Encyclopaedia of Mathematics and Physics

Robin Adams

Contents

Chapter 1

Set Theory

Proposition 1.1. *Every infinite subset of a countably infinite set is countable.*

PROOF:
$\langle 1\rangle 1.$ LET: $i : A \hookrightarrow \mathbb{N}$ be an infinite subset of $\mathbb{N}$.
$\langle 1\rangle 2.$ Define $j : \mathbb{N} \to A$ by: $j(k)$ is the element such that $i(j(k))$ is least such that $i(j(k)) \notin \{i(j(0)), \ldots, i(j(k-1))\}$.
$\langle 1\rangle 3.$ j is a bijection.
□

Proposition 1.2. *A countable union of countable sets is countable.*

PROOF:
$\langle 1\rangle 1.$ LET: (A_n) be a sequence of countable sets.
$\langle 1\rangle 2.$ For $n \in \mathbb{N}$, PICK an enumeration $(e_{nm})_m$ of A_n.
$\langle 1\rangle 3.$ LET: (p_k) be the following enumeration of $\mathbb{N} \times \mathbb{N}$:

$$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots$$

$\langle 1\rangle 4.$ $(e_{\pi_1(p_k)\pi_2(p_k)})_k$ is an enumeration of $\bigcup_n A_n$.
□

Theorem 1.3. $2^{\mathbb{N}}$ *is uncountable.*

PROOF:
$\langle 1\rangle 1.$ ASSUME: for a contradiction $f : \mathbb{N} \approx 2^{\mathbb{N}}$
$\langle 1\rangle 2.$ LET: $S = \{n \in \mathbb{N} : n \notin f(n)\}$
$\langle 1\rangle 3.$ For all n, we have $n \in S \Leftrightarrow n \notin f(n)$
$\langle 1\rangle 4.$ For all n we have $S \neq f(n)$.
$\langle 1\rangle 5.$ Q.E.D.
PROOF: This contradicts $\langle 1\rangle 1$.
□

Chapter 2

Relations

Definition 2.1 (Antisymmetric)**.** A relation R on a set A is *antisymmetric* iff, whenever xRy and yRx, then $x = y$.

Definition 2.2 (Transitive)**.** A relation R on a type A is *transitive* iff, whenever xRy and yRz, then xRz.

Chapter 3

Order Theory

Definition 3.1 (Linear Order)**.** A *linear order* on a set A is a binary relation $\leq$ on A that is transitive, antisymmetric and:

$$\forall x, y \in A. x \leq y \vee y \leq x \enspace .$$

A *linearly ordered set* is a pair $(A, \leq)$ where A is a set and $\leq$ is a binary relation on A.

We write $x < y$ for $x \leq y$ and $x \neq y$.

Definition 3.2 (Upper Bound)**.** Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is an *upper bound* in E iff $\forall x \in E. x \leq u$. We say E is *bounded above* iff it has an upper bound.

The *up-set* of E, denoted $E \uparrow$, is the set of upper bounds of E.

Definition 3.3 (Lower Bound)**.** Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then u is an *lower bound* in E iff $\forall x \in E. l \leq x$. We say E is *bounded below* iff it has a lower bound.

The *down-set* of E, denoted $E \downarrow$, is the set of lower bounds of E.

Definition 3.4 (Supremum)**.** Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is the *least upper bound* or *supremum* of E iff u is an upper bound for E and, for any upper bound u' for E, we have $u \leq u'$.

Definition 3.5 (Infimum)**.** Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then l is the *greatest lower bound* or *infimum* of E iff l is a lower bound for E and, for any lower bound l' for E, we have $l' \leq l$.

Definition 3.6 (Least Upper Bound Property)**.** A linearly ordered set S has the *least upper bound property* iff every nonempty subset of S that is bounded above has a least upper bound.

Proposition 3.7. *Let S be a linearly ordered set and $E \subseteq S$.*

1. *If $E \downarrow$ has a supremum l, then l is the infimum of E.*

2. If $E \uparrow$ has an infimum u, then U is the supremum of E.

PROOF:
$\langle 1 \rangle 1.$ If $E \downarrow$ has a supremum l, then l is the infimum of E.
$\langle 2 \rangle 1.$ l is a lower bound for E.
$\langle 3 \rangle 1.$ LET: $x \in E$
$\langle 3 \rangle 2.$ x is an upper bound for $E \downarrow$.
PROOF: For all $y \in E \downarrow$ we have $y \leq x$.
$\langle 3 \rangle 3.$ $l \leq x$
$\langle 2 \rangle 2.$ For any lower bound l' for E, we have $l' \leq l$.
PROOF: Since l is an upper bound for $E \downarrow$.
$\langle 1 \rangle 2.$ If $E \uparrow$ has an infimum u, then u is the supremum of E.
PROOF: Dual.
□

Corollary 3.7.1. *A linearly ordered set has the least upper bound property if and only if every nonempty set bounded below has an infimum.*

Definition 3.8 (Closed Downwards)**.** Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed downwards* iff, whenever $x \in E$ and $y < x$, then $y \in E$.

Definition 3.9 (Closed Upwards)**.** Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed upwards* iff, whenever $x \in E$ and $x < y$, then $y \in E$.

Definition 3.10 (Greatest)**.** Let S be a linearly ordered set and $u \in S$. Then u is *greatest* in S iff $\forall x \in S. x \leq u$.

Definition 3.11 (Least)**.** Let S be a linearly ordered set and $l \in S$. Then l is *least* in S iff $\forall x \in S. l \leq x$.

Proposition 3.12. *Let $\leq$ be a linear order on a set S and $E \subseteq S$. Then $\leq \cap E^2$ is a linear order on E.*

PROOF: Easy. □

Given a linearly ordered set $(S, \leq)$ and $E \subseteq S$, we write just E for the linearly ordered set $(E, \leq \cap E^2)$.

Definition 3.13 (Lexicographic Order)**.** Let A and B be linearly ordered sets. The *lexicographic order* or *dictionary order* on $A \times B$ is the order defined by

$$(a, b) \leq (a', b') \Leftrightarrow a = a' \vee (a < a' \wedge b \leq b') \ .$$

Proposition 3.14. *The lexicographic order is a linear order.*

Chapter 4

Field Theory

Definition 4.1 (Field)**.** A *field* F consists of a set F, two operations $+, \cdot : F^2 \to F$ and an element $0 \in F$ such that:

- $+$ is commutative.
- $+$ is associative.
- $\forall x \in F. x + 0 = x$
- $\forall x \in F. \exists y \in F. x + y = 0$
- $\cdot$ is commutative.
- $\cdot$ is associative.
- There exists $1 \in F$ such that $1 \neq 0$ and $\forall x \in F. x1 = x$ and $\forall x \in F. x \neq 0 \Rightarrow \exists y \in F. xy = 1$
- *Distributive Law* $\forall x, y, z \in F. x(y + z) = xy + xz$

Proposition 4.2. *In any field F, the element 0 is the unique element such that $\forall x \in F. x + 0 = x$.*

PROOF: If 0 and $0'$ both have this property then $0 = 0 + 0' = 0'$. □

Proposition 4.3. *In any field F, given $x \in F$, there is a unique $y \in F$ such that $x + y = 0$.*

PROOF: If $x + y = x + y' = 0$ then

$$\begin{aligned} y &= y + 0 \\ &= y + x + y' \\ &= 0 + y' \\ &= y' \end{aligned}$$

□

Definition 4.4. Let F be a field. Let $x \in F$. We denote by $-x$ the unique element of F such that $x + (-x) = 0$.

Given $x, y \in F$, we write $x - y$ for $x + (-y)$.

Proposition 4.5. *In any field F, if $x + y = x + z$ then $y = z$.*

PROOF: If $x + y = x + z$ we have

$$\begin{aligned} -x + x + y &= -x + x + z \\ \therefore 0 + y &= 0 + z \\ \therefore y &= z \end{aligned}$$ □

Proposition 4.6. *In any field F, we have $-(-x) = x$.*

PROOF: Since $x + (-x) = 0$. □

Proposition 4.7. *In any field F, the element 1 such that $\forall x \in F . x1 = x$ is unique.*

PROOF: If 1 and $1'$ both have this property then $1 = 1 \cdot 1' = 1'$. □

Proposition 4.8. *In any field F, given $x \in F$ with $x \neq 0$, the element y such that $xy = 1$ is unique.*

PROOF: If y and y' both have this property then we have

$$\begin{aligned} y &= y1 \\ &= yxy' \\ &= 1y' \\ &= y' \end{aligned}$$ □

Definition 4.9. In any field F, if $x \neq 0$, we write x^{-1} for the unique element such that $xx^{-1} = 1$.

We write x/y for xy^{-1}.

Proposition 4.10. *In any field F, if $xy = xz$ and $x \neq 0$ then $y = z$.*

PROOF:

$$\begin{aligned} y &= 1y \\ &= x^{-1}xy \\ &= x^{-1}xz \\ &= 1z \\ &= z \end{aligned}$$ □

Proposition 4.11. *In any field F, if $x \neq 0$ then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.*

PROOF: Since $xx^{-1} = 1$. □

Proposition 4.12. *In any field F, we have $x0 = 0$.*

PROOF:

$$\begin{aligned} x0 + 0 &= x0 \\ &= x(0+0) \\ &= x0 + x0 \\ \therefore 0 &= x0 \end{aligned} \qquad \square$$

Proposition 4.13. *In any field F, if $xy = 0$ then $x = 0$ or $y = 0$.*

PROOF: If $xy = 0$ and $x \neq 0$ then we have $y = x^{-1}xy = x^{-1}0 = 0$. $\square$

Proposition 4.14. *In any field F, we have $(-x)y = -(xy)$.*

PROOF:

$$\begin{aligned} xy + (-x)y &= (x + (-x))y \\ &= 0y \\ &= 0 && \text{(Proposition 4.12)} \square \end{aligned}$$

Corollary 4.14.1. *In any field F, we have $(-x)(-y) = xy$.*

PROOF:

$$\begin{aligned} (-x)(-y) &= -(x(-y)) \\ &= -(-(xy)) \\ &= xy && \text{(Proposition 4.6)} \square \end{aligned}$$

Proposition 4.15. *Let K be a field. Let $a, b \in K$. If $a^2 = b^2$ then $a = b$ or $a = -b$.*

PROOF:

$$a^2 - b^2 = 0$$
$$\therefore (a-b)(a+b) = 0$$

Hence either $a - b = 0$ or $a + b = 0$, and the conclusion follows. $\square$

4.1 Ordered Fields

Definition 4.16 (Ordered Field)**.** An *ordered field F* consists of a field F and a linear order $\leq$ on F such that:

- For all $x, y, z \in F$, if $y < z$ then $x + y < x + z$
- For all $x, y \in F$, if $x > 0$ and $y > 0$ then $xy > 0$.

We call x *positive* iff $x > 0$ and *negative* iff $x < 0$.

Example 4.17. $\mathbb{Q}$ is an ordered field.

Proposition 4.18. *In any ordered field, if x is positive then $-x$ is negative.*

PROOF: If $x > 0$ then $0 = x + (-x) > 0 = (-x) = -x$. $\square$

Proposition 4.19. *In any ordered field, if $y < z$ and x is positive then $xy < xz$.*

PROOF: If $y < z$ then we have

$$\begin{aligned} 0 &< z - y \\ \therefore 0 &< x(z - y) \\ &= xz - xy \\ \therefore xy &< xz \end{aligned}$$

∎

Proposition 4.20. *In any ordered field, if $y < z$ and x is negative then $xy > xz$.*

PROOF:
⟨1⟩1. $-x$ is positive.
⟨1⟩2. $(-x)y < (-x)z$
⟨1⟩3. $-(xy) < -(xz)$
⟨1⟩4. $xz < xy$
∎

Proposition 4.21. *In any ordered field, if $x \neq 0$ then $x^2 > 0$.*

PROOF:
⟨1⟩1. If $x > 0$ then $x^2 > 0$.
PROOF: Proposition 4.19.
⟨1⟩2. If $x < 0$ then $x^2 > 0$.
PROOF: Proposition 4.20.
∎

Corollary 4.21.1. *In any ordered field, we have $1 > 0$.*

Proposition 4.22. *In any ordered field, if x is positive then x^{-1} is positive.*

PROOF: If $x^{-1} < 0$ then we would have $1 = xx^{-1} < x0 = 0$ contradicting Corollary 4.21.1. ∎

Proposition 4.23. *In any ordered field, if $0 < x < y$ then $y^{-1} < x^{-1}$.*

PROOF:
⟨1⟩1. ASSUME: $0 < x < y$
⟨1⟩2. x^{-1} and y^{-1} are positive.
PROOF: Proposition 4.22.
⟨1⟩3. $xy^{-1} < yy^{-1} = 1$
⟨1⟩4. $y^{-1} = x^{-1}xy^{-1} < x^{-1}1 = x^{-1}$
∎

Lemma 4.24. *Let K be an ordered field. Let $b \in K$ with $b > 1$. Let n be a positive integer. Then*

$$b^n - 1 \geq n(b - 1)$$

PROOF:

$$\begin{aligned} b^n - 1 &= (b - 1)(b^{n-1} + b^{n-2} + \cdots + 1) \\ &\geq (b - 1)(1 + 1 + \cdots + 1) \\ &= n(b - 1) \end{aligned}$$

∎

Chapter 5

Real Analysis

5.1 Construction of the Real Numbers

Definition 5.1 (Cut)**.** A *cut* is a subset α of $\mathbb{Q}$ such that:

- $\emptyset \neq \alpha \neq \mathbb{Q}$
- α is closed downwards.
- α has no greatest element.

In this section, we write R for the set of all cuts.

