

Mathematics

Robin Adams

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Part I

Category Theory

Chapter 1

Foundations

This is a placeholder — I am not sure what foundation I want to use for this project yet. I will try to work in a way which is foundation-independent. What I do could be formalized in ZFC, ETCS, or some other system. I will assume the usual set theoretic constructions as needed. Sets will be defined up to bijection only.

Chapter 2

Categories

Definition 2.1 (Category). A *category* \mathcal{C} consists of:

- A class $|\mathcal{C}|$ of *objects*. We write $A \in \mathcal{C}$ for $A \in |\mathcal{C}|$.
- For any objects A, B , a set $\mathcal{C}[A, B]$ of *morphisms* from A to B . We write $f : A \rightarrow B$ for $f \in \mathcal{C}[A, B]$.
- For any object A , a morphism $\text{id}_A : A \rightarrow A$, the *identity* morphism on A .
- For any morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, a morphism $g \circ f : A \rightarrow C$, the *composite* of f and g .

such that:

Associativity Given $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Left Unit Law For any morphism $f : A \rightarrow B$, we have $\text{id}_B \circ f = f$.

Right Unit Law For any morphism $f : A \rightarrow B$, we have $f \circ \text{id}_A = f$.

Proposition 2.2. *The identity morphism on an object is unique.*

PROOF: If i and j are identity morphisms on A then $i = i \circ j = j$. \square

Example 2.3 (Category of Sets). The *category of sets* **Set** has objects all sets and morphisms all functions.

Definition 2.4 (Endomorphism). In a category \mathcal{C} , an *endomorphism* on an object A is a morphism $A \rightarrow A$. We write $\text{End}_{\mathcal{C}}(A)$ for $\mathcal{C}[A, A]$.

Definition 2.5 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with the same objects as \mathcal{C} and

$$\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$$

2.1 Preorders

Definition 2.6 (Preorder). A *preorder* on a set A is a relation \leq on A that is reflexive and transitive.

A *preordered set* is a pair (A, \leq) such that \leq is a preorder on A . We usually write A for the preordered set (A, \leq) .

We identify any preordered set A with the category whose objects are the elements of A , with one morphism $a \rightarrow b$ iff $a \leq b$, and no morphism $a \rightarrow b$ otherwise.

Example 2.7. For any ordinal α , let α be the preorder $\{\beta : \beta < \alpha\}$ under \leq .

Definition 2.8 (Discrete Preorder). We identify any set A with the *discrete* preorder $(A, =)$.

2.2 Monomorphisms and Epimorphisms

Definition 2.9 (Monomorphism). In a category, let $f : A \rightarrow B$. Then f is a *monomorphism* or *monic* iff, for every object X and morphism $x, y : X \rightarrow A$, if $fx = fy$ then $x = y$.

Definition 2.10 (Epimorphism). In a category, let $f : A \rightarrow B$. Then f is a *epimorphism* or *epi* iff, for every object X and morphism $x, y : B \rightarrow X$, if $xf = yf$ then $x = y$.

Proposition 2.11. *The composite of two monomorphism is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$ be monic.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$

$\langle 1 \rangle 3$. ASSUME: $g \circ f \circ x = g \circ f \circ y$

$\langle 1 \rangle 4$. $f \circ x = f \circ y$

$\langle 1 \rangle 5$. $x = y$

□

Proposition 2.12. *The composite of two epimorphisms is epi.*

PROOF: Dual. □

Proposition 2.13. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is monic then f is monic.*

PROOF: If $f \circ x = f \circ y$ then $g \circ f \circ x = g \circ f \circ y$ and so $x = y$. □

Proposition 2.14. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is epi then g is epi.*

PROOF: Dual. □

Proposition 2.15. *A function is a monomorphism in **Set** iff it is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is monic then f is injective.

$\langle 2 \rangle 1$. ASSUME: f is monic.

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 4$. LET: $\bar{x}, \bar{y} : 1 \rightarrow A$ be the functions such that $\bar{x}(*) = x$ and $\bar{y}(*) = y$

$\langle 2 \rangle 5$. $f \circ \bar{x} = f \circ \bar{y}$

$\langle 2 \rangle 6$. $\bar{x} = \bar{y}$

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. $x = y$

$\langle 1 \rangle 3$. If f is injective then f is monic.

$\langle 2 \rangle 1$. ASSUME: f is injective.

$\langle 2 \rangle 2$. LET: X be a set and $x, y : X \rightarrow A$.

$\langle 2 \rangle 3$. ASSUME: $f \circ x = f \circ y$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $t \in X$

PROVE: $x(t) = y(t)$

$\langle 2 \rangle 5$. $f(x(t)) = f(y(t))$

$\langle 2 \rangle 6$. $x(t) = y(t)$

PROOF: By $\langle 2 \rangle 1$.

□

Proposition 2.16. *A function is an epimorphism in **Set** iff it is surjective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is an epimorphism then f is surjective.

$\langle 2 \rangle 1$. ASSUME: f is an epimorphism.

$\langle 2 \rangle 2$. LET: $b \in B$

$\langle 2 \rangle 3$. LET: $x, y : B \rightarrow 2$ be defined by $x(b) = 1$ and $x(t) = 0$ for all other $t \in B$, $y(t) = 0$ for all $t \in B$.

$\langle 2 \rangle 4$. $x \neq y$

$\langle 2 \rangle 5$. $x \circ f \neq y \circ f$

$\langle 2 \rangle 6$. There exists $a \in A$ such that $f(a) = b$.

$\langle 1 \rangle 3$. If f is surjective then f is an epimorphism.

$\langle 2 \rangle 1$. ASSUME: f is surjective.

$\langle 2 \rangle 2$. LET: $x, y : B \rightarrow X$

$\langle 2 \rangle 3$. ASSUME: $x \circ f = y \circ f$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $b \in B$

PROVE: $x(b) = y(b)$

$\langle 2 \rangle 5$. PICK $a \in A$ such that $f(a) = b$

$\langle 2 \rangle 6$. $x(f(a)) = y(f(a))$

$\langle 2 \rangle 7$. $x(b) = y(b)$

□

Proposition 2.17. *In a preorder, every morphism is monic and epi.*

PROOF: Immediate from definitions. \square

2.3 Sections and Retractions

Definition 2.18 (Section, Retraction). In a category, let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $r \circ s = \text{id}_B$.

Proposition 2.19. *Every identity morphism is a section and retraction of itself.*

PROOF: Immediate from definitions. \square

Proposition 2.20. *Let $r, r' : A \rightarrow B$ and $s : B \rightarrow A$. If r is a retraction of s and r' is a section of s then $r = r'$.*

PROOF:

$$\begin{aligned} r &= r \circ \text{id}_A \\ &= r \circ s \circ r' \\ &= \text{id}_B \circ r' \\ &= r' \end{aligned} \quad \square$$

Proposition 2.21. *Let $r_1 : A \rightarrow B$, $r_2 : B \rightarrow C$, $s_1 : B \rightarrow A$ and $s_2 : C \rightarrow B$. If r_1 is a retraction of s_1 and r_2 is a retraction of s_2 then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= \text{id}_C \end{aligned} \quad \square$$

Proposition 2.22. *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $s : A \rightarrow B$ be a section of $r : B \rightarrow A$.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$ satisfy $sx = sy$.

$\langle 1 \rangle 3$. $rsx = rsy$

$\langle 1 \rangle 4$. $x = y$

\square

Proposition 2.23. *Every retraction is epi.*

PROOF: Dual. \square

Proposition 2.24. *In Set, every epimorphism has a retraction.*

PROOF: By the Axiom of Choice. \square

Example 2.25. It is not true in general that every monomorphism in any category has a section. nor that every epimorphism in any category has a retraction.

In the category **2**, the morphism $0 \leq 1$ is monic and epi but has no retraction or section.

2.4 Isomorphisms

Definition 2.26 (Isomorphism). In a category \mathcal{C} , a morphism $f : A \rightarrow B$ is an *isomorphism*, denoted $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$, the *inverse* of f , such that $f^{-1} \circ f = \text{id}_A$ and $f \circ f^{-1} = \text{id}_B$.

An *automorphism* on an object A is an isomorphism between A and itself. We write $\text{Aut}_{\mathcal{C}}(A)$ for the set of all automorphisms on A .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 2.27. *The inverse of an isomorphism is unique.*

PROOF: Proposition 2.20. \square

Proposition 2.28. *For any object A we have $\text{id}_A : A \cong A$ and $\text{id}_A^{-1} = \text{id}_A$.*

PROOF: Since $\text{id}_A \circ \text{id}_A = \text{id}_A$ by the Unit Laws. \square

Proposition 2.29. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Immediate from definitions. \square

Proposition 2.30. *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF: From Proposition 2.21. \square

Definition 2.31 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

2.5 Initial and Terminal Objects

Definition 2.32 (Initial Object). An object I in a category is *initial* iff, for any object X , there is exactly one morphism $I \rightarrow X$.

Example 2.33. The empty set is the initial object in **Set**.

Definition 2.34 (Terminal Object). An object T in a category is *terminal* iff, for any object X , there is exactly one morphism $X \rightarrow T$.

Example 2.35. Every singleton is terminal in **Set**.

Proposition 2.36. *If I and J are initial in a category, then there exists a unique isomorphism $I \cong J$.*

PROOF:

- $\langle 1 \rangle 1$. LET: i be the unique morphism $I \rightarrow J$.
- $\langle 1 \rangle 2$. LET: i^{-1} be the unique morphism $J \rightarrow I$.
- $\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_J$

PROOF: Since there is only one morphism $J \rightarrow J$.

- $\langle 1 \rangle 4$. $i^{-1} \circ i = \text{id}_I$

PROOF: Since there is only one morphism $I \rightarrow I$.
 \square

Proposition 2.37. *If S and T are terminal in a category, then there exists a unique isomorphism $S \cong T$.*

PROOF: Dual. \square

Chapter 3

Functors

Definition 3.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for any morphism $f : A \rightarrow B : \mathcal{C}$, a morphism $Ff : FA \rightarrow FB : \mathcal{D}$

such that:

- $F\text{id}_A = \text{id}_{FA}$
- $F(g \circ f) = Fg \circ Ff$

Definition 3.2 (Identity Functor). For any category \mathcal{C} , the *identity functor* $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is defined by

$$\begin{aligned} 1_{\mathcal{C}}A &= A \\ 1_{\mathcal{C}}f &= f \end{aligned}$$

Definition 3.3 (Constant Functor). Given categories \mathcal{C} , \mathcal{D} and an object $D \in \mathcal{D}$, the *constant functor* $K^{\mathcal{C}}D : \mathcal{C} \rightarrow \mathcal{D}$ is the functor defined by

$$\begin{aligned} K^{\mathcal{C}}DC &= D \\ K^{\mathcal{C}}Df &= \text{id}_D \end{aligned}$$

3.1 Comma Categories

Definition 3.4 (Comma Category). Let $F : \mathcal{C} \rightarrow \mathcal{E}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors. The *comma category* $F \downarrow G$ is the category with:

- objects all pairs (C, D, f) where $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f : FC \rightarrow GD : \mathcal{E}$

- morphisms $(u, v) : (C, D, f) \rightarrow (C', D', g)$ all pairs $u : C \rightarrow C' : \mathcal{C}$ and $v : D \rightarrow D' : \mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ \downarrow Fu & & \downarrow Gv \\ FC' & \xrightarrow{g} & GD' \end{array}$$

Definition 3.5 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category* over A , denoted \mathcal{C}/A , is the comma category $1_{\mathcal{C}} \downarrow K^1 A$.

Definition 3.6 (Coslice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *coslice category* over A , denoted $\mathcal{C} \backslash A$, is the comma category $K^1 A \downarrow 1_{\mathcal{C}}$.

Definition 3.7 (Pointed Sets). The *category of pointed sets* \mathbf{Set}_* is the coslice category $\mathbf{Set} \backslash 1$.

