Mathematics

Robin Adams

October 10, 2023

Contents

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*.

For any set A, let there be *elements* of A. We write $a \in A$ for: a is an element of A.

For any sets A and B, let there be a set B^A, whose elements are called *functions* from A to B. We write $f : A \to B$ for $f \in B^A$.

For any function $f : A \to B$ and element $a \in A$, let there be an element $f(a) \in B$, the *value* of the function f at the *argument* a.

1.2 Injections, Surjections and Bijections

Definition 1.2.1 (Injective)**.** A function $f : A \to B$ is *injective* or an *injection* iff, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

Definition 1.2.2 (Surjective)**.** A function $f : A \to B$ is *surjective* or a *surjection* iff, for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Definition 1.2.3 (Bijective)**.** A function $f : A \to B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$".

1.3 Axioms

Axiom Schema 1.3.1 (Choice). *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x \in X$ and $y \in Y$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a \in A$, there exists $b \in B$ such that $P[A, B, a, b]$. Then there exists a function $f : A \to B$ such that $\forall a \in A.P[A, B, a, f(a)]$.

Axiom 1.3.2 (Extensionality). *Let $f, g : A \to B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Definition 1.3.3 (Composition). Let $f : A \to B$ and $g : B \to C$. The *composite* $g \circ f : A \to C$ is the function such that, for all $a \in A$, we have

$$(g \circ f)(a) = g(f(a)) \ .$$

Axiom 1.3.4 (Pairing). *For any sets A and B, there exists a set $A \times B$, the* Cartesian product *of A and B, and functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that, for all $a \in A$ and $b \in B$, there exists a unique $(a, b) \in A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Axiom Schema 1.3.5 (Separation). *For every property $P[X, x]$ where X is a set variable and $x \in X$, the following is an axiom:*

For every set A, there exists a set $S = \{x \in A : P[A, x]\}$ and an injection $i : S \to A$ such that, for all $x \in A$, we have

$$(\exists y \in S.i(y) = x) \Leftrightarrow P[A, x] \ .$$

Axiom 1.3.6 (Infinity). *There exists a set $\mathbb{N}$, an element $0 \in \mathbb{N}$, and a function $s : \mathbb{N} \to \mathbb{N}$ such that:*

- $\forall n \in \mathbb{N}.s(n) \neq 0$
- $\forall m, n \in \mathbb{N}.s(m) = s(n) \Rightarrow m = n.$

Axiom Schema 1.3.7 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A, there exist sets B and Y and functions $p : B \to A$, and $m : B \times Y \Rightarrow \mathbb{N}$ such that:

- *m is injective.*
- $\forall b \in B.P[A, \{y \in Y : m(b, y) = 0\}, p(b)]$
- *For all $a \in A$, if $\exists Y.P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Axiom 1.3.8 (Universe). *There exists a set E, a set U and a function $el : E \to U$ such that the following holds.*

Let us say that a set A is small *iff there exists $u \in U$ such that $A \approx \{e \in E : el(e) = u\}$.*

- $\mathbb{N}$ *is small.*
- *For any* U*-small sets* A *and* B*, the set* B^A *is small.*
- *For any* U*-small sets* A *and* B*, the set* $A \times B$ *is small.*
- *Let* $f : A \to B$ *be a function. If* B *is small and* $\{a \in A : f(a) = b\}$ *is small for all* $b \in B$*, then* A *is small.*
- *If* $p : B \twoheadrightarrow A$ *is a surjective function such that* A *is small, then there exists a* U*-small set* C*, a surjection* $q : C \twoheadrightarrow A$*, and a function* $f : C \to B$ *such that* $q = p \circ f$*.*

Chapter 2

Sets and Functions

2.1 Composition

Proposition 2.1.1. *Given functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \ .$$

PROOF:
$\langle 1 \rangle 1.$ For all $x \in A$ we have $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$.
  $\langle 2 \rangle 1.$ LET: $x \in A$
  $\langle 2 \rangle 2.$ $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$
    PROOF:

$$\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) && \text{(Definition of composition)} \\
&= h(g(f(x))) && \text{(Definition of composition)} \\
&= (h \circ g)(f(x)) && \text{(Definition of composition)} \\
&= ((h \circ g) \circ f)(x) && \text{(Definition of composition)}
\end{aligned}$$

$\langle 1 \rangle 2.$ Q.E.D.
  PROOF: By the Axiom of Extensionality.
□

2.1.1 Injections

Proposition 2.1.2. *The composite of injective functions is injective.*

PROOF:
$\langle 1 \rangle 1.$ LET: A, B and C be sets.
$\langle 1 \rangle 2.$ LET: $f : A \to B$
$\langle 1 \rangle 3.$ LET: $g : B \to C$
$\langle 1 \rangle 4.$ ASSUME: g is injective.
$\langle 1 \rangle 5.$ ASSUME: f is injective.
$\langle 1 \rangle 6.$ LET: $x, y \in A$
$\langle 1 \rangle 7.$ ASSUME: $(g \circ f)(x) = (g \circ f)(y)$

PROVE: $x = y$
$\langle 1 \rangle 8.$ $g(f(x)) = g(f(y))$
PROOF:
$$\begin{aligned} g(f(x)) &= (g \circ f)(x) && \text{(definition of composition)} \\ &= (g \circ f)(y) && (\langle 1 \rangle 7) \\ &= g(f(y)) && \text{(definition of composition)} \end{aligned}$$
$\langle 1 \rangle 9.$ $f(x) = f(y)$
PROOF: $\langle 1 \rangle 4$, $\langle 1 \rangle 8$
$\langle 1 \rangle 10.$ $x = y$
PROOF: $\langle 1 \rangle 5$, $\langle 1 \rangle 9$
□

Proposition 2.1.3. *For functions $f : A \to B$ and $g : B \to C$, if $g \circ f$ is injective then f is injective.*

PROOF:
$\langle 1 \rangle 1.$ LET: A, B and C be sets.
$\langle 1 \rangle 2.$ LET: $f : A \to B$
$\langle 1 \rangle 3.$ LET: $g : B \to C$
$\langle 1 \rangle 4.$ ASSUME: $g \circ f$ is injective.
$\langle 1 \rangle 5.$ LET: $x, y \in A$
$\langle 1 \rangle 6.$ ASSUME: $f(x) = f(y)$
$\langle 1 \rangle 7.$ $(g \circ f)(x) = (g \circ f)(y)$
PROOF:
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{(definition of composition)} \\ &= g(f(y)) && (\langle 1 \rangle 6) \\ &= (g \circ f)(y) && \text{(definition of composition)} \end{aligned}$$
$\langle 1 \rangle 8.$ $x = y$
PROOF: $\langle 1 \rangle 4$, $\langle 1 \rangle 7$
□

Proposition 2.1.4. *Let $f : A \to B$ be injective. For every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: f is injective.
$\langle 1 \rangle 2.$ LET: X be a set.
$\langle 1 \rangle 3.$ LET: $x, y : X \to A$
$\langle 1 \rangle 4.$ ASSUME: $f \circ x = f \circ y$
$\langle 1 \rangle 5.$ $\forall t \in X . x(t) = y(t)$
$\langle 2 \rangle 1.$ LET: $t \in X$
$\langle 2 \rangle 2.$ $f(x(t)) = f(y(t))$
PROOF:
$$\begin{aligned} f(x(t)) &= (f \circ x)(t) && \text{(definition of composition)} \\ &= (f \circ y)(t) && (\langle 1 \rangle 4) \\ &= f(y(t)) && \text{(definition of composition)} \end{aligned}$$

$\langle 2\rangle 3.$ $x(t) = y(t)$
PROOF: $\langle 1\rangle 1$, $\langle 2\rangle 2$
$\langle 1\rangle 6.$ $x = y$
PROOF: Axiom of Extensionality, $\langle 1\rangle 5$
$\square$

2.1.2 Surjections

Proposition 2.1.5. *The composite of surjective functions is surjective.*

PROOF:
$\langle 1\rangle 1.$ LET: $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$ be injective.
$\langle 1\rangle 2.$ LET: $c \in C$
$\langle 1\rangle 3.$ PICK $b \in B$ such that $g(b) = c$.
$\langle 1\rangle 4.$ PICK $a \in A$ such that $f(a) = b$.
$\langle 1\rangle 5.$ $(g \circ f)(a) = c$
$\square$

Proposition 2.1.6. *Let $f : A \to B$. Then f is surjective if and only if, for any set X and functions $g, h : B \to X$, if $g \circ f = h \circ f$ then $g = h$.*

PROOF:
$\langle 1\rangle 1.$ If f is surjective then, for any set X and functions $g, h : B \to X$, if $g \circ f = h \circ f$ then $g = h$.
$\langle 2\rangle 1.$ ASSUME: f is surjective.
$\langle 2\rangle 2.$ LET: X be a set.
$\langle 2\rangle 3.$ LET: $g, h : B \to X$
$\langle 2\rangle 4.$ ASSUME: $g \circ f = h \circ f$
$\langle 2\rangle 5.$ LET: $b \in B$
PROVE: $g(b) = h(b)$
$\langle 2\rangle 6.$ PICK $a \in A$ such that $f(a) = b$
$\langle 2\rangle 7.$ $g(b) = h(b)$
PROOF: $g(b) = g(f(a)) = h(f(a)) = h(b)$
$\langle 1\rangle 2.$ If, for any set X and functions $g, h : B \to x$, if $g \circ f = h \circ f$ then $g = h$, then f is surjective.
$\langle 2\rangle 1.$ ASSUME: For any set X and functions $g, h : B \to X$, if $g \circ f = h \circ f$ then $g = h$.
$\langle 2\rangle 2.$ LET: $b \in B$
$\langle 2\rangle 3.$ LET: $h : B \to 2$ be the function that maps everything to 1.
$\langle 2\rangle 4.$ LET: $k : B \to 2$ be the function that maps b to 0 and everything else to 1.
$\langle 2\rangle 5.$ $h \neq k$
$\langle 2\rangle 6.$ $h \circ f \neq k \circ f$
$\langle 2\rangle 7.$ PICK $a \in A$ such that $h(f(a)) \neq k(f(a))$
$\langle 2\rangle 8.$ $f(a) = b$
$\square$

Proposition 2.1.7. *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is surjective then g is surjective.*

PROOF:
⟨1⟩1. LET: $c \in C$
⟨1⟩2. There exists $a \in A$ such that $g(f(a)) = c$.
⟨1⟩3. There exists $b \in B$ such that $g(b) = c$.
□

Proposition 2.1.8. *Let A and B be sets. If there exists an injective function $f : A \to B$, and A is nonempty, then there exists a surjective function $B \to A$.*

PROOF: Pick $a_0 \in A$. Define $g : B \to A$ by: $g(b)$ is the unique element in A such that $f(a) = b$ if there is such an a, otherwise $g(b) = a_0$. □

2.1.3 Bijections

Proposition 2.1.9. *The composite of bijections is a bijection.*

PROOF: Propositions 2.1.2 and 2.1.5. □

Proposition 2.1.10. *Let $f : A \to B$. Then f is bijective if and only if there exists a function $f^{-1} : B \to A$, the* inverse *of f, such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$, in which case the inverse is unique.*

PROOF:
⟨1⟩1. If f is bijective then there exists $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.
⟨2⟩1. ASSUME: f is bijective.
⟨2⟩2. PICK $g : B \to A$ such that $f \circ g = \mathrm{id}_B$
PROOF: Proposition 2.1.6.
⟨2⟩3. $f \circ g \circ f = f$
⟨2⟩4. $g \circ f = \mathrm{id}_A$
PROOF: Proposition 2.1.4.
⟨1⟩2. If there exists $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$, then f is bijective.
⟨2⟩1. LET: $f^{-1} : B \to A$ satisfy $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$
⟨2⟩2. f is injective.
PROOF: If $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$.
⟨2⟩3. f is surjective.
PROOF: Proposition 2.1.6.
⟨1⟩3. If $g, h : B \to A$ satisfy $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ and $h \circ f = \mathrm{id}_A$ then $g = h$.
PROOF: We have $g = g \circ f \circ h = h$.
□

Theorem 2.1.11 (Schroeder-Bernstein)**.** *Let A and B be sets. If there exist injections $A \to B$ and $B \to A$, then $A \approx B$.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : A \rightarrowtail B$ and $g : B \rightarrowtail A$ be injections.
$\langle 1 \rangle 2$. Define the subsets A_n of A by

$$A_0 := A - g(B)$$
$$A_{n+1} := g(f(A_n))$$

$\langle 1 \rangle 3$. Define $h : A \to B$ by

$$h(x) = \begin{cases} f(x) & \text{if } \exists n. x \in A_n \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

$\langle 1 \rangle 4$. h is injective.
  $\langle 2 \rangle 1$. LET: $x, y \in A$
  $\langle 2 \rangle 2$. ASSUME: $h(x) = h(y)$
  $\langle 2 \rangle 3$. CASE: $x \in A_m$ and $y \in A_n$.
    PROOF: Then $f(x) = f(y)$ so $x = y$ since f is injective.
  $\langle 2 \rangle 4$. CASE: $x \in A_m$ and there is no y such that $y \in A_n$.
    $\langle 3 \rangle 1$. $f(x) = g^{-1}(y)$
    $\langle 3 \rangle 2$. $y = g(f(x))$
    $\langle 3 \rangle 3$. $y \in A_{m+1}$
    $\langle 3 \rangle 4$. Q.E.D.
      PROOF: This is a contradiction.
  $\langle 2 \rangle 5$. CASE: $y \in A_n$ and there is no m such that $x \in A_m$.
    PROOF: Similar.
  $\langle 2 \rangle 6$. CASE: There is no m such that $x \in A_m$ and there is no n such that $y \in A_n$.
    PROOF: Then $g^{-1}(x) = g^{-1}(y)$ and so $x = y$.
$\langle 1 \rangle 5$. h is surjective.
  $\langle 2 \rangle 1$. LET: $y \in B$
  $\langle 2 \rangle 2$. CASE: $g(y) \in A_n$
    $\langle 3 \rangle 1$. $n \neq 0$
    $\langle 3 \rangle 2$. PICK $x \in A_{n-1}$ such that $g(y) = g(f(x))$
    $\langle 3 \rangle 3$. $y = f(x)$
    $\langle 3 \rangle 4$. $y = h(x)$
  $\langle 2 \rangle 3$. CASE: There is no n such that $g(y) \in A_n$.
    PROOF: Then $h(g(y)) = y$.
□

Proposition 2.1.12.

$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps f to $(\pi_1 \circ f, \pi_2 \circ f)$ is a bijection. □

Proposition 2.1.13.

$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function Φ such that $\Phi(f)(c)(b) = f(b, c)$ is a bijection. □

2.2 Identity Function

Definition 2.2.1 (Identity)**.** For any set A, the *identity* function $\mathrm{id}_A : A \to A$ is the function defined by $\mathrm{id}_A(a) = a$.

Proposition 2.2.2. *Let $f : A \to B$. Then $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.*

PROOF: Each is the function that maps a to $f(a)$. ∎

Proposition 2.2.3. *Let $f : A \to B$. Then f is surjective if and only if there exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 3$
⟨2⟩1. ASSUME: f is surjective.
⟨2⟩2. PICK $g : B \to A$ such that, for all $b \in B$, we have $f(g(b)) = b$.
PROOF: Axiom of Choice.
⟨2⟩3. $f \circ g = \mathrm{id}_B$.
⟨1⟩2. $3 \Rightarrow 2$
⟨2⟩1. LET: $g : B \to A$ such that $f \circ g = \mathrm{id}_B$
⟨2⟩2. LET: X be a set.
⟨2⟩3. LET: $h, k : B \to X$
⟨2⟩4. ASSUME: $h \circ f = k \circ f$
⟨2⟩5. $h = k$
PROOF: $h = h \circ f \circ g = k \circ f \circ g = k$
∎

2.2.1 The Empty Set

Theorem 2.2.4. *There exists a set which has no elements.*

PROOF:
⟨1⟩1. PICK a set A
PROOF: By the Axiom of Infinity, a set exists.
⟨1⟩2. LET: $S = \{x \in A : \bot\}$ with injection $i : S \to A$
PROOF: Axiom of Separation.
⟨1⟩3. S has no elements.
∎

Theorem 2.2.5. *If E and E' have no elements then $E \approx E'$.*

PROOF:
⟨1⟩1. LET: E and E' have no elements.
⟨1⟩2. PICK a function $F : E \to E'$.
PROOF: Axiom of Choice since vacuously $\forall x \in E. \exists y \in E'. \top$.
⟨1⟩3. F is injective.
PROOF: Vacuously, for all $x, y \in E$, if $F(x) = F(y)$ then $x = y$.
⟨1⟩4. F is surjective.
PROOF: Vacuously, for all $y \in E$, there exists $x \in E$ such that $F(x) = y$.

□

Definition 2.2.6 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

2.2.2 The Singleton

Theorem 2.2.7. *There exists a set that has exactly one element.*

PROOF:
⟨1⟩1. PICK a set A that has an element.
PROOF: By the Axiom of Infinity, there exists a set that has an element.
⟨1⟩2. PICK $a \in A$
⟨1⟩3. PICK a set S and injection $i : S \rightarrowtail A$ such that, for all $x \in A$, there exists $s \in S$ such that $s = x$ if and only if $x = a$
⟨1⟩4. S has exactly one element.
□

Theorem 2.2.8. *If A and B both have exactly one element then $A \approx B$.*

PROOF:
⟨1⟩1. LET: A and B both have exactly one element a and b respectively.
⟨1⟩2. LET: $F : A \to B$ be the function such that, for all $x \in A$, we have $(x = a \wedge F(x) = b)$
⟨1⟩3. F is a bijection.
□

Definition 2.2.9 (Singleton)**.** Let 1 be the set that has exactly one element. Let $*$ be its element.

Proposition 2.2.10. *Let $f : A \to B$. Assume that, for every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$. Then f is injective.*

PROOF: Take $X = 1$. □

2.2.3 Subsets

Definition 2.2.11 (Subset)**.** A *subset* of a set A consists of a set S and an injection $i : S \rightarrowtail A$. We write $(S, i) \subseteq A$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Proposition 2.2.12. *For any subset (S, i) of A we have $(S, i) = (S, i)$.*

PROOF: We have $\mathrm{id}_S : S \approx S$ and $i \circ \mathrm{id}_S = i$.

Proposition 2.2.13. *If $(S, i) = (T, j)$ then $(T, j) = (S, i)$.*

PROOF: If $\phi : S \approx T$ and $j \circ \phi = i$ then $\phi^{-1} : T \approx S$ and $i \circ \phi^{-1} = j$. □

Proposition 2.2.14. *If $(R, i) = (S, j)$ and $(S, j) = (T, k)$ then $(R, i) = (T, k)$.*

PROOF: If $\phi : R \approx S$ and $j \circ \phi = i$, and $\psi : S \approx T$ and $k \circ \psi = j$, then $\psi \circ \phi : R \approx T$ and $k \circ \psi \circ \phi = i$. □

Definition 2.2.15 (Membership)**.** Given $(S, i) \subseteq A$ and $a \in A$, we write $a \in (S, i)$ for $\exists s \in S.i(s) = a$.

Proposition 2.2.16. *If $a \in (S, i)$ and $(S, i) = (T, j)$ then $a \in (T, j)$.*

PROOF: If $i(s) = a$ then $j(\phi(s)) = a$. □

Definition 2.2.17 (Union)**.** Given subsets S and T of A, the *union* is the subset $\{x \in A : x \in S \vee x \in T\}$.

Definition 2.2.18 (Intersection)**.** Given subsets S and T of A, the *intersection* is the subset $\{x \in A : x \in S \wedge x \in T\}$.

Proposition 2.2.19 (Distributive Law)**.**

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

Proposition 2.2.20 (Distributive Law)**.**

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

Definition 2.2.21. Given a set A, we write $\varnothing$ for the subset $(\varnothing, !)$ where ! is the unique function $\varnothing \to A$.

Proposition 2.2.22.

$$S \cup \varnothing = S$$

Proposition 2.2.23.

$$S \cap \varnothing = S$$

Definition 2.2.24 (Inclusion)**.** Given subsets (S, i) and (T, j) of a set A, we write $(S, i) \subseteq (T, j)$ iff there exists $f : S \to T$ such that $j \circ f = i$.

Proposition 2.2.25.

$$\varnothing \subseteq S$$

Definition 2.2.26 (Disjoint)**.** Subsets S and T of A are *disjoint* iff $S \cap T = \varnothing$.

Definition 2.2.27 (Difference)**.** Given subsets S and T of A, the *difference* of S and T is $S - T = \{x \in A : x \in S \wedge x \notin T\}$.

Proposition 2.2.28 (De Morgan's Law)**.**

$$R - (S \cup T) = (R - S) \cap (R - T)$$

Proposition 2.2.29 (De Morgan's Law)**.**

$$R - (S \cap T) = (R - S) \cup (R - T)$$

2.2.4 Union

Definition 2.2.30 (Union)**.** Given $\mathcal{A} \in \mathcal{P}\mathcal{P}X$, its *union* is

$$\bigcup \mathcal{A} := \{x \in X : \exists S \in \mathcal{A}. x \in S\} \in \mathcal{P}X \ \ .$$

2.2.5 Intersection

Definition 2.2.31 (Intersection)**.** Given $\mathcal{A} \in \mathcal{P}\mathcal{P}X$, its *intersection* is

$$\bigcap \mathcal{A} := \{x \in X : \forall S \in \mathcal{A}. x \in S\} \in \mathcal{P}X \ \ .$$

2.2.6 Direct Image

Definition 2.2.32 (Direct Image)**.** Let $f : A \to B$. Let S be a subset of A. The *(direct) image* of S under f is the subset of B given by

$$f(S) := \{f(a) : a \in S\} \ \ .$$

Proposition 2.2.33.

1. *If* $S \subseteq T$ *then* $f(S) \subseteq f(T)$
2. $f(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f(S)$

Example 2.2.34. It is not true in general that $f(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f(S)$. Take f to be the only function $\{0,1\} \to \{0\}$, and $\mathcal{S} = \{\{0\},\{1\}\}$. Then $f(\bigcap \mathcal{S}) = \varnothing$ but $\bigcap_{S \in \mathcal{S}} f(S) = \{0\}$.

Example 2.2.35. It is not true in general that $f(S-T) = f(S) - f(T)$. Take f to be the only function $\{0,1\} \to \{0\}$, $S = \{0\}$ and $T = \{1\}$. Then $f(S-T) = \{0\}$ but $f(S) - f(T) = \varnothing$.

2.2.7 Inverse Image

Definition 2.2.36 (Inverse Image)**.** Let $f : A \to B$. Let S be a subset of B. The *inverse image* or *preimage* of S under f is the subset of A given by

$$f^{-1}(S) := \{x \in A : f(x) \in S\} \ \ .$$

Proposition 2.2.37.

1. *If* $S \subseteq T$ *then* $f^{-1}(S) \subseteq f^{-1}(T)$
2. $f^{-1}(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f^{-1}(S)$
3. $f^{-1}(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f^{-1}(S)$
4. $f^{-1}(S - T) = f^{-1}(S) - f^{-1}(T)$
5. $S \subseteq f^{-1}(f(S))$. *Equality holds if* f *is injective.*
6. $f(f^{-1}(T)) \subseteq T$. *Equality holds if* f *is surjective.*
7. $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$

2.3 Relations

Definition 2.3.1 (Relation)**.** Let A and B be sets. A *relation* R between A and B, $R : A \nrightarrow B$, is a subset of $A \times B$.

Given $a \in A$ and $b \in B$, we write aRb for $(a, b) \in R$.

A relation *on* a set A is a relation between A and A.

Definition 2.3.2 (Reflexive)**.** A relation R on a set A is *reflexive* iff $\forall a \in A.aRa$.

Definition 2.3.3 (Symmetric)**.** A relation R on a set A is *symmetric* iff, whenever xRy, then yRx.

Definition 2.3.4 (Transitive)**.** A relation R on a set A is *transitive* iff, whenever xRy and yRz, then xRz.

2.3.1 Equivalence Relations

Definition 2.3.5 (Equivalence Relation)**.** A relation R on a set A is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Definition 2.3.6 (Equivalence Class)**.** Let R be an equivalence relation on a set A and $a \in A$. The *equivalence class* of a with respect to R is

$$\{x \in A : xRa\} \ .$$

Proposition 2.3.7. *Two equivalence classes are either disjoint or equal.*

2.4 Power Set

Definition 2.4.1 (Power Set)**.** The *power set* of a set A is $\mathcal{P}A := 2^A$.

Given $S \in \mathcal{P}A$ and $a \in A$, we write $a \in A$ for $S(a) = 1$.

Definition 2.4.2 (Pairwise Disjoint)**.** Let $P \subseteq \mathcal{P}A$. We say the members of P are *pairwise disjoint* iff, for all $S, T \in P$, if $S \neq T$ then $S \cap T = \emptyset$.

2.4.1 Partitions

Definition 2.4.3 (Partition)**.** Let A be a set. A *partition* of A is a set $P \in \mathcal{P}\mathcal{P}A$ such that:

- $\bigcup P = A$
- Every member of P is nonempty.
- The members of P are pairwise disjoint.

2.5 Cartesian Product

Definition 2.5.1 (Cartesian Product)**.** Let A and B be sets. The *Cartesian product* of A and B, $A \times B$, is the tabulation of the relation $A \nrightarrow B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

2.6 Quotient Sets

Proposition 2.6.1. *Let $\sim$ be an equivalence relation on X. Then there exists a set $X/\sim$, the* quotient set *of X with respect to $\sim$, and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the* canonical projection*, such that, for all $x, y \in X$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to $\sim$, then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

2.7 Partitions

Definition 2.7.1 (Partition)**.** A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X.

2.8 Disjoint Union

Theorem 2.8.1. *For any sets A and B, there exists a set $A + B$, the* disjoint union *of A and B, and functions $\kappa_1 : A \to A + B$ and $\kappa_2 : B \to A + B$, the* injections*, such that, for every set X and functions $f : A \to X$ and $g : B \to X$, there exists a unique function $[f, g] : A + B \to X$ such that $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $A+B := \{p \in \mathcal{P}A \times \mathcal{P}B : \exists a \in A.p = (\{a\}, \emptyset) \vee \exists b \in B.p = (\emptyset, \{b\})\}$

Definition 2.8.2 (Restriction)**.** Let $f : A \to B$ and let (S, i) be a subset of A. The *restriction* of f to S is the function $f \restriction S : S \to B$ defined by $f \restriction S = f \circ i$.