Proposition 5.2. *R is linearly ordered by $\subseteq$.*

PROOF: The only difficult part is to prove that, for any cuts α and β, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.
$\langle 1 \rangle$1. ASSUME: $\alpha \nsubseteq \beta$
PROVE: $\beta \subseteq \alpha$
$\langle 1 \rangle$2. PICK $q \in \alpha$ such that $q \notin \beta$
$\langle 1 \rangle$3. LET: $r \in \beta$
$\langle 1 \rangle$4. $q \nless r$
$\langle 1 \rangle$5. $r < q$
$\langle 1 \rangle$6. $r \in \alpha$
$\square$

Proposition 5.3. *R has the least upper bound property.*

PROOF:
$\langle 1 \rangle$1. LET: $E \subseteq R$ be nonempty and bounded above.
$\langle 1 \rangle$2. LET: $s = \bigcup E$
PROVE: s is a cut.
$\langle 1 \rangle$3. $\emptyset \neq s$
PROOF: Since E is nonempty and every element of E is nonempty.
$\langle 1 \rangle$4. $s \neq \mathbb{Q}$

$\langle 2 \rangle 1.$ Pick an upper bound u for E.
$\langle 2 \rangle 2.$ Pick $q \notin u$
Prove: $q \notin s$
$\langle 2 \rangle 3.$ $\forall \alpha \in E. \alpha \subseteq u$
$\langle 2 \rangle 4.$ $s \subseteq u$
$\langle 2 \rangle 5.$ $q \notin s$
$\langle 1 \rangle 5.$ s is closed downwards.
$\langle 2 \rangle 1.$ Let: $q \in s$ and $r < q$.
$\langle 2 \rangle 2.$ Pick $\alpha \in E$ such that $q \in \alpha$.
$\langle 2 \rangle 3.$ $r \in \alpha$
$\langle 2 \rangle 4.$ $r \in s$
$\langle 1 \rangle 6.$ s has no greatest element.
$\langle 2 \rangle 1.$ Let: $q \in s$
$\langle 2 \rangle 2.$ Pick $\alpha \in E$ such that $q \in \alpha$.
$\langle 2 \rangle 3.$ Pick $r \in \alpha$ such that $q < r$.
$\langle 2 \rangle 4.$ $r \in s$
□

Definition 5.4 (Addition)**.** Given cuts α and β, we define

$$\alpha + \beta = \{q + r : q \in \alpha, r \in \beta\} .$$

Proposition 5.5. *Given cuts α and β, we have $\alpha + \beta$ is a cut.*

Proof:
$\langle 1 \rangle 1.$ $\alpha + \beta$ is nonempty.
Proof: Since α and β are nonempty.
$\langle 1 \rangle 2.$ $\alpha + \beta \neq \mathbb{Q}$
$\langle 2 \rangle 1.$ Pick $q \in \mathbb{Q} - \alpha$ and $r \in \mathbb{Q} - \beta$.
Prove: $q + r \notin \alpha + \beta$
$\langle 2 \rangle 2.$ Assume: for a contradiction $q + r \in \alpha + \beta$.
$\langle 2 \rangle 3.$ Pick $x \in \alpha$ and $y \in \beta$ such that $q + r = x + y$
$\langle 2 \rangle 4.$ $x < q$
$\langle 2 \rangle 5.$ $y < r$
$\langle 2 \rangle 6.$ $x + y < q + r$
$\langle 2 \rangle 7.$ Q.E.D.
Proof: This is a contradiction.
$\langle 1 \rangle 3.$ $\alpha + \beta$ is closed downwards.
$\langle 2 \rangle 1.$ Let: $q \in \alpha$, $r \in \beta$ and $x < q + r$
$\langle 2 \rangle 2.$ $x - q < r$
$\langle 2 \rangle 3.$ $x - q \in \beta$
$\langle 2 \rangle 4.$ $x \in \alpha + \beta$
$\langle 1 \rangle 4.$ $\alpha + \beta$ has no greatest element.
$\langle 2 \rangle 1.$ Let: $q \in \alpha$ and $r \in \beta$.
Prove: $q + r$ is not greatest in $\alpha + \beta$.
$\langle 2 \rangle 2.$ Pick $q' \in \alpha$ with $q < q'$ and $r' \in \beta$ with $r < r'$.
$\langle 2 \rangle 3.$ $q + r < q' + r' \in \alpha + \beta$

∎

Proposition 5.6. *Addition is commutative and associative on R.*

PROOF: Immediate from definitions and the fact that addition is commutative and associative on $\mathbb{Q}$. ∎

Definition 5.7. For any $q \in \mathbb{Q}$, let $q^* = \{r \in \mathbb{Q} : r < q\}$.

Proposition 5.8. *For any $q \in \mathbb{Q}$, we have q^* is a cut.*

PROOF:
⟨1⟩1. $q^* \neq \emptyset$
  PROOF: Since $q - 1 \in q^*$.
⟨1⟩2. $q^* \neq \mathbb{Q}$
  PROOF: Since $q \notin q^*$.
⟨1⟩3. q^* is closed downwards.
  PROOF: Immediate from definition.
⟨1⟩4. q^* has no greatest element.
  PROOF: For all $r \in q^*$ we have $r < (q + r)/2 \in q^*$.
∎

Proposition 5.9. *For any cut α we have $\alpha + 0^* = \alpha$.*

PROOF:
⟨1⟩1. $\alpha + 0^* \subseteq \alpha$
  ⟨2⟩1. LET: $q \in \alpha$ and $r \in 0^*$
    PROVE: $q + r \in \alpha$
  ⟨2⟩2. $r < 0$
  ⟨2⟩3. $q + r < q$
  ⟨2⟩4. $q + r \in \alpha$
⟨1⟩2. $\alpha \subseteq \alpha + 0^*$
  ⟨2⟩1. LET: $q \in \alpha$
  ⟨2⟩2. PICK $r \in \alpha$ such that $q < r$
  ⟨2⟩3. $q = r + (q - r) \in \alpha + 0^*$
∎

Proposition 5.10. *For any cut α, there exists a cut β such that $\alpha + \beta = 0$.*

PROOF:
⟨1⟩1. LET: $\beta = \{p \in \mathbb{Q} : \exists r > 0. -p - r \notin \alpha\}$
⟨1⟩2. β is a cut.
  ⟨2⟩1. $\beta \neq \emptyset$
    ⟨3⟩1. PICK $q \notin \alpha$
    ⟨3⟩2. $-q - 1 \in \beta$
  ⟨2⟩2. $\beta \neq \mathbb{Q}$
    ⟨3⟩1. PICK $q \in \alpha$
      PROVE: $-q \notin \beta$
    ⟨3⟩2. ASSUME: for a contradiction $-q \in \beta$

⟨3⟩3. PICK $r > 0$ such that $q - r \notin \alpha$
⟨3⟩4. $q - r < q$
⟨3⟩5. Q.E.D.
PROOF: This contradicts the fact that α is closed downwards.
⟨2⟩3. β is closed downwards.
⟨3⟩1. LET: $p \in \beta$ and $q < p$.
⟨3⟩2. PICK $r > 0$ such that $-p - r \notin \alpha$
⟨3⟩3. $-p - r < -q - r$
⟨3⟩4. $-q - r \notin \alpha$
⟨3⟩5. $q \in \beta$
⟨2⟩4. β has no greatest element.
⟨3⟩1. LET: $p \in \beta$
⟨3⟩2. PICK $r > 0$ such that $-p - r \notin \alpha$
⟨3⟩3. $-(p + r/2) - r/2 \notin \alpha$
⟨3⟩4. $p + r/2 \in \beta$
⟨1⟩3. $\alpha + \beta \subseteq 0^*$
⟨2⟩1. LET: $p \in \alpha$ and $q \in \beta$.
⟨2⟩2. PICK $r > 0$ such that $-q - r \notin \alpha$.
⟨2⟩3. $p < -q - r$
⟨2⟩4. $p + q < -r$
⟨2⟩5. $p + q < 0$
⟨2⟩6. $p + q \in 0^*$
⟨1⟩4. $0^* \subseteq \alpha + \beta$
⟨2⟩1. LET: $v \in 0^*$
⟨2⟩2. LET: $w = -v/2$
⟨2⟩3. $w > 0$
⟨2⟩4. PICK an integer n such that $nw \in \alpha$ and $(n + 1)w \notin \alpha$.
⟨2⟩5. LET: $p = -(n + 2)w$
⟨2⟩6. $p \in \beta$
⟨2⟩7. $v = nw + p$
⟨2⟩8. $v \in \alpha + \beta$
□

Proposition 5.11. *Given $\alpha, \beta, \gamma \in R$, if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

PROOF:
⟨1⟩1. $\alpha + \beta \subseteq \alpha + \gamma$
PROOF: Immediate from definitions.
⟨1⟩2. $\alpha + \beta \neq \alpha + \gamma$
PROOF: If $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$ by cancellation.
□

Definition 5.12. Given cuts α and β, define $\alpha\beta$ by:

$$\alpha\beta = \begin{cases} \{p \in \mathbb{Q} : \exists r \in \alpha . \exists s \in \beta (p \leq rs \wedge r > 0 \wedge s > 0\} & \text{if } \alpha > 0^* \text{ and } \beta > 0^* \\ (-\alpha)(-\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^* \\ -((-\alpha)\beta) & \text{if } \alpha < 0^* \text{ and } \beta > 0^* \\ -(\alpha(-\beta)) & \text{if } \alpha > 0^* \text{ and } \beta < 0^* \\ 0^* & \text{if } \alpha = 0^* \text{ or } \beta = 0^* \end{cases}$$

Proposition 5.13. *For any cuts α and β, we have $\alpha\beta$ is a cut.*

PROOF:
⟨1⟩1. If $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta$ is a cut.
  ⟨2⟩1. $\alpha\beta \neq \emptyset$
    ⟨3⟩1. PICK $q \in \alpha$ and $r \in \beta$ such that $q, r \notin 0^*$
    ⟨3⟩2. ASSUME: w.l.o.g. $0 < q$ and $0 < r$.
      PROOF: Since α and β have no greatest element.
    ⟨3⟩3. $qr \in \alpha\beta$
  ⟨2⟩2. $\alpha\beta \neq \mathbb{Q}$
    ⟨3⟩1. PICK $r \notin \alpha$ and $s \notin \beta$
      PROVE: $rs \notin \alpha\beta$
    ⟨3⟩2. ASSUME: for a contradiction $rs \in \alpha\beta$.
    ⟨3⟩3. PICK $r' \in \alpha$ and $s' \in \beta$ such that $rs \leq r's'$ and $r' > 0$ and $s' > 0$.
    ⟨3⟩4. $r' < r$ and $s' < s$
    ⟨3⟩5. $r's' < rs$
    ⟨3⟩6. Q.E.D.
      PROOF: This is a contradiction.
  ⟨2⟩3. $\alpha\beta$ is closed downwards.
    ⟨3⟩1. LET: $p \in \alpha\beta$ and $p' < p$
    ⟨3⟩2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$
    ⟨3⟩3. $p' \leq rs$
    ⟨3⟩4. $p' \in \alpha\beta$
  ⟨2⟩4. $\alpha\beta$ has no greatest element.
    ⟨3⟩1. LET: $p \in \alpha\beta$
    ⟨3⟩2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$.
    ⟨3⟩3. PICK $r' \in \alpha$ and $s' \in \beta$ with $r < r'$ and $s < s'$.
    ⟨3⟩4. $p < r's' \in \alpha\beta$
⟨1⟩2. For any cuts α and β, we have $\alpha\beta$ is a cut.
  PROOF: Since if α is a cut then $-\alpha$ is a cut.
□

Proposition 5.14. *For any cuts α and β we have $\alpha\beta = \beta\alpha$.*

PROOF: Easy from the definitions. □

Proposition 5.15. *For any cuts α, β and γ we have*

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma \ .$$

PROOF:
⟨1⟩1. CASE: α, β and γ are all positive.
PROOF: In this case $\alpha(\beta\gamma) = (\alpha\beta)\gamma = \{p \in \mathbb{Q} : \exists r \in \alpha.\exists s \in \beta.\exists t \in \gamma.(p \leq rst \wedge r > 0 \wedge s > 0 \wedge t > 0)\}$.
⟨1⟩2. CASE: One of α, β or γ is 0^*.
PROOF: Then $\alpha(\beta\gamma) = (\alpha\beta)\gamma = 0^*$.
⟨1⟩3. CASE: α and β are positive, γ is negative.
PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\
&= -(\alpha(\beta(-\gamma))) \\
&= -((\alpha\beta)(-\gamma)) && (\langle 1\rangle 1) \\
&= (\alpha\beta)\gamma
\end{aligned}$$

⟨1⟩4. CASE: α is positive, β is negative, γ is positive.
PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-((-\beta)\gamma)) \\
&= -(\alpha((-\beta)\gamma)) \\
&= -((\alpha(-\beta))\gamma) && (\langle 1\rangle 1) \\
&= (-(\alpha(-\beta)))\gamma \\
&= (\alpha\beta)\gamma
\end{aligned}$$

⟨1⟩5. CASE: α is positive, β and γ are negative.
PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha((-\beta)(-\gamma)) \\
&= (\alpha(-\beta))(-\gamma) && (\langle 1\rangle 1) \\
&= (-(\alpha\beta))(-\gamma) \\
&= (\alpha\beta)\gamma
\end{aligned}$$

⟨1⟩6. CASE: α is negative, β and γ are positive.
PROOF: Similar to ⟨1⟩3.
⟨1⟩7. CASE: α is negative, β is positive, γ is negative.
PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\
&= (-\alpha)(\beta(-\gamma)) \\
&= ((-\alpha)\beta)(-\gamma) && (\langle 1\rangle 1) \\
&= (-(\alpha\beta))(-\gamma) \\
&= (\alpha\beta)\gamma
\end{aligned}$$

⟨1⟩8. CASE: α and β are negative, γ is positive.
PROOF: Similar to ⟨1⟩5.
⟨1⟩9. CASE: α, β and γ are all negative.

