

# Summary of Halmos' Naive Set Theory

Robin Adams

August 26, 2023

# Contents

<b>1</b>	<b>Primitive Terms and Axioms</b>	<b>3</b>
<b>2</b>	<b>Basic Properties and Operations on Sets</b>	<b>5</b>
2.1	The Subset Relation . . . . .	5
2.2	Comprehension Notation . . . . .	5
2.3	The Empty Set . . . . .	6
2.4	Unordered Pairs . . . . .	6
2.5	Unions . . . . .	6
2.6	Intersections . . . . .	7
2.7	Unordered Triples . . . . .	7
2.8	Relative Complements . . . . .	8
2.9	Symmetric Difference . . . . .	10
2.10	Power Sets . . . . .	11
<b>3</b>	<b>Relations and Functions</b>	<b>13</b>
3.1	Ordered Pairs . . . . .	13
3.2	Relations . . . . .	14
3.3	Composition . . . . .	14
3.4	Inverses . . . . .	15
3.5	Equivalence Relations . . . . .	15
3.6	Functions . . . . .	16
3.7	Families . . . . .	17
3.8	Inverses and Composites of Functions . . . . .	19
3.9	Choice Functions . . . . .	20
<b>4</b>	<b>Equivalence</b>	<b>21</b>
<b>5</b>	<b>Order</b>	<b>22</b>
5.1	Well Orderings . . . . .	26
<b>6</b>	<b>Natural Numbers</b>	<b>31</b>
6.1	Natural Numbers . . . . .	31

<b>7</b>	<b>Ordinal Numbers</b>	<b>36</b>
7.1	Order on the Natural Numbers . . . . .	39
7.2	Finite Sets . . . . .	41
7.3	Ordinal Arithmetic . . . . .	45
7.4	Arithmetic on the Natural Numbers . . . . .	46

# Chapter 1

## Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*,  $\in$ . If  $x \in A$  we say that  $x$  *belongs to*  $A$ ,  $x$  is an *element* of  $A$ , or  $x$  is *contained in*  $A$ . If this does not hold we write  $x \notin A$ .

**Axiom 1.1** (Axiom of Extensionality). *Two sets are equal if and only if they have the same elements.*

**Axiom 1.2** (Axiom of Comprehension, Aussonderungsaxiom). *To every set  $A$  and to every condition  $S(x)$  there corresponds a set  $B$  whose elements are exactly those elements  $x$  of  $A$  for which  $S(x)$  holds.*

**Definition 1.3.** Given a set  $A$  and a condition  $S(x)$ , we write  $\{x \in A : S(x)\}$  for the set whose elements are exactly those elements  $x$  of  $A$  for which  $S(x)$  holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality.  $\square$

**Axiom 1.4** (Axiom of Pairing). *For any two sets, there exists a set that they both belong to.*

**Definition 1.5** ((Unordered) Pair). For any sets  $a$  and  $b$ , the *(unordered) pair*  $\{a, b\}$  is the set whose elements are just  $a$  and  $b$ .

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality.  $\square$

**Axiom 1.6** (Union Axiom). *For every set  $A$ , there exists a set that contains all the elements that belong to at least one element of  $A$ .*

**Definition 1.7** (Subset). Let  $A$  and  $B$  be sets. We say that  $A$  is a *subset* of  $B$ , or  $B$  *includes*  $A$ , and write  $A \subseteq B$  or  $B \supseteq A$ , iff every element of  $A$  is an element of  $B$ .

**Axiom 1.8** (Power Set Axiom). *For any set  $A$ , there exists a set that contains all the subsets of  $A$ .*

**Definition 1.9** (Empty). A set is *empty* iff it has no elements; otherwise it is *non-empty*.

**Axiom 1.10** (Axiom of Infinity). *There exists a set  $I$  such that:*

- $I$  has an element that is empty
- for all  $x \in I$ , there exists  $y \in I$  such that the elements of  $y$  are exactly  $x$  and the elements of  $x$ .

**Definition 1.11** (Ordered Pair). For any sets  $a$  and  $b$ , the *ordered pair*  $(a, b)$  is defined by

$$(a, b) := \{\{a\}, \{a, b\}\} .$$

**Definition 1.12** (Power Set). For any set  $A$ , the *power set* of  $A$ ,  $\mathcal{P}A$ , is the set whose elements are exactly the subsets of  $A$ .

PROOF: This exists by the Power Set Axiom and Axiom of Comprehension, and is unique by the Axiom of Extensionality.  $\square$

**Definition 1.13** (Cartesian Product). For any sets  $A$  and  $B$ , the *Cartesian product*  $A \times B$  is

$$A \times B := \{p \in \mathcal{P}\mathcal{P}(A \cup B) : \exists a \in A. \exists b \in B. p = (a, b)\} .$$

**Definition 1.14** (Relation). A *relation* is a set of ordered pairs.

If  $R$  is a relation, we write  $xRy$  for  $(x, y) \in R$ .

Given sets  $X$  and  $Y$ , a relation *between  $X$  and  $Y$*  is a subset of  $X \times Y$ .

Given a set  $X$ , a relation *on  $X$*  is a relation between  $X$  and  $X$ .

**Definition 1.15** (Function). Let  $X$  and  $Y$  be sets. A *function*, *map*, *mapping*, *transformation* or *operator*  $f$  from  $X$  to  $Y$ ,  $f : X \rightarrow Y$ , is a relation  $f$  between  $X$  and  $Y$  such that, for all  $x \in X$ , there exists a unique  $f(x) \in Y$ , called the *value* of  $f$  at the *argument*  $x$ , such that  $(x, f(x)) \in f$ .

**Definition 1.16** (Family). Let  $I$  and  $X$  be sets. A *family* of elements of  $X$  indexed by  $I$  is a function  $a : I \rightarrow X$ . We write  $a_i$  for  $a(i)$ , and  $\{a_i\}_{i \in I}$  for  $a$ .

**Definition 1.17** (Cartesian Product of a Family of Sets). Let  $\{A_i\}_{i \in I}$  be a family of sets. The *Cartesian product*  $\times_{i \in I} A_i$  is the set of all families  $\{a_i\}_{i \in I}$  such that  $\forall i \in I. a_i \in A_i$ .

We write  $A^I$  for  $\times_{i \in I} A$ .

**Axiom 1.18** (Axiom of Choice). *The Cartesian product of a non-empty family of non-empty sets is non-empty.*

**Axiom 1.19** (Axiom of substitution). *If  $S(a, b)$  is a sentence such that for each  $a$  in  $A$  the set  $\{b : S(a, b)\}$  can be formed, then there exists a function  $F$  with domain  $A$  such that  $F(a) = \{b : S(a, b)\}$  for each  $a$  in  $A$ .*

## Chapter 2

# Basic Properties and Operations on Sets

### 2.1 The Subset Relation

**Theorem 2.1.** *For any set  $A$ , we have  $A \subseteq A$ .*

PROOF: Every element of  $A$  is an element of  $A$ .  $\square$

**Theorem 2.2.** *For any sets  $A$ ,  $B$  and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ .*

PROOF: If every element of  $A$  is an element of  $B$ , and every element of  $B$  is an element of  $C$ , then every element of  $A$  is an element of  $C$ .  $\square$

**Theorem 2.3.** *For any sets  $A$  and  $B$ , if  $A \subseteq B$  and  $B \subseteq A$  then  $A = B$ .*

PROOF: If every element of  $A$  is an element of  $B$ , and every element of  $B$  is an element of  $A$ , then  $A$  and  $B$  have the same elements, and therefore are equal by the Axiom of Extensionality.  $\square$

**Definition 2.4** (Proper Subset). Let  $A$  and  $B$  be sets. We say that  $A$  is a *proper* subset of  $B$ , or  $B$  *properly* includes  $A$ , and write  $A \subsetneq B$  or  $B \supsetneq A$ , iff  $A \subseteq B$  and  $A \neq B$ .

### 2.2 Comprehension Notation

**Theorem 2.5.** *There is no set that contains every set.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be a set.

PROVE: There exists a set  $B$  such that  $B \notin A$ .

$\langle 1 \rangle 2$ . LET:  $B = \{x \in A : x \notin x\}$

$\langle 1 \rangle 3$ . If  $B \in A$  then we have  $B \in B$  if and only if  $B \notin B$ .

$\langle 1 \rangle 4$ .  $B \notin A$

$\square$

## 2.3 The Empty Set

**Theorem 2.6.** *There exists a set with no elements.*

PROOF: Immediate from the Axiom of Infinity.  $\square$

**Definition 2.7** (Empty Set). The *empty set*  $\emptyset$  is the set with no elements.

**Theorem 2.8.** *For any set  $A$  we have  $\emptyset \subset A$ .*

PROOF: Vacuous.  $\square$

## 2.4 Unordered Pairs

**Definition 2.9** (Singleton). For any set  $a$ , the *singleton*  $\{a\}$  is defined to be  $\{a, a\}$ .

## 2.5 Unions

**Definition 2.10** (Union). For any set  $\mathcal{C}$ , the *union* of  $\mathcal{C}$ ,  $\bigcup \mathcal{C}$ , is the set whose elements are the elements of the elements of  $\mathcal{C}$ .

We write  $\bigcup_{X \in \mathcal{A}} t[X]$  for  $\bigcup \{t[X] \mid X \in \mathcal{A}\}$ .

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality.  $\square$

**Proposition 2.11.**

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of  $\emptyset$ .  $\square$

**Proposition 2.12.** *For any set  $A$ , we have  $\bigcup \{A\} = A$ .*

PROOF: For any  $x$ , we have  $x$  is an element of an element of  $\{A\}$  if and only if  $x$  is an element of  $A$ .  $\square$

**Definition 2.13.** We write  $A \cup B$  for  $\bigcup \{A, B\}$ .

**Proposition 2.14.** *For any set  $A$ , we have  $A \cup \emptyset = A$ .*

PROOF:  $x \in A \cup \emptyset$  iff  $x \in A$  or  $x \in \emptyset$ , iff  $x \in A$ .  $\square$

**Proposition 2.15** (Idempotence). *For any set  $A$ , we have  $A \cup A = A$ .*

PROOF:  $x \in A$  or  $x \in A$  is equivalent to  $x \in A$ .  $\square$

**Proposition 2.16.** *For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A \cup B = B$ .*

PROOF: For any  $x$ , the statement "if  $x \in A$  then  $x \in B$ " is equivalent to " $x \in A$  or  $x \in B$  if and only if  $x \in B$ ".  $\square$

**Proposition 2.17.** *For any sets  $a$  and  $b$ , we have  $\{a\} \cup \{b\} = \{a, b\}$ .*

PROOF: Immediate from definitions.  $\square$

## 2.6 Intersections

**Definition 2.18** (Intersection). For any sets  $A$  and  $B$ , the *intersection*  $A \cap B$  is defined to be  $\{x \in A : x \in B\}$ .

**Proposition 2.19.** For any set  $A$ , we have  $A \cap \emptyset = \emptyset$ .

PROOF: There is no  $x$  such that  $x \in A$  and  $x \in \emptyset$ .  $\square$

**Proposition 2.20.** For any set  $A$ , we have

$$A \cap A = A .$$

PROOF: We have  $x \in A$  and  $x \in A$  if and only if  $x \in A$ .  $\square$

**Proposition 2.21.** For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A \cap B = A$ .

PROOF: For any  $x$ , the statement "if  $x \in A$  then  $x \in B$ " is equivalent to " $x \in A$  and  $x \in B$  if and only if  $x \in A$ ".  $\square$

**Proposition 2.22.** For any sets  $A$ ,  $B$  and  $C$ , we have  $C \subseteq A$  if and only if  $(A \cap B) \cup C = A \cap (B \cup C)$ .

PROOF: The statement "if  $x \in C$  then  $x \in A$ " is equivalent to the statement " $((x \in A \wedge x \in B) \vee x \in C) \Leftrightarrow (x \in A \wedge (x \in B \vee x \in C))$ ".  $\square$

**Definition 2.23** (Disjoint). Two sets  $A$  and  $B$  are *disjoint* if and only if  $A \cap B = \emptyset$ .

**Definition 2.24** (Pairwise Disjoint). Let  $A$  be a set. We say the elements of  $A$  are *pairwise disjoint* if and only if, for all  $x, y \in A$ , if  $x \cap y \neq \emptyset$  then  $x = y$ .

**Definition 2.25** (Intersection). For any nonempty set  $\mathcal{C}$ , the *intersection* of  $\mathcal{C}$ ,  $\bigcap \mathcal{C}$ , is the set that contains exactly those sets that belong to every element of  $\mathcal{C}$ .

We write  $\bigcap_{X \in \mathcal{A}} t[X]$  for  $\bigcap \{t[X] \mid X \in \mathcal{A}\}$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C}$  be a nonempty set.

$\langle 1 \rangle 2$ . There exists a set  $I$  whose elements are exactly the sets that belong to every element of  $\mathcal{C}$ .

PROOF: Pick  $A \in \mathcal{C}$ , and take  $I = \{x \in A : \forall X \in \mathcal{C}. x \in X\}$ .

$\langle 1 \rangle 3$ . For any sets  $I, J$ , if the elements of  $I$  and  $J$  are exactly the sets that belong to every element of  $\mathcal{C}$  then  $I = J$ .

PROOF: Axiom of Extensionality.

$\square$

## 2.7 Unordered Triples

**Definition 2.26** ((Unordered) Triple). Given sets  $a_1, \dots, a_n$ , define the (*unordered*) *n-tuple*  $\{a_1, \dots, a_n\}$  to be

$$\{a_1, \dots, a_n\} := \{a_1\} \cup \dots \cup \{a_n\} .$$



## 2.8 Relative Complements

**Definition 2.27** (Relative Complement). For any sets  $A$  and  $B$ , the *difference* or *relative complement*  $A - B$  is defined to be

$$A - B := \{x \in A : x \notin B\} .$$

**Proposition 2.28.** For any sets  $A$  and  $E$ , we have  $A \subseteq E$  if and only if

$$E - (E - A) = A$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $E$  be sets.

$\langle 1 \rangle 2$ . If  $A \subseteq E$  then  $E - (E - A) = A$

$\langle 2 \rangle 1$ . ASSUME:  $A \subseteq E$

$\langle 2 \rangle 2$ .  $E - (E - A) \subseteq A$

PROOF: If  $x \in E$  and  $x \notin E - A$  then  $x \in A$ .

$\langle 2 \rangle 3$ .  $A \subseteq E - (E - A)$

PROOF: If  $x \in A$  then  $x \in E$  and  $x \notin E - A$ .

$\langle 1 \rangle 3$ . If  $E - (E - A) = A$  then  $A \subseteq E$ .

