Mathematics

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September 18, 2023

Contents

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*. We write $A : \mathrm{Set}$ for: A is a set.

For any set A, let there be *elements* of A. We write $a : \mathrm{El}\,(A)$ for: a is an element of A.

For any sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B.

For any function $f : A \to B$ and element $a : \mathrm{El}\,(A)$, let there be an element $f(a) : \mathrm{El}\,(B)$, the *value* of the function f at the *argument* a.

For any sets A and B, let there be a set $A \times B$, the *Cartesian product* of A and B, and functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$, the *projections*.

For any elements $a : \mathrm{El}\,(A)$ and $b : \mathrm{El}\,(B)$, let there be an element $(a, b) : \mathrm{El}\,(A \times B)$, the *ordered pair* of a and b.

1.2 Axioms

Axiom 1.1 (Strong Extensionality)**.** *Let $i : A \to B$. Suppose that, for every $y : \mathrm{El}\,(B)$, there exists a unique $x : \mathrm{El}\,(A)$ such that $i(x) = y$. Then there exists a function $i^{-1} : B \to A$ such that $\forall x : \mathrm{El}\,(A)\,.i^{-1}(i(x)) = x$ and $\forall y : \mathrm{El}\,(B)\,.i(i^{-1}(y)) = y$.*

Axiom 1.2 (Pairing)**.**

- $\forall x : \mathrm{El}\,(A)\,.\forall y : \mathrm{El}\,(B)\,.\pi_1(x, y) = x$
- $\forall x : \mathrm{El}\,(A)\,.\forall y : \mathrm{El}\,(B)\,.\pi_2(x, y) = y$
- $\forall p : \mathrm{El}\,(A \times B)\,.p = (\pi_1(p), \pi_2(p))$

Definition 1.3 (Injective)**.** A function $f : A \to B$ is *injective* or an *injection* iff, for all $x, y : \mathrm{El}\,(A)$, if $f(x) = f(y)$ then $x = y$.

Axiom 1.4 (Separation)**.** *For every property $P[X, x]$ where X is a set variable and $x : \mathrm{El}\,(X)$, the following is an axiom:*

For every set A, there exists a set $S = \{x : \mathrm{El}\,(A) \mid P[A, x]\}$ and an injection $i : S \to A$ such that, for all $x : \mathrm{El}\,(A)$, we have

$$(\exists y : S.i(y) = x) \Leftrightarrow P[A, x] \ .$$

Axiom 1.5 (Infinity)**.** *There exists a set $\mathbb{N}$, an element $0 : \mathrm{El}\,(\mathbb{N})$, and a function $s : \mathbb{N} \to \mathbb{N}$ such that:*

- $\forall n : \mathrm{El}\,(\mathbb{N})\,.s(n) \neq 0$
- $\forall m, n : \mathrm{El}\,(\mathbb{N})\,.s(m) = s(n) \Rightarrow m = n.$

Axiom 1.6 (Choice)**.** *Let R be a set and $i : R \to A \times B$ an injection. Assume $\forall a : \mathrm{El}\,(A)\,.\exists r : \mathrm{El}\,(R)\,.\pi_1(i(r)) = a$. Then there exists a function $f : A \to B$ such that $\forall a : \mathrm{El}\,(A)\,.\exists r : \mathrm{El}\,(R)\,.i(r) = (a, f(a))$.*

1.3 Consequences of the Axioms

1.3.1 Definitions Used in the Axioms

Definition 1.7 (Equality of Relations)**.** Let $R, S : A \looparrowright B$. We say that R and S are *equal*, $R = S$, iff $\forall a : \mathrm{El}\,(A)\,.\forall b : \mathrm{El}\,(B)\,.aRb \Leftrightarrow aSb$.

Proposition 1.8. *Let $f, g : A \to B$. If $\forall x : \mathrm{El}\,(A)\,.f(x) = g(x)$ then $f = g$.*

PROOF: Since $xfy \Leftrightarrow y = f(x) \Leftrightarrow y = g(x) \Leftrightarrow xgy$. $\square$

Definition 1.9 (Injective)**.** A function $f : A \to B$ is *injective* iff, for all $x, y : \mathrm{El}\,(A)$, if $f(x) = f(y)$ then $x = y$.

Definition 1.10 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for all $y : \mathrm{El}\,(B)$, there exists $x : \mathrm{El}\,(A)$ such that $f(x) = y$.

