, a) \ .$$

16.1.1 Balls

Definition 16.1.5 ((Open) Ball)**.** Let X be a metric space. Let $x \in X$ and $r > 0$. The *(open) ball* with *centre* x and *radius* r is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} \ .$$

Definition 16.1.6 (Closed Ball)**.** Let X be a metric space. Let $x \in X$ and $r > 0$. The *closed ball* with *centre* x and *radius* r is

$$\overline{B(x,r)} = \{y \in X \mid d(x,y) < r\} \ .$$

Definition 16.1.7 (Metric Topology)**.** Let (X, d) be a metric space. The *metric topology* on X is the topology generated by the basis consisting of the balls.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. Every point is a member of some ball.
PROOF: Since $x \in B(x, 1)$.
$\langle 1 \rangle 2$. If B_1 and B_2 are balls and $x \in B_1 \cap B_2$, then there exists a ball B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.
$\langle 2 \rangle 1$. LET: $x \in B(a, \epsilon_1) \cap B(b, \epsilon_2)$
$\langle 2 \rangle 2$. LET: $\epsilon = \min(\epsilon_1 - d(x,a), \epsilon_2 - d(x,b))$
PROVE: $x \in B(x, \epsilon) \subseteq B(a, \epsilon_1) \cap B(b, \epsilon_2)$
$\langle 2 \rangle 3$. $B(x, \epsilon) \subseteq B(a, \epsilon_1)$
$\langle 3 \rangle 1$. LET: $y \in B(x, \epsilon)$
$\langle 3 \rangle 2$. $d(y, a) < \epsilon_1$
PROOF:

$$\begin{aligned} d(y,a) &\leqslant d(y,x) + d(x,a) && \text{(Triangle Inequality)} \\ &< \epsilon + d(x,a) && (\langle 3 \rangle 1) \\ &\leqslant \epsilon_1 && (\langle 2 \rangle 2) \end{aligned}$$

$\langle 2 \rangle 4$. $B(x, \epsilon) \subseteq B(b, \epsilon_2)$
PROOF: Similar.
□

Proposition 16.1.8. *The discrete metric on a set X induces the discrete topology.*

PROOF: Since $B(x, 1/2) = \{x\}$ for all $x \in X$. □

Proposition 16.1.9. *The standard metric on $\mathbb{R}$ induces the standard topology.*

PROOF:
$\langle 1 \rangle 1$. Every ball is open in the standard topology.
PROOF: Since $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$.
$\langle 1 \rangle 2$. Every open ray is open in the metric topology.
PROOF: If $x \in (a, +\infty)$ then $x \in B(x, x - a) \subseteq (a, +\infty)$. Similarly for $(-\infty, a)$.
□

Proposition 16.1.10. *Multiplication is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

PROOF:
$\langle 1 \rangle 1$. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$
$\langle 1 \rangle 2$. LET: $\delta = \min(\epsilon/(|x| + |y| + 1), 1)$
$\langle 1 \rangle 3$. LET: $(x', y') \in \mathbb{R}^2$ with $\rho((x, y), (x', y')) < \delta$

⟨1⟩4. $|x - x'|, |y - y'| < \delta$
⟨1⟩5. $|xy - x'y'| < \epsilon$
PROOF:

$$\begin{aligned} |xy - x'y'| &= |xy - xy' + xy - x'y - xy + x'y + xy' - x'y'| \\ &\leqslant |xy - xy'| + |xy - x'y| + |xy - x'y - xy' + xy'y| \quad = |x||y - y'| + |x - x'||y| + |x - x'||y - y'| \\ &< |x|\delta + |y|\delta + \delta^2 && (\langle 1\rangle 4) \\ &\leqslant |x|\delta + |y|\delta + \delta && (\langle 1\rangle 2) \\ &= (|x| + |y| + 1)\delta \\ &\leqslant \epsilon && (\langle 1\rangle 2) \end{aligned}$$

□

Corollary 16.1.10.1. *The unit circle S^1 is a closed subset of $\mathbb{R}^2$.*

PROOF: The function f that maps (x, y) to $x^2 + y^2$ is continuous, and $S^1 = f^{-1}(\{1\})$. □

Corollary 16.1.10.2. *The unit ball B^2 is a closed subset of $\mathbb{R}^2$.*

PROOF: The function f that maps (x, y) to $x^2 + y^2$ is continuous, and $B^2 = f^{-1}([0, 1])$. □

Proposition 16.1.11. *Let (a_n) and (b_n) be sequences of real numbers. Let $c, s, t \in \mathbb{R}$. Assume*

$$\sum_{n=0}^{\infty} a_n = s \text{ and } \sum_{n=0}^{\infty} b_n = t \enspace .$$

Then

$$\sum_{n=0}^{\infty} (ca_n + b_n) = cs + t \enspace .$$

PROOF:

$$\sum_{n=0}^{N} (ca_n + b_n) = c \sum_{n=0}^{N} a_n + \sum_{n=0}^{N} b_n \to cs + t \text{ as } n \to \infty \qquad \square$$

Proposition 16.1.12 (Comparison Test)**.** *Let (a_n) and (b_n) be sequences of real numbers. Assume $|a_n| \leqslant b_n$ for all n. Assume $\sum_{n=0}^{\infty} b_n$ converges. Then $\sum_{n=0}^{\infty} a_n$ converges.*

PROOF:
⟨1⟩1. For all n,
LET: $c_n = |a_n| + a_n$
⟨1⟩2. $\sum_{n=0}^{\infty} |a_n|$ converges.
PROOF: Since $(\sum_{n=0}^{N} |a_n|)_N$ is an increasing sequence of real numbers bounded above by $\sum_{n=0}^{\infty} b_n$.
⟨1⟩3. $\sum_{n=0}^{\infty} c_n$ converges.
PROOF: Since $(\sum_{n=0}^{N} c_n)_N$ is an increasing sequence of real numbers bounded above by $2 \sum_{n=0}^{\infty} a_n$.

⟨1⟩4. $\sum_{n=0}^{\infty} a_n$ converges.
PROOF: Since $a_n = c_n - |a_n|$.
□

Proposition 16.1.13. *Let X be a metric space. Let $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
⟨1⟩1. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
⟨2⟩1. ASSUME: U is open.
⟨2⟩2. LET: $x \in U$
⟨2⟩3. PICK a ball $B(a, \delta)$ such that $x \in B(a, \delta) \subseteq U$
⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$
PROVE: $B(x, \epsilon) \subseteq U$
⟨2⟩5. LET: $y \in B(x, \epsilon)$
⟨2⟩6. $y \in B(a, \delta)$
PROOF:

$$\begin{aligned} d(a,y) &\leqslant d(a,x) + d(x,y) && \text{(Triangle Inequality)} \\ &< d(a,x) + \epsilon && (\langle 2\rangle 5) \\ &= \delta \end{aligned}$$

⟨2⟩7. $y \in U$
PROOF: ⟨2⟩3
⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
PROOF: Immediate from definition of the metric topology.
□

Proposition 16.1.14. *Let X be a metric space. Let $a, b, c \in X$. Then*

$$|d(a,b) - d(a,c)| \leqslant d(b,c) \ .$$

PROOF:
⟨1⟩1. $d(a,b) - d(a,c) \leqslant d(b,c)$
PROOF: Triangle Inequality.
⟨1⟩2. $d(a,c) - d(a,b) \leqslant d(b,c)$
PROOF: Triangle Inequality.
□

Proposition 16.1.15. *Let (X, d) be a metric space. Then the metric topology on X is the coarsest topology such that $d : X^2 \to \mathbb{R}$ is continuous.*

PROOF:
⟨1⟩1. d is continuous with respect to the metric topology.
⟨2⟩1. LET: $(a, b) \in X^2$
⟨2⟩2. LET: V be a neighbourhood of $d(a, b)$.
⟨2⟩3. PICK $\epsilon > 0$ such that $(d(a, b) - \epsilon, d(a, b) + \epsilon) \subseteq V$.
⟨2⟩4. LET: $U = B(a, \epsilon/2) \times B(b, \epsilon/2)$
⟨2⟩5. LET: $(x, y) \in U$
⟨2⟩6. $|d(x, y) - d(a, b)| < \epsilon$

PROOF:

$$\begin{aligned}|d(x,y)-d(a,b)| &\leqslant |d(x,y)-d(a,y)|+|d(a,y)-d(a,b)| \\ &\leqslant d(a,x)+d(b,y) && \text{(Proposition 16.1.14)} \\ &< \epsilon\end{aligned}$$

⟨2⟩7. $d(x,y) \in V$

⟨1⟩2. If $\mathcal{T}$ is a topology on X with respect to which d is continuous then $\mathcal{T}$ is finer than the metric topology.

⟨2⟩1. LET: $\mathcal{T}$ be a topology on X with respect to which d is continuous.

⟨2⟩2. LET: $a \in X$ and $\epsilon > 0$.
PROVE: $B(a,\epsilon) \in \mathcal{T}$

⟨2⟩3. LET: $x \in B(a,\epsilon)$

⟨2⟩4. $(a,x) \in d^{-1}((0,\epsilon))$

⟨2⟩5. PICK $U, V \in \mathcal{T}$ such that $(a,x) \in U \times V \subseteq d^{-1}((0,\epsilon))$

⟨2⟩6. $x \in V \subseteq B(a,\epsilon)$

□

Proposition 16.1.16. *Let d and d' be two metrics on the same set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x,\delta) \subseteq B_d(x,\epsilon) \ .$$

PROOF:

⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$.

⟨2⟩1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$

⟨2⟩3. $x \in B_d(x,\epsilon) \in \mathcal{T}'$

⟨2⟩4. There exists $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$
PROOF: Proposition 16.1.13.

⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$.

⟨2⟩1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$.

⟨2⟩2. LET: $U \in \mathcal{T}$

⟨2⟩3. For all $x \in U$, there exists $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq U$

⟨3⟩1. LET: $x \in U$

⟨3⟩2. PICK $\epsilon > 0$ such that $B_d(x,\epsilon) \subseteq U$
PROOF: Proposition 16.1.13.

⟨3⟩3. PICK $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$.
PROOF: ⟨2⟩1

⟨3⟩4. $B_{d'}(x,\delta) \subseteq U$

⟨2⟩4. $U \in \mathcal{T}'$
PROOF: Proposition 16.1.13.

□

Definition 16.1.17 (Metrizable)**.** A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 16.1.18. $\mathbb{R}^2$ *under the dictionary order is metrizable.*

PROOF:
$\langle 1 \rangle 1$. LET: $d : (\mathbb{R}^2)^2 \to \mathbb{R}$ be defined by
$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} \min(|y_2 - y_1|, 1) & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$
$\langle 1 \rangle 2$. d is a metric.
$\langle 2 \rangle 1$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) \geqslant 0$.
PROOF: Immediate from definition.
$\langle 2 \rangle 2$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) = 0$ iff $x = y$.
PROOF: Immediate from definition.
$\langle 2 \rangle 3$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) = d(y, x)$.
PROOF: Immediate from definition.
$\langle 2 \rangle 4$. For all $x, y, z \in \mathbb{R}^2$ we have $d(x, z) \leqslant d(x, y) + d(y, z)$.
PROOF: Easy.
$\langle 1 \rangle 3$. The metric topology induced by d is finer than the order topology.
$\langle 2 \rangle 1$. LET: $a, b \in \mathbb{R}^2$
$\langle 2 \rangle 2$. LET: $x \in (a, b)$
$\langle 2 \rangle 3$. CASE: $\pi_1(x) = \pi_1(a) = \pi_1(b)$
$\langle 3 \rangle 1$. LET: $\epsilon = \min(\pi_2(x) - \pi_2(a), \pi_2(b) - \pi_2(x))$
$\langle 3 \rangle 2$. $B(x, \epsilon) \subseteq (a, b)$
$\langle 2 \rangle 4$. CASE: $\pi_1(a) = \pi_1(x) < \pi_1(b)$
$\langle 3 \rangle 1$. LET: $\epsilon = \pi_2(x) - \pi_2(a)$
$\langle 3 \rangle 2$. $B(x, \epsilon) \subseteq (a, b)$
$\langle 2 \rangle 5$. CASE: $\pi_1(a) < \pi_1(x) = \pi_1(b)$
PROOF: Similar.
$\langle 2 \rangle 6$. CASE: $\pi_1(a) < \pi_1(x) < \pi_1(b)$
PROOF: Then $B(x, \epsilon) \subseteq (a, b)$.
$\langle 1 \rangle 4$. The order topology is finer than the metric topology.
PROOF: Since $B((a, b), \epsilon) = ((a, b - \epsilon), (a, b + \epsilon))$ if $\epsilon \leqslant 1$, and $\mathbb{R}^2$ if $\epsilon > 1$.
□

Proposition 16.1.19. *Every metrizable space is first countable.*

PROOF: For any point a, the set $\{B(a, 1/n) : n \in \mathbb{Z}_+\}$ is a local basis at a. □

A metric space is compact if and only if it is sequentially compact.
A metric space is separable if and only if it is second countable.

Proposition 16.1.20. *Every compact metric space is bounded.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a compact metric space.
$\langle 1 \rangle 2$. ASSUME: w.l.o.g. X is nonempty.
$\langle 1 \rangle 3$. PICK $a \in X$
$\langle 1 \rangle 4$. $\{B(a, n) : n \in \mathbb{Z}_+\}$ is an open cover of X.
$\langle 1 \rangle 5$. PICK a finite subcover $\{B(a, n_1), \ldots, B(a, n_k)\}$.
$\langle 1 \rangle 6$. LET: $N = \max(n_1, \ldots, n_k)$

$\langle 1\rangle 7.$ $X = B(a, N)$
□

Example 16.1.21. The converse does not hold. An infinite discrete space is bounded but not compact.

16.1.2 Subspaces

Proposition 16.1.22. *Let (X, d) be a metric space and $Y \subseteq X$. Then $d \upharpoonright Y^2$ is a metric on Y that induces the subspace topology.*

PROOF:
$\langle 1\rangle 1.$ LET: $d' = d \upharpoonright Y^2 : Y^2 \to \mathbb{R}$
$\langle 1\rangle 2.$ d' is a metric.
  PROOF: Each of the axioms follows from the axiom in X.
$\langle 1\rangle 3.$ The metric topology induced by d' is finer than the subspace topology.
  $\langle 2\rangle 1.$ LET: U be open in X
    PROVE: $U \cap Y$ is open in the d'-topology.
  $\langle 2\rangle 2.$ LET: $y \in U \cap Y$
  $\langle 2\rangle 3.$ PICK $\epsilon > 0$ such that $B_d(y, \epsilon) \subseteq U$
  $\langle 2\rangle 4.$ $B_{d'}(y, \epsilon) \subseteq U \cap Y$
$\langle 1\rangle 4.$ The subspace topology is finer than the metric topology induced by d'.
  $\langle 2\rangle 1.$ LET: $y \in Y$ and $\epsilon > 0$
    PROVE: $B_{d'}(y, \epsilon)$ is open in the subspace topology.
  $\langle 2\rangle 2.$ $B_{d'}(y, \epsilon) = B_d(y, \epsilon) \cap Y$
□

16.1.3 Convergence

Proposition 16.1.23 (Sequence Lemma)**.** *Let X be a metric space. Let $A \subseteq X$. Let $l \in \overline{A}$. Then there exists a sequence in A that converges to l.*

PROOF:
$\langle 1\rangle 1.$ For $n \in \mathbb{N}$, PICK $a_n \in B(l, 1/(n+1)) \cap A$.
$\langle 1\rangle 2.$ $a_n \to l$ as $n \to \infty$.
□

Corollary 16.1.23.1. *$\mathbb{R}^\omega$ under the box topology is not first countable.*

PROOF:
$\langle 1\rangle 1.$ LET: A be the set of all sequences of positive reals.
$\langle 1\rangle 2.$ $0 \in \overline{A}$
$\langle 1\rangle 3.$ LET: (a_n) be a sequence in A
  PROVE: (a_n) does not converge to 0.
$\langle 1\rangle 4.$ For all $n \in \mathbb{N}$,
  LET: $a_n = (x_{nm})$
$\langle 1\rangle 5.$ LET: $B' = \prod_{n=0}^{\infty}(-x_{nn}, x_{nn})$
$\langle 1\rangle 6.$ B' is open in the box topology.