PROOF: Since  $E - (E - A) \subseteq E$ .

□

**Proposition 2.29.** For any set  $E$  we have

$$E - \emptyset = E$$

PROOF:  $x \in E$  if and only if  $x \in E$  and  $x \notin \emptyset$ . □

**Proposition 2.30.** For any set  $E$  we have

$$E - E = \emptyset .$$

PROOF: There is no  $x$  such that  $x \in E$  and  $x \notin E$ . □

**Proposition 2.31.** For any sets  $A$  and  $E$ , we have

$$A \cap (E - A) = \emptyset .$$

PROOF: There is no  $x$  such that  $x \in A$  and  $x \in E - A$ . □

**Proposition 2.32.** Let  $A$  and  $E$  be sets. Then  $A \subseteq E$  if and only if

$$A \cup (E - A) = E .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $E$  be sets.

$\langle 1 \rangle 2$ . If  $A \subseteq E$  then  $A \cup (E - A) = E$ .

$\langle 2 \rangle 1$ . ASSUME:  $A \subseteq E$

$\langle 2 \rangle 2$ .  $A \cup (E - A) \subseteq E$

PROOF: If  $x \in A$  or  $x \in E - A$  then  $x \in E$ .

$\langle 2 \rangle 3$ .  $E \subseteq A \cup (E - A)$

PROOF: If  $x \in E$  then either  $x \in A$  or  $x \notin A$ . In the latter case,  $x \in E - A$ .

$\langle 1 \rangle 3$ . If  $A \cup (E - A) = E$  then  $A \subseteq E$

PROOF: Since  $A \subseteq A \cup (E - A)$ .

□

**Proposition 2.33.** *Let  $A$ ,  $B$  and  $E$  be sets. Then:*

1. *If  $A \subseteq B$  then  $E - B \subseteq E - A$ .*
2. *If  $A \subseteq E$  and  $E - B \subseteq E - A$  then  $A \subseteq B$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$ ,  $B$  and  $E$  be sets.

$\langle 1 \rangle 2$ . If  $A \subseteq B$  then  $E - B \subseteq E - A$ .

PROOF: If  $A \subseteq B$ ,  $x \in E$  and  $x \notin B$ , then we have  $x \in E$  and  $x \notin A$ .

$\langle 1 \rangle 3$ . If  $A \subseteq E$  and  $E - B \subseteq E - A$  then  $A \subseteq B$ .

$\langle 2 \rangle 1$ . ASSUME:  $A \subseteq E$

$\langle 2 \rangle 2$ . ASSUME:  $E - B \subseteq E - A$

$\langle 2 \rangle 3$ . LET:  $x \in A$

$\langle 2 \rangle 4$ .  $x \in E$

$\langle 2 \rangle 5$ .  $x \notin E - A$

$\langle 2 \rangle 6$ .  $x \notin E - B$

$\langle 2 \rangle 7$ .  $x \in B$

□

**Example 2.34.** We cannot remove the hypothesis  $A \subseteq E$  in item 2 above. Let  $E = \emptyset$ ,  $A = \{\emptyset\}$  and  $B = \emptyset$ . Then  $E - B = E - A = \emptyset$  but  $A \not\subseteq B$ .

**Proposition 2.35** (De Morgan's Law). *For any sets  $A$ ,  $B$  and  $E$ , we have  $E - (A \cup B) = (E - A) \cap (E - B)$ .*

PROOF:  $(x \in E \wedge \neg(x \in A \vee x \in B)) \Leftrightarrow (x \in E \wedge x \notin A \wedge x \in E \wedge x \notin B)$ . □

**Proposition 2.36** (De Morgan's Law). *For any sets  $A$ ,  $B$  and  $E$ , we have  $E - (A \cap B) = (E - A) \cup (E - B)$ .*

PROOF:  $(x \in E \vee \neg(x \in A \wedge x \in B)) \Leftrightarrow (x \in E \wedge x \notin A) \vee (x \in E \wedge x \notin B)$ . □

**Proposition 2.37.** *For any sets  $A$ ,  $B$  and  $E$ , if  $A \subseteq E$  then*

$$A - B = A \cap (E - B) .$$

PROOF: If  $A \subseteq E$  then we have  $(x \in A \wedge x \notin B) \Leftrightarrow (x \in A \wedge x \in E \wedge x \notin B)$ . □

**Proposition 2.38.** *For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A - B = \emptyset$ .*

PROOF: Both are equivalent to the statement that there is no  $x$  such that  $x \in A$  and  $x \notin B$ . □

**Proposition 2.39.** *For any sets  $A$  and  $B$ , we have*

$$A - (A - B) = A \cap B .$$

PROOF:  $(x \in A \wedge \neg(x \in A \wedge x \notin B)) \Leftrightarrow x \in A \wedge x \in B$ .  $\square$

**Proposition 2.40.** *For any sets  $A$ ,  $B$  and  $C$ , we have*

$$A \cap (B - C) = (A \cap B) - (A \cap C) .$$

PROOF:  $(x \in A \wedge x \in B \wedge x \notin C) \Leftrightarrow (x \in A \wedge x \in B \wedge \neg(x \in A \wedge x \in C))$ .  $\square$

**Proposition 2.41.** *For any sets  $A$ ,  $B$ ,  $C$  and  $E$ , if  $(A \cap B) - C \subseteq E$  then we have*

$$A \cap B \subseteq (A \cap C) \cup (B \cap (E - C)) .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in A \cap B$

PROVE:  $x \in (A \cap C) \cup (B \cap (E - C))$

$\langle 1 \rangle 2$ . CASE:  $x \in C$

PROOF: Then  $x \in A \cap C$ .

$\langle 1 \rangle 3$ . CASE:  $x \notin C$

PROOF: Then  $x \in E$  and so  $x \in B \cap (E - C)$ .

$\square$

**Proposition 2.42.** *For any sets  $A$ ,  $B$ ,  $C$  and  $E$ , we have*

$$(A \cup C) \cap (B \cup (E - C)) \subseteq A \cup B .$$

PROOF: The statement  $(x \in A \vee x \in C) \wedge (x \in B \vee (x \in E \wedge x \notin C))$  implies  $x \in A \vee x \in B$ .  $\square$

**Proposition 2.43** (De Morgan's Law). *Let  $E$  be a set and  $\mathcal{C}$  a nonempty set. Then*

$$E - \bigcup_{X \in \mathcal{C}} X = \bigcap_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy.  $\square$

**Proposition 2.44** (De Morgan's Law). *Let  $E$  be a set and  $\mathcal{C}$  a nonempty set. Then*

$$E - \bigcap_{X \in \mathcal{C}} X = \bigcup_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy.  $\square$

## 2.9 Symmetric Difference

**Definition 2.45** (Symmetric Difference). For any sets  $A$  and  $B$ , the *symmetric difference*  $A + B$  is defined to be

$$A + B := (A - B) \cup (B - A) .$$

**Proposition 2.46.** *For any sets  $A$  and  $B$ , we have*

$$A + B = B + A$$

PROOF: From the commutativity of union.  $\square$

**Proposition 2.47.** *For any sets  $A$ ,  $B$  and  $C$ , we have*

$$A + (B + C) = (A + B) + C .$$

PROOF: Each is the set of all  $x$  that belong to either exactly one or all three of  $A$ ,  $B$  and  $C$ .  $\square$

**Proposition 2.48.** *For any set  $A$ , we have*

$$A + \emptyset = A .$$

PROOF:

$$\begin{aligned} A + \emptyset &= (A - \emptyset) \cup (\emptyset - A) \\ &= A \cup \emptyset \\ &= A \end{aligned}$$

$\square$

**Proposition 2.49.** *For any set  $A$  we have*

$$A + A = \emptyset .$$

PROOF:

$$\begin{aligned} A + A &= (A - A) \cup (A - A) \\ &= \emptyset \cup \emptyset \\ &= \emptyset \end{aligned}$$

$\square$

## 2.10 Power Sets

**Proposition 2.50.**

$$\mathcal{P}\emptyset = \{\emptyset\}$$

PROOF: The only subset of  $\emptyset$  is  $\emptyset$ .  $\square$

**Proposition 2.51.** *For any set  $a$ , we have*

$$\mathcal{P}\{a\} = \{\emptyset, \{a\}\} .$$

PROOF: The only subsets of  $\{a\}$  are  $\emptyset$  and  $\{a\}$ .  $\square$

**Proposition 2.52.** *For any sets  $a$  and  $b$ , we have*

$$\mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} .$$

PROOF: The only subsets of  $\{a, b\}$  are  $\emptyset$ ,  $\{a\}$ ,  $\{b\}$  and  $\{a, b\}$ .  $\square$

**Proposition 2.53.** *For any nonempty set  $\mathcal{C}$  we have*

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}X = \mathcal{P}\left(\bigcap \mathcal{C}\right) .$$

PROOF:

$$\begin{aligned} x \in \bigcup_{X \in \mathcal{C}} \mathcal{P}X &\Leftrightarrow \forall X \in \mathcal{C}. x \subseteq X \\ &\Leftrightarrow \forall X \in \mathcal{C}. \forall y \in x. y \in X \\ &\Leftrightarrow \forall y \in x. \forall X \in \mathcal{C}. y \in X \\ &\Leftrightarrow x \subseteq \bigcap \mathcal{C} \end{aligned}$$

□

**Proposition 2.54.** *For any set  $\mathcal{C}$  we have*

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}X \subseteq \mathcal{P}\bigcup \mathcal{C} .$$

PROOF: If there exists  $X \in \mathcal{C}$  such that  $x \subseteq X$  then  $x \subseteq \bigcup \mathcal{C}$ . □

**Proposition 2.55.** *For any set  $E$ , we have*

$$\bigcap \mathcal{P}E = \emptyset .$$

PROOF: Since  $\emptyset \in \mathcal{P}E$ . □

**Proposition 2.56.** *For any sets  $E$  and  $F$ , if  $E \subseteq F$  then  $\mathcal{P}E \subseteq \mathcal{P}F$ .*

PROOF: If  $E \subseteq F$  and  $X \subseteq E$  then  $X \subseteq F$ . □

## Chapter 3

# Relations and Functions

### 3.1 Ordered Pairs

**Proposition 3.1.** *For any sets  $a, b, x$  and  $y$ , if  $(a, b) = (x, y)$  then  $a = x$  and  $b = y$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $a, b, x$  and  $y$  be sets.

$\langle 1 \rangle 2$ . ASSUME:  $(a, b) = (x, y)$

$\langle 1 \rangle 3$ .  $a = x$

PROOF:  $\{a\} = \bigcap(a, b) = \bigcap(x, y) = \{x\}$ .

$\langle 1 \rangle 4$ .  $\{a, b\} = \{x, y\}$

$\langle 1 \rangle 5$ . CASE:  $a = b$

$\langle 2 \rangle 1$ .  $x = y$

PROOF: Since  $\{x, y\} = \{a, b\}$  is a singleton.

$\langle 2 \rangle 2$ .  $b = y$

PROOF:  $b = a = x = y$

$\langle 1 \rangle 6$ . CASE:  $a \neq b$

$\langle 2 \rangle 1$ .  $x \neq y$

PROOF: Since  $\{x, y\} = \{a, b\}$  is not a singleton.

$\langle 2 \rangle 2$ .  $b = y$

PROOF:  $\{b\} = \{a, b\} - \{a\} = \{x, y\} - \{x\} = \{y\}$ .

□

**Proposition 3.2.** *For any sets  $A, B$  and  $X$ , we have*

$$(A - B) \times X = (A \times X) - (B \times X) .$$

PROOF: Easy. □

**Proposition 3.3.** *For any sets  $A$  and  $B$ , we have  $A \times B = \emptyset$  if and only if  $A = \emptyset$  or  $B = \emptyset$ .*

PROOF: Easy. □

**Proposition 3.4.** For any sets  $A, B, X$  and  $Y$ , if  $A \subseteq X$  and  $B \subseteq Y$  then  $A \times B \subseteq X \times Y$ . The converse holds assuming  $A \neq \emptyset$  and  $B \neq \emptyset$ .

PROOF: Easy.  $\square$

## 3.2 Relations

**Definition 3.5** (Domain). The *domain* of a relation  $R$  is the set

$$\text{dom } R := \left\{ x \in \bigcup \bigcup R : \exists y. (x, y) \in R \right\} .$$

**Definition 3.6** (Range). The *range* of a relation  $R$  is the set

$$\text{ran } R := \left\{ y \in \bigcup \bigcup R : \exists x. (x, y) \in R \right\} .$$

**Definition 3.7** (Reflexive). Let  $R$  be a relation on  $X$ . Then  $R$  is *reflexive* iff, for all  $x \in X$ , we have  $xRx$ .

**Definition 3.8** (Symmetric). Let  $R$  be a relation on  $X$ . Then  $R$  is *symmetric* iff, whenever  $xRy$ , then  $yRx$ .

**Definition 3.9** (Antisymmetric). A relation  $R$  is *antisymmetric* iff, whenever  $xRy$  and  $yRx$ , then  $x = y$ .

**Definition 3.10** (Transitive). Let  $R$  be a relation on  $X$ . Then  $R$  is *transitive* iff, whenever  $xRy$  and  $yRz$ , then  $xRz$ .

**Definition 3.11** (Identity Relation). For any set  $X$ , the *identity relation*  $I_X$  on  $X$  is

$$I_X = \{(x, x) : x \in X\} .$$

## 3.3 Composition

**Definition 3.12** (Composition). Let  $R$  be a relation between  $X$  and  $Y$ , and  $S$  a relation between  $Y$  and  $Z$ . The *composite* or *relative product*  $S \circ R = SR$  is the relation between  $X$  and  $Z$  defined by

$$x(S \circ R)z \Leftrightarrow \exists y \in Y (xRy \wedge ySz) .$$

**Proposition 3.13.** Let  $R$  be a relation between  $X$  and  $Y$ ,  $S$  a relation between  $Y$  and  $Z$ , and  $T$  a relation between  $Z$  and  $W$ . Then

$$T(SR) = (TS)R .$$

PROOF: Easy.  $\square$

**Example 3.14.** Composition of relations is not commutative in general. Let  $X = \{a, b\}$  where  $a \neq b$ . Let  $R = \{(a, a), (b, a)\}$  and  $S = \{(a, b), (b, b)\}$ . Then  $SR = S$  but  $RS = R \neq S$ .

