Encyclopaedia of Mathematics and Physics

Robin Adams

Contents

Part I

Set Theory

Chapter 1

Foundations

1.1 The Theory of Semicategories

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B, and call A the *domain* of f and B the *codomain*.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $g \circ f : A \to C$, the *composite* of g and f.

Axiom 1.1 (Associativity)**.** *Given functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \ .$$

1.1.1 Identity Functions

Definition 1.2 (Identity Function)**.** Let A be a set. An *identity function* on A is a function $i : A \to A$ such that:

- For any set B and function $f : B \to A$, we have $i \circ f = f$.
- For any set B and function $f : A \to B$, we have $f \circ i = f$.

Proposition 1.3. *A set has at most one identity function.*

PROOF:
$\langle 1 \rangle$1. LET: A be a set.
$\langle 1 \rangle$2. LET: $i, j : A \to A$ be identity functions.
$\langle 1 \rangle$3. $i = j$
PROOF: If i and j both satisfy the conditions then $i = i \circ j = j$.
□

1.1.2 Monomorphisms and Epimorphisms

Definition 1.4 (Monomorphism)**.** We say a function $f : A \to B$ is a *monomorphism*, and write $f : A \rightarrowtail B$, iff, for any set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

Definition 1.5 (Epimorphism)**.** We say a function $f : A \to B$ is a *epimorphism*, and write $f : A \twoheadrightarrow B$, iff, for any set X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

1.2 The Theory of Categories

1.2.1 Minimalist Presentation

Axiom 1.6 (Identity Functions)**.** *Every set has an identity function.*

1.2.2 Practical Presentation

For any set A, let there be a function $\mathrm{id}_A : A \to A$.

Axiom 1.7 (Left Unit Law)**.** *For any function $f : A \to B$, we have $\mathrm{id}_B \circ f = f$.*

Axiom 1.8 (Right Unit Law)**.** *For any function $f : A \to B$, we have $f \circ \mathrm{id}_A = f$.*

1.2.3 Sections and Retractions

Definition 1.9 (Section, Retraction)**.** Let $r : A \to B$ and $s : B \to A$. We say that r is a *retraction* of s, and s is a *section* of r.

1.2.4 Bijections

Definition 1.10 (Bijection)**.** We say a function $f : A \to B$ is *bijective* or a *bijection*, and write $f : A \approx B$, iff there exists a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

1.2.5 Terminal Set

Definition 1.11 (Terminal Set)**.** A set T is *terminal* iff, for any set X, there is exactly one function $X \to T$.

Axiom 1.12 (Terminal Set)**.** *There exists a terminal set.*

Proposition 1.13. *For any terminal sets T and T', there is a unique bijection $T \approx T'$.*

PROOF:
$\langle 1 \rangle 1$. LET: i be the unique function $T \to T'$.
$\langle 1 \rangle 2$. LET: j be the unique function $T' \to T$.
$\langle 1 \rangle 3$. $i \circ j = \mathrm{id}_{T'}$

PROOF: Since there is only one function $T' \to T'$.

$\langle 1 \rangle 4.$ $j \circ i = \mathrm{id}_T$

PROOF: Since there is only one function $T \to T$.

□

Definition 1.14 (Terminal Set)**.** We denote the terminal set by 1.

Definition 1.15 (Element)**.** An *element* of a set A is a function $1 \to A$. We write $a \in A$ for $a : 1 \to A$. Given $a \in A$ and $f : A \to B$, we write $f(a)$ for $f \circ a$.

Axiom 1.16 (Extensionality)**.** *Let $f, g : A \to B$. Assume that, for all $a \in A$, if $f(a) = g(a)$ then $f = g$.*

Definition 1.17 (Injective)**.** We say a function $f : A \to B$ is *injective* or an *injection*, and we write $f : A \rightarrowtail B$, iff, for any $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

Definition 1.18 (Surjective)**.** We say a function $f : A \to B$ is *surjective* or a *surjection*, and we write $f : A \twoheadrightarrow B$, iff, for any $y \in B$, there exists $x \in A$ such that $f(x) = y$.

1.2.6 Empty Set

Axiom 1.19 (Empty Set)**.** *There exists a set with no elements.*

1.2.7 Products

Definition 1.20 (Product)**.** Let A, B and P be sets, and $\pi_1 : P \to A$, $\pi_2 : P \to B$. Then we say that (P, π_1, π_2) is a *product* of A and B iff, for any set X and functions $f : X \to A$ and $g : X \to B$, there exists a unique function $h : X \to A \times B$ such that

$$\pi_1 \circ h = f, \qquad \pi_2 \circ h = g \ .$$

Axiom 1.21 (Products)**.** *Any two sets have a product.*

Proposition 1.22. *If (P, p_1, p_2) and (Q, q_1, q_2) are products of A and B, then there exists a unique bijection $\phi : P \approx Q$ such that $q_1 \circ \phi = p_1$ and $q_2 \circ \phi = p_2$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $\phi : P \to Q$ be the unique function such that $q_1 \circ \phi = p_1$ and $q_2 \circ \phi = p_2$.

$\langle 1 \rangle 2.$ LET: $\phi^{-1} : Q \to P$ be the unique function such that $p_1 \circ \phi = q_1$ and $p_2 \circ \phi = q_2$.

$\langle 1 \rangle 3.$ $\phi \circ \phi^{-1} = \mathrm{id}_Q$

PROOF: Each is the unique $x : Q \to Q$ such that $q_1 \circ x = q_1$ and $q_2 \circ x = q_2$.

$\langle 1 \rangle 4.$ $\phi^{-1} \circ \phi = \mathrm{id}_P$

PROOF: Each is the unique $x : P \to P$ such that $p_1 \circ x = p_1$ and $p_2 \circ x = p_2$.

□

Definition 1.23. Given sets A and B, we write $A \times B$ for the product of A and B, with projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$. Given functions $f : A \to B$ and $g : A \to C$, we write $\langle f, g \rangle$ for the unique function $A \to B \times C$ such that

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g \ .$$

Definition 1.24. Given $f : A \to B$ and $g : C \to D$, we define $f \times g : A \times C \to B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle \ .$$

1.2.8 Function Sets

Definition 1.25 (Function Set)**.** Let A, B and F be sets, and let $\epsilon : F \times A \to B$. Then we say that F and ϵ form the *function set* from A to B, with ϵ the *evaluation function*, iff, for any set X and function $f : X \times A \to B$, there exists a unique function $g : X \to F$ such that

$$\epsilon \circ (g \times \mathrm{id}_A) = f \ .$$

Axiom 1.26 (Function Sets)**.** *Any two sets have a function set.*

Proposition 1.27. *Let $(F, \epsilon : F \times A \to B)$ and $(G, e : G \times A \to B)$ be function sets from A to B. Then there exists a unique bijection $\phi : F \approx G$ such that $e \circ (\phi \times \mathrm{id}_A) = \epsilon$.*

Definition 1.28. Given sets A and B, we write B^A for the function set from A to B, and $\epsilon : B^A \times A \to B$ for the evaluation function. Given $f : X \times A \to B$, we write λf for the unique function $X \to B^A$ such that

$$\epsilon \circ (\lambda f \times \mathrm{id}_A) = f \ .$$

1.2.9 Inverse Images

Definition 1.29 (Inverse Image)**.** Let A, B and I be sets. Let $f : A \to B$, $b \in B$ and $i : I \to A$. Then we say that I and i form the *inverse image* of b under f iff:

- $f \circ i = b \circ !_I$
- For any set X and function $j : X \to A$, if $f \circ j = b \circ !_X$, then there exists a unique $\overline{j} : X \to I$ such that $i \circ \overline{j} = j$.

Axiom 1.30 (Inverse Images)**.** *Given any sets A and B, function $f : A \to B$, and element $b \in B$, there exists an inverse image of b under f.*

Proposition 1.31. *If $(I, i : I \to A)$ and $(J, j : J \to A)$ are inverse images of $b \in B$ under $f : A \to B$, then there exists a unique isomorphism $\phi : I \approx J$ such that $j \circ \phi = i$.*

Definition 1.32. Let $f : A \to B$ and $b \in B$. We write $f^{-1}(b)$ and $i_{f,b} : f^{-1}(b) \to A$ for the inverse image of b under f.

1.2.10 Subset Classifiers

Definition 1.33 (Subset Classifier)**.** Let Ω be a set and $\top \in \Omega$. Then we say $(\Omega, \top)$ form a *subset classifier* iff, for any sets A and X and injection $j : A \rightarrowtail X$, there exists a unique $\chi : X \to \Omega$ such that (A, j) is the inverse image of $\top$ under χ.

Axiom 1.34 (Subset Classifier)**.** *There exists a subset classifier.*

Proposition 1.35. *If $(\Omega, \top)$ and $(\Omega', \top')$ are subset classifiers, then there exists a unique bijection $\phi : \Omega \approx \Omega'$ such that $\phi(\top) = \top'$.*

Definition 1.36. We write 2 and $\top \in 2$ for the subset classifier.

1.2.11 Natural Number Sets

Definition 1.37 (Natural Number Set)**.** Let N be a set, $z \in N$ and $s : N \to N$. Then we say (N, z, s) is a *natural number set* iff, for any set X, element $a \in X$ and function $f : X \to X$, there exists a unique $r : N \to X$ such that

$$r(z) = a, \qquad f \circ r = r \circ s \ .$$

Axiom 1.38 (Infinity)**.** *There exists a natural number set.*

Proposition 1.39. *If (N, z, s) and (N', z', s') are natural number sets, then there exists a unique bijection $\phi : N \approx N'$ such that $\phi(z) = z'$ and $s' \circ \phi = \phi \circ s$.*

Definition 1.40. We write $\mathbb{N}$, $0 \in \mathbb{N}$ and $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ for the natural number set.

1.2.12 The Axiom of Choice

Definition 1.41 (Axiom of Choice)**.** Every surjection is a retraction.

Chapter 2

Set Theory

Proposition 2.1. *Every infinite subset of a countably infinite set is countable.*

PROOF:
$\langle 1 \rangle 1$. LET: $i : A \hookrightarrow \mathbb{N}$ be an infinite subset of $\mathbb{N}$.
$\langle 1 \rangle 2$. Define $j : \mathbb{N} \to A$ by: $j(k)$ is the element such that $i(j(k))$ is least such that $i(j(k)) \notin \{i(j(0)), \ldots, i(j(k-1))\}$.
$\langle 1 \rangle 3$. j is a bijection.
□

Proposition 2.2. *A countable union of countable sets is countable.*

PROOF:
$\langle 1 \rangle 1$. LET: (A_n) be a sequence of countable sets.
$\langle 1 \rangle 2$. For $n \in \mathbb{N}$, PICK an enumeration $(e_{nm})_m$ of A_n.
$\langle 1 \rangle 3$. LET: (p_k) be the following enumeration of $\mathbb{N} \times \mathbb{N}$:

$$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots$$

$\langle 1 \rangle 4$. $(e_{\pi_1(p_k)\pi_2(p_k)})_k$ is an enumeration of $\bigcup_n A_n$.
□

Theorem 2.3. $2^{\mathbb{N}}$ *is uncountable.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $f : \mathbb{N} \approx 2^{\mathbb{N}}$
$\langle 1 \rangle 2$. LET: $S = \{n \in \mathbb{N} : n \notin f(n)\}$
$\langle 1 \rangle 3$. For all n, we have $n \in S \Leftrightarrow n \notin f(n)$
$\langle 1 \rangle 4$. For all n we have $S \neq f(n)$.
$\langle 1 \rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 1$.
□

Chapter 3

Relations

Definition 3.1 (Antisymmetric)**.** A relation R on a set A is *antisymmetric* iff, whenever xRy and yRx, then $x = y$.

Definition 3.2 (Transitive)**.** A relation R on a type A is *transitive* iff, whenever xRy and yRz, then xRz.

Chapter 4

Order Theory

Definition 4.1 (Linear Order)**.** A *linear order* on a set A is a binary relation $\leq$ on A that is transitive, antisymmetric and:

$$\forall x, y \in A. x \leq y \vee y \leq x \ \ .$$

A *linearly ordered set* is a pair $(A, \leq)$ where A is a set and $\leq$ is a binary relation on A.

We write $x < y$ for $x \leq y$ and $x \neq y$.

Definition 4.2 (Upper Bound)**.** Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is an *upper bound* in E iff $\forall x \in E. x \leq u$. We say E is *bounded above* iff it has an upper bound.

The *up-set* of E, denoted $E \uparrow$, is the set of upper bounds of E.

Definition 4.3 (Lower Bound)**.** Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then u is an *lower bound* in E iff $\forall x \in E. l \leq x$. We say E is *bounded below* iff it has a lower bound.

The *down-set* of E, denoted $E \downarrow$, is the set of lower bounds of E.

Definition 4.4 (Supremum)**.** Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is the *least upper bound* or *supremum* of E iff u is an upper bound for E and, for any upper bound u' for E, we have $u \leq u'$.

Definition 4.5 (Infimum)**.** Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then l is the *greatest lower bound* or *infimum* of E iff l is a lower bound for E and, for any lower bound l' for E, we have $l' \leq l$.

Definition 4.6 (Least Upper Bound Property)**.** A linearly ordered set S has the *least upper bound property* iff every nonempty subset of S that is bounded above has a least upper bound.

Proposition 4.7. *Let S be a linearly ordered set and $E \subseteq S$.*

1. *If $E \downarrow$ has a supremum l, then l is the infimum of E.*

2. If $E \uparrow$ has an infimum u, then U is the supremum of E.

PROOF:
⟨1⟩1. If $E \downarrow$ has a supremum l, then l is the infimum of E.
⟨2⟩1. l is a lower bound for E.
⟨3⟩1. LET: $x \in E$
⟨3⟩2. x is an upper bound for $E \downarrow$.
PROOF: For all $y \in E \downarrow$ we have $y \leq x$.
⟨3⟩3. $l \leq x$
⟨2⟩2. For any lower bound l' for E, we have $l' \leq l$.
PROOF: Since l is an upper bound for $E \downarrow$.
⟨1⟩2. If $E \uparrow$ has an infimum u, then u is the supremum of E.
PROOF: Dual.
□

Corollary 4.7.1. *A linearly ordered set has the least upper bound property if and only if every nonempty set bounded below has an infimum.*

Definition 4.8 (Closed Downwards)**.** Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed downwards* iff, whenever $x \in E$ and $y < x$, then $y \in E$.

Definition 4.9 (Closed Upwards)**.** Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed upwards* iff, whenever $x \in E$ and $x < y$, then $y \in E$.

Definition 4.10 (Greatest)**.** Let S be a linearly ordered set and $u \in S$. Then u is *greatest* in S iff $\forall x \in S . x \leq u$.

Definition 4.11 (Least)**.** Let S be a linearly ordered set and $l \in S$. Then l is *least* in S iff $\forall x \in S . l \leq x$.

Proposition 4.12. *Let $\leq$ be a linear order on a set S and $E \subseteq S$. Then $\leq \cap E^2$ is a linear order on E.*

PROOF: Easy. □

Given a linearly ordered set $(S, \leq)$ and $E \subseteq S$, we write just E for the linearly ordered set $(E, \leq \cap E^2)$.

Definition 4.13 (Lexicographic Order)**.** Let A and B be linearly ordered sets. The *lexicographic order* or *dictionary order* on $A \times B$ is the order defined by

$$(a, b) \leq (a', b') \Leftrightarrow a = a' \vee (a < a' \wedge b \leq b') \enspace .$$

Proposition 4.14. *The lexicographic order is a linear order.*

Chapter 5

Field Theory

Definition 5.1 (Field)**.** A *field* F consists of a set F, two operations $+, \cdot : F^2 \to F$ and an element $0 \in F$ such that:

- $+$ is commutative.
- $+$ is associative.
- $\forall x \in F. x + 0 = x$
- $\forall x \in F. \exists y \in F. x + y = 0$
- $\cdot$ is commutative.
- $\cdot$ is associative.
- There exists $1 \in F$ such that $1 \neq 0$ and $\forall x \in F. x1 = x$ and $\forall x \in F. x \neq 0 \Rightarrow \exists y \in F. xy = 1$
- *Distributive Law* $\forall x, y, z \in F. x(y + z) = xy + xz$

Proposition 5.2. *In any field F, the element 0 is the unique element such that $\forall x \in F. x + 0 = x$.*

Proof: If 0 and $0'$ both have this property then $0 = 0 + 0' = 0'$. □

Proposition 5.3. *In any field F, given $x \in F$, there is a unique $y \in F$ such that $x + y = 0$.*

Proof: If $x + y = x + y' = 0$ then

$$\begin{aligned} y &= y + 0 \\ &= y + x + y' \\ &= 0 + y' \\ &= y' \end{aligned}$$

□

Definition 5.4. Let F be a field. Let $x \in F$. We denote by $-x$ the unique element of F such that $x + (-x) = 0$.

Given $x, y \in F$, we write $x - y$ for $x + (-y)$.

Proposition 5.5. *In any field F, if $x + y = x + z$ then $y = z$.*

PROOF: If $x + y = x + z$ we have

$$\begin{aligned} -x + x + y &= -x + x + z \\ \therefore 0 + y &= 0 + z \\ \therefore y &= z \end{aligned}$$

∎

Proposition 5.6. *In any field F, we have $-(-x) = x$.*

PROOF: Since $x + (-x) = 0$. ∎

Proposition 5.7. *In any field F, the element 1 such that $\forall x \in F.x1 = x$ is unique.*

PROOF: If 1 and $1'$ both have this property then $1 = 1 \cdot 1' = 1'$. ∎

Proposition 5.8. *In any field F, given $x \in F$ with $x \neq 0$, the element y such that $xy = 1$ is unique.*

PROOF: If y and y' both have this property then we have

$$\begin{aligned} y &= y1 \\ &= yxy' \\ &= 1y' \\ &= y' \end{aligned}$$

∎

Definition 5.9. In any field F, if $x \neq 0$, we write x^{-1} for the unique element such that $xx^{-1} = 1$.

We write x/y for xy^{-1}.

Proposition 5.10. *In any field F, if $xy = xz$ and $x \neq 0$ then $y = z$.*

PROOF:

$$\begin{aligned} y &= 1y \\ &= x^{-1}xy \\ &= x^{-1}xz \\ &= 1z \\ &= z \end{aligned}$$

∎

Proposition 5.11. *In any field F, if $x \neq 0$ then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.*

PROOF: Since $xx^{-1} = 1$. ∎

Proposition 5.12. *In any field F, we have $x0 = 0$.*

PROOF:

$$\begin{aligned} x0 + 0 &= x0 \\ &= x(0+0) \\ &= x0 + x0 \\ \therefore 0 &= x0 \end{aligned} \qquad \square$$

Proposition 5.13. *In any field F, if $xy = 0$ then $x = 0$ or $y = 0$.*

PROOF: If $xy = 0$ and $x \neq 0$ then we have $y = x^{-1}xy = x^{-1}0 = 0$. $\square$

Proposition 5.14. *In any field F, we have $(-x)y = -(xy)$.*

PROOF:

$$\begin{aligned} xy + (-x)y &= (x + (-x))y \\ &= 0y \\ &= 0 && \text{(Proposition 5.12)} \square \end{aligned}$$

Corollary 5.14.1. *In any field F, we have $(-x)(-y) = xy$.*

PROOF:

$$\begin{aligned} (-x)(-y) &= -(x(-y)) \\ &= -(-(xy)) \\ &= xy && \text{(Proposition 5.6)} \square \end{aligned}$$

Proposition 5.15. *Let K be a field. Let $a, b \in K$. If $a^2 = b^2$ then $a = b$ or $a = -b$.*

PROOF:

$$\begin{aligned} a^2 - b^2 &= 0 \\ \therefore (a-b)(a+b) &= 0 \end{aligned}$$

Hence either $a - b = 0$ or $a + b = 0$, and the conclusion follows. $\square$

5.1 Ordered Fields

Definition 5.16 (Ordered Field)**.** An *ordered field* F consists of a field F and a linear order $\leq$ on F such that:

- For all $x, y, z \in F$, if $y < z$ then $x + y < x + z$
- For all $x, y \in F$, if $x > 0$ and $y > 0$ then $xy > 0$.

We call x *positive* iff $x > 0$ and *negative* iff $x < 0$.

Example 5.17. $\mathbb{Q}$ is an ordered field.

Proposition 5.18. *In any ordered field, if x is positive then $-x$ is negative.*

PROOF: If $x > 0$ then $0 = x + (-x) > 0 = (-x) = -x$. $\square$

Proposition 5.19. *In any ordered field, if $y < z$ and x is positive then $xy < xz$.*

Proof: If $y < z$ then we have

$$\begin{aligned} 0 &< z - y \\ \therefore 0 &< x(z-y) \\ &= xz - xy \\ \therefore xy &< xz \end{aligned}$$

∎

Proposition 5.20. *In any ordered field, if $y < z$ and x is negative then $xy > xz$.*

Proof:
⟨1⟩1. $-x$ is positive.
⟨1⟩2. $(-x)y < (-x)z$
⟨1⟩3. $-(xy) < -(xz)$
⟨1⟩4. $xz < xy$
∎

Proposition 5.21. *In any ordered field, if $x \neq 0$ then $x^2 > 0$.*

Proof:
⟨1⟩1. If $x > 0$ then $x^2 > 0$.
Proof: Proposition 5.19.
⟨1⟩2. If $x < 0$ then $x^2 > 0$.
Proof: Proposition 5.20.
∎

Corollary 5.21.1. *In any ordered field, we have $1 > 0$.*

Proposition 5.22. *In any ordered field, if x is positive then x^{-1} is positive.*

Proof: If $x^{-1} < 0$ then we would have $1 = xx^{-1} < x0 = 0$ contradicting Corollary 5.21.1. ∎

Proposition 5.23. *In any ordered field, if $0 < x < y$ then $y^{-1} < x^{-1}$.*

Proof:
⟨1⟩1. Assume: $0 < x < y$
⟨1⟩2. x^{-1} and y^{-1} are positive.
Proof: Proposition 5.22.
⟨1⟩3. $xy^{-1} < yy^{-1} = 1$
⟨1⟩4. $y^{-1} = x^{-1}xy^{-1} < x^{-1}1 = x^{-1}$
∎

Lemma 5.24. *Let K be an ordered field. Let $b \in K$ with $b > 1$. Let n be a positive integer. Then*

$$b^n - 1 \geq n(b-1)$$

Proof:

$$\begin{aligned} b^n - 1 &= (b-1)(b^{n-1} + b^{n-2} + \cdots + 1) \\ &\geq (b-1)(1 + 1 + \cdots + 1) \\ &= n(b-1) \end{aligned}$$

∎

Chapter 6

Real Analysis

6.1 Construction of the Real Numbers

Definition 6.1 (Cut)**.** A *cut* is a subset α of $\mathbb{Q}$ such that:

- $\emptyset \neq \alpha \neq \mathbb{Q}$
- α is closed downwards.
- α has no greatest element.

In this section, we write R for the set of all cuts.

Proposition 6.2. *R is linearly ordered by $\subseteq$.*

PROOF: The only difficult part is to prove that, for any cuts α and β, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.
$\langle 1 \rangle 1$. ASSUME: $\alpha \nsubseteq \beta$
PROVE: $\beta \subseteq \alpha$
$\langle 1 \rangle 2$. PICK $q \in \alpha$ such that $q \notin \beta$
$\langle 1 \rangle 3$. LET: $r \in \beta$
$\langle 1 \rangle 4$. $q \nless r$
$\langle 1 \rangle 5$. $r < q$
$\langle 1 \rangle 6$. $r \in \alpha$
□

Proposition 6.3. *R has the least upper bound property.*

PROOF:
$\langle 1 \rangle 1$. LET: $E \subseteq R$ be nonempty and bounded above.
$\langle 1 \rangle 2$. LET: $s = \bigcup E$
PROVE: s is a cut.
$\langle 1 \rangle 3$. $\emptyset \neq s$
PROOF: Since E is nonempty and every element of E is nonempty.
$\langle 1 \rangle 4$. $s \neq \mathbb{Q}$

$\langle 2\rangle 1.$ PICK an upper bound u for E.
$\langle 2\rangle 2.$ PICK $q \notin u$
PROVE: $q \notin s$
$\langle 2\rangle 3.$ $\forall \alpha \in E. \alpha \subseteq u$
$\langle 2\rangle 4.$ $s \subseteq u$
$\langle 2\rangle 5.$ $q \notin s$
$\langle 1\rangle 5.$ s is closed downwards.
$\langle 2\rangle 1.$ LET: $q \in s$ and $r < q$.
$\langle 2\rangle 2.$ PICK $\alpha \in E$ such that $q \in \alpha$.
$\langle 2\rangle 3.$ $r \in \alpha$
$\langle 2\rangle 4.$ $r \in s$
$\langle 1\rangle 6.$ s has no greatest element.
$\langle 2\rangle 1.$ LET: $q \in s$
$\langle 2\rangle 2.$ PICK $\alpha \in E$ such that $q \in \alpha$.
$\langle 2\rangle 3.$ PICK $r \in \alpha$ such that $q < r$.
$\langle 2\rangle 4.$ $r \in s$
□

Definition 6.4 (Addition)**.** Given cuts α and β, we define

$$\alpha + \beta = \{q + r : q \in \alpha, r \in \beta\} \ .$$

Proposition 6.5. *Given cuts α and β, we have $\alpha + \beta$ is a cut.*

PROOF:
$\langle 1\rangle 1.$ $\alpha + \beta$ is nonempty.
PROOF: Since α and β are nonempty.
$\langle 1\rangle 2.$ $\alpha + \beta \neq \mathbb{Q}$
$\langle 2\rangle 1.$ PICK $q \in \mathbb{Q} - \alpha$ and $r \in \mathbb{Q} - \beta$.
PROVE: $q + r \notin \alpha + \beta$
$\langle 2\rangle 2.$ ASSUME: for a contradiction $q + r \in \alpha + \beta$.
$\langle 2\rangle 3.$ PICK $x \in \alpha$ and $y \in \beta$ such that $q + r = x + y$
$\langle 2\rangle 4.$ $x < q$
$\langle 2\rangle 5.$ $y < r$
$\langle 2\rangle 6.$ $x + y < q + r$
$\langle 2\rangle 7.$ Q.E.D.
PROOF: This is a contradiction.
$\langle 1\rangle 3.$ $\alpha + \beta$ is closed downwards.
$\langle 2\rangle 1.$ LET: $q \in \alpha$, $r \in \beta$ and $x < q + r$
$\langle 2\rangle 2.$ $x - q < r$
$\langle 2\rangle 3.$ $x - q \in \beta$
$\langle 2\rangle 4.$ $x \in \alpha + \beta$
$\langle 1\rangle 4.$ $\alpha + \beta$ has no greatest element.
$\langle 2\rangle 1.$ LET: $q \in \alpha$ and $r \in \beta$.
PROVE: $q + r$ is not greatest in $\alpha + \beta$.
$\langle 2\rangle 2.$ PICK $q' \in \alpha$ with $q < q'$ and $r' \in \beta$ with $r < r'$.
$\langle 2\rangle 3.$ $q + r < q' + r' \in \alpha + \beta$

□

Proposition 6.6. *Addition is commutative and associative on R.*

PROOF: Immediate from definitions and the fact that addition is commutative and associative on $\mathbb{Q}$. □

Definition 6.7. For any $q \in \mathbb{Q}$, let $q^* = \{r \in \mathbb{Q} : r < q\}$.

