Mathematics

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Contents

1 Sets and Functions **5**
1.1 Primitive Terms 5
1.2 The Axioms 5

Chapter 1

Sets and Functions

1.1 Primitive Terms

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B, and call A the *domain* of f and B the *codomain*.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $g \circ f : A \to C$, the *composite* of f and g.

1.2 The Axioms

Axiom 1.1 (Associativity). *Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h(gf) = (hg)f \ .$$

Axiom 1.2 (Identity). *For any set A, there exists a function $i : A \to A$ such that:*

- *for any set B and function $f : A \to B$, we have $fi = f$*
- *for any set B and function $f : B \to A$, we have $if = f$.*

Proposition 1.3. *For any set A, there exists a unique function $i : A \to A$ such that:*

- *for any set B and function $f : A \to B$, we have $fi = f$*
- *for any set B and function $f : B \to A$, we have $if = f$.*

Proof: If i and j both satisfy these conditions then $i = ij = j$. $\square$

Definition 1.4 (Identity Function). For any set A, the *identity function* on A, id_A, is the unique function $A \to A$ such that:

- for any set B and function $f : A \to B$, we have $f\mathrm{id}_A = f$

- for any set B and function $f : B \to A$, we have $\mathrm{id}_B f = f$.

Definition 1.5 (Isomorphism)**.** A function $f : A \to B$ is an *isomorphism*, $f : A \cong B$, iff there exists a function $g : B \to A$ such that $fg = \mathrm{id}_B$ and $gf = \mathrm{id}_A$.

Axiom 1.6 (Terminal Set)**.** *There exists an* empty *set* $\varnothing$ *such that, for any set* A*, there exists exists exactly one function* $\varnothing \to A$*.*

Proposition 1.7. *If* S *and* T *are empty sets then there exists a unique isomorphism* $S \cong T$*.*

PROOF:
⟨1⟩1. LET: f be the unique function $S \to T$
⟨1⟩2. LET: f^{-1} be the unique function $T \to S$
⟨1⟩3. $ff^{-1} = \mathrm{id}_T$
PROOF: Each is the unique function $T \to T$.
⟨1⟩4. $f^{-1}f = \mathrm{id}_S$
PROOF: Each is the unique function $S \to S$.
□

Definition 1.8 (Empty Set)**.** Let $\varnothing$ be the set such that, for any set A, there exists exactly one function $\varnothing \to A$.

Axiom 1.9 (Terminal Set)**.** *There exists a* terminal *set* 1 *such that, for any set* A*, there exists exists exactly one function* $A \to 1$*.*

Proposition 1.10. *If* S *and* T *are terminal sets then there exists a unique isomorphism* $S \cong T$*.*

PROOF:
⟨1⟩1. LET: f be the unique function $S \to T$
⟨1⟩2. LET: f^{-1} be the unique function $T \to S$
⟨1⟩3. $ff^{-1} = \mathrm{id}_T$
PROOF: Each is the unique function $T \to T$.
⟨1⟩4. $f^{-1}f = \mathrm{id}_S$
PROOF: Each is the unique function $S \to S$.
□

Definition 1.11 (Terminal Set)**.** Let 1 be the set such that, for any set A, there exists exactly one function $A \to 1$.

Definition 1.12 (Element)**.** An *element* of a set A is a function $1 \to A$. We write $a \in A$ for $a : 1 \to A$.

Given $f : A \to B$ and $a \in A$, we write $f(a)$ for fa.

Axiom 1.13 (Extensionality)**.** *Let* A *and* B *be sets. Let* $f, g : A \to B$*. If* $\forall x \in A. f(x) = g(x)$ *then* $f = g$*.*

Axiom 1.14 (Non-degeneracy)**.** *The empty set* $\varnothing$ *has no elements.*

Definition 1.15 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Definition 1.16 (Injective)**.** A function $f : A \to B$ is *injective* iff, for all $x, x' \in A$, if $f(x) = f(x')$ then $x = x'$.

Definition 1.17 (Bijective)**.** A function is *bijective* iff it is injective and surjective.