**Proposition 3.15.** A relation  $R$  is transitive if and only if  $RR \subseteq R$ .

PROOF: Easy.  $\square$

### 3.4 Inverses

**Definition 3.16** (Inverse). Let  $R$  be a relation between  $X$  and  $Y$ . The *inverse* or *converse*  $R^{-1}$  is the relation between  $Y$  and  $X$  defined by

$$yR^{-1}x \Leftrightarrow xRy .$$

**Proposition 3.17.** *For any relation  $R$ , we have*

$$\text{dom } R^{-1} = \text{ran } R .$$

PROOF: Easy.  $\square$

**Proposition 3.18.** *For any relation  $R$ , we have*

$$\text{ran } R^{-1} = \text{dom } R .$$

PROOF: Easy.  $\square$

**Proposition 3.19.** *Let  $R$  be a relation between  $X$  and  $Y$ , and  $S$  a relation between  $Y$  and  $Z$ . Then*

$$(SR)^{-1} = R^{-1}S^{-1} .$$

PROOF: Easy.  $\square$

**Proposition 3.20.** *A relation  $R$  is symmetric if and only if  $R \subseteq R^{-1}$ .*

PROOF: Easy.  $\square$

**Proposition 3.21.** *Let  $R$  be a relation between  $X$  and  $Y$ . Then*

$$I_Y R = R I_X = R .$$

PROOF: Easy.  $\square$

**Proposition 3.22.** *A relation  $R$  on a set  $X$  is reflexive if and only if  $I_X \subseteq R$ .*

PROOF: Easy.  $\square$

**Proposition 3.23.** *Let  $R$  be a relation on a set  $X$ . Then  $R$  is antisymmetric iff  $R \cap R^{-1} \subseteq I_X$ .*

PROOF: Easy.  $\square$

### 3.5 Equivalence Relations

**Definition 3.24** (Equivalence Relation). Let  $R$  be a relation on  $X$ . Then  $R$  is an *equivalence relation* iff it is reflexive, symmetric and transitive.

**Definition 3.25** (Partition). Let  $X$  be a set. A *partition* of  $X$  is a pairwise disjoint set of nonempty subsets of  $X$  whose union is  $X$ .



**Definition 3.26** (Equivalence Class). Let  $R$  be an equivalence relation on  $X$ . Let  $x \in X$ . The *equivalence class* of  $x$  with respect to  $R$  is

$$x/R := \{y \in X : xRy\} .$$

We write  $X/R$  for the set of all equivalence classes with respect to  $R$ .

**Definition 3.27** (Induced). Let  $P$  be a partition of  $X$ . The relation *induced* by  $P$  is  $X/P$  where  $x(X/P)y$  iff there exists  $X \in P$  such that  $x \in X$  and  $y \in X$ .

**Theorem 3.28.** *Let  $R$  be an equivalence relation on  $X$ . Then  $X/R$  is a partition of  $X$  that induces the relation  $R$ .*

PROOF: Easy.  $\square$

**Theorem 3.29.** *Let  $P$  be a partition of  $X$ . Then  $X/P$  is an equivalence relation on  $X$ , and  $P = X/(X/P)$ .*

PROOF: Easy.  $\square$

## 3.6 Functions

**Definition 3.30** (Onto). Let  $f : X \rightarrow Y$ . We say  $f$  maps  $X$  *onto*  $Y$  iff  $\text{ran } f = Y$ .

**Definition 3.31** (Image). Let  $f : X \rightarrow Y$  and  $A \subseteq X$ . The *image* of  $A$  under  $f$  is

$$f(A) := \{f(x) : x \in A\} .$$

**Definition 3.32** (Inclusion Map). Let  $Y$  be a set and  $X \subseteq Y$ . Then the *inclusion map*  $i : X \hookrightarrow Y$  is the function defined by  $i(x) = x$  for all  $x \in X$ .

**Proposition 3.33.** *For any set  $X$ , the identity relation  $I_X$  is a function  $X \rightarrow X$ .*

PROOF: Easy.  $\square$

**Definition 3.34** (Restriction). Let  $f : Y \rightarrow Z$  and  $X \subseteq Y$ . The *restriction* of  $f$  to  $X$  is the function  $f \upharpoonright X : X \rightarrow Z$  defined by

$$(f \upharpoonright X)(x) = f(x) \quad (x \in X) .$$

Given sets  $X, Y$  and  $Z$  with  $X \subseteq Y$ , if  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$ , we say  $g$  is an *extension* of  $f$  to  $Y$  iff  $f = g \upharpoonright X$ .

**Definition 3.35** (Projection). Given sets  $X$  and  $Y$ , the *projection* maps  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  are defined by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y \quad (x \in X, y \in Y) .$$

**Definition 3.36** (Canonical Map). Let  $X$  be a set and  $R$  an equivalence relation on  $X$ . The *canonical map*  $\pi : X \rightarrow X/R$  is the map defined by  $\pi(x) = x/R$ .

**Definition 3.37** (One-to-One). A function  $f : X \rightarrow Y$  is *one-to-one*, or a *one-to-one correspondence*, iff, for all  $x, y \in X$ , if  $f(x) = f(y)$  then  $x = y$ .

**Proposition 3.38.** Let  $f : X \rightarrow Y$ . Then the following are equivalent:

1.  $f$  is one-to-one.
2. For all  $A, B \subseteq X$ , we have  $f(A \cap B) = f(A) \cap f(B)$ .
3. For all  $A \subseteq X$ , we have  $f(X - A) \subseteq Y - f(A)$ .

PROOF: Easy.  $\square$

**Proposition 3.39.** Let  $f : X \rightarrow Y$ . Then  $f$  maps  $X$  onto  $Y$  if and only if, for all  $A \subseteq X$ , we have  $Y - f(A) \subseteq f(X - A)$ .

PROOF: Easy.  $\square$

## 3.7 Families

**Proposition 3.40** (Generalized Associative Law for Unions). Let  $\{I_j\}_{j \in J}$  be a family of sets. Let  $K = \bigcup_{j \in J} I_j$ . Let  $\{A_k\}_{k \in K}$  be a family of sets indexed by  $K$ . Then

$$\bigcup_{k \in K} A_k = \bigcup_{j \in J} \bigcup_{i \in I_j} A_i .$$

PROOF: Easy.  $\square$

**Proposition 3.41** (Generalized Commutative Law for Unions). Let  $\{I_j\}_{j \in J}$  be a family of sets. Let  $f : J \rightarrow J$  be a one-to-one correspondence from  $J$  onto  $J$ . Then

$$\bigcup_{j \in J} I_j = \bigcup_{j \in J} I_{f(j)} .$$

PROOF: Easy.  $\square$

**Proposition 3.42** (Generalized Associative Law for Intersections). Let  $\{I_j\}_{j \in J}$  be a nonempty family of nonempty sets. Let  $K = \bigcup_{j \in J} I_j$ . Let  $\{A_k\}_{k \in K}$  be a family of sets indexed by  $K$ . Then

$$\bigcap_{k \in K} A_k = \bigcap_{j \in J} \bigcap_{i \in I_j} A_i .$$

PROOF: Easy.  $\square$

**Proposition 3.43** (Generalized Commutative Law for Intersections). Let  $\{I_j\}_{j \in J}$  be a nonempty family of sets. Let  $f : J \rightarrow J$  be a one-to-one correspondence from  $J$  onto  $J$ . Then

$$\bigcap_{j \in J} I_j = \bigcap_{j \in J} I_{f(j)} .$$

PROOF: Easy.  $\square$

**Proposition 3.44.** *Let  $B$  be a set and  $\{A_i\}_{i \in I}$  a family of sets. Then*

$$B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i)$$

PROOF: Easy.  $\square$

**Proposition 3.45.** *Let  $B$  be a set and  $\{A_i\}_{i \in I}$  a nonempty family of sets. Then*

$$B \cup \bigcap_{i \in I} A_i = \bigcap_{i \in I} (B \cup A_i)$$

PROOF: Easy.  $\square$

**Definition 3.46** (Projection). Let  $\{A_i\}_{i \in I}$  be a family of sets and  $i \in I$ . The projection function  $\pi_i : \times_{i \in I} A_i \rightarrow A_i$  is defined by  $\pi_i(a) = a_i$ .

**Proposition 3.47.** *Let  $\{A_i\}_{i \in I}$  and  $\{B_j\}_{j \in J}$  be families of sets. Then*

$$\left( \bigcup_{i \in I} A_i \right) \times \left( \bigcup_{j \in J} B_j \right) = \bigcup_{i \in I} \bigcup_{j \in J} (A_i \times B_j) .$$

PROOF: Easy.  $\square$

**Proposition 3.48.** *Let  $\{A_i\}_{i \in I}$  and  $\{B_j\}_{j \in J}$  be nonempty families of sets. Then*

$$\left( \bigcap_{i \in I} A_i \right) \times \left( \bigcap_{j \in J} B_j \right) = \bigcap_{i \in I} \bigcap_{j \in J} (A_i \times B_j) .$$

PROOF: Easy.  $\square$

**Proposition 3.49.** *Let  $f : X \rightarrow Y$ . Let  $\{A_i\}_{i \in I}$  be a family of subsets of  $X$ . Then*

$$f \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} f(A_i) .$$

PROOF: Easy.  $\square$

**Example 3.50.** It is not true in general that, if  $f : X \rightarrow Y$  and  $\{A_i\}_{i \in I}$  is a nonempty family of subsets of  $X$ , then  $f(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f(A_i)$ .

Take  $X = \{a, b\}$  and  $Y = \{c\}$  where  $a \neq b$ . Take  $I = \{i, j\}$  with  $i \neq j$ . Let  $A_i = \{a\}$  and  $A_j = \{b\}$ . Let  $f$  be the unique function  $X \rightarrow Y$ . Then  $f(\bigcap_{i \in I} A_i) = f(\emptyset) = \emptyset$  but  $\bigcap_{i \in I} f(A_i) = \{c\}$ .

### 3.8 Inverses and Composites of Functions

**Definition 3.51** (Inverse). Given a function  $f : X \rightarrow Y$ , the *inverse* of  $f$  is the function  $f^{-1} : \mathcal{P}Y \rightarrow \mathcal{P}X$  defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\} .$$

We call  $f^{-1}(B)$  the *inverse image* of  $B$  under  $f$ .

**Proposition 3.52.** *Let  $f : X \rightarrow Y$ . Then  $f$  maps  $X$  onto  $Y$  if and only if the inverse image of any nonempty subset of  $Y$  is nonempty.*

PROOF: Easy.  $\square$

**Proposition 3.53.** *Let  $f : X \rightarrow Y$ . Then  $f$  is one-to-one if and only if the inverse image of any singleton subset of  $Y$  is a singleton.*

PROOF: Easy.  $\square$

**Proposition 3.54.** *Let  $f : X \rightarrow Y$ . Let  $B \subseteq Y$ . Then*

$$f(f^{-1}(B)) \subseteq B .$$

PROOF: Easy.  $\square$

**Proposition 3.55.** *Let  $f : X \rightarrow Y$ . Let  $A \subseteq X$ . Then*

$$A \subseteq f^{-1}(f(A)) .$$

*Equality holds if  $f$  is one-to-one.*

PROOF: Easy.  $\square$

**Proposition 3.56.** *Let  $f : X \rightarrow Y$ . Let  $\{B_i\}_{i \in I}$  be a family of subsets of  $Y$ . Then*

$$f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i) .$$

PROOF: Easy.  $\square$

**Proposition 3.57.** *Let  $f : X \rightarrow Y$ . Let  $\{B_i\}_{i \in I}$  be a nonempty family of subsets of  $Y$ . Then*

$$f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i) .$$

PROOF: Easy.  $\square$

**Proposition 3.58.** *Let  $f : X \rightarrow Y$  and  $B \subseteq Y$ . Then  $f^{-1}(Y - B) = X - f^{-1}(B)$ .*

PROOF: Easy.  $\square$

**Proposition 3.59.** *Let  $f : X \rightarrow Y$  be one-to-one. Then the inverse of  $f$  as a relation,  $f^{-1}$ , is a function  $f^{-1} : \text{ran } f \rightarrow X$ , and for all  $y \in \text{ran } f$ , we have  $f^{-1}(y)$  is the unique  $x$  such that  $f(x) = y$ .*

PROOF: Easy.  $\square$

**Proposition 3.60.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . Then  $gf : X \rightarrow Z$  and, for all  $x \in X$ , we have*

$$(g \circ f)(x) = g(f(x)) .$$

PROOF: Easy.  $\square$

**Example 3.61.** Example 3.14 shows that function composition is not commutative in general.

**Proposition 3.62.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . Then*

$$(gf)^{-1} = f^{-1}g^{-1} : \mathcal{P}Z \rightarrow \mathcal{P}X .$$

PROOF: Easy.  $\square$

**Proposition 3.63.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$ . If  $gf = I_X$  then  $f$  is one-to-one and  $g$  maps  $Y$  onto  $X$ .*

PROOF: Easy.  $\square$

### 3.9 Choice Functions

**Definition 3.64** (Choice Function). A *choice function* for a set  $X$  is a function  $f : \mathcal{P}X - \{\emptyset\} \rightarrow X$  such that  $f(S) \in S$  for all  $S$ .

**Proposition 3.65.** *Every set has a choice function.*

PROOF: Given a nonempty set  $X$ , apply the Axiom of Choice to the family  $\{S\}_{S \in \mathcal{P}X - \{\emptyset\}}$ .  $\square$

**Proposition 3.66.** *For any relation  $R$ , there exists a function  $f \subseteq R$  such that  $\text{dom } f = \text{dom } R$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $R$  be a relation.

$\langle 1 \rangle 2$ . PICK a choice function  $g$  for  $\text{ran } R$ .

$\langle 1 \rangle 3$ . LET:  $f : \text{dom } R \rightarrow \text{ran } R$  be the function  $f(x) = g(\{y \in \text{ran } R : xRy\})$

$\langle 1 \rangle 4$ .  $f \subseteq R$  and  $\text{dom } f = \text{dom } R$ .

$\square$

**Proposition 3.67.** *If  $\mathcal{C}$  is a set of pairwise disjoint nonempty sets, then there exists a set  $A$  such that, for all  $C \in \mathcal{C}$ , we have  $A \cap C$  is a singleton.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f$  be a choice function for  $\bigcup \mathcal{C}$

$\langle 1 \rangle 2$ . LET:  $A = \{f(C) : C \in \mathcal{C}\}$

$\langle 1 \rangle 3$ . For all  $C \in \mathcal{C}$  we have  $A \cap C = \{f(C)\}$

$\square$

## Chapter 4

# Equivalence

**Definition 4.1** (Equivalent). Sets  $E$  and  $F$  are *equivalent*,  $E \sim F$ , iff there exists a one-to-one correspondence between them.