Proposition 6.8. *For any $q \in \mathbb{Q}$, we have q^* is a cut.*

PROOF:
$\langle 1 \rangle 1.$ $q^* \neq \emptyset$
  PROOF: Since $q - 1 \in q^*$.
$\langle 1 \rangle 2.$ $q^* \neq \mathbb{Q}$
  PROOF: Since $q \notin q^*$.
$\langle 1 \rangle 3.$ q^* is closed downwards.
  PROOF: Immediate from definition.
$\langle 1 \rangle 4.$ q^* has no greatest element.
  PROOF: For all $r \in q^*$ we have $r < (q + r)/2 \in q^*$.
□

Proposition 6.9. *For any cut α we have $\alpha + 0^* = \alpha$.*

PROOF:
$\langle 1 \rangle 1.$ $\alpha + 0^* \subseteq \alpha$
  $\langle 2 \rangle 1.$ LET: $q \in \alpha$ and $r \in 0^*$
    PROVE: $q + r \in \alpha$
  $\langle 2 \rangle 2.$ $r < 0$
  $\langle 2 \rangle 3.$ $q + r < q$
  $\langle 2 \rangle 4.$ $q + r \in \alpha$
$\langle 1 \rangle 2.$ $\alpha \subseteq \alpha + 0^*$
  $\langle 2 \rangle 1.$ LET: $q \in \alpha$
  $\langle 2 \rangle 2.$ PICK $r \in \alpha$ such that $q < r$
  $\langle 2 \rangle 3.$ $q = r + (q - r) \in \alpha + 0^*$
□

Proposition 6.10. *For any cut α, there exists a cut β such that $\alpha + \beta = 0$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\beta = \{p \in \mathbb{Q} : \exists r > 0. -p - r \notin \alpha\}$
$\langle 1 \rangle 2.$ β is a cut.
  $\langle 2 \rangle 1.$ $\beta \neq \emptyset$
    $\langle 3 \rangle 1.$ PICK $q \notin \alpha$
    $\langle 3 \rangle 2.$ $-q - 1 \in \beta$
  $\langle 2 \rangle 2.$ $\beta \neq \mathbb{Q}$
    $\langle 3 \rangle 1.$ PICK $q \in \alpha$
      PROVE: $-q \notin \beta$
    $\langle 3 \rangle 2.$ ASSUME: for a contradiction $-q \in \beta$

⟨3⟩3. PICK $r > 0$ such that $q - r \notin \alpha$
⟨3⟩4. $q - r < q$
⟨3⟩5. Q.E.D.
PROOF: This contradicts the fact that α is closed downwards.
⟨2⟩3. β is closed downwards.
⟨3⟩1. LET: $p \in \beta$ and $q < p$.
⟨3⟩2. PICK $r > 0$ such that $-p - r \notin \alpha$
⟨3⟩3. $-p - r < -q - r$
⟨3⟩4. $-q - r \notin \alpha$
⟨3⟩5. $q \in \beta$
⟨2⟩4. β has no greatest element.
⟨3⟩1. LET: $p \in \beta$
⟨3⟩2. PICK $r > 0$ such that $-p - r \notin \alpha$
⟨3⟩3. $-(p + r/2) - r/2 \notin \alpha$
⟨3⟩4. $p + r/2 \in \beta$
⟨1⟩3. $\alpha + \beta \subseteq 0^*$
⟨2⟩1. LET: $p \in \alpha$ and $q \in \beta$.
⟨2⟩2. PICK $r > 0$ such that $-q - r \notin \alpha$.
⟨2⟩3. $p < -q - r$
⟨2⟩4. $p + q < -r$
⟨2⟩5. $p + q < 0$
⟨2⟩6. $p + q \in 0^*$
⟨1⟩4. $0^* \subseteq \alpha + \beta$
⟨2⟩1. LET: $v \in 0^*$
⟨2⟩2. LET: $w = -v/2$
⟨2⟩3. $w > 0$
⟨2⟩4. PICK an integer n such that $nw \in \alpha$ and $(n + 1)w \notin \alpha$.
⟨2⟩5. LET: $p = -(n + 2)w$
⟨2⟩6. $p \in \beta$
⟨2⟩7. $v = nw + p$
⟨2⟩8. $v \in \alpha + \beta$
□

Proposition 6.11. *Given $\alpha, \beta, \gamma \in R$, if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

PROOF:
⟨1⟩1. $\alpha + \beta \subseteq \alpha + \gamma$
PROOF: Immediate from definitions.
⟨1⟩2. $\alpha + \beta \neq \alpha + \gamma$
PROOF: If $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$ by cancellation.
□

Definition 6.12. Given cuts α and β, define $\alpha\beta$ by:

$$\alpha\beta = \begin{cases} \{p \in \mathbb{Q} : \exists r \in \alpha . \exists s \in \beta (p \leq rs \wedge r > 0 \wedge s > 0\} & \text{if } \alpha > 0^* \text{ and } \beta > 0^* \\ (-\alpha)(-\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^* \\ -((-\alpha)\beta) & \text{if } \alpha < 0^* \text{ and } \beta > 0^* \\ -(\alpha(-\beta)) & \text{if } \alpha > 0^* \text{ and } \beta < 0^* \\ 0^* & \text{if } \alpha = 0^* \text{ or } \beta = 0^* \end{cases}$$

Proposition 6.13. *For any cuts α and β, we have $\alpha\beta$ is a cut.*

PROOF:
⟨1⟩1. If $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta$ is a cut.
  ⟨2⟩1. $\alpha\beta \neq \emptyset$
    ⟨3⟩1. PICK $q \in \alpha$ and $r \in \beta$ such that $q, r \notin 0^*$
    ⟨3⟩2. ASSUME: w.l.o.g. $0 < q$ and $0 < r$.
      PROOF: Since α and β have no greatest element.
    ⟨3⟩3. $qr \in \alpha\beta$
  ⟨2⟩2. $\alpha\beta \neq \mathbb{Q}$
    ⟨3⟩1. PICK $r \notin \alpha$ and $s \notin \beta$
      PROVE: $rs \notin \alpha\beta$
    ⟨3⟩2. ASSUME: for a contradiction $rs \in \alpha\beta$.
    ⟨3⟩3. PICK $r' \in \alpha$ and $s' \in \beta$ such that $rs \leq r's'$ and $r' > 0$ and $s' > 0$.
    ⟨3⟩4. $r' < r$ and $s' < s$
    ⟨3⟩5. $r's' < rs$
    ⟨3⟩6. Q.E.D.
      PROOF: This is a contradiction.
  ⟨2⟩3. $\alpha\beta$ is closed downwards.
    ⟨3⟩1. LET: $p \in \alpha\beta$ and $p' < p$
    ⟨3⟩2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$
    ⟨3⟩3. $p' \leq rs$
    ⟨3⟩4. $p' \in \alpha\beta$
  ⟨2⟩4. $\alpha\beta$ has no greatest element.
    ⟨3⟩1. LET: $p \in \alpha\beta$
    ⟨3⟩2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$.
    ⟨3⟩3. PICK $r' \in \alpha$ and $s' \in \beta$ with $r < r'$ and $s < s'$.
    ⟨3⟩4. $p < r's' \in \alpha\beta$
⟨1⟩2. For any cuts α and β, we have $\alpha\beta$ is a cut.
  PROOF: Since if α is a cut then $-\alpha$ is a cut.
□

Proposition 6.14. *For any cuts α and β we have $\alpha\beta = \beta\alpha$.*

PROOF: Easy from the definitions. □

Proposition 6.15. *For any cuts α, β and γ we have*

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma \ .$$

PROOF:
⟨1⟩1. CASE: α, β and γ are all positive.
PROOF: In this case $\alpha(\beta\gamma) = (\alpha\beta)\gamma = \{p \in \mathbb{Q} : \exists r \in \alpha.\exists s \in \beta.\exists t \in \gamma.(p \leq rst \wedge r > 0 \wedge s > 0 \wedge t > 0)\}$.
⟨1⟩2. CASE: One of α, β or γ is 0^*.
PROOF: Then $\alpha(\beta\gamma) = (\alpha\beta)\gamma = 0^*$.
⟨1⟩3. CASE: α and β are positive, γ is negative.
PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\ &= -(\alpha(\beta(-\gamma))) \\ &= -((\alpha\beta)(-\gamma)) && (\langle 1\rangle 1) \\ &= (\alpha\beta)\gamma \end{aligned}$$

⟨1⟩4. CASE: α is positive, β is negative, γ is positive.
PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-((-\beta)\gamma)) \\ &= -(\alpha((-\beta)\gamma)) \\ &= -((\alpha(-\beta))\gamma) && (\langle 1\rangle 1) \\ &= (-(\alpha(-\beta)))\gamma \\ &= (\alpha\beta)\gamma \end{aligned}$$

⟨1⟩5. CASE: α is positive, β and γ are negative.
PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha((-\beta)(-\gamma)) \\ &= (\alpha(-\beta))(-\gamma) && (\langle 1\rangle 1) \\ &= (-(\alpha\beta))(-\gamma) \\ &= (\alpha\beta)\gamma \end{aligned}$$

⟨1⟩6. CASE: α is negative, β and γ are positive.
PROOF: Similar to ⟨1⟩3.
⟨1⟩7. CASE: α is negative, β is positive, γ is negative.
PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\ &= (-\alpha)(\beta(-\gamma)) \\ &= ((-\alpha)\beta)(-\gamma) && (\langle 1\rangle 1) \\ &= (-(\alpha\beta))(-\gamma) \\ &= (\alpha\beta)\gamma \end{aligned}$$

⟨1⟩8. CASE: α and β are negative, γ is positive.
PROOF: Similar to ⟨1⟩5.
⟨1⟩9. CASE: α, β and γ are all negative.

PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-(-\beta)(-\gamma)) \\
&= -((-\alpha)((-\beta)(-\gamma))) \\
&= -(((-\alpha)(-\beta))(-\gamma)) && (\langle 1\rangle 1) \\
&= -((\alpha\beta)(-\gamma)) \\
&= (\alpha\beta)\gamma
\end{aligned}$$

□

Proposition 6.16. *For any cut α we have $\alpha 1^* = \alpha$.*

PROOF:
⟨1⟩1. CASE: α is positive.
⟨2⟩1. $\alpha 1^* \subseteq \alpha$
⟨2⟩2. $\alpha \subseteq \alpha 1^*$
⟨1⟩2. CASE: $\alpha = 0^*$
⟨1⟩3. CASE: α is negative.
□

Theorem 6.17. *There exists an ordered field with the least upper bound property.*

Proposition 6.18. *There is no rational p such that $p^2 = 2$.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $p^2 = 2$.
⟨1⟩2. PICK integers m, n not both even such that $p = m/n$.
⟨1⟩3. $m^2 = 2n^2$
⟨1⟩4. m is even.
⟨1⟩5. PICK an integer k such that $m = 2k$.
⟨1⟩6. $4k^2 = 2n^2$
⟨1⟩7. $2k^2 = n^2$
⟨1⟩8. n is even.
⟨1⟩9. Q.E.D.
PROOF: ⟨1⟩2, ⟨1⟩4 and ⟨1⟩8 form a contradiction.
□

Theorem 6.19. *Any two complete ordered fields are isomorphic.*

Definition 6.20. Let $\mathbb{R}$ be the complete ordered field. We call its elements *real numbers.*

6.2 Properties of the Real Numbers

Theorem 6.21. *$\mathbb{Q}$ is a subfield of $\mathbb{R}$.*

Theorem 6.22 (Archimedean Property)**.** *Let $x, y \in \mathbb{R}$ with $x > 0$. There exists a positive integer n such that $nx > y$.*

PROOF:
⟨1⟩1. LET: $A = \{nx : n \in \mathbb{Z}^+\}$
⟨1⟩2. ASSUME: for a contradiction there is no positive integer n such that $nx > y$.
⟨1⟩3. y is an upper bound for A.
⟨1⟩4. LET: $\alpha = \sup A$
⟨1⟩5. $\alpha - x$ is not an upper bound for A.
⟨1⟩6. PICK a positive integer m such that $\alpha - x < mx$
⟨1⟩7. $\alpha < (m+1)x \in A$
⟨1⟩8. Q.E.D.
PROOF: This contradicts ⟨1⟩4.
□

Theorem 6.23. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

PROOF:
⟨1⟩1. LET: $x, y \in \mathbb{R}$ with $x < y$
⟨1⟩2. PICK a positive integer n such that

$$n(y-x) > 1 \ .$$

PROOF: Archimedean property.
⟨1⟩3. PICK a positive integer m_1 such that $m_1 > nx$
PROOF: Archimedean property.
⟨1⟩4. PICK a positive integer m_2 such that $m_2 > -nx$
PROOF: Archimedean property.
⟨1⟩5. $-m_2 < nx < m_1$
⟨1⟩6. LET: m be the integer such that

$$m - 1 \leq nx < m \ .$$

⟨1⟩7. $nx < m \leq 1 + nx < ny$
⟨1⟩8. $x < m/n < y$
□

Theorem 6.24. *For every real number $x > 0$ and positive integer n, there exists a unique positive real number y such that $y^n = x$.*

PROOF:
⟨1⟩1. There exists a real $y > 0$ such that $y^n = x$.
⟨2⟩1. LET: $E = \{t \in \mathbb{R}^+ : t^n < x\}$
⟨2⟩2. LET: $y = \sup E$
⟨3⟩1. $E \neq \emptyset$
⟨4⟩1. LET: $t = x/(x+1)$
⟨4⟩2. $0 < t < 1$
⟨4⟩3. $t^n < t < x$
⟨4⟩4. $t \in E$
⟨3⟩2. $x + 1$ is an upper bound for E.
⟨4⟩1. LET: $t > x + 1$
⟨4⟩2. $t^n > t > x$
⟨4⟩3. $t \notin E$

$\langle 2\rangle 3.$ $y^n = x$
$\langle 3\rangle 1.$ $y^n \not< x$
$\langle 4\rangle 1.$ ASSUME: for a contradiction $y^n < x$.
$\langle 4\rangle 2.$ PICK h such that $0 < h < 1$ and
$$h < \frac{x - y^n}{n(y+1)^{n-1}} \quad .$$
$\langle 4\rangle 3.$ $(y+h)^n - y^n < x - y^n$
PROOF:
$$\begin{aligned}(y+h)^n - y^n &= ((y+h) - y) \sum_{i=0}^{n-1} (y+h)^{n-1-i} y^i \\ &= h \sum_{i=0}^{n-1} (y+h)^{n-1-i} y^i \\ &\leq hn(y+h)^{n-1} \\ &\leq hn(y+1)^{n-1} \\ &< x - y^n\end{aligned}$$
$\langle 4\rangle 4.$ $(y+h)^n < x$
$\langle 4\rangle 5.$ $y + h \in E$
$\langle 4\rangle 6.$ Q.E.D.
PROOF: This contradicts the fact that y is an upper bound for E.
$\langle 3\rangle 2.$ $y^n \not> x$
$\langle 4\rangle 1.$ ASSUME: for a contradiction $y^n > x$
$\langle 4\rangle 2.$ LET:
$$k = \frac{y^n - x}{ny^{n-1}}$$
$\langle 4\rangle 3.$ $0 < k < y$
$\langle 4\rangle 4.$ $y - k$ is an upper bound for E.
$\langle 5\rangle 1.$ LET: $t \geq y - k$
$\langle 5\rangle 2.$ $y^n - t^n \leq y^n - x$
PROOF:
$$\begin{aligned}y^n - t^n &\leq y^n - (y-k)^n \\ &= (y - (y-k)) \sum_{i=0}^{n-1} y^{n-i} (y-k)^i \\ &= k \sum_{i=0}^{n-1} y^{n-i} (y-k)^i \\ &\leq kny^{n-1} \\ &= y^n - x\end{aligned}$$
$\langle 5\rangle 3.$ $t^n \geq x$
$\langle 5\rangle 4.$ $t \notin E$
$\langle 4\rangle 5.$ Q.E.D.
PROOF: This contradicts the fact that y is the least upper bound of E.
$\langle 1\rangle 2.$ If y and y' are positive reals with $y^n = y'^n$ then $y = y'$.

PROOF: Since the function that sends y to y^n is strictly monotone.
∎

Definition 6.25 (nth Root)**.** Given any real number $x > 0$ and positive integer n, the *nth root* of x, denoted $x^{1/n}$, is the unique positive real such that

$$(x^{1/n})^n = x \ .$$

We write $\sqrt{x}$ for $x^{1/2}$.

Proposition 6.26. *Let a and b be positive real numbers and n a positive integer. Then*

$$(ab)^{1/n} = a^{1/n}b^{1/n} \ .$$

PROOF: Since $(a^{1/n}b^{1/n})^n = ab$. ∎

Lemma 6.27. *Let b be a real number with $b > 1$. Let n be a positive integer. Then*

$$b - 1 \geq n(b^{1/n} - 1) \ .$$

PROOF: From Lemma 5.24. ∎

Lemma 6.28. *Let b and t be real numbers with $b > 1$ and $t > 1$. For any positive integer n, if $n > \frac{b-1}{t-1}$ then $b^{1/n} < t$.*

PROOF:

$$\begin{aligned} b - 1 &\geq n(b^{1/n} - 1) \\ \therefore \frac{b-1}{n} &\geq b^{1/n} - 1 \\ \therefore t - 1 &> b^{1/n} - 1 \\ \therefore t &> b^{1/n} \end{aligned}$$

∎

Lemma 6.29. *Let b be a real number with $b > 0$. Let m, n, p, q be integers with $n > 0$ and $q > 0$. Assume $m/n = p/q$. Then*

$$(b^m)^{1/n} = (b^p)^{1/q} \ .$$

PROOF:
⟨1⟩1. $(b^m)^{1/n} = (b^{1/n})^m$
PROOF:

$$\begin{aligned} ((b^{1/n})^m)^n &= ((b^{1/n})^n)^m \\ &= b^m \end{aligned}$$

⟨1⟩2. $((b^m)^{1/n})^q = b^p$
PROOF:

$$\begin{aligned} ((b^m)^{1/n})^q &= (b^{1/n})^{mq} \\ &= (b^{1/n})^{np} \\ &= b^p \end{aligned}$$

∎

Definition 6.30. For a a positive real and q a rational number, we may therefore define a^q by

$$a^{m/n} = (a^m)^{1/n}$$

for m and n integers with $n > 0$.

Proposition 6.31. *Let a be a positive real and r, s rational numbers. Then*

$$a^{r+s} = a^r a^s \quad .$$

PROOF:

$$\begin{aligned} a^{m/n+p/q} &= a^{(mq+np)/nq} \\ &= (a^{mq+np})^{1/nq} \\ &= (a^{mq})^{1/nq}(a^{np})^{1/nq} \\ &= a^{m/n}a^{p/q} \end{aligned}$$

□

Proposition 6.32. *Let $b > 1$ be a real number and q a rational number. Then*

$$b^q = \sup\{b^t : t \in \mathbb{Q}, t \leq q\}$$

PROOF: It is the greatest element of this set. □

Definition 6.33. Let $b > 1$ be a real number and x a real number. Then

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t \leq x\} \quad .$$

Lemma 6.34. *Let b, w and y be real numbers with $b > 1$. Assume $b^w < y$. Then there exists a positive integer n such that $b^{w+1/n} < y$.*

PROOF:
⟨1⟩1. LET: $t = yb^{-w}$
⟨1⟩2. PICK a positive integer n such that $n > \frac{b-1}{t-1}$.
⟨1⟩3. $b^{1/n} < t$
 PROOF: Lemma 6.28.
⟨1⟩4. $b^{w+1/n} < y$
□

Lemma 6.35. *Let b, w and y be real numbers with $b > 1$. Assume $b^w > y$. Then there exists a positive integer n such that $b^{w-1/n} < y$.*

PROOF:
⟨1⟩1. LET: $t = b^w/y$
⟨1⟩2. PICK a positive integer n such that $n > \frac{b-1}{t-1}$
⟨1⟩3. $b^{1/n} < t$
 PROOF: Lemma 6.28.
⟨1⟩4. $y < b^{w-1/n}$
□

Proposition 6.36. *For b and x real numbers with $b > 1$ we have*

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t < x\} \ .$$

PROOF:
⟨1⟩1. b^x is an upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
⟨1⟩2. LET: u be any upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
PROVE: $b^x \leq u$
⟨1⟩3. LET: q be a rational number with $q \leq x$.
PROVE: $b^q \leq u$
⟨1⟩4. ASSUME: for a contradiction $b^q > u$.
⟨1⟩5. PICK a positive integer n such that $b^{q-1/n} > u$.
PROOF: Lemma 6.35.
⟨1⟩6. $b^{q-1/n} \leq u$
PROOF: ⟨1⟩2
⟨1⟩7. Q.E.D.
PROOF: This contradicts ⟨1⟩4.
□

Lemma 6.37. *Let A be a set of positive real numbers with supremum $a > 0$ and B a set of positive real numbers with supremum $b > 0$. Then ab is the supremum of $\{xy : x \in A, y \in B\}$.*

PROOF:
⟨1⟩1. For all $x \in A$ and $y \in B$ we have $xy \leq ab$.
⟨1⟩2. If u is any upper bound for $\{xy : x \in A, y \in B\}$ then $ab \leq u$.
⟨2⟩1. LET: u be an upper bound for $\{xy : x \in A, y \in B\}$.
⟨2⟩2. For all $x \in A$ we have u/x is an upper bound for B.
⟨2⟩3. For all $x \in A$ we have $b \leq u/x$
⟨2⟩4. For all $x \in A$ we have $x \leq u/b$
⟨2⟩5. $a \leq u/b$
⟨2⟩6. $ab \leq u$
□

Proposition 6.38. *Let $b, x, y \in \mathbb{R}$ with $b > 1$. Then*

$$b^{x+y} = b^x b^y \ .$$

PROOF:
⟨1⟩1. For any rational number $q < x + y$, there exist rational numbers $r < x$ and $s < y$ such that $q = r + s$.
⟨2⟩1. $q - x < y$
⟨2⟩2. PICK a rational t such that $q - x < t < y$
⟨2⟩3. $q = t + (q - t)$ and $t < y$, $q - t < x$
⟨1⟩2. $b^x b^y = b^{x+y}$

PROOF:

$$\begin{aligned} b^x b^y &= \sup\{b^q b^r : q, r \in \mathbb{Q}, q < x, r < y\} \\ &= \sup\{b^{q+r} : q, r \in \mathbb{Q}, q < x, r < y\} \\ &= \sup\{b^q : q \in \mathbb{Q}, q < x + y\} \\ &= b^{x+y} \end{aligned}$$

□

6.2.1 Logarithms

Proposition 6.39. *Let b and y be real numbers with $b > 1$ and $y > 0$. There exists a unique real x such that $b^x = y$.*

PROOF:
⟨1⟩1. LET: $x = \sup\{w : b^w < y\}$
PROVE: $b^x = y$
⟨2⟩1. $\{w : b^w < y\} \neq \emptyset$
PROOF: It contains 0.
⟨2⟩2. $\{w : b^w < y\}$ is bounded above.
⟨3⟩1. LET: n be the least integer such that

$$n \geq \frac{y-1}{b-1}$$

PROOF: Archimedean property.
⟨3⟩2. LET: w be a real number with $b^w < y$
PROVE: $w < n$
⟨3⟩3. $b^w < n(b-1) + 1$
⟨3⟩4. $b^w < b^n$
⟨3⟩5. $w < n$
⟨1⟩2. $b^x \leq y$
⟨2⟩1. ASSUME: for a contradiction $b^x > y$
⟨2⟩2. PICK a positive integer n such that $b^{x-1/n} > y$
PROOF: Lemma 6.35.
⟨2⟩3. PICK w such that $x - 1/n < w$ and $b^w < y$
PROOF: Since $x - 1/n$ is not an upper bound for $\{w : b^w < y\}$.
⟨2⟩4. $b^{x-1/n} < y$
⟨2⟩5. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
⟨1⟩3. $b^x \geq y$
⟨2⟩1. ASSUME: for a contradiction $b^x < y$.
⟨2⟩2. PICK a positive integer n such that $b^{x+1/n} < y$.
⟨2⟩3. $x + 1/n \leq x$
⟨2⟩4. Q.E.D.
PROOF: This is a contradiction.

□

Definition 6.40 (Logarithm)**.** Let b and y be real numbers with $b > 1$ and $y > 0$. The *logarithm* of y to *base* b, denoted $\log_b y$, is the unique real number such that

$$b^{\log_b y} = y \ .$$

6.2.2 Intervals

Definition 6.41 (Intervals)**.** Let $a, b \in \mathbb{R}$.

The *open interval* (a, b) is $\{x \in \mathbb{R} : a < x < b\}$.

The *closed interval* $[a, b]$ is $\{x \in \mathbb{R} : a \leq x \leq b\}$.

The *half-open intervals* $[a, b)$ and $(a, b]$ are defined by

$$\begin{aligned} [a, b) &:= \{x \in \mathbb{R} : a \leq x < b\} \\ (a, b] &:= \{x \in \mathbb{R} : a < x \leq b\} \end{aligned}$$

Definition 6.42 (k-cell)**.** Let k be a positive integer. A *k-cell* is a subset of $\mathbb{R}^k$ of the form

$$\{\vec{x} \in \mathbb{R}^k : \forall i = 1, \ldots, k. a_i \leq x_i \leq b_i\}$$

for some real numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$ with $a_i \leq b_i$ for each i.

6.2.3 The Cantor Set

Definition 6.43 (Cantor Set)**.** Define a sequence E_n of unions of intervals as follows:

- $E_0 = [0, 1]$

- E_{n+1} is formed from E_n by replacing every interval $[a, b]$ with $[a, (2a+b)/3]$ and $[(a + 2b)/3, b]$.

The *Cantor set* is $\bigcap_{n=0}^{\infty} E_n$.

6.3 The Extended Real Number System

Definition 6.44 (Extended Real Number System)**.** The *extended real number system* is the set $\mathbb{R} \cup \{+\infty, -\infty\}$.