Definition 1.11 (Bijective)**.** A function $f : A \to B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$".

1.3.2 Tabulations

Theorem 1.12. *Let $R : A \nrightarrow B$. Let $p : T \to A$ and $q : T \to B$ form a tabulation of R. Let $p' : T' \to A$ and $q' : T' \to B$ form a tabulation of R. Then there exists a unique bijection $f : T \approx T'$ such that $\forall t : \mathrm{El}(T) . p'(f(t)) = p(t)$ and $\forall t : \mathrm{El}(T) . q'(f(t)) = q(t)$.*

PROOF:
⟨1⟩1. LET: $f : T \nrightarrow T'$ be the relation such that tft' iff $p(t) = p'(t')$ and $q(t) = q'(t')$
PROOF: Axiom of Comprehension
⟨1⟩2. f is a function.
⟨2⟩1. LET: $x : \mathrm{El}(T)$
⟨2⟩2. $p(x) R q(x)$
PROOF: Since T is a tabulation of R.
⟨2⟩3. There exists a unique $y : \mathrm{El}(T')$ such that $p'(y) = p(x)$ and $q'(y) = q(x)$.
PROOF: Since T' is a tabulation of R.
⟨1⟩3. f is injective.
⟨2⟩1. LET: $x, y : \mathrm{El}(T)$
⟨2⟩2. ASSUME: $f(x) = f(y)$
⟨2⟩3. $p'(f(x)) = p'(f(y))$ and $q'(f(x)) = q'(f(y))$
⟨2⟩4. $p(x) = p(y)$ and $q(x) = q(y)$
⟨2⟩5. $x = y$
PROOF: Since T is a tabulation of R.
⟨1⟩4. f is surjective.
⟨2⟩1. LET: $y : \mathrm{El}(T')$
⟨2⟩2. $p'(y) R q'(y)$
PROOF: Since T' is a tabulation of R.
⟨2⟩3. There exists $x : \mathrm{El}(T)$ such that $p(x) = p'(y)$ and $q(x) = q'(y)$.
PROOF: Since T is a tabulation of R.
⟨1⟩5. If $g : T \approx T'$ satisfies $\forall t : \mathrm{El}(T) . p'(g(t)) = p(t)$ and $\forall t : \mathrm{El}(T) . q'(g(t)) = q(t)$.
⟨2⟩1. LET: $g : T \approx T'$ satisfy $\forall t : \mathrm{El}(T) . p'(g(t)) = p(t)$ and $\forall t : \mathrm{El}(T) . q'(g(t)) = q(t)$.
⟨2⟩2. For all $t : \mathrm{El}(T)$ we have $p'(f(t)) = p'(g(t))$ and $q'(f(t)) = q'(g(t))$.
⟨2⟩3. For all $t : \mathrm{El}(T)$ we have $f(t) = g(t)$.
$\square$

1.3.3 The Empty Set

Theorem 1.13. *There exists a set which has no elements.*

PROOF:
⟨1⟩1. PICK a set A
PROOF: By the Axiom of Infinity, a set exists.
⟨1⟩2. LET: $R : A \nrightarrow A$ be the relation such that, for all $x, y \in A$, we have $\neg(xRy)$

Proof: By the Axiom of Comprehension.
⟨1⟩3. Let: $|R|$ be the tabulation of R with projections $p, q : |R| \to A$.
Prove: $|R|$ has no elements.
Proof: By the Axiom of Tabulations.
⟨1⟩4. Assume: for a contradiction $r : \mathrm{El}(|R|)$
⟨1⟩5. $p(r)Rq(r)$
⟨1⟩6. Q.E.D.
Proof: This contradicts ⟨1⟩2.
□

Theorem 1.14. *If E and E' have no elements then $E \approx E'$.*

Proof:
⟨1⟩1. Let: E and E' have no elements.
⟨1⟩2. Let: $F : E \nrightarrow E'$ be the relation such that, for all $x : \mathrm{El}(E)$ and $y : \mathrm{El}(E')$, we have xFy.
Proof: Axiom of Comprehension.
⟨1⟩3. F is a function.
Proof: Vacuously, for all $x : \mathrm{El}(E)$, there exists a unique $y : \mathrm{El}(E')$ such that xFy.
⟨1⟩4. F is injective.
Proof: Vacuously, for all $x, y : \mathrm{El}(E)$, if $F(x) = F(y)$ then $x = y$.
⟨1⟩5. F is surjective.
Proof: Vacuously, for all $y : \mathrm{El}(E)$, there exists $x : \mathrm{El}(E)$ such that $F(x) = y$.
□