2.9 Natural Numbers

Theorem 2.9.1 (Principle of Recursive Definition)**.** *Let A be a set. Let F be the set of all functions $\{m \in \mathbb{N} : m < n\} \to A$ for some n. Let $\rho : F \to A$. Then there exists a unique $g : \mathbb{N} \to A$ such that, for all $n \in \mathbb{N}$, we have*

$$g(n) = \rho(g \restriction \{m \in \mathbb{N} : m < n\}) \ .$$

PROOF:
⟨1⟩1. Given a subset $B \subseteq \mathbb{N}$, let us say that a function $g : B \to A$ is *acceptable* iff, for all $n \in B$, we have
$$\forall m < n.m \in B$$
and
$$g(n) = \rho(g\restriction\{m \in \mathbb{N} : m < n\}) \ .$$
⟨1⟩2. For all $n \in \mathbb{N}$, there exists an acceptable function $\{m \in \mathbb{N} : m < n\} \to A$.
⟨2⟩1. LET: $P[n]$ be the property: There exists an acceptable function $\{m \in \mathbb{N} : m < n\} \to A$.
⟨2⟩2. $P[0]$
PROOF: The unique function $\emptyset \to A$ is acceptable.
⟨2⟩3. For any natural number n, if $P[n]$ then $P[n+1]$.
⟨3⟩1. ASSUME: $P[n]$
⟨3⟩2. PICK an acceptable $f : \{m \in \mathbb{N} : m < n\} \to A$.
⟨3⟩3. LET: $g : \{m \in \mathbb{N} : m < n+1\} \to A$ be the function
$$g(m) = \begin{cases} f(m) & \text{if } m < n \\ \rho(f) & \text{if } m = n \end{cases}$$
⟨3⟩4. g is acceptable.
⟨1⟩3. If $g : B \to A$ and $h : C \to A$ are acceptable, then g and h agree on $B \cap C$.
⟨1⟩4. Define $g : \mathbb{N} \to A$ by: $g(n) = a$ iff there exists an acceptable $h : \{m \in \mathbb{N} : m < n+1\}$ such that $h(n) = a$.
⟨1⟩5. g is acceptable.
⟨1⟩6. If $g' : \mathbb{N} \to A$ is acceptable then $g' = g$.
□

2.10 Finite and Infinite Sets

Definition 2.10.1 (Finite)**.** A set A is *finite* iff there exists $n \in \mathbb{N}$ such that $A \approx \{m \in \mathbb{N} : m < n\}$. In this case, we say A has *cardinality* n.

Proposition 2.10.2. *Let $n \in \mathbb{N}$. Let A be a set. Let $a_0 \in A$. Then $A \approx \{m \in \mathbb{N} : m < n+1\}$ if and only if $A - \{a_0\} \approx \{m \in \mathbb{N} : m < n\}$.*

Theorem 2.10.3. *Let A be a set. Suppose that $A \approx \{m \in \mathbb{N} : m < n\}$. Let B be a proper subset of A. Then $B \not\approx \{m \in \mathbb{N} : m < n\}$ but there exists $m < n$ such that $B \approx \{k \in \mathbb{N} : k < m\}$.*

PROOF:
⟨1⟩1. LET: $P[n]$ be the property: for every set A, if $Aapprox\{m \in \mathbb{N} : m < n\}$, then for every proper subset B of A, we have $B \not\approx \{m \in \mathbb{N} : m < n\}$ but there exists $m < n$ such that $B \approx \{k \in \mathbb{N} : k < m\}$.
⟨1⟩2. $P[0]$
PROOF: If $A \approx \{m \in \mathbb{N} : m < 0\}$ then A is empty and so has no proper subset.
⟨1⟩3. For every natural number n, if $P[n]$ then $P[n+1]$.
⟨2⟩1. LET: n be a natural number.
⟨2⟩2. ASSUME: $P[n]$

$\langle 2\rangle 3.$ Let: A be a set.
$\langle 2\rangle 4.$ Assume: $A \approx \{m \in \mathbb{N} : m < n+1\}$
$\langle 2\rangle 5.$ Let: B be a proper subset of A.
$\langle 2\rangle 6.$ Case: $B = \emptyset$
Proof: Then $B \not\approx \{m \in \mathbb{N} : m < n+1\}$ but $B \approx \{k \in \mathbb{N} : k < 0\}$.
$\langle 2\rangle 7.$ Case: $B \neq \emptyset$
$\langle 3\rangle 1.$ Pick $b_0 \in B$
$\langle 3\rangle 2.$ $A - \{b_0\} \approx \{m \in \mathbb{N} : m < n\}$
$\langle 3\rangle 3.$ $B - \{b_0\}$ is a proper subset of $A - \{b_0\}$
$\langle 3\rangle 4.$ $B - \{b_0\} \not\approx \{m \in \mathbb{N} : m < n\}$
$\langle 3\rangle 5.$ $B \approx \{m \in \mathbb{N} : m < n+1\}$
$\langle 3\rangle 6.$ Pick $m < n$ such that $B - \{b_0\} \approx \{k \in \mathbb{N} : k < m\}$
$\langle 3\rangle 7.$ $m + 1 < n + 1$
$\langle 3\rangle 8.$ $B \approx \{k \in \mathbb{N} : k < m+1\}$

$\square$

Corollary 2.10.3.1. *If A is finite then there is no bijection between A and a proper subset of A.*

Corollary 2.10.3.2. $\mathbb{N}$ *is infinite.*

Corollary 2.10.3.3. *The cardinality of a finite set is unique.*

Corollary 2.10.3.4. *A subset of a finite set is finite.*

Corollary 2.10.3.5. *If A is finite and B is a proper subset of A then $|B| < |A|$.*

Corollary 2.10.3.6. *Let A be a set. Then the following are equivalent:*

1. *A is finite.*
2. *There exists a surjection from an initial segment of $\mathbb{N}$ onto A.*
3. *There exists an injection from A to an initial segment of $\mathbb{N}$.*

Corollary 2.10.3.7. *A finite union of finite sets is finite.*

Corollary 2.10.3.8. *A finite Cartesian product of finite sets is finite.*

Theorem 2.10.4. *Let A be a set. The following are equivalent:*

1. *There exists an injective function $\mathbb{N} \rightarrowtail A$.*
2. *There exists a bijection between A and a proper subset of A.*
3. *A is infinite.*

Proof:
$\langle 1\rangle 1.$ $1 \Rightarrow 2$
$\langle 2\rangle 1.$ Let: $f : \mathbb{N} \rightarrowtail A$ be injective.
$\langle 2\rangle 2.$ Let: $s : \mathbb{N} \approx \mathbb{N} - \{0\}$ be the function $s(n) = n + 1$.
$\langle 2\rangle 3.$ $f \circ s \circ f^{-1} : A \approx A - \{f(0)\}$

⟨1⟩2. $2 \Rightarrow 3$
PROOF: Corollary 2.10.3.1.
⟨1⟩3. $3 \Rightarrow 1$
PROOF: Choose a function $f : \mathbb{N} \to A$ such that $f(n) \in A - \{f(m) : m < n\}$ for all n.
□

2.11 Countable Sets

Definition 2.11.1 (Countable)**.** A set A is *countably infinite* iff $A \approx \mathbb{N}$.

Proposition 2.11.2. *$\mathbb{N} \times \mathbb{N}$ is countably infinite.*

PROOF: Define $f : \mathbb{N} \times \mathbb{N} \approx \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leqslant x\}$ by
$$f(x, y) = (x + y, y)$$
Define $g : \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leqslant x\} \approx \mathbb{N}$ by
$$g(x, y) = x(x - 1)/2 + y\ .$$ □

Proposition 2.11.3. *Every infinite subset of $\mathbb{N}$ is countably infinite.*

PROOF:
⟨1⟩1. LET: C be an infinite subset of $\mathbb{N}$
⟨1⟩2. Define $h : \mathbb{Z} \to C$ by recursion thus: $h(n)$ is the smallest element of $C - \{h(m) : m < n\}$.
⟨1⟩3. h is injective.
PROOF: If $m < n$ then $h(m) \neq h(n)$ because $h(n) \in C - \{h(m) : m < n\}$.
⟨1⟩4. h is surjective.
⟨2⟩1. For all $n \in \mathbb{N}$ we have $n \leqslant h(n)$.
⟨2⟩2. LET: $c \in C$
⟨2⟩3. $c \leqslant h(c)$
⟨2⟩4. LET: n be least such that $c \leqslant h(n)$
⟨2⟩5. $c \in C - \{h(m) : m < n\}$
⟨2⟩6. $h(n) \leqslant c$
⟨2⟩7. $h(n) = c$
□

Definition 2.11.4 (Countable)**.** A set is *countable* iff it is either finite or countably infinite; otherwise it is *uncountable*.

Proposition 2.11.5. *Let B be a nonempty set. Then the following are equivalent.*

1. *B is countable.*
2. *There exists a surjection $\mathbb{N} \twoheadrightarrow B$.*
3. *There exists an injection $B \rightarrowtail \mathbb{N}$.*

PROOF:

$\langle 1\rangle 1.$ $1 \Rightarrow 2$
  $\langle 2\rangle 1.$ ASSUME: B is countable.
  $\langle 2\rangle 2.$ CASE: B is finite.
    $\langle 3\rangle 1.$ PICK a natural number n and bijection $f : \{m \in \mathbb{N} : m < n\} \approx B$
    $\langle 3\rangle 2.$ PICK $b \in B$
    $\langle 3\rangle 3.$ Extend f to a surjection $g : \mathbb{N} \twoheadrightarrow B$ by setting $g(m) = b$ for $m \geqslant n$.
  $\langle 2\rangle 3.$ CASE: B is countably infinite.
    PROOF: Then there exists a bijection $\mathbb{N} \approx B$.
$\langle 1\rangle 2.$ $2 \Rightarrow 3$
  PROOF: Given a surjection $f : \mathbb{N} \twoheadrightarrow B$, define $g : B \rightarrowtail \mathbb{N}$ by $g(b)$ is the smallest number such that $f(g(b)) = b$.
$\langle 1\rangle 3.$ $3 \Rightarrow 1$
  $\langle 2\rangle 1.$ LET: $f : B \rightarrowtail \mathbb{N}$ be injective.
  $\langle 2\rangle 2.$ $f(B)$ is countable.
  $\langle 2\rangle 3.$ $B \approx f(B)$
  $\langle 2\rangle 4.$ B is countable.
$\square$

Corollary 2.11.5.1. *A subset of a countable set is countable.*

Corollary 2.11.5.2. $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

PROOF: The function that maps (m, n) to $2^m 3^n$ is injective. $\square$

Corollary 2.11.5.3. *The Cartesian product of two countable sets is countable.*

Theorem 2.11.6. *A countable union of countable sets is countable.*

PROOF:
$\langle 1\rangle 1.$ LET: A be a set.
$\langle 1\rangle 2.$ LET: $\mathcal{B} \subseteq \mathcal{P}A$ be a countable set of countable sets such that $\bigcup \mathcal{B} = A$
$\langle 1\rangle 3.$ PICK a surjection $B : \mathbb{N} \twoheadrightarrow \mathcal{B}$
$\langle 1\rangle 4.$ ASSUME: w.l.o.g. each $B(n)$ is nonempty.
$\langle 1\rangle 5.$ For $n \in \mathbb{N}$, PICK a surjective function $g_n : \mathbb{N} \twoheadrightarrow B(n)$
$\langle 1\rangle 6.$ LET: $h : \mathbb{N} \times \mathbb{N} \to A$ be the function $h(m, n) = g_m(n)$
$\langle 1\rangle 7.$ h is surjective.
$\square$

Theorem 2.11.7. $2^{\mathbb{N}}$ *is uncountable.*

PROOF:
$\langle 1\rangle 1.$ LET: $f : \mathbb{N} \to 2^{\mathbb{N}}$
    PROVE: f is not surjective.
$\langle 1\rangle 2.$ Define $g : \mathbb{N} \to 2$ by $g(n) = 1 - f(n)(n)$.
$\langle 1\rangle 3.$ For all $n \in \mathbb{N}$ we have $g(n) \neq f(n)(n)$.
$\langle 1\rangle 4.$ For all $n \in \mathbb{N}$ we have $g \neq f(n)$.
$\square$

Theorem 2.11.8. *For any set A, there is no surjective function $A \to \mathcal{P}A$.*

PROOF:
$\langle 1\rangle 1$. LET: $f : A \to \mathcal{P}A$
$\langle 1\rangle 2$. LET: $S = \{x \in A : x \notin f(x)\}$
$\langle 1\rangle 3$. For all $a \in A$ we have $S \neq f(a)$
PROOF: We have $a \in S$ if and only if $a \notin f(a)$.
□

Corollary 2.11.8.1. *For any set A, there is no injective function $\mathcal{P}A \to A$.*

Chapter 3

Order Theory

3.1 Relations

Definition 3.1.1 (Reflexive)**.** A relation $R \subseteq A \times A$ is *reflexive* iff, for all $a \in A$, we have $(a, a) \in R$.

Definition 3.1.2 (Antisymmetric)**.** A relation $R \subseteq A \times A$ is *antisymmetric* iff, for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Definition 3.1.3 (Transitive)**.** A relation $R \subseteq A \times A$ is *transitive* iff, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Definition 3.1.4 (Partial Order)**.** A *partial order* on a set A is a relation on A that is reflexive, antisymmetric and transitive.

We say $(A, \leqslant)$ is a *partially ordered set* or *poset* iff $\leqslant$ is a partial order on A.

Definition 3.1.5 (Greatest)**.** Let A be a poset and $a \in A$. Then a is the *greatest* element iff $\forall x \in A. x \leqslant a$.

Definition 3.1.6 (Least)**.** Let A be a poset and $a \in A$. Then a is the *least* element iff $\forall x \in A. a \leqslant x$.

Definition 3.1.7 (Upper Bound)**.** Let A be a poset, $S \subseteq A$, and $u \in A$. Then u is an *upper bound* for S iff $\forall x \in S. x \leqslant u$. We say S is *bounded above* iff it has an upper bound.

Definition 3.1.8 (Lower Bound)**.** Let A be a poset, $S \subseteq A$, and $l \in A$. Then l is a *lower bound* for S iff $\forall x \in S. l \leqslant x$. We say S is *bounded below* iff it has a lower bound.

Definition 3.1.9 (Supremum)**.** Let A be a poset, $S \subseteq A$ and $s \in A$. Then s is the *supremum* or *least upper bound* for S iff s is the least element in the sub-poset of upper bounds for A.

Definition 3.1.10 (Supremum)**.** Let A be a poset, $S \subseteq A$ and $i \in A$. Then i is the *infimum* or *greatest lower bound* for S iff i is the greatest element in the sub-poset of lower bounds for A.

Definition 3.1.11 (Least Upper Bound Property)**.** A poset A has the *least upper bound property* iff every nonempty subset of A that is bounded above has a least upper bound.

Proposition 3.1.12. *Let A be a poset. Then A has the least upper bound property if and only if every nonempty subset of A that is bounded below has a greatest lower bound.*

PROOF:
$\langle 1 \rangle 1$. If A has the least upper bound property then every subset of A that is bounded below has a greatest lower bound.
  $\langle 2 \rangle 1$. ASSUME: A has the least upper bound property.
  $\langle 2 \rangle 2$. LET: $S \subseteq A$ be nonempty and bounded below.
  $\langle 2 \rangle 3$. LET: L be the set of lower bounds of S.
  $\langle 2 \rangle 4$. L is nonempty.
    PROOF: Because S is bounded below.
  $\langle 2 \rangle 5$. L is bounded above.
    PROOF: Pick an element $s \in S$. Then s is an upper bound for L.
  $\langle 2 \rangle 6$. LET: s be the supremum of L.
  $\langle 2 \rangle 7$. s is the greatest lower bound of S.
    $\langle 3 \rangle 1$. s is a lower bound of S.
      $\langle 4 \rangle 1$. LET: $x \in S$
      $\langle 4 \rangle 2$. x is an upper bound for L.
      $\langle 4 \rangle 3$. $s \leqslant x$
    $\langle 3 \rangle 2$. For any lower bound l of S we have $l \leqslant s$.
      PROOF: Immediate from $\langle 2 \rangle 6$.
$\langle 1 \rangle 2$. If every subset of A that is bounded below has a greatest lower bound, then A has the least upper bound property.
  PROOF: Dual.
□

3.1.1 Strict Partial Orders

Definition 3.1.13 (Strict Partial Order)**.** A *strict partial order* on a set A is a relation on A that is irreflexive and transitive.

Proposition 3.1.14.
1. *If $\leqslant$ is a partial order on A then $<$ is a strict partial order on A, where $x < y$ iff $x \leqslant y \wedge x \neq y$.*
2. *If $<$ is a strict partial order on A then $\leqslant$ is a partial order on A, where $x \leqslant y$ iff $x < y \vee x = y$.*
3. *These two relations are inverses of one another.*

3.1.2 Linear Orders

Definition 3.1.15 (Linear Order)**.** A *linear order* on a set A is a partial order $\leqslant$ on A such that, for all $x, y \in A$, we have $x \leqslant y$ or $y \leqslant x$.

A *linearly ordered set* is a pair $(X, \leqslant)$ such that X is a set and $\leqslant$ is a linear order on X.

Definition 3.1.16 (Open Interval)**.** Let X be a linearly ordered set and $a, b \in X$. The *open interval* (a, b) is the set

$$\{x \in X : a < x < b\} \ .$$

Definition 3.1.17 (Immediate Predecessor, Immediate Successor)**.** Let X be a linearly ordered set and $a, b \in X$. Then b is the *(immediate) successor* of a, and a is the *(immediate) predecessor* of b, iff $a < B$ and there is no x such that $a < x < b$.

Definition 3.1.18 (Dictionary Order)**.** Let A and B be linearly ordered sets. The *dictionary order* on $A \times B$ is the order defined by

$$(a, b) < (a', b') \Leftrightarrow a < a' \vee (a = a' \wedge b < b') \ .$$

Theorem 3.1.19 (Maximum Principle)**.** *Every poset has a maximal linearly ordered subset.*

Proof:
⟨1⟩1. Let: $(A, \leqslant)$ be a poset.
⟨1⟩2. Pick a well ordering $\preccurlyeq$ of A.
  Proof: Well Ordering Theorem.
⟨1⟩3. Let: $h : A \to 2$ be the function defined by $\preccurlyeq$-recursion thus:

$$h(a) = \begin{cases} 1 & \text{if } a \text{ is } \leqslant\text{-comparable with every } b \prec a \text{ such that } h(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

⟨1⟩4. Let: $B = \{x \in A : h(x) = 1\}$
  Prove: B is a maximal subset linearly ordered by $\leqslant$.
⟨1⟩5. B is linearly ordered by $\leqslant$.
  ⟨2⟩1. Let: $x, y \in B$
  ⟨2⟩2. Assume: w.l.o.g. $x \preccurlyeq y$
  ⟨2⟩3. y is $\leqslant$-comparable with x
⟨1⟩6. For any subset $C \subseteq A$ linearly ordered by $\leqslant$, if $B \subseteq C$ then $B = C$.
  ⟨2⟩1. Let: $x \in C$
  ⟨2⟩2. x is comparable with every $y \preccurlyeq x$ such that $h(x) = 1$
  ⟨2⟩3. $x \in B$
□

Theorem 3.1.20 (Zorn's Lemma)**.** *Let A be a poset. If every linearly ordered subset of A is bounded above, then A has a maximal element.*

Proof:

⟨1⟩1. Pick a maximal linearly ordered subset B of A.
Proof: Maximal Principle
⟨1⟩2. Pick an upper bound c for B.
Prove: c is maximal.
⟨1⟩3. Let: $x \in A$
⟨1⟩4. Assume: $c \leqslant x$
Prove: $x = c$
⟨1⟩5. x is an upper bound for B.
⟨1⟩6. $x \in B$
Proof: By the maximality of B, since $B \cup \{x\}$ is linearly ordered.
⟨1⟩7. $x \leqslant c$
Proof: ⟨1⟩2
⟨1⟩8. $x = c$
□

Corollary 3.1.20.1 (Kuratowski's Lemma)**.** *Let $\mathcal{A} \subseteq \mathcal{P}X$. Suppose that, for every subset $\mathcal{B} \subseteq \mathcal{A}$ that is linearly ordered by inclusion, we have $\bigcup \mathcal{B} \in \mathcal{A}$. Then $\mathcal{A}$ has a maximal element.*

Definition 3.1.21 (Closed Interval)**.** Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *closed interval* $[a, b]$ is

$$[a, b] := \{x \in X : a \leqslant x \leqslant b\} \ .$$

Definition 3.1.22 (Half-Open Interval)**.** Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *half-open intervals* $(a, b]$ and $[a, b)$ are defined by

$$(a, b] := \{x \in X : a < x \leqslant b\}$$
$$[a, b) := \{x \in X : a \leqslant x < b\}$$

Definition 3.1.23 (Open Ray)**.** Let X be a linearly ordered set and $a \in X$. The *open rays* $(a, +\infty)$ and $(-\infty, a)$ are defined by:

$$(a, +\infty) := \{x \in X : a < x\}$$
$$(-\infty, a) := \{x \in X : x < a\}$$

Definition 3.1.24 (Closed Ray)**.** Let X be a linearly ordered set and $a \in X$. The *closed rays* $[a, +\infty)$ and $(-\infty, a]$ are defined by:

$$[a, +\infty) := \{x \in X : a \leqslant x\}$$
$$(-\infty, a] := \{x \in X : x \leqslant a\}$$

3.1.3 Sets of Finite Type

Definition 3.1.25 (Finite Type)**.** Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ is of *finite type* if and only if, for any $B \subseteq X$, we have $B \in \mathcal{A}$ if and only if every finite subset of B is in $\mathcal{A}$.

Proposition 3.1.26 (Tukey's Lemma). *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$. If $\mathcal{A}$ is of finite type, then $\mathcal{A}$ has a maximal element.*

PROOF:
$\langle 1 \rangle 1$. For every subset $\mathcal{B} \subseteq \mathcal{A}$ that is linearly ordered by inclusion, we have $\bigcup \mathcal{B} \in \mathcal{A}$.
  $\langle 2 \rangle 1$. LET: $\mathcal{B} \subseteq \mathcal{A}$
  $\langle 2 \rangle 2$. ASSUME: $\mathcal{B}$ is linearly ordered by inclusion.
  $\langle 2 \rangle 3$. Every finite subset of $\bigcup \mathcal{B}$ is in $\mathcal{A}$
  $\langle 2 \rangle 4$. $\bigcup \mathcal{B} \in \mathcal{A}$
$\langle 1 \rangle 2$. Q.E.D.
  PROOF: Kuratowski's Lemma.
□

3.2 Well Orders

Definition 3.2.1 (Well Ordered Set). A *well ordered set* is a linearly ordered set such that every nonempty subset has a least element.

Proposition 3.2.2. *Any subset of a well ordered set is well ordered.*

Proposition 3.2.3. *The product of two well ordered sets is well ordered under the dictionary order.*

Theorem 3.2.4 (Well Ordering Theorem). *Every set has a well ordering.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a set.
$\langle 1 \rangle 2$. PICK a choice function $c : \mathcal{P}X - \{\emptyset\} \to X$
$\langle 1 \rangle 3$. Define a *tower* to be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well ordering of T, and

$$\forall x \in T. x = c(X - \{y \in T : y < x\}) \enspace .$$

$\langle 1 \rangle 4$. Given two towers, either they are equal or one is a section of the other.
  $\langle 2 \rangle 1$. LET: $(T_1, <_1)$ and $(T_2, <_2)$ be towers.
  $\langle 2 \rangle 2$. ASSUME: w.l.o.g. there exists a strictly monotone function $h : T_1 \to T_2$
  $\langle 2 \rangle 3$. $h(T_1)$ is either T_2 or a section of T_2
    PROOF: Proposition 3.2.11.
  $\langle 2 \rangle 4$. $\forall x \in T_1 . h(x) = x$
    $\langle 3 \rangle 1$. LET: $x \in T_1$
    $\langle 3 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall y < x. h(y) = y$
    $\langle 3 \rangle 3$. $h(x)$ is the least element of $T_2 - \{h(y) \in T_1 : y < x\}$
    $\langle 3 \rangle 4$. $h(x)$ is the least element of $T_2 - \{y \in T_1 : y < x\}$
      PROOF: $\langle 3 \rangle 2$
    $\langle 3 \rangle 5$. $h(x) = x$

PROOF:

$$\begin{aligned} h(x) &= c(X - \{y \in T_2 : y < h(x)\}) && (\langle 1\rangle 3) \\ &= c(X - \{y \in T_2 : y < x\}) && (\langle 3\rangle 4) \\ &= c(X - \{y \in T_1 : y < x\}) && (\langle 3\rangle 2) \\ &= x && (\langle 1\rangle 3) \end{aligned}$$

$\langle 1\rangle 5.$ If $(T, <)$ is a tower and $T \neq X$, then there exists a tower of which $(T, <)$ is a section.

PROOF: Let $T_1 = T \cup \{c(T)\}$ and $<_1$ be the extension of $<$ such that $x < c(T)$ for all $x \in T$.

$\langle 1\rangle 6.$ LET: $\mathbf{T} = \bigcup\{T : \exists R.(T, R) \text{ is a tower}\}$ and $\mathbf{R} = \bigcup\{R : \exists T.(T, R) \text{ is a tower}\}$

$\langle 1\rangle 7.$ $(\mathbf{T}, \mathbf{R})$ is a tower.

$\langle 2\rangle 1.$ $\mathbf{R}$ is irreflexive.

PROOF: Since for every tower $(T, <)$ we have $<$ is irreflexive.

$\langle 2\rangle 2.$ $\mathbf{R}$ is transitive.

$\langle 3\rangle 1.$ ASSUME: $x\mathbf{R}y$ and $y\mathbf{R}z$

$\langle 3\rangle 2.$ PICK towers $(T_1, <_1)$ and $(T_2, <_2)$ such that $x <_1 y$ and $y <_2 z$

$\langle 3\rangle 3.$ ASSUME: w.l.o.g. $(T_1, <_1)$ is either $(T_2, <_2)$ or a section of $(T_2, <_2)$

$\langle 3\rangle 4.$ $x <_2 y <_2 z$

$\langle 3\rangle 5.$ $x <_2 z$

$\langle 3\rangle 6.$ $x\mathbf{R}z$

$\langle 2\rangle 3.$ For all $x, y \in \mathbf{T}$, either $x\mathbf{R}y$ or $x = y$ or $y\mathbf{R}x$

PROOF: There exists a tower that has both x and y.