Part II

Group Theory

Chapter 4

Monoids

Definition 4.1 (Monoid). A *monoid* consists of a set M and a binary operation $\cdot : M^2 \rightarrow M$ such that:

- \cdot is associative
- There exists $e \in M$ such that, for all $x \in M$, we have $xe = ex = x$.

We identify a monoid M with the category with one object whose morphisms are the elements of M , with composition given by \cdot .

Proposition 4.2. *The identity in a group is unique.*

PROOF: Proposition 2.2.

Chapter 5

Groups

Definition 5.1 (Group). Let \mathcal{C} be a category with finite products. A *group (object)* in \mathcal{C} consists of an object $G \in \mathcal{C}$ and morphisms

$$m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G$$

such that the following diagrams commute.

$$\begin{array}{ccc} G^3 & \xrightarrow{m \times \text{id}_G} & G^2 \\ \downarrow \text{id}_G \times m & & \downarrow m \\ G^2 & \xrightarrow{m} & G \end{array}$$

$$\begin{array}{ccc} 1 \times G & \xrightarrow{e \times \text{id}_G} & G^2 \\ & \searrow \cong & \downarrow m \\ & & G \end{array} \quad \begin{array}{ccc} G \times 1 & \xrightarrow{\text{id}_G \times e} & G^2 \\ & \searrow \cong & \downarrow m \\ & & G \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{\Delta} & G^2 & \xrightarrow{\text{id}_G \times i} & G^2 \\ \downarrow & & & & \downarrow m \\ 1 & \xrightarrow{e} & G & & G \end{array} \quad \begin{array}{ccccc} G & \xrightarrow{\Delta} & G^2 & \xrightarrow{i \times \text{id}_G} & G^2 \\ \downarrow & & & & \downarrow m \\ 1 & \xrightarrow{e} & G & & G \end{array}$$

Definition 5.2 (Group). We write just 'group' for 'group in **Set**'. Thus, a *group* G consists of a set G and a binary operation $\cdot : G^2 \rightarrow G$ such that \cdot is associative, and there exists $e \in G$, the *identity* element of the group, such that:

- For all $x \in G$ we have $xe = ex = x$
- For all $x \in G$, there exists $x^{-1} \in G$, the *inverse* of x , such that $xx^{-1} = x^{-1}x = e$.

The *order* of a group G , denoted $|G|$, is the number of elements in G if G is finite; otherwise we write $|G| = \infty$.

Proposition 5.3. *The inverse of an element is unique.*

PROOF: If i and j are inverses of x then $i = ixj = j$. \square

Example 5.4. • The *trivial* group is $\{e\}$ under $ee = e$.

- \mathbb{Z} is a group under addition
- \mathbb{Q} is a group under addition
- $\mathbb{Q} - \{0\}$ is a group under multiplication
- \mathbb{R} is a group under addition
- $\mathbb{R} - \{0\}$ is a group under multiplication
- \mathbb{C} is a group under addition
- $\mathbb{C} - \{0\}$ is a group under multiplication
- $\{-1, 1\}$ is a group under multiplication
- The set of 2×2 real matrices with non-zero determinant is a group under matrix multiplication.
- For any category \mathcal{C} and object $A \in \mathcal{C}$, we have $\text{Aut}_{\mathcal{C}}(A)$ is a group under $gf = f \circ g$.
For A a set, we call $S_A = \text{Aut}_{\text{Set}}(A)$ the *symmetric group* or *group of permutations* of A .
- For $n \geq 3$, the *dihedral group* D_{2n} consists of the set of rigid motions that map the regular n -gon onto itself under composition.
- Let $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ under matrix multiplication.

Example 5.5. • The only group of order 1 is the trivial group.

- The only group of order 2 is \mathbb{Z}_2 .
- The only group of order 3 is \mathbb{Z}_3 .
- There are exactly two groups of order 4: \mathbb{Z}_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ under $(a, b)(c, d) = (ac, bd)$.

Proposition 5.6 (Cancellation). *Let G be a group. Let $a, g, h \in G$. If $ag = ah$ or $ga = ha$ then $g = h$.*

PROOF: If $ag = ah$ then $g = a^{-1}ag = a^{-1}ah = h$. Similarly if $ga = ha$. \square

Proposition 5.7. *Let G be a group and $g, h \in G$. Then $(gh)^{-1} = h^{-1}g^{-1}$.*

PROOF: Since $ghh^{-1}g^{-1} = e$. \square

Definition 5.8. Let G be a group. Let $g \in G$. We define $g^n \in G$ for all $n \in \mathbb{Z}$ as follows:

$$\begin{aligned} g^0 &= e \\ g^{n+1} &= g^n g & (n \geq 0) \\ g^{-n} &= (g^{-1})^n & (n > 0) \end{aligned}$$

Proposition 5.9. Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then

$$g^{m+n} = g^m g^n .$$

PROOF:

$\langle 1 \rangle 1$. For all $k \in \mathbb{Z}$ we have $g^{k+1} = g^k g$

$\langle 2 \rangle 1$. For all $k \geq 0$ we have $g^{k+1} = g^k g$

PROOF: Immediate from definition.

$\langle 2 \rangle 2$. $g^{-1+1} = g^{-1} g$

PROOF: Both are equal to e .

$\langle 2 \rangle 3$. For all $k > 1$ we have $g^{-k+1} = g^{-k} g$

PROOF:

$$\begin{aligned} g^{-k+1} &= (g^{-1})^{k-1} \\ &= (g^{-1})^{k-1} g^{-1} g \\ &= (g^{-1})^k g \\ &= g^{-k} g \end{aligned}$$

$\langle 1 \rangle 2$. For all $k \in \mathbb{Z}$ we have $g^{k-1} = g^k g^{-1}$

PROOF: Substitute $k = k - 1$ above and multiply by g^{-1} .

$\langle 1 \rangle 3$. $g^{m+0} = g^m g^0$

PROOF: Since $g^m g^0 = g^m e = g^m$.

$\langle 1 \rangle 4$. If $g^{m+n} = g^m g^n$ then $g^{m+n+1} = g^m g^{n+1}$

PROOF:

$$\begin{aligned} g^{m+n+1} &= g^{m+n} g & (\langle 1 \rangle 1) \\ &= g^m g^n g \\ &= g^m g^{n+1} & (\langle 1 \rangle 1) \end{aligned}$$

$\langle 1 \rangle 5$. If $g^{m+n} = g^m g^n$ then $g^{m+n-1} = g^m g^{n-1}$

PROOF:

$$\begin{aligned} g^{m+n-1} g &= g^{m+n} & (\langle 1 \rangle 1) \\ &= g^m g^n \\ \therefore g^{m+n-1} &= g^m g^n g^{-1} \\ &= g^m g^{n-1} & (\langle 1 \rangle 2) \end{aligned}$$

\square

Proposition 5.10. Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then

$$(g^m)^n = g^{mn} .$$

PROOF:

$\langle 1 \rangle 1. (g^m)^0 = g^0$

PROOF: Both sides are equal to e .

$\langle 1 \rangle 2. \text{ If } (g^m)^n = g^{mn} \text{ then } (g^m)^{n+1} = g^{m(n+1)}.$

PROOF:

$$(g^m)^{n+1} = (g^m)^n g^m \quad (\text{Proposition 5.9})$$

$$= g^{mn} g^m$$

$$= g^{mn+m}$$

(Proposition 5.9)

$\langle 1 \rangle 3. \text{ If } (g^m)^n = g^{mn} \text{ then } (g^m)^{n-1} = g^{m(n-1)}.$

PROOF:

$$(g^m)^n = g^{mn}$$

$$\therefore (g^m)^{n-1} g^m = g^{mn-m} g^m \quad (\text{Proposition 5.9})$$

$$\therefore (g^m)^{n-1} = g^{mn-m} \quad (\text{Cancellation})$$

□

Definition 5.11 (Commute). Let G be a group and $g, h \in G$. We say g and h *commute* iff $gh = hg$.

Definition 5.12. Let G be a group. Given $g \in G$ and $A \subseteq G$, we define

$$gA = \{ga : a \in A\}, \quad Ag = \{ag : a \in A\}.$$

Given sets $A, B \subseteq G$, we define

$$AB = \{ab : a \in A, b \in B\}.$$

5.1 Order of an Element

Definition 5.13 (Order). Let G be a group. Let $g \in G$. Then g has *finite order* iff there exists a positive integer n such that $g^n = e$. In this case, the *order* of g , denoted $|g|$, is the least positive integer n such that $g^n = e$.

If g does not have finite order, we write $|g| = \infty$.

Proposition 5.14. Let G be a group. Let $g \in G$ and n be a positive integer. If $g^n = e$ then $|g| \mid n$.

PROOF:

$\langle 1 \rangle 1. \text{ LET: } n = q|g| + d \text{ where } 0 \leq d < |g|$

PROOF: Division Algorithm.

$\langle 1 \rangle 2. g^d = e$

PROOF:

$$e = g^n$$

$$= g^{q|g|+d}$$

$$= (g^{|g|})^q g^d \quad (\text{Propositions 5.9, 5.10})$$

$$= e^q g^d$$

$$= g^d$$

$\langle 1 \rangle 3. d = 0$

PROOF: By minimality of $|g|$.

$\langle 1 \rangle 4. n = q|g|$

□

Corollary 5.14.1. *Let G be a group. Let $g \in G$ have finite order and $n \in \mathbb{Z}$. Then $g^n = e$ if and only if $|g| \mid n$.*

Proposition 5.15. *Let G be a group and $g \in G$. Then $|g| \leq |G|$.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: w.l.o.g. G is finite.

$\langle 1 \rangle 2.$ PICK i, j with $0 \leq i < j \leq |G|$ such that $g^i = g^j$.

PROOF: Otherwise $g^0, g^1, \dots, g^{|G|}$ would be $|G| + 1$ distinct elements of G .

$\langle 1 \rangle 3. g^{j-i} = e$

$\langle 1 \rangle 4. g$ has finite order and $|g| \leq |G|$

PROOF: Since $|g| \leq j - i \leq j \leq |G|$.

□

Proposition 5.16. *Let G be a group. Let $g \in G$ have finite order. Let $m \in \mathbb{N}$. Then*

$$|g^m| = \frac{\text{lcm}(m, |g|)}{m} = \frac{|g|}{\text{gcd}(m, |g|)}$$

PROOF: Since for any integer d we have

$$g^{md} = e \Leftrightarrow |g| \mid md \quad (\text{Corollary 5.14.1})$$

$$\Leftrightarrow \text{lcm}(m, |g|) \mid md$$

$$\Leftrightarrow \frac{\text{lcm}(m, |g|)}{m} \mid d \quad \square$$

and so $|g^m| = \frac{\text{lcm}(m, |g|)}{m}$ by Corollary 5.14.1. □

Corollary 5.16.1. *If g has odd order then $|g^2| = |g|$.*

Proposition 5.17. *Let G be a group. Let $g, h \in G$ have finite order. Assume $gh = hg$. Then $|gh|$ has finite order and*

$$|gh| \mid \text{lcm}(|g|, |h|)$$

PROOF: Since $(gh)^{\text{lcm}(|g|, |h|)} = g^{\text{lcm}(|g|, |h|)} h^{\text{lcm}(|g|, |h|)} = e$. □

Example 5.18. This example shows that we cannot remove the hypothesis that $gh = hg$.

In $\text{GL}_2(\mathbb{R})$, take

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then $|g| = 4$, $|h| = 3$ and $|gh| = \infty$.

Proposition 5.19. *Let G be a group and $g, h \in G$ have finite order. If $gh = hg$ and $\text{gcd}(|g|, |h|) = 1$ then $|gh| = |g||h|$.*

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } N = |gh|$$

$$\langle 1 \rangle 2. g^N = (h^{-1})^N$$

$$\langle 1 \rangle 3. g^{N|g|} = e$$

$$\langle 1 \rangle 4. |g^N| \mid |g|$$

$$\langle 1 \rangle 5. h^{-N|h|} = e$$

$$\langle 1 \rangle 6. |g^N| \mid |h|$$

$$\langle 1 \rangle 7. |g^N| = 1$$

PROOF: Since $\gcd(|g|, |h|) = 1$.

$$\langle 1 \rangle 8. g^N = e$$

$$\langle 1 \rangle 9. |g| \mid N$$

$$\langle 1 \rangle 10. h^{-N} = e$$

$$\langle 1 \rangle 11. |h| \mid N$$

$$\langle 1 \rangle 12. N = |g||h|$$

PROOF: Using Proposition 5.17.

