

# Mathematics

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Part I

Category Theory



# Chapter 1

## Foundations

This is a placeholder — I am not sure what foundation I want to use for this project yet. I will try to work in a way which is foundation-independent. What I do could be formalized in ZFC, ETCS, or some other system. I will assume the usual set theoretic constructions as needed. Sets will be defined up to bijection only.





# Chapter 2

## Categories

**Definition 2.1** (Category). A *category*  $\mathcal{C}$  consists of:

- A class  $|\mathcal{C}|$  of *objects*. We write  $A \in \mathcal{C}$  for  $A \in |\mathcal{C}|$ .
- For any objects  $A, B$ , a set  $\mathcal{C}[A, B]$  of *morphisms* from  $A$  to  $B$ . We write  $f : A \rightarrow B$  for  $f \in \mathcal{C}[A, B]$ .
- For any object  $A$ , a morphism  $\text{id}_A : A \rightarrow A$ , the *identity* morphism on  $A$ .
- For any morphisms  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , a morphism  $g \circ f : A \rightarrow C$ , the *composite* of  $f$  and  $g$ .

such that:

**Associativity** Given  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h : C \rightarrow D$ , we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

**Left Unit Law** For any morphism  $f : A \rightarrow B$ , we have  $\text{id}_B \circ f = f$ .

**Right Unit Law** For any morphism  $f : A \rightarrow B$ , we have  $f \circ \text{id}_A = f$ .

**Proposition 2.2.** *The identity morphism on an object is unique.*

PROOF: If  $i$  and  $j$  are identity morphisms on  $A$  then  $i = i \circ j = j$ .  $\square$

**Example 2.3** (Category of Sets). The *category of sets* **Set** has objects all sets and morphisms all functions.

**Definition 2.4** (Endomorphism). In a category  $\mathcal{C}$ , an *endomorphism* on an object  $A$  is a morphism  $A \rightarrow A$ . We write  $\text{End}_{\mathcal{C}}(A)$  for  $\mathcal{C}[A, A]$ .

**Definition 2.5** (Opposite Category). For any category  $\mathcal{C}$ , the *opposite* category  $\mathcal{C}^{\text{op}}$  is the category with the same objects as  $\mathcal{C}$  and

$$\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$$

## 2.1 Preorders

**Definition 2.6** (Preorder). A *preorder* on a set  $A$  is a relation  $\leq$  on  $A$  that is reflexive and transitive.

A *preordered set* is a pair  $(A, \leq)$  such that  $\leq$  is a preorder on  $A$ . We usually write  $A$  for the preordered set  $(A, \leq)$ .

We identify any preordered set  $A$  with the category whose objects are the elements of  $A$ , with one morphism  $a \rightarrow b$  iff  $a \leq b$ , and no morphism  $a \rightarrow b$  otherwise.

**Example 2.7.** For any ordinal  $\alpha$ , let  $\alpha$  be the preorder  $\{\beta : \beta < \alpha\}$  under  $\leq$ .

**Definition 2.8** (Discrete Preorder). We identify any set  $A$  with the *discrete* preorder  $(A, =)$ .

## 2.2 Monomorphisms and Epimorphisms

**Definition 2.9** (Monomorphism). In a category, let  $f : A \rightarrow B$ . Then  $f$  is a *monomorphism* or *monic* iff, for every object  $X$  and morphism  $x, y : X \rightarrow A$ , if  $fx = fy$  then  $x = y$ .

**Definition 2.10** (Epimorphism). In a category, let  $f : A \rightarrow B$ . Then  $f$  is a *epimorphism* or *epi* iff, for every object  $X$  and morphism  $x, y : B \rightarrow X$ , if  $xf = yf$  then  $x = y$ .

**Proposition 2.11.** *The composite of two monomorphism is monic.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be monic.

$\langle 1 \rangle 2$ . LET:  $x, y : X \rightarrow A$

$\langle 1 \rangle 3$ . ASSUME:  $g \circ f \circ x = g \circ f \circ y$

$\langle 1 \rangle 4$ .  $f \circ x = f \circ y$

$\langle 1 \rangle 5$ .  $x = y$

□

**Proposition 2.12.** *The composite of two epimorphisms is epi.*

PROOF: Dual. □

**Proposition 2.13.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $g \circ f$  is monic then  $f$  is monic.*

PROOF: If  $f \circ x = f \circ y$  then  $g \circ f \circ x = g \circ f \circ y$  and so  $x = y$ . □

**Proposition 2.14.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $g \circ f$  is epi then  $g$  is epi.*

PROOF: Dual. □

**Proposition 2.15.** *A function is a monomorphism in **Set** iff it is injective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$

$\langle 1 \rangle 2$ . If  $f$  is monic then  $f$  is injective.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is monic.

$\langle 2 \rangle 2$ . LET:  $x, y \in A$

$\langle 2 \rangle 3$ . ASSUME:  $f(x) = f(y)$

$\langle 2 \rangle 4$ . LET:  $\bar{x}, \bar{y} : 1 \rightarrow A$  be the functions such that  $\bar{x}(*) = x$  and  $\bar{y}(*) = y$

$\langle 2 \rangle 5$ .  $f \circ \bar{x} = f \circ \bar{y}$

$\langle 2 \rangle 6$ .  $\bar{x} = \bar{y}$

PROOF: By  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 7$ .  $x = y$

$\langle 1 \rangle 3$ . If  $f$  is injective then  $f$  is monic.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is injective.

$\langle 2 \rangle 2$ . LET:  $X$  be a set and  $x, y : X \rightarrow A$ .

$\langle 2 \rangle 3$ . ASSUME:  $f \circ x = f \circ y$

PROVE:  $x = y$

$\langle 2 \rangle 4$ . LET:  $t \in X$

PROVE:  $x(t) = y(t)$

$\langle 2 \rangle 5$ .  $f(x(t)) = f(y(t))$

$\langle 2 \rangle 6$ .  $x(t) = y(t)$

PROOF: By  $\langle 2 \rangle 1$ .

□

**Proposition 2.16.** *A function is an epimorphism in **Set** iff it is surjective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$

$\langle 1 \rangle 2$ . If  $f$  is an epimorphism then  $f$  is surjective.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is an epimorphism.

$\langle 2 \rangle 2$ . LET:  $b \in B$

$\langle 2 \rangle 3$ . LET:  $x, y : B \rightarrow 2$  be defined by  $x(b) = 1$  and  $x(t) = 0$  for all other  $t \in B$ ,  $y(t) = 0$  for all  $t \in B$ .

$\langle 2 \rangle 4$ .  $x \neq y$

$\langle 2 \rangle 5$ .  $x \circ f \neq y \circ f$

$\langle 2 \rangle 6$ . There exists  $a \in A$  such that  $f(a) = b$ .

$\langle 1 \rangle 3$ . If  $f$  is surjective then  $f$  is an epimorphism.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is surjective.

$\langle 2 \rangle 2$ . LET:  $x, y : B \rightarrow X$

$\langle 2 \rangle 3$ . ASSUME:  $x \circ f = y \circ f$

PROVE:  $x = y$

$\langle 2 \rangle 4$ . LET:  $b \in B$

PROVE:  $x(b) = y(b)$

$\langle 2 \rangle 5$ . PICK  $a \in A$  such that  $f(a) = b$

$\langle 2 \rangle 6$ .  $x(f(a)) = y(f(a))$

$\langle 2 \rangle 7$ .  $x(b) = y(b)$

□

**Proposition 2.17.** *In a preorder, every morphism is monic and epi.*

PROOF: Immediate from definitions.  $\square$

## 2.3 Sections and Retractions

**Definition 2.18** (Section, Retraction). In a category, let  $r : A \rightarrow B$  and  $s : B \rightarrow A$ . Then  $r$  is a *retraction* of  $s$ , and  $s$  is a *section* of  $r$ , iff  $r \circ s = \text{id}_B$ .

**Proposition 2.19.** *Every identity morphism is a section and retraction of itself.*

PROOF: Immediate from definitions.  $\square$

**Proposition 2.20.** *Let  $r, r' : A \rightarrow B$  and  $s : B \rightarrow A$ . If  $r$  is a retraction of  $s$  and  $r'$  is a section of  $s$  then  $r = r'$ .*

PROOF:

$$\begin{aligned} r &= r \circ \text{id}_A \\ &= r \circ s \circ r' \\ &= \text{id}_B \circ r' \\ &= r' \end{aligned} \quad \square$$

**Proposition 2.21.** *Let  $r_1 : A \rightarrow B$ ,  $r_2 : B \rightarrow C$ ,  $s_1 : B \rightarrow A$  and  $s_2 : C \rightarrow B$ . If  $r_1$  is a retraction of  $s_1$  and  $r_2$  is a retraction of  $s_2$  then  $r_2 \circ r_1$  is a retraction of  $s_1 \circ s_2$ .*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= \text{id}_C \end{aligned} \quad \square$$

**Proposition 2.22.** *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $s : A \rightarrow B$  be a section of  $r : B \rightarrow A$ .

$\langle 1 \rangle 2$ . LET:  $x, y : X \rightarrow A$  satisfy  $sx = sy$ .

$\langle 1 \rangle 3$ .  $rsx = rsy$

$\langle 1 \rangle 4$ .  $x = y$

$\square$

**Proposition 2.23.** *Every retraction is epi.*

PROOF: Dual.  $\square$

**Proposition 2.24.** *In Set, every epimorphism has a retraction.*

PROOF: By the Axiom of Choice.  $\square$

**Example 2.25.** It is not true in general that every monomorphism in any category has a section. nor that every epimorphism in any category has a retraction.

In the category **2**, the morphism  $0 \leq 1$  is monic and epi but has no retraction or section.

## 2.4 Isomorphisms

**Definition 2.26** (Isomorphism). In a category  $\mathcal{C}$ , a morphism  $f : A \rightarrow B$  is an *isomorphism*, denoted  $f : A \cong B$ , iff there exists a morphism  $f^{-1} : B \rightarrow A$ , the *inverse* of  $f$ , such that  $f^{-1} \circ f = \text{id}_A$  and  $f \circ f^{-1} = \text{id}_B$ .

An *automorphism* on an object  $A$  is an isomorphism between  $A$  and itself. We write  $\text{Aut}_{\mathcal{C}}(A)$  for the set of all automorphisms on  $A$ .

Objects  $A$  and  $B$  are *isomorphic*,  $A \cong B$ , iff there exists an isomorphism between them.

**Proposition 2.27.** *The inverse of an isomorphism is unique.*

PROOF: Proposition 2.20.  $\square$

**Proposition 2.28.** *For any object  $A$  we have  $\text{id}_A : A \cong A$  and  $\text{id}_A^{-1} = \text{id}_A$ .*

PROOF: Since  $\text{id}_A \circ \text{id}_A = \text{id}_A$  by the Unit Laws.  $\square$

**Proposition 2.29.** *If  $f : A \cong B$  then  $f^{-1} : B \cong A$  and  $(f^{-1})^{-1} = f$ .*

PROOF: Immediate from definitions.  $\square$

**Proposition 2.30.** *If  $f : A \cong B$  and  $g : B \cong C$  then  $g \circ f : A \cong C$  and  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .*

PROOF: From Proposition 2.21.  $\square$

**Definition 2.31** (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

## 2.5 Initial and Terminal Objects

**Definition 2.32** (Initial Object). An object  $I$  in a category is *initial* iff, for any object  $X$ , there is exactly one morphism  $I \rightarrow X$ .

**Example 2.33.** The empty set is the initial object in **Set**.

**Definition 2.34** (Terminal Object). An object  $T$  in a category is *terminal* iff, for any object  $X$ , there is exactly one morphism  $X \rightarrow T$ .

**Example 2.35.** Every singleton is terminal in **Set**.

**Proposition 2.36.** *If  $I$  and  $J$  are initial in a category, then there exists a unique isomorphism  $I \cong J$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $i$  be the unique morphism  $I \rightarrow J$ .
- $\langle 1 \rangle 2$ . LET:  $i^{-1}$  be the unique morphism  $J \rightarrow I$ .
- $\langle 1 \rangle 3$ .  $i \circ i^{-1} = \text{id}_J$

PROOF: Since there is only one morphism  $J \rightarrow J$ .

- $\langle 1 \rangle 4$ .  $i^{-1} \circ i = \text{id}_I$

PROOF: Since there is only one morphism  $I \rightarrow I$ .  
 $\square$

**Proposition 2.37.** *If  $S$  and  $T$  are terminal in a category, then there exists a unique isomorphism  $S \cong T$ .*

PROOF: Dual.  $\square$

## Chapter 3

# Functors

**Definition 3.1** (Functor). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of:

- for every object  $A \in \mathcal{C}$ , an object  $FA \in \mathcal{D}$
- for any morphism  $f : A \rightarrow B : \mathcal{C}$ , a morphism  $Ff : FA \rightarrow FB : \mathcal{D}$

such that:

- $F\text{id}_A = \text{id}_{FA}$
- $F(g \circ f) = Fg \circ Ff$

**Definition 3.2** (Identity Functor). For any category  $\mathcal{C}$ , the *identity functor*  $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  is defined by

$$\begin{aligned} 1_{\mathcal{C}}A &= A \\ 1_{\mathcal{C}}f &= f \end{aligned}$$

**Definition 3.3** (Constant Functor). Given categories  $\mathcal{C}$ ,  $\mathcal{D}$  and an object  $D \in \mathcal{D}$ , the *constant functor*  $K^{\mathcal{C}}D : \mathcal{C} \rightarrow \mathcal{D}$  is the functor defined by

$$\begin{aligned} K^{\mathcal{C}}DC &= D \\ K^{\mathcal{C}}Df &= \text{id}_D \end{aligned}$$

### 3.1 Comma Categories

**Definition 3.4** (Comma Category). Let  $F : \mathcal{C} \rightarrow \mathcal{E}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$  be functors. The *comma category*  $F \downarrow G$  is the category with:

- objects all pairs  $(C, D, f)$  where  $C \in \mathcal{C}$ ,  $D \in \mathcal{D}$  and  $f : FC \rightarrow GD : \mathcal{E}$

- morphisms  $(u, v) : (C, D, f) \rightarrow (C', D', g)$  all pairs  $u : C \rightarrow C' : \mathcal{C}$  and  $v : D \rightarrow D' : \mathcal{D}$  such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ \downarrow Fu & & \downarrow Gv \\ FC' & \xrightarrow{g} & GD' \end{array}$$

**Definition 3.5** (Slice Category). Let  $\mathcal{C}$  be a category and  $A \in \mathcal{C}$ . The *slice category* over  $A$ , denoted  $\mathcal{C}/A$ , is the comma category  $1_{\mathcal{C}} \downarrow K^1 A$ .

**Definition 3.6** (Coslice Category). Let  $\mathcal{C}$  be a category and  $A \in \mathcal{C}$ . The *coslice category* over  $A$ , denoted  $\mathcal{C} \backslash A$ , is the comma category  $K^1 A \downarrow 1_{\mathcal{C}}$ .

**Definition 3.7** (Pointed Sets). The *category of pointed sets*  $\mathbf{Set}_*$  is the coslice category  $\mathbf{Set} \backslash 1$ .



**Part II**

**Group Theory**



# Chapter 4

## Groups

**Definition 4.1** (Group). A *group*  $G$  consists of a set  $G$  and a binary operation  $\cdot : G^2 \rightarrow G$  such that  $\cdot$  is associative, and there exists  $e \in G$ , the *identity* element of the group, such that:

- For all  $x \in G$  we have  $xe = ex = x$
- For all  $x \in G$ , there exists  $x^{-1} \in G$ , the *inverse* of  $x$ , such that  $xx^{-1} = x^{-1}x = e$ .

We identify a group  $G$  with the category  $G$  with one object and morphisms the elements of  $G$ , with composition given by  $\cdot$ .

The *order* of a group  $G$ , denoted  $|G|$ , is the number of elements in  $G$  if  $G$  is finite; otherwise we write  $|G| = \infty$ .

**Proposition 4.2.** *The identity in a group is unique.*

PROOF: Proposition 2.2.

**Proposition 4.3.** *The inverse of an element is unique.*

PROOF: If  $i$  and  $j$  are inverses of  $x$  then  $i = ixj = j$ .  $\square$

**Example 4.4.** • The *trivial* group is  $\{e\}$  under  $ee = e$ .

- $\mathbb{Z}$  is a group under addition
- $\mathbb{Q}$  is a group under addition
- $\mathbb{Q} - \{0\}$  is a group under multiplication
- $\mathbb{R}$  is a group under addition
- $\mathbb{R} - \{0\}$  is a group under multiplication
- $\mathbb{C}$  is a group under addition
- $\mathbb{C} - \{0\}$  is a group under multiplication

- $\{-1, 1\}$  is a group under multiplication
- The set of  $2 \times 2$  real matrices with non-zero determinant is a group under matrix multiplication.
- For any category  $\mathcal{C}$  and object  $A \in \mathcal{C}$ , we have  $\text{Aut}_{\mathcal{C}}(A)$  is a group under  $gf = f \circ g$ .  
For  $A$  a set, we call  $S_A = \text{Aut}_{\text{Set}}(A)$  the *symmetric group* or *group of permutations* of  $A$ .
- For  $n \geq 3$ , the *dihedral group*  $D_{2n}$  consists of the set of rigid motions that map the regular  $n$ -gon onto itself under composition.
- Let  $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$  under matrix multiplication.

