

Mathematics

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Contents

I	Category Theory	5
1	Foundations	7
2	Categories	9
2.1	Preorders	10
2.2	Monomorphisms and Epimorphisms	10
2.3	Sections and Retractions	12
2.4	Isomorphisms	13
2.5	Initial and Terminal Objects	13
3	Functors	15
3.1	Comma Categories	15
II	Group Theory	17
4	Monoids	19
5	Groups	21
5.1	Order of an Element	24
5.2	Generators	27
6	Group Homomorphisms	29
6.1	Subgroups	31
6.2	Kernel	33
6.3	Inner Automorphisms	34
6.4	Direct Products	34
6.5	Free Groups	34
6.6	Normal Subgroups	37
6.7	Quotient Groups	38
6.8	Cosets	42
6.9	Congruence	46
6.10	Cyclic Groups	47
6.11	Euler's ϕ -function	47
6.12	Commutator Subgroup	48

6.13	Presentations	49
6.14	Index of a Subgroup	50
6.15	Cokernels	51
6.16	Cayley Graphs	51
7	Abelian Groups	53
7.1	The Category of Abelian Groups	59
7.2	Free Abelian Groups	60
7.3	Cokernels	62
8	Group Actions	65
8.1	Group Actions	65
8.2	Category of G -Sets	68
III	Ring Theory	71
9	Rings	73
9.1	Rngs	73
9.1.1	Commutative Rngs	74
9.2	Rings	75
9.3	Integral Domains	75
IV	Linear Algebra	77

Part I

Category Theory

Chapter 1

Foundations

This is a placeholder — I am not sure what foundation I want to use for this project yet. I will try to work in a way which is foundation-independent. What I do could be formalized in ZFC, ETCS, or some other system. I will assume the usual set theoretic constructions as needed. Sets will be defined up to bijection only.

Chapter 2

Categories

Definition 2.1 (Category). A *category* \mathcal{C} consists of:

- A class $|\mathcal{C}|$ of *objects*. We write $A \in \mathcal{C}$ for $A \in |\mathcal{C}|$.
- For any objects A, B , a set $\mathcal{C}[A, B]$ of *morphisms* from A to B . We write $f : A \rightarrow B$ for $f \in \mathcal{C}[A, B]$.
- For any object A , a morphism $\text{id}_A : A \rightarrow A$, the *identity* morphism on A .
- For any morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, a morphism $g \circ f : A \rightarrow C$, the *composite* of f and g .

such that:

Associativity Given $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Left Unit Law For any morphism $f : A \rightarrow B$, we have $\text{id}_B \circ f = f$.

Right Unit Law For any morphism $f : A \rightarrow B$, we have $f \circ \text{id}_A = f$.

Proposition 2.2. *The identity morphism on an object is unique.*

PROOF: If i and j are identity morphisms on A then $i = i \circ j = j$. \square

Example 2.3 (Category of Sets). The *category of sets* **Set** has objects all sets and morphisms all functions.

Definition 2.4 (Endomorphism). In a category \mathcal{C} , an *endomorphism* on an object A is a morphism $A \rightarrow A$. We write $\text{End}_{\mathcal{C}}(A)$ for $\mathcal{C}[A, A]$.

Definition 2.5 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with the same objects as \mathcal{C} and

$$\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$$

2.1 Preorders

Definition 2.6 (Preorder). A *preorder* on a set A is a relation \leq on A that is reflexive and transitive.

A *preordered set* is a pair (A, \leq) such that \leq is a preorder on A . We usually write A for the preordered set (A, \leq) .

We identify any preordered set A with the category whose objects are the elements of A , with one morphism $a \rightarrow b$ iff $a \leq b$, and no morphism $a \rightarrow b$ otherwise.

Example 2.7. For any ordinal α , let α be the preorder $\{\beta : \beta < \alpha\}$ under \leq .

Definition 2.8 (Discrete Preorder). We identify any set A with the *discrete* preorder $(A, =)$.

2.2 Monomorphisms and Epimorphisms

Definition 2.9 (Monomorphism). In a category, let $f : A \rightarrow B$. Then f is a *monomorphism* or *monic* iff, for every object X and morphism $x, y : X \rightarrow A$, if $fx = fy$ then $x = y$.

Definition 2.10 (Epimorphism). In a category, let $f : A \rightarrow B$. Then f is a *epimorphism* or *epi* iff, for every object X and morphism $x, y : B \rightarrow X$, if $xf = yf$ then $x = y$.

Proposition 2.11. *The composite of two monomorphism is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$ be monic.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$

$\langle 1 \rangle 3$. ASSUME: $g \circ f \circ x = g \circ f \circ y$

$\langle 1 \rangle 4$. $f \circ x = f \circ y$

$\langle 1 \rangle 5$. $x = y$

□

Proposition 2.12. *The composite of two epimorphisms is epi.*

PROOF: Dual. □

Proposition 2.13. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is monic then f is monic.*

PROOF: If $f \circ x = f \circ y$ then $gfx = gfy$ and so $x = y$. □

Proposition 2.14. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is epi then g is epi.*

PROOF: Dual. □

Proposition 2.15. *A function is a monomorphism in **Set** iff it is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is monic then f is injective.

$\langle 2 \rangle 1$. ASSUME: f is monic.

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 4$. LET: $\bar{x}, \bar{y} : 1 \rightarrow A$ be the functions such that $\bar{x}(*) = x$ and $\bar{y}(*) = y$

$\langle 2 \rangle 5$. $f \circ \bar{x} = f \circ \bar{y}$

$\langle 2 \rangle 6$. $\bar{x} = \bar{y}$

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. $x = y$

$\langle 1 \rangle 3$. If f is injective then f is monic.

$\langle 2 \rangle 1$. ASSUME: f is injective.

$\langle 2 \rangle 2$. LET: X be a set and $x, y : X \rightarrow A$.

$\langle 2 \rangle 3$. ASSUME: $f \circ x = f \circ y$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $t \in X$

PROVE: $x(t) = y(t)$

$\langle 2 \rangle 5$. $f(x(t)) = f(y(t))$

$\langle 2 \rangle 6$. $x(t) = y(t)$

PROOF: By $\langle 2 \rangle 1$.

□

Proposition 2.16. *A function is an epimorphism in **Set** iff it is surjective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is an epimorphism then f is surjective.

$\langle 2 \rangle 1$. ASSUME: f is an epimorphism.

$\langle 2 \rangle 2$. LET: $b \in B$

$\langle 2 \rangle 3$. LET: $x, y : B \rightarrow 2$ be defined by $x(b) = 1$ and $x(t) = 0$ for all other $t \in B$, $y(t) = 0$ for all $t \in B$.

$\langle 2 \rangle 4$. $x \neq y$

$\langle 2 \rangle 5$. $x \circ f \neq y \circ f$

$\langle 2 \rangle 6$. There exists $a \in A$ such that $f(a) = b$.

$\langle 1 \rangle 3$. If f is surjective then f is an epimorphism.

$\langle 2 \rangle 1$. ASSUME: f is surjective.

$\langle 2 \rangle 2$. LET: $x, y : B \rightarrow X$

$\langle 2 \rangle 3$. ASSUME: $x \circ f = y \circ f$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $b \in B$

PROVE: $x(b) = y(b)$

$\langle 2 \rangle 5$. PICK $a \in A$ such that $f(a) = b$

$\langle 2 \rangle 6$. $x(f(a)) = y(f(a))$

$\langle 2 \rangle 7$. $x(b) = y(b)$

□

Proposition 2.17. *In a preorder, every morphism is monic and epi.*

PROOF: Immediate from definitions. \square

2.3 Sections and Retractions

Definition 2.18 (Section, Retraction). In a category, let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $r \circ s = \text{id}_B$.

Proposition 2.19. *Every identity morphism is a section and retraction of itself.*

PROOF: Immediate from definitions. \square

Proposition 2.20. *Let $r, r' : A \rightarrow B$ and $s : B \rightarrow A$. If r is a retraction of s and r' is a section of s then $r = r'$.*

PROOF:

$$\begin{aligned} r &= r \circ \text{id}_A \\ &= r \circ s \circ r' \\ &= \text{id}_B \circ r' \\ &= r' \end{aligned} \quad \square$$

Proposition 2.21. *Let $r_1 : A \rightarrow B$, $r_2 : B \rightarrow C$, $s_1 : B \rightarrow A$ and $s_2 : C \rightarrow B$. If r_1 is a retraction of s_1 and r_2 is a retraction of s_2 then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= \text{id}_C \end{aligned} \quad \square$$

Proposition 2.22. *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $s : A \rightarrow B$ be a section of $r : B \rightarrow A$.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$ satisfy $sx = sy$.

$\langle 1 \rangle 3$. $rsx = rsy$

$\langle 1 \rangle 4$. $x = y$

\square

Proposition 2.23. *Every retraction is epi.*

PROOF: Dual. \square

Proposition 2.24. *In Set, every epimorphism has a retraction.*

PROOF: By the Axiom of Choice. \square

Example 2.25. It is not true in general that every monomorphism in any category has a section. nor that every epimorphism in any category has a retraction.

In the category **2**, the morphism $0 \leq 1$ is monic and epi but has no retraction or section.

2.4 Isomorphisms

Definition 2.26 (Isomorphism). In a category \mathcal{C} , a morphism $f : A \rightarrow B$ is an *isomorphism*, denoted $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$, the *inverse* of f , such that $f^{-1} \circ f = \text{id}_A$ and $f \circ f^{-1} = \text{id}_B$.

An *automorphism* on an object A is an isomorphism between A and itself. We write $\text{Aut}_{\mathcal{C}}(A)$ for the set of all automorphisms on A .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 2.27. *The inverse of an isomorphism is unique.*

PROOF: Proposition 2.20. \square

Proposition 2.28. *For any object A we have $\text{id}_A : A \cong A$ and $\text{id}_A^{-1} = \text{id}_A$.*

PROOF: Since $\text{id}_A \circ \text{id}_A = \text{id}_A$ by the Unit Laws. \square

Proposition 2.29. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Immediate from definitions. \square

Proposition 2.30. *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF: From Proposition 2.21. \square

Definition 2.31 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

2.5 Initial and Terminal Objects

Definition 2.32 (Initial Object). An object I in a category is *initial* iff, for any object X , there is exactly one morphism $I \rightarrow X$.

Example 2.33. The empty set is the initial object in **Set**.

Definition 2.34 (Terminal Object). An object T in a category is *terminal* iff, for any object X , there is exactly one morphism $X \rightarrow T$.

Example 2.35. Every singleton is terminal in **Set**.

Proposition 2.36. *If I and J are initial in a category, then there exists a unique isomorphism $I \cong J$.*

PROOF:

- $\langle 1 \rangle 1$. LET: i be the unique morphism $I \rightarrow J$.
- $\langle 1 \rangle 2$. LET: i^{-1} be the unique morphism $J \rightarrow I$.
- $\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_J$

PROOF: Since there is only one morphism $J \rightarrow J$.

- $\langle 1 \rangle 4$. $i^{-1} \circ i = \text{id}_I$

PROOF: Since there is only one morphism $I \rightarrow I$.
 \square

Proposition 2.37. *If S and T are terminal in a category, then there exists a unique isomorphism $S \cong T$.*

PROOF: Dual. \square

Chapter 3

Functors

Definition 3.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for any morphism $f : A \rightarrow B : \mathcal{C}$, a morphism $Ff : FA \rightarrow FB : \mathcal{D}$

such that:

- $F\text{id}_A = \text{id}_{FA}$
- $F(g \circ f) = Fg \circ Ff$

Definition 3.2 (Identity Functor). For any category \mathcal{C} , the *identity functor* $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is defined by

$$\begin{aligned} 1_{\mathcal{C}}A &= A \\ 1_{\mathcal{C}}f &= f \end{aligned}$$

Definition 3.3 (Constant Functor). Given categories \mathcal{C} , \mathcal{D} and an object $D \in \mathcal{D}$, the *constant functor* $K^{\mathcal{C}}D : \mathcal{C} \rightarrow \mathcal{D}$ is the functor defined by

$$\begin{aligned} K^{\mathcal{C}}DC &= D \\ K^{\mathcal{C}}Df &= \text{id}_D \end{aligned}$$

3.1 Comma Categories

Definition 3.4 (Comma Category). Let $F : \mathcal{C} \rightarrow \mathcal{E}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors. The *comma category* $F \downarrow G$ is the category with:

- objects all pairs (C, D, f) where $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f : FC \rightarrow GD : \mathcal{E}$

- morphisms $(u, v) : (C, D, f) \rightarrow (C', D', g)$ all pairs $u : C \rightarrow C' : \mathcal{C}$ and $v : D \rightarrow D' : \mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ \downarrow Fu & & \downarrow Gv \\ FC' & \xrightarrow{g} & GD' \end{array}$$

Definition 3.5 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category* over A , denoted \mathcal{C}/A , is the comma category $1_{\mathcal{C}} \downarrow K^1 A$.

Definition 3.6 (Coslice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *coslice category* over A , denoted $\mathcal{C} \backslash A$, is the comma category $K^1 A \downarrow 1_{\mathcal{C}}$.

Definition 3.7 (Pointed Sets). The *category of pointed sets* \mathbf{Set}_* is the coslice category $\mathbf{Set} \backslash 1$.

Part II

Group Theory

Chapter 4

Monoids

Definition 4.1 (Monoid). A *monoid* consists of a set M and a binary operation $\cdot : M^2 \rightarrow M$ such that:

- \cdot is associative
- There exists $e \in M$ such that, for all $x \in M$, we have $xe = ex = x$.

We identify a monoid M with the category with one object whose morphisms are the elements of M , with composition given by \cdot .

Proposition 4.2. *The identity in a group is unique.*

PROOF: Proposition 2.2.

Chapter 5

Groups

Definition 5.1 (Group). Let \mathcal{C} be a category with finite products. A *group (object)* in \mathcal{C} consists of an object $G \in \mathcal{C}$ and morphisms

$$m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G$$

such that the following diagrams commute.

$$\begin{array}{ccc}
 G^3 & \xrightarrow{m \times \text{id}_G} & G^2 \\
 \downarrow \text{id}_G \times m & & \downarrow m \\
 G^2 & \xrightarrow{m} & G
 \end{array}$$

$$\begin{array}{ccc}
 1 \times G & \xrightarrow{e \times \text{id}_G} & G^2 \\
 & \searrow \cong & \downarrow m \\
 & & G
 \end{array}
 \qquad
 \begin{array}{ccc}
 G \times 1 & \xrightarrow{\text{id}_G \times e} & G^2 \\
 & \searrow \cong & \downarrow m \\
 & & G
 \end{array}$$

$$\begin{array}{ccccc}
 G & \xrightarrow{\Delta} & G^2 & \xrightarrow{\text{id}_G \times i} & G^2 \\
 \downarrow & & & & \downarrow m \\
 1 & \xrightarrow{e} & G & & G
 \end{array}
 \qquad
 \begin{array}{ccccc}
 G & \xrightarrow{\Delta} & G^2 & \xrightarrow{i \times \text{id}_G} & G^2 \\
 \downarrow & & & & \downarrow m \\
 1 & \xrightarrow{e} & G & & G
 \end{array}$$

Definition 5.2 (Group). We write just 'group' for 'group in **Set**'. Thus, a *group* G consists of a set G and a binary operation $\cdot : G^2 \rightarrow G$ such that \cdot is associative, and there exists $e \in G$, the *identity* element of the group, such that:

- For all $x \in G$ we have $xe = ex = x$
- For all $x \in G$, there exists $x^{-1} \in G$, the *inverse* of x , such that $xx^{-1} = x^{-1}x = e$.

The *order* of a group G , denoted $|G|$, is the number of elements in G if G is finite; otherwise we write $|G| = \infty$.

Proposition 5.3. *The inverse of an element is unique.*

PROOF: If i and j are inverses of x then $i = ixj = j$. \square

Example 5.4. • The *trivial* group is $\{e\}$ under $ee = e$.