□

Proposition 5.20. *Let G be a finite group. Assume there is exactly one element $f \in G$ of order 2. Then the product of all the elements of G is f .*

PROOF: Let the elements of G be g_1, g_2, \dots, g_n . Apart from e and f , every element and its inverse are distinct elements of the list. Hence the product of the list is $ef = f$. □

Proposition 5.21. *Let G be a finite group of order n . Let m be the number of elements of G of order 2. Then $n - m$ is odd.*

PROOF: In the list of all elements that are not of order 2, every element and its inverse are distinct except for e . Hence the list has odd length. □

Corollary 5.21.1. *If a finite group has even order, then it contains an element of order 2.*

Proposition 5.22. *Let G be a group and $a, g \in G$. Then $|aga^{-1}| = |g|$.*

PROOF: Since

$$(aga^{-1})^n = e \Leftrightarrow ag^na^{-1} = e$$

$$\Leftrightarrow g^n = e$$

□

Proposition 5.23. *Let G be a group and $g, h \in G$. Then $|gh| = |hg|$.*

PROOF: Since $|gh| = |ghgg^{-1}| = |hg|$. □

Proposition 5.24. *Let G be a group of order n . Let k be relatively prime to n . Then every element in G has the form x^k for some x .*

$$\langle 1 \rangle 1. \text{ PICK integers } a \text{ and } b \text{ such that } an + bk = 1.$$

$$\langle 1 \rangle 2. \text{ LET: } g \in G$$

$$\langle 1 \rangle 3. g = (g^b)^k$$

PROOF:

$$\begin{aligned} g &= g \cdot (g^n)^{-a} & (g^n = e) \\ &= g^{1-an} \\ &= g^{bk} \end{aligned}$$

□

5.2 Generators

Definition 5.25 (Generator). Let G be a group and $a \in G$. We say a *generates* the group iff, for all $x \in G$, there exists an integer n such that $x^n = a$.

Example 5.26. $\text{SL}_2(\mathbb{Z})$ is generated by

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

PROOF:

$\langle 1 \rangle 1$. LET: $H = \langle s, t \rangle$

$\langle 1 \rangle 2$. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in H$.

PROOF: It is t^q .

$\langle 1 \rangle 3$. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \in H$.

PROOF:

$$\begin{aligned} st^{-q}s^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \end{aligned}$$

$\langle 1 \rangle 4$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & qa+b \\ c & qc+d \end{pmatrix}$$

$\langle 1 \rangle 5$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} a+qb & b \\ c+qd & d \end{pmatrix}$$

$\langle 1 \rangle 6$. For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, if c and d are both nonzero, then there exists $N \in H$ such that the bottom row of MN has one entry the same as M and one entry with smaller absolute value.

PROOF: From $\langle 1 \rangle 4$ and $\langle 1 \rangle 5$ taking $q = -1$.

$\langle 1 \rangle 7$. For any $M \in \text{SL}_2(\mathbb{Z})$, there exists $N \in H$ such that MN has a zero on the bottom row.

PROOF: Apply $\langle 1 \rangle 6$ repeatedly.

$\langle 1 \rangle 8$. Any matrix in $\text{SL}_2(\mathbb{Z})$ with a zero on the bottom row is in H .

$$\langle 2 \rangle 1. \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H$$

PROOF: $\langle 1 \rangle 2$

$$\langle 2 \rangle 2. \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \in H$$

PROOF: It is $s^2 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ since $s^2 = -I$.

$$\langle 2 \rangle 3. \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} \in H$$

PROOF: It is $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} s$.

$$\langle 2 \rangle 4. \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \in H$$

PROOF: It is $s^2 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s$.

$\langle 1 \rangle 9$. Every matrix in $\text{SL}_2(\mathbb{Z})$ is in H .

□

Chapter 6

Group Homomorphisms

Definition 6.1 (Homomorphism). Let G and H be groups. A (group) homomorphism $\phi : G \rightarrow H$ is a function such that, for all $x, y \in G$,

$$\phi(xy) = \phi(x)\phi(y) \text{ .}$$

Proposition 6.2. Let G and H be groups with identities e_G and e_H . Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\phi(e_G) = e_H$.

PROOF: Since $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$ and so $\phi(e_G) = e_H$ by Cancellation. \square

Proposition 6.3. Let $\phi : G \rightarrow H$ be a group homomorphism. For all $x \in G$ we have $\phi(x^{-1}) = \phi(x)^{-1}$.

PROOF: Since $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_H$. \square

Proposition 6.4. Let G, H and K be groups. If $\phi : G \rightarrow H$ and $\psi : H \rightarrow K$ are homomorphisms then $\psi \circ \phi : G \rightarrow K$ is a homomorphism.

PROOF: For $x, y \in G$ we have

$$\psi(\phi(xy)) = \psi(\phi(x)\phi(y)) = \psi(\phi(x))\psi(\phi(y)) \text{ .}$$

Proposition 6.5. Let G be a group. Then $\text{id}_G : G \rightarrow G$ is a group homomorphism.

PROOF: For $x, y \in G$ we have $\text{id}_G(xy) = xy = \text{id}_G(x)\text{id}_G(y)$. \square

Proposition 6.6. Let $\phi : G \rightarrow H$ be a group homomorphism. Let $g \in G$ have finite order. Then $|\phi(g)|$ divides $|g|$.

PROOF: Since $\phi(g)^{|g|} = \phi(g^{|g|}) = e$. \square

Definition 6.7 (Category of Groups). Let **Grp** be the category of groups and group homomorphisms.

Example 6.8. There are 49487365402 groups of order 1024 up to isomorphism.

Proposition 6.9. *A group homomorphism $\phi : G \rightarrow H$ is an isomorphism in **Grp** if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: ϕ is bijective.

PROVE: ϕ^{-1} is a group homomorphism.

$\langle 1 \rangle 2$. LET: $h, h' \in H$

$\langle 1 \rangle 3$. $\phi(\phi^{-1}(hh')) = \phi(\phi^{-1}(h)\phi^{-1}(h'))$

PROOF: Both are equal to hh' .

$\langle 1 \rangle 4$. $\phi^{-1}(hh') = \phi^{-1}(h)\phi^{-1}(h')$

□

Corollary 6.9.1.

$$D_6 \cong C_3$$

PROOF: The canonical homomorphism $D_6 \rightarrow C_3$ is bijective. □

Corollary 6.9.2.

$$(\mathbb{R}, +) \cong (\{x \in \mathbb{R} : x > 0\}, \cdot)$$

PROOF: The function that maps x to e^x is a bijective homomorphism. □

Proposition 6.10. *The trivial group is the zero object in **Grp**.*

PROOF: For any group G , the unique function $G \rightarrow \{e\}$ is a group homomorphism, and the only group homomorphism $\{e\} \rightarrow G$ maps e to e_G . □

Proposition 6.11. *For any groups G and H , the set $G \times H$ under $(g, h)(g', h') = (gg', hh')$ is the product of G and H in **Grp**.*

PROOF:

$\langle 1 \rangle 1$. $G \times H$ is a group.

$\langle 2 \rangle 1$. The multiplication is associative.

PROOF: Since $(g_1, h_1)((g_2, h_2)(g_3, h_3)) = ((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1g_2g_3, h_1h_2h_3)$.

$\langle 2 \rangle 2$. (e_G, e_H) is the identity.

PROOF: Since $(g, h)(e_G, e_H) = (e_G, e_H)(g, h) = (g, h)$.

$\langle 2 \rangle 3$. The inverse of (g, h) is (g^{-1}, h^{-1}) .

PROOF: Since $(g, h)(g^{-1}, h^{-1}) = (g^{-1}, h^{-1})(g, h) = (e_G, e_H)$.

$\langle 1 \rangle 2$. $\pi_1 : G \times H \rightarrow G$ is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$. $\pi_2 : G \times H \rightarrow H$ is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 4$. For any group homomorphism $\phi : K \rightarrow G$ and $\psi : K \rightarrow H$, the function $\langle \phi, \psi \rangle : K \rightarrow G \times H$ where $\langle \phi, \psi \rangle(k) = (\phi(k), \psi(k))$ is a group homomorphism.

PROOF:

$$\begin{aligned} \langle \phi, \psi \rangle(kk') &= (\phi(kk'), \psi(kk')) \\ &= (\phi(k)\phi(k'), \psi(k)\psi(k')) \\ &= (\phi(k), \psi(k))(\phi(k'), \psi(k')) \\ &= \langle \phi, \psi \rangle(k)\langle \phi, \psi \rangle(k') \end{aligned}$$

□

Proposition 6.12.

$$|\text{Aut}_{\mathbf{Grp}}(C_n)| = \phi(n)$$

PROOF: An automorphism α is determined by $\alpha(1)$ which is any element of order n , and g has order n iff $\gcd(g, n) = 1$. □

Example 6.13.

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}) \cong C_2$$

PROOF: The only automorphisms are the identity and multiplication by -1. □

6.1 Subgroups

Definition 6.14 (Subgroup). Let (G, \cdot) and $(H, *)$ be groups such that H is a subset of G . Then H is a *subgroup* of G iff the inclusion $i : H \hookrightarrow G$ is a group homomorphism.

Proposition 6.15. If $(H, *)$ is a subgroup of (G, \cdot) then $*$ is the restriction of \cdot to H .

PROOF: Given $x, y \in H$ we have

$$x * y = i(x * y) = i(x) \cdot i(y) = x \cdot y . \quad \square$$

Example 6.16. For any group G we have $\{e\}$ is a subgroup of G .

Proposition 6.17. Let G be a group. Let H be a subset of G . Then H is a subgroup of G iff H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$.

PROOF:

⟨1⟩1. If H is a subgroup of G then H is nonempty.

PROOF: Since every group has an identity element and so is nonempty.

⟨1⟩2. If H is a subgroup of G then, for all $x, y \in H$, we have $xy^{-1} \in H$.

PROOF: Easy.

⟨1⟩3. If H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$, then H is a subgroup of G .

⟨2⟩1. ASSUME: H is nonempty.

⟨2⟩2. ASSUME: $\forall x, y \in H. xy^{-1} \in H$

⟨2⟩3. $e \in H$

PROOF: Pick $x \in H$. We have $e = xx^{-1} \in H$.

⟨2⟩4. $\forall x \in H. x^{-1} \in H$

PROOF: Given $x \in H$ we have $x^{-1} = ex^{-1} \in H$.

⟨2⟩5. H is closed under the restriction of \cdot

PROOF: Given $x, y \in H$ we have $xy = x(y^{-1})^{-1} \in H$.

⟨2⟩6. H is a group under the restriction of \cdot

PROOF: Associativity is inherited from G and the existence of an identity element and inverses follows from ⟨2⟩3 and ⟨2⟩4.

⟨2⟩7. The inclusion $H \hookrightarrow G$ is a group homomorphism.

PROOF: For $x, y \in H$ we have $i(xy) = i(x)i(y) = xy$.

□

Corollary 6.17.1. *The intersection of a set of subgroups of G is a subgroup of G .*

Corollary 6.17.2. *Let $\phi : G \rightarrow H$ be a group homomorphism. Let K be a subgroup of H . Then $\phi^{-1}(K)$ is a subgroup of G .*

PROOF:

⟨1⟩1. $\phi^{-1}(K)$ is nonempty.

PROOF: Since $e \in \phi^{-1}(K)$.