We extend the ordering $\leq$ to the extended reals by defining

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

We extend $+$, $\cdot$ and $/$ to partial operations on the extended real by defining:

$$
\begin{array}{rr}
x + (+\infty) = +\infty & (x \in \mathbb{R}) \\
x + (-\infty) = -\infty & (x \in \mathbb{R}) \\
(+\infty) + x = +\infty & (x \in \mathbb{R}) \\
(+\infty) + (+\infty) \text{ is undefined} & \\
(+\infty) + (-\infty) \text{ is undefined} & \\
(-\infty) + x = -\infty & (x \in \mathbb{R}) \\
(-\infty) + (+\infty) \text{ is undefined} & \\
(-\infty) + (-\infty) \text{ is undefined} & \\
x \cdot (+\infty) = +\infty & (x \in \mathbb{R}) \\
x \cdot (-\infty) = -\infty & (x \in \mathbb{R}) \\
(+\infty) \cdot x = +\infty & (x \in \mathbb{R}) \\
(+\infty) \cdot (+\infty) \text{ is undefined} & \\
(+\infty) \cdot (-\infty) \text{ is undefined} & \\
(-\infty) \cdot x = -\infty & (x \in \mathbb{R}) \\
(-\infty) \cdot (+\infty) \text{ is undefined} & \\
(-\infty) \cdot (-\infty) \text{ is undefined} & \\
x/(+\infty) = 0 & (x \in \mathbb{R}) \\
x/(-\infty) = 0 & (x \in \mathbb{R}) \\
(+\infty)/x \text{ is undefined} & (x \in \mathbb{R}) \\
(+\infty)/(+\infty) \text{ is undefined} & \\
(+\infty)/(-\infty) \text{ is undefined} & \\
(-\infty)/x \text{ is undefined} & (x \in \mathbb{R}) \\
(-\infty)/(+\infty) \text{ is undefined} & \\
(-\infty)/(-\infty) \text{ is undefined} &
\end{array}
$$

Chapter 7

Complex Analysis

Definition 7.1 (Complex Numbers)**.** A *complex number* is a pair of real numbers. We write $\mathbb{C}$ for the set of complex numbers.

Define $+$ and $\cdot$ on $\mathbb{C}$ by:

$$\begin{aligned}(a,b)+(c,d) &= (a+c,b+d)\\ (a,b)(c,d) &= (ac-bd,ad+bc)\end{aligned}$$

Theorem 7.2. *The complex numbers form a field.*

Theorem 7.3. *The function that maps a to $(a,0)$ is an embedding of $\mathbb{R}$ in $\mathbb{C}$.*

Definition 7.4.

$$i = (0,1)$$

Lemma 7.5.

$$(a,b) = a+ib$$

Proof: Since $(a,0)+(0,1)(b,0)=(a,b)$. □

Lemma 7.6.

$$i^2 = -1$$

Proof: Immediate from definitions. □

Corollary 7.6.1. *There is no linear order on $\mathbb{C}$ that makes $\mathbb{C}$ into an ordered field.*

Definition 7.7 (Complex Conjugate)**.** For any complex number z, the *complex conjugate* $\overline{z}$ is defined by

$$\overline{a+ib} = a-ib \qquad (a,b\in\mathbb{R}) \ .$$

Definition 7.8 (Real Part)**.** For any complex number z, the *real part* of z, denoted $\mathrm{Re}(z)$, is defined by

$$\mathrm{Re}(a+ib) = a \qquad (a,b\in\mathbb{R}) \ .$$

Definition 7.9 (Imaginary Part)**.** For any complex number z, the *imaginar part* of z, denoted $\mathrm{Im}(z)$, is defined by

$$\mathrm{Im}(a+ib) = b \qquad (a, b \in \mathbb{R}) \ .$$

Theorem 7.10. *For all $z, w \in \mathbb{C}$ we have*

$$\overline{z+w} = \overline{z} + \overline{w} \ .$$

PROOF:

$$\begin{aligned} \overline{(a+ib)+(c+id)} &= \overline{(a+c)+i(b+d)} \\ &= (a+c) - i(b+d) \\ &= (a-ib) + (c-id) \\ &= \overline{a+ib} + \overline{c+id} \end{aligned}$$

□

Theorem 7.11. *For all $z, w \in \mathbb{C}$ we have*

$$\overline{zw} = \overline{z} \cdot \overline{w} \ .$$

PROOF:

$$\begin{aligned} \overline{(a+ib)(c+id)} &= \overline{(ac-bd)+i(ad+bc)} \\ &= (ac-bd) - i(ad+bc) \\ &= (a-ib)(c-id) \\ &= \overline{a+ib} \cdot \overline{c+id} \end{aligned}$$

□

Theorem 7.12. *For all $z \in \mathbb{C}$ we have*

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}) \ .$$

PROOF:

$$\begin{aligned} (a+ib) + \overline{a+ib} &= (a+ib) + (a-ib) \\ &= 2a \\ &= 2\,\mathrm{Re}(a+ib) \end{aligned}$$

□

Theorem 7.13. *For all $z \in \mathbb{C}$ we have*

$$\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z}) \ .$$

PROOF:

$$\begin{aligned} (a+ib) - \overline{a+ib} &= (a+ib) - (a-ib) \\ &= 2ib \\ &= 2i\,\mathrm{Im}(a+ib) \end{aligned}$$

□

Theorem 7.14. *For all $z \in \mathbb{C}$ we have $z\overline{z}$ is a non-negative real.*

PROOF:
$$\begin{aligned}(a+ib)(\overline{a+ib}) &= (a+ib)(a-ib)\\ &= a^2+b^2 \qquad\qquad \square\end{aligned}$$

Theorem 7.15. *For any $z \in \mathbb{C}$, if $z\overline{z} = 0$ then $z = 0$.*

PROOF: Let $z = a + ib$. Then $z\overline{z} = a^2 + b^2 = 0$ iff $a = b = 0$. $\square$

Definition 7.16 (Absolute Value)**.** For $z \in \mathbb{C}$, the *absolute value* of z is

$$|z| = (z\overline{z})^{1/2} \quad .$$

Proposition 7.17. *For x a non-negative real we have $|x| = x$.*

PROOF: Since $|x| = \sqrt{x^2} = x$. $\square$

Proposition 7.18. *For x a negative real we have $|x| = -x$.*

PROOF: Since $|x| = \sqrt{x^2} = -x$. $\square$

Theorem 7.19. *For any complex number z we have $|z| \geq 0$.*

PROOF: Immediate from definition. $\square$

Theorem 7.20. *For any complex number z, if $|z| = 0$ then $z = 0$.*

PROOF: From Theorem 7.15. $\square$

Theorem 7.21. *For any complex number z we have*

$$|\overline{z}| = |z| \quad .$$

PROOF: Immediate from definitions. $\square$

Theorem 7.22. *For any complex numbers z and w we have*

$$|zw| = |z||w| \quad .$$

PROOF:
$$\begin{aligned}|zw| &= \sqrt{zw\overline{zw}}\\ &= \sqrt{z\overline{z}}\sqrt{w\overline{w}} && \text{(Proposition 6.26)}\\ &= |z||w| && \square\end{aligned}$$

Theorem 7.23. *For any complex number z we have*

$$|\operatorname{Re} z| \leq |z|$$

PROOF: Let $z = a + ib$. Then
$$|\operatorname{Re} z| = \sqrt{a^2} \leq \sqrt{a^2+b^2} \ .\square$$

Theorem 7.24. *For any complex numbers z and w we have*

$$|z + w| \leq |z| + |w| \quad .$$

PROOF:

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \\
&= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 && \text{(Theorem 7.12)} \\
&\leq |z|^2 + 2|z\overline{w}| + |w|^2 && \text{(Theorem 7.23)} \\
&= |z|^2 + 2|z||w| + |w|^2 && \text{(Theorem 7.22)} \\
&= (|z| + |w|)^2 && \square
\end{aligned}$$

Theorem 7.25 (Schwarz Inequality)**.** *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be complex numbers. Then*

$$\left|\sum_{j=1}^n a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2 \quad .$$

PROOF:
⟨1⟩1. LET: $A = \sum_{j=1}^n |a_j|^2$
⟨1⟩2. LET: $B = \sum_{j=1}^n |b_j|^2$
⟨1⟩3. LET: $C = \sum_{j=1}^n a_j \overline{b_j}$
⟨1⟩4. ASSUME: w.l.o.g. $B > 0$
PROOF: If $B = 0$ then $b_1 = \cdots = b_n = 0$ and both sides of the inequality are 0.
⟨1⟩5. $\sum_{j=1}^n |Ba_j - Cb_j|^2 = B(AB - |C|^2)$
PROOF:

$$\begin{aligned}
\sum_{j=1}^n |Ba_j - Cb_j|^2 &= \sum_{j=1}^n (Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j}) \\
&= B^2 \sum_{j=1}^n |a_j|^2 - B\overline{C} \sum_{j=1}^n a_j \overline{b_j} - BC \sum_{j=1}^n \overline{a_j} b_j + |C|^2 \sum_{j=1}^n |b_j|^2 \\
&= B^2 A - 2B|C|^2 + B|C|^2 \\
&= B(AB - |C|^2)
\end{aligned}$$

⟨1⟩6. $B(AB - |C|^2) \geq 0$
⟨1⟩7. $AB \geq |C|^2$
$\square$

Proposition 7.26. *For any non-zero complex number w, there are exactly two complex numbers z such that $z^2 = w$.*

PROOF:
⟨1⟩1. There are at most two complex numbers z such that $z^2 = w$.
PROOF: Proposition 5.15.
⟨1⟩2. There are at least two complex numbers z such that $z^2 = w$.
⟨2⟩1. LET: $w = u + iv$
⟨2⟩2. LET: $a = \sqrt{\frac{|w|+u}{2}}$
⟨2⟩3. LET: $b = \sqrt{\frac{|w|-u}{2}}$

⟨2⟩4. Case: $v \geq 0$
⟨3⟩1. Let: $z = a + ib$
⟨3⟩2. $z^2 = w$
Proof:

$$\begin{aligned} z^2 &= (a+ib)^2 \\ &= a^2 - b^2 + 2iab \\ &= u + i\sqrt{|w|^2 - u^2} \\ &= u + iv \\ &= w \end{aligned}$$

⟨3⟩3. $(-z)^2 = w$
⟨2⟩5. Case: $v \leq 0$
⟨3⟩1. Let: $z = a - ib$
⟨3⟩2. $z^2 = w$
Proof:

$$\begin{aligned} z^2 &= (a-ib)^2 \\ &= a^2 - b^2 - 2iab \\ &= u - i\sqrt{|w|^2 - u^2} \\ &= u - i|v| \\ &= w \end{aligned}$$

⟨3⟩3. $(-z)^2 = w$
□

7.1 Algebraic Numbers

Definition 7.27 (Algebraic)**.** A complex number z is *algebraic* iff there exist integers a_0, a_1, …, a_n not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \;;$$

otherwise, it is *transcendental*.

Proposition 7.28. *The set of algebraic numbers is countable.*

Proof: There are countably many finite sequences of integers $(a_0, a_1, \ldots, a_n)$, and for each one, there are only finitely many complex numbers z such that $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$. □

Part II

Category Theory

Chapter 8

Categories

Definition 8.1 (Category)**.** A *category* $\mathcal{C}$ consists of:

- a preset $|\mathcal{C}|$ of *objects*;
- for any objects A and B, a set $\mathcal{C}[A, B]$ of *morphisms* from A to B. We write $f : A \to B$ for $f \in \mathcal{C}[A, B]$, and call A the *source* of f and B the *target*.
- for any object A, a morphism $\mathrm{id}_A : A \to A$, the *identity* morphism on A
- for any morphisms $f : A \to B$ and $g : B \to C$, a morphism $g \circ f : A \to C$, the *composite* of f and g.

such that:

Associativity For any morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Left Unit Law For any morphism $f : A \to B$, we have

$$\mathrm{id}_B \circ f = f$$

Right Unit Law For any morphism $f : A \to B$, we have

$$f \circ \mathrm{id}_A = f$$

Example 8.2. For any preset of sets $\mathcal{U}$, the category $\mathbf{Set}_{\mathcal{U}}$ with objects $\mathcal{U}$ and morphisms all functions is a category.

8.1 Monomorphisms and Epimorphisms

Definition 8.3 (Monomorphism)**.** We say a morphism $f : A \to B$ is *monic* or a *monomorphism*, and write $f : A \rightarrowtail B$, iff, for any object X and morphisms $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

Definition 8.4 (Epimorphism)**.** We say a morphism $f : A \to B$ is *epi* or a *epimorphism*, and write $f : A \twoheadrightarrow B$, iff, for any object X and morphisms $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

8.2 Sections and Retractions

Definition 8.5 (Section, Retraction)**.** Let $r : A \to B$ and $s : B \to A$. Then we say s is a *section* of r, and r is a *retraction* of s, iff $r \circ s = \mathrm{id}_B$.

Proposition 8.6. *If $r : A \to B$ is a retraction of $f : B \to A$ and $s : A \to B$ is a section of f, then $r = s$.*

Proof:

$$\begin{aligned} r &= r \circ \mathrm{id}_A && \text{(Right Unit Law)} \\ &= r \circ f \circ s && (s \text{ is a section of } f) \\ &= \mathrm{id}_B \circ s && (r \text{ is a retraction of } f) \\ &= s && \text{(Left Unit Law)} \quad \square \end{aligned}$$

Proposition 8.7. *If $s_1 : A \to B$ is a section of $r_1 : B \to A$ and $s_2 : B \to C$ is a section of $r_2 : C \to B$ then $s_2 \circ s_1$ is a section of $r_1 \circ r_2$.*

Proof:

$$\begin{aligned} r_1 \circ r_2 \circ s_2 \circ s_1 &= r_1 \circ \mathrm{id}_B \circ s_1 \\ &= r_1 \circ s_1 \\ &= \mathrm{id}_A \qquad \square \end{aligned}$$

Proposition 8.8. *Every retraction is epi.*

Proof: If r is a retraction of s and $x \circ r = y \circ r$ then
$$x = xrs = yrs = y \;. \square$$

Proposition 8.9. *Every section is monic.*

Proof: Similar. $\square$

8.3 Isomorphisms

Definition 8.10 (Isomorphism)**.** We say a morphism $f : A \to B$ is an *isomorphism*, and write $f : A \cong B$, iff there exists a morphism $g : B \to A$, the *inverse* of f, such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 8.11. *The inverse of an isomorphism is unique.*

PROOF: From Proposition 8.6. □

Proposition 8.12. $\mathrm{id}_A : A \cong A$ *with* $\mathrm{id}_A^{-1} = \mathrm{id}_A$.

PROOF: Immediate from the Unit Laws. □

Proposition 8.13. *If* $f : A \cong B$ *then* $f^{-1} : B \cong A$ *and* $(f^{-1})^{-1} = f$.

PROOF: Immediate from definitions. □

Proposition 8.14. *If* $f : A \cong B$ *and* $g : B \cong C$ *then* $g \circ f : A \cong C$ *and* $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

PROOF: From Proposition 8.7. □

Proposition 8.15. *Every monic retraction is iso.*

PROOF:
$\langle 1 \rangle 1$. LET: $r : A \to B$ be a monic retraction of $s : B \to A$.
$\langle 1 \rangle 2$. $rsr = r$
$\langle 1 \rangle 3$. $sr = \mathrm{id}_A$
□

Proposition 8.16. *Every epi section is iso.*

PROOF: Dual. □

8.4 Idempotent Morphisms

Definition 8.17 (Idempotent)**.** A morphism $e : A \to A$ is *idempotent* iff $e \circ e = e$.

Proposition 8.18. *If* $r : A \to B$ *is a retraction of* $s : B \to A$ *then* $s \circ r$ *is idempotent.*

PROOF: Since $srsr = s\mathrm{id}_B r = sr$. □

Definition 8.19 (Splitting)**.** A *splitting* of an idempotent endomorphism $e : A \to A$ consists of an object B and morphisms $r : A \to B$, $s : B \to A$ such that $rs = \mathrm{id}_B$ and $sr = e$.

We say e *splits*, or is a *split idempotent*, iff it has a splitting.

Proposition 8.20. *If* $r : A \to B$, $s : B \to A$ *and* $r' : A \to B'$, $s' : B' \to A$ *are splittings of* $e : A \to A$*, then there exists a unique isomorphism* $\phi : B \cong B'$ *such that the following diagram commutes.*

$$\begin{tikzcd}
 & B \arrow[dd, "\phi"] \arrow[dr, "s"] & \\
A \arrow[ur, "r"] \arrow[dr, "r'"'] & & A \\
 & B' \arrow[ur, "s'"'] &
\end{tikzcd}$$

PROOF:
⟨1⟩1. LET: $\phi = r' \circ s$
⟨1⟩2. LET: $\phi^{-1} = r \circ s'$
⟨1⟩3. $\phi \circ \phi^{-1} = \mathrm{id}_{B'}$
PROOF:

$$\begin{aligned} \phi\phi^{-1} &= r'srs' \\ &= r'es' \\ &= r's'r's' \\ &= \mathrm{id}_{B'}\mathrm{id}_{B'} \\ &= \mathrm{id}_{B'} \end{aligned}$$

⟨1⟩4. $\phi^{-1} \circ \phi = \mathrm{id}_B$
PROOF:

$$\begin{aligned} \phi^{-1}\phi &= rs'r's \\ &= res \\ &= rsrs \\ &= \mathrm{id}_B\mathrm{id}_B \\ &= \mathrm{id}_B \end{aligned}$$

⟨1⟩5. $\phi \circ r = r'$
PROOF:

$$\begin{aligned} \phi r &= r'sr \\ &= r'e \\ &= r's'r' \\ &= r' \end{aligned}$$

⟨1⟩6. $s' \circ \phi = s$
PROOF:

$$\begin{aligned} s'\phi &= s'r's \\ &= es \\ &= srs \\ &= s \end{aligned}$$

⟨1⟩7. If $\psi : B \cong B'$ satisfies $\psi r = r'$ and $s'\psi = s$ then $\psi = \phi$.
PROOF:

$$\begin{aligned} \psi &= \psi rs \\ &= r's \\ &= \phi \end{aligned}$$

□

Hence we may talk about *the* splitting of a split idempotent.

Proposition 8.21. *If $e : A \to A$ is idempotent and either monic or epi, then $e = \mathrm{id}_A$.*

PROOF: We have $ee = e$ and so $e = \mathrm{id}_A$ by cancellation. □

8.5 Involutions

Definition 8.22 (Involution)**.** An *involution* on an object A is an automorphism $f : A \cong A$ such that $ff = f$.

Chapter 9

Terminal Objects

Definition 9.1 (Terminal Object)**.** An object T is *terminal* iff, for any object X, there is exactly one morphism $X \to T$.

Proposition 9.2. *If T and T' are terminal objects, then there exists a unique isomorphism $T \cong T'$.*

PROOF:
$\langle 1 \rangle 1$. LET: ϕ be the unique morphism $T \to T'$.
$\langle 1 \rangle 2$. LET: ϕ^{-1} be the unique morphism $T' \to T$.
$\langle 1 \rangle 3$. $\phi \circ \phi^{-1} = \mathrm{id}_{T'}$
  PROOF: Since there is only one morphism $T' \to T'$.
$\langle 1 \rangle 4$. $\phi^{-1} \circ \phi = \mathrm{id}_T$
  PROOF: Since there is only one morphism $T \to T$.
$\square$

Chapter 10

Initial Objects

Definition 10.1 (Initial Object)**.** An object I is *initial* iff, for any object X, there is exactly one morphism $I \to X$.

Proposition 10.2. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:
$\langle 1 \rangle 1$. LET: ϕ be the unique morphism $I \to I'$.
$\langle 1 \rangle 2$. LET: ϕ^{-1} be the unique morphism $I' \to I$.
$\langle 1 \rangle 3$. $\phi \circ \phi^{-1} = \mathrm{id}_{I'}$
  PROOF: Since there is only one morphism $I' \to I'$.
$\langle 1 \rangle 4$. $\phi^{-1} \circ \phi = \mathrm{id}_I$
  PROOF: Since there is only one morphism $I \to I$.
□

Chapter 11

Functors

Definition 11.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for every morphism $f : A \to B$ in $\mathcal{C}$, a morphism $Ff : FA \to FB$ in $\mathcal{D}$

such that:

$$F\mathrm{id}_A = \mathrm{id}_{FA}$$
$$F(g \circ f) = Fg \circ Ff$$

Part III

Abstract Algebra

Chapter 12

Monoid Theory

Definition 12.1. We identify every monoid M with the category with one object and set of morphisms M, with composition given by the multiplication in M.

Part IV

Linear Algebra

Chapter 13

Vector Spaces

13.1 Convex Sets

Definition 13.1 (Convex)**.** Let $E \subseteq \mathbb{R}^k$. Then E is *convex* iff, for all $\vec{x}, \vec{y} \in E$ and $\lambda \in (0,1)$,

$$\lambda\vec{x} + (1-\lambda)\vec{y} \in E \enspace .$$

Proposition 13.2. *Every k-cell is convex.*

PROOF:
⟨1⟩1. LET: $C = \{\vec{x} \in \mathbb{R}^k : \forall i.a_i \leq x_i \leq b_i\}$ be a k-cell.
⟨1⟩2. LET: $\vec{x}, \vec{y} \in C$ and $\lambda \in (0,1)$.
PROVE: $\lambda\vec{x} + (1-\lambda)\vec{y} \in C$
⟨1⟩3. For each i we have $a_i \leq \lambda x_i + (1-\lambda)y_i \leq b_i$
PROOF: Since $\lambda a_1 + (1-\lambda)a_i \leq \lambda x_i + (1-\lambda)y_i \leq \lambda b_i + (1-\lambda)b_i$.
∎

13.2 Linear Transformations

Definition 13.3 (Norm)**.** For $A \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$, define the *norm* of A to be

$$\|A\| := \{\|A\vec{x}\| : \vec{x} \in \mathbb{R}^n, \|\vec{x}\| = 1\} \enspace .$$

We prove that this always exists.

PROOF: Since for $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1^2 + \cdots + x_n^; = 1$ we have

$$\begin{aligned}
\|A(x_1, \ldots, x_n)\| &= \|\sum_{i=1}^n x_i A\vec{e_i}\| \\
&\leq \sum_{i=1}^n |x_i| \|A\vec{e_i}\| \\
&\leq \sum_{i=1}^n \|A\vec{e_i}\|
\end{aligned}$$

∎

Proposition 13.4. *Given $A, B \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$, we have*

$$\|A + B\| \leq \|A\| + \|B\|$$

PROOF: Since $\|A\vec{x} + B\vec{x}\| \leq \|A\vec{x}\| + \|B\vec{x}\|$. $\square$

Proposition 13.5. *Given $A \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m$ and $c \in \mathbb{R}$, we have*

$$\|cA\| = |c|\|A\| \enspace .$$

PROOF: Since $\|cA\vec{x}\| = |c|\|A\vec{x}\|$. $\square$

Proposition 13.6. *Given $A \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$ and $B \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^m, \mathbb{R}^k]$, we have*

$$\|BA\| \leq \|B\|\|A\| \enspace .$$

PROOF: Since $\|BA\vec{x}\| \leq \|B\|\|A\vec{x}\| \leq \|B\|\|A\|\|\vec{x}\|$. $\square$

Lemma 13.7. *Let $A, B \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$ with A invertible. Let $\alpha = 1/\|A^{-1}\|$ and $\beta = \|B - A\|$. Then, for all $\vec{x} \in \mathbb{R}^n$, we have*

$$(\alpha - \beta)\|\vec{x}\| \leq \|B\vec{x}\| \enspace .$$

PROOF:

$$\begin{aligned}
\alpha\|\vec{x}\| &= \alpha\|A^{-1}A\vec{x}\| \\
&\leq \alpha\|A^{-1}\|\|A\vec{x}\| \\
&= \|A\vec{x}\| \\
&\leq \|(A - B)\vec{x}\| + \|B\vec{x}\| \\
&\leq \beta\|\vec{x}\| + \|B\vec{x}\|
\end{aligned}$$

$\square$

Proposition 13.8. *Let $A, B \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^n]$. If A is invertible and*

$$\|B - A\|\|A^{-1}\| < 1$$

then B is invertible.

PROOF:
⟨1⟩1. LET: $\alpha = 1/\|A^{-1}\|$
⟨1⟩2. LET: $\beta = \|B - A\|$
⟨1⟩3. $\beta < \alpha$
⟨1⟩4. For all $\vec{x} \in \mathbb{R}^n$ we have $\alpha\|\vec{x}\| \leq \beta\|\vec{x}\| + \|B\vec{x}\|$
PROOF: Lemma 12.7.
⟨1⟩5. For all $\vec{x} \in \mathbb{R}^n$ we have

$$(\alpha - \beta)\|\vec{x}\| \leq \|B\vec{x}\| \enspace .$$

⟨1⟩6. $\ker B = 0$
PROOF: Since $\alpha - \beta > 0$ so if $\|\vec{x}\| > 0$ then $\|B\vec{x}\| > 0$.
$\square$

Chapter 14

Real Inner Product Spaces

Definition 14.1 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^k$, define the *inner product* $\vec{x} \cdot \vec{y}$ by

$$(x_1, \ldots, x_k) \cdot (y_1, \ldots, y_k) = x_1 y_1 + \cdots + x_k y_k \ .$$

Definition 14.2 (Norm)**.** Define the *norm* of a vector $\vec{x} \in \mathbb{R}^k$ by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} \ .$$

Proposition 14.3.

$$\|\vec{x}\| \geq 0$$

PROOF: Immediate from the definition. □

Proposition 14.4. *If* $\|\vec{x}\| = 0$ *then* $\vec{x} = \vec{0}$.

PROOF: If $\|\vec{x}\| = 0$ then $x_1^2 + \cdots + x_n^2 = 0$ so $x_1 = \cdots = x_n = 0$. □

Proposition 14.5. *For* $\alpha \in \mathbb{R}$ *and* $\vec{x} \in \mathbb{R}^k$,

$$\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| \ .$$

PROOF: Easy. □

Proposition 14.6. *For* $\vec{x}, \vec{y} \in \mathbb{R}^k$, *we have*

$$\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\| \|\vec{y}\| \ .$$

PROOF: By the Schwarz inequality. □

Proposition 14.7. *For* $\vec{x}, \vec{y} \in \mathbb{R}^k$ *we have*

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \ .$$

PROOF:

$$\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \\
&= \vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Proposition 13.6)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2
\end{aligned}$$

∎

Corollary 14.7.1. *For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ we have*

$$\|\vec{x}-\vec{z}\| \leq \|\vec{x}-\vec{y}\| + \|\vec{y}-\vec{z}\| \ .$$

Definition 14.8 (Bounded Function)**.** Let E be a set. Let $f : E \to \mathbb{R}^k$. Then f is *bounded* iff $f(E)$ is bounded.