- \mathbb{Z} is a group under addition
- \mathbb{Q} is a group under addition
- $\mathbb{Q} - \{0\}$ is a group under multiplication
- \mathbb{R} is a group under addition
- $\mathbb{R} - \{0\}$ is a group under multiplication
- \mathbb{C} is a group under addition
- $\mathbb{C} - \{0\}$ is a group under multiplication
- $\{-1, 1\}$ is a group under multiplication
- For any category \mathcal{C} and object $A \in \mathcal{C}$, we have $\text{Aut}_{\mathcal{C}}(A)$ is a group under $gf = f \circ g$.
For A a set, we call $S_A = \text{Aut}_{\text{Set}}(A)$ the *symmetric group* or *group of permutations* of A .
- For $n \geq 3$, the *dihedral group* D_{2n} consists of the set of rigid motions that map the regular n -gon onto itself under composition.
- Let $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ under matrix multiplication.

Example 5.5. • The only group of order 1 is the trivial group.

- The only group of order 2 is \mathbb{Z}_2 .
- The only group of order 3 is \mathbb{Z}_3 .
- There are exactly two groups of order 4: \mathbb{Z}_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ under $(a, b)(c, d) = (ac, bd)$.

Proposition 5.6 (Cancellation). *Let G be a group. Let $a, g, h \in G$. If $ag = ah$ or $ga = ha$ then $g = h$.*

PROOF: If $ag = ah$ then $g = a^{-1}ag = a^{-1}ah = h$. Similarly if $ga = ha$. \square

Proposition 5.7. *Let G be a group and $g, h \in G$. Then $(gh)^{-1} = h^{-1}g^{-1}$.*

PROOF: Since $ghh^{-1}g^{-1} = e$. \square

Definition 5.8. Let G be a group. Let $g \in G$. We define $g^n \in G$ for all $n \in \mathbb{Z}$ as follows:

$$\begin{aligned} g^0 &= e \\ g^{n+1} &= g^n g & (n \geq 0) \\ g^{-n} &= (g^{-1})^n & (n > 0) \end{aligned}$$

Proposition 5.9. Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then

$$g^{m+n} = g^m g^n .$$

PROOF:

$\langle 1 \rangle 1$. For all $k \in \mathbb{Z}$ we have $g^{k+1} = g^k g$

$\langle 2 \rangle 1$. For all $k \geq 0$ we have $g^{k+1} = g^k g$

PROOF: Immediate from definition.

$\langle 2 \rangle 2$. $g^{-1+1} = g^{-1} g$

PROOF: Both are equal to e .

$\langle 2 \rangle 3$. For all $k > 1$ we have $g^{-k+1} = g^{-k} g$

PROOF:

$$\begin{aligned} g^{-k+1} &= (g^{-1})^{k-1} \\ &= (g^{-1})^{k-1} g^{-1} g \\ &= (g^{-1})^k g \\ &= g^{-k} g \end{aligned}$$

$\langle 1 \rangle 2$. For all $k \in \mathbb{Z}$ we have $g^{k-1} = g^k g^{-1}$

PROOF: Substitute $k = k - 1$ above and multiply by g^{-1} .

$\langle 1 \rangle 3$. $g^{m+0} = g^m g^0$

PROOF: Since $g^m g^0 = g^m e = g^m$.

$\langle 1 \rangle 4$. If $g^{m+n} = g^m g^n$ then $g^{m+n+1} = g^m g^{n+1}$

PROOF:

$$\begin{aligned} g^{m+n+1} &= g^{m+n} g & (\langle 1 \rangle 1) \\ &= g^m g^n g \\ &= g^m g^{n+1} & (\langle 1 \rangle 1) \end{aligned}$$

$\langle 1 \rangle 5$. If $g^{m+n} = g^m g^n$ then $g^{m+n-1} = g^m g^{n-1}$

PROOF:

$$\begin{aligned} g^{m+n-1} g &= g^{m+n} & (\langle 1 \rangle 1) \\ &= g^m g^n \\ \therefore g^{m+n-1} &= g^m g^n g^{-1} \\ &= g^m g^{n-1} & (\langle 1 \rangle 2) \end{aligned}$$

□

Proposition 5.10. Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then

$$(g^m)^n = g^{mn} .$$

PROOF:

$\langle 1 \rangle 1. (g^m)^0 = g^0$

PROOF: Both sides are equal to e .

$\langle 1 \rangle 2. \text{ If } (g^m)^n = g^{mn} \text{ then } (g^m)^{n+1} = g^{m(n+1)}.$

PROOF:

$$(g^m)^{n+1} = (g^m)^n g^m \quad (\text{Proposition 5.9})$$

$$= g^{mn} g^m$$

$$= g^{mn+m}$$

(Proposition 5.9)

$\langle 1 \rangle 3. \text{ If } (g^m)^n = g^{mn} \text{ then } (g^m)^{n-1} = g^{m(n-1)}.$

PROOF:

$$(g^m)^n = g^{mn}$$

$$\therefore (g^m)^{n-1} g^m = g^{mn-m} g^m \quad (\text{Proposition 5.9})$$

$$\therefore (g^m)^{n-1} = g^{mn-m} \quad (\text{Cancellation})$$

□

Definition 5.11 (Commute). Let G be a group and $g, h \in G$. We say g and h *commute* iff $gh = hg$.

Definition 5.12. Let G be a group. Given $g \in G$ and $A \subseteq G$, we define

$$gA = \{ga : a \in A\}, \quad Ag = \{ag : a \in A\}.$$

Given sets $A, B \subseteq G$, we define

$$AB = \{ab : a \in A, b \in B\}.$$

5.1 Order of an Element

Definition 5.13 (Order). Let G be a group. Let $g \in G$. Then g has *finite order* iff there exists a positive integer n such that $g^n = e$. In this case, the *order* of g , denoted $|g|$, is the least positive integer n such that $g^n = e$.

If g does not have finite order, we write $|g| = \infty$.

Proposition 5.14. Let G be a group. Let $g \in G$ and n be a positive integer. If $g^n = e$ then $|g| \mid n$.

PROOF:

$\langle 1 \rangle 1. \text{ LET: } n = q|g| + d \text{ where } 0 \leq d < |g|$

PROOF: Division Algorithm.

$\langle 1 \rangle 2. g^d = e$

PROOF:

$$e = g^n$$

$$= g^{q|g|+d}$$

$$= (g^{|g|})^q g^d$$

(Propositions 5.9, 5.10)

$$= e^q g^d$$

$$= g^d$$

$\langle 1 \rangle 3. d = 0$

PROOF: By minimality of $|g|$.

$\langle 1 \rangle 4. n = q|g|$

□

Corollary 5.14.1. *Let G be a group. Let $g \in G$ have finite order and $n \in \mathbb{Z}$. Then $g^n = e$ if and only if $|g| \mid n$.*

Proposition 5.15. *Let G be a group and $g \in G$. Then $|g| \leq |G|$.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: w.l.o.g. G is finite.

$\langle 1 \rangle 2.$ PICK i, j with $0 \leq i < j \leq |G|$ such that $g^i = g^j$.

PROOF: Otherwise $g^0, g^1, \dots, g^{|G|}$ would be $|G| + 1$ distinct elements of G .

$\langle 1 \rangle 3. g^{j-i} = e$

$\langle 1 \rangle 4. g$ has finite order and $|g| \leq |G|$

PROOF: Since $|g| \leq j - i \leq j \leq |G|$.

□

Proposition 5.16. *Let G be a group. Let $g \in G$ have finite order. Let $m \in \mathbb{N}$. Then*

$$|g^m| = \frac{\text{lcm}(m, |g|)}{m} = \frac{|g|}{\text{gcd}(m, |g|)}$$

PROOF: Since for any integer d we have

$$g^{md} = e \Leftrightarrow |g| \mid md \quad (\text{Corollary 5.14.1})$$

$$\Leftrightarrow \text{lcm}(m, |g|) \mid md$$

$$\Leftrightarrow \frac{\text{lcm}(m, |g|)}{m} \mid d \quad \square$$

and so $|g^m| = \frac{\text{lcm}(m, |g|)}{m}$ by Corollary 5.14.1. □

Corollary 5.16.1. *If g has odd order then $|g^2| = |g|$.*

Proposition 5.17. *Let G be a group. Let $g, h \in G$ have finite order. Assume $gh = hg$. Then $|gh|$ has finite order and*

$$|gh| \mid \text{lcm}(|g|, |h|)$$

PROOF: Since $(gh)^{\text{lcm}(|g|, |h|)} = g^{\text{lcm}(|g|, |h|)} h^{\text{lcm}(|g|, |h|)} = e$. □

Example 5.18. This example shows that we cannot remove the hypothesis that $gh = hg$.

In $\text{GL}_2(\mathbb{R})$, take

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then $|g| = 4$, $|h| = 3$ and $|gh| = \infty$.

Proposition 5.19. *Let G be a group and $g, h \in G$ have finite order. If $gh = hg$ and $\text{gcd}(|g|, |h|) = 1$ then $|gh| = |g||h|$.*

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } N = |gh|$$

$$\langle 1 \rangle 2. g^N = (h^{-1})^N$$

$$\langle 1 \rangle 3. g^{N|g|} = e$$

$$\langle 1 \rangle 4. |g^N| \mid |g|$$

$$\langle 1 \rangle 5. h^{-N|h|} = e$$

$$\langle 1 \rangle 6. |g^N| \mid |h|$$

$$\langle 1 \rangle 7. |g^N| = 1$$

PROOF: Since $\gcd(|g|, |h|) = 1$.

$$\langle 1 \rangle 8. g^N = e$$

$$\langle 1 \rangle 9. |g| \mid N$$

$$\langle 1 \rangle 10. h^{-N} = e$$

$$\langle 1 \rangle 11. |h| \mid N$$

$$\langle 1 \rangle 12. N = |g||h|$$

PROOF: Using Proposition 5.17.

□

Proposition 5.20. *Let G be a finite group. Assume there is exactly one element $f \in G$ of order 2. Then the product of all the elements of G is f .*

PROOF: Let the elements of G be g_1, g_2, \dots, g_n . Apart from e and f , every element and its inverse are distinct elements of the list. Hence the product of the list is $ef = f$. □

Proposition 5.21. *Let G be a finite group of order n . Let m be the number of elements of G of order 2. Then $n - m$ is odd.*

PROOF: In the list of all elements that are not of order 2, every element and its inverse are distinct except for e . Hence the list has odd length. □

Corollary 5.21.1. *If a finite group has even order, then it contains an element of order 2.*

Proposition 5.22. *Let G be a group and $a, g \in G$. Then $|aga^{-1}| = |g|$.*

PROOF: Since

$$(aga^{-1})^n = e \Leftrightarrow ag^na^{-1} = e$$

$$\Leftrightarrow g^n = e$$

□

Proposition 5.23. *Let G be a group and $g, h \in G$. Then $|gh| = |hg|$.*

PROOF: Since $|gh| = |ghgg^{-1}| = |hg|$. □

Proposition 5.24. *Let G be a group of order n . Let k be relatively prime to n . Then every element in G has the form x^k for some x .*

$$\langle 1 \rangle 1. \text{ PICK integers } a \text{ and } b \text{ such that } an + bk = 1.$$

$$\langle 1 \rangle 2. \text{ LET: } g \in G$$

$$\langle 1 \rangle 3. g = (g^b)^k$$

PROOF:

$$\begin{aligned} g &= g \cdot (g^n)^{-a} & (g^n = e) \\ &= g^{1-an} \\ &= g^{bk} \end{aligned}$$

□

5.2 Generators

Definition 5.25 (Generator). Let G be a group and $a \in G$. We say a *generates* the group iff, for all $x \in G$, there exists an integer n such that $x^n = a$.

Example 5.26. $\text{SL}_2(\mathbb{Z})$ is generated by

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

PROOF:

$\langle 1 \rangle 1$. LET: $H = \langle s, t \rangle$

$\langle 1 \rangle 2$. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in H$.

PROOF: It is t^q .

$\langle 1 \rangle 3$. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \in H$.

PROOF:

$$\begin{aligned} st^{-q}s^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \end{aligned}$$

$\langle 1 \rangle 4$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & qa+b \\ c & qc+d \end{pmatrix}$$

$\langle 1 \rangle 5$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} a+qb & b \\ c+qd & d \end{pmatrix}$$

$\langle 1 \rangle 6$. For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, if c and d are both nonzero, then there exists $N \in H$ such that the bottom row of MN has one entry the same as M and one entry with smaller absolute value.

PROOF: From $\langle 1 \rangle 4$ and $\langle 1 \rangle 5$ taking $q = -1$.

$\langle 1 \rangle 7$. For any $M \in \text{SL}_2(\mathbb{Z})$, there exists $N \in H$ such that MN has a zero on the bottom row.

PROOF: Apply $\langle 1 \rangle 6$ repeatedly.

$\langle 1 \rangle 8$. Any matrix in $\text{SL}_2(\mathbb{Z})$ with a zero on the bottom row is in H .

$$\langle 2 \rangle 1. \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H$$

PROOF: $\langle 1 \rangle 2$

$$\langle 2 \rangle 2. \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \in H$$

PROOF: It is $s^2 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ since $s^2 = -I$.

$$\langle 2 \rangle 3. \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} \in H$$

PROOF: It is $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} s$.

$$\langle 2 \rangle 4. \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \in H$$

PROOF: It is $s^2 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s$.

$\langle 1 \rangle 9$. Every matrix in $\text{SL}_2(\mathbb{Z})$ is in H .

□

Chapter 6

Group Homomorphisms

Definition 6.1 (Homomorphism). Let G and H be groups. A (group) homomorphism $\phi : G \rightarrow H$ is a function such that, for all $x, y \in G$,

$$\phi(xy) = \phi(x)\phi(y) \text{ .}$$

Proposition 6.2. Let G and H be groups with identities e_G and e_H . Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\phi(e_G) = e_H$.

PROOF: Since $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$ and so $\phi(e_G) = e_H$ by Cancellation. \square

Proposition 6.3. Let $\phi : G \rightarrow H$ be a group homomorphism. For all $x \in G$ we have $\phi(x^{-1}) = \phi(x)^{-1}$.

PROOF: Since $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_H$. \square

Proposition 6.4. Let G, H and K be groups. If $\phi : G \rightarrow H$ and $\psi : H \rightarrow K$ are homomorphisms then $\psi \circ \phi : G \rightarrow K$ is a homomorphism.

PROOF: For $x, y \in G$ we have

$$\psi(\phi(xy)) = \psi(\phi(x)\phi(y)) = \psi(\phi(x))\psi(\phi(y)) \text{ .}$$

Proposition 6.5. Let G be a group. Then $\text{id}_G : G \rightarrow G$ is a group homomorphism.

PROOF: For $x, y \in G$ we have $\text{id}_G(xy) = xy = \text{id}_G(x)\text{id}_G(y)$. \square

Proposition 6.6. Let $\phi : G \rightarrow H$ be a group homomorphism. Let $g \in G$ have finite order. Then $|\phi(g)|$ divides $|g|$.

PROOF: Since $\phi(g)^{|g|} = \phi(g^{|g|}) = e$. \square

Definition 6.7 (Category of Groups). Let **Grp** be the category of groups and group homomorphisms.

Example 6.8. There are 49487365402 groups of order 1024 up to isomorphism.

Proposition 6.9. *A group homomorphism $\phi : G \rightarrow H$ is an isomorphism in **Grp** if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: ϕ is bijective.

PROVE: ϕ^{-1} is a group homomorphism.

$\langle 1 \rangle 2$. LET: $h, h' \in H$

$\langle 1 \rangle 3$. $\phi(\phi^{-1}(hh')) = \phi(\phi^{-1}(h)\phi^{-1}(h'))$

PROOF: Both are equal to hh' .

$\langle 1 \rangle 4$. $\phi^{-1}(hh') = \phi^{-1}(h)\phi^{-1}(h')$

□

Corollary 6.9.1.

$$D_6 \cong C_3$$

PROOF: The canonical homomorphism $D_6 \rightarrow C_3$ is bijective. □

Corollary 6.9.2.

$$(\mathbb{R}, +) \cong (\{x \in \mathbb{R} : x > 0\}, \cdot)$$

PROOF: The function that maps x to e^x is a bijective homomorphism. □

Proposition 6.10. *The trivial group is the zero object in **Grp**.*

PROOF: For any group G , the unique function $G \rightarrow \{e\}$ is a group homomorphism, and the only group homomorphism $\{e\} \rightarrow G$ maps e to e_G . □

Proposition 6.11. *For any groups G and H , the set $G \times H$ under $(g, h)(g', h') = (gg', hh')$ is the product of G and H in **Grp**.*

PROOF:

$\langle 1 \rangle 1$. $G \times H$ is a group.

$\langle 2 \rangle 1$. The multiplication is associative.