$\langle 2\rangle 4.$ Every nonempty subset of $\mathbf{T}$ has an $\mathbf{R}$-least element.

$\langle 3\rangle 1.$ LET: $A \subseteq \mathbf{T}$ be nonempty.

$\langle 3\rangle 2.$ PICK $a \in A$

$\langle 3\rangle 3.$ PICK a tower $(T, <)$ such that $a \in T$.

$\langle 3\rangle 4.$ LET: b be the $<$-least element of $A \cap T$

PROVE: b is $\mathbf{R}$-least in A.

$\langle 3\rangle 5.$ LET: $x \in A$

$\langle 3\rangle 6.$ Etc.

$\langle 2\rangle 5.$ $\forall x \in \mathbf{T}.x = c(X - \{y \in \mathbf{T} : y\mathbf{R}x\})$

$\langle 1\rangle 8.$ $\mathbf{T} = X$

$\langle 1\rangle 9.$ $\mathbf{R}$ is a well ordering of X.

□

Proposition 3.2.5. *There exists a well-ordered set with a largest element Ω such that $(-\infty, \Omega)$ is uncountable but, for all $\alpha < \Omega$, we have $(-\infty, \alpha)$ is countable.*

PROOF:

$\langle 1\rangle 1.$ PICK an uncountable well ordered set B.

$\langle 1\rangle 2.$ LET: $C = 2 \times B$ under the dictionary order.

$\langle 1\rangle 3.$ LET: Ω be the least element of C such that $(-\infty, \Omega)$ is uncountable.

$\langle 1\rangle 4.$ LET: $A = (-\infty, \Omega]$

$\langle 1\rangle 5.$ A is a well ordered set with largest element Ω such that $(-\infty, \Omega)$ is uncountable but, for all $\alpha < \Omega$, we have $(-\infty, \alpha)$ is countable.

□

Proposition 3.2.6. *Every well ordered set has the least upper bound property.*

PROOF: For any subset that is bounded above, the set of upper bounds is nonempty, hence has a least element. □

Proposition 3.2.7. *In a well ordered set, every element that is not greatest has a successor.*

PROOF: If a is not greatest, then $\{x : x > a\}$ is nonempty, hence has a least element. □

Theorem 3.2.8 (Transfinite Induction)**.** *Let J be a well ordered set. Let $S \subseteq J$. Assume that, for every $\alpha \in J$, if $\forall x < \alpha . x \in S$ then $\alpha in S$. Then $S = J$.*

PROOF: Otherwise $J - S$ would be a nonempty subset of J with no least element. □

Proposition 3.2.9. *Let I be a well ordered set. Let $\{A_i\}_{i \in I}$ be a family of well ordered sets. Define $<$ on $\coprod_{i \in I} A_i$ by: $\kappa_i(a) < \kappa_j(b)$ iff either $i < j$, or $i = j$ and $a < b$ in A_i. Then $<$ well orders $\coprod_{i \in I} A_i$.*

PROOF: Easy. □

Theorem 3.2.10 (Principle of Transfinite Recursion)**.** *Let J be a well ordered set. Let C be a set. Let $\mathcal{F}$ be the set of all functions from a section of J into C. Let $\rho : \mathcal{F} \to C$. Then there exists a unique function $h : J \to C$ such that, for all $\alpha \in J$, we have*

$$h(\alpha) = \rho(h \restriction (-\infty, \alpha)) \ .$$

PROOF:

⟨1⟩1. For a function h mapping either a section of J or all of J into C, let us say h is *acceptable* iff, for all $x \in \operatorname{dom} h$, we have $(-\infty, x) \subseteq \operatorname{dom} h$ and $h(x) = \rho(h \restriction (-\infty, x))$.

⟨1⟩2. If h and k are acceptable functions then $h(x) = k(x)$ for all x in both domains.

⟨2⟩1. LET: $x \in J$

⟨2⟩2. ASSUME: as transfinite induction hypothesis that, for all $y < x$ and any acceptable functions h and k with $y \in \operatorname{dom} h \cap \operatorname{dom} k$, we have $h(y) = k(y)$

⟨2⟩3. LET: h and k be acceptable functions with $x \in \operatorname{dom} h \cap \operatorname{dom} k$

⟨2⟩4. $h \restriction (-\infty, x) = k \restriction (-\infty, x)$

PROOF: By ⟨2⟩2.

⟨2⟩5. $h(x) = k(x)$

PROOF: By ⟨2⟩3, each is the least element of the set in ⟨2⟩4.

⟨1⟩3. For $\alpha \in J$, if there exists an acceptable function $(-\infty, \alpha) \to C$, then there exists an acceptable function $(-\infty, \alpha] \to C$.

⟨2⟩1. LET: $\alpha \in J$

⟨2⟩2. LET: $f : (-\infty, \alpha) \to C$ be acceptable.
⟨2⟩3. LET: $g : (-\infty, \alpha] \to C$ be the function given by

$$g(x) = \begin{cases} f(x) & \text{if } x < \alpha \\ \rho(f) & \text{if } x = \alpha \end{cases}$$

⟨2⟩4. g is acceptable.
⟨1⟩4. Let $K \subseteq J$. Assume that, for all $\alpha \in K$, there exists an acceptable function $(-\infty, \alpha) \to C$. Then there exists an acceptable function $\bigcup_{\alpha \in K}(-\infty, \alpha) \to C$.
⟨2⟩1. Define $f : \bigcup_{\alpha \in K}(-\infty, \alpha) \to C$ by: $f(x) = y$ iff there exists $\alpha \in K$ and $g : (-\infty, \alpha) \to C$ acceptable such that $g(x) = y$.
⟨1⟩5. For every $\beta \in J$, there exists an acceptable function $(-\infty, \beta) \to C$
⟨2⟩1. LET: $\beta \in J$
⟨2⟩2. ASSUME: as transfinite induction hypothesis that, for all $\alpha < \beta$, there exists an acceptable function $(-\infty, \alpha) \to C$
⟨2⟩3. CASE: β has a predecessor
⟨3⟩1. LET: α be the predecessor of β.
⟨3⟩2. There exists an acceptable function $(-\infty, \alpha) \to C$.
⟨3⟩3. There exists an acceptable function $(-\infty, \beta) \to C$.
PROOF: By ⟨1⟩3 since $(-\infty, \beta) = (-\infty, \alpha]$.
⟨2⟩4. CASE: β has no predecessor.
PROOF: The result follows by ⟨1⟩4 since $(-\infty, \beta) = \bigcup_{\alpha < \beta}(-\infty, \alpha)$.
⟨1⟩6. There exists an acceptable function $J \to C$.
⟨2⟩1. CASE: J has a greatest element.
⟨3⟩1. LET: g be greatest.
⟨3⟩2. There exists an acceptable function $(-\infty, g) \to C$.
PROOF: ⟨1⟩5
⟨3⟩3. There exists an acceptable function $J \to C$.
PROOF: By ⟨1⟩3 since $J = (-\infty, g]$.
⟨2⟩2. CASE: J has no greatest element.
PROOF: By ⟨1⟩4 since $J = \bigcup_{\alpha \in J}(-\infty, \alpha)$.
□

Corollary 3.2.10.1 (Cardinal Comparability)**.** *Let A and B be sets. Then either $A \preccurlyeq B$ or $B \preccurlyeq A$.*

PROOF: Choose well orderings of A and B. Then either there exists a surjection $A \twoheadrightarrow B$, or there exists an injective function $h : A \rightarrowtail B$ defined by transfinite recursion by $h(x)$ is the least element of $B - h((-\infty, x))$. □

Proposition 3.2.11. *Let J and E be well ordered sets. Let $h : J \to E$. Then the following are equivalent.*

1. *h is strictly monotone and $h(J)$ is either E or a section of E.*
2. *For all $\alpha \in J$, we have $h(\alpha)$ is the least element of $E - h((-\infty, \alpha))$.*

PROOF:

$\langle 1\rangle 1.$ $1 \Rightarrow 2$
$\langle 2\rangle 1.$ Assume: 1
$\langle 2\rangle 2.$ $h(J)$ is closed downwards.
$\langle 2\rangle 3.$ Let: $\alpha \in J$
$\langle 2\rangle 4.$ $h(\alpha) \in E - h((-\infty, \alpha))$
Proof: If $\beta < \alpha$ then $h(\beta) < h(\alpha)$.
$\langle 2\rangle 5.$ For all $y \in E - h((-\infty, \alpha))$ we have $h(\alpha) \leqslant y$
$\langle 3\rangle 1.$ Assume: for a contradiction $y < h(\alpha)$
$\langle 3\rangle 2.$ $y \in h(J)$
$\langle 3\rangle 3.$ Pick $\beta \in J$ such that $h(\beta) = y$
$\langle 3\rangle 4.$ $h(\beta) < h(\alpha)$
$\langle 3\rangle 5.$ $\beta < \alpha$
$\langle 3\rangle 6.$ Q.E.D.
Proof: This contradicts the fact that $y \notin h((-\infty, \alpha))$.
$\langle 1\rangle 2.$ $2 \Rightarrow 1$
$\langle 2\rangle 1.$ Assume: 2
$\langle 2\rangle 2.$ h is strictly monotone.
$\langle 3\rangle 1.$ Let: $\alpha, \beta \in J$ with $\alpha < \beta$
$\langle 3\rangle 2.$ $h(\alpha) \neq h(\beta)$
Proof: Because $h(\beta) \in E - h((-\infty, \beta))$.
$\langle 3\rangle 3.$ $h(\alpha) \leqslant h(\beta)$
Proof:Because $h(\alpha)$ is least in $E - h((-\infty, \alpha))$.
$\langle 3\rangle 4.$ $h(\alpha) < h(\beta)$
$\langle 2\rangle 3.$ $h(J)$ is either E or a section of E.
$\langle 3\rangle 1.$ Assume: $h(J) \neq E$
$\langle 3\rangle 2.$ Let: e be least in $E - h(J)$
Prove: $h(J) = (-\infty, e)$
$\langle 3\rangle 3.$ $h(J) \subseteq (-\infty, e)$
$\langle 4\rangle 1.$ Let: $\alpha \in J$
$\langle 4\rangle 2.$ $h(\alpha) \neq e$
Proof: $e \notin h(J)$
$\langle 4\rangle 3.$ $h(\alpha) \leqslant e$
Proof: Since $h(\alpha)$ is least in $E - h((-\infty, \alpha))$.
$\langle 4\rangle 4.$ $h(\alpha) < e$
$\langle 3\rangle 4.$ $(-\infty, e) \subseteq h(J)$
Proof: If $e' < e$ then $e' \in h(J)$ by leastness of e.

$\square$

Chapter 4

Category Theory

4.1 Categories

Definition 4.1.1. A *category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \mathrm{Ob}(\mathcal{C})$.
- for any objects X and Y, a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y. We write $f : X \to Y$ for $f \in \mathcal{C}[X, Y]$.
- for any objects X, Y and Z, a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \to \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X, there exists a morphism $\mathrm{id}_X : X \to X$, the *identity morphism* on X, such that:
  - for any object Y and morphism $f : Y \to X$ we have $\mathrm{id}_X \circ f = f$
  - for any object Y and morphism $f : X \to Y$ we have $f \circ \mathrm{id}_X = f$

We write the composite of morphism $f_1, \ldots, f_n$ as $f_n \circ \cdots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 4.1.2. Let **Set** be the category of small sets and functions.

Definition 4.1.3. Let **LPos** be the category of linearly ordered sets and monotone functions.

Proposition 4.1.4. *Any finite linearly ordered set is isomorphic to* $\{m \in \mathbb{N} : m < n\}$ *for some* n*.*

PROOF:
$\langle 1 \rangle 1$. Every finite nonempty linearly ordered set has a greatest element.

⟨2⟩1. Let: $P[n]$ be the property: for any linearly ordered set A, if there exists a bijection $A \approx \{m \in \mathbb{N} : m < n\}$ and A is nonempty then A has a greatest element.
⟨2⟩2. $P[0]$
Proof: Vacuous.
⟨2⟩3. $\forall n \in \mathbb{N}.P[n] \Rightarrow P[n+1]$
⟨3⟩1. Let: $n \in \mathbb{N}$
⟨3⟩2. Assume: $P[n]$
⟨3⟩3. Let: A be a nonempty linearly ordered set.
⟨3⟩4. Let: $f : A \approx \{m \in \mathbb{N} : m < n+1\}$
⟨3⟩5. Let: $a = f^{-1}(n)$
⟨3⟩6. $f \restriction (A - \{a\}) : A - \{a\} \approx \{m \in \mathbb{N} : m < n\}$
⟨3⟩7. Assume: w.l.o.g. a is not greatest in A.
⟨3⟩8. Let: b be greatest in $A - \{a\}$
Proof: ⟨3⟩2
⟨3⟩9. b is greatest in A.
⟨1⟩2. Let: $P[n]$ be the property: for any linearly ordered set A, if there exists a bijection $A \approx \{m \in \mathbb{N} : m < n\}$ then there exists an isomorphism in **LPos** $A \cong \{m \in \mathbb{N} : m < n\}$.
⟨1⟩3. $P[0]$
Proof: If there exists a bijection $A \approx \emptyset$ then A is empty and so the unique function $A \to \emptyset$ is an order isomorphism.
⟨1⟩4. For every natural number n, if $P[n]$ then $P[n+1]$.
⟨2⟩1. Let: n be a natural number.
⟨2⟩2. Assume: $P[n]$
⟨2⟩3. Let: A be a linearly ordered set.
⟨2⟩4. Assume: A has $n+1$ elements.
⟨2⟩5. Let: a be the greatest element in A.
⟨2⟩6. Let: $f : A - \{a\} \cong \{m \in \mathbb{N} : m < n\}$ be an order isomorphism.
Proof: ⟨2⟩2
⟨2⟩7. Define $g : A \to \{m \in \mathbb{N} : m < n+1\}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ n & \text{if } x = a \end{cases}$$

⟨2⟩8. g is an order isomorphism.
⟨1⟩5. $\forall n \in \mathbb{N}.P[n]$
□

Corollary 4.1.4.1. *Any finite linearly ordered set is well ordered.*

Proposition 4.1.5. *Let J and E be well ordered sets. Suppose there is a strictly monotone map $J \to E$. Then J is isomorphic either to E or a section of E.*

Proof:
⟨1⟩1. Let: $k : J \to E$ be strictly monotone.
⟨1⟩2. Assume: w.l.o.g. E is nonempty.
⟨1⟩3. Pick $e_0 \in E$

⟨1⟩4. LET: $h : J \to E$ be the function defined by transfinite recursion thus:

$$h(\alpha) = \begin{cases} \text{the least element in } E - h((-\infty, \alpha)) & \text{if } h((-\infty, \alpha)) \neq E \\ e_0 & \text{if } h((-\infty, \alpha)) = E \end{cases}$$

⟨1⟩5. $\forall \alpha \in J. h(\alpha) \leqslant k(\alpha)$
  ⟨2⟩1. LET: $\alpha \in J$
  ⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha. h(\beta) \leqslant k(\beta)$.
  ⟨2⟩3. $\forall \beta < \alpha. h(\beta) < k(\alpha)$
  ⟨2⟩4. $h((-\infty, \alpha)) \neq E$
  ⟨2⟩5. $h(\alpha)$ is the least element in $E - h((-\infty, \alpha))$.
  ⟨2⟩6. $k(\alpha) \in E - h((-\infty, \alpha))$
  ⟨2⟩7. $h(\alpha) \leqslant k(\alpha)$
⟨1⟩6. $\forall \alpha \in J. h((-\infty, \alpha)) \neq E$
  PROOF: For $\beta < \alpha$ we have $h(\beta) \leqslant k(\beta) < k(\alpha)$ so $k(\alpha) \notin h((-\infty, \alpha))$.
⟨1⟩7. For all $\alpha \in J$, we have $h(\alpha)$ is the least element of $E - h((-\infty, \alpha))$.
⟨1⟩8. h is strictly monotone and $h(J)$ is either E or a section of E.
  PROOF: Proposition 3.2.11.
□

Proposition 4.1.6. *If A and B are well ordered sets, then exactly one of the following conditions hold: $A \cong B$, or A is isomorphic to a section of B, or B is isomorphic to a section of A.*

PROOF:
⟨1⟩1. At least one of the conditions holds.
  ⟨2⟩1. B is isomorphic to either $A + B$ or a section of $A + B$.
  ⟨2⟩2. CASE: $B \cong A + B$
    ⟨3⟩1. LET: ϕ be the isomorphism $B \cong A + B$
    ⟨3⟩2. LET: b_0 be the least element in B.
    ⟨3⟩3. A is isomorphic to the section $(-\infty, \phi^{-1}(\kappa_2(b_0)))$ of B.
  ⟨2⟩3. CASE: $a \in A$ and $B \cong (-\infty, \kappa_1(a))$
    PROOF: Then B is isomorphic to the section $(-\infty, a)$ of A.
  ⟨2⟩4. CASE: $b \in B$ and $\phi : B \cong (-\infty, \kappa_2(b))$
    ⟨3⟩1. CASE: b is least in B.
      PROOF: Then $A \cong B$.
    ⟨3⟩2. CASE: b is not least in B.
      ⟨4⟩1. LET: b_0 be least in B.
      ⟨4⟩2. A is isomorphic to the section $(-\infty, \phi^{-1}(\kappa_2(b_0)))$ of B.
⟨1⟩2. At most one of the conditions holds.
  PROOF: Since a well ordered set cannot be isomorphic to a section of itself.
□

Theorem 4.1.7. *There exists a well ordered set, unique up to order isomorphism, that is uncountable but such that every section is countable.*

PROOF:
⟨1⟩1. There exists a well ordered set that is uncountable but such that every section is countable.

⟨2⟩1. PICK a well ordered set A with an element $\Omega \in A$ such that $(-\infty, \Omega)$ is uncountable but $\forall \alpha < \Omega.(-\infty, \alpha)$ is countable.
⟨2⟩2. LET: $(-\infty, Omega)$ is uncountable but every section is countable.
⟨1⟩2. If A and B are uncountable well ordered sets such that every section is countable, then $A \cong B$.
PROOF: Since it cannot be that one of A and B is isomorphic to a section of the other.
□

Definition 4.1.8 (Minimal Uncountable Well Ordered Set)**.** The *minimal uncountable well ordered set* Ω is the well ordered set that is uncountable but such that every section is countable.

We write $\overline{\Omega}$ for the well ordered set $\Omega \cup \{\Omega\}$ where Ω is greatest.

Proposition 4.1.9. *Every countable subset of* Ω *is bounded above.*

PROOF:
⟨1⟩1. LET: A be a countable subset of Ω.
⟨1⟩2. For all $a \in A$ we have $(-\infty, a)$ is countable.
⟨1⟩3. $\bigcup_{a \in A}(-\infty, a)$ is countable.
⟨1⟩4. $\bigcup_{a \in A}(-\infty, a) \neq \Omega$
⟨1⟩5. PICK $x \in \Omega - \bigcup_{a \in A}(-\infty, a)$
⟨1⟩6. x is an upper bound for A.
□

Proposition 4.1.10. Ω *has no greatest element.*

PROOF: For any $\alpha \in \Omega$ we have $(-\infty, \alpha]$ is countable and hence not the whole of Ω. □

Proposition 4.1.11. *There are uncountably many elements of* Ω *that have no predecessor.*

PROOF:
⟨1⟩1. LET: A be the set of all elements of Ω that have no predecessor.
⟨1⟩2. LET: $f : A \times \mathbb{N} \to \Omega$ be the function that maps (a, n) to the nth successor of a.
⟨1⟩3. f is surjective.
⟨2⟩1. ASSUME: for a contradiction $x \in \Omega$ and there is no element $a \in A$ and $n \in \mathbb{N}$ such that x is the nth successor of a.
⟨2⟩2. LET: x_n be the nth predecessor of x for $n \in \mathbb{N}$.
⟨2⟩3. $\{x_n : n \in \mathbb{N}\}$ is a nonempty subset of Ω with no least element.
⟨1⟩4. $A \times \mathbb{N}$ is uncountable.
⟨1⟩5. A is uncountable.
□

Definition 4.1.12. We identify a poset $(A, \leqslant)$ with the category with:

- set of objects A

- for $a, b \in A$, the set of homomorphisms is $\{x \in 1 : a \leqslant b\}$

Proposition 4.1.13. *A category is a poset iff, for any two objects, there exists at most one morphism between them.*

Proposition 4.1.14. *The identity morphism on an object is unique.*

PROOF:
⟨1⟩1. LET: $\mathcal{C}$ be a category.
⟨1⟩2. LET: $A \in \mathcal{C}$
⟨1⟩3. LET: $i, j : A \to A$ be identity morphisms on A.
⟨1⟩4. $i = j$
PROOF:

$$\begin{aligned} i &= i \circ j && (j \text{ is an identity on } A) \\ &= j && (i \text{ is an identity on } A) \end{aligned}$$

□

Proposition 4.1.15. *Let A be a linearly ordered set. Then A is well ordered if and only if it does not contain a subset of order type $\mathbb{N}^{\mathrm{op}}$.*

PROOF:
⟨1⟩1. If A is well ordered then it does not contain a subset of order type $\mathbb{N}^{\mathrm{op}}$.
PROOF: A subset of order type $\mathbb{N}^{\mathrm{op}}$ would be a subset with no least element.
⟨1⟩2. If A is not well ordered then it contains a subset of order type $\mathbb{N}^{\mathrm{op}}$.
⟨2⟩1. ASSUME: A is not well ordered.
⟨2⟩2. PICK a nonempty subset S with no least element.
⟨2⟩3. PICK $a_0 \in S$
⟨2⟩4. Extend to a sequence (a_n) in S such that $a_{n+1} < a_n$ for all n.
⟨2⟩5. $\{a_n : n \in \mathbb{N}\}$ has order type $\mathbb{N}^{\mathrm{op}}$.

□

Corollary 4.1.15.1. *Let A be a linearly ordered set. If every countable subset of A is well ordered, then A is well ordered.*

Definition 4.1.16. Given $f : A \to B$ and an object C, define the function $f^* : \mathcal{C}[B, C] \to \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 4.1.17. Given $f : A \to B$ and an object C, define the function $f_* : \mathcal{C}[C, A] \to \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

4.1.1 Monomorphisms

Definition 4.1.18 (Monomorphism)**.** Let $f : A \to B$. Then f is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for any object X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

4.1.2 Epimorphisms

Definition 4.1.19 (Epimorphism)**.** Let $f : A \to B$. Then f is *epic* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for any object X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

4.1.3 Sections and Retractions

Definition 4.1.20 (Section, Retraction)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $rs = \mathrm{id}_B$.

Proposition 4.1.21. *Let $f : A \to B$ and $r, s : B \to A$. If r is a retraction of f and s is a section of f then $r = s$.*

Proof:

$$\begin{aligned} r &= r\mathrm{id}_B && \text{(Unit Law)} \\ &= rfs && (s \text{ is a section of } f) \\ &= \mathrm{id}_A s && (r \text{ is a retraction of } f) \\ &= s && \text{(Unit Law)} \end{aligned}$$ ∎

Proposition 4.1.22. *Every section is monic.*

Proof:
⟨1⟩1. Let: $s : B \to A$ be a section of $r : A \to B$.
⟨1⟩2. Let: X be an object and $x, y : X \to B$
⟨1⟩3. Assume: $s \circ x = s \circ y$
⟨1⟩4. $x = y$
Proof: $x = r \circ s \circ x = r \circ s \circ y = y$.
∎

Proposition 4.1.23. *Every retraction is epic.*

Proof: Dual. ∎

4.1.4 Isomorphisms

Definition 4.1.24 (Isomorphism)**.** A morphism $f : A \to B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \to A$ that is both a retraction and section of f.

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 4.1.25. *The inverse of an isomorphism is unique.*

Proof: From Proposition 4.1.21. ∎

Proposition 4.1.26. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

Proof: Since $ff^{-1} = \mathrm{id}_B$ and $f^{-1}f = \mathrm{id}_A$. ∎

Isomorphism.
Define the opposite category.
Slice categories

Definition 4.1.27. Let $\mathcal{C}$ be a category and $B \in \mathcal{C}$. The category $\mathcal{C}_B^B$ of objects *over and under B* is the category with:

- objects all triples (X, u, p) such that $u : B \to X$ and $p : X \to B$
- morphisms $f : (X, u, p) \to (Y, u', p')$ all morphisms $f : X \to Y$ such that $fu = u'$ and $p'f = p$.

Proposition 4.1.28.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B)\backslash \mathrm{id}_B \cong (\mathcal{C}\backslash B)/\mathrm{id}_B$$

$(B, \mathrm{id}_B, \mathrm{id}_B)$ is the zero object in $\mathcal{C}_B^B$.

4.1.5 Initial Objects

Definition 4.1.29 (Initial Object)**.** An object I is *initial* iff, for any object X, there exists exactly one morphism $I \to X$.

Proposition 4.1.30. *The empty set is initial in* **Set**.

PROOF: For any set A, the nowhere-defined function is the unique function $\emptyset \to A$. □

Proposition 4.1.31. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:
$\langle 1 \rangle 1$. LET: $i : I \to I'$ be the unique morphism $I \to I'$.
$\langle 1 \rangle 2$. LET: $i^{-1} : I' \to I$ be the unique morphism $I' \to I$.
$\langle 1 \rangle 3$. $ii^{-1} = \mathrm{id}_{I'}$
PROOF: There is only one morphism $I' \to I'$.
$\langle 1 \rangle 4$. $i^{-1}i = \mathrm{id}_I$
PROOF: There is only one morphism $I \to I$.
□

4.1.6 Terminal Objects

Definition 4.1.32 (Terminal Object)**.** An object T is *terminal* iff, for any object X, there exists exactly one morphism $X \to T$.

Proposition 4.1.33. *1 is terminal in* **Set**.

PROOF: For any set A, the constant function to $*$ is the only function $A \to 1$. □

Proposition 4.1.34. *If T and T' are terminal objects, then there exists a unique isomorphism $T \cong T'$.*

PROOF: Dual to Proposition 4.1.31. □

4.1.7 Zero Objects

Definition 4.1.35 (Zero Object)**.** An object Z is a *zero object* iff it is an initial object and a terminal object.

Definition 4.1.36 (Zero Morphism)**.** Let $\mathcal{C}$ be a category with a zero object Z. Let $A, B \in \mathcal{C}$. The *zero morphism* $A \to B$ is the unique morphism $A \to Z \to B$.