⟨1⟩7. $0 \in B'$
⟨1⟩8. For all n we have $a_n \notin B'$
□

Corollary 16.1.23.2. *If J is an uncountable set then $\mathbb{R}^J$ under the product topology is not first countable.*

PROOF:
⟨1⟩1. LET: $A = \{x \in \mathbb{R}^J : \pi_j(x) = 1$ for all but finitely many $j \in J\}$
⟨1⟩2. $0 \in \overline{A}$
⟨1⟩3. LET: (a_n) be a sequence in A.
PROVE: (a_n) does not converge to 0.
⟨1⟩4. For $n \in \mathbb{N}$,
LET: $J_n = \{j \in J : \pi_j(a_n) \neq 1\}$
⟨1⟩5. $\bigcup_{n \in \mathbb{N}} J_n$ is countable.
⟨1⟩6. PICK $\beta \in J - \bigcup_{n \in \mathbb{N}} J_n$
⟨1⟩7. $\forall n \in \mathbb{N}.\pi_\beta(a_n) = 1$
⟨1⟩8. LET: $U = {\pi_\beta}^{-1}((-1, 1))$
⟨1⟩9. $0 \in U$
⟨1⟩10. $\forall n \in \mathbb{N}.a_n \notin U$
⟨1⟩11. (a_n) does not converge to 0.
□

16.1.4 Continuous Functions

Proposition 16.1.24. *Let X and Y be metric spaces. Let $f : X \to Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:
⟨1⟩1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
⟨2⟩1. ASSUME: f is continuous.
⟨2⟩2. LET: $x \in X$
⟨2⟩3. LET: $\epsilon > 0$
⟨2⟩4. $x \in f^{-1}(B(f(x), \epsilon)$
⟨2⟩5. There exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon)$.
⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.
⟨2⟩1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
⟨2⟩2. LET: V be open in Y
⟨2⟩3. LET: $x \in f^{-1}(V)$
⟨2⟩4. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
⟨2⟩5. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
⟨2⟩6. $B(x, \delta) \subseteq f^{-1}(V)$
□

Proposition 16.1.25. *Let X be a metrizable space and Y a topological space. Let $f : X \to Y$. Assume that, for every sequence (x_n) in X and $l \in X$, if $x_n \to l$ as $n \to \infty$ then $f(x_n) \to f(l)$ as $n \to \infty$. Then f is continuous.*

PROOF:
⟨1⟩1. LET: $A \subseteq X$
PROVE: $f(\overline{A}) \subseteq \overline{f(A)}$
⟨1⟩2. LET: $l \in \overline{A}$
PROVE: $f(l) \in \overline{f(A)}$
⟨1⟩3. PICK a sequence (x_n) in A such that $x_n \to l$ as $n \to \infty$.
⟨1⟩4. $f(x_n) \to f(l)$ as $n \to \infty$.
⟨1⟩5. $f(l) \in \overline{f(A)}$
∎

Proposition 16.1.26. *The function $i : \mathbb{R} - \{0\} \to \mathbb{R}$ that maps x to x^{-1} is continuous.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}$ with $a < b$
PROVE: $i^{-1}((a, b))$ is open.
⟨1⟩2. CASE: $0 < a$
PROOF: $i^{-1}((a, b)) = (b^{-1}, a^{-1})$
⟨1⟩3. CASE: $a = 0$
PROOF: $i^{-1}((a, b)) = (b^{-1}, +\infty)$
⟨1⟩4. CASE: $a < 0 < b$
PROOF: $i^{-1}((a, b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$
⟨1⟩5. CASE: $b = 0$
PROOF: $i^{-1}((a, b)) = (-\infty, a^{-1})$
⟨1⟩6. CASE: $b < 0$
PROOF: $i^{-1}((a, b)) = (b^{-1}, a^{-1})$
∎

Proposition 16.1.27. *Subtraction is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

PROOF: Since $a - b = a + (-1)b$ and both addition and multiplication are continuous. ∎

Proposition 16.1.28. *Division is a continuous function $\mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R}$.*

PROOF: Since both multiplication and the function that maps x to x^{-1} are continuous. ∎

Proposition 16.1.29. *Let X be a metric space. Let $A \subseteq X$ be nonempty. The function $d(-, A) : X \to \mathbb{R}$ is continuous.*

PROOF:
⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$.
⟨1⟩2. LET: $\delta = \epsilon/2$
⟨1⟩3. LET: $y \in X$ with $d(x, y) < \delta$.

⟨1⟩4. $d(x,A) < d(y,A) + \epsilon$
⟨2⟩1. $\forall a \in A. d(x,a) < d(y,a) + \delta$
PROOF:
$$\begin{aligned} d(x,a) &\leqslant d(y,a) + d(x,y) && \text{(Triangle Inequality)} \\ &< d(y,a) + \delta && (\langle 1\rangle 3) \end{aligned}$$
⟨2⟩2. $\forall a \in A. d(x,A) < d(y,a) + \delta$
⟨2⟩3. $\forall a \in A. d(x,A) - \delta < d(y,a)$
⟨2⟩4. $d(x,A) - \delta \leqslant d(y,A)$
⟨2⟩5. $d(x,A) \leqslant d(y,A) + \delta$
⟨2⟩6. $d(x,A) < d(y,A) + \epsilon$
⟨1⟩5. $d(y,A) < d(x,A) + \epsilon$
PROOF: Similar.
⟨1⟩6. $|d(x,A) - d(y,A)| < \epsilon$
□

Proposition 16.1.30. *Addition is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

PROOF:
⟨1⟩1. LET: $(x,y) \in \mathbb{R}^2$ and $\epsilon > 0$
⟨1⟩2. LET: $\delta = \epsilon/2$
⟨1⟩3. LET: $(x',y') \in (x-\delta, x+\delta) \times (y-\delta, y+\delta)$
⟨1⟩4. $|x-x'|, |y-y'| < \delta$
⟨1⟩5. $|(x+y)-(x'+y')| < \epsilon$
PROOF:
$$\begin{aligned} |(x+y)-(x'+y')| &\leqslant |x-x'| + |y-y'| \\ &< \delta + \delta && (\langle 1\rangle 4) \\ &= \epsilon && (\langle 1\rangle 2) \end{aligned}$$
□

16.1.5 First Countable Spaces

Proposition 16.1.31. *Every metrizable space is first countable.*

PROOF: For any point x, the set $\{B(x,1/n) : n \in \mathbb{Z}_+\}$ is a countable basis at x.
□

Corollary 16.1.31.1. *$\mathbb{R}^\omega$ under the box topology is not metrizable.*

Corollary 16.1.31.2. *If J is an uncountable set then $\mathbb{R}^J$ under the product topology is not metrizable.*

16.1.6 Hausdorff Spaces

Proposition 16.1.32. *Every metric space is Hausdorff.*

PROOF:
⟨1⟩1. LET: X be a metric space.

$\langle 1\rangle 2$. LET: $x, y \in X$ with $x \neq y$.
$\langle 1\rangle 3$. LET: $\epsilon = d(x, y)$
$\langle 1\rangle 4$. $B(x, \epsilon/2)$ and $B(y, \epsilon/2)$ are disjoint neighbourhoods of x and y.
$\square$

16.1.7 Bounded Sets

Definition 16.1.33 (Bounded)**.** Let X be a metric space. Let $A \subseteq X$. Then A is *bounded* iff there exists M such that $\forall x, y \in A.d(x, y) \leqslant M$. Its *diameter* is then defined to be

$$\operatorname{diam} A := \sup\{d(x, y) : x, y \in A\} \ .$$

16.1.8 Uniform Convergence

Definition 16.1.34 (Uniform Convergence)**.** Let X be a set and Y a metric space. Let (f_n) be a sequence of functions $X \to Y$, and $f : X \to Y$. Then (f_n) *converges uniformly* to f iff, for all $\epsilon > 0$, there exists N such that

$$\forall n \geqslant N.\forall x \in X.d(f_n(x), f(x)) < \epsilon \ .$$

Example 16.1.35. For $n \in \mathbb{N}$ define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = x^n$. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = 0$ for $x < 1$, $f(1) = 1$. Then f_n converges pointwise to f, but does not converge uniformly to f.

We prove that, for all N, there exists $n \geqslant N$ and $x \in [0, 1]$ such that $|x^n - f(x)| \geqslant 1/2$. Take $n = N$ and x to be the Nth root of $3/4$.

Example 16.1.36. For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} \ .$$

Then for all $x \in \mathbb{R}$ we have $f_n(x) \to 0$ as $n \to \infty$, but (f_n) does not converge uniformly to 0.

We prove that, for all N, there exists $n \geqslant N$ and $x \in \mathbb{R}$ such that $|f_n(x)| \geqslant 1/2$. Take $n = N$ and $x = 1/N$. We have $f_N(1/N) = 1$.

Theorem 16.1.37 (Uniform Limit Theorem)**.** *Let X be a topological space and Y a metric space. Let (f_n) be a sequence of functions $X \to Y$, and $f : X \to Y$. If every f_n is continuous and (f_n) converges uniformly to f, then f is continuous.*

PROOF:
$\langle 1\rangle 1$. LET: V be open in Y.
$\langle 1\rangle 2$. LET: $x_0 \in f^{-1}(V)$
PROVE: There exists a neighbourhood U of x_0 such that $f(U) \subseteq V$.
$\langle 1\rangle 3$. LET: $y_0 = f(x_0)$
$\langle 1\rangle 4$. PICK $\epsilon > 0$ such that $B(y_0, \epsilon) \subseteq V$.

⟨1⟩5. PICK N such that $\forall n \geqslant N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/3$.
⟨1⟩6. PICK a neighbourhood U of x_0 such that $f_N(U_2) \subseteq B(f_N(x_0), \epsilon/3)$.
PROVE: $f(U) \subseteq V$
⟨1⟩7. LET: $y \in U$
⟨1⟩8. $d(f(y), y_0) < \epsilon$
PROOF:

$$\begin{aligned} d(f(y), y_0) &\leqslant d(f(y), f_N(y)) + d(f_N(y), f_N(x_0)) + d(f_N(x_0), y_0) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1\rangle 5,\ \langle 1\rangle 6) l \\ &= \epsilon \end{aligned}$$

⟨1⟩9. $f(y) in V$
PROOF: ⟨1⟩4
□

Proposition 16.1.38. *Let X be a topological space. Let Y be a metric space. Let f_n be a sequence of functions $X \to Y$ and $f : X \to Y$. Let x_n be a sequence of points in X and $l \in X$. If f_n converges uniformly to f, x_n converges to l, and f is continuous, then $f_n(x_n)$ converges to $f(l)$.*

PROOF:
⟨1⟩1. f is continuous.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK $\delta > 0$ such that $\forall y \in X. d(y, l) < \delta \Rightarrow d(f(y), f(l)) < \epsilon/2$
⟨1⟩4. PICK N such that $\forall n \geqslant N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/2$ and $\forall n \geqslant N. d(x_n, l) < \delta$
⟨1⟩5. For all $n \geqslant N$ we have $d(f_n(x_n), f(l)) < \epsilon$
PROOF:

$$\begin{aligned} d(f_n(x_n), f(l)) &\leqslant d(f_n(x_n), f(x_n)) + d(f(x_n), f(l)) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

Theorem 16.1.39 (Weierstrass M-Test)**.** *Let X be a set. Let (f_n) be a sequence of functions $X \to \mathbb{R}$. Let (M_n) be a sequence of real numbers. For $n \in \mathbb{N}$, let*

$$s_n(x) = \sum_{i=0}^{n} f_i(x) \ .$$

Assume that $\forall n \in \mathbb{N}. \forall x \in X. |f_n(x)| \leqslant M_n$. Assume that $\sum_{n=0}^{\infty} M_n$ converges. Then (s_n) uniformly converges to s where $s(x) = \sum_{n=0}^{\infty} f_n(x)$.

PROOF:
⟨1⟩1. For all $x \in X$ we have $\sum_{n=0}^{\infty} f_n(x)$ converges.
PROOF: By the Comparison Test.
⟨1⟩2. For $n \in \mathbb{N}$,
LET: $r_n = \sum_{i=n+1}^{\infty} M_i$.
⟨1⟩3. For all $k, n \in \mathbb{N}$ and $x \in X$, if $k > n$ then $|s_k(x) - s_n(x)| \leqslant r_n$.

PROOF:

$$\begin{aligned}
|s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^{k} f_i(x) \right| \\
&\leqslant \sum_{i=n+1}^{k} |f_i(x)| \\
&\leqslant \sum_{i=n+1}^{k} M_i \\
&\leqslant \sum_{i=n+1}^{\infty} M_i \\
&= r_n
\end{aligned}$$

⟨1⟩4. For all $n \in \mathbb{N}$ we have $|s(x) - s_n(x)| \leqslant r_n$.
PROOF: Taking the limit $k \to \infty$ in ⟨1⟩3.
⟨1⟩5. (s_n) converges uniformly to s.
PROOF: We have $\overline{\rho}(s_n, s) \leqslant r_n$ and so $\overline{\rho}(s_n, s) \to 0$ as $n \to \infty$ by the Sandwich Theorem.
□

Theorem 16.1.40. *Let X be a compact space. Let $f_n : X \to \mathbb{R}$ be a monotone increasing sequence of continuous functions, and $f : X \to \mathbb{R}$ be continuous. If f_n converges to f pointwise, then f_n converges to f uniformly.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. For all $x \in X$, there exists a neighbourhood U of x and an integer N such that, for all $y \in U$, we have $|f_N(y) - f(y)| < \epsilon$
⟨2⟩1. LET: $x \in X$
⟨2⟩2. PICK N such that $\forall n \geqslant N. |f_n(x) - f(x)| < \epsilon/3$.
⟨2⟩3. PICK neighbourhoods U_1, U_2 of x such that $\forall y \in U_1. |f_N(y) - f_N(x)| < \epsilon/3$ and $\forall y \in U. |f(y) - f(x)| < \epsilon/3$.
⟨2⟩4. LET: $U = U_1 \cap U_2$
⟨2⟩5. LET: $y \in U$
⟨2⟩6. $|f_N(y) - f(y)| < \epsilon$
PROOF:

$$\begin{aligned}
|f_N(y) - f(y)| &\leqslant |f_N(y) - f_N(x)| + |f_N(x) - f(x)| + |f(x) - f(y)| && \text{(Triangle Inequality)} \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 2\rangle 2,\ \langle 2\rangle 3) \\
&= \epsilon
\end{aligned}$$

⟨1⟩3. PICK open sets $U_1, \ldots, U_k$ that cover X such that, for all i, there exists N_i such that $\forall y \in U_i. |f_{N_i}(y) - f(y)| < \epsilon$
⟨1⟩4. LET: $N = \max(N_1, \ldots, N_k)$
⟨1⟩5. For all $x \in X$ we have $|f_N(x) - f(x)| < \epsilon$
⟨1⟩6. For all $n \geqslant N$ and $x \in X$ we have $|f_n(x) - f(x)| < \epsilon$
□

Example 16.1.41. We cannot remove the requirement that (f_n) is monotone increasing.

For all $n \in \mathbb{Z}_+$, define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} \ .$$

Then $f_n(x) \to 0$ as $n \to \infty$ for all x, but the convergence is not uniform.

Example 16.1.42. We cannot remove the requirement that X is compact.

For all $n \in \mathbb{Z}_+$, define $f_n : [0,1) \to \mathbb{R}$ by $f_n(x) = -x^n$. Then (f_n) is monotone increasing and $f_n(x) \to 0$ as $n \to \infty$ for all x, but the convergence is not uniform.

16.1.9 Standard Bounded Metric

Definition 16.1.43 (Standard Bounded Metric)**.** Let (X, d) be a metric space. The *standard bounded metric* corresponding to d is

$$\overline{d}(x,y) := \min(d(x,y), 1) \ .$$

Proposition 16.1.44. *The standard bounded metric associated with d induces the same topology as d.*

PROOF:
⟨1⟩1. LET: (X, d) be a metric space.
⟨1⟩2. Every d-ball is open under the topology induced by $\overline{d}$.
  ⟨2⟩1. LET: $a \in X$ and $\epsilon > 0$
  ⟨2⟩2. LET: $x \in B_d(a, \epsilon)$
  ⟨2⟩3. LET: $\delta = \min(\epsilon - d(a,x), 1/2)$
  ⟨2⟩4. $B_{\overline{d}}(x, \delta) \subseteq B_d(a, \epsilon)$
⟨1⟩3. Every $\overline{d}$-ball is open under the topology induced by d.
  PROOF: Since $B_{\overline{d}}(a, \epsilon) = B_d(a, \epsilon)$ if $\epsilon \leqslant 1$, and X if $\epsilon > 1$.
□

16.1.10 Product Spaces

Proposition 16.1.45. *The product of a countable family of metrizable spaces is metrizable.*

PROOF:
⟨1⟩1. LET: (X_n, d_n) be a sequence of metric spaces.
⟨1⟩2. For $n \in \mathbb{N}$,
  LET: $\overline{d_n}$ be the standard bounded metric associated with d_n.
⟨1⟩3. LET: $X = \prod_{n \in \mathbb{N}} X_n$
⟨1⟩4. Define $D : X^2 \to \mathbb{R}$ by $D(x,y) = \sup_{n \in \mathbb{N}} \overline{d_n}(\pi_n(x), \pi_n(y))/(n+1)$.
⟨1⟩5. D is a metric on X.
  ⟨2⟩1. For all $x, y \in X$ we have $D(x,y) \geqslant 0$.

⟨2⟩2. For all $x, y \in X$ we have $D(x, y) = 0$ iff $x = y$.
⟨2⟩3. For all $x, y \in X$ we have $D(x, y) = D(y, x)$.
⟨2⟩4. For all $x, y, z \in X$ we have $D(x, z) \leqslant D(x, y) + D(y, z)$.
⟨1⟩6. The product topology is finer than the metric topology induced by D.
⟨2⟩1. LET: $a \in X$ and $\epsilon > 0$.
⟨2⟩2. LET: $x \in B(a, \epsilon)$
⟨2⟩3. LET: $\delta = \epsilon - D(a, x)$
⟨2⟩4. PICK $N \in \mathbb{N}$ such that $1/(N + 1) < \delta$
⟨2⟩5. $x \in \prod_{n=0}^{N} B_{\overline{d_n}}(\pi_n(a), n\delta) \times \prod_{n=N+1}^{\infty} \subseteq B(a, \epsilon)$
⟨1⟩7. The metric topology induced by D is finer than the product topology.
⟨2⟩1. LET: $n \in \mathbb{N}$ and U be an open set in X_n.
PROVE: ${\pi_n}^{-1}(U)$ is open in the metric topology.
⟨2⟩2. LET: $x \in {\pi_n}^{-1}(U)$
⟨2⟩3. PICK $\epsilon > 0$ such that $B_{\overline{d_n}}(\pi_n(x), \epsilon) \subseteq U$
⟨2⟩4. $B(x, \epsilon/(n + 1)) \subseteq {\pi_n}^{-1}(U)$
□

Definition 16.1.46. For $n \geqslant 1$, the *unit ball* B^n is the closed ball $\overline{B(0, 1)}$ in $\mathbb{R}^n$ under the Euclidean metric.