PROOF:

$$\begin{aligned}\alpha(\beta\gamma) &= \alpha(-(-\beta)(-\gamma)) \\ &= -((-\alpha)((-\beta)(-\gamma))) \\ &= -(((-\alpha)(-\beta))(-\gamma)) && (\langle 1\rangle 1) \\ &= -((\alpha\beta)(-\gamma)) \\ &= (\alpha\beta)\gamma\end{aligned}$$

□

Proposition 5.16. *For any cut α we have $\alpha 1^* = \alpha$.*

PROOF:
$\langle 1\rangle 1$. CASE: α is positive.
  $\langle 2\rangle 1$. $\alpha 1^* \subseteq \alpha$
  $\langle 2\rangle 2$. $\alpha \subseteq \alpha 1^*$
$\langle 1\rangle 2$. CASE: $\alpha = 0^*$
$\langle 1\rangle 3$. CASE: α is negative.
□

Theorem 5.17. *There exists an ordered field with the least upper bound property.*

Proposition 5.18. *There is no rational p such that $p^2 = 2$.*

PROOF:
$\langle 1\rangle 1$. ASSUME: for a contradiction $p^2 = 2$.
$\langle 1\rangle 2$. PICK integers m, n not both even such that $p = m/n$.
$\langle 1\rangle 3$. $m^2 = 2n^2$
$\langle 1\rangle 4$. m is even.
$\langle 1\rangle 5$. PICK an integer k such that $m = 2k$.
$\langle 1\rangle 6$. $4k^2 = 2n^2$
$\langle 1\rangle 7$. $2k^2 = n^2$
$\langle 1\rangle 8$. n is even.
$\langle 1\rangle 9$. Q.E.D.
  PROOF: $\langle 1\rangle 2$, $\langle 1\rangle 4$ and $\langle 1\rangle 8$ form a contradiction.
□

Theorem 5.19. *Any two complete ordered fields are isomorphic.*

Definition 5.20. Let $\mathbb{R}$ be the complete ordered field. We call its elements *real numbers.*

5.2 Properties of the Real Numbers

Theorem 5.21. *$\mathbb{Q}$ is a subfield of $\mathbb{R}$.*

Theorem 5.22 (Archimedean Property)**.** *Let $x, y \in \mathbb{R}$ with $x > 0$. There exists a positive integer n such that $nx > y$.*

PROOF:
⟨1⟩1. LET: $A = \{nx : n \in \mathbb{Z}^+\}$
⟨1⟩2. ASSUME: for a contradiction there is no positive integer n such that $nx > y$.
⟨1⟩3. y is an upper bound for A.
⟨1⟩4. LET: $\alpha = \sup A$
⟨1⟩5. $\alpha - x$ is not an upper bound for A.
⟨1⟩6. PICK a positive integer m such that $\alpha - x < mx$
⟨1⟩7. $\alpha < (m+1)x \in A$
⟨1⟩8. Q.E.D.
PROOF: This contradicts ⟨1⟩4.
□

Theorem 5.23. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

PROOF:
⟨1⟩1. LET: $x, y \in \mathbb{R}$ with $x < y$
⟨1⟩2. PICK a positive integer n such that
$$n(y - x) > 1 \ .$$
PROOF: Archimedean property.
⟨1⟩3. PICK a positive integer m_1 such that $m_1 > nx$
PROOF: Archimedean property.
⟨1⟩4. PICK a positive integer m_2 such that $m_2 > -nx$
PROOF: Archimedean property.
⟨1⟩5. $-m_2 < nx < m_1$
⟨1⟩6. LET: m be the integer such that
$$m - 1 \leq nx < m \ .$$
⟨1⟩7. $nx < m \leq 1 + nx < ny$
⟨1⟩8. $x < m/n < y$
□

Theorem 5.24. *For every real number $x > 0$ and positive integer n, there exists a unique positive real number y such that $y^n = x$.*

PROOF:
⟨1⟩1. There exists a real $y > 0$ such that $y^n = x$.
⟨2⟩1. LET: $E = \{t \in \mathbb{R}^+ : t^n < x\}$
⟨2⟩2. LET: $y = \sup E$
⟨3⟩1. $E \neq \emptyset$
⟨4⟩1. LET: $t = x/(x+1)$
⟨4⟩2. $0 < t < 1$
⟨4⟩3. $t^n < t < x$
⟨4⟩4. $t \in E$
⟨3⟩2. $x + 1$ is an upper bound for E.
⟨4⟩1. LET: $t > x + 1$
⟨4⟩2. $t^n > t > x$
⟨4⟩3. $t \notin E$

⟨2⟩3. $y^n = x$
⟨3⟩1. $y^n \not< x$
⟨4⟩1. ASSUME: for a contradiction $y^n < x$.
⟨4⟩2. PICK h such that $0 < h < 1$ and
$$h < \frac{x - y^n}{n(y+1)^{n-1}} \enspace .$$
⟨4⟩3. $(y+h)^n - y^n < x - y^n$
PROOF:
$$\begin{aligned}
(y+h)^n - y^n &= ((y+h) - y) \sum_{i=0}^{n-1} (y+h)^{n-1-i} y^i \\
&= h \sum_{i=0}^{n-1} (y+h)^{n-1-i} y^i \\
&\leq hn(y+h)^{n-1} \\
&\leq hn(y+1)^{n-1} \\
&< x - y^n
\end{aligned}$$
⟨4⟩4. $(y+h)^n < x$
⟨4⟩5. $y + h \in E$
⟨4⟩6. Q.E.D.
PROOF: This contradicts the fact that y is an upper bound for E.
⟨3⟩2. $y^n \not> x$
⟨4⟩1. ASSUME: for a contradiction $y^n > x$
⟨4⟩2. LET:
$$k = \frac{y^n - x}{ny^{n-1}}$$
⟨4⟩3. $0 < k < y$
⟨4⟩4. $y - k$ is an upper bound for E.
⟨5⟩1. LET: $t \geq y - k$
⟨5⟩2. $y^n - t^n \leq y^n - x$
PROOF:
$$\begin{aligned}
y^n - t^n &\leq y^n - (y-k)^n \\
&= (y - (y-k)) \sum_{i=0}^{n-1} y^{n-i} (y-k)^i \\
&= k \sum_{i=0}^{n-1} y^{n-i} (y-k)^i \\
&\leq kny^{n-1} \\
&= y^n - x
\end{aligned}$$
⟨5⟩3. $t^n \geq x$
⟨5⟩4. $t \notin E$
⟨4⟩5. Q.E.D.
PROOF: This contradicts the fact that y is the least upper bound of E.
⟨1⟩2. If y and y' are positive reals with $y^n = y'^n$ then $y = y'$.

PROOF: Since the function that sends y to y^n is strictly monotone.
∎

Definition 5.25 (nth Root)**.** Given any real number $x > 0$ and positive integer n, the *nth root* of x, denoted $x^{1/n}$, is the unique positive real such that

$$(x^{1/n})^n = x \quad .$$

We write $\sqrt{x}$ for $x^{1/2}$.

Proposition 5.26. *Let a and b be positive real numbers and n a positive integer. Then*

$$(ab)^{1/n} = a^{1/n} b^{1/n} \quad .$$

PROOF: Since $(a^{1/n} b^{1/n})^n = ab$. ∎

Lemma 5.27. *Let b be a real number with $b > 1$. Let n be a positive integer. Then*

$$b - 1 \geq n(b^{1/n} - 1) \quad .$$

PROOF: From Lemma 4.24. ∎

Lemma 5.28. *Let b and t be real numbers with $b > 1$ and $t > 1$. For any positive integer n, if $n > \frac{b-1}{t-1}$ then $b^{1/n} < t$.*

PROOF:

$$\begin{aligned} b - 1 &\geq n(b^{1/n} - 1) \\ \therefore \frac{b-1}{n} &\geq b^{1/n} - 1 \\ \therefore t - 1 &> b^{1/n} - 1 \\ \therefore t &> b^{1/n} \end{aligned}$$ ∎

Lemma 5.29. *Let b be a real number with $b > 0$. Let m, n, p, q be integers with $n > 0$ and $q > 0$. Assume $m/n = p/q$. Then*

$$(b^m)^{1/n} = (b^p)^{1/q} \quad .$$

PROOF:
⟨1⟩1. $(b^m)^{1/n} = (b^{1/n})^m$
PROOF:

$$\begin{aligned} ((b^{1/n})^m)^n &= ((b^{1/n})^n)^m \\ &= b^m \end{aligned}$$

⟨1⟩2. $((b^m)^{1/n})^q = b^p$
PROOF:

$$\begin{aligned} ((b^m)^{1/n})^q &= (b^{1/n})^{mq} \\ &= (b^{1/n})^{np} \\ &= b^p \end{aligned}$$

∎

Definition 5.30. For a a positive real and q a rational number, we may therefore define a^q by

$$a^{m/n} = (a^m)^{1/n}$$

for m and n integers with $n > 0$.

Proposition 5.31. *Let a be a positive real and r, s rational numbers. Then*

$$a^{r+s} = a^r a^s \ .$$

PROOF:

$$\begin{aligned} a^{m/n+p/q} &= a^{(mq+np)/nq} \\ &= (a^{mq+np})^{1/nq} \\ &= (a^{mq})^{1/nq}(a^{np})^{1/nq} \\ &= a^{m/n}a^{p/q} \end{aligned}$$

□

Proposition 5.32. *Let $b > 1$ be a real number and q a rational number. Then*

$$b^q = \sup\{b^t : t \in \mathbb{Q}, t \leq q\}$$

PROOF: It is the greatest element of this set. □

Definition 5.33. Let $b > 1$ be a real number and x a real number. Then

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t \leq x\} \ .$$

Lemma 5.34. *Let b, w and y be real numbers with $b > 1$. Assume $b^w < y$. Then there exists a positive integer n such that $b^{w+1/n} < y$.*

PROOF:
$\langle 1 \rangle 1$. LET: $t = yb^{-w}$
$\langle 1 \rangle 2$. PICK a positive integer n such that $n > \frac{b-1}{t-1}$.
$\langle 1 \rangle 3$. $b^{1/n} < t$
 PROOF: Lemma 5.28.
$\langle 1 \rangle 4$. $b^{w+1/n} < y$
□

Lemma 5.35. *Let b, w and y be real numbers with $b > 1$. Assume $b^w > y$. Then there exists a positive integer n such that $b^{w-1/n} < y$.*

PROOF:
$\langle 1 \rangle 1$. LET: $t = b^w/y$
$\langle 1 \rangle 2$. PICK a positive integer n such that $n > \frac{b-1}{t-1}$
$\langle 1 \rangle 3$. $b^{1/n} < t$
 PROOF: Lemma 5.28.
$\langle 1 \rangle 4$. $y < b^{w-1/n}$
□

Proposition 5.36. *For b and x real numbers with $b > 1$ we have*

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t < x\} \ .$$

PROOF:
⟨1⟩1. b^x is an upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
⟨1⟩2. LET: u be any upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
PROVE: $b^x \leq u$
⟨1⟩3. LET: q be a rational number with $q \leq x$.
PROVE: $b^q \leq u$
⟨1⟩4. ASSUME: for a contradiction $b^q > u$.
⟨1⟩5. PICK a positive integer n such that $b^{q-1/n} > u$.
PROOF: Lemma 5.35.
⟨1⟩6. $b^{q-1/n} \leq u$
PROOF: ⟨1⟩2
⟨1⟩7. Q.E.D.
PROOF: This contradicts ⟨1⟩4.
□

Lemma 5.37. *Let A be a set of positive real numbers with supremum $a > 0$ and B a set of positive real numbers with supremum $b > 0$. Then ab is the supremum of $\{xy : x \in A, y \in B\}$.*

PROOF:
⟨1⟩1. For all $x \in A$ and $y \in B$ we have $xy \leq ab$.
⟨1⟩2. If u is any upper bound for $\{xy : x \in A, y \in B\}$ then $ab \leq u$.
⟨2⟩1. LET: u be an upper bound for $\{xy : x \in A, y \in B\}$.
⟨2⟩2. For all $x \in A$ we have u/x is an upper bound for B.
⟨2⟩3. For all $x \in A$ we have $b \leq u/x$
⟨2⟩4. For all $x \in A$ we have $x \leq u/b$
⟨2⟩5. $a \leq u/b$
⟨2⟩6. $ab \leq u$
□

Proposition 5.38. *Let $b, x, y \in \mathbb{R}$ with $b > 1$. Then*

$$b^{x+y} = b^x b^y \ .$$

PROOF:
⟨1⟩1. For any rational number $q < x + y$, there exist rational numbers $r < x$ and $s < y$ such that $q = r + s$.
⟨2⟩1. $q - x < y$
⟨2⟩2. PICK a rational t such that $q - x < t < y$
⟨2⟩3. $q = t + (q - t)$ and $t < y$, $q - t < x$
⟨1⟩2. $b^x b^y = b^{x+y}$

PROOF:

$$\begin{aligned} b^x b^y &= \sup\{b^q b^r : q, r \in \mathbb{Q}, q < x, r < y\} \\ &= \sup\{b^{q+r} : q, r \in \mathbb{Q}, q < x, r < y\} \\ &= \sup\{b^q : q \in \mathbb{Q}, q < x + y\} \\ &= b^{x+y} \end{aligned}$$

□

5.2.1 Logarithms

Proposition 5.39. *Let b and y be real numbers with $b > 1$ and $y > 0$. There exists a unique real x such that $b^x = y$.*

PROOF:
⟨1⟩1. LET: $x = \sup\{w : b^w < y\}$
PROVE: $b^x = y$
⟨2⟩1. $\{w : b^w < y\} \neq \emptyset$
PROOF: It contains 0.
⟨2⟩2. $\{w : b^w < y\}$ is bounded above.
⟨3⟩1. LET: n be the least integer such that

$$n \geq \frac{y - 1}{b - 1}$$

PROOF: Archimedean property.
⟨3⟩2. LET: w be a real number with $b^w < y$
PROVE: $w < n$
⟨3⟩3. $b^w < n(b - 1) + 1$
⟨3⟩4. $b^w < b^n$
⟨3⟩5. $w < n$
⟨1⟩2. $b^x \leq y$
⟨2⟩1. ASSUME: for a contradiction $b^x > y$
⟨2⟩2. PICK a positive integer n such that $b^{x-1/n} > y$
PROOF: Lemma 5.35.
⟨2⟩3. PICK w such that $x - 1/n < w$ and $b^w < y$
PROOF: Since $x - 1/n$ is not an upper bound for $\{w : b^w < y\}$.
⟨2⟩4. $b^{x-1/n} < y$
⟨2⟩5. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
⟨1⟩3. $b^x \geq y$
⟨2⟩1. ASSUME: for a contradiction $b^x < y$.
⟨2⟩2. PICK a positive integer n such that $b^{x+1/n} < y$.
⟨2⟩3. $x + 1/n \leq x$
⟨2⟩4. Q.E.D.
PROOF: This is a contradiction.