⟨1⟩2. LET: $x, y \in \phi^{-1}(K)$

⟨1⟩3. $\phi(x), \phi(y) \in K$

⟨1⟩4. $\phi(x)\phi(y)^{-1} \in K$

⟨1⟩5. $\phi(xy^{-1}) \in K$

⟨1⟩6. $xy^{-1} \in \phi^{-1}(K)$

□

Corollary 6.17.3. *Let $\phi : G \rightarrow H$ be a group homomorphism. Let K be a subgroup of G . Then $\phi(K)$ is a subgroup of H .*

PROOF:

⟨1⟩1. LET: $x, y \in \phi(K)$

⟨1⟩2. PICK $a, b \in K$ such that $x = \phi(a)$ and $y = \phi(b)$

⟨1⟩3. $xy^{-1} = \phi(ab^{-1})$

⟨1⟩4. $xy^{-1} \in \phi(K)$

□

Proposition 6.18. *Let G be a subgroup of \mathbb{Z} . Then there exists $d \geq 0$ such that $G = d\mathbb{Z}$.*

PROOF:

⟨1⟩1. ASSUME: w.l.o.g. $G \neq \{0\}$

PROOF: Since $\{0\} = 0\mathbb{Z}$.

⟨1⟩2. LET: d be the least positive element of G .

PROVE: $G = d\mathbb{Z}$

PROOF: If $n \in G$ then $-n \in G$ so G must contain a positive element.

⟨1⟩3. $G \subseteq d\mathbb{Z}$

⟨2⟩1. LET: $n \in G$

⟨2⟩2. LET: q and r be the integers such that $n = qd + r$ and $0 \leq r < d$.

⟨2⟩3. $r \in G$

PROOF: Since $r = n - qd$.

⟨2⟩4. $r = 0$

PROOF: By minimality of d .

⟨2⟩5. $n = qd \in d\mathbb{Z}$

⟨1⟩4. $d\mathbb{Z} \subseteq G$

□

6.2 Kernel

Definition 6.19 (Kernel). Let $\phi : G \rightarrow H$ be a group homomorphism. The *kernel* of ϕ is

$$\ker \phi = \{g \in G : \phi(g) = e\} .$$

Proposition 6.20. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\ker \phi$ is a subgroup of G .*

PROOF: Corollary 6.17.2. \square

Proposition 6.21. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then the inclusion $i : \ker \phi \hookrightarrow G$ is terminal in the category of pairs $(K, \alpha : K \rightarrow G)$ such that $\phi \circ \alpha = 0$.*

PROOF:

$\langle 1 \rangle 1.$ $\phi \circ i = 0$

$\langle 1 \rangle 2.$ For any group K and homomorphism $\alpha : K \rightarrow G$ such that $\phi \circ \alpha = 0$, there exists a unique homomorphism $\beta : K \rightarrow \ker \phi$ such that $i \circ \beta = \alpha$.

\square

Proposition 6.22. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then the following are equivalent:*

1. ϕ is monic.
2. $\ker \phi = \{e\}$
3. ϕ is injective.

PROOF:

$\langle 1 \rangle 1.$ $1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: ϕ is monic.

$\langle 2 \rangle 2.$ LET: $i : \ker \phi \hookrightarrow G$, $j : \{e\} \hookrightarrow \ker \phi \hookrightarrow G$ be the inclusions.

$\langle 2 \rangle 3.$ $\phi \circ i = \phi \circ j$

$\langle 2 \rangle 4.$ $i = j$

$\langle 1 \rangle 2.$ $2 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: $\ker \phi = \{e\}$

$\langle 2 \rangle 2.$ LET: $x, y \in G$

$\langle 2 \rangle 3.$ ASSUME: $\phi(x) = \phi(y)$

$\langle 2 \rangle 4.$ $\phi(xy^{-1}) = e$

$\langle 2 \rangle 5.$ $xy^{-1} \in \ker \phi$

$\langle 2 \rangle 6.$ $xy^{-1} = e$

$\langle 2 \rangle 7.$ $x = y$

$\langle 1 \rangle 3.$ $3 \Rightarrow 1$

PROOF: Easy.

\square

Proposition 6.23. *A group homomorphism is an epimorphism if and only if it is surjective.*

6.3 Inner Automorphisms

Proposition 6.24. *Let G be a group and $g \in G$. The function $\gamma_g : G \rightarrow G$ defined by $\gamma_g(a) = gag^{-1}$ is an automorphism on G .*

PROOF:

$\langle 1 \rangle 1$. γ_g is a homomorphism.

PROOF:

$$\begin{aligned}\gamma_g(ab) &= gabg^{-1} \\ &= gag^{-1}gbg^{-1} \\ &= \gamma_g(a)\gamma_g(b)\end{aligned}$$

$\langle 1 \rangle 2$. γ_g is injective.

PROOF: By Cancellation.

$\langle 1 \rangle 3$. γ_g is surjective.

PROOF: Given $b \in G$, we have $\gamma_g(g^{-1}bg) = b$.

□

Definition 6.25 (Inner Automorphism). Let G be a group. An *inner automorphism* on G is a function of the form $\gamma_g(a) = gag^{-1}$ for some $g \in G$.

We write $\text{Inn}(G)$ for the set of inner automorphisms of G .

Proposition 6.26. *Let G be a group. The function $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$ that maps g to γ_g is a group homomorphism.*

PROOF: Since $\gamma_{gh}(a) = ghah^{-1}g^{-1} = \gamma_g(\gamma_h(a))$. □

Corollary 6.26.1. *$\text{Inn}(G)$ is a subgroup of $\text{Aut}_{\mathbf{Grp}}(G)$.*

6.4 Direct Products

Definition 6.27 (Direct Product). The *direct product* of groups G and H is their product in \mathbf{Grp} .

6.5 Free Groups

Proposition 6.28. *Let A be a set. Let \mathcal{F}^A be the category whose objects are pairs (G, j) where G is a group and j is a function $A \rightarrow G$, with morphisms $f : (G, j) \rightarrow (H, k)$ the group homomorphisms $f : G \rightarrow H$ such that $f \circ j = k$. Then \mathcal{F}^A has an initial object.*

PROOF:

$\langle 1 \rangle 1$. LET: $W(A)$ be the set of words in the alphabet whose elements are the elements of A together with $\{a^{-1} : a \in A\}$.

$\langle 1 \rangle 2$. LET: $r : W(A) \rightarrow W(A)$ be the function that, given a word w , removes the first pair of letters of the form aa^{-1} or $a^{-1}a$; if there is no such pair, then $r(w) = w$.

$\langle 1 \rangle 3$. Let us say that a word w is a *reduced word* iff $r(w) = w$.

$\langle 1 \rangle 4$. For any word w of length n , we have $r^{\lceil \frac{n}{2} \rceil}(w)$ is a reduced word.

PROOF: Since we cannot remove more than $n/2$ pairs of letters from w .

$\langle 1 \rangle 5$. LET: $R : W(A) \rightarrow W(A)$ be the function $R(w) = r^{\lceil \frac{n}{2} \rceil}(w)$, where n is the length of w .

$\langle 1 \rangle 6$. LET: $F(A)$ be the set of reduced words.

$\langle 1 \rangle 7$. Define $\cdot : F(A)^2 \rightarrow F(A)$ by $w \cdot w' = R(ww')$

$\langle 1 \rangle 8$. \cdot is associative.

PROOF: Both $w_1 \cdot (w_2 \cdot w_3)$ and $(w_1 \cdot w_2) \cdot w_3$ are equal to $R(w_1w_2w_3)$.

$\langle 1 \rangle 9$. The empty word is the identity element in $F(A)$

$\langle 1 \rangle 10$. The inverse of $a_1^{\pm 1}a_2^{\pm 1} \dots a_n^{\pm 1}$ is $a_n^{\mp 1} \dots a_2^{\mp 1}a_1^{\mp 1}$.

$\langle 1 \rangle 11$. LET: $j : A \rightarrow F(A)$ be the function that maps a to the word a of length

$\langle 1 \rangle 12$. LET: G be any group and $k : A \rightarrow G$ any function.

$\langle 1 \rangle 13$. The only morphism $f : (F(A), j) \rightarrow (G, k)$ in \mathcal{F}^A is $f(a_1^{\pm 1}a_2^{\pm 1} \dots a_n^{\pm 1}) = k(a_1)^{\pm 1}k(a_2)^{\pm 1} \dots k(a_n)^{\pm 1}$.

□

Definition 6.29 (Free Group). For any set A , the *free group* on A is the initial object $(F(A), i)$ in \mathcal{F}^A .

Proposition 6.30. $i : A \rightarrow F(A)$ is injective.

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in A$

$\langle 1 \rangle 2$. ASSUME: $x \neq y$

PROVE: $i(x) \neq i(y)$

$\langle 1 \rangle 3$. LET: $f : A \rightarrow C_2$ be the function that maps x to 0 and all other elements of A to 1.

$\langle 1 \rangle 4$. LET: $\phi : F(A) \rightarrow C_2$ be the group homomorphism such that $f = \phi \circ i$.

$\langle 1 \rangle 5$. $f(x) \neq f(y)$

$\langle 1 \rangle 6$. $\phi(i(x)) \neq \phi(i(y))$

$\langle 1 \rangle 7$. $i(x) \neq i(y)$

□

Proposition 6.31.

$$F(0) \cong \{e\}$$

PROOF: For any set A , the unique group homomorphism $\{e\} \rightarrow A$ makes the following diagram commute.

$$\begin{array}{ccc} \{e\} & \longrightarrow & A \\ \uparrow & \nearrow & \\ \emptyset & & \end{array}$$

Proposition 6.32. The free group on 1 is \mathbb{Z} with the injection mapping 0 to 1.

PROOF: Given any group G and function $a : 1 \rightarrow G$, the required unique homomorphism $\phi : \mathbb{Z} \rightarrow G$ is defined by $\phi(n) = a(0)^n$. □

Proposition 6.33. *For any sets A and B , we have that $F(A + B)$ is the coproduct of $F(A)$ and $F(B)$ in **Grp**.*

$$\begin{array}{ccccc}
 & & G & & \\
 & \nearrow f & \uparrow k & \nwarrow g & \\
 F(A) & \xrightarrow{\kappa_1} & F(A+B) & \xleftarrow{\kappa_2} & F(B) \\
 \uparrow i_A & & \uparrow j & & \uparrow i_B \\
 A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
 \end{array}$$

PROOF:

- $\langle 1 \rangle 1$. LET: $i_A : A \rightarrow F(A)$, $i_B : B \rightarrow F(B)$, $j : A + B \rightarrow F(A + B)$ be the canonical injections.
- $\langle 1 \rangle 2$. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
- $\langle 1 \rangle 3$. LET: G be any group and $f : F(A) \rightarrow G$, $g : F(B) \rightarrow G$ any group homomorphisms.
- $\langle 1 \rangle 4$. LET: $h : A + B \rightarrow G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.
- $\langle 1 \rangle 5$. LET: $k : F(A + B) \rightarrow G$ be the unique group homomorphism such that $k \circ j = h$.
- $\langle 1 \rangle 6$. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
- $\langle 1 \rangle 7$. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$. \square

Definition 6.34 (Subgroup Generated by a Group). Let G be a group and A a subset of G . Let $\phi : F(A) \rightarrow G$ be the unique group homomorphism such that $\phi(a) = a$ for all $a \in A$. The subgroup *generated* by A is

$$\langle A \rangle := \text{im } \phi$$

$$\begin{array}{ccc}
 F(A) & \xrightarrow{\phi} & G \\
 \uparrow & \nearrow & \\
 A & &
 \end{array}$$

Proposition 6.35. *Let G be a group and A a subset of G . Then $\langle A \rangle$ is the set of all elements of the form $a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$ (where $n \geq 0$) such that $a_1, \dots, a_n \in A$.*

PROOF: Immediate from definitions. \square

Corollary 6.35.1. *Let G be a group and $g \in G$. Then*

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\} .$$

Proposition 6.36. *Let G be a group and A a subset of G . Then $\langle A \rangle$ is the intersection of all the subgroups of G that include A .*

PROOF: Easy. \square

Definition 6.37 (Finitely Generated). Let G be a group. Then G is *finitely generated* iff there exists a finite subset A of G such that $G = \langle A \rangle$.