Proposition 4.1.37. *There is no zero object in* **Set***.*

PROOF: Since $\emptyset \not\approx 1$. □

4.1.8 Triads

Definition 4.1.38 (Triad)**.** Let $\mathcal{C}$ be a category. A *triad* consists of objects X, Y, M and morphisms $\alpha : X \to M$, $\beta : Y \to M$. We call M the *codomain* of the triad.

4.1.9 Cotriads

Definition 4.1.39 (Cotriad)**.** Let $\mathcal{C}$ be a category. A *cotriad* consists of objects X, Y, W and morphisms $\xi : W \to X$, $\eta : W \to Y$. We call W the *domain* of the triad.

4.1.10 Pullbacks

Definition 4.1.40 (Pullback)**.** A diagram

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

is a *pullback* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : Z \to X$ and $g : Z \to Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \to W$ such that $\xi h = f$ and $\eta h = g$.

In this case we also say that η is the *pullback* of β along α.

Proposition 4.1.41. *If $\xi : W \to X$ and $\eta : W \to Y$ form a pullback of $\alpha : X \to M$ and $\beta : Y \to M$, and $\xi' : W' \to X$ and $\eta' : W' \to Y$ also form the pullback of α and β, then there exists a unique isomorphism $\phi : W \cong W'$ such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.*

PROOF:
⟨1⟩1. LET: $\phi : W \to W'$ be the unique morphism such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.
⟨1⟩2. LET: $\phi^{-1} : W' \to W$ be the unique morphism such that $\eta\phi^{-1} = \eta'$ and $\xi\phi^{-1} = \xi'$.
⟨1⟩3. $\phi\phi^{-1} = \mathrm{id}_{W'}$

PROOF: Each is the unique $x : W' \to W'$ such that $\eta' x = \eta'$ and $\xi' x = \xi'$.
⟨1⟩4. $\phi^{-1}\phi = \mathrm{id}_W$
PROOF: Each is the unique $x : W \to W$ such that $\eta x = \eta$ and $\xi x = \xi$.
□

Proposition 4.1.42. *For any morphism $h : A \to B$, the following diagram is a pullback diagram.*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

PROOF:
⟨1⟩1. LET: Z be an object.
⟨1⟩2. LET: $f : Z \to B$ and $g : Z \to A$ satisfy $\mathrm{id}_B f = hg$
⟨1⟩3. $g : Z \to B$ is the unique morphism such that $\mathrm{id}_A g = g$ and $hg = f$.
□

Proposition 4.1.43. *The pullback of an isomorphism is an isomorphism.*

PROOF:
⟨1⟩1. LET:

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pullback diagram.
⟨1⟩2. ASSUME: β is an isomorphism.
⟨1⟩3. LET: ξ^{-1} be the unique morphism $X \to W$ such that $\xi\xi^{-1} = \mathrm{id}_X$ and $\eta\xi^{-1} = \beta^{-1}\alpha$.
PROOF: This exists since $\alpha \mathrm{id}_X = \beta\beta^{-1}\alpha = \alpha$.
⟨1⟩4. $\xi^{-1}\xi = \mathrm{id}_W$
PROOF: Each is the unique $x : W \to W$ such that $\xi x = \xi$ and $\eta x = \eta$.
□

Proposition 4.1.44. *Let $\beta : (Y, y) \to (M, m)$ and $\alpha : (X, x) \to (M, m)$ in $\mathcal{C}\backslash A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pullback in $\mathcal{C}$. Let $w : A \to W$ be the unique morphism such that $\xi w = x$ and $\eta w = y$. Then $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ is the pullback of β and α in $\mathcal{C}\backslash A$.

Proof:
⟨1⟩1. Let: $(Z, z) \in \mathcal{C} \backslash A$
⟨1⟩2. Let: $f : (Z, z) \to (X, x)$ and $g : (Z, z) \to (Y, y)$ satisfy $\alpha f = \beta g$.
⟨1⟩3. Let: $h : Z \to W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.
⟨1⟩4. $hz = w$
⟨2⟩1. $\xi hz = \xi w$
Proof:

$$\begin{aligned} \xi hz &= fz && (\langle 1\rangle 3) \\ &= x && (\langle 1\rangle 2) \\ &= \xi w \end{aligned}$$

⟨2⟩2. $\eta hz = \eta w$
Proof: Similar.
⟨1⟩5. $h : (Z, z) \to (W, w)$
□

Proposition 4.1.45. *Let $\beta : (Y, y) \to (M, m)$ and $\alpha : (X, x) \to (M, m)$ in $\mathcal{C}/A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pullback in $\mathcal{C}$. Let $w = x\xi : W \to A$. Then $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ form a pullback of α and β in $\mathcal{C}/A$.

Proof:
⟨1⟩1. $\eta : (W, w) \to (Y, y)$
Proof:

$$\begin{aligned} y\eta &= m\beta\eta \\ &= m\alpha\xi \\ &= x\xi \\ &= w \end{aligned}$$

⟨1⟩2. Let: $(Z, z) \in \mathcal{C}/A$
⟨1⟩3. Let: $f : (Z, z) \to (X, x)$ and $g : (Z, z) \to (Y, y)$ satisfy $\alpha f = \beta g$.
⟨1⟩4. Let: $h : Z \to W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.
⟨1⟩5. $h : (Z, z) \to (W, w)$
Proof:

$$\begin{aligned} wh &= x\xi h \\ &= xf && (\langle 1\rangle 4) \\ &= z && (\langle 1\rangle 3) \end{aligned}$$

□

Proposition 4.1.46. *In **Set**, let $\alpha : X \to M$ and $\beta : Y \to M$. Let $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$ with inclusion $i : W \to X \times Y$. Let $\xi = \pi_1 i : W \to X$ and $\eta : \pi_2 i : W \to Y$. Then ξ and η form the pullback of α and β.*

PROOF:
⟨1⟩1. $\alpha\xi = \beta\eta$
PROOF: For $w \in W$, if $i(w) = (x, y)$ then then $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$.
⟨1⟩2. For every set Z and functions $f : Z \to X$, $g : Z \to Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \to W$ such that $\xi h = f$ and $\eta h = g$
PROOF: For $z \in Z$, let $h(z)$ be the unique element of W such that $i(h(z)) = (f(z), g(z))$.
□

Pullback lemma

4.1.11 Pushouts

Definition 4.1.47 (Pushout)**.** A diagram

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd} \tag{4.1}$$

is a *pushout* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : X \to Z$ and $g : Y \to Z$ such that $f\xi = g\eta$, there exists a unique $h : M \to Z$ such that $h\alpha = f$ and $h\beta = g$.

We also say that β is the *pushout* of ξ along η.

Proposition 4.1.48. *If $\alpha : X \to M$ and $\beta : Y \to M$ form a pushout of $\xi : W \to X$ and $\eta : W \to Y$, and $\alpha' : X \to M'$ and $\beta' : Y \to M'$ also form a pushout of ξ and η, then there exists a unique isomorphism $\phi : M \cong M'$ such that $\phi\alpha = \alpha'$ and $\phi\beta = \beta'$.*

PROOF: Dual to Proposition 4.1.41. □

Proposition 4.1.49. *For any morphism $h : A \to B$, the following diagram is a pushout diagram.*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

PROOF: Dual to Proposition 4.1.42.

Proposition 4.1.50. *The diagram (4.1) is a pushout in $\mathcal{C}$ iff it is a pullback in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. □

Proposition 4.1.51. *The pushout of an isomorphism is an isomorphism.*

PROOF: Dual to Proposition 4.1.43. ∎

Proposition 4.1.52. *Let $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ in $\mathcal{C}\backslash A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pushout in $\mathcal{C}$. Let $m := \alpha x : A \to M$. Then $\alpha : (X, x) \to (M, m)$ and $\beta : (Y, y) \to (M, m)$ is the pushout of ξ and η in $\mathcal{C}\backslash A$.

PROOF: Dual to Proposition 4.1.45. ∎

Proposition 4.1.53. *Let $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ in $\mathcal{C}/A$. Let*

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & M \end{tikzcd}$$

be a pushout in $\mathcal{C}$. Let $m : M \to A$ be the unique morphism such that $m\alpha = x$ and $m\beta = y$. Then $\alpha : (X, x) \to (M, m)$ and $\beta : (Y, y) \to (M, m)$ is the pushout of ξ and η in $\mathcal{C}\backslash A$.

PROOF: Dual to Proposition 4.1.44. ∎

Proposition 4.1.54. Set *has pushouts.*

PROOF:
⟨1⟩1. LET: $\xi : W \to X$ and $\eta : W \to Y$.
⟨1⟩2. LET: $\sim$ be the equivalence relation on $X + Y$ generated by $\xi(w) \sim \eta(w)$ for all $w \in W$
⟨1⟩3. LET: $M = (X + Y)/\sim$ with canonical projection $\pi : X + Y \twoheadrightarrow M$.
⟨1⟩4. LET: $\alpha = \pi \circ \kappa_1 : X \to M$
⟨1⟩5. LET: $\beta = \pi \circ \kappa_2 : Y \to M$
⟨1⟩6. LET: Z be any set, $f : X \to Z$ and $g : Y \to Z$.
⟨1⟩7. ASSUME: $f\xi = g\eta$
⟨1⟩8. LET: $h : X + Y \to Z$ be the function defined by $h(x) = f(x)$ and $h(y) = g(y)$ for $x \in X$ and $y \in Y$
⟨1⟩9. h respects $\sim$
PROOF: For $w \in W$ we have

$$\begin{aligned} h(\xi(w)) &= f(\xi(w)) && (\langle 1\rangle 8) \\ &= g(\eta(w)) && (\langle 1\rangle 7) \\ &= h(\eta(w)) && (\langle 1\rangle 8) \end{aligned}$$

⟨1⟩10. LET: $\overline{h} : M \to Z$ be the induced function.
⟨1⟩11. $\overline{h}\alpha = f$

Proof:

$$\begin{aligned}\overline{h}(\alpha(x)) &= \overline{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x)\end{aligned}$$

⟨1⟩12. $\overline{h}\beta = g$
Proof: Similar.
⟨1⟩13. For all $k : M \to Z$, if $k\alpha = f$ and $k\beta = g$ then $k = \overline{h}$.
Proof:

$$\begin{aligned}k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\ &= f(x) \\ k(\pi(\kappa_2(y))) &= k(\beta(y)) \\ &= g(y) \\ \therefore k \circ \pi &= h \\ \therefore k &= \overline{h}\end{aligned}$$

□

Definition 4.1.55. Let $u : A \rightarrowtail X$ be an injection. The *pointed set obtained from X by collapsing* (A, u), denoted $X/(A, u)$, is the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"] & 1 \arrow[d, "*"'] \\ X \arrow[r] & X/(A,u) \end{tikzcd}$$

Proposition 4.1.56. *In* $\mathbf{Set}_*$, *any two morphisms* $1 \to X$ *and* $1 \to Y$ *have a pushout.*

Proof: The pushout of $a : (1, *) \to (X, x)$ and $b : (1, *) \to (Y, y)$ is $(X+Y/\sim, x)$ where $\sim$ is the equivalence relation generated by $x \sim y$. □

Definition 4.1.57 (Wedge)**.** The *wedge* of pointed sets X and Y, $X \vee Y$, is the pushout of the unique morphism $1 \to X$ and $1 \to Y$.

Definition 4.1.58 (Smash)**.** Let X and Y be pointed sets. Let $\xi : X \vee Y \to X$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, equal, bend left] & \\ Y \arrow[r] \arrow[drr, bend right, "0"'] & X \vee Y \arrow[dr, "\xi"] & \\ & & X \end{tikzcd}$$

Let $\eta : X \vee Y \to Y$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[drr, bend left, "0"] & & \\
Y \arrow[r] \arrow[drrr, bend right, equal] & X \vee Y \arrow[drr, "\eta"] & & \\
& & & Y
\end{tikzcd}$$

Let $\zeta = \langle \xi, \eta \rangle : X \vee Y \to X \times Y$. The *smash* of X and Y, $X \wedge Y$, is the result of collapsing $X \times Y$ with respect to ζ.

Pushout lemma

4.1.12 Subcategories

Definition 4.1.59 (Subcategory)**.** A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ consists of:

- a subset $\mathrm{Ob}(\mathcal{C}')$ of $\mathcal{C}$
- for all $A, B \in \mathrm{Ob}(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \mathrm{Ob}(\mathcal{C}')$, we have $\mathrm{id}_A \in \mathcal{C}'[A, A]$
- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \mathrm{Ob}(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

4.1.13 Opposite Category

Definition 4.1.60 (Opposite Category)**.** For any category $\mathcal{C}$, the *opposite* category $\mathcal{C}^{\mathrm{op}}$ is the category with

- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$
- $\mathcal{C}^{\mathrm{op}}[A, B] = \mathcal{C}[B, A]$
- Given $f \in \mathcal{C}^{\mathrm{op}}[A, B]$ and $g \in \mathcal{C}^{\mathrm{op}}[B, C]$, their composite in $\mathcal{C}^{\mathrm{op}}$ is $f \circ g$, where $\circ$ is composition in $\mathcal{C}$.

Proposition 4.1.61. *An object is initial in $\mathcal{C}$ iff it is terminal in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. □

Proposition 4.1.62. *An object is terminal in $\mathcal{C}$ iff it is initial in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. □

Corollary 4.1.62.1. *If T and T' are terminal objects in $\mathcal{C}$ then there exists a unique isomorphism $T \cong T'$.*

4.1.14 Groupoids

Definition 4.1.63 (Groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

4.1.15 Concrete Categories

Definition 4.1.64 (Concrete Category)**.** A *concrete category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}(\mathcal{C})$ of *objects*
- for any object $A \in \mathrm{Ob}(\mathcal{C})$, a set $|A|$
- for any objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$
- for any object A we have $\mathrm{id}_{|A|} \in \mathcal{C}[A, A]$.

4.1.16 Power of Categories

Definition 4.1.65. Let $\mathcal{C}$ be a category and J a set. The category $\mathcal{C}^J$ is the category with:

- objects all J-indexed families of objects of $\mathcal{C}$
- morphisms $\{X_j\}_{j \in J} \to \{Y_j\}_{j \in J}$ all families $\{f_j\}_{j \in J}$ where $f_j : X_j \to Y_j$

4.1.17 Arrow Category

Definition 4.1.66 (Arrow Category)**.** Let $\mathcal{C}$ be a category. The *arrow category* $\mathcal{C}^{\to}$ is the category with:

- objects all triples (A, B, f) where $f : A \to B$ in $\mathcal{C}$
- morphisms $(A, B, f) \to (C, D, g)$ all pairs $(u : A \to C, v : B \to D)$ such that $vf = gu$.

4.1.18 Slice Category

Definition 4.1.67 (Slice Category)**.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category under* A, $\mathcal{C} \backslash A$, is the category with:

- objects all pairs (B, f) where $B \in \mathcal{C}$ and $f : A \to B$
- morphisms $(B, f) \to (C, g)$ are morphisms $u : B \to C$ such that $uf = g$.

We identify this with the subcategory of $\mathcal{C}^{\to}$ formed by mapping (B, f) to (A, B, f) and u to (id_A, u).

Proposition 4.1.68. *If $s : (B, f) \to (C, g)$ in $\mathcal{C}\backslash A$, then any retraction of s in $\mathcal{C}$ is a retraction of s in $\mathcal{C}\backslash A$.*

PROOF:
⟨1⟩1. LET: $r : C \to B$ be a retraction of s in $\mathcal{C}$.
⟨1⟩2. $rg = f$
PROOF: $rg = rsf = f$.
⟨1⟩3. $r : (C, g) \to (B, f)$ in $\mathcal{C}\backslash A$
⟨1⟩4. $rs = \mathrm{id}_{(B,f)}$
PROOF: Because composition is inherited from $\mathcal{C}$.
□

Proposition 4.1.69. id_A *is the initial object in $\mathcal{C}\backslash A$.*

PROOF: For any $(B, f) \in \mathcal{C}\backslash A$, we have f is the only morphism $A \to B$ such that $f\mathrm{id}_A = f$. □

Proposition 4.1.70. *If A is terminal in $\mathcal{C}$ then* id_A *is the zero object in $\mathcal{C}\backslash A$.*

PROOF: For any $(B, f) \in \mathcal{C}\backslash A$, the unique morphism $! : B \to A$ is the unique morphism such that $!f = \mathrm{id}_A$. □

Definition 4.1.71 (Pointed Sets)**.** The *category of pointed sets* is $\mathbf{Set}\backslash 1$.

Definition 4.1.72. Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category over A*, $\mathcal{C}/A$, is the category with:

- objects all pairs (B, f) with $f : B \to A$
- morphisms $u : (B, f) \to (C, g)$ all morphisms $u : B \to C$ such that $gu = f$.

Proposition 4.1.73. *Let $u : (B, f) \to (C, g) : \mathcal{C}/A$. Any section of u in $\mathcal{C}$ is a section of u in $\mathcal{C}/A$.*

PROOF: Dual to Proposition 4.1.68. □

Proposition 4.1.74. id_A *is terminal in $\mathcal{C}/A$.*

PROOF: Dual to Proposition 4.1.69. □

Proposition 4.1.75. *If A is initial in $\mathcal{C}$ then* id_A *is the zero object in $\mathcal{C}/A$.*

PROOF: Dual to Proposition 4.1.70. □

Definition 4.1.76. Let $A \in \mathcal{C}$. The category of objects *over and under A*, written $\mathcal{C}_A^A$, is the category with:

- objects all triples (X, u, p) where $u : A \to X$, $p : X \to A$ and $pu = \mathrm{id}_A$
- morphism $f : (X, u, p) \to (Y, v, q)$ all morphisms $f : X \to Y$ such that $fu = v$ and $qf = p$

Proposition 4.1.77. $(A, \mathrm{id}_A, \mathrm{id}_A)$ *is the zero object in $\mathcal{C}_A^A$.*

Proof: For any object (X, u, p), we have p is the unique morphism $(X, u, p) \to (A, \mathrm{id}_A, \mathrm{id}_A)$, and u is the unique morphism $(A, \mathrm{id}_A, \mathrm{id}_A) \to (X, u, p)$. □

Definition 4.1.78 (Fibre Collapsing)**.** Let B be a set. Let $u : (A, a) \to (X, x)$ in $\mathbf{Set}/B$. Form the pushout

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"] & B \arrow[d, "j"'] \\
X \arrow[r, "i"'] & C
\end{tikzcd}$$

Let $c : C \to B$ be the unique morphism such that $cj = \mathrm{id}_B$ and $ci = x$. Then $(C, j, c) \in \mathbf{Set}_B^B$ is called the set over and under B obtained from X by *fibre collapsing* with respect to u. If (A, u) is a subset of X, we denote this set over and under B by $X/_B(A, u)$.

Definition 4.1.79 (Fibre Wedge)**.** Let B be a small set. Let $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$. The *fibre wedge* of X and Y is the pushout of u_X and u_Y:

$$\begin{tikzcd}
B \arrow[r, "u_X"] \arrow[d, "u_Y"] & X \arrow[d] \\
Y \arrow[r] & X \vee_B Y
\end{tikzcd}$$

Definition 4.1.80 (Fibre Smash)**.** Let $X, Y \in \mathbf{Set}_B^B$. Let $\xi : X \vee_B Y \to X$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, equal, bend left] & \\
Y \arrow[r] \arrow[drr, bend right, "0"'] & X \vee_B Y \arrow[dr, "\xi"] & \\
& & X
\end{tikzcd}$$

Let $\eta : X \vee_B Y \to Y$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "0"] & \\
Y \arrow[r] \arrow[drr, equal, bend right] & X \vee_B Y \arrow[dr, "\eta"] & \\
& & Y
\end{tikzcd}$$

Let $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \to X \times Y$. The *fibre smash* of X and Y, $X \wedge_B Y$, is the result of collapsing $X \times Y$ with respect to ζ.

Proposition 4.1.81. **Set** *has products and coproducts.*

Proposition 4.1.82. *Let $\mathcal{C}$ be a category. Let $\{X_\alpha\}_{\alpha\in I}$ be a family of objects in $\mathcal{C}$ and $Z \in \mathcal{C}$. Let $\coprod_{\alpha\in I} X_\alpha$ be the coproduct of $\{X_\alpha\}_{\alpha\in I}$. Then*

$$\mathcal{C}[\coprod_{\alpha\in I} X_\alpha, Z] \approx \prod_{\alpha\in I} \mathcal{C}[X_\alpha, Z] \ .$$

Proposition 4.1.83. *Let $\mathcal{C}$ be a category. Let $\{X_\alpha\}_{\alpha\in I}$ be a family of objects in $\mathcal{C}$ and $Z \in \mathcal{C}$. Let $\prod_{\alpha\in I} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha\in I}$. Then*

$$\mathcal{C}[Z, \prod_{\alpha\in I} X_\alpha] \approx \prod_{\alpha\in I} \mathcal{C}[Z, X_\alpha] \ .$$

Proposition 4.1.84. *A product in $\mathcal{C}$ constitutes a product in $\mathcal{C}\backslash A$.*

Proposition 4.1.85. *A coproduct in $\mathcal{C}$ constitutes a product in $\mathcal{C}/A$.*

4.2 Functors

Definition 4.2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- a function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$
- for every morphism $f : A \to B$ in $\mathcal{C}$, a morphism $Ff : FA \to FB$ in $\mathcal{D}$

such that:

- for all $A \in \mathrm{Ob}(C)$ we have $F\mathrm{id}_A = \mathrm{id}_{FA}$
- for any morphism $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, we have $F(g \circ f) = Fg \circ Ff$

Proposition 4.2.2. *Functors preserve isomorphisms.*

Proof:
⟨1⟩1. Let: $F : \mathcal{C} \to \mathcal{D}$ be a functor.
⟨1⟩2. Let: $f : A \cong B$ in $\mathcal{C}$
⟨1⟩3. $Ff^{-1} \circ Ff = \mathrm{id}_{FA}$
Proof:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\mathrm{id}_A \\ &= \mathrm{id}_{FA} \end{aligned}$$

⟨1⟩4. $Ff \circ Ff^{-1} = \mathrm{id}_{FB}$
Proof:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\mathrm{id}_B \\ &= \mathrm{id}_{FB} \end{aligned}$$

∎

Definition 4.2.3 (Identity Functor)**.** For any category $\mathcal{C}$, the *identity* functor on $\mathcal{C}$ is the functor $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ defined by

$$\begin{aligned} I_{\mathcal{C}} A &:= A && (A \in \mathcal{C}) \\ I_{\mathcal{C}} f &:= f && (f : A \to B \text{ in } \mathcal{C}) \end{aligned}$$

Proposition 4.2.4. *Let $F : \mathcal{C} \to \mathcal{D}$. If $r : A \to B$ is a retraction of $s : B \to A$ in $\mathcal{C}$ then Fr is a retraction of Fs.*

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\mathrm{id}_B \\ &= \mathrm{id}_{FB} \end{aligned}$$

∎

Corollary 4.2.4.1. *Let $F : \mathcal{C} \to \mathcal{D}$. If $\phi : A \cong B$ is an isomorphism in $\mathcal{C}$ then $F\phi : FA \cong FB$ is an isomorphism in $\mathcal{D}$ with $(F\phi)^{-1} = F\phi^{-1}$.*

Definition 4.2.5 (Composition of Functors)**.** Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, the *composite* functor $GF : \mathcal{C} \to \mathcal{E}$ is defined by

$$\begin{aligned} (GF)A &= G(FA) && (A \in \mathcal{C}) \\ (GF)f &= G(Ff) && (f : A \to B : \mathcal{C}) \end{aligned}$$

Definition 4.2.6 (Category of Categories)**.** Let **Cat** be the category of small categories and functors.

Definition 4.2.7 (Isomorphism of Categories)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an *isomorphism of categories* iff there exists a functor $F^{-1} : \mathcal{D} \to \mathcal{C}$, the *inverse* of F, such that $FF^{-1} = I_{\mathcal{D}}$ and $F^{-1}F = I_{\mathcal{C}}$.

Categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, iff there exists an isomorphism between them.

Proposition 4.2.8. *If A is initial in $\mathcal{C}$ then $\mathcal{C} \backslash A \cong \mathcal{C}$.*

PROOF:

$\langle 1 \rangle 1.$ Define $F : \mathcal{C} \backslash A \to \mathcal{C}$ by

$$\begin{aligned} F(B, f) &= B \\ F(u : (B, f) \to (C, g)) &= u \end{aligned}$$

$\langle 1 \rangle 2.$ Define $G : \mathcal{C} \to \mathcal{C} \backslash A$ by

$$GB = (B, !_B) \qquad \text{where } !_B \text{ is the unique morphism } A \to B$$

$$G(u : B \to C) = u : (B, !_B) \to (C, !_C)$$

$\langle 1 \rangle 3.$ $FG = \mathrm{id}_{\mathcal{C}}$

$\langle 1 \rangle 4.$ $GF = \mathrm{id}_{\mathcal{C} \backslash A}$

PROOF: Since $GF(B, f) = (B, !_B) = (B, f)$ because the morphism $A \to B$ is unique.

∎

Proposition 4.2.9. *If A is terminal in $\mathcal{C}$ then $\mathcal{C}/A \cong \mathcal{C}$.*

PROOF: Dual. $\square$

Proposition 4.2.10.

$$\mathcal{C}_A^A \cong (\mathcal{C}/A)\backslash(A, \mathrm{id}_A) \cong (\mathcal{C}\backslash A)/(A, \mathrm{id}_A)$$

PROOF:
$\langle 1 \rangle 1$. Define a functor $F : \mathcal{C}_A^A \to (\mathcal{C}/A)\backslash(A, \mathrm{id}_A)$.
  $\langle 2 \rangle 1$. Given $A \xrightarrow{u} X \xrightarrow{p} A$ in $\mathcal{C}_A^A$, let $F(X, u, p) = ((X, p), u)$
  $\langle 2 \rangle 2$. Given $f : (A \xrightarrow{u} X \xrightarrow{p} A) \to (A \xrightarrow{v} Y \xrightarrow{q} A)$, let $Ff = f$.
$\langle 1 \rangle 2$. Define a functor $G : (\mathcal{C}/A)\backslash(A, \mathrm{id}_A) \to \mathcal{C}_A^A$.
$\langle 1 \rangle 3$. Define a functor $H : \mathcal{C}_A^A \to (\mathcal{C}\backslash A)/(A, \mathrm{id}_A)$.
$\langle 1 \rangle 4$. Define a functor $K : (\mathcal{C}\backslash A)/(A, \mathrm{id}_A) \to \mathcal{C}_A^A$.
$\square$

Definition 4.2.11 (Forgetful Functor)**.** For any concrete category $\mathcal{C}$, define the *forgetful* functor $U : \mathcal{C} \to \mathbf{Set}$ by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

Definition 4.2.12 (Switching Functor)**.** For any category $\mathcal{C}$, define the *switching functor* $T : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

Definition 4.2.13 (Reduction)**.** Let $\Phi : \mathbf{Set} \to \mathbf{Set}$ be a functor. The *reduction* of Φ is the functor $\Phi^* : \mathbf{Set}_* \to \mathbf{Set}_*$ defined by: $\Phi^*(X, a)$ is the collapse of $\Phi(X)$ with respect to $\Phi(a) : \Phi(1) \rightarrowtail \Phi(X)$.