□

Definition 5.40 (Logarithm)**.** Let b and y be real numbers with $b > 1$ and $y > 0$. The *logarithm* of y to *base* b, denoted $\log_b y$, is the unique real number

such that
$$b^{\log_b y} = y \ .$$

5.2.2 Intervals

Definition 5.41 (Intervals)**.** Let $a, b \in \mathbb{R}$.
The *open interval* (a, b) is $\{x \in \mathbb{R} : a < x < b\}$.
The *closed interval* $[a, b]$ is $\{x \in \mathbb{R} : a \leq x \leq b\}$.
The *half-open intervals* $[a, b)$ and $(a, b]$ are defined by

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

Proposition 5.42. *Let (I_n) be a sequence of closed intervals with $I_0 \supseteq I_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} I_n$ is nonempty.*

PROOF:
$\langle 1 \rangle 1.$ LET: $I_n = [a_n, b_n]$
$\langle 1 \rangle 2.$ LET: $x = \sup_n a_n$
PROOF: $\{a_n : n \in \mathbb{N}\}$ is bounded above by b_0.
$\langle 1 \rangle 3.$ $x \in \bigcap_{n=0}^{\infty} I_n$
PROOF: For all n we have $a_n \leq x \leq b_n$ since b_n is an upper bound for $\{a_n : n \in \mathbb{N}\}$.
□

Definition 5.43 (k-cell)**.** Let k be a positive integer. A *k-cell* is a subset of $\mathbb{R}^k$ of the form
$$\{\vec{x} \in \mathbb{R}^k : \forall i = 1, \ldots, k. a_i \leq x_i \leq b_i\}$$
for some real numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$ with $a_i \leq b_i$ for each i.

Proposition 5.44. *Let (I_n) be a sequence of k-cells such that $I_0 \supseteq I_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $I_n = J_{n1} \times \cdots \times J_{nk}$ where each J_{ni} is a closed interval.
$\langle 1 \rangle 2.$ For $i = 1, \ldots, k$, PICK $a_i \in \bigcap_{n=0}^{\infty} J_{ni}$.
$\langle 1 \rangle 3.$ $(a_1, \ldots, a_k) \in \bigcap_{n=0}^{\infty} I_n$
□

5.2.3 The Cantor Set

Definition 5.45 (Cantor Set)**.** Define a sequence E_n of unions of intervals as follows:

- $E_0 = [0, 1]$
- E_{n+1} is formed from E_n by replacing every interval $[a, b]$ with $[a, (2a+b)/3]$ and $[(a + 2b)/3, b]$.

The *Cantor set* is $\bigcap_{n=0}^{\infty} E_n$.

5.3 The Extended Real Number System

Definition 5.46 (Extended Real Number System)**.** The *extended real number system* is the set $\mathbb{R} \cup \{+\infty, -\infty\}$.

We extend the ordering $\leq$ to the extended reals by defining

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

We extend $+$, $\cdot$ and $/$ to partial operations on the extended real by defining:

$$\begin{aligned}
x + (+\infty) &= +\infty && (x \in \mathbb{R})\\
x + (-\infty) &= -\infty && (x \in \mathbb{R})\\
(+\infty) + x &= +\infty && (x \in \mathbb{R})\\
(+\infty) + (+\infty) &\text{ is undefined} &&\\
(+\infty) + (-\infty) &\text{ is undefined} &&\\
(-\infty) + x &= -\infty && (x \in \mathbb{R})\\
(-\infty) + (+\infty) &\text{ is undefined} &&\\
(-\infty) + (-\infty) &\text{ is undefined} &&\\
x \cdot (+\infty) &= +\infty && (x \in \mathbb{R})\\
x \cdot (-\infty) &= -\infty && (x \in \mathbb{R})\\
(+\infty) \cdot x &= +\infty && (x \in \mathbb{R})\\
(+\infty) \cdot (+\infty) &\text{ is undefined} &&\\
(+\infty) \cdot (-\infty) &\text{ is undefined} &&\\
(-\infty) \cdot x &= -\infty && (x \in \mathbb{R})\\
(-\infty) \cdot (+\infty) &\text{ is undefined} &&\\
(-\infty) \cdot (-\infty) &\text{ is undefined} &&\\
x/(+\infty) &= 0 && (x \in \mathbb{R})\\
x/(-\infty) &= 0 && (x \in \mathbb{R})\\
(+\infty)/x &\text{ is undefined} && (x \in \mathbb{R})\\
(+\infty)/(+\infty) &\text{ is undefined} &&\\
(+\infty)/(-\infty) &\text{ is undefined} &&\\
(-\infty)/x &\text{ is undefined} && (x \in \mathbb{R})\\
(-\infty)/(+\infty) &\text{ is undefined} &&\\
(-\infty)/(-\infty) &\text{ is undefined} &&
\end{aligned}$$

Chapter 6

Complex Analysis

Definition 6.1 (Complex Numbers)**.** A *complex number* is a pair of real numbers. We write $\mathbb{C}$ for the set of complex numbers.

Define $+$ and $\cdot$ on $\mathbb{C}$ by:

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b)(c, d) &= (ac - bd, ad + bc)\end{aligned}$$

Theorem 6.2. *The complex numbers form a field.*

Theorem 6.3. *The function that maps a to $(a, 0)$ is an embedding of $\mathbb{R}$ in $\mathbb{C}$.*

Definition 6.4.

$$i = (0, 1)$$

Lemma 6.5.

$$(a, b) = a + ib$$

Proof: Since $(a, 0) + (0, 1)(b, 0) = (a, b)$. $\square$

Lemma 6.6.

$$i^2 = -1$$

Proof: Immediate from definitions. $\square$

Corollary 6.6.1. *There is no linear order on $\mathbb{C}$ that makes $\mathbb{C}$ into an ordered field.*

Definition 6.7 (Complex Conjugate)**.** For any complex number z, the *complex conjugate* $\overline{z}$ is defined by

$$\overline{a + ib} = a - ib \qquad (a, b \in \mathbb{R}) \ .$$

Definition 6.8 (Real Part)**.** For any complex number z, the *real part* of z, denoted $\mathrm{Re}(z)$, is defined by

$$\mathrm{Re}(a + ib) = a \qquad (a, b \in \mathbb{R}) \ .$$

Definition 6.9 (Imaginary Part)**.** For any complex number z, the *imaginar part* of z, denoted $\mathrm{Im}(z)$, is defined by

$$\mathrm{Im}(a+ib) = b \qquad (a, b \in \mathbb{R}) \ .$$

Theorem 6.10. *For all $z, w \in \mathbb{C}$ we have*

$$\overline{z+w} = \overline{z} + \overline{w} \ .$$

PROOF:

$$\begin{aligned} \overline{(a+ib)+(c+id)} &= \overline{(a+c)+i(b+d)} \\ &= (a+c) - i(b+d) \\ &= (a-ib) + (c-id) \\ &= \overline{a+ib} + \overline{c+id} \end{aligned}$$

∎

Theorem 6.11. *For all $z, w \in \mathbb{C}$ we have*

$$\overline{zw} = \overline{z} \cdot \overline{w} \ .$$

PROOF:

$$\begin{aligned} \overline{(a+ib)(c+id)} &= \overline{(ac-bd)+i(ad+bc)} \\ &= (ac-bd) - i(ad+bc) \\ &= (a-ib)(c-id) \\ &= \overline{a+ib} \cdot \overline{c+id} \end{aligned}$$

∎

Theorem 6.12. *For all $z \in \mathbb{C}$ we have*

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}) \ .$$

PROOF:

$$\begin{aligned} (a+ib) + \overline{a+ib} &= (a+ib) + (a-ib) \\ &= 2a \\ &= 2\,\mathrm{Re}(a+ib) \end{aligned}$$

∎

Theorem 6.13. *For all $z \in \mathbb{C}$ we have*

$$\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z}) \ .$$

PROOF:

$$\begin{aligned} (a+ib) - \overline{a+ib} &= (a+ib) - (a-ib) \\ &= 2ib \\ &= 2i\,\mathrm{Im}(a+ib) \end{aligned}$$

∎

Theorem 6.14. *For all $z \in \mathbb{C}$ we have $z\overline{z}$ is a non-negative real.*

PROOF:
$$\begin{aligned}(a+ib)(\overline{a+ib}) &= (a+ib)(a-ib)\\ &= a^2+b^2\end{aligned}\qquad\square$$

Theorem 6.15. *For any $z \in \mathbb{C}$, if $z\overline{z} = 0$ then $z = 0$.*

PROOF: Let $z = a + ib$. Then $z\overline{z} = a^2 + b^2 = 0$ iff $a = b = 0$. $\square$

Definition 6.16 (Absolute Value)**.** For $z \in \mathbb{C}$, the *absolute value* of z is

$$|z| = (z\overline{z})^{1/2} \quad .$$

Proposition 6.17. *For x a non-negative real we have $|x| = x$.*

PROOF: Since $|x| = \sqrt{x^2} = x$. $\square$

Proposition 6.18. *For x a negative real we have $|x| = -x$.*

PROOF: Since $|x| = \sqrt{x^2} = -x$. $\square$

Theorem 6.19. *For any complex number z we have $|z| \geq 0$.*

PROOF: Immediate from definition. $\square$

Theorem 6.20. *For any complex number z, if $|z| = 0$ then $z = 0$.*

PROOF: From Theorem 6.15. $\square$

Theorem 6.21. *For any complex number z we have*

$$|\overline{z}| = |z| \quad .$$

PROOF: Immediate from definitions. $\square$

Theorem 6.22. *For any complex numbers z and w we have*

$$|zw| = |z||w| \quad .$$

PROOF:
$$\begin{aligned}|zw| &= \sqrt{zw\overline{zw}}\\ &= \sqrt{z\overline{z}}\sqrt{w\overline{w}} && \text{(Proposition 5.26)}\\ &= |z||w|\end{aligned}\qquad\square$$

Theorem 6.23. *For any complex number z we have*

$$|\operatorname{Re} z| \leq |z|$$

PROOF: Let $z = a + ib$. Then
$$|\operatorname{Re} z| = \sqrt{a^2} \leq \sqrt{a^2+b^2} \ .\square$$

Theorem 6.24. *For any complex numbers z and w we have*

$$|z + w| \leq |z| + |w| \quad .$$

PROOF:

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \\
&= |z|^2 + 2\,\mathrm{Re}(z\overline{w}) + |w|^2 && \text{(Theorem 6.12)} \\
&\leq |z|^2 + 2|z\overline{w}| + |w|^2 && \text{(Theorem 6.23)} \\
&= |z|^2 + 2|z||w| + |w|^2 && \text{(Theorem 6.22)} \\
&= (|z|+|w|)^2
\end{aligned}$$

□

Theorem 6.25 (Schwarz Inequality)**.** *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be complex numbers. Then*

$$\left|\sum_{j=1}^n a_j\overline{b_j}\right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2 \enspace .$$

PROOF:
⟨1⟩1. LET: $A = \sum_{j=1}^n |a_j|^2$
⟨1⟩2. LET: $B = \sum_{j=1}^n |b_j|^2$
⟨1⟩3. LET: $C = \sum_{j=1}^n a_j\overline{b_j}$
⟨1⟩4. ASSUME: w.l.o.g. $B > 0$
PROOF: If $B = 0$ then $b_1 = \cdots = b_n = 0$ and both sides of the inequality are 0.
⟨1⟩5. $\sum_{j=1}^n |Ba_j - Cb_j|^2 = B(AB - |C|^2)$
PROOF:

$$\begin{aligned}
\sum_{j=1}^n |Ba_j - Cb_j|^2 &= \sum_{j=1}^n (Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j}) \\
&= B^2\sum_{j=1}^n |a_j|^2 - B\overline{C}\sum_{j=1}^n a_j\overline{b_j} - BC\sum_{j=1}^n \overline{a_j}b_j + |C|^2\sum_{j=1}^n |b_j|^2 \\
&= B^2A - 2B|C|^2 + B|C|^2 \\
&= B(AB - |C|^2)
\end{aligned}$$

⟨1⟩6. $B(AB - |C|^2) \geq 0$
⟨1⟩7. $AB \geq |C|^2$
□

Proposition 6.26. *For any non-zero complex number w, there are exactly two complex numbers z such that $z^2 = w$.*

PROOF:
⟨1⟩1. There are at most two complex numbers z such that $z^2 = w$.
PROOF: Proposition 4.15.
⟨1⟩2. There are at least two complex numbers z such that $z^2 = w$.
⟨2⟩1. LET: $w = u + iv$
⟨2⟩2. LET: $a = \sqrt{\frac{|w|+u}{2}}$
⟨2⟩3. LET: $b = \sqrt{\frac{|w|-u}{2}}$

⟨2⟩4. CASE: $v \geq 0$
⟨3⟩1. LET: $z = a + ib$
⟨3⟩2. $z^2 = w$
PROOF:

$$\begin{aligned} z^2 &= (a+ib)^2 \\ &= a^2 - b^2 + 2iab \\ &= u + i\sqrt{|w|^2 - u^2} \\ &= u + iv \\ &= w \end{aligned}$$

⟨3⟩3. $(-z)^2 = w$
⟨2⟩5. CASE: $v \leq 0$
⟨3⟩1. LET: $z = a - ib$
⟨3⟩2. $z^2 = w$
PROOF:

$$\begin{aligned} z^2 &= (a-ib)^2 \\ &= a^2 - b^2 - 2iab \\ &= u - i\sqrt{|w|^2 - u^2} \\ &= u - i|v| \\ &= w \end{aligned}$$

⟨3⟩3. $(-z)^2 = w$
□

6.1 Algebraic Numbers

Definition 6.27 (Algebraic)**.** A complex number z is *algebraic* iff there exist integers a_0, a_1, ..., a_n not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \ ;$$

otherwise, it is *transcendental*.

