# Encyclopaedia of Mathematics and Physics

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# Contents

# Part I

# Set Theory

# Chapter 1

# Foundations

## 1.1 The Theory of Semicategories

Let there be *sets*.

Given sets $A$ and $B$, let there be *functions* from $A$ to $B$. We write $f : A \to B$ iff $f$ is a function from $A$ to $B$, and call $A$ the *domain* of $f$ and $B$ the *codomain*.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $g \circ f : A \to C$, the *composite* of $g$ and $f$.

**Axiom 1.1** (Associativity). *Given functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \quad .$$

### 1.1.1 Identity Functions

**Definition 1.2** (Identity Function). Let $A$ be a set. An *identity function* on $A$ is a function $i : A \to A$ such that:

- For any set $B$ and function $f : B \to A$, we have $i \circ f = f$.

- For any set $B$ and function $f : A \to B$, we have $f \circ i = f$.

**Proposition 1.3.** *A set has at most one identity function.*

PROOF:
⟨1⟩1. LET: $A$ be a set.
⟨1⟩2. LET: $i, j : A \to A$ be identity functions.
⟨1⟩3. $i = j$
PROOF: If $i$ and $j$ both satisfy the conditions then $i = i \circ j = j$.
□

### 1.1.2 Monomorphisms and Epimorphisms

**Definition 1.4** (Monomorphism)**.** We say a function $f : A \to B$ is a *monomorphism*, and write $f : A \rightarrowtail B$, iff, for any set $X$ and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

**Definition 1.5** (Epimorphism)**.** We say a function $f : A \to B$ is a *epimorphism*, and write $f : A \twoheadrightarrow B$, iff, for any set $X$ and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

## 1.2 The Theory of Categories

### 1.2.1 Minimalist Presentation

**Axiom 1.6** (Identity Functions)**.** *Every set has an identity function.*

### 1.2.2 Practical Presentation

For any set $A$, let there be a function $\mathrm{id}_A : A \to A$.

**Axiom 1.7** (Left Unit Law)**.** *For any function $f : A \to B$, we have $\mathrm{id}_B \circ f = f$.*

**Axiom 1.8** (Right Unit Law)**.** *For any function $f : A \to B$, we have $f \circ \mathrm{id}_A = f$.*

### 1.2.3 Sections and Retractions

**Definition 1.9** (Section, Retraction)**.** Let $r : A \to B$ and $s : B \to A$. We say that $r$ is a *retraction* of $s$, and $s$ is a *section* of $r$.

### 1.2.4 Bijections

**Definition 1.10** (Bijection)**.** We say a function $f : A \to B$ is *bijective* or a *bijection*, and write $f : A \approx B$, iff there exists a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

### 1.2.5 Terminal Set

**Definition 1.11** (Terminal Set)**.** A set $T$ is *terminal* iff, for any set $X$, there is exactly one function $X \to T$.

**Axiom 1.12** (Terminal Set)**.** *There exists a terminal set.*

**Proposition 1.13.** *For any terminal sets $T$ and $T'$, there is a unique bijection $T \approx T'$.*

PROOF:
$\langle 1 \rangle 1$. LET: $i$ be the unique function $T \to T'$.
$\langle 1 \rangle 2$. LET: $j$ be the unique function $T' \to T$.
$\langle 1 \rangle 3$. $i \circ j = \mathrm{id}_{T'}$

PROOF: Since there is only one function $T' \to T'$.

$\langle 1 \rangle 4.$ $j \circ i = \mathrm{id}_T$

PROOF: Since there is only one function $T \to T$.

□

**Definition 1.14** (Terminal Set)**.** We denote the terminal set by 1.

**Definition 1.15** (Element)**.** An *element* of a set $A$ is a function $1 \to A$. We write $a \in A$ for $a : 1 \to A$. Given $a \in A$ and $f : A \to B$, we write $f(a)$ for $f \circ a$.

**Axiom 1.16** (Extensionality)**.** *Let $f, g : A \to B$. Assume that, for all $a \in A$, if $f(a) = g(a)$ then $f = g$.*

**Definition 1.17** (Injective)**.** We say a function $f : A \to B$ is *injective* or an *injection*, and we write $f : A \rightarrowtail B$, iff, for any $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

**Definition 1.18** (Surjective)**.** We say a function $f : A \to B$ is *surjective* or a *surjection*, and we write $f : A \twoheadrightarrow B$, iff, for any $y \in B$, there exists $x \in A$ such that $f(x) = y$.

### 1.2.6 Empty Set

**Axiom 1.19** (Empty Set)**.** *There exists a set with no elements.*

### 1.2.7 Products

**Definition 1.20** (Product)**.** Let $A$, $B$ and $P$ be sets, and $\pi_1 : P \to A$, $\pi_2 : P \to B$. Then we say that $(P, \pi_1, \pi_2)$ is a *product* of $A$ and $B$ iff, for any set $X$ and functions $f : X \to A$ and $g : X \to B$, there exists a unique function $h : X \to A \times B$ such that

$$\pi_1 \circ h = f, \qquad \pi_2 \circ h = g \enspace .$$

**Axiom 1.21** (Products)**.** *Any two sets have a product.*

**Proposition 1.22.** *If $(P, p_1, p_2)$ and $(Q, q_1, q_2)$ are products of $A$ and $B$, then there exists a unique bijection $\phi : P \approx Q$ such that $q_1 \circ \phi = p_1$ and $q_2 \circ \phi = p_2$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $\phi : P \to Q$ be the unique function such that $q_1 \circ \phi = p_1$ and $q_2 \circ \phi = p_2$.

$\langle 1 \rangle 2.$ LET: $\phi^{-1} : Q \to P$ be the unique function such that $p_1 \circ \phi = q_1$ and $p_2 \circ \phi = q_2$.

$\langle 1 \rangle 3.$ $\phi \circ \phi^{-1} = \mathrm{id}_Q$

PROOF: Each is the unique $x : Q \to Q$ such that $q_1 \circ x = q_1$ and $q_2 \circ x = q_2$.

$\langle 1 \rangle 4.$ $\phi^{-1} \circ \phi = \mathrm{id}_P$

PROOF: Each is the unique $x : P \to P$ such that $p_1 \circ x = p_1$ and $p_2 \circ x = p_2$.

□

**Definition 1.23.** Given sets $A$ and $B$, we write $A \times B$ for the product of $A$ and $B$, with projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$. Given functions $f : A \to B$ and $g : A \to C$, we write $\langle f, g \rangle$ for the unique function $A \to B \times C$ such that

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g \ .$$

**Definition 1.24.** Given $f : A \to B$ and $g : C \to D$, we define $f \times g : A \times C \to B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle \ .$$

### 1.2.8 Function Sets

**Definition 1.25** (Function Set)**.** Let $A$, $B$ and $F$ be sets, and let $\epsilon : F \times A \to B$. Then we say that $F$ and $\epsilon$ form the *function set* from $A$ to $B$, with $\epsilon$ the *evaluation function*, iff, for any set $X$ and function $f : X \times A \to B$, there exists a unique function $g : X \to F$ such that

$$\epsilon \circ (g \times \mathrm{id}_A) = f \ .$$

**Axiom 1.26** (Function Sets)**.** *Any two sets have a function set.*

**Proposition 1.27.** *Let $(F, \epsilon : F \times A \to B)$ and $(G, e : G \times A \to B)$ be function sets from $A$ to $B$. Then there exists a unique bijection $\phi : F \approx G$ such that $e \circ (\phi \times \mathrm{id}_A) = \epsilon$.*

**Definition 1.28.** Given sets $A$ and $B$, we write $B^A$ for the function set from $A$ to $B$, and $\epsilon : B^A \times A \to B$ for the evaluation function. Given $f : X \times A \to B$, we write $\lambda f$ for the unique function $X \to B^A$ such that

$$\epsilon \circ (\lambda f \times \mathrm{id}_A) = f \ .$$

### 1.2.9 Inverse Images

**Definition 1.29** (Inverse Image)**.** Let $A$, $B$ and $I$ be sets. Let $f : A \to B$, $b \in B$ and $i : I \to A$. Then we say that $I$ and $i$ form the *inverse image* of $b$ under $f$ iff:

- $f \circ i = b \circ !_I$
- For any set $X$ and function $j : X \to A$, if $f \circ j = b \circ !_X$, then there exists a unique $\overline{j} : X \to I$ such that $i \circ \overline{j} = j$.

**Axiom 1.30** (Inverse Images)**.** *Given any sets $A$ and $B$, function $f : A \to B$, and element $b \in B$, there exists an inverse image of $b$ under $f$.*

**Proposition 1.31.** *If $(I, i : I \to A)$ and $(J, j : J \to A)$ are inverse images of $b \in B$ under $f : A \to B$, then there exists a unique isomorphism $\phi : I \approx J$ such that $j \circ \phi = i$.*

**Definition 1.32.** Let $f : A \to B$ and $b \in B$. We write $f^{-1}(b)$ and $i_{f,b} : f^{-1}(b) \to A$ for the inverse image of $b$ under $f$.

### 1.2.10 Subset Classifiers

**Definition 1.33** (Subset Classifier)**.** Let $\Omega$ be a set and $\top \in \Omega$. Then we say $(\Omega, \top)$ form a *subset classifier* iff, for any sets $A$ and $X$ and injection $j : A \rightarrowtail X$, there exists a unique $\chi : X \to \Omega$ such that $(A, j)$ is the inverse image of $\top$ under $\chi$.

**Axiom 1.34** (Subset Classifier)**.** *There exists a subset classifier.*

**Proposition 1.35.** *If $(\Omega, \top)$ and $(\Omega', \top')$ are subset classifiers, then there exists a unique bijection $\phi : \Omega \approx \Omega'$ such that $\phi(\top) = \top'$.*

**Definition 1.36.** We write 2 and $\top \in 2$ for the subset classifier.

### 1.2.11 Natural Number Sets

**Definition 1.37** (Natural Number Set)**.** Let $N$ be a set, $z \in N$ and $s : N \to N$. Then we say $(N, z, s)$ is a *natural number set* iff, for any set $X$, element $a \in X$ and function $f : X \to X$, there exists a unique $r : N \to X$ such that

$$r(z) = a, \qquad f \circ r = r \circ s \ .$$

**Axiom 1.38** (Infinity)**.** *There exists a natural number set.*

**Proposition 1.39.** *If $(N, z, s)$ and $(N', z', s')$ are natural number sets, then there exists a unique bijection $\phi : N \approx N'$ such that $\phi(z) = z'$ and $s' \circ \phi = \phi \circ s$.*

**Definition 1.40.** We write $\mathbb{N}$, $0 \in \mathbb{N}$ and $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ for the natural number set.

### 1.2.12 The Axiom of Choice

**Definition 1.41** (Axiom of Choice)**.** Every surjection is a retraction.

# Chapter 2

# Set Theory

**Proposition 2.1.** *Every infinite subset of a countably infinite set is countable.*

Proof:
$\langle 1\rangle 1$. Let: $i : A \hookrightarrow \mathbb{N}$ be an infinite subset of $\mathbb{N}$.
$\langle 1\rangle 2$. Define $j : \mathbb{N} \to A$ by: $j(k)$ is the element such that $i(j(k))$ is least such that $i(j(k)) \notin \{i(j(0)), \ldots, i(j(k-1))\}$.
$\langle 1\rangle 3$. $j$ is a bijection.
□

**Proposition 2.2.** *A countable union of countable sets is countable.*

Proof:
$\langle 1\rangle 1$. Let: $(A_n)$ be a sequence of countable sets.
$\langle 1\rangle 2$. For $n \in \mathbb{N}$, Pick an enumeration $(e_{nm})_m$ of $A_n$.
$\langle 1\rangle 3$. Let: $(p_k)$ be the following enumeration of $\mathbb{N} \times \mathbb{N}$:

$$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots$$

$\langle 1\rangle 4$. $(e_{\pi_1(p_k)\pi_2(p_k)})_k$ is an enumeration of $\bigcup_n A_n$.
□

**Theorem 2.3.** $2^{\mathbb{N}}$ *is uncountable.*

Proof:
$\langle 1\rangle 1$. Assume: for a contradiction $f : \mathbb{N} \approx 2^{\mathbb{N}}$
$\langle 1\rangle 2$. Let: $S = \{n \in \mathbb{N} : n \notin f(n)\}$
$\langle 1\rangle 3$. For all $n$, we have $n \in S \Leftrightarrow n \notin f(n)$
$\langle 1\rangle 4$. For all $n$ we have $S \neq f(n)$.
$\langle 1\rangle 5$. Q.E.D.
Proof: This contradicts $\langle 1\rangle 1$.
□

# Chapter 3

# Relations

**Definition 3.1** (Antisymmetric)**.** A relation $R$ on a set $A$ is *antisymmetric* iff, whenever $xRy$ and $yRx$, then $x = y$.

**Definition 3.2** (Transitive)**.** A relation $R$ on a type $A$ is *transitive* iff, whenever $xRy$ and $yRz$, then $xRz$.

# Chapter 4

# Order Theory

**Definition 4.1** (Linear Order)**.** A *linear order* on a set $A$ is a binary relation $\leq$ on $A$ that is transitive, antisymmetric and:

$$\forall x, y \in A. x \leq y \vee y \leq x \quad .$$

A *linearly ordered set* is a pair $(A, \leq)$ where $A$ is a set and $\leq$ is a binary relation on $A$.

We write $x < y$ for $x \leq y$ and $x \neq y$.

**Definition 4.2** (Upper Bound)**.** Let $S$ be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then $u$ is an *upper bound* in $E$ iff $\forall x \in E. x \leq u$. We say $E$ is *bounded above* iff it has an upper bound.

The *up-set* of $E$, denoted $E \uparrow$, is the set of upper bounds of $E$.

**Definition 4.3** (Lower Bound)**.** Let $S$ be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then $u$ is an *lower bound* in $E$ iff $\forall x \in E. l \leq x$. We say $E$ is *bounded below* iff it has a lower bound.

The *down-set* of $E$, denoted $E \downarrow$, is the set of lower bounds of $E$.

**Definition 4.4** (Supremum)**.** Let $S$ be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then $u$ is the *least upper bound* or *supremum* of $E$ iff $u$ is an upper bound for $E$ and, for any upper bound $u'$ for $E$, we have $u \leq u'$.

**Definition 4.5** (Infimum)**.** Let $S$ be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then $l$ is the *greatest lower bound* or *infimum* of $E$ iff $l$ is a lower bound for $E$ and, for any lower bound $l'$ for $E$, we have $l' \leq l$.

**Definition 4.6** (Least Upper Bound Property)**.** A linearly ordered set $S$ has the *least upper bound property* iff every nonempty subset of $S$ that is bounded above has a least upper bound.

**Proposition 4.7.** *Let $S$ be a linearly ordered set and $E \subseteq S$.*

1. *If $E \downarrow$ has a supremum $l$, then $l$ is the infimum of $E$.*

*2. If $E\uparrow$ has an infimum $u$, then $U$ is the supremum of $E$.*

PROOF:
⟨1⟩1. If $E\downarrow$ has a supremum $l$, then $l$ is the infimum of $E$.
⟨2⟩1. $l$ is a lower bound for $E$.
⟨3⟩1. LET: $x \in E$
⟨3⟩2. $x$ is an upper bound for $E\downarrow$.
PROOF: For all $y \in E\downarrow$ we have $y \leq x$.
⟨3⟩3. $l \leq x$
⟨2⟩2. For any lower bound $l'$ for $E$, we have $l' \leq l$.
PROOF: Since $l$ is an upper bound for $E\downarrow$.
⟨1⟩2. If $E\uparrow$ has an infimum $u$, then $u$ is the supremum of $E$.
PROOF: Dual.
□

**Corollary 4.7.1.** *A linearly ordered set has the least upper bound property if and only if every nonempty set bounded below has an infimum.*

**Definition 4.8** (Closed Downwards)**.** Let $S$ be a linearly ordered set and $E \subseteq S$. Then $E$ is *closed downwards* iff, whenever $x \in E$ and $y < x$, then $y \in E$.

**Definition 4.9** (Closed Upwards)**.** Let $S$ be a linearly ordered set and $E \subseteq S$. Then $E$ is *closed upwards* iff, whenever $x \in E$ and $x < y$, then $y \in E$.

**Definition 4.10** (Greatest)**.** Let $S$ be a linearly ordered set and $u \in S$. Then $u$ is *greatest* in $S$ iff $\forall x \in S.x \leq u$.

**Definition 4.11** (Least)**.** Let $S$ be a linearly ordered set and $l \in S$. Then $l$ is *least* in $S$ iff $\forall x \in S.l \leq x$.

**Proposition 4.12.** *Let $\leq$ be a linear order on a set $S$ and $E \subseteq S$. Then $\leq \cap E^2$ is a linear order on $E$.*

PROOF: Easy. □

Given a linearly ordered set $(S, \leq)$ and $E \subseteq S$, we write just $E$ for the linearly ordered set $(E, \leq \cap E^2)$.

**Definition 4.13** (Lexicographic Order)**.** Let $A$ and $B$ be linearly ordered sets. The *lexicographic order* or *dictionary order* on $A \times B$ is the order defined by

$$(a, b) \leq (a', b') \Leftrightarrow a = a' \vee (a < a' \wedge b \leq b') \ .$$

**Proposition 4.14.** *The lexicographic order is a linear order.*

# Chapter 5

# Field Theory

**Definition 5.1** (Field)**.** A *field* $F$ consists of a set $F$, two operations $+, \cdot : F^2 \to F$ and an element $0 \in F$ such that:

- $+$ is commutative.
- $+$ is associative.
- $\forall x \in F. x + 0 = x$
- $\forall x \in F. \exists y \in F. x + y = 0$
- $\cdot$ is commutative.
- $\cdot$ is associative.
- There exists $1 \in F$ such that $1 \neq 0$ and $\forall x \in F. x1 = x$ and $\forall x \in F. x \neq 0 \Rightarrow \exists y \in F. xy = 1$
- *Distributive Law* $\forall x, y, z \in F. x(y + z) = xy + xz$

**Proposition 5.2.** *In any field $F$, the element 0 is the unique element such that $\forall x \in F. x + 0 = x$.*

Proof: If 0 and $0'$ both have this property then $0 = 0 + 0' = 0'$. □

**Proposition 5.3.** *In any field $F$, given $x \in F$, there is a unique $y \in F$ such that $x + y = 0$.*

Proof: If $x + y = x + y' = 0$ then

$$\begin{aligned} y &= y + 0 \\ &= y + x + y' \\ &= 0 + y' \\ &= y' \end{aligned}$$

□

**Definition 5.4.** Let $F$ be a field. Let $x \in F$. We denote by $-x$ the unique element of $F$ such that $x + (-x) = 0$.

Given $x, y \in F$, we write $x - y$ for $x + (-y)$.

**Proposition 5.5.** *In any field $F$, if $x + y = x + z$ then $y = z$.*

PROOF: If $x + y = x + z$ we have

$$\begin{aligned} -x + x + y &= -x + x + z \\ \therefore 0 + y &= 0 + z \\ \therefore y &= z \end{aligned}$$

∎

**Proposition 5.6.** *In any field $F$, we have $-(-x) = x$.*

PROOF: Since $x + (-x) = 0$. ∎

**Proposition 5.7.** *In any field $F$, the element 1 such that $\forall x \in F . x1 = x$ is unique.*

PROOF: If 1 and $1'$ both have this property then $1 = 1 \cdot 1' = 1'$. ∎

**Proposition 5.8.** *In any field $F$, given $x \in F$ with $x \neq 0$, the element $y$ such that $xy = 1$ is unique.*

PROOF: If $y$ and $y'$ both have this property then we have

$$\begin{aligned} y &= y1 \\ &= yxy' \\ &= 1y' \\ &= y' \end{aligned}$$

∎

**Definition 5.9.** In any field $F$, if $x \neq 0$, we write $x^{-1}$ for the unique element such that $xx^{-1} = 1$.

We write $x/y$ for $xy^{-1}$.

**Proposition 5.10.** *In any field $F$, if $xy = xz$ and $x \neq 0$ then $y = z$.*

PROOF:

$$\begin{aligned} y &= 1y \\ &= x^{-1}xy \\ &= x^{-1}xz \\ &= 1z \\ &= z \end{aligned}$$

∎

**Proposition 5.11.** *In any field $F$, if $x \neq 0$ then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.*

PROOF: Since $xx^{-1} = 1$. ∎

**Proposition 5.12.** *In any field $F$, we have $x0 = 0$.*

PROOF:
$$\begin{aligned} x0 + 0 &= x0 \\ &= x(0+0) \\ &= x0 + x0 \\ \therefore 0 &= x0 \end{aligned} \qquad \square$$

**Proposition 5.13.** *In any field $F$, if $xy = 0$ then $x = 0$ or $y = 0$.*

PROOF: If $xy = 0$ and $x \neq 0$ then we have $y = x^{-1}xy = x^{-1}0 = 0$. $\square$

**Proposition 5.14.** *In any field $F$, we have $(-x)y = -(xy)$.*

PROOF:
$$\begin{aligned} xy + (-x)y &= (x + (-x))y \\ &= 0y \\ &= 0 && \text{(Proposition 5.12)}\square \end{aligned}$$

**Corollary 5.14.1.** *In any field $F$, we have $(-x)(-y) = xy$.*

PROOF:
$$\begin{aligned} (-x)(-y) &= -(x(-y)) \\ &= -(-(xy)) \\ &= xy && \text{(Proposition 5.6)}\square \end{aligned}$$

**Proposition 5.15.** *Let $K$ be a field. Let $a, b \in K$. If $a^2 = b^2$ then $a = b$ or $a = -b$.*

PROOF:
$$\begin{aligned} a^2 - b^2 &= 0 \\ \therefore (a-b)(a+b) &= 0 \end{aligned}$$
Hence either $a - b = 0$ or $a + b = 0$, and the conclusion follows. $\square$

## 5.1 Ordered Fields

**Definition 5.16** (Ordered Field)**.** An *ordered field $F$* consists of a field $F$ and a linear order $\leq$ on $F$ such that:

- For all $x, y, z \in F$, if $y < z$ then $x + y < x + z$
- For all $x, y \in F$, if $x > 0$ and $y > 0$ then $xy > 0$.

We call $x$ *positive* iff $x > 0$ and *negative* iff $x < 0$.

**Example 5.17.** $\mathbb{Q}$ is an ordered field.

**Proposition 5.18.** *In any ordered field, if $x$ is positive then $-x$ is negative.*

PROOF: If $x > 0$ then $0 = x + (-x) > 0 = (-x) = -x$. $\square$

**Proposition 5.19.** *In any ordered field, if $y < z$ and $x$ is positive then $xy < xz$.*

Proof: If $y < z$ then we have

$$\begin{aligned} 0 &< z - y \\ \therefore 0 &< x(z-y) \\ &= xz - xy \\ \therefore xy &< xz \end{aligned}$$

∎

**Proposition 5.20.** *In any ordered field, if $y < z$ and $x$ is negative then $xy > xz$.*

Proof:
⟨1⟩1. $-x$ is positive.
⟨1⟩2. $(-x)y < (-x)z$
⟨1⟩3. $-(xy) < -(xz)$
⟨1⟩4. $xz < xy$
∎

**Proposition 5.21.** *In any ordered field, if $x \neq 0$ then $x^2 > 0$.*

Proof:
⟨1⟩1. If $x > 0$ then $x^2 > 0$.
Proof: Proposition 5.19.
⟨1⟩2. If $x < 0$ then $x^2 > 0$.
Proof: Proposition 5.20.
∎

**Corollary 5.21.1.** *In any ordered field, we have $1 > 0$.*

**Proposition 5.22.** *In any ordered field, if $x$ is positive then $x^{-1}$ is positive.*

Proof: If $x^{-1} < 0$ then we would have $1 = xx^{-1} < x0 = 0$ contradicting Corollary 5.21.1. ∎

**Proposition 5.23.** *In any ordered field, if $0 < x < y$ then $y^{-1} < x^{-1}$.*

Proof:
⟨1⟩1. Assume: $0 < x < y$
⟨1⟩2. $x^{-1}$ and $y^{-1}$ are positive.
Proof: Proposition 5.22.
⟨1⟩3. $xy^{-1} < yy^{-1} = 1$
⟨1⟩4. $y^{-1} = x^{-1}xy^{-1} < x^{-1}1 = x^{-1}$
∎

**Lemma 5.24.** *Let $K$ be an ordered field. Let $b \in K$ with $b > 1$. Let $n$ be a positive integer. Then*

$$b^n - 1 \geq n(b-1)$$

Proof:

$$\begin{aligned} b^n - 1 &= (b-1)(b^{n-1} + b^{n-2} + \cdots + 1) \\ &\geq (b-1)(1 + 1 + \cdots + 1) \\ &= n(b-1) \end{aligned}$$

∎

# Chapter 6

# Real Analysis

## 6.1 Construction of the Real Numbers

**Definition 6.1** (Cut)**.** A *cut* is a subset $\alpha$ of $\mathbb{Q}$ such that:

- $\emptyset \neq \alpha \neq \mathbb{Q}$
- $\alpha$ is closed downwards.
- $\alpha$ has no greatest element.

In this section, we write $R$ for the set of all cuts.