Theorem 16.1.47. *Let n be a positive integer. Let A be a subspace of $\mathbb{R}^n$. Then A is compact if and only if it is closed and bounded.*

PROOF:
⟨1⟩1. If A is compact then A is closed.
PROOF: Theorem 15.31.10.
⟨1⟩2. If A is compact then A is bounded.
PROOF: Proposition 16.1.20.
⟨1⟩3. If A is closed and bounded then A is compact.
⟨2⟩1. ASSUME: A is closed and bounded.
⟨2⟩2. PICK M such that $\forall \vec{x}, \vec{y} \in A. \rho(\vec{x}, \vec{y}) < M$.
⟨2⟩3. ASSUME: w.l.o.g. $A \neq \emptyset$
⟨2⟩4. PICK $\vec{a} \in A$
⟨2⟩5. $A \subseteq \prod_{i=1}^{n} [a_i - M, a_i + M]$
⟨2⟩6. A is compact.
PROOF: Theorem 15.31.7.

Corollary 16.1.47.1. *For $n \geqslant 1$, the unit sphere S^{n-1} and the unit ball B^n are compact.*

16.2 Uniform Metric

Definition 16.2.1 (Uniform Metric)**.** Let J be a nonempty set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(x, y) = \sup_{j \in J} \overline{d}(x_j, y_j)$$

where $\overline{d}$ is the standard bounded metric associated with the standard metric on $\mathbb{R}$.

On $\mathbb{R}^n$ we call the uniform metric the *square metric* and denote it by ρ.

The topology it induces is called the *uniform topology*.

We prove this is a metric.

PROOF:
⟨1⟩1. For all $x, y \in \mathbb{R}^\omega$ we have $\overline{\rho}(x, y) \geqslant 0$.
PROOF: Pick $j_0 \in J$. Then

$$\begin{aligned}\overline{\rho}(x, y) &= \sup_j \overline{d}(x_j, y_j) \\ &\geqslant \overline{d}(x_{j_0}, y_{j_0}) \\ &\geqslant 0\end{aligned}$$

⟨1⟩2. For all $x, y \in \mathbb{R}^\omega$ we have $\overline{\rho}(x, y) = 0$ iff $x = y$.
PROOF:

$$\begin{aligned}\overline{\rho}(x, y) = 0 &\Leftrightarrow \sup_j \overline{d}(x_j, y_j) = 0 \\ &\Leftrightarrow \forall j. \overline{d}(x_j, y_j) = 0 \\ &\Leftrightarrow \forall j. x_j = y_j \\ &\Leftrightarrow x = y\end{aligned}$$

⟨1⟩3. For all $x, y \in \mathbb{R}^\omega$ we have $\overline{\rho}(x, y) = \overline{\rho}(y, x)$.
PROOF:

$$\begin{aligned}\overline{\rho}(x, y) &= \sup_j \overline{d}(x_j, y_j) \\ &= \sup_j \overline{d}(y_j, x_j) \\ &= \overline{\rho}(y, x)\end{aligned}$$

⟨1⟩4. For all $x, y, z \in \mathbb{R}^\omega$ we have $\overline{\rho}(x, z) \leqslant \overline{\rho}(x, y) + \overline{\rho}(y, z)$.
PROOF:

$$\begin{aligned}\overline{\rho}(x, z) &= \sup_j \overline{d}(x_j, z_j) \\ &\leqslant \sup_j (\overline{d}(x_j, y_j) + \overline{d}(y_j, z_j)) \\ &\leqslant \sup_j \overline{d}(x_j, y_j) + \sup_j \overline{d}(y_j, z_j) \\ &= \overline{\rho}(x, y) + \overline{\rho}(y, z)\end{aligned}$$

□

Proposition 16.2.2. *The uniform topology is finer than the product topology. It is strictly finer iff J is infinite.*

PROOF:
⟨1⟩1. The uniform topology is finer than the product topology.
⟨2⟩1. LET: U be open in $\mathbb{R}$ and $j \in J$
PROVE: ${\pi_j}^{-1}(U)$ is open in the uniform topology.
⟨2⟩2. LET: $x \in {\pi_j}^{-1}(U)$

⟨2⟩3. $\pi_j(x) \in U$
⟨2⟩4. Pick $\epsilon > 0$ such that $B_{\overline{d}}(\pi_j(x), \epsilon) \subseteq U$
⟨2⟩5. $B_{\overline{\rho}}(x, \epsilon) \subseteq {\pi_j}^{-1}(U)$
⟨1⟩2. If J is finite then the uniform topology is equal to the product topology.
Proof: In $\mathbb{R}^n$, the uniform topology is the square topology.
⟨1⟩3. If J is infinite then the uniform topology is not equal to the product topology.
Proof: If J is infinite then $B(0, 1)$ is not open in the product topology.
□

Proposition 16.2.3. *The uniform topology is coarser than the box topology. It is strictly coarser iff J is infinite.*

Proof:
⟨1⟩1. The uniform topology is coarser than the box topology.
⟨2⟩1. Let: U be open in the uniform topology.
Prove: U is open in the box topology.
⟨2⟩2. Let: $x \in U$
⟨2⟩3. Pick $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
⟨2⟩4. $\prod_{j \in J}(x_j - \epsilon, x_j + \epsilon) \subseteq U$
⟨1⟩2. If J is finite then the uniform topology is equal to the box topology.
Proof: On $\mathbb{R}^n$, the uniform metric is the square metric.
⟨1⟩3. If J is infinite then the uniform topology is not equal to the box topology.
⟨2⟩1. Assume: J is infinite.
⟨2⟩2. Pick a sequence (j_n) of distinct elements in J.
⟨2⟩3. Let: $U = \prod_j U_j$ where $J_{j_n} = (-1/(n+1), 1/(n+1))$ for $n \in \mathbb{N}$ and $J_j = (-1, 1)$ for all other j.
⟨2⟩4. U is not open in the uniform topology.
□

Proposition 16.2.4. *The uniform topology on $\mathbb{R}^\infty$ is strictly finer than the product topology.*

Proof: The set of all sequences $(x_n) \in \mathbb{R}^\infty$ such that $\forall n. |x_n| < 1$ is open in the uniform topology but not in the product topology. □

Proposition 16.2.5. *The uniform topology on $\mathbb{R}^\infty$ is strictly coarser than the box topology.*

Proof: The set of sequences $(x_n) \in \mathbb{R}^\infty$ such that $\forall n. |x_n| < 1/n$ is open in the box topology but not in the uniform topology. □

Proposition 16.2.6. *The uniform topology on the Hilbert cube is the same as the product topology.*

Proof:
⟨1⟩1. Let: (x_n) be in the Hilbert cube H and $\epsilon > 0$.
Prove: $B((x_n), \epsilon) \cap H$ is open in the product topology.
⟨1⟩2. Pick N such that $1/N < \epsilon$

$\langle 1\rangle 3.$ $B((x_n), \epsilon) = (\prod_{n=0}^{N}(x_n - \epsilon, x_n + \epsilon) \times \prod_{n=N+1}^{\infty}[0, 1/(n+1)]) \cap H$
□

Corollary 16.2.6.1. *The uniform topology on the Hilbert cube is strictly finer than the box topology.*

Proposition 16.2.7. *Let X be a set and Y a metric space. Let (f_n) be a sequence of functions $X \to Y$, and $f : X \to Y$. Then (f_n) converges uniformly to f iff (f_n) converges to f in Y^X under the uniform topology.*

PROOF:
$\langle 1\rangle 1.$ If (f_n) converges uniformly to f then (f_n) converges to f in Y^X under the uniform topology.
  $\langle 2\rangle 1.$ ASSUME: (f_n) converges uniformly to f.
  $\langle 2\rangle 2.$ LET: $\epsilon > 0$
  $\langle 2\rangle 3.$ PICK N such that $\forall n \geqslant N.\forall x \in X.d(f_n(x), f(x)) < \epsilon/2$
  $\langle 2\rangle 4.$ $\forall n \geqslant N.\overline{\rho}(f_n, f) \leqslant \epsilon/2$
  $\langle 2\rangle 5.$ $\forall n \geqslant N.\overline{\rho}(f_n, f) < \epsilon$
$\langle 1\rangle 2.$ If (f_n) converges to f in Y^X under the uniform topology then (f_n) converges uniformly to f.
  $\langle 2\rangle 1.$ ASSUME: (f_n) converges to f in Y^X under the uniform topology.
  $\langle 2\rangle 2.$ LET: $\epsilon > 0$
  $\langle 2\rangle 3.$ PICK N such that $\forall n \geqslant N.\overline{\rho}(f_n, f) < \epsilon$
  $\langle 2\rangle 4.$ $\forall n \geqslant N.\forall x \in X.d(f_n(x), f(x)) < \epsilon$
□

Proposition 16.2.8. *In $\mathbb{R}^\omega$ under the uniform topology, $\vec{x}$ and $\vec{y}$ lie in the same component if and only if $\vec{x} - \vec{y}$ is bounded.*

PROOF:
$\langle 1\rangle 1.$ The set of bounded sequences form a component of $\mathbb{R}^\omega$.
  $\langle 2\rangle 1.$ LET: B be the set of bounded sequences.
  $\langle 2\rangle 2.$ B is connected.
    $\langle 3\rangle 1.$ LET: $\vec{x} \in B$
      PROVE: The straight line path $p : [0, 1] \to \mathbb{R}^\omega$ from 0 to $\vec{x}$ is continuous.
    $\langle 3\rangle 2.$ LET: $t \in [0, 1]$ and $\epsilon > 0$
    $\langle 3\rangle 3.$ PICK $B > 0$ such that $\forall n.|x_n| < B$
    $\langle 3\rangle 4.$ LET: $\delta = \epsilon/B$
    $\langle 3\rangle 5.$ LET: $s \in [0, 1]$ with $|s - t| < \delta$
    $\langle 3\rangle 6.$ For all n we have $|p(s)_n - p(t)_n| < \epsilon/2$
      PROOF:

$$\begin{aligned} |p(s)_n - p(t)_n| &= |s - t||x_n| \\ &< \delta B \\ &= \epsilon \end{aligned}$$

    $\langle 3\rangle 7.$ $\overline{\rho}(p(s), p(t)) \leqslant \epsilon/2$
    $\langle 3\rangle 8.$ $\overline{\rho}(p(s), p(t)) < \epsilon$

⟨2⟩3. B is maximally connected.
PROOF: Since $(B, \mathbb{R}^\omega - B)$ form a separation of $\mathbb{R}^\omega$.
⟨1⟩2. For any $\vec{y} \in \mathbb{R}^\omega$, the component containing $\vec{y}$ is $\{\vec{x} \in \mathbb{R}^\omega : \vec{x} - \vec{y} \text{ is bounded}\}$.
PROOF: Since the function that maps $\vec{x}$ to $\vec{x} + \vec{y}$ is a homeomorphism between $\mathbb{R}^\omega$ and itself.
□

16.2.1 Products

Definition 16.2.9 (Euclidean Metric)**.** Let X and Y be metric spaces. The *Euclidean metric* on $X \times Y$ is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} \ .$$

We write $X \times Y$ for the set $X \times Y$ under this metric.

We prove this is a metric.

PROOF:
⟨1⟩1. $d((x_1, y_1), (x_2, y_2)) \geqslant 0$
PROOF: Immediate from definition.
⟨1⟩2. $d((x_1, y_1), (x_2, y_2)) = 0$ iff $(x_1, y_1) = (x_2, y_2)$
PROOF: $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = 0$ iff $d(x_1, x_2) = d(y_1, y_2) = 0$ iff $x_1 = x_2$ and $y_1 = y_2$.
⟨1⟩3. $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$
PROOF: Since $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = \sqrt{d(x_2, x_1)^2 + d(y_2, y_1)^2}$.
⟨1⟩4. The triangle inequality holds.
PROOF:

$$\begin{aligned}
&(d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)))^2 \\
=&d((x_1, y_1), (x_2, y_2))^2 + 2d((x_1, y_1), (x_2, y_2))d((x_2, y_2), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3))^2 \\
=&d(x_1, x_2)^2 + d(y_1, y_2)^2 + 2\sqrt{(d(x_1, x_2)^2 + d(y_1, y_2)^2)(d(x_2, x_3)^2 + d(y_2, y_3)^2)} + d(x_2, x_3)^2 + d(y_2, y_3)^2 \\
\geqslant&d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(y_1, y_2)^2 + d(y_2, y_3)^+ 2(d(x_1, x_2)d(x_2, x_3) + d(y_1, y_2)d(y_2, y_3)) \\
&\text{(Cauchy-Schwarz)} \\
=&(d(x_1, x_2) + d(x_2, x_3))^2 + (d(y_1, y_2) + d(y_2, y_3))^2 \\
\geqslant&d(x_1, x_3)^2 + d(y_1, y_3)^2 \\
=&d((x_1, y_1), (x_3, y_3))^2
\end{aligned}$$

□

Proposition 16.2.10. *Let X and Y be metric spaces. The Euclidean metric on $X \times Y$ induces the product topology on $X \times Y$.*

PROOF:
⟨1⟩1. Every open ball is open in the product topology.
⟨2⟩1. LET: $(x, y) \in B((a, b), \epsilon)$
PROVE: $B(x, \sqrt{\epsilon}) \times B(y, \sqrt{\epsilon}) \subseteq B((a, b), \epsilon)$
⟨2⟩2. LET: $x' \in B(x, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$ and $y' \in B(y, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$

PROVE: $d((x', y'), (a, b)) < \epsilon$

$\langle 2\rangle 3.$ $d((x', y'), (x, y)) < \epsilon - d((x, y), (a, b))$

PROOF:

$$\begin{aligned} d((x', y'), (x, y)) &= \sqrt{d(x', x)^2 + d(y', y)^2} \\ &< \sqrt{(\epsilon - d((x, y), (a, b)))^2/2 + (\epsilon - d((x, y), (a, b))^2/2} \\ &= \epsilon - d((x, y), (a, b)) \end{aligned}$$

$\langle 2\rangle 4.$ $d((x', y'), (a, b)) < \epsilon$

PROOF:

$$\begin{aligned} d((x', y'), (a, b)) &\leqslant d((x', y'), (x, y)) + d((x, y), (a, b)) && \text{(Triangle Inequality)} \\ &< \epsilon && (\langle 2\rangle 3) \end{aligned}$$

$\langle 1\rangle 2.$ If U is open in X and V is open in Y then $U \times V$ is open under the Euclidean metric.

$\langle 2\rangle 1.$ LET: $(x, y) \in U \times V$

$\langle 2\rangle 2.$ PICK $\delta, \epsilon > 0$ such that $B(x, \delta) \subseteq U$ and $B(y, \epsilon) \subseteq V$

PROVE: $(B((x, y), \min(\delta, \epsilon)) \subseteq U \times V$

$\langle 2\rangle 3.$ LET: $(x', y') \in B((x, y), \min(\delta, \epsilon))$

$\langle 2\rangle 4.$ $d(x', x) < \delta$

$\langle 3\rangle 1.$ $d((x', y'), (x, y)) < \min(\delta, \epsilon)$

$\langle 3\rangle 2.$ $d(x', x)^2 + d(y', y)^2 < \delta^2$

$\langle 3\rangle 3.$ $d(x', x)^2 < \delta^2$

$\langle 2\rangle 5.$ $d(y', y) < \epsilon$

PROOF: Similar.

$\langle 2\rangle 6.$ $(x', y') \in U \times V$

□

Proposition 16.2.11. *The square metric on $\mathbb{R}^n$ induces the product topology.*

PROOF:

$\langle 1\rangle 1.$ LET: d be the Euclidean metric on $\mathbb{R}^n$ and ρ the square metric.

$\langle 1\rangle 2.$ For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_d(x, \delta) \subseteq B_\rho(x, \epsilon)$

PROOF: If $d(x, y) < \epsilon$ then $\rho(x, y) < \epsilon$.

$\langle 1\rangle 3.$ For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_\rho(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: If $\rho(x, y) < \epsilon/\sqrt{n}$ then $d(x, y) < \epsilon$.

$\langle 1\rangle 4.$ d and ρ induce the same topology.

PROOF: Proposition 16.1.16.

□

16.2.2 Connected Spaces

Example 16.2.12. The space $\mathbb{R}^\omega$ under the uniform topology is disconnected. The set of bounded sequences and the set of unbounded sequences form a separation.

16.3 Isometric Embeddings

Definition 16.3.1 (Isometric Embedding)**.** Let X and Y be metric spaces. Let $f : X \to Y$. Then f is an *isometric embedding* of X in Y iff, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 16.3.2. *Every isometric embedding is an embedding.*

PROOF:
⟨1⟩1. LET: X and Y be metric spaces.
⟨1⟩2. LET: $f : X \to Y$ be an isometric embedding.
⟨1⟩3. f is injective.
⟨1⟩4. The subspace topology induced by f is finer than the metric topology.
  ⟨2⟩1. LET: $x \in X$ and $\epsilon > 0$
    PROVE: $B(x, \epsilon)$ is open in the subspace topology.
  ⟨2⟩2. $B(x, \epsilon) = f^{-1}(B(f(x), \epsilon))$
⟨1⟩5. The metric topology is finer than the subspace topology induced by f.
  ⟨2⟩1. LET: V be open in Y
    PROVE: $f^{-1}(V)$ is open in X
  ⟨2⟩2. LET: $x \in f^{-1}(V)$
  ⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
  ⟨2⟩4. $B(x, \epsilon) \subseteq f^{-1}(V)$
□

16.4 Lebesgue Numbers

Definition 16.4.1 (Lebesgue Number)**.** Let X be a metric space. Let $\mathcal{A}$ be an open covering of X. A *Lebesgue number* for $\mathcal{A}$ is a real number $\delta > 0$ such that every bounded set with diameter $< \delta$ is included in some member of $\mathcal{A}$.