14.1 Balls

Definition 14.9 (Closed Ball)**.** Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *closed ball* with *centre* $\vec{x}$ and *radius* r is

$$\{y \in \mathbb{R}^k : \|y - x\| \leq r\} \ .$$

Proposition 14.10. *Every closed ball is convex.*

PROOF:

$\langle 1\rangle 1.$ LET: B be the closed ball with center $\vec{a}$ and radius r.

$\langle 1\rangle 2.$ LET: $\vec{x}, \vec{y} \in B$

$\langle 1\rangle 3.$ LET: $\lambda \in (0,1)$

$\langle 1\rangle 4.$ $\lambda\vec{x} + (1-\lambda)\vec{y} \in B$

PROOF:

$$\begin{aligned}
\|\lambda\vec{x} + (1-\lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x}-\vec{a}) + (1-\lambda)(\vec{y}-\vec{a})\| \\
&= \lambda\|\vec{x}-\vec{a}\| + (1-\lambda)\|\vec{y}-\vec{a}\| \\
&\leq \lambda r + (1-\lambda) r \\
&= r
\end{aligned}$$

∎

∎

Chapter 15

Complex Inner Product Spaces

Definition 15.1 (Inner Product)**.** Let V be a complex vector space. An *inner product* on V is a function $\langle\ ,\ \rangle : V^2 \to \mathbb{C}$ such that, for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

- $\langle y, x\rangle = \overline{\langle x, y\rangle}$
- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
- $\langle \alpha x, y\rangle = \overline{\alpha}\langle x, y\rangle$
- $\langle x, x\rangle \geq 0$
- If $\langle x, x\rangle = 0$ then $x = 0$.

An *inner product space* consists of a complex vector space V and an inner product on V.

Definition 15.2 (Norm)**.** Let V be an inner product space and $x \in V$. The *norm* of x is

$$\|x\| = \sqrt{\langle x, x\rangle}\ .$$

Proposition 15.3. *An inner product space is a metric space under*

$$d(x, y) = \|x - y\|\ .$$

Definition 15.4 (Bounded)**.** Let V_1 and V_2 be inner product spaces and $T : V_1 \to V_2$ a linear transformation. Then T is *bounded* iff $\{\|T(x)\| : \|x\| = 1\}$ is bounded above.

Proposition 15.5. *Every linear transformation between finite dimensional inner product spaces is bounded.*

Definition 15.6 (Outer Product)**.** Let V be an inner product space and $|\psi\rangle, |\phi\rangle \in V$. The *outer product* of $|\psi\rangle$ and $|\phi\rangle$ is

$$|\psi\rangle\langle\phi| : V \to V\ .$$

15.1 Hilbert Spaces

Definition 15.7 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Theorem 15.8 (Completeness Relation)**.** *Let $\mathcal{H}$ be a Hilbert space. Let $\{|e_n\rangle\}_{n\in\mathbb{N}}$ be a countable orthonormal basis for $\mathcal{H}$. Then*

$$\sum_{n=0}^{\infty} |e_n\rangle \langle e_n| = I \quad .$$

PROOF:
⟨1⟩1. LET: $|\psi\rangle \in \mathcal{H}$
⟨1⟩2. LET: $|\psi\rangle = \sum_{n=0}^{\infty} \alpha_n |e_n\rangle$
⟨1⟩3. $\sum_{n=0}^{\infty} \langle e_n|\phi\rangle |e_n\rangle = |\psi\rangle$
PROOF:

$$\begin{aligned}\sum_{n=0}^{\infty} \langle e_n|\phi\rangle |e_n\rangle &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \alpha_m \langle e_n|e_m\rangle |e_n\rangle \\ &= \sum_{n=0}^{\infty} \alpha_n |e_n\rangle \\ &= |\psi\rangle \end{aligned}$$

□

□

Definition 15.9 (Separable)**.** A Hilbert space is *separable* iff it has a countable dense orthonormal basis.

Chapter 16

Lie Algebras

Definition 16.1 (Lie Algebra)**.** Let K be a field. A *Lie algebra* $\mathcal{L}$ over K consists of a vector space $\mathcal{L}$ over K and an operation

$$[\ ,\ ] : \mathcal{L}^2 \to \mathcal{L}\ ,$$

the *Lie bracket* or *commutator*, such that, for all $x, y, z \in \mathcal{L}$ and $\alpha \in K$:

$$\begin{aligned} [x+y, z] &= [x, z] + [y, z] \\ [x, y+z] &= [x, y] + [x, z] \\ [\alpha x, y] &= \alpha[x, y] \\ [x, x] &= 0 \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \qquad \text{(Jacobi identity)} \end{aligned}$$

Lemma 16.2. *If K has characteristic 0 then the condition $[x, x] = 0$ can be replaced with $[x, y] = -[y, x]$.*

Proposition 16.3. *The commutator is determind by its values on any basis for $\mathcal{L}$.*

Example 16.4. $\mathbb{R}^3$ with the cross product is a real Lie algebra.

Example 16.5. For any $n \geq 0$, we have $GL(n, K)$ is a Lie algebra over K under

$$[A, B] = AB - BA\ .$$

Definition 16.6 (Linear Lie Algebra)**.** A *linear Lie algebra* over K is a Lie algebra over K that is a subalgebra of $GL(n, K)$ for some n.

Example 16.7 (Special Linear Algebra)**.** The *special Linear algebra* $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \mathrm{tr} = 0\}$ is a real linear Lie algebra.

Example 16.8 (Orthogonal Lie Algebra)**.** The *orthogonal Lie algebra* $SO(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A \text{ is skew-symmetric}\}$ is a real linear Lie algebra.

Example 16.9. Let $u(n)$ be the set of all skew-Hermitian $n \times n$-matrices as a real Lie algebra.

Let $su(n) = u(n) \cap SL(n, \mathbb{R})$.

Proposition 16.10. *$SU(2)$ is spanned by the* Pauli matrices

$$\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which satisfy

$$[\sigma_x, \sigma_y] = \sigma_z$$
$$[\sigma_y, \sigma_z] = \sigma_x$$
$$[\sigma_z, \sigma_x] = \sigma_y$$

16.1 Lie Algebar Homomorphisms

Definition 16.11 (Homomorphism)**.** Let L_1 and L_2 be Lie algebras over the same field. A *Lie algebra homomorphism* $\phi : L_1 \to L_2$ is a linear transformation such that

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in L_1$.

Lemma 16.12. *Every bijective Lie algebra homomorphism is an isomorphism.*

Definition 16.13 (Representation)**.** Let L be a real (complex) Lie algebra. A *representation* of L is a Lie algebra homomorphism $L \to GL(n, \mathbb{R})$ $(GL(n, \mathbb{C}))$ for some n.

Example 16.14. The linear transformation $\mathbb{R}^3 \to su(2)$ defined by

$$i \mapsto \sigma_x, j \mapsto \sigma_y, k \mapsto \sigma_z$$

is a representation of $\mathbb{R}^3$.

Part V

Topology

Chapter 17

Metric Spaces

Definition 17.1 (Metric)**.** A *metric* on a set X is a function $d : X^2 \to \mathbb{R}$ such that, for all $x, y, z \in X$:

- $d(x, y) \geq 0$
- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- **Triangle Inequality** $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* X consists of a set X and a metric on X.

Example 17.2. $\mathbb{R}^k$ is a metric space under $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$. The triangle inequality is Corollary 13.7.1.

Example 17.3. For any set X, the *discrete* metric on X is defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Example 17.4. $\mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$ is a metric space under $d(A, B) = \|A - B\|$.

Proposition 17.5. *Let (X, d) be a metric space and Y a subset of X. Then $d \restriction Y^2$ is a metric on Y.*

Proof: Easy. $\square$

17.1 Balls

Definition 17.6 (Open Ball)**.** Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *open ball* with *centre* $\vec{x}$ and *radius* r is

$$\{y \in \mathbb{R}^k : \|y - x\| < r\} \ .$$

Proposition 17.7. *Every open ball in $\mathbb{R}^k$ is convex.*

PROOF:
$\langle 1\rangle 1.$ LET: B be the open ball with center $\vec{a}$ and radius r.
$\langle 1\rangle 2.$ LET: $\vec{x}, \vec{y} \in B$
$\langle 1\rangle 3.$ LET: $\lambda \in (0,1)$
$\langle 1\rangle 4.$ $\lambda\vec{x} + (1-\lambda)\vec{y} \in B$
PROOF:

$$\begin{aligned} \|\lambda\vec{x} + (1-\lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1-\lambda)(\vec{y} - \vec{a})\| \\ &= \lambda\|\vec{x} - \vec{a}\| + (1-\lambda)\|\vec{y} - \vec{a}\| \\ &< \lambda r + (1-\lambda)r \\ &= r \end{aligned}$$

□

□

17.2 Limit Points

Definition 17.8 (Limit Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is a *limit point* of E iff every open ball with centre p contains a point of E other than p.

Proposition 17.9. *Let X be a metric space. Let $E \subseteq X$. Let p be a limit point of E. Then every neighbourhood of p contains infinitely many points of E.*

PROOF:
$\langle 1\rangle 1.$ ASSUME: for a contradiction N is a neighbourhood of p that contains only finitely many points $q_1, \ldots, q_n$ of $E - \{p\}$.
$\langle 1\rangle 2.$ LET: $r = \min(q_1, \ldots, q_n)$
$\langle 1\rangle 3.$ LET: B be the open ball with centre p and radius r.
$\langle 1\rangle 4.$ B is a neighbourhood of p that contains no points of E other than p.
□

Corollary 17.9.1. *A finite set has no limit points.*

Definition 17.10 (Isolated Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *isolated point* of E iff $p \in E$ and p is not a limit point of E.

17.3 Closed Sets

Definition 17.11 (Closed Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *closed* iff every limit point of E is a member of E.

17.4 Interior Points

Definition 17.12 (Interior Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *interior point* of E iff there exists an open ball B with centre p such that $B \subseteq E$.

Definition 17.13 (Interior)**.** The *interior* of a set E, denoted E°, is the set of all its interior points.

Proposition 17.14. *The interior of E is the largest open set that is included in E.*

PROOF:
$\langle 1 \rangle 1$. LET: I be the interior of E.
$\langle 1 \rangle 2$. I is open.
  $\langle 2 \rangle 1$. LET: $p \in I$
  $\langle 2 \rangle 2$. PICK an open ball B with centre p such that $B \subseteq E$.
  $\langle 2 \rangle 3$. $B \subseteq I$
    $\langle 3 \rangle 1$. LET: $q \in B$
    $\langle 3 \rangle 2$. There exists an open ball B' with centre q such that $B' \subseteq B$.
    $\langle 3 \rangle 3$. There exists an open ball B' with centre q such that $B' \subseteq E$.
    $\langle 3 \rangle 4$. $q \in I$
$\langle 1 \rangle 3$. If J is any open set and $J \subseteq E$ then $J \subseteq I$.
  $\langle 2 \rangle 1$. LET: J be an open set.
  $\langle 2 \rangle 2$. ASSUME: $J \subseteq E$
  $\langle 2 \rangle 3$. For all $p \in J$, there exists an open ball B with centre p such that $B \subseteq J$.
  $\langle 2 \rangle 4$. For all $p \in J$, there exists an open ball B with centre p such that $B \subseteq E$.
  $\langle 2 \rangle 5$. $p \in I$
$\square$

17.5 Open Sets

Definition 17.15 (Open Sets)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *open* iff every point in E is an interior point of E.

Proposition 17.16. *Every open ball is open.*

PROOF:
$\langle 1 \rangle 1$. LET: B be an open ball with centre c and radius r.
$\langle 1 \rangle 2$. LET: $x \in B$
$\langle 1 \rangle 3$. LET: $\epsilon = r - d(x, c)$
$\langle 1 \rangle 4$. LET: B' be the open ball with centre x and radius ϵ.
  PROVE: $B' \subseteq B$
$\langle 1 \rangle 5$. LET: $y \in B'$
$\langle 1 \rangle 6$. $d(y, c) < r$
  PROOF:

$$\begin{aligned} d(y,c) &\leq d(y,x) + d(x,c) && \text{(Triangle Inequality)} \\ &< \epsilon + d(x,c) && (\langle 1 \rangle 5) \\ &= r && (\langle 1 \rangle 3) \end{aligned}$$

$\square$

Proposition 17.17. *A set is open if and only if its complement is closed.*

PROOF:
⟨1⟩1. LET: $E \subseteq X$
⟨1⟩2. If E is open then $X - E$ is closed.
  ⟨2⟩1. ASSUME: E is open.
  ⟨2⟩2. LET: p be a limit point of $X - E$.
     PROVE: $p \in X - E$
  ⟨2⟩3. ASSUME: for a contradiction $p \in E$.
  ⟨2⟩4. PICK an open ball B with centre p such that $B \subseteq E$.
  ⟨2⟩5. B contains a point of $X - E$.
   PROOF: ⟨2⟩2
  ⟨2⟩6. Q.E.D.
   PROOF: This contradicts ⟨2⟩4.
⟨1⟩3. If $X - E$ is closed then E is open.
  ⟨2⟩1. ASSUME: $X - E$ is closed.
  ⟨2⟩2. LET: $p \in E$
  ⟨2⟩3. ASSUME: for a contradiction no open ball with centre p is a subset of E.
  ⟨2⟩4. Every open ball with centre p intersects $X - E$.
  ⟨2⟩5. p is a limit point of $X - E$.
  ⟨2⟩6. $p \in X - E$
   PROOF: ⟨2⟩1
  ⟨2⟩7. Q.E.D.
   PROOF: This contradicts ⟨2⟩2.
□

Corollary 17.17.1. *A set is closed if and only if its complement is open.*

Proposition 17.18. *The union of a set of open sets is open.*

PROOF:
⟨1⟩1. LET: $\mathcal{U}$ be a set of open sets.
⟨1⟩2. LET: $p \in \bigcup \mathcal{U}$
⟨1⟩3. PICK $U \in \mathcal{U}$ such that $p \in U$.
⟨1⟩4. PICK an open ball B with centre p such that $B \subseteq U$.
⟨1⟩5. $B \subseteq \bigcup \mathcal{U}$
□

Corollary 17.18.1. *The intersection of a set of closed sets is closed.*

Proposition 17.19. *The intersection of two open sets is open.*

PROOF:
⟨1⟩1. LET: U and V be open.
⟨1⟩2. LET: $p \in U \cap V$
⟨1⟩3. PICK open balls B_1 and B_2 with centre p such that $B_1 \subseteq U$ and $B_2 \subseteq V$.
⟨1⟩4. ASSUME: w.l.o.g. the radius of B_1 is $\leq$ the radius of B_2.

⟨1⟩5. $B_1 \subseteq U \cap V$
□

Corollary 17.19.1. *The union of two closed sets is closed.*

Example 17.20. The intersection of a set of open sets is not necessarily open.

For every positive integer n, we have $(-1/n, 1/n)$ is open in $\mathbb{R}$, but $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$ is not open.

Theorem 17.21. *Let X be a metric space. Let $Y \subseteq X$ and $E \subseteq Y$. Then E is open in Y if and only if there exists an open subset G of X such that $E = G \cap Y$.*

PROOF:
⟨1⟩1. If E is open in Y then there exists an open subset G of X such that $E = G \cap Y$.
⟨2⟩1. ASSUME: E is open in Y.
⟨2⟩2. For $p \in E$, PICK $r_p > 0$ such that the open ball in Y with centre p and radius r_p is included in E.
⟨2⟩3. For $p \in E$,
LET: V_p be the open ball in X with centre p and radius r_p.
⟨2⟩4. LET: $G = \bigcup_{p \in E} V_p$
⟨2⟩5. G is open in Y.
PROOF: Proposition 16.18.
⟨2⟩6. $E = G \cap Y$
⟨3⟩1. $E \subseteq G \cap Y$
⟨4⟩1. LET: $p \in E$
⟨4⟩2. $p \in V_p$
⟨4⟩3. $p \in G$
⟨3⟩2. $G \cap Y \subseteq E$
⟨4⟩1. LET: $x \in G \cap Y$
⟨4⟩2. PICK $p \in E$ such that $x \in V_p$
⟨4⟩3. $d(x, p) < r_p$
⟨4⟩4. $x \in E$
⟨1⟩2. For any open subset G of X, we have $G \cap Y$ is open in Y.
⟨2⟩1. LET: G be an open subset of X.
⟨2⟩2. LET: $p \in G \cap Y$
⟨2⟩3. PICK $r > 0$ such that the open ball in X with centre p and radius r is included in G.
⟨2⟩4. The open ball in Y with centre p and radius r is included in $G \cap Y$.
□

Proposition 17.22. *The set Ω of all invertible linear transformations is an open set in $\mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^n]$.*

PROOF: For $A \in \Omega$ we have $B(A, 1/\|A^{-1}\|) \subseteq \Omega$ by Proposition 12.8. □

17.6 Perfect Sets

Definition 17.23 (Perfect Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *perfect* iff E is closed and every point in E is a limit point of E.

17.7 Bounded Sets

Definition 17.24 (Bounded Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *bounded* iff there exists a real number M and $q \in X$ such that, for all $p \in E$, we have $d(p, q) < M$.

Definition 17.25 (Diameter)**.** Let X be a metric space and $E \subseteq X$ be bounded. Then the *diameter* of E is $\sup\{d(x, y) : x, y \in E\}$.

Proposition 17.26. *Let X be a metric space. Let $E \subseteq X$ be bounded. Then $\overline{E}$ is bounded and*

$$\operatorname{diam} \overline{E} = \operatorname{diam} E \quad .$$

PROOF:
⟨1⟩1. $\operatorname{diam} E$ is an upper bound for $\{d(x, y) : x, y \in \overline{E}\}$.
  ⟨2⟩1. LET: $x, y \in \overline{E}$
  ⟨2⟩2. For all $\epsilon > 0$ we have $d(x, y) < \operatorname{diam} E + \epsilon$.
    ⟨3⟩1. LET: $\epsilon > 0$
    ⟨3⟩2. PICK $x', y' \in E$ such that $d(x', x) < \epsilon/2$ and $d(y', y) < \epsilon/2$
    ⟨3⟩3. $d(x', y') < \operatorname{diam} E$
    ⟨3⟩4. $d(x, y) < \operatorname{diam} E + \epsilon$
  ⟨2⟩3. $d(x, y) \leq \operatorname{diam} E$
⟨1⟩2. $\operatorname{diam} \overline{E}$ is an upper bound for $\{d(x, y) : x, y \in E\}$.
  PROOF: This follows since $E \subseteq \overline{E}$.
□

17.8 Dense Sets

Definition 17.27 (Dense Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *dense* iff every point of X is either a limit point of E or a point of E, or both.

17.9 Closure

Definition 17.28 (Closure)**.** Let X be a metric space. Let $E \subseteq X$. Then the *closure* of E, denoted $\overline{E}$, is the union of E and the set of limit points of E.

Proposition 17.29. *$\overline{E}$ is the smallest closed set that includes E.*

PROOF:
⟨1⟩1. $\overline{E}$ is closed.
  ⟨2⟩1. LET: p be a limit point of $\overline{E}$.

⟨2⟩2. Assume: $p \notin E$
Prove: p is a limit point of E.
⟨2⟩3. Let: B be the open ball with centre p and radius r.
Prove: B intersects E.
⟨2⟩4. Pick a point $q \in B \cap \overline{E}$.
⟨2⟩5. Pick an open ball B' with centre q such that $B' \subseteq B$.
⟨2⟩6. Pick a point $r \in E \cap B'$
⟨2⟩7. $r \in E \cap B$
⟨1⟩2. If C is closed and $E \subseteq C$ then $\overline{E} \subseteq C$.
⟨2⟩1. Assume: C is closed.
⟨2⟩2. Assume: $E \subseteq C$
⟨2⟩3. Let: $p \in \overline{E}$
⟨2⟩4. Assume: for a contradiction $p \notin C$
⟨2⟩5. p is a limit point of C.
⟨3⟩1. Let: B be an open ball with centre p.
⟨3⟩2. B intersects E.
⟨3⟩3. B intersects C.
⟨3⟩4. B intersects C in a point other than p.
Proof: ⟨2⟩3
⟨2⟩6. Q.E.D.
Proof: This contradicts ⟨2⟩1.
□

Corollary 17.29.1. *E is closed if and only if $E = \overline{E}$.*

Theorem 17.30. *Let E be a nonempty set of real numbers bounded above. Then $\sup E \in \overline{E}$.*

Proof:
⟨1⟩1. Assume: $\sup E \notin E$
Prove: $\sup E$ is a limit point of E.
⟨1⟩2. Let: B be an open ball with centre $\sup E$ and radius r.
⟨1⟩3. There exists $x \in E$ such that $x > \sup E - r$.
⟨1⟩4. E intersects B in a point other than p.
□

Proposition 17.31.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proof:
⟨1⟩1. $\overline{A} \cup \overline{B}$ is a closed set that includes $A \cup B$.
⟨1⟩2. If C is a closed set that includes $A \cup B$ then $\overline{A} \cup \overline{B} \subseteq C$.
□

Example 17.32. It is not true in general. that $\overline{\bigcup \mathcal{A}} = \bigcup_{A \in \mathcal{A}} \overline{A}$.

In $\mathbb{R}$, let $\mathcal{A} = \{\{1/n\} : n \in \mathbb{Z}^+\}$. Then

$$\overline{\bigcup \mathcal{A}} = \{1/n : n \in \mathbb{Z}^+\} \cup \{0\}$$
$$\bigcup_{A \in \mathcal{A}} \overline{A} = \{1/n : n \in \mathbb{Z}^+\}$$

Proposition 17.33.

$$X - E^\circ = \overline{X - E}$$

Proof:

$$\begin{aligned} p \in X - E^\circ &\Leftrightarrow p \notin E^\circ \\ &\Leftrightarrow \forall B \text{ an open ball with centre } p. B \nsubseteq E \\ &\Leftrightarrow \forall B \text{ an open ball with centre } p. B \text{ intersects} X - E \\ &\Leftrightarrow p \in \overline{X - E} \end{aligned}$$

∎

17.10 Compact Sets

Definition 17.34 (Open Cover)**.** Let X be a metric space. Let $E \subseteq X$. An *open cover* of E is a set $\mathcal{U}$ of open sets such that $E \subseteq \bigcup \mathcal{U}$.

Definition 17.35 (Compact Set)**.** Let X be a metric space. Let $K \subseteq X$. Then K is *compact* iff every open cover of K includes a finite subcover.

Proposition 17.36. *Every finite set is compact.*

Proof: Easy. ∎

Theorem 17.37. *Let X be a metric space. Let $Y \subseteq X$ and $K \subseteq Y$. Then K is compact in Y if and only if K is compact in X.*

Proof:
$\langle 1 \rangle 1$. If K is compact in Y then K is compact in X.
  $\langle 2 \rangle 1$. Assume: K is compact in Y.
  $\langle 2 \rangle 2$. Let: $\mathcal{U}$ be an open cover of K in X.
  $\langle 2 \rangle 3$. $\{U \cap Y : U \in \mathcal{U}\}$ is an open cover of K in Y.
  $\langle 2 \rangle 4$. Pick a finite subcover $\{U_1 \cap Y, \dots, U_n \cap Y\}$
  $\langle 2 \rangle 5$. $\{U_1, \dots, U_n\}$ is a finite subset of $\mathcal{U}$ that is an open cover of K is X.
$\langle 1 \rangle 2$. If K is compact in X then K is compact in Y.
  $\langle 2 \rangle 1$. Assume: K is compact in X.
  $\langle 2 \rangle 2$. Let: $\mathcal{U}$ be an open cover of K in Y.
  $\langle 2 \rangle 3$. $\{U \text{ open in } X : U \cap Y \in \mathcal{U}\}$ is an open cover of K in X.
  $\langle 2 \rangle 4$. Pick a finite subcover $\{U_1, \dots, U_n\}$.
  $\langle 2 \rangle 5$. $\{U_1 \cap Y, \dots, U_n \cap Y\}$ is a subset of $\mathcal{U}$ that is an open cover of E in Y.
∎

Proposition 17.38. *Every compact set is closed.*

PROOF:
⟨1⟩1. LET: E be compact.
⟨1⟩2. LET: $p \in X - E$
PROVE: There exists an open ball with centre p that is a subset of $X - E$.
⟨1⟩3. For all $q \in E$, there exist disjoint open balls B with centre q and B' with centre p.
⟨1⟩4. The set of open balls B such that there exists a disjoint open ball B' with centre p is an open cover of E.
⟨1⟩5. PICK a finite subcover $\{B_1, \ldots, B_n\}$.
⟨1⟩6. For $i = 1, \ldots, n$, PICK an open ball B'_i with centre p such that $B_i \cap B'_i = \emptyset$.
⟨1⟩7. $B'_1 \cap \cdots \cap B'_n$ is an open ball with centre p that is a subset of $X - E$.
□

Proposition 17.39. *Every closed subset of a compact set is compact.*

PROOF:
⟨1⟩1. LET: E be compact and $C \subseteq E$ be closed.
⟨1⟩2. LET: $\mathcal{U}$ be an open cover of C.
⟨1⟩3. $\mathcal{U} \cup \{X - C\}$ is an open cover of E.
⟨1⟩4. PICK a finite subcover $\{U_1, \ldots, U_n\}$ or $\{U_1, \ldots, U_n, X - C\}$.
⟨1⟩5. $\{U_1, \ldots, U_n\}$ covers C.
□

Corollary 17.39.1. *The intersection of a compact set and a closed set is compact.*

Proposition 17.40. *Let $\mathcal{K}$ be a nonempty set of compact sets. If every nonempty finite subset of $\mathcal{K}$ has nonempty intersection, then $\bigcap \mathcal{K}$ is nonempty.*

PROOF:
⟨1⟩1. PICK $K \in \mathcal{K}$
⟨1⟩2. ASSUME: $\bigcap \mathcal{K} = \emptyset$
⟨1⟩3. $\{X - K' : K' \in \mathcal{K}\}$ is an open cover of K.
⟨1⟩4. PICK a finite subcover $\{X - K_1, \ldots, X - K_n\}$.
⟨1⟩5. There exists $p \in K \cap K_1 \cap \cdots \cap K_n$
⟨1⟩6. Q.E.D.
PROOF: ⟨1⟩4 and ⟨1⟩5 form a contradiction.
□

Corollary 17.40.1. *Let (K_n) be a sequence of nonempty compact sets such that $K_0 \supseteq K_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.*

Theorem 17.41. *Let X be a metric space and $E \subseteq X$. Then E is compact if and only if every infinite subset of E has a limit point in E.*

PROOF:
⟨1⟩1. If E is compact then every infinite subset of E has a limit point in E.
⟨2⟩1. ASSUME: E is compact.
⟨2⟩2. LET: $A \subseteq E$ be infinite.

