Mathematics

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September 3, 2023

Contents

Chapter 1

Primitive Notions and Axioms

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ for 'f is a function from A to B'. We call A the *domain* of A, and B the *codomain*.

Given sets A, B and C, and functions $f : A \to B$ and $g : B \to C$, let there be a function $gf = g \circ f : A \to C$, the *composite* of f and g.

Axiom 1.1 (Associativity)**.** *For any functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \ .$$

Axiom 1.2 (Identity)**.** *For any set A, there exists a function $\mathrm{id}_A : A \to A$, called an* identity *function on A, such that:*

- *for every set B and function $f : A \to B$, we have $f \circ \mathrm{id}_A = f$;*
- *for every set B and function $f : B \to A$, we have $\mathrm{id}_A \circ f = f$.*

Axiom 1.3 (Terminal Set)**.** *There exists a set 1 such that, for any set A, there exists a unique function $A \to 1$.*

Definition 1.4 (Element)**.** For any set A, an *element* of A is a function $a : 1 \to A$. We write $a \in A$.

Axiom 1.5 (Empty Set)**.** *There exists a set 0 such that there is no function $1 \to 0$.*

Axiom 1.6 (Extensionality)**.** *Let A and B be sets. Let $f, g : A \to B$. If, for all $x : 1 \to A$, we have $f \circ x = g \circ x$, then $f = g$.*

Axiom 1.7 (Products)**.** *Let A and B be sets. There exists a set $A \times B$ and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ such that, for every set X and* *functions* $f : X \to A$*,* $g : X \to B$*, there exists a unique function* $\langle f, g\rangle : X \to A \times B$ *such that*

$$\pi_1 \circ \langle f, g\rangle = f, \qquad \pi_2 \circ \langle f, g\rangle = g \ .$$

Axiom 1.8 (Function Sets)**.** *Let* A *and* B *be sets. There exists a set* A^B *and function* $\epsilon : A^B \times B \to A$ *such that, for any set* X *and function* $f : X \times B \to A$*, there exists a unique function* $\lambda f : X \to A^B$ *such that*

$$f = \epsilon \circ \langle \lambda f \circ \pi_1, \pi_2\rangle \ .$$

Definition 1.9 (Inverse Image)**.** Let A, X and Y be sets. Let $f : X \to Y$, $a \in Y$ and $j : A \to X$. Then j is the *inverse image* of a under f if and only if:

- $f \circ j = a\circ !_A$
- for every set I and function $q : I \to X$ such that $f \circ q = a\circ !_I$, there exists a unique $\overline{q} : I \to A$ such that $q = j \circ \overline{q}$.

Axiom 1.10 (Inverse Images)**.** *For any sets* X *and* Y*, function* $f : X \to Y$ *and element* $a \in Y$*, there exists a set* $f^{-1}(a)$ *and function* $j : f^{-1}(a) \to X$ *such that* j *is the inverse image of* a *under* f*.*

Definition 1.11 (Injective)**.** A function $f : A \to B$ is *injective* iff, for every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

Definition 1.12 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for every set X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

Axiom 1.13 (Subset Classifier)**.** *There exists a set* 2 *and function* $\top : 1 \to 2$ *such that, for every injective function* $f : A \to X$*, there exists a unique function* $\chi : X \to 2$ *such that* f *is the inverse image of* $\top$ *under* χ*.*

Axiom 1.14 (Natural Numbers)**.** *There exists a set* $\mathbb{N}$*, an element* $0 \in \mathbb{N}$ *and a function* $s : \mathbb{N} \to \mathbb{N}$ *such that, for every set* X*, element* $a \in X$ *and function* $r : X \to X$*, there exists a unique function* $x : \mathbb{N} \to X$ *such that* $x \circ 0 = a$ *and* $x \circ s = r \circ x$*.*

Axiom 1.15 (Choice)**.** *For every surjective function* $r : X \to Y$*, there exists* $s : Y \to X$ *such that* $r \circ s$ *is an identity function on* X*.*