**Proposition 4.2.** *For any set  $X$ , equivalence is an equivalence relation on  $\mathcal{P}X$ .*

PROOF: Easy.

**Theorem 4.3** (Schröder-Bernstein). *Let  $X$  and  $Y$  be sets. If there exist injective functions  $X \rightarrow Y$  and  $Y \rightarrow X$ , then  $X \sim Y$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  be one-to-one.
- $\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $X \cap Y = \emptyset$
- $\langle 1 \rangle 3$ . For  $x \in X$ , let us say that  $x$  is the *parent* of  $f(x)$ ; and for  $y \in Y$ , let us say that  $y$  is the *parent* of  $g(y)$ .
- $\langle 1 \rangle 4$ . For  $z \in X \cup Y$ , let the set of *descendants* of  $z$  be the intersection of all the subsets  $S$  of  $X \cup Y$  such that  $z \in S$  and, if  $t \in S$  and  $t$  is the parent of  $u$  then  $u \in S$ .
- $\langle 1 \rangle 5$ . LET:  $X_X$  be the set of all elements of  $X$  that are descendants of the elements of  $X$  that have no parent.
- $\langle 1 \rangle 6$ . LET:  $X_Y$  be the set of all elements of  $X$  that are descendants of the elements of  $Y$  that have no parent.
- $\langle 1 \rangle 7$ . LET:  $X_\infty = X - X_X - X_Y$
- $\langle 1 \rangle 8$ . LET:  $Y_X$  be the set of all elements of  $Y$  that are descendants of the elements of  $X$  that have no parent.
- $\langle 1 \rangle 9$ . LET:  $Y_Y$  be the set of all elements of  $Y$  that are descendants of the elements of  $Y$  that have no parent.
- $\langle 1 \rangle 10$ . LET:  $Y_\infty = Y - Y_X - Y_Y$
- $\langle 1 \rangle 11$ .  $f|X_X : X_X \sim Y_X$
- $\langle 1 \rangle 12$ .  $g|Y_Y : Y_Y \sim X_Y$
- $\langle 1 \rangle 13$ .  $f|X_\infty : X_\infty \sim Y_\infty$
- $\langle 1 \rangle 14$ . Define  $h : X \rightarrow Y$  by  $h(x) = g^{-1}(x)$  if  $x \in X_Y$ , and  $f(x)$  if not.

$\langle 1 \rangle_{15}$ .  $h : X \sim Y$   
 $\square$

## Chapter 5

# Order

**Definition 5.1** (Partial Order). A *partial order* on a set  $X$  is a relation on  $X$  that is reflexive, antisymmetric and transitive.

A *partially ordered set* or *poset* is a pair  $(X, \leq)$  such that  $\leq$  is a partial order on  $X$ . We write  $X$  for the poset  $(X, \leq)$ .

Given a partial order  $\leq$ , we write  $\geq$  for the inverse of  $\leq$ .

We write  $x < y$  or  $y > x$  for  $x \leq y \wedge x \neq y$ . When this holds, we say  $x$  is *less than*  $y$ , *smaller than*  $y$ , or a *predecessor* of  $y$ ; and  $y$  is *greater than*  $x$ , *larger than*  $x$ , or a *successor* of  $x$ .

**Proposition 5.2.** *For any set  $X$ , the relation  $\subseteq$  is a partial order on  $\mathcal{P}X$ .*

PROOF: Easy.  $\square$

**Proposition 5.3.** *In a poset, we never have  $x < y$  and  $y < x$ .*

PROOF: We would then have  $x \leq y$  and  $y \leq x$  hence  $x = y$  by antisymmetry. But if  $x < y$  or  $y < x$  then  $x \neq y$ .  $\square$

**Proposition 5.4.** *The relation  $<$  is transitive.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $x < y$  and  $y < z$

$\langle 1 \rangle 2$ .  $x \leq y$  and  $y \leq z$

$\langle 1 \rangle 3$ .  $x \leq z$

PROOF: Since  $\leq$  is transitive.

$\langle 1 \rangle 4$ .  $x \neq z$

PROOF: By Proposition 5.3.

$\square$

**Proposition 5.5.** *Let  $<$  be a transitive relation on  $X$  such that we never have  $x < y$  and  $y < x$ . Define  $\leq$  by:  $x \leq y$  iff  $x < y$  or  $x = y$ . Then  $\leq$  is a partial order on  $X$ .*

PROOF:



$\langle 1 \rangle 1.$   $\leq$  is reflexive.

PROOF: By definition.

$\langle 1 \rangle 2.$   $\leq$  is asymmetric.

PROOF: If  $x \leq y$  and  $y \leq x$ , we must have  $x = y$ , because otherwise we would have  $x < y$  and  $y < x$ .

$\langle 1 \rangle 3.$   $\leq$  is transitive.

$\langle 2 \rangle 1.$  LET:  $x \leq y$  and  $y \leq z$

$\langle 2 \rangle 2.$  CASE:  $x = y$

PROOF: We have  $y \leq z$  so  $x \leq z$ .

$\langle 2 \rangle 3.$  CASE:  $y = z$

PROOF: We have  $x \leq y$  so  $x \leq z$ .

$\langle 2 \rangle 4.$  CASE:  $x < y$  and  $y < z$

PROOF: We have  $x < z$  by transitivity, so  $x \leq z$ .

□

**Definition 5.6** ((Strict) Initial Segment). Let  $X$  be a poset and  $a \in X$ . The *(strict) initial segment* determined by  $a$  is

$$s(a) := \{x \in X : x < a\} .$$

**Definition 5.7** (Weak Initial Segment). Let  $X$  be a poset and  $a \in X$ . The *weak initial segment* determined by  $a$  is

$$\bar{s}(a) := \{x \in X : x \leq a\} .$$

**Definition 5.8** (Immediate Successor). Let  $X$  be a poset and  $x, y \in X$ . Then  $y$  is the *immediate successor* of  $x$ , and  $x$  is the *immediate predecessor* of  $y$ , iff  $x < y$  and there is no  $z$  such that  $x < z < y$ .

**Definition 5.9** (Least). Let  $X$  be a partial order and  $a \in X$ . Then  $a$  is *least* in  $X$  iff  $\forall x \in X. a \leq x$ .

**Proposition 5.10.** *A poset has at most one least element.*

PROOF: If  $a$  and  $b$  are least then  $a \leq b$  and  $b \leq a$ , hence  $a = b$ . □

**Definition 5.11** (Greatest). Let  $X$  be a partial order and  $a \in X$ . Then  $a$  is *greatest* in  $X$  iff  $\forall x \in X. x \leq a$ .

**Proposition 5.12.** *A poset has at most one greatest element.*

PROOF: If  $a$  and  $b$  are greatest then  $a \leq b$  and  $b \leq a$ , hence  $a = b$ . □

**Definition 5.13** (Minimal). Let  $X$  be a poset and  $a \in X$ . Then  $a$  is *minimal* iff there is no  $x \in X$  such that  $x < a$ .

**Definition 5.14** (Maximal). Let  $X$  be a poset and  $a \in X$ . Then  $a$  is *maximal* iff there is no  $x \in X$  such that  $a < x$ .

**Definition 5.15** (Lower Bound). Let  $X$  be a poset. Let  $E \subseteq X$  and  $a \in X$ . Then  $a$  is a *lower bound* for  $E$  iff  $\forall x \in E. a \leq x$ .

**Definition 5.16** (Upper Bound). Let  $X$  be a poset. Let  $E \subseteq X$  and  $a \in X$ . Then  $a$  is an *upper bound* for  $E$  iff  $\forall x \in E. x \leq a$ .

**Definition 5.17** (Greatest Lower Bound, Infimum). Let  $X$  be a poset. Let  $E \subseteq X$  and  $a \in X$ . Then  $a$  is the *greatest lower bound* or *infimum* for  $E$  iff  $a$  is the greatest element in the set of lower bounds for  $E$ .

**Definition 5.18** (Least Upper Bound, Supremum). Let  $X$  be a poset. Let  $E \subseteq X$  and  $a \in X$ . Then  $a$  is the *least upper bound* or *supremum* for  $E$  iff  $a$  is the least element in the set of upper bounds for  $E$ .

**Definition 5.19** (Total Order). A partial order  $\leq$  on a set  $X$  is a *total order*, *simple order* or *linear order* iff, for all  $x, y \in X$ , either  $x \leq y$  or  $y \leq x$ . We then call the poset  $(X, \leq)$  a *linearly ordered set* or a *chain*.

**Proposition 5.20.** Let  $R$  be a partial order on  $X$ . Then  $R$  is total if and only if  $X^2 \subseteq R \cup R^{-1}$ .

PROOF: Easy.  $\square$

**Proposition 5.21.** For any set  $X$ , the relation  $\subseteq$  is a total order on  $X$  iff  $X$  is either  $\emptyset$  or a singleton.

PROOF: Easy.  $\square$

**Theorem 5.22** (Zorn's Lemma). Let  $X$  be a poset such that every chain in  $X$  has an upper bound. Then  $X$  has a maximal element.

PROOF:

$\langle 1 \rangle 1$ . PICK a choice function  $f$  for  $X$ .

$\langle 1 \rangle 2$ . LET:  $\mathcal{X}$  be the set of chains in  $X$ .

$\langle 1 \rangle 3$ . For all  $A \in \mathcal{X}$ ,

LET:  $\hat{A} = \{x \in X : A \cup \{x\} \in \mathcal{X}\}$

$\langle 1 \rangle 4$ . LET:  $g : \mathcal{X} \rightarrow \mathcal{X}$  be the function

$$g(A) = \begin{cases} A \cup \{f(\hat{A} - A)\} & \text{if } \hat{A} - A \neq \emptyset \\ A & \text{if } \hat{A} - A = \emptyset \end{cases}$$

$\langle 1 \rangle 5$ . For  $\mathcal{T} \subseteq \mathcal{X}$ , let us say  $\mathcal{T}$  is a *tower* iff:

- $\emptyset \in \mathcal{T}$
- $\forall A \in \mathcal{T}. g(A) \in \mathcal{T}$
- For every chain  $\mathcal{C}$  in  $\mathcal{T}$ , we have  $\bigcup \mathcal{C} \in \mathcal{T}$

$\langle 1 \rangle 6$ . LET:  $\mathcal{T}_0$  be the intersection of the set of all towers.

PROOF: The set of all towers is nonempty since  $\mathcal{X}$  is a tower.

$\langle 1 \rangle 7$ . LET:  $A = \bigcup \mathcal{T}_0$

$\langle 1 \rangle 8$ .  $A$  is a chain in  $X$ .

$\langle 2 \rangle 1$ .  $\mathcal{T}_0$  is a chain under  $\subseteq$

$\langle 3 \rangle 1$ . Given  $C \in \mathcal{T}_0$ , let us say that  $C$  is *comparable* iff, for all  $A \in \mathcal{T}_0$ , either  $A \subseteq C$  or  $C \subseteq A$ .

$\langle 3 \rangle 2$ . For all  $A, C \in \mathcal{T}_0$ , if  $C$  is comparable and  $A \subsetneq C$  then  $g(A) \subseteq C$ .  
 PROOF: Since  $g(A) - A$  has at most one element, so if  $A \subsetneq C \subseteq g(A)$  then  $C = g(A)$ .  
 $\langle 3 \rangle 3$ . For  $C \in \mathcal{T}_0$  comparable,  
 LET:  $\mathcal{U}_C = \{A \in \mathcal{T}_0 : A \subseteq C \vee g(C) \subseteq A\}$   
 $\langle 3 \rangle 4$ . For  $C \in \mathcal{T}_0$  comparable,  $\mathcal{U}_C$  is a tower.  
 $\langle 4 \rangle 1$ . LET:  $C \in \mathcal{T}_0$  be comparable  
 $\langle 4 \rangle 2$ .  $\emptyset \in \mathcal{U}_C$   
 PROOF: Since  $\emptyset \subseteq C$ .  
 $\langle 4 \rangle 3$ .  $\forall A \in \mathcal{U}_C. g(A) \in \mathcal{U}_C$   
 PROOF: By  $\langle 1 \rangle 8$ .  
 $\langle 4 \rangle 4$ . For every chain  $\mathcal{C} \subseteq \mathcal{U}_C$  we have  $\bigcup \mathcal{C} \in \mathcal{U}_C$   
 $\langle 5 \rangle 1$ . LET:  $\mathcal{C} \subseteq \mathcal{U}_C$  be a chain.  
 $\langle 5 \rangle 2$ . CASE:  $\exists A \in \mathcal{C}. g(C) \subseteq A$   
 PROOF: Then  $g(C) \subseteq \bigcup \mathcal{C}$   
 $\langle 5 \rangle 3$ . CASE:  $\forall A \in \mathcal{C}. A \subseteq C$   
 PROOF: Then  $\bigcup \mathcal{C} \subseteq C$ .  
 $\langle 3 \rangle 5$ . For  $C \in \mathcal{T}_0$  comparable,  $\mathcal{U}_C = \mathcal{T}_0$ .  
 $\langle 3 \rangle 6$ . For  $C \in \mathcal{T}_0$  comparable we have  $g(C)$  is comparable.  
 PROOF: Since for all  $A \in \mathcal{T}_0$  either  $A \subseteq C \subseteq g(C)$  or  $g(C) \subseteq A$ .  
 $\langle 3 \rangle 7$ . The set of comparable sets in  $\mathcal{T}_0$  is a tower.  
 $\langle 4 \rangle 1$ .  $\emptyset$  is comparable.  
 PROOF:  $\forall A \in \mathcal{T}_0. \emptyset \subseteq A$   
 $\langle 4 \rangle 2$ . For all  $C \in \mathcal{T}_0$ , if  $A$  is comparable then  $g(C)$  is comparable.  
 PROOF:  $\langle 3 \rangle 6$   
 $\langle 4 \rangle 3$ . For every chain  $\mathcal{C} \subseteq \mathcal{T}_0$  of comparable sets, we have  $\bigcup \mathcal{C}$  is comparable.  
 $\langle 5 \rangle 1$ . LET:  $\mathcal{C} \subseteq \mathcal{T}_0$  be a chain of comparable sets.  
 $\langle 5 \rangle 2$ . LET:  $A \in \mathcal{T}_0$   
 $\langle 5 \rangle 3$ . CASE: there exists  $C \in \mathcal{C}$  such that  $A \subseteq C$   
 PROOF: Then  $A \subseteq \bigcup \mathcal{C}$ .  
 $\langle 5 \rangle 4$ . CASE: for all  $C \in \mathcal{C}$  we have  $C \subseteq A$   
 PROOF: Then  $\bigcup \mathcal{C} \subseteq A$ .  
 $\langle 3 \rangle 8$ . Every set in  $\mathcal{T}_0$  is comparable.  
 $\langle 2 \rangle 2$ . LET:  $x, y \in A$   
 $\langle 2 \rangle 3$ . PICK  $A, C \in \mathcal{T}_0$  such that  $x \in A$  and  $y \in C$   
 $\langle 2 \rangle 4$ . ASSUME: w.l.o.g.  $A \subseteq C$   
 $\langle 2 \rangle 5$ .  $x, y \in C$   
 $\langle 2 \rangle 6$ .  $x \leq y$  or  $y \leq x$   
 PROOF: Since  $C \in \mathcal{X}$  so  $C$  is a chain.  
 $\langle 1 \rangle 9$ . PICK an upper bound  $u$  for  $A$ .  
 $\langle 1 \rangle 10$ .  $A \in \mathcal{T}_0$   
 PROOF: Since  $\mathcal{T}_0$  is a chain in  $\mathcal{T}_0$  so  $\bigcup \mathcal{T}_0 \in \mathcal{T}_0$ .  
 $\langle 1 \rangle 11$ .  $g(A) \in \mathcal{T}_0$   
 $\langle 1 \rangle 12$ .  $g(A) \subseteq A$   
 $\langle 1 \rangle 13$ .  $g(A) = A$

⟨1⟩14.  $\hat{A} - A = \emptyset$

⟨1⟩15.  $u \in A$

PROOF: Since  $A \cup \{u\}$  is a chain so  $u \in \hat{A}$  and therefore  $u \in A$ .

⟨1⟩16.  $u$  is maximal in  $X$ .

⟨2⟩1. LET:  $x \in X$

⟨2⟩2. ASSUME:  $u \leq x$

⟨2⟩3.  $A \cup \{x\}$  is a chain.

⟨2⟩4.  $x \in A$

⟨2⟩5.  $x \leq u$

⟨2⟩6.  $x = u$

□

**Definition 5.23** (Cofinal). Let  $X$  be a poset and  $A \subseteq X$ . Then  $A$  is *cofinal* iff, for all  $x \in X$ , there exists  $a \in A$  such that  $x \leq a$ .

**Definition 5.24** (Similar). Two posets  $X$  and  $Y$  are *similar*,  $X \cong Y$  iff there exists an order preserving one-to-one correspondence  $f$  between them. We write  $f : X \cong Y$  and call  $f$  a *similarity*.

**Proposition 5.25.** Let  $X$  and  $Y$  be posets. Let  $f$  be a one-to-one correspondence between  $X$  and  $Y$ . Then  $f$  is a similarity if and only if, for all  $x, y \in X$ , we have  $x < y$  iff  $f(x) < f(y)$ .

PROOF: Easy. □

**Proposition 5.26.** For any poset  $X$  we have  $I_X : X \cong X$ .

PROOF: Easy. □

**Proposition 5.27.** If  $f : X \cong Y$  then  $f^{-1} : Y \cong X$ .

PROOF: Easy. □

**Proposition 5.28.** If  $f : X \cong Y$  and  $g : Y \cong Z$  then  $g \circ f : X \cong Z$ .

PROOF: Easy. □

**Corollary 5.28.1.** For any set  $E$ , similarity is an equivalence relation on the set of all posets that are subsets of  $E$ .

## 5.1 Well Orderings

**Definition 5.29** (Well Ordered Set). A poset  $X$  is *well ordered*, and its ordering is a *well ordering*, iff every nonempty subset of  $X$  has a least element.

**Proposition 5.30.** Every well ordered set is totally ordered.

PROOF: For all  $x$  and  $y$  we have  $\{x, y\}$  has a least element, so  $x \leq y$  or  $y \leq x$ . □

**Theorem 5.31** (Transfinite Induction). *Let  $X$  be a well ordered set. Let  $S \subseteq X$  satisfy:*

$$\forall x \in X (\forall y < x. y \in S) \Rightarrow x \in S .$$

*Then  $S = X$ .*

PROOF: We have  $X - S$  has no least element, so  $X - S = \emptyset$ .  $\square$

**Definition 5.32** (Continuation). Let  $A$  and  $B$  be well ordered sets. Then  $B$  is a *continuation* of  $A$  iff there exists  $b \in B$  such that  $A = s(b)$  and the order on  $A$  is the restriction of the order on  $B$  to  $A$ .

**Proposition 5.33.** *Let  $\mathcal{C}$  be a set of well ordered sets that is totally ordered under continuation. Then there exists a unique well ordering on  $\bigcup \mathcal{C}$  such that  $\bigcup \mathcal{C}$  is a continuation of every element of  $\mathcal{C}$ .*

PROOF: Define  $\leq$  on  $\bigcup \mathcal{C}$  by:  $x \leq y$  iff there exists  $C \in \mathcal{C}$  such that  $x, y \in C$  and  $x \leq y$  in  $C$ .  $\square$

**Proposition 5.34.** *Every totally ordered set has a cofinal well ordered subset.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a totally ordered set.

$\langle 1 \rangle 2$ . LET:  $\mathcal{C}$  be the poset of all well ordered subsets of  $X$  under continuation.

$\langle 1 \rangle 3$ . Every chain in  $\mathcal{C}$  has an upper bound.

PROOF: Proposition 5.33.

$\langle 1 \rangle 4$ . PICK a maximal element  $C$  of  $\mathcal{C}$

PROVE:  $C$  is cofinal

PROOF: Zorn's Lemma

$\langle 1 \rangle 5$ . LET:  $x \in X$

$\langle 1 \rangle 6$ . We cannot have  $\forall c \in C. c < x$

PROOF: Then  $C \cup \{x\}$  would be a larger chain.

$\langle 1 \rangle 7$ .  $\exists c \in C. x \leq c$

$\square$

**Theorem 5.35** (Well Ordering Theorem). *Every set can be well ordered.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a set.

$\langle 1 \rangle 2$ . LET:  $\mathcal{W}$  be the poset of all well ordered subsets of  $X$  under continuation.

$\langle 1 \rangle 3$ . Every chain in  $\mathcal{W}$  has an upper bound.

PROOF: Proposition 5.33.

$\langle 1 \rangle 4$ . PICK a maximal  $M \in \mathcal{W}$

PROOF: Zorn's Lemma

$\langle 1 \rangle 5$ .  $M = X$

PROOF: If  $x \in X - M$  then  $M \cup \{x\}$  with  $x$  as the greatest element is a continuation of  $M$ .

$\square$

**Theorem 5.36** (Transfinite Recursion). *Let  $W$  be a well ordered set and  $X$  a set. Let  $S$  be the set of all functions  $f$  such that  $\text{ran } f \subseteq X$ , and there exists  $a \in W$  such that  $\text{dom } f = s(a)$ . Then there exists a unique function  $U : W \rightarrow X$  such that*

$$\forall a \in W. U(a) = f(U \upharpoonright s(a)) .$$

PROOF:

$\langle 1 \rangle 1$ . Let us say that a subset  $A \subseteq W \times X$  is *f-closed* iff, whenever  $a \in W$  and  $t : s(a) \rightarrow X$  satisfies  $\forall c < a. (c, t(c)) \in A$ , then  $(a, f(t)) \in A$ .

$\langle 1 \rangle 2$ . LET:  $U$  be the intersection of the set of *f-closed* subsets of  $W \times X$

PROOF: This set is nonempty since  $W \times X$  is *f-closed*.

$\langle 1 \rangle 3$ .  $U$  is *f-closed*.

$\langle 1 \rangle 4$ .  $U$  is a function.

$\langle 2 \rangle 1$ . LET:  $P(a)$  be the property: there is at most one  $x \in X$  such that  $(a, x) \in U$

$\langle 2 \rangle 2$ . LET:  $a \in W$

$\langle 2 \rangle 3$ . ASSUME: as transfinite induction hypothesis  $\forall c < a. P(c)$

$\langle 2 \rangle 4$ . LET:  $(a, x), (a, y) \in U$

$\langle 2 \rangle 5$ .  $x = f(U \upharpoonright c)$

PROOF: If not then  $U - \{(a, x)\}$  would be *f-closed*.

$\langle 2 \rangle 6$ .  $y = f(U \upharpoonright c)$

$\langle 2 \rangle 7$ .  $x = y$

$\langle 1 \rangle 5$ .  $\text{dom } U = W$

$\langle 2 \rangle 1$ . LET:  $a \in W$

$\langle 2 \rangle 2$ . ASSUME: as transfinite induction hypothesis  $\forall c < a. c \in \text{dom } U$

$\langle 2 \rangle 3$ .  $(a, f(U \upharpoonright s(a))) \in U$

$\langle 1 \rangle 6$ . If  $U' : W \rightarrow X$  and  $\forall a \in W. U'(a) = f(U' \upharpoonright s(a))$ , then  $U' = U$ .

PROOF: Prove  $U'(a) = U(a)$  by transfinite induction on  $a$ .

□

**Proposition 5.37.** *Let  $X$  be a well ordered set and  $f$  a similarity between  $X$  and a subset of  $X$ . Then, for all  $a \in X$ , we have  $a \leq f(a)$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $a \in X$

$\langle 1 \rangle 2$ . ASSUME: as transfinite induction hypothesis  $\forall c < a. c \leq f(c)$

$\langle 1 \rangle 3$ . ASSUME: for a contradiction  $f(a) < a$

$\langle 1 \rangle 4$ .  $f(a) \leq f(f(a))$

PROOF:  $\langle 1 \rangle 2$

$\langle 1 \rangle 5$ .  $f(f(a)) < f(a)$

PROOF: From  $\langle 1 \rangle 3$  since  $f$  is a similarity.

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 5.38.** *Let  $X$  and  $Y$  be well ordered sets. Then there is at most one similarity between them.*

PROOF:

- ⟨1⟩1. LET:  $f, g : X \cong Y$   
           PROVE:  $\forall a \in X. f(a) = g(a)$
- ⟨1⟩2. LET:  $a \in X$
- ⟨1⟩3. ASSUME: as transfinite induction hypothesis  $\forall c < a. f(c) = g(c)$
- ⟨1⟩4.  $f(a)$  is the least element of  $Y - \{f(c) : c < a\}$
- ⟨1⟩5.  $g(a)$  is the least element of  $Y - \{g(c) : c < a\}$
- ⟨1⟩6.  $f(a) = g(a)$

□

**Proposition 5.39.** *A well ordered set is not similar to any of its initial segments.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a well ordered set.
- ⟨1⟩2. ASSUME: for a contradiction  $f : X \cong s(a)$  for some  $a \in X$
- ⟨1⟩3.  $f(a) < a$
- ⟨1⟩4. Q.E.D.

PROOF: This contradicts Proposition 5.37.

□

**Theorem 5.40** (Comparability Theorem). *Given well ordered sets  $X$  and  $Y$ , either  $X \cong Y$ , or  $X$  is similar to an initial segment of  $Y$ , or  $Y$  is similar to an initial segment of  $X$ .*

PROOF:

- ⟨1⟩1. LET:  $X_0 = \{a \in X : \exists b \in Y. s(a) \cong s(b)\}$
- ⟨1⟩2. LET:  $U : X_0 \rightarrow Y$  be the function: for  $a \in X_0$ , we have  $U(a)$  is the unique element in  $Y$  such that  $s(a) \cong s(U(a))$
- ⟨1⟩3. LET:  $Y_0 = \text{ran } U$
- ⟨1⟩4. Either  $X_0 = X$  or there exists  $a \in X$  such that  $X_0 = s(a)$ 
  - ⟨2⟩1. ASSUME:  $X_0 \neq X$
  - ⟨2⟩2. LET:  $a$  be the least element of  $X - X_0$
  - ⟨2⟩3. LET:  $x \in X_0$   
           PROVE:  $x < a$
  - ⟨2⟩4. PICK  $f : s(x) \cong s(U(x))$
  - ⟨2⟩5. ASSUME: for a contradiction  $a < x$
  - ⟨2⟩6.  $f \upharpoonright s(a) : s(a) \cong s(f(a))$
  - ⟨2⟩7.  $a \in X_0$
  - ⟨2⟩8. Q.E.D.

PROOF: This is a contradiction.

- ⟨1⟩5. Either  $Y_0 = Y$  or there exists  $b \in Y$  such that  $Y_0 = s(b)$

PROOF: Similar.

- ⟨1⟩6. CASE:  $X_0 = X$  and  $Y_0 = Y$

PROOF: Then  $U : X \cong Y$ .

- ⟨1⟩7. CASE:  $X_0 = X$  and  $Y_0 \neq Y$

PROOF: Then  $U : X \cong s(b)$  where  $Y_0 = s(b)$ .

$\langle 1 \rangle 8$ . CASE:  $X_0 \neq X$  and  $Y_0 = Y$

PROOF: Then  $U : s(a) \cong Y$  where  $X_0 = s(a)$ .

$\langle 1 \rangle 9$ . CASE:  $X_0 \neq X$  and  $Y_0 \neq Y$

$\langle 2 \rangle 1$ . LET:  $X_0 = s(a)$  and  $Y_0 = s(b)$

$\langle 2 \rangle 2$ .  $U : s(a) \cong s(b)$

$\langle 2 \rangle 3$ .  $a \in X_0$

$\langle 2 \rangle 4$ . Q.E.D.

PROOF: This is a contradiction.

□

**Corollary 5.40.1.** *Let  $X$  be a well ordered set. Then any subset  $A$  of  $X$  is either similar to  $X$  or to an initial segment of  $X$ .*

PROOF: We cannot have  $X$  is similar to an initial segment of  $A$ , say  $f : X \cong \{x \in A : x < a\}$ , because then we would have  $f(a) < a$  contradicting Proposition 5.37. □



# Chapter 6

## Natural Numbers

### 6.1 Natural Numbers

**Definition 6.1** (Successor). The *successor* of a set  $x$ ,  $x^+$ , is defined by

$$x^+ := x \cup \{x\} .$$

**Definition 6.2.** We define

$$\begin{aligned} 0 &= \emptyset \\ 1 &= 0^+ \\ 2 &= 1^+ \end{aligned}$$

etc.

**Definition 6.3** (Characteristic Function). Let  $X$  be a set and  $A \subseteq X$ . The *characteristic function* of  $A$  is the function  $\chi_A : X \rightarrow 2$  defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

**Theorem 6.4.** Let  $X$  be a set. The function  $\chi : \mathcal{P}X \rightarrow 2^X$  that maps a subset  $A$  of  $X$  to  $\chi_A$  is a one-to-one correspondence.

PROOF: Easy.  $\square$

**Definition 6.5.** The set  $\omega$  of *natural numbers* is the set such that:

- $0 \in \omega$
- For all  $n \in \omega$  we have  $n^+ \in \omega$
- For any set  $X$ , if  $0 \in X$  and  $\forall n \in X. n^+ \in X$  then  $\omega \subseteq X$

PROOF: To show this exists, pick a set  $A$  such that  $0 \in A$  and  $\forall n \in A. n^+ \in A$  (by the Axiom of Infinity), and let  $\omega = \bigcap \{X \in \mathcal{P}A : 0 \in X \wedge \forall n \in X. n^+ \in X\}$ .  
 $\square$

**Definition 6.6** (Sequence). A *finite sequence* is a family whose index set is a natural number. An *infinite sequence* is a family whose index set is  $\omega$ .

Given a finite sequence of sets  $\{A_i\}_{i \in n^+}$ , we write  $\bigcup_{i=0}^n A_i$  for  $\bigcup_{i \in n^+} A_i$ . Given an infinite sequence of sets  $\{A_i\}_{i \in \omega}$ , we write  $\bigcup_{i=0}^{\infty} A_i$  for  $\bigcup_{i \in \omega} A_i$ .

We make similar definitions for  $\bigcap$  and  $\times$ .

**Proposition 6.7.** For any natural numbers  $m$  and  $n$ , if  $m \in n$  then  $m^+ \in n^+$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the property  $\forall m \in n. m^+ \in n^+$

$\langle 1 \rangle 2$ .  $P(0)$

PROOF: Vacuous.

$\langle 1 \rangle 3$ . For any natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$

$\langle 2 \rangle 3$ . LET:  $m \in n^+$

$\langle 2 \rangle 4$ .  $m \in n$  or  $m = n$

$\langle 2 \rangle 5$ .  $m^+ \in n^+$  or  $m^+ = n^+$

PROOF:  $\langle 2 \rangle 2$

$\langle 2 \rangle 6$ . CASE:  $m^+ \in n^{++}$

$\square$

**Theorem 6.8** (Principle of Mathematical Induction). For any subset  $S$  of  $\omega$ , if  $0 \in S$  and  $\forall n \in S. n^+ \in S$ , then  $S = \omega$ .

PROOF: From the definition of  $\omega$ .  $\square$

**Proposition 6.9.**

$$\forall n \in \omega. \forall x \in n. n \not\subseteq x$$

PROOF:

$\langle 1 \rangle 1$ .  $\forall x \in 0. 0 \not\subseteq x$

PROOF: Vacuous.

$\langle 1 \rangle 2$ . For any natural number  $n$ , if  $\forall x \in n. n \not\subseteq x$  then  $\forall x \in n^+. n^+ \not\subseteq x$ .

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $\forall x \in n. n \not\subseteq x$

$\langle 2 \rangle 3$ . LET:  $x \in n^+$

$\langle 2 \rangle 4$ . ASSUME: for a contradiction  $n^+ \subseteq x$

$\langle 2 \rangle 5$ .  $x \in n$  or  $x = n$

$\langle 2 \rangle 6$ . CASE:  $x \in n$

PROOF: Then we have  $n \subseteq n^+ \subseteq x$  contradicting  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 7$ . CASE:  $x = n$

PROOF: Then we have  $n \in n^+ \subseteq x = n$  and  $n \subseteq n$  contradicting  $\langle 2 \rangle 2$ .

$\square$

**Corollary 6.9.1.** *For any natural number  $n$  we have  $n \notin n$ .*

**Corollary 6.9.2.** *For any natural number  $n$  we have  $n \neq n^+$ .*

**Definition 6.10** (Transitive Set). A set  $E$  is a *transitive set* iff, whenever  $x \in y \in E$ , then  $x \in E$ .

**Proposition 6.11.** *Every natural number is a transitive set.*

PROOF:

$\langle 1 \rangle 1$ . 0 is a transitive set.

PROOF: Vacuously, if  $x \in y \in 0$  then  $x \in 0$ .

$\langle 1 \rangle 2$ . For any natural number  $n$ , if  $n$  is a transitive set, then  $n^+$  is a transitive set.

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $n$  is a transitive set.

$\langle 2 \rangle 3$ . LET:  $x \in y \in n^+$

$\langle 2 \rangle 4$ .  $y \in n$  or  $y = n$

$\langle 2 \rangle 5$ . CASE:  $y \in n$

$\langle 3 \rangle 1$ .  $x \in n$

PROOF:  $\langle 2 \rangle 2$ ,  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 5$ .

$\langle 3 \rangle 2$ .  $x \in n^+$

$\langle 2 \rangle 6$ . CASE:  $y = n$

$\langle 3 \rangle 1$ .  $x \in n$

PROOF:  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 6$

$\langle 3 \rangle 2$ .  $x \in n^+$

□

**Proposition 6.12.** *For any natural numbers  $m$  and  $n$ , if  $m^+ = n^+$  then  $m = n$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $m$  and  $n$  be natural numbers.

$\langle 1 \rangle 2$ . ASSUME:  $m^+ = n^+$

$\langle 1 \rangle 3$ .  $m \in m^+ = n^+$

$\langle 1 \rangle 4$ .  $m \in n$  or  $m = n$

$\langle 1 \rangle 5$ .  $n \in n^+ = m^+$

$\langle 1 \rangle 6$ .  $n \in m$  or  $n = m$

$\langle 1 \rangle 7$ . We cannot have  $m \in n$  and  $n \in m$

$\langle 2 \rangle 1$ . ASSUME: for a contradiction  $m \in n$  and  $n \in m$

$\langle 2 \rangle 2$ .  $m \in m$

PROOF: Since  $m$  is a transitive set (Proposition 6.11).

$\langle 2 \rangle 3$ . Q.E.D.

PROOF: This contradicts Proposition 6.9.

$\langle 1 \rangle 8$ .  $m = n$

□

**Theorem 6.13** (Recursion Theorem). *Let  $X$  be a set. Let  $a \in X$ . Let  $f : X \rightarrow X$ . There exists a function  $u : \omega \rightarrow X$  such that  $u(0) = a$  and, for all  $n \in \omega$ , we have  $u(n^+) = f(u(n))$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C} = \{A \in \mathcal{P}(\omega \times X) : (0, a) \in A \wedge \forall n \in \omega. \forall x \in X. (n, x) \in A \Rightarrow (n^+, f(x)) \in A\}$

$\langle 1 \rangle 2$ .  $\mathcal{C} \neq \emptyset$

PROOF:  $\omega \times X \in \mathcal{C}$

$\langle 1 \rangle 3$ . LET:  $u = \bigcap \mathcal{C}$

$\langle 1 \rangle 4$ .  $u \in \mathcal{C}$

$\langle 1 \rangle 5$ .  $u$  is a function.

$\langle 2 \rangle 1$ . LET:  $P(n)$  be the property:  $\forall x, y \in X. (n, x) \in u \wedge (n, y) \in u \Rightarrow x = y$

$\langle 2 \rangle 2$ .  $P(0)$

$\langle 3 \rangle 1$ .  $\forall x \in X. (0, x) \in u \Rightarrow x = a$

PROOF: If  $(0, x) \in u$  and  $x \neq a$  then  $u - \{(0, x)\} \in \mathcal{C}$  and so  $u - \{(0, x)\} \subseteq u$ , which is impossible.

$\langle 2 \rangle 3$ . For every natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .

$\langle 3 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 3 \rangle 2$ . ASSUME:  $P(n)$

$\langle 3 \rangle 3$ . LET:  $x, y \in X$

$\langle 3 \rangle 4$ . ASSUME:  $(n^+, x), (n^+, y) \in u$

$\langle 3 \rangle 5$ . PICK  $x', y' \in X$  such that  $(n, x') \in u$ ,  $(n, y') \in u$  and  $f(x') = x$  and  $f(y') = y$

PROOF: If no such  $x'$  exists then  $u - \{(n^+, x)\} \in \mathcal{C}$  and so  $u - \{(n^+, x)\} \subseteq u$  which is impossible. Similarly for  $y'$ .

$\langle 3 \rangle 6$ .  $x' = y'$

PROOF:  $\langle 3 \rangle 2$

$\langle 3 \rangle 7$ .  $x = y$

□

**Proposition 6.14.** *For any natural number  $n$ , either  $n = 0$  or there exists a natural number  $m$  such that  $n = m^+$ .*

PROOF: Easy induction on  $n$ . □

**Proposition 6.15.**  *$\omega$  is a transitive set.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the property  $\forall x \in n. x \in \omega$

$\langle 1 \rangle 2$ .  $P(0)$

PROOF: Vacuous.

$\langle 1 \rangle 3$ . For any natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$

$\langle 2 \rangle 3$ . LET:  $x \in n^+$

$\langle 2 \rangle 4$ .  $x \in n$  or  $x = n$

$\langle 2 \rangle 5$ . CASE:  $x \in n$

PROOF: Then  $x \in \omega$  by  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 6$ . CASE:  $x = n$

PROOF: Then  $x \in \omega$  by  $\langle 2 \rangle 1$ .

□

**Proposition 6.16.** *For any natural number  $n$  and any nonempty subset  $E \subseteq n$ , there exists  $k \in E$  such that  $\forall m \in E. k = m \vee k \in m$ .*

PROOF:

⟨1⟩1. LET:  $P(n)$  be the property: for any nonempty subset  $E \subseteq n$ , there exists  $k \in E$  such that  $\forall m \in E. k = m \vee k \in m$

⟨1⟩2.  $P(0)$

PROOF: Vacuous as there is no nonempty subset of 0.

⟨1⟩3. For any natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .

⟨2⟩1. LET:  $n$  be a natural number.

⟨2⟩2. ASSUME:  $P(n)$

⟨2⟩3. LET:  $E$  be a nonempty subset of  $n^+$

⟨2⟩4. CASE:  $E - \{n\} = \emptyset$

PROOF: Then  $E = \{n\}$  so take  $k = n$ .

⟨2⟩5. CASE:  $E - \{n\} \neq \emptyset$

⟨3⟩1. PICK  $k \in E - \{n\}$  such that  $\forall m \in E - \{n\}. k = m \vee k \in m$

PROOF: By ⟨2⟩2.

⟨3⟩2.  $\forall m \in E. k = m \vee k \in m$

PROOF: Since  $k \in n$ .

□

## Chapter 7

# Ordinal Numbers

**Definition 7.1** (Ordinal (Number)). An *ordinal (number)* is a well ordered set  $\alpha$  such that  $\forall \xi \in \alpha. s(\xi) = \xi$ .

Given ordinals  $\alpha, \beta$ , we write  $\alpha < \beta$  iff  $\alpha \in \beta$ .

**Proposition 7.2.** *Every natural number is an ordinal.*

PROOF: Easy.  $\square$

**Proposition 7.3.**  $\omega$  is an ordinal.

PROOF: Easy.  $\square$

**Proposition 7.4.** If  $\alpha$  is an ordinal number then so is  $\alpha^+$ .

PROOF: Easy.  $\square$

**Proposition 7.5.** Let  $\alpha$  be an ordinal and  $\eta, \xi \in \alpha$ . Then  $\eta < \xi$  if and only if  $\eta \in \xi$ .

PROOF: Easy.  $\square$

**Proposition 7.6.** Every ordinal is a transitive set.

PROOF: Easy.  $\square$

**Proposition 7.7.** Every element of an ordinal is an ordinal.

PROOF: Easy.  $\square$

**Proposition 7.8.** Similar ordinals are equal.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\alpha, \beta$  be ordinals.

$\langle 1 \rangle 2$ . LET:  $f : \alpha \cong \beta$  be a similarity.

PROVE:  $\forall \xi \in \alpha. f(\xi) = \xi$

$\langle 1 \rangle 3$ . LET:  $\xi \in \alpha$

$\langle 1 \rangle 4$ . ASSUME: as transfinite induction hypothesis  $\forall \eta < \xi. f(\eta) = \eta$   
 $\langle 1 \rangle 5$ .  $f(\xi) \subseteq \xi$   
 $\langle 2 \rangle 1$ . LET:  $\eta \in f(\xi)$   
 $\langle 2 \rangle 2$ . PICK  $\zeta \in \alpha$  such that  $f(\zeta) = \eta$   
 $\langle 2 \rangle 3$ .  $\zeta \in \xi$   
PROOF: Since  $f(\zeta) \in f(\xi)$  and  $f$  is a similarity.  
 $\langle 2 \rangle 4$ .  $f(\zeta) = \zeta$   
PROOF:  $\langle 1 \rangle 4$   
 $\langle 2 \rangle 5$ .  $\eta = \zeta$   
PROOF:  $\langle 2 \rangle 2, \langle 2 \rangle 4$   
 $\langle 2 \rangle 6$ .  $\eta \in \xi$   
PROOF:  $\langle 2 \rangle 3, \langle 2 \rangle 5$   
 $\langle 1 \rangle 6$ .  $\xi \subseteq f(\xi)$   
 $\langle 2 \rangle 1$ . LET:  $\eta \in \xi$   
 $\langle 2 \rangle 2$ .  $\eta = f(\eta) \in f(\xi)$   
 $\langle 1 \rangle 7$ .  $f(\xi) = \xi$   
 $\square$

**Proposition 7.9.** *Let  $\alpha$  and  $\beta$  be ordinals. Then the following are equivalent.*

1.  $\alpha \in \beta$
2.  $\alpha \subsetneq \beta$
3.  $\beta$  is a continuation of  $\alpha$ .