Proposition 6.38. *Every subgroup of a finitely generated free group is free.*

PROOF: TODO.

Proposition 6.39. *$F(2)$ includes subgroups isomorphic to the free group on arbitrarily many generators.*

PROOF: TODO

Proposition 6.40.

$$[F(2), F(2)] \cong F(\mathbb{Z})$$

PROOF: TODO

6.6 Normal Subgroups

Definition 6.41 (Normal Subgroup). A subgroup N of G is *normal* iff, for all $g \in G$ and $n \in N$, we have $gng^{-1} \in N$.

Proposition 6.42. *Let G be a group and N a subgroup of G . Then the following are equivalent.*

1. N is normal.
2. $\forall g \in G. gNg^{-1} \subseteq N$
3. $\forall g \in G. gNg^{-1} = N$
4. $\forall g \in G. gN \subseteq Ng$
5. $\forall g \in G. gN = Ng$

PROOF:

$\langle 1 \rangle 1. 1 \Leftrightarrow 2$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: If 2 holds then we have $gNg^{-1} \subseteq N$ and $g^{-1}Ng \subseteq N$ hence $N = gNg^{-1}$.

$\langle 1 \rangle 3. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 4. 2 \Leftrightarrow 4$

PROOF: Easy.

$\langle 1 \rangle 5. 3 \Leftrightarrow 5$

PROOF: Easy.

□

Proposition 6.43. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\ker \phi$ is a normal subgroup of G .*

PROOF: Given $g \in G$ and $n \in \ker \phi$ we have

$$\begin{aligned}\phi(gng^{-1}) &= \phi(g)\phi(n)\phi(g)^{-1} \\ &= \phi(g)\phi(g)^{-1} \\ &= e\end{aligned}$$

and so $gng^{-1} \in \ker \phi$. □

6.7 Quotient Groups

Definition 6.44. Let G be a group. Let \sim be an equivalence relation on G . Then we say that \sim is *compatible* with the group operation on G iff, for all $a, a', g \in G$, if $a \sim a'$ then $ga \sim ga'$ and $ag \sim a'g$.

Proposition 6.45. *Let G be a group. Let \sim be an equivalence relation on G . Then there exists an operation $\cdot : (G/\sim)^2 \rightarrow G/\sim$ such that*

$$\forall a, b \in G. [a][b] = [ab]$$

iff \sim is compatible with the group operation on G . In this case, G/\sim is a group under \cdot and the canonical function $\pi : G \rightarrow G/\sim$ is a group homomorphism, and is universal with respect to group homomorphisms $\phi : G \rightarrow G'$ such that if $a \sim a'$ then $\phi(a) = \phi(a')$.

PROOF: Easy. □

Definition 6.46 (Quotient Group). Let G be a group. Let \sim be an equivalence relation on G that is compatible with the group operation on G . Then G/\sim is the *quotient group* of G by \sim under $[a][b] = [ab]$.

Proposition 6.47. *Let G be a group and H a subgroup of G . Then H is normal if and only if there exists a group K and homomorphism $\phi : G \rightarrow K$ such that $H = \ker \phi$.*

PROOF: One direction is given by Proposition 6.43. For the other direction, take $K = G/H$ and ϕ to be the canonical map $G \rightarrow G/H$. □

Definition 6.48 (Modular Group). The *modular group* $\text{PSL}_2(\mathbb{Z})$ is $\text{SL}_2(\mathbb{Z})/\{I, -I\}$.

Proposition 6.49. $\text{PSL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

PROOF: By Example 5.26.

Proposition 6.50 (Roger Alperin). $\text{PSL}_2(\mathbb{Z})$ is presented by $(x, y|x^2, y^3)$.

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\langle 1 \rangle 2. \text{ LET: } y = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$\langle 1 \rangle 3.$ Define an action of $\text{PSL}_2(\mathbb{Z})$ on $\mathbb{R} - \mathbb{Q}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} r = \frac{ar + b}{cr + d}.$$

$\langle 2 \rangle 1.$ Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z})$ and r irrational we have $\frac{ar+b}{cr+d}$ is irrational.

$\langle 3 \rangle 1.$ ASSUME: for a contradiction $\frac{ar+b}{cr+d} = \frac{p}{q}$ where p and q are integers with $q > 0$.

$$\langle 3 \rangle 2. aqr + bq = cpr + dp$$

$$\langle 3 \rangle 3. (aq - cp)r = dp - bq$$

$$\langle 3 \rangle 4. aq = cp = dp - bq = 0$$

$$\langle 3 \rangle 5. adq - cdp = 0$$

$$\langle 3 \rangle 6. cdp - cbq = 0$$

$$\langle 3 \rangle 7. (ad - cb)q = 0$$

PROOF: Since $ad - cb = 1$.

$$\langle 3 \rangle 8. q = 0$$

$$\langle 3 \rangle 9. \text{ Q.E.D.}$$

PROOF: This contradicts $\langle 3 \rangle 1$.

$$\langle 2 \rangle 2. -Ir = r$$

PROOF: Since $-Ir = \frac{-r}{-1} = r$.

$$\langle 2 \rangle 3. \text{ Given } A, B \in \text{PSL}_2(\mathbb{Z}) \text{ we have } A(Br) = (AB)r.$$

PROOF:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} r \right] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{er + f}{gr + h} \\ &= \frac{a \frac{er+f}{gr+h} + b}{c \frac{er+f}{gr+h} + d} \\ &= \frac{a(er + f) + b(gr + h)}{c(er + f) + d(gr + h)} \\ &= \frac{(ae + bg)r + (af + bh)}{(ce + dg)r + (cf + dh)} \\ &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} r \\ &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right] r \end{aligned}$$

$$\langle 1 \rangle 4.$$

$$yr = 1 - \frac{1}{r}$$

$$\langle 1 \rangle 5.$$

$$y^{-1}r = \frac{1}{1-r}$$

PROOF: Since $y^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

(1)6.

PROOF: Since $yx = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. $yxr = 1 + r$

(1)7.

$$y^{-1}xr = \frac{r}{1+r}$$

PROOF: Since $y^{-1}x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

(1)8. If $r > -1$ is positive then yxr is positive.

(1)9. If r is positive then $y^{-1}xr$ is positive.

(1)10. If $r < -1$ then $y^{-1}xr$ is positive.

(1)11. If r is negative then yr is positive.

(1)12. If r is negative then $y^{-1}r$ is positive.

(1)13. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)$$

with one or more factors can equal the identity.

PROOF: If the last factor is (yx) , then the product maps numbers in $(-1, 0)$ to positive numbers. If the last factor is $(y^{-1}x)$, then the product maps numbers < -1 to positive numbers.

(1)14. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)y^{\pm 1}$$

with one or more factors can equal the identity.

PROOF: The product maps negative numbers to positive numbers.

(1)15. $\text{PSL}_2(\mathbb{Z})$ is presented by $(x, y|x^2, y^3)$.

□

Corollary 6.50.1. $\text{PSL}_2(\mathbb{Z})$ is the coproduct of C_2 and C_3 in **Grp**.

Theorem 6.51. Every group homomorphism $\phi : G \rightarrow H$ may be decomposed as

$$G \longrightarrow G/\ker \phi \xrightarrow{\cong} \text{im } \phi \longrightarrow H$$

PROOF: Easy. □

Corollary 6.51.1 (First Isomorphism Theorem). Let $\phi : G \rightarrow H$ be a surjective group homomorphism. Then $H \cong G/\ker \phi$.

Proposition 6.52. Let H_1 be a normal subgroup of G_1 and H_2 a normal subgroup of G_2 . Then $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$, and

$$\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}.$$

PROOF: $\pi \times \pi : G_1 \times G_2 \twoheadrightarrow G_1/H_1 \times G_2/H_2$ is a surjective homomorphism with kernel $H_1 \times H_2$. □

Example 6.53.

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

PROOF: Map a real number r to $(\cos r, \sin r)$. The result is a surjective group homomorphism with kernel \mathbb{Z} . \square

Proposition 6.54. *Let H be a normal subgroup of a group G . For every subgroup K of G that includes H , we have H is a normal subgroup of K , and K/H is a subgroup of G/H . The mapping*

$$u : \{\text{subgroups of } G \text{ including } H\} \rightarrow \{\text{subgroups of } G/H\}$$

with $u(K) = K/H$ is a poset isomorphism.

PROOF:

- $\langle 1 \rangle 1$. If K is a subgroup of G that includes H then H is normal in K .
- $\langle 1 \rangle 2$. If K is a subgroup of G that includes H then K/H is a subgroup of G/H .
- $\langle 1 \rangle 3$. If $H \subseteq K_1 \subseteq K_2$ then $K_1/H \subseteq K_2/H$.
- $\langle 1 \rangle 4$. If $K_1/H = K_2/H$ then $K_1 = K_2$
 - $\langle 2 \rangle 1$. ASSUME: $K_1/H = K_2/H$
 - $\langle 2 \rangle 2$. $K_1 \subseteq K_2$
 - $\langle 3 \rangle 1$. LET: $k \in K_1$
 - $\langle 3 \rangle 2$. $kH \in K_2/H$
 - $\langle 3 \rangle 3$. PICK $k' \in K_2$ such that $kH = k'H$
 - $\langle 3 \rangle 4$. $kk'^{-1} \in H$
 - $\langle 3 \rangle 5$. $kk'^{-1} \in K_2$
 - $\langle 3 \rangle 6$. $k \in K_2$
 - $\langle 2 \rangle 3$. $K_2 \subseteq K_1$
- PROOF: Similar.
- $\langle 1 \rangle 5$. For any subgroup L of G/H , there exists a subgroup K of G that includes H such that $L = K/H$.
 - $\langle 2 \rangle 1$. LET: L be a subgroup of G/H .
 - $\langle 2 \rangle 2$. LET: $K = \{k \in G : kH \in L\}$
 - $\langle 2 \rangle 3$. K is a subgroup of G .
 - PROOF: Given $k, k' \in K$ we have $kH, k'H \in L$ hence $kk'^{-1}H \in L$ and so $kk'^{-1} \in K$.
 - $\langle 2 \rangle 4$. $H \subseteq K$
 - PROOF: For all $h \in H$ we have $hH = H \in L$.
 - $\langle 2 \rangle 5$. $L = K/H$
 - PROOF: By definition.

\square

Proposition 6.55 (Third Isomorphism Theorem). *Let H be a normal subgroup of a group G . Let N be a subgroup of G that includes H . Then N/H is normal in G/H if and only if N is normal in G , in which case*

$$\frac{G/H}{N/H} \cong \frac{G}{N}$$

PROOF:

- ⟨1⟩1. If N/H is normal in G/H then N is normal in G .
 - ⟨2⟩1. ASSUME: N/H is normal in G/H .
 - ⟨2⟩2. LET: $g \in G$ and $n \in N$.
 - ⟨2⟩3. $gng^{-1}H \in N/H$
 - ⟨2⟩4. PICK $n' \in N$ such that $gng^{-1}H = n'H$
 - ⟨2⟩5. $gng^{-1}n'^{-1} \in H$
 - ⟨2⟩6. $gng^{-1}n'^{-1} \in N$
 - ⟨2⟩7. $gng^{-1} \in N$
- ⟨1⟩2. If N is normal in G then N/H is normal in G/H and $(G/H)/(N/H) \cong G/N$.
 - ⟨2⟩1. ASSUME: N is normal in G .
 - ⟨2⟩2. LET: $\phi : G/H \rightarrow G/N$ be the homomorphism $\phi(gH) = gN$
 - ⟨3⟩1. If $gH = g'H$ then $gN = g'N$
 PROOF: If $gg'^{-1} \in H$ then $gg'^{-1} \in N$.
 - ⟨3⟩2. $\phi((gH)(g'H)) = \phi(gH)\phi(g'H)$
 PROOF: Both are $gg'N$.
 - ⟨2⟩3. ϕ is surjective.
 - ⟨2⟩4. $\ker \phi = N/H$
 - ⟨2⟩5. $(G/H)/(N/H) \cong G/N$
 PROOF: By the First Isomorphism Theorem.