Definition 4.2.14. Extend the wedge $\vee$ to a functor $\mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ by defining, given $f : X \to X'$ and $g : Y \to Y'$, thene $f \vee g$ is the unique morphism that makes the following diagram commute.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[d] \arrow[dr, "f"] & \\
Y \arrow[r] \arrow[dr, "g"'] & X \vee Y \arrow[dr, "f \vee g"] & X' \arrow[d] \\
& Y' \arrow[r] & X' \vee Y'
\end{tikzcd}$$

Definition 4.2.15. Extend smash to a functor $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ as follows. Given $f : X \to X'$ and $g : Y \to Y'$, let $f \wedge g : X \wedge Y \to X' \wedge Y'$ be the unique morphism such that the following diagram commutes.

$$\begin{tikzcd}
X \vee Y \arrow[r] \arrow[d] \arrow[ddr] & 1 \arrow[d] \arrow[ddr, equals, bend left] & \\
X \times Y \arrow[r] \arrow[ddr, "f \times g"'] & X \wedge Y \arrow[ddr] & \\
& X' \vee Y' \arrow[r] \arrow[d] & 1 \arrow[d] \\
& X' \times Y' \arrow[r] & X' \wedge Y'
\end{tikzcd}$$

Definition 4.2.16 (Reduction)**.** Let B be a small set. Let $\Phi_B : \mathbf{Set}/B \to \mathbf{Set}/B$ be a functor. The *reduction* of Φ_B is the functor $\Phi_B^B : \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ defined as follows.

For $(X, u : B \to X, p : X \to B) \in \mathbf{Set}_B^B$, let $\Phi_B^B(X)$ be the set over and under B obtained from $\Phi_B(X)$ by collapsing with respect to $\Phi_B(u) : \Phi_B(B) \to \Phi_B(X)$.

Definition 4.2.17. Extend $\vee_B$ to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$.

Definition 4.2.18. Extend $\wedge_B$ to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$.

Definition 4.2.19 (Faithful)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* iff, for any objects $A, B \in \mathcal{C}$ and morphisms $f, g : A \to B : \mathcal{C}$, if $Ff = Fg$ then $f = g$.

Definition 4.2.20 (Full)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* iff, for any objects $A, B \in \mathcal{C}$ and morphism $g : FA \to FB : \mathcal{D}$, there exists $f : A \to B : \mathcal{C}$ such that $Ff = g$.

Definition 4.2.21 (Fully Faithful)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* iff it is full and faithful.

Definition 4.2.22 (Full Embedding)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is a *full embedding* iff it is fully faithful and injective on objects.

4.3 Natural Transformations

Definition 4.3.1 (Natural Transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\tau : F \Rightarrow G$ is a family of morphisms $\{\tau_X : FX \to GX\}_{X \in \mathcal{C}}$ such that, for every morphism $f : X \to Y : \mathcal{C}$, we have $Gf \circ \tau_X = \tau_Y \circ Ff$.

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\tau_X"'] & FY \arrow[d, "\tau_Y"] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}$$

Definition 4.3.2 (Natural Isomorphism)**.** A natural transformation $\tau : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ is a *natural isomorphism*, $\tau : F \cong G$, iff for all $X \in \mathcal{C}$, τ_X is an isomorphism $FX \cong GX$.

Functors F and G are *naturally isomorphic*, $F \cong G$, iff there exists a natural isomorphism between them.

Definition 4.3.3 (Inverse)**.** Let $\tau : F \cong G$. The *inverse* natural isomorphism $\tau^{-1} : G \cong F$ is defined by $(\tau^{-1})_X = {\tau_X}^{-1}$.

4.4 Bifunctors

Definition 4.4.1 (Commutative)**.** A bifunctor $\square : \mathcal{C}^2 \to \mathcal{C}$ is *commutative* iff $\square \cong \square \circ T$, where $T : \mathcal{C}^2 \to \mathcal{C}^2$ is the swap functor.

Proposition 4.4.2. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is commutative.*

Proof: Since the pushout of f and g is the pushout of g and f. ∎

Proposition 4.4.3. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is commutative.*

Proof: In the diagram defining $X \wedge Y$, construct the isomorphism between the version with X and Y and the version with Y with X for every object. ∎

Proposition 4.4.4. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is commutative.*

Proposition 4.4.5. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is commutative.*

Definition 4.4.6 (Associative)**.** A bifunctor $\square$ is *associative* iff $\square \circ (\square \times \mathrm{id}) \cong \square \circ (\mathrm{id} \times \square)$.

Proposition 4.4.7. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is associative.*

Proof: Since $X \vee (Y \vee Z)$ and $(X \vee Y) \vee Z$ are both the pushout of the unique morphisms $1 \to X$, $1 \to Y$ and $1 \to Z$. ∎

Proposition 4.4.8. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is associative.*

Proof: Draw isomorphisms between the diagrams for $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$. ∎

Product and coproduct are commutative and associative.

Proposition 4.4.9. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is associative.*

Proposition 4.4.10. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is associative.*

Proposition 4.4.11. *Let $\mathcal{C}$ be a category with binary coproducts. Let $\square : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a bifunctor. Then $\square$* distributes *over $+$ iff the canonical morphism*

$$(X \square Z) + (Y \square Z) \to (X + Y) \square Z$$

is an isomorphism for all X, Y, Z.

Proposition 4.4.12. *In a category with binary products and binary coproducts, then $\times$ distributes over $+$.*

Proposition 4.4.13. *In* $\mathbf{Set}/*$*, we have $\times$ does not distribute over $\vee$.*

Proposition 4.4.14. *In* $\mathbf{Set}/*$*, we have $\wedge$ distributes over $\vee$.*

Proposition 4.4.15. *In* $\mathbf{Set}/B$*, we have $\times_B$ distributes over $+_B$.*

Proposition 4.4.16. *In* $\mathbf{Set}/B^B$*, we have $\wedge_B$ distributes over $\vee_B$.*

4.5 Functor Categories

Definition 4.5.1 (Functor Category)**.** Given categories $\mathcal{C}$ and $\mathcal{D}$, define the *functor category* $\mathcal{C}^{\mathcal{D}}$ to be the category with objects the functors from $\mathcal{D}$ to $\mathcal{C}$ and morphisms the natural transformations.

Definition 4.5.2 (Yoneda Embedding)**.** Let $\mathcal{C}$ be a category. The *Yoneda embedding* $Y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the functor that maps an object A to $\mathcal{C}[-, A]$ and morphisms similarly.

Theorem 4.5.3 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}[\mathcal{C}[-, X], F] \cong FX$$

that maps $\tau : \mathcal{C}[-, X] \Rightarrow F$ to $\tau_X(\mathrm{id}_X)$.

Proof:
⟨1⟩1. ϕ is natural in X.
  Proof:
  ⟨2⟩1. Let: $f : X \to Y : \mathcal{C}$
  ⟨2⟩2. Let: $\tau : \mathcal{C}[-, X] \Rightarrow F$
  ⟨2⟩3. $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$
    Proof:

$$\begin{aligned}
\phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\mathrm{id}_Y \circ f) \\
&= \tau_Y(f) \\
&= \tau_Y(f \circ \mathrm{id}_X) \\
&= Ff(\tau_X(\mathrm{id}_X)) && (\tau \text{ natural}) \\
&= Ff(\phi(\tau))
\end{aligned}$$

⟨1⟩2. ϕ is natural in F.
  ⟨2⟩1. Let: $\alpha : F \Rightarrow G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$
  ⟨2⟩2. Let: $\tau : \mathcal{C}[-, X] \Rightarrow F$
  ⟨2⟩3. $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$
    Proof: $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\mathrm{id}_X)) = \alpha_X(\phi(\tau))$
⟨1⟩3. Each ϕ_{XF} is injective.
  ⟨2⟩1. Let: $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$
  ⟨2⟩2. Assume: $\phi(\sigma) = \phi(\tau)$

⟨2⟩3. LET: $f : Y \to X$
⟨2⟩4. $\sigma_Y(f) = \tau_Y(f)$
PROOF:

$$\begin{aligned} \sigma_Y(f) &= \sigma_Y(\mathrm{id}_X \circ f) \\ &= Ff(\sigma_X(\mathrm{id}_X)) && (\sigma \text{ is natural}) \\ &= Ff(\tau_X(\mathrm{id}_X)) && (\langle 2\rangle 2) \\ &= \tau_Y(\mathrm{id}_X \circ f) && (\tau \text{ is natural}) \\ &= \tau_Y(f) \end{aligned}$$

⟨1⟩4. Each ϕ_{XF} is surjective.
⟨2⟩1. LET: $X \in \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$
⟨2⟩2. LET: $a \in FX$
⟨2⟩3. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$ be given by $\tau_Y(g) = Fg(a)$ for $g : Y \to X$
⟨2⟩4. τ is natural.
⟨3⟩1. LET: $h : Y \to Z : \mathcal{C}$
PROVE: $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \mathrm{id}_X]$
⟨3⟩2. LET: $g : Z \to X$
⟨3⟩3. $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$
PROOF:

$$\begin{aligned} \tau_Y(g \circ h) &= F(g \circ h)(a) \\ &= Fh(Fg(a)) \\ &= Fh(\tau_Z(g)) \end{aligned}$$

⟨2⟩5. $\phi(\tau) = a$
PROOF:

$$\begin{aligned} \phi_X(\tau) &= \tau_X(\mathrm{id}_X) \\ &= F\mathrm{id}_X(a) \\ &= a \end{aligned}$$

□

Corollary 4.5.3.1. *The Yoneda embedding is fully faithful.*

Corollary 4.5.3.2. *Given objects A and B in $\mathcal{C}$, we have $A \cong B$ if and only if $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$.*

Chapter 5

The Real Numbers

Theorem 5.0.1. *The following hold in the real numbers:*

1. $x + (y + z) = (x + y) + z$
2. $x(yz) = (xy)z$
3. $x + y = y + x$
4. $xy = yx$
5. $x + 0 = x$
6. $x1 = x$
7. $x + (-x) = 0$
8. *If* $x \neq 0$ *then* $x \cdot (1/x) = 1$
9. $x(y + z) = xy + xz$
10. *If* $x > y$ *then* $x + z > y + z$.
11. *If* $x > y$ *and* $z > 0$ *then* $xz > yz$.
12. $\mathbb{R}$ *has the least upper bound property.*
13. *If* $x < y$ *then there exists* z *such that* $x < z < y$.

Definition 5.0.2 (Subtraction)**.** We write $x - y$ for $x + (-y)$.

Definition 5.0.3. Given real numbers x and y with $y \neq 0$, we write x/y for xy^{-1}.

Theorem 5.0.4. *For any real numbers x and y, if $x + y = x$ then $y = 0$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $x, y \in \mathbb{R}$

⟨1⟩2. Assume: $x + y = x$
⟨1⟩3. $y = 0$
Proof:

$$\begin{aligned} y &= y + 0 && \text{(Definition of zero)} \\ &= y + (x + (-x)) && \text{(Definition of } -x) \\ &= (y + x) + (-x) && \text{(Associativity of Addition)} \\ &= (x + y) + (-x) && \text{(Commutativity of Addition)} \\ &= x + (-x) && (\langle 1\rangle 2) \\ &= 0 && \text{(Definition of } -x) \end{aligned}$$

□

Theorem 5.0.5.

$$\forall x \in \mathbb{R}. 0x = 0$$

Proof:
⟨1⟩1. Let: $x \in \mathbb{R}$
⟨1⟩2. $xx + 0x = xx$
Proof:

$$\begin{aligned} xx + 0x &= (x + 0)x && \text{(Distributive Law)} \\ &= xx && \text{(Definition of 0)} \end{aligned}$$

⟨1⟩3. $0x = 0$
Proof: Theorem 5.0.4, ⟨1⟩2.
□

Theorem 5.0.6.

$$-0 = 0$$

Proof: Since $0 + 0 = 0$. □

Theorem 5.0.7.

$$\forall x \in \mathbb{R}. -(-x) = x$$

Proof: Since $-x + x = 0$. □

Theorem 5.0.8.

$$\forall x, y \in \mathbb{R}. x(-y) = -(xy)$$

Proof:

$$\begin{aligned} x(-y) + xy &= x((-y) + y) && \text{(Distributive Law)} \\ &= x0 && \text{(Definition of } -y) \\ &= 0 && \text{(Theorem 5.0.5)} \end{aligned}$$ □

Theorem 5.0.9.

$$\forall x \in \mathbb{R}. (-1)x = -x$$

Proof:

$$\begin{aligned} (-1)x &= -(1 \cdot x) && \text{(Theorem 5.0.8)} \\ &= -x && \text{(Definition of 1)} \end{aligned}$$ □

5.0.1 Subtraction

Theorem 5.0.10.
$$\forall x, y, z \in \mathbb{R}. x(y - z) = xy - xz$$

PROOF:
$$\begin{aligned} x(y - z) &= x(y + (-z)) && \text{(Definition of subtraction)} \\ &= xy + x(-z) && \text{(Distributive Law)} \\ &= xy + (-(xz)) && \text{(Theorem 5.0.8)} \\ &= xy - xz && \text{(Definition of subtraction)} \ \square \end{aligned}$$

Theorem 5.0.11.
$$\forall x, y \in \mathbb{R}. -(x + y) = -x - y$$

PROOF:
$$\begin{aligned} -(x + y) &= (-1)(x + y) && \text{(Theorem 5.0.9)} \\ &= (-1)x + (-1)y && \text{(Distributive Law)} \\ &= -x + (-y) && \text{(Theorem 5.0.9)} \\ &= -x - y && \text{(Definition of subtraction)} \ \square \end{aligned}$$

Theorem 5.0.12.
$$\forall x, y \in \mathbb{R}. -(x - y) = -x + y$$

PROOF:
$$\begin{aligned} -(x - y) &= -(x + (-y)) && \text{(Definition of subtraction)} \\ &= -x - (-y) && \text{(Theorem 5.0.11)} \\ &= -x + (-(-y)) && \text{(Definition of subtraction)} \\ &= -x + y && \text{(Theorem 5.0.7)} \ \square \end{aligned}$$

Definition 5.0.13 (Reciprocal)**.** Given $x \in \mathbb{R}$ with $x \neq 0$, the *reciprocal* of x, $1/x$, is the unique real number such that $x \cdot 1/x = 1$.

Theorem 5.0.14. *For any real numbers x and y, if $x \neq 0$ and $xy = x$ then $y = 1$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $x, y \in \mathbb{R}$
$\langle 1 \rangle 2.$ ASSUME: $x \neq 0$
$\langle 1 \rangle 3.$ ASSUME: $xy = x$
$\langle 1 \rangle 4.$ $y = 1$
PROOF:
$$\begin{aligned} y &= y1 && \text{(Definition of 1)} \\ &= y(x \cdot 1/x) && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\ &= (yx)1/x && \text{(Associativity of Multiplication)} \\ &= (xy)1/x && \text{(Commutativity of Multiplication)} \\ &= x \cdot 1/x && (\langle 1 \rangle 3) \\ &= 1 && \text{(Definition of } 1/x, \langle 1 \rangle 2) \end{aligned}$$

$\square$

Definition 5.0.15 (Quotient)**.** Given real numbers x and y with $y \neq 0$, the *quotient* x/y is defined by

$$x/y = x \cdot 1/y \ .$$

Theorem 5.0.16. *For any real number x, if $x \neq 0$ then $x/x = 1$.*

PROOF: Immediate from definitions. $\square$

Theorem 5.0.17.

$$\forall x \in \mathbb{R}. x/1 = x$$

PROOF:
$\langle 1 \rangle 1$. LET: $x \in \mathbb{R}$
$\langle 1 \rangle 2$. $1/1 = 1$
PROOF: Since $1 \cdot 1 = 1$.
$\langle 1 \rangle 3$. $x/1 = x$
PROOF: Since $x/1 = x \cdot 1/1 = x \cdot 1 = x$.
$\square$

Theorem 5.0.18. *For any real numbers x and y, if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.*

PROOF:
$\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$
$\langle 1 \rangle 2$. ASSUME: $xy = 0$ and $x \neq 0$
PROVE: $y = 0$
$\langle 1 \rangle 3$. $y = 0$
PROOF:

$$\begin{aligned} y &= 1y && \text{(Definition of 1)} \\ &= (1/x)xy && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\ &= (1/x)0 && (\langle 1 \rangle 2) \\ &= 0 && \text{(Theorem 5.0.5)} \end{aligned}$$

$\square$

Theorem 5.0.19. *For any real numbers y and z, if $y \neq 0$ and $z \neq 0$ then $(1/y)(1/z) = 1/(yz)$.*

PROOF: Since $yz(1/y)(1/z) = 1 \cdot 1 = 1$. $\square$

Corollary 5.0.19.1. *For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have $(x/y)(z/w) = (xz)/(yw)$.*

Theorem 5.0.20. *For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have*

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + yz}{yw}$$

PROOF:

$$\begin{aligned} yw\left(\frac{x}{y}+\frac{z}{w}\right) &= yw\frac{x}{y}+yw\frac{z}{w} \\ &= wx+yz \end{aligned} \qquad \square$$

Theorem 5.0.21. *For any real number x, if $x \neq 0$ then $1/x \neq 0$.*

PROOF: Since $x \cdot 1/x = 1 \neq 0$. $\square$

Theorem 5.0.22. *For any real numbers w, z, if $w \neq 0 \neq z$ then $1/(w/z) = z/w$.*

PROOF: Since $(z/w)(w/z) = (wz)/(wz) = 1$. $\square$

Theorem 5.0.23. *For any real numbers a, x and y, if $y \neq 0$ then $(ax)/y = a(x/y)$*

PROOF: Since $ya(x/y) = ax$. $\square$

Theorem 5.0.24. *For any real numbers x and y, if $y \neq 0$ then $(-x)/y = x/(-y) = -(x/y)$.*

PROOF:
$\langle 1\rangle 1.$ $(-x)/y = -(x/y)$
  PROOF: Take $a = -1$ in Theorem 5.0.23.
$\langle 1\rangle 2.$ $x/(-y) = -(x/y)$
  PROOF: Since $(-y)(-(x/y)) = y(x/y) = x$.
$\square$

Theorem 5.0.25. *For any real numbers x, y, z and w, if $x > y$ and $w > z$ then $x + w > y + z$.*

PROOF: We have $y + z < x + z < x + w$ by Monotonicity of Addition twice. $\square$

Corollary 5.0.25.1. *For any real numbers x and y, if $x > 0$ and $y > 0$ then $x + y > 0$.*

Theorem 5.0.26. *For any real numbers x and y, if $x > 0$ and $y > 0$ then $xy > 0$.*

PROOF:

$$\begin{aligned} xy &> 0y && \text{(Monotonicity of Multiplication)} \\ &= 0 && \text{(Theorem 5.0.5)} \end{aligned} \qquad \square$$

Theorem 5.0.27. *For any real number x, we have $x > 0$ iff $-x < 0$.*

PROOF:
$\langle 1\rangle 1.$ If $0 < x$ then $-x < 0$
  PROOF: By Monotonicity of Addition adding $-x$ to both sides.
$\langle 1\rangle 2.$ If $-x < 0$ then $0 < x$
  PROOF: By Monotonicity of Addition adding x to both sides.
$\square$

Theorem 5.0.28. *For any real numbers x and y, we have $x > y$ iff $-x < -y$.*

PROOF:
$\langle 1 \rangle 1$. If $y < x$ then $-x < -y$.
  PROOF: By Monotonicity of Addition adding $-x - y$ to both sides.
$\langle 1 \rangle 2$. If $-x < -y$ then $y < x$.
  PROOF: By Monotonicity of Addition adding $x + y$ to both sides.
□

Theorem 5.0.29. *For any real numbers x, y and z, if $x > y$ and $z < 0$ then $xz < yz$.*

PROOF:
$\langle 1 \rangle 1$. LET: x, y and z be real numbers.
$\langle 1 \rangle 2$. ASSUME: $x > y$
$\langle 1 \rangle 3$. ASSUME: $z < 0$
$\langle 1 \rangle 4$. $-z > 0$
  PROOF: Theorem 5.0.27, $\langle 1 \rangle 3$.
$\langle 1 \rangle 5$. $x(-z) > y(-z)$
  PROOF: $\langle 1 \rangle 2$, $\langle 1 \rangle 4$, Monotonicity of Multiplication.
$\langle 1 \rangle 6$. $-(xz) > -(yz)$
  PROOF: Theorem 5.0.8, $\langle 1 \rangle 5$.
$\langle 1 \rangle 7$. $xz < yz$
  PROOF: Theorem 5.0.27, $\langle 1 \rangle 6$.
□

Theorem 5.0.30. *For any real number x, if $x \neq 0$ then $xx > 0$.*

PROOF:
$\langle 1 \rangle 1$. If $x > 0$ then $xx > 0$
  PROOF: By Monotonicity of Multiplication.
$\langle 1 \rangle 2$. If $x < 0$ then $xx > 0$
  PROOF: Theorem 5.0.29.
□

Theorem 5.0.31.

$$0 < 1$$

PROOF: By Theorem 5.0.30 since $1 = 1 \cdot 1$. □

Definition 5.0.32 (Positive)**.** A real number x is *positive* iff $x > 0$.
We write $\mathbb{R}_+$ for the set of positive reals.

Theorem 5.0.33. *For any real numbers x and y, we have xy is positive if and only if x and y are both positive or both negative.*

PROOF: By the Monotonicity of Multiplication and Theorem 5.0.29. □

Corollary 5.0.33.1. *For any real number x, if $x > 0$ then $1/x > 0$.*

PROOF: Since $x \cdot 1/x = 1$ is positive. □

Theorem 5.0.34. *For any real numbers x and y, if $x > y > 0$ then $1/x < 1/y$.*

PROOF: If $1/y \leqslant 1/x$ then $1 < 1$ by Monotonicity of Multiplication. □

Theorem 5.0.35. *For any real numbers x and y, if $x < y$ then $x < (x+y)/2 < y$.*

PROOF: We have $2x < x+y$ and $x+y < 2y$ by Monotonicity of Addition, hence $x < (x+y)/2 < y$ by Monotonicity of Multiplication since $1/2 > 0$. □

Corollary 5.0.35.1. $\mathbb{R}$ *is a linear continuum.*

Definition 5.0.36 (Negative)**.** A real number x is *negative* iff $x < 0$.

We write $\overline{\mathbb{R}_+}$ for the set of nonnegative reals.

Theorem 5.0.37. *For every positive real number a, there exists a unique positive real $\sqrt{a}$ such that $\sqrt{a}^2 = a$.*

PROOF:

$\langle 1 \rangle 1$. LET: a be a positive real.

$\langle 1 \rangle 2$. For any real numbers x and h, if $0 \leqslant h < 1$, then
$$(x+h)^2 < x^2 + h(2x+1) \ .$$

$\langle 2 \rangle 1$. LET: x and h be real numbers.

$\langle 2 \rangle 2$. ASSUME: $0 \leqslant h < 1$

$\langle 2 \rangle 3$. $(x+h)^2 < x^2 + h(2x+1)$

PROOF:
$$\begin{aligned} (x+h)^2 &= x^2 + 2hx + h^2 \\ &< x^2 + 2hx + h && (\langle 2 \rangle 2) \\ &= x^2 + h(2x+1) \end{aligned}$$

$\langle 1 \rangle 3$. For any real numbers x and h, if $h > 0$ then
$$(x-h)^2 > x^2 - 2hx \ .$$

$\langle 2 \rangle 1$. LET: x and h be real numbers.

$\langle 2 \rangle 2$. ASSUME: $h > 0$

$\langle 2 \rangle 3$. $(x-h)^2 > x^2 - 2hx$

PROOF:
$$\begin{aligned} (x-h)^2 &= x^2 - 2hx + h^2 \\ &> x^2 - 2hx && (\langle 2 \rangle 2) \end{aligned}$$

$\langle 1 \rangle 4$. For any positive real x, if $x^2 < a$ then there exists $h > 0$ such that $(x+h)^2 < a$.

$\langle 2 \rangle 1$. LET: x be a positive real.

$\langle 2 \rangle 2$. ASSUME: $x^2 < a$

$\langle 2 \rangle 3$. LET: $h = \min((a - x^2)/(2x+1), 1/2)$

$\langle 2 \rangle 4$. $0 < h < 1$

$\langle 2 \rangle 5$. $(x+h)^2 < a$

PROOF:
$$\begin{aligned} (x+h)^2 &< x^2 + h(2x+1) && (\langle 1 \rangle 2) \\ &\leqslant a \end{aligned}$$

$\langle 1\rangle 5$. For any positive real x, if $x^2 > a$ then there exists $h > 0$ such that $(x-h)^2 > a$.

$\langle 2\rangle 1$. LET: x be a positive real.

$\langle 2\rangle 2$. ASSUME: $x^2 > a$

$\langle 2\rangle 3$. LET: $h = (x^2 - a)/2x$

$\langle 2\rangle 4$. $h > 0$

$\langle 2\rangle 5$. $(x-h)^2 > a$

PROOF:

$$\begin{aligned} (x-h)^2 &> x^2 - 2hx \\ &= a && (\langle 2\rangle 3) \end{aligned}$$

$\langle 1\rangle 6$. LET: $B = \{x \in \mathbb{R} : x^2 < a\}$

$\langle 1\rangle 7$. B is bounded above.

PROOF: If $a \geqslant 1$ then a is an upper bound. If $a < 1$ then 1 is an upper bound.

$\langle 1\rangle 8$. B contains at least one positive real.

PROOF: If $a \geqslant 1$ then $1 \in B$. If $a < 1$ then $a \in B$.