Definition 1.15 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

1.3.4 The Singleton

Theorem 1.16. *There exists a set that has exactly one element.*

Proof:
⟨1⟩1. Pick a set A that has an element.
Proof: By the Axiom of Infinity, there exists a set that has an element.
⟨1⟩2. Pick $a : \mathrm{El}(A)$
⟨1⟩3. Let: $R : A \nrightarrow A$ be the relation such that, for all $x, y : \mathrm{El}(A)$, we have xRy if and only if $x = y = a$.
Proof: By the Axiom of Comprehension.
⟨1⟩4. Let: $|R|$ be the tabulation of R with projections $p, q : |R| \to A$.
Prove: $|R|$ has exactly one element.
Proof: By the Axiom of Tabulations.
⟨1⟩5. Let: $r : \mathrm{El}(|R|)$ be the element such that $p(r) = q(r) = a$
Proof: Since aRa by ⟨1⟩3.
⟨1⟩6. Let: $s : \mathrm{El}(|R|)$
Prove: $s = r$

⟨1⟩7. $p(s)Rq(s)$
  PROOF: By the Axiom of Tabulations.
⟨1⟩8. $p(s) = q(s) = a$
  PROOF: By ⟨1⟩3.
⟨1⟩9. $p(s) = p(r)$ and $q(s) = q(r)$
  PROOF: By ⟨1⟩5.
⟨1⟩10. $s = r$
  PROOF: By the Axiom of Tabulations.
□

Theorem 1.17. *If A and B both have exactly one element then $A \approx B$.*

PROOF:
⟨1⟩1. LET: A and B both have exactly one element.
⟨1⟩2. LET: $F : A \looparrowright B$ be the relation such that, for all $x : \mathrm{El}(A)$ and $y : \mathrm{El}(B)$, we have xFy.
⟨1⟩3. F is a function.
  PROOF: If xFy and xFy' then $y = y'$ because B has only one element.
⟨1⟩4. F is injective.
  PROOF: If $F(x) = F(x')$ then $x = x'$ because A has only one element.
⟨1⟩5. F is surjective.
  ⟨2⟩1. LET: $y : \mathrm{El}(B)$
  ⟨2⟩2. LET: x be the element of A.
  ⟨2⟩3. $F(x) = y$
□

Definition 1.18 (Singleton)**.** Let 1 be the set that has exactly one element. Let $*$ be its element.

1.3.5 Subsets

Definition 1.19 (Subset)**.** A *subset* of a set A is a relation $1 \looparrowright S$.
Given $S : 1 \looparrowright S$ and $a : \mathrm{El}(A)$, we write $a \in S$ for $*Sa$.

Theorem Schema 1.20. *For any property $P[X, x]$ where X is a set variable and $x : \mathrm{El}(X)$, the following is a theorem:*
For any set A, there exists a set B and injection $i : B \to A$ such that, for all $x : \mathrm{El}(A)$, we have $P[A, x]$ if and only if there exists $b : \mathrm{El}(B)$ such that $i(b) = x$.

PROOF:
⟨1⟩1. LET: $S : 1 \looparrowright A$ be the relation such that, for all $e : \mathrm{El}(1)$ and $a : \mathrm{El}(A)$, we have eSa if and only if $P[A, a]$.
  PROOF: Axiom of Comprehension.
⟨1⟩2. LET: B be the tabulation of S with projections $p : B \to 1$ and $i : B \to A$.
  PROOF: Axiom of Tabulations.
⟨1⟩3. i is injective.
  ⟨2⟩1. LET: $r, s : \mathrm{El}(B)$