**Example 4.5.** • The only group of order 1 is the trivial group.

- The only group of order 2 is  $\mathbb{Z}_2$ .
- The only group of order 3 is  $\mathbb{Z}_3$ .
- There are exactly two groups of order 4:  $\mathbb{Z}_4$  and  $\mathbb{Z}_2 \times \mathbb{Z}_2$  under  $(a, b)(c, d) = (ac, bd)$ .

**Proposition 4.6** (Cancellation). *Let  $G$  be a group. Let  $a, g, h \in G$ . If  $ag = ah$  or  $ga = ha$  then  $g = h$ .*

PROOF: If  $ag = ah$  then  $g = a^{-1}ag = a^{-1}ah = h$ . Similarly if  $ga = ha$ .  $\square$

**Proposition 4.7.** *Let  $G$  be a group and  $g, h \in G$ . Then  $(gh)^{-1} = h^{-1}g^{-1}$ .*

PROOF: Since  $ghh^{-1}g^{-1} = e$ .  $\square$

**Definition 4.8.** Let  $G$  be a group. Let  $g \in G$ . We define  $g^n \in G$  for all  $n \in \mathbb{Z}$  as follows:

$$\begin{aligned} g^0 &= e \\ g^{n+1} &= g^n g & (n \geq 0) \\ g^{-n} &= (g^{-1})^n & (n > 0) \end{aligned}$$

**Proposition 4.9.** *Let  $G$  be a group. Let  $g \in G$  and  $m, n \in \mathbb{Z}$ . Then*

$$g^{m+n} = g^m g^n .$$

PROOF:

- $\langle 1 \rangle$ 1. For all  $k \in \mathbb{Z}$  we have  $g^{k+1} = g^k g$   
 $\langle 2 \rangle$ 1. For all  $k \geq 0$  we have  $g^{k+1} = g^k g$

PROOF: Immediate from definition.

$\langle 2 \rangle 2. g^{-1+1} = g^{-1}g$

PROOF: Both are equal to  $e$ .

$\langle 2 \rangle 3. \text{ For all } k > 1 \text{ we have } g^{-k+1} = g^{-k}g$

PROOF:

$$\begin{aligned} g^{-k+1} &= (g^{-1})^{k-1} \\ &= (g^{-1})^{k-1}g^{-1}g \\ &= (g^{-1})^k g \\ &= g^{-k}g \end{aligned}$$

$\langle 1 \rangle 2. \text{ For all } k \in \mathbb{Z} \text{ we have } g^{k-1} = g^k g^{-1}$

PROOF: Substitute  $k = k - 1$  above and multiply by  $g^{-1}$ .

$\langle 1 \rangle 3. g^{m+0} = g^m g^0$

PROOF: Since  $g^m g^0 = g^m e = g^m$ .

$\langle 1 \rangle 4. \text{ If } g^{m+n} = g^m g^n \text{ then } g^{m+n+1} = g^m g^{n+1}$

PROOF:

$$g^{m+n+1} = g^{m+n}g \quad (\langle 1 \rangle 1)$$

$$= g^m g^n g$$

$$= g^m g^{n+1} \quad (\langle 1 \rangle 1)$$

$\langle 1 \rangle 5. \text{ If } g^{m+n} = g^m g^n \text{ then } g^{m+n-1} = g^m g^{n-1}$

PROOF:

$$g^{m+n-1}g = g^{m+n} \quad (\langle 1 \rangle 1)$$

$$= g^m g^n$$

$$\therefore g^{m+n-1} = g^m g^n g^{-1}$$

$$= g^m g^{n-1} \quad (\langle 1 \rangle 2)$$

□

**Proposition 4.10.** *Let  $G$  be a group. Let  $g \in G$  and  $m, n \in \mathbb{Z}$ . Then*

$$(g^m)^n = g^{mn}.$$

PROOF:

$\langle 1 \rangle 1. (g^m)^0 = g^0$

PROOF: Both sides are equal to  $e$ .

$\langle 1 \rangle 2. \text{ If } (g^m)^n = g^{mn} \text{ then } (g^m)^{n+1} = g^{m(n+1)}.$

PROOF:

$$(g^m)^{n+1} = (g^m)^n g^m \quad (\text{Proposition 4.9})$$

$$= g^{mn} g^m$$

$$= g^{mn+m} \quad (\text{Proposition 4.9})$$

$\langle 1 \rangle 3. \text{ If } (g^m)^n = g^{mn} \text{ then } (g^m)^{n-1} = g^{m(n-1)}.$

PROOF:

$$(g^m)^n = g^{mn}$$

$$\therefore (g^m)^{n-1} g^m = g^{mn-m} g^m \quad (\text{Proposition 4.9})$$

$$\therefore (g^m)^{n-1} = g^{mn-m} \quad (\text{Cancellation})$$

□

**Definition 4.11** (Commute). Let  $G$  be a group and  $g, h \in G$ . We say  $g$  and  $h$  *commute* iff  $gh = hg$ .

**Definition 4.12.** Let  $G$  be a group. Given  $g \in G$  and  $A \subseteq G$ , we define

$$gA = \{ga : a \in A\}, \quad Ag = \{ag : a \in A\}.$$

Given sets  $A, B \subseteq G$ , we define

$$AB = \{ab : a \in A, b \in B\}.$$

## 4.1 Order of an Element

**Definition 4.13** (Order). Let  $G$  be a group. Let  $g \in G$ . Then  $g$  has *finite order* iff there exists a positive integer  $n$  such that  $g^n = e$ . In this case, the *order* of  $g$ , denoted  $|g|$ , is the least positive integer  $n$  such that  $g^n = e$ .

If  $g$  does not have finite order, we write  $|g| = \infty$ .

**Proposition 4.14.** Let  $G$  be a group. Let  $g \in G$  and  $n$  be a positive integer. If  $g^n = e$  then  $|g| \mid n$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $n = q|g| + d$  where  $0 \leq d < |g|$

PROOF: Division Algorithm.

$\langle 1 \rangle 2$ .  $g^d = e$

PROOF:

$$\begin{aligned} e &= g^n \\ &= g^{q|g|+d} \\ &= (g^{|g|})^q g^d && \text{(Propositions 4.9, 4.10)} \\ &= e^q g^d \\ &= g^d \end{aligned}$$

$\langle 1 \rangle 3$ .  $d = 0$

PROOF: By minimality of  $|g|$ .

$\langle 1 \rangle 4$ .  $n = q|g|$

□

**Corollary 4.14.1.** Let  $G$  be a group. Let  $g \in G$  have finite order and  $n \in \mathbb{Z}$ . Then  $g^n = e$  if and only if  $|g| \mid n$ .

**Proposition 4.15.** Let  $G$  be a group and  $g \in G$ . Then  $|g| \leq |G|$ .

PROOF:

$\langle 1 \rangle 1$ . ASSUME: w.l.o.g.  $G$  is finite.

$\langle 1 \rangle 2$ . PICK  $i, j$  with  $0 \leq i < j \leq |G|$  such that  $g^i = g^j$ .

PROOF: Otherwise  $g^0, g^1, \dots, g^{|G|}$  would be  $|G| + 1$  distinct elements of  $G$ .

$\langle 1 \rangle 3. g^{j-i} = e$

$\langle 1 \rangle 4. g$  has finite order and  $|g| \leq |G|$

PROOF: Since  $|g| \leq j - i \leq j \leq |G|$ .

□

**Proposition 4.16.** *Let  $G$  be a group. Let  $g \in G$  have finite order. Let  $m \in \mathbb{N}$ . Then*

$$|g^m| = \frac{\text{lcm}(m, |g|)}{m} = \frac{|g|}{\text{gcd}(m, |g|)}$$

PROOF: Since for any integer  $d$  we have

$$g^{md} = e \Leftrightarrow |g| \mid md \quad (\text{Corollary 4.14.1})$$

$$\Leftrightarrow \text{lcm}(m, |g|) \mid md$$

$$\Leftrightarrow \frac{\text{lcm}(m, |g|)}{m} \mid d \quad \square$$

and so  $|g^m| = \frac{\text{lcm}(m, |g|)}{m}$  by Corollary 4.14.1. □

**Corollary 4.16.1.** *If  $g$  has odd order then  $|g^2| = |g|$ .*

**Proposition 4.17.** *Let  $G$  be a group. Let  $g, h \in G$  have finite order. Assume  $gh = hg$ . Then  $|gh|$  has finite order and*

$$|gh| \mid \text{lcm}(|g|, |h|)$$

PROOF: Since  $(gh)^{\text{lcm}(|g|, |h|)} = g^{\text{lcm}(|g|, |h|)} h^{\text{lcm}(|g|, |h|)} = e$ . □

**Example 4.18.** This example shows that we cannot remove the hypothesis that  $gh = hg$ .

In  $\text{GL}_2(\mathbb{R})$ , take

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then  $|g| = 4$ ,  $|h| = 3$  and  $|gh| = \infty$ .

**Proposition 4.19.** *Let  $G$  be a group and  $g, h \in G$  have finite order. If  $gh = hg$  and  $\text{gcd}(|g|, |h|) = 1$  then  $|gh| = |g||h|$ .*

PROOF:

$\langle 1 \rangle 1.$  LET:  $N = |gh|$

$\langle 1 \rangle 2. g^N = (h^{-1})^N$

$\langle 1 \rangle 3. g^{N|g|} = e$

$\langle 1 \rangle 4. |g^N| \mid |g|$

$\langle 1 \rangle 5. h^{-N|h|} = e$

$\langle 1 \rangle 6. |g^N| \mid |h|$

$\langle 1 \rangle 7. |g^N| = 1$

PROOF: Since  $\text{gcd}(|g|, |h|) = 1$ .

$\langle 1 \rangle 8. g^N = e$

$\langle 1 \rangle 9. |g| \mid N$

⟨1⟩10.  $h^{-N} = e$

⟨1⟩11.  $|h| \mid N$

⟨1⟩12.  $N = |g||h|$

PROOF: Using Proposition 4.17.

□

**Proposition 4.20.** *Let  $G$  be a finite group. Assume there is exactly one element  $f \in G$  of order 2. Then the product of all the elements of  $G$  is  $f$ .*

PROOF: Let the elements of  $G$  be  $g_1, g_2, \dots, g_n$ . Apart from  $e$  and  $f$ , every element and its inverse are distinct elements of the list. Hence the product of the list is  $ef = f$ . □

**Proposition 4.21.** *Let  $G$  be a finite group of order  $n$ . Let  $m$  be the number of elements of  $G$  of order 2. Then  $n - m$  is odd.*

PROOF: In the list of all elements that are not of order 2, every element and its inverse are distinct except for  $e$ . Hence the list has odd length. □

**Corollary 4.21.1.** *If a finite group has even order, then it contains an element of order 2.*

**Proposition 4.22.** *Let  $G$  be a group and  $a, g \in G$ . Then  $|aga^{-1}| = |g|$ .*

PROOF: Since

$$\begin{aligned} (aga^{-1})^n = e &\Leftrightarrow ag^na^{-1} = e \\ &\Leftrightarrow g^n = e \end{aligned} \quad \square$$

**Proposition 4.23.** *Let  $G$  be a group and  $g, h \in G$ . Then  $|gh| = |hg|$ .*

PROOF: Since  $|gh| = |ghgg^{-1}| = |hg|$ . □

**Proposition 4.24.** *Let  $G$  be a group of order  $n$ . Let  $k$  be relatively prime to  $n$ . Then every element in  $G$  has the form  $x^k$  for some  $x$ .*

⟨1⟩1. PICK integers  $a$  and  $b$  such that  $an + bk = 1$ .

⟨1⟩2. LET:  $g \in G$

⟨1⟩3.  $g = (g^b)^k$

PROOF:

$$\begin{aligned} g &= g \cdot (g^n)^{-a} & (g^n = e) \\ &= g^{1-an} \\ &= g^{bk} \end{aligned}$$

□

## 4.2 Generators

**Definition 4.25** (Generator). Let  $G$  be a group and  $a \in G$ . We say  $a$  *generates* the group iff, for all  $x \in G$ , there exists an integer  $n$  such that  $x^n = a$ .



**Example 4.26.**  $\text{SL}_2(\mathbb{Z})$  is generated by

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $H = \langle s, t \rangle$

$\langle 1 \rangle 2$ . For all  $q \in \mathbb{Z}$  we have  $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in H$ .

PROOF: It is  $t^q$ .

$\langle 1 \rangle 3$ . For all  $q \in \mathbb{Z}$  we have  $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \in H$ .

PROOF:

$$\begin{aligned} st^{-q}s^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \end{aligned}$$

$\langle 1 \rangle 4$ .

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & qa+b \\ c & qc+d \end{pmatrix}$$

$\langle 1 \rangle 5$ .

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} a+qb & b \\ c+qd & d \end{pmatrix}$$

$\langle 1 \rangle 6$ . For any  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ , if  $c$  and  $d$  are both nonzero, then there exists  $N \in H$  such that the bottom row of  $MN$  has one entry the same as  $M$  and one entry with smaller absolute value.

PROOF: From  $\langle 1 \rangle 4$  and  $\langle 1 \rangle 5$  taking  $q = -1$ .

$\langle 1 \rangle 7$ . For any  $M \in \text{SL}_2(\mathbb{Z})$ , there exists  $N \in H$  such that  $MN$  has a zero on the bottom row.

PROOF: Apply  $\langle 1 \rangle 6$  repeatedly.

$\langle 1 \rangle 8$ . Any matrix in  $\text{SL}_2(\mathbb{Z})$  with a zero on the bottom row is in  $H$ .

$\langle 2 \rangle 1$ .  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H$

PROOF:  $\langle 1 \rangle 2$

$\langle 2 \rangle 2$ .  $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \in H$

PROOF: It is  $s^2 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$  since  $s^2 = -I$ .

$\langle 2 \rangle 3$ .  $\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} \in H$

PROOF: It is  $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} s$ .

$$\langle 2 \rangle 4. \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \in H$$

PROOF: It is  $s^2 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s$ .

$\langle 1 \rangle 9$ . Every matrix in  $\mathrm{SL}_2(\mathbb{Z})$  is in  $H$ .  
 $\square$

## Chapter 5

# Group Homomorphisms

**Definition 5.1** (Homomorphism). Let  $G$  and  $H$  be groups. A (group) homomorphism  $\phi : G \rightarrow H$  is a function such that, for all  $x, y \in G$ ,

$$\phi(xy) = \phi(x)\phi(y) \ .$$

**Proposition 5.2.** Let  $G$  and  $H$  be groups with identities  $e_G$  and  $e_H$ . Let  $\phi : G \rightarrow H$  be a group homomorphism. Then  $\phi(e_G) = e_H$ .

PROOF: Since  $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$  and so  $\phi(e_G) = e_H$  by Cancellation.  $\square$

**Proposition 5.3.** Let  $\phi : G \rightarrow H$  be a group homomorphism. For all  $x \in G$  we have  $\phi(x^{-1}) = \phi(x)^{-1}$ .

PROOF: Since  $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_H$ .  $\square$

**Proposition 5.4.** Let  $G, H$  and  $K$  be groups. If  $\phi : G \rightarrow H$  and  $\psi : H \rightarrow K$  are homomorphisms then  $\psi \circ \phi : G \rightarrow K$  is a homomorphism.

PROOF: For  $x, y \in G$  we have

$$\psi(\phi(xy)) = \psi(\phi(x)\phi(y)) = \psi(\phi(x))\psi(\phi(y)) \ .$$

**Proposition 5.5.** Let  $G$  be a group. Then  $\text{id}_G : G \rightarrow G$  is a group homomorphism.

PROOF: For  $x, y \in G$  we have  $\text{id}_G(xy) = xy = \text{id}_G(x)\text{id}_G(y)$ .  $\square$

**Proposition 5.6.** Let  $\phi : G \rightarrow H$  be a group homomorphism. Let  $g \in G$  have finite order. Then  $|\phi(g)|$  divides  $|g|$ .

PROOF: Since  $\phi(g)^{|g|} = \phi(g^{|g|}) = e$ .  $\square$

**Definition 5.7** (Category of Groups). Let **Grp** be the category of groups and group homomorphisms.

**Example 5.8.** There are 49487365402 groups of order 1024 up to isomorphism.

**Proposition 5.9.** *A group homomorphism  $\phi : G \rightarrow H$  is an isomorphism in **Grp** if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\phi$  is bijective.