⟨2⟩3. Assume: for a contradiction E has no limit point in K.
⟨2⟩4. For all $p \in K$, there exists an open ball B with centre p such that B does not intersect E outside p.
⟨2⟩5. The set of open balls that intersect E in at most one point is an open cover for K.
⟨2⟩6. Pick a finite subcover $B_1, \ldots, B_n$.
⟨2⟩7. E has at most n points.
⟨2⟩8. Q.E.D.
Proof: This contradicts the fact that E is finite.
⟨1⟩2. If every infinite subset of K has a limit point in K then K is compact.
⟨2⟩1. Assume: Every infinite subset of K has a limit point in K.
⟨2⟩2. Let: $\mathcal{U}$ be an open cover of K.
⟨2⟩3. Assume: w.l.o.g. $\mathcal{U}$ is countable.
Proof: We may replace $\mathcal{U}$ with the set of all open balls B with centres in $\mathbb{Q}^2$ and rational radius such that there exists $U \in \mathcal{U}$ such that $B \subseteq U$.
⟨2⟩4. Pick an enumeration $\mathcal{U} = \{G_n : n \in \mathbb{N}\}$.
⟨2⟩5. For $n \in \mathbb{N}$,
Let: $F_n = \bigcup_{i=0}^{n} G_n$.
⟨2⟩6. For all $n \in \mathbb{N}$, we have $K - F_n \neq \emptyset$.
Proof: Since $\{G_0, \ldots, G_n\}$ does not cover K.
⟨2⟩7. $\bigcap_{n=0}^{\infty} F_n = \emptyset$
Proof: Since $\{G_n : n \in \mathbb{N}\}$ covers K.
⟨2⟩8. For $n \in \mathbb{N}$, Pick $a_n \in K - F_n$
⟨2⟩9. Let: $E = \{a_n : n \in \mathbb{N}\}$
⟨2⟩10. E is infinite.
⟨3⟩1. Let: $n \in \mathbb{N}$
Prove: there exists m such that $a_m \notin \{a_0, a_1, \ldots, a_n\}$.
⟨3⟩2. For $i = 0, \ldots, n$, Pick k_i such that $a_i \in G_{k_i}$.
⟨3⟩3. Let: $m = \max(k_0, \ldots, k_n)$
⟨3⟩4. Assume: for a contradiction $a_m = a_i$ for some $i = 0, \ldots, n$
⟨3⟩5. $a_i \in G_{k_i}$
⟨3⟩6. $a_i \notin F_m$
⟨3⟩7. Q.E.D.
Proof: This is a contradiction since $k_i \leq m$.
⟨2⟩11. Pick a limit point l for E in K.
Proof: From ⟨2⟩1.
⟨2⟩12. Pick n such that $l \in G_n$.
⟨2⟩13. Pick an open ball B with centre l such that $B \subseteq G_n$
⟨2⟩14. $B \cap E$ is infinite.
Proof: Proposition 16.9.
⟨2⟩15. Pick $m \geq$ ‘n such that $a_m \in B$.
⟨2⟩16. $a_m \in G_n$
⟨2⟩17. Q.E.D.
Proof: This is a contradiction since $a_m \notin F_m$.

□

Theorem 17.42 (Heine-Borel)**.** *Let $E \subseteq \mathbb{R}^k$. Then E is compact if and only if it is closed and bounded.*

PROOF:
⟨1⟩1. If E is compact then E is closed.
  PROOF: Proposition 16.38.
⟨1⟩2. If E is compact then E is bounded.
  PROOF: Otherwise $\{(-N, N)^k : N \in \mathbb{Z}^+\}$ would be an open cover of E with no finite subcover.
⟨1⟩3. If E is closed and bounded then E is compact.
  ⟨2⟩1. ASSUME: E is closed and bounded.
  ⟨2⟩2. PICK $\vec{c}$ and M such that $\forall \vec{x} \in E. \|\vec{x} - \vec{c}\| < M$.
  ⟨2⟩3. $E \subseteq \prod_{i=1}^{k} [c_i - M, c_i + M]$
  ⟨2⟩4. E is compact.
    PROOF: Proposition 16.39.
□

Corollary 17.42.1 (Weierstrass's Theorem)**.** *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point.*

PROOF: It is a bounded infinite subset of some k-cell and therefore has a limit point in that k-cell. □

Example 17.43. It is not true that, in any metric space, a set is compact if and only if it is closed and bounded.

In $\mathbb{Q}$, the set $\{p \in \mathbb{Q} : 2 < p^2 < 3\}$ is closed and bounded but not compact.

Theorem 17.44. *Every nonempty perfect set in $\mathbb{R}^k$ is uncountable.*

PROOF:
⟨1⟩1. LET: P be a nonempty perfect set in $\mathbb{R}^k$.
⟨1⟩2. P is infinite.
  PROOF: Corollary 16.9.1.
⟨1⟩3. ASSUME: for a contradiction P is countable.
⟨1⟩4. PICK an enumeration $P = \{x_n : n \in \mathbb{N}\}$.
⟨1⟩5. PICK a sequence (V_n) of open balls such that, for all n, we have $\overline{V_{n+1}} \subseteq V_n$ and $x_n \notin \overline{V_{n+1}}$ and $V_n \cap P \neq \emptyset$
  ⟨2⟩1. ASSUME: as induction hypothesis we have picked $V_0, \ldots, V_{n-1}$ that satisfy these conditions.
  ⟨2⟩2. PICK $p \in P \cap V_n$ such that $p \neq x_n$
    PROOF: We cannot have $P \cap V_n = \{x_n\}$ because then V_n would be a neighbourhood of x_n that only intersects P at x_n.
  ⟨2⟩3. PICK an open ball B with centre p such that $B \subseteq V_n \cap P - \{x_n\}$
  ⟨2⟩4. LET: V_{n+1} be the open ball with centre p and half the radius of B.
  ⟨2⟩5. $\overline{V_{n+1}} \subseteq V_n$
    PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq V_n$.
  ⟨2⟩6. $x_n \notin \overline{V_{n+1}}$
    PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq P - \{x_n\}$.

⟨2⟩7. $V_{n+1} \cap P \neq \emptyset$
PROOF: Since $p \in V_{n+1} \cap P$.
⟨1⟩6. For $n \in \mathbb{N}$,
LET: $K_n = \overline{V_n} \cap P$.
⟨1⟩7. For all $n \in \mathbb{N}$, K_n is compact.
PROOF: By the Heine-Borel Theorem.
⟨1⟩8. $\bigcap_{n=0}^{\infty} K_n \cap P = \emptyset$
PROOF: Since for each n we have $x_n \notin K_{n+1}$.
⟨1⟩9. $\bigcap_{n=0}^{\infty} K_n = \emptyset$
PROOF: Since $\bigcap_{n=0}^{\infty} K_n \subseteq P$.
⟨1⟩10. Q.E.D.
PROOF: This contradicts Proposition 16.40.
□

Corollary 17.44.1. *For any $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a, b]$ is uncountable.*

Corollary 17.44.2. $\mathbb{R}$ *is uncountable.*

Corollary 17.44.3. *The set of transcendental numbers is uncountable.*

PROOF: Since the set of algebraic numbers is countable. □

Example 17.45. The Cantor set is a perfect set in $\mathbb{R}$ that does not include any open interval.

PROOF:
⟨1⟩1. LET: (E_n) be the sequence of unions of closed intervals from the definition of the Cantor set, and C be the Cantor set.
⟨1⟩2. $C \neq \emptyset$
PROOF: Since $0 \in C$.
⟨1⟩3. C is closed.
PROOF: Each E_n is closed and C is their intersection.
⟨1⟩4. Every point of C is a limit point of C.
⟨2⟩1. LET: $p \in C$
⟨2⟩2. LET: B be an open ball with centre p and radius r.
⟨2⟩3. PICK n such that each of the intervals that make up E_n has length $< r/2$.
⟨2⟩4. LET: I be the interval in E_n that contains p.
⟨2⟩5. $I \subseteq B$
⟨2⟩6. The endpoint of I that is not p is in $P \cap B$.
⟨1⟩5. C does not include any open interval.
⟨2⟩1. LET: (α, β) be any open interval.
⟨2⟩2. PICK m such that $3^{-m} < (\beta - \alpha)/6$
⟨2⟩3. PICK k such that $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq (\alpha, \beta)$
⟨2⟩4. $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq P$
⟨2⟩5. $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \cap E_m = \emptyset$
⟨2⟩6. Q.E.D.

PROOF: This is a contradiction.

□

Corollary 17.45.1. *The Cantor set is uncountable.*

Proposition 17.46. *Let X be a metric space. Let (K_n) be a sequence of compact sets in X such that $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$. Assume* $\operatorname{diam} K_n \to 0$ *as* $n \to \infty$. *Then $\bigcap_{n=0}^{\infty} K_n$ is a singleton.*

PROOF:
⟨1⟩1. $\bigcap_n K_n \neq \emptyset$
PROOF: Corollary 16.40.1.
⟨1⟩2. $\bigcap_n K_n$ has no more than one point.
⟨2⟩1. ASSUME: for a contradiction $a, b \in \bigcap_n K_n$ with $a \neq b$.
⟨2⟩2. LET: $\epsilon = d(a, b)$
⟨2⟩3. PICK n such that $\operatorname{diam} K_n < \epsilon$
⟨2⟩4. $a, b \in K_n$
⟨2⟩5. Q.E.D.
PROOF: This is a contradiction.

□

17.11 Connected Sets

Definition 17.47 (Separated)**.** Let X be a metric space. Let $A, B \subseteq X$. Then A and B are *separated* iff $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Proposition 17.48. *Any two disjoint open sets are separated.*

PROOF:
⟨1⟩1. LET: A and B be disjoint open sets.
⟨1⟩2. ASSUME: for a contradiction $p \in \overline{A} \cap B$.
⟨1⟩3. B is a neighbourhood of p.
⟨1⟩4. B intersects A.

□

Definition 17.49 (Connected)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *connected* iff E is not the union of two nonempty separated sets.

Theorem 17.50. *A subset E of the real line is connected if and only if it is convex.*

PROOF:
⟨1⟩1. If E is connected then E is convex.
⟨2⟩1. ASSUME: E is connected.
⟨2⟩2. LET: $x, y \in E$
⟨2⟩3. LET: $z \in (x, y)$
⟨2⟩4. $z \in E$
PROOF: Otherwise $E \cap (-\infty, z)$ and $E \cap (z, +\infty)$ would be a separation of E.

⟨1⟩2. If E is convex then E is connected.
⟨2⟩1. Assume: E is convex.
⟨2⟩2. Assume: for a contradiction $E = A \cup B$ where A and B are nonempty and separated.
⟨2⟩3. Pick $a \in A$ and $b \in B$.
⟨2⟩4. Assume: w.l.o.g. $a < b$
⟨2⟩5. Let: $z = \sup(A \cap [a, b])$
⟨2⟩6. $z \in \overline{A}$
⟨2⟩7. $z \notin B$
⟨2⟩8. $z < b$
⟨2⟩9. Case: $z \in A$
⟨3⟩1. $z \notin \overline{B}$
⟨3⟩2. Pick $z_1 \in (z, b)$ such that $z_1 \notin B$
⟨3⟩3. $a < z_1 < b$
⟨3⟩4. $z_1 \notin E$
Proof: We have $z_1 \notin A$ from ⟨2⟩5 since $z_1 \in [a, b]$ and $z_1 > z$, and $z_1 \notin B$ from ⟨3⟩2.
⟨3⟩5. Q.E.D.
Proof: This contradicts ⟨2⟩1.
⟨2⟩10. Case: $z \notin A$
Proof: Then $a < z < b$ and $z \notin E$ contradicting ⟨2⟩1.
□

Proposition 17.51. *Every connected metric space with more than one point is uncountable.*

Proof:
⟨1⟩1. Let: X be a connected metric space with more than one points.
⟨1⟩2. Pick distinct points $p, q \in X$.
⟨1⟩3. Let: $\epsilon = d(p, q)$
⟨1⟩4. For every $r \in (0, \epsilon)$, there exists a point $x \in X$ such that $d(p, x) = r$.
Proof: Otherwise $\{x \in X : d(p, x) < r\}$ and $\{x \in X : d(p, x) > r\}$ would form a separation of X.
□

Proposition 17.52. *The closure of a connected set is connected.*

Proof:
⟨1⟩1. Let: X be a metric space.
⟨1⟩2. Let: E be a connected subspace of X.
⟨1⟩3. Assume: for a contradiction A and B form a separation of $\overline{E}$
Prove: $A \cap E$ and $B \cap E$ form a separation of E.
⟨1⟩4. $A \cap E \neq \emptyset$
⟨2⟩1. Assume: for a contradiction $A \cap E = \emptyset$
⟨2⟩2. $E \subseteq B$
⟨2⟩3. $\overline{E} \subseteq \overline{B}$
⟨2⟩4. $A \subseteq \overline{B}$

⟨2⟩5. $A \cap \overline{B} = A \neq \emptyset$
⟨2⟩6. Q.E.D.
Proof: This contradicts ⟨1⟩3.
⟨1⟩5. $B \cap E \neq \emptyset$
Proof: Similar.
⟨1⟩6. $\overline{A \cap E} \cap B \cap E = \emptyset$
Proof: Since $\overline{A \cap E} \cap B \cap E \subseteq \overline{A} \cap B$.
⟨1⟩7. $A \cap E \cap \overline{B \cap E} = \emptyset$
Proof: Similar.
□

Example 17.53. The interior of a connected set is not necessarily connected.

Two touching discs in $\mathbb{R}^2$ form a connected set but the interior is disconnected.

Proposition 17.54. *Every convex set in $\mathbb{R}^k$ is connected.*

Proof:
⟨1⟩1. Let: E be a convex set in $\mathbb{R}^k$.
⟨1⟩2. Assume: for a contradiction A and B form a separation of E.
⟨1⟩3. Pick $\vec{a} \in A$ and $\vec{b} \in B$.
⟨1⟩4. Define $p : [0, 1] \to \mathbb{R}^k$ by $p(t) = (1 - t)\vec{a} + t\vec{b}$.
⟨1⟩5. $p^{-1}(A)$ and $p^{-1}(B)$ are separated sets in $\mathbb{R}$.
⟨1⟩6. Pick $x \in [0, 1]$ such that $x \notin p^{-1}(A)$ and $x \notin p^{-1}(B)$.
Proof: There exists such an x since $[0, 1]$ is connected.
⟨1⟩7. $p(x) \in E$
Proof: Since E is convex.
⟨1⟩8. $p(x) \notin A \cup B$
⟨1⟩9. Q.E.D.
Proof: This contradicts ⟨1⟩2.
□

17.12 Separable Spaces

Definition 17.55 (Separable)**.** A metric space is *separable* iff it has a countable dense subset.

Example 17.56. $\mathbb{R}^k$ is separable since $\mathbb{Q}^k$ is dense.

Proposition 17.57. *Every compact metric space is separable.*

Proof:
⟨1⟩1. Let: X be a compact metric space.
⟨1⟩2. For $n \in \mathbb{Z}^+$, pick finitely many points $a_{n1}, \ldots, a_{nr_n}$ such that $\{B(a_{ni}, 1/n) : 1 \leq i \leq r_n\}$ covers X.
Proof: Since $\{B(x, 1/n) : x \in X\}$ covers X.
⟨1⟩3. $\{a_{ni} : n \in \mathbb{Z}^+, 1 \leq i \leq r_n\}$ is dense.

⟨2⟩1. LET: U be an open set and $p \in U$.
⟨2⟩2. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$.
⟨2⟩3. PICK n such that $1/n < \epsilon$.
⟨2⟩4. PICK i such that $p \in B(a_{ni}, 1/n)$
⟨2⟩5. $a_{ni} \in U$
□

17.13 Bases

Definition 17.58 (Basis)**.** A *basis* for a metric space X is a set $\mathcal{B}$ of open sets such that, for every open set U and point $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.

Proposition 17.59. *Every separable metric space has a countable basis.*

PROOF:
⟨1⟩1. LET: X be a separable metric space.
⟨1⟩2. PICK a countable dense set D in X.
⟨1⟩3. LET: $\mathcal{B} = \{B(p, \epsilon) : p \in D, \epsilon \in \mathbb{Q}^+\}$
PROVE: $\mathcal{B}$ is a basis.
⟨1⟩4. LET: U be an open set in X and $p \in U$
⟨1⟩5. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$
⟨1⟩6. PICK $q \in B(p, \epsilon) \cap D$
⟨1⟩7. PICK a rational δ such that $d(p, q) < \delta < \epsilon$.
⟨1⟩8. $B(q, \delta) \in \mathcal{B}$ and $B(q, \delta) \subseteq U$.
□

17.14 Condensation Points

Definition 17.60 (Condensation Point)**.** Let X be a metric space, $p \in X$ and $E \subseteq X$. Then p is a *condensation point* of E iff every neighbourhood of p contains uncountably many points in E.

Proposition 17.61. *Let X be a metric space. Let $E \subseteq X$. Let P be the set of condensation points of E. Then P is perfect.*

PROOF:
⟨1⟩1. P is closed.
⟨2⟩1. LET: $p \in X - P$
⟨2⟩2. PICK a neighbourhood U of p that contains only countably many points of E.
⟨2⟩3. For every $x \in U$, we have that U is a neighbourhood of x that contains only countably many points of E.
⟨2⟩4. $p \in U \subseteq X - P$
⟨1⟩2. Every point in P is a limit point of P.
PROOF: Immediate from definitions.

□

Proposition 17.62. *Let X be a metric space with a countable basis. Let $E \subseteq X$ be uncountable. Let P be the set of condensation points of E. Then $E - P$ is countable.*

PROOF:
⟨1⟩1. PICK a countable basis $\mathcal{B}$ for X.
⟨1⟩2. LET: $W = \bigcup\{B \in \mathcal{B} : E \cap B \text{ is countable}\}$
⟨1⟩3. $P = X - W$
  ⟨2⟩1. $P \subseteq X - W$
    ⟨3⟩1. ASSUME: for a contradiction $p \in P \cap W$
    ⟨3⟩2. PICK $B \in \mathcal{B}$ such that $p \in B$ and $E \cap B$ is countable.
    ⟨3⟩3. $E \cap B$ is uncountable.
    ⟨3⟩4. Q.E.D.
      PROOF: This is a contradiction.
  ⟨2⟩2. $X - W \subseteq P$
    ⟨3⟩1. LET: $p \in X - W$
    ⟨3⟩2. LET: U be a neighbourhood of p.
    ⟨3⟩3. PICK $B \in \mathcal{B}$ such that $p \in B \subseteq U$.
    ⟨3⟩4. $E \cap B$ is uncountable.
      PROOF: Since $p \notin W$.
    ⟨3⟩5. $E \cap W$ is uncountable.
⟨1⟩4. $E - P = E \cap W$
⟨1⟩5. $E - P$ is countable.
□

Corollary 17.62.1. *Every closed subset of a metric space with a countable basis is the union of a perfect set and a countable set.*

PROOF:
⟨1⟩1. LET: X be a metric space with a countable basis.
⟨1⟩2. LET: E be a closed subset of X.
⟨1⟩3. LET: P be the set of condensation points of E.
⟨1⟩4. $E - P$ is countable.
  PROOF: Proposition 16.62.
⟨1⟩5. $P \cap E$ is perfect.
  ⟨2⟩1. $P \cap E$ is closed.
    PROOF: Proposition 16.61.
  ⟨2⟩2. Every point in $P \cap E$ is a limit point of $P \cap E$.
    ⟨3⟩1. LET: $l \in P \cap E$
    ⟨3⟩2. LET: U be a neighbourhood of l.
    ⟨3⟩3. PICK $x \in P \cap U$
    ⟨3⟩4. U is a neighbourhood of x.
    ⟨3⟩5. U contains uncountably many points of E.
    ⟨3⟩6. U intersects $P \cap E$
      PROOF: It cannot be that every point in U and E is not in P since $E - P$ is countable.

$\square$

Corollary 17.62.2. *Let X be a metric space with a countable basis. Then every countable set in X has an isolated point.*

Chapter 18

Convergence

Definition 18.1 (Converge)**.** Let X be a metric space. Let (p_n) be a sequence in X and $l \in X$. Then we say (p_n) *converges* to the *limit* l, and write

$$p_n \to l \text{ as } n \to \infty \enspace ,$$

iff for every $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon$.

We say (p_n) *diverges* iff it does not converge to any limit.

Proposition 18.2. *A sequence has at most one limit.*

PROOF:
⟨1⟩1. ASSUME: $p_n \to l$ and $p_n \to m$ as $n \to \infty$.
⟨1⟩2. ASSUME: for a contradiction $l \neq m$.
⟨1⟩3. LET: $\epsilon = d(l, m)/2$
⟨1⟩4. There exists N such that $\forall n \geq N . d(p_n, l) < \epsilon$ and $d(p_n, m) < \epsilon$
⟨1⟩5. $d(l, m) < 2\epsilon$
⟨1⟩6. Q.E.D.
PROOF: This is a contradiction.
□

Proposition 18.3. *Every convergent sequence is bounded.*

PROOF:
⟨1⟩1. LET: $p_n \to l$ as $n \to \infty$
⟨1⟩2. PICK N such that $\forall n \geq N . d(p_n, l) < 1$
⟨1⟩3. LET: $M = \max(d(p_0, l), \ldots, d(p_{N-1}, l), 1)$
⟨1⟩4. For all n, we have $d(p_n, l) \leq M$.
□

Proposition 18.4. *If l is a limit point of E, then there exists a sequence in E that converges to l.*

PROOF:

⟨1⟩1. For $n \in \mathbb{Z}^+$, Pick a point $a_n \in E$ such that $d(a_n, l) < 1/n$.
Proof: Since $B(l, 1/n)$ intersects E.
⟨1⟩2. $a_n \to l$ as $n \to \infty$.
□

Corollary 18.4.1. *Every sequence in a compact metric space has a convergent subsequence.*

Proof: By Theorem 16.41. □

Proposition 18.5. *Assume $s_n \to s$ and $t_n \to t$ in $\mathbb{R}^k$. Then $s_n + t_n \to s + t$.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. Pick N such that, for all $n \geq N$, we have $\|s_n - s\| < \epsilon/2$ and $\|t_n - t\| < \epsilon/2$.
⟨1⟩3. For all $n \geq N$ we have $\|(s_n + t_n) - (s + t)\| < \epsilon$.
Proof: Since $\|(s_n + t_n) - (s + t)\| \leq \|s_n - s\| + \|t_n - t\|$.
□

Lemma 18.6. *If $s_n \to s$ as $n \to \infty$ in $\mathbb{C}$, and $c \in \mathbb{C}$, then $cs_n \to cs$ as $n \to \infty$.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. Assume: w.l.o.g. $c \neq 0$
⟨1⟩3. Pick N such that $\forall n \geq N.|s_n - s| < \epsilon/|c|$.
⟨1⟩4. $\forall n \geq N.|cs_n - cs| < \epsilon$
□

Proposition 18.7. *If $s_n \to s$ and $t_n \to t$ in $\mathbb{C}$ then $s_n t_n \to st$.*

Proof:
⟨1⟩1. $(s_n - s)(t_n - t) \to 0$ as $n \to \infty$
⟨2⟩1. Let: $\epsilon > 0$
⟨2⟩2. Pick N such that, for all $n \geq N$, we have $|s_n - s| < \sqrt{\epsilon}$ and $|t_n - t| < \sqrt{\epsilon}$.
⟨2⟩3. For all $n \geq N$ we have $|(s_n - s)(t_n - t)| < \epsilon$
⟨1⟩2. $s_n t_n - st \to 0$ as $n \to \infty$
Proof:

$$\begin{aligned} s_n t_n - st &= (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s) \\ &\to 0 \qquad \text{as } n \to \infty \end{aligned}$$

□

Proposition 18.8. *If $s_n \to s$ as $n \to \infty$ in $\mathbb{C}$, and every s_n and s is nonzero, then $1/s_n \to 1/s$ as $n \to \infty$.*

Proof:
⟨1⟩1. Pick m such that, for all $n \geq m$, we have $|s_n - s| < \frac{1}{2}|s|$.
⟨1⟩2. $\forall n \geq m.|s_n| > \frac{1}{2}|s|$
⟨1⟩3. Let: $\epsilon > 0$

⟨1⟩4. Pick $N > m$ such that, for all $n \geq N$, we have
$$|s_n - s| < \frac{1}{2}|s|^2\epsilon \ .$$
⟨1⟩5. For all $n \geq N$, we have
$$\left|\frac{1}{s_n} - \frac{1}{s}\right| < \epsilon \ .$$
Proof:
$$\begin{aligned}\left|\frac{1}{s_n} - \frac{1}{s}\right| &= \frac{|s_n - s|}{|s_n||s|} \\ &< \frac{|s|^2\epsilon}{2|s_n||s|} \\ &= \frac{|s|\epsilon}{2|s_n|} \\ &< \epsilon\end{aligned}$$

□

Theorem 18.9. *Let $(\vec{x_n})$ be a sequence in $\mathbb{R}^k$ and $\vec{l} \in \mathbb{R}^k$. Then $\vec{x_n} \to \vec{l}$ as $n \to \infty$ iff, for $i = 1, \ldots, k$, we have $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ as $n \to \infty$.*

Proof:
⟨1⟩1. If $\vec{x_n} \to \vec{l}$ then $\pi_i(\vec{x_n}) \to \pi_i(l)$.
⟨2⟩1. $\|\vec{x_n} - \vec{l}\| \to 0$ as $n \to \infty$.
⟨2⟩2. $\sqrt{\sum_{i=1}^k (\pi_i(\vec{x_n}) - \pi_i(l))^2} \to 0$ as $n \to \infty$.
⟨2⟩3. $\sum_{i=1}^k (\pi_i(\vec{x_n}) - \pi_i(l))^2 \to 0$ as $n \to \infty$.
⟨2⟩4. $(\pi_i(\vec{x_n}) - \pi_i(l))^2 \to 0$ as $n \to \infty$
⟨2⟩5. $\pi_i(\vec{x_n}) - \pi_i(l) \to 0$ as $n \to \infty$.
⟨1⟩2. If $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ for every i then $\vec{x_n} \to l$.
⟨2⟩1. Assume: $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ for every i.
⟨2⟩2. $\vec{x_n} \to \vec{l}$
Proof:
$$\begin{aligned}\|\vec{x_n} - \vec{l}\|^2 &= \sum_{i=1}^k (\pi_i(\vec{x_n}) - \pi_i(\vec{l}))^2 \\ &\to 0\end{aligned}$$

□

Corollary 18.9.1. *If $\beta_n \to \beta$ in $\mathbb{R}$ and $\vec{x_n} \to \vec{l}$ in $\mathbb{R}^k$, then $\beta_n \vec{x_n} \to \beta\vec{l}$.*

Proposition 18.10. *If $\vec{x_n} \to \vec{x}$ and $\vec{y_n} \to \vec{y}$ in $\mathbb{R}^k$, then $\vec{x_n} \cdot \vec{y_n} \to \vec{x} \cdot \vec{y}$.*

PROOF:

$$\begin{aligned}\vec{x_n} \cdot \vec{y_n} &= \sum_{i=1}^{k} \pi_i(\vec{x_n})\pi_i(\vec{y_n}) \\ &\to \sum_{i=1}^{k} \pi_i(\vec{x})\pi_i(\vec{y}) \\ &= \vec{x} \cdot \vec{y}\end{aligned}$$

∎

Proposition 18.11. *Let (p_n) be a sequence in the metric space X. The set E^* of all limits of convergent subsequences is a closed set.*

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. $\{p_n : n \in \mathbb{N}\}$ is infinite.
⟨1⟩2. LET: q be a limit point of E^*.
PROVE: $q \in E^*$
⟨1⟩3. PICK an integer n_0 such that $q \neq p_{n_0}$.
⟨1⟩4. Extend a strictly increasing sequence of integers (n_i) such that, for all i, we have $d(q, p_{n_i}) \leq 2^i d(q, p_{n_0})$.
⟨2⟩1. ASSUME: as induction hypothesis we have picked $n_0 < n_1 < \cdots < n_i$ such that, for $0 \leq j \leq i$, we have $d(q, p_{n_j}) \leq 2^j d(q, p_{n_0})$.
⟨2⟩2. PICK $x \in E^*$ such that $d(x, q) < 2^{-(i+2)}\delta$
⟨2⟩3. There exists a subsequence of (p_n) that converges to x.
⟨2⟩4. There exists $n_{i+1} > n_i$ such that $d(p_{n_{i+1}}, x) < 2^{-(i+2)}\delta$.
⟨2⟩5. $d(p_{n_{i+1}}, q) < 2^{-(i+1)}\delta$
⟨1⟩5. $p_{n_i} \to q$ as $i \to \infty$.
⟨1⟩6. $q \in E^*$
∎

Theorem 18.12. *Every monotonically increasing sequence in $\mathbb{R}$ that is bounded above converges to its supremum.*

PROOF:
⟨1⟩1. LET: (s_n) be a monotonically increasing sequence with supremum s.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK S such that $|s_N - s| < \epsilon$
⟨1⟩4. For all $n \geq N$, we have $s - \epsilon < s - s_N \leq s - s_n \leq s$.
⟨1⟩5. $\forall n \geq N. |s_n - s| < \epsilon$
∎

Theorem 18.13. *Every monotonically decreasing sequence in $\mathbb{R}$ that is bounded below converges to its infimum.*

PROOF: Similar. ∎

Proposition 18.14 (Sandwich Theorem)**.** *Let (a_n), (b_n) and (c_n) be sequences of real numbers and $l \in \mathbb{R}$. Assume $\forall n. a_n \leq b_n \leq c_n$ and $a_n \to l$ and $c_n \to l$. Then $b_n \to l$.*

PROOF:
$\langle 1\rangle 1$. LET: $\epsilon > 0$
$\langle 1\rangle 2$. PICK N such that, for all $n \geq N$, we have $|a_n - l| < \epsilon$ and $|c_n - l| < \epsilon$.
$\langle 1\rangle 3$. $\forall n \geq N . |b_n - l| < \epsilon$
□

Theorem 18.15. *For any real $p > 0$ we have*

$$\frac{1}{(n+1)^p} \to 0$$

as $n \to \infty$.