Lemma 16.4.2 (Lebesgue Number Lemma)**.** *In a compact metric space, every open cover has a Lebesgue number.*

PROOF:
⟨1⟩1. LET: X be a compact metric space.
⟨1⟩2. LET: $\mathcal{A}$ be an open cover of X.
⟨1⟩3. ASSUME: w.l.o.g. $X \notin \mathcal{A}$
⟨1⟩4. PICK $A_1, \ldots, A_n \in \mathcal{A}$ that cover X.
⟨1⟩5. For $i = 1, \ldots, n$,
  LET: $C_i = X - A_i$.
⟨1⟩6. LET: $f : X \to \mathbb{R}$ be the function

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i) \quad .$$

  PROOF: Each C_i is nonempty by ⟨1⟩3.
⟨1⟩7. $\forall x \in X . f(x) > 0$
  ⟨2⟩1. LET: $x \in X$

⟨2⟩2. Pick i such that $x \in A_i$
⟨2⟩3. Pick $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A_i$
⟨2⟩4. $d(x, C_i) \geqslant \epsilon$
⟨2⟩5. $f(x) \geqslant \epsilon/n$
⟨1⟩8. Let: δ be the minimum value of $f(X)$.
Prove: δ is a Lebesgue number for $\mathcal{A}$.
Proof: $f(X)$ has a least element by the Extreme Value Theorem.
⟨1⟩9. Let: $B \subseteq X$ have diameter $< \delta$
⟨1⟩10. Pick $x_0 \in B$
⟨1⟩11. $B \subseteq B(x_0, \delta)$
⟨1⟩12. Let: m be such that $d(x_0, C_m) = \max(d(x_0, C_1), \ldots, d(x_0, C_n))$
⟨1⟩13. $d(x_0, C_m) \geqslant \delta$
Proof:

$$\begin{aligned}
\delta &\leqslant f(x_0) && (\langle 1\rangle 8) \\
&= \frac{1}{n} \sum_{i=1}^{n} d(x_0, C_i) && (\langle 1\rangle 6) \\
&\leqslant \frac{1}{n} \sum_{i=1}^{n} d(x_0, C_m) && (\langle 1\rangle 12) \\
&= d(x_0, C_m)
\end{aligned}$$

⟨1⟩14. $B \subseteq A_m$
Proof: ⟨1⟩11, ⟨1⟩13
□

16.5 Uniform Continuity

Definition 16.5.1 (Uniformly Continuous)**.** Let X and Y be metric spaces. Let $f : X \to Y$. Then f is *uniformly continuous* iff, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Theorem 16.5.2 (Uniform Continuity Theorem)**.** *Every continuous function from a compact metric space to a metric space is uniformly continuous.*

Proof:
⟨1⟩1. Let: X be a compact metric space.
⟨1⟩2. Let: Y be a metric space.
⟨1⟩3. Let: $f : X \to Y$ be continuous.
⟨1⟩4. Let: $\epsilon > 0$
⟨1⟩5. Pick a Lebesgue number δ for $\{f^{-1}(B(y, \epsilon/2)) : y \in Y\}$.
⟨1⟩6. Let: $x, x' \in X$
⟨1⟩7. Assume: $d(x, x') < \delta$
⟨1⟩8. Pick $y \in Y$ such that $\{x, x'\} \subseteq f^{-1}(B(y, \epsilon/2))$
⟨1⟩9. $d(f(x), f(x')) < \epsilon$

PROOF:

$$\begin{aligned} d(f(x), f(x')) &\leqslant d(f(x), y) + d(y, f(x')) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1 \rangle 8) \\ &= \epsilon \end{aligned}$$

□

16.6 Complete Metric Spaces

Definition 16.6.1 (Complete)**.** A metric space is *complete* iff every Cauchy sequence converges.

Example 16.6.2. $\mathbb{R}$ is complete.

Proposition 16.6.3. *The product of two complete metric spaces is complete.*

Proposition 16.6.4. *Every compact metric space is complete.*

Proposition 16.6.5. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 16.6.6 (Completion)**.** Let X be a metric space. A *completion* of X is a complete metric space $\hat{X}$ and injection $i : X \rightarrowtail \hat{X}$ such that:

- The metric on X is the restriction of the metric on $\hat{X}$
- X is dense in $\hat{X}$.

Proposition 16.6.7. *Let $i_1 : X \to Y_1$ and $i_2 : X \to Y_2$ be completions of X. Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

PROOF: Define $\phi(\lim_{n\to\infty} i_1(x_n)) = \lim_{n\to\infty} i_2(x_n)$. □

Theorem 16.6.8. *Every metric space has a completion.*

PROOF: Let $\hat{X}$ be the set of Cauchy sequences in X quotiented by $\sim$ where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \to 0$. □

16.7 Manifolds

Definition 16.7.1 (Manifold)**.** An *n-dimensional manifold* is a second countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$.

Chapter 17

Homotopy Theory

17.1 Homotopies

Definition 17.1.1 (Homotopy)**.** Let X and Y be topological spaces. Let $f, g : X \to Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0,1] \to Y$ such that

- $\forall x \in X.h(x,0) = f(x)$
- $\forall x \in X.h(x,1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \to Y$.

Proposition 17.1.2. *Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.*

Definition 17.1.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

A *homotopy functor* is a functor $\mathbf{Top} \to \mathcal{C}$ that factors through the canonical functor $\mathbf{Top} \to \mathbf{HTop}$.

Definition 17.1.4. A functor $F : \mathbf{Top} \to \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \to Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

17.2 Homotopy Equivalence

Definition 17.2.1 (Homotopy Equivalence)**.** Let X and Y be topological spaces. A *homotopy equivalence* between X and Y, $f : X \simeq Y$, is a continuous function $f : X \to Y$ such that there exists a continuous function $g : Y \to X$, the *homotopy inverse* to f, such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Definition 17.2.2 (Contractible)**.** A topological space X is *contractible* iff $X \simeq 1$.

Example 17.2.3. $\mathbb{R}^n$ is contractible.

Example 17.2.4. D^n is contractible.

Definition 17.2.5 (Deformation Retract)**.** Let X be a topological space and A a subspace of X. A retraction $\rho : X \to A$ is a *deformation retraction* iff $i \circ \rho \simeq \mathrm{id}_X$, where i is the inclusion $A \rightarrowtail X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 17.2.6 (Strong Deformation Retract)**.** Let X be a topological space and A a subspace of X. A *strong deformation retraction* $\rho : X \to A$ is a continuous function such that there exists a homotopy $h : X \times [0,1] \to X$ between $i \circ \rho$ and id_X such that, for all $a \in X$ and $t \in [0,1]$, we have $h(a,t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 17.2.7. $\{0\}$ is a strong deformation retract of $\mathbb{R}^n$ and of D^n.

Example 17.2.8. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 17.2.9. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 17.2.10. For any topological space X, the singleton consisting of the vertex is a strong deformation retract of the cone over X.

Chapter 18

Simplicial Complexes

Definition 18.0.1 (Simplex)**.** A *k-dimensional simplex* or *k-simplex* in $\mathbb{R}^n$ is the convex hull $s(x_0, \ldots, x_k)$ of $k + 1$ points in general position.

Definition 18.0.2 (Face)**.** A *sub-simplex* or *face* of $s(x_0, \ldots, x_k)$ is the convex hull of a subset of $\{x_0, \ldots, x_k\}$.

Definition 18.0.3 (Simplicial Complex)**.** A *simplicial complex* in $\mathbb{R}^n$ is a set K of simplices such that:

- for every simplex s in K, every face of s is in K.
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2.
- K is locally finite, i.e. every point of $\mathbb{R}^n$ has a neighbourhood that only intersects finitely many elements of K.

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of $\mathbb{R}^n$.

18.1 Cell Decompositions

Definition 18.1.1 (n-cell)**.** An *n-cell* is a topological space homeomorphic to $\mathbb{R}^n$.

Definition 18.1.2 (Cell Decomposition)**.** Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n-cells.

Definition 18.1.3 (n-skeleton)**.** Given a cell decomposition of X, the *n-skeleton* X^n is the union of all the cells of dimension $\leqslant n$.

18.2 CW-complexes

Definition 18.2.1 (CW-Complex)**.** A *CW-complex* consists of a topological space X and a cell decomposition $\mathcal{E}$ of X such that:

1. *Characteristic Maps* For every n-cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \to X$ such that $\Phi((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in $\mathcal{E}$ of dimension $< n$.

2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have $\overline{e}$ intersects only finitely many other cells in $\mathcal{E}$.

3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \overline{e}$ is closed.

Proposition 18.2.2. *If a cell decomposition $\mathcal{E}$ satisfies the Characteristic Maps axiom, then for every n-cell $e \in \mathcal{E}$ we have $\overline{e} = \Phi_e(D^n)$. Therefore $\overline{e}$ is compact and $\overline{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

PROOF:
⟨1⟩1. $e \subseteq \Phi_e(D^n) \subseteq \overline{e}$
PROOF:

$$\begin{aligned} e &= \Phi_e((D^n)^\circ) \\ &\subseteq \Phi_e(D^n) \\ &= \Phi_e(\overline{(D^n)^\circ}) \\ &\subseteq \overline{\Phi_e((D^n)^\circ)} \\ &= \overline{e} \end{aligned}$$

⟨1⟩2. $\Phi_e(D^n)$ is compact.
PROOF: Because D^n is compact.
⟨1⟩3. $\Phi_e(D^n)$ is closed.
⟨1⟩4. $\Phi_e(D^n) = \overline{e}$
□

Chapter 19

Topological Groups

19.1 Topological Groups

Definition 19.1.1 (Topological Group)**.** A *topological group* is a group G with a topology such that the function $G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.

Example 19.1.2. $\mathbb{Z}$ is a topological group under addition.

PROOF: The function that sends (x, y) to xy^{-1} is continuous because the topology on $\mathbb{Z}$ is discrete. □

Example 19.1.3. $\mathbb{R}$ is a topological group under addition.

PROOF: From Propositions 16.1.30 and 16.1.10. □

Example 19.1.4. $\mathbb{R}_+$ is a topological group under multiplication.

PROOF: From Propositions 16.1.10 and 16.1.26. □

Example 19.1.5. S^1 as a subspace of $\mathbb{C}$ is a topological group under multiplication.

PROOF:
⟨1⟩1. LET: $f : S^1 \to S^1$ be the function $f(x, y) = xy^{-1}$
⟨1⟩2. LET: U be an open set in S^1
PROVE: $f^{-1}(U)$ is open in $(S^1)^2$
⟨1⟩3. LET: $(x, y) \in f^{-1}(U)$
⟨1⟩4. $xy^{-1} \in U$
⟨1⟩5. LET: $x = e^{i\phi}$ and $y = e^{i\psi}$
⟨1⟩6. $xy^{-1} = e^{i(\phi-\psi)} \in U$
⟨1⟩7. PICK $\epsilon > 0$ such that, for all t, if $|\phi - \psi - t| < \epsilon$ then $e^{it} \in U$
⟨1⟩8. $(x, y) \in \{e^{it} : |\phi - t| < \epsilon/2\} \times \{e^{it} : |\psi - t| < \epsilon/2\} \subseteq f^{-1}(U)$
□

Example 19.1.6. $GL(n, \mathbb{R})$ is a topological group considered as a subspace of $\mathbb{R}^{n^2}$.

PROOF: Since the calculations for matrix multiplication and inverse are compositions of continuous functions. ∎

Example 19.1.7. $GL(n, \mathbb{C})$ is a topological group.

PROOF: Similar. ∎

Proposition 19.1.8. *Let G be a group with a topology. Then G is a topological group if and only if the functions $m : G^2 \to G$ that sends (x, y) to xy and the function $i : G \to G$ that sends x to x^{-1} are continuous.*

PROOF:
⟨1⟩1. If G is a topological group then i is continuous.
  PROOF: Since $x^{-1} = ex^{-1}$.
⟨1⟩2. If G is a topological group then m is continuous.
  PROOF: Since $xy = x(y^{-1})^{-1}$.
⟨1⟩3. If m and i are continuous then G is a topological group.
  PROOF: Since $xy^{-1} = m(x, i(y))$.
∎

Proposition 19.1.9. *Let G be a topological group. Let $\alpha \in G$. The function that maps x to αx is a homeomorphism between G and itself.*

PROOF:
⟨1⟩1. For any $\alpha \in G$, the function that maps x to αx is continuous.
  PROOF: From the definition of topological group.
⟨1⟩2. For any $\alpha \in G$, the function that maps x to αx is a homeomorphism between G and itself.
  PROOF: Its inverse is the function that maps x to $\alpha^{-1}x$.
∎

Corollary 19.1.9.1. *Every topological group is homogeneous.*

Proposition 19.1.10. *Let G be a topological group. Let $\alpha \in G$. The function that maps x to $x\alpha$ is a homeomorphism between G and itself.*

PROOF: Similar. ∎

19.1.1 Subgroups

Proposition 19.1.11. *Any subgroup of a topological group is a topological group under the subspace topology.*

PROOF: Since the restriction of continuous functions is continuous. ∎

Proposition 19.1.12. *Let G be a topological group and H a subgroup of G. Then $\overline{H}$ is a topological group under the subspace topology.*

PROOF:
⟨1⟩1. LET: $x, y \in \overline{H}$

PROVE: $xy^{-1} \in \overline{H}$
⟨1⟩2. LET: U be a neighbourhood of xy^{-1}.
PROVE: U intersects H.
⟨1⟩3. LET: $f : G^2 \to G$ be the function that maps (x, y) to xy^{-1}.
⟨1⟩4. $f^{-1}(U)$ is a neighbourhood of (x, y)
⟨1⟩5. PICK neighbourhoods V of x and W of y such that $V \times W \subseteq f^{-1}(U)$.
⟨1⟩6. PICK elements $x' \in V \cap H$ and $y' \in W \cap H$
⟨1⟩7. $x'y'^{-1} \in U \cap H$
□

Proposition 19.1.13. *Let G be a topological group. The component of G that contains e is a normal subgroup of G.*

PROOF:
⟨1⟩1. LET: C be the component that contains e.
⟨1⟩2. For all $x \in G$, we have Cx is the component of G that contains x.
PROOF: Since right multiplication by x is a homeomorphism between G and itself.
⟨1⟩3. C is a subgroup of G.
⟨2⟩1. LET: $g, h \in C$
⟨2⟩2. $C = Ch$
PROOF: ⟨1⟩2
⟨2⟩3. PICK $x \in C$ such that $xh = g$
⟨2⟩4. $x = gh^{-1}$
⟨2⟩5. $gh^{-1} \in C$
⟨1⟩4. C is a normal subgroup of G.
⟨2⟩1. LET: $g \in G$ and $h \in C$.
PROVE: $ghg^{-1} \in C$
⟨2⟩2. $C = Ch^{-1}$
⟨2⟩3. $Cg = Ch^{-1}g$
⟨2⟩4. $g \in Ch^{-1}g$
⟨2⟩5. PICK $x \in C$ such that $g = xh^{-1}g$
⟨2⟩6. $x = ghg^{-1}$
⟨2⟩7. $ghg^{-1} \in C$
□

19.1.2 Left Cosets

Proposition 19.1.14. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Let $\alpha \in G$. Define $f_\alpha : G/H \to G/H$ by*

$$f_\alpha(xH) = \alpha x H \quad .$$

Then f_α is a homeomorphism.

PROOF:
⟨1⟩1. For all $\alpha \in G$ we have f_α is well defined.

⟨2⟩1. LET: $x, y \in G$
⟨2⟩2. ASSUME: $xH = yH$
PROVE: $\alpha xH = \alpha yH$
⟨2⟩3. $x^{-1}y \in H$
⟨2⟩4. $x^{-1}\alpha^{-1}\alpha y \in H$
⟨2⟩5. $\alpha xH = \alpha yH$
⟨1⟩2. For all $\alpha \in G$ we have f_α is injective.
⟨2⟩1. LET: $x, y \in G$
⟨2⟩2. ASSUME: $\alpha xH = \alpha yH$
PROVE: $xH = yH$
⟨2⟩3. $\alpha x^{-1}\alpha y \in H$
⟨2⟩4. $x^{-1}y \in H$
⟨2⟩5. $xH = yH$
⟨1⟩3. For all $\alpha \in G$ we have f_α is surjective.
PROOF: For all $x \in G$ we have $xH = f_\alpha(\alpha^{-1}xH)$.
⟨1⟩4. For all $\alpha \in G$ we have f_α is continuous.
⟨2⟩1. LET: V be open in G/H
⟨2⟩2. $\pi^{-1}({f_\alpha}^{-1}(V))$ is open in G.
PROOF: It is ${g_\alpha}^{-1}(\pi^{-1}(V))$ where $g_\alpha : V \to V$ is the homeomorphism $g_\alpha(x) = \alpha x$.
⟨2⟩3. ${f_\alpha}^{-1}(V)$ is open in G/H.
⟨1⟩5. For all $\alpha \in G$ we have ${f_\alpha}^{-1}$ is continuous.
PROOF: It is $f_{\alpha^{-1}}$.
□

Corollary 19.1.14.1. *Let G be a topological group and H a subgroup of G. Then G/H is a homogeneous space.*

Proposition 19.1.15. *Let G be a T_1 topological group and H a closed subgroup of G. Then G/H is T_1.*

PROOF:
⟨1⟩1. LET: $x \in G$
PROVE: xH is closed.
⟨1⟩2. $\pi^{-1}(xH)$ is closed in G.
PROOF: It is $f_x(H)$ and f_x is a homeomorphism.
⟨1⟩3. xH is closed in G/H.
□

Proposition 19.1.16. *Let G be a topological group and H a subgroup of G. Then the canonical map $\pi : G \twoheadrightarrow G/H$ is an open map.*

PROOF:
⟨1⟩1. LET: U be open in G.
⟨1⟩2. $\forall h \in H.Uh$ is open in G.
PROOF: Since the function that maps g to gh is an automorphism of G.
⟨1⟩3. UH is open in G
PROOF: It is $\bigcup_{h \in H} Uh$.