Proposition 6.28. *The set of algebraic numbers is countable.*

PROOF: There are countably many finite sequences of integers $(a_0, a_1, \ldots, a_n)$, and for each one, there are only finitely many complex numbers z such that $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$. □

Part I

Linear Algebra

Chapter 7

Vector Spaces

7.1 Convex Sets

Definition 7.1 (Convex)**.** Let $E \subseteq \mathbb{R}^k$. Then E is *convex* iff, for all $\vec{x}, \vec{y} \in E$ and $\lambda \in (0, 1)$,

$$\lambda \vec{x} + (1 - \lambda)\vec{y} \in E \ .$$

Proposition 7.2. *Every k-cell is convex.*

PROOF:
$\langle 1 \rangle 1$. LET: $C = \{\vec{x} \in \mathbb{R}^k : \forall i . a_i \leq x_i \leq b_i\}$ be a k-cell.
$\langle 1 \rangle 2$. LET: $\vec{x}, \vec{y} \in C$ and $\lambda \in (0, 1)$.
PROVE: $\lambda \vec{x} + (1 - \lambda)\vec{y} \in C$
$\langle 1 \rangle 3$. For each i we have $a_i \leq \lambda x_i + (1 - \lambda) y_i \leq b_i$
PROOF: Since $\lambda a_1 + (1 - \lambda) a_i \leq \lambda x_i + (1 - \lambda) y_i \leq \lambda b_i + (1 - \lambda) b_i$.
$\square$

Chapter 8

Real Inner Product Spaces

Definition 8.1 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^k$, define the *inner product* $\vec{x} \cdot \vec{y}$ by

$$(x_1, \ldots, x_k) \cdot (y_1, \ldots, y_k) = x_1 y_1 + \cdots + x_k y_k \quad .$$

Definition 8.2 (Norm)**.** Define the *norm* of a vector $\vec{x} \in \mathbb{R}^k$ by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} \quad .$$

Proposition 8.3.

$$\|\vec{x}\| \geq 0$$

Proof: Immediate from the definition. □

Proposition 8.4. *If* $\|\vec{x}\| = 0$ *then* $\vec{x} = \vec{0}$.

Proof: If $\|\vec{x}\| = 0$ then $x_1^2 + \cdots + x_n^2 = 0$ so $x_1 = \cdots = x_n = 0$. □

Proposition 8.5. *For* $\alpha \in \mathbb{R}$ *and* $\vec{x} \in \mathbb{R}^k$*,*

$$\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| \quad .$$

Proof: Easy. □

Proposition 8.6. *For* $\vec{x}, \vec{y} \in \mathbb{R}^k$*, we have*

$$\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\| \|\vec{y}\| \quad .$$

Proof: By the Schwarz inequality. □

Proposition 8.7. *For* $\vec{x}, \vec{y} \in \mathbb{R}^k$ *we have*

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \quad .$$

PROOF:

$$\begin{aligned}\|\vec{x}+\vec{y}\|^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \\ &= \vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Proposition 8.6)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square\end{aligned}$$

Corollary 8.7.1. *For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ we have*

$$\|\vec{x}-\vec{z}\| \leq \|\vec{x}-\vec{y}\| + \|\vec{y}-\vec{z}\| \enspace .$$

8.1 Balls

Definition 8.8 (Closed Ball)**.** Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *closed ball* with *centre* $\vec{x}$ and *radius* r is

$$\{y \in \mathbb{R}^k : \|y - x\| \leq r\} \enspace .$$

Proposition 8.9. *Every closed ball is convex.*

PROOF:
⟨1⟩1. LET: B be the closed ball with center $\vec{a}$ and radius r.
⟨1⟩2. LET: $\vec{x}, \vec{y} \in B$
⟨1⟩3. LET: $\lambda \in (0,1)$
⟨1⟩4. $\lambda\vec{x} + (1-\lambda)\vec{y} \in B$
PROOF:

$$\begin{aligned}\|\lambda\vec{x} + (1-\lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x}-\vec{a}) + (1-\lambda)(\vec{y}-\vec{a})\| \\ &= \lambda\|\vec{x}-\vec{a}\| + (1-\lambda)\|\vec{y}-\vec{a}\| \\ &\leq \lambda r + (1-\lambda) r \\ &= r && \square\end{aligned}$$

$\square$

Chapter 9

Complex Inner Product Spaces

Definition 9.1 (Inner Product)**.** Let V be a complex vector space. An *inner product* on V is a function $\langle\ ,\ \rangle : V^2 \to \mathbb{C}$ such that, for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

- $\langle y, x\rangle = \overline{\langle x, y\rangle}$
- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
- $\langle \alpha x, y\rangle = \overline{\alpha}\langle x, y\rangle$
- $\langle x, x\rangle \geq 0$
- If $\langle x, x\rangle = 0$ then $x = 0$.

An *inner product space* consists of a complex vector space V and an inner product on V.

Definition 9.2 (Norm)**.** Let V be an inner product space and $x \in V$. The *norm* of x is

$$\|x\| = \sqrt{\langle x, x\rangle}\ .$$

Proposition 9.3. *An inner product space is a metric space under*

$$d(x, y) = \|x - y\|\ .$$

Definition 9.4 (Bounded)**.** Let V_1 and V_2 be inner product spaces and $T : V_1 \to V_2$ a linear transformation. Then T is *bounded* iff $\{\|T(x)\| : \|x\| = 1\}$ is bounded above.

Proposition 9.5. *Every linear transformation between finite dimensional inner product spaces is bounded.*

Definition 9.6 (Outer Product)**.** Let V be an inner product space and $|\psi\rangle, |\phi\rangle \in V$. The *outer product* of $|\psi\rangle$ and $|\phi\rangle$ is

$$|\psi\rangle\langle\phi| : V \to V\ .$$

9.1 Hilbert Spaces

Definition 9.7 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Theorem 9.8 (Completeness Relation)**.** *Let $\mathcal{H}$ be a Hilbert space. Let $\{|e_n\rangle\}_{n\in\mathbb{N}}$ be a countable orthonormal basis for $\mathcal{H}$. Then*

$$\sum_{n=0}^{\infty} |e_n\rangle \langle e_n| = I \ .$$

PROOF:
$\langle 1\rangle 1.$ LET: $|\psi\rangle \in \mathcal{H}$
$\langle 1\rangle 2.$ LET: $|\psi\rangle = \sum_{n=0}^{\infty} \alpha_n |e_n\rangle$
$\langle 1\rangle 3.$ $\sum_{n=0}^{\infty} \langle e_n|\phi\rangle |e_n\rangle = |\psi\rangle$
PROOF:

$$\begin{aligned}\sum_{n=0}^{\infty} \langle e_n|\phi\rangle |e_n\rangle &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \alpha_m \langle e_n|e_m\rangle |e_n\rangle \\ &= \sum_{n=0}^{\infty} \alpha_n |e_n\rangle \\ &= |\psi\rangle \end{aligned}$$

□

□

Definition 9.9 (Separable)**.** A Hilbert space is *separable* iff it has a countable dense orthonormal basis.

Chapter 10

Lie Algebras

Definition 10.1 (Lie Algebra)**.** Let K be a field. A *Lie algebra* $\mathcal{L}$ over K consists of a vector space $\mathcal{L}$ over K and an operation

$$[\ ,\ ] : \mathcal{L}^2 \to \mathcal{L}\ ,$$

the *Lie bracket* or *commutator*, such that, for all $x, y, z \in \mathcal{L}$ and $\alpha \in K$:

$$\begin{aligned}
[x + y, z] &= [x, z] + [y, z] \\
[x, y + z] &= [x, y] + [x, z] \\
[\alpha x, y] &= \alpha[x, y] \\
[x, x] &= 0 \\
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \qquad \text{(Jacobi identity)}
\end{aligned}$$

Lemma 10.2. *If K has characteristic 0 then the condition $[x, x] = 0$ can be replaced with $[x, y] = -[y, x]$.*

Proposition 10.3. *The commutator is determind by its values on any basis for $\mathcal{L}$.*

Example 10.4. $\mathbb{R}^3$ with the cross product is a real Lie algebra.

Example 10.5. For any $n \geq 0$, we have $GL(n, K)$ is a Lie algebra over K under

$$[A, B] = AB - BA\ .$$

Definition 10.6 (Linear Lie Algebra)**.** A *linear Lie algebra* over K is a Lie algebra over K that is a subalgebra of $GL(n, K)$ for some n.

Example 10.7 (Special Linear Algebra)**.** The *special Linear algebra* $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \mathrm{tr} = 0\}$ is a real linear Lie algebra.

Example 10.8 (Orthogonal Lie Algebra)**.** The *orthogonal Lie algebra* $SO(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A \text{ is skew-symmetric}\}$ is a real linear Lie algebra.

Example 10.9. Let $u(n)$ be the set of all skew-Hermitian $n \times n$-matrices as a real Lie algebra.

Let $su(n) = u(n) \cap SL(n, \mathbb{R})$.

Proposition 10.10. *$SU(2)$ is spanned by the* Pauli matrices

$$\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which satisfy

$$[\sigma_x, \sigma_y] = \sigma_z$$
$$[\sigma_y, \sigma_z] = \sigma_x$$
$$[\sigma_z, \sigma_x] = \sigma_y$$

10.1 Lie Algebar Homomorphisms

Definition 10.11 (Homomorphism)**.** Let L_1 and L_2 be Lie algebras over the same field. A *Lie algebra homomorphism* $\phi : L_1 \to L_2$ is a linear transformation such that

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in L_1$.

Lemma 10.12. *Every bijective Lie algebra homomorphism is an isomorphism.*

Definition 10.13 (Representation)**.** Let L be a real (complex) Lie algebra. A *representation* of L is a Lie algebra homomorphism $L \to GL(n, \mathbb{R})$ $(GL(n, \mathbb{C}))$ for some n.

Example 10.14. The linear transformation $\mathbb{R}^3 \to su(2)$ defined by

$$i \mapsto \sigma_x, j \mapsto \sigma_y, k \mapsto \sigma_z$$

is a representation of $\mathbb{R}^3$.

Part II

Topology

Chapter 11

Metric Spaces

Definition 11.1 (Metric)**.** A *metric* on a set X is a function $d : X^2 \to \mathbb{R}$ such that, for all $x, y, z \in X$:

- $d(x, y) \geq 0$
- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- **Triangle Inequality** $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* X consists of a set X and a metric on X.

Example 11.2. $\mathbb{R}^k$ is a metric space under $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$. The triangle inequality is Corollary 8.7.1.

Example 11.3. For any set X, the *discrete* metric on X is defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Proposition 11.4. *Let (X, d) be a metric space and Y a subset of X. Then $d \upharpoonright Y^2$ is a metric on Y.*

Proof: Easy. □

11.1 Balls

Definition 11.5 (Open Ball)**.** Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *open ball* with *centre* $\vec{x}$ and *radius* r is

$$\{y \in \mathbb{R}^k : \|y - x\| < r\} \ .$$

Proposition 11.6. *Every open ball in $\mathbb{R}^k$ is convex.*

PROOF:
⟨1⟩1. LET: B be the open ball with center $\vec{a}$ and radius r.
⟨1⟩2. LET: $\vec{x}, \vec{y} \in B$
⟨1⟩3. LET: $\lambda \in (0, 1)$
⟨1⟩4. $\lambda\vec{x} + (1 - \lambda)\vec{y} \in B$
PROOF:

$$\begin{aligned} \|\lambda\vec{x} + (1-\lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1-\lambda)(\vec{y} - \vec{a})\| \\ &= \lambda\|\vec{x} - \vec{a}\| + (1-\lambda)\|\vec{y} - \vec{a}\| \\ &< \lambda r + (1-\lambda)r \\ &= r \end{aligned}$$

□

□

11.2 Limit Points

Definition 11.7 (Limit Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is a *limit point* of E iff every open ball with centre p contains a point of E other than p.

Proposition 11.8. *Let X be a metric space. Let $E \subseteq X$. Let p be a limit point of E. Then every neighbourhood of p contains infinitely many points of E.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction N is a neighbourhood of p that contains only finitely many points $q_1, \ldots, q_n$ of $E - \{p\}$.
⟨1⟩2. LET: $r = \min(q_1, \ldots, q_n)$
⟨1⟩3. LET: B be the open ball with centre p and radius r.
⟨1⟩4. B is a neighbourhood of p that contains no points of E other than p.
□

Corollary 11.8.1. *A finite set has no limit points.*

Definition 11.9 (Isolated Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *isolated point* of E iff $p \in E$ and p is not a limit point of E.

11.3 Closed Sets

Definition 11.10 (Closed Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *closed* iff every limit point of E is a member of E.

11.4 Interior Points

Definition 11.11 (Interior Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *interior point* of E iff there exists an open ball B with centre p such that $B \subseteq E$.

Definition 11.12 (Interior)**.** The *interior* of a set E, denoted E°, is the set of all its interior points.