PROVE:  $\phi^{-1}$  is a group homomorphism.

$\langle 1 \rangle 2$ . LET:  $h, h' \in H$

$\langle 1 \rangle 3$ .  $\phi(\phi^{-1}(hh')) = \phi(\phi^{-1}(h)\phi^{-1}(h'))$

PROOF: Both are equal to  $hh'$ .

$\langle 1 \rangle 4$ .  $\phi^{-1}(hh') = \phi^{-1}(h)\phi^{-1}(h')$

□

**Corollary 5.9.1.**

$$D_6 \cong C_3$$

PROOF: The canonical homomorphism  $D_6 \rightarrow C_3$  is bijective. □

**Corollary 5.9.2.**

$$(\mathbb{R}, +) \cong (\{x \in \mathbb{R} : x > 0\}, \cdot)$$

PROOF: The function that maps  $x$  to  $e^x$  is a bijective homomorphism. □

**Proposition 5.10.** *The trivial group is the zero object in **Grp**.*

PROOF: For any group  $G$ , the unique function  $G \rightarrow \{e\}$  is a group homomorphism, and the only group homomorphism  $\{e\} \rightarrow G$  maps  $e$  to  $e_G$ . □

**Proposition 5.11.** *For any groups  $G$  and  $H$ , the set  $G \times H$  under  $(g, h)(g', h') = (gg', hh')$  is the product of  $G$  and  $H$  in **Grp**.*

PROOF:

$\langle 1 \rangle 1$ .  $G \times H$  is a group.

$\langle 2 \rangle 1$ . The multiplication is associative.

PROOF: Since  $(g_1, h_1)((g_2, h_2)(g_3, h_3)) = ((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1g_2g_3, h_1h_2h_3)$ .

$\langle 2 \rangle 2$ .  $(e_G, e_H)$  is the identity.

PROOF: Since  $(g, h)(e_G, e_H) = (e_G, e_H)(g, h) = (g, h)$ .

$\langle 2 \rangle 3$ . The inverse of  $(g, h)$  is  $(g^{-1}, h^{-1})$ .

PROOF: Since  $(g, h)(g^{-1}, h^{-1}) = (g^{-1}, h^{-1})(g, h) = (e_G, e_H)$ .

$\langle 1 \rangle 2$ .  $\pi_1 : G \times H \rightarrow G$  is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$ .  $\pi_2 : G \times H \rightarrow H$  is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 4$ . For any group homomorphism  $\phi : K \rightarrow G$  and  $\psi : K \rightarrow H$ , the function  $\langle \phi, \psi \rangle : K \rightarrow G \times H$  where  $\langle \phi, \psi \rangle(k) = (\phi(k), \psi(k))$  is a group homomorphism.

PROOF:

$$\begin{aligned} \langle \phi, \psi \rangle(kk') &= (\phi(kk'), \psi(kk')) \\ &= (\phi(k)\phi(k'), \psi(k)\psi(k')) \\ &= (\phi(k), \psi(k))(\phi(k'), \psi(k')) \\ &= \langle \phi, \psi \rangle(k)\langle \phi, \psi \rangle(k') \end{aligned}$$

□

**Proposition 5.12.**

$$|\text{Aut}_{\mathbf{Grp}}(C_n)| = \phi(n)$$

PROOF: An automorphism  $\alpha$  is determined by  $\alpha(1)$  which is any element of order  $n$ , and  $g$  has order  $n$  iff  $\gcd(g, n) = 1$ . □

**Example 5.13.**

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}) \cong C_2$$

PROOF: The only automorphisms are the identity and multiplication by -1. □

## 5.1 Subgroups

**Definition 5.14** (Subgroup). Let  $(G, \cdot)$  and  $(H, *)$  be groups such that  $H$  is a subset of  $G$ . Then  $H$  is a *subgroup* of  $G$  iff the inclusion  $i : H \hookrightarrow G$  is a group homomorphism.

**Proposition 5.15.** If  $(H, *)$  is a subgroup of  $(G, \cdot)$  then  $*$  is the restriction of  $\cdot$  to  $H$ .

PROOF: Given  $x, y \in H$  we have

$$x * y = i(x * y) = i(x) \cdot i(y) = x \cdot y . \quad \square$$

**Example 5.16.** For any group  $G$  we have  $\{e\}$  is a subgroup of  $G$ .

**Proposition 5.17.** Let  $G$  be a group. Let  $H$  be a subset of  $G$ . Then  $H$  is a subgroup of  $G$  iff  $H$  is nonempty and, for all  $x, y \in H$ , we have  $xy^{-1} \in H$ .

PROOF:

⟨1⟩1. If  $H$  is a subgroup of  $G$  then  $H$  is nonempty.

PROOF: Since every group has an identity element and so is nonempty.

⟨1⟩2. If  $H$  is a subgroup of  $G$  then, for all  $x, y \in H$ , we have  $xy^{-1} \in H$ .

PROOF: Easy.

⟨1⟩3. If  $H$  is nonempty and, for all  $x, y \in H$ , we have  $xy^{-1} \in H$ , then  $H$  is a subgroup of  $G$ .

⟨2⟩1. ASSUME:  $H$  is nonempty.

⟨2⟩2. ASSUME:  $\forall x, y \in H. xy^{-1} \in H$

⟨2⟩3.  $e \in H$

PROOF: Pick  $x \in H$ . We have  $e = xx^{-1} \in H$ .

⟨2⟩4.  $\forall x \in H. x^{-1} \in H$

PROOF: Given  $x \in H$  we have  $x^{-1} = ex^{-1} \in H$ .

⟨2⟩5.  $H$  is closed under the restriction of  $\cdot$

PROOF: Given  $x, y \in H$  we have  $xy = x(y^{-1})^{-1} \in H$ .

⟨2⟩6.  $H$  is a group under the restriction of  $\cdot$

PROOF: Associativity is inherited from  $G$  and the existence of an identity element and inverses follows from ⟨2⟩3 and ⟨2⟩4.

⟨2⟩7. The inclusion  $H \hookrightarrow G$  is a group homomorphism.

PROOF: For  $x, y \in H$  we have  $i(xy) = i(x)i(y) = xy$ .

□

**Corollary 5.17.1.** *The intersection of a set of subgroups of  $G$  is a subgroup of  $G$ .*

**Corollary 5.17.2.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Let  $K$  be a subgroup of  $H$ . Then  $\phi^{-1}(K)$  is a subgroup of  $G$ .*

PROOF:

⟨1⟩1.  $\phi^{-1}(K)$  is nonempty.

PROOF: Since  $e \in \phi^{-1}(K)$ .

⟨1⟩2. LET:  $x, y \in \phi^{-1}(K)$

⟨1⟩3.  $\phi(x), \phi(y) \in K$

⟨1⟩4.  $\phi(x)\phi(y)^{-1} \in K$

⟨1⟩5.  $\phi(xy^{-1}) \in K$

⟨1⟩6.  $xy^{-1} \in \phi^{-1}(K)$

□

**Corollary 5.17.3.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Let  $K$  be a subgroup of  $G$ . Then  $\phi(K)$  is a subgroup of  $H$ .*

PROOF:

⟨1⟩1. LET:  $x, y \in \phi(K)$

⟨1⟩2. PICK  $a, b \in K$  such that  $x = \phi(a)$  and  $y = \phi(b)$

⟨1⟩3.  $xy^{-1} = \phi(ab^{-1})$

⟨1⟩4.  $xy^{-1} \in \phi(K)$

□

**Proposition 5.18.** *Let  $G$  be a subgroup of  $\mathbb{Z}$ . Then there exists  $d \geq 0$  such that  $G = d\mathbb{Z}$ .*

PROOF:

⟨1⟩1. ASSUME: w.l.o.g.  $G \neq \{0\}$

PROOF: Since  $\{0\} = 0\mathbb{Z}$ .

⟨1⟩2. LET:  $d$  be the least positive element of  $G$ .

PROVE:  $G = d\mathbb{Z}$

PROOF: If  $n \in G$  then  $-n \in G$  so  $G$  must contain a positive element.

⟨1⟩3.  $G \subseteq d\mathbb{Z}$

⟨2⟩1. LET:  $n \in G$

⟨2⟩2. LET:  $q$  and  $r$  be the integers such that  $n = qd + r$  and  $0 \leq r < d$ .

⟨2⟩3.  $r \in G$

PROOF: Since  $r = n - qd$ .

⟨2⟩4.  $r = 0$

PROOF: By minimality of  $d$ .

⟨2⟩5.  $n = qd \in d\mathbb{Z}$

⟨1⟩4.  $d\mathbb{Z} \subseteq G$

□

## 5.2 Kernel

**Definition 5.19** (Kernel). Let  $\phi : G \rightarrow H$  be a group homomorphism. The *kernel* of  $\phi$  is

$$\ker \phi = \{g \in G : \phi(g) = e\} .$$

**Proposition 5.20.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Then  $\ker \phi$  is a subgroup of  $G$ .*

PROOF: Corollary 5.17.2.  $\square$

**Proposition 5.21.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Then the inclusion  $i : \ker \phi \hookrightarrow G$  is terminal in the category of pairs  $(K, \alpha : K \rightarrow G)$  such that  $\phi \circ \alpha = 0$ .*

PROOF:

$\langle 1 \rangle 1.$   $\phi \circ i = 0$

$\langle 1 \rangle 2.$  For any group  $K$  and homomorphism  $\alpha : K \rightarrow G$  such that  $\phi \circ \alpha = 0$ , there exists a unique homomorphism  $\beta : K \rightarrow \ker \phi$  such that  $i \circ \beta = \alpha$ .

$\square$

**Proposition 5.22.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Then the following are equivalent:*

1.  $\phi$  is monic.
2.  $\ker \phi = \{e\}$
3.  $\phi$  is injective.

PROOF:

$\langle 1 \rangle 1.$   $1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $\phi$  is monic.

$\langle 2 \rangle 2.$  LET:  $i : \ker \phi \hookrightarrow G$ ,  $j : \{e\} \hookrightarrow \ker \phi \hookrightarrow G$  be the inclusions.

$\langle 2 \rangle 3.$   $\phi \circ i = \phi \circ j$

$\langle 2 \rangle 4.$   $i = j$

$\langle 1 \rangle 2.$   $2 \Rightarrow 3$

$\langle 2 \rangle 1.$  ASSUME:  $\ker \phi = \{e\}$

$\langle 2 \rangle 2.$  LET:  $x, y \in G$

$\langle 2 \rangle 3.$  ASSUME:  $\phi(x) = \phi(y)$

$\langle 2 \rangle 4.$   $\phi(xy^{-1}) = e$

$\langle 2 \rangle 5.$   $xy^{-1} \in \ker \phi$

$\langle 2 \rangle 6.$   $xy^{-1} = e$

$\langle 2 \rangle 7.$   $x = y$

$\langle 1 \rangle 3.$   $3 \Rightarrow 1$

PROOF: Easy.

$\square$

**Proposition 5.23.** *A group homomorphism is an epimorphism if and only if it is surjective.*

### 5.3 Inner Automorphisms

**Proposition 5.24.** *Let  $G$  be a group and  $g \in G$ . The function  $\gamma_g : G \rightarrow G$  defined by  $\gamma_g(a) = gag^{-1}$  is an automorphism on  $G$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\gamma_g$  is a homomorphism.

PROOF:

$$\begin{aligned}\gamma_g(ab) &= gabg^{-1} \\ &= gag^{-1}gbg^{-1} \\ &= \gamma_g(a)\gamma_g(b)\end{aligned}$$

$\langle 1 \rangle 2$ .  $\gamma_g$  is injective.

PROOF: By Cancellation.

$\langle 1 \rangle 3$ .  $\gamma_g$  is surjective.

PROOF: Given  $b \in G$ , we have  $\gamma_g(g^{-1}bg) = b$ .

□

**Definition 5.25** (Inner Automorphism). Let  $G$  be a group. An *inner automorphism* on  $G$  is a function of the form  $\gamma_g(a) = gag^{-1}$  for some  $g \in G$ .

We write  $\text{Inn}(G)$  for the set of inner automorphisms of  $G$ .

**Proposition 5.26.** *Let  $G$  be a group. The function  $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$  that maps  $g$  to  $\gamma_g$  is a group homomorphism.*

PROOF: Since  $\gamma_{gh}(a) = ghah^{-1}g^{-1} = \gamma_g(\gamma_h(a))$ . □

**Corollary 5.26.1.**  $\text{Inn}(G)$  is a subgroup of  $\text{Aut}_{\mathbf{Grp}}(G)$ .

### 5.4 Direct Products

**Definition 5.27** (Direct Product). The *direct product* of groups  $G$  and  $H$  is their product in  $\mathbf{Grp}$ .

### 5.5 Free Groups

**Proposition 5.28.** *Let  $A$  be a set. Let  $\mathcal{F}^A$  be the category whose objects are pairs  $(G, j)$  where  $G$  is a group and  $j$  is a function  $A \rightarrow G$ , with morphisms  $f : (G, j) \rightarrow (H, k)$  the group homomorphisms  $f : G \rightarrow H$  such that  $f \circ j = k$ . Then  $\mathcal{F}^A$  has an initial object.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $W(A)$  be the set of words in the alphabet whose elements are the elements of  $A$  together with  $\{a^{-1} : a \in A\}$ .

$\langle 1 \rangle 2$ . LET:  $r : W(A) \rightarrow W(A)$  be the function that, given a word  $w$ , removes the first pair of letters of the form  $aa^{-1}$  or  $a^{-1}a$ ; if there is no such pair, then  $r(w) = w$ .



⟨1⟩3. Let us say that a word  $w$  is a *reduced word* iff  $r(w) = w$ .

⟨1⟩4. For any word  $w$  of length  $n$ , we have  $r^{\lceil \frac{n}{2} \rceil}(w)$  is a reduced word.

PROOF: Since we cannot remove more than  $n/2$  pairs of letters from  $w$ .

⟨1⟩5. LET:  $R : W(A) \rightarrow W(A)$  be the function  $R(w) = r^{\lceil \frac{n}{2} \rceil}(w)$ , where  $n$  is the length of  $w$ .

⟨1⟩6. LET:  $F(A)$  be the set of reduced words.

⟨1⟩7. Define  $\cdot : F(A)^2 \rightarrow F(A)$  by  $w \cdot w' = R(ww')$

⟨1⟩8.  $\cdot$  is associative.

PROOF: Both  $w_1 \cdot (w_2 \cdot w_3)$  and  $(w_1 \cdot w_2) \cdot w_3$  are equal to  $R(w_1w_2w_3)$ .

⟨1⟩9. The empty word is the identity element in  $F(A)$

⟨1⟩10. The inverse of  $a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}$  is  $a_n^{\mp 1} \dots a_2^{\mp 1} a_1^{\mp 1}$ .

⟨1⟩11. LET:  $j : A \rightarrow F(A)$  be the function that maps  $a$  to the word  $a$  of length

⟨1⟩12. LET:  $G$  be any group and  $k : A \rightarrow G$  any function.

⟨1⟩13. The only morphism  $f : (F(A), j) \rightarrow (G, k)$  in  $\mathcal{F}^A$  is  $f(a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}) = k(a_1)^{\pm 1} k(a_2)^{\pm 1} \dots k(a_n)^{\pm 1}$ .

□

**Definition 5.29** (Free Group). For any set  $A$ , the *free group* on  $A$  is the initial object  $(F(A), i)$  in  $\mathcal{F}^A$ .

**Proposition 5.30.**  $i : A \rightarrow F(A)$  is injective.

PROOF:

⟨1⟩1. LET:  $x, y \in A$

⟨1⟩2. ASSUME:  $x \neq y$

PROVE:  $i(x) \neq i(y)$

⟨1⟩3. LET:  $f : A \rightarrow C_2$  be the function that maps  $x$  to 0 and all other elements of  $A$  to 1.

⟨1⟩4. LET:  $\phi : F(A) \rightarrow C_2$  be the group homomorphism such that  $f = \phi \circ i$ .