⟨1⟩4. $UH = \pi^{-1}(\pi(U))$
PROOF:
$$\begin{aligned}\pi^{-1}(\pi(U)) &= \{x \in G : \exists y \in U.xH = yH\} \\ &= \{x \in G : \exists y \in U.x^{-1}y \in H\} \\ &= \{x \in G : \exists y \in U.\exists h \in H.y^{-1}x = h\} \\ &= \{x \in G : \exists y \in U.\exists h \in H.x = yh\} \\ &= UH\end{aligned}$$
⟨1⟩5. $\pi^{-1}(\pi(U))$ is open in G.
⟨1⟩6. $\pi(U)$ is open in G/H.
□

Proposition 19.1.17. *Let G be a topological group. Let H be a normal subgroup of G. Then G/H is a topological group.*

PROOF:
⟨1⟩1. LET: $f : G^2 \to G$ be the map $f(x, y) = xy^{-1}$
⟨1⟩2. LET: $g : (G/H)^2 \to G/H$ be the map $g(xH, yH) = xy^{-1}H$
⟨1⟩3. $g \circ (\pi \times \pi) = \pi \circ f : G^2 \to G/H$
⟨1⟩4. $g \circ (\pi \times \pi)$ is continuous.
PROOF: Since π and f are continuous.
⟨1⟩5. π is an open quotient map.
PROOF: Proposition 19.1.16.
⟨1⟩6. $\pi \times \pi$ is an open quotient map.
PROOF: Corollary 15.10.7.1.
⟨1⟩7. g is continuous.
PROOF: Theorem 15.10.3.
□

19.1.3 Homogeneous Spaces

Definition 19.1.18 (Homogeneous Space)**.** A *homogeneous space* is a topological space of the form G/H, where G is a topological group and H is a normal subgroup of G, under the quotient topology.

Proposition 19.1.19. *Let G be a topological group and H a normal subgroup of G. Then G/H is Hausdorff if and only if H is closed.*

PROOF: See Bourbaki, N., General Topology. III.12 □

19.2 Symmetric Neighbourhoods

Definition 19.2.1 (Symmetric Neighbourhood)**.** Let G be a topological group. Let V be a neighbourhood of e. Then V is *symmetric* iff $V = V^{-1}$.

Proposition 19.2.2. *Let G be a topological group. Let U be a neighbourhood of e. Then there exists a symmetric neighbourhood V of e such that $VV \subseteq U$.*

PROOF:
⟨1⟩1. PICK a neighbourhood V' of e such that $V'V' \subseteq U$.
 ⟨2⟩1. LET: $m : G^2 \to G$ be the function $m(x, y) = xy$
 ⟨2⟩2. $m^{-1}(U)$ is open in G^2
 ⟨2⟩3. $(e, e) \in m^{-1}(U)$
 ⟨2⟩4. PICK neighbourhoods V_1, V_2 of e such that $V_1 \times V_2 \subseteq m^{-1}(U)$
 ⟨2⟩5. LET: $V' = V_1 \cap V_2$
⟨1⟩2. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
 ⟨2⟩1. LET: $f : G^2 \to G$ be the function $m(x, y) = xy^{-1}$
 ⟨2⟩2. $f^{-1}(V')$ is open in G^2
 ⟨2⟩3. $(e, e) \in m^{-1}(V')$
 ⟨2⟩4. PICK neighbourhoods W_1, W_2 of e such that $W_1 \times W_2 \subseteq f^{-1}(V')$
 ⟨2⟩5. LET: $W = W_1 \cap W_2$
⟨1⟩3. LET: $V = WW^{-1}$
⟨1⟩4. V is a neighbourhood of e.
⟨1⟩5. V is symmetric.
⟨1⟩6. $VV \subseteq U$
□

Proposition 19.2.3. *Every T_1 topological group is regular.*

PROOF:
⟨1⟩1. LET: G be a T_1 topological group.
⟨1⟩2. LET: A be a closed set in G and $x \in G - A$.
⟨1⟩3. $G - Ax^{-1}$ is a neighbourhood of e.
⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq G - Ax^{-1}$.
⟨1⟩5. LET: $U = VA$ and $U' = Vx$
⟨1⟩6. U and U' are disjoint open sets with $A \subseteq U$ and $x \in U'$.
□

Proposition 19.2.4. *Let G be a T_1 topological group. Let H be a closed subgroup of G. Then G/H is regular.*

PROOF:
⟨1⟩1. LET: A be a closed set in G/H and $xH \in G/H - A$.
⟨1⟩2. $G - \pi^{-1}(A)x^{-1}$ is a neighbourhood of e.
⟨1⟩3. PICK a symmetric neighbourhood V of e such that $VV \subseteq G - \pi^{-1}(A)x^{-1}$.
⟨1⟩4. LET: $U = \pi(V)A$ and $U' = \pi(V)(xH)$.
⟨1⟩5. U and U' are disjoint open sets with $A \subseteq U$ and $xH \in U'$
 ⟨2⟩1. ASSUME: for a contradiction $U \cap U' \neq \emptyset$.
 ⟨2⟩2. PICK $v_1, v_2 \in V$ and $a \in G$ such that $aH \in A$ and $v_1 aH = v_2 xH$.
 ⟨2⟩3. $a^{-1}{v_1}^{-1}v2x \in H$
 ⟨2⟩4. ${v_1}^{-1}v_2 \in \pi^{-1}(A)x^{-1}$
 ⟨2⟩5. Q.E.D.
  PROOF: This contradicts ⟨1⟩3.
□

Proposition 19.2.5. *Let G be a topological group. Let A and B be subspaces of G. If A is closed and B is compact, then AB is closed.*

PROOF:
⟨1⟩1. For all $c \in G - AB$, there exists a neighbourhood W of c such that $WB^{-1} \cap A = \varnothing$.
⟨2⟩1. LET: $c \in G - AB$
⟨2⟩2. LET: $\phi : G^2 \to G$ be the function $\phi(x, y) = xy^{-1}$
⟨2⟩3. $\{c\} \times B \subseteq \phi^{-1}(G - A)$
⟨2⟩4. PICK a neighbourhood W of c such that $W \times B \subseteq \phi^{-1}(G - A)$
PROOF: Tube Lemma.
⟨2⟩5. $WB^{-1} \cap A = \varnothing$
⟨1⟩2. For all $c \in G - AB$, there exists a neighbourhood W of c such that $W \subseteq G - AB$.
□

Corollary 19.2.5.1. *Let G be a topological group. Let H be a compact subgroup of G. Let $p : G \twoheadrightarrow G/H$ be the quotient map. Then p is a perfect map.*

PROOF: The only thing remaining to prove is that, for all $gH \in G/H$, we have $p^{-1}(gH)$ is compact. This holds because $p^{-1}(gH) = gH$ is homeomorphic to H. □

Corollary 19.2.5.2. *Let G be a topological group. Let H be a compact subgroup of G. If G/H is compact then G is compact.*

19.3 Continuous Actions

Definition 19.3.1 (Continuous Action)**.** Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \to X$ such that:

- $\forall x \in X. ex = x$
- $\forall g, h \in G. \forall x \in X. g(hx) = (gh)x$

A *G-space* consists of a topological space X and a continuous action of G on X.

Definition 19.3.2 (Orbit)**.** Let X be a G-space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 19.3.3. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:
⟨1⟩1. LET: $f_3 : S^2/SO(2) \to [-1, 1]$ be the function induced by $\pi_3 : S^2 \to [-1, 1]$

$\langle 1 \rangle 2$. f_3 is bijective.
$\langle 1 \rangle 3$. $S^2/SO(2)$ is compact.
Proof: It is the continuous image of S^2 which is compact.
$\langle 1 \rangle 4$. $[-1, 1]$ is Hausdorff.
$\langle 1 \rangle 5$. f_3 is a homeomorphism.
□

Definition 19.3.4 (Stabilizer)**.** Let X be a G-space and $x \in X$. The *stabilizer* of x is $G_x := \{g \in G : gx = x\}$.

Proposition 19.3.5. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx.*

Proof:
$\langle 1 \rangle 1$. If $gG_x = hG_x$ then $gx = hx$.
$\langle 2 \rangle 1$. Assume: $gG_x = hG_x$
$\langle 2 \rangle 2$. $g^{-1}h \in G_x$
$\langle 2 \rangle 3$. $g^{-1}hx = x$
$\langle 2 \rangle 4$. $gx = hx$
$\langle 1 \rangle 2$. If $gx = hx$ then $gG_x = hG_x$.
Proof: Similar.
$\langle 1 \rangle 3$. The function is continuous.
Proof: Theorem 15.10.3.
□

Chapter 20

Topological Vector Spaces

Definition 20.0.1 (Topological Vector Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Substraction is a continuous function $E^2 \to E$
- Multiplication is a continuous function $K \times E \to E$

Proposition 20.0.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. □

Theorem 20.0.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 □

Proposition 20.0.4. *Let E be a topological vector space and E_0 a subspace of E. Then $\overline{E_0}$ is a subspace of E.*

Definition 20.0.5. Let E be a topological vector space. The topological space *associated* with E is $E/\overline{\{0\}}$.

20.1 Cauchy Sequences

Definition 20.1.1 (Cauchy Sequence)**.** Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geqslant n_0 . x_n - x_m \in U$.

Definition 20.1.2 (Complete Topological Vector Space)**.** A topological vector space is *complete* iff every Cauchy sequence converges.

20.2 Seminorms

Definition 20.2.1 (Seminorm)**.** Let E be a vector space over K. A *seminorm* on E is a function $\| \, \| : E \to \mathbb{R}$ such that:

1. $\forall x \in E . \|x\| \geqslant 0$
2. $\forall \alpha \in K . \forall x \in E . \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y :\in E . \|x + y\| \leqslant \|x\| + \|y\|$

Example 20.2.2. The function that maps $(x_1, \ldots, x_n)$ to $|x_i|$ is a seminorm on $\mathbb{R}^n$.

Definition 20.2.3. Let E be a vector space over K. Let Λ be a set of seminorms on E. The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x \in E$.

Proposition 20.2.4. *E is a topological vector space under this topology. It is Hausdorff iff, for all $x \in E$, if $\forall \lambda \in \Lambda . \lambda(x) = 0$ then $x = 0$.*

20.3 Fréchet Spaces

Definition 20.3.1 (Pre-Fréchet Space)**.** A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 20.3.2. *Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \, \|_n : n \in \mathbb{Z}^+\}$. Then*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 20.3.3 (Fréchet Space)**.** A *Fréchet space* is a complete pre-Fréchet space.

20.4 Normed Spaces

Definition 20.4.1 (Normed Space)**.** Let E be a vector space over K. A *norm* on E is a function $\| \, \| : E \to \mathbb{R}$ is a seminorm such that, $\forall x \in E . \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 20.4.2. *If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E.*

Definition 20.4.3 (p-norm)**.** For any $p \geqslant 1$, the *p-norm* on $\mathbb{R}^n$ is defined by

$$\|\vec{x}\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} .$$

We prove this is a norm.

PROOF:
⟨1⟩1. For all $\vec{x} \in \mathbb{R}^n$ we have $\|\vec{x}\|_p \geqslant 0$
PROOF: Immediate from definition.
⟨1⟩2. For all $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ we have $\|\alpha\vec{x}\|_p = |\alpha|\|\vec{x}\|_p$
PROOF:

$$\begin{aligned}\|\alpha(x_1, \ldots, x_n)\| &= \|(\alpha x_1, \ldots, \alpha x_n)\| \\ &= \left(\sum_{i=1}^n (\alpha x_i)^p\right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \\ &= |\alpha|\|\vec{x}\|_p\end{aligned}$$

⟨1⟩3. The triangle inequality holds.
PROOF:

$$\begin{aligned}\|\vec{x}+\vec{y}\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\ &= \sum_{i=1}^n |x_i + y_i||x_i + y_i|^{p-1} \\ &\leqslant \sum_{i=1}^n (|x_i| + |y_i|)|x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i||x_i + y_i|^{p-1} \\ &\leqslant \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{p-1}{p}} \quad \text{(Hölder's Inequality)} \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p)\|\vec{x}+\vec{y}\|^{p-1}\end{aligned}$$

Assuming w.l.o.g. $\|\vec{x}+\vec{y}\|^{p-1} \neq 0$ (using **??**) we have $\|\vec{x}+\vec{y}\|_p \leqslant \|\vec{x}\|_p + \|\vec{y}\|_p$.
⟨1⟩4. For any $\vec{x} \in \mathbb{R}^n$, we have $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.
PROOF:$\sum_{i=1}^n x_i^p = 0$ iff $x_1 = \cdots = x_n = 0$.
□

Proposition 20.4.4. *The p-norm on $\mathbb{R}^n$ induces the product topology.*

PROOF:
⟨1⟩1. LET: d be the metric induced by the p-norm and ρ the square metric on $\mathbb{R}^n$.
⟨1⟩2. The metric topology is finer than the product topology.
⟨2⟩1. LET: $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$
⟨2⟩2. LET: $\delta = \epsilon / n^{\frac{1}{p}}$
PROVE: $B_\rho(\vec{x}, \delta) \subseteq B_d(\vec{x}, \epsilon)$
⟨2⟩3. LET: $\vec{y} \in B_\rho(\vec{x}, \delta)$
⟨2⟩4. $\forall i . |x_i - y_i| < \delta$
⟨2⟩5. $d(\vec{x}, \vec{y}) < \epsilon$
PROOF:

$$\begin{aligned} d(\vec{x}, \vec{y}) &= \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}} \\ &< \left(\sum_{i=1}^n \delta^p \right)^{\frac{1}{p}} && (\langle 2 \rangle 4) \\ &= n^{\frac{1}{p}} \delta \\ &= \epsilon && (\langle 2 \rangle 2) \end{aligned}$$

⟨1⟩3. The product topology is finer than the metric topology.
⟨2⟩1. LET: $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$
⟨2⟩2. LET: $\vec{y} \in B_d(\vec{x}, \epsilon)$
⟨2⟩3. $d(\vec{x}, \vec{y}) < \epsilon$
⟨2⟩4. $\sum_{i=1}^n |x_i - y_i|^p < \epsilon^p$
⟨2⟩5. $\forall i . |x_i - y_i|^p < \epsilon^p$
⟨2⟩6. $\forall i . |x_i - y_i| < \epsilon$
⟨2⟩7. $\rho(\vec{x}, \vec{y}) < \epsilon$
□

Definition 20.4.5 (Sup-norm)**.** The *sup-norm* on $\mathbb{R}^n$ is defined by

$$\|(x_1, \ldots, x_n)\|_\infty := \max(|x_1|, \ldots, |x_n|) \ .$$

Proposition 20.4.6. *The 2-norm on $\mathbb{R}^n$ induces the standard metric.*

PROOF: Immediate from definitions. □

Definition 20.4.7. For $p \geqslant 1$, the normed space l_p is the set of all sequences (x_n) in $\mathbb{R}$ such that $\sum_{n=1}^\infty x_n^p$ converges, under

$$\|(x_n)\|_p := \left(\sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} \ .$$

Proposition 20.4.8. *The spaces l_p for $p \geqslant 1$ are all homeomorphic.*

PROOF: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. http://dx.doi.org/10.1007/BF01075865.

Proposition 20.4.9. *The metric topology on l_2 is strictly finer than the uniform topology.*

PROOF:
⟨1⟩1. LET: d be the metric induced by the l^2-norm and $\overline{\rho}$ the uniform topology.
⟨1⟩2. The metric topology is finer than the uniform topology.
⟨2⟩1. LET: $x \in l_2$
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. LET: $\delta = \epsilon/2$
⟨2⟩4. LET: $y \in B_d(x, \delta)$
⟨2⟩5. $\sum_{n=0}^{\infty}(x_n - y_n)^2 < \delta^2$
⟨2⟩6. $\forall n.(x_n - y_n)^2 < \delta^2$
⟨2⟩7. $\forall n.|x_n - y_n| < \delta$
⟨2⟩8. $\forall n.\overline{d}(x_n, y_n) < \delta$
⟨2⟩9. $\overline{\rho}(x, y) \leqslant \delta$
⟨2⟩10. $\overline{\rho}(x, y) < \epsilon$
⟨2⟩11. $y \in B_{\overline{\rho}}(x, \epsilon)$
⟨1⟩3. The metric topology is not the same as the uniform topology.
⟨2⟩1. ASSUME: for a contradiction $B_d(0, 1)$ is open in the uniform topology.
⟨2⟩2. PICK $\epsilon > 0$ such that $B_{\overline{\rho}}(0, \epsilon) \subseteq B_d(0, 1)$
⟨2⟩3. PICK an integer N such that $1/N < \epsilon^2/4$
⟨2⟩4. LET: (x_n) be the sequence with $x_n = \epsilon/2$ for $n < N$ and $x_n = 0$ for $n \geqslant N$
⟨2⟩5. $(x_n) \in l_2$
⟨2⟩6. $(x_n) \in B_{\overline{\rho}}(0, \epsilon)$
PROOF: Since $\overline{\rho}((x_n), 0) = \epsilon/2$.
⟨2⟩7. $d((x_n), 0) > 1$
PROOF:

$$\begin{aligned} d((x_n), 0)^2 &= \sum_{n=0}^{\infty} x_n^2 \\ &= N\epsilon^2/4 \\ &> 1 \end{aligned}$$

□

Proposition 20.4.10. *The metric topology on l_2 is strictly coarser than the box topology.*

PROOF:
⟨1⟩1. The box topology is finer than the metric topology.
⟨2⟩1. LET: $(x_n) \in l_2$ and $\epsilon > 0$.
⟨2⟩2. LET: $(y_n) \in B((x_n), \epsilon)$
⟨2⟩3. PICK a sequence of real numbers (δ_n) such that $\sum_{n=0}^{\infty} \delta_n^2 < (\epsilon - d((x_n), (y_n)))^2$
⟨2⟩4. LET: $U = \prod_n (y_n - \delta_n, y_n + \delta_n)$
PROVE: $U \subseteq B((x_n), \epsilon)$
⟨2⟩5. LET: $(z_n) \in U$
⟨2⟩6. $d((z_n), (y_n)) < \epsilon - d((x_n), (y_n))$

PROOF:

$$\begin{aligned} d((z_n),(y_n))^2 &= \sum_{n=0}^{\infty}(z_n - y_n)^2 \\ &< \sum_{n=0}^{\infty}\delta_n^2 \\ &< (\epsilon - d((x_n),(y_n)))^2 \end{aligned}$$

⟨2⟩7. $d((z_n),(x_n)) < \epsilon$

⟨1⟩2. The box topology is not equal to the metric topology.

⟨2⟩1. LET: $U = \prod_n(-1/n, 1/n)$

⟨2⟩2. ASSUME: for a contradiction U is open in the metric topology.

⟨2⟩3. PICK $\epsilon > 0$ such that $B(0,\epsilon) \subseteq U$

⟨2⟩4. PICK N such that $1/N < \epsilon/2$.

⟨2⟩5. LET: (x_n) be the sequence with $x_N = \epsilon/2$ and $x_n = 0$ for all other n.

⟨2⟩6. $d((x_n), 0) = \epsilon/2$

⟨2⟩7. $(x_n) \notin U$

□

Proposition 20.4.11. *The l^2-topology on $\mathbb{R}^\infty$ is strictly finer than the uniform topology.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction $B_d(0,1) \cap \mathbb{R}^\infty$ is open in the uniform topology.

⟨1⟩2. PICK $\epsilon > 0$ such that $B_{\overline{\rho}}(0,\epsilon) \cap \mathbb{R}^\infty \subseteq B_d(0,1) \cap \mathbb{R}^\infty$

⟨1⟩3. PICK an integer N such that $1/N < \epsilon^2/4$

⟨1⟩4. LET: (x_n) be the sequence with $x_n = \epsilon/2$ for $n < N$ and $x_n = 0$ for $n \geqslant N$

⟨1⟩5. $(x_n) \in \mathbb{R}^\infty$

⟨1⟩6. $(x_n) \in B_{\overline{\rho}}(0,\epsilon)$

PROOF: Since $\overline{\rho}((x_n), 0) = \epsilon/2$.

⟨1⟩7. $d((x_n), 0) > 1$

PROOF:

$$\begin{aligned} d((x_n),0)^2 &= \sum_{n=0}^{\infty} x_n^2 \\ &= N\epsilon^2/4 \\ &> 1 \end{aligned}$$

□

Proposition 20.4.12. *The l^2-topology on $\mathbb{R}^\infty$ is strictly coarser than the box topology.*

⟨1⟩1. LET: $U = \prod_n(-1/n, 1/n) \cap \mathbb{R}^\infty$

⟨1⟩2. ASSUME: for a contradiction U is open in the metric topology.

⟨1⟩3. PICK $\epsilon > 0$ such that $B(0,\epsilon) \cap \mathbb{R}^\infty \subseteq U \cap \mathbb{R}^\infty$

⟨1⟩4. PICK N such that $1/N < \epsilon/2$.

⟨1⟩5. LET: (x_n) be the sequence with $x_N = \epsilon/2$ and $x_n = 0$ for all other n.
⟨1⟩6. $d((x_n), 0) = \epsilon/2$
⟨1⟩7. $(x_n) \notin U$
□

Proposition 20.4.13. *The l^2-topology on the Hilbert cube the same as the product topology.*

PROOF:
⟨1⟩1. For every $(x_n) \in H$ and $\epsilon > 0$, there exists a neighbourhood U of (x_n) in the product topology such that $U \subseteq B((x_n), \epsilon)$.
⟨2⟩1. LET: $(x_n) \in H$
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK N such that $\sum_{i=N+1}^{\infty} 1/i^2 < \epsilon^2/2$
⟨2⟩4. LET: $B' = (\prod_{i=0}^{N}(x_i - \epsilon/\sqrt{2N}, x_i + \epsilon/\sqrt{2N}) \times \prod_{i=N+1}^{\infty}[0, 1/(i+1)]) \cap H$
PROVE: $B' \subseteq B((x_n), \epsilon)$
⟨2⟩5. LET: $(y_n) \in B'$
⟨2⟩6. $d((x_n), (y_n)) < \epsilon$
PROOF:

$$\begin{aligned} d((x_n), (y_n))^2 &= \sum_{i=0}^{\infty} |x_n - y_n|^2 \\ &< \sum_{i=0}^{N} \epsilon^2/2N + \sum_{i=N+1}^{\infty} 1/(i+1)1/(i+1)^2 \\ &< \epsilon^2/2 + \epsilon^2/2 \\ &= \epsilon^2 \end{aligned}$$

⟨1⟩2. The product topology is finer than the l^2-topology.
⟨2⟩1. LET: $(x_n) \in H$ and $\epsilon > 0$
PROVE: $B((x_n), \epsilon) \cap H$ is open in the product topology.
⟨2⟩2. LET: $(y_n) \in B((x_n), \epsilon)$
⟨2⟩3. PICK a neighbourhood U of (y_n) in the product topology such that $U \subseteq B((y_n), \epsilon - d((x_n), (y_n)))$
⟨2⟩4. $U \subseteq B((x_n), \epsilon)$
□

Definition 20.4.14. Let l_∞ be the set of all bounded sequences in $\mathbb{R}$ under

$$\|(x_n)\| := \sup_n |x_n|$$

Proposition 20.4.15. *For all $p \geqslant 1$ we have l_p is not homeomorphic to l_∞.*

Proposition 20.4.16. *Let $\| \ \|$ be a seminorm on the vector space E. Then $\| \ \|$ defines a norm on $E/\overline{\{0\}}$.*

Proposition 20.4.17. *Let E and F be normed spaces. Any continuous linear map $E \to F$ is uniformly continuous.*

Definition 20.4.18. For $p \geqslant 1$. let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt{p}\int_{\mathbb{R}^n} |f(x)|^p dx$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\overline{\{0\}}\ .$$

20.5 Unit Ball

Proposition 20.5.1. *Let n be a positive integer. Every open ball $B(\vec{x}, \epsilon)$ in $\mathbb{R}^n$ is path connected.*

PROOF:
⟨1⟩1. LET: $\vec{y}, \vec{z} \in B(\vec{x}, \epsilon)$
⟨1⟩2. LET: $\vec{p} : [0,1] \to B(\vec{x}, \epsilon)$ be the path $\vec{p}(t) = (1-t)\vec{y} + t\vec{z}$.
  ⟨2⟩1. LET: $t \in [0,1]$
    PROVE: $\vec{p}(t) \in B(\vec{x}, \epsilon)$
  ⟨2⟩2. $d(\vec{p}(t), \vec{x}) < \epsilon$
   PROOF:

$$\begin{aligned} d(\vec{p}(t), \vec{x}) &= \|(1-t)\vec{y} + t\vec{z} - \vec{x}\| \\ &= \|(1-t)(\vec{y} - \vec{x}) + t(\vec{z} - \vec{x})\| \\ &\leqslant (1-t)\|\vec{y} - \vec{x}\| + t\|\vec{z} - \vec{x}\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3. $\vec{p}$ is a path from $\vec{x}$ to $\vec{y}$.
□

Proposition 20.5.2. *Let n be a positive integer. Every closed ball $B(\vec{x}, \epsilon)$ in $\mathbb{R}^n$ is path connected.*

PROOF:
⟨1⟩1. LET: $\vec{y}, \vec{z} \in \overline{B(\vec{x}, \epsilon)}$
⟨1⟩2. LET: $\vec{p} : [0,1] \to \overline{B(\vec{x}, \epsilon)}$ be the path $\vec{p}(t) = (1-t)\vec{y} + t\vec{z}$.
  ⟨2⟩1. LET: $t \in [0,1]$
    PROVE: $\vec{p}(t) \in \overline{B(\vec{x}, \epsilon)}$
  ⟨2⟩2. $d(\vec{p}(t), \vec{x}) \leqslant \epsilon$
   PROOF:

$$\begin{aligned} d(\vec{p}(t), \vec{x}) &= \|(1-t)\vec{y} + t\vec{z} - \vec{x}\| \\ &= \|(1-t)(\vec{y} - \vec{x}) + t(\vec{z} - \vec{x})\| \\ &\leqslant (1-t)\|\vec{y} - \vec{x}\| + t\|\vec{z} - \vec{x}\| \\ &\leqslant (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3. $\vec{p}$ is a path from $\vec{x}$ to $\vec{y}$.
□

20.6 Unit Sphere

Definition 20.6.1 (Unit Sphere)**.** Let n be a positive integer. The *unit sphere* S^{n-1} is

$$S^{n-1} := \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = 1\} \ .$$

Proposition 20.6.2. *For $n > 1$. the unit sphere S^{n-1} is path connected.*

PROOF: The map $g : \mathbb{R}^n - \{\vec{0}\} \twoheadrightarrow S^{n-1}$ defined by $g(\vec{x}) = \vec{x}/\|\vec{x}\|$ is continuous and surjective. Hence S^{n-1} is the continuous image of a path connected space.
□

20.7 Inner Product Spaces

Definition 20.7.1 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n \ .$$

Proposition 20.7.2.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF:

$$\begin{aligned} \vec{x} \cdot (\vec{y} + \vec{z}) &= x_1(y_1 + z_1) + \cdots + x_n(y_n + z_n) \\ &= x_1 y_1 + x_1 z_1 + \cdots + x_n y_n + x_n z_n \\ &= \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z} \end{aligned}$$ □

Proposition 20.7.3. *For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have*

$$|\vec{x} \cdot \vec{y}| \leqslant \|\vec{x}\| \|\vec{y}\| \ .$$

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. $\vec{x} \neq \vec{0} \neq \vec{y}$
⟨1⟩2. LET: $a = 1/\|x\|$
⟨1⟩3. LET: $b = 1/\|y\|$
⟨1⟩4. $\|a\vec{x} + b\vec{y}\| \geqslant 0$
⟨1⟩5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geqslant 0$
⟨1⟩6. $ab\vec{x} \cdot \vec{y} \geqslant -1$
⟨1⟩7. $\|a\vec{x} - b\vec{y}\| \geqslant 0$
⟨1⟩8. $ab\vec{x} \cdot \vec{y} \leqslant 1$
⟨1⟩9. $|\vec{x} \cdot \vec{y}| \leqslant 1/ab$
□

Proposition 20.7.4. *Let (x_n), (y_n) be sequences of real numbers. If $\sum_{n=0}^{\infty} x_n^2$ and $\sum_{n=0}^{\infty} y_n^2$ converge then $\sum_{n=0}^{\infty} |x_n y_n|$ converges.*

PROOF:

$$\begin{aligned} \sum_{n=0}^{N} |x_n y_n| &\leqslant \sqrt{\sum_{n=0}^{N} x_n^2 \sum_{n=0}^{N} y_n^2} && \text{(Proposition 20.7.3)} \\ &\leqslant \sqrt{\sum_{n=0}^{\infty} x_n^2 \sum_{n=0}^{\infty} y_n^2} && \square \end{aligned}$$

Proposition 20.7.5. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on E.*

20.8 Banach Spaces

Definition 20.8.1 (Banach Space)**.** A *Banach space* is a complete normed space.

Example 20.8.2. For any topological space X, the set $C(X)$ of bounded continuous functions $X \to \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 20.8.3. *The completion of a normed space is a Banach space.*

Proposition 20.8.4. *Let E and F be normed spaces. Let $f : E \to F$ be a continuous linear map. Then the extension to the completions $\hat{E} \to \hat{F}$ is linear.*

Proposition 20.8.5. *$L^p(\mathbb{R}^n)$ is a Banach space.*

Proposition 20.8.6. *$C(\mathbb{R})$ is first countable but not second countable.*

PROOF: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n. Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.
It is first countable because it is metrizable. $\square$

20.9 Hilbert Spaces

Definition 20.9.1 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Example 20.9.2. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \to \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g\rangle = \frac{1}{\pi} \int_{-\pi} \pi f(x) g(x) dx \quad .$$

Proposition 20.9.3. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

20.10 Locally Convex Spaces

Definition 20.10.1 (Locally Convex Space)**.** A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 20.10.2. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Proposition 20.10.3. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Example 20.10.4. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \to \mathbb{R}$ is continuous as a map $E' \to \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 20.10.5 (Thom Space)**.** Let E be a vector bundle with a Riemannian metric, $DE = \{x \in E : \|x\| \leqslant 1\}$ its disc bundle and $SE := \{v \in E : \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE.

Part VII

Probability Theory

Chapter 21

Discrete Random Variables

Definition 21.0.1 (Discrete random variable)**.** Let Ω be a countable set. A *discrete random variable* X that takes values in Ω is a function

$$X : \Omega \to [0, 1]$$

such that

$$\sum_{a \in \Omega} X(a) = 1 \ .$$

We write $P(X = a)$ for $X(a)$, and call this the *probability* that X takes value a.

Definition 21.0.2 (Expected Value)**.** Let X be a discrete random variable that takes values in a real vector space Ω. The *expected value* of X is

$$\langle X \rangle = \sum_{a \in \Omega} P(X = a) a \ .$$

Definition 21.0.3 (Variance)**.** The *variance* of X is

$$\langle (X - \langle X \rangle)^2 \rangle \ .$$

Proposition 21.0.4. *The variance of X is $\langle X^2 \rangle - \langle X \rangle^2$.*

PROOF:

$$\begin{aligned}
\langle (X - \langle X \rangle)^2 \rangle &= \sum_{a \in \Omega} (a - \langle X \rangle)^2 P(X = a) \\
&= \sum_{a \in \Omega} (a^2 - 2a\langle X \rangle + \langle X \rangle^2) P(X = a) \\
&= \sum_{a \in \Omega} a^2 P(X = a) - 2\langle X \rangle \sum_{a \in \Omega} a P(X = a) + \langle X \rangle^2 \sum_{a \in \Omega} P(X = a) \\
&= \langle X^2 \rangle - 2\langle X \rangle^2 + \langle X \rangle^2 \\
&= \langle X^2 \rangle - \langle X \rangle^2
\end{aligned}$$

$\square$

Corollary 21.0.4.1.

$$\langle X^2 \rangle \geqslant \langle X \rangle^2$$

Proof: For all $a \in \Omega$ we have $(a - \langle X \rangle)^2 \geqslant 0$, so the variance of X must be $\geqslant 0$. □

Definition 21.0.5 (Standard Deviation)**.** The *standard deviation* of X, denoted σ_X, is the square root of the variance.

Chapter 22

Continuous Random Variables

Definition 22.0.1 (Continuous random variable)**.** A *continuous random variable* X that takes values in $\mathbb{R}$ is an integrable function

$$\rho : \mathbb{R} \to [0, 1]$$

such that

$$\int \rho = 1 \ .$$

Given a measurable set $S \subseteq \mathbb{R}$, *probability* that X takes a value in S is

$$\int_S \rho \ .$$

Example 22.0.2. A *Gaussian distribution* is

$$\rho(x) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2}$$

for some λ and a.

Definition 22.0.3 (Expected Value)**.** Let X be a discrete random variable that takes values in a real vector space Ω. The *expected value* of X is

$$\langle X \rangle = \int x\rho(x)dx$$

Example 22.0.4. The Gaussian distribution

$$\rho(x) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2}$$

has expected value a.

Definition 22.0.5 (Variance)**.** The *variance* of X is

$$\langle (X - \langle X \rangle)^2 \rangle \ .$$

Proposition 22.0.6. *The variance of X is $\langle X^2 \rangle - \langle X \rangle^2$.*

PROOF:

$$\begin{aligned}
\langle (X - \langle X \rangle)^2 \rangle &= \int (x - \langle X \rangle)^2 \rho(x) dx \\
&= \int (x^2 - 2x\langle X \rangle + \langle X \rangle^2) \rho(x) dx \\
&= \int x^2 \rho(x) dx - 2\langle X \rangle \int x \rho(x) dx + \langle X \rangle^2 \int \rho(x) dx \\
&= \langle X^2 \rangle - 2\langle X \rangle^2 + \langle X \rangle^2 \\
&= \langle X^2 \rangle - \langle X \rangle^2
\end{aligned}$$

□

Corollary 22.0.6.1.

$$\langle X^2 \rangle \geqslant \langle X \rangle^2$$

PROOF: For all $x \in \mathbb{R}$ we have $(x - \langle X \rangle)^2 \geqslant 0$, so the variance of X must be $\geqslant 0$. □

Example 22.0.7. The Gaussian distribution

$$\rho(x) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2}$$

has variance $\frac{1}{2\lambda}$.

Definition 22.0.8 (Standard Deviation)**.** The *standard deviation* of X, denoted σ_X, is the square root of the variance.

Example 22.0.9. The Gaussian distribution

$$\rho(x) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2}$$

has standard deviation $1/\sqrt{2\lambda}$.

Part VIII

Quantum Theory

Chapter 23

The Postulates of Quantum Mechanics

Axiom 23.0.1 (Principle of Superposition)**.** *For any physical system S, there exists a complex inner product space V such that the states of S correspond to the one-dimensional subspaces of V. We call the elements of this subspace the* state vectors *that correspond to the state.*

Definition 23.0.2 (Observable)**.** An *observable* of a physical system is a physical quantity that can be measured by an experiment whose value is a real number.