⟨2⟩2. ASSUME: $i(r) = i(s)$
⟨2⟩3. $p(r) = p(s)$
PROOF: Since 1 has only one element.
⟨2⟩4. $r = s$
PROOF: Axiom of Tabulations.
⟨1⟩4. For all $x : \mathrm{El}(A)$, we have $P[A, x]$ if and only if there exists $b : \mathrm{El}(B)$ such that $i(b) = x$.
⟨2⟩1. LET: $x : \mathrm{El}(A)$
⟨2⟩2. If $P[A, x]$ then there exists $b : \mathrm{El}(B)$ such that $i(b) = x$
⟨3⟩1. ASSUME: $P[A, x]$
⟨3⟩2. $*Sx$
PROOF: ⟨1⟩1
⟨3⟩3. There exists $b : \mathrm{El}(B)$ such that $p(b) = *$ and $i(b) = x$
PROOF: Axiom of Tabulations.
⟨2⟩3. For all $b : \mathrm{El}(B)$ we have $P[A, i(b)]$
⟨3⟩1. LET: $b : \mathrm{El}(B)$
⟨3⟩2. $p(b)Si(b)$
PROOF: Axiom of Tabulations.
⟨3⟩3. $P[A, i(b)]$
PROOF: ⟨1⟩1

□

1.4 Composition

Definition 1.21 (Composite)**.** Let $\phi : A \nrightarrow B$ and $\psi : B \nrightarrow C$. The *composite* $\psi \circ \phi : A \nrightarrow C$ is the relation such that $a(\psi \circ \phi)c$ iff there exists b such that $a\phi b$ and $b\psi c$.

Definition 1.22 (Identity)**.** For any set A, the *identity* function $\mathrm{id}_A : A \to A$ is the function defined by $\mathrm{id}_A(a) = a$.

Theorem 1.23. *Composition of relations is associative, and the identity function is an identity for composition. The composite of functions is a function. The composite of injective functions is injective. The composite of surjective functions is surjective. The composite of bijections is a bijection. A function $f : A \to B$ is a bijection iff there exists a function $f^{-1} : B \to A$ such that $f^{-1}f = \mathrm{id}_A$ and $ff^{-1} = \mathrm{id}_B$, in which case f^{-1} is unique.*

1.5 Axioms Part Two

Axiom 1.24 (Power Set)**.** *For any set A, there exists a set $\mathcal{P}A$, the* power set *of A, and a relation $\in : A \nrightarrow \mathcal{P}A$, called* membership*, such that, for any subset S of A, there exists a unique $\overline{S} \in \mathcal{P}A$ such that, for all $x \in A$, we have $x \in \overline{S}$ if and only if $x \in S$.*

We usually write just S for $\overline{S}$.

Axiom Schema 1.25 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A, there exists a set B, a function $p : B \to A$, a set Y and a relation $M : B \looparrowright Y$ such that:

- $\forall b \in B. P[A, \{y \in Y : bMy\}, p(b)]$
- *For all $a \in A$, if $\exists Y. P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 1.26 (Universe). Let $E : U \looparrowright X$ be a relation. Let us say that a set A is *small* iff there exists $u \in U$ such that $A \approx \{x \in X : uEx\}$.

Then (U, X, E) form a *universe* if and only if:

- $\mathbb{N}$ is U-small.
- For any U-small sets A and B and relation $R : A \looparrowright B$, the tabulation of R is U-small.
- If A is U-small then so is $\mathcal{P}A$
- Let $f : A \to B$ be a function. If B is U-small and $f^{-1}(b)$ is U-small for all $b \in B$, then A is U-small.
- If $p : B \twoheadrightarrow A$ is a surjective function such that A is U-small, then there exists a U-small set C, a surjection $q : C \twoheadrightarrow A$, and a function $f : C \to B$ such that $q = pf$.

Axiom 1.27 (Universe). *There exists a universe.*

Let $E : U \looparrowright X$ be a universe. We shall say a set is *small* iff it is U-small, and *large* otherwise.

1.6 Cartesian Product

Definition 1.28 (Cartesian Product). Let A and B be sets. The *Cartesian product* of A and B, $A \times B$, is the tabulation of the relation $A \looparrowright B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

Chapter 2

Topology

2.1 Topological Spaces

Definition 2.1 (Topological Space)**.** Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say $(X, \mathcal{O})$ is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call $\mathcal{O}$ the *topology* of the toplogical space, and call its elements *open* sets. We shall often write X for the topological space $(X, \mathcal{O})$.

Definition 2.2 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 2.3. *A set B is open if and only if $X - B$ is closed.*

Proposition 2.4. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that $\mathcal{C}$ is the set of closed sets if and only if:*

- *For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$*
- *For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*
- $\emptyset \in \mathcal{C}$

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Definition 2.5 (Neighbourhood)**.** Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x, and x is an *interior* point of U, iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 2.6. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 2.7. *Let X be a set and $\mathcal{N} : X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that, for all $x \in X$, we have $\mathcal{N}_x$ is the set of neighbourhoods of x, if and only if:*

- *For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$*
- *For all $x \in X$ we have $X \in \mathcal{N}_x$*
- *For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$*
- *For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$*
- *For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M.M \in \mathcal{N}_y$.*

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{U : \forall x \in U.U \in \mathcal{N}_x\}$.

Definition 2.8 (Exterior Point)**.** Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x.

Definition 2.9 (Boundary Point)**.** Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B.

Definition 2.10 (Interior)**.** Let X be a topological space and $B \subseteq X$. The *interior* of B, B°, is the set of all interior points of B.

Proposition 2.11. *The interior of B is the union of all the open sets included in B.*

Definition 2.12 (Closure)**.** Let X be a topological space and $B \subseteq X$. The *closure* of B, $\overline{B}$, is the set of all points that are not exterior points of B.

Proposition 2.13. *The closure of B is the intersection of all the closed sets that include B.*

Proposition 2.14. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 2.15 (Kuratowski Closure Axioms)**.** *Let X be a set and $\overline{} : \mathcal{P}X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ such that, for all $B \subseteq X$, $\overline{B}$ is the closure of B, if and only if:*

- $\overline{\emptyset} = \emptyset$
- *For all $A \subseteq X$ we have $A \subseteq \overline{A}$*
- *For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$*
- *For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

In this case, $\mathcal{O}$ is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

2.1.1 Subspaces

Definition 2.16 (Subspace)**.** Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

2.1.2 Topological Disjoint Union

Definition 2.17. Let X and Y be topological spaces. The *disjoint union* is $X + Y$ where $U \subseteq X + Y$ is open if and only if ${\kappa_1}^{-1}(U)$ is open in X and ${\kappa_2}^{-1}(U)$ is open in Y.

2.1.3 Product Topology

Definition 2.18. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the set of all subsets $W \subseteq X \times Y$ such that, for all $(x, y) \in W$, there exist neighbourhoods U of x in X and V of y in Y such that $U \times V \subseteq W$.

2.1.4 Bases

Definition 2.19 (Basis)**.** Let X be a topological space. A *basis* for the topology on X is a set of open sets $\mathcal{B}$ such that every open set is the union of a subset of $\mathcal{B}$.

2.1.5 Subbases

Definition 2.20 (Subbasis)**.** Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of $\mathcal{S}$.

2.2 Continuous Functions

Definition 2.21 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* iff, for every open set V in Y, the inverse image $f^{-1}(V)$ is open in X.

Proposition 2.22.

1. id_X *is continuous*
2. *The composite of two continuous functions is continuous.*
3. *If $f : X \to Y$ is continuous and $X_0 \subseteq X$ then $f{\restriction}X_0 : X_0 \to Y$ is continuous.*
4. *If $f : X + Y \to Z$, then f is continuous iff $f \circ \kappa_1 : X \to Z$ and $f \circ \kappa_2 : Y \to Z$ are continuous.*
5. *If $f : Z \to X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.*

Definition 2.23 (Homeomorphism)**.** Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

2.3 Convergence

Definition 2.24 (Convergence)**.** Let X be a topological space. Let (x_n) be a sequence in X. A point $a : \mathrm{El}(X)$ is a *limit* of the sequence iff, for every neighbourhood U of a, there exists n_0 such that $\forall n \geqslant n_0 . x_n \in U$.

2.4 Connected Spaces

Definition 2.25 (Connected)**.** A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 2.26. *The continuous image of a connected space is connected.*

Proposition 2.27. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \varnothing$, and A and B are connected, then X is connected.*

Proposition 2.28. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 2.29 (Path-connected)**.** A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \to X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 2.30. *The continuous image of a path connected space is path connected.*

Proposition 2.31. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \varnothing$, and A and B are path connected, then X is path connected.*

Proposition 2.32. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

2.5 Hausdorff Spaces

Definition 2.33 (Hausdorff)**.** A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 2.34. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 2.35.

1. *Every subspace of a Hausdorff space is Hausdorff.*
2. *The disjoint union of two Hausdorff spaces is Hausdorff.*
3. *The product of two Hausdorff spaces is Hausdorff.*

2.6 Compactness

Definition 2.36 (Compact)**.** A topological space is *compact* iff every open cover has a finite subcover.