⟨1⟩5.  $f(x) \neq f(y)$

⟨1⟩6.  $\phi(i(x)) \neq \phi(i(y))$

⟨1⟩7.  $i(x) \neq i(y)$

□

**Proposition 5.31.**

$$F(0) \cong \{e\}$$

PROOF: For any set  $A$ , the unique group homomorphism  $\{e\} \rightarrow A$  makes the following diagram commute.

$$\begin{array}{ccc} \{e\} & \longrightarrow & A \\ \uparrow & \nearrow & \\ \emptyset & & \end{array}$$

**Proposition 5.32.** The free group on 1 is  $\mathbb{Z}$  with the injection mapping 0 to 1.

PROOF: Given any group  $G$  and function  $a : 1 \rightarrow G$ , the required unique homomorphism  $\phi : \mathbb{Z} \rightarrow G$  is defined by  $\phi(n) = a(0)^n$ . □

**Proposition 5.33.** *For any sets  $A$  and  $B$ , we have that  $F(A + B)$  is the coproduct of  $F(A)$  and  $F(B)$  in **Grp**.*

$$\begin{array}{ccccc}
 & & G & & \\
 & f \nearrow & \uparrow k & \nwarrow g & \\
 F(A) & \xrightarrow{\kappa_1} & F(A+B) & \xleftarrow{\kappa_2} & F(B) \\
 i_A \uparrow & & j \uparrow & & i_B \uparrow \\
 A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
 \end{array}$$

PROOF:

- $\langle 1 \rangle 1$ . LET:  $i_A : A \rightarrow F(A)$ ,  $i_B : B \rightarrow F(B)$ ,  $j : A + B \rightarrow F(A + B)$  be the canonical injections.
- $\langle 1 \rangle 2$ . LET:  $\kappa_1, \kappa_2$  be the unique group homomorphisms that make the diagram above commute.
- $\langle 1 \rangle 3$ . LET:  $G$  be any group and  $f : F(A) \rightarrow G$ ,  $g : F(B) \rightarrow G$  any group homomorphisms.
- $\langle 1 \rangle 4$ . LET:  $h : A + B \rightarrow G$  be the unique function such that  $h \circ k_1 = f \circ i_A$  and  $h \circ k_2 = g \circ i_B$ .
- $\langle 1 \rangle 5$ . LET:  $k : F(A + B) \rightarrow G$  be the unique group homomorphism such that  $k \circ j = h$ .
- $\langle 1 \rangle 6$ .  $k$  is the unique group homomorphism such that  $k \circ \kappa_1 \circ i_A = f \circ i_A$  and  $k \circ \kappa_2 \circ i_B = g \circ i_B$ .
- $\langle 1 \rangle 7$ .  $k$  is the unique group homomorphism such that  $k \circ \kappa_1 = f$  and  $k \circ \kappa_2 = g$ .  $\square$

**Definition 5.34** (Subgroup Generated by a Group). Let  $G$  be a group and  $A$  a subset of  $G$ . Let  $\phi : F(A) \rightarrow G$  be the unique group homomorphism such that  $\phi(a) = a$  for all  $a \in A$ . The subgroup *generated* by  $A$  is

$$\langle A \rangle := \text{im } \phi$$

$$\begin{array}{ccc}
 F(A) & \xrightarrow{\phi} & G \\
 \uparrow & \nearrow & \\
 A & & 
 \end{array}$$

**Proposition 5.35.** *Let  $G$  be a group and  $A$  a subset of  $G$ . Then  $\langle A \rangle$  is the set of all elements of the form  $a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$  (where  $n \geq 0$ ) such that  $a_1, \dots, a_n \in A$ .*

PROOF: Immediate from definitions.  $\square$

**Corollary 5.35.1.** *Let  $G$  be a group and  $g \in G$ . Then*

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\} .$$

**Proposition 5.36.** *Let  $G$  be a group and  $A$  a subset of  $G$ . Then  $\langle A \rangle$  is the intersection of all the subgroups of  $G$  that include  $A$ .*

PROOF: Easy.  $\square$

**Definition 5.37** (Finitely Generated). Let  $G$  be a group. Then  $G$  is *finitely generated* iff there exists a finite subset  $A$  of  $G$  such that  $G = \langle A \rangle$ .

**Proposition 5.38.** *Every subgroup of a finitely generated free group is free.*

PROOF: TODO.

**Proposition 5.39.**  *$F(2)$  includes subgroups isomorphic to the free group on arbitrarily many generators.*

PROOF: TODO

**Proposition 5.40.**

$$[F(2), F(2)] \cong F(\mathbb{Z})$$

PROOF: TODO

## 5.6 Normal Subgroups

**Definition 5.41** (Normal Subgroup). A subgroup  $N$  of  $G$  is *normal* iff, for all  $g \in G$  and  $n \in N$ , we have  $gng^{-1} \in N$ .

**Proposition 5.42.** *Let  $G$  be a group and  $N$  a subgroup of  $G$ . Then the following are equivalent.*

1.  $N$  is normal.
2.  $\forall g \in G. gNg^{-1} \subseteq N$
3.  $\forall g \in G. gNg^{-1} = N$
4.  $\forall g \in G. gN \subseteq Ng$
5.  $\forall g \in G. gN = Ng$

PROOF:

$\langle 1 \rangle 1. 1 \Leftrightarrow 2$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: If 2 holds then we have  $gNg^{-1} \subseteq N$  and  $g^{-1}Ng \subseteq N$  hence  $N = gNg^{-1}$ .

$\langle 1 \rangle 3. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 4. 2 \Leftrightarrow 4$

PROOF: Easy.

$\langle 1 \rangle 5. 3 \Leftrightarrow 5$

PROOF: Easy.

□

**Proposition 5.43.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Then  $\ker \phi$  is a normal subgroup of  $G$ .*

PROOF: Given  $g \in G$  and  $n \in \ker \phi$  we have

$$\begin{aligned}\phi(gng^{-1}) &= \phi(g)\phi(n)\phi(g)^{-1} \\ &= \phi(g)\phi(g)^{-1} \\ &= e\end{aligned}$$

and so  $gng^{-1} \in \ker \phi$ . □

## 5.7 Quotient Groups

**Definition 5.44.** Let  $G$  be a group. Let  $\sim$  be an equivalence relation on  $G$ . Then we say that  $\sim$  is *compatible* with the group operation on  $G$  iff, for all  $a, a', g \in G$ , if  $a \sim a'$  then  $ga \sim ga'$  and  $ag \sim a'g$ .

**Proposition 5.45.** *Let  $G$  be a group. Let  $\sim$  be an equivalence relation on  $G$ . Then there exists an operation  $\cdot : (G/\sim)^2 \rightarrow G/\sim$  such that*

$$\forall a, b \in G. [a][b] = [ab]$$

*iff  $\sim$  is compatible with the group operation on  $G$ . In this case,  $G/\sim$  is a group under  $\cdot$  and the canonical function  $\pi : G \rightarrow G/\sim$  is a group homomorphism, and is universal with respect to group homomorphisms  $\phi : G \rightarrow G'$  such that if  $a \sim a'$  then  $\phi(a) = \phi(a')$ .*

PROOF: Easy. □

**Definition 5.46** (Quotient Group). Let  $G$  be a group. Let  $\sim$  be an equivalence relation on  $G$  that is compatible with the group operation on  $G$ . Then  $G/\sim$  is the *quotient group* of  $G$  by  $\sim$  under  $[a][b] = [ab]$ .

**Proposition 5.47.** *Let  $G$  be a group and  $H$  a subgroup of  $G$ . Then  $H$  is normal if and only if there exists a group  $K$  and homomorphism  $\phi : G \rightarrow K$  such that  $H = \ker \phi$ .*

PROOF: One direction is given by Proposition 5.43. For the other direction, take  $K = G/H$  and  $\phi$  to be the canonical map  $G \rightarrow G/H$ . □

**Definition 5.48** (Modular Group). The *modular group*  $\text{PSL}_2(\mathbb{Z})$  is  $\text{SL}_2(\mathbb{Z})/\{I, -I\}$ .

**Proposition 5.49.**  $\text{PSL}_2(\mathbb{Z})$  is generated by  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ .

PROOF: By Example 4.26.

**Theorem 5.50.** *Every group homomorphism  $\phi : G \rightarrow H$  may be decomposed as*

$$G \longrightarrow G/\ker \phi \xrightarrow{\cong} \text{im } \phi \longrightarrow H$$

PROOF: Easy.  $\square$

**Corollary 5.50.1** (First Isomorphism Theorem). *Let  $\phi : G \rightarrow H$  be a surjective group homomorphism. Then  $H \cong G/\ker \phi$ .*

**Proposition 5.51.** *Let  $H_1$  be a normal subgroup of  $G_1$  and  $H_2$  a normal subgroup of  $G_2$ . Then  $H_1 \times H_2$  is a normal subgroup of  $G_1 \times G_2$ , and*

$$\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}.$$

PROOF:  $\pi \times \pi : G_1 \times G_2 \rightarrow G_1/H_1 \times G_2/H_2$  is a surjective homomorphism with kernel  $H_1 \times H_2$ .  $\square$

**Example 5.52.**

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

PROOF: Map a real number  $r$  to  $(\cos r, \sin r)$ . The result is a surjective group homomorphism with kernel  $\mathbb{Z}$ .  $\square$

**Proposition 5.53.** *Let  $H$  be a normal subgroup of a group  $G$ . For every subgroup  $K$  of  $G$  that includes  $H$ , we have  $H$  is a normal subgroup of  $K$ , and  $K/H$  is a subgroup of  $G/H$ . The mapping*

$$u : \{\text{subgroups of } G \text{ including } H\} \rightarrow \{\text{subgroups of } G/H\}$$

*with  $u(K) = K/H$  is a poset isomorphism.*

PROOF:

- $\langle 1 \rangle 1$ . If  $K$  is a subgroup of  $G$  that includes  $H$  then  $H$  is normal in  $K$ .
- $\langle 1 \rangle 2$ . If  $K$  is a subgroup of  $G$  that includes  $H$  then  $K/H$  is a subgroup of  $G/H$ .
- $\langle 1 \rangle 3$ . If  $H \subseteq K_1 \subseteq K_2$  then  $K_1/H \subseteq K_2/H$ .
- $\langle 1 \rangle 4$ . If  $K_1/H = K_2/H$  then  $K_1 = K_2$
- $\langle 2 \rangle 1$ . ASSUME:  $K_1/H = K_2/H$
- $\langle 2 \rangle 2$ .  $K_1 \subseteq K_2$
- $\langle 3 \rangle 1$ . LET:  $k \in K_1$
- $\langle 3 \rangle 2$ .  $kH \in K_2/H$
- $\langle 3 \rangle 3$ . PICK  $k' \in K_2$  such that  $kH = k'H$
- $\langle 3 \rangle 4$ .  $kk'^{-1} \in H$
- $\langle 3 \rangle 5$ .  $kk'^{-1} \in K_2$
- $\langle 3 \rangle 6$ .  $k \in K_2$
- $\langle 2 \rangle 3$ .  $K_2 \subseteq K_1$

PROOF: Similar.

- $\langle 1 \rangle 5$ . For any subgroup  $L$  of  $G/H$ , there exists a subgroup  $K$  of  $G$  that includes  $H$  such that  $L = K/H$ .

⟨2⟩1. LET:  $L$  be a subgroup of  $G/H$ .

⟨2⟩2. LET:  $K = \{k \in G : kH \in L\}$

⟨2⟩3.  $K$  is a subgroup of  $G$ .

PROOF: Given  $k, k' \in K$  we have  $kH, k'H \in L$  hence  $kk'^{-1}H \in L$  and so  $kk'^{-1} \in K$ .

⟨2⟩4.  $H \subseteq K$

PROOF: For all  $h \in H$  we have  $hH = H \in L$ .

⟨2⟩5.  $L = K/H$

PROOF: By definition.

□

**Proposition 5.54** (Third Isomorphism Theorem). *Let  $H$  be a normal subgroup of a group  $G$ . Let  $N$  be a subgroup of  $G$  that includes  $H$ . Then  $N/H$  is normal in  $G/H$  if and only if  $N$  is normal in  $G$ , in which case*

$$\frac{G/H}{N/H} \cong \frac{G}{N}$$

PROOF:

⟨1⟩1. If  $N/H$  is normal in  $G/H$  then  $N$  is normal in  $G$ .

⟨2⟩1. ASSUME:  $N/H$  is normal in  $G/H$ .

⟨2⟩2. LET:  $g \in G$  and  $n \in N$ .

⟨2⟩3.  $gng^{-1}H \in N/H$

⟨2⟩4. PICK  $n' \in N$  such that  $gng^{-1}H = n'H$

⟨2⟩5.  $gng^{-1}n'^{-1} \in H$

⟨2⟩6.  $gng^{-1}n'^{-1} \in N$

⟨2⟩7.  $gng^{-1} \in N$

⟨1⟩2. If  $N$  is normal in  $G$  then  $N/H$  is normal in  $G/H$  and  $(G/H)/(N/H) \cong G/N$ .

⟨2⟩1. ASSUME:  $N$  is normal in  $G$ .

⟨2⟩2. LET:  $\phi : G/H \rightarrow G/N$  be the homomorphism  $\phi(gH) = gN$

⟨3⟩1. If  $gH = g'H$  then  $gN = g'N$

PROOF: If  $gg'^{-1} \in H$  then  $gg'^{-1} \in N$ .

⟨3⟩2.  $\phi((gH)(g'H)) = \phi(gH)\phi(g'H)$

PROOF: Both are  $gg'N$ .

⟨2⟩3.  $\phi$  is surjective.

⟨2⟩4.  $\ker \phi = N/H$

⟨2⟩5.  $(G/H)/(N/H) \cong G/N$

PROOF: By the First Isomorphism Theorem.

□

**Proposition 5.55** (Second Isomorphism Theorem). *Let  $H$  and  $K$  be subgroups of a group  $G$ . Assume that  $H$  is normal in  $G$ . Then:*

1.  $HK$  is a subgroup of  $G$ , and  $H$  is normal in  $HK$ .

2.  $H \cap K$  is normal in  $K$ , and

$$\frac{HK}{H} \cong \frac{K}{H \cap K}.$$

PROOF:

(1)1.  $HK$  is a subgroup of  $G$ .

PROOF: Since  $hkh'k' = hh'(h'^{-1}kh')k' \in HK$ .

(1)2.  $H$  is normal in  $HK$ .

(1)3.  $H \cap K$  is normal in  $K$  and  $HK/H \cong K/(H \cap K)$

PROOF: The function that maps  $k$  to  $kH$  is a surjective homomorphism  $K \twoheadrightarrow HK/H$  with kernel  $H \cap K$ . Surjectivity follows because  $hkh = hkh^{-1}H$ .

□

See also Proposition 5.70 for a result that holds even if  $H$  is not normal.

## 5.8 Cosets

**Proposition 5.56.** *Let  $G$  be a group. Let  $\sim$  be an equivalence relation on  $G$  such that, for all  $a, b, g \in G$ , if  $a \sim b$  then  $ga \sim gb$ . Let  $H = \{h \in G : h \sim e\}$ . Then  $H$  is a subgroup of  $G$  and, for all  $a, b \in G$ , we have*

$$a \sim b \Leftrightarrow a^{-1}b \in H \Leftrightarrow aH = bH .$$

PROOF:

(1)1.  $e \in H$

(1)2. For all  $x, y \in H$  we have  $xy^{-1} \in H$ .

(2)1. ASSUME:  $x \sim e$  and  $y \sim e$ .

(2)2.  $e \sim y^{-1}$

PROOF: Since  $yy^{-1} \sim ey^{-1}$ .

(2)3.  $xy^{-1} \sim e$

PROOF: Since  $xy^{-1} \sim ey^{-1} \sim e$ .

(1)3. If  $a \sim b$  then  $a^{-1}b \in H$ .

PROOF: If  $a \sim b$  then  $a^{-1}b \sim a^{-1}a = e$ .

(1)4. If  $a^{-1}b \in H$  then  $aH = bH$ .

(2)1. ASSUME:  $a^{-1}b \in H$

(2)2.  $bH \subseteq aH$

PROOF: For any  $h \in H$  we have  $bh = aa^{-1}bh \in aH$ .

(2)3.  $aH \subseteq bH$

PROOF: Similar since  $b^{-1}a \in H$ .

(1)5. If  $aH = bH$  then  $a \sim b$ .

(2)1. ASSUME:  $aH = bH$

(2)2. PICK  $h \in H$  such that  $a = bh$ .

(2)3.  $b^{-1}a = h$

(2)4.  $b^{-1}a \in H$

(2)5.  $b^{-1}a \sim e$

(2)6.  $a \sim b$

PROOF:  $a = bb^{-1}a \sim be = b$ .

□

**Definition 5.57** (Coset). Let  $G$  be a group and  $H$  a subgroup of  $G$ . A *left coset* of  $H$  is a set of the form  $aH$  for  $a \in G$ . A *right coset* of  $H$  is a set of the

form  $Ha$  for some  $a \in G$ .

We write  $G/H$  for the set of all left cosets of  $H$ , and  $G \backslash H$  for the set of all right cosets of  $H$ .