PROOF:
$\langle 1\rangle 1$. LET: $\epsilon > 0$
$\langle 1\rangle 2$. PICK N such that $N > (1/\epsilon)^{1/p}$.
$\langle 1\rangle 3$. LET: $n \geq N$
$\langle 1\rangle 4$. $1/n^p < \epsilon$
□

Theorem 18.16. *For any real $p > 0$ we have*

$$p^{\frac{1}{n+1}} \to 1$$

as $n \to \infty$.

PROOF:
$\langle 1\rangle 1$. CASE: $p > 1$
  $\langle 2\rangle 1$. For $n \in \mathbb{N}$
    LET: $x_n = p^{\frac{1}{n+1}} - 1$.
  $\langle 2\rangle 2$. $\forall n \in \mathbb{N} . x_n > 0$
  $\langle 2\rangle 3$. $\forall n \in \mathbb{N}$.

$$1 + (n+1)x_n \leq p \ .$$

  PROOF: Since $1 + (n+1)x_n \leq (1 + x_n)^{n+1}$ by the Binomial Theorem.
  $\langle 2\rangle 4$. $\forall n \in \mathbb{N}$.

$$0 < x_n \leq \frac{p-1}{n+1} \ .$$

  $\langle 2\rangle 5$. $x_n \to 0$ as $n \to \infty$.
    PROOF: Sandwich Theorem.
$\langle 1\rangle 2$. CASE: $p = 1$
  PROOF: Trivial.
$\langle 1\rangle 3$. CASE: $p < 1$
  PROOF: Then $p^{1/(n+1)} = 1/((1/p)^{1/(n+1)}) \to 1/1 = 1$ by $\langle 1\rangle 1$.
□

Theorem 18.17.

$$(n+1)^{1/(n+1)} \to 1 \textit{ as } n \to \infty$$

PROOF:

⟨1⟩1. For $n \in \mathbb{N}$,
LET: $x_n = (n+1)^{1/(n+1)} - 1$.
⟨1⟩2. $\forall n \in \mathbb{N}. x_n \geq 0$
⟨1⟩3. $\forall n \in \mathbb{N}$

$$n + 1 \geq \frac{n(n+1)}{2} x_n^2 \ .$$

PROOF: Since $(1 + x_n)^{n+1} \geq \frac{n(n+1)}{2} x_n^2$ by the Binomial Theorem.
⟨1⟩4. $\forall n \geq 1$

$$0 \leq x_n \leq \sqrt{\frac{2}{n}}$$

⟨1⟩5. $x_n \to 0$ as $n \to \infty$.
PROOF: Sandwich Theorem.
□

Theorem 18.18. *Let p and α be real numbers with $p > 0$. Then*

$$\frac{n^\alpha}{(1+p)^n} \to 0 \ \textit{as}\ n \to \infty \ .$$

PROOF:
⟨1⟩1. PICK a positive integer k such that $k > \alpha$.
PROOF: Archimedean Property.
⟨1⟩2. $\forall n > 2k$

$$(1+p)^n > \frac{n^k p^k}{2^k k!} \ .$$

$$\begin{aligned} (1+p)^n &> \binom{n}{k} p^k && \text{(Binomial Theorem)} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k!} p^k \\ &> \frac{n^k p^k}{2^k k!} && (n > 2k \text{ so if } n-k < i \leq n \text{ then } i > n/2) \end{aligned}$$

⟨1⟩3. $\forall n > 2k$

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k} \ .$$

⟨1⟩4. $n^{\alpha-k} \to 0$ as $n \to \infty$
PROOF: Theorem 17.15.
⟨1⟩5. $\frac{n^\alpha}{(1+p)^n} \to 0$ as $n \to \infty$.
PROOF: Sandwich Theorem.
□

Corollary 18.18.1. *For any real number x with $|x| < 1$ we have $x^n \to 0$ as $n \to \infty$.*

PROOF: Taking $\alpha = 0$. □

18.1 Cauchy Sequences

Definition 18.19 (Cauchy Sequence)**.** Let (p_n) be a sequence in the metric space X. Then (p_n) is a *Cauchy sequence* iff, for every $\epsilon > 0$, there exists N such that, for all $m, n \geq N$, we have $d(p_m, p_n) < \epsilon$.

Proposition 18.20. *Let (p_n) be a sequence in the metric space X and let $E_N = \{p_n : n \geq N\}$ for all N. Then (p_n) is a Cauchy sequence if and only if* $\operatorname{diam} E_N \to 0$ *as $N \to \infty$.*

PROOF: Immediate from definitions. □

Theorem 18.21. *Every convergent sequence is Cauchy.*

PROOF:
⟨1⟩1. LET: (p_n) be a convergent sequence with limit l.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK N such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon/2$
⟨1⟩4. $\forall m, n \geq N . d(p_m, p_n) < \epsilon$
□

18.2 Complete Metric Spaces

Definition 18.22 (Complete Metric Space)**.** A metric space is *complete* iff every Cauchy sequence converges.

Theorem 18.23. *Every compact metric space is complete.*

PROOF:
⟨1⟩1. LET: X be a compact metric space.
⟨1⟩2. LET: (p_n) be a Cauchy sequence in X.
⟨1⟩3. For $N \in \mathbb{N}$,
LET: $E_N = \{p_n : n \geq N\}$.
⟨1⟩4. $\operatorname{diam} \overline{E_N} \to 0$ as $N \to \infty$.
⟨1⟩5. For all N, every $\overline{E_N}$ is compact.
PROOF: Proposition 16.39.
⟨1⟩6. For all N we have $\overline{E_N} \supseteq \overline{E_{N+1}}$.
⟨1⟩7. LET: l be the unique point in $\bigcap_{N=0}^{\infty} \overline{E_N}$
PROVE: $p_n \to l$ as $n \to \infty$.
PROOF: Proposition 16.46.
⟨1⟩8. LET: $\epsilon > 0$
⟨1⟩9. PICK N_0 such that $\forall N \geq N_0 . \operatorname{diam} \overline{E_N} < \epsilon$.
⟨1⟩10. $\forall q \in E_N . d(l, q) < \epsilon$
⟨1⟩11. $\forall n \geq N . d(l, p_n) < \epsilon$
□

Corollary 18.23.1. *Let X be a metric space. If every closed bounded set in X is compact, then X is complete.*

PROOF:
⟨1⟩1. LET: S be a Cauchy sequence in X.
⟨1⟩2. S is bounded.
⟨1⟩3. $\overline{S}$ is closed and bounded.
⟨1⟩4. $\overline{S}$ is compact.
⟨1⟩5. S is a Cauchy sequence in $\overline{S}$.
⟨1⟩6. S converges.
□

Corollary 18.23.2. *For every natural number k, we have $\mathbb{R}^k$ is complete.*

Corollary 18.23.3. *Every closed subspace of a complete metric space is complete.*

Proposition 18.24. *Let X be a complete metric space. Let (E_n) be a sequence of nonempty closed bounded sets in X with*

$$E_0 \supseteq E_1 \supseteq \cdots$$

and $\operatorname{diam} E_n \to 0$ *as $n \to \infty$. Then $\bigcap_{n=0}^{\infty} E_n$ consists of exactly one point.*

PROOF:
⟨1⟩1. LET: $K = \bigcap_{n=0}^{\infty} E_n$
⟨1⟩2. K has at least one point.
  ⟨2⟩1. For each n, PICK $a_n \in E_n$
  ⟨2⟩2. (a_n) is Cauchy.
    ⟨3⟩1. LET: $\epsilon > 0$
    ⟨3⟩2. PICK N such that $\forall n \geq N. \operatorname{diam} E_n < \epsilon$
    ⟨3⟩3. $\forall m, n \geq N. d(a_m, a_n) < \epsilon$
  ⟨2⟩3. LET: $l = \lim_{n\to\infty} a_n$
  ⟨2⟩4. $l \in K$
    ⟨3⟩1. LET: $n \in \mathbb{N}$
    ⟨3⟩2. For all $m \geq n$ we have $a_m \in E_n$
    ⟨3⟩3. $l \in E_n$
⟨1⟩3. K has at most one point.
  ⟨2⟩1. ASSUME: for a contradiction $a, b \in K$ such that $a \neq b$
  ⟨2⟩2. PICK n such that $\operatorname{diam} E_n < d(a, b)$
  ⟨2⟩3. $a, b \in E_n$
  ⟨2⟩4. Q.E.D.
    PROOF: This is a contradiction.
□

Theorem 18.25 (Baire's Theorem)**.** *Let X be a complete metric space. Let (G_n) be a sequence of dense open subsets of X. Then $\bigcap_{n=0}^{\infty} G_n$ is not empty.*

PROOF:
⟨1⟩1. PICK a sequence (E_n) of open balls such that $E_0 \supseteq E_1 \supseteq \cdots$ and $\operatorname{diam} E_n \leq 1/2^n$ and $\overline{E_n} \subseteq G_n$.

$\langle 2\rangle 1.$ Assume: as induction hypothesis we have chosen $E_0, \ldots, E_n$ with centres $c_0, \ldots, c_n$.
$\langle 2\rangle 2.$ Pick $x \in E_n \cap G_{n+1}$
$\langle 2\rangle 3.$ Pick $0 < \epsilon \leq 1/2^{n+2}$ such that $B(x, \epsilon) \subseteq E_n \cap G_{n+1}$
$\langle 2\rangle 4.$ Let: $E_{n+1} = B(x, \epsilon/2)$
$\langle 2\rangle 5.$ $E_{n+1} \subseteq E_n$
$\langle 2\rangle 6.$ $\operatorname{diam} E_{n+1} \leq 1/2^{n+1}$
$\langle 2\rangle 7.$ $\overline{E_{n+1}} \subseteq G_{n+1}$
$\langle 1\rangle 2.$ Let: $\bigcap_{n=0}^{\infty} \overline{E_n} = \{p\}$
Proof: Proposition 17.24.
$\langle 1\rangle 3.$ $p \in \bigcap_{n=0}^{\infty} G_n$
□

18.3 Divergent Sequences

Definition 18.26. Let (s_n) be a sequence in $\mathbb{R}$. Then we say s_n *diverges to* $+\infty$, and write

$$s_n \to +\infty \text{ as } n \to \infty \enspace ,$$

iff for every real number M, there exists an integer N such that

$$\forall n \geq N. s_n \geq M \enspace .$$

We say s_n *diverges to* $-\infty$, and write

$$s_n \to -\infty \text{ as } n \to \infty \enspace ,$$

iff for every real number M, there exists an integer N such that

$$\forall n \geq N. s_n \leq M \enspace .$$

Definition 18.27 (Limit Supremum, Limit Infimum)**.** Let (s_n) be a sequence in $\mathbb{R}$. Let E be the set of all $l \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that there exists a subsequence of (s_n) that converges to l.

The *limit supremum* of (s_n), denoted

$$\limsup_{n \to \infty} s_n \enspace ,$$

is the supremum of E in the extended reals.

The *limit infimum* of (s_n), denoted

$$\liminf_{n \to \infty} s_n \enspace ,$$

is the infimum of E in the extended reals.

Proof: The set E is always nonempty because: if (s_n) is unbounded above then $+\infty \in E$; if it is unbounded below then $-\infty \in E$; and if it is bounded above and below then there is a real number in E by Corollary 17.4.1. □

Theorem 18.28. *Let (s_n) be a sequence in $\mathbb{R}$. Then there exists a subsequence of (s_n) that converges or diverges to* $\limsup_{n\to\infty} s_n$

PROOF:
$\langle 1\rangle 1$. CASE: $\limsup_n s_n = +\infty$
PROOF: (s_n) is unbounded above and so has a subsequence that diverges to $+\infty$.
$\langle 1\rangle 2$. CASE: $\limsup_n s_n \in \mathbb{R}$
PROOF: Then $\limsup s_n$ is in the set of limits of subsequences of (s_n) by Proposition 17.11.
$\langle 1\rangle 3$. CASE: $\limsup_n s_n = -\infty$
PROOF: (s_n) is unbounded below and so has a subsequence that diverges to $-\infty$.
$\square$

Theorem 18.29. *Let (s_n) be a sequence in $\mathbb{R}$. Then there exists a subsequence of (s_n) that converges or diverges to* $\liminf_{n\to\infty} s_n$

PROOF: Similar. $\square$

Theorem 18.30. *Let (s_n) be a sequence in $\mathbb{R}$. If $x > \limsup_n s_n$, then there exists N such that $\forall n \geq N. s_n < x$.*

PROOF: If not, we could choose a subsequence of (s_n) that converges to a value $\geq x$, contradicting the definition of $\limsup_n s_n$. $\square$

Theorem 18.31. *Let (s_n) be a sequence in $\mathbb{R}$. If $x < \liminf_n s_n$, then there exists N such that $\forall n \geq N. s_n > x$.*

PROOF: Similar. $\square$

Theorem 18.32. *Let (s_n) be a sequence in $\mathbb{R}$. Let s^* be an extended real such that:*

- *There exists a subsequence of (s_n) that converges or diverges to s^*.*
- *For any $x > s^*$, there exists N such that $\forall n \geq N. s_n < x$.*

Then $s^* = \limsup_n s_n$.

PROOF:
$\langle 1\rangle 1$. LET: E be the set of subsequential limits of (s_n).
$\langle 1\rangle 2$. s^* is an upper bound for E.
$\langle 2\rangle 1$. LET: $x \in E$
$\langle 2\rangle 2$. ASSUME: for a contradiction $x > s^*$.
$\langle 2\rangle 3$. $s^* \in \mathbb{R}$
$\langle 2\rangle 4$. LET: $y = x$ if $x \in \mathbb{R}$, or $s^* + 1$ if $x = +\infty$
$\langle 2\rangle 5$. There exists N such that $\forall n \geq N. s_n < y$.
$\langle 2\rangle 6$. Q.E.D.
PROOF: This contradicts the fact that some subsequence of (s_n) converges or diverges to x.

⟨1⟩3. If u is an upper bound for E then $s^* \leq u$.
□

Theorem 18.33. *Let (s_n) be a sequence in $\mathbb{R}$. Let s^* be an extended real such that:*

- *There exists a subsequence of (s_n) that converges or diverges to s^*.*
- *For any $x < s^*$, there exists N such that $\forall n \geq N . s_n > x$.*

Then $s^* = \liminf_n s_n$.

PROOF: Similar. □

Proposition 18.34. *Let (s_n) be a sequence of real numbers and $l \in \mathbb{R}$. Then (s_n) converges to l iff $\limsup_n s_n = \liminf_n s_n = l$.*

PROOF:
⟨1⟩1. If (s_n) converges to l then $\limsup_n s_n = \liminf_n s_n = l$.
  PROOF: If (s_n) converges to l then every subsequence of (s_n) converges to l.
⟨1⟩2. If $\limsup_n s_n = \liminf_n s_n = l$ then (s_n) converges to l.
  ⟨2⟩1. ASSUME: $\limsup_n s_n = \liminf_n s_n = l$
  ⟨2⟩2. For all $\epsilon > 0$, there exists N such that $\forall n \geq N . l - \epsilon < s_n < l + \epsilon$.
    PROOF: Theorem 17.32 and 17.33.
  ⟨2⟩3. $s_n \to l$ as $n \to \infty$.
□

Theorem 18.35. *Let (s_n) and (t_n) be sequences of real numbers and $N \in \mathbb{N}$. Assume $\forall n \geq N . s_n \leq t_n$. Then*

$$\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n \quad .$$

PROOF:
⟨1⟩1. For any subsequence (t_{n_r}) of (t_n) that converges or diverges to $\pm\infty$, we have $\liminf_n s_n \leq \lim_r t_{n_r}$
  ⟨2⟩1. LET: (t_{n_r}) be a subsequence of (t_n) with limit l.
  ⟨2⟩2. PICK m such that a subsequence of (s_{n_r}) has limit m.
  ⟨2⟩3. $\forall r . s_{n_r} \leq t_{n_r}$
  ⟨2⟩4. $m \leq l$
  ⟨2⟩5. $\liminf_n s_n \leq l$
⟨1⟩2. $\liminf_n s_n \leq \liminf_n t_n$
□

Theorem 18.36. *Let (s_n) and (t_n) be sequences of real numbers and $N \in \mathbb{N}$. Assume $\forall n \geq N . s_n \leq t_n$. Then*

$$\limsup_{n \to \infty} s_n \leq \limsup_{n \to \infty} t_n \quad .$$

PROOF: Similar. □

Theorem 18.37. *For any sequence (c_n) of positive real numbers, we have*

$$\limsup_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n} \ .$$

PROOF:
⟨1⟩1. LET: $\alpha = \limsup_n c_{n+1}/c_n$
⟨1⟩2. ASSUME: w.l.o.g. $\alpha < +\infty$
⟨1⟩3. For all $\beta > \alpha$ we have $\limsup_{n\to\infty} c_n^{1/n} \leq \beta$.
⟨2⟩1. LET: $\beta > \alpha$
⟨2⟩2. PICK N such that, for all $n \geq N$, we have

$$\frac{c_{n+1}}{c_n} \leq \beta \ .$$

PROOF: Theorem 17.30.
⟨2⟩3. For all $k \geq 0$ we have

$$c_{N+k+1} \leq \beta c_{N+k} \ .$$

⟨2⟩4. For all $n \geq N$ we have

$$c_n \leq c_N \beta^{-N} \beta^n \ .$$

PROOF: Induction on n.
⟨2⟩5. For all $n \geq N$ we have

$$c_n^{1/n} \leq (c_N \beta^{-N})^{1/n} \beta \ .$$

⟨2⟩6.

$$\limsup_{n\to\infty} c_n^{1/n} \leq \beta$$

PROOF:

$$\begin{aligned} \limsup_{n\to\infty} c_n^{1/n} &\leq \limsup_{n\to\infty} (c_N \beta^{-N})^{1/n} \beta && \text{(Theorem 17.36)} \\ &= \beta && \text{(Theorem 17.16)} \end{aligned}$$

⟨1⟩4.

$$\limsup_{n\to\infty} c_n^{1/n} \leq \alpha$$

□

Theorem 18.38. *For any sequence (c_n) of positive real numbers, we have*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} c_n^{1/n} \ .$$

PROOF: Similar. □

Proposition 18.39. *Let (a_n) and (b_n) be sequences of reals. Assume that it is not the case that one of $\limsup_n a_n$, $\limsup_n b_n$ is $+\infty$ and the other is $-\infty$. Then*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \ .$$

18.4 Infinite Series

Definition 18.40. Let (a_n) be a sequence in $\mathbb{R}^k$ and $s \in \mathbb{R}^k$. We say the *infinite series* $\sum_{n=0}^{\infty} a_n$ *converges* to s, and write

$$\sum_{n=0}^{\infty} a_n = s$$

iff

$$\sum_{n=0}^{N} a_n \to s \text{ as } N \to \infty \ .$$

If $(\sum_{n=0}^{N} a_n)$ diverges, we say the infinite series $\sum_{n=0}^{\infty} a_n$ diverges.

Theorem 18.41. *Let (a_n) be a sequence in $\mathbb{R}^k$. Then $\sum_{n=0}^{\infty} a_n$ converges if and only if, for all $\epsilon > 0$, there exists N such that, for all $m, n \geq N$,*

$$\left\| \sum_{i=m}^{n} a_i \right\| \leq \epsilon \ .$$

PROOF: This is what it means for $(\sum_{i=0}^{n} a_i)$ to be a Cauchy sequence. ∎

Corollary 18.41.1. *If $\sum_{n=0}^{\infty} a_n$ converges then $a_n \to 0$ as $n \to \infty$.*

Theorem 18.42. *A series of nonnegative reals converges if and only if its partial sums form a bounded sequence.*

PROOF: Its partial sums are a monotonically increasing sequence, and so converge if and only if they are bounded above. ∎

Theorem 18.43 (Comparison Test)**.** *Let (a_n) be a sequence in $\mathbb{R}^k$ and (c_n) a sequence of real numbers. If there exists N such that $\forall n \geq N. \|a_n\| \leq c_n$, and if $\sum_n c_n$ converges, then $\sum_n a_n$ converges.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N such that $\forall n \geq N. \|a_n\| \leq c_n$ and $\forall m, n \geq N. \sum_{k=m}^{n} c_k < \epsilon$.
⟨1⟩3. $\forall m, n \geq N. \|\sum_{k=m}^{n} a_k\| \leq \epsilon$
∎

Corollary 18.43.1. *Let (a_n) and (d_n) be sequences of real numbers. If there exists N such that $\forall n \geq N. a_n \geq d_n \geq 0$, and if $\sum_n d_n$ diverges, then $\sum_n a_n$ diverges.*

Theorem 18.44 (Geometric Series)**.** *For x a real number with $0 \leq x < 1$ we have*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \ .$$

PROOF:Since $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x} \to \frac{1}{1-x}$ as $n \to \infty$. □

Theorem 18.45. *For x a real number with $x \geq 1$ we have $\sum_{n=0}^{\infty} x^n$ diverges.*

PROOF: If $x = 1$ then $\sum_{n=0}^{N} x^n = N + 1$. If $x > 1$ then $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$. Both of these sequences diverge. □

Theorem 18.46. *Let (a_n) be a monotonically decreasing sequence of nonnegative real numbers. Then $\sum_n a_n$ converges if and only if $\sum_n 2^n a_{2^n}$ converges.*

PROOF:
⟨1⟩1. For $N \in \mathbb{N}$,
 LET: $s_N = \sum_{n=0}^{N} a_n$.
⟨1⟩2. For $N \in \mathbb{N}$,
 LET: $t_N = \sum_{n=0}^{N} 2^n a_{2^n}$.
⟨1⟩3. For natural number N and k with $N < 2^k$ we have $s_N \leq a_0 + t_{k-1}$.
 PROOF:

$$\begin{aligned} s_N &\leq \sum_{n=0}^{2^k-1} a_n \\ &= a_0 + \sum_{i=0}^{k-1} \sum_{n=2^i} 2^{i+1} - 1a_n \\ &\leq a_0 + \sum_{i=0}^{k-1} 2^i a_{2^i} \\ &= a_0 + t_{k-1} \end{aligned}$$

⟨1⟩4. For natural number N and k with $N > 2^k$ we have $t_k < 2s_N$.
 PROOF:

$$\begin{aligned} s_N &\geq \sum_{n=1}^{2^k} a_n \\ &\geq \sum_{i=0}^{k} \sum_{n=2^i+1} 2^{i+1} a_n \\ &\geq \sum_{i=0}^{k} 2^i a_{2^{i+1}} \\ &= (1/2)t_k \end{aligned}$$

⟨1⟩5. (s_N) converges if and only if (t_k) converges.
□

Theorem 18.47. *If p is a real number with $p > 1$ then $\sum_n 1/n^p$ converges.*

PROOF: Since

$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}$$

which converges since $2^{1-p} < 1$. □

Theorem 18.48. *If p is a real number with $p \leq 1$ then $\sum_n 1/n^p$ diverges.*

PROOF: If $p \leq 0$ then $1/n^p$ does not converge to 0.
If $0 < p \leq 1$ we have
$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}$$
which diverges since $2^{1-p} \geq 1$. □

Theorem 18.49. *Let p be a real number. The series*
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$
converges if and only if $p > 1$.