□

Proposition 6.56 (Second Isomorphism Theorem). *Let H and K be subgroups of a group G . Assume that H is normal in G . Then:*

- 1. HK is a subgroup of G , and H is normal in HK .
- 2. $H \cap K$ is normal in K , and

$$\frac{HK}{H} \cong \frac{K}{H \cap K}.$$

PROOF:

- ⟨1⟩1. HK is a subgroup of G .
 PROOF: Since $hkh'k' = hh'(h'^{-1}kh')k' \in HK$.
- ⟨1⟩2. H is normal in HK .
- ⟨1⟩3. $H \cap K$ is normal in K and $HK/H \cong K/(H \cap K)$
 PROOF: The function that maps k to kH is a surjective homomorphism $K \twoheadrightarrow HK/H$ with kernel $H \cap K$. Surjectivity follows because $hkh = hkh^{-1}H$.

□

See also Proposition 6.71 for a result that holds even if H is not normal.

6.8 Cosets

Proposition 6.57. *Let G be a group. Let \sim be an equivalence relation on G such that, for all $a, b, g \in G$, if $a \sim b$ then $ga \sim gb$. Let $H = \{h \in G : h \sim e\}$.*

Then H is a subgroup of G and, for all $a, b \in G$, we have

$$a \sim b \Leftrightarrow a^{-1}b \in H \Leftrightarrow aH = bH .$$

PROOF:

$\langle 1 \rangle 1.$ $e \in H$

$\langle 1 \rangle 2.$ For all $x, y \in H$ we have $xy^{-1} \in H$.

$\langle 2 \rangle 1.$ ASSUME: $x \sim e$ and $y \sim e$.

$\langle 2 \rangle 2.$ $e \sim y^{-1}$

PROOF: Since $yy^{-1} \sim ey^{-1}$.

$\langle 2 \rangle 3.$ $xy^{-1} \sim e$

PROOF: Since $xy^{-1} \sim ey^{-1} \sim e$.

$\langle 1 \rangle 3.$ If $a \sim b$ then $a^{-1}b \in H$.

PROOF: If $a \sim b$ then $a^{-1}b \sim a^{-1}a = e$.

$\langle 1 \rangle 4.$ If $a^{-1}b \in H$ then $aH = bH$.

$\langle 2 \rangle 1.$ ASSUME: $a^{-1}b \in H$

$\langle 2 \rangle 2.$ $bH \subseteq aH$

PROOF: For any $h \in H$ we have $bh = aa^{-1}bh \in aH$.

$\langle 2 \rangle 3.$ $aH \subseteq bH$

PROOF: Similar since $b^{-1}a \in H$.

$\langle 1 \rangle 5.$ If $aH = bH$ then $a \sim b$.

$\langle 2 \rangle 1.$ ASSUME: $aH = bH$

$\langle 2 \rangle 2.$ PICK $h \in H$ such that $a = bh$.

$\langle 2 \rangle 3.$ $b^{-1}a = h$

$\langle 2 \rangle 4.$ $b^{-1}a \in H$

$\langle 2 \rangle 5.$ $b^{-1}a \sim e$

$\langle 2 \rangle 6.$ $a \sim b$

PROOF: $a = bb^{-1}a \sim be = b$.

□

Definition 6.58 (Coset). Let G be a group and H a subgroup of G . A *left coset* of H is a set of the form aH for $a \in G$. A *right coset* of H is a set of the form Ha for some $a \in G$.

We write G/H for the set of all left cosets of H , and $G \backslash H$ for the set of all right cosets of H .

Proposition 6.59.

$$G/H \cong G \backslash H$$

PROOF: The function that maps aH to Ha^{-1} is a bijection. □

Proposition 6.60. Let G be a group and H a subgroup of G . Define \sim_H on G by: $a \sim b$ iff $a^{-1}b \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations \sim on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ga \sim gb$. The equivalence class of a is aH .

PROOF:

$\langle 1 \rangle 1.$ For any subgroup H , we have \sim_H is an equivalence relation on G .

⟨2⟩1. \sim is reflexive.

PROOF: For any $a \in G$ we have $a^{-1}a = e \in H$.

⟨2⟩2. \sim is symmetric.

PROOF: If $a^{-1}b \in H$ then $b^{-1}a \in H$.

⟨2⟩3. \sim is transitive.

PROOF: If $a^{-1}b \in H$ and $b^{-1}c \in H$ then $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$.

⟨1⟩2. If $a \sim_H b$ then $ga \sim_H gb$.

PROOF: If $a^{-1}b \in H$ then $(ga)^{-1}(gb) = a^{-1}g^{-1}gb = a^{-1}b \in H$.

⟨1⟩3. For any equivalence relation \sim on G such that, whenever $a \sim b$, then $ga \sim gb$, there exists a subgroup H such that $\sim = \sim_H$.

PROOF: Proposition 6.57.

⟨1⟩4. The \sim_H -equivalence class of a is aH .

PROOF:

$$\begin{aligned} a \sim b &\Leftrightarrow a^{-1}b \in H \\ &\Leftrightarrow \exists h \in H. a^{-1}b = h \\ &\Leftrightarrow \exists h \in H. b = ah \\ &\Leftrightarrow b \in aH \end{aligned}$$

□

Proposition 6.61. Let G be a group and H a subgroup of G . Define \sim_H on G by: $a \sim b$ iff $ab^{-1} \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations \sim on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ag \sim bg$. The equivalence class of a is Ha .

PROOF: Similar. □

Proposition 6.62. Let G be a group and H be a subgroup of G . Define \sim_L and \sim_R on G by:

$$a \sim_L b \Leftrightarrow a^{-1}b \in H, \quad a \sim_R b \Leftrightarrow ab^{-1} \in H.$$

Then $\sim_L = \sim_R$ if and only if H is normal.

PROOF:

⟨1⟩1. If $\sim_L = \sim_R$ then H is normal.

⟨2⟩1. ASSUME: $\sim_L = \sim_R$

⟨2⟩2. LET: $h \in H$ and $g \in G$

⟨2⟩3. $g \sim_L gh^{-1}$

⟨2⟩4. $g \sim_R gh^{-1}h$

⟨2⟩5. $ghg^{-1} \in H$

⟨1⟩2. If H is normal then $\sim_L = \sim_R$.

⟨2⟩1. ASSUME: H is normal.

⟨2⟩2. If $a \sim_L b$ then $a \sim_R b$.

⟨3⟩1. ASSUME: $a \sim_L b$

⟨3⟩2. $a^{-1}b \in H$

⟨3⟩3. $aa^{-1}ba^{-1} \in H$

⟨3⟩4. $ba^{-1} \in H$

- $\langle 3 \rangle 5. a \sim_R b$
 $\langle 2 \rangle 3. \text{ If } a \sim_R b \text{ then } a \sim_L b.$

PROOF: Similar.

□

Corollary 6.62.1. *Let G be a group and H be a normal subgroup of G . Define \sim on G by $a \sim b$ iff $a^{-1}b \in H$. Then G/\sim is a group under $[a][b] = [ab]$.*

Definition 6.63 (Quotient Group). Let G be a group and H be a normal subgroup of G . The *quotient group* G/H is G/\sim where $a \sim b$ iff $a^{-1}b \in H$, under $[a][b] = [ab]$ or $(aH)(bH) = abH$.

Corollary 6.63.1. *Let H be a normal subgroup of a group G . For every group homomorphism $\phi : G \rightarrow G'$ such that $H \subseteq \ker \phi$, there exists a unique group homomorphism $\bar{\phi} : G/H \rightarrow G'$ such that the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ & \searrow \pi & \nearrow \bar{\phi} \\ & G/H & \end{array}$$

Proposition 6.64. $\mathbb{Z}/n\mathbb{Z}$ has exactly n elements.

PROOF: Every integer is congruent to one of $0, 1, \dots, n-1$ by the division algorithm, and no two of them are congruent to one another, since if $0 \leq i < j < n$ then $0 < j - i < n$. □

Proposition 6.65. *Let m and n be integers with $n > 0$. The order of m in $\mathbb{Z}/n\mathbb{Z}$ is $\frac{n}{\gcd(m,n)}$.*

PROOF: By Proposition 5.16 since the order of 1 is n . □

Proposition 6.66. *The integer m generates $\mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(m, n) = 1$.*

PROOF: By Proposition 6.65. □

Corollary 6.66.1. *If p is prime then every non-zero element in $\mathbb{Z}/p\mathbb{Z}$ is a generator.*

Proposition 6.67.

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$$

PROOF: Every permutation of $\{(1, 0), (0, 1), (1, 1)\}$ gives an automorphism of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

Example 6.68. Not all monomorphisms split in \mathbf{Grp} .

Define $\phi : \mathbb{Z}/3\mathbb{Z} \rightarrow S_3$ by

$$\phi(0) = \text{id}_3, \quad \phi(1) = (1 \ 3 \ 2), \quad \phi(2) = (1 \ 2 \ 3) .$$

Then ϕ is monic but has no retraction.

For if $r : S_3 \rightarrow \mathbb{Z}/3\mathbb{Z}$ is a retraction, then we would have

$$r(1\ 2) + r(2\ 3) = 1, \quad r(2\ 3) + r(1\ 2) = 2$$

which is impossible.

Proposition 6.69. *Let G be a group, H a subgroup of G , and $g \in G$. The function that maps h to gh is a bijection $H \cong gH$.*

PROOF: By Cancellation. \square

Proposition 6.70. *Let G be a group, H a subgroup of G , and $g \in G$. The function that maps h to hg is a bijection $H \cong Hg$.*

PROOF: By Cancellation. \square

Proposition 6.71. *Let H and K be finite subgroups of a group G . Then*

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

PROOF:

$\langle 1 \rangle 1$. LET: $f : \{hK : h \in H\} \rightarrow H/(H \cap K)$ be the function $f(hK) = h(H \cap K)$

PROOF: This is well-defined because if $hK = h'K$ then $h^{-1}h' \in H \cap K$ so $h(H \cap K) = h'(H \cap K)$.

$\langle 1 \rangle 2$. f is injective.

PROOF: If $h(H \cap K) = h'(H \cap K)$ then $hK = h'K$.

$\langle 1 \rangle 3$. f is surjective.

PROOF: Clear.

$\langle 1 \rangle 4$.

$$\frac{|HK|}{|K|} = \frac{|H|}{|H \cap K|}$$

\square

6.9 Congruence

Definition 6.72 (Congruence). Given integers a, b, n with n positive, we say a is *congruent to b modulo n* , and write $a \equiv b \pmod{n}$, iff $a + n\mathbb{Z} = b + n\mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$.

Proposition 6.73. *Given integers a, b, n with n positive, we have $a \equiv b \pmod{n}$ iff $n \mid a - b$.*

PROOF: By Proposition 6.57. \square

Proposition 6.74. *If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then $a + b \equiv a' + b' \pmod{n}$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid (a' + b') - (a + b)$. \square

Proposition 6.75. *If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then $ab \equiv a'b' \pmod{n}$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid a'b' - ab = a'(b' - b) + (a' - a)b$. \square

6.10 Cyclic Groups

Definition 6.76 (Cyclic Group). The *cyclic* groups are \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$ for positive integers n .

Proposition 6.77. *If m and n are positive integers with $\gcd(m, n) = 1$ then $C_{mn} \cong C_m \times C_n$.*

PROOF: The function that maps x to $(x \bmod m, x \bmod n)$ is an isomorphism. \square

Proposition 6.78. *Let G be a group and $g \in G$. Then $\langle g \rangle$ is cyclic.*

PROOF: If g has finite order then $\langle g \rangle \cong C_{|g|}$, otherwise $\langle g \rangle \cong \mathbb{Z}$. \square

Proposition 6.79. *Every finitely generated subgroup of \mathbb{Q} is cyclic.*

PROOF:

$\langle 1 \rangle 1$. LET: $G = \langle a_1/b, \dots, a_n/b \rangle$ where a_1, \dots, a_n, b are integers with $b > 0$

$\langle 1 \rangle 2$. LET: $a = \gcd(a_1, \dots, a_n)$

$\langle 1 \rangle 3$. $G = \langle a/b \rangle$

\square

Corollary 6.79.1. \mathbb{Q} is not finitely generated.

6.11 Euler's ϕ -function

Definition 6.80. For n a positive integer, let $(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$.

PROOF: We prove this is well-defined.

$\langle 1 \rangle 1$. If $m \equiv m' \pmod{n}$ and $\gcd(m, n) = 1$ then $\gcd(m', n) = 1$.

$\langle 2 \rangle 1$. PICK integers a, b such that $am + bn = 1$

$\langle 2 \rangle 2$. PICK an integer c such that $m' - m = cn$

$\langle 2 \rangle 3$. $am' + (b - ac)n = 1$

\square

Example 6.81. For any positive integer n , the set

$$(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$$

is a group under multiplication.

PROOF:

$\langle 1 \rangle 1$. If $\gcd(m_1, n) = \gcd(m_2, n) = 1$ then $\gcd(m_1 m_2, n) = 1$

$\langle 2 \rangle 1$. PICK integers a, b, c, d such that $am_1 + bn = cm_2 + dn = 1$

$\langle 2 \rangle 2$. $acm_1 m_2 + (bcm_2 + d)n = 1$

$\langle 1 \rangle 2$. Multiplication is associative.

$\langle 1 \rangle 3$. 1 is the identity element.

$\langle 1 \rangle 4$. Every element has an inverse.

- (2)1. LET: $a \in (\mathbb{Z}/n\mathbb{Z})^*$
 (2)2. PICK integers b, c such that $ab + cn = 1$
 (2)3. $ab = 1$ in $(\mathbb{Z}/n\mathbb{Z})^*$

□

Definition 6.82. For n a positive integer, let $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$.

Proposition 6.83. If n is an odd positive integer then $\phi(2n) = \phi(n)$.

PROOF:

- (1)1. LET: n be an odd positive integer.
 (1)2. For any integer m , if $\gcd(m, n) = 1$ then $\gcd(2m + n, 2n) = 1$
 PROOF: For p a prime, if $p \mid 2m + n$ and $p \mid 2n$ then $p \neq 2$ (since $2m + n$ is odd) so $p \mid n$ and hence $p \mid m$, which is a contradiction.
 (1)3. For any integer r , if $\gcd(r, 2n) = 1$ then $\gcd(\frac{r+n}{2}, n) = 1$
 PROOF: If $p \mid n$ and $p \mid \frac{r+n}{2}$ then $p \mid r + n$ so $p \mid r$ which is a contradiction.
 (1)4. The function that maps m to $2m + n$ is a bijection between $(\mathbb{Z}/n\mathbb{Z})^*$ and $(\mathbb{Z}/2n\mathbb{Z})^*$.

□

Theorem 6.84. For any positive integer n we have

$$\sum_{m>0, m|n} \phi(m) = n.$$

PROOF:

- (1)1. Define $\chi : \{0, 1, \dots, n-1\} \rightarrow \{(m, d) : m > 0, m \mid n, d \text{ generates } \langle n/m \rangle\}$
 by: $\chi(x) = (\gcd(x, n), x)$.
 (1)2. χ is injective.
 (1)3. χ is surjective.
 PROOF: Given (m, d) such that d generates $\langle n/m \rangle$ we have $\chi(d) = (m, d)$.
 (1)4. $n = \sum_{m>0, m|n} \phi(m)$
 PROOF: Since $\langle n/m \rangle \cong C_m$ and so has $\phi(m)$ generators.

□

Proposition 6.85. For any positive integers a and n , we have $n \mid \phi(a^n - 1)$.

PROOF: Since the order of a is n in $(\mathbb{Z}/(a^n - 1)\mathbb{Z})^*$. □

Theorem 6.86 (Euler's Theorem). For any coprime integers a and n we have $a^{\phi(n)} \equiv a \pmod{n}$.

PROOF: Immediate from Lagrange's Theorem. □

6.12 Commutator Subgroup

Definition 6.87 (Commutator Subgroup). Let G be a group. The *commutator subgroup* $[G, G]$ is the subgroup generated by the elements of the form $aba^{-1}b^{-1}$.

Proposition 6.88. *The commutator subgroup is normal.*

PROOF: Since

$$\begin{aligned} & ga_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_nb_na_n^{-1}b_n^{-1}g^{-1} \\ &= (ga_1g^{-1})(gb_1g^{-1})(ga_1g^{-1})^{-1}(gb_1g^{-1})^{-1} \cdots (ga_ng^{-1})(gb_ng^{-1})(ga_ng^{-1})^{-1}(gb_ng^{-1})^{-1}. \quad \square \end{aligned}$$

6.13 Presentations

Definition 6.89 (Presentation). A *presentation* of a group G is a pair (A, R) where A is a set and $R \subseteq F(A)$ is a set of words such that

$$G \cong F(A)/N(R)$$

where $N(R)$ is the smallest normal subgroup of $F(A)$ that includes R .

Example 6.90. The free group on a set A is presented by (A, \emptyset) .

Example 6.91. S_3 is presented by $(x, y|x^2, y^3, xyxy)$.

Example 6.92. $(a, b | a^2, b^2, (ab)^n)$ is a presentation of D_{2n} .

Proposition 6.93 (Word Problem). *Let (A, R) be a presentation of the group G . Let $w_1, w_2 \in F(A)$ be two words. Then it is undecidable in general if $w_1N(R) = w_2N(R)$ in G .*

Definition 6.94 (Finitely Presented). A group is *finitely presented* iff it has a presentation (A, R) where both A and R are finite.

Proposition 6.95. *Let $(A|R)$ be a presentation of G and $(A'|R')$ a presentation of H . Assume w.l.o.g. A and A' are disjoint. Then the group $G * G'$ presented by $(A \cup A' | R \cup R')$ is the coproduct of G and G' in **Grp**.*

$$\begin{array}{ccccc} A & \longrightarrow & A \cup A' & \longleftarrow & A' \\ \downarrow & & \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(A \cup A') & \longleftarrow & F(A') \\ \downarrow & & \downarrow & & \downarrow \\ G & \xrightarrow{\kappa_1} & G * G' & \xleftarrow{\kappa_2} & G' \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: $\kappa_1 : G \rightarrow G * G'$ and $\kappa_2 : G' \rightarrow G * G'$ be the unique homomorphisms that make the diagram above commute.

$\langle 1 \rangle 2$. LET: $\phi : G \rightarrow H$ and $\psi : G' \rightarrow H$ be any homomorphisms.

$\langle 1 \rangle 3$. LET: $[\phi, \psi] : F(A \cup A') \rightarrow H$ be the unique homomorphism such that ...

$\langle 1 \rangle 4$. $R \cup R' \subseteq \ker[\phi, \psi]$

$\langle 1 \rangle 5$. $[\phi, \psi]$ factors uniquely through the morphism $F(A \cup A') \rightarrow G * G'$

\square

6.14 Index of a Subgroup

Definition 6.96 (Index). Let G be a group and H a subgroup of G . The *index* of H in G , denoted $|G : H|$, is the number of left cosets of H in G if this is finite, otherwise ∞ .

Theorem 6.97 (Lagrange's Theorem). *Let G be a finite group and H a subgroup of G . Then*

$$|G| = |G : H||H| .$$

PROOF: G/H is a partition of G into $|G : H|$ subsets, each of size $|H|$. \square

Corollary 6.97.1. *For p a prime number, the only group of order p is C_p .*

PROOF: Let G be a group of order p and $g \in G$ with $g \neq e$. Then $|\langle g \rangle|$ divides p and is not 1, hence is p , that is, $G = \langle g \rangle$. \square

Corollary 6.97.2 (Fermat's Little Theorem). *Let p be a prime number and a any integer. Then $a^p \equiv a \pmod{p}$.*

PROOF: If $p \mid a$ then $a^p \equiv a \equiv 0 \pmod{p}$. Otherwise, we have $a^{p-1} \equiv 1 \pmod{p}$ by applying Lagrange's Theorem to $(\mathbb{Z}/p\mathbb{Z})^*$. \square

Example 6.98. It is not true that, if $n \mid |G|$, then G has a subgroup of order n . The group A_4 has order 12 but no subgroup of order 6.

Theorem 6.99 (Cauchy's Theorem). *Let G be a finite group. If p is prime and $p \mid |G|$ then G has a subgroup of order p .*

Proposition 6.100. *Let G be a group. Let K be a subgroup of G and H a subgroup of K . If $|G : H|$, $|G : K|$ and $|K : H|$ are all finite then*

$$|G : H| = |G : K||K : H| .$$

PROOF:

$\langle 1 \rangle 1$. LET: $G/K = \{g_1K, g_2K, \dots, g_mK\}$

$\langle 1 \rangle 2$. LET: $K/H = \{k_1H, k_2H, \dots, k_nH\}$

$\langle 1 \rangle 3$. $G/H = \{g_ik_jH : 1 \leq i \leq m, 1 \leq j \leq n\}$

$\langle 2 \rangle 1$. LET: $g \in G$

$\langle 2 \rangle 2$. PICK i such that $gK = g_iK$

$\langle 2 \rangle 3$. $g^{-1}g_i \in K$

$\langle 2 \rangle 4$. PICK j such that $g^{-1}g_iH = k_jH$

$\langle 2 \rangle 5$. $g^{-1}g_ik_j \in H$

$\langle 2 \rangle 6$. $gH = g_ik_jH$

$\langle 1 \rangle 4$. If $g_ik_jH = g_{i'}k_{j'}H$ then $i = i'$ and $j = j'$.

$\langle 2 \rangle 1$. ASSUME: $g_ik_jH = g_{i'}k_{j'}H$

$\langle 2 \rangle 2$. $g_iK = g_{i'}K$

$\langle 2 \rangle 3$. $i = i'$

$\langle 2 \rangle 4$. $k_jH = k_{j'}H$

$\langle 2 \rangle 5$. $j = j'$

\square

6.15 Cokernels

Proposition 6.101. *Let $\phi : G \rightarrow H$ be a homomorphism between groups. Then there exists a group K and homomorphism $\pi : H \rightarrow K$ that is initial with respect to all homomorphism $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.*

PROOF:

- $\langle 1 \rangle 1$. LET: N be the intersection of all the normal subgroups of H that include $\text{im } \phi$.
 - $\langle 1 \rangle 2$. LET: $K = H/N$ and π be the canonical homomorphism.
 - $\langle 1 \rangle 3$. LET: $\pi \circ \phi = 0$
 - $\langle 1 \rangle 4$. LET: $\alpha : H \rightarrow L$ satisfy $\alpha \circ \phi = 0$
 - $\langle 1 \rangle 5$. $\text{im } \phi \subseteq \ker \alpha$
 - $\langle 1 \rangle 6$. $N \subseteq \ker \alpha$
 - $\langle 1 \rangle 7$. There exists a unique $\bar{\alpha} : H/\text{im } \phi \rightarrow L$ such that $\bar{\alpha} \circ \pi = \alpha$
-

Definition 6.102 (Cokernel). For any homomorphism $\phi : G \rightarrow H$ in **Grp**, the *cokernel* of ϕ is the group $\text{coker } \phi$ and homomorphism $\pi : H \rightarrow \text{coker } \phi$ that is initial among homomorphisms $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.

Example 6.103. It is not true that a homomorphism with trivial cokernel is epi. The inclusion $\langle (1\ 2) \rangle \hookrightarrow S_3$ has trivial cokernel but is not epi.

6.16 Cayley Graphs

Definition 6.104 (Cayley Graph). Let G be a finitely generated group. Let A be a finite set of generators for G . The *Cayley graph* of G with respect to A is the directed graph whose vertices are the elements of G , with an edge $g_1 \rightarrow g_2$ labelled by $a \in A$ iff $g_2 = g_1 a$.

Proposition 6.105. *G is the free group on A iff the Cayley graph with respect to A is a tree.*

PROOF: Both are equivalent to saying that the product of two different strings of elements of A and/or their inverses are not equal. □

Chapter 7

Abelian Groups

Definition 7.1 (Abelian Group). A group is *Abelian* iff any two elements commute.