**Proposition 6.2.** *$R$ is linearly ordered by $\subseteq$.*

PROOF: The only difficult part is to prove that, for any cuts $\alpha$ and $\beta$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.
$\langle 1 \rangle 1$. ASSUME: $\alpha \nsubseteq \beta$
PROVE: $\beta \subseteq \alpha$
$\langle 1 \rangle 2$. PICK $q \in \alpha$ such that $q \notin \beta$
$\langle 1 \rangle 3$. LET: $r \in \beta$
$\langle 1 \rangle 4$. $q \nless r$
$\langle 1 \rangle 5$. $r < q$
$\langle 1 \rangle 6$. $r \in \alpha$
□

**Proposition 6.3.** *$R$ has the least upper bound property.*

PROOF:
$\langle 1 \rangle 1$. LET: $E \subseteq R$ be nonempty and bounded above.
$\langle 1 \rangle 2$. LET: $s = \bigcup E$
PROVE: $s$ is a cut.
$\langle 1 \rangle 3$. $\emptyset \neq s$
PROOF: Since $E$ is nonempty and every element of $E$ is nonempty.
$\langle 1 \rangle 4$. $s \neq \mathbb{Q}$

$\langle 2 \rangle 1.$ PICK an upper bound $u$ for $E$.
$\langle 2 \rangle 2.$ PICK $q \notin u$
PROVE: $q \notin s$
$\langle 2 \rangle 3.$ $\forall \alpha \in E. \alpha \subseteq u$
$\langle 2 \rangle 4.$ $s \subseteq u$
$\langle 2 \rangle 5.$ $q \notin s$
$\langle 1 \rangle 5.$ $s$ is closed downwards.
$\langle 2 \rangle 1.$ LET: $q \in s$ and $r < q$.
$\langle 2 \rangle 2.$ PICK $\alpha \in E$ such that $q \in \alpha$.
$\langle 2 \rangle 3.$ $r \in \alpha$
$\langle 2 \rangle 4.$ $r \in s$
$\langle 1 \rangle 6.$ $s$ has no greatest element.
$\langle 2 \rangle 1.$ LET: $q \in s$
$\langle 2 \rangle 2.$ PICK $\alpha \in E$ such that $q \in \alpha$.
$\langle 2 \rangle 3.$ PICK $r \in \alpha$ such that $q < r$.
$\langle 2 \rangle 4.$ $r \in s$
□

**Definition 6.4** (Addition)**.** Given cuts $\alpha$ and $\beta$, we define

$$\alpha + \beta = \{q + r : q \in \alpha, r \in \beta\} \ .$$

**Proposition 6.5.** *Given cuts $\alpha$ and $\beta$, we have $\alpha + \beta$ is a cut.*

PROOF:
$\langle 1 \rangle 1.$ $\alpha + \beta$ is nonempty.
PROOF: Since $\alpha$ and $\beta$ are nonempty.
$\langle 1 \rangle 2.$ $\alpha + \beta \neq \mathbb{Q}$
$\langle 2 \rangle 1.$ PICK $q \in \mathbb{Q} - \alpha$ and $r \in \mathbb{Q} - \beta$.
PROVE: $q + r \notin \alpha + \beta$
$\langle 2 \rangle 2.$ ASSUME: for a contradiction $q + r \in \alpha + \beta$.
$\langle 2 \rangle 3.$ PICK $x \in \alpha$ and $y \in \beta$ such that $q + r = x + y$
$\langle 2 \rangle 4.$ $x < q$
$\langle 2 \rangle 5.$ $y < r$
$\langle 2 \rangle 6.$ $x + y < q + r$
$\langle 2 \rangle 7.$ Q.E.D.
PROOF: This is a contradiction.
$\langle 1 \rangle 3.$ $\alpha + \beta$ is closed downwards.
$\langle 2 \rangle 1.$ LET: $q \in \alpha$, $r \in \beta$ and $x < q + r$
$\langle 2 \rangle 2.$ $x - q < r$
$\langle 2 \rangle 3.$ $x - q \in \beta$
$\langle 2 \rangle 4.$ $x \in \alpha + \beta$
$\langle 1 \rangle 4.$ $\alpha + \beta$ has no greatest element.
$\langle 2 \rangle 1.$ LET: $q \in \alpha$ and $r \in \beta$.
PROVE: $q + r$ is not greatest in $\alpha + \beta$.
$\langle 2 \rangle 2.$ PICK $q' \in \alpha$ with $q < q'$ and $r' \in \beta$ with $r < r'$.
$\langle 2 \rangle 3.$ $q + r < q' + r' \in \alpha + \beta$

□

**Proposition 6.6.** *Addition is commutative and associative on $R$.*

PROOF: Immediate from definitions and the fact that addition is commutative and associative on $\mathbb{Q}$. □

**Definition 6.7.** For any $q \in \mathbb{Q}$, let $q^* = \{r \in \mathbb{Q} : r < q\}$.

**Proposition 6.8.** *For any $q \in \mathbb{Q}$, we have $q^*$ is a cut.*

PROOF:
$\langle 1 \rangle 1.$ $q^* \neq \emptyset$
  PROOF: Since $q - 1 \in q^*$.
$\langle 1 \rangle 2.$ $q^* \neq \mathbb{Q}$
  PROOF: Since $q \notin q^*$.
$\langle 1 \rangle 3.$ $q^*$ is closed downwards.
  PROOF: Immediate from definition.
$\langle 1 \rangle 4.$ $q^*$ has no greatest element.
  PROOF: For all $r \in q^*$ we have $r < (q + r)/2 \in q^*$.
□

**Proposition 6.9.** *For any cut $\alpha$ we have $\alpha + 0^* = \alpha$.*

PROOF:
$\langle 1 \rangle 1.$ $\alpha + 0^* \subseteq \alpha$
  $\langle 2 \rangle 1.$ LET: $q \in \alpha$ and $r \in 0^*$
    PROVE: $q + r \in \alpha$
  $\langle 2 \rangle 2.$ $r < 0$
  $\langle 2 \rangle 3.$ $q + r < q$
  $\langle 2 \rangle 4.$ $q + r \in \alpha$
$\langle 1 \rangle 2.$ $\alpha \subseteq \alpha + 0^*$
  $\langle 2 \rangle 1.$ LET: $q \in \alpha$
  $\langle 2 \rangle 2.$ PICK $r \in \alpha$ such that $q < r$
  $\langle 2 \rangle 3.$ $q = r + (q - r) \in \alpha + 0^*$
□

**Proposition 6.10.** *For any cut $\alpha$, there exists a cut $\beta$ such that $\alpha + \beta = 0$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\beta = \{p \in \mathbb{Q} : \exists r > 0. -p - r \notin \alpha\}$
$\langle 1 \rangle 2.$ $\beta$ is a cut.
  $\langle 2 \rangle 1.$ $\beta \neq \emptyset$
    $\langle 3 \rangle 1.$ PICK $q \notin \alpha$
    $\langle 3 \rangle 2.$ $-q - 1 \in \beta$
  $\langle 2 \rangle 2.$ $\beta \neq \mathbb{Q}$
    $\langle 3 \rangle 1.$ PICK $q \in \alpha$
      PROVE: $-q \notin \beta$
    $\langle 3 \rangle 2.$ ASSUME: for a contradiction $-q \in \beta$

⟨3⟩3. PICK $r > 0$ such that $q - r \notin \alpha$
⟨3⟩4. $q - r < q$
⟨3⟩5. Q.E.D.
PROOF: This contradicts the fact that $\alpha$ is closed downwards.
⟨2⟩3. $\beta$ is closed downwards.
⟨3⟩1. LET: $p \in \beta$ and $q < p$.
⟨3⟩2. PICK $r > 0$ such that $-p - r \notin \alpha$
⟨3⟩3. $-p - r < -q - r$
⟨3⟩4. $-q - r \notin \alpha$
⟨3⟩5. $q \in \beta$
⟨2⟩4. $\beta$ has no greatest element.
⟨3⟩1. LET: $p \in \beta$
⟨3⟩2. PICK $r > 0$ such that $-p - r \notin \alpha$
⟨3⟩3. $-(p + r/2) - r/2 \notin \alpha$
⟨3⟩4. $p + r/2 \in \beta$
⟨1⟩3. $\alpha + \beta \subseteq 0^*$
⟨2⟩1. LET: $p \in \alpha$ and $q \in \beta$.
⟨2⟩2. PICK $r > 0$ such that $-q - r \notin \alpha$.
⟨2⟩3. $p < -q - r$
⟨2⟩4. $p + q < -r$
⟨2⟩5. $p + q < 0$
⟨2⟩6. $p + q \in 0^*$
⟨1⟩4. $0^* \subseteq \alpha + \beta$
⟨2⟩1. LET: $v \in 0^*$
⟨2⟩2. LET: $w = -v/2$
⟨2⟩3. $w > 0$
⟨2⟩4. PICK an integer $n$ such that $nw \in \alpha$ and $(n + 1)w \notin \alpha$.
⟨2⟩5. LET: $p = -(n + 2)w$
⟨2⟩6. $p \in \beta$
⟨2⟩7. $v = nw + p$
⟨2⟩8. $v \in \alpha + \beta$
□

**Proposition 6.11.** *Given $\alpha, \beta, \gamma \in R$, if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

PROOF:
⟨1⟩1. $\alpha + \beta \subseteq \alpha + \gamma$
PROOF: Immediate from definitions.
⟨1⟩2. $\alpha + \beta \neq \alpha + \gamma$
PROOF: If $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$ by cancellation.
□

**Definition 6.12.** Given cuts $\alpha$ and $\beta$, define $\alpha\beta$ by:

$$\alpha\beta = \begin{cases} \{p \in \mathbb{Q} : \exists r \in \alpha . \exists s \in \beta (p \leq rs \wedge r > 0 \wedge s > 0\} & \text{if } \alpha > 0^* \text{ and } \beta > 0^* \\ (-\alpha)(-\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^* \\ -((-\alpha)\beta) & \text{if } \alpha < 0^* \text{ and } \beta > 0^* \\ -(\alpha(-\beta)) & \text{if } \alpha > 0^* \text{ and } \beta < 0^* \\ 0^* & \text{if } \alpha = 0^* \text{ or } \beta = 0^* \end{cases}$$

**Proposition 6.13.** *For any cuts $\alpha$ and $\beta$, we have $\alpha\beta$ is a cut.*

PROOF:
⟨1⟩1. If $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta$ is a cut.
  ⟨2⟩1. $\alpha\beta \neq \emptyset$
    ⟨3⟩1. PICK $q \in \alpha$ and $r \in \beta$ such that $q, r \notin 0^*$
    ⟨3⟩2. ASSUME: w.l.o.g. $0 < q$ and $0 < r$.
      PROOF: Since $\alpha$ and $\beta$ have no greatest element.
    ⟨3⟩3. $qr \in \alpha\beta$
  ⟨2⟩2. $\alpha\beta \neq \mathbb{Q}$
    ⟨3⟩1. PICK $r \notin \alpha$ and $s \notin \beta$
      PROVE: $rs \notin \alpha\beta$
    ⟨3⟩2. ASSUME: for a contradiction $rs \in \alpha\beta$.
    ⟨3⟩3. PICK $r' \in \alpha$ and $s' \in \beta$ such that $rs \leq r's'$ and $r' > 0$ and $s' > 0$.
    ⟨3⟩4. $r' < r$ and $s' < s$
    ⟨3⟩5. $r's' < rs$
    ⟨3⟩6. Q.E.D.
      PROOF: This is a contradiction.
  ⟨2⟩3. $\alpha\beta$ is closed downwards.
    ⟨3⟩1. LET: $p \in \alpha\beta$ and $p' < p$
    ⟨3⟩2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$
    ⟨3⟩3. $p' \leq rs$
    ⟨3⟩4. $p' \in \alpha\beta$
  ⟨2⟩4. $\alpha\beta$ has no greatest element.
    ⟨3⟩1. LET: $p \in \alpha\beta$
    ⟨3⟩2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$.
    ⟨3⟩3. PICK $r' \in \alpha$ and $s' \in \beta$ with $r < r'$ and $s < s'$.
    ⟨3⟩4. $p < r's' \in \alpha\beta$
⟨1⟩2. For any cuts $\alpha$ and $\beta$, we have $\alpha\beta$ is a cut.
  PROOF: Since if $\alpha$ is a cut then $-\alpha$ is a cut.
□

**Proposition 6.14.** *For any cuts $\alpha$ and $\beta$ we have $\alpha\beta = \beta\alpha$.*

PROOF: Easy from the definitions. □

**Proposition 6.15.** *For any cuts $\alpha$, $\beta$ and $\gamma$ we have*

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma \ .$$

PROOF:
$\langle 1\rangle 1$. CASE: $\alpha$, $\beta$ and $\gamma$ are all positive.
PROOF: In this case $\alpha(\beta\gamma) = (\alpha\beta)\gamma = \{p \in \mathbb{Q} : \exists r \in \alpha.\exists s \in \beta.\exists t \in \gamma.(p \leq rst \wedge r > 0 \wedge s > 0 \wedge t > 0)\}$.
$\langle 1\rangle 2$. CASE: One of $\alpha$, $\beta$ or $\gamma$ is $0^*$.
PROOF: Then $\alpha(\beta\gamma) = (\alpha\beta)\gamma = 0^*$.
$\langle 1\rangle 3$. CASE: $\alpha$ and $\beta$ are positive, $\gamma$ is negative.
PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\
&= -(\alpha(\beta(-\gamma))) \\
&= -((\alpha\beta)(-\gamma)) && (\langle 1\rangle 1) \\
&= (\alpha\beta)\gamma
\end{aligned}$$

$\langle 1\rangle 4$. CASE: $\alpha$ is positive, $\beta$ is negative, $\gamma$ is positive.
PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-((-\beta)\gamma)) \\
&= -(\alpha((-\beta)\gamma)) \\
&= -((\alpha(-\beta))\gamma) && (\langle 1\rangle 1) \\
&= (-(\alpha(-\beta)))\gamma \\
&= (\alpha\beta)\gamma
\end{aligned}$$

$\langle 1\rangle 5$. CASE: $\alpha$ is positive, $\beta$ and $\gamma$ are negative.
PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha((-\beta)(-\gamma)) \\
&= (\alpha(-\beta))(-\gamma) && (\langle 1\rangle 1) \\
&= (-(\alpha\beta))(-\gamma) \\
&= (\alpha\beta)\gamma
\end{aligned}$$

$\langle 1\rangle 6$. CASE: $\alpha$ is negative, $\beta$ and $\gamma$ are positive.
PROOF: Similar to $\langle 1\rangle 3$.
$\langle 1\rangle 7$. CASE: $\alpha$ is negative, $\beta$ is positive, $\gamma$ is negative.
PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\
&= (-\alpha)(\beta(-\gamma)) \\
&= ((-\alpha)\beta)(-\gamma) && (\langle 1\rangle 1) \\
&= (-(\alpha\beta))(-\gamma) \\
&= (\alpha\beta)\gamma
\end{aligned}$$

$\langle 1\rangle 8$. CASE: $\alpha$ and $\beta$ are negative, $\gamma$ is positive.
PROOF: Similar to $\langle 1\rangle 5$.
$\langle 1\rangle 9$. CASE: $\alpha$, $\beta$ and $\gamma$ are all negative.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(-\beta)(-\gamma)) \\ &= -((-\alpha)((-\beta)(-\gamma))) \\ &= -(((-\alpha)(-\beta))(-\gamma)) && (\langle 1\rangle 1) \\ &= -((\alpha\beta)(-\gamma)) \\ &= (\alpha\beta)\gamma \end{aligned}$$

□

**Proposition 6.16.** *For any cut $\alpha$ we have $\alpha 1^* = \alpha$.*

PROOF:
⟨1⟩1. CASE: $\alpha$ is positive.
  ⟨2⟩1. $\alpha 1^* \subseteq \alpha$
  ⟨2⟩2. $\alpha \subseteq \alpha 1^*$
⟨1⟩2. CASE: $\alpha = 0^*$
⟨1⟩3. CASE: $\alpha$ is negative.
□

**Theorem 6.17.** *There exists an ordered field with the least upper bound property.*

**Proposition 6.18.** *There is no rational $p$ such that $p^2 = 2$.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $p^2 = 2$.
⟨1⟩2. PICK integers $m$, $n$ not both even such that $p = m/n$.
⟨1⟩3. $m^2 = 2n^2$
⟨1⟩4. $m$ is even.
⟨1⟩5. PICK an integer $k$ such that $m = 2k$.
⟨1⟩6. $4k^2 = 2n^2$
⟨1⟩7. $2k^2 = n^2$
⟨1⟩8. $n$ is even.
⟨1⟩9. Q.E.D.
  PROOF: ⟨1⟩2, ⟨1⟩4 and ⟨1⟩8 form a contradiction.
□

**Theorem 6.19.** *Any two complete ordered fields are isomorphic.*

**Definition 6.20.** Let $\mathbb{R}$ be the complete ordered field. We call its elements *real numbers.*

## 6.2 Properties of the Real Numbers

**Theorem 6.21.** *$\mathbb{Q}$ is a subfield of $\mathbb{R}$.*

**Theorem 6.22** (Archimedean Property)**.** *Let $x, y \in \mathbb{R}$ with $x > 0$. There exists a positive integer $n$ such that $nx > y$.*

PROOF:
⟨1⟩1. LET: $A = \{nx : n \in \mathbb{Z}^+\}$
⟨1⟩2. ASSUME: for a contradiction there is no positive integer $n$ such that $nx > y$.
⟨1⟩3. $y$ is an upper bound for $A$.
⟨1⟩4. LET: $\alpha = \sup A$
⟨1⟩5. $\alpha - x$ is not an upper bound for $A$.
⟨1⟩6. PICK a positive integer $m$ such that $\alpha - x < mx$
⟨1⟩7. $\alpha < (m+1)x \in A$
⟨1⟩8. Q.E.D.
PROOF: This contradicts ⟨1⟩4.
□

**Theorem 6.23.** *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

PROOF:
⟨1⟩1. LET: $x, y \in \mathbb{R}$ with $x < y$
⟨1⟩2. PICK a positive integer $n$ such that

$$n(y - x) > 1 \ .$$

PROOF: Archimedean property.
⟨1⟩3. PICK a positive integer $m_1$ such that $m_1 > nx$
PROOF: Archimedean property.
⟨1⟩4. PICK a positive integer $m_2$ such that $m_2 > -nx$
PROOF: Archimedean property.
⟨1⟩5. $-m_2 < nx < m_1$
⟨1⟩6. LET: $m$ be the integer such that

$$m - 1 \leq nx < m \ .$$

⟨1⟩7. $nx < m \leq 1 + nx < ny$
⟨1⟩8. $x < m/n < y$
□

**Theorem 6.24.** *For every real number $x > 0$ and positive integer $n$, there exists a unique positive real number $y$ such that $y^n = x$.*

PROOF:
⟨1⟩1. There exists a real $y > 0$ such that $y^n = x$.
⟨2⟩1. LET: $E = \{t \in \mathbb{R}^+ : t^n < x\}$
⟨2⟩2. LET: $y = \sup E$
⟨3⟩1. $E \neq \emptyset$
⟨4⟩1. LET: $t = x/(x+1)$
⟨4⟩2. $0 < t < 1$
⟨4⟩3. $t^n < t < x$
⟨4⟩4. $t \in E$
⟨3⟩2. $x + 1$ is an upper bound for $E$.
⟨4⟩1. LET: $t > x + 1$
⟨4⟩2. $t^n > t > x$
⟨4⟩3. $t \notin E$

$\langle 2\rangle 3.$ $y^n = x$

$\langle 3\rangle 1.$ $y^n \not< x$

$\langle 4\rangle 1.$ ASSUME: for a contradiction $y^n < x$.

$\langle 4\rangle 2.$ PICK $h$ such that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y+1)^{n-1}} \ .$$

$\langle 4\rangle 3.$ $(y+h)^n - y^n < x - y^n$

PROOF:

$$\begin{aligned}
(y+h)^n - y^n &= ((y+h) - y) \sum_{i=0}^{n-1} (y+h)^{n-1-i} y^i \\
&= h \sum_{i=0}^{n-1} (y+h)^{n-1-i} y^i \\
&\leq hn(y+h)^{n-1} \\
&\leq hn(y+1)^{n-1} \\
&< x - y^n
\end{aligned}$$

$\langle 4\rangle 4.$ $(y+h)^n < x$

$\langle 4\rangle 5.$ $y + h \in E$

$\langle 4\rangle 6.$ Q.E.D.

PROOF: This contradicts the fact that $y$ is an upper bound for $E$.

$\langle 3\rangle 2.$ $y^n \not> x$

$\langle 4\rangle 1.$ ASSUME: for a contradiction $y^n > x$

$\langle 4\rangle 2.$ LET:

$$k = \frac{y^n - x}{ny^{n-1}}$$

$\langle 4\rangle 3.$ $0 < k < y$

$\langle 4\rangle 4.$ $y - k$ is an upper bound for $E$.

$\langle 5\rangle 1.$ LET: $t \geq y - k$

$\langle 5\rangle 2.$ $y^n - t^n \leq y^n - x$

PROOF:

$$\begin{aligned}
y^n - t^n &\leq y^n - (y-k)^n \\
&= (y - (y-k)) \sum_{i=0}^{n-1} y^{n-i} (y-k)^i \\
&= k \sum_{i=0}^{n-1} y^{n-i} (y-k)^i \\
&\leq kny^{n-1} \\
&= y^n - x
\end{aligned}$$

$\langle 5\rangle 3.$ $t^n \geq x$

$\langle 5\rangle 4.$ $t \notin E$

$\langle 4\rangle 5.$ Q.E.D.

PROOF: This contradicts the fact that $y$ is the least upper bound of $E$.

$\langle 1\rangle 2.$ If $y$ and $y'$ are positive reals with $y^n = y'^n$ then $y = y'$.

PROOF: Since the function that sends $y$ to $y^n$ is strictly monotone.
∎

**Definition 6.25** ($n$th Root)**.** Given any real number $x > 0$ and positive integer $n$, the *nth root* of $x$, denoted $x^{1/n}$, is the unique positive real such that

$$(x^{1/n})^n = x \ .$$

We write $\sqrt{x}$ for $x^{1/2}$.

**Proposition 6.26.** *Let $a$ and $b$ be positive real numbers and $n$ a positive integer. Then*

$$(ab)^{1/n} = a^{1/n} b^{1/n} \ .$$

PROOF: Since $(a^{1/n} b^{1/n})^n = ab$. ∎

**Lemma 6.27.** *Let $b$ be a real number with $b > 1$. Let $n$ be a positive integer. Then*

$$b - 1 \geq n(b^{1/n} - 1) \ .$$

PROOF: From Lemma 5.24. ∎

**Lemma 6.28.** *Let $b$ and $t$ be real numbers with $b > 1$ and $t > 1$. For any positive integer $n$, if $n > \frac{b-1}{t-1}$ then $b^{1/n} < t$.*

PROOF:

$$\begin{aligned} b - 1 &\geq n(b^{1/n} - 1) \\ \therefore \frac{b-1}{n} &\geq b^{1/n} - 1 \\ \therefore t - 1 &> b^{1/n} - 1 \\ \therefore t &> b^{1/n} \end{aligned}$$

∎

**Lemma 6.29.** *Let $b$ be a real number with $b > 0$. Let $m$, $n$, $p$, $q$ be integers with $n > 0$ and $q > 0$. Assume $m/n = p/q$. Then*

$$(b^m)^{1/n} = (b^p)^{1/q} \ .$$

PROOF:
⟨1⟩1. $(b^m)^{1/n} = (b^{1/n})^m$
PROOF:

$$\begin{aligned} ((b^{1/n})^m)^n &= ((b^{1/n})^n)^m \\ &= b^m \end{aligned}$$

⟨1⟩2. $((b^m)^{1/n})^q = b^p$
PROOF:

$$\begin{aligned} ((b^m)^{1/n})^q &= (b^{1/n})^{mq} \\ &= (b^{1/n})^{np} \\ &= b^p \end{aligned}$$

∎

**Definition 6.30.** For $a$ a positive real and $q$ a rational number, we may therefore define $a^q$ by

$$a^{m/n} = (a^m)^{1/n}$$

for $m$ and $n$ integers with $n > 0$.

**Proposition 6.31.** *Let $a$ be a positive real and $r$, $s$ rational numbers. Then*

$$a^{r+s} = a^r a^s \quad .$$

PROOF:

$$\begin{aligned} a^{m/n+p/q} &= a^{(mq+np)/nq} \\ &= (a^{mq+np})^{1/nq} \\ &= (a^{mq})^{1/nq}(a^{np})^{1/nq} \\ &= a^{m/n}a^{p/q} \end{aligned}$$

□

**Proposition 6.32.** *Let $b > 1$ be a real number and $q$ a rational number. Then*

$$b^q = \sup\{b^t : t \in \mathbb{Q}, t \leq q\}$$

PROOF: It is the greatest element of this set. □

**Definition 6.33.** Let $b > 1$ be a real number and $x$ a real number. Then

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t \leq x\} \quad .$$

**Lemma 6.34.** *Let $b$, $w$ and $y$ be real numbers with $b > 1$. Assume $b^w < y$. Then there exists a positive integer $n$ such that $b^{w+1/n} < y$.*

PROOF:
$\langle 1 \rangle 1$. LET: $t = yb^{-w}$
$\langle 1 \rangle 2$. PICK a positive integer $n$ such that $n > \frac{b-1}{t-1}$.
$\langle 1 \rangle 3$. $b^{1/n} < t$
  PROOF: Lemma 6.28.
$\langle 1 \rangle 4$. $b^{w+1/n} < y$
□

**Lemma 6.35.** *Let $b$, $w$ and $y$ be real numbers with $b > 1$. Assume $b^w > y$. Then there exists a positive integer $n$ such that $b^{w-1/n} < y$.*

PROOF:
$\langle 1 \rangle 1$. LET: $t = b^w/y$
$\langle 1 \rangle 2$. PICK a positive integer $n$ such that $n > \frac{b-1}{t-1}$
$\langle 1 \rangle 3$. $b^{1/n} < t$
  PROOF: Lemma 6.28.
$\langle 1 \rangle 4$. $y < b^{w-1/n}$
□

**Proposition 6.36.** *For $b$ and $x$ real numbers with $b > 1$ we have*

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t < x\} \ .$$

PROOF:
⟨1⟩1. $b^x$ is an upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
⟨1⟩2. LET: $u$ be any upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
PROVE: $b^x \leq u$
⟨1⟩3. LET: $q$ be a rational number with $q \leq x$.
PROVE: $b^q \leq u$
⟨1⟩4. ASSUME: for a contradiction $b^q > u$.
⟨1⟩5. PICK a positive integer $n$ such that $b^{q-1/n} > u$.
PROOF: Lemma 6.35.
⟨1⟩6. $b^{q-1/n} \leq u$
PROOF: ⟨1⟩2
⟨1⟩7. Q.E.D.
PROOF: This contradicts ⟨1⟩4.
□

**Lemma 6.37.** *Let $A$ be a set of positive real numbers with supremum $a > 0$ and $B$ a set of positive real numbers with supremum $b > 0$. Then $ab$ is the supremum of $\{xy : x \in A, y \in B\}$.*

PROOF:
⟨1⟩1. For all $x \in A$ and $y \in B$ we have $xy \leq ab$.
⟨1⟩2. If $u$ is any upper bound for $\{xy : x \in A, y \in B\}$ then $ab \leq u$.
⟨2⟩1. LET: $u$ be an upper bound for $\{xy : x \in A, y \in B\}$.
⟨2⟩2. For all $x \in A$ we have $u/x$ is an upper bound for $B$.
⟨2⟩3. For all $x \in A$ we have $b \leq u/x$
⟨2⟩4. For all $x \in A$ we have $x \leq u/b$
⟨2⟩5. $a \leq u/b$
⟨2⟩6. $ab \leq u$
□

**Proposition 6.38.** *Let $b, x, y \in \mathbb{R}$ with $b > 1$. Then*

$$b^{x+y} = b^x b^y \ .$$

PROOF:
⟨1⟩1. For any rational number $q < x + y$, there exist rational numbers $r < x$ and $s < y$ such that $q = r + s$.
⟨2⟩1. $q - x < y$
⟨2⟩2. PICK a rational $t$ such that $q - x < t < y$
⟨2⟩3. $q = t + (q - t)$ and $t < y$, $q - t < x$
⟨1⟩2. $b^x b^y = b^{x+y}$

PROOF:

$$\begin{aligned} b^x b^y &= \sup\{b^q b^r : q, r \in \mathbb{Q}, q < x, r < y\} \\ &= \sup\{b^{q+r} : q, r \in \mathbb{Q}, q < x, r < y\} \\ &= \sup\{b^q : q \in \mathbb{Q}, q < x + y\} \\ &= b^{x+y} \end{aligned}$$

□

### 6.2.1 Logarithms

**Proposition 6.39.** *Let $b$ and $y$ be real numbers with $b > 1$ and $y > 0$. There exists a unique real $x$ such that $b^x = y$.*

PROOF:
$\langle 1 \rangle 1$. LET: $x = \sup\{w : b^w < y\}$
PROVE: $b^x = y$
$\langle 2 \rangle 1$. $\{w : b^w < y\} \neq \emptyset$
PROOF: It contains 0.
$\langle 2 \rangle 2$. $\{w : b^w < y\}$ is bounded above.
$\langle 3 \rangle 1$. LET: $n$ be the least integer such that

$$n \geq \frac{y-1}{b-1}$$

PROOF: Archimedean property.
$\langle 3 \rangle 2$. LET: $w$ be a real number with $b^w < y$
PROVE: $w < n$
$\langle 3 \rangle 3$. $b^w < n(b-1) + 1$
$\langle 3 \rangle 4$. $b^w < b^n$
$\langle 3 \rangle 5$. $w < n$
$\langle 1 \rangle 2$. $b^x \leq y$
$\langle 2 \rangle 1$. ASSUME: for a contradiction $b^x > y$
$\langle 2 \rangle 2$. PICK a positive integer $n$ such that $b^{x-1/n} > y$
PROOF: Lemma 6.35.
$\langle 2 \rangle 3$. PICK $w$ such that $x - 1/n < w$ and $b^w < y$
PROOF: Since $x - 1/n$ is not an upper bound for $\{w : b^w < y\}$.
$\langle 2 \rangle 4$. $b^{x-1/n} < y$
$\langle 2 \rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 2 \rangle 2$.
$\langle 1 \rangle 3$. $b^x \geq y$
$\langle 2 \rangle 1$. ASSUME: for a contradiction $b^x < y$.
$\langle 2 \rangle 2$. PICK a positive integer $n$ such that $b^{x+1/n} < y$.
$\langle 2 \rangle 3$. $x + 1/n \leq x$
$\langle 2 \rangle 4$. Q.E.D.
PROOF: This is a contradiction.

□

**Definition 6.40** (Logarithm)**.** Let $b$ and $y$ be real numbers with $b > 1$ and $y > 0$. The *logarithm* of $y$ to *base* $b$, denoted $\log_b y$, is the unique real number such that

$$b^{\log_b y} = y \ .$$

### 6.2.2 Intervals

**Definition 6.41** (Intervals)**.** Let $a, b \in \mathbb{R}$.

The *open interval* $(a, b)$ is $\{x \in \mathbb{R} : a < x < b\}$.

The *closed interval* $[a, b]$ is $\{x \in \mathbb{R} : a \leq x \leq b\}$.

The *half-open intervals* $[a, b)$ and $(a, b]$ are defined by

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

**Definition 6.42** ($k$-cell)**.** Let $k$ be a positive integer. A *$k$-cell* is a subset of $\mathbb{R}^k$ of the form

$$\{\vec{x} \in \mathbb{R}^k : \forall i = 1, \ldots, k.a_i \leq x_i \leq b_i\}$$

for some real numbers $a_1$, $\ldots$, $a_k$, $b_1$, $\ldots$, $b_k$ with $a_i \leq b_i$ for each $i$.

### 6.2.3 The Cantor Set

**Definition 6.43** (Cantor Set)**.** Define a sequence $E_n$ of unions of intervals as follows:

- $E_0 = [0, 1]$
- $E_{n+1}$ is formed from $E_n$ by replacing every interval $[a, b]$ with $[a, (2a+b)/3]$ and $[(a + 2b)/3, b]$.

The *Cantor set* is $\bigcap_{n=0}^{\infty} E_n$.

## 6.3 The Extended Real Number System

**Definition 6.44** (Extended Real Number System)**.** The *extended real number system* is the set $\mathbb{R} \cup \{+\infty, -\infty\}$.

We extend the ordering $\leq$ to the extended reals by defining

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

We extend $+$, $\cdot$ and $/$ to partial operations on the extended real by defining:

$$
\begin{aligned}
x + (+\infty) &= +\infty && (x \in \mathbb{R}) \\
x + (-\infty) &= -\infty && (x \in \mathbb{R}) \\
(+\infty) + x &= +\infty && (x \in \mathbb{R}) \\
(+\infty) + (+\infty) &\text{ is undefined} \\
(+\infty) + (-\infty) &\text{ is undefined} \\
(-\infty) + x &= -\infty && (x \in \mathbb{R}) \\
(-\infty) + (+\infty) &\text{ is undefined} \\
(-\infty) + (-\infty) &\text{ is undefined} \\
x \cdot (+\infty) &= +\infty && (x \in \mathbb{R}) \\
x \cdot (-\infty) &= -\infty && (x \in \mathbb{R}) \\
(+\infty) \cdot x &= +\infty && (x \in \mathbb{R}) \\
(+\infty) \cdot (+\infty) &\text{ is undefined} \\
(+\infty) \cdot (-\infty) &\text{ is undefined} \\
(-\infty) \cdot x &= -\infty && (x \in \mathbb{R}) \\
(-\infty) \cdot (+\infty) &\text{ is undefined} \\
(-\infty) \cdot (-\infty) &\text{ is undefined} \\
x/(+\infty) &= 0 && (x \in \mathbb{R}) \\
x/(-\infty) &= 0 && (x \in \mathbb{R}) \\
(+\infty)/x &\text{ is undefined} && (x \in \mathbb{R}) \\
(+\infty)/(+\infty) &\text{ is undefined} \\
(+\infty)/(-\infty) &\text{ is undefined} \\
(-\infty)/x &\text{ is undefined} && (x \in \mathbb{R}) \\
(-\infty)/(+\infty) &\text{ is undefined} \\
(-\infty)/(-\infty) &\text{ is undefined}
\end{aligned}
$$

# Chapter 7

# Complex Analysis

**Definition 7.1** (Complex Numbers)**.** A *complex number* is a pair of real numbers. We write $\mathbb{C}$ for the set of complex numbers.

Define $+$ and $\cdot$ on $\mathbb{C}$ by:

$$\begin{aligned}(a,b)+(c,d) &= (a+c,b+d)\\ (a,b)(c,d) &= (ac-bd,ad+bc)\end{aligned}$$

**Theorem 7.2.** *The complex numbers form a field.*

**Theorem 7.3.** *The function that maps $a$ to $(a,0)$ is an embedding of $\mathbb{R}$ in $\mathbb{C}$.*

**Definition 7.4.**

$$i = (0,1)$$

**Lemma 7.5.**

$$(a,b) = a + ib$$

Proof: Since $(a,0)+(0,1)(b,0)=(a,b)$. $\square$

**Lemma 7.6.**

$$i^2 = -1$$

Proof: Immediate from definitions. $\square$

**Corollary 7.6.1.** *There is no linear order on $\mathbb{C}$ that makes $\mathbb{C}$ into an ordered field.*

**Definition 7.7** (Complex Conjugate)**.** For any complex number $z$, the *complex conjugate* $\overline{z}$ is defined by

$$\overline{a+ib} = a - ib \qquad (a,b\in\mathbb{R}) \ .$$

**Definition 7.8** (Real Part)**.** For any complex number $z$, the *real part* of $z$, denoted $\mathrm{Re}(z)$, is defined by

$$\mathrm{Re}(a+ib) = a \qquad (a,b\in\mathbb{R}) \ .$$

**Definition 7.9** (Imaginary Part)**.** For any complex number $z$, the *imaginar part* of $z$, denoted $\mathrm{Im}(z)$, is defined by

$$\mathrm{Im}(a+ib) = b \qquad (a, b \in \mathbb{R}) \ .$$

**Theorem 7.10.** *For all $z, w \in \mathbb{C}$ we have*

$$\overline{z+w} = \overline{z} + \overline{w} \ .$$

PROOF:

$$\begin{aligned} \overline{(a+ib)+(c+id)} &= \overline{(a+c)+i(b+d)} \\ &= (a+c) - i(b+d) \\ &= (a-ib) + (c-id) \\ &= \overline{a+ib} + \overline{c+id} \end{aligned}$$

□

**Theorem 7.11.** *For all $z, w \in \mathbb{C}$ we have*

$$\overline{zw} = \overline{z} \cdot \overline{w} \ .$$

PROOF:

$$\begin{aligned} \overline{(a+ib)(c+id)} &= \overline{(ac-bd)+i(ad+bc)} \\ &= (ac-bd) - i(ad+bc) \\ &= (a-ib)(c-id) \\ &= \overline{a+ib} \cdot \overline{c+id} \end{aligned}$$

□

**Theorem 7.12.** *For all $z \in \mathbb{C}$ we have*

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}) \ .$$

PROOF:

$$\begin{aligned} (a+ib) + \overline{a+ib} &= (a+ib) + (a-ib) \\ &= 2a \\ &= 2\,\mathrm{Re}(a+ib) \end{aligned}$$

□

**Theorem 7.13.** *For all $z \in \mathbb{C}$ we have*

$$\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z}) \ .$$

PROOF:

$$\begin{aligned} (a+ib) - \overline{a+ib} &= (a+ib) - (a-ib) \\ &= 2ib \\ &= 2i\,\mathrm{Im}(a+ib) \end{aligned}$$

□

**Theorem 7.14.** *For all $z \in \mathbb{C}$ we have $z\overline{z}$ is a non-negative real.*

PROOF:

$$\begin{aligned}(a+ib)(\overline{a+ib}) &= (a+ib)(a-ib)\\ &= a^2+b^2\end{aligned}$$ □

**Theorem 7.15.** *For any $z \in \mathbb{C}$, if $z\overline{z} = 0$ then $z = 0$.*

PROOF: Let $z = a + ib$. Then $z\overline{z} = a^2 + b^2 = 0$ iff $a = b = 0$. □

**Definition 7.16** (Absolute Value)**.** For $z \in \mathbb{C}$, the *absolute value* of $z$ is

$$|z| = (z\overline{z})^{1/2} \quad .$$

**Proposition 7.17.** *For $x$ a non-negative real we have $|x| = x$.*

PROOF: Since $|x| = \sqrt{x^2} = x$. □

**Proposition 7.18.** *For $x$ a negative real we have $|x| = -x$.*

PROOF: Since $|x| = \sqrt{x^2} = -x$. □

**Theorem 7.19.** *For any complex number $z$ we have $|z| \geq 0$.*

PROOF: Immediate from definition. □

**Theorem 7.20.** *For any complex number $z$, if $|z| = 0$ then $z = 0$.*

PROOF: From Theorem 7.15. □

**Theorem 7.21.** *For any complex number $z$ we have*

$$|\overline{z}| = |z| \quad .$$

PROOF: Immediate from definitions. □

**Theorem 7.22.** *For any complex numbers $z$ and $w$ we have*

$$|zw| = |z||w| \quad .$$

PROOF:

$$\begin{aligned}|zw| &= \sqrt{zw\overline{zw}}\\ &= \sqrt{z\overline{z}}\sqrt{w\overline{w}} && \text{(Proposition 6.26)}\\ &= |z||w|\end{aligned}$$ □

**Theorem 7.23.** *For any complex number $z$ we have*

$$|\operatorname{Re} z| \leq |z|$$

PROOF: Let $z = a + ib$. Then

$$|\operatorname{Re} z| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} \quad .$$ □

**Theorem 7.24.** *For any complex numbers $z$ and $w$ we have*

$$|z + w| \leq |z| + |w| \quad .$$

Proof:

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\ &= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \\ &= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 && \text{(Theorem 7.12)} \\ &\leq |z|^2 + 2|z\overline{w}| + |w|^2 && \text{(Theorem 7.23)} \\ &= |z|^2 + 2|z||w| + |w|^2 && \text{(Theorem 7.22)} \\ &= (|z|+|w|)^2 \end{aligned}$$

□

**Theorem 7.25** (Schwarz Inequality)**.** *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be complex numbers. Then*

$$\left|\sum_{j=1}^n a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2 \enspace .$$

Proof:
⟨1⟩1. Let: $A = \sum_{j=1}^n |a_j|^2$
⟨1⟩2. Let: $B = \sum_{j=1}^n |b_j|^2$
⟨1⟩3. Let: $C = \sum_{j=1}^n a_j \overline{b_j}$
⟨1⟩4. Assume: w.l.o.g. $B > 0$
Proof: If $B = 0$ then $b_1 = \cdots = b_n = 0$ and both sides of the inequality are 0.
⟨1⟩5. $\sum_{j=1}^n |Ba_j - Cb_j|^2 = B(AB - |C|^2)$
Proof:

$$\begin{aligned} \sum_{j=1}^n |Ba_j - Cb_j|^2 &= \sum_{j=1}^n (Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j}) \\ &= B^2 \sum_{j=1}^n |a_j|^2 - B\overline{C} \sum_{j=1}^n a_j \overline{b_j} - BC \sum_{j=1}^n \overline{a_j} b_j + |C|^2 \sum_{j=1}^n |b_j|^2 \\ &= B^2 A - 2B|C|^2 + B|C|^2 \\ &= B(AB - |C|^2) \end{aligned}$$

⟨1⟩6. $B(AB - |C|^2) \geq 0$
⟨1⟩7. $AB \geq |C|^2$
□

**Proposition 7.26.** *For any non-zero complex number $w$, there are exactly two complex numbers $z$ such that $z^2 = w$.*

Proof:
⟨1⟩1. There are at most two complex numbers $z$ such that $z^2 = w$.
Proof: Proposition 5.15.
⟨1⟩2. There are at least two complex numbers $z$ such that $z^2 = w$.
⟨2⟩1. Let: $w = u + iv$
⟨2⟩2. Let: $a = \sqrt{\frac{|w|+u}{2}}$
⟨2⟩3. Let: $b = \sqrt{\frac{|w|-u}{2}}$

⟨2⟩4. CASE: $v \geq 0$
⟨3⟩1. LET: $z = a + ib$
⟨3⟩2. $z^2 = w$
PROOF:

$$\begin{aligned} z^2 &= (a+ib)^2 \\ &= a^2 - b^2 + 2iab \\ &= u + i\sqrt{|w|^2 - u^2} \\ &= u + iv \\ &= w \end{aligned}$$

⟨3⟩3. $(-z)^2 = w$
⟨2⟩5. CASE: $v \leq 0$
⟨3⟩1. LET: $z = a - ib$
⟨3⟩2. $z^2 = w$
PROOF:

$$\begin{aligned} z^2 &= (a-ib)^2 \\ &= a^2 - b^2 - 2iab \\ &= u - i\sqrt{|w|^2 - u^2} \\ &= u - i|v| \\ &= w \end{aligned}$$

⟨3⟩3. $(-z)^2 = w$
□

## 7.1 Algebraic Numbers

**Definition 7.27** (Algebraic)**.** A complex number $z$ is *algebraic* iff there exist integers $a_0$, $a_1$, ..., $a_n$ not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \;;$$

otherwise, it is *transcendental*.

**Proposition 7.28.** *The set of algebraic numbers is countable.*

PROOF: There are countably many finite sequences of integers $(a_0, a_1, \ldots, a_n)$, and for each one, there are only finitely many complex numbers $z$ such that $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$. □

# Part II

# Category Theory

# Chapter 8

# Categories

**Definition 8.1** (Category)**.** A *category* $\mathcal{C}$ consists of:

- a preset $|\mathcal{C}|$ of *objects*;
- for any objects $A$ and $B$, a set $\mathcal{C}[A, B]$ of *morphisms* from $A$ to $B$. We write $f : A \to B$ for $f \in \mathcal{C}[A, B]$, and call $A$ the *source* of $f$ and $B$ the *target*.
- for any object $A$, a morphism $\mathrm{id}_A : A \to A$, the *identity* morphism on $A$
- for any morphisms $f : A \to B$ and $g : B \to C$, a morphism $g \circ f : A \to C$, the *composite* of $f$ and $g$.

such that:

**Associativity** For any morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

**Left Unit Law** For any morphism $f : A \to B$, we have

$$\mathrm{id}_B \circ f = f$$

**Right Unit Law** For any morphism $f : A \to B$, we have

$$f \circ \mathrm{id}_A = f$$

**Example 8.2.** For any preset of sets $\mathcal{U}$, the category $\mathbf{Set}_{\mathcal{U}}$ with objects $\mathcal{U}$ and morphisms all functions is a category.

## 8.1 Isomorphisms

**Definition 8.3** (Isomorphism)**.** We say a morphism $f : A \to B$ is an *isomorphism*, and write $f : A \cong B$, iff there exists a morphism $g : B \to A$, the *inverse* of $f$, such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

Objects $A$ and $B$ are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

# Part III

# Linear Algebra

# Chapter 9

# Vector Spaces

## 9.1 Convex Sets

**Definition 9.1** (Convex)**.** Let $E \subseteq \mathbb{R}^k$. Then $E$ is *convex* iff, for all $\vec{x}, \vec{y} \in E$ and $\lambda \in (0, 1)$,

$$\lambda \vec{x} + (1 - \lambda)\vec{y} \in E \ .$$

**Proposition 9.2.** *Every $k$-cell is convex.*

PROOF:
$\langle 1 \rangle 1$. LET: $C = \{\vec{x} \in \mathbb{R}^k : \forall i. a_i \leq x_i \leq b_i\}$ be a $k$-cell.
$\langle 1 \rangle 2$. LET: $\vec{x}, \vec{y} \in C$ and $\lambda \in (0, 1)$.
PROVE: $\lambda \vec{x} + (1 - \lambda)\vec{y} \in C$
$\langle 1 \rangle 3$. For each $i$ we have $a_i \leq \lambda x_i + (1 - \lambda) y_i \leq b_i$
PROOF: Since $\lambda a_1 + (1 - \lambda) a_i \leq \lambda x_i + (1 - \lambda) y_i \leq \lambda b_i + (1 - \lambda) b_i$.
$\square$

## 9.2 Linear Transformations

**Definition 9.3** (Norm)**.** For $A \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$, define the *norm* of $A$ to be

$$\|A\| := \{\|A\vec{x}\| : \vec{x} \in \mathbb{R}^n, \|\vec{x}\| = 1\} \ .$$

We prove that this always exists.

PROOF: Since for $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1^2 + \cdots + x_n^2 = 1$ we have

$$\begin{aligned} \|A(x_1, \ldots, x_n)\| &= \|\sum_{i=1}^{n} x_i A\vec{e_i}\| \\ &\leq \sum_{i=1}^{n} |x_i| \|A\vec{e_i}\| \\ &\leq \sum_{i=1}^{n} \|A\vec{e_i}\| \end{aligned}$$

$\square$

**Proposition 9.4.** *Given $A, B \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$, we have*

$$\|A + B\| \leq \|A\| + \|B\|$$

PROOF: Since $\|A\vec{x} + B\vec{x}\| \leq \|A\vec{x}\| + \|B\vec{x}\|$. □

**Proposition 9.5.** *Given $A \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m$ and $c \in \mathbb{R}$, we have*

$$\|cA\| = |c|\|A\| \ .$$

PROOF: Since $\|cA\vec{x}\| = |c|\|A\vec{x}\|$. □

**Proposition 9.6.** *Given $A \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$ and $B \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^m, \mathbb{R}^k]$, we have*

$$\|BA\| \leq \|B\|\|A\| \ .$$

PROOF: Since $\|BA\vec{x}\| \leq \|B\|\|A\vec{x}\| \leq \|B\|\|A\|\|\vec{x}\|$. □

**Lemma 9.7.** *Let $A, B \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$ with $A$ invertible. Let $\alpha = 1/\|A^{-1}\|$ and $\beta = \|B - A\|$. Then, for all $\vec{x} \in \mathbb{R}^n$, we have*

$$(\alpha - \beta)\|\vec{x}\| \leq \|B\vec{x}\| \ .$$

PROOF:

$$\begin{aligned} \alpha\|\vec{x}\| &= \alpha\|A^{-1}A\vec{x}\| \\ &\leq \alpha\|A^{-1}\|\|A\vec{x}\| \\ &= \|A\vec{x}\| \\ &\leq \|(A - B)\vec{x}\| + \|B\vec{x}\| \\ &\leq \beta\|\vec{x}\| + \|B\vec{x}\| \end{aligned}$$ □

**Proposition 9.8.** *Let $A, B \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^n]$. If $A$ is invertible and*

$$\|B - A\|\|A^{-1}\| < 1$$

*then $B$ is invertible.*

PROOF:
⟨1⟩1. LET: $\alpha = 1/\|A^{-1}\|$
⟨1⟩2. LET: $\beta = \|B - A\|$
⟨1⟩3. $\beta < \alpha$
⟨1⟩4. For all $\vec{x} \in \mathbb{R}^n$ we have $\alpha\|\vec{x}\| \leq \beta\|\vec{x}\| + \|B\vec{x}\|$
  PROOF: Lemma 9.7.
⟨1⟩5. For all $\vec{x} \in \mathbb{R}^n$ we have

$$(\alpha - \beta)\|\vec{x}\| \leq \|B\vec{x}\| \ .$$

⟨1⟩6. $\ker B = 0$
  PROOF: Since $\alpha - \beta > 0$ so if $\|\vec{x}\| > 0$ then $\|B\vec{x}\| > 0$.
□

# Chapter 10

# Real Inner Product Spaces

**Definition 10.1** (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^k$, define the *inner product* $\vec{x} \cdot \vec{y}$ by

$$(x_1, \ldots, x_k) \cdot (y_1, \ldots, y_k) = x_1 y_1 + \cdots + x_k y_k \ .$$

**Definition 10.2** (Norm)**.** Define the *norm* of a vector $\vec{x} \in \mathbb{R}^k$ by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} \ .$$

**Proposition 10.3.**

$$\|\vec{x}\| \geq 0$$

Proof: Immediate from the definition. □

**Proposition 10.4.** *If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$.*

Proof: If $\|\vec{x}\| = 0$ then $x_1^2 + \cdots + x_n^2 = 0$ so $x_1 = \cdots = x_n = 0$. □

**Proposition 10.5.** *For $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^k$,*

$$\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| \ .$$

Proof: Easy. □

**Proposition 10.6.** *For $\vec{x}, \vec{y} \in \mathbb{R}^k$, we have*

$$\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\| \|\vec{y}\| \ .$$

Proof: By the Schwarz inequality. □

**Proposition 10.7.** *For $\vec{x}, \vec{y} \in \mathbb{R}^k$ we have*

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \ .$$

PROOF:

$$\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \\
&= \vec{x}\cdot\vec{x}+2\vec{x}\cdot\vec{y}+\vec{y}\cdot\vec{y} \\
&\leq \|\vec{x}\|^2+2\|\vec{x}\|\|\vec{y}\|+\|\vec{y}\|^2 && \text{(Proposition 10.6)} \\
&= (\|\vec{x}\|+\|\vec{y}\|)^2
\end{aligned}$$

□

**Corollary 10.7.1.** *For $\vec{x},\vec{y},\vec{z}\in\mathbb{R}^k$ we have*

$$\|\vec{x}-\vec{z}\| \leq \|\vec{x}-\vec{y}\|+\|\vec{y}-\vec{z}\| \ .$$

**Definition 10.8** (Bounded Function)**.** Let $E$ be a set. Let $f:E\to\mathbb{R}^k$. Then $f$ is *bounded* iff $f(E)$ is bounded.

## 10.1 Balls

**Definition 10.9** (Closed Ball)**.** Let $\vec{x}\in\mathbb{R}^k$ and $r>0$. The *closed ball* with *centre* $\vec{x}$ and *radius* $r$ is

$$\{y\in\mathbb{R}^k : \|y-x\|\leq r\} \ .$$

**Proposition 10.10.** *Every closed ball is convex.*

PROOF:
⟨1⟩1. LET: $B$ be the closed ball with center $\vec{a}$ and radius $r$.
⟨1⟩2. LET: $\vec{x},\vec{y}\in B$
⟨1⟩3. LET: $\lambda\in(0,1)$
⟨1⟩4. $\lambda\vec{x}+(1-\lambda)\vec{y}\in B$
PROOF:

$$\begin{aligned}
\|\lambda\vec{x}+(1-\lambda)\vec{y}-\vec{a}\| &= \|\lambda(\vec{x}-\vec{a})+(1-\lambda)(\vec{y}-\vec{a})\| \\
&= \lambda\|\vec{x}-\vec{a}\|+(1-\lambda)\|\vec{y}-\vec{a}\| \\
&\leq \lambda r+(1-\lambda)r \\
&= r
\end{aligned}$$

□

□

# Chapter 11

# Complex Inner Product Spaces

**Definition 11.1** (Inner Product)**.** Let $V$ be a complex vector space. An *inner product* on $V$ is a function $\langle\ ,\ \rangle : V^2 \to \mathbb{C}$ such that, for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

- $\langle y, x\rangle = \overline{\langle x, y\rangle}$
- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
- $\langle \alpha x, y\rangle = \overline{\alpha}\langle x, y\rangle$
- $\langle x, x\rangle \geq 0$
- If $\langle x, x\rangle = 0$ then $x = 0$.

An *inner product space* consists of a complex vector space $V$ and an inner product on $V$.

**Definition 11.2** (Norm)**.** Let $V$ be an inner product space and $x \in V$. The *norm* of $x$ is

$$\|x\| = \sqrt{\langle x, x\rangle}\ .$$

**Proposition 11.3.** *An inner product space is a metric space under*

$$d(x, y) = \|x - y\|\ .$$

**Definition 11.4** (Bounded)**.** Let $V_1$ and $V_2$ be inner product spaces and $T : V_1 \to V_2$ a linear transformation. Then $T$ is *bounded* iff $\{\|T(x)\| : \|x\| = 1\}$ is bounded above.

**Proposition 11.5.** *Every linear transformation between finite dimensional inner product spaces is bounded.*

**Definition 11.6** (Outer Product)**.** Let $V$ be an inner product space and $|\psi\rangle, |\phi\rangle \in V$. The *outer product* of $|\psi\rangle$ and $|\phi\rangle$ is

$$|\psi\rangle\langle\phi| : V \to V\ .$$

## 11.1 Hilbert Spaces

**Definition 11.7** (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

**Theorem 11.8** (Completeness Relation)**.** *Let $\mathcal{H}$ be a Hilbert space. Let $\{|e_n\rangle\}_{n\in\mathbb{N}}$ be a countable orthonormal basis for $\mathcal{H}$. Then*

$$\sum_{n=0}^{\infty} |e_n\rangle \langle e_n| = I \ .$$

PROOF:
$\langle 1\rangle 1.$ LET: $|\psi\rangle \in \mathcal{H}$
$\langle 1\rangle 2.$ LET: $|\psi\rangle = \sum_{n=0}^{\infty} \alpha_n |e_n\rangle$
$\langle 1\rangle 3.$ $\sum_{n=0}^{\infty} \langle e_n|\phi\rangle |e_n\rangle = |\psi\rangle$
PROOF:

$$\begin{aligned} \sum_{n=0}^{\infty} \langle e_n|\phi\rangle |e_n\rangle &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_m \langle e_n|e_m\rangle |e_n\rangle \\ &= \sum_{n=0}^{\infty} \alpha_n |e_n\rangle \\ &= |\psi\rangle \end{aligned}$$

□

□

**Definition 11.9** (Separable)**.** A Hilbert space is *separable* iff it has a countable dense orthonormal basis.

# Chapter 12

# Lie Algebras

**Definition 12.1** (Lie Algebra)**.** Let $K$ be a field. A *Lie algebra* $\mathcal{L}$ over $K$ consists of a vector space $\mathcal{L}$ over $K$ and an operation

$$[\ ,\ ] : \mathcal{L}^2 \to \mathcal{L} \ ,$$

the *Lie bracket* or *commutator*, such that, for all $x, y, z \in \mathcal{L}$ and $\alpha \in K$:

$$\begin{aligned}
[x+y, z] &= [x,z] + [y,z] \\
[x, y+z] &= [x,y] + [x,z] \\
[\alpha x, y] &= \alpha[x,y] \\
[x,x] &= 0 \\
[x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= 0 \qquad \text{(Jacobi identity)}
\end{aligned}$$

**Lemma 12.2.** *If $K$ has characteristic 0 then the condition $[x,x] = 0$ can be replaced with $[x,y] = -[y,x]$.*

**Proposition 12.3.** *The commutator is determind by its values on any basis for $\mathcal{L}$.*

**Example 12.4.** $\mathbb{R}^3$ with the cross product is a real Lie algebra.

**Example 12.5.** For any $n \geq 0$, we have $GL(n,K)$ is a Lie algebra over $K$ under

$$[A,B] = AB - BA \ .$$

**Definition 12.6** (Linear Lie Algebra)**.** A *linear Lie algebra* over $K$ is a Lie algebra over $K$ that is a subalgebra of $GL(n,K)$ for some $n$.

**Example 12.7** (Special Linear Algebra)**.** The *special Linear algebra* $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \mathrm{tr} = 0\}$ is a real linear Lie algebra.

**Example 12.8** (Orthogonal Lie Algebra)**.** The *orthogonal Lie algebra* $SO(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : A \text{ is skew-symmetric}\}$ is a real linear Lie algebra.

**Example 12.9.** Let $u(n)$ be the set of all skew-Hermitian $n \times n$-matrices as a real Lie algebra.

Let $su(n) = u(n) \cap SL(n, \mathbb{R})$.

**Proposition 12.10.** *$SU(2)$ is spanned by the* Pauli matrices

$$\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

*which satisfy*

$$[\sigma_x, \sigma_y] = \sigma_z$$
$$[\sigma_y, \sigma_z] = \sigma_x$$
$$[\sigma_z, \sigma_x] = \sigma_y$$

## 12.1 Lie Algebar Homomorphisms

**Definition 12.11** (Homomorphism)**.** Let $L_1$ and $L_2$ be Lie algebras over the same field. A *Lie algebra homomorphism* $\phi : L_1 \to L_2$ is a linear transformation such that

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in L_1$.

**Lemma 12.12.** *Every bijective Lie algebra homomorphism is an isomorphism.*

**Definition 12.13** (Representation)**.** Let $L$ be a real (complex) Lie algebra. A *representation* of $L$ is a Lie algebra homomorphism $L \to GL(n, \mathbb{R})$ $(GL(n, \mathbb{C}))$ for some $n$.

**Example 12.14.** The linear transformation $\mathbb{R}^3 \to su(2)$ defined by

$$i \mapsto \sigma_x, j \mapsto \sigma_y, k \mapsto \sigma_z$$

is a representation of $\mathbb{R}^3$.

# Part IV

# Topology

# Chapter 13

# Metric Spaces

**Definition 13.1** (Metric)**.** A *metric* on a set $X$ is a function $d : X^2 \to \mathbb{R}$ such that, for all $x, y, z \in X$:

- $d(x, y) \geq 0$
- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- **Triangle Inequality** $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* $X$ consists of a set $X$ and a metric on $X$.

**Example 13.2.** $\mathbb{R}^k$ is a metric space under $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$. The triangle inequality is Corollary 10.7.1.

**Example 13.3.** For any set $X$, the *discrete* metric on $X$ is defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

**Example 13.4.** $\mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$ is a metric space under $d(A, B) = \|A - B\|$.

**Proposition 13.5.** *Let $(X, d)$ be a metric space and $Y$ a subset of $X$. Then $d \restriction Y^2$ is a metric on $Y$.*

Proof: Easy. □

## 13.1 Balls

**Definition 13.6** (Open Ball)**.** Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *open ball* with *centre* $\vec{x}$ and *radius* $r$ is

$$\{y \in \mathbb{R}^k : \|y - x\| < r\} \ .$$

**Proposition 13.7.** *Every open ball in $\mathbb{R}^k$ is convex.*

PROOF:
⟨1⟩1. LET: $B$ be the open ball with center $\vec{a}$ and radius $r$.
⟨1⟩2. LET: $\vec{x}, \vec{y} \in B$
⟨1⟩3. LET: $\lambda \in (0,1)$
⟨1⟩4. $\lambda\vec{x} + (1-\lambda)\vec{y} \in B$
PROOF:

$$\begin{aligned}\|\lambda\vec{x} + (1-\lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1-\lambda)(\vec{y} - \vec{a})\| \\ &= \lambda\|\vec{x} - \vec{a}\| + (1-\lambda)\|\vec{y} - \vec{a}\| \\ &< \lambda r + (1-\lambda) r \\ &= r\end{aligned}$$

□

□

## 13.2 Limit Points

**Definition 13.8** (Limit Point)**.** Let $X$ be a metric space. Let $E \subseteq X$ and $p \in X$. Then $p$ is a *limit point* of $E$ iff every open ball with centre $p$ contains a point of $E$ other than $p$.

**Proposition 13.9.** *Let $X$ be a metric space. Let $E \subseteq X$. Let $p$ be a limit point of $E$. Then every neighbourhood of $p$ contains infinitely many points of $E$.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $N$ is a neighbourhood of $p$ that contains only finitely many points $q_1, \ldots, q_n$ of $E - \{p\}$.
⟨1⟩2. LET: $r = \min(q_1, \ldots, q_n)$
⟨1⟩3. LET: $B$ be the open ball with centre $p$ and radius $r$.
⟨1⟩4. $B$ is a neighbourhood of $p$ that contains no points of $E$ other than $p$.
□

**Corollary 13.9.1.** *A finite set has no limit points.*

**Definition 13.10** (Isolated Point)**.** Let $X$ be a metric space. Let $E \subseteq X$ and $p \in X$. Then $p$ is an *isolated point* of $E$ iff $p \in E$ and $p$ is not a limit point of $E$.

## 13.3 Closed Sets

**Definition 13.11** (Closed Set)**.** Let $X$ be a metric space. Let $E \subseteq X$. Then $E$ is *closed* iff every limit point of $E$ is a member of $E$.

## 13.4 Interior Points

**Definition 13.12** (Interior Point)**.** Let $X$ be a metric space. Let $E \subseteq X$ and $p \in X$. Then $p$ is an *interior point* of $E$ iff there exists an open ball $B$ with centre $p$ such that $B \subseteq E$.

**Definition 13.13** (Interior)**.** The *interior* of a set $E$, denoted $E^\circ$, is the set of all its interior points.

**Proposition 13.14.** *The interior of $E$ is the largest open set that is included in $E$.*

PROOF:
⟨1⟩1. LET: $I$ be the interior of $E$.
⟨1⟩2. $I$ is open.
 ⟨2⟩1. LET: $p \in I$
 ⟨2⟩2. PICK an open ball $B$ with centre $p$ such that $B \subseteq E$.
 ⟨2⟩3. $B \subseteq I$
  ⟨3⟩1. LET: $q \in B$
  ⟨3⟩2. There exists an open ball $B'$ with centre $q$ such that $B' \subseteq B$.
  ⟨3⟩3. There exists an open ball $B'$ with centre $q$ such that $B' \subseteq E$.
  ⟨3⟩4. $q \in I$
⟨1⟩3. If $J$ is any open set and $J \subseteq E$ then $J \subseteq I$.
 ⟨2⟩1. LET: $J$ be an open set.
 ⟨2⟩2. ASSUME: $J \subseteq E$
 ⟨2⟩3. For all $p \in J$, there exists an open ball $B$ with centre $p$ such that $B \subseteq J$.
 ⟨2⟩4. For all $p \in J$, there exists an open ball $B$ with centre $p$ such that $B \subseteq E$.
 ⟨2⟩5. $p \in I$
□

## 13.5 Open Sets

**Definition 13.15** (Open Sets)**.** Let $X$ be a metric space. Let $E \subseteq X$. Then $E$ is *open* iff every point in $E$ is an interior point of $E$.

**Proposition 13.16.** *Every open ball is open.*

PROOF:
⟨1⟩1. LET: $B$ be an open ball with centre $c$ and radius $r$.
⟨1⟩2. LET: $x \in B$
⟨1⟩3. LET: $\epsilon = r - d(x, c)$
⟨1⟩4. LET: $B'$ be the open ball with centre $x$ and radius $\epsilon$.
 PROVE: $B' \subseteq B$
⟨1⟩5. LET: $y \in B'$
⟨1⟩6. $d(y, c) < r$
 PROOF:

$$\begin{aligned} d(y,c) &\leq d(y,x) + d(x,c) && \text{(Triangle Inequality)} \\ &< \epsilon + d(x,c) && (\langle 1\rangle 5) \\ &= r && (\langle 1\rangle 3) \end{aligned}$$

□

**Proposition 13.17.** *A set is open if and only if its complement is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: $E \subseteq X$
$\langle 1 \rangle 2$. If $E$ is open then $X - E$ is closed.
  $\langle 2 \rangle 1$. ASSUME: $E$ is open.
  $\langle 2 \rangle 2$. LET: $p$ be a limit point of $X - E$.
      PROVE: $p \in X - E$
  $\langle 2 \rangle 3$. ASSUME: for a contradiction $p \in E$.
  $\langle 2 \rangle 4$. PICK an open ball $B$ with centre $p$ such that $B \subseteq E$.
  $\langle 2 \rangle 5$. $B$ contains a point of $X - E$.
    PROOF: $\langle 2 \rangle 2$
  $\langle 2 \rangle 6$. Q.E.D.
    PROOF: This contradicts $\langle 2 \rangle 4$.
$\langle 1 \rangle 3$. If $X - E$ is closed then $E$ is open.
  $\langle 2 \rangle 1$. ASSUME: $X - E$ is closed.
  $\langle 2 \rangle 2$. LET: $p \in E$
  $\langle 2 \rangle 3$. ASSUME: for a contradiction no open ball with centre $p$ is a subset of $E$.
  $\langle 2 \rangle 4$. Every open ball with centre $p$ intersects $X - E$.
  $\langle 2 \rangle 5$. $p$ is a limit point of $X - E$.
  $\langle 2 \rangle 6$. $p \in X - E$
    PROOF: $\langle 2 \rangle 1$
  $\langle 2 \rangle 7$. Q.E.D.
    PROOF: This contradicts $\langle 2 \rangle 2$.
$\square$

**Corollary 13.17.1.** *A set is closed if and only if its complement is open.*

**Proposition 13.18.** *The union of a set of open sets is open.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{U}$ be a set of open sets.
$\langle 1 \rangle 2$. LET: $p \in \bigcup \mathcal{U}$
$\langle 1 \rangle 3$. PICK $U \in \mathcal{U}$ such that $p \in U$.
$\langle 1 \rangle 4$. PICK an open ball $B$ with centre $p$ such that $B \subseteq U$.
$\langle 1 \rangle 5$. $B \subseteq \bigcup \mathcal{U}$
$\square$

**Corollary 13.18.1.** *The intersection of a set of closed sets is closed.*

**Proposition 13.19.** *The intersection of two open sets is open.*

PROOF:
$\langle 1 \rangle 1$. LET: $U$ and $V$ be open.
$\langle 1 \rangle 2$. LET: $p \in U \cap V$
$\langle 1 \rangle 3$. PICK open balls $B_1$ and $B_2$ with centre $p$ such that $B_1 \subseteq U$ and $B_2 \subseteq V$.
$\langle 1 \rangle 4$. ASSUME: w.l.o.g. the radius of $B_1$ is $\leq$ the radius of $B_2$.

$\langle 1 \rangle 5.$ $B_1 \subseteq U \cap V$
□

**Corollary 13.19.1.** *The union of two closed sets is closed.*

**Example 13.20.** The intersection of a set of open sets is not necessarily open.

For every positive integer $n$, we have $(-1/n, 1/n)$ is open in $\mathbb{R}$, but $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$ is not open.

**Theorem 13.21.** *Let $X$ be a metric space. Let $Y \subseteq X$ and $E \subseteq Y$. Then $E$ is open in $Y$ if and only if there exists an open subset $G$ of $X$ such that $E = G \cap Y$.*

PROOF:
$\langle 1 \rangle 1.$ If $E$ is open in $Y$ then there exists an open subset $G$ of $X$ such that $E = G \cap Y$.
$\langle 2 \rangle 1.$ ASSUME: $E$ is open in $Y$.
$\langle 2 \rangle 2.$ For $p \in E$, PICK $r_p > 0$ such that the open ball in $Y$ with centre $p$ and radius $r_p$ is included in $E$.
$\langle 2 \rangle 3.$ For $p \in E$,
LET: $V_p$ be the open ball in $X$ with centre $p$ and radius $r_p$.
$\langle 2 \rangle 4.$ LET: $G = \bigcup_{p \in E} V_p$
$\langle 2 \rangle 5.$ $G$ is open in $Y$.
PROOF: Proposition 13.18.
$\langle 2 \rangle 6.$ $E = G \cap Y$
$\langle 3 \rangle 1.$ $E \subseteq G \cap Y$
$\langle 4 \rangle 1.$ LET: $p \in E$
$\langle 4 \rangle 2.$ $p \in V_p$
$\langle 4 \rangle 3.$ $p \in G$
$\langle 3 \rangle 2.$ $G \cap Y \subseteq E$
$\langle 4 \rangle 1.$ LET: $x \in G \cap Y$
$\langle 4 \rangle 2.$ PICK $p \in E$ such that $x \in V_p$
$\langle 4 \rangle 3.$ $d(x, p) < r_p$
$\langle 4 \rangle 4.$ $x \in E$
$\langle 1 \rangle 2.$ For any open subset $G$ of $X$, we have $G \cap Y$ is open in $Y$.
$\langle 2 \rangle 1.$ LET: $G$ be an open subset of $X$.
$\langle 2 \rangle 2.$ LET: $p \in G \cap Y$
$\langle 2 \rangle 3.$ PICK $r > 0$ such that the open ball in $X$ with centre $p$ and radius $r$ is included in $G$.
$\langle 2 \rangle 4.$ The open ball in $Y$ with centre $p$ and radius $r$ is included in $G \cap Y$.
□

**Proposition 13.22.** *The set $\Omega$ of all invertible linear transformations is an open set in $\mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^n]$.*

PROOF: For $A \in \Omega$ we have $B(A, 1/\|A^{-1}\|) \subseteq \Omega$ by Proposition 9.8. □

## 13.6 Perfect Sets

**Definition 13.23** (Perfect Set)**.** Let $X$ be a metric space. Let $E \subseteq X$. Then $E$ is *perfect* iff $E$ is closed and every point in $E$ is a limit point of $E$.

## 13.7 Bounded Sets

**Definition 13.24** (Bounded Set)**.** Let $X$ be a metric space. Let $E \subseteq X$. Then $E$ is *bounded* iff there exists a real number $M$ and $q \in X$ such that, for all $p \in E$, we have $d(p, q) < M$.

**Definition 13.25** (Diameter)**.** Let $X$ be a metric space and $E \subseteq X$ be bounded. Then the *diameter* of $E$ is $\sup\{d(x, y) : x, y \in E\}$.

**Proposition 13.26.** *Let $X$ be a metric space. Let $E \subseteq X$ be bounded. Then $\overline{E}$ is bounded and*

$$\operatorname{diam} \overline{E} = \operatorname{diam} E \quad .$$

PROOF:
⟨1⟩1. $\operatorname{diam} E$ is an upper bound for $\{d(x, y) : x, y \in \overline{E}\}$.
  ⟨2⟩1. LET: $x, y \in \overline{E}$
  ⟨2⟩2. For all $\epsilon > 0$ we have $d(x, y) < \operatorname{diam} E + \epsilon$.
    ⟨3⟩1. LET: $\epsilon > 0$
    ⟨3⟩2. PICK $x', y' \in E$ such that $d(x', x) < \epsilon/2$ and $d(y', y) < \epsilon/2$
    ⟨3⟩3. $d(x', y') < \operatorname{diam} E$
    ⟨3⟩4. $d(x, y) < \operatorname{diam} E + \epsilon$
  ⟨2⟩3. $d(x, y) \leq \operatorname{diam} E$
⟨1⟩2. $\operatorname{diam} \overline{E}$ is an upper bound for $\{d(x, y) : x, y \in E\}$.
  PROOF: This follows since $E \subseteq \overline{E}$.
□

## 13.8 Dense Sets

**Definition 13.27** (Dense Set)**.** Let $X$ be a metric space. Let $E \subseteq X$. Then $E$ is *dense* iff every point of $X$ is either a limit point of $E$ or a point of $E$, or both.

## 13.9 Closure

**Definition 13.28** (Closure)**.** Let $X$ be a metric space. Let $E \subseteq X$. Then the *closure* of $E$, denoted $\overline{E}$, is the union of $E$ and the set of limit points of $E$.

**Proposition 13.29.** *$\overline{E}$ is the smallest closed set that includes $E$.*

PROOF:
⟨1⟩1. $\overline{E}$ is closed.
  ⟨2⟩1. LET: $p$ be a limit point of $\overline{E}$.

$\langle 2\rangle 2.$ ASSUME: $p \notin E$
PROVE: $p$ is a limit point of $E$.
$\langle 2\rangle 3.$ LET: $B$ be the open ball with centre $p$ and radius $r$.
PROVE: $B$ intersects $E$.
$\langle 2\rangle 4.$ PICK a point $q \in B \cap \overline{E}$.
$\langle 2\rangle 5.$ PICK an open ball $B'$ with centre $q$ such that $B' \subseteq B$.
$\langle 2\rangle 6.$ PICK a point $r \in E \cap B'$
$\langle 2\rangle 7.$ $r \in E \cap B$
$\langle 1\rangle 2.$ If $C$ is closed and $E \subseteq C$ then $\overline{E} \subseteq C$.
$\langle 2\rangle 1.$ ASSUME: $C$ is closed.
$\langle 2\rangle 2.$ ASSUME: $E \subseteq C$
$\langle 2\rangle 3.$ LET: $p \in \overline{E}$
$\langle 2\rangle 4.$ ASSUME: for a contradiction $p \notin C$
$\langle 2\rangle 5.$ $p$ is a limit point of $C$.
$\langle 3\rangle 1.$ LET: $B$ be an open ball with centre $p$.
$\langle 3\rangle 2.$ $B$ intersects $E$.
$\langle 3\rangle 3.$ $B$ intersects $C$.
$\langle 3\rangle 4.$ $B$ intersects $C$ in a point other than $p$.
PROOF: $\langle 2\rangle 3$
$\langle 2\rangle 6.$ Q.E.D.
PROOF: This contradicts $\langle 2\rangle 1$.
□

**Corollary 13.29.1.** *$E$ is closed if and only if $E = \overline{E}$.*

**Theorem 13.30.** *Let $E$ be a nonempty set of real numbers bounded above. Then $\sup E \in \overline{E}$.*

PROOF:
$\langle 1\rangle 1.$ ASSUME: $\sup E \notin E$
PROVE: $\sup E$ is a limit point of $E$.
$\langle 1\rangle 2.$ LET: $B$ be an open ball with centre $\sup E$ and radius $r$.
$\langle 1\rangle 3.$ There exists $x \in E$ such that $x > \sup E - r$.
$\langle 1\rangle 4.$ $E$ intersects $B$ in a point other than $p$.
□

**Proposition 13.31.**

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
$\langle 1\rangle 1.$ $\overline{A} \cup \overline{B}$ is a closed set that includes $A \cup B$.
$\langle 1\rangle 2.$ If $C$ is a closed set that includes $A \cup B$ then $\overline{A} \cup \overline{B} \subseteq C$.
□

**Example 13.32.** It is not true in general. that $\overline{\bigcup \mathcal{A}} = \bigcup_{A \in \mathcal{A}} \overline{A}$.

In $\mathbb{R}$, let $\mathcal{A} = \{\{1/n\} : n \in \mathbb{Z}^+\}$. Then

$$\overline{\bigcup \mathcal{A}} = \{1/n : n \in \mathbb{Z}^+\} \cup \{0\}$$
$$\bigcup_{A \in \mathcal{A}} \overline{A} = \{1/n : n \in \mathbb{Z}^+\}$$

**Proposition 13.33.**

$$X - E^\circ = \overline{X - E}$$

Proof:

$$\begin{aligned} p \in X - E^\circ &\Leftrightarrow p \notin E^\circ \\ &\Leftrightarrow \forall B \text{ an open ball with centre } p. B \nsubseteq E \\ &\Leftrightarrow \forall B \text{ an open ball with centre } p. B \text{ intersects} X - E \\ &\Leftrightarrow p \in \overline{X - E} \end{aligned}$$

∎

## 13.10 Compact Sets

**Definition 13.34** (Open Cover)**.** Let $X$ be a metric space. Let $E \subseteq X$. An *open cover* of $E$ is a set $\mathcal{U}$ of open sets such that $E \subseteq \bigcup \mathcal{U}$.

**Definition 13.35** (Compact Set)**.** Let $X$ be a metric space. Let $K \subseteq X$. Then $K$ is *compact* iff every open cover of $K$ includes a finite subcover.

**Proposition 13.36.** *Every finite set is compact.*

Proof: Easy. ∎

**Theorem 13.37.** *Let $X$ be a metric space. Let $Y \subseteq X$ and $K \subseteq Y$. Then $K$ is compact in $Y$ if and only if $K$ is compact in $X$.*

Proof:
$\langle 1 \rangle 1.$ If $K$ is compact in $Y$ then $K$ is compact in $X$.
  $\langle 2 \rangle 1.$ Assume: $K$ is compact in $Y$.
  $\langle 2 \rangle 2.$ Let: $\mathcal{U}$ be an open cover of $K$ in $X$.
  $\langle 2 \rangle 3.$ $\{U \cap Y : U \in \mathcal{U}\}$ is an open cover of $K$ in $Y$.
  $\langle 2 \rangle 4.$ Pick a finite subcover $\{U_1 \cap Y, \ldots, U_n \cap Y\}$
  $\langle 2 \rangle 5.$ $\{U_1, \ldots, U_n\}$ is a finite subset of $\mathcal{U}$ that is an open cover of $K$ is $X$.
$\langle 1 \rangle 2.$ If $K$ is compact in $X$ then $K$ is compact in $Y$.
  $\langle 2 \rangle 1.$ Assume: $K$ is compact in $X$.
  $\langle 2 \rangle 2.$ Let: $\mathcal{U}$ be an open cover of $K$ in $Y$.
  $\langle 2 \rangle 3.$ $\{U \text{ open in } X : U \cap Y \in \mathcal{U}\}$ is an open cover of $K$ in $X$.
  $\langle 2 \rangle 4.$ Pick a finite subcover $\{U_1, \ldots, U_n\}$.
  $\langle 2 \rangle 5.$ $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ is a subset of $\mathcal{U}$ that is an open cover of $E$ in $Y$.
∎

**Proposition 13.38.** *Every compact set is closed.*

PROOF:
⟨1⟩1. LET: $E$ be compact.
⟨1⟩2. LET: $p \in X - E$
PROVE: There exists an open ball with centre $p$ that is a subset of $X - E$.
⟨1⟩3. For all $q \in E$, there exist disjoint open balls $B$ with centre $q$ and $B'$ with centre $p$.
⟨1⟩4. The set of open balls $B$ such that there exists a disjoint open ball $B'$ with centre $p$ is an open cover of $E$.
⟨1⟩5. PICK a finite subcover $\{B_1, \ldots, B_n\}$.
⟨1⟩6. For $i = 1, \ldots, n$, PICK an open ball $B'_i$ with centre $p$ such that $B_i \cap B'_i = \emptyset$.
⟨1⟩7. $B'_1 \cap \cdots \cap B'_n$ is an open ball with centre $p$ that is a subset of $X - E$.
□

**Proposition 13.39.** *Every closed subset of a compact set is compact.*

PROOF:
⟨1⟩1. LET: $E$ be compact and $C \subseteq E$ be closed.
⟨1⟩2. LET: $\mathcal{U}$ be an open cover of $C$.
⟨1⟩3. $\mathcal{U} \cup \{X - C\}$ is an open cover of $E$.
⟨1⟩4. PICK a finite subcover $\{U_1, \ldots, U_n\}$ or $\{U_1, \ldots, U_n, X - C\}$.
⟨1⟩5. $\{U_1, \ldots, U_n\}$ covers $C$.
□

**Corollary 13.39.1.** *The intersection of a compact set and a closed set is compact.*

**Proposition 13.40.** *Let $\mathcal{K}$ be a nonempty set of compact sets. If every nonempty finite subset of $\mathcal{K}$ has nonempty intersection, then $\bigcap \mathcal{K}$ is nonempty.*

PROOF:
⟨1⟩1. PICK $K \in \mathcal{K}$
⟨1⟩2. ASSUME: $\bigcap \mathcal{K} = \emptyset$
⟨1⟩3. $\{X - K' : K' \in \mathcal{K}\}$ is an open cover of $K$.
⟨1⟩4. PICK a finite subcover $\{X - K_1, \ldots, X - K_n\}$.
⟨1⟩5. There exists $p \in K \cap K_1 \cap \cdots \cap K_n$
⟨1⟩6. Q.E.D.
PROOF: ⟨1⟩4 and ⟨1⟩5 form a contradiction.
□

**Corollary 13.40.1.** *Let $(K_n)$ be a sequence of nonempty compact sets such that $K_0 \supseteq K_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.*

**Theorem 13.41.** *Let $X$ be a metric space and $E \subseteq X$. Then $E$ is compact if and only if every infinite subset of $E$ has a limit point in $E$.*

PROOF:
⟨1⟩1. If $E$ is compact then every infinite subset of $E$ has a limit point in $E$.
⟨2⟩1. ASSUME: $E$ is compact.
⟨2⟩2. LET: $A \subseteq E$ be infinite.

$\langle 2\rangle 3$. ASSUME: for a contradiction $E$ has no limit point in $K$.
$\langle 2\rangle 4$. For all $p \in K$, there exists an open ball $B$ with centre $p$ such that $B$ does not intersect $E$ outside $p$.
$\langle 2\rangle 5$. The set of open balls that intersect $E$ in at most one point is an open cover for $K$.
$\langle 2\rangle 6$. PICK a finite subcover $B_1, \ldots, B_n$.
$\langle 2\rangle 7$. $E$ has at most $n$ points.
$\langle 2\rangle 8$. Q.E.D.
PROOF: This contradicts the fact that $E$ is finite.
$\langle 1\rangle 2$. If every infinite subset of $K$ has a limit point in $K$ then $K$ is compact.
$\langle 2\rangle 1$. ASSUME: Every infinite subset of $K$ has a limit point in $K$.
$\langle 2\rangle 2$. LET: $\mathcal{U}$ be an open cover of $K$.
$\langle 2\rangle 3$. ASSUME: w.l.o.g. $\mathcal{U}$ is countable.
PROOF: We may replace $\mathcal{U}$ with the set of all open balls $B$ with centres in $\mathbb{Q}^2$ and rational radius such that there exists $U \in \mathcal{U}$ such that $B \subseteq U$.
$\langle 2\rangle 4$. PICK an enumeration $\mathcal{U} = \{G_n : n \in \mathbb{N}\}$.
$\langle 2\rangle 5$. For $n \in \mathbb{N}$,
LET: $F_n = \bigcup_{i=0}^{n} G_n$.
$\langle 2\rangle 6$. For all $n \in \mathbb{N}$, we have $K - F_n \neq \emptyset$.
PROOF: Since $\{G_0, \ldots, G_n\}$ does not cover $K$.
$\langle 2\rangle 7$. $\bigcap_{n=0}^{\infty} F_n = \emptyset$
PROOF: Since $\{G_n : n \in \mathbb{N}\}$ covers $K$.
$\langle 2\rangle 8$. For $n \in \mathbb{N}$, PICK $a_n \in K - F_n$
$\langle 2\rangle 9$. LET: $E = \{a_n : n \in \mathbb{N}\}$
$\langle 2\rangle 10$. $E$ is infinite.
$\langle 3\rangle 1$. LET: $n \in \mathbb{N}$
PROVE: there exists $m$ such that $a_m \notin \{a_0, a_1, \ldots, a_n\}$.
$\langle 3\rangle 2$. For $i = 0, \ldots, n$, PICK $k_i$ such that $a_i \in G_{k_i}$.
$\langle 3\rangle 3$. LET: $m = \max(k_0, \ldots, k_n)$
$\langle 3\rangle 4$. ASSUME: for a contradiction $a_m = a_i$ for some $i = 0, \ldots, n$
$\langle 3\rangle 5$. $a_i \in G_{k_i}$
$\langle 3\rangle 6$. $a_i \notin F_m$
$\langle 3\rangle 7$. Q.E.D.
PROOF: This is a contradiction since $k_i \leq m$.
$\langle 2\rangle 11$. PICK a limit point $l$ for $E$ in $K$.
PROOF: From $\langle 2\rangle 1$.
$\langle 2\rangle 12$. PICK $n$ such that $l \in G_n$.
$\langle 2\rangle 13$. PICK an open ball $B$ with centre $l$ such that $B \subseteq G_n$
$\langle 2\rangle 14$. $B \cap E$ is infinite.
PROOF: Proposition 13.9.
$\langle 2\rangle 15$. PICK $m \geq$ ‘$n$ such that $a_m \in B$.
$\langle 2\rangle 16$. $a_m \in G_n$
$\langle 2\rangle 17$. Q.E.D.
PROOF: This is a contradiction since $a_m \notin F_m$.
□

**Theorem 13.42** (Heine-Borel). *Let $E \subseteq \mathbb{R}^k$. Then $E$ is compact if and only if it is closed and bounded.*

PROOF:
$\langle 1 \rangle$1. If $E$ is compact then $E$ is closed.
 PROOF: Proposition 13.38.
$\langle 1 \rangle$2. If $E$ is compact then $E$ is bounded.
 PROOF: Otherwise $\{(-N,N)^k : N \in \mathbb{Z}^+\}$ would be an open cover of $E$ with no finite subcover.
$\langle 1 \rangle$3. If $E$ is closed and bounded then $E$ is compact.
 $\langle 2 \rangle$1. ASSUME: $E$ is closed and bounded.
 $\langle 2 \rangle$2. PICK $\vec{c}$ and $M$ such that $\forall \vec{x} \in E . \|\vec{x} - \vec{c}\| < M$.
 $\langle 2 \rangle$3. $E \subseteq \prod_{i=1}^k [c_i - M, c_i + M]$
 $\langle 2 \rangle$4. $E$ is compact.
  PROOF: Proposition 13.39.
□

**Corollary 13.42.1** (Weierstrass's Theorem). *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point.*

PROOF: It is a bounded infinite subset of some $k$-cell and therefore has a limit point in that $k$-cell. □

**Example 13.43.** It is not true that, in any metric space, a set is compact if and only if it is closed and bounded.

In $\mathbb{Q}$, the set $\{p \in \mathbb{Q} : 2 < p^2 < 3\}$ is closed and bounded but not compact.

**Theorem 13.44.** *Every nonempty perfect set in $\mathbb{R}^k$ is uncountable.*

PROOF:
$\langle 1 \rangle$1. LET: $P$ be a nonempty perfect set in $\mathbb{R}^k$.
$\langle 1 \rangle$2. $P$ is infinite.
 PROOF: Corollary 13.9.1.
$\langle 1 \rangle$3. ASSUME: for a contradiction $P$ is countable.
$\langle 1 \rangle$4. PICK an enumeration $P = \{x_n : n \in \mathbb{N}\}$.
$\langle 1 \rangle$5. PICK a sequence $(V_n)$ of open balls such that, for all $n$, we have $\overline{V_{n+1}} \subseteq V_n$ and $x_n \notin \overline{V_{n+1}}$ and $V_n \cap P \neq \emptyset$
 $\langle 2 \rangle$1. ASSUME: as induction hypothesis we have picked $V_0, \ldots, V_{n-1}$ that satisfy these conditions.
 $\langle 2 \rangle$2. PICK $p \in P \cap V_n$ such that $p \neq x_n$
  PROOF: We cannot have $P \cap V_n = \{x_n\}$ because then $V_n$ would be a neighbourhood of $x_n$ that only intersects $P$ at $x_n$.
 $\langle 2 \rangle$3. PICK an open ball $B$ with centre $p$ such that $B \subseteq V_n \cap P - \{x_n\}$
 $\langle 2 \rangle$4. LET: $V_{n+1}$ be the open ball with centre $p$ and half the radius of $B$.
 $\langle 2 \rangle$5. $\overline{V_{n+1}} \subseteq V_n$
  PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq V_n$.
 $\langle 2 \rangle$6. $x_n \notin \overline{V_{n+1}}$
  PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq P - \{x_n\}$.

⟨2⟩7. $V_{n+1} \cap P \neq \emptyset$
PROOF: Since $p \in V_{n+1} \cap P$.
⟨1⟩6. For $n \in \mathbb{N}$,
LET: $K_n = \overline{V_n} \cap P$.
⟨1⟩7. For all $n \in \mathbb{N}$, $K_n$ is compact.
PROOF: By the Heine-Borel Theorem.
⟨1⟩8. $\bigcap_{n=0}^{\infty} K_n \cap P = \emptyset$
PROOF: Since for each $n$ we have $x_n \notin K_{n+1}$.
⟨1⟩9. $\bigcap_{n=0}^{\infty} K_n = \emptyset$
PROOF: Since $\bigcap_{n=0}^{\infty} K_n \subseteq P$.
⟨1⟩10. Q.E.D.
PROOF: This contradicts Proposition 13.40.
□

**Corollary 13.44.1.** *For any $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a, b]$ is uncountable.*

**Corollary 13.44.2.** *$\mathbb{R}$ is uncountable.*

**Corollary 13.44.3.** *The set of transcendental numbers is uncountable.*

PROOF: Since the set of algebraic numbers is countable. □

**Example 13.45.** The Cantor set is a perfect set in $\mathbb{R}$ that does not include any open interval.

PROOF:
⟨1⟩1. LET: $(E_n)$ be the sequence of unions of closed intervals from the definition of the Cantor set, and $C$ be the Cantor set.
⟨1⟩2. $C \neq \emptyset$
PROOF: Since $0 \in C$.
⟨1⟩3. $C$ is closed.
PROOF: Each $E_n$ is closed and $C$ is their intersection.
⟨1⟩4. Every point of $C$ is a limit point of $C$.
⟨2⟩1. LET: $p \in C$
⟨2⟩2. LET: $B$ be an open ball with centre $p$ and radius $r$.
⟨2⟩3. PICK $n$ such that each of the intervals that make up $E_n$ has length $< r/2$.
⟨2⟩4. LET: $I$ be the interval in $E_n$ that contains $p$.
⟨2⟩5. $I \subseteq B$
⟨2⟩6. The endpoint of $I$ that is not $p$ is in $P \cap B$.
⟨1⟩5. $C$ does not include any open interval.
⟨2⟩1. LET: $(\alpha, \beta)$ be any open interval.
⟨2⟩2. PICK $m$ such that $3^{-m} < (\beta - \alpha)/6$
⟨2⟩3. PICK $k$ such that $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq (\alpha, \beta)$
⟨2⟩4. $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq P$
⟨2⟩5. $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \cap E_m = \emptyset$
⟨2⟩6. Q.E.D.

PROOF: This is a contradiction.

∎

**Corollary 13.45.1.** *The Cantor set is uncountable.*

**Proposition 13.46.** *Let $X$ be a metric space. Let $(K_n)$ be a sequence of compact sets in $X$ such that $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$. Assume* $\operatorname{diam} K_n \to 0$ *as $n \to \infty$. Then $\bigcap_{n=0}^{\infty} K_n$ is a singleton.*

PROOF:
⟨1⟩1. $\bigcap_n K_n \neq \emptyset$
PROOF: Corollary 13.40.1.
⟨1⟩2. $\bigcap_n K_n$ has no more than one point.
⟨2⟩1. ASSUME: for a contradiction $a, b \in \bigcap_n K_n$ with $a \neq b$.
⟨2⟩2. LET: $\epsilon = d(a, b)$
⟨2⟩3. PICK $n$ such that $\operatorname{diam} K_n < \epsilon$
⟨2⟩4. $a, b \in K_n$
⟨2⟩5. Q.E.D.
PROOF: This is a contradiction.

∎

## 13.11 Connected Sets

**Definition 13.47** (Separated)**.** Let $X$ be a metric space. Let $A, B \subseteq X$. Then $A$ and $B$ are *separated* iff $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

**Proposition 13.48.** *Any two disjoint open sets are separated.*

PROOF:
⟨1⟩1. LET: $A$ and $B$ be disjoint open sets.
⟨1⟩2. ASSUME: for a contradiction $p \in \overline{A} \cap B$.
⟨1⟩3. $B$ is a neighbourhood of $p$.
⟨1⟩4. $B$ intersects $A$.

∎

**Definition 13.49** (Connected)**.** Let $X$ be a metric space. Let $E \subseteq X$. Then $E$ is *connected* iff $E$ is not the union of two nonempty separated sets.

**Theorem 13.50.** *A subset $E$ of the real line is connected if and only if it is convex.*

PROOF:
⟨1⟩1. If $E$ is connected then $E$ is convex.
⟨2⟩1. ASSUME: $E$ is connected.
⟨2⟩2. LET: $x, y \in E$
⟨2⟩3. LET: $z \in (x, y)$
⟨2⟩4. $z \in E$
PROOF: Otherwise $E \cap (-\infty, z)$ and $E \cap (z, +\infty)$ would be a separation of $E$.

$\langle 1 \rangle 2$. If $E$ is convex then $E$ is connected.
$\langle 2 \rangle 1$. Assume: $E$ is convex.
$\langle 2 \rangle 2$. Assume: for a contradiction $E = A \cup B$ where $A$ and $B$ are nonempty and separated.
$\langle 2 \rangle 3$. Pick $a \in A$ and $b \in B$.
$\langle 2 \rangle 4$. Assume: w.l.o.g. $a < b$
$\langle 2 \rangle 5$. Let: $z = \sup(A \cap [a, b])$
$\langle 2 \rangle 6$. $z \in \overline{A}$
$\langle 2 \rangle 7$. $z \notin B$
$\langle 2 \rangle 8$. $z < b$
$\langle 2 \rangle 9$. Case: $z \in A$
$\langle 3 \rangle 1$. $z \notin \overline{B}$
$\langle 3 \rangle 2$. Pick $z_1 \in (z, b)$ such that $z_1 \notin B$
$\langle 3 \rangle 3$. $a < z_1 < b$
$\langle 3 \rangle 4$. $z_1 \notin E$
Proof: We have $z_1 \notin A$ from $\langle 2 \rangle 5$ since $z_1 \in [a, b]$ and $z_1 > z$, and $z_1 \notin B$ from $\langle 3 \rangle 2$.
$\langle 3 \rangle 5$. Q.E.D.
Proof: This contradicts $\langle 2 \rangle 1$.
$\langle 2 \rangle 10$. Case: $z \notin A$
Proof: Then $a < z < b$ and $z \notin E$ contradicting $\langle 2 \rangle 1$.
□

**Proposition 13.51.** *Every connected metric space with more than one point is uncountable.*

Proof:
$\langle 1 \rangle 1$. Let: $X$ be a connected metric space with more than one points.
$\langle 1 \rangle 2$. Pick distinct points $p, q \in X$.
$\langle 1 \rangle 3$. Let: $\epsilon = d(p, q)$
$\langle 1 \rangle 4$. For every $r \in (0, \epsilon)$, there exists a point $x \in X$ such that $d(p, x) = r$.
Proof: Otherwise $\{x \in X : d(p, x) < r\}$ and $\{x \in X : d(p, x) > r\}$ would form a separation of $X$.
□

**Proposition 13.52.** *The closure of a connected set is connected.*

Proof:
$\langle 1 \rangle 1$. Let: $X$ be a metric space.
$\langle 1 \rangle 2$. Let: $E$ be a connected subspace of $X$.
$\langle 1 \rangle 3$. Assume: for a contradiction $A$ and $B$ form a separation of $\overline{E}$
Prove: $A \cap E$ and $B \cap E$ form a separation of $E$.
$\langle 1 \rangle 4$. $A \cap E \neq \emptyset$
$\langle 2 \rangle 1$. Assume: for a contradiction $A \cap E = \emptyset$
$\langle 2 \rangle 2$. $E \subseteq B$
$\langle 2 \rangle 3$. $\overline{E} \subseteq \overline{B}$
$\langle 2 \rangle 4$. $A \subseteq \overline{B}$

⟨2⟩5. $A \cap \overline{B} = A \neq \emptyset$
⟨2⟩6. Q.E.D.
PROOF: This contradicts ⟨1⟩3.
⟨1⟩5. $B \cap E \neq \emptyset$
PROOF: Similar.
⟨1⟩6. $\overline{A \cap E} \cap B \cap E = \emptyset$
PROOF: Since $\overline{A \cap E} \cap B \cap E \subseteq \overline{A} \cap B$.
⟨1⟩7. $A \cap E \cap \overline{B \cap E} = \emptyset$
PROOF: Similar.
□

**Example 13.53.** The interior of a connected set is not necessarily connected.

Two touching discs in $\mathbb{R}^2$ form a connected set but the interior is disconnected.

**Proposition 13.54.** *Every convex set in $\mathbb{R}^k$ is connected.*

PROOF:
⟨1⟩1. LET: $E$ be a convex set in $\mathbb{R}^k$.
⟨1⟩2. ASSUME: for a contradiction $A$ and $B$ form a separation of $E$.
⟨1⟩3. PICK $\vec{a} \in A$ and $\vec{b} \in B$.
⟨1⟩4. Define $p : [0, 1] \to \mathbb{R}^k$ by $p(t) = (1 - t)\vec{a} + t\vec{b}$.
⟨1⟩5. $p^{-1}(A)$ and $p^{-1}(B)$ are separated sets in $\mathbb{R}$.
⟨1⟩6. PICK $x \in [0, 1]$ such that $x \notin p^{-1}(A)$ and $x \notin p^{-1}(B)$.
PROOF: There exists such an $x$ since $[0, 1]$ is connected.
⟨1⟩7. $p(x) \in E$
PROOF: Since $E$ is convex.
⟨1⟩8. $p(x) \notin A \cup B$
⟨1⟩9. Q.E.D.
PROOF: This contradicts ⟨1⟩2.
□

## 13.12 Separable Spaces

**Definition 13.55** (Separable)**.** A metric space is *separable* iff it has a countable dense subset.

**Example 13.56.** $\mathbb{R}^k$ is separable since $\mathbb{Q}^k$ is dense.

**Proposition 13.57.** *Every compact metric space is separable.*

PROOF:
⟨1⟩1. LET: $X$ be a compact metric space.
⟨1⟩2. For $n \in \mathbb{Z}^+$, pick finitely many points $a_{n1}, \ldots, a_{nr_n}$ such that $\{B(a_{ni}, 1/n) : 1 \leq i \leq r_n\}$ covers $X$.
PROOF: Since $\{B(x, 1/n) : x \in X\}$ covers $X$.
⟨1⟩3. $\{a_{ni} : n \in \mathbb{Z}^+, 1 \leq i \leq r_n\}$ is dense.

⟨2⟩1. LET: $U$ be an open set and $p \in U$.
⟨2⟩2. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$.
⟨2⟩3. PICK $n$ such that $1/n < \epsilon$.
⟨2⟩4. PICK $i$ such that $p \in B(a_{ni}, 1/n)$
⟨2⟩5. $a_{ni} \in U$
□

## 13.13 Bases

**Definition 13.58** (Basis)**.** A *basis* for a metric space $X$ is a set $\mathcal{B}$ of open sets such that, for every open set $U$ and point $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.

**Proposition 13.59.** *Every separable metric space has a countable basis.*

PROOF:
⟨1⟩1. LET: $X$ be a separable metric space.
⟨1⟩2. PICK a countable dense set $D$ in $X$.
⟨1⟩3. LET: $\mathcal{B} = \{B(p, \epsilon) : p \in D, \epsilon \in \mathbb{Q}^+\}$
PROVE: $\mathcal{B}$ is a basis.
⟨1⟩4. LET: $U$ be an open set in $X$ and $p \in U$
⟨1⟩5. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$
⟨1⟩6. PICK $q \in B(p, \epsilon) \cap D$
⟨1⟩7. PICK a rational $\delta$ such that $d(p, q) < \delta < \epsilon$.
⟨1⟩8. $B(q, \delta) \in \mathcal{B}$ and $B(q, \delta) \subseteq U$.
□

## 13.14 Condensation Points

**Definition 13.60** (Condensation Point)**.** Let $X$ be a metric space, $p \in X$ and $E \subseteq X$. Then $p$ is a *condensation point* of $E$ iff every neighbourhood of $p$ contains uncountably many points in $E$.

**Proposition 13.61.** *Let $X$ be a metric space. Let $E \subseteq X$. Let $P$ be the set of condensation points of $E$. Then $P$ is perfect.*

PROOF:
⟨1⟩1. $P$ is closed.
⟨2⟩1. LET: $p \in X - P$
⟨2⟩2. PICK a neighbourhood $U$ of $p$ that contains only countably many points of $E$.
⟨2⟩3. For every $x \in U$, we have that $U$ is a neighbourhood of $x$ that contains only countably many points of $E$.
⟨2⟩4. $p \in U \subseteq X - P$
⟨1⟩2. Every point in $P$ is a limit point of $P$.
PROOF: Immediate from definitions.

□

**Proposition 13.62.** *Let $X$ be a metric space with a countable basis. Let $E \subseteq X$ be uncountable. Let $P$ be the set of condensation points of $E$. Then $E - P$ is countable.*

PROOF:
⟨1⟩1. PICK a countable basis $\mathcal{B}$ for $X$.
⟨1⟩2. LET: $W = \bigcup \{B \in \mathcal{B} : E \cap B \text{ is countable}\}$
⟨1⟩3. $P = X - W$
  ⟨2⟩1. $P \subseteq X - W$
    ⟨3⟩1. ASSUME: for a contradiction $p \in P \cap W$
    ⟨3⟩2. PICK $B \in \mathcal{B}$ such that $p \in B$ and $E \cap B$ is countable.
    ⟨3⟩3. $E \cap B$ is uncountable.
    ⟨3⟩4. Q.E.D.
      PROOF: This is a contradiction.
  ⟨2⟩2. $X - W \subseteq P$
    ⟨3⟩1. LET: $p \in X - W$
    ⟨3⟩2. LET: $U$ be a neighbourhood of $p$.
    ⟨3⟩3. PICK $B \in \mathcal{B}$ such that $p \in B \subseteq U$.
    ⟨3⟩4. $E \cap B$ is uncountable.
      PROOF: Since $p \notin W$.
    ⟨3⟩5. $E \cap W$ is uncountable.
⟨1⟩4. $E - P = E \cap W$
⟨1⟩5. $E - P$ is countable.
□

**Corollary 13.62.1.** *Every closed subset of a metric space with a countable basis is the union of a perfect set and a countable set.*

PROOF:
⟨1⟩1. LET: $X$ be a metric space with a countable basis.
⟨1⟩2. LET: $E$ be a closed subset of $X$.
⟨1⟩3. LET: $P$ be the set of condensation points of $E$.
⟨1⟩4. $E - P$ is countable.
  PROOF: Proposition 13.62.
⟨1⟩5. $P \cap E$ is perfect.
  ⟨2⟩1. $P \cap E$ is closed.
    PROOF: Proposition 13.61.
  ⟨2⟩2. Every point in $P \cap E$ is a limit point of $P \cap E$.
    ⟨3⟩1. LET: $l \in P \cap E$
    ⟨3⟩2. LET: $U$ be a neighbourhood of $l$.
    ⟨3⟩3. PICK $x \in P \cap U$
    ⟨3⟩4. $U$ is a neighbourhood of $x$.
    ⟨3⟩5. $U$ contains uncountably many points of $E$.
    ⟨3⟩6. $U$ intersects $P \cap E$
      PROOF: It cannot be that every point in $U$ and $E$ is not in $P$ since $E - P$ is countable.

□

**Corollary 13.62.2.** *Let $X$ be a metric space with a countable basis. Then every countable set in $X$ has an isolated point.*

# Chapter 14

# Convergence

**Definition 14.1** (Converge)**.** Let $X$ be a metric space. Let $(p_n)$ be a sequence in $X$ and $l \in X$. Then we say $(p_n)$ *converges* to the *limit* $l$, and write

$$p_n \to l \text{ as } n \to \infty \quad ,$$

iff for every $\epsilon > 0$, there exists an integer $N$ such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon$.

We say $(p_n)$ *diverges* iff it does not converge to any limit.

**Proposition 14.2.** *A sequence has at most one limit.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $p_n \to l$ and $p_n \to m$ as $n \to \infty$.
$\langle 1 \rangle 2$. ASSUME: for a contradiction $l \neq m$.
$\langle 1 \rangle 3$. LET: $\epsilon = d(l, m)/2$
$\langle 1 \rangle 4$. There exists $N$ such that $\forall n \geq N . d(p_n, l) < \epsilon$ and $d(p_n, m) < \epsilon$
$\langle 1 \rangle 5$. $d(l, m) < 2\epsilon$
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This is a contradiction.
□

**Proposition 14.3.** *Every convergent sequence is bounded.*

PROOF:
$\langle 1 \rangle 1$. LET: $p_n \to l$ as $n \to \infty$
$\langle 1 \rangle 2$. PICK $N$ such that $\forall n \geq N . d(p_n, l) < 1$
$\langle 1 \rangle 3$. LET: $M = \max(d(p_0, l), \ldots, d(p_{N-1}, l), 1)$
$\langle 1 \rangle 4$. For all $n$, we have $d(p_n, l) \leq M$.
□

**Proposition 14.4.** *If $l$ is a limit point of $E$, then there exists a sequence in $E$ that converges to $l$.*

PROOF:

⟨1⟩1. For $n \in \mathbb{Z}^+$, Pick a point $a_n \in E$ such that $d(a_n, l) < 1/n$.
Proof: Since $B(l, 1/n)$ intersects $E$.
⟨1⟩2. $a_n \to l$ as $n \to \infty$.
□

**Corollary 14.4.1.** *Every sequence in a compact metric space has a convergent subsequence.*

Proof: By Theorem 13.41. □

**Proposition 14.5.** *Assume $s_n \to s$ and $t_n \to t$ in $\mathbb{R}^k$. Then $s_n + t_n \to s + t$.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. Pick $N$ such that, for all $n \geq N$, we have $\|s_n - s\| < \epsilon/2$ and $\|t_n - t\| < \epsilon/2$.
⟨1⟩3. For all $n \geq N$ we have $\|(s_n + t_n) - (s + t)\| < \epsilon$.
Proof: Since $\|(s_n + t_n) - (s + t)\| \leq \|s_n - s\| + \|t_n - t\|$.
□

**Lemma 14.6.** *If $s_n \to s$ as $n \to \infty$ in $\mathbb{C}$, and $c \in \mathbb{C}$, then $cs_n \to cs$ as $n \to \infty$.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. Assume: w.l.o.g. $c \neq 0$
⟨1⟩3. Pick $N$ such that $\forall n \geq N . |s_n - s| < \epsilon / |c|$.
⟨1⟩4. $\forall n \geq N . |cs_n - cs| < \epsilon$
□

**Proposition 14.7.** *If $s_n \to s$ and $t_n \to t$ in $\mathbb{C}$ then $s_n t_n \to st$.*

Proof:
⟨1⟩1. $(s_n - s)(t_n - t) \to 0$ as $n \to \infty$
⟨2⟩1. Let: $\epsilon > 0$
⟨2⟩2. Pick $N$ such that, for all $n \geq N$, we have $|s_n - s| < \sqrt{\epsilon}$ and $|t_n - t| < \sqrt{\epsilon}$.
⟨2⟩3. For all $n \geq N$ we have $|(s_n - s)(t_n - t)| < \epsilon$
⟨1⟩2. $s_n t_n - st \to 0$ as $n \to \infty$
Proof:

$$\begin{aligned} s_n t_n - st &= (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s) \\ &\to 0 && \text{as } n \to \infty \end{aligned}$$

□

**Proposition 14.8.** *If $s_n \to s$ as $n \to \infty$ in $\mathbb{C}$, and every $s_n$ and $s$ is nonzero, then $1/s_n \to 1/s$ as $n \to \infty$.*

Proof:
⟨1⟩1. Pick $m$ such that, for all $n \geq m$, we have $|s_n - s| < \frac{1}{2}|s|$.
⟨1⟩2. $\forall n \geq m . |s_n| > \frac{1}{2}|s|$
⟨1⟩3. Let: $\epsilon > 0$

$\langle 1 \rangle 4$. PICK $N > m$ such that, for all $n \geq N$, we have

$$|s_n - s| < \frac{1}{2}|s|^2\epsilon \ .$$

$\langle 1 \rangle 5$. For all $n \geq N$, we have

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| < \epsilon \ .$$

PROOF:

$$\begin{aligned}\left|\frac{1}{s_n} - \frac{1}{s}\right| &= \frac{|s_n - s|}{|s_n||s|} \\ &< \frac{|s|^2\epsilon}{2|s_n||s|} \\ &= \frac{|s|\epsilon}{2|s_n|} \\ &< \epsilon\end{aligned}$$

□

**Theorem 14.9.** *Let $(\vec{x_n})$ be a sequence in $\mathbb{R}^k$ and $\vec{l} \in \mathbb{R}^k$. Then $\vec{x_n} \to \vec{l}$ as $n \to \infty$ iff, for $i = 1, \ldots, k$, we have $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. If $\vec{x_n} \to \vec{l}$ then $\pi_i(\vec{x_n}) \to \pi_i(l)$.
  $\langle 2 \rangle 1$. $\|\vec{x_n} - \vec{l}\| \to 0$ as $n \to \infty$.
  $\langle 2 \rangle 2$. $\sqrt{\sum_{i=1}^k (\pi_i(\vec{x_n}) - \pi_i(l))^2} \to 0$ as $n \to \infty$.
  $\langle 2 \rangle 3$. $\sum_{i=1}^k (\pi_i(\vec{x_n}) - \pi_i(l))^2 \to 0$ as $n \to \infty$.
  $\langle 2 \rangle 4$. $(\pi_i(\vec{x_n}) - \pi_i(l))^2 \to 0$ as $n \to \infty$
  $\langle 2 \rangle 5$. $\pi_i(\vec{x_n}) - \pi_i(l) \to 0$ as $n \to \infty$.
$\langle 1 \rangle 2$. If $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ for every $i$ then $\vec{x_n} \to l$.
  $\langle 2 \rangle 1$. ASSUME: $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ for every $i$.
  $\langle 2 \rangle 2$. $\vec{x_n} \to \vec{l}$
    PROOF:

$$\begin{aligned}\|\vec{x_n} - \vec{l}\|^2 &= \sum_{i=1}^k (\pi_i(\vec{x_n}) - \pi_i(\vec{l}))^2 \\ &\to 0\end{aligned}$$

□

**Corollary 14.9.1.** *If $\beta_n \to \beta$ in $\mathbb{R}$ and $\vec{x_n} \to \vec{l}$ in $\mathbb{R}^k$, then $\beta_n\vec{x_n} \to \beta\vec{l}$.*

**Proposition 14.10.** *If $\vec{x_n} \to \vec{x}$ and $\vec{y_n} \to \vec{y}$ in $\mathbb{R}^k$, then $\vec{x_n} \cdot \vec{y_n} \to \vec{x} \cdot \vec{y}$.*

PROOF:

$$\begin{aligned}\vec{x_n} \cdot \vec{y_n} &= \sum_{i=1}^{k} \pi_i(\vec{x_n})\pi_i(\vec{y_n}) \\ &\to \sum_{i=1}^{k} \pi_i(\vec{x})\pi_i(\vec{y}) \\ &= \vec{x} \cdot \vec{y}\end{aligned}$$

□

**Proposition 14.11.** *Let $(p_n)$ be a sequence in the metric space $X$. The set $E^*$ of all limits of convergent subsequences is a closed set.*

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. $\{p_n : n \in \mathbb{N}\}$ is infinite.
⟨1⟩2. LET: $q$ be a limit point of $E^*$.
PROVE: $q \in E^*$
⟨1⟩3. PICK an integer $n_0$ such that $q \neq p_{n_0}$.
⟨1⟩4. Extend a strictly increasing sequence of integers $(n_i)$ such that, for all $i$, we have $d(q, p_{n_i}) \leq 2^i d(q, p_{n_0})$.
⟨2⟩1. ASSUME: as induction hypothesis we have picked $n_0 < n_1 < \cdots < n_i$ such that, for $0 \leq j \leq i$, we have $d(q, p_{n_j}) \leq 2^j d(q, p_{n_0})$.
⟨2⟩2. PICK $x \in E^*$ such that $d(x, q) < 2^{-(i+2)}\delta$
⟨2⟩3. There exists a subsequence of $(p_n)$ that converges to $x$.
⟨2⟩4. There exists $n_{i+1} > n_i$ such that $d(p_{n_{i+1}}, x) < 2^{-(i+2)}\delta$.
⟨2⟩5. $d(p_{n_{i+1}}, q) < 2^{-(i+1)}\delta$
⟨1⟩5. $p_{n_i} \to q$ as $i \to \infty$.
⟨1⟩6. $q \in E^*$
□

**Theorem 14.12.** *Every monotonically increasing sequence in $\mathbb{R}$ that is bounded above converges to its supremum.*

PROOF:
⟨1⟩1. LET: $(s_n)$ be a monotonically increasing sequence with supremum $s$.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK $S$ such that $|s_N - s| < \epsilon$
⟨1⟩4. For all $n \geq N$, we have $s - \epsilon < s - s_N \leq s - s_n \leq s$.
⟨1⟩5. $\forall n \geq N. |s_n - s| < \epsilon$
□

**Theorem 14.13.** *Every monotonically decreasing sequence in $\mathbb{R}$ that is bounded below converges to its infimum.*

PROOF: Similar. □

**Proposition 14.14** (Sandwich Theorem)**.** *Let $(a_n)$, $(b_n)$ and $(c_n)$ be sequences of real numbers and $l \in \mathbb{R}$. Assume $\forall n. a_n \leq b_n \leq c_n$ and $a_n \to l$ and $c_n \to l$. Then $b_n \to l$.*

PROOF:
$\langle 1\rangle 1.$ LET: $\epsilon > 0$
$\langle 1\rangle 2.$ PICK $N$ such that, for all $n \geq N$, we have $|a_n - l| < \epsilon$ and $|c_n - l| < \epsilon$.
$\langle 1\rangle 3.$ $\forall n \geq N . |b_n - l| < \epsilon$
□

**Theorem 14.15.** *For any real $p > 0$ we have*

$$\frac{1}{(n+1)^p} \to 0$$

*as $n \to \infty$.*

PROOF:
$\langle 1\rangle 1.$ LET: $\epsilon > 0$
$\langle 1\rangle 2.$ PICK $N$ such that $N > (1/\epsilon)^{1/p}$.
$\langle 1\rangle 3.$ LET: $n \geq N$
$\langle 1\rangle 4.$ $1/n^p < \epsilon$
□

**Theorem 14.16.** *For any real $p > 0$ we have*

$$p^{\frac{1}{n+1}} \to 1$$

*as $n \to \infty$.*

PROOF:
$\langle 1\rangle 1.$ CASE: $p > 1$
  $\langle 2\rangle 1.$ For $n \in \mathbb{N}$
    LET: $x_n = p^{\frac{1}{n+1}} - 1$.
  $\langle 2\rangle 2.$ $\forall n \in \mathbb{N} . x_n > 0$
  $\langle 2\rangle 3.$ $\forall n \in \mathbb{N}$.

$$1 + (n+1)x_n \leq p \ .$$

  PROOF: Since $1 + (n+1)x_n \leq (1 + x_n)^{n+1}$ by the Binomial Theorem.
  $\langle 2\rangle 4.$ $\forall n \in \mathbb{N}$.

$$0 < x_n \leq \frac{p-1}{n+1} \ .$$

  $\langle 2\rangle 5.$ $x_n \to 0$ as $n \to \infty$.
    PROOF: Sandwich Theorem.
$\langle 1\rangle 2.$ CASE: $p = 1$
  PROOF: Trivial.
$\langle 1\rangle 3.$ CASE: $p < 1$
  PROOF: Then $p^{1/(n+1)} = 1/((1/p)^{1/(n+1)}) \to 1/1 = 1$ by $\langle 1\rangle 1$.
□

**Theorem 14.17.**

$$(n+1)^{1/(n+1)} \to 1 \text{ as } n \to \infty$$

PROOF:

⟨1⟩1. For $n \in \mathbb{N}$,
LET: $x_n = (n+1)^{1/(n+1)} - 1$.
⟨1⟩2. $\forall n \in \mathbb{N}. x_n \geq 0$
⟨1⟩3. $\forall n \in \mathbb{N}$

$$n+1 \geq \frac{n(n+1)}{2} x_n^2 \quad .$$

PROOF: Since $(1+x_n)^{n+1} \geq \frac{n(n+1)}{2} x_n^2$ by the Binomial Theorem.
⟨1⟩4. $\forall n \geq 1$

$$0 \leq x_n \leq \sqrt{\frac{2}{n}}$$

⟨1⟩5. $x_n \to 0$ as $n \to \infty$.
PROOF: Sandwich Theorem.
□

**Theorem 14.18.** *Let $p$ and $\alpha$ be real numbers with $p > 0$. Then*

$$\frac{n^\alpha}{(1+p)^n} \to 0 \text{ as } n \to \infty \quad .$$

PROOF:
⟨1⟩1. PICK a positive integer $k$ such that $k > \alpha$.
PROOF: Archimedean Property.
⟨1⟩2. $\forall n > 2k$

$$(1+p)^n > \frac{n^k p^k}{2^k k!} \quad .$$

$$\begin{aligned} (1+p)^n &> \binom{n}{k} p^k && \text{(Binomial Theorem)} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k!} p^k \\ &> \frac{n^k p^k}{2^k k!} && (n > 2k \text{ so if } n-k < i \leq n \text{ then } i > n/2) \end{aligned}$$

⟨1⟩3. $\forall n > 2k$

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k} \quad .$$

⟨1⟩4. $n^{\alpha-k} \to 0$ as $n \to \infty$
PROOF: Theorem 14.15.
⟨1⟩5. $\frac{n^\alpha}{(1+p)^n} \to 0$ as $n \to \infty$.
PROOF: Sandwich Theorem.
□

**Corollary 14.18.1.** *For any real number $x$ with $|x| < 1$ we have $x^n \to 0$ as $n \to \infty$.*

PROOF: Taking $\alpha = 0$. □

## 14.1 Cauchy Sequences

**Definition 14.19** (Cauchy Sequence)**.** Let $(p_n)$ be a sequence in the metric space $X$. Then $(p_n)$ is a *Cauchy sequence* iff, for every $\epsilon > 0$, there exists $N$ such that, for all $m, n \geq N$, we have $d(p_m, p_n) < \epsilon$.

**Proposition 14.20.** *Let $(p_n)$ be a sequence in the metric space $X$ and let $E_N = \{p_n : n \geq N\}$ for all $N$. Then $(p_n)$ is a Cauchy sequence if and only if $\operatorname{diam} E_N \to 0$ as $N \to \infty$.*

PROOF: Immediate from definitions. □

**Theorem 14.21.** *Every convergent sequence is Cauchy.*

PROOF:
⟨1⟩1. LET: $(p_n)$ be a convergent sequence with limit $l$.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK $N$ such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon/2$
⟨1⟩4. $\forall m, n \geq N. d(p_m, p_n) < \epsilon$
□

## 14.2 Complete Metric Spaces

**Definition 14.22** (Complete Metric Space)**.** A metric space is *complete* iff every Cauchy sequence converges.

**Theorem 14.23.** *Every compact metric space is complete.*

PROOF:
⟨1⟩1. LET: $X$ be a compact metric space.
⟨1⟩2. LET: $(p_n)$ be a Cauchy sequence in $X$.
⟨1⟩3. For $N \in \mathbb{N}$,
LET: $E_N = \{p_n : n \geq N\}$.
⟨1⟩4. $\operatorname{diam} \overline{E_N} \to 0$ as $N \to \infty$.
⟨1⟩5. For all $N$, every $\overline{E_N}$ is compact.
PROOF: Proposition 13.39.
⟨1⟩6. For all $N$ we have $\overline{E_N} \supseteq \overline{E_{N+1}}$.
⟨1⟩7. LET: $l$ be the unique point in $\bigcap_{N=0}^{\infty} \overline{E_N}$
PROVE: $p_n \to l$ as $n \to \infty$.
PROOF: Proposition 13.46.
⟨1⟩8. LET: $\epsilon > 0$
⟨1⟩9. PICK $N_0$ such that $\forall N \geq N_0. \operatorname{diam} \overline{E_N} < \epsilon$.
⟨1⟩10. $\forall q \in E_N . d(l, q) < \epsilon$
⟨1⟩11. $\forall n \geq N . d(l, p_n) < \epsilon$
□

**Corollary 14.23.1.** *Let $X$ be a metric space. If every closed bounded set in $X$ is compact, then $X$ is complete.*

PROOF:
$\langle 1\rangle 1$. LET: $S$ be a Cauchy sequence in $X$.
$\langle 1\rangle 2$. $S$ is bounded.
$\langle 1\rangle 3$. $\overline{S}$ is closed and bounded.
$\langle 1\rangle 4$. $\overline{S}$ is compact.
$\langle 1\rangle 5$. $S$ is a Cauchy sequence in $\overline{S}$.
$\langle 1\rangle 6$. $S$ converges.
□

**Corollary 14.23.2.** *For every natural number $k$, we have $\mathbb{R}^k$ is complete.*

**Corollary 14.23.3.** *Every closed subspace of a complete metric space is complete.*

**Proposition 14.24.** *Let $X$ be a complete metric space. Let $(E_n)$ be a sequence of nonempty closed bounded sets in $X$ with*

$$E_0 \supseteq E_1 \supseteq \cdots$$

*and* $\operatorname{diam} E_n \to 0$ *as $n \to \infty$. Then $\bigcap_{n=0}^{\infty} E_n$ consists of exactly one point.*

PROOF:
$\langle 1\rangle 1$. LET: $K = \bigcap_{n=0}^{\infty} E_n$
$\langle 1\rangle 2$. $K$ has at least one point.
  $\langle 2\rangle 1$. For each $n$, PICK $a_n \in E_n$
  $\langle 2\rangle 2$. $(a_n)$ is Cauchy.
    $\langle 3\rangle 1$. LET: $\epsilon > 0$
    $\langle 3\rangle 2$. PICK $N$ such that $\forall n \geq N. \operatorname{diam} E_n < \epsilon$
    $\langle 3\rangle 3$. $\forall m, n \geq N. d(a_m, a_n) < \epsilon$
  $\langle 2\rangle 3$. LET: $l = \lim_{n\to\infty} a_n$
  $\langle 2\rangle 4$. $l \in K$
    $\langle 3\rangle 1$. LET: $n \in \mathbb{N}$
    $\langle 3\rangle 2$. For all $m \geq n$ we have $a_m \in E_n$
    $\langle 3\rangle 3$. $l \in E_n$
$\langle 1\rangle 3$. $K$ has at most one point.
  $\langle 2\rangle 1$. ASSUME: for a contradiction $a, b \in K$ such that $a \neq b$
  $\langle 2\rangle 2$. PICK $n$ such that $\operatorname{diam} E_n < d(a, b)$
  $\langle 2\rangle 3$. $a, b \in E_n$
  $\langle 2\rangle 4$. Q.E.D.
    PROOF: This is a contradiction.
□

**Theorem 14.25** (Baire's Theorem)**.** *Let $X$ be a complete metric space. Let $(G_n)$ be a sequence of dense open subsets of $X$. Then $\bigcap_{n=0}^{\infty} G_n$ is not empty.*

PROOF:
$\langle 1\rangle 1$. PICK a sequence $(E_n)$ of open balls such that $E_0 \supseteq E_1 \supseteq \cdots$ and $\operatorname{diam} E_n \leq 1/2^n$ and $\overline{E_n} \subseteq G_n$.

⟨2⟩1. ASSUME: as induction hypothesis we have chosen $E_0, \ldots, E_n$ with centres $c_0, \ldots, c_n$.
⟨2⟩2. PICK $x \in E_n \cap G_{n+1}$
⟨2⟩3. PICK $0 < \epsilon \leq 1/2^{n+2}$ such that $B(x, \epsilon) \subseteq E_n \cap G_{n+1}$
⟨2⟩4. LET: $E_{n+1} = B(x, \epsilon/2)$
⟨2⟩5. $E_{n+1} \subseteq E_n$
⟨2⟩6. $\operatorname{diam} E_{n+1} \leq 1/2^{n+1}$
⟨2⟩7. $\overline{E_{n+1}} \subseteq G_{n+1}$
⟨1⟩2. LET: $\bigcap_{n=0}^{\infty} \overline{E_n} = \{p\}$
PROOF: Proposition 14.24.
⟨1⟩3. $p \in \bigcap_{n=0}^{\infty} G_n$
□

## 14.3 Divergent Sequences

**Definition 14.26.** Let $(s_n)$ be a sequence in $\mathbb{R}$. Then we say $s_n$ *diverges to* $+\infty$, and write

$$s_n \to +\infty \text{ as } n \to \infty \quad ,$$

iff for every real number $M$, there exists an integer $N$ such that

$$\forall n \geq N. s_n \geq M \quad .$$

We say $s_n$ *diverges to* $-\infty$, and write

$$s_n \to -\infty \text{ as } n \to \infty \quad ,$$

iff for every real number $M$, there exists an integer $N$ such that

$$\forall n \geq N. s_n \leq M \quad .$$

**Definition 14.27** (Limit Supremum, Limit Infimum)**.** Let $(s_n)$ be a sequence in $\mathbb{R}$. Let $E$ be the set of all $l \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that there exists a subsequence of $(s_n)$ that converges to $l$.

The *limit supremum* of $(s_n)$, denoted

$$\limsup_{n \to \infty} s_n \quad ,$$

is the supremum of $E$ in the extended reals.

The *limit infimum* of $(s_n)$, denoted

$$\liminf_{n \to \infty} s_n \quad ,$$

is the infimum of $E$ in the extended reals.

PROOF: The set $E$ is always nonempty because: if $(s_n)$ is unbounded above then $+\infty \in E$; if it is unbounded below then $-\infty \in E$; and if it is bounded above and below then there is a real number in $E$ by Corollary 14.4.1. □

**Theorem 14.28.** *Let $(s_n)$ be a sequence in $\mathbb{R}$. Then there exists a subsequence of $(s_n)$ that converges or diverges to $\limsup_{n\to\infty} s_n$*

PROOF:
⟨1⟩1. CASE: $\limsup_n s_n = +\infty$
PROOF: $(s_n)$ is unbounded above and so has a subsequence that diverges to $+\infty$.
⟨1⟩2. CASE: $\limsup_n s_n \in \mathbb{R}$
PROOF: Then $\limsup s_n$ is in the set of limits of subsequences of $(s_n)$ by Proposition 14.11.
⟨1⟩3. CASE: $\limsup_n s_n = -\infty$
PROOF: $(s_n)$ is unbounded below and so has a subsequence that diverges to $-\infty$.
□

**Theorem 14.29.** *Let $(s_n)$ be a sequence in $\mathbb{R}$. Then there exists a subsequence of $(s_n)$ that converges or diverges to $\liminf_{n\to\infty} s_n$*

PROOF: Similar. □

**Theorem 14.30.** *Let $(s_n)$ be a sequence in $\mathbb{R}$. If $x > \limsup_n s_n$, then there exists $N$ such that $\forall n \geq N . s_n < x$.*

PROOF: If not, we could choose a subsequence of $(s_n)$ that converges to a value $\geq x$, contradicting the definition of $\limsup_n s_n$. □

**Theorem 14.31.** *Let $(s_n)$ be a sequence in $\mathbb{R}$. If $x < \liminf_n s_n$, then there exists $N$ such that $\forall n \geq N . s_n > x$.*

PROOF: Similar. □

**Theorem 14.32.** *Let $(s_n)$ be a sequence in $\mathbb{R}$. Let $s^*$ be an extended real such that:*

- *There exists a subsequence of $(s_n)$ that converges or diverges to $s^*$.*
- *For any $x > s^*$, there exists $N$ such that $\forall n \geq N . s_n < x$.*

*Then $s^* = \limsup_n s_n$.*

PROOF:
⟨1⟩1. LET: $E$ be the set of subsequential limits of $(s_n)$.
⟨1⟩2. $s^*$ is an upper bound for $E$.
⟨2⟩1. LET: $x \in E$
⟨2⟩2. ASSUME: for a contradiction $x > s^*$.
⟨2⟩3. $s^* \in \mathbb{R}$
⟨2⟩4. LET: $y = x$ if $x \in \mathbb{R}$, or $s^* + 1$ if $x = +\infty$
⟨2⟩5. There exists $N$ such that $\forall n \geq N . s_n < y$.
⟨2⟩6. Q.E.D.
PROOF: This contradicts the fact that some subsequence of $(s_n)$ converges or diverges to $x$.

$\langle 1 \rangle 3$. If $u$ is an upper bound for $E$ then $s^* \leq u$.
$\square$

**Theorem 14.33.** *Let $(s_n)$ be a sequence in $\mathbb{R}$. Let $s^*$ be an extended real such that:*

- *There exists a subsequence of $(s_n)$ that converges or diverges to $s^*$.*
- *For any $x < s^*$, there exists $N$ such that $\forall n \geq N . s_n > x$.*

*Then $s^* = \liminf_n s_n$.*

PROOF: Similar. $\square$

**Proposition 14.34.** *Let $(s_n)$ be a sequence of real numbers and $l \in \mathbb{R}$. Then $(s_n)$ converges to $l$ iff $\limsup_n s_n = \liminf_n s_n = l$.*

PROOF:
$\langle 1 \rangle 1$. If $(s_n)$ converges to $l$ then $\limsup_n s_n = \liminf_n s_n = l$.
  PROOF: If $(s_n)$ converges to $l$ then every subsequence of $(s_n)$ converges to $l$.
$\langle 1 \rangle 2$. If $\limsup_n s_n = \liminf_n s_n = l$ then $(s_n)$ converges to $l$.
  $\langle 2 \rangle 1$. ASSUME: $\limsup_n s_n = \liminf_n s_n = l$
  $\langle 2 \rangle 2$. For all $\epsilon > 0$, there exists $N$ such that $\forall n \geq N . l - \epsilon < s_n < l + \epsilon$.
    PROOF: Theorem 14.32 and 14.33.
  $\langle 2 \rangle 3$. $s_n \to l$ as $n \to \infty$.
$\square$

**Theorem 14.35.** *Let $(s_n)$ and $(t_n)$ be sequences of real numbers and $N \in \mathbb{N}$. Assume $\forall n \geq N . s_n \leq t_n$. Then*

$$\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n \quad .$$

PROOF:
$\langle 1 \rangle 1$. For any subsequence $(t_{n_r})$ of $(t_n)$ that converges or diverges to $\pm\infty$, we have $\liminf_n s_n \leq \lim_r t_{n_r}$
  $\langle 2 \rangle 1$. LET: $(t_{n_r})$ be a subsequence of $(t_n)$ with limit $l$.
  $\langle 2 \rangle 2$. PICK $m$ such that a subsequence of $(s_{n_r})$ has limit $m$.
  $\langle 2 \rangle 3$. $\forall r . s_{n_r} \leq t_{n_r}$
  $\langle 2 \rangle 4$. $m \leq l$
  $\langle 2 \rangle 5$. $\liminf_n s_n \leq l$
$\langle 1 \rangle 2$. $\liminf_n s_n \leq \liminf_n t_n$
$\square$

**Theorem 14.36.** *Let $(s_n)$ and $(t_n)$ be sequences of real numbers and $N \in \mathbb{N}$. Assume $\forall n \geq N . s_n \leq t_n$. Then*

$$\limsup_{n \to \infty} s_n \leq \limsup_{n \to \infty} t_n \quad .$$

PROOF: Similar. $\square$

**Theorem 14.37.** *For any sequence $(c_n)$ of positive real numbers, we have*

$$\limsup_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n} \quad .$$

PROOF:
⟨1⟩1. LET: $\alpha = \limsup_n c_{n+1}/c_n$
⟨1⟩2. ASSUME: w.l.o.g. $\alpha < +\infty$
⟨1⟩3. For all $\beta > \alpha$ we have $\limsup_{n\to\infty} c_n^{1/n} \leq \beta$.
⟨2⟩1. LET: $\beta > \alpha$
⟨2⟩2. PICK $N$ such that, for all $n \geq N$, we have

$$\frac{c_{n+1}}{c_n} \leq \beta \quad .$$

PROOF: Theorem 14.30.
⟨2⟩3. For all $k \geq 0$ we have

$$c_{N+k+1} \leq \beta c_{N+k} \quad .$$

⟨2⟩4. For all $n \geq N$ we have

$$c_n \leq c_N \beta^{-N} \beta^n \quad .$$

PROOF: Induction on $n$.
⟨2⟩5. For all $n \geq N$ we have

$$c_n^{1/n} \leq (c_N \beta^{-N})^{1/n} \beta \quad .$$

⟨2⟩6.

$$\limsup_{n\to\infty} c_n^{1/n} \leq \beta$$

PROOF:

$$\begin{aligned}\limsup_{n\to\infty} c_n^{1/n} &\leq \limsup_{n\to\infty} (c_N \beta^{-N})^{1/n} \beta && \text{(Theorem 14.36)} \\ &= \beta && \text{(Theorem 14.16)}\end{aligned}$$

⟨1⟩4.

$$\limsup_{n\to\infty} c_n^{1/n} \leq \alpha$$

□

**Theorem 14.38.** *For any sequence $(c_n)$ of positive real numbers, we have*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} c_n^{1/n} \quad .$$

PROOF: Similar. □

**Proposition 14.39.** *Let $(a_n)$ and $(b_n)$ be sequences of reals. Assume that it is not the case that one of $\limsup_n a_n$, $\limsup_n b_n$ is $+\infty$ and the other is $-\infty$. Then*

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \quad .$$

## 14.4 Infinite Series

**Definition 14.40.** Let $(a_n)$ be a sequence in $\mathbb{R}^k$ and $s \in \mathbb{R}^k$. We say the *infinite series* $\sum_{n=0}^{\infty} a_n$ *converges* to $s$, and write

$$\sum_{n=0}^{\infty} a_n = s$$

iff

$$\sum_{n=0}^{N} a_n \to s \text{ as } N \to \infty \quad .$$

If $(\sum_{n=0}^{N} a_n)$ diverges, we say the infinite series $\sum_{n=0}^{\infty} a_n$ diverges.

**Theorem 14.41.** *Let $(a_n)$ be a sequence in $\mathbb{R}^k$. Then $\sum_{n=0}^{\infty} a_n$ converges if and only if, for all $\epsilon > 0$, there exists $N$ such that, for all $m, n \geq N$,*

$$\left\| \sum_{i=m}^{n} a_i \right\| \leq \epsilon \quad .$$

Proof: This is what it means for $(\sum_{i=0}^{n} a_i)$ to be a Cauchy sequence. ☐

**Corollary 14.41.1.** *If $\sum_{n=0}^{\infty} a_n$ converges then $a_n \to 0$ as $n \to \infty$.*

**Theorem 14.42.** *A series of nonnegative reals converges if and only if its partial sums form a bounded sequence.*

Proof: Its partial sums are a monotonically increasing sequence, and so converge if and only if they are bounded above. ☐

**Theorem 14.43** (Comparison Test)**.** *Let $(a_n)$ be a sequence in $\mathbb{R}^k$ and $(c_n)$ a sequence of real numbers. If there exists $N$ such that $\forall n \geq N. \|a_n\| \leq c_n$, and if $\sum_n c_n$ converges, then $\sum_n a_n$ converges.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. Pick $N$ such that $\forall n \geq N. \|a_n\| \leq c_n$ and $\forall m, n \geq N. \sum_{k=m}^{n} c_k < \epsilon$.
⟨1⟩3. $\forall m, n \geq N. \|\sum_{k=m}^{n} a_k\| \leq \epsilon$
☐

**Corollary 14.43.1.** *Let $(a_n)$ and $(d_n)$ be sequences of real numbers. If there exists $N$ such that $\forall n \geq N. a_n \geq d_n \geq 0$, and if $\sum_n d_n$ diverges, then $\sum_n a_n$ diverges.*

**Theorem 14.44** (Geometric Series)**.** *For $x$ a real number with $0 \leq x < 1$ we have*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad .$$

PROOF:Since $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x} \to \frac{1}{1-x}$ as $n \to \infty$. □

**Theorem 14.45.** *For $x$ a real number with $x \geq 1$ we have $\sum_{n=0}^{\infty} x^n$ diverges.*

PROOF: If $x = 1$ then $\sum_{n=0}^{N} x^n = N + 1$. If $x > 1$ then $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$. Both of these sequences diverge. □

**Theorem 14.46.** *Let $(a_n)$ be a monotonically decreasing sequence of nonnegative real numbers. Then $\sum_n a_n$ converges if and only if $\sum_n 2^n a_{2^n}$ converges.*

PROOF:
⟨1⟩1. For $N \in \mathbb{N}$,
LET: $s_N = \sum_{n=0}^{N} a_n$.
⟨1⟩2. For $N \in \mathbb{N}$,
LET: $t_N = \sum_{n=0}^{N} 2^n a_{2^n}$.
⟨1⟩3. For natural number $N$ and $k$ with $N < 2^k$ we have $s_N \leq a_0 + t_{k-1}$.
PROOF:

$$\begin{aligned} s_N &\leq \sum_{n=0}^{2^k-1} a_n \\ &= a_0 + \sum_{i=0}^{k-1} \sum_{n=2^i} 2^{i+1} - 1 a_n \\ &\leq a_0 + \sum_{i=0}^{k-1} 2^i a_{2^i} \\ &= a_0 + t_{k-1} \end{aligned}$$

⟨1⟩4. For natural number $N$ and $k$ with $N > 2^k$ we have $t_k < 2s_N$.
PROOF:

$$\begin{aligned} s_N &\geq \sum_{n=1}^{2^k} a_n \\ &\geq \sum_{i=0}^{k} \sum_{n=2^i+1} 2^{i+1} a_n \\ &\geq \sum_{i=0}^{k} 2^i a_{2^{i+1}} \\ &= (1/2) t_k \end{aligned}$$

⟨1⟩5. $(s_N)$ converges if and only if $(t_k)$ converges.
□

**Theorem 14.47.** *If $p$ is a real number with $p > 1$ then $\sum_n 1/n^p$ converges.*

PROOF: Since

$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}$$

which converges since $2^{1-p} < 1$. □

**Theorem 14.48.** *If $p$ is a real number with $p \leq 1$ then $\sum_n 1/n^p$ diverges.*

PROOF: If $p \leq 0$ then $1/n^p$ does not converge to 0.
If $0 < p \leq 1$ we have
$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}$$
which diverges since $2^{1-p} \geq 1$. $\square$

**Theorem 14.49.** *Let $p$ be a real number. The series*
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$
*converges if and only if $p > 1$.*

PROOF:
$$\begin{aligned} 2^k \frac{1}{2^k(\ln 2^k)^p} &= \frac{1}{(k \ln 2)^p} \\ &= \frac{1}{(\ln 2)^p} \cdot \frac{1}{k^p} \end{aligned}$$
and this series converges iff $\sum_k \frac{1}{k^p}$ converges iff $p > 1$. $\square$

**Theorem 14.50** (Root Test)**.** *Let $(a_n)_{n \geq 1}$ be a sequence in $\mathbb{R}^k$. Let $\alpha = \limsup_{n \to \infty} \|a_n\|^{1/n}$.*

1. *If $\alpha < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $\alpha > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.*

PROOF:
$\langle 1 \rangle 1.$ If $\alpha < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.
  $\langle 2 \rangle 1.$ ASSUME: $\alpha < 1$
  $\langle 2 \rangle 2.$ PICK $\beta$ such that $\alpha < \beta < 1$
  $\langle 2 \rangle 3.$ PICK $N$ such that $\forall n \geq N. \|a_n\|^{1/n} < \beta$
    PROOF: Theorem 14.30.
  $\langle 2 \rangle 4.$ $\forall n \geq N. \|a_n\| < \beta^n$
  $\langle 2 \rangle 5.$ $\sum_{n=1}^{\infty} \beta^n$ converges.
    PROOF: Theorem 14.44.
  $\langle 2 \rangle 6.$ $\sum_{n=1}^{\infty} a_n$ converges.
    PROOF: Comparison Test.
$\langle 1 \rangle 2.$ If $\alpha > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.
  $\langle 2 \rangle 1.$ ASSUME: $\alpha > 1$
  $\langle 2 \rangle 2.$ There exists a sequence of positive integers $(n_k)$ such that $\|a_{n_k}\|^{1/n_k} \to \alpha$ as $k \to \infty$.
    PROOF: Theorem 14.28.
  $\langle 2 \rangle 3.$ There are infinitely many $n$ such that $\|a_n\| > 1$.
  $\langle 2 \rangle 4.$ $a_n \nrightarrow 0$ as $n \to \infty$.
  $\langle 2 \rangle 5.$ $\sum_{n=1}^{\infty} a_n$ diverges.

PROOF: Corollary 14.41.1.
∎

**Example 14.51.** If $a_n = 1/n$ then $|a_n|^{1/n} \to 1$ and $\sum_n a_n$ diverges.
If $a_n = 1/n^2$ then $|a_n|^{1/n} \to 1$ and $\sum_n a_n$ converges.

**Theorem 14.52** (Ratio Test)**.** *Let $(a_n)_{n \geq 0}$ be a sequence in $\mathbb{R}^k$.*

1. *If*
$$\limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$$
*then $\sum_{n=0}^{\infty} a_n$ converges.*

2. *If there exists $N$ such that $\forall n \geq N . \frac{\|a_{n+1}\|}{\|a_n\|} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.*

PROOF:
$\langle 1 \rangle 1$. If $\limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$ then $\sum_{n=0}^{\infty} a_n$ converges.
$\langle 2 \rangle 1$. ASSUME: $\limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$
$\langle 2 \rangle 2$. $\limsup_{n \to \infty} \|a_n\|^{1/n} < 1$
PROOF: Theorem 14.37.
$\langle 2 \rangle 3$. $\sum_{n=0}^{\infty} a_n$ converges.
PROOF: Root Test
$\langle 1 \rangle 2$. If there exists $N$ such that $\forall n \geq N . \frac{\|a_{n+1}\|}{\|a_n\|} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.
PROOF: Since $a_n \nrightarrow 0$ as $n \to \infty$.
∎

**Example 14.53.** If $a_n = 1/n$ then $a_{n+1}/a_n \to 1$ and $\sum_n a_n$ diverges.
If $a_n = 1/n^2$ then $a_{n+1}/a_n \to 1$ and $\sum_n a_n$ converges.

## 14.5 The Number $e$

**Lemma 14.54.** *The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.*

PROOF:
$$\begin{aligned} \sum_{n=0}^{N} \frac{1}{n!} &\leq 1 + \sum_{n=1}^{N} \frac{1}{2^{n-1}} \\ &< 3 \end{aligned}$$
∎

**Definition 14.55.** The number $e$ is
$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \enspace .$$

**Theorem 14.56.**
$$\left(1 + \frac{1}{n}\right)^n \to e \text{ as } n \to \infty$$

PROOF:
⟨1⟩1. For $n \in \mathbb{N}$,
LET: $s_n = \sum_{k=0}^{n} \frac{1}{k!}$
⟨1⟩2. For $n \in \mathbb{Z}^+$,
LET: $t_n = \left(1 + \frac{1}{n}\right)^n$
⟨1⟩3. For $n \in \mathbb{Z}^+$ we have

$$t_n = \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \quad .$$

PROOF:

$$\begin{aligned} t_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} && \text{(Binomial Theorem)} \\ &= \sum_{k=0}^{n} \frac{\prod_{i=0}^{k-1}(n-i)}{k!} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=0}^{k-1} \frac{n-i}{n} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \end{aligned}$$

⟨1⟩4. For $n \in \mathbb{Z}^+$ we have $t_n \leq s_n$.
⟨1⟩5. $\limsup_{n\to\infty} t_n \leq e$
⟨1⟩6. For $m, n \in \mathbb{Z}^+$ with $n \geq m$ we have

$$t_n \geq \sum_{k=0}^{m} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \quad .$$

⟨1⟩7. For $m \in \mathbb{Z}^+$ we have

$$\liminf_{n\to\infty} t_n \geq \sum_{k=0}^{m} \frac{1}{k!} \quad .$$

⟨1⟩8. For $m \in \mathbb{Z}^+$ we have

$$s_m \leq \liminf_{n\to\infty} t_n \quad .$$

⟨1⟩9.

$$e \leq \liminf_{n\to\infty} t_n$$

⟨1⟩10. $t_n \to e$ as $n \to \infty$.
PROOF: From ⟨1⟩5 and ⟨1⟩9.
□

**Theorem 14.57.** *$e$ is irrational.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $e = p/q$ where $p$ and $q$ are positive integers.
⟨1⟩2. For $n \in \mathbb{N}$,

LET: $s_n = \sum_{k=0}^{n} \frac{1}{k!}$.
⟨1⟩3. For $n \in \mathbb{Z}^+$ we have

$$0 < e - s_n < \frac{1}{n!n} \quad .$$

PROOF:

$$\begin{aligned} e - s_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &< \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \frac{1}{(n+1)^k} \\ &= \frac{1}{n!n} \end{aligned}$$

⟨1⟩4.

$$0 < q!(e - s_q) < \frac{1}{q}$$

⟨1⟩5. $q!e$ is an integer.
⟨1⟩6. $q!(e - s_q)$ is an integer.
⟨1⟩7. There exists an integer between 0 and 1.
⟨1⟩8. Q.E.D.
PROOF: This is a contradiction.
□

**Theorem 14.58.** *$e$ is transcendental.*

PROOF: See I. M. Niven. *Irrational Numbers* p. 25. □

## 14.6 Power Series

**Definition 14.59** (Power Series)**.** Let $(c_n)$ be a sequence of complex numbers. The *power series* with *coefficients* $(c_n)$ is the function that maps a complex number $z$ to the series

$$\sum_{n=0}^{\infty} c_n z^n \quad .$$

**Definition 14.60** (Radius of Convergence)**.** Let $(c_n)$ be a sequence of complex numbers. Let

$$\begin{aligned} \alpha &:= \limsup_{n \to \infty} |c_n|^{1/n} \\ R &:= \frac{1}{\alpha} \end{aligned}$$

where $R = +\infty$ if $\alpha = 0$ and $R = 0$ if $\alpha = +\infty$. Then $R$ is called the *radius of convergence* of the power series $\sum_n c_n z^n$.

**Theorem 14.61.** *Let $R$ be the radius of convergence of $\sum_n c_n z^n$.*

1. *If $|z| < R$ then $\sum_{n=0}^{\infty} c_n z^n$ converges.*

2. *If $|z| > R$ then $\sum_{n=0}^{\infty} c_n z^n$ diverges.*

PROOF:
⟨1⟩1. For $z \in \mathbb{C}$ and $n \in \mathbb{N}$,
LET: $a_n(z) = c_n z^n$
⟨1⟩2.

$$\limsup_{n\to\infty} |a_n(z)|^{1/n} = |z|/R$$

⟨1⟩3. If $|z| < R$ then $\sum_{n=0}^{\infty} a_n(z)$ converges.
PROOF: Root Test.
⟨1⟩4. If $|z| > R$ then $\sum_{n=0}^{\infty} a_n(z)$ diverges.
PROOF: Root Test.
□

## 14.7 Summation by Parts

**Theorem 14.62.** *Let $(a_n)$, $(b_n)$ be two sequences in $\mathbb{R}^k$. Let*

$$A_n = \sum_{k=0}^{n} a_k \qquad (n \geq -1) \enspace .$$

*Let $p$ and $q$ be integers with $0 \leq p \leq q$. Then*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \enspace .$$

PROOF:

$$\begin{aligned}
\sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \\
&= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\
&= \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p
\end{aligned}$$

□

**Theorem 14.63.** *Let $(a_n)$ be a sequence in $\mathbb{R}^k$ and $(b_n)$ be a sequence of real numbers. Assume that:*

1. *The partial sums $\sum_{n=0}^{N} a_n$ form a bounded sequence.*

2. *$(b_n)$ is monotone decreasing.*

3. *$b_n \to 0$ as $n \to \infty$.*

*Then $\sum_{n=0}^{\infty} a_n b_n$ converges.*

PROOF:

⟨1⟩1. Pick $M$ such that, for all $N$, we have $\| \sum_{n=0}^{N} a_n \| \leq M$.
⟨1⟩2. Let: $\epsilon > 0$
⟨1⟩3. Pick $N$ such that $b_N \leq \epsilon / 2M$.
⟨1⟩4. Let: $N \leq p \leq q$
⟨1⟩5. For any integer $k$,
Let: $A_k = \sum_{n=0}^{k} a_n$.
⟨1⟩6. $\| \sum_{n=p}^{q} a_n b_n \| \leq \epsilon$
Proof:

$$\begin{aligned}
\left\| \sum_{n=p}^{q} a_n b_n \right\| &= \left\| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right\| \quad \text{(Summation by Parts)} \\
&\leq M \left\| \sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q - b_p \right\| \\
&= 2M b_p \\
&\leq 2M b_N \\
&\leq \epsilon
\end{aligned}$$

⟨1⟩7. Q.E.D.
Proof: Cauchy criterion.
□

**Corollary 14.63.1** (Alternating Series)**.** *Let $(c_n)$ be a sequence of real numbers. Assume that*

1. *$(|c_n|)$ is monotone decreasing.*
2. *$c_n \geq 0$ for all odd $n$, and $c_n \leq 0$ for all even $n$.*
3. *$c_n \to 0$ as $n \to \infty$*

*Then $\sum_{n=0}^{\infty} c_n$ converges.*

Proof: Take $a_n = (-1)^{n+1}$ and $b_n = |c_n|$. □

**Theorem 14.64.** *Let $\sum_n c_n z^n$ be a power series with radius of convergence 1. Suppose $(c_n)$ is monotone decreasing with limit 0. Then $\sum_n c_n z^n$ converges at every point on the circle $|z| = 1$ except possibly $z = 1$.*

Proof:
⟨1⟩1. Let: $z$ be a complex number with $|z| = 1$ and $z \neq 1$.
⟨1⟩2. For $n \in \mathbb{N}$,
Let: $a_n = z^n$.
⟨1⟩3. For $n \in \mathbb{N}$,
Let: $b_n = c_n$.
⟨1⟩4. The partial sums $\sum_{n=0}^{N} a_n$ form a bounded sequence.

PROOF:

$$\begin{aligned}\left|\sum_{n=0}^{N} a_n\right| &= \left|\sum_{n=0}^{N} z^n\right| \\ &= \left|\frac{1-z^{N+1}}{1-z}\right| \\ &\leq \frac{2}{|1-z|}\end{aligned}$$

⟨1⟩5. $(b_n)$ is monotone decreasing with limit 0.
⟨1⟩6. Q.E.D.
PROOF: Theorem 14.63.
□

## 14.8 Absolute Convergence

**Definition 14.65** (Absolute Convergence)**.** Let $(a_n)$ be a sequence in $\mathbb{R}^k$. Then the series $\sum_{n=0}^{\infty} a_n$ *converges absolutely* iff $\sum_{n=0}^{\infty} \|a_n\|$ converges.

**Theorem 14.66.** *If $\sum_{n=0}^{\infty} a_n$ converges absolutely then $\sum_{n=0}^{\infty} a_n$ converges.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK $N$ such that, for all $p, q \geq N$, we have

$$\sum_{n=p}^{q} \|a_n\| \leq \epsilon \ .$$

⟨1⟩3. For $p, q \geq N$, we have

$$\left\|\sum_{n=p}^{q} a_n\right\| \leq \epsilon \ .$$

S
⟨1⟩4. Q.E.D.
PROOF: Cauchy criterion.
□

## 14.9 Addition and Multiplication of Series

**Theorem 14.67.** *If $\sum_n a_n = A$ and $\sum_n b_n = B$ then $\sum_n (a_n + b_n) = A + B$.*

PROOF:

$$\begin{aligned}\sum_{n=0}^{N}(a_n + b_n) &= \sum_{n=0}^{N} a_n + \sum_{n=0}^{N} b_n \\ &\to A + B \qquad \text{as } N \to \infty\end{aligned}$$ □

**Theorem 14.68.** *If $\sum_n a_n = A$ then $\sum_n (ca_n) = cA$.*

PROOF:

$$\begin{aligned}\sum_{n=0}^{N} ca_n &= c\sum_{n=0}^{N} a_n \\ &\to cA \qquad\qquad \text{as } N \to \infty\Box\end{aligned}$$

**Definition 14.69** (Cauchy Product)**.** The *(Cauchy) product* of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} \quad .$$

**Theorem 14.70.** *Let $(a_n)$ and $(b_n)$ be sequences of complex numbers. Assume:*

1. *$\sum_{n=0}^{\infty} a_n$ converges absolutely.*
2. *$\sum_{n=0}^{\infty} b_n$ converges.*

*For $n \in \mathbb{N}$, let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) \quad .$$

PROOF:
⟨1⟩1. LET:

$$A = \sum_{n=0}^{\infty} a_n$$

⟨1⟩2. LET:

$$B = \sum_{n=0}^{\infty} b_n$$

⟨1⟩3. For $n \in \mathbb{N}$,
LET:

$$A_n = \sum_{k=0}^{n} a_k \quad .$$

⟨1⟩4. For $n \in \mathbb{N}$,
LET:

$$B_n = \sum_{k=0}^{n} b_k \quad .$$

⟨1⟩5. For $n \in \mathbb{N}$,
LET:

$$C_n = \sum_{k=0}^{n} a_k b_{n-k} \quad .$$

⟨1⟩6. For $n \in \mathbb{N}$,
LET:

$$\beta_n = B_n - B$$

⟨1⟩7. For $n \in \mathbb{N}$,

$$C_n = A_n B + \sum_{k=0}^{n} a_k \beta_{n-k} \quad .$$

⟨1⟩8. For $n \in \mathbb{N}$,
LET:

$$\gamma_n = \sum_{k=0}^{n} a_k \beta_{n-k}$$

⟨1⟩9. $A_n B \to AB$ as $n \to \infty$.
⟨1⟩10. $\gamma_n \to 0$ as $n \to \infty$.
⟨2⟩1. LET: $\alpha = \sum_{n=0}^{\infty} |a_n|$
⟨2⟩2. For all $\epsilon > 0$ we have $\limsup_n |\gamma_n| \leq \epsilon\alpha$.
⟨3⟩1. LET: $\epsilon > 0$
⟨3⟩2. PICK $N$ such that $\forall n \geq N . |\beta_n| \leq \epsilon$.
⟨3⟩3. For all $n \geq N$ we have $|\gamma_n| \leq \left|\sum_{k=0}^{N} \beta_k \alpha_{n-k}\right| + \epsilon\alpha$.
PROOF:

$$\begin{aligned} |\gamma_n| &\leq \left|\sum_{k=0}^{N} \beta_k \alpha_{n-k}\right| + \left|\sum_{k=N+1}^{n} \beta_k a_{n-k}\right| \\ &\leq \left|\sum_{k=0}^{N} \beta_k \alpha_{n-k}\right| + \epsilon\alpha \end{aligned}$$

⟨3⟩4.

$$\limsup_{n\to\infty} |\gamma_n| \leq \epsilon\alpha$$

⟨2⟩3. $\limsup_n \gamma_n = 0$
⟨1⟩11. $C_n \to AB$ as $n \to \infty$.
□

**Theorem 14.71** (Abel)**.** *Let $(a_n)$ and $(b_n)$ be sequences of complex numbers. Let*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

*for all $n$. If the series $\sum_n a_n$, $\sum_n b_n$ and $\sum_n c_n$ all converge, then*

$$\sum_n c_n = \left(\sum_n a_n\right)\left(\sum_n b_n\right) \quad .$$

**Proposition 14.72.** *The Cauchy product of two absolutely convergent series is absolutely convergent.*

PROOF:
⟨1⟩1. LET: $\sum_n a_n$ and $\sum_n b_n$ be two absolutely convergent series.
⟨1⟩2. LET: $c_n = \sum_{k=0}^{n} a_k b_{n-k}$
⟨1⟩3. $\sum_n |c_n|$ converges.

PROOF:

$$\begin{aligned}\sum_{n=0}^{\infty}|c_n| &= \sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} a_k b_{n-k}\right| \\ &\leq \sum_{n=0}^{\infty}\sum_{k=0}^{n}|a_k||b_{n-k}|\end{aligned}$$

which converges by Theorem 14.70.

## 14.10 Rearrangements

**Definition 14.73** (Rearrangement)**.** A *rearrangement* of a sequence $(a_n)$ is a sequence $(a_{\phi(n)})$ for some bijection $\phi : \mathbb{N} \approx \mathbb{N}$.

**Theorem 14.74** (Riemann)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a series that converges but not absolutely. Let $\alpha$ and $\beta$ be extended reals with $\alpha \leq \beta$. Then there exists a rearrangement of $\sum_n a_n$ with partial sums $s'_n$ such that*

$$\limsup_{n\to\infty} s'_n = \alpha, \qquad \liminf_{n\to\infty} s'_n = \beta \ .$$

PROOF:

⟨1⟩1. For $n \in \mathbb{Z}^+$,
LET:

$$p_n = \frac{|a_n| + a_n}{2} \ .$$

⟨1⟩2. For $n \in \mathbb{Z}^+$,
LET:

$$q_n = \frac{|a_n| - a_n}{2} \ .$$

⟨1⟩3. $\forall n \in \mathbb{Z}^+ . p_n - q_n = a_n$
⟨1⟩4. $\forall n \in \mathbb{Z}^+ . p_n + q_n = |a_n|$
⟨1⟩5. $\forall n \in \mathbb{Z}^+ . p_n \geq 0$
⟨1⟩6. $\forall n \in \mathbb{Z}^+ . q_n \geq 0$
⟨1⟩7. $\sum_n p_n$ and $\sum_n q_n$ both diverge.
⟨2⟩1. It is not the case than $\sum_n p_n$ and $\sum_n q_n$ both converge.
PROOF: This would imply that $\sum_n |a_n|$ converges by ⟨1⟩4.
⟨2⟩2. It is not the case that $\sum_n p_n$ converges and $\sum_n q_n$ diverges.
PROOF: This would imply that $\sum_n a_n$ diverges by ⟨1⟩3.
⟨2⟩3. It is not the case that $\sum_n p_n$ diverges and $\sum_n q_n$ converges.
PROOF: This would imply that $\sum_n a_n$ diverges by ⟨1⟩3.
⟨1⟩8. LET: $(P_n)$ be the subsequence of $(a_n)$ consisting of the non-negative terms.
⟨1⟩9. LET: $(Q_n)$ be the subsequence of $(|a_n|)$ consisting only of the terms such that $a_n$ is negative.
⟨1⟩10. $\sum_n P_n$ diverges.
PROOF: It is the series $\sum_n p_n$ with the zero terms removed.
⟨1⟩11. $\sum_n Q_n$ diverges.
PROOF: It is the series $\sum_n q_n$ with the zero terms removed.

⟨1⟩12. PICK sequences of real numbers $(\alpha_n)$, $(\beta_n)$ such that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha_n < \beta_n$ for all $n$, and $\beta_1 > 0$.

⟨1⟩13. PICK strictly increasing sequences of natural numbers $(m_n)_{n \geq 1}$, $(k_n)_{n \geq 1}$ such that, for all $n$,

$$\sum_{i=1}^{n-1} \left( \sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) + \sum_{j=m_{n-1}+1}^{m_n} P_j > \beta_n$$

$$\sum_{i=1}^{n} \left( \sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) < \alpha_n$$

and $m_n$ and $k_n$ are the smallest integers that make these inequalities true.

PROOF: Given the choice of $m_1, \ldots, m_n$ and $k_1, \ldots, k_n$, there must exist such an $m_{n+1}$ by ⟨1⟩10, and then there must exist such a $k_{n+1}$ by ⟨1⟩11.

⟨1⟩14. For $n \in \mathbb{Z}^+$,
LET: $x_n = \sum_{i=1}^{n-1} \left( \sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) + \sum_{j=m_{n-1}+1}^{m_n} P_j$

⟨1⟩15. For $n \in \mathbb{Z}^+$,
LET: $y_n = \sum_{i=1}^{n} \left( \sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right)$

⟨1⟩16. For $n \in \mathbb{Z}^+$ we have

$$|x_n - \beta_n| \leq P_{m_n} \quad .$$

PROOF: By minimality of $m_n$.

⟨1⟩17. For $n \in \mathbb{Z}^+$ we have

$$|y_n - \alpha_n| \leq Q_{k_n} \quad .$$

PROOF: By minimality of $k_n$.

⟨1⟩18. $P_n \to 0$ as $n \to \infty$.

PROOF: Since $a_n \to 0$ as $n \to \infty$.

⟨1⟩19. $Q_n \to 0$ as $n \to \infty$.

PROOF: Since $a_n \to 0$ as $n \to \infty$.

⟨1⟩20. $x_n \to \beta$ as $n \to \infty$.

PROOF: ⟨1⟩16, ⟨1⟩18

⟨1⟩21. $y_n \to \alpha$ as $n \to \infty$.

PROOF: ⟨1⟩17, ⟨1⟩19

⟨1⟩22. No number less than $\alpha$ or greater than $\beta$ is a subsequential limit of the partial sums of the series $P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots$

PROOF: Since every partial sum after the $m_n + k_n$ term is between $\alpha_n - Q_{k_n}$ and $\beta_n + P_{m_n}$.

□

**Theorem 14.75.** *If $\sum_n a_n$ is a series in $\mathbb{R}^k$ that converges absolutely to $s$, then every rearrangement of $\sum_n a_n$ converges to $s$.*

PROOF:

⟨1⟩1. LET: $\sum_n a'_n = \sum_n a_{k_n}$ be a rearrangement with partial sums $s'_n$.

⟨1⟩2. LET: $\epsilon > 0$

⟨1⟩3. PICK $N$ such that, for all $m \geq n \geq N$, we have

$$\sum_{i=n}^{m} \|a_i\| \leq \epsilon/3 \ .$$

⟨1⟩4. PICK $p$ such that $\{1, \ldots, N\} \subseteq \{k_1, k_2, \ldots, k_p\}$.
⟨1⟩5. For all $n > p$ we have $\|s_n - s'_n\| \leq \epsilon$.
PROOF:

$$\begin{aligned}
\|s_n - s'_n\| &= \left\| \sum_{i=1}^{N} a_i + \sum_{i=N+1}^{n} a_i - \sum_{i=1}^{p} a_{k_i} - \sum_{i=p+1}^{n} a_{k_i} \right\| \\
&\leq \left\| \sum_{i=N+1}^{n} a_i \right\| + \left\| \sum_{\substack{1 \leq i \leq p \\ k_i > N}} a_{k_i} \right\| + \left\| \sum_{i=p+1}^{n} a_{k_i} \right\| \\
&\leq \epsilon && (\langle 1 \rangle 3)
\end{aligned}$$

⟨1⟩6. $s'_n \to s$ as $n \to \infty$.
□

## 14.11 Completion of a Metric Space

**Definition 14.76** (Completion)**.** Let $X$ be a metric space. Let $X^*$ be the set of all Cauchy sequences in $X$, quotiented by: $(p_n) \sim (q_n)$ iff $d(p_n, q_n) \to 0$. Define the distance function on $X^*$ by:

$$\Delta((p_n), (q_n)) = \lim_{n \to \infty} d(p_n, q_n) \ .$$

Then the metric space $X^*$ is called the *completion* of $X$.

**Theorem 14.77.** *The completion of $X^*$ is a complete metric space, and $X$ is a dense subspace under the embedding that maps $p \in X$ to the constant sequence $(p)$.*

**Example 14.78.** $\mathbb{R}$ is the completion of $\mathbb{Q}$.

# Chapter 15

# Continuity

## 15.1 Limit of a Function

**Definition 15.1** (Limit)**.** Let $X$ and $Y$ be metric spaces. Let $E \subseteq X$ and $f : E \to Y$. Let $p$ be a limit point of $E$ and $q \in Y$. Then we say $q$ is the *limit* of $f$ at $p$, and write

$$f(x) \to q \text{ as } x \to p, \text{ or } \lim_{x \to p} f(x) = q \ ,$$

iff for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in E$, if $0 < d(x,p) < \delta$ then $d(f(x), q) < \epsilon$.

**Theorem 15.2.** *Let $X$ and $Y$ be metric spaces. Let $E \subseteq X$ and $f : E \to Y$. Let $p$ be a limit point of $E$ and $q \in Y$. Then $f(x) \to q$ as $x \to p$ if and only if, for every sequence $(p_n)$ in $E - \{p\}$ with limit $p$, we have $f(p_n) \to q$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. If $f(x) \to q$ as $x \to p$ then, for every sequence $(p_n)$ in $E - \{p\}$ with limit $p$, we have $f(p_n) \to q$ as $n \to \infty$.
  $\langle 2 \rangle 1$. ASSUME: $f(x) \to q$ as $x \to p$.
  $\langle 2 \rangle 2$. LET: $(p_n)$ be a sequence in $E - \{p\}$ with limit $p$.
  $\langle 2 \rangle 3$. LET: $\epsilon > 0$
  $\langle 2 \rangle 4$. PICK $\delta > 0$ such that, for all $x \in E$, if $0 < d(x,p) < \delta$ then $d(f(x), q) < \epsilon$.
  $\langle 2 \rangle 5$. PICK $N$ such that, for all $n \geq N$, we have $d(p_n, p) < \delta$
  $\langle 2 \rangle 6$. $\forall n \geq N. d(f(p_n), q) < \epsilon$
$\langle 1 \rangle 2$. If, for every sequence $(p_n)$ in $E - \{p\}$ with limit $p$, we have $f(p_n) \to q$ as $n \to \infty$, then $f(x) \to q$ as $x \to p$.
  $\langle 2 \rangle 1$. ASSUME: $f(x) \nrightarrow q$ as $x \to p$.
  $\langle 2 \rangle 2$. PICK $\epsilon > 0$ such that, for all $\delta > 0$, there exists a $x \in E$ such that $0 < d(x,p) < \delta$ and $d(f(x), q) \geq \epsilon$.
  $\langle 2 \rangle 3$. For all $n \in \mathbb{Z}^+$, PICK $p_n \in E$ such that $0 < d(p_n, p) < 1/n$ and $d(f(p_n), q) \geq \epsilon$.

⟨2⟩4. $p_n \to p$ as $n \to \infty$.
⟨2⟩5. $f(p_n) \not\to q$ as $n \to \infty$.
□

**Corollary 15.2.1.** *A function has at most one limit at any point.*

**Theorem 15.3.** *Let $X$ be a metric space, $E \subseteq X$, and $p$ a limit point of $E$. Let $f, g : E \to \mathbb{R}^k$. Assume $f(x) \to a$ as $x \to p$ and $g(x) \to b$ as $x \to p$. Then*

$$f(x) + g(x) \to a + b \text{ as } x \to p \ .$$

PROOF:
⟨1⟩1. LET: $(p_n)$ be a sequence in $E$ that converges to $p$.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $g(p_n) \to b$ as $n \to \infty$.
⟨1⟩4. $f(p_n) + g(p_n) \to a + b$ as $n \to \infty$.
PROOF: Proposition 14.5.
⟨1⟩5. Q.E.D.
PROOF: Theorem 15.2.
□

**Theorem 15.4.** *Let $X$ be a metric space, $E \subseteq X$, and $p$ a limit point of $E$. Let $f, g : E \to \mathbb{C}$. Assume $f(x) \to a$ as $x \to p$ and $g(x) \to b$ as $x \to p$. Then*

$$f(x)g(x) \to ab \text{ as } x \to p \ .$$

PROOF:
⟨1⟩1. LET: $(p_n)$ be a sequence in $E$ that converges to $p$.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $g(p_n) \to b$ as $n \to \infty$.
⟨1⟩4. $f(p_n)g(p_n) \to ab$ as $n \to \infty$.
PROOF: Proposition 14.7.
⟨1⟩5. Q.E.D.
PROOF: Theorem 15.2.
□

**Theorem 15.5.** *Let $X$ be a metric space, $E \subseteq X$, and $p$ a limit point of $E$. Let $f : E \to \mathbb{C} - \{0\}$. Assume $f(x) \to a \neq 0$ as $x \to p$. Then*

$$f(x)^{-1} \to a^{-1} \text{ as } x \to p \ .$$

PROOF:
⟨1⟩1. LET: $(p_n)$ be a sequence in $E$ that converges to $p$.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $f(p_n)^{-1} \to a^{-1}$ as $n \to \infty$.
PROOF: Proposition 14.8.
⟨1⟩4. Q.E.D.
PROOF: Theorem 15.2.

∎

**Theorem 15.6.** *Let $X$ be a metric space, $E \subseteq X$, and $p$ a limit point of $E$. Let $f, g : E \to \mathbb{R}^k$. Assume $f(x) \to a$ as $x \to p$ and $g(x) \to b$ as $x \to p$. Then*

$$f(x) \cdot g(x) \to a \cdot b \text{ as } x \to p \ .$$

Proof:
⟨1⟩1. Let: $(p_n)$ be a sequence in $E$ that converges to $p$.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $g(p_n) \to b$ as $n \to \infty$.
⟨1⟩4. $f(p_n) \cdot g(p_n) \to a \cdot b$ as $n \to \infty$.
Proof: Proposition 14.10.
⟨1⟩5. Q.E.D.
Proof: Theorem 15.2.
∎

## 15.2 Continuous Functions

**Definition 15.7** (Continuous)**.** Let $X$ be a metric space, $E \subseteq X$ and $p \in E$. Then $f$ is *continuous* at $p$ iff, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in E$, if $d(x,p) < \delta$ then

$$d(f(x), f(p)) < \epsilon \ .$$

$f$ is *continuous* or *continuous on $E$* iff $f$ is continuous at every point.

**Theorem 15.8.** *Let $X$ be a metric space, $E \subseteq X$ and $p \in E$ be a limit point of $E$. Then $f$ is continuous at $p$ iff $f(x) \to f(p)$ as $x \to p$.*

Proof: Easy. ∎

**Theorem 15.9.** *Let $X$, $Y$ and $Z$ be metric spaces. Let $E \subseteq X$. Let $f : E \to Y$ and $g : f(E) \to Z$. Let $p \in E$. If $f$ is continuous at $p$ and $g$ is continuous at $f(p)$ then $g \circ f$ is continuous at $p$.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. Pick $\delta_1 > 0$ such that, for all $y \in f(E)$, if $d(y, f(p)) < \delta_1$ then $d(g(y), g(f(p))) < \epsilon$.
⟨1⟩3. Pick $\delta_2 > 0$ such that, for all $x \in E$, if $d(x,p) < \delta_2$ then $d(f(x), f(p)) < \delta_1$.
⟨1⟩4. For all $x \in E$, if $d(x,p) < \delta_2$ then $d(g(f(x)), g(f(p))) < \epsilon$.
∎

**Theorem 15.10.** *Let $X$ and $Y$ be metric spaces. Let $f : X \to Y$. Then $f$ is continuous if and only if, for every open set $V \subseteq Y$, we have $f^{-1}(V)$ is open in $X$.*

PROOF:
⟨1⟩1. If $f$ is continuous then, for every open set $V$ in $Y$, we have $f^{-1}(V)$ is open in $X$.
⟨2⟩1. ASSUME: $f$ is continuous.
⟨2⟩2. LET: $V$ be an open set in $Y$.
PROVE: $f^{-1}(V)$ is open.
⟨2⟩3. LET: $x \in f^{-1}(V)$
⟨2⟩4. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$.
⟨2⟩5. PICK $\delta > 0$ such that, for all $x' \in X$, if $d(x', x) < \delta$ then $d(f(x'), f(x)) < \epsilon$.
⟨2⟩6. $B(x, \delta) \subseteq f^{-1}(V)$
⟨1⟩2. If, for every open set $V$ in $Y$, we have $f^{-1}(V)$ is open in $X$, then $f$ is continuous.
⟨2⟩1. ASSUME: For every open set $V$ in $Y$, we have $f^{-1}(V)$ is open in $X$.
⟨2⟩2. LET: $p \in X$
⟨2⟩3. LET: $\epsilon > 0$
⟨2⟩4. $f^{-1}(B(f(p), \epsilon))$ is open in $X$.
⟨2⟩5. PICK $\delta > 0$ such that $B(p, \delta) \subseteq f^{-1}(B(f(p), \epsilon))$.
⟨2⟩6. For all $x \in X$, if $d(x, p) < \delta$ then $d(f(x), f(p)) < \epsilon$.
□

**Corollary 15.10.1.** *Let $X$ and $Y$ be metric spaces. Let $f : X \to Y$. Then $f$ is continuous if and only if, for every closed set $C$ in $Y$, we have $f^{-1}(C)$ is closed in $X$.*

**Theorem 15.11.** *Let $X$ be a metric space. Let $f : X \to \mathbb{R}^k$. Then $f$ is continuous if and only if, for $i = 1, \ldots, k$, we have $\pi_i \circ f$ is continuous.*

PROOF:
⟨1⟩1. Each $\pi_i$ is continuous.
⟨2⟩1. LET: $\vec{p} \in \mathbb{R}^k$
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. LET: $\vec{q} \in \mathbb{R}^k$
⟨2⟩4. ASSUME: $\|\vec{p} - \vec{q}\| < \epsilon$
⟨2⟩5. $|p_i - q_i| < \epsilon$
⟨1⟩2. If, for all $i$, we have $\pi_i \circ f$ is continuous, then $f$ is continuous.
⟨2⟩1. ASSUME: For all $i$, we have $\pi_i \circ f$ is continuous.
⟨2⟩2. LET: $p \in X$
⟨2⟩3. LET: $\epsilon > 0$
⟨2⟩4. For $i = 1, \ldots, k$, PICK $\delta_i > 0$ such that, for all $x \in X$, we have if $d(x, p) < \delta_i$ then $|\pi_i(f(p)) - \pi_i(f(x))| < \epsilon/\sqrt{k}$
⟨2⟩5. LET: $\delta = \min(\delta_1, \ldots, \delta_k)$
⟨2⟩6. LET: $q \in X$ with $d(p, q) < \delta$.
⟨2⟩7. $\|f(p) - f(q)\| < \epsilon$

PROOF:

$$\begin{aligned} \|f(p) - f(q)\| &= \sqrt{\sum_{i=1}^{k} |\pi_i(f(p)) - \pi_i(f(q))|^2} \\ &< \sqrt{\sum_{i=1}^{k} \epsilon^2/k} \\ &= \epsilon \end{aligned}$$

∎

**Theorem 15.12.** *Let $X$ be a compact metric space and $Y$ a metric space. Let $f : X \to Y$ be continuous. Then $f(X)$ is compact.*

PROOF:
⟨1⟩1. LET: $\mathcal{V}$ be an open cover of $f(X)$.
⟨1⟩2. $\{f^{-1}(V) : V \in \mathcal{V}\}$ is an open cover of $X$.
⟨1⟩3. PICK a finite subcover $\{f^{-1}(V_1), \ldots, f^{-1}(V_n)\}$.
⟨1⟩4. $\{V_1, \ldots, V_n\}$ covers $Y$.
∎

**Corollary 15.12.1.** *Every continuous function from a compact metric space to $\mathbb{R}^k$ is bounded.*

**Example 15.13.** If $E \subseteq \mathbb{R}$ is not compact, then there exists a continuous function $E \to \mathbb{R}$ that is not bounded.

PROOF:
⟨1⟩1. CASE: $E$ is bounded.
  ⟨2⟩1. PICK a limit point $x_0$ of $E$ that is not in $E$.
  ⟨2⟩2. Define $f : E \to \mathbb{R}$ by $f(x) = 1/(x - x_0)$.
  ⟨2⟩3. $f$ is continuous and unbounded.
⟨1⟩2. CASE: $E$ is unbounded.
  PROOF: The inclusion $E \hookrightarrow \mathbb{R}$ is continuous and unbounded.
∎

**Theorem 15.14** (Extreme Values Theorem)**.** *Let $X$ be a compact metric space. Let $f : X \to \mathbb{R}$. Let $M = \sup f(X)$ and $m = \inf f(X)$. Then there exist $p, q \in X$ such that $f(p) = M$ and $f(q) \in m$.*

PROOF: Since $f(X)$ is compact and hence closed. ∎

**Example 15.15.** For any $E \subseteq \mathbb{R}$ that is not compact, there exists a continuous and bounded function $E \to \mathbb{R}$ that does not attain its supremum.

PROOF:
⟨1⟩1. CASE: $E$ is bounded.
  ⟨2⟩1. PICK a limit point $x_0$ for $E$ such that $x_0 \notin E$.
  ⟨2⟩2. Define $g : E \to \mathbb{R}$ by $g(x) = 1/(1 + (x - x_0)^2)$.

⟨2⟩3. $g$ is continuous and bounded but does not attain its supremum 1.
⟨1⟩2. CASE: $E$ is unbounded.
PROOF: Then $h(x) = x^2/(1+x^2)$ is continuous and bounded but does not attain its supremum 1.
□

**Theorem 15.16.** *Let $X$ be a compact metric space and $Y$ a metric space. Let $f : X \approx Y$ be a continuous bijection. Then $f^{-1}$ is continuous.*

PROOF:
⟨1⟩1. LET: $V$ be open in $X$.
⟨1⟩2. $X - V$ is compact.
⟨1⟩3. $f(X - V)$ is compact.
⟨1⟩4. $Y - f(V)$ is compact.
⟨1⟩5. $Y - f(V)$ is closed.
⟨1⟩6. $f(V)$ is open.
□

**Example 15.17.** This example shows we cannot remove the hypothesis of compactness of $X$, even if $Y$ is compact.

Let $X = [0, 2\pi)$. Let $f : X \to S^1$ be the function $f(t) = (\cos t, \sin t)$. Then $f$ is a continuous bijection $X \approx S^1$, but the inverse $f^{-1}$ is not continuous.

**Proposition 15.18.** *The continuous image of a connected metric space is connected.*

PROOF:
⟨1⟩1. LET: $X$ be a connected metric space and $Y$ a metric space.
⟨1⟩2. LET: $f : X \twoheadrightarrow Y$ be a continuous surjection.
⟨1⟩3. ASSUME: for a contradiction $A$ and $B$ form a separation of $Y$.
⟨1⟩4. $f^{-1}(A)$ and $f^{-1}(B)$ form a separation of $X$.
⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1.
□

**Corollary 15.18.1** (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < c < f(b)$ or $f(a) > c > f(b)$, then there exists a real number $x \in (a, b)$ such that $f(x) = c$.*

PROOF: Since $f([a, b])$ is connected. □

**Example 15.19.** The converse does not hold. Let $f : [-1, 1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For all $a, b \in [-1, 1]$ with $a < b$, and all $c$ with $f(a) < c < f(b)$, there exists $x \in (a, b)$ such that $f(x) = c$. Nevertheless, $f$ is discontinuous at 0.

**Proposition 15.20.** *Let $\Omega$ be the set of all invertible linear transformations in $\mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^n]$. Then the function that sends $A$ to $A^{-1}$ is a continuous function $\Omega \to \Omega$.*

PROOF:
$\langle 1\rangle 1$. LET: $\epsilon > 0$ and $A \in \Omega$
$\langle 1\rangle 2$. LET: $\alpha = 1/\|A^{-1}\|$
$\langle 1\rangle 3$. LET: $\delta = \alpha^2\epsilon/(1+\alpha\epsilon)$
$\langle 1\rangle 4$. LET: $B \in \Omega$ with $\|B - A\| < \delta$.
$\langle 1\rangle 5$. $\|B^{-1}\| \leq (\alpha - \delta)^{-1}$
$\langle 2\rangle 1$. For all $\vec{y} \in \mathbb{R}^n$ we have $(\alpha - \delta)\|B^{-1}\vec{y}\| \leq \|\vec{y}\|$.
PROOF:

$$\begin{aligned} (\alpha - \delta)\|B^{-1}\vec{y}\| &< (\alpha - \|B - A\|)\|B^{-1}\vec{y}\| && (\langle 1\rangle 4) \\ &\leq \|BB^{-1}\vec{y}\| && (\text{Lemma 9.7}) \\ &= \|\vec{y}\| \end{aligned}$$

$\langle 1\rangle 6$. $\|B^{-1} - A^{-1}\| < \epsilon$
PROOF:

$$\begin{aligned} \|B^{-1} - A^{-1}\| &\leq \|B^{-1}\|\|B - A\|\|A^{-1}\| && (\text{since } B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}) \\ &< \frac{\delta}{\alpha(\alpha - \delta)} && (\langle 1\rangle 2,\ \langle 1\rangle 4,\ \langle 1\rangle 5) \\ &= \epsilon && (\langle 1\rangle 3) \end{aligned}$$

$\square$

## 15.3 Limits from the Left and the Right

**Definition 15.21** (Limit from the Left)**.** Let $f : (a, b) \to \mathbb{R}$. Let $c \in (a, b]$ and $q \in \mathbb{R}$. Then we say $q$ is the *limit* as $f$ approaches $c$ *from the left*, and write

$$f(x) \to q \text{ as } x \to c-$$

or

$$\lim_{x \to c-} f(x) = q$$

iff, for every sequence $(t_n)$ in $(a, c)$ such that $t_n \to c$ as $n \to \infty$, we have $f(t_n) \to q$ as $n \to \infty$.

**Definition 15.22** (Limit from the Right)**.** Let $f : (a, b) \to \mathbb{R}$. Let $c \in [a, b)$ and $q \in \mathbb{R}$. Then we say $q$ is the *limit* as $f$ approaches $c$ *from the right*, and write

$$f(x) \to q \text{ as } x \to c+$$

or

$$\lim_{x \to c+} f(x) = q$$

iff, for every sequence $(t_n)$ in $(c, b)$ such that $t_n \to c$ as $n \to \infty$, we have $f(t_n) \to q$ as $n \to \infty$.

**Proposition 15.23.** *Let $f : (a, b) \to \mathbb{R}$. Let $c \in (a, b)$ and $q \in \mathbb{R}$. Then $f(x) \to q$ as $x \to c$ iff $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$.*

PROOF:
⟨1⟩1. If $f(x) \to q$ as $x \to c$ then $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$.
  PROOF: Theorem 15.2.
⟨1⟩2. If $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$ then $f(x) \to q$ as $x \to c$.
  ⟨2⟩1. ASSUME: $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$.
  ⟨2⟩2. ASSUME: for a contradiction $f(x) \nrightarrow q$ as $x \to c$.
  ⟨2⟩3. PICK a sequence $(p_n)$ such that $p_n \to c$ as $n \to \infty$, $f(p_n) \nrightarrow q$ as $n \to \infty$, and $p_n \neq c$ for all $n$.
  ⟨2⟩4. CASE: There are only finitely many $n$ such that $p_n > c$.
    ⟨3⟩1. LET: $(q_n)$ be the subsequence of $(p_n)$ consisting of all the terms such that $p_n < c$.
    ⟨3⟩2. $q_n \to c$ as $n \to \infty$.
    ⟨3⟩3. $f(q_n) \nrightarrow q$ as $n \to \infty$.
    ⟨3⟩4. Q.E.D.
      PROOF: This contradicts ⟨2⟩1.
  ⟨2⟩5. CASE: There are only finitely many $n$ such that $p_n < c$.
    PROOF: Similar.
  ⟨2⟩6. CASE: There are infinitely many $n$ such that $p_n > c$ and infinitely many $n$ such that $p_n < c$.
    ⟨3⟩1. LET: $(q_n)$ the subsequence of $(p_n)$ consisting of all the terms such that $p_n > c$, and $(r_n)$ the subsequence consisting of all the terms such that $p_n < c$.
    ⟨3⟩2. $q_n \to c$ as $n \to \infty$ and $r_n \to c$ as $n \to \infty$.
    ⟨3⟩3. It is not the case that $f(q_n) \to q$ as $n \to \infty$ and $f(r_n) \to q$ as $n \to \infty$.
      PROOF: If $f(q_n) \to q$ as $n \to \infty$ and $f(r_n) \to q$ as $n \to \infty$ then $f(p_n) \to q$ as $n \to \infty$.
    ⟨3⟩4. Q.E.D.
      PROOF: This contradicts ⟨2⟩1.

□

**Proposition 15.24.** *Let $f : (a, b) \to \mathbb{R}$ be monotonic. Then, for all $c \in (a, b)$ we have $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ both exist, and*

$$\sup_{a<x<c} f(x) = \lim_{x\to c-} f(x) \leq f(c) \leq \lim_{x\to c+} f(x) = \inf_{c<x<b} f(x) \ .$$

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. $f$ is monotonically increasing on $(a, b)$.
⟨1⟩2. $f(x) \to \sup_{a<x<c} f(x)$ as $x \to c-$.
  ⟨2⟩1. LET: $(t_n)$ be a sequence in $(a, c)$ such that $t_n \to c$ as $n \to \infty$.
    PROVE: $f(t_n) \to \sup_{a<x<c} f(x)$ as $n \to \infty$.
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. PICK $x \in (a, c)$ such that $f(x)$
⟨1⟩3. $f(x) \to \inf_{c<x<b} f(x)$ as $x \to c+$.
  PROOF: Similar.

□

## 15.4 Discontinuities

**Definition 15.25** (Simple Discontinuity)**.** Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. We say $f$ has a *simple discontinuity* or *discontinuity of the first kind* at $c$ iff $f$ is discontinuous at $c$ but $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ both exist.

**Definition 15.26** (Discontinuity of the Second Kind)**.** Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. We say $f$ has a *discontinuity of the second kind* at $c$ iff $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ do not both exist.

## 15.5 Uniform Continuity

**Definition 15.27** (Uniformly Continuous)**.** Let $X$ and $Y$ be metric spaces. Let $f : X \to Y$. Then $f$ is *uniformly continuous* iff, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $p, q \in X$, if $d(p, q) < \delta$ then $d(f(p), f(q)) < \epsilon$.

**Theorem 15.28.** *Let $X$ be a compact metric space and $Y$ a metric space. Let $f : X \to Y$. If $f$ is continuous then $f$ is uniformly continuous.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. For all $p \in X$, Pick $\phi(p) > 0$ such that, for all $q \in X$, if $d(p, q) < \phi(x)$ then $d(f(p), f(q)) < \epsilon/2$.
⟨1⟩3. For all $p \in X$,
Let: $J(p) = B(p, \phi(x)/2)$.
⟨1⟩4. $\{J(p) : p \in X\}$ is an open cover of $X$.
⟨1⟩5. Pick a finite subcover $\{J(p_1), \ldots, J(p_n)\}$.
⟨1⟩6. Let: $\delta = \min(\phi(p_1), \ldots, \phi(p_n))/2$
⟨1⟩7. Let: $p, q \in X$ with $d(p, q) < \delta$.
⟨1⟩8. Pick $m$ such that $p \in J(p_m)$.
⟨1⟩9. $d(p, p_m) < \phi(p_m)/2$
⟨1⟩10. $d(q, p_m) < \phi(p_m)$
⟨1⟩11. $d(f(p), f(q)) < \epsilon$
□

**Example 15.29.** Let $E \subseteq \mathbb{R}$ be bounded but not compact. Then there exists a continuous function $E \to \mathbb{R}$ that is not uniformly continuous.

Proof: Pick a limit point $x_0$ for $E$ that is not in $E$. Then the function $f(x) = 1/(x - x_0)$ is continuous but not uniformly continuous. □

**Proposition 15.30.** *Every linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous.*

Proof: Since $\|A\vec{x} - A\vec{y}\| \leq \|A\|\|\vec{x} - \vec{y}\|$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. □

# Part V

# Analysis

# Chapter 16

# Differentiation

**Definition 16.1** (Derivative)**.** Let $E$ be an open set in $\mathbb{R}^n$. Let $f : E \to \mathbb{R}^m$. Let $\vec{x} \in E$. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then we say $A$ is the *derivative* of $f$ at $\vec{x}$, and write $f'(\vec{x}) = A$, iff

$$\frac{\|f(\vec{x}+\vec{h}) - f(\vec{x}) - A(\vec{h})\|}{\|\vec{h}\|} \to 0 \text{ as } \vec{h} \to \vec{0} \ .$$

We say $f$ is *differentiable* at $\vec{x}$ iff $f$ has a derivative at $\vec{x}$.

We say $f$ is *differentiable* on $E$ iff $f$ is differentiable at every point in $E$. In this case, the *differential* or *total derivative* $f'$ is the function that maps a point $\vec{x}$ to the derivative at that point.

**Proposition 16.2.** *A function has at most one derivative at any point.*

PROOF:
⟨1⟩1. ASSUME: $A_1$ and $A_2$ are derivatives of $f$ at $\vec{x}$.
⟨1⟩2. LET: $B = A_1 - A_2$
⟨1⟩3. For all $\vec{h}$ such that $\vec{x} + \vec{h} \in E$ we have

$$\|B(\vec{h})\| \leq \|f(\vec{x}+\vec{h}) - f(\vec{x}) - A_1(\vec{h})\| + \|f(\vec{x}+\vec{h}) - f(\vec{x}) - A_2(\vec{h})\|$$

⟨1⟩4. $\|B(\vec{h})\|/\|\vec{h}\| \to 0$ as $\vec{h} \to \vec{0}$.
⟨1⟩5. For $\vec{h} \neq \vec{0}$ such that $\vec{x} + \vec{h} \in E$ we have

$$\frac{\|B(t\vec{h})\|}{\|t\vec{h}\|} \to 0 \text{ as } t \to 0 \ .$$

⟨1⟩6. For $\vec{h} \neq \vec{0}$ such that $\vec{x} + \vec{h} \in E$ we have

$$\frac{\|B(\vec{h})\|}{\|\vec{h}\|} \to 0 \text{ as } t \to 0 \ .$$

PROOF: Since $B$ is linear.
⟨1⟩7. For $\vec{h} \neq \vec{0}$ such that $\vec{x} + \vec{h} \in E$ we have $B\vec{h} = \vec{0}$
⟨1⟩8. $B = 0$
⟨1⟩9. $A_1 = A_2$
□

**Proposition 16.3.** *A linear transformation is its own derivative.*

PROOF: If $A$ is linear then
$$\frac{\|A(\vec{x}+\vec{h})-A(\vec{x})-A(\vec{h})\|}{\|\vec{h}\|}=0$$ □

**Theorem 16.4** (Chain Rule)**.** *Let $E$ be an open set in $\mathbb{R}^n$ and $U$ an open set in $\mathbb{R}^m$. Let $f:E\to U$ and $g:U\to\mathbb{R}^k$. Let $\vec{x_0}\in E$. If $f$ is differentiable at $\vec{x_0}$ and $g$ is differentiable at $f(\vec{x_0})$, then $g\circ f$ is differentiable at $\vec{x_0}$ and*

$$(g\circ f)'(\vec{x_0})=g'(f(\vec{x_0}))\circ f'(\vec{x_0}) \quad .$$

PROOF:
⟨1⟩1. LET: $\vec{y_0}=f(\vec{x_0})$
⟨1⟩2. LET: $A=f'(\vec{x_0})$
⟨1⟩3. LET: $B=g'(\vec{y_0})$
⟨1⟩4. For $\vec{h}$ such that $\vec{x_0}+\vec{h}\in E$,
LET: $u(\vec{h})=f(\vec{x_0}+\vec{h})-f(\vec{x_0})+A(\vec{h}) \quad .$
⟨1⟩5. For $\vec{k}$ such that $\vec{y_0}+\vec{k}\in U$,
LET: $v(\vec{k})=g(\vec{y_0}+\vec{k})-g(\vec{y_0})+B(\vec{k}) \quad .$
⟨1⟩6. For $\vec{h}$ non-zero such that $\vec{x_0}+\vec{h}\in E$,
LET: $\epsilon(\vec{h})=\|u(\vec{h})\|/\|\vec{h}\| \quad .$
⟨1⟩7. For $\vec{k}$ non-zero such that $\vec{y_0}+\vec{k}\in U$,
LET: $\eta(\vec{k})=\|v(\vec{k})\|/\|\vec{k}\| \quad .$
⟨1⟩8. $\epsilon(\vec{h})\to 0$ as $\vec{h}\to\vec{0}$
PROOF: ⟨1⟩2, ⟨1⟩4, ⟨1⟩6.
⟨1⟩9. $\eta(\vec{k})\to 0$ as $\vec{k}\to\vec{0}$
PROOF: ⟨1⟩3, ⟨1⟩5, ⟨1⟩7
⟨1⟩10. For $\vec{h}$ such that $\vec{x_0}+\vec{h}\in E$,
LET: $k(\vec{h})=f(\vec{x_0}+\vec{h})-f(\vec{x_0}) \quad .$

# Part VI

# More Algebra

# Chapter 17

# Lie Groups

**Definition 17.1** (Lie Group)**.** A *Lie group $G$* is a group $G$ that is also an analytic differentiable manifold such that the group operation and inverse operation are analytic.

A *homomorphism of Lie groups* is a group homomorphism that is an analytic function.

**Lemma 17.2.** *Every bijective Lie group homomorphism is an isomorphism.*

**Definition 17.3** (Unitary Group)**.** The *unitary group $U(n)$* is the Lie group of all $n \times n$ unitary matrices.

**Definition 17.4** (Special Unitary Group)**.** The *special unitary group $SU(n)$* is the Lie group of all $n \times n$ unitary matrices with determinant 1.

**Definition 17.5** (Lie Subgroup)**.** Let $G$ be a Lie group. A *Lie subgroup* of $G$ is a subgroup that is also an analytic submanifold of $G$.

**Example 17.6.** $U(n)$ and $SU(n)$ are Lie subgroups of $GL(n, \mathbb{C})$.