**Proposition 5.58.**

$$G/H \cong G \backslash H$$

PROOF: The function that maps  $aH$  to  $Ha^{-1}$  is a bijection.  $\square$

**Proposition 5.59.** *Let  $G$  be a group and  $H$  a subgroup of  $G$ . Define  $\sim_H$  on  $G$  by:  $a \sim b$  iff  $a^{-1}b \in H$ . This defines a one-to-one correspondence between the subgroups of  $G$  and the equivalence relations  $\sim$  on  $G$  such that, for all  $a, b, g \in G$ , if  $a \sim b$ , then  $ga \sim gb$ . The equivalence class of  $a$  is  $aH$ .*

PROOF:

(1)1. For any subgroup  $H$ , we have  $\sim_H$  is an equivalence relation on  $G$ .

(2)1.  $\sim$  is reflexive.

PROOF: For any  $a \in G$  we have  $a^{-1}a = e \in H$ .

(2)2.  $\sim$  is symmetric.

PROOF: If  $a^{-1}b \in H$  then  $b^{-1}a \in H$ .

(2)3.  $\sim$  is transitive.

PROOF: If  $a^{-1}b \in H$  and  $b^{-1}c \in H$  then  $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$ .

(1)2. If  $a \sim_H b$  then  $ga \sim_H gb$ .

PROOF: If  $a^{-1}b \in H$  then  $(ga)^{-1}(gb) = a^{-1}g^{-1}gb = a^{-1}b \in H$ .

(1)3. For any equivalence relation  $\sim$  on  $G$  such that, whenever  $a \sim b$ , then  $ga \sim gb$ , there exists a subgroup  $H$  such that  $\sim = \sim_H$ .

PROOF: Proposition 5.56.

(1)4. The  $\sim_H$ -equivalence class of  $a$  is  $aH$ .

PROOF:

$$\begin{aligned} a \sim b &\Leftrightarrow a^{-1}b \in H \\ &\Leftrightarrow \exists h \in H. a^{-1}b = h \\ &\Leftrightarrow \exists h \in H. b = ah \\ &\Leftrightarrow b \in aH \end{aligned}$$

$\square$

**Proposition 5.60.** *Let  $G$  be a group and  $H$  a subgroup of  $G$ . Define  $\sim_H$  on  $G$  by:  $a \sim b$  iff  $ab^{-1} \in H$ . This defines a one-to-one correspondence between the subgroups of  $G$  and the equivalence relations  $\sim$  on  $G$  such that, for all  $a, b, g \in G$ , if  $a \sim b$ , then  $ag \sim bg$ . The equivalence class of  $a$  is  $Ha$ .*

PROOF: Similar.  $\square$

**Proposition 5.61.** *Let  $G$  be a group and  $H$  be a subgroup of  $G$ . Define  $\sim_L$  and  $\sim_R$  on  $G$  by:*

$$a \sim_L b \Leftrightarrow a^{-1}b \in H, \quad a \sim_R b \Leftrightarrow ab^{-1} \in H.$$

*Then  $\sim_L = \sim_R$  if and only if  $H$  is normal.*



PROOF:

$\langle 1 \rangle 1$ . If  $\sim_L = \sim_R$  then  $H$  is normal.

$\langle 2 \rangle 1$ . ASSUME:  $\sim_L = \sim_R$

$\langle 2 \rangle 2$ . LET:  $h \in H$  and  $g \in G$

$\langle 2 \rangle 3$ .  $g \sim_L gh^{-1}$

$\langle 2 \rangle 4$ .  $g \sim_R gh^{-1}h$

$\langle 2 \rangle 5$ .  $ghg^{-1} \in H$

$\langle 1 \rangle 2$ . If  $H$  is normal then  $\sim_L = \sim_R$ .

$\langle 2 \rangle 1$ . ASSUME:  $H$  is normal.

$\langle 2 \rangle 2$ . If  $a \sim_L b$  then  $a \sim_R b$ .

$\langle 3 \rangle 1$ . ASSUME:  $a \sim_L b$

$\langle 3 \rangle 2$ .  $a^{-1}b \in H$

$\langle 3 \rangle 3$ .  $aa^{-1}ba^{-1} \in H$

$\langle 3 \rangle 4$ .  $ba^{-1} \in H$

$\langle 3 \rangle 5$ .  $a \sim_R b$

$\langle 2 \rangle 3$ . If  $a \sim_R b$  then  $a \sim_L b$ .

PROOF: Similar.

□

**Corollary 5.61.1.** Let  $G$  be a group and  $H$  be a normal subgroup of  $G$ . Define  $\sim$  on  $G$  by  $a \sim b$  iff  $a^{-1}b \in H$ . Then  $G/\sim$  is a group under  $[a][b] = [ab]$ .

**Definition 5.62** (Quotient Group). Let  $G$  be a group and  $H$  be a normal subgroup of  $G$ . The quotient group  $G/H$  is  $G/\sim$  where  $a \sim b$  iff  $a^{-1}b \in H$ , under  $[a][b] = [ab]$  or  $(aH)(bH) = abH$ .

**Corollary 5.62.1.** Let  $H$  be a normal subgroup of a group  $G$ . For every group homomorphism  $\phi : G \rightarrow G'$  such that  $H \subseteq \ker \phi$ , there exists a unique group homomorphism  $\bar{\phi} : G/H \rightarrow G'$  such that the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ & \searrow \pi & \nearrow \bar{\phi} \\ & G/H & \end{array}$$

**Proposition 5.63.**  $\mathbb{Z}/n\mathbb{Z}$  has exactly  $n$  elements.

PROOF: Every integer is congruent to one of  $0, 1, \dots, n-1$  by the division algorithm, and no two of them are congruent to one another, since if  $0 \leq i < j < n$  then  $0 < j-i < n$ . □

**Proposition 5.64.** Let  $m$  and  $n$  be integers with  $n > 0$ . The order of  $m$  in  $\mathbb{Z}/n\mathbb{Z}$  is  $\frac{n}{\gcd(m,n)}$ .

PROOF: By Proposition 4.16 since the order of 1 is  $n$ . □

**Proposition 5.65.** The integer  $m$  generates  $\mathbb{Z}/n\mathbb{Z}$  if and only if  $\gcd(m, n) = 1$ .

PROOF: By Proposition 5.64. □

**Corollary 5.65.1.** *If  $p$  is prime then every non-zero element in  $\mathbb{Z}/p\mathbb{Z}$  is a generator.*

**Proposition 5.66.**

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$$

PROOF: Every permutation of  $\{(1,0), (0,1), (1,1)\}$  gives an automorphism of  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .  $\square$

**Example 5.67.** Not all monomorphisms split in  $\mathbf{Grp}$ .

Define  $\phi : \mathbb{Z}/3\mathbb{Z} \rightarrow S_3$  by

$$\phi(0) = \text{id}_3, \quad \phi(1) = (1\ 3\ 2), \quad \phi(2) = (1\ 2\ 3) .$$

Then  $\phi$  is monic but has no retraction.

For if  $r : S_3 \rightarrow \mathbb{Z}/3\mathbb{Z}$  is a retraction, then we would have

$$r(1\ 2) + r(2\ 3) = 1, \quad r(2\ 3) + r(1\ 2) = 2$$

which is impossible.

**Proposition 5.68.** *Let  $G$  be a group,  $H$  a subgroup of  $G$ , and  $g \in G$ . The function that maps  $h$  to  $gh$  is a bijection  $H \cong gH$ .*

PROOF: By Cancellation.  $\square$

**Proposition 5.69.** *Let  $G$  be a group,  $H$  a subgroup of  $G$ , and  $g \in G$ . The function that maps  $h$  to  $hg$  is a bijection  $H \cong Hg$ .*

PROOF: By Cancellation.  $\square$

**Proposition 5.70.** *Let  $H$  and  $K$  be finite subgroups of a group  $G$ . Then*

$$|HK| = \frac{|H||K|}{|H \cap K|} .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : \{hK : h \in H\} \rightarrow H/(H \cap K)$  be the function  $f(hK) = h(H \cap K)$

PROOF: This is well-defined because if  $hK = h'K$  then  $h^{-1}h' \in H \cap K$  so  $h(H \cap K) = h'(H \cap K)$ .

$\langle 1 \rangle 2$ .  $f$  is injective.

PROOF: If  $h(H \cap K) = h'(H \cap K)$  then  $hK = h'K$ .

$\langle 1 \rangle 3$ .  $f$  is surjective.

PROOF: Clear.

$\langle 1 \rangle 4$ .

$$\frac{|HK|}{|K|} = \frac{|H|}{|H \cap K|}$$

$\square$

## 5.9 Congruence

**Definition 5.71** (Congruence). Given integers  $a, b, n$  with  $n$  positive, we say  $a$  is congruent to  $b$  modulo  $n$ , and write  $a \equiv b \pmod{n}$ , iff  $a + n\mathbb{Z} = b + n\mathbb{Z}$  in  $\mathbb{Z}/n\mathbb{Z}$ .

**Proposition 5.72.** Given integers  $a, b, n$  with  $n$  positive, we have  $a \equiv b \pmod{n}$  iff  $n \mid a - b$ .

PROOF: By Proposition 5.56.  $\square$

**Proposition 5.73.** If  $a \equiv a' \pmod{n}$  and  $b \equiv b' \pmod{n}$  then  $a + b \equiv a' + b' \pmod{n}$ .

PROOF: If  $n \mid a' - a$  and  $n \mid b' - b$  then  $n \mid (a' + b') - (a + b)$ .  $\square$

**Proposition 5.74.** If  $a \equiv a' \pmod{n}$  and  $b \equiv b' \pmod{n}$  then  $ab \equiv a'b' \pmod{n}$ .

PROOF: If  $n \mid a' - a$  and  $n \mid b' - b$  then  $n \mid a'b' - ab = a'(b' - b) + (a' - a)b$ .  $\square$

## 5.10 Cyclic Groups

**Definition 5.75** (Cyclic Group). The *cyclic* groups are  $\mathbb{Z}$  and  $\mathbb{Z}/n\mathbb{Z}$  for positive integers  $n$ .

**Proposition 5.76.** If  $m$  and  $n$  are positive integers with  $\gcd(m, n) = 1$  then  $C_{mn} \cong C_m \times C_n$ .

PROOF: The function that maps  $x$  to  $(x \bmod m, x \bmod n)$  is an isomorphism.  $\square$

**Proposition 5.77.** Let  $G$  be a group and  $g \in G$ . Then  $\langle g \rangle$  is cyclic.

PROOF: If  $g$  has finite order then  $\langle g \rangle \cong C_{|g|}$ , otherwise  $\langle g \rangle \cong \mathbb{Z}$ .  $\square$

**Proposition 5.78.** Every finitely generated subgroup of  $\mathbb{Q}$  is cyclic.

PROOF:

$\langle 1 \rangle$ 1. LET:  $G = \langle a_1/b, \dots, a_n/b \rangle$  where  $a_1, \dots, a_n, b$  are integers with  $b > 0$

$\langle 1 \rangle$ 2. LET:  $a = \gcd(a_1, \dots, a_n)$

$\langle 1 \rangle$ 3.  $G = \langle a/b \rangle$

$\square$

**Corollary 5.78.1.**  $\mathbb{Q}$  is not finitely generated.

## 5.11 Euler's $\phi$ -function

**Definition 5.79.** For  $n$  a positive integer, let  $(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$ .

PROOF: We prove this is well-defined.

$\langle 1 \rangle$ 1. If  $m \equiv m' \pmod{n}$  and  $\gcd(m, n) = 1$  then  $\gcd(m', n) = 1$ .

- $\langle 2 \rangle 1$ . PICK integers  $a, b$  such that  $am + bn = 1$
- $\langle 2 \rangle 2$ . PICK an integer  $c$  such that  $m' - m = cn$
- $\langle 2 \rangle 3$ .  $am' + (b - ac)n = 1$

□

**Example 5.80.** For any positive integer  $n$ , the set

$$(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$$

is a group under multiplication.

PROOF:

- $\langle 1 \rangle 1$ . If  $\gcd(m_1, n) = \gcd(m_2, n) = 1$  then  $\gcd(m_1 m_2, n) = 1$
- $\langle 2 \rangle 1$ . PICK integers  $a, b, c, d$  such that  $am_1 + bn = cm_2 + dn = 1$
- $\langle 2 \rangle 2$ .  $acm_1 m_2 + (bcm_2 + d)n = 1$
- $\langle 1 \rangle 2$ . Multiplication is associative.
- $\langle 1 \rangle 3$ . 1 is the identity element.
- $\langle 1 \rangle 4$ . Every element has an inverse.
- $\langle 2 \rangle 1$ . LET:  $a \in (\mathbb{Z}/n\mathbb{Z})^*$
- $\langle 2 \rangle 2$ . PICK integers  $b, c$  such that  $ab + cn = 1$
- $\langle 2 \rangle 3$ .  $ab = 1$  in  $(\mathbb{Z}/n\mathbb{Z})^*$

□

**Definition 5.81.** For  $n$  a positive integer, let  $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$ .

**Proposition 5.82.** If  $n$  is an odd positive integer then  $\phi(2n) = \phi(n)$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $n$  be an odd positive integer.
- $\langle 1 \rangle 2$ . For any integer  $m$ , if  $\gcd(m, n) = 1$  then  $\gcd(2m + n, 2n) = 1$   
 PROOF: For  $p$  a prime, if  $p \mid 2m + n$  and  $p \mid 2n$  then  $p \neq 2$  (since  $2m + n$  is odd) so  $p \mid n$  and hence  $p \mid m$ , which is a contradiction.
- $\langle 1 \rangle 3$ . For any integer  $r$ , if  $\gcd(r, 2n) = 1$  then  $\gcd(\frac{r+n}{2}, n) = 1$   
 PROOF: If  $p \mid n$  and  $p \mid \frac{r+n}{2}$  then  $p \mid r + n$  so  $p \mid r$  which is a contradiction.
- $\langle 1 \rangle 4$ . The function that maps  $m$  to  $2m + n$  is a bijection between  $(\mathbb{Z}/n\mathbb{Z})^*$  and  $(\mathbb{Z}/2n\mathbb{Z})^*$ .

□

**Theorem 5.83.** For any positive integer  $n$  we have

$$\sum_{m>0, m|n} \phi(m) = n .$$

PROOF:

- $\langle 1 \rangle 1$ . Define  $\chi : \{0, 1, \dots, n-1\} \rightarrow \{(m, d) : m > 0, m \mid n, d \text{ generates } \langle n/m \rangle\}$   
 by:  $\chi(x) = (\gcd(x, n), x)$ .
- $\langle 1 \rangle 2$ .  $\chi$  is injective.
- $\langle 1 \rangle 3$ .  $\chi$  is surjective.

PROOF: Given  $(m, d)$  such that  $d$  generates  $\langle n/m \rangle$  we have  $\chi(d) = (m, d)$ .

(1)4.  $n = \sum_{m>0, m|n} \phi(m)$

PROOF: Since  $\langle n/m \rangle \cong C_m$  and so has  $\phi(m)$  generators.

□

**Proposition 5.84.** *For any positive integers  $a$  and  $n$ , we have  $n \mid \phi(a^n - 1)$ .*

PROOF: Since the order of  $a$  is  $n$  in  $(\mathbb{Z}/(a^n - 1)\mathbb{Z})^*$ . □

**Theorem 5.85** (Euler's Theorem). *For any coprime integers  $a$  and  $n$  we have  $a^{\phi(n)} \equiv a \pmod{n}$ .*

PROOF: Immediate from Lagrange's Theorem. □

## 5.12 Commutator Subgroup

**Definition 5.86** (Commutator Subgroup). Let  $G$  be a group. The *commutator subgroup*  $[G, G]$  is the subgroup generated by the elements of the form  $aba^{-1}b^{-1}$ .

**Proposition 5.87.** *The commutator subgroup is normal.*

PROOF: Since

$$\begin{aligned} & ga_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_nb_na_n^{-1}b_n^{-1}g^{-1} \\ &= (ga_1g^{-1})(gb_1g^{-1})(ga_1g^{-1})^{-1}(gb_1g^{-1})^{-1} \cdots (ga_ng^{-1})(gb_ng^{-1})(ga_ng^{-1})^{-1}(gb_ng^{-1})^{-1}. \end{aligned} \quad \square$$

## 5.13 Presentations

**Definition 5.88** (Presentation). A *presentation* of a group  $G$  is a pair  $(A, R)$  where  $A$  is a set and  $R \subseteq F(A)$  is a set of words such that

$$G \cong F(A)/N(R)$$

where  $N(R)$  is the smallest normal subgroup of  $F(A)$  that includes  $R$ .

**Example 5.89.** The free group on a set  $A$  is presented by  $(A, \emptyset)$ .

**Example 5.90.**  $S_3$  is presented by  $(x, y|x^2, y^3, xyxy)$ .

**Example 5.91.**  $(a, b | a^2, b^2, (ab)^n)$  is a presentation of  $D_{2n}$ .

**Proposition 5.92** (Word Problem). *Let  $(A, R)$  be a presentation of the group  $G$ . Let  $w_1, w_2 \in F(A)$  be two words. Then it is undecidable in general if  $w_1N(R) = w_2N(R)$  in  $G$ .*

**Definition 5.93** (Finitely Presented). A group is *finitely presented* iff it has a presentation  $(A, R)$  where both  $A$  and  $R$  are finite.