PROOF: Since $(g_1, h_1)((g_2, h_2)(g_3, h_3)) = ((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1g_2g_3, h_1h_2h_3)$.

$\langle 2 \rangle 2$. (e_G, e_H) is the identity.

PROOF: Since $(g, h)(e_G, e_H) = (e_G, e_H)(g, h) = (g, h)$.

$\langle 2 \rangle 3$. The inverse of (g, h) is (g^{-1}, h^{-1}) .

PROOF: Since $(g, h)(g^{-1}, h^{-1}) = (g^{-1}, h^{-1})(g, h) = (e_G, e_H)$.

$\langle 1 \rangle 2$. $\pi_1 : G \times H \rightarrow G$ is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$. $\pi_2 : G \times H \rightarrow H$ is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 4$. For any group homomorphism $\phi : K \rightarrow G$ and $\psi : K \rightarrow H$, the function $\langle \phi, \psi \rangle : K \rightarrow G \times H$ where $\langle \phi, \psi \rangle(k) = (\phi(k), \psi(k))$ is a group homomorphism.

PROOF:

$$\begin{aligned} \langle \phi, \psi \rangle(kk') &= (\phi(kk'), \psi(kk')) \\ &= (\phi(k)\phi(k'), \psi(k)\psi(k')) \\ &= (\phi(k), \psi(k))(\phi(k'), \psi(k')) \\ &= \langle \phi, \psi \rangle(k)\langle \phi, \psi \rangle(k') \end{aligned}$$

□

Proposition 6.12.

$$|\text{Aut}_{\mathbf{Grp}}(C_n)| = \phi(n)$$

PROOF: An automorphism α is determined by $\alpha(1)$ which is any element of order n , and g has order n iff $\gcd(g, n) = 1$. □

Example 6.13.

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}) \cong C_2$$

PROOF: The only automorphisms are the identity and multiplication by -1. □

6.1 Subgroups

Definition 6.14 (Subgroup). Let (G, \cdot) and $(H, *)$ be groups such that H is a subset of G . Then H is a *subgroup* of G iff the inclusion $i : H \hookrightarrow G$ is a group homomorphism.

Proposition 6.15. If $(H, *)$ is a subgroup of (G, \cdot) then $*$ is the restriction of \cdot to H .

PROOF: Given $x, y \in H$ we have

$$x * y = i(x * y) = i(x) \cdot i(y) = x \cdot y . \quad \square$$

Example 6.16. For any group G we have $\{e\}$ is a subgroup of G .

Proposition 6.17. Let G be a group. Let H be a subset of G . Then H is a subgroup of G iff H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$.

PROOF:

⟨1⟩1. If H is a subgroup of G then H is nonempty.

PROOF: Since every group has an identity element and so is nonempty.

⟨1⟩2. If H is a subgroup of G then, for all $x, y \in H$, we have $xy^{-1} \in H$.

PROOF: Easy.

⟨1⟩3. If H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$, then H is a subgroup of G .

⟨2⟩1. ASSUME: H is nonempty.

⟨2⟩2. ASSUME: $\forall x, y \in H. xy^{-1} \in H$

⟨2⟩3. $e \in H$

PROOF: Pick $x \in H$. We have $e = xx^{-1} \in H$.

⟨2⟩4. $\forall x \in H. x^{-1} \in H$

PROOF: Given $x \in H$ we have $x^{-1} = ex^{-1} \in H$.

⟨2⟩5. H is closed under the restriction of \cdot

PROOF: Given $x, y \in H$ we have $xy = x(y^{-1})^{-1} \in H$.

⟨2⟩6. H is a group under the restriction of \cdot

PROOF: Associativity is inherited from G and the existence of an identity element and inverses follows from ⟨2⟩3 and ⟨2⟩4.

⟨2⟩7. The inclusion $H \hookrightarrow G$ is a group homomorphism.

PROOF: For $x, y \in H$ we have $i(xy) = i(x)i(y) = xy$.

□

Corollary 6.17.1. *The intersection of a set of subgroups of G is a subgroup of G .*

Corollary 6.17.2. *Let $\phi : G \rightarrow H$ be a group homomorphism. Let K be a subgroup of H . Then $\phi^{-1}(K)$ is a subgroup of G .*

PROOF:

⟨1⟩1. $\phi^{-1}(K)$ is nonempty.

PROOF: Since $e \in \phi^{-1}(K)$.

⟨1⟩2. LET: $x, y \in \phi^{-1}(K)$

⟨1⟩3. $\phi(x), \phi(y) \in K$

⟨1⟩4. $\phi(x)\phi(y)^{-1} \in K$

⟨1⟩5. $\phi(xy^{-1}) \in K$

⟨1⟩6. $xy^{-1} \in \phi^{-1}(K)$

□

Corollary 6.17.3. *Let $\phi : G \rightarrow H$ be a group homomorphism. Let K be a subgroup of G . Then $\phi(K)$ is a subgroup of H .*

PROOF:

⟨1⟩1. LET: $x, y \in \phi(K)$

⟨1⟩2. PICK $a, b \in K$ such that $x = \phi(a)$ and $y = \phi(b)$

⟨1⟩3. $xy^{-1} = \phi(ab^{-1})$

⟨1⟩4. $xy^{-1} \in \phi(K)$

□

Proposition 6.18. *Let G be a subgroup of \mathbb{Z} . Then there exists $d \geq 0$ such that $G = d\mathbb{Z}$.*

PROOF:

⟨1⟩1. ASSUME: w.l.o.g. $G \neq \{0\}$

PROOF: Since $\{0\} = 0\mathbb{Z}$.

⟨1⟩2. LET: d be the least positive element of G .

PROVE: $G = d\mathbb{Z}$

PROOF: If $n \in G$ then $-n \in G$ so G must contain a positive element.

⟨1⟩3. $G \subseteq d\mathbb{Z}$

⟨2⟩1. LET: $n \in G$

⟨2⟩2. LET: q and r be the integers such that $n = qd + r$ and $0 \leq r < d$.

⟨2⟩3. $r \in G$

PROOF: Since $r = n - qd$.

⟨2⟩4. $r = 0$

PROOF: By minimality of d .

⟨2⟩5. $n = qd \in d\mathbb{Z}$

⟨1⟩4. $d\mathbb{Z} \subseteq G$

□

6.2 Kernel

Definition 6.19 (Kernel). Let $\phi : G \rightarrow H$ be a group homomorphism. The *kernel* of ϕ is

$$\ker \phi = \{g \in G : \phi(g) = e\} .$$

Proposition 6.20. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\ker \phi$ is a subgroup of G .*

PROOF: Corollary 6.17.2. \square

Proposition 6.21. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then the inclusion $i : \ker \phi \hookrightarrow G$ is terminal in the category of pairs $(K, \alpha : K \rightarrow G)$ such that $\phi \circ \alpha = 0$.*

PROOF:

$\langle 1 \rangle 1.$ $\phi \circ i = 0$

$\langle 1 \rangle 2.$ For any group K and homomorphism $\alpha : K \rightarrow G$ such that $\phi \circ \alpha = 0$, there exists a unique homomorphism $\beta : K \rightarrow \ker \phi$ such that $i \circ \beta = \alpha$.

\square

Proposition 6.22. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then the following are equivalent:*

1. ϕ is monic.
2. $\ker \phi = \{e\}$
3. ϕ is injective.

PROOF:

$\langle 1 \rangle 1.$ $1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: ϕ is monic.

$\langle 2 \rangle 2.$ LET: $i : \ker \phi \hookrightarrow G$, $j : \{e\} \hookrightarrow \ker \phi \hookrightarrow G$ be the inclusions.

$\langle 2 \rangle 3.$ $\phi \circ i = \phi \circ j$

$\langle 2 \rangle 4.$ $i = j$

$\langle 1 \rangle 2.$ $2 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: $\ker \phi = \{e\}$

$\langle 2 \rangle 2.$ LET: $x, y \in G$

$\langle 2 \rangle 3.$ ASSUME: $\phi(x) = \phi(y)$

$\langle 2 \rangle 4.$ $\phi(xy^{-1}) = e$

$\langle 2 \rangle 5.$ $xy^{-1} \in \ker \phi$

$\langle 2 \rangle 6.$ $xy^{-1} = e$

$\langle 2 \rangle 7.$ $x = y$

$\langle 1 \rangle 3.$ $3 \Rightarrow 1$

PROOF: Easy.

\square

Proposition 6.23. *A group homomorphism is an epimorphism if and only if it is surjective.*

6.3 Inner Automorphisms

Proposition 6.24. *Let G be a group and $g \in G$. The function $\gamma_g : G \rightarrow G$ defined by $\gamma_g(a) = gag^{-1}$ is an automorphism on G .*

PROOF:

$\langle 1 \rangle 1$. γ_g is a homomorphism.

PROOF:

$$\begin{aligned}\gamma_g(ab) &= gabg^{-1} \\ &= gag^{-1}gbg^{-1} \\ &= \gamma_g(a)\gamma_g(b)\end{aligned}$$

$\langle 1 \rangle 2$. γ_g is injective.

PROOF: By Cancellation.

$\langle 1 \rangle 3$. γ_g is surjective.

PROOF: Given $b \in G$, we have $\gamma_g(g^{-1}bg) = b$.

□

Definition 6.25 (Inner Automorphism). Let G be a group. An *inner automorphism* on G is a function of the form $\gamma_g(a) = gag^{-1}$ for some $g \in G$.

We write $\text{Inn}(G)$ for the set of inner automorphisms of G .

Proposition 6.26. *Let G be a group. The function $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$ that maps g to γ_g is a group homomorphism.*

PROOF: Since $\gamma_{gh}(a) = ghah^{-1}g^{-1} = \gamma_g(\gamma_h(a))$. □

Corollary 6.26.1. $\text{Inn}(G)$ is a subgroup of $\text{Aut}_{\mathbf{Grp}}(G)$.

6.4 Direct Products

Definition 6.27 (Direct Product). The *direct product* of groups G and H is their product in \mathbf{Grp} .

6.5 Free Groups

Proposition 6.28. *Let A be a set. Let \mathcal{F}^A be the category whose objects are pairs (G, j) where G is a group and j is a function $A \rightarrow G$, with morphisms $f : (G, j) \rightarrow (H, k)$ the group homomorphisms $f : G \rightarrow H$ such that $f \circ j = k$. Then \mathcal{F}^A has an initial object.*

PROOF:

$\langle 1 \rangle 1$. LET: $W(A)$ be the set of words in the alphabet whose elements are the elements of A together with $\{a^{-1} : a \in A\}$.

$\langle 1 \rangle 2$. LET: $r : W(A) \rightarrow W(A)$ be the function that, given a word w , removes the first pair of letters of the form aa^{-1} or $a^{-1}a$; if there is no such pair, then $r(w) = w$.

$\langle 1 \rangle 3$. Let us say that a word w is a *reduced word* iff $r(w) = w$.

$\langle 1 \rangle 4$. For any word w of length n , we have $r^{\lceil \frac{n}{2} \rceil}(w)$ is a reduced word.

PROOF: Since we cannot remove more than $n/2$ pairs of letters from w .

$\langle 1 \rangle 5$. LET: $R : W(A) \rightarrow W(A)$ be the function $R(w) = r^{\lceil \frac{n}{2} \rceil}(w)$, where n is the length of w .

$\langle 1 \rangle 6$. LET: $F(A)$ be the set of reduced words.

$\langle 1 \rangle 7$. Define $\cdot : F(A)^2 \rightarrow F(A)$ by $w \cdot w' = R(ww')$

$\langle 1 \rangle 8$. \cdot is associative.

PROOF: Both $w_1 \cdot (w_2 \cdot w_3)$ and $(w_1 \cdot w_2) \cdot w_3$ are equal to $R(w_1w_2w_3)$.

$\langle 1 \rangle 9$. The empty word is the identity element in $F(A)$

$\langle 1 \rangle 10$. The inverse of $a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}$ is $a_n^{\mp 1} \dots a_2^{\mp 1} a_1^{\mp 1}$.

$\langle 1 \rangle 11$. LET: $j : A \rightarrow F(A)$ be the function that maps a to the word a of length

$\langle 1 \rangle 12$. LET: G be any group and $k : A \rightarrow G$ any function.

$\langle 1 \rangle 13$. The only morphism $f : (F(A), j) \rightarrow (G, k)$ in \mathcal{F}^A is $f(a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}) = k(a_1)^{\pm 1} k(a_2)^{\pm 1} \dots k(a_n)^{\pm 1}$.

□

Definition 6.29 (Free Group). For any set A , the *free group* on A is the initial object $(F(A), i)$ in \mathcal{F}^A .

Proposition 6.30. $i : A \rightarrow F(A)$ is injective.

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in A$

$\langle 1 \rangle 2$. ASSUME: $x \neq y$

PROVE: $i(x) \neq i(y)$

$\langle 1 \rangle 3$. LET: $f : A \rightarrow C_2$ be the function that maps x to 0 and all other elements of A to 1.

$\langle 1 \rangle 4$. LET: $\phi : F(A) \rightarrow C_2$ be the group homomorphism such that $f = \phi \circ i$.

$\langle 1 \rangle 5$. $f(x) \neq f(y)$

$\langle 1 \rangle 6$. $\phi(i(x)) \neq \phi(i(y))$

$\langle 1 \rangle 7$. $i(x) \neq i(y)$

□

Proposition 6.31.

$$F(0) \cong \{e\}$$

PROOF: For any set A , the unique group homomorphism $\{e\} \rightarrow A$ makes the following diagram commute.

$$\begin{array}{ccc} \{e\} & \longrightarrow & A \\ \uparrow & \nearrow & \\ \emptyset & & \end{array}$$

Proposition 6.32. The free group on 1 is \mathbb{Z} with the injection mapping 0 to 1.

PROOF: Given any group G and function $a : 1 \rightarrow G$, the required unique homomorphism $\phi : \mathbb{Z} \rightarrow G$ is defined by $\phi(n) = a(0)^n$. □

Proposition 6.33. *For any sets A and B , we have that $F(A + B)$ is the coproduct of $F(A)$ and $F(B)$ in **Grp**.*

$$\begin{array}{ccccc}
 & & G & & \\
 & f \nearrow & \uparrow k & \nwarrow g & \\
 F(A) & \xrightarrow{\kappa_1} & F(A+B) & \xleftarrow{\kappa_2} & F(B) \\
 i_A \uparrow & & j \uparrow & & i_B \uparrow \\
 A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
 \end{array}$$

PROOF:

- <1>1. LET: $i_A : A \rightarrow F(A)$, $i_B : B \rightarrow F(B)$, $j : A + B \rightarrow F(A + B)$ be the canonical injections.
- <1>2. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
- <1>3. LET: G be any group and $f : F(A) \rightarrow G$, $g : F(B) \rightarrow G$ any group homomorphisms.
- <1>4. LET: $h : A + B \rightarrow G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.
- <1>5. LET: $k : F(A + B) \rightarrow G$ be the unique group homomorphism such that $k \circ j = h$.
- <1>6. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
- <1>7. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$. \square

Definition 6.34 (Subgroup Generated by a Group). Let G be a group and A a subset of G . Let $\phi : F(A) \rightarrow G$ be the unique group homomorphism such that $\phi(a) = a$ for all $a \in A$. The subgroup *generated* by A is

$$\langle A \rangle := \text{im } \phi$$

$$\begin{array}{ccc}
 F(A) & \xrightarrow{\phi} & G \\
 \uparrow & \nearrow & \\
 A & &
 \end{array}$$

Proposition 6.35. *Let G be a group and A a subset of G . Then $\langle A \rangle$ is the set of all elements of the form $a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$ (where $n \geq 0$) such that $a_1, \dots, a_n \in A$.*

PROOF: Immediate from definitions. \square

Corollary 6.35.1. *Let G be a group and $g \in G$. Then*

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\} .$$

Proposition 6.36. *Let G be a group and A a subset of G . Then $\langle A \rangle$ is the intersection of all the subgroups of G that include A .*

PROOF: Easy. \square

Definition 6.37 (Finitely Generated). Let G be a group. Then G is *finitely generated* iff there exists a finite subset A of G such that $G = \langle A \rangle$.

Proposition 6.38. *Every subgroup of a finitely generated free group is free.*

PROOF: TODO.