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 3$   
PROOF: If  $\alpha \in \beta$  then  $\alpha = s(\alpha)$ .  
 $\langle 1 \rangle 2$ .  $3 \Rightarrow 2$   
PROOF: Immediate from definitions.  
 $\langle 1 \rangle 3$ .  $2 \Rightarrow 1$   
 $\langle 2 \rangle 1$ . LET:  $\gamma$  be the least element of  $\beta$  such that  $\gamma \notin \alpha$   
 $\langle 2 \rangle 2$ .  $\alpha \subseteq \gamma$   
 $\langle 3 \rangle 1$ . LET:  $\eta \in \alpha$   
 $\langle 3 \rangle 2$ .  $\eta \subseteq \alpha$   
 $\langle 3 \rangle 3$ .  $\gamma \notin \eta$   
 $\langle 3 \rangle 4$ .  $\eta \in \gamma$  or  $\eta = \gamma$   
 $\langle 3 \rangle 5$ .  $\eta \neq \gamma$   
PROOF: Since  $\eta \in \alpha$  and  $\gamma \notin \alpha$ .  
 $\langle 3 \rangle 6$ .  $\eta \in \gamma$   
 $\langle 2 \rangle 3$ .  $\gamma \subseteq \alpha$   
PROOF: For all  $\eta \in \gamma$  we have  $\eta \in \alpha$  by leastness of  $\gamma$ .  
 $\langle 2 \rangle 4$ .  $\gamma = \alpha$   
 $\langle 2 \rangle 5$ .  $\alpha \in \beta$   
 $\square$

**Proposition 7.10.** *For any ordinal numbers  $\alpha$  and  $\beta$ , either  $\alpha = \beta$ , or  $\alpha < \beta$ , or  $\beta < \alpha$ .*

PROOF:

- ⟨1⟩1. Either  $\alpha = \beta$ , or  $\alpha$  is similar to an initial segment of  $\beta$ , or  $\beta$  is similar to an initial segment of  $\alpha$ .
- ⟨1⟩2. CASE:  $\alpha$  is similar to an initial segment of  $\beta$ .
  - ⟨2⟩1. PICK  $\eta \in \beta$  such that  $\alpha \sim s(\eta)$
  - ⟨2⟩2.  $\alpha \sim \eta$
  - ⟨2⟩3.  $\alpha = \eta$
  - PROOF: Proposition 7.8.
  - ⟨2⟩4.  $\alpha \in \beta$
- ⟨1⟩3. CASE:  $\beta$  is similar to an initial segment of  $\alpha$ .  
 PROOF: Then  $\beta \in \alpha$  similarly.

□

**Proposition 7.11.** *Every set of ordinals is well ordered by  $<$ .*

PROOF:

- ⟨1⟩1. LET:  $E$  be a set of ordinals.
- ⟨1⟩2. LET:  $A$  be a nonempty subset of  $E$ .
- ⟨1⟩3. PICK  $\alpha \in A$
- ⟨1⟩4. CASE:  $\alpha \cap A = \emptyset$   
 PROOF: Then  $\alpha$  is least in  $A$ .
- ⟨1⟩5. CASE:  $\alpha \cap A \neq \emptyset$   
 PROOF: Then  $\alpha \cap A$  has a least element, which is least in  $A$ .

□

**Definition 7.12** (Limit Ordinal). A *limit ordinal* is an ordinal number that is not 0 and not  $\alpha^+$  for any ordinal  $\alpha$ .

**Proposition 7.13.** *For any set  $E$  of ordinal numbers,  $\bigcup E$  is an ordinal and is the supremum of  $E$ .*

PROOF: Proposition 5.33. □

**Theorem 7.14** (Burali-Forti Paradox). *There is no set whose members are exactly the ordinal numbers.*

PROOF: For any set of ordinals  $E$ , we have  $(\bigcup E)^+$  is an ordinal that is not in  $E$ . □

**Theorem 7.15** (Counting Theorem). *Every well ordered set is similar to a unique ordinal.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a well ordered set.
- ⟨1⟩2. There exists an ordinal  $\alpha$  such that  $X \cong \alpha$ .
  - ⟨2⟩1. For all  $a \in X$ , there exists a unique ordinal  $\alpha$  such that  $s(a) \cong \alpha$ 
    - ⟨3⟩1. LET:  $a \in X$
    - ⟨3⟩2. ASSUME: as transfinite induction hypothesis that, for all  $b < a$ , there exists a unique ordinal  $\beta$  such that  $s(b) \cong \beta$



$\langle 3 \rangle 3$ . LET:  $\alpha = \{\beta : \beta \text{ is an ordinal} \wedge \exists b < a. s(b) \cong \beta\}$   
 PROOF: This is a set by the Axiom of Substitution.  
 $\langle 3 \rangle 4$ .  $\alpha$  is an ordinal  
 $\langle 4 \rangle 1$ . LET:  $\gamma \in \beta \in \alpha$   
 $\langle 4 \rangle 2$ . PICK  $b < a$  and  $f : s(b) \cong \beta$   
 $\langle 4 \rangle 3$ . PICK  $c < b$  such that  $f(c) = \gamma$   
 $\langle 4 \rangle 4$ .  $f \upharpoonright s(c) : s(c) \cong \gamma$   
 $\langle 3 \rangle 5$ .  $s(a) \cong \alpha$   
 PROOF: The function  $f : s(a) \rightarrow \alpha$  defined by  $f(b)$  is the ordinal such that  $s(b) \cong f(b)$  is a similarity.  
 $\langle 3 \rangle 6$ .  $\alpha$  is unique.  
 PROOF: Proposition 7.8.  
 $\langle 2 \rangle 2$ . LET:  $\alpha = \{\beta : \beta \text{ is an ordinal} \wedge \exists a \in X. s(a) \cong \beta\}$   
 PROOF: This is a set by the Axiom of Substitution.  
 $\langle 2 \rangle 3$ .  $\alpha$  is an ordinal.  
 PROOF: Similar.  
 $\langle 2 \rangle 4$ .  $X \cong \alpha$   
 PROOF: Similar.  
 $\langle 1 \rangle 3$ . For any ordinals  $\alpha$  and  $\beta$ , if  $X \cong \alpha$  and  $X \cong \beta$  then  $\alpha = \beta$ .  
 PROOF: Proposition 7.8.  
 $\square$

## 7.1 Order on the Natural Numbers

**Proposition 7.16.** *For natural numbers  $m, n$  and  $k$ , if  $m < n$  then  $m + k < n + k$ .*

PROOF:  
 $\langle 1 \rangle 1$ . LET:  $m, n \in \omega$   
 $\langle 1 \rangle 2$ . ASSUME:  $m < n$   
 $\langle 1 \rangle 3$ .  $m + 0 < n + 0$   
 $\langle 1 \rangle 4$ .  $\forall k \in \omega. m + k < n + k \Rightarrow m + k^+ < n + k^+$   
 PROOF: By Proposition 6.7.  
 $\square$

**Proposition 7.17.** *For natural numbers  $m, n$  and  $k$ , if  $m < n$  and  $k \neq 0$  then  $mk < nk$ .*

PROOF:  
 $\langle 1 \rangle 1$ . LET:  $m, n \in \omega$   
 $\langle 1 \rangle 2$ . ASSUME:  $m < n$   
 $\langle 1 \rangle 3$ .  $m1 < n1$   
 $\langle 1 \rangle 4$ . For all  $k \in \omega$ , if  $k \neq 0$  and  $mk < nk$  then  $m(k + 1) < n(k + 1)$

PROOF:

$$\begin{aligned}
m(k+1) &= mk + m \\
&< mk + n && \text{(Proposition 7.16)} \\
&< nk + n && \text{(Proposition 7.16)} \\
&= n(k+1)
\end{aligned}$$

□

**Proposition 7.18.** *Let  $n$  be a natural number. Let  $X$  be a proper subset of  $n$ . Then there exists  $m < n$  such that  $X \sim m$ .*

PROOF:

⟨1⟩1. LET:  $P(n)$  be the property: for every proper subset  $X \subsetneq n$ , there exists  $m < n$  such that  $X \sim m$ .

⟨1⟩2.  $P(0)$

PROOF: Vacuous.

⟨1⟩3.  $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

⟨2⟩1. LET:  $n \in \omega$

⟨2⟩2. ASSUME:  $P(n)$

⟨2⟩3. LET:  $X$  be a proper subset of  $n+1$

⟨2⟩4. CASE:  $X - \{n\} = n$

PROOF: Then  $X = n$  so  $X \sim n < n+1$ .

⟨2⟩5. CASE:  $X - \{n\} \subsetneq n$

⟨3⟩1. PICK  $m < n$  such that  $X - \{n\} \sim m$

⟨3⟩2.  $X \sim m$  or  $X \sim m+1$

PROOF: If  $n \in X$  then  $X \sim m+1$ . If  $n \notin X$  then  $X \sim m$ .

□

**Proposition 7.19.** *For every natural number  $n$ , we have  $n$  is not equivalent to a proper subset of  $n$ .*

PROOF:

⟨1⟩1. LET:  $P(n)$  be the property: every one-to-one function  $n \rightarrow n$  is onto.

⟨1⟩2.  $P(0)$

PROOF: The only function  $0 \rightarrow 0$  is  $\emptyset$ .

⟨1⟩3.  $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

⟨2⟩1. LET:  $n \in \omega$

⟨2⟩2. ASSUME:  $P(n)$

⟨2⟩3. ASSUME:  $f : n+1 \rightarrow n+1$  is one-to-one.

⟨2⟩4. LET:  $g : n \rightarrow n$  be the function

$$g(k) = \begin{cases} f(k) & \text{if } f(k) < n \\ f(n) & \text{if } f(k) = n \end{cases}$$

PROOF: If  $k < n$  and  $f(k) = n$  then  $f(n) < n$  since  $f$  is one-to-one.

⟨2⟩5.  $g$  is one-to-one.

⟨3⟩1. LET:  $k, l < n$

⟨3⟩2. ASSUME:  $g(k) = g(l)$

⟨3⟩3. CASE:  $f(k) < n$  and  $f(l) < n$

PROOF: Then  $f(k) = g(k) = g(l) = f(l)$  so  $k = l$  since  $f$  is one-to-one.

⟨3⟩4. CASE:  $f(k) < n$  and  $f(l) = n$   
PROOF: Then  $f(k) = g(k) = g(l) = f(n)$  contradicting the fact that  $f$  is one-to-one.

⟨3⟩5. CASE:  $f(k) = n$  and  $f(l) < n$   
PROOF: Similar.

⟨3⟩6. CASE:  $f(k) = n$  and  $f(l) = n$   
PROOF: Then  $k = l$  since  $f$  is one-to-one.

⟨2⟩6.  $g$  maps  $n$  onto  $n$ .  
PROOF: ⟨2⟩2

⟨2⟩7.  $f$  maps  $n + 1$  onto  $n + 1$ .

⟨3⟩1. LET:  $l < n + 1$

⟨3⟩2. CASE:  $l < n$   
⟨4⟩1. PICK  $k < n$  such that  $g(k) = l$   
⟨4⟩2.  $f(k) = l$  or  $f(n) = l$

⟨3⟩3. CASE:  $l = n$   
⟨4⟩1. CASE:  $f(n) = n$   
PROOF: Then  $l \in \text{ran } f$  as required.

⟨4⟩2. CASE:  $f(n) < n$   
⟨5⟩1. PICK  $k < n$  such that  $g(k) = f(n)$   
⟨5⟩2.  $f(k) = n$

□

**Corollary 7.19.1.** *Equivalent natural numbers are equal.*

**Definition 7.20** (Lexicographical Order). The *lexicographical* order on  $\omega \times \omega$  is the relation  $S$  defined by  $(a, b)S(x, y)$  iff  $a < x$  or  $(a = x \text{ and } b < y)$ .

**Proposition 7.21.** *The lexicographical order is a well ordering on  $\omega \times \omega$ .*

PROOF: Easy. □

## 7.2 Finite Sets

**Definition 7.22** (Finite). A set is *finite* iff it is equivalent to a natural number; otherwise, it is *infinite*.

**Proposition 7.23.** *No finite set is equivalent to one of its proper subsets.*

PROOF: From Proposition 7.19. □

**Proposition 7.24.**  *$\omega$  is infinite.*

PROOF: Since the function that maps  $n$  to  $n + 1$  is a one-to-one correspondence between  $\omega$  and  $\omega - \{0\}$ . □

**Proposition 7.25.** *Every subset of a finite set is finite.*

PROOF: Proposition 7.18. □

**Definition 7.26** (Number of Elements). For any finite set  $E$ , the *number of elements* in  $E$ ,  $\sharp(E)$ , is the unique natural number such that  $E \sim \sharp(E)$ .

**Proposition 7.27.** *Let  $E$  and  $F$  be finite sets. If  $E \subseteq F$  then  $\sharp(E) \leq \sharp(F)$ .*

PROOF: Proposition 7.18.  $\square$

**Proposition 7.28.** *Let  $E$  and  $F$  be disjoint finite sets. Then  $E \cup F$  is finite and  $\sharp(E \cup F) = \sharp(E) + \sharp(F)$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the statement:  $n \in \omega$  and for any  $m \in \omega$ , if  $E \sim m$ ,  $F \sim n$  and  $E \cap F = \emptyset$ , then  $E \cup F \sim m + n$

$\langle 1 \rangle 2$ .  $P(0)$

$\langle 2 \rangle 1$ . LET:  $m \in \omega$

$\langle 2 \rangle 2$ . LET:  $E \sim m$  and  $F \sim 0$

$\langle 2 \rangle 3$ .  $F = \emptyset$

$\langle 2 \rangle 4$ .  $E \cup F = E \sim m = m + 0$

$\langle 1 \rangle 3$ .  $\forall n \in \omega. P(n) \Rightarrow P(n + 1)$

$\langle 2 \rangle 1$ . LET:  $n \in \omega$

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$

$\langle 2 \rangle 3$ . LET:  $m \in \omega$

$\langle 2 \rangle 4$ . LET:  $E \sim m$  and  $F \sim n + 1$

$\langle 2 \rangle 5$ . ASSUME:  $E \cap F = \emptyset$

$\langle 2 \rangle 6$ . PICK  $f \in F$

$\langle 2 \rangle 7$ .  $F - \{f\} \sim n$

$\langle 2 \rangle 8$ .  $E \cap (F - \{f\}) = \emptyset$

$\langle 2 \rangle 9$ .  $E \cup (F - \{f\}) \sim m + n$

PROOF:  $\langle 2 \rangle 2$

$\langle 2 \rangle 10$ .  $E \cup F \sim m + n + 1$

$\square$

**Corollary 7.28.1.** *The union of two finite sets is finite.*

PROOF: Since, if  $E$  and  $F$  are finite, then  $E \cup F = (E - F) \cup (E \cap F) \cup (F - E)$  and these are finite and disjoint.  $\square$

**Proposition 7.29.** *If  $E$  and  $F$  are finite sets then  $E \times F$  is finite and  $\sharp(E \times F) = \sharp(E)\sharp(F)$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the statement:  $n \in \omega$  and for all  $m \in \omega$ , if  $E \sim m$  and  $F \sim n$  then  $E \times F \sim mn$

$\langle 1 \rangle 2$ .  $P(0)$

PROOF: If  $F \sim 0$  then  $F = \emptyset$  so  $E \times F = \emptyset \sim 0$ .

$\langle 1 \rangle 3$ .  $\forall n \in \omega. P(n) \Rightarrow P(n + 1)$

$\langle 2 \rangle 1$ . LET:  $n \in \omega$

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$

$\langle 2 \rangle 3$ . LET:  $m \in \omega$

- ⟨2⟩4. ASSUME:  $E \sim m$  and  $F \sim n + 1$
- ⟨2⟩5. PICK  $f \in F$
- ⟨2⟩6.  $F - \{f\} \sim n$
- ⟨2⟩7.  $E \times (F - \{f\}) \sim mn$
- ⟨2⟩8.  $E \times F = (E \times (F - \{f\})) \cup (E \times \{f\})$
- ⟨2⟩9.  $E \times \{f\} \sim m$
- ⟨2⟩10.  $E \times F \sim mn + m$

PROOF: Proposition 7.28.