Definition 23.0.3 (Eigenstate)**.** Let S be a physical system. Let E be an observable of S. Let $\alpha \in \mathbb{R}$. A state s of S is an *eigenstate* of E with *eigenvalue* α iff the probability is 1 that, if we measure E when the system is in state s, then the outcome will be α.

Axiom 23.0.4. *Let E be an observable of a physical system S. Let $\alpha \in \mathbb{R}$. The state vectors corresponding to the eigenstates of E form a subspace of the Hilbert space associated with S, called the* eigenspace *of α.*

Axiom 23.0.5 (Projection Postulate)**.** *Let S be a physical system with associated state space V. Let E be an observable of S. Let V_α be the eigenspace of V with eigenvalue α. Let $P_\alpha : V \to V_\alpha$ be the projection operator. If S is in a state with normalised state vector $|\psi\rangle$, and we measure E on S, then the probability of getting the value α is*

$$p_E(\alpha|\psi) = \frac{\langle\psi|P_\alpha|\psi\rangle}{\langle\psi|\psi\rangle}$$

and after the experiment, the state of the system is $P_\alpha|\psi\rangle$.

Proposition 23.0.6. *Let E be an observable of a physical system S. Then eigenstates of E with different eigenvalues are orthogonal.*

PROOF:

⟨1⟩1. LET: $|\psi\rangle$ a normalised eigenstate with eigenvalue β and $\alpha \neq \beta$.
⟨1⟩2. The probability of measuring α if the system is in state $|\psi\rangle$ is 0.
⟨1⟩3. $\langle\psi|P_\alpha|\psi\rangle = 0$
⟨1⟩4. $P_\alpha|\psi\rangle = 0$
⟨1⟩5. $|\psi\rangle$ is orthogonal to every eigenvector with eigenvalue α.
□

Proposition 23.0.7. *Let E be an observable of a physical system S with state space V. Assume E has countably many outcomes α_1, α_2, … and, for any state vector $|\psi\rangle$, we have*

$$\sum_{n=1}^{\infty} P_{\alpha_n} |\psi\rangle \text{ converges.}$$

Then the eigenvectors of E span V.

PROOF:
⟨1⟩1. LET: $|\psi\rangle \in V$
⟨1⟩2. LET: $|\psi'\rangle = |\psi\rangle - \sum_{n=1}^{\infty} P_{\alpha_n} |\psi\rangle$
⟨1⟩3. ASSUME: for a contradiction $|\psi'\rangle \neq 0$
⟨1⟩4. LET: $|\phi\rangle = |\psi'\rangle / \| |\psi'\rangle \|$
⟨1⟩5. For all n we have $P_{\alpha_n} |\phi\rangle = 0$
PROOF:

$$\begin{aligned} P_{\alpha_n} |\psi'\rangle &= P_{\alpha_n} |\psi\rangle - \sum_{m=1}^{\infty} P_{\alpha_n} P_{\alpha_m} |\psi\rangle \\ &= P_{\alpha_n} |\psi\rangle - P_{\alpha_n} |\psi\rangle && \text{(Proposition 23.0.6)} \\ &= 0 \end{aligned}$$

⟨1⟩6. For all n, the probability of measuring α_n if the system is in state $|\phi\rangle$ is 0.
⟨1⟩7. Q.E.D.
PROOF: This is a contradiction.
□

Definition 23.0.8. For any observable E of a physical system with state space V, pick an orthonormal basis $\mathcal{B}$ of normalised eigenvectors of E. The operator that represents E is $\hat{E} : V \to V$ where, for $|\phi\rangle \in \mathcal{B}$, if α is the eigenvalue of $|\phi\rangle$ then

$$\hat{E} |\phi\rangle = \alpha |\phi\rangle \quad .$$

Proposition 23.0.9. *For any observable E, the operator $\hat{E}$ is Hermitian.*

PROOF:
⟨1⟩1. LET: $\hat{\phi} = \sum_n c_n \hat{\psi}_n$ and $\hat{\psi} = \sum_n d_n \hat{\psi}_n$ be any two state vectors, where the $\hat{\psi}_n$ are the basis vectors with eigenvalues α_n.
⟨1⟩2. $\langle\phi|\hat{E}|\psi\rangle = \overline{\langle\psi|\hat{E}|\phi\rangle}$

PROOF:

$$\begin{aligned}
\langle\phi|\hat{E}|\psi\rangle &= \sum_m \sum_n \overline{c_m} d_n \langle\psi_m|\hat{E}|\psi_n\rangle \\
&= \sum_m \sum_n \overline{c_m} d_n \alpha_n \langle\psi_m|\psi_n\rangle \\
&= \sum_n \overline{c_n} d_n \alpha_n \\
\langle\psi|\hat{E}|\phi\rangle &= \sum_n \overline{d_n} c_n \alpha_n & \text{similarly}
\end{aligned}$$

∎

Definition 23.0.10. Let E be an observable and $f : \mathbb{R} \to \mathbb{R}$. Then $f(E)$ is the observable that consists of performing the experiment to obtain the value of E and then applying f to it.

Proposition 23.0.11. $\widehat{E^n} = \hat{E}^n$

PROOF: Since if $|\phi\rangle$ is an eigenstate with eigenvalue α then $\hat{E}^n |\phi\rangle = \alpha^n |\phi\rangle$. ∎

Corollary 23.0.11.1. *For any real polynomial p we have $\widehat{p(E)} = p(\hat{E})$.*

23.1 Observables as a Random Variable

Definition 23.1.1 (Uncertainty)**.** Let E be an observable. The *uncertainty* of E in a given state, ΔE, is the standard deviation of the value of E when measured.

Proposition 23.1.2. *Let $|\psi\rangle$ be a normalised state. Let E be an observable. Then the expected value of E in state $|\psi\rangle$ is*

$$\langle E\rangle = \langle\psi|\hat{E}|\psi\rangle$$

PROOF:
⟨1⟩1. PICK an orthonormal basis $\mathcal{B}$ of eigenstates for E.
⟨1⟩2. LET: $\psi = \sum_i c_i |\psi_i\rangle$ where each $|\psi_i\rangle \in \mathcal{B}$ has eigenvalue α_i.
⟨1⟩3. $\langle E\rangle = \langle\psi|\hat{E}|\psi\rangle$
PROOF:

$$\begin{aligned}
\langle E\rangle &= \sum_\alpha \alpha\langle\psi|P_\alpha|\psi\rangle \\
&= \sum_\alpha \alpha \sum_i |c_i|^2 \langle\psi_i|P_\alpha|\psi_i\rangle \\
&= \sum_i |c_i|^2 \sum_\alpha \alpha\langle\psi_i|P_\alpha|\psi_i\rangle \\
&= \sum_i |c_i|^2 \alpha_i \\
&= \langle\psi|\hat{E}|\psi\rangle
\end{aligned}$$

□

Corollary 23.1.2.1. *Let $|\psi\rangle$ be a normalised state. Let E be an observable. Then the variance of E in state $|\psi\rangle$ is*

$$(\Delta E)^2 = \langle\psi|\hat{E}^2|\psi\rangle - \langle\psi|\hat{E}|\psi\rangle^2$$

Theorem 23.1.3 (Generalised Uncertainty Principle)**.** *Let A and B be observables. Then in any state,*

$$\Delta A \Delta B \geqslant \frac{1}{2}|\langle i[A,B]\rangle| \enspace .$$

PROOF:
⟨1⟩1. LET: $|\psi\rangle$ be any state.
⟨1⟩2. LET: $A_1 = A - \langle A\rangle$
⟨1⟩3. LET: $B_1 = B - \langle B\rangle$
⟨1⟩4. For $x \in \mathbb{R}$,
LET: $|\phi(x)\rangle = \hat{A}_1 |\psi\rangle + ix\hat{B}_1 |\psi\rangle$
⟨1⟩5. For all $x \in \mathbb{R}$ we have $\langle\phi(x)| = \langle\psi| \hat{A}_1 - ix \langle\psi| \hat{B}_1$
⟨1⟩6. For all $x \in \mathbb{R}$ we have $\langle\phi(x)|\phi(x)\rangle = \langle\psi|\hat{A_1}^2|\psi\rangle - x\langle\psi|i[\hat{A}_1, \hat{B}_1]|\psi\rangle + x^2\langle\psi|\hat{B_1}^2|\psi\rangle$
⟨1⟩7. $[\hat{A}_1, \hat{B}_1] = [\hat{A}, \hat{B}]$
PROOF:

$$\begin{aligned}
[\hat{A}_1, \hat{B}_1] &= (A - \langle A\rangle)(B - \langle B\rangle) - (B - \langle B\rangle)(A - \langle A\rangle) \\
&= AB - \langle B\rangle A - \langle A\rangle B + \langle A\rangle\langle B\rangle \\
&- BA + \langle A\rangle B + \langle B\rangle A - \langle A\rangle\langle B\rangle \\
&= AB - BA \\
&= [A, B]
\end{aligned}$$

⟨1⟩8. For all $x \in \mathbb{R}$ we have $\langle\phi(x)|\phi(x)\rangle = \Delta A^2 - x\langle[A,B]\rangle + x^2\Delta B^2$
⟨1⟩9. For all $x \in \mathbb{R}$ we have $\langle\phi(x)|\phi(x)\rangle \geqslant 0$
⟨1⟩10. For all $x \in \mathbb{R}$ we have $\Delta A^2 - x\langle x[A,B]\rangle + x^2\Delta B^2 \geqslant 0$.
⟨1⟩11. $\langle i[A,B]\rangle^2 \leqslant 4\Delta A^2 \Delta B^2$
⟨1⟩12. $\Delta A \Delta B \geqslant \langle i[A,B]\rangle$
□

23.2 Compatible Observables

Definition 23.2.1 (Compatible)**.** Two observables A and B on a physical system are *compatible* iff, when we measure A then B then A, the second measurement of A always yields the same value as the first.

Proposition 23.2.2. *Let A and B be observables of a physical system S with state space V. Then the following are equivalent.*

1. *A and B are compatible.*

2. $\hat{A}\hat{B} = \hat{B}\hat{A}$

3. *There exists a basis for V whose elements are eigenvectors of both A and B.*

PROOF:
⟨1⟩1. $1 \Rightarrow 3$
⟨2⟩1. ASSUME: A and B are compatible.
⟨2⟩2. PICK orthonormal bases $\mathcal{B}_A$ and $\mathcal{B}_B$ whose elements are eigenvectors of $\hat{A}$ and of $\hat{B}$ respectively.
⟨2⟩3. For $|\psi\rangle \in \mathcal{B}_A$,
LET: $|\psi\rangle = c_{\psi 1} |\phi_{\psi 1}\rangle + \cdots + c_{\psi n_\psi} |\phi_{\psi n_\psi}\rangle$ where each $c_{\psi i}$ is nonzero and each $|\phi_{\psi i}\rangle \in \mathcal{B}_B$
⟨2⟩4. LET: α be the $\hat{A}$-eigenvalue of $|\psi\rangle$
⟨2⟩5. Each $|\phi_{\psi i}\rangle$ is an eigenvalue of $\hat{A}$ with eigenvalue α.
PROOF: There is a nonzero probability that, if we perform A then B, we will obtain value α for A and then be in state $|\phi_{\psi i}\rangle$. If we perform A in this state, the value is certain to be α.
⟨2⟩6. $\{|\phi_{\psi i}\rangle : \psi \in \mathcal{B}_A, 1 \leqslant i \leqslant n_\psi\}$ is a basis consisting of eigenvectors of both $\hat{A}$ and $\hat{B}$.
⟨1⟩2. $3 \Rightarrow 2$
⟨2⟩1. ASSUME: $\mathcal{B}$ is a basis for V whose elements are eigenvectors of both A and B
⟨2⟩2. For all $|\phi\rangle \in \mathcal{B}$ we have $\hat{A}\hat{B}|\phi\rangle = \hat{B}\hat{A}|\phi\rangle$
⟨2⟩3. For all $|\phi\rangle$ we have $\hat{A}\hat{B}|\phi\rangle = \hat{B}\hat{A}|\phi\rangle$
⟨1⟩3. $2 \Rightarrow 1$
⟨2⟩1. ASSUME: $\hat{A}\hat{B} = \hat{B}\hat{A}$
⟨2⟩2. ASSUME: We perform A and obtain the value α, leaving the system in state $|\phi\rangle$.
⟨2⟩3. $|\phi\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue α.
⟨2⟩4. $\hat{A}\hat{B}|\phi\rangle = \alpha$

$hatB\,|\phi\rangle$

PROOF:

$$\begin{aligned} \hat{A}\hat{B}|\phi\rangle &= \hat{B}\hat{A}|\phi\rangle \\ &= \alpha\hat{B}|\phi\rangle \end{aligned}$$

⟨2⟩5. $\hat{B}|\phi\rangle$ is an eigenvector of A with eigenvalue α.
⟨2⟩6. If we perform B then A, we are certain to get the value α.
□

Definition 23.2.3 (Complete)**.** A set $\mathcal{E}$ of compatible observables is *complete* iff, for any states $|\phi\rangle$ and $|\psi\rangle$, if for all $E \in \mathcal{E}$ we have that $|\phi\rangle$ and $|\psi\rangle$ are eigenvectors of E with the same eigenvalue, then $|\phi\rangle = |\psi\rangle$.

Chapter 24

The Wave Function

24.1 The Schrödinger Equation

Definition 24.1.1 (Planck's constant)**.** *Planck's constant* is

$$h = 6.62607015 \times 10^{-3}\text{Js} \ .$$

Definition 24.1.2 (Reduced Planck's constant)**.** The *reduced Planck's constant* is

$$\hbar = h/2\pi \ .$$

Consider a particle of mass m moving in one dimension under a force given by a potential energy function $V(x,t) : \mathbb{R} \times [0,+\infty) \to [0,+\infty]$. Associated with the particle is a *wave function* $\Psi(x,t) : \mathbb{R} \times [0,+\infty) \to \mathbb{C}$ that is differentiable in t, twice differentiable in x, satisfies the *(time-dependent) Schrödinger equation*: for all x and t,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\Psi(x,t)$$

and satisfies

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = 1 \ .$$

Proposition 24.1.3.

$$\frac{\partial}{\partial t}|\Psi(x,t)|^2 = \frac{i\hbar}{2m} \frac{\partial}{\partial x} \left(\Psi^*(x,t) \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi(x,t) \right)$$

PROOF:
$\langle 1 \rangle 1.$

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V(x,t)\Psi(x,t)$$

PROOF: Schrödinger equation.

⟨1⟩2.
$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V(x,t)\Psi(x,t)^*$$
PROOF: Taking complex conjugates in ⟨1⟩1.

⟨1⟩3.
$$\frac{\partial}{\partial t}|\Psi(x,t)|^2 = \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\Psi^*(x,t)\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi(x,t)\right)$$
PROOF:
$$\begin{aligned}\frac{\partial}{\partial t}|\Psi(x,t)|^2 &= \frac{\partial}{\partial t}(\Psi(x,t)^*\Psi(x,t)) \\ &= \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi \\ &= \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right) && (\langle 1\rangle 1, \langle 1\rangle 2) \\ &= \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\Psi^*(x,t)\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi(x,t)\right)\end{aligned}$$
□

Proposition 24.1.4. *For all $t \in [0, +\infty)$ we have*
$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1 \quad .$$

PROOF:
⟨1⟩1.
$$\frac{d}{dt}\int_{-\infty}^{\infty}|\Psi(x,t)|^2 dx = 0$$
PROOF:
$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}|\Psi(x,t)|^2 dx &= \int_{-\infty}^{\infty}\frac{\partial}{\partial t}|\Psi(x,t)|^2 dx \\ &= \frac{i\hbar}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right]_{-\infty}^{\infty} && \text{(Proposition 24.1.3)} \\ &= 0 && (\Psi \to 0 \text{ as } x \to \pm\infty)\end{aligned}$$
⟨2⟩1. $\int_{-\infty}^{\infty}|\Psi(x,t)|^2 dx$ is constant.
□

24.2 Statistical Interpretation

Born's statistical interpretation of the wave function:

The *position* on the particle at time t is a random variable x with probability density function $|\Psi(x,t)|^2$ at time t.

Proposition 24.2.1. *The expected value of position is*
$$\langle x\rangle = \int_{-\infty}^{+\infty} x|\Psi(x,t)|^2 dx = \int_{-\infty}^{+\infty}\Psi(x,t)^* x\Psi(x,t)dx$$

PROOF: Immediate from definitions. □

24.3 Momentum

Associated with any *observable* quantity Q is a linear operator

$$\hat{Q} : \mathcal{C}(\mathbb{R}, \mathbb{C}) \to \mathcal{C}(\mathbb{R}, \mathbb{C}) \ .$$

The *expected value* of Q at time t is then

$$\langle Q \rangle = \int_{-\infty}^{+\infty} \Psi(x,t)^* \hat{Q}(\lambda x.\Psi(x,t))dx \ .$$

Position x is represented by the operator $\hat{x}$, multiplication by x.

Momentum p is represented by the operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$.