$\langle 1\rangle 9$. LET: $b = \sup B$

$\langle 1\rangle 10$. $b^2 = a$

$\langle 2\rangle 1$. $b^2 \geqslant a$

$\langle 3\rangle 1$. ASSUME: for a contradiction $b^2 < a$

$\langle 3\rangle 2$. PICK $h > 0$ such that $(b+h)^2 < a$

PROOF: $\langle 1\rangle 4$

$\langle 3\rangle 3$. $b + h \in B$

$\langle 3\rangle 4$. Q.E.D.

PROOF: This contradicts $\langle 1\rangle 9$.

$\langle 2\rangle 2$. $b^2 \leqslant a$

$\langle 3\rangle 1$. ASSUME: for a contradiction $b^2 > a$

$\langle 3\rangle 2$. PICK $h > 0$ such that $(b-h)^2 > a$

PROOF: $\langle 1\rangle 5$

$\langle 3\rangle 3$. PICK $x \in B$ such that $b - h < x$

PROOF: $\langle 1\rangle 9$

$\langle 3\rangle 4$. $(b-h)^2 < x^2 < a$

$\langle 3\rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 3\rangle 2$

$\langle 1\rangle 11$. For any positive reals b and c, if $b^2 = c^2$ then $b = c$.

$\langle 2\rangle 1$. LET: b and c be positive reals.

$\langle 2\rangle 2$. ASSUME: $b^2 = c^2$

$\langle 2\rangle 3$. $b^2 - c^2 = 0$

$\langle 2\rangle 4$. $(b-c)(b+c) = 0$

$\langle 2\rangle 5$. $b - c = 0$ or $b + c = 0$

$\langle 2\rangle 6$. $b + c \neq 0$

PROOF: Since $b + c > 0$

$\langle 2\rangle 7$. $b - c = 0$

$\langle 2\rangle 8$. $b = c$

□

Theorem 5.0.38. *The set of real numbers is uncountable.*

Chapter 6

Integers and Rationals

6.1 Positive Integers

Definition 6.1.1 (Inductive)**.** A set of real numbers A is *inductive* iff $1 \in A$ and $\forall x \in A . x + 1 \in A$.

Definition 6.1.2 (Positive Integer)**.** The set $\mathbb{Z}_+$ of *positive integers* is the intersection of the set of inductive sets.

Proposition 6.1.3. *Every positive integer is positive.*

PROOF: The set of positive reals is inductive. □

Proposition 6.1.4. *1 is the least element of* $\mathbb{Z}_+$.

PROOF: Since $\{x \in \mathbb{R} : x \geqslant 1\}$ is inductive. □

Proposition 6.1.5. $\mathbb{Z}_+$ *is inductive.*

PROOF: 1 is an element of every inductive set, and for all $x \in \mathbb{R}$, if x is an element of every inductive set then so is $x + 1$. □

Theorem 6.1.6 (Principle of Induction)**.** *If A is an inductive set of positive integers then* $A = \mathbb{Z}_+$.

PROOF: Immediate from definitions. □

Theorem 6.1.7 (Well-Ordering Property)**.** $\mathbb{Z}_+$ *is well ordered.*

PROOF: Construct the obvious order isomorphism $\omega \cong \mathbb{Z}_+$. □

Theorem 6.1.8 (Archimedean Ordering Property)**.** *The set $\mathbb{Z}_+$ is unbounded above.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $\mathbb{Z}_+$ is bounded above.

$\langle 1 \rangle 2$. LET:
$s = \sup \mathbb{Z}_+$
$\langle 1 \rangle 3$. PICK $n \in \mathbb{Z}_+$ such that $s - 1 < n$
$\langle 1 \rangle 4$. $s < n + 1$
$\langle 1 \rangle 5$. Q.E.D.
PROOF: $\langle 1 \rangle 2$ and $\langle 1 \rangle 4$ form a contradiction.
□

6.1.1 Exponentiation

Definition 6.1.9. For a a real number and n a positive integer, define the real number a^n recursively as follows:

$$a^1 = a$$
$$a^{n+1} = a^n a$$

Theorem 6.1.10. *For all $a \in \mathbb{R}$ and $m, n \in mathbbZ_+$, we have*

$$a^n a^m = a^{n+m}$$

PROOF:
$\langle 1 \rangle 1$. LET: $P(m)$ be the property $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. a^n a^m = a^{n+m}$
$\langle 1 \rangle 2$. $P(1)$
PROOF: $a^n a^1 = a^n a = a^{n+1}$.
$\langle 1 \rangle 3$. $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$
$\langle 2 \rangle 1$. LET: m be a positive integer.
$\langle 2 \rangle 2$. ASSUME: $P(m)$
$\langle 2 \rangle 3$. LET: $a \in \mathbb{R}$
$\langle 2 \rangle 4$. LET: $n \in \mathbb{Z}_+$
$\langle 2 \rangle 5$. $a^n a^{m+1} = a^{n+m+1}$
PROOF:

$$\begin{aligned} a^n a^{m+1} &= a^n a^m a \\ &= a^{n+m} a && (\langle 2 \rangle 2) \\ &= a^{n+m+1} \end{aligned}$$

$\langle 1 \rangle 4$. Q.E.D.
PROOF: By induction.
□

Theorem 6.1.11. *For all $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}_+$,*

$$(a^n)^m = a^{nm} \quad .$$

PROOF:
$\langle 1 \rangle 1$. LET: $P(m)$ be the property $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. (a^n)^m = a^{nm}$.
$\langle 1 \rangle 2$. $P(1)$
PROOF: $(a^n)^1 = a^n = a^{n \cdot 1}$

$\langle 1 \rangle 3.$ $\forall m \in \mathbb{Z}_+ . P(m) \Rightarrow P(m+1)$
PROOF:

$$\begin{aligned}(a^n)^{m+1} &= (a^n)^m a^n \\ &= a^{nm} a^n \\ &= a^{nm+n} && \text{(Theorem 6.1.10)} \\ &= a^{n(m+1)}\end{aligned}$$

□

Theorem 6.1.12. *For any real numbers a and b and positive integer m,*

$$a^m b^m = (ab)^m \enspace .$$

PROOF: Induction on m. □

6.2 Integers

Definition 6.2.1 (Integer)**.** The set $\mathbb{Z}$ of *integers* is

$$\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \{-x : x \in \mathbb{Z}_+\} \enspace .$$

Proposition 6.2.2. *The sum, difference and product of two integers is an integer.*

PROOF: Easy. □

Example 6.2.3. $1/2$ is not an integer.

Proposition 6.2.4. *For any integer n, there is no integer a such that $n < a < n+1$.*

PROOF:
$\langle 1 \rangle 1.$ For any positive integer n, there is no integer a such that $n < a < n+1$.
$\langle 2 \rangle 1.$ There is no integer a such that $1 < a < 2$.
$\langle 3 \rangle 1.$ There is no positive integer a such that $1 < a < 2$.
$\langle 4 \rangle 1.$ We do not have $1 < 1 < 2$.
$\langle 4 \rangle 2.$ For any positive integer n, we do not have $1 < n+1 < 2$.
PROOF: Since $n \geqslant 1$ so $n+1 \geqslant 2$.
$\langle 3 \rangle 2.$ We do not have $1 < 0 < 2$.
$\langle 3 \rangle 3.$ For any positive integer a, we do not have $1 < -a < 2$.
PROOF: Since $-a < 0 < 1$.
$\langle 2 \rangle 2.$ For any positive integer n, if there is no integer a such that $n < a < n+1$, then there is no integer a such that $n+1 < a < n+2$.
PROOF: If $n+1 < a < n+2$ then $n < a-1 < n+1$.
$\langle 1 \rangle 2.$ There is no integer a such that $0 < a < 1$.
PROOF: If $0 < a < 1$ then $1 < a+1 < 2$.
$\langle 1 \rangle 3.$ For any positive integer n, there is no integer a such that $-n < a < -n+1$.
PROOF: If $-n < a < -n+1$ then $n-1 < -a < n$.

□

Theorem 6.2.5. *Every nonempty subset of $\mathbb{Z}$ bounded above has a largest element.*

PROOF:
$\langle 1 \rangle 1$. LET: S be a nonempty subset of $\mathbb{Z}$ bounded above.
$\langle 1 \rangle 2$. LET: u be an upper bound for S.
$\langle 1 \rangle 3$. PICK an integer $n > u$
PROOF: Archimedean property.
$\langle 1 \rangle 4$. LET: k be the least positive integer such that $n - k \in S$.
$\langle 2 \rangle 1$. PICK $m \in S$
$\langle 2 \rangle 2$. $n - m$ is a positive integer.
$\langle 2 \rangle 3$. There exists a positive integer k such that $n - k \in S$.
$\langle 1 \rangle 5$. $n - k$ is the greatest element in S.
$\langle 2 \rangle 1$. LET: $m \in S$
$\langle 2 \rangle 2$. $n - m \geqslant k$
$\langle 2 \rangle 3$. $m \leqslant n - k$

□

Theorem 6.2.6. *For any real number x, if x is not an integer then there exists a unique integer n such that $n < x < n + 1$.*

PROOF:
$\langle 1 \rangle 1$. $\{n \in \mathbb{Z} : n < x\}$ is a nonempty set of integers bounded above.
$\langle 2 \rangle 1$. PICK $m > -x$
PROOF: Archimedean property.
$\langle 2 \rangle 2$. $-m < x$
$\langle 2 \rangle 3$. $\{n \in \mathbb{Z} : n < x\}$ is nonempty.
$\langle 1 \rangle 2$. LET: n be the greatest integer such that $n < x$
$\langle 1 \rangle 3$. $x < n + 1$
$\langle 1 \rangle 4$. If n' is an integer with $n' < x < n' + 1$ then $n' = n$.
PROOF: We have $n' < n + 1$ so $n' \leqslant n$, and $n < n' + 1$ so $n \leqslant n'$.

□

Definition 6.2.7 (Even)**.** An integer n is *even* iff $n/2$ is an integer; otherwise, n is *odd*.

Theorem 6.2.8. *If the integer m is odd then there exists an integer n such that $m = 2n + 1$.*

PROOF:
$\langle 1 \rangle 1$. LET: n be the integer such that $n < m/2 < n + 1$
PROOF: Theorem 6.2.6.
$\langle 1 \rangle 2$. $2n < m < 2n + 2$
$\langle 1 \rangle 3$. $m = 2n + 1$

□

Theorem 6.2.9. *The product of two odd integers is odd.*

PROOF: $(2m+1)(2n+1) = 2(2mn+m+n)+1$. □

Corollary 6.2.9.1. *If p is an odd integer and n is a positive integer then p^n is an odd integer.*

Definition 6.2.10 (Exponentiation)**.** Extend the definition of exponentiation so a^n is defined for:

- all real numbers a and non-negative integers n
- all non-zero real numbers a and integers n

as follows:

$$\begin{aligned} a^0 &= 1 \\ a^{-n} &= 1/a^n && (n \text{ a positive integer}) \end{aligned}$$

Theorem 6.2.11 (Laws of Exponents)**.** *For all non-zero reals a and b and integers m and n,*

$$\begin{aligned} a^n a^m &= a^{n+m} \\ (a^n)^m &= a^{nm} \\ a^m b^m &= (ab)^m \end{aligned}$$

PROOF: Easy. □

Theorem 6.2.12. *$\mathbb{Z}$ is countable.*

PROOF: The function that maps an integer n to $2n$ if $n \geqslant 0$ and $-1-2n$ if $n < 0$ is a bijection $\mathbb{Z} \approx \mathbb{N}$. □

6.3 Rational Numbers

Definition 6.3.1 (Rational Number)**.** The set $\mathbb{Q}$ of *rational numbers* is the set of all real numbers that are the quotient of two integers. A real that is not rational is *irrational*.

Theorem 6.3.2. *$\sqrt{2}$ is irrational.*

PROOF:
⟨1⟩1. For any positive rational a, there exist positive integers m and n not both even such that $a = m/n$.
 ⟨2⟩1. LET: a be a positive rational.
 ⟨2⟩2. LET: n be the least positive integer such that na is a positive integer.
 ⟨2⟩3. LET: $m = na$
 ⟨2⟩4. ASSUME: for a contradiction m and n are both even.
 ⟨2⟩5. $m/2 = (n/2)a$
 ⟨2⟩6. Q.E.D.

PROOF: This contradicts the leastness of n ($\langle 2\rangle 2$).
$\langle 1\rangle 2$. ASSUME: for a contradiction $\sqrt{2}$ is rational.
$\langle 1\rangle 3$. PICK positive integers m and n not both even such that $\sqrt{2} = m/n$.
$\langle 1\rangle 4$. $m^2 = 2n^2$
$\langle 1\rangle 5$. m^2 is even.
$\langle 1\rangle 6$. m is even.
PROOF: Theorem 6.2.9.
$\langle 1\rangle 7$. LET: $k = m/2$
$\langle 1\rangle 8$. $4k^2 = 2n^2$
$\langle 1\rangle 9$. $n^2 = 2k^2$
$\langle 1\rangle 10$. n^2 is even.
$\langle 1\rangle 11$. n is even.
PROOF: Theorem 6.2.9.
$\langle 1\rangle 12$. Q.E.D.
PROOF: $\langle 1\rangle 3$, $\langle 1\rangle 6$ and $\langle 1\rangle 11$ form a contradiction.
□

Theorem 6.3.3. $\mathbb{Q}$ *is countably infinite.*

PROOF: The function $\mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ that maps (m, n) to $m/(n+1)$ is a surjection.
□

6.4 Algebraic Numbers

Definition 6.4.1 (Algebraic Number)**.** A real number r is *algebraic* iff there exists a natural number n and rational numbers a_0, a_1, ..., a_{n-1} such that

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Otherwise, r is *transcendental*.

Proposition 6.4.2. *The set of algebraic numbers is countably infinite.*

PROOF: There are countably many finite sequences of rational numbers, and each corresponding polynomial has only finitely many roots. □

Corollary 6.4.2.1. *The set of transcendental numbers is uncountable.*

Chapter 7

Monoid Theory

Definition 7.0.1 (Monoid)**.** A *monoid* is a category with one object.

Definition 7.0.2. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. The monoid $\mathrm{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \to X$ under composition.

Proposition 7.0.3. *For any functor $F : \mathcal{C} \to \mathcal{D}$ and $X \in \mathcal{C}$, we have that $F : \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{D}}(FX)$ is a monoid homomorphism.*

PROOF: Since $F\mathrm{id}_X = \mathrm{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$. $\square$

Chapter 8

Group Theory

Definition 8.0.1. Let **Grp** be the category of small groups and group homomorphisms.

Definition 8.0.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G.

Proposition 8.0.3. *The trivial group is a zero object in* **Grp**.

PROOF: Easy. □

The zero morphism $G \to H$ maps every element in G to e.

Definition 8.0.4. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. We write $\mathrm{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Proposition 8.0.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $X \in \mathcal{C}$. Then $F : \mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{D}}(FX)$ is a group homomorphism.*

PROOF: Since $F\mathrm{id}_X = \mathrm{id}_{FX}$, $F(g \circ f) = Fg \circ Ff$, and $Ff^{-1} = (Ff)^{-1}$. □

Proposition 8.0.6. **Grp** *has products.*

Definition 8.0.7 (Free Product)**.** The product of a family of groups in **Grp** is called the *free product*.

Proposition 8.0.8. **Ab** *has products given by direct sums.*

Chapter 9

Ring Theory

Definition 9.0.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Definition 9.0.2 (Spectrum)**.** Let R be a commutative ring. The *spectrum* of R, $\operatorname{spec} R$, is the set of all prime ideals of R.

Definition 9.0.3 (Zariski Topology)**.** Let R be a commutative ring. The *Zariski topology* on $\operatorname{spec} R$ is the topology where the closed sets are the sets of the form

$$VE := \{p \in \operatorname{spec} R : E \subseteq p\}$$

for any $E \in \mathcal{P}R$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$
$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{A} \in \mathcal{C}$
$\langle 2 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{C}$
$\langle 2 \rangle 2$. LET: $E = \bigcup \{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$
PROVE: $VE = \bigcap \mathcal{A}$
$\langle 2 \rangle 3$. For all $p \in \operatorname{spec} R$, if $E \subseteq p$ then $p \in \bigcap \mathcal{A}$
$\langle 3 \rangle 1$. LET: $p \in \operatorname{spec} R$
$\langle 3 \rangle 2$. ASSUME: $E \subseteq p$
$\langle 3 \rangle 3$. LET: $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$
$\langle 3 \rangle 4$. $E' \subseteq E$
$\langle 3 \rangle 5$. $E' \subseteq p$
$\langle 3 \rangle 6$. $p \in VE'$
$\langle 2 \rangle 4$. For all $p \in \operatorname{spec} R$, if $p \in \bigcap \mathcal{A}$ then $E \subseteq p$
$\langle 3 \rangle 1$. LET: $p \in \bigcap \mathcal{A}$
$\langle 3 \rangle 2$. For all $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$ we have $E' \subseteq p$
$\langle 3 \rangle 3$. $E \subseteq p$
$\langle 1 \rangle 3$. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.
PROOF: Since $VE \cup VE' = V(E \cap E')$

$\langle 1 \rangle 4.$ $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 1.$ $VR = \emptyset$

PROOF: If $p \in VR$ then $R \subseteq p$ contradicting the fact that p is a prime ideal.

$\square$

Definition 9.0.4. For any ring R, let $R - \mathbf{Mod}$ be the category of small R-modules and R-module homomorphisms.

Proposition 9.0.5. $R - \mathbf{Mod}$ *has products and coproducts.*

Chapter 10

Field Theory

Proposition 10.0.1. Field *does not have binary products.*

PROOF: There cannot be a field K with field homomorphisms $K \to \mathbb{Z}_2$ and $K \to \mathbb{Z}_3$, because its characteristic would be both 2 and 3. □

Chapter 11

Linear Algebra

Definition 11.0.1 (Span)**.** Let V be a vector space and $A \subseteq V$. The *span* of A is the set of all linear combinations of elements of A.

Definition 11.0.2 (Independent)**.** Let V be a vector space and $A \subseteq V$. Then A is *linearly independent* iff, whenever

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

where $v_1, \ldots, v_n \in A$, then

$$\alpha_1 = \cdots = \alpha_n = 0 \ .$$

Proposition 11.0.3. *Let V be a vector space, $A \subseteq V$ and $v \in V$. If A is linearly independent and $v \notin \operatorname{span} A$, then $A \cup \{v\}$ is independent.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\alpha_1 v_1 + \cdots + \alpha_n v_n + \beta v = 0$ where $v_1, \ldots, v_n \in A$
$\langle 1 \rangle 2.$ $\beta = 0$
  PROOF: Otherwise $v = (\alpha_1/\beta)v_1 + \cdots + (\alpha_n/\beta)v_n \in \operatorname{span} A$.
$\langle 1 \rangle 3.$ $\alpha_1 = \cdots = \alpha_n = 0$
  PROOF: Since A is linearly independent.
□

Theorem 11.0.4. *Every vector space has a basis.*

PROOF:
$\langle 1 \rangle 1.$ LET: V be a vector space.
$\langle 1 \rangle 2.$ PICK a maximal linearly independent set $\mathcal{B}$.
  PROOF: By Tukey's Lemma.
$\langle 1 \rangle 3.$ $\operatorname{span} \mathcal{B} = V$
  PROOF: Proposition 11.0.3.
□

Definition 11.0.5. For any field K, we write $\mathbf{Vect}_K$ for $K - \mathbf{Mod}$.

Dual space functor $\mathbf{Vect}_K^{\mathrm{op}} \to \mathbf{Vect}_K$.

Chapter 12

Topology

12.1 Topological Spaces

Definition 12.1.1 (Topological Space)**.** Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say $(X, \mathcal{O})$ is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call $\mathcal{O}$ the *topology* of the toplogical space, and call its elements *open* sets. We shall often write X for the topological space $(X, \mathcal{O})$.

Definition 12.1.2 (Discrete Topology)**.** For any set X, the power set $\mathcal{P}X$ is called the *discrete* topology on X.

Proposition 12.1.3. *For any set X, the discrete topology on X is a topology on X.*

Definition 12.1.4 (Indiscrete Topology)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Proposition 12.1.5. *For any set X, the indiscrete topology on X is a topology on X.*

Definition 12.1.6 (Cofinite Topology)**.** For any set X, the *cofinite* topology is $\{X - U : U \subseteq X \text{ is finite}\}$.

Definition 12.1.7 (Cocountable Topology)**.** For any set X, the *cocountable* topology is $\{X - U : U \subseteq X \text{ is countable}\}$.

Definition 12.1.8 (Sierpiński Two-Point Space)**.** The *Sierpiński two-point space* is $\{0, 1\}$ under the topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Proposition 12.1.9. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

Proposition 12.1.10. *The intersection of a set of topologies on a set X is a topology on X.*

Definition 12.1.11 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 12.1.12. *A set B is open if and only if $X - B$ is closed.*

Proposition 12.1.13. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that $\mathcal{C}$ is the set of closed sets if and only if:*

- *For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$*
- *For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*
- *$\emptyset \in \mathcal{C}$*

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Theorem 12.1.14. *There are infinitely many primes.*

Furstenberg's proof:
PROOF:
⟨1⟩1. For $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$,
LET: $S(a, b) := \{an + b : n \in \mathbb{N}\}$
⟨1⟩2. LET: $\mathcal{T}$ be the topology generated by the basis $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$
⟨2⟩1. For every $n \in \mathbb{Z}$, there exist a, b such that $n \in S(a, b)$.
PROOF: $n \in S(n, 0)$
⟨2⟩2. If $n \in S(a_1, b_1) \cap S(a_2, b_2)$ then there exist a_3, b_3 such that $n \in S(a_3, b_3) \subseteq S_(a_1, b_1) \cap S(a_2, b_2)$
⟨3⟩1. LET: $d = \mathrm{lcm}(a_1, a_2)$
PROVE: $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$
⟨3⟩2. LET: $d = a_1 k = a_2 l$
⟨3⟩3. LET: $n = a_1 c + b_1 = a_2 d + b_2$
⟨3⟩4. LET: $z \in \mathbb{Z}$
PROVE: $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$
⟨3⟩5. $dz + n \in S(a_1, b_1)$
PROOF:

$$\begin{aligned} dz + n &= a_1 kz + a_1 c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

⟨3⟩6. $dz + n \in S(a_2, b_2)$
PROOF: Similar.
⟨1⟩3. For all $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$ we have $S(a, b)$ is closed.
⟨2⟩1. LET: $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$
⟨2⟩2. LET: $n \in \mathbb{Z} - S(a, b)$
⟨2⟩3. $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$

⟨3⟩1. LET: $x \in S(a, n)$
⟨3⟩2. ASSUME: for a contradiction $x \in S(a, b)$
⟨3⟩3. PICK m such that $x = am + b$
⟨3⟩4. PICK l such that $x = al + n$
⟨3⟩5. $n = a(m - l) + b$
⟨3⟩6. $n \in S(a, b)$
⟨3⟩7. Q.E.D.
PROOF: This contradicts ⟨2⟩2.

⟨1⟩4.
$$\mathbb{Z} - \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$$
PROOF: Since every integer except 1 and -1 is divisible by a prime.

⟨1⟩5. No nonempty finite set is open.
⟨2⟩1. LET: U be a nonempty open set
⟨2⟩2. PICK $n \in U$
⟨2⟩3. There exist a, b such that $n \in S(a, b) \subseteq U$
⟨2⟩4. U is infinite.

⟨1⟩6. $\mathbb{Z} - \{1, -1\}$ is not closed.
⟨1⟩7. $\bigcup_{p \text{ prime}} S(p, 0)$ is not closed.
⟨1⟩8. The union of finitely many closed sets is closed.
⟨1⟩9. There are infinitely many primes.
□

Definition 12.1.15 (Neighbourhood)**.** Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x, and x is an *interior* point of U, iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 12.1.16. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 12.1.17. *Let X be a set and $\mathcal{N} : X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that, for all $x \in X$, we have $\mathcal{N}_x$ is the set of neighbourhoods of x, if and only if:*

- *For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$*
- *For all $x \in X$ we have $X \in \mathcal{N}_x$*
- *For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$*
- *For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$*
- *For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M . M \in \mathcal{N}_y$.*

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{U : \forall x \in U . U \in \mathcal{N}_x\}$.

Definition 12.1.18 (Exterior Point)**.** Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x.

Definition 12.1.19 (Boundary Point)**.** Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B.

Definition 12.1.20 (Interior)**.** Let X be a topological space and $B \subseteq X$. The *interior* of B, B°, is the set of all interior points of B.

Proposition 12.1.21. *The interior of B is the union of all the open sets included in B.*

Definition 12.1.22 (Closure)**.** Let X be a topological space and $B \subseteq X$. The *closure* of B, $\overline{B}$, is the set of all points that are not exterior points of B.

Proposition 12.1.23. *The closure of B is the intersection of all the closed sets that include B.*

Proposition 12.1.24. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 12.1.25 (Kuratowski Closure Axioms)**.** *Let X be a set and $\overline{} : \mathcal{P}X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ such that, for all $B \subseteq X$, $\overline{B}$ is the closure of B, if and only if:*

- $\overline{\emptyset} = \emptyset$
- *For all $A \subseteq X$ we have $A \subseteq \overline{A}$*
- *For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$*
- *For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

In this case, $\mathcal{O}$ is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

Definition 12.1.26 (Finer, Coarser)**.** Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the set X. Then $\mathcal{T}$ is *coarser*, *smaller* or *weaker* than $\mathcal{T}'$, or $\mathcal{T}'$ is *finer*, *larger* or *weaker* than $\mathcal{T}$, iff $\mathcal{T} \subseteq \mathcal{T}'$.

12.1.1 Subspaces

Definition 12.1.27 (Subspace)**.** Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

Example 12.1.28. The *unit sphere* S^2 is $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a subspace of $\mathbb{R}^3$.

Theorem 12.1.29. *Let X be a topological space and (Y, i) a subset of X. Then the subspace topology on Y is the unique topology such that, for every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f : Z \to X$ is continuous.*

Proof:

$\langle 1\rangle 1$. If we give Y the subspace topology then, for every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f$ is continuous.

$\langle 2\rangle 1$. Given Y the subspace topology.

$\langle 2\rangle 2$. LET: Z be a topological space.

$\langle 2\rangle 3$. LET: $f : Z \to Y$

$\langle 2\rangle 4$. If f is continuous then $i \circ f$ is continuous.

PROOF: Since i is continuous.

$\langle 2\rangle 5$. If $i \circ f$ is continuous then f is continuous.

$\langle 3\rangle 1$. ASSUME: $i \circ f$ is continuous.

$\langle 3\rangle 2$. LET: U be open in Y.