PROOF:
$$\begin{aligned} 2^k \frac{1}{2^k(\ln 2^k)^p} &= \frac{1}{(k \ln 2)^p} \\ &= \frac{1}{(\ln 2)^p} \cdot \frac{1}{k^p} \end{aligned}$$
and this series converges iff $\sum_k \frac{1}{k^p}$ converges iff $p > 1$. □

Theorem 18.50 (Root Test)**.** *Let $(a_n)_{n \geq 1}$ be a sequence in $\mathbb{R}^k$. Let $\alpha = \limsup_{n \to \infty} \|a_n\|^{1/n}$.*

1. *If $\alpha < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $\alpha > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.*

PROOF:
$\langle 1 \rangle 1$. If $\alpha < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.
  $\langle 2 \rangle 1$. ASSUME: $\alpha < 1$
  $\langle 2 \rangle 2$. PICK β such that $\alpha < \beta < 1$
  $\langle 2 \rangle 3$. PICK N such that $\forall n \geq N . \|a_n\|^{1/n} < \beta$
    PROOF: Theorem 17.30.
  $\langle 2 \rangle 4$. $\forall n \geq N . \|a_n\| < \beta^n$
  $\langle 2 \rangle 5$. $\sum_{n=1}^{\infty} \beta^n$ converges.
    PROOF: Theorem 17.44.
  $\langle 2 \rangle 6$. $\sum_{n=1}^{\infty} a_n$ converges.
    PROOF: Comparison Test.
$\langle 1 \rangle 2$. If $\alpha > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.
  $\langle 2 \rangle 1$. ASSUME: $\alpha > 1$
  $\langle 2 \rangle 2$. There exists a sequence of positive integers (n_k) such that $\|a_{n_k}\|^{1/n_k} \to \alpha$ as $k \to \infty$.
    PROOF: Theorem 17.28.
  $\langle 2 \rangle 3$. There are infinitely many n such that $\|a_n\| > 1$.
  $\langle 2 \rangle 4$. $a_n \not\to 0$ as $n \to \infty$.
  $\langle 2 \rangle 5$. $\sum_{n=1}^{\infty} a_n$ diverges.

PROOF: Corollary 17.41.1.
∎

Example 18.51. If $a_n = 1/n$ then $|a_n|^{1/n} \to 1$ and $\sum_n a_n$ diverges.
If $a_n = 1/n^2$ then $|a_n|^{1/n} \to 1$ and $\sum_n a_n$ converges.

Theorem 18.52 (Ratio Test)**.** *Let $(a_n)_{n \geq 0}$ be a sequence in $\mathbb{R}^k$.*

1. *If*
$$\limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$$
then $\sum_{n=0}^{\infty} a_n$ converges.

2. *If there exists N such that $\forall n \geq N . \frac{\|a_{n+1}\|}{\|a_n\|} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.*

PROOF:
⟨1⟩1. If $\limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$ then $\sum_{n=0}^{\infty} a_n$ converges.
⟨2⟩1. ASSUME: $\limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$
⟨2⟩2. $\limsup_{n \to \infty} \|a_n\|^{1/n} < 1$
PROOF: Theorem 17.37.
⟨2⟩3. $\sum_{n=0}^{\infty} a_n$ converges.
PROOF: Root Test
⟨1⟩2. If there exists N such that $\forall n \geq N . \frac{\|a_{n+1}\|}{\|a_n\|} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.
PROOF: Since $a_n \nrightarrow 0$ as $n \to \infty$.
∎

Example 18.53. If $a_n = 1/n$ then $a_{n+1}/a_n \to 1$ and $\sum_n a_n$ diverges.
If $a_n = 1/n^2$ then $a_{n+1}/a_n \to 1$ and $\sum_n a_n$ converges.

18.5 The Number e

Lemma 18.54. *The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.*

PROOF:
$$\begin{aligned} \sum_{n=0}^{N} \frac{1}{n!} &\leq 1 + \sum_{n=1}^{N} \frac{1}{2^{n-1}} \\ &< 3 \end{aligned}$$
∎

Definition 18.55. The number e is
$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \quad .$$

Theorem 18.56.
$$\left(1 + \frac{1}{n}\right)^n \to e \text{ as } n \to \infty$$

PROOF:
⟨1⟩1. For $n \in \mathbb{N}$,
LET: $s_n = \sum_{k=0}^{n} \frac{1}{k!}$
⟨1⟩2. For $n \in \mathbb{Z}^+$,
LET: $t_n = \left(1 + \frac{1}{n}\right)^n$
⟨1⟩3. For $n \in \mathbb{Z}^+$ we have

$$t_n = \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \quad .$$

PROOF:

$$\begin{aligned} t_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} && \text{(Binomial Theorem)} \\ &= \sum_{k=0}^{n} \frac{\prod_{i=0}^{k-1}(n-i)}{k!} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=0}^{k-1} \frac{n-i}{n} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \end{aligned}$$

⟨1⟩4. For $n \in \mathbb{Z}^+$ we have $t_n \leq s_n$.
⟨1⟩5. $\limsup_{n\to\infty} t_n \leq e$
⟨1⟩6. For $m, n \in \mathbb{Z}^+$ with $n \geq m$ we have

$$t_n \geq \sum_{k=0}^{m} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \quad .$$

⟨1⟩7. For $m \in \mathbb{Z}^+$ we have

$$\liminf_{n\to\infty} t_n \geq \sum_{k=0}^{m} \frac{1}{k!} \quad .$$

⟨1⟩8. For $m \in \mathbb{Z}^+$ we have

$$s_m \leq \liminf_{n\to\infty} t_n \quad .$$

⟨1⟩9.

$$e \leq \liminf_{n\to\infty} t_n$$

⟨1⟩10. $t_n \to e$ as $n \to \infty$.
PROOF: From ⟨1⟩5 and ⟨1⟩9.
□

Theorem 18.57. *e is irrational.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $e = p/q$ where p and q are positive integers.
⟨1⟩2. For $n \in \mathbb{N}$,

LET: $s_n = \sum_{k=0}^{n} \frac{1}{k!}$.
⟨1⟩3. For $n \in \mathbb{Z}^+$ we have
$$0 < e - s_n < \frac{1}{n!n} \quad .$$
PROOF:
$$\begin{aligned} e - s_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &< \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \frac{1}{(n+1)^k} \\ &= \frac{1}{n!n} \end{aligned}$$
⟨1⟩4.
$$0 < q!(e - s_q) < \frac{1}{q}$$
⟨1⟩5. $q!e$ is an integer.
⟨1⟩6. $q!(e - s_q)$ is an integer.
⟨1⟩7. There exists an integer between 0 and 1.
⟨1⟩8. Q.E.D.
PROOF: This is a contradiction.
□

Theorem 18.58. *e is transcendental.*

PROOF: See I. M. Niven. *Irrational Numbers* p. 25. □

18.6 Power Series

Definition 18.59 (Power Series)**.** Let (c_n) be a sequence of complex numbers. The *power series* with *coefficients* (c_n) is the function that maps a complex number z to the series
$$\sum_{n=0}^{\infty} c_n z^n \quad .$$

Definition 18.60 (Radius of Convergence)**.** Let (c_n) be a sequence of complex numbers. Let
$$\begin{aligned} \alpha &:= \limsup_{n \to \infty} |c_n|^{1/n} \\ R &:= \frac{1}{\alpha} \end{aligned}$$
where $R = +\infty$ if $\alpha = 0$ and $R = 0$ if $\alpha = +\infty$. Then R is called the *radius of convergence* of the power series $\sum_n c_n z^n$.

Theorem 18.61. *Let R be the radius of convergence of $\sum_n c_n z^n$.*

1. *If $|z| < R$ then $\sum_{n=0}^{\infty} c_n z^n$ converges.*

2. *If* $|z| > R$ *then* $\sum_{n=0}^{\infty} c_n z^n$ *diverges.*

PROOF:
⟨1⟩1. For $z \in \mathbb{C}$ and $n \in \mathbb{N}$,
LET: $a_n(z) = c_n z^n$
⟨1⟩2.

$$\limsup_{n\to\infty} |a_n(z)|^{1/n} = |z|/R$$

⟨1⟩3. If $|z| < R$ then $\sum_{n=0}^{\infty} a_n(z)$ converges.
PROOF: Root Test.
⟨1⟩4. If $|z| > R$ then $\sum_{n=0}^{\infty} a_n(z)$ diverges.
PROOF: Root Test.
□

18.7 Summation by Parts

Theorem 18.62. *Let* (a_n), (b_n) *be two sequences in* $\mathbb{R}^k$. *Let*

$$A_n = \sum_{k=0}^{n} a_k \qquad (n \geq -1) \ .$$

Let p *and* q *be integers with* $0 \leq p \leq q$. *Then*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \ .$$

PROOF:

$$\begin{aligned}\sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\end{aligned}$$

□

Theorem 18.63. *Let* (a_n) *be a sequence in* $\mathbb{R}^k$ *and* (b_n) *be a sequence of real numbers. Assume that:*

1. *The partial sums* $\sum_{n=0}^{N} a_n$ *form a bounded sequence.*

2. (b_n) *is monotone decreasing.*

3. $b_n \to 0$ *as* $n \to \infty$.

Then $\sum_{n=0}^{\infty} a_n b_n$ *converges.*

PROOF:

⟨1⟩1. PICK M such that, for all N, we have $\| \sum_{n=0}^{N} a_n \| \leq M$.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK N such that $b_N \leq \epsilon / 2M$.
⟨1⟩4. LET: $N \leq p \leq q$
⟨1⟩5. For any integer k,
LET: $A_k = \sum_{n=0}^{k} a_n$.
⟨1⟩6. $\| \sum_{n=p}^{q} a_n b_n \| \leq \epsilon$
PROOF:

$$\begin{aligned} \left\| \sum_{n=p}^{q} a_n b_n \right\| &= \left\| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right\| \quad \text{(Summation by Parts)} \\ &\leq M \left\| \sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q - b_p \right\| \\ &= 2M b_p \\ &\leq 2M b_N \\ &\leq \epsilon \end{aligned}$$

⟨1⟩7. Q.E.D.
PROOF: Cauchy criterion.
□

Corollary 18.63.1 (Alternating Series)**.** *Let (c_n) be a sequence of real numbers. Assume that*

1. *$(|c_n|)$ is monotone decreasing.*
2. *$c_n \geq 0$ for all odd n, and $c_n \leq 0$ for all even n.*
3. *$c_n \to 0$ as $n \to \infty$*

Then $\sum_{n=0}^{\infty} c_n$ converges.

PROOF: Take $a_n = (-1)^{n+1}$ and $b_n = |c_n|$. □

Theorem 18.64. *Let $\sum_n c_n z^n$ be a power series with radius of convergence 1. Suppose (c_n) is monotone decreasing with limit 0. Then $\sum_n c_n z^n$ converges at every point on the circle $|z| = 1$ except possibly $z = 1$.*

PROOF:
⟨1⟩1. LET: z be a complex number with $|z| = 1$ and $z \neq 1$.
⟨1⟩2. For $n \in \mathbb{N}$,
LET: $a_n = z^n$.
⟨1⟩3. For $n \in \mathbb{N}$,
LET: $b_n = c_n$.
⟨1⟩4. The partial sums $\sum_{n=0}^{N} a_n$ form a bounded sequence.

PROOF:

$$\left|\sum_{n=0}^{N} a_n\right| = \left|\sum_{n=0}^{N} z^n\right|$$
$$= \left|\frac{1 - z^{N+1}}{1 - z}\right|$$
$$\leq \frac{2}{|1 - z|}$$

⟨1⟩5. (b_n) is monotone decreasing with limit 0.
⟨1⟩6. Q.E.D.
PROOF: Theorem 17.63.
□

18.8 Absolute Convergence

Definition 18.65 (Absolute Convergence)**.** Let (a_n) be a sequence in $\mathbb{R}^k$. Then the series $\sum_{n=0}^{\infty} a_n$ *converges absolutely* iff $\sum_{n=0}^{\infty} \|a_n\|$ converges.

Theorem 18.66. *If $\sum_{n=0}^{\infty} a_n$ converges absolutely then $\sum_{n=0}^{\infty} a_n$ converges.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N such that, for all $p, q \geq N$, we have

$$\sum_{n=p}^{q} \|a_n\| \leq \epsilon \ .$$

⟨1⟩3. For $p, q \geq N$, we have

$$\left\|\sum_{n=p}^{q} a_n\right\| \leq \epsilon \ .$$

S
⟨1⟩4. Q.E.D.
PROOF: Cauchy criterion.
□

18.9 Addition and Multiplication of Series

Theorem 18.67. *If $\sum_n a_n = A$ and $\sum_n b_n = B$ then $\sum_n (a_n + b_n) = A + B$.*

PROOF:

$$\sum_{n=0}^{N} (a_n + b_n) = \sum_{n=0}^{N} a_n + \sum_{n=0}^{N} b_n$$
$$\to A + B \qquad \text{as } N \to \infty$$
□

Theorem 18.68. *If $\sum_n a_n = A$ then $\sum_n (ca_n) = cA$.*

PROOF:

$$\sum_{n=0}^{N} ca_n = c \sum_{n=0}^{N} a_n$$
$$\to cA \qquad \text{as } N \to \infty \square$$

Definition 18.69 (Cauchy Product)**.** The *(Cauchy) product* of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} \quad .$$

Theorem 18.70. *Let (a_n) and (b_n) be sequences of complex numbers. Assume:*

1. *$\sum_{n=0}^{\infty} a_n$ converges absolutely.*
2. *$\sum_{n=0}^{\infty} b_n$ converges.*

For $n \in \mathbb{N}$, let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) \quad .$$

PROOF:
⟨1⟩1. LET:

$$A = \sum_{n=0}^{\infty} a_n$$

⟨1⟩2. LET:

$$B = \sum_{n=0}^{\infty} b_n$$

⟨1⟩3. For $n \in \mathbb{N}$,
LET:

$$A_n = \sum_{k=0}^{n} a_k \quad .$$

⟨1⟩4. For $n \in \mathbb{N}$,
LET:

$$B_n = \sum_{k=0}^{n} b_k \quad .$$

⟨1⟩5. For $n \in \mathbb{N}$,
LET:

$$C_n = \sum_{k=0}^{n} a_k b_{n-k} \quad .$$

⟨1⟩6. For $n \in \mathbb{N}$,
LET:

$$\beta_n = B_n - B$$

⟨1⟩7. For $n \in \mathbb{N}$,

$$C_n = A_n B + \sum_{k=0}^{n} a_k \beta_{n-k} \quad .$$

⟨1⟩8. For $n \in \mathbb{N}$,
LET:

$$\gamma_n = \sum_{k=0}^{n} a_k \beta_{n-k}$$

⟨1⟩9. $A_n B \to AB$ as $n \to \infty$.
⟨1⟩10. $\gamma_n \to 0$ as $n \to \infty$.
⟨2⟩1. LET: $\alpha = \sum_{n=0}^{\infty} |a_n|$
⟨2⟩2. For all $\epsilon > 0$ we have $\limsup_n |\gamma_n| \leq \epsilon\alpha$.
⟨3⟩1. LET: $\epsilon > 0$
⟨3⟩2. PICK N such that $\forall n \geq N . |\beta_n| \leq \epsilon$.
⟨3⟩3. For all $n \geq N$ we have $|\gamma_n| \leq \left|\sum_{k=0}^{N} \beta_k \alpha_{n-k}\right| + \epsilon\alpha$.
PROOF:

$$\begin{aligned} |\gamma_n| &\leq \left|\sum_{k=0}^{N} \beta_k \alpha_{n-k}\right| + \left|\sum_{k=N+1}^{n} \beta_k a_{n-k}\right| \\ &\leq \left|\sum_{k=0}^{N} \beta_k \alpha_{n-k}\right| + \epsilon\alpha \end{aligned}$$

⟨3⟩4.

$$\limsup_{n \to \infty} |\gamma_n| \leq \epsilon\alpha$$

⟨2⟩3. $\limsup_n \gamma_n = 0$
⟨1⟩11. $C_n \to AB$ as $n \to \infty$.
□

Theorem 18.71 (Abel)**.** *Let (a_n) and (b_n) be sequences of complex numbers. Let*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

for all n. If the series $\sum_n a_n$, $\sum_n b_n$ and $\sum_n c_n$ all converge, then

$$\sum_n c_n = \left(\sum_n a_n\right)\left(\sum_n b_n\right) \quad .$$

Proposition 18.72. *The Cauchy product of two absolutely convergent series is absolutely convergent.*

PROOF:
⟨1⟩1. LET: $\sum_n a_n$ and $\sum_n b_n$ be two absolutely convergent series.
⟨1⟩2. LET: $c_n = \sum_{k=0}^{n} a_k b_{n-k}$
⟨1⟩3. $\sum_n |c_n|$ converges.

PROOF:

$$\sum_{n=0}^{\infty} |c_n| = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} a_k b_{n-k} \right|$$
$$\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k||b_{n-k}|$$

which converges by Theorem 17.70.

18.10 Rearrangements

Definition 18.73 (Rearrangement)**.** A *rearrangement* of a sequence (a_n) is a sequence $(a_{\phi(n)})$ for some bijection $\phi : \mathbb{N} \approx \mathbb{N}$.

Theorem 18.74 (Riemann)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a series that converges but not absolutely. Let α and β be extended reals with $\alpha \leq \beta$. Then there exists a rearrangement of $\sum_n a_n$ with partial sums s'_n such that*

$$\limsup_{n\to\infty} s'_n = \alpha, \qquad \liminf_{n\to\infty} s'_n = \beta \ .$$

PROOF:
⟨1⟩1. For $n \in \mathbb{Z}^+$,
LET:
$$p_n = \frac{|a_n| + a_n}{2} \ .$$
⟨1⟩2. For $n \in \mathbb{Z}^+$,
LET:
$$q_n = \frac{|a_n| - a_n}{2} \ .$$
⟨1⟩3. $\forall n \in \mathbb{Z}^+ . p_n - q_n = a_n$
⟨1⟩4. $\forall n \in \mathbb{Z}^+ . p_n + q_n = |a_n|$
⟨1⟩5. $\forall n \in \mathbb{Z}^+ . p_n \geq 0$
⟨1⟩6. $\forall n \in \mathbb{Z}^+ . q_n \geq 0$
⟨1⟩7. $\sum_n p_n$ and $\sum_n q_n$ both diverge.
⟨2⟩1. It is not the case than $\sum_n p_n$ and $\sum_n q_n$ both converge.
PROOF: This would imply that $\sum_n |a_n|$ converges by ⟨1⟩4.
⟨2⟩2. It is not the case that $\sum_n p_n$ converges and $\sum_n q_n$ diverges.
PROOF: This would imply that $\sum_n a_n$ diverges by ⟨1⟩3.
⟨2⟩3. It is not the case that $\sum_n p_n$ diverges and $\sum_n q_n$ converges.
PROOF: This would imply that $\sum_n a_n$ diverges by ⟨1⟩3.
⟨1⟩8. LET: (P_n) be the subsequence of (a_n) consisting of the non-negative terms.
⟨1⟩9. LET: (Q_n) be the subsequence of $(|a_n|)$ consisting only of the terms such that a_n is negative.
⟨1⟩10. $\sum_n P_n$ diverges.
PROOF: It is the series $\sum_n p_n$ with the zero terms removed.
⟨1⟩11. $\sum_n Q_n$ diverges.
PROOF: It is the series $\sum_n q_n$ with the zero terms removed.

⟨1⟩12. PICK sequences of real numbers (α_n), (β_n) such that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha_n < \beta_n$ for all n, and $\beta_1 > 0$.

⟨1⟩13. PICK strictly increasing sequences of natural numbers $(m_n)_{n \geq 1}$, $(k_n)_{n \geq 1}$ such that, for all n,

$$\sum_{i=1}^{n-1} \left(\sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) + \sum_{j=m_{n-1}+1}^{m_n} P_j > \beta_n$$

$$\sum_{i=1}^{n} \left(\sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) < \alpha_n$$

and m_n and k_n are the smallest integers that make these inequalities true.

PROOF: Given the choice of $m_1, \ldots, m_n$ and $k_1, \ldots, k_n$, there must exist such an m_{n+1} by ⟨1⟩10, and then there must exist such a k_{n+1} by ⟨1⟩11.

⟨1⟩14. For $n \in \mathbb{Z}^+$,

LET: $x_n = \sum_{i=1}^{n-1} \left(\sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) + \sum_{j=m_{n-1}+1}^{m_n} P_j$

⟨1⟩15. For $n \in \mathbb{Z}^+$,

LET: $y_n = \sum_{i=1}^{n} \left(\sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right)$

⟨1⟩16. For $n \in \mathbb{Z}^+$ we have

$$|x_n - \beta_n| \leq P_{m_n} \quad .$$

PROOF: By minimality of m_n.

⟨1⟩17. For $n \in \mathbb{Z}^+$ we have

$$|y_n - \alpha_n| \leq Q_{k_n} \quad .$$

PROOF: By minimality of k_n.

⟨1⟩18. $P_n \to 0$ as $n \to \infty$.

PROOF: Since $a_n \to 0$ as $n \to \infty$.

⟨1⟩19. $Q_n \to 0$ as $n \to \infty$.

PROOF: Since $a_n \to 0$ as $n \to \infty$.

⟨1⟩20. $x_n \to \beta$ as $n \to \infty$.

PROOF: ⟨1⟩16, ⟨1⟩18

⟨1⟩21. $y_n \to \alpha$ as $n \to \infty$.

PROOF: ⟨1⟩17, ⟨1⟩19

⟨1⟩22. No number less than α or greater than β is a subsequential limit of the partial sums of the series $P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots$

PROOF: Since every partial sum after the $m_n + k_n$ term is between $\alpha_n - Q_{k_n}$ and $\beta_n + P_{m_n}$.

□

Theorem 18.75. *If $\sum_n a_n$ is a series in $\mathbb{R}^k$ that converges absolutely to s, then every rearrangement of $\sum_n a_n$ converges to s.*

PROOF:

⟨1⟩1. LET: $\sum_n a'_n = \sum_n a_{k_n}$ be a rearrangement with partial sums s'_n.

⟨1⟩2. LET: $\epsilon > 0$

⟨1⟩3. PICK N such that, for all $m \geq n \geq N$, we have

$$\sum_{i=n}^{m} \|a_i\| \leq \epsilon/3 \ .$$

⟨1⟩4. PICK p such that $\{1, \ldots, N\} \subseteq \{k_1, k_2, \ldots, k_p\}$.

⟨1⟩5. For all $n > p$ we have $\|s_n - s'_n\| \leq \epsilon$.

PROOF:

$$\begin{aligned} \|s_n - s'_n\| &= \left\| \sum_{i=1}^{N} a_i + \sum_{i=N+1}^{n} a_i - \sum_{i=1}^{p} a_{k_i} - \sum_{i=p+1}^{n} a_{k_i} \right\| \\ &\leq \left\| \sum_{i=N+1}^{n} a_i \right\| + \left\| \sum_{\substack{1 \leq i \leq p \\ k_i > N}} a_{k_i} \right\| + \left\| \sum_{i=p+1}^{n} a_{k_i} \right\| \\ &\leq \epsilon && (\langle 1 \rangle 3) \end{aligned}$$

⟨1⟩6. $s'_n \to s$ as $n \to \infty$.

□

18.11 Completion of a Metric Space

Definition 18.76 (Completion)**.** Let X be a metric space. Let X^* be the set of all Cauchy sequences in X, quotiented by: $(p_n) \sim (q_n)$ iff $d(p_n, q_n) \to 0$. Define the distance function on X^* by:

$$\Delta((p_n), (q_n)) = \lim_{n \to \infty} d(p_n, q_n) \ .$$

Then the metric space X^* is called the *completion* of X.

Theorem 18.77. *The completion of X^* is a complete metric space, and X is a dense subspace under the embedding that maps $p \in X$ to the constant sequence (p).*

Example 18.78. $\mathbb{R}$ is the completion of $\mathbb{Q}$.

Chapter 19

Continuity

19.1 Limit of a Function

Definition 19.1 (Limit)**.** Let X and Y be metric spaces. Let $E \subseteq X$ and $f : E \to Y$. Let p be a limit point of E and $q \in Y$. Then we say q is the *limit* of f at p, and write

$$f(x) \to q \text{ as } x \to p, \text{ or } \lim_{x \to p} f(x) = q \ ,$$

iff for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in E$, if $0 < d(x,p) < \delta$ then $d(f(x), q) < \epsilon$.

Theorem 19.2. *Let X and Y be metric spaces. Let $E \subseteq X$ and $f : E \to Y$. Let p be a limit point of E and $q \in Y$. Then $f(x) \to q$ as $x \to p$ if and only if, for every sequence (p_n) in $E - \{p\}$ with limit p, we have $f(p_n) \to q$ as $n \to \infty$.*

Proof:
$\langle 1 \rangle 1$. If $f(x) \to q$ as $x \to p$ then, for every sequence (p_n) in $E - \{p\}$ with limit p, we have $f(p_n) \to q$ as $n \to \infty$.
  $\langle 2 \rangle 1$. Assume: $f(x) \to q$ as $x \to p$.
  $\langle 2 \rangle 2$. Let: (p_n) be a sequence in $E - \{p\}$ with limit p.
  $\langle 2 \rangle 3$. Let: $\epsilon > 0$
  $\langle 2 \rangle 4$. Pick $\delta > 0$ such that, for all $x \in E$, if $0 < d(x,p) < \delta$ then $d(f(x), q) < \epsilon$.
  $\langle 2 \rangle 5$. Pick N such that, for all $n \geq N$, we have $d(p_n, p) < \delta$
  $\langle 2 \rangle 6$. $\forall n \geq N. d(f(p_n), q) < \epsilon$
$\langle 1 \rangle 2$. If, for every sequence (p_n) in $E - \{p\}$ with limit p, we have $f(p_n) \to q$ as $n \to \infty$, then $f(x) \to q$ as $x \to p$.
  $\langle 2 \rangle 1$. Assume: $f(x) \nrightarrow q$ as $x \to p$.
  $\langle 2 \rangle 2$. Pick $\epsilon > 0$ such that, for all $\delta > 0$, there exists a $x \in E$ such that $0 < d(x,p) < \delta$ and $d(f(x), q) \geq \epsilon$.
  $\langle 2 \rangle 3$. For all $n \in \mathbb{Z}^+$, Pick $p_n \in E$ such that $0 < d(p_n, p) < 1/n$ and $d(f(p_n), q) \geq \epsilon$.