Proposition 11.13. *The interior of E is the largest open set that is included in E.*

PROOF:
$\langle 1 \rangle 1$. LET: I be the interior of E.
$\langle 1 \rangle 2$. I is open.
  $\langle 2 \rangle 1$. LET: $p \in I$
  $\langle 2 \rangle 2$. PICK an open ball B with centre p such that $B \subseteq E$.
  $\langle 2 \rangle 3$. $B \subseteq I$
    $\langle 3 \rangle 1$. LET: $q \in B$
    $\langle 3 \rangle 2$. There exists an open ball B' with centre q such that $B' \subseteq B$.
    $\langle 3 \rangle 3$. There exists an open ball B' with centre q such that $B' \subseteq E$.
    $\langle 3 \rangle 4$. $q \in I$
$\langle 1 \rangle 3$. If J is any open set and $J \subseteq E$ then $J \subseteq I$.
  $\langle 2 \rangle 1$. LET: J be an open set.
  $\langle 2 \rangle 2$. ASSUME: $J \subseteq E$
  $\langle 2 \rangle 3$. For all $p \in J$, there exists an open ball B with centre p such that $B \subseteq J$.
  $\langle 2 \rangle 4$. For all $p \in J$, there exists an open ball B with centre p such that $B \subseteq E$.
  $\langle 2 \rangle 5$. $p \in I$
$\square$

11.5 Open Sets

Definition 11.14 (Open Sets)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *open* iff every point in E is an interior point of E.

Proposition 11.15. *Every open ball is open.*

PROOF:
$\langle 1 \rangle 1$. LET: B be an open ball with centre c and radius r.
$\langle 1 \rangle 2$. LET: $x \in B$
$\langle 1 \rangle 3$. LET: $\epsilon = r - d(x, c)$
$\langle 1 \rangle 4$. LET: B' be the open ball with centre x and radius ϵ.
  PROVE: $B' \subseteq B$
$\langle 1 \rangle 5$. LET: $y \in B'$
$\langle 1 \rangle 6$. $d(y, c) < r$
  PROOF:

$$\begin{aligned} d(y,c) &\leq d(y,x) + d(x,c) && \text{(Triangle Inequality)} \\ &< \epsilon + d(x,c) && (\langle 1 \rangle 5) \\ &= r && (\langle 1 \rangle 3) \end{aligned}$$

$\square$

Proposition 11.16. *A set is open if and only if its complement is closed.*

PROOF:
⟨1⟩1. LET: $E \subseteq X$
⟨1⟩2. If E is open then $X - E$ is closed.
  ⟨2⟩1. ASSUME: E is open.
  ⟨2⟩2. LET: p be a limit point of $X - E$.
    PROVE: $p \in X - E$
  ⟨2⟩3. ASSUME: for a contradiction $p \in E$.
  ⟨2⟩4. PICK an open ball B with centre p such that $B \subseteq E$.
  ⟨2⟩5. B contains a point of $X - E$.
    PROOF: ⟨2⟩2
  ⟨2⟩6. Q.E.D.
    PROOF: This contradicts ⟨2⟩4.
⟨1⟩3. If $X - E$ is closed then E is open.
  ⟨2⟩1. ASSUME: $X - E$ is closed.
  ⟨2⟩2. LET: $p \in E$
  ⟨2⟩3. ASSUME: for a contradiction no open ball with centre p is a subset of E.
  ⟨2⟩4. Every open ball with centre p intersects $X - E$.
  ⟨2⟩5. p is a limit point of $X - E$.
  ⟨2⟩6. $p \in X - E$
    PROOF: ⟨2⟩1
  ⟨2⟩7. Q.E.D.
    PROOF: This contradicts ⟨2⟩2.
□

Corollary 11.16.1. *A set is closed if and only if its complement is open.*

Proposition 11.17. *The union of a set of open sets is open.*

PROOF:
⟨1⟩1. LET: $\mathcal{U}$ be a set of open sets.
⟨1⟩2. LET: $p \in \bigcup \mathcal{U}$
⟨1⟩3. PICK $U \in \mathcal{U}$ such that $p \in U$.
⟨1⟩4. PICK an open ball B with centre p such that $B \subseteq U$.
⟨1⟩5. $B \subseteq \bigcup \mathcal{U}$
□

Corollary 11.17.1. *The intersection of a set of closed sets is closed.*

Proposition 11.18. *The intersection of two open sets is open.*

PROOF:
⟨1⟩1. LET: U and V be open.
⟨1⟩2. LET: $p \in U \cap V$
⟨1⟩3. PICK open balls B_1 and B_2 with centre p such that $B_1 \subseteq U$ and $B_2 \subseteq V$.
⟨1⟩4. ASSUME: w.l.o.g. the radius of B_1 is $\leq$ the radius of B_2.
⟨1⟩5. $B_1 \subseteq U \cap V$
□

Corollary 11.18.1. *The union of two closed sets is closed.*

Example 11.19. The intersection of a set of open sets is not necessarily open.

For every positive integer n, we have $(-1/n, 1/n)$ is open in $\mathbb{R}$, but $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$ is not open.

Theorem 11.20. *Let X be a metric space. Let $Y \subseteq X$ and $E \subseteq Y$. Then E is open in Y if and only if there exists an open subset G of X such that $E = G \cap Y$.*

PROOF:
$\langle 1 \rangle 1$. If E is open in Y then there exists an open subset G of X such that $E = G \cap Y$.
  $\langle 2 \rangle 1$. ASSUME: E is open in Y.
  $\langle 2 \rangle 2$. For $p \in E$, PICK $r_p > 0$ such that the open ball in Y with centre p and radius r_p is included in E.
  $\langle 2 \rangle 3$. For $p \in E$,
    LET: V_p be the open ball in X with centre p and radius r_p.
  $\langle 2 \rangle 4$. LET: $G = \bigcup_{p \in E} V_p$
  $\langle 2 \rangle 5$. G is open in Y.
    PROOF: Proposition 11.17.
  $\langle 2 \rangle 6$. $E = G \cap Y$
    $\langle 3 \rangle 1$. $E \subseteq G \cap Y$
      $\langle 4 \rangle 1$. LET: $p \in E$
      $\langle 4 \rangle 2$. $p \in V_p$
      $\langle 4 \rangle 3$. $p \in G$
    $\langle 3 \rangle 2$. $G \cap Y \subseteq E$
      $\langle 4 \rangle 1$. LET: $x \in G \cap Y$
      $\langle 4 \rangle 2$. PICK $p \in E$ such that $x \in V_p$
      $\langle 4 \rangle 3$. $d(x, p) < r_p$
      $\langle 4 \rangle 4$. $x \in E$
$\langle 1 \rangle 2$. For any open subset G of X, we have $G \cap Y$ is open in Y.
  $\langle 2 \rangle 1$. LET: G be an open subset of X.
  $\langle 2 \rangle 2$. LET: $p \in G \cap Y$
  $\langle 2 \rangle 3$. PICK $r > 0$ such that the open ball in X with centre p and radius r is included in G.
  $\langle 2 \rangle 4$. The open ball in Y with centre p and radius r is included in $G \cap Y$.
$\square$

11.6 Perfect Sets

Definition 11.21 (Perfect Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *perfect* iff E is closed and every point in E is a limit point of E.

11.7 Bounded Sets

Definition 11.22 (Bounded Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *bounded* iff there exists a real number M and $q \in X$ such that, for all $p \in E$, we have $d(p, q) < M$.

11.8 Dense Sets

Definition 11.23 (Dense Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *dense* iff every point of X is either a limit point of E or a point of E, or both.

11.9 Closure

Definition 11.24 (Closure)**.** Let X be a metric space. Let $E \subseteq X$. Then the *closure* of E, denoted $\overline{E}$, is the union of E and the set of limit points of E.

Proposition 11.25. *$\overline{E}$ is the smallest closed set that includes E.*

PROOF:
⟨1⟩1. $\overline{E}$ is closed.
  ⟨2⟩1. LET: p be a limit point of $\overline{E}$.
  ⟨2⟩2. ASSUME: $p \notin E$
      PROVE: p is a limit point of E.
  ⟨2⟩3. LET: B be the open ball with centre p and radius r.
      PROVE: B intersects E.
  ⟨2⟩4. PICK a point $q \in B \cap \overline{E}$.
  ⟨2⟩5. PICK an open ball B' with centre q such that $B' \subseteq B$.
  ⟨2⟩6. PICK a point $r \in E \cap B'$
  ⟨2⟩7. $r \in E \cap B$
⟨1⟩2. If C is closed and $E \subseteq C$ then $\overline{E} \subseteq C$.
  ⟨2⟩1. ASSUME: C is closed.
  ⟨2⟩2. ASSUME: $E \subseteq C$
  ⟨2⟩3. LET: $p \in \overline{E}$
  ⟨2⟩4. ASSUME: for a contradiction $p \notin C$
  ⟨2⟩5. p is a limit point of C.
    ⟨3⟩1. LET: B be an open ball with centre p.
    ⟨3⟩2. B intersects E.
    ⟨3⟩3. B intersects C.
    ⟨3⟩4. B intersects C in a point other than p.
      PROOF: ⟨2⟩3
  ⟨2⟩6. Q.E.D.
    PROOF: This contradicts ⟨2⟩1.
□

Corollary 11.25.1. *E is closed if and only if $E = \overline{E}$.*

Theorem 11.26. *Let E be a nonempty set of real numbers bounded above. Then* $\sup E \in \overline{E}$.

PROOF:
⟨1⟩1. ASSUME: $\sup E \notin E$
PROVE: $\sup E$ is a limit point of E.
⟨1⟩2. LET: B be an open ball with centre $\sup E$ and radius r.
⟨1⟩3. There exists $x \in E$ such that $x > \sup E - r$.
⟨1⟩4. E intersects B in a point other than p.
□

Proposition 11.27.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
⟨1⟩1. $\overline{A} \cup \overline{B}$ is a closed set that includes $A \cup B$.
⟨1⟩2. If C is a closed set that includes $A \cup B$ then $\overline{A} \cup \overline{B} \subseteq C$.
□

Example 11.28. It is not true in general. that $\overline{\bigcup \mathcal{A}} = \bigcup_{A \in \mathcal{A}} \overline{A}$.

In $\mathbb{R}$, let $\mathcal{A} = \{\{1/n\} : n \in \mathbb{Z}^+\}$. Then

$$\overline{\bigcup \mathcal{A}} = \{1/n : n \in \mathbb{Z}^+\} \cup \{0\}$$
$$\bigcup_{A \in \mathcal{A}} \overline{A} = \{1/n : n \in \mathbb{Z}^+\}$$

Proposition 11.29.

$$X - E^\circ = \overline{X - E}$$

PROOF:

$$\begin{aligned} p \in X - E^\circ &\Leftrightarrow p \notin E^\circ \\ &\Leftrightarrow \forall B \text{ an open ball with centre } p . B \nsubseteq E \\ &\Leftrightarrow \forall B \text{ an open ball with centre } p . B \text{ intersects} X - E \\ &\Leftrightarrow p \in \overline{X - E} \end{aligned}$$

□

11.10 Compact Sets

Definition 11.30 (Open Cover)**.** Let X be a metric space. Let $E \subseteq X$. An *open cover* of E is a set $\mathcal{U}$ of open sets such that $E \subseteq \bigcup \mathcal{U}$.

Definition 11.31 (Compact Set)**.** Let X be a metric space. Let $K \subseteq X$. Then K is *compact* iff every open cover of K includes a finite subcover.

Proposition 11.32. *Every finite set is compact.*

PROOF: Easy. □

Theorem 11.33. *Let X be a metric space. Let $Y \subseteq X$ and $K \subseteq Y$. Then K is compact in Y if and only if K is compact in X.*

PROOF:
$\langle 1 \rangle 1$. If K is compact in Y then K is compact in X.
  $\langle 2 \rangle 1$. ASSUME: K is compact in Y.
  $\langle 2 \rangle 2$. LET: $\mathcal{U}$ be an open cover of K in X.
  $\langle 2 \rangle 3$. $\{U \cap Y : U \in \mathcal{U}\}$ is an open cover of K in Y.
  $\langle 2 \rangle 4$. PICK a finite subcover $\{U_1 \cap Y, \ldots, U_n \cap Y\}$
  $\langle 2 \rangle 5$. $\{U_1, \ldots, U_n\}$ is a finite subset of $\mathcal{U}$ that is an open cover of K is X.
$\langle 1 \rangle 2$. If K is compact in X then K is compact in Y.
  $\langle 2 \rangle 1$. ASSUME: K is compact in X.
  $\langle 2 \rangle 2$. LET: $\mathcal{U}$ be an open cover of K in Y.
  $\langle 2 \rangle 3$. $\{U \text{ open in } X : U \cap Y \in \mathcal{U}\}$ is an open cover of K in X.
  $\langle 2 \rangle 4$. PICK a finite subcover $\{U_1, \ldots, U_n\}$.
  $\langle 2 \rangle 5$. $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ is a subset of $\mathcal{U}$ that is an open cover of E in Y.
□

Proposition 11.34. *Every compact set is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: E be compact.
$\langle 1 \rangle 2$. LET: $p \in X - E$
  PROVE: There exists an open ball with centre p that is a subset of $X - E$.
$\langle 1 \rangle 3$. For all $q \in E$, there exist disjoint open balls B with centre q and B' with centre p.
$\langle 1 \rangle 4$. The set of open balls B such that there exists a disjoint open ball B' with centre p is an open cover of E.
$\langle 1 \rangle 5$. PICK a finite subcover $\{B_1, \ldots, B_n\}$.
$\langle 1 \rangle 6$. For $i = 1, \ldots, n$, PICK an open ball B_i' with centre p such that $B_i \cap B_i' = \emptyset$.
$\langle 1 \rangle 7$. $B_1' \cap \cdots \cap B_n'$ is an open ball with centre p that is a subset of $X - E$.
□

Proposition 11.35. *Every closed subset of a compact set is compact.*

PROOF:
$\langle 1 \rangle 1$. LET: E be compact and $C \subseteq E$ be closed.
$\langle 1 \rangle 2$. LET: $\mathcal{U}$ be an open cover of C.
$\langle 1 \rangle 3$. $\mathcal{U} \cup \{X - C\}$ is an open cover of E.
$\langle 1 \rangle 4$. PICK a finite subcover $\{U_1, \ldots, U_n\}$ or $\{U_1, \ldots, U_n, X - C\}$.
$\langle 1 \rangle 5$. $\{U_1, \ldots, U_n\}$ covers C.
□

Corollary 11.35.1. *The intersection of a compact set and a closed set is compact.*

Proposition 11.36. *Let $\mathcal{K}$ be a nonempty set of compact sets. If every nonempty finite subset of $\mathcal{K}$ has nonempty intersection, then $\bigcap \mathcal{K}$ is nonempty.*

PROOF:
⟨1⟩1. PICK $K \in \mathcal{K}$
⟨1⟩2. ASSUME: $\bigcap \mathcal{K} = \emptyset$
⟨1⟩3. $\{X - K' : K' \in \mathcal{K}\}$ is an open cover of K.
⟨1⟩4. PICK a finite subcover $\{X - K_1, \ldots, X - K_n\}$.
⟨1⟩5. There exists $p \in K \cap K_1 \cap \cdots \cap K_n$
⟨1⟩6. Q.E.D.
  PROOF: ⟨1⟩4 and ⟨1⟩5 form a contradiction.
□

Corollary 11.36.1. *Let (K_n) be a sequence of nonempty compact sets such that $K_0 \supseteq K_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.*

Theorem 11.37. *Let X be a metric space and $E \subseteq X$. Then E is compact if and only if every infinite subset of E has a limit point in E.*

PROOF:
⟨1⟩1. If E is compact then every infinite subset of E has a limit point in E.
  ⟨2⟩1. ASSUME: E is compact.
  ⟨2⟩2. LET: $A \subseteq E$ be infinite.
  ⟨2⟩3. ASSUME: for a contradiction E has no limit point in K.
  ⟨2⟩4. For all $p \in K$, there exists an open ball B with centre p such that B does not intersect E outside p.
  ⟨2⟩5. The set of open balls that intersect E in at most one point is an open cover for K.
  ⟨2⟩6. PICK a finite subcover $B_1, \ldots, B_n$.
  ⟨2⟩7. E has at most n points.
  ⟨2⟩8. Q.E.D.
    PROOF: This contradicts the fact that E is finite.
⟨1⟩2. If every infinite subset of K has a limit point in K then K is compact.
  ⟨2⟩1. ASSUME: Every infinite subset of K has a limit point in K.
  ⟨2⟩2. LET: $\mathcal{U}$ be an open cover of K.
  ⟨2⟩3. ASSUME: w.l.o.g. $\mathcal{U}$ is countable.
    PROOF: We may replace $\mathcal{U}$ with the set of all open balls B with centres in $\mathbb{Q}^2$ and rational radius such that there exists $U \in \mathcal{U}$ such that $B \subseteq U$.
  ⟨2⟩4. PICK an enumeration $\mathcal{U} = \{G_n : n \in \mathbb{N}\}$.
  ⟨2⟩5. For $n \in \mathbb{N}$,
    LET: $F_n = \bigcup_{i=0}^{n} G_n$.
  ⟨2⟩6. For all $n \in \mathbb{N}$, we have $K - F_n \neq \emptyset$.
    PROOF: Since $\{G_0, \ldots, G_n\}$ does not cover K.
  ⟨2⟩7. $\bigcap_{n=0}^{\infty} F_n = \emptyset$
    PROOF: Since $\{G_n : n \in \mathbb{N}\}$ covers K.
  ⟨2⟩8. For $n \in \mathbb{N}$, PICK $a_n \in K - F_n$
  ⟨2⟩9. LET: $E = \{a_n : n \in \mathbb{N}\}$
  ⟨2⟩10. E is infinite.
    ⟨3⟩1. LET: $n \in \mathbb{N}$
      PROVE: there exists m such that $a_m \notin \{a_0, a_1, \ldots, a_n\}$.

⟨3⟩2. For $i = 0, \ldots, n$, PICK k_i such that $a_i \in G_{k_i}$.
⟨3⟩3. LET: $m = \max(k_0, \ldots, k_n)$
⟨3⟩4. ASSUME: for a contradiction $a_m = a_i$ for some $i = 0, \ldots, n$
⟨3⟩5. $a_i \in G_{k_i}$
⟨3⟩6. $a_i \notin F_m$
⟨3⟩7. Q.E.D.
PROOF: This is a contradiction since $k_i \leq m$.
⟨2⟩11. PICK a limit point l for E in K.
PROOF: From ⟨2⟩1.
⟨2⟩12. PICK n such that $l \in G_n$.
⟨2⟩13. PICK an open ball B with centre l such that $B \subseteq G_n$
⟨2⟩14. $B \cap E$ is infinite.
PROOF: Proposition 11.8.
⟨2⟩15. PICK $m \geq$ ‘n such that $a_m \in B$.
⟨2⟩16. $a_m \in G_n$
⟨2⟩17. Q.E.D.
PROOF: This is a contradiction since $a_m \notin F_m$.
□

Theorem 11.38 (Heine-Borel)**.** *Let $E \subseteq \mathbb{R}^k$. Then E is compact if and only if it is closed and bounded.*

PROOF:
⟨1⟩1. If E is compact then E is closed.
PROOF: Proposition 11.34.
⟨1⟩2. If E is compact then E is bounded.
PROOF: Otherwise $\{(-N, N)^k : N \in \mathbb{Z}^+\}$ would be an open cover of E with no finite subcover.
⟨1⟩3. If E is closed and bounded then E is compact.
⟨2⟩1. ASSUME: E is closed and bounded.
⟨2⟩2. PICK $\vec{c}$ and M such that $\forall \vec{x} \in E. \|\vec{x} - \vec{c}\| < M$.
⟨2⟩3. $E \subseteq \prod_{i=1}^k [c_i - M, c_i + M]$
⟨2⟩4. E is compact.
PROOF: Proposition 11.35.
□

Corollary 11.38.1 (Weierstrass's Theorem)**.** *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point.*

PROOF: It is a bounded infinite subset of some k-cell and therefore has a limit point in that k-cell. □

Example 11.39. It is not true that, in any metric space, a set is compact if and only if it is closed and bounded.

In $\mathbb{Q}$, the set $\{p \in \mathbb{Q} : 2 < p^2 < 3\}$ is closed and bounded but not compact.

Theorem 11.40. *Every nonempty perfect set in $\mathbb{R}^k$ is uncountable.*

PROOF:

⟨1⟩1. LET: P be a nonempty perfect set in $\mathbb{R}^k$.
⟨1⟩2. P is infinite.
PROOF: Corollary 11.8.1.
⟨1⟩3. ASSUME: for a contradiction P is countable.
⟨1⟩4. PICK an enumeration $P = \{x_n : n \in \mathbb{N}\}$.
⟨1⟩5. PICK a sequence (V_n) of open balls such that, for all n, we have $\overline{V_{n+1}} \subseteq V_n$ and $x_n \notin \overline{V_{n+1}}$ and $V_n \cap P \neq \emptyset$
⟨2⟩1. ASSUME: as induction hypothesis we have picked $V_0, \ldots, V_{n-1}$ that satisfy these conditions.
⟨2⟩2. PICK $p \in P \cap V_n$ such that $p \neq x_n$
PROOF: We cannot have $P \cap V_n = \{x_n\}$ because then V_n would be a neighbourhood of x_n that only intersects P at x_n.
⟨2⟩3. PICK an open ball B with centre p such that $B \subseteq V_n \cap P - \{x_n\}$
⟨2⟩4. LET: V_{n+1} be the open ball with centre p and half the radius of B.
⟨2⟩5. $\overline{V_{n+1}} \subseteq V_n$
PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq V_n$.
⟨2⟩6. $x_n \notin \overline{V_{n+1}}$
PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq P - \{x_n\}$.
⟨2⟩7. $V_{n+1} \cap P \neq \emptyset$
PROOF: Since $p \in V_{n+1} \cap P$.
⟨1⟩6. For $n \in \mathbb{N}$,
LET: $K_n = \overline{V_n} \cap P$.
⟨1⟩7. For all $n \in \mathbb{N}$, K_n is compact.
PROOF: By the Heine-Borel Theorem.
⟨1⟩8. $\bigcap_{n=0}^{\infty} K_n \cap P = \emptyset$
PROOF: Since for each n we have $x_n \notin K_{n+1}$.
⟨1⟩9. $\bigcap_{n=0}^{\infty} K_n = \emptyset$
PROOF: Since $\bigcap_{n=0}^{\infty} K_n \subseteq P$.
⟨1⟩10. Q.E.D.
PROOF: This contradicts Proposition 11.36.
□

Corollary 11.40.1. *For any $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a, b]$ is uncountable.*

Corollary 11.40.2. *$\mathbb{R}$ is uncountable.*

Corollary 11.40.3. *The set of transcendental numbers is uncountable.*

PROOF: Since the set of algebraic numbers is countable. □

Example 11.41. The Cantor set is a perfect set in $\mathbb{R}$ that does not include any open interval.

PROOF:
⟨1⟩1. LET: (E_n) be the sequence of unions of closed intervals from the definition of the Cantor set, and C be the Cantor set.
⟨1⟩2. $C \neq \emptyset$

PROOF: Since $0 \in C$.
$\langle 1 \rangle 3$. C is closed.
PROOF: Each E_n is closed and C is their intersection.
$\langle 1 \rangle 4$. Every point of C is a limit point of C.
$\langle 2 \rangle 1$. LET: $p \in C$
$\langle 2 \rangle 2$. LET: B be an open ball with centre p and radius r.
$\langle 2 \rangle 3$. PICK n such that each of the intervals that make up E_n has length $< r/2$.
$\langle 2 \rangle 4$. LET: I be the interval in E_n that contains p.
$\langle 2 \rangle 5$. $I \subseteq B$
$\langle 2 \rangle 6$. The endpoint of I that is not p is in $P \cap B$.
$\langle 1 \rangle 5$. C does not include any open interval.
$\langle 2 \rangle 1$. LET: (α, β) be any open interval.
$\langle 2 \rangle 2$. PICK m such that $3^{-m} < (\beta - \alpha)/6$
$\langle 2 \rangle 3$. PICK k such that $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq (\alpha, \beta)$
$\langle 2 \rangle 4$. $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq P$
$\langle 2 \rangle 5$. $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \cap E_m = \emptyset$
$\langle 2 \rangle 6$. Q.E.D.
PROOF: This is a contradiction.
□

Corollary 11.41.1. *The Cantor set is uncountable.*

11.11 Connected Sets

Definition 11.42 (Separated)**.** Let X be a metric space. Let $A, B \subseteq X$. Then A and B are *separated* iff $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Proposition 11.43. *Any two disjoint open sets are separated.*

PROOF:
$\langle 1 \rangle 1$. LET: A and B be disjoint open sets.
$\langle 1 \rangle 2$. ASSUME: for a contradiction $p \in \overline{A} \cap B$.
$\langle 1 \rangle 3$. B is a neighbourhood of p.
$\langle 1 \rangle 4$. B intersects A.
□

Definition 11.44 (Connected)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *connected* iff E is not the union of two nonempty separated sets.

Theorem 11.45. *A subset E of the real line is connected if and only if it is convex.*

PROOF:
$\langle 1 \rangle 1$. If E is connected then E is convex.
$\langle 2 \rangle 1$. ASSUME: E is connected.
$\langle 2 \rangle 2$. LET: $x, y \in E$

⟨2⟩3. LET: $z \in (x, y)$
⟨2⟩4. $z \in E$
PROOF: Otherwise $E \cap (-\infty, z)$ and $E \cap (z, +\infty)$ would be a separation of E.
⟨1⟩2. If E is convex then E is connected.
⟨2⟩1. ASSUME: E is convex.
⟨2⟩2. ASSUME: for a contradiction $E = A \cup B$ where A and B are nonempty and separated.
⟨2⟩3. PICK $a \in A$ and $b \in B$.
⟨2⟩4. ASSUME: w.l.o.g. $a < b$
⟨2⟩5. LET: $z = \sup(A \cap [a, b])$
⟨2⟩6. $z \in \overline{A}$
⟨2⟩7. $z \notin B$
⟨2⟩8. $z < b$
⟨2⟩9. CASE: $z \in A$
⟨3⟩1. $z \notin \overline{B}$
⟨3⟩2. PICK $z_1 \in (z, b)$ such that $z_1 \notin B$
⟨3⟩3. $a < z_1 < b$
⟨3⟩4. $z_1 \notin E$
PROOF: We have $z_1 \notin A$ from ⟨2⟩5 since $z_1 \in [a, b]$ and $z_1 > z$, and $z_1 \notin B$ from ⟨3⟩2.
⟨3⟩5. Q.E.D.
PROOF: This contradicts ⟨2⟩1.
⟨2⟩10. CASE: $z \notin A$
PROOF: Then $a < z < b$ and $z \notin E$ contradicting ⟨2⟩1.
□

Proposition 11.46. *Every connected metric space with more than one point is uncountable.*

PROOF:
⟨1⟩1. LET: X be a connected metric space with more than one points.
⟨1⟩2. PICK distinct points $p, q \in X$.
⟨1⟩3. LET: $\epsilon = d(p, q)$
⟨1⟩4. For every $r \in (0, \epsilon)$, there exists a point $x \in X$ such that $d(p, x) = r$.
PROOF: Otherwise $\{x \in X : d(p, x) < r\}$ and $\{x \in X : d(p, x) > r\}$ would form a separation of X.
□

Proposition 11.47. *The closure of a connected set is connected.*

PROOF:
⟨1⟩1. LET: X be a metric space.
⟨1⟩2. LET: E be a connected subspace of X.
⟨1⟩3. ASSUME: for a contradiction A and B form a separation of $\overline{E}$
PROVE: $A \cap E$ and $B \cap E$ form a separation of E.
⟨1⟩4. $A \cap E \neq \emptyset$

⟨2⟩1. ASSUME: for a contradiction $A \cap E = \emptyset$
⟨2⟩2. $E \subseteq B$
⟨2⟩3. $\overline{E} \subseteq \overline{B}$
⟨2⟩4. $A \subseteq \overline{B}$
⟨2⟩5. $A \cap \overline{B} = A \neq \emptyset$
⟨2⟩6. Q.E.D.
PROOF: This contradicts ⟨1⟩3.
⟨1⟩5. $B \cap E \neq \emptyset$
PROOF: Similar.
⟨1⟩6. $\overline{A \cap E} \cap B \cap E = \emptyset$
PROOF: Since $\overline{A \cap E} \cap B \cap E \subseteq \overline{A} \cap B$.
⟨1⟩7. $A \cap E \cap \overline{B \cap E} = \emptyset$
PROOF: Similar.
□

Example 11.48. The interior of a connected set is not necessarily connected.