$\langle 3\rangle 3$. $f^{-1}(i^{-1}(i(U))$ is open in Z.

$\langle 3\rangle 4$. $f^{-1}(U)$ is open in Z.

$\langle 1\rangle 2$. If, for every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f$ is continuous.

$\langle 2\rangle 1$. ASSUME: For every topological space Z and function $f : Z \to Y$, we have f is continuous if and only if $i \circ f$ is continuous.

$\langle 2\rangle 2$. i is continuous.

$\langle 2\rangle 3$. For every open set U in X, we have $i^{-1}(X)$ is open in Y

$\langle 2\rangle 4$. LET: Z be the set Y under the subspace topology and $f : Z \to Y$ the identity function.

$\langle 2\rangle 5$. $i \circ f$ is continuous.

$\langle 2\rangle 6$. f is continuous.

$\langle 2\rangle 7$. Every set open in Y is open in Z.

□

12.1.2 Topological Disjoint Union

Definition 12.1.30 (Coproduct Topology)**.** Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The *coproduct topology* on $\coprod_{\alpha \in A} X_\alpha$ is

$$\mathcal{T} = \left\{ \coprod_{\alpha \in A} U_\alpha : \{U_\alpha\}_{\alpha \in A} \text{ is a family with } U_\alpha \text{ open in } X_\alpha \text{ for all } \alpha \right\} .$$

We prove this is a topology.

PROOF:

$\langle 1\rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF:

$$\bigcup_{i \in I} \coprod_{\alpha \in A} U_{i\alpha} = \coprod_{\alpha \in A} \bigcup_{i \in I} U_{i\alpha}$$

$\langle 1\rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF:

$$\coprod_{\alpha \in A} U_\alpha \cap \coprod_{\alpha \in A} V_\alpha = \coprod_{\alpha \in A} (U_\alpha \cap V_\alpha)$$

$\langle 1\rangle 3$. $\coprod_{\alpha \in A} X_\alpha \in \mathcal{T}$

PROOF: Since every X_α is open in X_α.
□

Proposition 12.1.31. *The coproduct topology is the finest topology on $\coprod_{\alpha \in A} X_\alpha$ such that every injection $\kappa_\alpha : X_\alpha \to \coprod_{\alpha \in A} X_\alpha$ is continuous.*

PROOF:
⟨1⟩1. LET: $P = \coprod_{\alpha \in A} X_\alpha$
⟨1⟩2. LET: $\mathcal{T}_c$ be the coproduct topology.
⟨1⟩3. LET: $\mathcal{T}$ be any topology on P
⟨1⟩4. For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \to (P, \mathcal{T}_c)$ is continuous.
⟨2⟩1. LET: $\alpha \in A$
⟨2⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with each U_α open in X_α.
⟨2⟩3. For all $\alpha \in A$, we have ${\kappa_\alpha}^{-1}(\coprod_{\alpha \in A} U_\alpha)$ is open in X_α.
PROOF: Since ${\kappa_\alpha}^{-1}(\coprod_{\alpha \in A} U_\alpha) = U_\alpha$.
⟨1⟩5. If, for all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \to (P, \mathcal{T})$ is continuous, then $\mathcal{T} \subseteq \mathcal{T}_c$.
⟨2⟩1. ASSUME: For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \to (P, \mathcal{T})$ is continuous.
⟨2⟩2. LET: $U \in \mathcal{T}$
⟨2⟩3. For all $\alpha \in a$, we have ${\kappa_\alpha}^{-1}(U)$ is open in X_α.
⟨2⟩4. $U = \coprod_{\alpha \in A} {\kappa_\alpha}^{-1}(U) \in \mathcal{T}_c$
□

Theorem 12.1.32. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The coproduct topology is the unique topology on $\coprod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : \coprod_{\alpha \in A} X_\alpha \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.*

PROOF:
⟨1⟩1. LET: $X = \coprod_{\alpha \in A} X_\alpha$
⟨1⟩2. LET: $\mathcal{T}_c$ be the coproduct topology.
⟨1⟩3. For every topological space Z and function $f : (X, \mathcal{T}_c) \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
⟨2⟩1. LET: Z be a topological space.
⟨2⟩2. LET: $f : X \to Z$
⟨2⟩3. If f is continuous then $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
PROOF: Because the composite of two continuous functions is continuous.
⟨2⟩4. If $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous then f is continuous.
⟨3⟩1. ASSUME: $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
⟨3⟩2. LET: U be open in Z
⟨3⟩3. For all $\alpha \in A$ we have ${\kappa_\alpha}^{-1}(f^{-1}(U))$ is open in X_α
⟨3⟩4. $f^{-1}(U) = \coprod_{\alpha \in A} {\kappa_\alpha}^{-1}(f^{-1}(U))$
⟨3⟩5. $f^{-1}(U)$ is open in X
⟨1⟩4. For any topology $\mathcal{T}$ on X, if for every topological space Z and function $f : (X, \mathcal{T}) \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous, then $\mathcal{T} = \mathcal{T}_c$.
⟨2⟩1. LET: $\mathcal{T}$ be a topology on X.

$\langle 2\rangle 2$. ASSUME: For every topological space Z and function $f : (X, \mathcal{T}) \to Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.

$\langle 2\rangle 3$. $\mathcal{T} \subseteq \mathcal{T}_c$

$\langle 3\rangle 1$. For all $\alpha \in A$ we have $\kappa_\alpha : X_\alpha \to (X, \mathcal{T})$ is continuous.

PROOF: From $\langle 2\rangle 1$ since id_X is continuous.

$\langle 3\rangle 2$. $\mathcal{T} \subseteq \mathcal{T}_c$

PROOF: Proposition 12.1.31.

$\langle 2\rangle 4$. $\mathcal{T}_c \subseteq \mathcal{T}$

$\langle 3\rangle 1$. LET: $f : (X, \mathcal{T}) \to (X, \mathcal{T}_c)$ be the identity function.

$\langle 3\rangle 2$. $f \circ \kappa_\alpha$ is continuous for all α.

$\langle 3\rangle 3$. f is continuous.

PROOF: $\langle 2\rangle 1$

$\langle 3\rangle 4$. $\mathcal{T}_c \subseteq \mathcal{T}$

□

12.1.3 Product Topology

Definition 12.1.33 (Product Topology)**.** Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

Proposition 12.1.34. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The product topology on $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis $\mathcal{B} = \{\prod_{\alpha \in A} U_\alpha :$ for all $\alpha \in A, U_\alpha$ is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A\}$.*

PROOF:

$\langle 1\rangle 1$. $\mathcal{B}$ is a basis for a topology.

$\langle 1\rangle 2$. LET: $\mathcal{T}$ be the topology generated by $\mathcal{B}$.

$\langle 1\rangle 3$. LET: $\mathcal{T}_p$ be the product topology.

$\langle 1\rangle 4$. $\mathcal{T} \subseteq \mathcal{T}_p$

$\langle 2\rangle 1$. LET: $B \in \mathcal{B}$

$\langle 2\rangle 2$. LET: $B = \prod_{\alpha \in A} U_\alpha$ with each U_α open in X_α and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$

$\langle 2\rangle 3$. $B = {\pi_{\alpha_1}}^{-1}(U_{\alpha_1}) \cap \cdots \cap {\pi_{\alpha_n}}^{-1}(U_{\alpha_n})$

$\langle 2\rangle 4$. $B \in \mathcal{T}_p$

$\langle 1\rangle 5$. $\mathcal{T}_p \subseteq \mathcal{T}$

$\langle 2\rangle 1$. For every $\alpha \in A$ we have π_α is continuous.

PROOF: Since $\pi^{-1}(U)$ is open for every U open in X_α.

□

Theorem 12.1.35. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Then the product topology on $\prod_{\alpha \in A} X_\alpha$ is the unique topology such that, for every topological space Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f : Z \to X_\alpha$ is continuous.*

PROOF:

⟨1⟩1. If we give $\prod_{\alpha \in A} X_\alpha$ the product topology, then for every topological space Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.

⟨2⟩1. Give $\prod_{\alpha \in A} X_\alpha$ the product topology.

⟨2⟩2. LET: Z be a topological space.

⟨2⟩3. LET: $f : Z \to \prod_{\alpha \in A} X_\alpha$

⟨2⟩4. If f is continuous then, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.

PROOF: Since the composite of two continuous functions is continuous.

⟨2⟩5. If, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then f is continuous.

⟨3⟩1. ASSUME: For all $\alpha \in A$ we have $\pi_\alpha \circ f$ is continuous.

⟨3⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with U_α open in X_α such that $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \ldots, \alpha_n$.

⟨3⟩3. For all α we have $f^{-1}({\pi_\alpha}^{-1}(U_\alpha))$ is open in Z.

⟨3⟩4. $f^{-1}(\prod_\alpha U_\alpha)$ is open in Z

PROOF: Since $f^{-1}(\prod_\alpha U_\alpha) = f^{-1}({\pi_{\alpha_1}}^{-1}(U_{\alpha_1})) \cap \cdots \cap f^{-1}({\pi_{\alpha_n}}^{-1}(U_{\alpha_n}))$.

⟨1⟩2. If $\mathcal{T}$ is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological pace Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then $\mathcal{T}$ is the product topology.

⟨2⟩1. ASSUME: $\mathcal{T}$ is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological pace Z and function $f : Z \to \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.

⟨2⟩2. LET: $\mathcal{T}_p$ be the product topology.

⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_p$

⟨3⟩1. LET: $Z = (\prod_\alpha X_\alpha, \mathcal{T}_p)$

⟨3⟩2. LET: $f : Z \to \prod_\alpha X_\alpha$ be the identity function

⟨3⟩3. For all α we have $\pi_\alpha \circ f$ is continuous.

⟨3⟩4. f is continuous.

PROOF: ⟨2⟩1

⟨3⟩5. Every set open in $\mathcal{T}$ is open in $\mathcal{T}_p$

⟨2⟩4. $\mathcal{T}_p \subseteq \mathcal{T}$

⟨3⟩1. $\mathrm{id}_{\prod_\alpha X_\alpha}$ is continuous.

⟨3⟩2. For all α we have π_α is continuous.

PROOF: ⟨2⟩1

⟨3⟩3. $\mathcal{T}_p \subseteq \mathcal{T}$

PROOF: Since $\mathcal{T}_p$ is the coarsest topology such that every π_α is continuous.

□

Example 12.1.36. It is not true that, for any function $f : \prod_{\alpha \in A} X_\alpha \to Y$, if f is continuous in every variable separately then f is continuous.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

Then f is continuous in x and in y, but is not continuous.

Proposition 12.1.37. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology on $\prod_{i \in I} Y_i$ as a subspace of $\prod_{i \in I} X_i$.*

PROOF:
⟨1⟩1. Given $\prod_{i \in I} Y_i$ the subspace topology.
⟨1⟩2. LET: $\iota : \prod_{i \in I} Y_i$ be the inclusion.
⟨1⟩3. LET: Z be any topological space.
⟨1⟩4. LET: $f : Z \to \prod_{i \in I} Y_i$
⟨1⟩5. f is continuous if and only if, for all $i \in I$, we have $\pi_i \circ f$ is continuous.
PROOF:

$$\begin{aligned} f \text{ is continuous} &\Leftrightarrow \iota \circ f : Z \to \prod_{i \in I} X_i \text{ is continuous} && \text{(Theorem 12.1.29)} \\ &\Leftrightarrow \forall i \in I . \pi_i \circ \iota \circ f : Z \to X_i \text{ is continuous} && \text{(Theorem 12.1.35)} \\ &\Leftrightarrow \forall i \in I . \iota_i \circ \pi_i \circ f : Z \to X_i \text{ is continuous} \\ &\Leftrightarrow \forall i \in I . \pi_i \circ f : Z \to Y_i \text{ is continuous} && \text{(Theorem 12.1.29)} \end{aligned}$$

where ι_i is the inclusion $Y_i \to X_i$.
□

12.1.4 Bases

Definition 12.1.38 (Basis)**.** Let X be a topological space. A *basis* for the topology on X is a set of open sets $\mathcal{B}$ such that every open set is the union of a subset of $\mathcal{B}$. The elements of $\mathcal{B}$ are called *basic open neighbourhoods* of their elements.

Proposition 12.1.39. *Let X be a set. The set of all one-element subsets of X is a basis for the discrete topology on X.*

Proposition 12.1.40. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Then the topology on X is the coarsest topology that includes $\mathcal{B}$.*

Definition 12.1.41 (Order Topology)**.** Let X be a linearly ordered set. The *order topology* on X is the topology generated by the open interval (a, b) as well as the open rays $(a, +\infty)$ and $(-\infty, b)$ for $a, b \in X$.

The *standard topology* on $\mathbb{R}$ is the order topology.

Proposition 12.1.42. *Let X be a linearly ordered set. Then the order topology is generated by the basis consisting of:*

- *all open intervals (a, b)*
- *all intervals of the form $[\bot, b)$ where $\bot$ is the least element of X, if any*
- *all intervals of the form $(a, \top]$ where $\top$ is the greatest element of X, if any.*

Proposition 12.1.43. *Let X be a linearly ordered set. The open rays in X form a subbasis for the order topology.*

Definition 12.1.44 (Lower Limit Topology)**.** The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals $[a, b)$.

We write $\mathbb{R}_l$ for $\mathbb{R}$ under the lower limit topology.

Definition 12.1.45 (K-topology)**.** Let $K = \{1/n : n \in \mathbb{Z}_+\}$. The *$K$-topology* on $\mathbb{R}$ is the topology generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) - K$.

We write $\mathbb{R}_K$ for $\mathbb{R}$ under the K -topology.

Proposition 12.1.46. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proposition 12.1.47. *Let X be a topological space and $\mathcal{B} \subseteq X$. Assume that, for every open set U and element $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then $\mathcal{B}$ is a basis for the topology on X.*

Proposition 12.1.48. *Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a basis for a topology on X if and only if:*

1. $\bigcup \mathcal{B} = X$
2. *For all $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.*

In this case, the topology is unique and is the set of all unions of subsets of $\mathcal{B}$. We call it the topology generated *by $\mathcal{B}$.*

Proposition 12.1.49. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if, for every $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Corollary 12.1.49.1. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$ but are not comparable to one another.*

12.1.5 Subbases

Definition 12.1.50 (Subbasis)**.** Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of $\mathcal{S}$.

Proposition 12.1.51. *Let X be a set and $\mathcal{S} \subseteq X$. Then $\mathcal{S}$ is a subbasis for a topology on X if and only if $\bigcup \mathcal{S} = X$, in which case the topology is unique and is the set of all unions of finite intersections of elements of $\mathcal{S}$.*

Proposition 12.1.52. *Let X be a topological space. Let $\mathcal{S}$ be a subbasis for the topology on X. Then the topology on X is the coarsest topology that includes $\mathcal{S}$.*

Definition 12.1.53 (Space with Basepoint)**.** A *space with basepoint* is a pair (X, x) where X is a topological space and $x \in X$.

12.1.6 Countability Axioms

Definition 12.1.54 (Neighbourhood Basis)**.** Let X be a topological space and $x_0 \in X$. A *neighbourhood basis* of x_0 is a set $\mathcal{U}$ of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of $\mathcal{U}$.

Definition 12.1.55 (First Countable)**.** A topological space is *first countable* iff every point has a countable neighbourhood basis.

Definition 12.1.56 (Second Countable)**.** A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.
A subspace of a first countable space is first countable.
A subspace of a second countable space is second countable.
$\mathbb{R}^n$ is second countable.
An uncountable discrete space is first countable but not second countable.

Proposition 12.1.57. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.*

PROOF:
⟨1⟩1. For all $\lambda \in \Lambda$, PICK U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.
⟨1⟩2. For all $\lambda \in \Lambda$, PICK $x_\lambda \in U_\lambda$.
⟨1⟩3. ASSUME: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda \in \Lambda}$.
⟨1⟩4. PICK $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$
⟨1⟩5. There is no $U \in \lambda$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$
⟨1⟩6. Q.E.D.
PROOF: This is a contradiction.
□

12.2 Continuous Functions

Definition 12.2.1 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* iff, for every open set V in Y, the inverse image $f^{-1}(V)$ is open in X.

Proposition 12.2.2.

1. *id_X is continuous*
2. *The composite of two continuous functions is continuous.*
3. *If $f : X \to Y$ is continuous and $X_0 \subseteq X$ then $f \restriction X_0 : X_0 \to Y$ is continuous.*
4. *If $f : X + Y \to Z$, then f is continuous iff $f \circ \kappa_1 : X \to Z$ and $f \circ \kappa_2 : Y \to Z$ are continuous.*
5. *If $f : Z \to X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.*

Proposition 12.2.3. *Let X and Y be topological spaces. Let $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
  PROOF: Since every element of $\mathcal{B}$ is open in Y.
⟨1⟩2. If, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X, then f is continuous.
  ⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
  ⟨2⟩2. LET: U be open in Y.
  ⟨2⟩3. LET: $x \in f^{-1}(U)$
  ⟨2⟩4. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq U$.
  ⟨2⟩5. $x \in f^{-1}(B) \subseteq f^{-1}(U)$
□

Definition 12.2.4 (Homeomorphism)**.** Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Definition 12.2.5 (Retraction)**.** Let X be a topological space and A a subspace of X. A continuous function $\rho : X \to A$ is a *retraction* iff $\rho \restriction A = \mathrm{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 12.2.6. Let **Top** be the category of small topological spaces and continuous functions.

Proposition 12.2.7. $\emptyset$ *is initial in* **Top**.

Proposition 12.2.8. 1 *is terminal in* **Top**.

Forgetful functor **Top** → **Set**.
Basepoint preserving continuous functor.

Proposition 12.2.9. *Let $(X, \mathcal{T})$ be a topological space. Let S be the Sierpiński two-point space. Define $\Phi : \mathcal{T} \to \mathbf{Top}[X, S]$ by $\Phi(U)(x) = 1$ iff $x \in U$. Then Φ is a bijection.*

PROOF:
⟨1⟩1. For all $U \in \mathcal{T}$ we have $\Phi(U)$ is continuous.
  ⟨2⟩1. LET: $U \in \mathcal{T}$
  ⟨2⟩2. $\Phi(U)(\{1\})$ is open.
    PROOF: Since $\Phi(U)(\{1\}) = U$.
⟨1⟩2. Φ is injective.
  PROOF: If $\Phi(U) = \Phi(V)$ then we have $\forall x(x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$.
⟨1⟩3. Φ is surjective.
  PROOF: Given $f : X \to S$ continuous we have $\Phi(f^{-1}(1)) = f$.
□

12.2.1 Paths

Definition 12.2.10 (Path)**.** A *path* in a topological space X is a continuous function $[0,1] \to X$.

12.2.2 Loops

Definition 12.2.11 (Loop)**.** A *loop* in a topological space X is a path $\alpha : [0,1] \to X$ such that $\alpha(0) = \alpha(1)$.

12.3 Convergence

Definition 12.3.1 (Convergence)**.** Let X be a topological space. Let (x_n) be a sequence in X. A point $a \in X$ is a *limit* of the sequence iff, for every neighbourhood U of a, there exists n_0 such that $\forall n \geqslant n_0 . x_n \in U$.

Convergence in a product space is pointwise convergence.

If $f : X \to Y$ is continuous and $x_n \to l$ in X then $f(x_n) \to f(l)$ in Y.

Example 12.3.2. The converse does not hold.

Let X be the set of all continuous functions $[0,1] \to [-1,1]$ under the product topology. Let $i : X \to L^2([0,1])$ be the inclusion.

If $f_n \to f$ then $i(f_n) \to i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K_\epsilon . \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0,1]} U_\lambda$ where $U_\lambda = [-1,1]$ except for $\lambda = \lambda_1, \ldots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \ldots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 12.3.3. *The converse does hold for first countable spaces. If $f : X \to Y$ where X is first countable, and Y is a topological space, and whenever $x_n \to x$ then $f(x_n) \to f(x)$, then f is continuous.*

12.4 Subspaces

Definition 12.4.1 (Subspace)**.** Let X be a topological space, Y a set, and $f : Y \to X$. The *subspace topology* on Y induced by f is $\mathcal{T} = \{i^{-1}(U) : U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF: Since $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$.

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: Since $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.

$\langle 1 \rangle 3$. $Y \in \mathcal{T}$

PROOF: Since $Y = f^{-1}(X)$.

∎

Proposition 12.4.2. *Let X be a topological space, Y a set and $f : Y \to X$ a function. Then the subspace topology on Y is the coarsest topology such that f is continuous.*

PROOF: Immediate from definition. ∎

12.5 Embedding

Definition 12.5.1 (Embedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *embedding* iff f is injective and the topology on X is the subspace induced by f.

12.6 Quotient Spaces

Definition 12.6.1 (Quotient Topology)**.** Let X be a topological space, S a set, and $\pi : X \twoheadrightarrow S$ be a surjection. The *quotient topology* on S induced by π is $\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
  PROOF: Since $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}$.
⟨1⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.
  PROOF: Since $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$.
⟨1⟩3. $X \in \mathcal{T}$
  PROOF: Since $X = \pi^{-1}(Y)$.
∎

Proposition 12.6.2. *Let X be a topological space, S a set and $\pi : X \twoheadrightarrow S$ a surjection. Then the quotient topology on S is the finest topology such that π is continuous.*

PROOF: Immediate from definitions. ∎

Definition 12.6.3 (Quotient Map)**.** Let X and S be topological spaces and $\pi : X \to S$. Then π is a *quotient map* iff π is surjective and the topology on S is the quotient topology induced by π.

Theorem 12.6.4. *Let X be a topological space, let S be a set, and let $\pi : X \twoheadrightarrow S$ be surjective. Then the quotient topology on S is the unique topology such that, for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.*

PROOF:

⟨1⟩1. If S is given the quotient topology, then for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.
  ⟨2⟩1. Give S the quotient topology.
  ⟨2⟩2. LET: Z be a topological space.
  ⟨2⟩3. LET: $f : S \to Z$
  ⟨2⟩4. If f is continuous then $f \circ \pi$ is continuous.
  PROOF: The composite of two continuous functions is continuous.
  ⟨2⟩5. If $f \circ \pi$ is continuous then f is continuous.
    ⟨3⟩1. ASSUME: $f \circ \pi$ is continuous.
    ⟨3⟩2. LET: U be open in Z.
    ⟨3⟩3. $\pi^{-1}(f^{-1}(U))$ is open in X.
    ⟨3⟩4. $f^{-1}(U)$ is open in S.
⟨1⟩2. If S is given a topology such that, for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous, then that topology is the quotient topology.
  ⟨2⟩1. Give S a topology such that, for every topological space Z and function $f : S \to Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.
  ⟨2⟩2. LET: $U \subseteq S$
  ⟨2⟩3. If $\pi^{-1}(U)$ is open in X then U is open in S.
    ⟨3⟩1. LET: Z be S under the quotient topology induced by π.
    ⟨3⟩2. LET: $f : S \to Z$ be the identity function.
    ⟨3⟩3. $f \circ \pi$ is continuous.
    ⟨3⟩4. f is continuous.
    PROOF: ⟨2⟩1
    ⟨3⟩5. U is open in Z.
    ⟨3⟩6. U is open in X.
  ⟨2⟩4. If U is open in S then $\pi^{-1}(U)$ is open in X.
  PROOF: Since π is continuous (taking $Z = S$ and $f = \mathrm{id}_S$ in ⟨2⟩1).
□

Corollary 12.6.4.1. *Let $\pi : X \twoheadrightarrow S$ be a quotient map. Let Z be a topological space. Let $f : X \to Z$ be continuous. Then there exists a continuous map $g : S \to Z$ such that $f = g \circ \pi$ if and only if, for all $s \in S$, we have f is constant on $\pi^{-1}(s)$.*

Proposition 12.6.5. *Let Z be a topological space. Define $\pi : [0,1] \to S^1$ by $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$. Given any continuous function $f : S^1 \to Z$, we have $f \circ \pi$ is a loop in Z. This defines a bijection between* **Top**$[S^1, Z]$ *and the set of loops in Z.*

PROOF: Since π is a quotient map. □

Definition 12.6.6 (Projective Space)**.** The *projective space* $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by $\sim$ where $x \sim \lambda x$ for all $x \in \mathbb{R}^{n+1} - \{0\}$ and $\lambda \in \mathbb{R}$.

Definition 12.6.7 (Torus)**.** The *torus* T is the quotient of $[0,1]^2$ by $\sim$ where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$.

Definition 12.6.8 (Möbius Band)**.** The *Möbius band* is the quotient of $[0,1]^2$ by $\sim$ where $(0,y) \sim (1, 1-y)$.

Definition 12.6.9 (Klein Bottle)**.** The *Klein bottle* is the quotient of $[0,1]^2$ by $\sim$ where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,1-y)$.

Proposition 12.6.10. *$\mathbb{RP}^2$ is the quotient of $[0,1]^2$ by $\sim$ where $(x,0) \sim (1-x,1)$ and $(0,y) \sim (1,1-y)$.*

PROOF:TODO

Example 12.6.11. Let $\{X_i\}_{i\in I}$ be a family of topological spaces and $\{Y_i\}_{i \in I}$ a family of sets. Let $q_i : X_i \twoheadrightarrow Y_i$ be a surjective function for all $i \in I$. Give each Y_i the quotient topology. It is not true in general that the product topology on $\prod_{i\in I} Y_i$ is the same as the quotient topology induced by $\prod_{i \in I} q_i : \prod_{i\in I} X_i \twoheadrightarrow \prod_{i \in I} Y_i$.

PROOF:
⟨1⟩1. LET: $X^* = \mathbb{R} - \mathbb{Z}_+ + \{b\}$ be the quotient space obtained from $\mathbb{R}$ by identifying the subset $\mathbb{Z}_+$ to the point b.
⟨1⟩2. LET: $p : \mathbb{R} \to X^*$ be the quotient map.
PROVE: $p \times \mathrm{id}_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \to X^* \times \mathbb{Q}$ is not a quotient map.
⟨1⟩3. For $n \in \mathbb{Z}_+$,
LET: $c_n = \sqrt{2}/n$
⟨1⟩4. For $n \in \mathbb{Z}_+$,
LET: $U_n = \{(x,y) \in \mathbb{Q} \times \mathbb{R} : n - 1/4 < x < n + 1/4$ and $((y > x + c_n - n$ and $y > -x + c_n + n)$ or $(y < x + c_n - n$ and $y < -x + c_n + n))\}$
⟨1⟩5. For all $n \in \mathbb{Z}_+$, U_n is open in $\mathbb{R} \times \mathbb{Q}$
⟨1⟩6. For all $n \in \mathbb{Z}_+$ we have $\{n\} \times \mathbb{Q} \subseteq U_n$
⟨1⟩7. LET: $U = \bigcup_{n \in \mathbb{Z}_+} U_n$
⟨1⟩8. U is open in $\mathbb{R} \times \mathbb{Q}$.
⟨1⟩9. U is saturated with respect to $p \times \mathrm{id}_{\mathbb{Q}}$.
⟨1⟩10. LET: $U' = (p \times \mathrm{id}_{\mathbb{Q}})(U)$
⟨1⟩11. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$.