⟨2⟩4. $p_n \to p$ as $n \to \infty$.
⟨2⟩5. $f(p_n) \nrightarrow q$ as $n \to \infty$.
□

Corollary 19.2.1. *A function has at most one limit at any point.*

Theorem 19.3. *Let X be a metric space, $E \subseteq X$, and p a limit point of E. Let $f, g : E \to \mathbb{R}^k$. Assume $f(x) \to a$ as $x \to p$ and $g(x) \to b$ as $x \to p$. Then*

$$f(x) + g(x) \to a + b \text{ as } x \to p \ .$$

PROOF:
⟨1⟩1. LET: (p_n) be a sequence in E that converges to p.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $g(p_n) \to b$ as $n \to \infty$.
⟨1⟩4. $f(p_n) + g(p_n) \to a + b$ as $n \to \infty$.
PROOF: Proposition 17.5.
⟨1⟩5. Q.E.D.
PROOF: Theorem 18.2.
□

Theorem 19.4. *Let X be a metric space, $E \subseteq X$, and p a limit point of E. Let $f, g : E \to \mathbb{C}$. Assume $f(x) \to a$ as $x \to p$ and $g(x) \to b$ as $x \to p$. Then*

$$f(x)g(x) \to ab \text{ as } x \to p \ .$$

PROOF:
⟨1⟩1. LET: (p_n) be a sequence in E that converges to p.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $g(p_n) \to b$ as $n \to \infty$.
⟨1⟩4. $f(p_n)g(p_n) \to ab$ as $n \to \infty$.
PROOF: Proposition 17.7.
⟨1⟩5. Q.E.D.
PROOF: Theorem 18.2.
□

Theorem 19.5. *Let X be a metric space, $E \subseteq X$, and p a limit point of E. Let $f : E \to \mathbb{C} - \{0\}$. Assume $f(x) \to a \neq 0$ as $x \to p$. Then*

$$f(x)^{-1} \to a^{-1} \text{ as } x \to p \ .$$

PROOF:
⟨1⟩1. LET: (p_n) be a sequence in E that converges to p.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $f(p_n)^{-1} \to a^{-1}$ as $n \to \infty$.
PROOF: Proposition 17.8.
⟨1⟩4. Q.E.D.
PROOF: Theorem 18.2.

∎

Theorem 19.6. *Let X be a metric space, $E \subseteq X$, and p a limit point of E. Let $f, g : E \to \mathbb{R}^k$. Assume $f(x) \to a$ as $x \to p$ and $g(x) \to b$ as $x \to p$. Then*

$$f(x) \cdot g(x) \to a \cdot b \text{ as } x \to p \ .$$

PROOF:
⟨1⟩1. LET: (p_n) be a sequence in E that converges to p.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $g(p_n) \to b$ as $n \to \infty$.
⟨1⟩4. $f(p_n) \cdot g(p_n) \to a \cdot b$ as $n \to \infty$.
PROOF: Proposition 17.10.
⟨1⟩5. Q.E.D.
PROOF: Theorem 18.2.
∎

19.2 Continuous Functions

Definition 19.7 (Continuous)**.** Let X be a metric space, $E \subseteq X$ and $p \in E$. Then f is *continuous* at p iff, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in E$, if $d(x, p) < \delta$ then

$$d(f(x), f(p)) < \epsilon \ .$$

f is *continuous* or *continuous on E* iff f is continuous at every point.

Theorem 19.8. *Let X be a metric space, $E \subseteq X$ and $p \in E$ be a limit point of E. Then f is continuous at p iff $f(x) \to f(p)$ as $x \to p$.*

PROOF: Easy. ∎

Theorem 19.9. *Let X, Y and Z be metric spaces. Let $E \subseteq X$. Let $f : E \to Y$ and $g : f(E) \to Z$. Let $p \in E$. If f is continuous at p and g is continuous at $f(p)$ then $g \circ f$ is continuous at p.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK $\delta_1 > 0$ such that, for all $y \in f(E)$, if $d(y, f(p)) < \delta_1$ then $d(g(y), g(f(p))) < \epsilon$.
⟨1⟩3. PICK $\delta_2 > 0$ such that, for all $x \in E$, if $d(x, p) < \delta_2$ then $d(f(x), f(p)) < \delta_1$.
⟨1⟩4. For all $x \in E$, if $d(x, p) < \delta_2$ then $d(g(f(x)), g(f(p))) < \epsilon$.
∎

Theorem 19.10. *Let X and Y be metric spaces. Let $f : X \to Y$. Then f is continuous if and only if, for every open set $V \subseteq Y$, we have $f^{-1}(V)$ is open in X.*

PROOF:
$\langle 1\rangle 1$. If f is continuous then, for every open set V in Y, we have $f^{-1}(V)$ is open in X.
$\langle 2\rangle 1$. ASSUME: f is continuous.
$\langle 2\rangle 2$. LET: V be an open set in Y.
PROVE: $f^{-1}(V)$ is open.
$\langle 2\rangle 3$. LET: $x \in f^{-1}(V)$
$\langle 2\rangle 4$. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$.
$\langle 2\rangle 5$. PICK $\delta > 0$ such that, for all $x' \in X$, if $d(x', x) < \delta$ then $d(f(x'), f(x)) < \epsilon$.
$\langle 2\rangle 6$. $B(x, \delta) \subseteq f^{-1}(V)$
$\langle 1\rangle 2$. If, for every open set V in Y, we have $f^{-1}(V)$ is open in X, then f is continuous.
$\langle 2\rangle 1$. ASSUME: For every open set V in Y, we have $f^{-1}(V)$ is open in X.
$\langle 2\rangle 2$. LET: $p \in X$
$\langle 2\rangle 3$. LET: $\epsilon > 0$
$\langle 2\rangle 4$. $f^{-1}(B(f(p), \epsilon))$ is open in X.
$\langle 2\rangle 5$. PICK $\delta > 0$ such that $B(p, \delta) \subseteq f^{-1}(B(f(p), \epsilon))$.
$\langle 2\rangle 6$. For all $x \in X$, if $d(x, p) < \delta$ then $d(f(x), f(p)) < \epsilon$.
□

Corollary 19.10.1. *Let X and Y be metric spaces. Let $f : X \to Y$. Then f is continuous if and only if, for every closed set C in Y, we have $f^{-1}(C)$ is closed in X.*

Theorem 19.11. *Let X be a metric space. Let $f : X \to \mathbb{R}^k$. Then f is continuous if and only if, for $i = 1, \ldots, k$, we have $\pi_i \circ f$ is continuous.*

PROOF:
$\langle 1\rangle 1$. Each π_i is continuous.
$\langle 2\rangle 1$. LET: $\vec{p} \in \mathbb{R}^k$
$\langle 2\rangle 2$. LET: $\epsilon > 0$
$\langle 2\rangle 3$. LET: $\vec{q} \in \mathbb{R}^k$
$\langle 2\rangle 4$. ASSUME: $\|\vec{p} - \vec{q}\| < \epsilon$
$\langle 2\rangle 5$. $|p_i - q_i| < \epsilon$
$\langle 1\rangle 2$. If, for all i, we have $\pi_i \circ f$ is continuous, then f is continuous.
$\langle 2\rangle 1$. ASSUME: For all i, we have $\pi_i \circ f$ is continuous.
$\langle 2\rangle 2$. LET: $p \in X$
$\langle 2\rangle 3$. LET: $\epsilon > 0$
$\langle 2\rangle 4$. For $i = 1, \ldots, k$, PICK $\delta_i > 0$ such that, for all $x \in X$, we have if $d(x, p) < \delta_i$ then $|\pi_i(f(p)) - \pi_i(f(x))| < \epsilon/\sqrt{k}$
$\langle 2\rangle 5$. LET: $\delta = \min(\delta_1, \ldots, \delta_k)$
$\langle 2\rangle 6$. LET: $q \in X$ with $d(p, q) < \delta$.
$\langle 2\rangle 7$. $\|f(p) - f(q)\| < \epsilon$

PROOF:

$$\begin{aligned}\|f(p)-f(q)\| &= \sqrt{\sum_{i=1}^{k} |\pi_i(f(p)) - \pi_i(f(q))|^2} \\ &< \sqrt{\sum_{i=1}^{k} \epsilon^2/k} \\ &= \epsilon\end{aligned}$$

□

Theorem 19.12. *Let X be a compact metric space and Y a metric space. Let $f : X \to Y$ be continuous. Then $f(X)$ is compact.*

PROOF:
⟨1⟩1. LET: $\mathcal{V}$ be an open cover of $f(X)$.
⟨1⟩2. $\{f^{-1}(V) : V \in \mathcal{V}\}$ is an open cover of X.
⟨1⟩3. PICK a finite subcover $\{f^{-1}(V_1), \ldots, f^{-1}(V_n)\}$.
⟨1⟩4. $\{V_1, \ldots, V_n\}$ covers Y.
□

Corollary 19.12.1. *Every continuous function from a compact metric space to $\mathbb{R}^k$ is bounded.*

Example 19.13. If $E \subseteq \mathbb{R}$ is not compact, then there exists a continuous function $E \to \mathbb{R}$ that is not bounded.

PROOF:
⟨1⟩1. CASE: E is bounded.
  ⟨2⟩1. PICK a limit point x_0 of E that is not in E.
  ⟨2⟩2. Define $f : E \to \mathbb{R}$ by $f(x) = 1/(x - x_0)$.
  ⟨2⟩3. f is continuous and unbounded.
⟨1⟩2. CASE: E is unbounded.
  PROOF: The inclusion $E \hookrightarrow \mathbb{R}$ is continuous and unbounded.
□

Theorem 19.14 (Extreme Values Theorem)**.** *Let X be a compact metric space. Let $f : X \to \mathbb{R}$. Let $M = \sup f(X)$ and $m = \inf f(X)$. Then there exist $p, q \in X$ such that $f(p) = M$ and $f(q) \in m$.*

PROOF: Since $f(X)$ is compact and hence closed. □

Example 19.15. For any $E \subseteq \mathbb{R}$ that is not compact, there exists a continuous and bounded function $E \to \mathbb{R}$ that does not attain its supremum.

PROOF:
⟨1⟩1. CASE: E is bounded.
  ⟨2⟩1. PICK a limit point x_0 for E such that $x_0 \notin E$.
  ⟨2⟩2. Define $g : E \to \mathbb{R}$ by $g(x) = 1/(1 + (x - x_0)^2)$.

$\langle 2\rangle 3$. g is continuous and bounded but does not attain its supremum 1.
$\langle 1\rangle 2$. CASE: E is unbounded.
PROOF: Then $h(x) = x^2/(1+x^2)$ is continuous and bounded but does not attain its supremum 1.
□

Theorem 19.16. *Let X be a compact metric space and Y a metric space. Let $f : X \approx Y$ be a continuous bijection. Then f^{-1} is continuous.*

PROOF:
$\langle 1\rangle 1$. LET: V be open in X.
$\langle 1\rangle 2$. $X - V$ is compact.
$\langle 1\rangle 3$. $f(X - V)$ is compact.
$\langle 1\rangle 4$. $Y - f(V)$ is compact.
$\langle 1\rangle 5$. $Y - f(V)$ is closed.
$\langle 1\rangle 6$. $f(V)$ is open.
□

Example 19.17. This example shows we cannot remove the hypothesis of compactness of X, even if Y is compact.

Let $X = [0, 2\pi)$. Let $f : X \to S^1$ be the function $f(t) = (\cos t, \sin t)$. Then f is a continuous bijection $X \approx S^1$, but the inverse f^{-1} is not continuous.

Proposition 19.18. *The continuous image of a connected metric space is connected.*

PROOF:
$\langle 1\rangle 1$. LET: X be a connected metric space and Y a metric space.
$\langle 1\rangle 2$. LET: $f : X \twoheadrightarrow Y$ be a continuous surjection.
$\langle 1\rangle 3$. ASSUME: for a contradiction A and B form a separation of Y.
$\langle 1\rangle 4$. $f^{-1}(A)$ and $f^{-1}(B)$ form a separation of X.
$\langle 1\rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 1$.
□

Corollary 19.18.1 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < c < f(b)$ or $f(a) > c > f(b)$, then there exists a real number $x \in (a, b)$ such that $f(x) = c$.*

PROOF: Since $f([a, b])$ is connected. □

Example 19.19. The converse does not hold. Let $f : [-1, 1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For all $a, b \in [-1, 1]$ with $a < b$, and all c with $f(a) < c < f(b)$, there exists $x \in (a, b)$ such that $f(x) = c$. Nevertheless, f is discontinuous at 0.

Proposition 19.20. *Let Ω be the set of all invertible linear transformations in $\mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^n]$. Then the function that sends A to A^{-1} is a continuous function $\Omega \to \Omega$.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$ and $A \in \Omega$
⟨1⟩2. LET: $\alpha = 1/\|A^{-1}\|$
⟨1⟩3. LET: $\delta = \alpha^2\epsilon/(1 + \alpha\epsilon)$
⟨1⟩4. LET: $B \in \Omega$ with $\|B - A\| < \delta$.
⟨1⟩5. $\|B^{-1}\| \leq (\alpha - \delta)^{-1}$
⟨2⟩1. For all $\vec{y} \in \mathbb{R}^n$ we have $(\alpha - \delta)\|B^{-1}\vec{y}\| \leq \|\vec{y}\|$.
PROOF:

$$\begin{aligned}(\alpha - \delta)\|B^{-1}\vec{y}\| &< (\alpha - \|B - A\|)\|B^{-1}\vec{y}\| && (\langle 1\rangle 4)\\ &\leq \|BB^{-1}\vec{y}\| && (\text{Lemma 12.7})\\ &= \|\vec{y}\|\end{aligned}$$

⟨1⟩6. $\|B^{-1} - A^{-1}\| < \epsilon$
PROOF:

$$\begin{aligned}\|B^{-1} - A^{-1}\| &\leq \|B^{-1}\|\|B - A\|\|A^{-1}\| && (\text{since } B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1})\\ &< \frac{\delta}{\alpha(\alpha - \delta)} && (\langle 1\rangle 2,\ \langle 1\rangle 4,\ \langle 1\rangle 5)\\ &= \epsilon && (\langle 1\rangle 3)\end{aligned}$$

□

19.3 Limits from the Left and the Right

Definition 19.21 (Limit from the Left)**.** Let $f : (a, b) \to \mathbb{R}$. Let $c \in (a, b]$ and $q \in \mathbb{R}$. Then we say q is the *limit* as f approaches c *from the left*, and write

$$f(x) \to q \text{ as } x \to c-$$

or

$$\lim_{x \to c-} f(x) = q$$

iff, for every sequence (t_n) in (a, c) such that $t_n \to c$ as $n \to \infty$, we have $f(t_n) \to q$ as $n \to \infty$.

Definition 19.22 (Limit from the Right)**.** Let $f : (a, b) \to \mathbb{R}$. Let $c \in [a, b)$ and $q \in \mathbb{R}$. Then we say q is the *limit* as f approaches c *from the right*, and write

$$f(x) \to q \text{ as } x \to c+$$

or

$$\lim_{x \to c+} f(x) = q$$

iff, for every sequence (t_n) in (c, b) such that $t_n \to c$ as $n \to \infty$, we have $f(t_n) \to q$ as $n \to \infty$.

Proposition 19.23. *Let $f : (a,b) \to \mathbb{R}$. Let $c \in (a,b)$ and $q \in \mathbb{R}$. Then $f(x) \to q$ as $x \to c$ iff $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$.*

PROOF:
⟨1⟩1. If $f(x) \to q$ as $x \to c$ then $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$.
PROOF: Theorem 18.2.
⟨1⟩2. If $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$ then $f(x) \to q$ as $x \to c$.
⟨2⟩1. ASSUME: $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$.
⟨2⟩2. ASSUME: for a contradiction $f(x) \nrightarrow q$ as $x \to c$.
⟨2⟩3. PICK a sequence (p_n) such that $p_n \to c$ as $n \to \infty$, $f(p_n) \nrightarrow q$ as $n \to \infty$, and $p_n \neq c$ for all n.
⟨2⟩4. CASE: There are only finitely many n such that $p_n > c$.
⟨3⟩1. LET: (q_n) be the subsequence of (p_n) consisting of all the terms such that $p_n < c$.
⟨3⟩2. $q_n \to c$ as $n \to \infty$.
⟨3⟩3. $f(q_n) \nrightarrow q$ as $n \to \infty$.
⟨3⟩4. Q.E.D.
PROOF: This contradicts ⟨2⟩1.
⟨2⟩5. CASE: There are only finitely many n such that $p_n < c$.
PROOF: Similar.
⟨2⟩6. CASE: There are infinitely many n such that $p_n > c$ and infinitely many n such that $p_n < c$.
⟨3⟩1. LET: (q_n) the subsequence of (p_n) consisting of all the terms such that $p_n > c$, and (r_n) the subsequence consisting of all the terms such that $p_n < c$.
⟨3⟩2. $q_n \to c$ as $n \to \infty$ and $r_n \to c$ as $n \to \infty$.
⟨3⟩3. It is not the case that $f(q_n) \to q$ as $n \to \infty$ and $f(r_n) \to q$ as $n \to \infty$.
PROOF: If $f(q_n) \to q$ as $n \to \infty$ and $f(r_n) \to q$ as $n \to \infty$ then $f(p_n) \to q$ as $n \to \infty$.
⟨3⟩4. Q.E.D.
PROOF: This contradicts ⟨2⟩1.
□

Proposition 19.24. *Let $f : (a,b) \to \mathbb{R}$ be monotonic. Then, for all $c \in (a,b)$ we have $\lim_{x \to c-} f(x)$ and $\lim_{x \to c+} f(x)$ both exist, and*

$$\sup_{a<x<c} f(x) = \lim_{x \to c-} f(x) \leq f(c) \leq \lim_{x \to c+} f(x) = \inf_{c<x<b} f(x) \ .$$

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. f is monotonically increasing on (a,b).
⟨1⟩2. $f(x) \to \sup_{a<x<c} f(x)$ as $x \to c-$.
⟨2⟩1. LET: (t_n) be a sequence in (a,c) such that $t_n \to c$ as $n \to \infty$.
PROVE: $f(t_n) \to \sup_{a<x<c} f(x)$ as $n \to \infty$.
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK $x \in (a,c)$ such that $f(x)$
⟨1⟩3. $f(x) \to \inf_{c<x<b} f(x)$ as $x \to c+$.
PROOF: Similar.
□

19.4 Discontinuities

Definition 19.25 (Simple Discontinuity)**.** Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. We say f has a *simple discontinuity* or *discontinuity of the first kind* at c iff f is discontinuous at c but $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ both exist.

Definition 19.26 (Discontinuity of the Second Kind)**.** Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. We say f has a *discontinuity of the second kind* at c iff $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ do not both exist.

19.5 Uniform Continuity

Definition 19.27 (Uniformly Continuous)**.** Let X and Y be metric spaces. Let $f : X \to Y$. Then f is *uniformly continuous* iff, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $p, q \in X$, if $d(p, q) < \delta$ then $d(f(p), f(q)) < \epsilon$.

Theorem 19.28. *Let X be a compact metric space and Y a metric space. Let $f : X \to Y$. If f is continuous then f is uniformly continuous.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. For all $p \in X$, PICK $\phi(p) > 0$ such that, for all $q \in X$, if $d(p, q) < \phi(x)$ then $d(f(p), f(q)) < \epsilon/2$.
⟨1⟩3. For all $p \in X$,
LET: $J(p) = B(p, \phi(x)/2)$.
⟨1⟩4. $\{J(p) : p \in X\}$ is an open cover of X.
⟨1⟩5. PICK a finite subcover $\{J(p_1), \ldots, J(p_n)\}$.
⟨1⟩6. LET: $\delta = \min(\phi(p_1), \ldots, \phi(p_n))/2$
⟨1⟩7. LET: $p, q \in X$ with $d(p, q) < \delta$.
⟨1⟩8. PICK m such that $p \in J(p_m)$.
⟨1⟩9. $d(p, p_m) < \phi(p_m)/2$
⟨1⟩10. $d(q, p_m) < \phi(p_m)$
⟨1⟩11. $d(f(p), f(q)) < \epsilon$
$\square$

Example 19.29. Let $E \subseteq \mathbb{R}$ be bounded but not compact. Then there exists a continuous function $E \to \mathbb{R}$ that is not uniformly continuous.

PROOF: Pick a limit point x_0 for E that is not in E. Then the function $f(x) = 1/(x - x_0)$ is continuous but not uniformly continuous. $\square$

Proposition 19.30. *Every linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous.*

PROOF: Since $\|A\vec{x} - A\vec{y}\| \leq \|A\|\|\vec{x} - \vec{y}\|$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. $\square$

Part VI

Analysis

Chapter 20

Differentiation

Definition 20.1 (Derivative)**.** Let E be an open set in $\mathbb{R}^n$. Let $f : E \to \mathbb{R}^m$. Let $\vec{x} \in E$. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then we say A is the *derivative* of f at $\vec{x}$, and write $f'(\vec{x}) = A$, iff

$$\frac{\|f(\vec{x}+\vec{h}) - f(\vec{x}) - A(\vec{h})\|}{\|\vec{h}\|} \to 0 \text{ as } \vec{h} \to \vec{0} \enspace .$$

We say f is *differentiable* at $\vec{x}$ iff f has a derivative at $\vec{x}$.

We say f is *differentiable* on E iff f is differentiable at every point in E. In this case, the *differential* or *total derivative* f' is the function that maps a point $\vec{x}$ to the derivative at that point.

Proposition 20.2. *A function has at most one derivative at any point.*

Proof:
⟨1⟩1. Assume: A_1 and A_2 are derivatives of f at $\vec{x}$.
⟨1⟩2. Let: $B = A_1 - A_2$
⟨1⟩3. For all $\vec{h}$ such that $\vec{x} + \vec{h} \in E$ we have
$$\|B(\vec{h})\| \leq \|f(\vec{x}+\vec{h}) - f(\vec{x}) - A_1(\vec{h})\| + \|f(\vec{x}+\vec{h}) - f(\vec{x}) - A_2(\vec{h})\|$$
⟨1⟩4. $\|B(\vec{h})\|/\|\vec{h}\| \to 0$ as $\vec{h} \to \vec{0}$.
⟨1⟩5. For $\vec{h} \neq \vec{0}$ such that $\vec{x} + \vec{h} \in E$ we have
$$\frac{\|B(t\vec{h})\|}{\|t\vec{h}\|} \to 0 \text{ as } t \to 0 \enspace .$$
⟨1⟩6. For $\vec{h} \neq \vec{0}$ such that $\vec{x} + \vec{h} \in E$ we have
$$\frac{\|B(\vec{h})\|}{\|\vec{h}\|} \to 0 \text{ as } t \to 0 \enspace .$$
Proof: Since B is linear.
⟨1⟩7. For $\vec{h} \neq \vec{0}$ such that $\vec{x} + \vec{h} \in E$ we have $B\vec{h} = \vec{0}$
⟨1⟩8. $B = 0$
⟨1⟩9. $A_1 = A_2$
□

Proposition 20.3. *A linear transformation is its own derivative.*

PROOF: If A is linear then

$$\frac{\|A(\vec{x}+\vec{h})-A(\vec{x})-A(\vec{h})\|}{\|\vec{h}\|}=0$$

□

Theorem 20.4 (Chain Rule)**.** *Let E be an open set in $\mathbb{R}^n$ and U an open set in $\mathbb{R}^m$. Let $f : E \to U$ and $g : U \to \mathbb{R}^k$. Let $\vec{x_0} \in E$. If f is differentiable at $\vec{x_0}$ and g is differentiable at $f(\vec{x_0})$, then $g \circ f$ is differentiable at $\vec{x_0}$ and*

$$(g \circ f)'(\vec{x_0}) = g'(f(\vec{x_0})) \circ f'(\vec{x_0}) \quad .$$

PROOF:

⟨1⟩1. LET: $\vec{y_0} = f(\vec{x_0})$
⟨1⟩2. LET: $A = f'(\vec{x_0})$
⟨1⟩3. LET: $B = g'(\vec{y_0})$
⟨1⟩4. For $\vec{h}$ such that $\vec{x_0} + \vec{h} \in E$,
LET: $u(\vec{h}) = f(\vec{x_0} + \vec{h}) - f(\vec{x_0}) + A(\vec{h}) \quad .$
⟨1⟩5. For $\vec{k}$ such that $\vec{y_0} + \vec{k} \in U$,
LET: $v(\vec{k}) = g(\vec{y_0} + \vec{k}) - g(\vec{y_0}) + B(\vec{k}) \quad .$
⟨1⟩6. For $\vec{h}$ non-zero such that $\vec{x_0} + \vec{h} \in E$,
LET: $\epsilon(\vec{h}) = \|u(\vec{h})\|/\|\vec{h}\| \quad .$
⟨1⟩7. For $\vec{k}$ non-zero such that $\vec{y_0} + \vec{k} \in U$,
LET: $\eta(\vec{k}) = \|v(\vec{k})\|/\|\vec{k}\| \quad .$
⟨1⟩8. $\epsilon(\vec{h}) \to 0$ as $\vec{h} \to \vec{0}$
PROOF: ⟨1⟩2, ⟨1⟩4, ⟨1⟩6.
⟨1⟩9. $\eta(\vec{k}) \to 0$ as $\vec{k} \to \vec{0}$
PROOF: ⟨1⟩3, ⟨1⟩5, ⟨1⟩7
⟨1⟩10. For $\vec{h}$ such that $\vec{x_0} + \vec{h} \in E$,
LET: $k(\vec{h}) = f(\vec{x_0} + \vec{h}) - f(\vec{x_0}) \quad .$

Part VII

More Algebra

Chapter 21

Lie Groups

Definition 21.1 (Lie Group)**.** A *Lie group* G is a group G that is also an analytic differentiable manifold such that the group operation and inverse operation are analytic.

A *homomorphism of Lie groups* is a group homomorphism that is an analytic function.

Lemma 21.2. *Every bijective Lie group homomorphism is an isomorphism.*

Definition 21.3 (Unitary Group)**.** The *unitary group* $U(n)$ is the Lie group of all $n \times n$ unitary matrices.

Definition 21.4 (Special Unitary Group)**.** The *special unitary group* $SU(n)$ is the Lie group of all $n \times n$ unitary matrices with determinant 1.

Definition 21.5 (Lie Subgroup)**.** Let G be a Lie group. A *Lie subgroup* of G is a subgroup that is also an analytic submanifold of G.

Example 21.6. $U(n)$ and $SU(n)$ are Lie subgroups of $GL(n, \mathbb{C})$.