Proposition 2.37. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P. Then $X \in P$.*

PROOF:
$\langle 1 \rangle 1.$ P is an open cover of X
$\langle 1 \rangle 2.$ PICK a finite subcover $U_1, \ldots, U_n \in P$
$\langle 1 \rangle 3.$ $X = U_1 \cup \cdots \cup U_n \in P$
$\square$

Corollary 2.37.1. *Let f be a compact space and $f : X \to \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. $\square$

Proposition 2.38. *The continuous image of a compact space is compact.*

Proposition 2.39. *A closed subspace of a compact space is compact.*

Proposition 2.40. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. *X and Y are compact.*
2. *$X + Y$ is compact.*
3. *$X \times Y$ is compact.*

Proposition 2.41. *A compact subspace of a Hausdorff space is closed.*

Proposition 2.42. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

2.7 Metric Spaces

Definition 2.43 (Metric Space)**.** Let X be a set and $d : X^2 \to \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geqslant 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leqslant d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d). We often write X for the metric space (X, d).

Definition 2.44 (Topology of a Metric Space)**.** Let (X, d) be a metric space. The topology *induced* by the metric d is defined by: for $V \subseteq X$, we have V is open if and only if, for all $x \in V$, there exists $\epsilon > 0$ such that $\{y \in X : d(x, y) < \epsilon\} \subseteq V$.

Definition 2.45 (Metrizable)**.** A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 2.46. *Every metrizable space is Hausdorff.*

Chapter 3

Topological Vector Spaces

Definition 3.1 (Topological Vector Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Substraction is a continuous function $E^2 \to E$
- Multiplication is a continuous function $K \times E \to E$

Theorem 3.2. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 □

3.1 Cauchy Sequences

Definition 3.3 (Cauchy Sequence)**.** Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geqslant n_0 . x_n - x_m \in U$.

Definition 3.4 (Complete Topological Vector Space)**.** A topological vector space is *complete* iff every Cauchy sequence converges.

3.2 Seminorms

Definition 3.5 (Seminorm)**.** Let E be a vector space over K. A *seminorm* on E is a function $\| \ \| : E \to \mathbb{R}$ such that:

1. $\forall x : \mathrm{El}\,(E) \, . \|x\| \geqslant 0$
2. $\forall \alpha : \mathrm{El}\,(K) \, . \forall x : \mathrm{El}\,(E) \, . \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y : \mathrm{El}\,(E) \, . \|x + y\| \leqslant \|x\| + \|y\|$

Example 3.6. The function that maps $(x_1, \ldots, x_n)$ to $|x_i|$ is a seminorm on $\mathbb{R}^n$.

Definition 3.7. Let E be a vector space over K. Let Λ be a set of seminorms on E. The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x : \mathrm{El}\,(E)$.

Proposition 3.8. *E is a topological vector space under this topology. It is Hausdorff iff, for all $x : \mathrm{El}\,(E)$, if $\forall \lambda \in \Lambda. \lambda(x) = 0$ then $x = 0$.*

3.3 Fréchet Spaces

Definition 3.9 (Pre-Fréchet Space)**.** A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 3.10. *Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \ \|_n : n \in \mathbb{Z}^+\}$. Then*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 3.11 (Fréchet Space)**.** A *Fréchet space* is a complete pre-Fréchet space.

3.4 Normed Spaces

Definition 3.12 (Normed Space)**.** Let E be a vector space over K. A *norm* on E is a function $\| \ \| : E \to \mathbb{R}$ is a seminorm such that, $\forall x \in E. \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 3.13. *If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E.*

3.5 Inner Product Spaces

Proposition 3.14. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on E.*

3.6 Banach Spaces

Definition 3.15 (Banach Space)**.** A *Banach space* is a complete normed space.

Example 3.16. For any topological space X, the set $C(X)$ of bounded continuous functions $X \to \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

3.7 Hilbert Spaces

Definition 3.17 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Example 3.18. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \to \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi} \pi f(x) g(x) dx \ \ .$$

3.8 Locally Convex Spaces

Definition 3.19 (Locally Convex Space)**.** A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 3.20. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

Proof: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Proposition 3.21. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

Proof: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Example 3.22. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \to \mathbb{R}$ is continuous as a map $E' \to \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.