**Proposition 5.94.** *Let  $(A|R)$  be a presentation of  $G$  and  $(A'|R')$  a presentation of  $H$ . Assume w.l.o.g.  $A$  and  $A'$  are disjoint. Then the group  $G * G'$  presented by  $(A \cup A' | R \cup R')$  is the coproduct of  $G$  and  $G'$  in  $\mathbf{Grp}$ .*

$$\begin{array}{ccccc}
A & \longrightarrow & A \cup A' & \longleftarrow & A' \\
\downarrow & & \downarrow & & \downarrow \\
F(A) & \longrightarrow & F(A \cup A') & \longleftarrow & F(A') \\
\downarrow & & \downarrow & & \downarrow \\
G & \xrightarrow{\kappa_1} & G * G' & \xleftarrow{\kappa_2} & G'
\end{array}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\kappa_1 : G \rightarrow G * G'$  and  $\kappa_2 : G' \rightarrow G * G'$  be the unique homomorphisms that make the diagram above commute.

$\langle 1 \rangle 2$ . LET:  $\phi : G \rightarrow H$  and  $\psi : G' \rightarrow H$  be any homomorphisms.

$\langle 1 \rangle 3$ . LET:  $[\phi, \psi] : F(A \cup A') \rightarrow H$  be the unique homomorphism such that ...

$\langle 1 \rangle 4$ .  $R \cup R' \subseteq \ker[\phi, \psi]$

$\langle 1 \rangle 5$ .  $[\phi, \psi]$  factors uniquely through the morphism  $F(A \cup A') \rightarrow G * G'$

□

## 5.14 Index of a Subgroup

**Definition 5.95** (Index). Let  $G$  be a group and  $H$  a subgroup of  $G$ . The *index* of  $H$  in  $G$ , denoted  $|G : H|$ , is the number of left cosets of  $H$  in  $G$  if this is finite, otherwise  $\infty$ .

**Theorem 5.96** (Lagrange's Theorem). Let  $G$  be a finite group and  $H$  a subgroup of  $G$ . Then

$$|G| = |G : H||H| .$$

PROOF:  $G/H$  is a partition of  $G$  into  $|G : H|$  subsets, each of size  $|H|$ . □

**Corollary 5.96.1**. For  $p$  a prime number, the only group of order  $p$  is  $C_p$ .

PROOF: Let  $G$  be a group of order  $p$  and  $g \in G$  with  $g \neq e$ . Then  $|\langle g \rangle|$  divides  $p$  and is not 1, hence is  $p$ , that is,  $G = \langle g \rangle$ . □

**Corollary 5.96.2** (Fermat's Little Theorem). Let  $p$  be a prime number and  $a$  any integer. Then  $a^p \equiv a \pmod{p}$ .

PROOF: If  $p \mid a$  then  $a^p \equiv a \equiv 0 \pmod{p}$ . Otherwise, we have  $a^{p-1} \equiv 1 \pmod{p}$  by applying Lagrange's Theorem to  $(\mathbb{Z}/p\mathbb{Z})^*$ . □

**Example 5.97**. It is not true that, if  $n \mid |G|$ , then  $G$  has a subgroup of order  $n$ . The group  $A_4$  has order 12 but no subgroup of order 6.

**Theorem 5.98** (Cauchy's Theorem). Let  $G$  be a finite group. If  $p$  is prime and  $p \mid |G|$  then  $G$  has a subgroup of order  $p$ .

**Proposition 5.99**. Let  $G$  be a group. Let  $K$  be a subgroup of  $G$  and  $H$  a subgroup of  $K$ . If  $|G : H|$ ,  $|G : K|$  and  $|K : H|$  are all finite then

$$|G : H| = |G : K||K : H| .$$

PROOF:

- $\langle 1 \rangle 1$ . LET:  $G/K = \{g_1K, g_2K, \dots, g_mK\}$
- $\langle 1 \rangle 2$ . LET:  $K/H = \{k_1H, k_2H, \dots, k_nH\}$
- $\langle 1 \rangle 3$ .  $G/H = \{g_ik_jH : 1 \leq i \leq m, 1 \leq j \leq n\}$ 
  - $\langle 2 \rangle 1$ . LET:  $g \in G$
  - $\langle 2 \rangle 2$ . PICK  $i$  such that  $gK = g_iK$
  - $\langle 2 \rangle 3$ .  $g^{-1}g_i \in K$
  - $\langle 2 \rangle 4$ . PICK  $j$  such that  $g^{-1}g_iH = k_jH$
  - $\langle 2 \rangle 5$ .  $g^{-1}g_ik_j \in H$
  - $\langle 2 \rangle 6$ .  $gH = g_ik_jH$
- $\langle 1 \rangle 4$ . If  $g_ik_jH = g_{i'}k_{j'}H$  then  $i = i'$  and  $j = j'$ .
  - $\langle 2 \rangle 1$ . ASSUME:  $g_ik_jH = g_{i'}k_{j'}H$
  - $\langle 2 \rangle 2$ .  $g_iK = g_{i'}K$
  - $\langle 2 \rangle 3$ .  $i = i'$
  - $\langle 2 \rangle 4$ .  $k_jH = k_{j'}H$
  - $\langle 2 \rangle 5$ .  $j = j'$

□

## 5.15 Cokernels

**Proposition 5.100.** *Let  $\phi : G \rightarrow H$  be a homomorphism between groups. Then there exists a group  $K$  and homomorphism  $\pi : H \rightarrow K$  that is initial with respect to all homomorphism  $\alpha : H \rightarrow L$  such that  $\alpha \circ \phi = 0$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $N$  be the intersection of all the normal subgroups of  $H$  that include  $\text{im } \phi$ .
- $\langle 1 \rangle 2$ . LET:  $K = H/N$  and  $\pi$  be the canonical homomorphism.
- $\langle 1 \rangle 3$ . LET:  $\pi \circ \phi = 0$
- $\langle 1 \rangle 4$ . LET:  $\alpha : H \rightarrow L$  satisfy  $\alpha \circ \phi = 0$
- $\langle 1 \rangle 5$ .  $\text{im } \phi \subseteq \ker \alpha$
- $\langle 1 \rangle 6$ .  $N \subseteq \ker \alpha$
- $\langle 1 \rangle 7$ . There exists a unique  $\bar{\alpha} : H/\text{im } \phi \rightarrow L$  such that  $\bar{\alpha} \circ \pi = \alpha$

□

**Definition 5.101** (Cokernel). For any homomorphism  $\phi : G \rightarrow H$  in **Grp**, the *cokernel* of  $\phi$  is the group  $\text{coker } \phi$  and homomorphism  $\pi : H \rightarrow \text{coker } \phi$  that is initial among homomorphisms  $\alpha : H \rightarrow L$  such that  $\alpha \circ \phi = 0$ .

**Example 5.102.** It is not true that a homomorphism with trivial cokernel is epi. The inclusion  $\langle (1 \ 2) \rangle \hookrightarrow S_3$  has trivial cokernel but is not epi.

## 5.16 Cayley Graphs

**Definition 5.103** (Cayley Graph). Let  $G$  be a finitely generated group. Let  $A$  be a finite set of generators for  $G$ . The *Cayley graph* of  $G$  with respect to  $A$  is

the directed graph whose vertices are the elements of  $G$ , with an edge  $g_1 \rightarrow g_2$  labelled by  $a \in A$  iff  $g_2 = g_1 a$ .

**Proposition 5.104.**  *$G$  is the free group on  $A$  iff the Cayley graph with respect to  $A$  is a tree.*

PROOF: Both are equivalent to saying that the product of two different strings of elements of  $A$  and/or their inverses are not equal.  $\square$



## Chapter 6

# Abelian Groups

**Definition 6.1** (Abelian Group). A group is *Abelian* iff any two elements commute.

In an Abelian group  $G$ , we often denote the group operation by  $+$ , the identity element by  $0$  and the inverse of an element  $g$  by  $-g$ . We write  $ng$  for  $g^n$  ( $g \in G, n \in \mathbb{Z}$ ).

**Example 6.2.** Every group of order  $\leq 4$  is Abelian.

**Example 6.3.** For any positive integer  $n$ , we have  $\mathbb{Z}/n\mathbb{Z}$  is an Abelian group under addition.

**Example 6.4.**  $S_n$  is not Abelian for  $n \geq 3$ . If  $x = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$  and  $y = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$  then  $xy = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$  and  $yx = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ .

**Example 6.5.** There are 42 Abelian groups of order 1024 up to isomorphism.

**Proposition 6.6.** Let  $G$  be a group. If  $g^2 = e$  for all  $g \in G$  then  $G$  is Abelian.

PROOF: For any  $g, h \in G$  we have

$$ghgh = e$$

$$\therefore hgh = g \quad (\text{multiplying on the left by } g)$$

$$\therefore hg = gh \quad (\text{multiplying on the right by } h) \square$$

**Proposition 6.7.** Let  $G$  be a group. Then  $G$  is Abelian if and only if the function that maps  $g$  to  $g^{-1}$  is a group homomorphism.

PROOF:

(1)1. If  $G$  is Abelian then the function that maps  $g$  to  $g^{-1}$  is a group homomorphism.

PROOF: Since  $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$ .

(1)2. If the function that maps  $g$  to  $g^{-1}$  is a group homomorphism then  $G$  is Abelian.

PROOF: Since  $gh = (g^{-1})^{-1}(h^{-1})^{-1} = (g^{-1}h^{-1})^{-1} = hg$ .  
 $\square$

**Proposition 6.8.** *Let  $G$  be a group. Then  $G$  is Abelian if and only if the function that maps  $g$  to  $g^2$  is a group homomorphism.*

PROOF:

$\langle 1 \rangle 1$ . If  $G$  is Abelian then the function that maps  $g$  to  $g^2$  is a group homomorphism.

PROOF: Since  $(gh)^2 = g^2h^2$ .

$\langle 1 \rangle 2$ . If the function that maps  $g$  to  $g^2$  is a group homomorphism then  $G$  is Abelian.

PROOF: Since we have  $(gh)^2 = ghgh = g^2h^2$  and so  $hg = gh$ .

$\square$

**Proposition 6.9.** *Let  $G$  be a group. Then  $G$  is Abelian if and only if the homomorphism  $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$  is the trivial homomorphism.*

PROOF:

$\langle 1 \rangle 1$ . If  $G$  is Abelian then  $\gamma$  is trivial.

PROOF: Since  $\gamma_g(a) = gag^{-1} = a$ .

$\langle 1 \rangle 2$ . If  $\gamma$  is trivial then  $G$  is Abelian.

PROOF: If  $\gamma_g(a) = gag^{-1} = a$  for all  $g$  and  $a$  then  $ga = ag$  for all  $g, a$ .

$\square$

**Proposition 6.10.** *Let  $G$  be an Abelian group. Let  $g, h \in G$ . If  $g$  has maximal finite order in  $G$ , and  $h$  has finite order, then  $|h| \mid |g|$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $|h| \nmid |g|$ .

$\langle 1 \rangle 2$ . PICK a prime  $p$  such that  $|g| = p^m r$ ,  $|h| = p^n s$  where  $p \nmid r$ ,  $p \nmid s$  and  $m < n$ .

$\langle 1 \rangle 3$ .  $|g^{p^m} h^s| = p^n r$

PROOF: Proposition 4.19.

$\langle 1 \rangle 4$ .  $|g| < |g^{p^m} h^s|$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This contradicts the maximality of  $|g|$ .

$\square$

**Proposition 6.11.** *If  $p$  is prime then  $(\mathbb{Z}/p\mathbb{Z})^*$  is cyclic.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $g$  be an element of maximal order in  $(\mathbb{Z}/p\mathbb{Z})^*$ .

$\langle 1 \rangle 2$ . For all  $h \in (\mathbb{Z}/p\mathbb{Z})^*$  we have  $h^{|g|} = 1$ .

PROOF: Proposition 6.10.

$\langle 1 \rangle 3$ . There are at most  $|g|$  elements  $x$  such that  $x^{|g|} = 1$  in  $\mathbb{Z}/p\mathbb{Z}$

$\langle 1 \rangle 4$ .  $p - 1 \leq |g|$

$\langle 1 \rangle 5$ .  $|g| = p - 1$

$\langle 1 \rangle 6$ .  $g$  generates  $(\mathbb{Z}/p\mathbb{Z})^*$ .

□

**Example 6.12.**  $(\mathbb{Z}/12\mathbb{Z})^*$  is not cyclic. Its elements are 1, 5, 7 and 11 with orders 1, 2, 2 and 2.

**Theorem 6.13** (Wilson's Theorem). *A positive integer  $p$  is prime if and only if  $(p-1)! \equiv 1 \pmod{p}$ .*

$\langle 1 \rangle 1$ . If  $p$  is prime then  $(p-1)! \equiv 1 \pmod{p}$ .

$\langle 2 \rangle 1$ . ASSUME:  $p$  is prime.

$\langle 2 \rangle 2$ .  $(p-1)!$  is the product of all the elements of  $(\mathbb{Z}/p\mathbb{Z})^*$

$\langle 2 \rangle 3$ . The only element of  $(\mathbb{Z}/p\mathbb{Z})^*$  with order 2 is  $-1$ .

$\langle 2 \rangle 4$ .  $(p-1)! \equiv -1 \pmod{p}$

PROOF: Proposition 4.20.

$\langle 1 \rangle 2$ . If  $(p-1)! \equiv -1 \pmod{p}$  then  $p$  is prime.

$\langle 2 \rangle 1$ . ASSUME: (

$(p-1)! \equiv -1 \pmod{p}$ )

$\langle 2 \rangle 2$ . LET:  $d$  be a proper divisor of  $p$ .

PROVE:  $d = 1$

$\langle 2 \rangle 3$ .  $d \mid (p-1)!$

$\langle 2 \rangle 4$ .  $d \mid 1$

PROOF: Since  $d \mid p \mid (p-1)! + 1$ .

$\langle 2 \rangle 5$ .  $d = 1$

□

**Proposition 6.14.** *If  $p$  and  $q$  are distinct odd primes then  $(\mathbb{Z}/pq\mathbb{Z})^*$  is not cyclic.*

PROOF:

$\langle 1 \rangle 1$ .  $|(\mathbb{Z}/pq\mathbb{Z})^*| = (p-1)(q-1)$

$\langle 1 \rangle 2$ . LET:  $g \in (\mathbb{Z}/pq\mathbb{Z})^*$

PROVE:  $g$  does not have order  $(p-1)(q-1)$

$\langle 1 \rangle 3$ .  $g^{(p-1)(q-1)/2} \equiv 1 \pmod{p}$

$\langle 1 \rangle 4$ .  $g^{(p-1)(q-1)/2} \equiv 1 \pmod{q}$

$\langle 1 \rangle 5$ .  $pq \mid g^{(p-1)(q-1)/2} - 1$

$\langle 1 \rangle 6$ .  $g^{(p-1)(q-1)/2} \equiv 1 \pmod{pq}$

$\langle 1 \rangle 7$ .  $|g| \mid (p-1)(q-1)/2$

□

**Proposition 6.15.** *For any prime  $p$ , we have  $\text{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : \text{Aut}_{\mathbf{Grp}}(C_p) \rightarrow (\mathbb{Z}/p\mathbb{Z})^*$  be the function  $\phi(\alpha) = \alpha(1)$ .

PROOF:  $\alpha(1)$  has order  $p$  in  $C_p$  and so is coprime with  $p$ .

$\langle 1 \rangle 2$ .  $\phi$  is a homomorphism.

PROOF:  $\phi(\alpha \circ \beta) = \alpha(\beta(1)) = \alpha(\beta(1)1) = \beta(1)\alpha(1) = \phi(\alpha)\phi(\beta)$

$\langle 1 \rangle 3$ .  $\phi$  is injective.

PROOF: If  $\phi(\alpha) = \phi(\beta)$  then for any  $n$  we have  $\alpha(n) = n\alpha(1) = n\phi(\alpha) = n\phi(\beta) = n\beta(1) = \beta(n)$ .

$\langle 1 \rangle 4.$   $\phi$  is surjective.

PROOF: For any  $r \in (\mathbb{Z}/p\mathbb{Z})^*$  we have  $r = \phi(\alpha)$  where  $\alpha(n) = nr \bmod p$ .