Two touching discs in $\mathbb{R}^2$ form a connected set but the interior is disconnected.

Proposition 11.49. *Every convex set in $\mathbb{R}^k$ is connected.*

PROOF:
⟨1⟩1. LET: E be a convex set in $\mathbb{R}^k$.
⟨1⟩2. ASSUME: for a contradiction A and B form a separation of E.
⟨1⟩3. PICK $\vec{a} \in A$ and $\vec{b} \in B$.
⟨1⟩4. Define $p : [0, 1] \to \mathbb{R}^k$ by $p(t) = (1 - t)\vec{a} + t\vec{b}$.
⟨1⟩5. $p^{-1}(A)$ and $p^{-1}(B)$ are separated sets in $\mathbb{R}$.
⟨1⟩6. PICK $x \in [0, 1]$ such that $x \notin p^{-1}(A)$ and $x \notin p^{-1}(B)$.
PROOF: There exists such an x since $[0, 1]$ is connected.
⟨1⟩7. $p(x) \in E$
PROOF: Since E is convex.
⟨1⟩8. $p(x) \notin A \cup B$
⟨1⟩9. Q.E.D.
PROOF: This contradicts ⟨1⟩2.
□

11.12 Separable Spaces

Definition 11.50 (Separable)**.** A metric space is *separable* iff it has a countable dense subset.

Example 11.51. $\mathbb{R}^k$ is separable since $\mathbb{Q}^k$ is dense.

Proposition 11.52. *Every compact metric space is separable.*

PROOF:
⟨1⟩1. LET: X be a compact metric space.

⟨1⟩2. For $n \in \mathbb{Z}^+$, pick finitely many points $a_{n1}, \ldots, a_{nr_n}$ such that $\{B(a_{ni}, 1/n) : 1 \leq i \leq r_n\}$ covers X.
PROOF: Since $\{B(x, 1/n) : x \in X\}$ covers X.
⟨1⟩3. $\{a_{ni} : n \in \mathbb{Z}^+, 1 \leq i \leq r_n\}$ is dense.
⟨2⟩1. LET: U be an open set and $p \in U$.
⟨2⟩2. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$.
⟨2⟩3. PICK n such that $1/n < \epsilon$.
⟨2⟩4. PICK i such that $p \in B(a_{ni}, 1/n)$
⟨2⟩5. $a_{ni} \in U$
□

11.13 Bases

Definition 11.53 (Basis)**.** A *basis* for a metric space X is a set $\mathcal{B}$ of open sets such that, for every open set U and point $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.

Proposition 11.54. *Every separable metric space has a countable basis.*

PROOF:
⟨1⟩1. LET: X be a separable metric space.
⟨1⟩2. PICK a countable dense set D in X.
⟨1⟩3. LET: $\mathcal{B} = \{B(p, \epsilon) : p \in D, \epsilon \in \mathbb{Q}^+\}$
PROVE: $\mathcal{B}$ is a basis.
⟨1⟩4. LET: U be an open set in X and $p \in U$
⟨1⟩5. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$
⟨1⟩6. PICK $q \in B(p, \epsilon) \cap D$
⟨1⟩7. PICK a rational δ such that $d(p, q) < \delta < \epsilon$.
⟨1⟩8. $B(q, \delta) \in \mathcal{B}$ and $B(q, \delta) \subseteq U$.
□

11.14 Condensation Points

Definition 11.55 (Condensation Point)**.** Let X be a metric space, $p \in X$ and $E \subseteq X$. Then p is a *condensation point* of E iff every neighbourhood of p contains uncountably many points in E.

Proposition 11.56. *Let X be a metric space. Let $E \subseteq X$. Let P be the set of condensation points of E. Then P is perfect.*

PROOF:
⟨1⟩1. P is closed.
⟨2⟩1. LET: $p \in X - P$
⟨2⟩2. PICK a neighbourhood U of p that contains only countably many points of E.
⟨2⟩3. For every $x \in U$, we have that U is a neighbourhood of x that contains only countably many points of E.

⟨2⟩4. $p \in U \subseteq X - P$
⟨1⟩2. Every point in P is a limit point of P.
PROOF: Immediate from definitions.
□

Proposition 11.57. *Let X be a metric space with a countable basis. Let $E \subseteq X$ be uncountable. Let P be the set of condensation points of E. Then $E - P$ is countable.*

PROOF:
⟨1⟩1. PICK a countable basis $\mathcal{B}$ for X.
⟨1⟩2. LET: $W = \bigcup\{B \in \mathcal{B} : E \cap B \text{ is countable}\}$
⟨1⟩3. $P = X - W$
⟨2⟩1. $P \subseteq X - W$
⟨3⟩1. ASSUME: for a contradiction $p \in P \cap W$
⟨3⟩2. PICK $B \in \mathcal{B}$ such that $p \in B$ and $E \cap B$ is countable.
⟨3⟩3. $E \cap B$ is uncountable.
⟨3⟩4. Q.E.D.
PROOF: This is a contradiction.
⟨2⟩2. $X - W \subseteq P$
⟨3⟩1. LET: $p \in X - W$
⟨3⟩2. LET: U be a neighbourhood of p.
⟨3⟩3. PICK $B \in \mathcal{B}$ such that $p \in B \subseteq U$.
⟨3⟩4. $E \cap B$ is uncountable.
PROOF: Since $p \notin W$.
⟨3⟩5. $E \cap W$ is uncountable.
⟨1⟩4. $E - P = E \cap W$
⟨1⟩5. $E - P$ is countable.
□

Corollary 11.57.1. *Every closed subset of a metric space with a countable basis is the union of a perfect set and a countable set.*

PROOF:
⟨1⟩1. LET: X be a metric space with a countable basis.
⟨1⟩2. LET: E be a closed subset of X.
⟨1⟩3. LET: P be the set of condensation points of E.
⟨1⟩4. $E - P$ is countable.
PROOF: Proposition 11.57.
⟨1⟩5. $P \cap E$ is perfect.
⟨2⟩1. $P \cap E$ is closed.
PROOF: Proposition 11.56.
⟨2⟩2. Every point in $P \cap E$ is a limit point of $P \cap E$.
⟨3⟩1. LET: $l \in P \cap E$
⟨3⟩2. LET: U be a neighbourhood of l.
⟨3⟩3. PICK $x \in P \cap U$
⟨3⟩4. U is a neighbourhood of x.

$\langle 3\rangle 5$. U contains uncountably many points of E.

$\langle 3\rangle 6$. U intersects $P \cap E$

PROOF: It cannot be that every point in U and E is not in P since $E - P$ is countable.

$\square$

Corollary 11.57.2. *Let X be a metric space with a countable basis. Then every countable set in X has an isolated point.*

Chapter 12

Convergence

Definition 12.1 (Converge)**.** Let X be a metric space. Let (p_n) be a sequence in X and $l \in X$. Then we say (p_n) *converges* to the *limit* l, and write

$$p_n \to l \text{ as } n \to \infty \enspace ,$$

iff for every $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon$.

We say (p_n) *diverges* iff it does not converge to any limit.

Proposition 12.2. *A sequence has at most one limit.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $p_n \to l$ and $p_n \to m$ as $n \to \infty$.
$\langle 1 \rangle 2$. ASSUME: for a contradiction $l \neq m$.
$\langle 1 \rangle 3$. LET: $\epsilon = d(l, m)/2$
$\langle 1 \rangle 4$. There exists N such that $\forall n \geq N . d(p_n, l) < \epsilon$ and $d(p_n, m) < \epsilon$
$\langle 1 \rangle 5$. $d(l, m) < 2\epsilon$
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This is a contradiction.
□

Proposition 12.3. *Every convergent sequence is bounded.*

PROOF:
$\langle 1 \rangle 1$. LET: $p_n \to l$ as $n \to \infty$
$\langle 1 \rangle 2$. PICK N such that $\forall n \geq N . d(p_n, l) < 1$
$\langle 1 \rangle 3$. LET: $M = \max(d(p_0, l), \ldots, d(p_{N-1}, l), 1)$
$\langle 1 \rangle 4$. For all n, we have $d(p_n, l) \leq M$.
□

Proposition 12.4. *If l is a limit point of E, then there exists a sequence in E that converges to l.*

PROOF:

⟨1⟩1. For $n \in \mathbb{Z}^+$, Pick a point $a_n \in E$ such that $d(a_n, l) < 1/n$.
Proof: Since $B(l, 1/n)$ intersects E.
⟨1⟩2. $a_n \to l$ as $n \to \infty$.
□

Proposition 12.5. *Assume $s_n \to s$ and $t_n \to t$ in $\mathbb{C}$. Then $s_n + t_n \to s + t$.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. Pick N such that, for all $n \geq N$, we have $d(s_n, s) < \epsilon/2$ and $d(t_n, t) < \epsilon/2$.
⟨1⟩3. For all $n \geq N$ we have $d(s_n + t_n, s + t) < \epsilon$.
□

Lemma 12.6. *If $s_n \to s$ as $n \to \infty$ in $\mathbb{C}$, and $c \in \mathbb{C}$, then $cs_n \to cs$ as $n \to \infty$.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. Assume: w.l.o.g. $c \neq 0$
⟨1⟩3. Pick N such that $\forall n \geq N. |s_n - s| < \epsilon/|c|$.
⟨1⟩4. $\forall n \geq N. |cs_n - cs| < \epsilon$
□

Proposition 12.7. *If $s_n \to s$ and $t_n \to t$ in $\mathbb{C}$ then $s_n t_n \to st$.*

Proof:
⟨1⟩1. $(s_n - s)(t_n - t) \to 0$ as $n \to \infty$
⟨2⟩1. Let: $\epsilon > 0$
⟨2⟩2. Pick N such that, for all $n \geq N$, we have $|s_n - s| < \sqrt{\epsilon}$ and $|t_n - t| < \sqrt{\epsilon}$.
⟨2⟩3. For all $n \geq N$ we have $|(s_n - s)(t_n - t)| < \epsilon$
⟨1⟩2. $s_n t_n - st \to 0$ as $n \to \infty$
Proof:

$$\begin{aligned} s_n t_n - st &= (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s) \\ &\to 0 \qquad \text{as } n \to \infty \end{aligned}$$

□

Proposition 12.8. *If $s_n \to s$ as $n \to \infty$ in $\mathbb{C}$, and every s_n and s is nonzero, then $1/s_n \to 1/s$ as $n \to \infty$.*

Proof:
⟨1⟩1. Pick m such that, for all $n \geq m$, we have $|s_n - s| < \frac{1}{2}|s|$.
⟨1⟩2. $\forall n \geq m. |s_n| > \frac{1}{2}|s|$
⟨1⟩3. Let: $\epsilon > 0$
⟨1⟩4. Pick $N > m$ such that, for all $n \geq N$, we have

$$|s_n - s| < \frac{1}{2}|s|^2\epsilon \ .$$

⟨1⟩5. For all $n \geq N$, we have

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| < \epsilon \ .$$

PROOF:

$$\begin{aligned}\left|\frac{1}{s_n}-\frac{1}{s}\right| &= \frac{|s_n-s|}{|s_n||s|} \\ &< \frac{|s|^2\epsilon}{2|s_n||s|} \\ &= \frac{|s|\epsilon}{2|s_n|} \\ &< \epsilon\end{aligned}$$

□

Theorem 12.9. *Assume $\vec{x_n} \to \vec{l}$ as $n \to \infty$ in $\mathbb{R}^k$. Let $\vec{x_n} = (\alpha_{n1}, \dots, \alpha_{nk})$ and $\vec{l} = (\alpha_1, \dots, \alpha_k)$. Then $\alpha_{ni} \to \alpha_i$ as $n \to \infty$.*

PROOF:
$\langle 1\rangle 1.$ $\|\vec{x_n} - \vec{l}\| \to 0$ as $n \to \infty$.
$\langle 1\rangle 2.$ $\sqrt{\sum_{i=1}^k (\alpha_{ni} - \alpha_i)^2} \to 0$ as $n \to \infty$.
$\langle 1\rangle 3.$ $\sum_{i=1}^k (\alpha_{ni} - \alpha_i)^2 \to 0$ as $n \to \infty$.
$\langle 1\rangle 4.$ $(\alpha_{ni} - \alpha_i)^2 \to 0$ as $n \to \infty$
$\langle 1\rangle 5.$ $\alpha_{ni} - \alpha_i \to 0$ as $n \to \infty$.
□

Part III

More Algebra

Chapter 13

Lie Groups

Definition 13.1 (Lie Group)**.** A *Lie group G* is a group G that is also an analytic differentiable manifold such that the group operation and inverse operation are analytic.

A *homomorphism of Lie groups* is a group homomorphism that is an analytic function.

Lemma 13.2. *Every bijective Lie group homomorphism is an isomorphism.*

Definition 13.3 (Unitary Group)**.** The *unitary group $U(n)$* is the Lie group of all $n \times n$ unitary matrices.

Definition 13.4 (Special Unitary Group)**.** The *special unitary group $SU(n)$* is the Lie group of all $n \times n$ unitary matrices with determinant 1.

Definition 13.5 (Lie Subgroup)**.** Let G be a Lie group. A *Lie subgroup* of G is a subgroup that is also an analytic submanifold of G.

Example 13.6. $U(n)$ and $SU(n)$ are Lie subgroups of $GL(n, \mathbb{C})$.