Kinetic energy T is represented by

$$\hat{T} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \ .$$

Proposition 24.3.1 (Ehrenfest's Theorem)**.** *1.*

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt}$$

2.

$$\frac{d\langle p \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle$$

PROOF:

$$
\begin{aligned}
m\frac{d\langle x\rangle}{dt} &= m\frac{d}{dt}\int_{-\infty}^{+\infty} x|\Psi^2|dx \\
&= m\int_{-\infty}^{+\infty} x\frac{\partial}{\partial t}|\Psi^2|dx \\
&= \frac{i\hbar}{2}\int_{-\infty}^{+\infty} x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx && \text{(Proposition 24.1.3)} \\
&= -\frac{i\hbar}{2}\int_{-\infty}^{+\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx \\
&+ \left[x\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]_{-\infty}^{+\infty} && \text{(integrating by parts)} \\
&= -\frac{i\hbar}{2}\int_{-\infty}^{+\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx \\
&= -i\hbar\int_{-\infty}^{+\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx && \text{(integrating by parts)} \\
&= \langle p\rangle \\
\frac{d\langle p\rangle}{dt} &= -i\hbar\frac{d}{dt}\int_{-\infty}^{+\infty}\Psi(x,t)^*\frac{\partial\Psi}{\partial x}dx \\
&= -i\hbar\int_{-\infty}^{+\infty}\frac{\partial}{\partial t}\left(\Psi(x,t)^*\frac{\partial\Psi}{\partial x}\right)dx \\
&= -\imath\hbar\int_{-\infty}^{+\infty}\left(\Psi^*\frac{\partial^2\Psi}{\partial t\partial x} + \frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x}\right)dx \\
\frac{\partial\Psi}{\partial t} &= \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi && \text{(Schrödinger equation)} \\
\therefore \frac{\partial^2\Psi}{\partial x\partial t} &= \frac{i\hbar}{2m}\frac{\partial^3\Psi}{\partial x^3} - \frac{i}{\hbar}\frac{\partial V}{\partial x}\Psi - \frac{i}{\hbar}V\frac{\partial\Psi}{\partial x} \\
\frac{\partial\Psi^*}{\partial t} &= -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^* \\
\therefore \Psi^*\frac{\partial^2\Psi}{\partial t\partial x} + \frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x} &= \frac{i\hbar}{2m}\Psi^*\frac{\partial^3\Psi}{\partial x^3} - \frac{i}{\hbar}\frac{\partial V}{\partial x}\Psi^*\Psi - \frac{i}{\hbar}V\Psi^*\frac{\partial\Psi}{\partial x} \\
&- \frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x} + \frac{i}{\hbar}V\Psi^*\frac{\partial\Psi}{\partial x} \\
\therefore \frac{d\langle p\rangle}{dt} &= \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\left(\Psi^*\frac{\partial^3\Psi}{\partial x^3} - \frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\right)dx \\
&+ \int_{-\infty}^{+\infty}\Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi dx \\
&= -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\left(\frac{\partial\Psi^*}{\partial x}\frac{\partial^2\Psi}{\partial x^3} + \frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\right)dx \\
&+ \left[\Psi^*\frac{\partial^2\Psi}{\partial x^2}\right]_{-\infty}^{+\infty} + \left\langle -\frac{\partial V}{\partial x}\right\rangle \\
&= -\frac{\hbar^2}{2m}\left[\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}\right]_{-\infty}^{+\infty} + \left\langle \bullet - \frac{\partial V}{\partial x}\right\rangle \\
&= \left\langle -\frac{\partial V}{\partial x}\right\rangle
\end{aligned}
$$

Proposition 24.3.2 (Canonical Commutation Relation)**.**

$$[\hat{x}, \hat{p}] = i\hbar$$

PROOF:

$$\begin{aligned}
[\hat{x}, \hat{p}]\psi &= -i\hbar x \frac{d\psi}{dx} + i\hbar \frac{d}{dx}(x\psi) \\
&= -i\hbar (x\frac{d\psi}{dx} - x\frac{d\psi}{dx} - \psi) \\
&= i\hbar\psi
\end{aligned}$$

□

24.4 The Time-Independent Schrödinger Equation

Definition 24.4.1 (Hamiltonian)**.** Assume that the potential V does not vary with t. The *Hamiltonian* or *total energy* H is the quantity with operator

$$\begin{aligned}
\hat{H} &= \frac{\hat{p}^2}{2m} + V(t)\hat{I} \\
&= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\hat{I}
\end{aligned}$$

Definition 24.4.2 (Time-independent Schrödinger equation)**.** Assume that the potential V does not vary with t. Let $E \geqslant 0$. The *time-independent Schrödinger equation* with *energy* E is

$$\hat{H}\psi = E\psi$$

i.e.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x) \ .$$

Proposition 24.4.3. *Let* $\psi : \mathbb{R} \to \mathbb{C}$*. Then*

$$\Psi(x,t) = \psi(x)e^{-\frac{iEt}{\hbar}}$$

is a solution to the time-dependent Schrödinger equation iff ψ *is a solution to the time-independent Schrödinger equation.*

PROOF:

$$\begin{aligned}
i\hbar\frac{\partial\Psi}{\partial t} &= i\hbar\psi(-\frac{iE}{\hbar})e^{-\frac{iEt}{\hbar}} \\
&= E\psi e^{-\frac{iEt}{\hbar}} \\
-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi &= e^{-\frac{iEt}{\hbar}}\left(-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi\right)
\end{aligned}$$

and these are equal iff the time-independent equation holds. □

Proposition 24.4.4 (Solutions to the Time-Independent Equation are Stationary States)**.** *Let the wave function of the particle be*

$$\Psi(x,t) = \psi(x)e^{-\frac{iEt}{\hbar}}$$

For any quantity Q, the expectation value $\langle Q \rangle$ is constant in t.

Proof:

$$\begin{aligned}
\langle Q \rangle &= \int_{-\infty}^{+\infty} \Psi^*(x,t)\hat{Q}(\lambda x.\Psi(x,t))(x)dx \\
&= \int_{-\infty}^{+\infty} \psi^*(x)e^{\frac{iEt}{\hbar}}\hat{Q}(\lambda x.\psi(x)e^{-\frac{iEt}{\hbar}})dx \\
&= \int_{-\infty}^{+\infty} \psi^*(x)\hat{Q}(\psi)(x)dx
\end{aligned}$$

since $\hat{Q}$ is linear. □

Corollary 24.4.4.1. *If the wave function is given by $\psi(x)e^{-\frac{iEt}{\hbar}}$ then $\langle p \rangle = 0$.*

Proposition 24.4.5. *If the wave function is given by $\psi(x)e^{-\frac{iEt}{\hbar}}$ then $\langle H \rangle = E$ and $\sigma_H = 0$.*

Proof:

$$\begin{aligned}
\langle H \rangle &= \int \psi^* \hat{H} \psi dx \\
&= \int \psi^* E \psi dx && \text{(time-independent Schrödinger equation)} \\
&= E \int |\psi|^2 dx \\
&= E \\
\langle H^2 \rangle &= \int \psi^* \hat{H}^2 \psi dx \\
&= E^2 \int \psi^* \psi dx \\
&= E^2 \\
\therefore \sigma_H^2 &= \langle H^2 \rangle - \langle H \rangle^2 \\
&= 0
\end{aligned}$$

□

Example 24.4.6 (The Infinite Square Well)**.** The *infinite square well* with size a ($a > 0$) is a particle moving under the potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant a, \\ \infty & \text{otherwise} \end{cases}$$

The normalizable solutions to the time-independent equation are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x$$

with associated energy

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \ .$$

24.5 The Quantum Harmonic Oscillator

The *quantum harmonic oscillator* with *frequency* ω is a particle of mass m moving under the potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 \ .$$

Proposition 24.5.1. *The Hamiltonian operator for the quantum harmonic oscillator is*

$$\hat{H} = \frac{1}{2m}\left(\hat{p}^2 + (m\omega x)^2\right) \ .$$

PROOF: Immediate from definitions. □

Definition 24.5.2 (Ladder Operators)**.** The *raising operator* $\hat{a_+}$ is

$$\hat{a_+} = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})$$

The *lowering operator* $\hat{a_+}$ is

$$\hat{a_-} = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x})$$

Together, these are called the *ladder operators*.

Proposition 24.5.3.

$$[\hat{a_-}, \hat{a_+}] = 1$$

Proposition 24.5.4.

$$\hat{H} = \hbar\omega(\hat{a_-}\hat{a_+} - \frac{1}{2}) = \hbar\omega(\hat{a_+}\hat{a_-} + \frac{1}{2})$$

Proposition 24.5.5. *If ψ is a solution to the time-independent Schrödinger equation with energy E, then $\hat{a_+}\psi$ is a solution with energy $E + \hbar\omega$, and $\hat{a_-}\psi$ is a solution with energy $E - \hbar\omega$.*

Proposition 24.5.6. *For any integrable functions $f, g : \mathbb{R} \to \mathbb{C}$,*

$$\int_{-\infty}^{+\infty} f^*(\hat{a_\pm} g)dx = \int_{-\infty}^{+\infty} (\hat{a_\mp} f)^* g dx$$

PROOF:

$$\begin{aligned}
\int_{-\infty}^{+\infty} f^*(\hat{a_\pm} g)dx &= \frac{1}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{+\infty} f^*(\mp\hbar\frac{d}{dx} + m\omega x)g\,dx \\
&= \frac{1}{\sqrt{2\hbar m\omega}} \left[\mp \int f^* \frac{dg}{dx} dx + \int m\omega f^* x g\,dx\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}} \left[\pm \int \frac{df^*}{dx} g\,dx + \int m\omega f^* x g\,dx\right] \quad \text{(integrating by parts)} \\
&= \frac{1}{\sqrt{2\hbar m\omega}} \int((\pm\hbar\frac{d}{dx} + m\omega x)f^*)g\,dx \\
&= \int (\hat{a_m p} f)^* g\,dx
\end{aligned}$$

∎

Proposition 24.5.7. *The normalized solutions to the time-independent Schrödinger equation are*

$$\psi_n(x) = \frac{1}{\sqrt{n!}}(\hat{a_+})^n \psi_0$$

with energies

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

where

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} \quad .$$

PROOF:

⟨1⟩1. $\psi(x) = e^{-\frac{m\omega}{2\hbar}x^2}$ is a solution with energy $\frac{1}{2}\hbar m\omega$.

PROOF:

$$\begin{aligned}
\frac{d\psi}{dx} &= -\frac{m\omega}{\hbar}x\psi \\
\therefore \frac{d^2\psi}{dx^2} &= -\frac{m\omega}{\hbar}\left(x\frac{d\psi}{dx} + \psi\right) \\
&= \frac{m^2\omega^2}{\hbar^2}x^2\psi - \frac{m\omega}{\hbar}\psi \\
\therefore \hat{H}\psi &= -\frac{\hbar^2}{2m}\frac{m^2\omega^2}{\hbar^2}x^2\psi - \frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\psi + \frac{1}{2}m\omega^2 x^2\psi \\
&= \frac{1}{2}\hbar m\omega\psi
\end{aligned}$$

⟨1⟩2. For this ψ we have $\int |\psi|^2 dx = \sqrt{\frac{\pi\hbar}{m\omega}}$.

PROOF:

$$\begin{aligned}
\int |\psi|^2 dx &= \int e^{-\frac{m\omega}{\hbar}x^2} dx \\
&= \sqrt{\frac{\pi\hbar}{m\omega}}
\end{aligned}$$

⟨1⟩3. ψ_0 is a normalized solution.

⟨1⟩4. For all n we have $(\hat{a_+})^n\psi_0$ is a solution with energy E_n.

PROOF: Proposition 24.5.5.

$\langle 1 \rangle 5.$ $\hat{a_-}\hat{a_+}\chi_n = (n+1)\chi - n$
PROOF: Since from Proposition 24.5.4 we have

$$\hbar\omega\hat{a_-}\hat{a_+}\chi_n - \frac{1}{2}\chi_n = (n + \frac{1}{2})\hbar\omega\chi_n$$

$\langle 1 \rangle 6.$ For all n, we have $\int |(\hat{a_+})^n \psi_0|^2 dx = n!$.
PROOF:
$\langle 2 \rangle 1.$ LET: $\chi_n = (\hat{a_+})^n \psi_0$
$\langle 2 \rangle 2.$ ASSUME: as induction hypothesis $\int |\chi_n|^2 dx = n!$.
$\langle 2 \rangle 3.$ $\int |\chi_{n+1}|^2 dx = (n+1)!$
PROOF:

$$\begin{aligned}
\int |\chi_{n+1}|^2 dx &= \int (\hat{a_+}\chi_n)^*(\hat{a_+}\chi_n) dx \\
&= \int \chi_n^* (\hat{a_-}\hat{a_+}\chi_n) dx && \text{(Proposition 24.5.6)} \\
&= (n+1) \int \chi_n^* \chi_n dx && (\langle 1 \rangle 5) \\
&= (n+1)n! && (\langle 1 \rangle 3) \\
&= (n+1)!
\end{aligned}$$

$\langle 1 \rangle 7.$ For all n, ψ_n is a normalized solution.
$\langle 1 \rangle 8.$ For all $n > 0$, $\hat{a_-}\psi_n = \sqrt{n}\psi_{n-1}$
PROOF: Using $\langle 1 \rangle 5$.
$\langle 1 \rangle 9.$ For any non-zero solution ψ, if $\hat{a_-}\psi$ has energy $\leqslant 0$ then ψ is a constant multiplied by ψ_0.
PROOF:
$\langle 2 \rangle 1.$ ASSUME: $\hat{a_-}\psi$ has energy $\leqslant 0$
$\langle 2 \rangle 2.$ $\hat{a_-}\psi = 0$
$\langle 2 \rangle 3.$

$$\hbar\frac{d\psi}{dx} + m\omega x\psi = 0$$

$\langle 2 \rangle 4.$

$$\frac{1}{\psi}\frac{d\psi}{dx} = -\frac{m\omega}{\hbar}x$$

$\langle 2 \rangle 5.$ $\ln\psi = -\frac{m\omega}{2\hbar}x^2$ plus a constant.
$\langle 2 \rangle 6.$ $\psi = e^{-\frac{m\omega}{2\hbar}x^2}$ multiplied by a constant.
$\langle 1 \rangle 10.$ For any solution ψ with energy > 0, there exists n such that ψ is a constant multiplied by ψ_n.
$\langle 2 \rangle 1.$ LET: n be least such that $(\hat{a_-})^{n+1}\psi$ has non-positive energy.
$\langle 2 \rangle 2.$ $(\hat{a_-})^n\psi$ is a constant multiplied by ψ_0.
$\langle 2 \rangle 3.$ ψ is a constant multiplied by $(\hat{a_+})^n\psi_0$.
$\langle 2 \rangle 4.$ ψ is a constant multiplied by ψ_n.
□

Definition 24.5.8. We call ψ_0 the *ground state* of the quantum harmonic oscillator, and the other ψ_n the *excited states*.

Proposition 24.5.9.

$$x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a_+} + \hat{a_-})$$

PROOF: Straightforward calculation. □

Proposition 24.5.10.

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a_+} - \hat{a_-})$$

PROOF: Straightforward calculation. □

Proposition 24.5.11. *If $m \neq n$ then*

$$\int \psi_m^* \psi_n dx = 0$$

PROOF: We have

$$\int \psi_m^* \hat{a_+}\hat{a_-}\psi_n dx = n \int \psi_m^* \psi_n dx$$

and

$$\begin{aligned} \int \psi_m^* \hat{a_+}\hat{a_-}\psi_n dx &= \int \hat{a_+}\hat{a_-}\psi_m^* \psi_n dx && \text{(Proposition 24.5.6)} \\ &= m \int \psi_m^* \psi_n dx \end{aligned}$$

Therefore either $m = n$ or $\int \psi_m^* \psi_n dx = 0$. □

Proposition 24.5.12. *For the nth excited state of the quantum harmonic oscillator we have:*

$$\begin{aligned} \langle x \rangle &= 0 \\ \langle p \rangle &= 0 \\ \sigma_x &= \sqrt{\left(n + \frac{1}{2}\right)\frac{\hbar}{m\omega}} \\ \sigma_p &= \sqrt{\left(n + \frac{1}{2}\right)\hbar m\omega} \\ \langle T \rangle &= \frac{1}{2}\hbar\omega\left(n + \frac{1}{2}\right) \\ \langle V \rangle &= \frac{1}{2}\hbar\omega\left(n + \frac{1}{2}\right) \end{aligned}$$

PROOF: These follow from

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{+\infty} \psi_n^* x^2 \psi_n dx \\
&= \frac{\hbar}{2m\omega} \int_{-\infty}^{+\infty} \psi_n^* [\hat{a_+}^2 + \hat{a_+}\hat{a_-} + \hat{a_-}\hat{a_+} + \hat{a_-}^2] \psi_n dx \\
&= \frac{\hbar}{2m\omega} \left[\int_{-\infty}^{+\infty} a\psi_n^* \psi_{n+2} dx + n \int \psi_n^* \psi_n dx + (n+1) \int \psi_n^* \psi_n dx + \int b\psi_n^* \psi_{n-2} dx \right] \\
&= \frac{\hbar}{2m\omega}(2n+1)
\end{aligned}$$

and $\langle p^2 \rangle = \frac{\hbar m\omega}{2}(2n+1)$ similary. □

Proposition 24.5.13. *The nth excited state is*

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\frac{\xi^2}{2}}$$

where H_n is the nth Hermite polynomial and

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x \quad .$$

PROOF: From Proposition 24.5.7 by induction. □

24.6 The Free Particle

Proposition 24.6.1. *The solutions to the time-independent Schrödinger equation with $V = 0$ are*

$$A e^{i(kx - \frac{\hbar k^2}{2m} t)}$$

Proposition 24.6.2. *The normalized solutions to the time-dependent Schrödinger equation with $V = 0$ are the* wave packets

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)} dk$$

where $\phi : \mathbb{R} \to \mathbb{C}$ is a function such that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) dk = 1 \quad .$$

Proposition 24.6.3. *Let a particle moving under potential $V = 0$ have initial wave function $\Psi(x, 0)$. Then we have*

$$Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)} dk$$

where

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x, 0) e^{-ikx} dx \quad .$$

PROOF: We have $\phi(k)$ is the inverse Fourier transform of $\Phi(x, t)$ hence of $\Phi(x, 0)$.
$\square$