$\langle 1 \rangle 5.$   $(\mathbb{Z}/p\mathbb{Z})^* \cong C_{p-1}$

□

**Proposition 6.16.** *Given a set  $A$  and an Abelian group  $H$ , the set  $H^A$  is an Abelian group under*

$$(\phi + \psi)(a) = \phi(a) + \psi(a) \quad (\phi, \psi \in H^A, a \in A) .$$

PROOF:

$\langle 1 \rangle 1.$   $\phi + (\psi + \chi) = (\phi + \psi) + \chi$

$\langle 1 \rangle 2.$   $\phi + \psi = \psi + \phi$

$\langle 1 \rangle 3.$  LET:  $0 : A \rightarrow H$  be the function  $0(a) = 0$ .

$\langle 1 \rangle 4.$   $\phi + 0 = 0 + \phi = \phi$

$\langle 1 \rangle 5.$  Given  $\phi : A \rightarrow H$ , define  $-\phi : A \rightarrow H$  by  $(-\phi)(a) = -(\phi(a))$ .

$\langle 1 \rangle 6.$   $\phi + (-\phi) = (-\phi) + \phi = 0$

□

**Proposition 6.17.** *Given a group  $G$  and an Abelian group  $H$ , the set  $\mathbf{Grp}[G, H]$  is a subgroup of  $H^G$ .*

PROOF:

$\langle 1 \rangle 1.$  Given  $\phi, \psi : G \rightarrow H$  group homomorphisms, we have  $\phi - \psi$  is a group homomorphism.

PROOF:

$$\begin{aligned} (\phi - \psi)(g + g') &= \phi(g + g') - \psi(g + g') \\ &= \phi(g) + \phi(g') - \psi(g) - \psi(g') \\ &= \phi(g) - \psi(g) + \phi(g') - \psi(g') \\ &= (\phi - \psi)(g) + (\phi - \psi)(g') \end{aligned}$$

□

**Proposition 6.18.** *Let  $G$  be a group. The following are equivalent.*

1.  $\text{Inn}(G)$  is cyclic.

2.  $\text{Inn}(G)$  is trivial.

3.  $G$  is Abelian.

PROOF:

$\langle 1 \rangle 1.$   $1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $\text{Inn}(G) = \langle \gamma_g \rangle$

$\langle 2 \rangle 2.$   $g$  commutes with every element of  $G$

$\langle 3 \rangle 1.$  LET:  $x \in G$

$\langle 3 \rangle 2.$  PICK  $n \in \mathbb{Z}$  such that  $\gamma_x = \gamma_g^n$

$\langle 3 \rangle 3.$   $\forall y \in G. xyx^{-1} = g^n yg^{-n}$

$\langle 3 \rangle 4.$   $xgx^{-1} = g$

$\langle 2 \rangle 3.$   $\gamma_g = \text{id}_G$   
 $\langle 1 \rangle 2.$   $2 \Rightarrow 3$   
 $\langle 2 \rangle 1.$  ASSUME:  $\forall g \in G. \gamma_g = \text{id}_G$   
 $\langle 2 \rangle 2.$  LET:  $x, y \in G$   
 $\langle 2 \rangle 3.$   $\gamma_x(y) = y$   
 $\langle 2 \rangle 4.$   $xyx^{-1} = y$   
 $\langle 2 \rangle 5.$   $xy = yx$   
 $\langle 1 \rangle 3.$   $3 \Rightarrow 2$   
 PROOF: If  $xy = yx$  for all  $x, y$  then  $\gamma_x(y) = y$  for all  $x, y$ .  
 $\langle 1 \rangle 4.$   $2 \Rightarrow 1$   
 PROOF: Easy.  
 $\square$

**Corollary 6.18.1.** *If  $\text{Aut}_{\text{Grp}}(G)$  is cyclic then  $G$  is Abelian.*

**Proposition 6.19.** *Every subgroup of an Abelian group is normal.*

PROOF: Let  $G$  be an Abelian group and  $N$  a subgroup of  $G$ . Given  $g \in G$  and  $n \in N$  we have  $gng^{-1} = n \in N$ .  $\square$

**Proposition 6.20.** *For any group  $G$ , the group  $G/[G, G]$  is Abelian.*

PROOF: For any  $g, h \in G$  we have  
 $gh(hg)^{-1} \in [G, G]$

$$\therefore gh[G, G] = hg[G, G] \quad \square$$

**Proposition 6.21.** *Let  $G$  be a finite Abelian group. Let  $p$  be a prime divisor of  $|G|$ . Then  $G$  has an element of order  $p$ .*

PROOF:

$\langle 1 \rangle 1.$  ASSUME: as induction hypothesis the result holds for all groups smaller than  $G$ .  
 $\langle 1 \rangle 2.$  PICK  $g \in G - \{0\}$ .  
 $\langle 1 \rangle 3.$  PICK an element  $h \in \langle g \rangle$  with prime order  $q$ .  
 $\langle 1 \rangle 4.$  CASE:  $q = p$   
 PROOF:  $h$  is the required element.  
 $\langle 1 \rangle 5.$  CASE:  $q \neq p$   
 $\langle 2 \rangle 1.$  PICK  $r \in G$  such that  $r + \langle h \rangle$  has order  $p$  in  $G/\langle h \rangle$ .  
 PROOF: By induction hypothesis since  $|G/\langle h \rangle| = |G|/q$ .  
 $\langle 2 \rangle 2.$   $pr \in \langle h \rangle$   
 $\langle 2 \rangle 3.$  PICK  $k$  such that  $pr = kh$   
 $\langle 2 \rangle 4.$   $pqr = e$   
 $\langle 2 \rangle 5.$   $qr$  has order  $p$ .  
 $\square$

**Corollary 6.21.1.** *For  $n$  an odd integer, any Abelian group of order  $2n$  has exactly one element of order 2.*

PROOF: If  $x$  and  $y$  are distinct elements of order 2 then  $\langle x, y \rangle = \{e, x, y, xy\}$  has size 4 and so  $4 \mid 2n$  which is a contradiction.  $\square$

**Example 6.22.** It is not true that, if  $G$  is a finite group and  $d \mid |G|$ , then  $G$  has an element of order  $d$ . The quaternion group has no element of order 4.

**Proposition 6.23.** *If  $G$  is a finite Abelian group and  $d \mid |G|$  then  $G$  has a subgroup of size  $d$ .*

PROOF:

- $\langle 1 \rangle 1$ . ASSUME: as induction hypothesis the result is true for all  $d' < d$ .
- $\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $d \neq 1$ .
- $\langle 1 \rangle 3$ . PICK a prime  $p$  such that  $p \mid d$ .
- $\langle 1 \rangle 4$ . PICK an element  $g \in G$  of order  $p$ .
- $\langle 1 \rangle 5$ .  $d/p \mid |G/\langle g \rangle|$
- $\langle 1 \rangle 6$ . PICK a subgroup  $H$  of  $G/\langle g \rangle$  of size  $d/p$ .
- $\langle 1 \rangle 7$ .  $\pi^{-1}(H)$  is a subgroup of  $G$  of size  $d$ .

□

## 6.1 The Category of Abelian Groups

**Definition 6.24** (Category of Abelian Groups). Let **Ab** be the full subcategory of **Grp** whose objects are the Abelian groups.

**Definition 6.25** (Direct Sum). Given Abelian groups  $G$  and  $H$ , we also call the direct product of  $G$  and  $H$  the *direct sum* and denote it  $G \oplus H$ .

**Proposition 6.26.** *Given Abelian groups  $G$  and  $H$ , the direct sum  $G \oplus H$  is the coproduct of  $G$  and  $H$  in **Ab**.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\kappa_1 : G \rightarrow G \oplus H$  be the group homomorphism  $\kappa_1(g) = (g, e_H)$ .
- $\langle 1 \rangle 2$ . LET:  $\kappa_2 : H \rightarrow G \oplus H$  be the group homomorphism  $\kappa_2(h) = (e_G, h)$ .
- $\langle 1 \rangle 3$ . Given group homomorphism  $\phi : G \rightarrow K$  and  $\psi : H \rightarrow K$ , define  $[\phi, \psi] : G \oplus H \rightarrow K$  by  $[\phi, \psi](g, h) = \phi(g) + \psi(h)$ .
- $\langle 1 \rangle 4$ .  $[\phi, \psi]$  is a group homomorphism.

PROOF:

$$\begin{aligned}
 [\phi, \psi]((g, h) + (g', h')) &= [\phi, \psi](g + g', h + h') \\
 &= \phi(g + g') + \psi(h + h') \\
 &= \phi(g) + \phi(g') + \psi(h) + \psi(h') \\
 &= \phi(g) + \psi(h) + \phi(g') + \psi(h') \\
 &= [\phi, \psi](g, h) + [\phi, \psi](g', h')
 \end{aligned}$$

- $\langle 1 \rangle 5$ .  $[\phi, \psi] \circ \kappa_1 = \phi$

PROOF:

$$\begin{aligned}
 [\phi, \psi](\kappa_1(g)) &= [\phi, \psi](g, e_H) \\
 &= \phi(g) + \psi(e_H) \\
 &= \phi(g) + e_K \\
 &= \phi(g)
 \end{aligned}$$

(1)6.  $[\phi, \psi] \circ \kappa_2 = \psi$

PROOF: Similar.

(1)7. If  $f : G \oplus H \rightarrow K$  is a group homomorphism with  $f \circ \kappa_1 = \phi$  and  $f \circ \kappa_2 = \psi$  then  $f = [\phi, \psi]$ .

PROOF:

$$\begin{aligned} f(g, h) &= f((g, e_H) + (e_G, h)) \\ &= f(\kappa_1(g)) + f(\kappa_2(h)) \\ &= \phi(g) + \psi(h) \end{aligned}$$

□

**Theorem 6.27.** *Every finitely generated Abelian group is a direct sum of cyclic groups.*

PROOF: TODO □

## 6.2 Free Abelian Groups

**Proposition 6.28.** *Let  $A$  be a set. Let  $\mathcal{F}^A$  be the category whose objects are pairs  $(G, j)$  where  $G$  is an Abelian group and  $j$  is a function  $A \rightarrow G$ , with morphisms  $f : (G, j) \rightarrow (H, k)$  the group homomorphisms  $f : G \rightarrow H$  such that  $f \circ j = k$ . Then  $\mathcal{F}^A$  has an initial object.*

PROOF:

(1)1. LET:  $\mathbb{Z}^{\oplus A}$  be the subgroup of  $\mathbb{Z}^A$  consisting of all functions  $\alpha : A \rightarrow \mathbb{Z}$  such that  $\alpha(a) = 0$  for only finitely many  $a \in A$ .

(1)2. LET:  $i : A \rightarrow \mathbb{Z}^{\oplus A}$  be the function such that  $i(a)(b) = 1$  if  $a = b$  and 0 if  $a \neq b$ .

(1)3. LET:  $G$  be any Abelian group and  $j : A \rightarrow G$  any function.

(1)4. The unique homomorphism  $\phi : \mathbb{Z}^{\oplus A} \rightarrow G$  required is defined by  $\phi(\alpha) = \sum_{a \in A} \alpha(a)j(a)$

□

**Definition 6.29** (Free Abelian Group). For any set  $A$ , the *free Abelian group* on  $A$  is the initial object  $(F^{ab}(A), i)$  in  $\mathcal{F}^A$ .

**Proposition 6.30.** *For any sets  $A$  and  $B$ , we have that  $F^{ab}(A + B)$  is the coproduct of  $F^{ab}(A)$  and  $F^{ab}(B)$  in **Grp**.*

$$\begin{array}{ccccc} & & G & & \\ & \nearrow f & \uparrow k & \nwarrow g & \\ F^{ab}(A) & \xrightarrow{\kappa_1} & F^{ab}(A+B) & \xleftarrow{\kappa_2} & F^{ab}(B) \\ \uparrow i_A & & \uparrow j & & \uparrow i_B \\ A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B \end{array}$$

PROOF:

- ⟨1⟩1. LET:  $i_A : A \rightarrow F^{ab}(A)$ ,  $i_B : B \rightarrow F^{ab}(B)$ ,  $j : A + B \rightarrow F^{ab}(A + B)$  be the canonical injections.
- ⟨1⟩2. LET:  $\kappa_1, \kappa_2$  be the unique group homomorphisms that make the diagram above commute.
- ⟨1⟩3. LET:  $G$  be any group and  $f : F^{ab}(A) \rightarrow G$ ,  $g : F^{ab}(B) \rightarrow G$  any group homomorphisms.
- ⟨1⟩4. LET:  $h : A + B \rightarrow G$  be the unique function such that  $h \circ \kappa_1 = f \circ i_A$  and  $h \circ \kappa_2 = g \circ i_B$ .
- ⟨1⟩5. LET:  $k : F^{ab}(A + B) \rightarrow G$  be the unique group homomorphism such that  $k \circ j = h$ .
- ⟨1⟩6.  $k$  is the unique group homomorphism such that  $k \circ \kappa_1 \circ i_A = f \circ i_A$  and  $k \circ \kappa_2 \circ i_B = g \circ i_B$ .
- ⟨1⟩7.  $k$  is the unique group homomorphism such that  $k \circ \kappa_1 = f$  and  $k \circ \kappa_2 = g$ .

□

**Proposition 6.31.** *For  $A$  and  $B$  finite sets, if  $F^{ab}(A) \cong F^{ab}(B)$  then  $A \cong B$ .*

PROOF:

- ⟨1⟩1. For any set  $C$ , define  $\sim$  on  $F^{ab}(C)$  by:  $f \sim f'$  iff there exists  $g \in F^{ab}(C)$  such that  $f - f' = 2g$ .
- ⟨1⟩2. For any set  $C$ ,  $\sim$  is an equivalence relation on  $F^{ab}(C)$ .
- ⟨1⟩3. For any set  $C$ , we have  $F^{ab}(C) / \sim$  is finite if and only if  $C$  is finite, in which case  $|F^{ab}(C) / \sim| = 2^{|C|}$ .

PROOF: There is a bijection between  $F^{ab}(C) / \sim$  and the finite subsets of  $C$ , which maps  $f$  to  $\{c \in C : f(c) \text{ is odd}\}$ .

- ⟨1⟩4. If  $F^{ab}(A) \cong F^{ab}(B)$  then  $A \cong B$ .

PROOF: If  $|F^{ab}(A) / \sim| = |F^{ab}(B) / \sim|$  then  $2^{|A|} = 2^{|B|}$  and so  $|A| = |B|$ .

□

**Proposition 6.32.** *Let  $G$  be an Abelian group. Then  $G$  is finitely generated if and only if there exists a surjective homomorphism  $\mathbb{Z}^{\oplus n} \rightarrow G$  for some  $n$ .*

PROOF:

- ⟨1⟩1. If  $G$  is finitely generated then there exists a surjective homomorphism  $\mathbb{Z}^{\oplus n} \rightarrow G$  for some  $n$ .

PROOF: Let  $G = \langle a_1, \dots, a_n \rangle$ . Define  $\phi : \mathbb{Z}^{\oplus n} \rightarrow G$  by  $\phi(i_1, \dots, i_n) = i_1 \cdot a_1 + \dots + i_n \cdot a_n$ .

- ⟨1⟩2. If there exists a surjective homomorphism  $\phi : \mathbb{Z}^{\oplus n} \rightarrow G$  for some  $n$  then  $G$  is finitely generated.

PROOF:  $G$  is generated by  $\phi(1, 0, \dots, 0)$ ,  $\phi(0, 1, 0, \dots, 0)$ ,  $\dots$ ,  $\phi(0, \dots, 0, 1)$ .

□

**Proposition 6.33.** *Let  $A$  be a set. Let  $i : A \hookrightarrow F(A)$  be the free group on  $A$ . Then  $\pi \circ i : A \rightarrow F(A) / [F(A), F(A)]$  is the free Abelian group on  $A$ .*



$$\begin{array}{ccc}
 & F(A)/[F(A), F(A)] & \\
 \uparrow \pi & \searrow h & \\
 F(A) & \xrightarrow{g} & G \\
 \uparrow i & \nearrow f & \\
 A & & 
 \end{array}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $G$  be an Abelian group and  $f : A \rightarrow G$  a function.

$\langle 1 \rangle 2$ . LET:  $g : F(A) \rightarrow G$  be the unique group homomorphism such that  $g \circ i = f$ .

$\langle 1 \rangle 3$ .  $[F(A), F(A)] \subseteq \ker g$

PROOF: For all  $x, y \in F(A)$  we have  $g(xyx^{-1}y^{-1}) = g(x) + g(y) - g(x) - g(y) = 0$ .

$\langle 1 \rangle 4$ . LET:  $h : F(A)/[F(A), F(A)] \rightarrow G$  be the unique group homomorphism such that  $h \circ \pi = g$ .

$\langle 1 \rangle 5$ .  $h$  is the unique group homomorphism such that  $h \circ \pi \circ i = f$ .