Proposition 6.39. *$F(2)$ includes subgroups isomorphic to the free group on arbitrarily many generators.*

PROOF: TODO

Proposition 6.40.

$$[F(2), F(2)] \cong F(\mathbb{Z})$$

PROOF: TODO

6.6 Normal Subgroups

Definition 6.41 (Normal Subgroup). A subgroup N of G is *normal* iff, for all $g \in G$ and $n \in N$, we have $gng^{-1} \in N$.

Proposition 6.42. *Let G be a group and N a subgroup of G . Then the following are equivalent.*

1. N is normal.
2. $\forall g \in G. gNg^{-1} \subseteq N$
3. $\forall g \in G. gNg^{-1} = N$
4. $\forall g \in G. gN \subseteq Ng$
5. $\forall g \in G. gN = Ng$

PROOF:

$\langle 1 \rangle 1. 1 \Leftrightarrow 2$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: If 2 holds then we have $gNg^{-1} \subseteq N$ and $g^{-1}Ng \subseteq N$ hence $N = gNg^{-1}$.

$\langle 1 \rangle 3. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 4. 2 \Leftrightarrow 4$

PROOF: Easy.

$\langle 1 \rangle 5. 3 \Leftrightarrow 5$

PROOF: Easy.

□

Proposition 6.43. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\ker \phi$ is a normal subgroup of G .*

PROOF: Given $g \in G$ and $n \in \ker \phi$ we have

$$\begin{aligned}\phi(gng^{-1}) &= \phi(g)\phi(n)\phi(g)^{-1} \\ &= \phi(g)\phi(g)^{-1} \\ &= e\end{aligned}$$

and so $gng^{-1} \in \ker \phi$. □

6.7 Quotient Groups

Definition 6.44. Let G be a group. Let \sim be an equivalence relation on G . Then we say that \sim is *compatible* with the group operation on G iff, for all $a, a', g \in G$, if $a \sim a'$ then $ga \sim ga'$ and $ag \sim a'g$.

Proposition 6.45. *Let G be a group. Let \sim be an equivalence relation on G . Then there exists an operation $\cdot : (G/\sim)^2 \rightarrow G/\sim$ such that*

$$\forall a, b \in G. [a][b] = [ab]$$

iff \sim is compatible with the group operation on G . In this case, G/\sim is a group under \cdot and the canonical function $\pi : G \rightarrow G/\sim$ is a group homomorphism, and is universal with respect to group homomorphisms $\phi : G \rightarrow G'$ such that if $a \sim a'$ then $\phi(a) = \phi(a')$.

PROOF: Easy. □

Definition 6.46 (Quotient Group). Let G be a group. Let \sim be an equivalence relation on G that is compatible with the group operation on G . Then G/\sim is the *quotient group* of G by \sim under $[a][b] = [ab]$.

Proposition 6.47. *Let G be a group and H a subgroup of G . Then H is normal if and only if there exists a group K and homomorphism $\phi : G \rightarrow K$ such that $H = \ker \phi$.*

PROOF: One direction is given by Proposition 6.43. For the other direction, take $K = G/H$ and ϕ to be the canonical map $G \rightarrow G/H$. □

Definition 6.48 (Modular Group). The *modular group* $\text{PSL}_2(\mathbb{Z})$ is $\text{SL}_2(\mathbb{Z})/\{I, -I\}$.

Proposition 6.49. $\text{PSL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

PROOF: By Example 5.26.

Proposition 6.50 (Roger Alperin). $\text{PSL}_2(\mathbb{Z})$ is presented by $(x, y|x^2, y^3)$.

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\langle 1 \rangle 2. \text{ LET: } y = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$\langle 1 \rangle 3.$ Define an action of $\text{PSL}_2(\mathbb{Z})$ on $\mathbb{R} - \mathbb{Q}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} r = \frac{ar + b}{cr + d}.$$

$\langle 2 \rangle 1.$ Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z})$ and r irrational we have $\frac{ar+b}{cr+d}$ is irrational.

$\langle 3 \rangle 1.$ ASSUME: for a contradiction $\frac{ar+b}{cr+d} = \frac{p}{q}$ where p and q are integers with $q > 0$.

$$\langle 3 \rangle 2. aqr + bq = cpr + dp$$

$$\langle 3 \rangle 3. (aq - cp)r = dp - bq$$

$$\langle 3 \rangle 4. aq = cp = dp - bq = 0$$

$$\langle 3 \rangle 5. adq - cdp = 0$$

$$\langle 3 \rangle 6. cdp - cbq = 0$$

$$\langle 3 \rangle 7. (ad - cb)q = 0$$

PROOF: Since $ad - cb = 1$.

$$\langle 3 \rangle 8. q = 0$$

$$\langle 3 \rangle 9. \text{ Q.E.D.}$$

PROOF: This contradicts $\langle 3 \rangle 1$.

$$\langle 2 \rangle 2. -Ir = r$$

PROOF: Since $-Ir = \frac{-r}{-1} = r$.

$$\langle 2 \rangle 3. \text{ Given } A, B \in \text{PSL}_2(\mathbb{Z}) \text{ we have } A(Br) = (AB)r.$$

PROOF:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} r \right] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{er + f}{gr + h} \\ &= \frac{a \frac{er+f}{gr+h} + b}{c \frac{er+f}{gr+h} + d} \\ &= \frac{a(er + f) + b(gr + h)}{c(er + f) + d(gr + h)} \\ &= \frac{(ae + bg)r + (af + bh)}{(ce + dg)r + (cf + dh)} \\ &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} r \\ &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right] r \end{aligned}$$

$$\langle 1 \rangle 4.$$

$$yr = 1 - \frac{1}{r}$$

$$\langle 1 \rangle 5.$$

$$y^{-1}r = \frac{1}{1-r}$$

PROOF: Since $y^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

(1)6.

PROOF: Since $yx = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. $yxr = 1 + r$

(1)7.

$$y^{-1}xr = \frac{r}{1+r}$$

PROOF: Since $y^{-1}x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

(1)8. If $r > -1$ is positive then yxr is positive.

(1)9. If r is positive then $y^{-1}xr$ is positive.

(1)10. If $r < -1$ then $y^{-1}xr$ is positive.

(1)11. If r is negative then yr is positive.

(1)12. If r is negative then $y^{-1}r$ is positive.

(1)13. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)$$

with one or more factors can equal the identity.

PROOF: If the last factor is (yx) , then the product maps numbers in $(-1, 0)$ to positive numbers. If the last factor is $(y^{-1}x)$, then the product maps numbers < -1 to positive numbers.

(1)14. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)y^{\pm 1}$$

with one or more factors can equal the identity.

PROOF: The product maps negative numbers to positive numbers.

(1)15. $\text{PSL}_2(\mathbb{Z})$ is presented by $(x, y|x^2, y^3)$.

□

Corollary 6.50.1. $\text{PSL}_2(\mathbb{Z})$ is the coproduct of C_2 and C_3 in **Grp**.

Theorem 6.51. Every group homomorphism $\phi : G \rightarrow H$ may be decomposed as

$$G \longrightarrow G/\ker \phi \xrightarrow{\cong} \text{im } \phi \longrightarrow H$$

PROOF: Easy. □

Corollary 6.51.1 (First Isomorphism Theorem). Let $\phi : G \rightarrow H$ be a surjective group homomorphism. Then $H \cong G/\ker \phi$.

Proposition 6.52. Let H_1 be a normal subgroup of G_1 and H_2 a normal subgroup of G_2 . Then $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$, and

$$\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}.$$

PROOF: $\pi \times \pi : G_1 \times G_2 \twoheadrightarrow G_1/H_1 \times G_2/H_2$ is a surjective homomorphism with kernel $H_1 \times H_2$. □

Example 6.53.

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

PROOF: Map a real number r to $(\cos r, \sin r)$. The result is a surjective group homomorphism with kernel \mathbb{Z} . \square

Proposition 6.54. *Let H be a normal subgroup of a group G . For every subgroup K of G that includes H , we have H is a normal subgroup of K , and K/H is a subgroup of G/H . The mapping*

$$u : \{\text{subgroups of } G \text{ including } H\} \rightarrow \{\text{subgroups of } G/H\}$$

with $u(K) = K/H$ is a poset isomorphism.

PROOF:

- $\langle 1 \rangle 1$. If K is a subgroup of G that includes H then H is normal in K .
- $\langle 1 \rangle 2$. If K is a subgroup of G that includes H then K/H is a subgroup of G/H .
- $\langle 1 \rangle 3$. If $H \subseteq K_1 \subseteq K_2$ then $K_1/H \subseteq K_2/H$.
- $\langle 1 \rangle 4$. If $K_1/H = K_2/H$ then $K_1 = K_2$
 - $\langle 2 \rangle 1$. ASSUME: $K_1/H = K_2/H$
 - $\langle 2 \rangle 2$. $K_1 \subseteq K_2$
 - $\langle 3 \rangle 1$. LET: $k \in K_1$
 - $\langle 3 \rangle 2$. $kH \in K_2/H$
 - $\langle 3 \rangle 3$. PICK $k' \in K_2$ such that $kH = k'H$
 - $\langle 3 \rangle 4$. $kk'^{-1} \in H$
 - $\langle 3 \rangle 5$. $kk'^{-1} \in K_2$
 - $\langle 3 \rangle 6$. $k \in K_2$
 - $\langle 2 \rangle 3$. $K_2 \subseteq K_1$
- PROOF: Similar.
- $\langle 1 \rangle 5$. For any subgroup L of G/H , there exists a subgroup K of G that includes H such that $L = K/H$.
 - $\langle 2 \rangle 1$. LET: L be a subgroup of G/H .
 - $\langle 2 \rangle 2$. LET: $K = \{k \in G : kH \in L\}$
 - $\langle 2 \rangle 3$. K is a subgroup of G .
 - PROOF: Given $k, k' \in K$ we have $kH, k'H \in L$ hence $kk'^{-1}H \in L$ and so $kk'^{-1} \in K$.
 - $\langle 2 \rangle 4$. $H \subseteq K$
 - PROOF: For all $h \in H$ we have $hH = H \in L$.
 - $\langle 2 \rangle 5$. $L = K/H$
 - PROOF: By definition.

\square

Proposition 6.55 (Third Isomorphism Theorem). *Let H be a normal subgroup of a group G . Let N be a subgroup of G that includes H . Then N/H is normal in G/H if and only if N is normal in G , in which case*

$$\frac{G/H}{N/H} \cong \frac{G}{N}$$

PROOF:

- ⟨1⟩1. If N/H is normal in G/H then N is normal in G .
 - ⟨2⟩1. ASSUME: N/H is normal in G/H .
 - ⟨2⟩2. LET: $g \in G$ and $n \in N$.
 - ⟨2⟩3. $gng^{-1}H \in N/H$
 - ⟨2⟩4. PICK $n' \in N$ such that $gng^{-1}H = n'H$
 - ⟨2⟩5. $gng^{-1}n'^{-1} \in H$
 - ⟨2⟩6. $gng^{-1}n'^{-1} \in N$
 - ⟨2⟩7. $gng^{-1} \in N$
- ⟨1⟩2. If N is normal in G then N/H is normal in G/H and $(G/H)/(N/H) \cong G/N$.
 - ⟨2⟩1. ASSUME: N is normal in G .
 - ⟨2⟩2. LET: $\phi : G/H \rightarrow G/N$ be the homomorphism $\phi(gH) = gN$
 - ⟨3⟩1. If $gH = g'H$ then $gN = g'N$
 PROOF: If $gg'^{-1} \in H$ then $gg'^{-1} \in N$.
 - ⟨3⟩2. $\phi((gH)(g'H)) = \phi(gH)\phi(g'H)$
 PROOF: Both are $gg'N$.
 - ⟨2⟩3. ϕ is surjective.
 - ⟨2⟩4. $\ker \phi = N/H$
 - ⟨2⟩5. $(G/H)/(N/H) \cong G/N$
 PROOF: By the First Isomorphism Theorem.

□

Proposition 6.56 (Second Isomorphism Theorem). *Let H and K be subgroups of a group G . Assume that H is normal in G . Then:*

- 1. HK is a subgroup of G , and H is normal in HK .
- 2. $H \cap K$ is normal in K , and

$$\frac{HK}{H} \cong \frac{K}{H \cap K}.$$

PROOF:

- ⟨1⟩1. HK is a subgroup of G .
 PROOF: Since $hkh'k' = hh'(h'^{-1}kh')k' \in HK$.
- ⟨1⟩2. H is normal in HK .
- ⟨1⟩3. $H \cap K$ is normal in K and $HK/H \cong K/(H \cap K)$
 PROOF: The function that maps k to kH is a surjective homomorphism $K \twoheadrightarrow HK/H$ with kernel $H \cap K$. Surjectivity follows because $hkh = hkh^{-1}H$.

□

See also Proposition 6.71 for a result that holds even if H is not normal.

6.8 Cosets

Proposition 6.57. *Let G be a group. Let \sim be an equivalence relation on G such that, for all $a, b, g \in G$, if $a \sim b$ then $ga \sim gb$. Let $H = \{h \in G : h \sim e\}$.*

Then H is a subgroup of G and, for all $a, b \in G$, we have

$$a \sim b \Leftrightarrow a^{-1}b \in H \Leftrightarrow aH = bH .$$

PROOF:

$\langle 1 \rangle 1.$ $e \in H$

$\langle 1 \rangle 2.$ For all $x, y \in H$ we have $xy^{-1} \in H$.

$\langle 2 \rangle 1.$ ASSUME: $x \sim e$ and $y \sim e$.

$\langle 2 \rangle 2.$ $e \sim y^{-1}$

PROOF: Since $yy^{-1} \sim ey^{-1}$.

$\langle 2 \rangle 3.$ $xy^{-1} \sim e$

PROOF: Since $xy^{-1} \sim ey^{-1} \sim e$.

$\langle 1 \rangle 3.$ If $a \sim b$ then $a^{-1}b \in H$.

PROOF: If $a \sim b$ then $a^{-1}b \sim a^{-1}a = e$.

$\langle 1 \rangle 4.$ If $a^{-1}b \in H$ then $aH = bH$.

$\langle 2 \rangle 1.$ ASSUME: $a^{-1}b \in H$

$\langle 2 \rangle 2.$ $bH \subseteq aH$

PROOF: For any $h \in H$ we have $bh = aa^{-1}bh \in aH$.

$\langle 2 \rangle 3.$ $aH \subseteq bH$

PROOF: Similar since $b^{-1}a \in H$.

$\langle 1 \rangle 5.$ If $aH = bH$ then $a \sim b$.

$\langle 2 \rangle 1.$ ASSUME: $aH = bH$

$\langle 2 \rangle 2.$ PICK $h \in H$ such that $a = bh$.

$\langle 2 \rangle 3.$ $b^{-1}a = h$

$\langle 2 \rangle 4.$ $b^{-1}a \in H$

$\langle 2 \rangle 5.$ $b^{-1}a \sim e$

$\langle 2 \rangle 6.$ $a \sim b$

PROOF: $a = bb^{-1}a \sim be = b$.

□

Definition 6.58 (Coset). Let G be a group and H a subgroup of G . A *left coset* of H is a set of the form aH for $a \in G$. A *right coset* of H is a set of the form Ha for some $a \in G$.

We write G/H for the set of all left cosets of H , and $G \backslash H$ for the set of all right cosets of H .

Proposition 6.59.

$$G/H \cong G \backslash H$$

PROOF: The function that maps aH to Ha^{-1} is a bijection. □

Proposition 6.60. Let G be a group and H a subgroup of G . Define \sim_H on G by: $a \sim b$ iff $a^{-1}b \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations \sim on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ga \sim gb$. The equivalence class of a is aH .

PROOF:

$\langle 1 \rangle 1.$ For any subgroup H , we have \sim_H is an equivalence relation on G .

⟨2⟩1. \sim is reflexive.

PROOF: For any $a \in G$ we have $a^{-1}a = e \in H$.

⟨2⟩2. \sim is symmetric.

PROOF: If $a^{-1}b \in H$ then $b^{-1}a \in H$.

⟨2⟩3. \sim is transitive.

PROOF: If $a^{-1}b \in H$ and $b^{-1}c \in H$ then $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$.

⟨1⟩2. If $a \sim_H b$ then $ga \sim_H gb$.

PROOF: If $a^{-1}b \in H$ then $(ga)^{-1}(gb) = a^{-1}g^{-1}gb = a^{-1}b \in H$.

⟨1⟩3. For any equivalence relation \sim on G such that, whenever $a \sim b$, then $ga \sim gb$, there exists a subgroup H such that $\sim = \sim_H$.

PROOF: Proposition 6.57.

⟨1⟩4. The \sim_H -equivalence class of a is aH .

PROOF:

$$\begin{aligned} a \sim b &\Leftrightarrow a^{-1}b \in H \\ &\Leftrightarrow \exists h \in H. a^{-1}b = h \\ &\Leftrightarrow \exists h \in H. b = ah \\ &\Leftrightarrow b \in aH \end{aligned}$$

□

Proposition 6.61. Let G be a group and H a subgroup of G . Define \sim_H on G by: $a \sim b$ iff $ab^{-1} \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations \sim on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ag \sim bg$. The equivalence class of a is Ha .

PROOF: Similar. □

Proposition 6.62. Let G be a group and H be a subgroup of G . Define \sim_L and \sim_R on G by:

$$a \sim_L b \Leftrightarrow a^{-1}b \in H, \quad a \sim_R b \Leftrightarrow ab^{-1} \in H.$$

Then $\sim_L = \sim_R$ if and only if H is normal.

PROOF:

⟨1⟩1. If $\sim_L = \sim_R$ then H is normal.

⟨2⟩1. ASSUME: $\sim_L = \sim_R$

⟨2⟩2. LET: $h \in H$ and $g \in G$

⟨2⟩3. $g \sim_L gh^{-1}$

⟨2⟩4. $g \sim_R gh^{-1}h$

⟨2⟩5. $ghg^{-1} \in H$

⟨1⟩2. If H is normal then $\sim_L = \sim_R$.

⟨2⟩1. ASSUME: H is normal.

⟨2⟩2. If $a \sim_L b$ then $a \sim_R b$.

⟨3⟩1. ASSUME: $a \sim_L b$

⟨3⟩2. $a^{-1}b \in H$

⟨3⟩3. $aa^{-1}ba^{-1} \in H$

⟨3⟩4. $ba^{-1} \in H$

- $\langle 3 \rangle 5. a \sim_R b$
 $\langle 2 \rangle 3. \text{ If } a \sim_R b \text{ then } a \sim_L b.$

PROOF: Similar.

□

Corollary 6.62.1. *Let G be a group and H be a normal subgroup of G . Define \sim on G by $a \sim b$ iff $a^{-1}b \in H$. Then G/\sim is a group under $[a][b] = [ab]$.*

Definition 6.63 (Quotient Group). Let G be a group and H be a normal subgroup of G . The *quotient group* G/H is G/\sim where $a \sim b$ iff $a^{-1}b \in H$, under $[a][b] = [ab]$ or $(aH)(bH) = abH$.

Corollary 6.63.1. *Let H be a normal subgroup of a group G . For every group homomorphism $\phi : G \rightarrow G'$ such that $H \subseteq \ker \phi$, there exists a unique group homomorphism $\bar{\phi} : G/H \rightarrow G'$ such that the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ & \searrow \pi & \nearrow \bar{\phi} \\ & G/H & \end{array}$$

Proposition 6.64. $\mathbb{Z}/n\mathbb{Z}$ has exactly n elements.

PROOF: Every integer is congruent to one of $0, 1, \dots, n-1$ by the division algorithm, and no two of them are congruent to one another, since if $0 \leq i < j < n$ then $0 < j - i < n$. □

Proposition 6.65. *Let m and n be integers with $n > 0$. The order of m in $\mathbb{Z}/n\mathbb{Z}$ is $\frac{n}{\gcd(m,n)}$.*

PROOF: By Proposition 5.16 since the order of 1 is n . □

Proposition 6.66. *The integer m generates $\mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(m, n) = 1$.*

PROOF: By Proposition 6.65. □

Corollary 6.66.1. *If p is prime then every non-zero element in $\mathbb{Z}/p\mathbb{Z}$ is a generator.*

Proposition 6.67.

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$$

PROOF: Every permutation of $\{(1, 0), (0, 1), (1, 1)\}$ gives an automorphism of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

Example 6.68. Not all monomorphisms split in \mathbf{Grp} .

Define $\phi : \mathbb{Z}/3\mathbb{Z} \rightarrow S_3$ by

$$\phi(0) = \text{id}_3, \quad \phi(1) = (1 \ 3 \ 2), \quad \phi(2) = (1 \ 2 \ 3) .$$

Then ϕ is monic but has no retraction.

For if $r : S_3 \rightarrow \mathbb{Z}/3\mathbb{Z}$ is a retraction, then we would have

$$r(1\ 2) + r(2\ 3) = 1, \quad r(2\ 3) + r(1\ 2) = 2$$

which is impossible.

Proposition 6.69. *Let G be a group, H a subgroup of G , and $g \in G$. The function that maps h to gh is a bijection $H \cong gH$.*

PROOF: By Cancellation. \square

Proposition 6.70. *Let G be a group, H a subgroup of G , and $g \in G$. The function that maps h to hg is a bijection $H \cong Hg$.*

PROOF: By Cancellation. \square

Proposition 6.71. *Let H and K be finite subgroups of a group G . Then*

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

PROOF:

$\langle 1 \rangle 1$. LET: $f : \{hK : h \in H\} \rightarrow H/(H \cap K)$ be the function $f(hK) = h(H \cap K)$

PROOF: This is well-defined because if $hK = h'K$ then $h^{-1}h' \in H \cap K$ so $h(H \cap K) = h'(H \cap K)$.

$\langle 1 \rangle 2$. f is injective.

PROOF: If $h(H \cap K) = h'(H \cap K)$ then $hK = h'K$.

$\langle 1 \rangle 3$. f is surjective.

PROOF: Clear.

$\langle 1 \rangle 4$.

$$\frac{|HK|}{|K|} = \frac{|H|}{|H \cap K|}$$

\square

6.9 Congruence

Definition 6.72 (Congruence). Given integers a, b, n with n positive, we say a is *congruent to b modulo n* , and write $a \equiv b \pmod{n}$, iff $a + n\mathbb{Z} = b + n\mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$.

Proposition 6.73. *Given integers a, b, n with n positive, we have $a \equiv b \pmod{n}$ iff $n \mid a - b$.*

PROOF: By Proposition 6.57. \square

Proposition 6.74. *If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then $a + b \equiv a' + b' \pmod{n}$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid (a' + b') - (a + b)$. \square

Proposition 6.75. *If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then $ab \equiv a'b' \pmod{n}$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid a'b' - ab = a'(b' - b) + (a' - a)b$. \square

6.10 Cyclic Groups

Definition 6.76 (Cyclic Group). The *cyclic* groups are \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$ for positive integers n .

Proposition 6.77. *If m and n are positive integers with $\gcd(m, n) = 1$ then $C_{mn} \cong C_m \times C_n$.*

PROOF: The function that maps x to $(x \bmod m, x \bmod n)$ is an isomorphism. \square

Proposition 6.78. *Let G be a group and $g \in G$. Then $\langle g \rangle$ is cyclic.*

PROOF: If g has finite order then $\langle g \rangle \cong C_{|g|}$, otherwise $\langle g \rangle \cong \mathbb{Z}$. \square

Proposition 6.79. *Every finitely generated subgroup of \mathbb{Q} is cyclic.*

PROOF:

$\langle 1 \rangle 1$. LET: $G = \langle a_1/b, \dots, a_n/b \rangle$ where a_1, \dots, a_n, b are integers with $b > 0$

$\langle 1 \rangle 2$. LET: $a = \gcd(a_1, \dots, a_n)$

$\langle 1 \rangle 3$. $G = \langle a/b \rangle$

\square

Corollary 6.79.1. \mathbb{Q} is not finitely generated.

6.11 Euler's ϕ -function

Definition 6.80. For n a positive integer, let $(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$.

PROOF: We prove this is well-defined.

$\langle 1 \rangle 1$. If $m \equiv m' \pmod{n}$ and $\gcd(m, n) = 1$ then $\gcd(m', n) = 1$.

$\langle 2 \rangle 1$. PICK integers a, b such that $am + bn = 1$

$\langle 2 \rangle 2$. PICK an integer c such that $m' - m = cn$

$\langle 2 \rangle 3$. $am' + (b - ac)n = 1$

\square

Example 6.81. For any positive integer n , the set

$$(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$$

is a group under multiplication.

PROOF:

$\langle 1 \rangle 1$. If $\gcd(m_1, n) = \gcd(m_2, n) = 1$ then $\gcd(m_1 m_2, n) = 1$

$\langle 2 \rangle 1$. PICK integers a, b, c, d such that $am_1 + bn = cm_2 + dn = 1$

$\langle 2 \rangle 2$. $acm_1 m_2 + (bcm_2 + d)n = 1$

$\langle 1 \rangle 2$. Multiplication is associative.

$\langle 1 \rangle 3$. 1 is the identity element.

$\langle 1 \rangle 4$. Every element has an inverse.

- ⟨2⟩1. LET: $a \in (\mathbb{Z}/n\mathbb{Z})^*$
 ⟨2⟩2. PICK integers b, c such that $ab + cn = 1$
 ⟨2⟩3. $ab = 1$ in $(\mathbb{Z}/n\mathbb{Z})^*$

□

Definition 6.82. For n a positive integer, let $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$.

Proposition 6.83. If n is an odd positive integer then $\phi(2n) = \phi(n)$.

PROOF:

- ⟨1⟩1. LET: n be an odd positive integer.
 ⟨1⟩2. For any integer m , if $\gcd(m, n) = 1$ then $\gcd(2m + n, 2n) = 1$
 PROOF: For p a prime, if $p \mid 2m + n$ and $p \mid 2n$ then $p \neq 2$ (since $2m + n$ is odd) so $p \mid n$ and hence $p \mid m$, which is a contradiction.
 ⟨1⟩3. For any integer r , if $\gcd(r, 2n) = 1$ then $\gcd(\frac{r+n}{2}, n) = 1$
 PROOF: If $p \mid n$ and $p \mid \frac{r+n}{2}$ then $p \mid r + n$ so $p \mid r$ which is a contradiction.
 ⟨1⟩4. The function that maps m to $2m + n$ is a bijection between $(\mathbb{Z}/n\mathbb{Z})^*$ and $(\mathbb{Z}/2n\mathbb{Z})^*$.

□

Theorem 6.84. For any positive integer n we have

$$\sum_{m>0, m|n} \phi(m) = n.$$

PROOF:

- ⟨1⟩1. Define $\chi : \{0, 1, \dots, n-1\} \rightarrow \{(m, d) : m > 0, m \mid n, d \text{ generates } \langle n/m \rangle\}$
 by: $\chi(x) = (\gcd(x, n), x)$.
 ⟨1⟩2. χ is injective.
 ⟨1⟩3. χ is surjective.
 PROOF: Given (m, d) such that d generates $\langle n/m \rangle$ we have $\chi(d) = (m, d)$.
 ⟨1⟩4. $n = \sum_{m>0, m|n} \phi(m)$
 PROOF: Since $\langle n/m \rangle \cong C_m$ and so has $\phi(m)$ generators.

□

Proposition 6.85. For any positive integers a and n , we have $n \mid \phi(a^n - 1)$.

PROOF: Since the order of a is n in $(\mathbb{Z}/(a^n - 1)\mathbb{Z})^*$. □

Theorem 6.86 (Euler's Theorem). For any coprime integers a and n we have $a^{\phi(n)} \equiv a \pmod{n}$.

PROOF: Immediate from Lagrange's Theorem. □

6.12 Commutator Subgroup

Definition 6.87 (Commutator Subgroup). Let G be a group. The *commutator subgroup* $[G, G]$ is the subgroup generated by the elements of the form $aba^{-1}b^{-1}$.

Proposition 6.88. *The commutator subgroup is normal.*

PROOF: Since

$$\begin{aligned} & ga_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_nb_na_n^{-1}b_n^{-1}g^{-1} \\ &= (ga_1g^{-1})(gb_1g^{-1})(ga_1g^{-1})^{-1}(gb_1g^{-1})^{-1} \cdots (ga_ng^{-1})(gb_ng^{-1})(ga_ng^{-1})^{-1}(gb_ng^{-1})^{-1}. \quad \square \end{aligned}$$

6.13 Presentations

Definition 6.89 (Presentation). A *presentation* of a group G is a pair (A, R) where A is a set and $R \subseteq F(A)$ is a set of words such that

$$G \cong F(A)/N(R)$$

where $N(R)$ is the smallest normal subgroup of $F(A)$ that includes R .

Example 6.90. The free group on a set A is presented by (A, \emptyset) .

Example 6.91. S_3 is presented by $(x, y|x^2, y^3, xyxy)$.

Example 6.92. $(a, b | a^2, b^2, (ab)^n)$ is a presentation of D_{2n} .

Proposition 6.93 (Word Problem). *Let (A, R) be a presentation of the group G . Let $w_1, w_2 \in F(A)$ be two words. Then it is undecidable in general if $w_1N(R) = w_2N(R)$ in G .*

Definition 6.94 (Finitely Presented). A group is *finitely presented* iff it has a presentation (A, R) where both A and R are finite.

Proposition 6.95. *Let $(A|R)$ be a presentation of G and $(A'|R')$ a presentation of H . Assume w.l.o.g. A and A' are disjoint. Then the group $G * G'$ presented by $(A \cup A' | R \cup R')$ is the coproduct of G and G' in \mathbf{Grp} .*

$$\begin{array}{ccccc} A & \longrightarrow & A \cup A' & \longleftarrow & A' \\ \downarrow & & \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(A \cup A') & \longleftarrow & F(A') \\ \downarrow & & \downarrow & & \downarrow \\ G & \xrightarrow{\kappa_1} & G * G' & \xleftarrow{\kappa_2} & G' \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: $\kappa_1 : G \rightarrow G * G'$ and $\kappa_2 : G' \rightarrow G * G'$ be the unique homomorphisms that make the diagram above commute.

$\langle 1 \rangle 2$. LET: $\phi : G \rightarrow H$ and $\psi : G' \rightarrow H$ be any homomorphisms.

$\langle 1 \rangle 3$. LET: $[\phi, \psi] : F(A \cup A') \rightarrow H$ be the unique homomorphism such that ...

$\langle 1 \rangle 4$. $R \cup R' \subseteq \ker[\phi, \psi]$

$\langle 1 \rangle 5$. $[\phi, \psi]$ factors uniquely through the morphism $F(A \cup A') \rightarrow G * G'$

\square

6.14 Index of a Subgroup

Definition 6.96 (Index). Let G be a group and H a subgroup of G . The *index* of H in G , denoted $|G : H|$, is the number of left cosets of H in G if this is finite, otherwise ∞ .

Theorem 6.97 (Lagrange's Theorem). *Let G be a finite group and H a subgroup of G . Then*

$$|G| = |G : H||H| .$$

PROOF: G/H is a partition of G into $|G : H|$ subsets, each of size $|H|$. \square

Corollary 6.97.1. *For p a prime number, the only group of order p is C_p .*

PROOF: Let G be a group of order p and $g \in G$ with $g \neq e$. Then $|\langle g \rangle|$ divides p and is not 1, hence is p , that is, $G = \langle g \rangle$. \square

Corollary 6.97.2 (Fermat's Little Theorem). *Let p be a prime number and a any integer. Then $a^p \equiv a \pmod{p}$.*

PROOF: If $p \mid a$ then $a^p \equiv a \equiv 0 \pmod{p}$. Otherwise, we have $a^{p-1} \equiv 1 \pmod{p}$ by applying Lagrange's Theorem to $(\mathbb{Z}/p\mathbb{Z})^*$. \square

Example 6.98. It is not true that, if $n \mid |G|$, then G has a subgroup of order n . The group A_4 has order 12 but no subgroup of order 6.

Theorem 6.99 (Cauchy's Theorem). *Let G be a finite group. If p is prime and $p \mid |G|$ then G has a subgroup of order p .*

Proposition 6.100. *Let G be a group. Let K be a subgroup of G and H a subgroup of K . If $|G : H|$, $|G : K|$ and $|K : H|$ are all finite then*

$$|G : H| = |G : K||K : H| .$$

PROOF:

$\langle 1 \rangle 1$. LET: $G/K = \{g_1K, g_2K, \dots, g_mK\}$

$\langle 1 \rangle 2$. LET: $K/H = \{k_1H, k_2H, \dots, k_nH\}$

$\langle 1 \rangle 3$. $G/H = \{g_ik_jH : 1 \leq i \leq m, 1 \leq j \leq n\}$

$\langle 2 \rangle 1$. LET: $g \in G$

$\langle 2 \rangle 2$. PICK i such that $gK = g_iK$

$\langle 2 \rangle 3$. $g^{-1}g_i \in K$

$\langle 2 \rangle 4$. PICK j such that $g^{-1}g_iH = k_jH$

$\langle 2 \rangle 5$. $g^{-1}g_ik_j \in H$

$\langle 2 \rangle 6$. $gH = g_ik_jH$

$\langle 1 \rangle 4$. If $g_ik_jH = g_{i'}k_{j'}H$ then $i = i'$ and $j = j'$.

$\langle 2 \rangle 1$. ASSUME: $g_ik_jH = g_{i'}k_{j'}H$

$\langle 2 \rangle 2$. $g_iK = g_{i'}K$

$\langle 2 \rangle 3$. $i = i'$

$\langle 2 \rangle 4$. $k_jH = k_{j'}H$

$\langle 2 \rangle 5$. $j = j'$

\square

6.15 Cokernels

Proposition 6.101. *Let $\phi : G \rightarrow H$ be a homomorphism between groups. Then there exists a group K and homomorphism $\pi : H \rightarrow K$ that is initial with respect to all homomorphism $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.*

PROOF:

- (1)1. LET: N be the intersection of all the normal subgroups of H that include $\text{im } \phi$.
 - (1)2. LET: $K = H/N$ and π be the canonical homomorphism.
 - (1)3. LET: $\pi \circ \phi = 0$
 - (1)4. LET: $\alpha : H \rightarrow L$ satisfy $\alpha \circ \phi = 0$
 - (1)5. $\text{im } \phi \subseteq \ker \alpha$
 - (1)6. $N \subseteq \ker \alpha$
 - (1)7. There exists a unique $\bar{\alpha} : H/\text{im } \phi \rightarrow L$ such that $\bar{\alpha} \circ \pi = \alpha$
-

Definition 6.102 (Cokernel). For any homomorphism $\phi : G \rightarrow H$ in **Grp**, the *cokernel* of ϕ is the group $\text{coker } \phi$ and homomorphism $\pi : H \rightarrow \text{coker } \phi$ that is initial among homomorphisms $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.

Example 6.103. It is not true that a homomorphism with trivial cokernel is epi. The inclusion $\langle (1\ 2) \rangle \hookrightarrow S_3$ has trivial cokernel but is not epi.

6.16 Cayley Graphs

Definition 6.104 (Cayley Graph). Let G be a finitely generated group. Let A be a finite set of generators for G . The *Cayley graph* of G with respect to A is the directed graph whose vertices are the elements of G , with an edge $g_1 \rightarrow g_2$ labelled by $a \in A$ iff $g_2 = g_1 a$.

Proposition 6.105. *G is the free group on A iff the Cayley graph with respect to A is a tree.*

PROOF: Both are equivalent to saying that the product of two different strings of elements of A and/or their inverses are not equal. □

Chapter 7

Abelian Groups

Definition 7.1 (Abelian Group). A group is *Abelian* iff any two elements commute.

In an Abelian group G , we often denote the group operation by $+$, the identity element by 0 and the inverse of an element g by $-g$. We write ng for g^n ($g \in G, n \in \mathbb{Z}$).

Example 7.2. Every group of order ≤ 4 is Abelian.

Example 7.3. For any positive integer n , we have $\mathbb{Z}/n\mathbb{Z}$ is an Abelian group under addition.

Example 7.4. S_n is not Abelian for $n \geq 3$. If $x = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ then $xy = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $yx = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$.

Example 7.5. There are 42 Abelian groups of order 1024 up to isomorphism.

Proposition 7.6. Let G be a group. If $g^2 = e$ for all $g \in G$ then G is Abelian.

PROOF: For any $g, h \in G$ we have

$$ghgh = e$$

$$\therefore hgh = g \quad (\text{multiplying on the left by } g)$$

$$\therefore hg = gh \quad (\text{multiplying on the right by } h) \square$$

Proposition 7.7. Let G be a group. Then G is Abelian if and only if the function that maps g to g^{-1} is a group homomorphism.

PROOF:

(1)1. If G is Abelian then the function that maps g to g^{-1} is a group homomorphism.

PROOF: Since $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$.

(1)2. If the function that maps g to g^{-1} is a group homomorphism then G is Abelian.

PROOF: Since $gh = (g^{-1})^{-1}(h^{-1})^{-1} = (g^{-1}h^{-1})^{-1} = hg$.
 \square

Proposition 7.8. *Let G be a group. Then G is Abelian if and only if the function that maps g to g^2 is a group homomorphism.*

PROOF:

$\langle 1 \rangle 1$. If G is Abelian then the function that maps g to g^2 is a group homomorphism.

PROOF: Since $(gh)^2 = g^2h^2$.

$\langle 1 \rangle 2$. If the function that maps g to g^2 is a group homomorphism then G is Abelian.

PROOF: Since we have $(gh)^2 = ghgh = g^2h^2$ and so $hg = gh$.

\square

Proposition 7.9. *Let G be a group. Then G is Abelian if and only if the homomorphism $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$ is the trivial homomorphism.*

PROOF:

$\langle 1 \rangle 1$. If G is Abelian then γ is trivial.

PROOF: Since $\gamma_g(a) = gag^{-1} = a$.

$\langle 1 \rangle 2$. If γ is trivial then G is Abelian.

PROOF: If $\gamma_g(a) = gag^{-1} = a$ for all g and a then $ga = ag$ for all g, a .

\square

Proposition 7.10. *Let G be an Abelian group. Let $g, h \in G$. If g has maximal finite order in G , and h has finite order, then $|h| \mid |g|$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $|h| \nmid |g|$.

$\langle 1 \rangle 2$. PICK a prime p such that $|g| = p^m r$, $|h| = p^n s$ where $p \nmid r$, $p \nmid s$ and $m < n$.

$\langle 1 \rangle 3$. $|g^{p^m} h^s| = p^n r$

PROOF: Proposition 5.19.

$\langle 1 \rangle 4$. $|g| < |g^{p^m} h^s|$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts the maximality of $|g|$.

\square

Proposition 7.11. *If p is prime then $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.*

PROOF:

$\langle 1 \rangle 1$. LET: g be an element of maximal order in $(\mathbb{Z}/p\mathbb{Z})^*$.

$\langle 1 \rangle 2$. For all $h \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $h^{|g|} = 1$.

PROOF: Proposition 7.10.

$\langle 1 \rangle 3$. There are at most $|g|$ elements x such that $x^{|g|} = 1$ in $\mathbb{Z}/p\mathbb{Z}$

$\langle 1 \rangle 4$. $p - 1 \leq |g|$

$\langle 1 \rangle 5$. $|g| = p - 1$

$\langle 1 \rangle 6$. g generates $(\mathbb{Z}/p\mathbb{Z})^*$.

□

Example 7.12. $(\mathbb{Z}/12\mathbb{Z})^*$ is not cyclic. Its elements are 1, 5, 7 and 11 with orders 1, 2, 2 and 2.

Theorem 7.13 (Wilson's Theorem). *A positive integer p is prime if and only if $(p-1)! \equiv 1 \pmod{p}$.*

⟨1⟩1. If p is prime then $(p-1)! \equiv 1 \pmod{p}$.

⟨2⟩1. ASSUME: p is prime.

⟨2⟩2. $(p-1)!$ is the product of all the elements of $(\mathbb{Z}/p\mathbb{Z})^*$

⟨2⟩3. The only element of $(\mathbb{Z}/p\mathbb{Z})^*$ with order 2 is -1 .

⟨2⟩4. $(p-1)! \equiv -1 \pmod{p}$

PROOF: Proposition 5.20.

⟨1⟩2. If $(p-1)! \equiv -1 \pmod{p}$ then p is prime.

⟨2⟩1. ASSUME: (

$(p-1)! \equiv -1 \pmod{p}$)

⟨2⟩2. LET: d be a proper divisor of p .

PROVE: $d = 1$

⟨2⟩3. $d \mid (p-1)!$

⟨2⟩4. $d \mid 1$

PROOF: Since $d \mid p \mid (p-1)! + 1$.

⟨2⟩5. $d = 1$

□

Proposition 7.14. *If p and q are distinct odd primes then $(\mathbb{Z}/pq\mathbb{Z})^*$ is not cyclic.*

PROOF:

⟨1⟩1. $|(\mathbb{Z}/pq\mathbb{Z})^*| = (p-1)(q-1)$

⟨1⟩2. LET: $g \in (\mathbb{Z}/pq\mathbb{Z})^*$

PROVE: g does not have order $(p-1)(q-1)$

⟨1⟩3. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{p}$

⟨1⟩4. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{q}$

⟨1⟩5. $pq \mid g^{(p-1)(q-1)/2} - 1$

⟨1⟩6. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{pq}$

⟨1⟩7. $|g| \mid (p-1)(q-1)/2$

□

Proposition 7.15. *For any prime p , we have $\text{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$.*

PROOF:

⟨1⟩1. LET: $\phi : \text{Aut}_{\mathbf{Grp}}(C_p) \rightarrow (\mathbb{Z}/p\mathbb{Z})^*$ be the function $\phi(\alpha) = \alpha(1)$.

PROOF: $\alpha(1)$ has order p in C_p and so is coprime with p .

⟨1⟩2. ϕ is a homomorphism.

PROOF: $\phi(\alpha \circ \beta) = \alpha(\beta(1)) = \alpha(\beta(1)1) = \beta(1)\alpha(1) = \phi(\alpha)\phi(\beta)$

⟨1⟩3. ϕ is injective.

PROOF: If $\phi(\alpha) = \phi(\beta)$ then for any n we have $\alpha(n) = n\alpha(1) = n\phi(\alpha) = n\phi(\beta) = n\beta(1) = \beta(n)$.

$\langle 1 \rangle 4.$ ϕ is surjective.

PROOF: For any $r \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $r = \phi(\alpha)$ where $\alpha(n) = nr \bmod p$.

$\langle 1 \rangle 5.$ $(\mathbb{Z}/p\mathbb{Z})^* \cong C_{p-1}$

□

Proposition 7.16. *Given a set A and an Abelian group H , the set H^A is an Abelian group under*

$$(\phi + \psi)(a) = \phi(a) + \psi(a) \quad (\phi, \psi \in H^A, a \in A) .$$

PROOF:

$\langle 1 \rangle 1.$ $\phi + (\psi + \chi) = (\phi + \psi) + \chi$

$\langle 1 \rangle 2.$ $\phi + \psi = \psi + \phi$

$\langle 1 \rangle 3.$ LET: $0 : A \rightarrow H$ be the function $0(a) = 0$.

$\langle 1 \rangle 4.$ $\phi + 0 = 0 + \phi = \phi$

$\langle 1 \rangle 5.$ Given $\phi : A \rightarrow H$, define $-\phi : A \rightarrow H$ by $(-\phi)(a) = -(\phi(a))$.

$\langle 1 \rangle 6.$ $\phi + (-\phi) = (-\phi) + \phi = 0$

□

Proposition 7.17. *Given a group G and an Abelian group H , the set $\mathbf{Grp}[G, H]$ is a subgroup of H^G .*

PROOF:

$\langle 1 \rangle 1.$ Given $\phi, \psi : G \rightarrow H$ group homomorphisms, we have $\phi - \psi$ is a group homomorphism.

PROOF:

$$\begin{aligned} (\phi - \psi)(g + g') &= \phi(g + g') - \psi(g + g') \\ &= \phi(g) + \phi(g') - \psi(g) - \psi(g') \\ &= \phi(g) - \psi(g) + \phi(g') - \psi(g') \\ &= (\phi - \psi)(g) + (\phi - \psi)(g') \end{aligned}$$

□

Proposition 7.18. *Let G be a group. The following are equivalent.*

1. $\text{Inn}(G)$ is cyclic.
2. $\text{Inn}(G)$ is trivial.
3. G is Abelian.

PROOF:

$\langle 1 \rangle 1.$ $1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: $\text{Inn}(G) = \langle \gamma_g \rangle$

$\langle 2 \rangle 2.$ g commutes with every element of G

$\langle 3 \rangle 1.$ LET: $x \in G$

$\langle 3 \rangle 2.$ PICK $n \in \mathbb{Z}$ such that $\gamma_x = \gamma_g^n$

$\langle 3 \rangle 3.$ $\forall y \in G. xyx^{-1} = g^n yg^{-n}$

$\langle 3 \rangle 4.$ $xgx^{-1} = g$

$\langle 2 \rangle 3.$ $\gamma_g = \text{id}_G$
 $\langle 1 \rangle 2.$ $2 \Rightarrow 3$
 $\langle 2 \rangle 1.$ ASSUME: $\forall g \in G. \gamma_g = \text{id}_G$
 $\langle 2 \rangle 2.$ LET: $x, y \in G$
 $\langle 2 \rangle 3.$ $\gamma_x(y) = y$
 $\langle 2 \rangle 4.$ $xyx^{-1} = y$
 $\langle 2 \rangle 5.$ $xy = yx$
 $\langle 1 \rangle 3.$ $3 \Rightarrow 2$
 PROOF: If $xy = yx$ for all x, y then $\gamma_x(y) = y$ for all x, y .
 $\langle 1 \rangle 4.$ $2 \Rightarrow 1$
 PROOF: Easy.
 \square

Corollary 7.18.1. *If $\text{Aut}_{\text{Grp}}(G)$ is cyclic then G is Abelian.*

Proposition 7.19. *Every subgroup of an Abelian group is normal.*

PROOF: Let G be an Abelian group and N a subgroup of G . Given $g \in G$ and $n \in N$ we have $gng^{-1} = n \in N$. \square

Proposition 7.20. *For any group G , the group $G/[G, G]$ is Abelian.*

PROOF: For any $g, h \in G$ we have
 $gh(hg)^{-1} \in [G, G]$

$$\therefore gh[G, G] = hg[G, G] \quad \square$$

Proposition 7.21. *Let G be a finite Abelian group. Let p be a prime divisor of $|G|$. Then G has an element of order p .*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: as induction hypothesis the result holds for all groups smaller than G .
 $\langle 1 \rangle 2.$ PICK $g \in G - \{0\}$.
 $\langle 1 \rangle 3.$ PICK an element $h \in \langle g \rangle$ with prime order q .
 $\langle 1 \rangle 4.$ CASE: $q = p$
 PROOF: h is the required element.
 $\langle 1 \rangle 5.$ CASE: $q \neq p$
 $\langle 2 \rangle 1.$ PICK $r \in G$ such that $r + \langle h \rangle$ has order p in $G/\langle h \rangle$.
 PROOF: By induction hypothesis since $|G/\langle h \rangle| = |G|/q$.
 $\langle 2 \rangle 2.$ $pr \in \langle h \rangle$
 $\langle 2 \rangle 3.$ PICK k such that $pr = kh$
 $\langle 2 \rangle 4.$ $pqr = e$
 $\langle 2 \rangle 5.$ qr has order p .
 \square

Corollary 7.21.1. *For n an odd integer, any Abelian group of order $2n$ has exactly one element of order 2.*

PROOF: If x and y are distinct elements of order 2 then $\langle x, y \rangle = \{e, x, y, xy\}$ has size 4 and so $4 \mid 2n$ which is a contradiction. \square

Example 7.22. It is not true that, if G is a finite group and $d \mid |G|$, then G has an element of order d . The quaternion group has no element of order 4.

Proposition 7.23. *If G is a finite Abelian group and $d \mid |G|$ then G has a subgroup of size d .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: as induction hypothesis the result is true for all $d' < d$.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $d \neq 1$.

$\langle 1 \rangle 3$. PICK a prime p such that $p \mid d$.

$\langle 1 \rangle 4$. PICK an element $g \in G$ of order p .

$\langle 1 \rangle 5$. $d/p \mid |G/\langle g \rangle|$

$\langle 1 \rangle 6$. PICK a subgroup H of $G/\langle g \rangle$ of size d/p .

$\langle 1 \rangle 7$. $\pi^{-1}(H)$ is a subgroup of G of size d .