12.7 Connected Spaces

Definition 12.7.1 (Connected)**.** A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 12.7.2. *The continuous image of a connected space is connected.*

Proposition 12.7.3. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are connected, then X is connected.*

Proposition 12.7.4. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 12.7.5 (Path-connected)**.** A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0,1] \to X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 12.7.6. *The continuous image of a path connected space is path connected.*

Proposition 12.7.7. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are path connected, then X is path connected.*

Proposition 12.7.8. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

12.8 Hausdorff Spaces

Definition 12.8.1 (Hausdorff)**.** A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 12.8.2. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 12.8.3. *1. Every subspace of a Hausdorff space is Hausdorff.*

2. The disjoint union of two Hausdorff spaces is Hausdorff.

3. The product of two Hausdorff spaces is Hausdorff.

Proposition 12.8.4. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \to B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X, then $f = g$.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.
$\langle 1 \rangle 2.$ PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.
$\langle 1 \rangle 3.$ PICK $x \in f^{-1}(U) \cap g^{-1}(V)$
$\langle 1 \rangle 4.$ $f(x) = g(x) \in U \cap V$
$\langle 1 \rangle 5.$ Q.E.D.
PROOF: This is a contradiction.
□

Proposition 12.8.5. *Let X and Y be metric spaces. Let $f : X \to Y$ be uniformly continuous. Let $\hat{X}$ and $\hat{Y}$ be the completions of X and Y. Then f extends uniquely to a continuous map $\hat{X} \to \hat{Y}$.*

PROOF: The extension maps $\lim_{n \to \infty} x_n$ to $\lim_{n \to \infty} f(x_n)$. □

12.9 Separable Spaces

Definition 12.9.1 (Separable)**.** A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

12.10 Sequential Compactness

Definition 12.10.1 (Sequentially Compact)**.** A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

12.11 Compactness

Definition 12.11.1 (Compact)**.** A topological space is *compact* iff every open cover has a finite subcover.

Proposition 12.11.2. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P. Then $X \in P$.*

PROOF:
$\langle 1 \rangle 1.$ P is an open cover of X
$\langle 1 \rangle 2.$ PICK a finite subcover $U_1, \ldots, U_n \in P$
$\langle 1 \rangle 3.$ $X = U_1 \cup \cdots \cup U_n \in P$
□

Corollary 12.11.2.1. *Let f be a compact space and $f : X \to \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. □

Proposition 12.11.3. *The continuous image of a compact space is compact.*

Proposition 12.11.4. *A closed subspace of a compact space is compact.*

Proposition 12.11.5. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. *X and Y are compact.*
2. *$X + Y$ is compact.*
3. *$X \times Y$ is compact.*

Proposition 12.11.6. *A compact subspace of a Hausdorff space is closed.*

Proposition 12.11.7. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 12.11.8. *A first countable compact space is sequentially compact.*

12.12 Quotient Spaces

Definition 12.12.1 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. The *quotient topology* on $X/\sim$ is defined by: $U \in \mathcal{P}X$ is open in $X/\sim$ if and only if $\pi^{-1}(U)$ is open in X.

Proposition 12.12.2. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $f : X/\sim \to Y$. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proposition 12.12.3. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $\phi : Y \to X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \to X$ such that $\pi \circ \Phi = \phi \restriction U$. Then ϕ is continuous.

Proposition 12.12.4. *A quotient of a connected space is connected.*

Proposition 12.12.5. *A quotient of a path connected space is path connected.*

Proposition 12.12.6. *Let X be a topological space and $\sim$ an equivalence relation on X. If $X/\sim$ is Hausdorff then every equivalence class of $\sim$ is closed in X.*

Definition 12.12.7. Let X be a topological space and $A_1, \ldots, A_r \subseteq X$. Then $X/A_1, \ldots, A_r$ is the quotient space of X with respect to $\sim$ where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 12.12.8 (Cone)**.** Let X be a topological space. The *cone over X* is the space $(X \times [0,1])/(X \times \{1\})$.

Definition 12.12.9 (Suspension)**.** Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1,1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 12.12.10 (Wedge Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 12.12.11 (Smash Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 12.12.12. $D^n/S^{n-1} \cong S^n$

PROOF:
$\langle 1 \rangle$1. LET: $\phi : D^n/S^{n-1} \to S^n$ be the function induced by the map $D^n \to S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.
$\langle 1 \rangle$2. ϕ is a bijection.
$\langle 1 \rangle$3. ϕ is a homeomorphism.
PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.
□

12.13 Gluing

Definition 12.13.1 (Gluing)**.** Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \to Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X+Y)/\sim$, where $\sim$ is the equivalence relation generated by $x \sim \phi(x)$ for all $x \in X$.

Proposition 12.13.2. *Y is a subspace of $Y \cup_\phi X$.*

Definition 12.13.3. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0,1])/\alpha$ is the quotient space of $X \times [0,1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x \in X$.

Definition 12.13.4 (Möbius Strip)**.** The *Möbius strip* is $([-1,1] \times [0,1])/\alpha$ where $\alpha(x) = -x$.

Definition 12.13.5 (Klein Bottle)**.** The *Klein bottle* is $(S^1 \times [0,1])/\alpha$ where $\alpha(z) = \overline{z}$.

Proposition 12.13.6. *Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\mathrm{id}_{\partial M}} M \cong K$.*

PROOF:
⟨1⟩1. LET: $f : ([-1,1] \times [0,1]) + ([-1,1] \times [0,1]) \to S^1 \times [0,1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{\pi i \theta/2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-\pi i \theta/2}, t)$.
⟨1⟩2. f induces a bijection $M \cup_{\mathrm{id}_{\partial M}} M \approx K$
⟨1⟩3. f is a homeomorphism.
□

12.14 Metric Spaces

Definition 12.14.1 (Metric Space)**.** Let X be a set and $d : X^2 \to \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geqslant 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leqslant d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d). We often write X for the metric space (X, d).

Definition 12.14.2 (Ball)**.** Let X be a metric space. Let $x \in X$ and $r > 0$. The *ball* with *centre* x and *radius* r is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} \ .$$

Definition 12.14.3 (Metric Topology)**.** Let (X, d) be a metric space. The *metric topology* on X is the topology generated by the basis consisting of the balls.

Definition 12.14.4 (Metrizable)**.** A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 12.14.5. *Every metrizable space is Hausdorff.*

Every metrizable space is first countable.
A metric space is compact if and only if it is sequentially compact.
A metric space is separable if and only if it is second countable.

12.14.1 Products

Definition 12.14.6 (Euclidean Metric)**.** Let X and Y be metric spaces. The *Euclidean metric* on $X \times Y$ is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} \ .$$

We write $X \times Y$ for the set $X \times Y$ under this metric.

We prove this is a metric.

Proof:
⟨1⟩1. $d((x_1, y_1), (x_2, y_2)) \geqslant 0$
Proof: Immediate from definition.
⟨1⟩2. $d((x_1, y_1), (x_2, y_2)) = 0$ iff $(x_1, y_1) = (x_2, y_2)$
Proof: $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = 0$ iff $d(x_1, x_2) = d(y_1, y_2) = 0$ iff $x_1 = x_2$ and $y_1 = y_2$.
⟨1⟩3. $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$
Proof: Since $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = \sqrt{d(x_2, x_1)^2 + d(y_2, y_1)^2}$.
⟨1⟩4. The triangle inequality holds.
Proof:

$$
\begin{aligned}
& (d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)))^2 \\
=& d((x_1, y_1), (x_2, y_2))^2 + 2d((x_1, y_1), (x_2, y_2))d((x_2, y_2), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3))^2 \\
=& d(x_1, x_2)^2 + d(y_1, y_2)^2 + 2\sqrt{(d(x_1, x_2)^2 + d(y_1, y_2)^2)(d(x_2, x_3)^2 + d(y_2, y_3)^2)} + d(x_2, x_3)^2 + d(y_2, y_3)^2 \\
\geqslant& d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(y_1, y_2)^2 + d(y_2, y_3)^{+}2(d(x_1, x_2)d(x_2, x_3) + d(y_1, y_2)d(y_2, y_3)) \\
& \text{(Cauchy-Schwarz)} \\
=& (d(x_1, x_2) + d(x_2, x_3))^2 + (d(y_1, y_2) + d(y_2, y_3))^2 \\
\geqslant& d(x_1, x_3)^2 + d(y_1, y_3)^2 \\
=& d((x_1, y_1), (x_3, y_3))^2
\end{aligned}
$$

□

Proposition 12.14.7. *Let X and Y be metric spaces. The Euclidean metric on $X \times Y$ induces the product topology on $X \times Y$.*

Proof:
⟨1⟩1. Every open ball is open in the product topology.
⟨2⟩1. Let: $(x, y) \in B((a, b), \epsilon)$
Prove: $B(x, \sqrt{\epsilon}) \times B(y, \sqrt{\epsilon}) \subseteq B((a, b), \epsilon)$
⟨2⟩2. Let: $x' \in B(x, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$ and $y' \in B(y, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$
Prove: $d((x', y'), (a, b)) < \epsilon$
⟨2⟩3. $d((x', y'), (x, y)) < \epsilon - d((x, y), (a, b))$
Proof:

$$
\begin{aligned}
d((x', y'), (x, y)) &= \sqrt{d(x', x)^2 + d(y', y)^2} \\
&< \sqrt{(\epsilon - d((x, y), (a, b)))^2/2 + (\epsilon - d((x, y), (a, b))^2/2} \\
&= \epsilon - d((x, y), (a, b))
\end{aligned}
$$

⟨2⟩4. $d((x',y'),(a,b)) < \epsilon$
PROOF:
$$\begin{aligned} d((x',y'),(a,b)) &\leqslant d((x',y'),(x,y)) + d((x,y),(a,b)) && \text{(Triangle Inequality)} \\ &< \epsilon && (\langle 2\rangle 3) \end{aligned}$$

⟨1⟩2. If U is open in X and V is open in Y then $U \times V$ is open under the Euclidean metric.
⟨2⟩1. LET: $(x,y) \in U \times V$
⟨2⟩2. PICK $\delta, \epsilon > 0$ such that $B(x,\delta) \subseteq U$ and $B(y,\epsilon) \subseteq V$
PROVE: $(B((x,y),\min(\delta,\epsilon)) \subseteq U \times V$
⟨2⟩3. LET: $(x',y') \in B((x,y),\min(\delta,\epsilon))$
⟨2⟩4. $d(x',x) < \delta$
⟨3⟩1. $d((x',y'),(x,y)) < \min(\delta,\epsilon)$
⟨3⟩2. $d(x',x)^2 + d(y',y)^2 < \delta^2$
⟨3⟩3. $d(x',x)^2 < \delta^2$
⟨2⟩5. $d(y',y) < \epsilon$
PROOF: Similar.
⟨2⟩6. $(x',y') \in U \times V$
□

12.15 Complete Metric Spaces

Definition 12.15.1 (Complete)**.** A metric space is *complete* iff every Cauchy sequence converges.

Example 12.15.2. $\mathbb{R}$ is complete.

Proposition 12.15.3. *The product of two complete metric spaces is complete.*

Proposition 12.15.4. *Every compact metric space is complete.*

Proposition 12.15.5. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 12.15.6 (Completion)**.** Let X be a metric space. A *completion* of X is a complete metric space $\hat{X}$ and injection $i : X \rightarrowtail \hat{X}$ such that:

- The metric on X is the restriction of the metric on $\hat{X}$
- X is dense in $\hat{X}$.

Proposition 12.15.7. *Let $i_1 : X \to Y_1$ and $i_2 : X \to Y_2$ be completions of X. Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

PROOF: Define $\phi(\lim_{n\to\infty} i_1(x_n)) = \lim_{n\to\infty} i_2(x_n)$. □

Theorem 12.15.8. *Every metric space has a completion.*

PROOF: Let $\hat{X}$ be the set of Cauchy sequences in X quotiented by $\sim$ where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \to 0$. □

12.16 Manifolds

Definition 12.16.1 (Manifold)**.** An *n-dimensional manifold* is a second countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$.

Chapter 13

Homotopy Theory

13.1 Homotopies

Definition 13.1.1 (Homotopy)**.** Let X and Y be topological spaces. Let $f, g : X \to Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0, 1] \to Y$ such that

- $\forall x \in X.h(x, 0) = f(x)$
- $\forall x \in X.h(x, 1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \to Y$.

Proposition 13.1.2. *Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.*

Definition 13.1.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

A *homotopy functor* is a functor $\mathbf{Top} \to \mathcal{C}$ that factors through the canonical functor $\mathbf{Top} \to \mathbf{HTop}$.

Definition 13.1.4. A functor $F : \mathbf{Top} \to \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \to Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

13.2 Homotopy Equivalence

Definition 13.2.1 (Homotopy Equivalence)**.** Let X and Y be topological spaces. A *homotopy equivalence* between X and Y, $f : X \simeq Y$, is a continuous function $f : X \to Y$ such that there exists a continuous function $g : Y \to X$, the *homotopy inverse* to f, such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Definition 13.2.2 (Contractible)**.** A topological space X is *contractible* iff $X \simeq 1$.

Example 13.2.3. $\mathbb{R}^n$ is contractible.

Example 13.2.4. D^n is contractible.

Definition 13.2.5 (Deformation Retract)**.** Let X be a topological space and A a subspace of X. A retraction $\rho : X \to A$ is a *deformation retraction* iff $i \circ \rho \simeq \mathrm{id}_X$, where i is the inclusion $A \rightarrowtail X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 13.2.6 (Strong Deformation Retract)**.** Let X be a topological space and A a subspace of X. A *strong deformation retraction* $\rho : X \to A$ is a continuous function such that there exists a homotopy $h : X \times [0, 1] \to X$ between $i \circ \rho$ and id_X such that, for all $a \in X$ and $t \in [0, 1]$, we have $h(a, t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 13.2.7. $\{0\}$ is a strong deformation retract of $\mathbb{R}^n$ and of D^n.

Example 13.2.8. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 13.2.9. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 13.2.10. For any topological space X, the singleton consisting of the vertex is a strong deformation retract of the cone over X.

Chapter 14

Simplicial Complexes

Definition 14.0.1 (Simplex)**.** A *k-dimensional simplex* or *k-simplex* in $\mathbb{R}^n$ is the convex hull $s(x_0, \ldots, x_k)$ of $k+1$ points in general position.

Definition 14.0.2 (Face)**.** A *sub-simplex* or *face* of $s(x_0, \ldots, x_k)$ is the convex hull of a subset of $\{x_0, \ldots, x_k\}$.

Definition 14.0.3 (Simplicial Complex)**.** A *simplicial complex* in $\mathbb{R}^n$ is a set K of simplices such that:

- for every simplex s in K, every face of s is in K.
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2.
- K is locally finite, i.e. every point of $\mathbb{R}^n$ has a neighbourhood that only intersects finitely many elements of K.

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of $\mathbb{R}^n$.

14.1 Cell Decompositions

Definition 14.1.1 (n-cell)**.** An *n-cell* is a topological space homeomorphic to $\mathbb{R}^n$.

Definition 14.1.2 (Cell Decomposition)**.** Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n-cells.

Definition 14.1.3 (n-skeleton)**.** Given a cell decomposition of X, the *n-skeleton* X^n is the union of all the cells of dimension $\leqslant n$.

14.2 CW-complexes

Definition 14.2.1 (CW-Complex)**.** A *CW-complex* consists of a topological space X and a cell decomposition $\mathcal{E}$ of X such that:

1. *Characteristic Maps* For every n-cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \to X$ such that $\Phi((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in $\mathcal{E}$ of dimension $< n$.

2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have $\overline{e}$ intersects only finitely many other cells in $\mathcal{E}$.

3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \overline{e}$ is closed.

Proposition 14.2.2. *If a cell decomposition $\mathcal{E}$ satisfies the Characteristic Maps axiom, then for every n-cell $e \in \mathcal{E}$ we have $\overline{e} = \Phi_e(D^n)$. Therefore $\overline{e}$ is compact and $\overline{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

PROOF:
⟨1⟩1. $e \subseteq \Phi_e(D^n) \subseteq \overline{e}$
PROOF:

$$\begin{aligned} e &= \Phi_e((D^n)^\circ) \\ &\subseteq \Phi_e(D^n) \\ &= \Phi_e(\overline{(D^n)^\circ}) \\ &\subseteq \overline{\Phi_e((D^n)^\circ)} \\ &= \overline{e} \end{aligned}$$

⟨1⟩2. $\Phi_e(D^n)$ is compact.
PROOF: Because D^n is compact.
⟨1⟩3. $\Phi_e(D^n)$ is closed.
⟨1⟩4. $\Phi_e(D^n) = \overline{e}$
□

Chapter 15

Topological Groups

Definition 15.0.1 (Topological Group)**.** A *topological group* is a group G with a topology such that the function $G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.

Example 15.0.2. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 15.0.3. *Any subgroup of a topological group is a topological group under the subspace topology.*

Definition 15.0.4 (Homogeneous Space)**.** A *homogeneous space* is a topological space of the form G/H, where G is a topological group and H is a normal subgroup of G, under the quotient topology.

Proposition 15.0.5. *Let G be a topological group and H a normal subgroup of G. Then G/H is Hausdorff if and only if H is closed.*

Proof: See Bourbaki, N., General Topology. III.12 $\square$

15.1 Continuous Actions

Definition 15.1.1 (Continuous Action)**.** Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \to X$ such that:

- $\forall x \in X . ex = x$
- $\forall g, h \in G . \forall x \in X . g(hx) = (gh)x$

A *G-space* consists of a topological space X and a continuous action of G on X.

Definition 15.1.2 (Orbit)**.** Let X be a G-space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 15.1.3. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:
$\langle 1 \rangle$1. LET: $f_3 : S^2/SO(2) \to [-1, 1]$ be the function induced by $\pi_3 : S^2 \to [-1, 1]$
$\langle 1 \rangle$2. f_3 is bijective.
$\langle 1 \rangle$3. $S^2/SO(2)$ is compact.
PROOF: It is the continuous image of S^2 which is compact.
$\langle 1 \rangle$4. $[-1, 1]$ is Hausdorff.
$\langle 1 \rangle$5. f_3 is a homeomorphism.
□

Definition 15.1.4 (Stabilizer)**.** Let X be a G-space and $x \in X$. The *stabilizer* of x is $G_x := \{g \in G : gx = x\}$.

Proposition 15.1.5. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx.*

PROOF:
$\langle 1 \rangle$1. If $gG_x = hG_x$ then $gx = hx$.
$\langle 2 \rangle$1. ASSUME: $gG_x = hG_x$
$\langle 2 \rangle$2. $g^{-1}h \in G_x$
$\langle 2 \rangle$3. $g^{-1}hx = x$
$\langle 2 \rangle$4. $gx = hx$
$\langle 1 \rangle$2. If $gx = hx$ then $gG_x = hG_x$.
PROOF: Similar.
$\langle 1 \rangle$3. The function is continuous.
PROOF: Proposition 12.12.2.
□

Chapter 16

Topological Vector Spaces

Definition 16.0.1 (Topological Vector Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Substraction is a continuous function $E^2 \to E$
- Multiplication is a continuous function $K \times E \to E$

Proposition 16.0.2. *Every topological vector space is a topological group under addition.*

Proof: Immediate from the definition. □

Theorem 16.0.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

Proof: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 □

Proposition 16.0.4. *Let E be a topological vector space and E_0 a subspace of E. Then $\overline{E_0}$ is a subspace of E.*

Definition 16.0.5. Let E be a topological vector space. The topological space *associated* with E is $E/\overline{\{0\}}$.

16.1 Cauchy Sequences

Definition 16.1.1 (Cauchy Sequence)**.** Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geqslant n_0 . x_n - x_m \in U$.

Definition 16.1.2 (Complete Topological Vector Space)**.** A topological vector space is *complete* iff every Cauchy sequence converges.

16.2 Seminorms

Definition 16.2.1 (Seminorm)**.** Let E be a vector space over K. A *seminorm* on E is a function $\| \, \| : E \to \mathbb{R}$ such that:

1. $\forall x \in E. \|x\| \geqslant 0$
2. $\forall \alpha \in K. \forall x \in E. \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y :\in E. \|x + y\| \leqslant \|x\| + \|y\|$

Example 16.2.2. The function that maps $(x_1, \dots, x_n)$ to $|x_i|$ is a seminorm on $\mathbb{R}^n$.

Definition 16.2.3. Let E be a vector space over K. Let Λ be a set of seminorms on E. The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x \in E$.

Proposition 16.2.4. *E is a topological vector space under this topology. It is Hausdorff iff, for all $x \in E$, if $\forall \lambda \in \Lambda. \lambda(x) = 0$ then $x = 0$.*

16.3 Fréchet Spaces

Definition 16.3.1 (Pre-Fréchet Space)**.** A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 16.3.2. *Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \, \|_n : n \in \mathbb{Z}^+\}$. Then*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 16.3.3 (Fréchet Space)**.** A *Fréchet space* is a complete pre-Fréchet space.

16.4 Normed Spaces

Definition 16.4.1 (Normed Space)**.** Let E be a vector space over K. A *norm* on E is a function $\| \, \| : E \to \mathbb{R}$ is a seminorm such that, $\forall x \in E. \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 16.4.2. *If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E.*

Definition 16.4.3 (p-norm)**.** For any $p \geqslant 1$, the *p-norm* on $\mathbb{R}^n$ is defined by

$$\|\vec{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} .$$

We prove this is a norm.

PROOF:
⟨1⟩1. For all $\vec{x} \in \mathbb{R}^n$ we have $\|\vec{x}\|_p \geqslant 0$
PROOF: Immediate from definition.
⟨1⟩2. For all $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ we have $\|\alpha\vec{x}\|_p = |\alpha|\|\vec{x}\|_p$
PROOF:

$$\begin{aligned} \|\alpha(x_1, \ldots, x_n)\| &= \|(\alpha x_1, \ldots, \alpha x_n)\| \\ &= \left(\sum_{i=1}^{n} (\alpha x_i)^p \right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|\vec{x}\|_p \end{aligned}$$

⟨1⟩3. The triangle inequality holds.
PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|_p^p &= \sum_{i=1}^{n} |x_i + y_i|^p \\ &= \sum_{i=1}^{n} |x_i + y_i||x_i + y_i|^{p-1} \\ &\leqslant \sum_{i=1}^{n} (|x_i| + |y_i|)|x_i + y_i|^{p-1} \\ &= \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1} \\ &\leqslant \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{p-1}{p}} && \text{(Hölder's Inequality)} \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p)\|\vec{x} + \vec{y}\|^{p-1} \end{aligned}$$

Assuming w.l.o.g. $\|\vec{x} + \vec{y}\|^{p-1} \neq 0$ (using **??**) we have $\|\vec{x} + \vec{y}\|_p \leqslant \|\vec{x}\|_p + \|\vec{y}\|_p$.
⟨1⟩4. For any $\vec{x} \in \mathbb{R}^n$, we have $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.
PROOF:$\sum_{i=1}^{n} x_i^p = 0$ iff $x_1 = \cdots = x_n = 0$.
□

Definition 16.4.4 (Sup-norm)**.** The *sup-norm* on $\mathbb{R}^n$ is defined by

$$\|(x_1, \ldots, x_n)\|_\infty := \max(|x_1|, \ldots, |x_n|) \ .$$

Proposition 16.4.5. *The 2-norm on $\mathbb{R}^n$ induces the standard metric.*

PROOF: Immediate from definitions. □

Definition 16.4.6. For $p \geqslant 1$, the normed space l_p is the set of all sequences (x_n) in $\mathbb{R}$ such that $\sum_{n=1}^\infty x_n^p$ converges, under

$$\|(x_n)\|_p := \left(\sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} \ .$$

Proposition 16.4.7. *The spaces l_p for $p \geqslant 1$ are all homeomorphic.*

PROOF: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. http://dx.doi.org/10.1007/BF01075865.

Definition 16.4.8. Let l_∞ be the set of all bounded sequences in $\mathbb{R}$ under

$$\|(x_n)\| := \sup_n |x_n|$$

Proposition 16.4.9. *For all $p \geqslant 1$ we have l_p is not homeomorphic to l_∞.*

Proposition 16.4.10. *Let $\| \ \|$ be a seminorm on the vector space E. Then $\| \ \|$ defines a norm on $E/\overline{\{0\}}$.*

Proposition 16.4.11. *Let E and F be normed spaces. Any continuous linear map $E \to F$ is uniformly continuous.*

Definition 16.4.12. For $p \geqslant 1$. let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt{p} \int_{\mathbb{R}^n} |f(x)|^p dx$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\overline{\{0\}} \ .$$

16.5 Inner Product Spaces

Proposition 16.5.1. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E.*

16.6 Banach Spaces

Definition 16.6.1 (Banach Space)**.** A *Banach space* is a complete normed space.

Example 16.6.2. For any topological space X, the set $C(X)$ of bounded continuous functions $X \to \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 16.6.3. *The completion of a normed space is a Banach space.*

Proposition 16.6.4. *Let E and F be normed spaces. Let $f : E \to F$ be a continuous linear map. Then the extension to the completions $\hat{E} \to \hat{F}$ is linear.*

Proposition 16.6.5. $L^p(\mathbb{R}^n)$ *is a Banach space.*

Proposition 16.6.6. $C(\mathbb{R})$ *is first countable but not second countable.*

Proof: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n. Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.
It is first countable because it is metrizable. □

16.7 Hilbert Spaces

Definition 16.7.1 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Example 16.7.2. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \to \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi} \pi f(x) g(x) dx \ .$$

Proposition 16.7.3. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

16.8 Locally Convex Spaces

Definition 16.8.1 (Locally Convex Space)**.** A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 16.8.2. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

Proof: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Proposition 16.8.3. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Example 16.8.4. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \to \mathbb{R}$ is continuous as a map $E' \to \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 16.8.5 (Thom Space)**.** Let E be a vector bundle with a Riemannian metric, $DE = \{x \in E : \|x\| \leqslant 1\}$ its disc bundle and $SE := \{v \in E : \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE.