Mathematics

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Contents

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*. We write $A : \mathrm{Set}$ for: A is a set.

For any set A, let there be *elements* of A. We write $a : \mathrm{El}(A)$ for: a is an element of A.

For any sets A and B, let there be *relations* between A and B. We write $R : A \looparrowright B$ for: R is a relation between A and B.

For any set A and elements $a, b : \mathrm{El}(A)$, let there be a proposition that a and b are *equal*, $a = b$.

For any relation $R : A \looparrowright B$ and elements $a : \mathrm{El}(A)$, $b : \mathrm{El}(B)$, let there be a proposition aRb, that R *holds* between a and b.

1.2 Definitions Used in the Axioms

Definition 1.1 (Function)**.** Let A and B be sets and $F : A \looparrowright B$. Then F is a *function* from A to B, $F : A \to B$, if and only if, for all $x \in A$, there exists a unique $y \in B$ such that xFy. We denote this unique y by $F(x)$.

Definition 1.2 (Injective)**.** A function $f : A \to B$ is *injective* iff, for all $x, y : \mathrm{El}(A)$, if $f(x) = f(y)$ then $x = y$.

Definition 1.3 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for all $y : \mathrm{El}(B)$, there exists $x : \mathrm{El}(A)$ such that $f(x) = y$.

Definition 1.4 (Bijective)**.** A function $f : A \to B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two subsets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$".

1.3 Axioms

Axiom Schema 1.5 (Comprehension)**.** *For any formula $\phi[X, Y, x, y]$ where X and Y are set variables and $x \in X$ and $y \in Y$, the following is an axiom:*

For any sets A and B, there exists a relation R such that, for all $a \in A$ and $b \in B$, we have aRb if and only if $\phi[A, B, a, b]$.

Axiom 1.6 (Tabulations)**.** *For any sets A and B and relation $R : A \nrightarrow B$, there exists a set $|R|$, a* tabulation *of R, and functions $p : |R| \to A$ and $q : |R| \to B$ such that:*

- *For all $x : \mathrm{El}\,(A)$ and $y : \mathrm{El}\,(B)$, we have xRy if and only if there exists $r : \mathrm{El}\,(|R|)$ such that $p(r) = x$ and $q(r) = y$*
- *For all $r, s : \mathrm{El}\,(|R|)$, if $p(r) = p(s)$ and $q(r) = q(s)$ then $r = s$.*

Axiom 1.7 (Infinity)**.** *There exists a set $\mathbb{N}$, an element $0 : \mathrm{El}\,(\mathbb{N})$, and a function $s : \mathbb{N} \to \mathbb{N}$ such that:*

- $\forall n : \mathrm{El}\,(\mathbb{N})\,.s(n) \neq 0$
- $\forall m, n : \mathrm{El}\,(\mathbb{N})\,.s(m) = s(n) \Rightarrow m = n$.

1.4 Consequences of the Axioms

1.4.1 The Empty Set

Theorem 1.8. *There exists a set which has no elements.*

PROOF:
$\langle 1 \rangle 1$. PICK a set A
  PROOF: By the Axiom of Infinity, a set exists.
$\langle 1 \rangle 2$. LET: $R : A \nrightarrow A$ be the relation such that, for all $x, y \in A$, we have $\neg(xRy)$
  PROOF: By the Axiom of Comprehension.
$\langle 1 \rangle 3$. LET: $|R|$ be the tabulation of R with projections $p, q : |R| \to A$.
  PROVE: $|R|$ has no elements.
  PROOF: By the Axiom of Tabulations.
$\langle 1 \rangle 4$. ASSUME: for a contradiction $r : \mathrm{El}\,(|R|)$
$\langle 1 \rangle 5$. $p(r)Rq(r)$
$\langle 1 \rangle 6$. Q.E.D.
  PROOF: This contradicts $\langle 1 \rangle 2$.
□

Theorem 1.9. *If E and E' have no elements then $E \approx E'$.*

PROOF:
⟨1⟩1. LET: E and E' have no elements.
⟨1⟩2. LET: $F : E \nrightarrow E'$ be the relation such that, for all $x : \mathrm{El}(E)$ and $y : \mathrm{El}(E')$, we have xFy.
PROOF: Axiom of Comprehension.
⟨1⟩3. F is a function.
PROOF: Vacuously, for all $x : \mathrm{El}(E)$, there exists a unique $y : \mathrm{El}(E')$ such that xFy.
⟨1⟩4. F is injective.
PROOF: Vacuously, for all $x, y : \mathrm{El}(E)$, if $F(x) = F(y)$ then $x = y$.
⟨1⟩5. F is surjective.
PROOF: Vacuously, for all $y : \mathrm{El}(E)$, there exists $x : \mathrm{El}(E)$ such that $F(x) = y$.
□

Definition 1.10 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

1.4.2 The Singleton

Theorem 1.11. *There exists a set that has exactly one element.*

PROOF:
⟨1⟩1. PICK a set A that has an element.
PROOF: By the Axiom of Infinity, there exists a set that has an element.
⟨1⟩2. PICK $a : \mathrm{El}(A)$
⟨1⟩3. LET: $R : A \nrightarrow A$ be the relation such that, for all $x, y : \mathrm{El}(A)$, we have xRy if and only if $x = y = a$.
PROOF: By the Axiom of Comprehension.
⟨1⟩4. LET: $|R|$ be the tabulation of R with projections $p, q : |R| \to A$.
PROVE: $|R|$ has exactly one element.
PROOF: By the Axiom of Tabulations.
⟨1⟩5. LET: $r : \mathrm{El}(|R|)$ be the element such that $p(r) = q(r) = a$
PROOF: Since aRa by ⟨1⟩3.
⟨1⟩6. LET: $s : \mathrm{El}(|R|)$
PROVE: $s = r$
⟨1⟩7. $p(s)Rq(s)$
PROOF: By the Axiom of Tabulations.
⟨1⟩8. $p(s) = q(s) = a$
PROOF: By ⟨1⟩3.
⟨1⟩9. $p(s) = p(r)$ and $q(s) = q(r)$
PROOF: By ⟨1⟩5.
⟨1⟩10. $s = r$
PROOF: By the Axiom of Tabulations.
□

Theorem 1.12. *If A and B both have exactly one element then $A \approx B$.*

PROOF:
$\langle 1 \rangle 1.$ LET: A and B both have exactly one element.
$\langle 1 \rangle 2.$ LET: $F : A \looparrowright B$ be the relation such that, for all $x : \mathrm{El}(A)$ and $y : \mathrm{El}(B)$, we have xFy.
$\langle 1 \rangle 3.$ F is a function.
PROOF: If xFy and xFy' then $y = y'$ because B has only one element.
$\langle 1 \rangle 4.$ F is injective.
PROOF: If $F(x) = F(x')$ then $x = x'$ because A has only one element.
$\langle 1 \rangle 5.$ F is surjective.
$\langle 2 \rangle 1.$ LET: $y : \mathrm{El}(B)$
$\langle 2 \rangle 2.$ LET: x be the element of A.
$\langle 2 \rangle 3.$ $F(x) = y$
□

Definition 1.13 (Singleton)**.** Let 1 be the set that has exactly one element. Let $*$ be its element.