□

**Corollary 6.33.1.** *Let  $A$  and  $B$  be sets. Let  $F(A)$  and  $F(B)$  be the free groups on  $A$  and  $B$  respectively. If  $F(A) \cong F(B)$  then  $A \cong B$ .*

PROOF: Proposition 6.31. □

## 6.3 Cokernels

**Proposition 6.34.** *Let  $\phi : G \rightarrow H$  be a homomorphism between Abelian groups. Then there exists an Abelian group  $K$  and homomorphism  $\pi : H \rightarrow K$  that is initial with respect to all homomorphism  $\alpha : H \rightarrow L$  such that  $\alpha \circ \phi = 0$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $K = H/\text{im } \phi$  and  $\pi$  be the canonical homomorphism.

$\langle 1 \rangle 2$ . LET:  $\pi \circ \phi = 0$

$\langle 1 \rangle 3$ . LET:  $\alpha : H \rightarrow L$  satisfy  $\alpha \circ \phi = 0$

$\langle 1 \rangle 4$ .  $\text{im } \phi \subseteq \ker \alpha$

$\langle 1 \rangle 5$ . There exists a unique  $\bar{\alpha} : H/\text{im } \phi \rightarrow L$  such that  $\bar{\alpha} \circ \pi = \alpha$

□

**Definition 6.35** (Cokernel). For any homomorphism  $\phi : G \rightarrow H$  in **Ab**, the *cokernel* of  $\phi$  is the Abelian group  $\text{coker } \phi$  and homomorphism  $\pi : H \rightarrow \text{coker } \phi$  that is initial among homomorphisms  $\alpha : H \rightarrow L$  such that  $\alpha \circ \phi = 0$ .

**Proposition 6.36.**  $\pi : H \rightarrow \text{coker } \phi$  is initial among functions  $f : H \rightarrow X$  such that, for all  $x, y \in H$ , if  $x + \text{im } \phi = y + \text{im } \phi$  then  $f(x) = f(y)$ .

PROOF: Easy. □

**Proposition 6.37.** *Let  $\phi : G \rightarrow H$  be a homomorphism of Abelian groups. Then the following are equivalent.*

- $\phi$  is an epimorphism.
- $\text{coker } \phi$  is trivial.
- $\phi$  is surjective.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $\phi$  is epi.

$\langle 2 \rangle 2.$  LET:  $\pi : H \rightarrow \text{coker } \phi$  be the canonical homomorphism.

$\langle 2 \rangle 3.$   $\pi \circ \phi = 0 \circ \phi$

$\langle 2 \rangle 4.$   $\pi = 0$

$\langle 2 \rangle 5.$   $\text{coker } \phi = \text{im } \pi$  is trivial.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: If  $\text{coker } \phi = H / \text{im } \phi$  is trivial then  $\text{im } \phi = H$ .

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: If it is surjective then it is epi in **Set**.

□

## Chapter 7

# Group Actions

### 7.1 Group Actions

**Definition 7.1** (Action). Let  $G$  be a group. Let  $A$  be an object of a category  $\mathcal{C}$ . A (left) action of  $G$  on  $A$  is a group homomorphism  $G \rightarrow \text{Aut}_{\mathcal{C}}(A)$ .

It is *faithful* or *effective* iff it is injective.

**Proposition 7.2.** Let  $A$  be a set. An action of the group  $G$  on the set  $A$  is given by a function  $\cdot : G \times A \rightarrow A$  such that

- $\forall a \in A. ea = a$
- $\forall g, h \in G. \forall a \in A. (gh)a = g(ha)$

PROOF: Just unfolding definitions.  $\square$

**Example 7.3.** Left multiplication defines a faithful action of any group on its own underlying set.

In fact, for any subgroup  $H$  of a group  $G$ , left multiplication defines an action of  $G$  on  $G/H$ .

**Corollary 7.3.1** (Cayley's Theorem). Every group  $G$  is a subgroup of a symmetric group, namely  $\text{Aut}_{\text{Set}}(G)$ .

**Example 7.4.** Conjugation  $g * h = ghg^{-1}$  is an action of any group on its own underlying set.

**Definition 7.5** (Transitive). An action of a group  $G$  on a set  $A$  is *transitive* iff, for all  $a, b \in A$ , there exists  $g \in G$  such that  $ga = b$ .

**Example 7.6.** Left multiplication of a group  $G$  is a transitive action of  $G$  on  $G$ .

**Definition 7.7** (Orbit). Given an action of a group  $G$  on a set  $A$  and  $a \in A$ , the *orbit* of  $a$  is

$$\text{O}_G(a) := \{ga : g \in G\} .$$

**Proposition 7.8.** *Given an action of a group  $G$  on a set  $A$ , the orbits form a partition of  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . Every element of  $A$  is in some orbit.

PROOF: Since  $a \in O_G(a)$ .

$\langle 1 \rangle 2$ . Distinct orbits are disjoint.

$\langle 2 \rangle 1$ . LET:  $a \in O_G(b) \cap O_G(c)$

$\langle 2 \rangle 2$ . PICK  $g, h \in G$  such that  $a = gb = hc$ .

$\langle 2 \rangle 3$ .  $O_G(b) \subseteq O_G(c)$

PROOF: For all  $k \in G$  we have  $kb = kg^{-1}hc$ .

$\langle 2 \rangle 4$ .  $O_G(c) \subseteq O_G(b)$

PROOF: Similar.

□

**Proposition 7.9.** *Given an action of a group  $G$  on a set  $A$  and  $a \in A$ , the action is transitive on  $O_G(a)$ .*

PROOF:

$\langle 1 \rangle 1$ . The restriction of the action is an action on  $O_G(a)$ .

PROOF: Since  $g(ha) = (gh)a$ , the action maps  $O_G(a)$  to itself.

$\langle 1 \rangle 2$ . The restricted action is transitive.

PROOF: Given  $ga, ha \in O_G(a)$ , we have  $ha = (hg^{-1})(ga)$ .

□

**Definition 7.10** (Stabilizer Subgroup). Given an action of a group  $G$  on a set  $A$  and  $a \in A$ , the *stabilizer subgroup* of  $a$  is

$$\text{Stab}_G(a) := \{g \in G : ga = a\} .$$

**Proposition 7.11.** *Stabilizer subgroups are subgroups.*

PROOF: If  $g, h \in \text{Stab}_G(a)$  then  $gh^{-1}a = a$  so  $gh^{-1} \in \text{Stab}_G(a)$ . □

**Proposition 7.12.** *Let  $G$  act on a set  $A$ . Let  $a \in A$  and  $g \in G$ . Then*

$$\text{Stab}_G(ga) = g\text{Stab}_G(a)g^{-1} .$$

PROOF:

$$h \in \text{Stab}_G(ga) \Leftrightarrow hga = ga$$

$$\Leftrightarrow g^{-1}hga = a$$

$$\Leftrightarrow g^{-1}hg \in \text{Stab}_G(a)$$

$$\Leftrightarrow h \in g\text{Stab}_G(a)g^{-1}$$

□

**Corollary 7.12.1.** *Let  $G$  be an action on a set  $A$  and  $a \in A$ . If  $\text{Stab}_G(a)$  is normal in  $G$ , then for any  $b \in O_G(a)$  we have  $\text{Stab}_G(a) = \text{Stab}_G(b)$ .*

**Definition 7.13** (Free). An action of a group  $G$  on a set  $A$  is *free* iff, whenever  $ga = a$ , then  $g = e$ .

**Example 7.14.** The action of left multiplication is free.

**Proposition 7.15.** *Let  $G$  be a finite group. Let  $H$  be a subgroup of  $G$  of index  $p$  where  $p$  is the smallest prime that divides  $|G|$ . Then  $H$  is normal in  $G$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\sigma : G \rightarrow \text{Aut}_{\text{Set}}(G/H)$  be the action of left multiplication.

$\langle 1 \rangle 2$ .  $G/\ker \sigma$  is a subgroup of  $\text{Aut}_{\text{Set}}(G/H)$ .

$\langle 1 \rangle 3$ .  $|G : \ker \sigma|$  divides  $p!$ .

$\langle 1 \rangle 4$ .  $\ker \sigma \subseteq H$

$\langle 2 \rangle 1$ . LET:  $g \in \ker \sigma$

$\langle 2 \rangle 2$ .  $\sigma(g)(H) = H$

$\langle 2 \rangle 3$ .  $gH = H$

$\langle 2 \rangle 4$ .  $g \in H$

$\langle 1 \rangle 5$ .  $\ker \sigma = H$

$\langle 2 \rangle 1$ .  $|\ker \sigma| \mid |H|$

PROOF: By Lagrange's Theorem and  $\langle 1 \rangle 4$ .

$\langle 2 \rangle 2$ . LET:  $|H| = k|\ker \sigma|$

$\langle 2 \rangle 3$ .  $|G| = pk|\ker \sigma|$

$\langle 2 \rangle 4$ .  $pk \mid p!$

PROOF: By  $\langle 1 \rangle 3$ .

$\langle 2 \rangle 5$ .  $k \mid (p-1)!$

$\langle 2 \rangle 6$ . Every prime that divides  $k$  is  $< p$ .

PROOF: From  $\langle 2 \rangle 5$ .

$\langle 2 \rangle 7$ . Every prime that divides  $k$  is  $\geq p$ .

PROOF: By  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 8$ .  $k = 1$

$\langle 2 \rangle 9$ .  $|H| = |\ker \sigma|$

□

**Corollary 7.15.1.** *Any subgroup of index 2 is normal.*

## 7.2 Category of $G$ -Sets

**Definition 7.16.** Given a group  $G$ , let  $G - \text{Set}$  be the category with:

- objects all pairs  $(A, \rho)$  such that  $A$  is a set and  $\rho : G \times A \rightarrow A$  is an action of  $G$  on  $A$ ;
- morphisms  $f : (A, \rho) \rightarrow (B, \sigma)$  are functions  $f : A \rightarrow B$  that are  $(G-)$ equivariant, i.e.

$$\forall g \in G. \forall a \in A. f(\rho(g, a)) = \sigma(g, f(a)) .$$

**Proposition 7.17.** *A  $G$ -equivariant function  $f : A \rightarrow B$  is an isomorphism in  $G - \text{Set}$  if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$  be  $G$ -equivariant and bijective.

PROVE:  $f^{-1}$  is  $G$ -equivariant.

$\langle 1 \rangle 2$ . LET:  $g \in G$  and  $b \in B$

$\langle 1 \rangle 3$ .  $f^{-1}(gb) = gf^{-1}(b)$

PROOF:

$$\begin{aligned} f(f^{-1}(gb)) &= gb \\ &= gf(f^{-1}(b)) \\ &= f(gf^{-1}(b)) \end{aligned}$$

□

**Proposition 7.18.** *Let  $G$  be a group and  $A$  a transitive  $G$ -set. Let  $a \in A$ . Then  $A$  is isomorphic to  $G/\text{Stab}_G(a)$  under left multiplication.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : G/\text{Stab}_G(a) \rightarrow A$  be the function  $f(g\text{Stab}_G(a)) = ga$ .

$\langle 2 \rangle 1$ . ASSUME:  $g\text{Stab}_G(a) = h\text{Stab}_G(a)$

PROVE:  $ga = ha$

$\langle 2 \rangle 2$ .  $g^{-1}h \in \text{Stab}_G(a)$

$\langle 2 \rangle 3$ .  $g^{-1}ha = a$

$\langle 2 \rangle 4$ .  $ha = ga$

$\langle 1 \rangle 2$ .  $f$  is  $G$ -equivariant.

PROOF: Since  $f(gh\text{Stab}_G(a)) = gha = gf(h\text{Stab}_G(a))$ .

$\langle 1 \rangle 3$ .  $f$  is injective.

PROOF: If  $ga = ha$  then  $g^{-1}h \in \text{Stab}_G(a)$  so  $g\text{Stab}_G(a) = h\text{Stab}_G(a)$ .

$\langle 1 \rangle 4$ .  $f$  is surjective.

PROOF: Since for all  $b \in A$  there exists  $g \in G$  such that  $ga = b$ .

□

**Corollary 7.18.1.** *If  $O$  is an orbit of the action of a finite group  $G$  on a set  $A$ , then  $O$  is finite and  $|O|$  divides  $|G|$ .*

**Corollary 7.18.2.** *Let  $H$  be a subgroup of  $G$  and  $g \in G$ . Then*

$$G/H \cong G/(gHg^{-1})$$

in  $G$  – **Set**.

PROOF: Taking  $A = G/H$  and  $a = gH$ . □

**Proposition 7.19.** *Given a family of  $G$ -sets  $\{A_i\}_{i \in I}$ , we have  $\prod_{i \in I} A_i$  is their product in  $G$  – **Set** under*

$$g\{a_i\}_{i \in I} = \{ga_i\}_{i \in I} .$$

PROOF: Easy. □

**Proposition 7.20.** *Given a family of  $G$ -sets  $\{A_i\}_{i \in I}$ , we have  $\prod_{i \in I} A_i$  is their product in  $G$  – **Set** under*

$$g(i, a_i) = (i, ga_i) .$$

PROOF: Easy.  $\square$

**Proposition 7.21.** *Every finite  $G$ -set is a coproduct of  $G$ -sets of the form  $G/H$ .*

PROOF: If  $O(a_1), \dots, O(a_n)$  are the orbits of the  $G$ -set  $A$ , then  $G$  is the coproduct of  $G/\text{Stab}_G(a_1), \dots, G/\text{Stab}_G(a_n)$ .  $\square$





**Part III**

**Linear Algebra**



**Definition 7.22.** Let  $\text{GL}_n(\mathbb{R})$  be the group of invertible  $n \times n$  real matrices.  
 $\text{GL}_n(\mathbb{R})$  acts on  $\mathbb{R}^n$  by matrix multiplication.

**Definition 7.23.** Let  $\text{GL}_n(\mathbb{C})$  be the group of invertible  $n \times n$  complex matrices.  
 $\text{GL}_n(\mathbb{C})$  acts on  $\mathbb{C}^n$  by matrix multiplication.

**Definition 7.24.** Let  $\text{SL}_n(\mathbb{R}) = \{M \in \text{GL}_n(\mathbb{R}) : \det M = 1\}$ .

**Proposition 7.25.**  $\text{SL}_n(\mathbb{R})$  is a normal subgroup of  $\text{GL}_n(\mathbb{R})$ .

PROOF: If  $\det M = 1$  then  $\det(AMA^{-1}) = (\det A)(\det M)(\det A)^{-1} = 1$ .  $\square$

**Proposition 7.26.**

$$\text{GL}_n(\mathbb{R})/\text{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$$

**Definition 7.27.** Let  $\text{SL}_n(\mathbb{C}) = \{M \in \text{GL}_n(\mathbb{C}) : \det M = 1\}$ .

**Definition 7.28.** Let  $\text{O}_n(\mathbb{R}) = \{M \in \text{GL}_n(\mathbb{R}) : MM^T = M^T M = I_n\}$ .

**Proposition 7.29.** The action of  $\text{O}_n(\mathbb{R})$  on  $\mathbb{R}^n$  preserves lengths and angles.

**Definition 7.30.** Let  $\text{SO}_n(\mathbb{R}) = \{M \in \text{O}_n(\mathbb{R}) : \det M = 1\}$ .

**Definition 7.31.** Let  $\text{U}_n(\mathbb{C}) = \{M \in \text{GL}_n(\mathbb{C}) : MM^\dagger = M^\dagger M = I_n\}$ .

**Definition 7.32.** Let  $\text{SU}_n(\mathbb{C}) = \{M \in \text{U}_n(\mathbb{C}) : \det M = 1\}$ .

**Proposition 7.33.** Every matrix in  $\text{SU}_2(\mathbb{C})$  can be written in the form

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

for some  $a, b, c, d \in \mathbb{R}$  with  $a^2 + b^2 + c^2 + d^2 = 1$ .

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SU}_2(\mathbb{C})$$

$$\langle 1 \rangle 2. M^{-1} = M^\dagger$$

$$\langle 1 \rangle 3. \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

$$\langle 1 \rangle 4. \text{ LET: } \alpha = a+bi \text{ and } \beta = c+di.$$

$$\langle 1 \rangle 5. \delta = \bar{\alpha} = a-bi$$

$$\langle 1 \rangle 6. \gamma = -\bar{\beta} = -c+di$$

$$\langle 1 \rangle 7. \det M = a^2 + b^2 + c^2 + d^2 = 1$$

$\square$

**Corollary 7.33.1.**  $\text{SU}_2(\mathbb{C})$  is simply connected.

**Corollary 7.33.2.**

$$\text{SO}_3(\mathbb{R}) \cong \text{SU}_2(\mathbb{C})/\{I, -I\}$$

PROOF: The function that maps  $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$  to  $\begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{pmatrix}$

is a surjective homomorphism with kernel  $\{I, -I\}$ .  $\square$

**Corollary 7.33.3.** The fundamental group of  $\text{SO}_3(\mathbb{R})$  is  $C_2$ .