□

Proposition 7.24. *Let (G, \cdot) be a group. Let $\circ : G^2 \rightarrow G$ be a group homomorphism such that (G, \circ) is a group. Then \circ and \cdot coincide, and G is Abelian.*

PROOF:

$\langle 1 \rangle 1$. For all $g_1, g_2, h_1, h_2 \in G$ we have

$$(g_1 g_2) \circ (h_1 h_2) = (g_1 \circ h_1)(g_2 \circ h_2)$$

$\langle 1 \rangle 2$. $e \circ e = e$

PROOF:

$$\begin{aligned} e \circ e &= (ee) \circ (ee) \\ &= (e \circ e)(e \circ e) \end{aligned}$$

Hence $e \circ e = e$ by Cancellation.

$\langle 1 \rangle 3$. e is the identity of (G, \circ)

$\langle 1 \rangle 4$. For all $g, h \in G$ we have

$$g \circ h = gh$$

PROOF:

$$\begin{aligned} g \circ h &= (ge) \circ (eh) \\ &= (g \circ e)(e \circ h) \\ &= gh \end{aligned}$$

$\langle 1 \rangle 5$. For all $g, h \in G$ we have $gh = hg$.

PROOF:

$$\begin{aligned} gh &= (e \circ g)(h \circ e) \\ &= (eh) \circ (ge) \\ &= h \circ g \\ &= hg \end{aligned}$$

□

Corollary 7.24.1. *If $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$ is a group object in **Grp** then m is the multiplication of G , $e(*)$ is the identity of G , $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Grp** where $e(*) = e$ and $i(g) = g^{-1}$.*

7.1 The Category of Abelian Groups

Definition 7.25 (Category of Abelian Groups). Let **Ab** be the full subcategory of **Grp** whose objects are the Abelian groups.

Proposition 7.26. *If $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$ is a group object in **Ab** then m is the multiplication of G , $e(*)$ is the identity of G , $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Ab** where $e(*) = e$ and $i(g) = g^{-1}$.*

PROOF: Immediate from Corollary 7.24.1. \square

Definition 7.27 (Direct Sum). Given Abelian groups G and H , we also call the direct product of G and H the *direct sum* and denote it $G \oplus H$.

Proposition 7.28. *Given Abelian groups G and H , the direct sum $G \oplus H$ is the coproduct of G and H in **Ab**.*

PROOF:

$\langle 1 \rangle 1$. LET: $\kappa_1 : G \rightarrow G \oplus H$ be the group homomorphism $\kappa_1(g) = (g, e_H)$.

$\langle 1 \rangle 2$. LET: $\kappa_2 : H \rightarrow G \oplus H$ be the group homomorphism $\kappa_2(h) = (e_G, h)$.

$\langle 1 \rangle 3$. Given group homomorphism $\phi : G \rightarrow K$ and $\psi : H \rightarrow K$, define $[\phi, \psi] : G \oplus H \rightarrow K$ by $[\phi, \psi](g, h) = \phi(g) + \psi(h)$.

$\langle 1 \rangle 4$. $[\phi, \psi]$ is a group homomorphism.

PROOF:

$$\begin{aligned} [\phi, \psi]((g, h) + (g', h')) &= [\phi, \psi](g + g', h + h') \\ &= \phi(g + g') + \psi(h + h') \\ &= \phi(g) + \phi(g') + \psi(h) + \psi(h') \\ &= \phi(g) + \psi(h) + \phi(g') + \psi(h') \\ &= [\phi, \psi](g, h) + [\phi, \psi](g', h') \end{aligned}$$

$\langle 1 \rangle 5$. $[\phi, \psi] \circ \kappa_1 = \phi$

PROOF:

$$\begin{aligned} [\phi, \psi](\kappa_1(g)) &= [\phi, \psi](g, e_h) \\ &= \phi(g) + \psi(e_h) \\ &= \phi(g) + e_K \\ &= \phi(g) \end{aligned}$$

$\langle 1 \rangle 6$. $[\phi, \psi] \circ \kappa_2 = \psi$

PROOF: Similar.

$\langle 1 \rangle 7$. If $f : G \oplus H \rightarrow K$ is a group homomorphism with $f \circ \kappa_1 = \phi$ and $f \circ \kappa_2 = \psi$ then $f = [\phi, \psi]$.

PROOF:

$$\begin{aligned} f(g, h) &= f((g, e_H) + (e_G, h)) \\ &= f(\kappa_1(g)) + f(\kappa_2(h)) \\ &= \phi(g) + \psi(h) \end{aligned}$$

□

Theorem 7.29. *Every finitely generated Abelian group is a direct sum of cyclic groups.*

PROOF: TODO □

7.2 Free Abelian Groups

Proposition 7.30. *Let A be a set. Let \mathcal{F}^A be the category whose objects are pairs (G, j) where G is an Abelian group and j is a function $A \rightarrow G$, with morphisms $f : (G, j) \rightarrow (H, k)$ the group homomorphisms $f : G \rightarrow H$ such that $f \circ j = k$. Then \mathcal{F}^A has an initial object.*

PROOF:

- ⟨1⟩1. LET: $\mathbb{Z}^{\oplus A}$ be the subgroup of \mathbb{Z}^A consisting of all functions $\alpha : A \rightarrow \mathbb{Z}$ such that $\alpha(a) = 0$ for only finitely many $a \in A$.
- ⟨1⟩2. LET: $i : A \rightarrow \mathbb{Z}^{\oplus A}$ be the function such that $i(a)(b) = 1$ if $a = b$ and 0 if $a \neq b$.
- ⟨1⟩3. LET: G be any Abelian group and $j : A \rightarrow G$ any function.
- ⟨1⟩4. The unique homomorphism $\phi : \mathbb{Z}^{\oplus A} \rightarrow G$ required is defined by $\phi(\alpha) = \sum_{a \in A} \alpha(a)j(a)$

□

Definition 7.31 (Free Abelian Group). For any set A , the *free Abelian group* on A is the initial object $(F^{ab}(A), i)$ in \mathcal{F}^A .

Proposition 7.32. *For any sets A and B , we have that $F^{ab}(A + B)$ is the coproduct of $F^{ab}(A)$ and $F^{ab}(B)$ in **Grp**.*

$$\begin{array}{ccccc}
 & & G & & \\
 & f \nearrow & \uparrow k & \nwarrow g & \\
 F^{ab}(A) & \xrightarrow{\kappa_1} & F^{ab}(A+B) & \xleftarrow{\kappa_2} & F^{ab}(B) \\
 i_A \uparrow & & j \uparrow & & i_B \uparrow \\
 A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
 \end{array}$$

PROOF:

- ⟨1⟩1. LET: $i_A : A \rightarrow F^{ab}(A)$, $i_B : B \rightarrow F^{ab}(B)$, $j : A + B \rightarrow F^{ab}(A + B)$ be the canonical injections.
- ⟨1⟩2. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
- ⟨1⟩3. LET: G be any group and $f : F^{ab}(A) \rightarrow G$, $g : F^{ab}(B) \rightarrow G$ any group homomorphisms.
- ⟨1⟩4. LET: $h : A + B \rightarrow G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.

- (1)5. LET: $k : F^{\text{ab}}(A + B) \rightarrow G$ be the unique group homomorphism such that $k \circ j = h$.
- (1)6. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
- (1)7. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.
 \square

Proposition 7.33. *For A and B finite sets, if $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$ then $A \cong B$.*

PROOF:

- (1)1. For any set C , define \sim on $F^{\text{ab}}(C)$ by: $f \sim f'$ iff there exists $g \in F^{\text{ab}}(C)$ such that $f - f' = 2g$.
- (1)2. For any set C , \sim is an equivalence relation on $F^{\text{ab}}(C)$.
- (1)3. For any set C , we have $F^{\text{ab}}(C) / \sim$ is finite if and only if C is finite, in which case $|F^{\text{ab}}(C) / \sim| = 2^{|C|}$.
- PROOF: There is a bijection between $F^{\text{ab}}(C) / \sim$ and the finite subsets of C , which maps f to $\{c \in C : f(c) \text{ is odd}\}$.
- (1)4. If $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$ then $A \cong B$.
- PROOF: If $|F^{\text{ab}}(A) / \sim| = |F^{\text{ab}}(B) / \sim|$ then $2^{|A|} = 2^{|B|}$ and so $|A| = |B|$.
 \square

Proposition 7.34. *Let G be an Abelian group. Then G is finitely generated if and only if there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n .*

PROOF:

- (1)1. If G is finitely generated then there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n .
- PROOF: Let $G = \langle a_1, \dots, a_n \rangle$. Define $\phi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$ by $\phi(i_1, \dots, i_n) = i_1 \cdot a_1 + \dots + i_n \cdot a_n$.
- (1)2. If there exists a surjective homomorphism $\phi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n then G is finitely generated.
- PROOF: G is generated by $\phi(1, 0, \dots, 0), \phi(0, 1, 0, \dots, 0), \dots, \phi(0, \dots, 0, 1)$.
 \square

Proposition 7.35. *Let A be a set. Let $i : A \hookrightarrow F(A)$ be the free group on A . Then $\pi \circ i : A \rightarrow F(A)/[F(A), F(A)]$ is the free Abelian group on A .*

$$\begin{array}{ccc}
 & F(A)/[F(A), F(A)] & \\
 \uparrow \pi & \searrow h & \\
 F(A) & \xrightarrow{g} & G \\
 \uparrow i & \nearrow f & \\
 A & &
 \end{array}$$

PROOF:

- (1)1. LET: G be an Abelian group and $f : A \rightarrow G$ a function.

- $\langle 1 \rangle 2$. LET: $g : F(A) \rightarrow G$ be the unique group homomorphism such that $g \circ i = f$.
- $\langle 1 \rangle 3$. $[F(A), F(A)] \subseteq \ker g$
 PROOF: For all $x, y \in F(A)$ we have $g(xy x^{-1} y^{-1}) = g(x) + g(y) - g(x) - g(y) = 0$.
- $\langle 1 \rangle 4$. LET: $h : F(A)/[F(A), F(A)]$ be the unique group homomorphism such that $h \circ \pi = g$.
- $\langle 1 \rangle 5$. h is the unique group homomorphism such that $h \circ \pi \circ i = f$.
 \square

Corollary 7.35.1. *Let A and B be sets. Let $F(A)$ and $F(B)$ be the free groups on A and B respectively. If $F(A) \cong F(B)$ then $A \cong B$.*

PROOF: Proposition 7.33. \square

7.3 Cokernels

Proposition 7.36. *Let $\phi : G \rightarrow H$ be a homomorphism between Abelian groups. Then there exists an Abelian group K and homomorphism $\pi : H \rightarrow K$ that is initial with respect to all homomorphism $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.*

PROOF:

- $\langle 1 \rangle 1$. LET: $K = H/\text{im } \phi$ and π be the canonical homomorphism.
- $\langle 1 \rangle 2$. LET: $\pi \circ \phi = 0$
- $\langle 1 \rangle 3$. LET: $\alpha : H \rightarrow L$ satisfy $\alpha \circ \phi = 0$
- $\langle 1 \rangle 4$. $\text{im } \phi \subseteq \ker \alpha$
- $\langle 1 \rangle 5$. There exists a unique $\bar{\alpha} : H/\text{im } \phi \rightarrow L$ such that $\bar{\alpha} \circ \pi = \alpha$
 \square

Definition 7.37 (Cokernel). For any homomorphism $\phi : G \rightarrow H$ in **Ab**, the *cokernel* of ϕ is the Abelian group $\text{coker } \phi$ and homomorphism $\pi : H \rightarrow \text{coker } \phi$ that is initial among homomorphisms $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.

Proposition 7.38. $\pi : H \rightarrow \text{coker } \phi$ is initial among functions $f : H \rightarrow X$ such that, for all $x, y \in H$, if $x + \text{im } \phi = y + \text{im } \phi$ then $f(x) = f(y)$.

PROOF: Easy. \square

Proposition 7.39. *Let $\phi : G \rightarrow H$ be a homomorphism of Abelian groups. Then the following are equivalent.*

- ϕ is an epimorphism.
- $\text{coker } \phi$ is trivial.
- ϕ is surjective.

PROOF:

- $\langle 1 \rangle 1$. $1 \Rightarrow 2$

- $\langle 2 \rangle 1$. ASSUME: ϕ is epi.
- $\langle 2 \rangle 2$. LET: $\pi : H \rightarrow \text{coker } \phi$ be the canonical homomorphism.
- $\langle 2 \rangle 3$. $\pi \circ \phi = 0 \circ \phi$
- $\langle 2 \rangle 4$. $\pi = 0$
- $\langle 2 \rangle 5$. $\text{coker } \phi = \text{im } \pi$ is trivial.
- $\langle 1 \rangle 2$. $2 \Rightarrow 3$
 PROOF: If $\text{coker } \phi = H / \text{im } \phi$ is trivial then $\text{im } \phi = H$.
- $\langle 1 \rangle 3$. $3 \Rightarrow 1$
 PROOF: If it is surjective then it is epi in **Set**.

□

Chapter 8

Group Actions

8.1 Group Actions

Definition 8.1 (Action). Let G be a group. Let A be an object of a category \mathcal{C} . A (left) action of G on A is a group homomorphism $G \rightarrow \text{Aut}_{\mathcal{C}}(A)$.

It is *faithful* or *effective* iff it is injective.

Proposition 8.2. Let A be a set. An action of the group G on the set A is given by a function $\cdot : G \times A \rightarrow A$ such that

- $\forall a \in A. ea = a$
- $\forall g, h \in G. \forall a \in A. (gh)a = g(ha)$

PROOF: Just unfolding definitions. \square

Example 8.3. Left multiplication defines a faithful action of any group on its own underlying set.

In fact, for any subgroup H of a group G , left multiplication defines an action of G on G/H .

Corollary 8.3.1 (Cayley's Theorem). Every group G is a subgroup of a symmetric group, namely $\text{Aut}_{\text{Set}}(G)$.

Example 8.4. Conjugation $g * h = ghg^{-1}$ is an action of any group on its own underlying set.

Definition 8.5 (Transitive). An action of a group G on a set A is *transitive* iff, for all $a, b \in A$, there exists $g \in G$ such that $ga = b$.

Example 8.6. Left multiplication of a group G is a transitive action of G on G .

Definition 8.7 (Orbit). Given an action of a group G on a set A and $a \in A$, the *orbit* of a is

$$\text{O}_G(a) := \{ga : g \in G\} .$$

Proposition 8.8. *Given an action of a group G on a set A , the orbits form a partition of A .*

PROOF:

$\langle 1 \rangle 1$. Every element of A is in some orbit.

PROOF: Since $a \in O_G(a)$.

$\langle 1 \rangle 2$. Distinct orbits are disjoint.

$\langle 2 \rangle 1$. LET: $a \in O_G(b) \cap O_G(c)$

$\langle 2 \rangle 2$. PICK $g, h \in G$ such that $a = gb = hc$.

$\langle 2 \rangle 3$. $O_G(b) \subseteq O_G(c)$

PROOF: For all $k \in G$ we have $kb = kg^{-1}hc$.

$\langle 2 \rangle 4$. $O_G(c) \subseteq O_G(b)$

PROOF: Similar.

□

Proposition 8.9. *Given an action of a group G on a set A and $a \in A$, the action is transitive on $O_G(a)$.*

PROOF:

$\langle 1 \rangle 1$. The restriction of the action is an action on $O_G(a)$.

PROOF: Since $g(ha) = (gh)a$, the action maps $O_G(a)$ to itself.

$\langle 1 \rangle 2$. The restricted action is transitive.

PROOF: Given $ga, ha \in O_G(a)$, we have $ha = (hg^{-1})(ga)$.

□

Definition 8.10 (Stabilizer Subgroup). Given an action of a group G on a set A and $a \in A$, the *stabilizer subgroup* of a is

$$\text{Stab}_G(a) := \{g \in G : ga = a\} .$$

Proposition 8.11. *Stabilizer subgroups are subgroups.*

PROOF: If $g, h \in \text{Stab}_G(a)$ then $gh^{-1}a = a$ so $gh^{-1} \in \text{Stab}_G(a)$. □

Proposition 8.12. *Let G act on a set A . Let $a \in A$ and $g \in G$. Then*

$$\text{Stab}_G(ga) = g\text{Stab}_G(a)g^{-1} .$$

PROOF:

$$h \in \text{Stab}_G(ga) \Leftrightarrow hga = ga$$

$$\Leftrightarrow g^{-1}hga = a$$

$$\Leftrightarrow g^{-1}hg \in \text{Stab}_G(a)$$

$$\Leftrightarrow h \in g\text{Stab}_G(a)g^{-1}$$

□

Corollary 8.12.1. *Let G be an action on a set A and $a \in A$. If $\text{Stab}_G(a)$ is normal in G , then for any $b \in O_G(a)$ we have $\text{Stab}_G(a) = \text{Stab}_G(b)$.*

Definition 8.13 (Free). An action of a group G on a set A is *free* iff, whenever $ga = a$, then $g = e$.