□

**Proposition 7.30.** *For any finite sets  $E$  and  $F$ , we have  $E^F$  is finite and  $\sharp(E^F) = \sharp(E)^{\sharp(F)}$ .*

PROOF:

- ⟨1⟩1. LET:  $P(n)$  be the property:  $n \in \omega$  and for all  $m \in \omega$ , if  $E \sim m$  and  $F \sim n$  then  $E^F \sim m^n$
- ⟨1⟩2.  $P(0)$   
PROOF: Since  $E^\emptyset = \{\emptyset\} \sim 1$
- ⟨1⟩3.  $\forall n \in \omega. P(n) \Rightarrow P(n + 1)$ 
  - ⟨2⟩1. LET:  $n \in \omega$
  - ⟨2⟩2. ASSUME:  $P(n)$
  - ⟨2⟩3. LET:  $m \in \omega$
  - ⟨2⟩4. LET:  $E \sim m$  and  $F \sim n + 1$
  - ⟨2⟩5. PICK  $f \in F$
  - ⟨2⟩6.  $F - \{f\} \sim n$
  - ⟨2⟩7. LET:  $\phi : E^F \rightarrow E^{F - \{f\}} \times E$  be the function  $\phi(g) = (g \upharpoonright (F - \{f\}), g(f))$
  - ⟨2⟩8.  $\phi$  is a one-to-one correspondence
  - ⟨2⟩9.  $\sharp(E^F) = m^{n+1}$

PROOF:

$$\begin{aligned}
 \sharp(E^F) &= \sharp(E^{F - \{f\}} \times E) \\
 &= \sharp(E^{F - \{f\}}) \sharp(E) && \text{(Proposition 7.29)} \\
 &= m^n m && (\langle 2 \rangle 2, \langle 2 \rangle 4) \\
 &= m^{n+1}
 \end{aligned}$$

□

**Corollary 7.30.1.** *If  $E$  is finite then  $\mathcal{P}E$  is finite and  $\sharp(\mathcal{P}E) = 2^{\sharp(E)}$ .*

**Proposition 7.31.** *The union of a finite set of finite sets is finite.*

PROOF:

- ⟨1⟩1. LET:  $P(n)$  be the property: for any set  $E$ , if  $E \sim n$  and every element of  $E$  is finite, then  $\bigcup E$  is finite.
- ⟨1⟩2.  $P(0)$   
PROOF: Since  $\bigcup \emptyset = \emptyset$  is finite.
- ⟨1⟩3.  $\forall n \in \omega. P(n) \Rightarrow P(n + 1)$ 
  - ⟨2⟩1. LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$   
 $\langle 2 \rangle 3$ . LET:  $E \sim n + 1$   
 $\langle 2 \rangle 4$ . PICK  $X \in E$   
 $\langle 2 \rangle 5$ .  $E - \{X\} \sim n$   
 $\langle 2 \rangle 6$ .  $\bigcup(E - \{X\})$  is finite.  
 PROOF:  $\langle 2 \rangle 2$   
 $\langle 2 \rangle 7$ .  $\bigcup E = \bigcup(E - \{X\}) \cup X$   
 $\langle 2 \rangle 8$ .  $\bigcup E$  is finite.  
 PROOF: Corollary 7.28.1.

□

**Proposition 7.32.** *Every nonempty finite set of natural numbers has a greatest element.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the property: for every  $E \subseteq \mathbb{N}$ , if  $E \sim n$  then  $E$  has a greatest element.  
 $\langle 1 \rangle 2$ .  $P(1)$   
 PROOF: Since  $k$  is the greatest element of  $\{k\}$ .  
 $\langle 1 \rangle 3$ .  $\forall n \geq 1. P(n) \Rightarrow P(n + 1)$   
 $\langle 2 \rangle 1$ . LET:  $n \geq 1$   
 $\langle 2 \rangle 2$ . ASSUME:  $P(n)$   
 $\langle 2 \rangle 3$ . ASSUME:  $E \subseteq \omega$  and  $E \sim n + 1$   
 $\langle 2 \rangle 4$ . PICK  $k \in E$   
 $\langle 2 \rangle 5$ . LET:  $l$  be the greatest element of  $E - \{k\}$   
 $\langle 2 \rangle 6$ . Either  $k$  or  $l$  is greatest in  $E$ .

□

**Proposition 7.33.** *Every infinite set has a subset equivalent to  $\omega$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be an infinite set.  
 $\langle 1 \rangle 2$ . PICK a choice function  $f$  for  $X$ .  
 $\langle 1 \rangle 3$ . LET:  $\mathcal{C}$  be the set of all finite subsets of  $X$ .  
 $\langle 1 \rangle 4$ . For all  $A \in \mathcal{C}$  we have  $X - A \in \text{dom } f$ .  
 PROOF: For all  $A \in \mathcal{C}$  we have  $X - A \neq \emptyset$ .  
 $\langle 1 \rangle 5$ . LET:  $U : \omega \rightarrow \mathcal{C}$  be the function defined recursively by  $U(0) = \emptyset$  and  $U(n + 1) = U(n) \cup \{f(X - U(n))\}$  for all  $n \in \omega$ .  
 $\langle 1 \rangle 6$ . LET:  $v : \omega \rightarrow X$  be the function  $v(n) = f(X - U(n))$   
 PROVE:  $v$  is one-to-one.  
 $\langle 1 \rangle 7$ .  $\forall n \in \omega. v(n) \notin U(n)$   
 PROOF: Since  $v(n) = f(X - U(n)) \in X - U(n)$ .  
 $\langle 1 \rangle 8$ .  $\forall n \in \omega. v(n) \in U(n + 1)$   
 $\langle 1 \rangle 9$ .  $\forall m, n \in \omega. n \leq m \Rightarrow U(n) \subseteq U(m)$   
 PROOF: Since  $U(n) \subseteq U(n + 1)$  for all  $n$ .  
 $\langle 1 \rangle 10$ .  $\forall m, n \in \omega. n < m \Rightarrow v(n) \neq v(m)$   
 PROOF: Since  $v(n) \in U(m)$  and  $v(m) \notin U(m)$ .

□

**Corollary 7.33.1.** *A set is infinite if and only if it is equivalent to a proper subset.*

## 7.3 Ordinal Arithmetic

**Definition 7.34** (Addition). Let  $I$  be a well ordered set and  $(\alpha_i)_{i \in I}$  be a sequence of ordinals. Choose a well ordered set  $A_i$  such that  $A_i \cong \alpha_i$  for each  $i \in I$ , and assume the sets  $A_i$  are pairwise disjoint. The *sum*  $\sum_{i \in I} \alpha_i$  is the ordinal of the well ordered set  $\bigcup_{i \in I} A_i$ , where:

- for  $x, y \in A_i$ , we have  $x <_{\bigcup_{i \in I} A_i} y$  if and only if  $x <_{A_i} y$
- for  $x \in A_i$  and  $y \in A_j$  with  $i \neq j$ , we have  $x <_{\bigcup_{i \in I} A_i} y$  iff  $i <_I j$

We write  $\alpha + \beta$  for  $\sum_{i \in 2} \gamma_i$  where  $\gamma_0 = \alpha$  and  $\gamma_1 = \beta$ .

**Proposition 7.35.**

$$\begin{aligned}\alpha + 0 &= \alpha \\ 0 + \alpha &= \alpha \\ \alpha + 1 &= \alpha^+ \\ \alpha + (\beta + \gamma) &= (\alpha + \beta) + \gamma\end{aligned}$$

PROOF: Easy. □

**Proposition 7.36.** *For any ordinals  $\alpha$  and  $\beta$ , we have  $\alpha < \beta$  if and only if there exists  $\gamma \neq 0$  such that  $\beta = \alpha + \gamma$ .*

PROOF: Easy. □

**Proposition 7.37.**

$$1 + \omega = \omega$$

PROOF: Easy. □

**Definition 7.38** (Multiplication). Given ordinals  $\alpha$  and  $\beta$ , the *product*  $\alpha\beta$  is the ordinal of  $\alpha \times \beta$  under the *reverse lexicographic order*:  $(a, b) < (c, d)$  iff  $b < d$  or  $(b = d \text{ and } a < c)$ .

**Proposition 7.39.**

$$\begin{aligned}\alpha 0 &= 0 \\ 0 \alpha &= 0 \\ \alpha 1 &= \alpha \\ 1 \alpha &= \alpha \\ \alpha(\beta \gamma) &= (\alpha \beta) \gamma \\ \alpha(\beta + \gamma) &= \alpha \beta + \alpha \gamma\end{aligned}$$

PROOF: Easy.  $\square$

**Proposition 7.40.** *For ordinals  $\alpha$  and  $\beta$ , if  $\alpha\beta = 0$  then  $\alpha = 0$  or  $\beta = 0$ .*

PROOF: Easy.  $\square$

**Example 7.41.** The commutative law fails:

$$2\omega = \omega \neq \omega 2$$

PROOF: Easy.  $\square$

**Example 7.42.** The right distributive law fails:

$$(1 + 1)\omega = \omega \neq 1\omega + 1\omega = \omega 2$$

**Definition 7.43** (Exponentiation). Given ordinals  $\alpha$  and  $\beta$ , define the ordinal  $\alpha^\beta$  by

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta \alpha \\ \alpha^\lambda &= \bigcup_{\beta < \lambda} \alpha^\beta \quad (\lambda \text{ a limit ordinal}) \end{aligned}$$

**Proposition 7.44.**

$$\begin{aligned} 0^\alpha &= 0 & (\alpha \geq 1) \\ 1^\gamma &= 1 \\ \alpha^{\beta+\gamma} &= \alpha^\beta \alpha^\gamma \\ \alpha^{\beta\gamma} &= (\alpha^\beta)^\gamma \end{aligned}$$

PROOF: Easy.  $\square$

**Example 7.45.**  $(\alpha\beta)^\gamma$  is different from  $\alpha^\gamma\beta^\gamma$  in general:

$$(2 \cdot 2)^\omega = \omega \neq 2^\omega 2^\omega = \omega^2 .$$

## 7.4 Arithmetic on the Natural Numbers

**Proposition 7.46.** *For all  $m, n \in \omega$ , we have*

$$m + n = n + m .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(m)$  be the property  $\forall n \in \omega. m + n = n + m$

$\langle 1 \rangle 2$ .  $P(0)$

$\langle 2 \rangle 1$ . LET:  $Q(n)$  be the property  $0 + n = n + 0$

$\langle 2 \rangle 2$ .  $Q(0)$



PROOF: Trivial.

$\langle 2 \rangle 3$ .  $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$ . LET:  $n \in \omega$

$\langle 3 \rangle 2$ . ASSUME:  $Q(n)$

$\langle 3 \rangle 3$ .  $0 + n^+ = n^+ + 0$

PROOF:

$$\begin{aligned} 0 + n^+ &= (0 + n)^+ \\ &= (n + 0)^+ & (\langle 3 \rangle 2) \\ &= n^+ \\ &= n^+ + 0 \end{aligned}$$

$\langle 1 \rangle 3$ .  $\forall m \in \omega. P(m) \Rightarrow P(m^+)$

$\langle 2 \rangle 1$ . LET:  $m \in \omega$

$\langle 2 \rangle 2$ . ASSUME:  $P(m)$

$\langle 2 \rangle 3$ . LET:  $Q(n)$  be the property  $m^+ + n = n + m^+$

$\langle 2 \rangle 4$ .  $Q(0)$

PROOF:  $\langle 1 \rangle 2$

$\langle 2 \rangle 5$ .  $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$ . LET:  $n \in \omega$

$\langle 3 \rangle 2$ . ASSUME:  $Q(n)$

$\langle 3 \rangle 3$ .  $Q(n^+)$

PROOF:

$$\begin{aligned} m^+ + n^+ &= (m^+ + n)^+ \\ &= (n + m^+)^+ & (\langle 3 \rangle 2) \\ &= (n + m)^{++} \\ &= (m + n)^{++} & (\langle 2 \rangle 2) \\ &= (m + n^+)^+ \\ &= (n^+ + m)^+ & (\langle 2 \rangle 2) \\ &= n^+ + m^+ \end{aligned}$$

□

**Proposition 7.47.** *For all  $m, n \in \omega$ , we have*

$$mn = nm \text{ .}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(m)$  be the statement  $\forall n \in \omega. mn = nm$

$\langle 1 \rangle 2$ .  $P(0)$

$\langle 2 \rangle 1$ . LET:  $Q(n)$  be the statement  $0n = n0$

$\langle 2 \rangle 2$ .  $Q(0)$

PROOF: Trivial.

$\langle 2 \rangle 3$ .  $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$ . LET:  $n \in \omega$

$\langle 3 \rangle 2$ . ASSUME:  $Q(n)$

$\langle 3 \rangle 3. Q(n^+)$

PROOF:

$$\begin{aligned}
 0n^+ &= 0n + 0 \\
 &= 0n \\
 &= n0 & (\langle 3 \rangle 2) \\
 &= 0 \\
 &= n^+0
 \end{aligned}$$

$\langle 1 \rangle 3. \forall m \in \omega. P(m) \Rightarrow P(m^+)$

$\langle 2 \rangle 1. \text{LET: } m \in \omega$

$\langle 2 \rangle 2. \text{ASSUME: } P(m)$

$\langle 2 \rangle 3. \text{LET: } Q(n) \text{ be the statement } m^+n = nm^+$

$\langle 2 \rangle 4. Q(0)$

PROOF:  $\langle 1 \rangle 2$

$\langle 2 \rangle 5. \forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1. \text{LET: } n \in \omega$

$\langle 3 \rangle 2. \text{ASSUME: } Q(n)$

$\langle 3 \rangle 3. Q(n^+)$

PROOF:

$$\begin{aligned}
 m^+n^+ &= m^+n + m^+ \\
 &= (m^+n + m)^+ \\
 &= (nm^+ + m)^+ & (\langle 3 \rangle 2) \\
 &= (nm + n + m)^+ \\
 &= (mn + m + n)^+ & (\langle 2 \rangle 2, \text{Proposition ??, Proposition 7.46}) \\
 &= (mn^+ + n)^+ \\
 &= (n^+m + n)^+ & (\langle 2 \rangle 2) \\
 &= n^+m + n^+ \\
 &= n^+m^+
 \end{aligned}$$

□