Example 8.14. The action of left multiplication is free.

Proposition 8.15. *Let G be a group. Let H be a subgroup of G of finite index n . Then H includes a subgroup K that is normal in G and such that $|G : K|$ divides $\gcd(|G|, n!)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\sigma : G \rightarrow \text{Aut}_{\text{Set}}(G/H)$ be the action of left multiplication.

$\langle 1 \rangle 2$. LET: $K = \ker \sigma$

$\langle 1 \rangle 3$. $K \subseteq H$

$\langle 2 \rangle 1$. LET: $g \in K$

$\langle 2 \rangle 2$. $\sigma(g)(H) = H$

$\langle 2 \rangle 3$. $gH = H$

$\langle 2 \rangle 4$. $g \in H$

$\langle 1 \rangle 4$. K is normal in G .

PROOF: Proposition 6.43.

$\langle 1 \rangle 5$. $|G : K| \mid |G|$

PROOF: Lagrange's Theorem.

$\langle 1 \rangle 6$. $|G : K| \mid n!$

PROOF: Since G/K is a subgroup of $\text{Aut}_{\text{Set}}(G/H)$.

□

Corollary 8.15.1. *Let G be a finite group. Let H be a subgroup of G of index p where p is the smallest prime that divides $|G|$. Then H is normal in G .*

PROOF:

$\langle 1 \rangle 1$. PICK a subgroup K of H normal in G such that $|G : K|$ divides $\gcd(|G|, p!)$.

$\langle 1 \rangle 2$. $|G : K|$ divides p .

$\langle 1 \rangle 3$. $|G : H| |H : K|$ divides p .

$\langle 1 \rangle 4$. $|H : K| = 1$

$\langle 1 \rangle 5$. $H = K$

$\langle 1 \rangle 6$. H is normal.

□

Corollary 8.15.2. *Any subgroup of index 2 is normal.*

Proposition 8.16. *Let G be a group with finite set of generators A . Then left multiplication defines a free action of G on its Cayley graph.*

PROOF: Easy since if $g_2 = g_1 a$ then $hg_2 = hg_1 a$. □

Corollary 8.16.1. *A free group acts freely on a tree.*

Theorem 8.17. *If a group G acts freely on a tree then G is free.*

Corollary 8.17.1. *Every subgroup of the free group on a finite set is free.*

PROOF: If H is a subgroup of $F(A)$ then left multiplication defines a free action of H on the Cayley graph of $F(A)$, which is a tree. □

8.2 Category of G -Sets

Definition 8.18. Given a group G , let $G - \mathbf{Set}$ be the category with:

- objects all pairs (A, ρ) such that A is a set and $\rho : G \times A \rightarrow A$ is an action of G on A ;
- morphisms $f : (A, \rho) \rightarrow (B, \sigma)$ are functions $f : A \rightarrow B$ that are $(G-)$ equivariant, i.e.

$$\forall g \in G. \forall a \in A. f(\rho(g, a)) = \sigma(g, f(a)) .$$

Proposition 8.19. *A G -equivariant function $f : A \rightarrow B$ is an isomorphism in $G - \mathbf{Set}$ if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ be G -equivariant and bijective.

PROVE: f^{-1} is G -equivariant.

$\langle 1 \rangle 2$. LET: $g \in G$ and $b \in B$

$\langle 1 \rangle 3$. $f^{-1}(gb) = gf^{-1}(b)$

PROOF:

$$\begin{aligned} f(f^{-1}(gb)) &= gb \\ &= gf(f^{-1}(b)) \\ &= f(gf^{-1}(b)) \end{aligned}$$

□

Proposition 8.20. *Let G be a group and A a transitive G -set. Let $a \in A$. Then A is isomorphic to $G/\text{Stab}_G(a)$ under left multiplication.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : G/\text{Stab}_G(a) \rightarrow A$ be the function $f(g\text{Stab}_G(a)) = ga$.

$\langle 2 \rangle 1$. ASSUME: $g\text{Stab}_G(a) = h\text{Stab}_G(a)$

PROVE: $ga = ha$

$\langle 2 \rangle 2$. $g^{-1}h \in \text{Stab}_G(a)$

$\langle 2 \rangle 3$. $g^{-1}ha = a$

$\langle 2 \rangle 4$. $ha = ga$

$\langle 1 \rangle 2$. f is G -equivariant.

PROOF: Since $f(gh\text{Stab}_G(a)) = gha = gf(h\text{Stab}_G(a))$.

$\langle 1 \rangle 3$. f is injective.

PROOF: If $ga = ha$ then $g^{-1}h \in \text{Stab}_G(a)$ so $g\text{Stab}_G(a) = h\text{Stab}_G(a)$.

$\langle 1 \rangle 4$. f is surjective.

PROOF: Since for all $b \in A$ there exists $g \in G$ such that $ga = b$.

□

Corollary 8.20.1. *If O is an orbit of the action of a finite group G on a set A , then O is finite and $|O|$ divides $|G|$.*

Corollary 8.20.2. *Let H be a subgroup of G and $g \in G$. Then*

$$G/H \cong G/(gHg^{-1})$$

in $G - \mathbf{Set}$.

PROOF: Taking $A = G/H$ and $a = gH$. \square

Proposition 8.21. *Given a family of G -sets $\{A_i\}_{i \in I}$, we have $\prod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g\{a_i\}_{i \in I} = \{ga_i\}_{i \in I}.$$

PROOF: Easy. \square

Proposition 8.22. *Given a family of G -sets $\{A_i\}_{i \in I}$, we have $\coprod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g(i, a_i) = (i, ga_i).$$

PROOF: Easy. \square

Proposition 8.23. *Every finite G -set is a coproduct of G -sets of the form G/H .*

PROOF: If $O(a_1), \dots, O(a_n)$ are the orbits of the G -set A , then G is the coproduct of $G/\text{Stab}_G(a_1), \dots, G/\text{Stab}_G(a_n)$. \square

Proposition 8.24. *For any group G we have $G \cong \text{Aut}_{G-\mathbf{Set}}(G)$ (considering G as a G -set under left multiplication).*

PROOF:

$\langle 1 \rangle 1$. Define $\phi : G \rightarrow \text{Aut}_{G-\mathbf{Set}}(G)$ by $\phi(g)(g') = g'g^{-1}$.

$\langle 2 \rangle 1$. LET: $g \in G$

PROVE: $\lambda g' \in G.g'g^{-1}$ is an automorphism of G in $G - \mathbf{Set}$.

$\langle 2 \rangle 2$. $\phi(g)$ is G -equivariant.

PROOF: Since $\phi(g)(h_1h_2) = h_1h_2g^{-1} = h_1\phi(g)(h_2)$.

$\langle 2 \rangle 3$. $\phi(g)$ is injective.

PROOF: By Cancellation.

$\langle 2 \rangle 4$. $\phi(g)$ is surjective.

PROOF: For any $h \in G$ we have $h = \phi(g)(hg)$.

$\langle 1 \rangle 2$. ϕ is a group homomorphism.

PROOF: $\phi(g_1g_2)(h) = hg_2^{-1}g_1^{-1} = \phi(g_1)(\phi(g_2)(h))$.

$\langle 1 \rangle 3$. ϕ is injective.

PROOF: If $\phi(g) = \phi(g')$ then $g = \phi(g)(e) = \phi(g')(e) = g'$.

$\langle 1 \rangle 4$. ϕ is surjective.

$\langle 2 \rangle 1$. LET: $\sigma \in \text{Aut}_{G-\mathbf{Set}}(G)$

$\langle 2 \rangle 2$. LET: $g = \sigma(e)$

PROVE: $\sigma = \phi(g^{-1})$

$\langle 2 \rangle 3$. $\sigma(h) = hg$

PROOF: $\sigma(h) = \sigma(hg) = h\sigma(e) = hg$.

\square

Part III

Ring Theory

Chapter 9

Rings

9.1 Rngs

Definition 9.1 (Ring). A *rng* consists of a set R and binary operations $+, \cdot : R^2 \rightarrow R$ such that:

- $(R, +)$ is an Abelian group
- \cdot is associative.
- The *distributive properties* hold: for all $r, s, t \in R$ we have

$$(r + s)t = rt + st, \quad r(s + t) = rs + rt .$$

Example 9.2. • The *zero rng* is $\{0\}$.

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} are rngs.
- $2\mathbb{Z}$ is a rng.
- Given a rng R and natural number n , then the set $\mathfrak{gl}_n(R)$ of all $n \times n$ matrices with entries in R is a rng under matrix addition and matrix multiplication.
- For any set S , the power set $\mathcal{P}S$ is a rng under $A + B = (A \cup B) - (A \cap B)$ and $AB = A \cap B$.
- Given a rng R and a set S , then R^S is a rng under $(f + g)(s) = f(s) + g(s)$ and $(fg)(s) = f(s)g(s)$ for all $f, g \in R^S$ and $s \in S$.
- The set $\mathfrak{sl}_n(\mathbb{R}) = \{M \in \mathfrak{gl}_n(\mathbb{R}) : \text{tr } M = 0\}$ is a rng.
- The set $\mathfrak{sl}_n(\mathbb{C}) = \{M \in \mathfrak{gl}_n(\mathbb{C}) : \text{tr } M = 0\}$ is a rng.
- $\mathbb{Z}/n\mathbb{Z}$ is a rng.

Proposition 9.3. *In any rng R we have*

$$\forall x \in R. x0 = 0x = 0 \text{ .}$$

PROOF:

$$\begin{aligned} x0 &= x(0 + 0) \\ &= x0 + x0 \end{aligned}$$

and so $x0 = 0$ by Cancellation. Similarly $0x = 0$. \square

Definition 9.4 (Zero Divisor). Let R be a rng and $a \in R$.

Then a is a *left-zero-divisor* iff there exists $b \in R - \{0\}$ such that $ab = 0$.

The element a is a *right-zero-divisor* iff there exists $b \in R - \{0\}$ such that $ba = 0$.

Example 9.5. 0 is a left- and right-zero-divisor in every non-zero rng.

The zero rng is the only ring with no zero-divisors.

Proposition 9.6. *Let R be a rng and $a \in R$. Then a is not a left-zero-divisor if and only if left multiplication by a is an injective function $R \rightarrow R$.*

PROOF:

$\langle 1 \rangle 1$. If a is not a left-zero-divisor then left multiplication by a is injective.

$\langle 2 \rangle 1$. ASSUME: a is not a left-zero-divisor.

$\langle 2 \rangle 2$. LET: $ab = ac$

$\langle 2 \rangle 3$. $a(b - c) = 0$

$\langle 2 \rangle 4$. $b - c = 0$

$\langle 2 \rangle 5$. $b = c$

$\langle 1 \rangle 2$. If a is a left-zero-divisor then left multiplication by a is not injective.

$\langle 2 \rangle 1$. PICK $b \neq 0$ such that $ab = 0$.

$\langle 2 \rangle 2$. $ab = a0$ but $b \neq 0$

\square

9.1.1 Commutative Rngs

Definition 9.7 (Commutative). A rng R is *commutative* iff $\forall x, y \in R. xy = yx$.

Example 9.8. • The zero rng is commutative.

- \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are commutative.
- $2\mathbb{Z}$ is commutative.
- $\mathfrak{gl}_2(\mathbb{R})$ is not commutative.
- For any set S , the rng $\mathcal{P}S$ is commutative.
- If R is commutative then R^S is commutative.

9.2 Rings

Definition 9.9 (Ring). A *ring* R is a rng such that there exists $1 \in R$, the *multiplicative identity*, such that

$$\forall x \in R. x1 = 1x = x \quad .$$

Example 9.10. • The zero rng is a ring with $1 = 0$.

- \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are rngs.
- $2\mathbb{Z}$ is not a ring.
- If R is a ring then $\mathfrak{gl}_n(R)$ is a ring.
- For any set S , the rng $\mathcal{P}S$ is a ring with $1 = S$.
- If R is a ring then R^S is a ring.
- $\mathfrak{sl}_n(\mathbb{R})$ is not a ring for $n > 0$.
- $\mathfrak{sl}_n(\mathbb{C})$ is not a ring for $n > 0$.
- $\mathfrak{so}_n(\mathbb{R}) = \{M \in \mathfrak{sl}_n(\mathbb{R}) : M + M^T = 0\}$ is not a ring.
- $\mathbb{Z}/n\mathbb{Z}$ is a ring.

Proposition 9.11. *In any ring R , if $0 = 1$ then R is the zero ring.*

PROOF: For any $x \in R$ we have $x = 1x = 0x = 0$. \square

Proposition 9.12. *In any ring we have $(-1)x = -x$.*

PROOF: Since

$$\begin{aligned} x + (-1)x &= 1x + (-1)x \\ &= (1 + (-1))x \\ &= 0x \\ &= 0 \end{aligned} \quad \square$$

9.3 Integral Domains

Definition 9.13 (Integral Domain). An *integral domain* is a non-trivial commutative ring with no nonzero zero-divisors.

Example 9.14. \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are integral domains.

Part IV

Linear Algebra

Definition 9.15. Let $\text{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ real matrices.

$\text{GL}_n(\mathbb{R})$ acts on \mathbb{R}^n by matrix multiplication.

Definition 9.16. Let $\text{GL}_n(\mathbb{C})$ be the group of invertible $n \times n$ complex matrices.

$\text{GL}_n(\mathbb{C})$ acts on \mathbb{C}^n by matrix multiplication.

Definition 9.17. Let $\text{SL}_n(\mathbb{R}) = \{M \in \text{GL}_n(\mathbb{R}) : \det M = 1\}$.

Proposition 9.18. $\text{SL}_n(\mathbb{R})$ is a normal subgroup of $\text{GL}_n(\mathbb{R})$.

PROOF: If $\det M = 1$ then $\det(AMA^{-1}) = (\det A)(\det M)(\det A)^{-1} = 1$. \square

Proposition 9.19.

$$\text{GL}_n(\mathbb{R})/\text{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$$

Definition 9.20. Let $\text{SL}_n(\mathbb{C}) = \{M \in \text{GL}_n(\mathbb{C}) : \det M = 1\}$.

Definition 9.21. Let $\text{O}_n(\mathbb{R}) = \{M \in \text{GL}_n(\mathbb{R}) : MM^T = M^T M = I_n\}$.

Proposition 9.22. The action of $\text{O}_n(\mathbb{R})$ on \mathbb{R}^n preserves lengths and angles.

Definition 9.23. Let $\text{SO}_n(\mathbb{R}) = \{M \in \text{O}_n(\mathbb{R}) : \det M = 1\}$.

Definition 9.24. Let $\text{U}_n(\mathbb{C}) = \{M \in \text{GL}_n(\mathbb{C}) : MM^\dagger = M^\dagger M = I_n\}$.

Definition 9.25. Let $\text{SU}_n(\mathbb{C}) = \{M \in \text{U}_n(\mathbb{C}) : \det M = 1\}$.

Proposition 9.26. Every matrix in $\text{SU}_2(\mathbb{C})$ can be written in the form

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 + c^2 + d^2 = 1$.

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SU}_2(\mathbb{C})$$

$$\langle 1 \rangle 2. M^{-1} = M^\dagger$$

$$\langle 1 \rangle 3. \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

$$\langle 1 \rangle 4. \text{ LET: } \alpha = a+bi \text{ and } \beta = c+di.$$

$$\langle 1 \rangle 5. \delta = \bar{\alpha} = a-bi$$

$$\langle 1 \rangle 6. \gamma = -\bar{\beta} = -c+di$$

$$\langle 1 \rangle 7. \det M = a^2 + b^2 + c^2 + d^2 = 1$$

\square

Corollary 9.26.1. $\text{SU}_2(\mathbb{C})$ is simply connected.

Corollary 9.26.2.

$$\text{SO}_3(\mathbb{R}) \cong \text{SU}_2(\mathbb{C})/\{I, -I\}$$

PROOF: The function that maps $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$ to $\begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{pmatrix}$

is a surjective homomorphism with kernel $\{I, -I\}$. \square

Corollary 9.26.3. The fundamental group of $\text{SO}_3(\mathbb{R})$ is C_2 .