In an Abelian group G , we often denote the group operation by $+$, the identity element by 0 and the inverse of an element g by $-g$. We write ng for g^n ($g \in G, n \in \mathbb{Z}$).

Example 7.2. Every group of order ≤ 4 is Abelian.

Example 7.3. For any positive integer n , we have $\mathbb{Z}/n\mathbb{Z}$ is an Abelian group under addition.

Example 7.4. S_n is not Abelian for $n \geq 3$. If $x = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ then $xy = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $yx = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$.

Example 7.5. There are 42 Abelian groups of order 1024 up to isomorphism.

Proposition 7.6. Let G be a group. If $g^2 = e$ for all $g \in G$ then G is Abelian.

PROOF: For any $g, h \in G$ we have

$$ghgh = e$$

$$\therefore hgh = g \quad (\text{multiplying on the left by } g)$$

$$\therefore hg = gh \quad (\text{multiplying on the right by } h) \square$$

Proposition 7.7. Let G be a group. Then G is Abelian if and only if the function that maps g to g^{-1} is a group homomorphism.

PROOF:

(1)1. If G is Abelian then the function that maps g to g^{-1} is a group homomorphism.

PROOF: Since $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$.

(1)2. If the function that maps g to g^{-1} is a group homomorphism then G is Abelian.

PROOF: Since $gh = (g^{-1})^{-1}(h^{-1})^{-1} = (g^{-1}h^{-1})^{-1} = hg$.
 \square

Proposition 7.8. *Let G be a group. Then G is Abelian if and only if the function that maps g to g^2 is a group homomorphism.*

PROOF:

$\langle 1 \rangle 1$. If G is Abelian then the function that maps g to g^2 is a group homomorphism.

PROOF: Since $(gh)^2 = g^2h^2$.

$\langle 1 \rangle 2$. If the function that maps g to g^2 is a group homomorphism then G is Abelian.

PROOF: Since we have $(gh)^2 = ghgh = g^2h^2$ and so $hg = gh$.

\square

Proposition 7.9. *Let G be a group. Then G is Abelian if and only if the homomorphism $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$ is the trivial homomorphism.*

PROOF:

$\langle 1 \rangle 1$. If G is Abelian then γ is trivial.

PROOF: Since $\gamma_g(a) = gag^{-1} = a$.

$\langle 1 \rangle 2$. If γ is trivial then G is Abelian.

PROOF: If $\gamma_g(a) = gag^{-1} = a$ for all g and a then $ga = ag$ for all g, a .

\square

Proposition 7.10. *Let G be an Abelian group. Let $g, h \in G$. If g has maximal finite order in G , and h has finite order, then $|h| \mid |g|$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $|h| \nmid |g|$.

$\langle 1 \rangle 2$. PICK a prime p such that $|g| = p^m r$, $|h| = p^n s$ where $p \nmid r$, $p \nmid s$ and $m < n$.

$\langle 1 \rangle 3$. $|g^{p^m} h^s| = p^n r$

PROOF: Proposition 5.19.

$\langle 1 \rangle 4$. $|g| < |g^{p^m} h^s|$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts the maximality of $|g|$.

\square

Proposition 7.11. *If p is prime then $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.*

PROOF:

$\langle 1 \rangle 1$. LET: g be an element of maximal order in $(\mathbb{Z}/p\mathbb{Z})^*$.

$\langle 1 \rangle 2$. For all $h \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $h^{|g|} = 1$.

PROOF: Proposition 7.10.

$\langle 1 \rangle 3$. There are at most $|g|$ elements x such that $x^{|g|} = 1$ in $\mathbb{Z}/p\mathbb{Z}$

$\langle 1 \rangle 4$. $p - 1 \leq |g|$

$\langle 1 \rangle 5$. $|g| = p - 1$

$\langle 1 \rangle 6$. g generates $(\mathbb{Z}/p\mathbb{Z})^*$.

□

Example 7.12. $(\mathbb{Z}/12\mathbb{Z})^*$ is not cyclic. Its elements are 1, 5, 7 and 11 with orders 1, 2, 2 and 2.

Theorem 7.13 (Wilson's Theorem). *A positive integer p is prime if and only if $(p-1)! \equiv 1 \pmod{p}$.*

⟨1⟩1. If p is prime then $(p-1)! \equiv 1 \pmod{p}$.

⟨2⟩1. ASSUME: p is prime.

⟨2⟩2. $(p-1)!$ is the product of all the elements of $(\mathbb{Z}/p\mathbb{Z})^*$

⟨2⟩3. The only element of $(\mathbb{Z}/p\mathbb{Z})^*$ with order 2 is -1 .

⟨2⟩4. $(p-1)! \equiv -1 \pmod{p}$

PROOF: Proposition 5.20.

⟨1⟩2. If $(p-1)! \equiv -1 \pmod{p}$ then p is prime.

⟨2⟩1. ASSUME: (

$(p-1)! \equiv -1 \pmod{p}$)

⟨2⟩2. LET: d be a proper divisor of p .

PROVE: $d = 1$

⟨2⟩3. $d \mid (p-1)!$

⟨2⟩4. $d \mid 1$

PROOF: Since $d \mid p \mid (p-1)! + 1$.

⟨2⟩5. $d = 1$

□

Proposition 7.14. *If p and q are distinct odd primes then $(\mathbb{Z}/pq\mathbb{Z})^*$ is not cyclic.*

PROOF:

⟨1⟩1. $|(\mathbb{Z}/pq\mathbb{Z})^*| = (p-1)(q-1)$

⟨1⟩2. LET: $g \in (\mathbb{Z}/pq\mathbb{Z})^*$

PROVE: g does not have order $(p-1)(q-1)$

⟨1⟩3. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{p}$

⟨1⟩4. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{q}$

⟨1⟩5. $pq \mid g^{(p-1)(q-1)/2} - 1$

⟨1⟩6. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{pq}$

⟨1⟩7. $|g| \mid (p-1)(q-1)/2$

□

Proposition 7.15. *For any prime p , we have $\text{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$.*

PROOF:

⟨1⟩1. LET: $\phi : \text{Aut}_{\mathbf{Grp}}(C_p) \rightarrow (\mathbb{Z}/p\mathbb{Z})^*$ be the function $\phi(\alpha) = \alpha(1)$.

PROOF: $\alpha(1)$ has order p in C_p and so is coprime with p .

⟨1⟩2. ϕ is a homomorphism.

PROOF: $\phi(\alpha \circ \beta) = \alpha(\beta(1)) = \alpha(\beta(1)1) = \beta(1)\alpha(1) = \phi(\alpha)\phi(\beta)$

⟨1⟩3. ϕ is injective.

PROOF: If $\phi(\alpha) = \phi(\beta)$ then for any n we have $\alpha(n) = n\alpha(1) = n\phi(\alpha) = n\phi(\beta) = n\beta(1) = \beta(n)$.

$\langle 1 \rangle 4.$ ϕ is surjective.

PROOF: For any $r \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $r = \phi(\alpha)$ where $\alpha(n) = nr \bmod p$.

$\langle 1 \rangle 5.$ $(\mathbb{Z}/p\mathbb{Z})^* \cong C_{p-1}$

□

Proposition 7.16. *Given a set A and an Abelian group H , the set H^A is an Abelian group under*

$$(\phi + \psi)(a) = \phi(a) + \psi(a) \quad (\phi, \psi \in H^A, a \in A) .$$

PROOF:

$\langle 1 \rangle 1.$ $\phi + (\psi + \chi) = (\phi + \psi) + \chi$

$\langle 1 \rangle 2.$ $\phi + \psi = \psi + \phi$

$\langle 1 \rangle 3.$ LET: $0 : A \rightarrow H$ be the function $0(a) = 0$.

$\langle 1 \rangle 4.$ $\phi + 0 = 0 + \phi = \phi$

$\langle 1 \rangle 5.$ Given $\phi : A \rightarrow H$, define $-\phi : A \rightarrow H$ by $(-\phi)(a) = -(\phi(a))$.

$\langle 1 \rangle 6.$ $\phi + (-\phi) = (-\phi) + \phi = 0$

□

Proposition 7.17. *Given a group G and an Abelian group H , the set $\mathbf{Grp}[G, H]$ is a subgroup of H^G .*

PROOF:

$\langle 1 \rangle 1.$ Given $\phi, \psi : G \rightarrow H$ group homomorphisms, we have $\phi - \psi$ is a group homomorphism.

PROOF:

$$\begin{aligned} (\phi - \psi)(g + g') &= \phi(g + g') - \psi(g + g') \\ &= \phi(g) + \phi(g') - \psi(g) - \psi(g') \\ &= \phi(g) - \psi(g) + \phi(g') - \psi(g') \\ &= (\phi - \psi)(g) + (\phi - \psi)(g') \end{aligned}$$

□

Proposition 7.18. *Let G be a group. The following are equivalent.*

1. $\text{Inn}(G)$ is cyclic.

2. $\text{Inn}(G)$ is trivial.

3. G is Abelian.

PROOF:

$\langle 1 \rangle 1.$ $1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: $\text{Inn}(G) = \langle \gamma_g \rangle$

$\langle 2 \rangle 2.$ g commutes with every element of G

$\langle 3 \rangle 1.$ LET: $x \in G$

$\langle 3 \rangle 2.$ PICK $n \in \mathbb{Z}$ such that $\gamma_x = \gamma_g^n$

$\langle 3 \rangle 3.$ $\forall y \in G. xyx^{-1} = g^n yg^{-n}$

$\langle 3 \rangle 4.$ $xgx^{-1} = g$

$\langle 2 \rangle 3.$ $\gamma_g = \text{id}_G$
 $\langle 1 \rangle 2.$ $2 \Rightarrow 3$
 $\langle 2 \rangle 1.$ ASSUME: $\forall g \in G. \gamma_g = \text{id}_G$
 $\langle 2 \rangle 2.$ LET: $x, y \in G$
 $\langle 2 \rangle 3.$ $\gamma_x(y) = y$
 $\langle 2 \rangle 4.$ $xyx^{-1} = y$
 $\langle 2 \rangle 5.$ $xy = yx$
 $\langle 1 \rangle 3.$ $3 \Rightarrow 2$
 PROOF: If $xy = yx$ for all x, y then $\gamma_x(y) = y$ for all x, y .
 $\langle 1 \rangle 4.$ $2 \Rightarrow 1$
 PROOF: Easy.
 \square

Corollary 7.18.1. *If $\text{Aut}_{\text{Grp}}(G)$ is cyclic then G is Abelian.*

Proposition 7.19. *Every subgroup of an Abelian group is normal.*

PROOF: Let G be an Abelian group and N a subgroup of G . Given $g \in G$ and $n \in N$ we have $gng^{-1} = n \in N$. \square

Proposition 7.20. *For any group G , the group $G/[G, G]$ is Abelian.*

PROOF: For any $g, h \in G$ we have
 $gh(hg)^{-1} \in [G, G]$

$$\therefore gh[G, G] = hg[G, G] \quad \square$$

Proposition 7.21. *Let G be a finite Abelian group. Let p be a prime divisor of $|G|$. Then G has an element of order p .*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: as induction hypothesis the result holds for all groups smaller than G .
 $\langle 1 \rangle 2.$ PICK $g \in G - \{0\}$.
 $\langle 1 \rangle 3.$ PICK an element $h \in \langle g \rangle$ with prime order q .
 $\langle 1 \rangle 4.$ CASE: $q = p$
 PROOF: h is the required element.
 $\langle 1 \rangle 5.$ CASE: $q \neq p$
 $\langle 2 \rangle 1.$ PICK $r \in G$ such that $r + \langle h \rangle$ has order p in $G/\langle h \rangle$.
 PROOF: By induction hypothesis since $|G/\langle h \rangle| = |G|/q$.
 $\langle 2 \rangle 2.$ $pr \in \langle h \rangle$
 $\langle 2 \rangle 3.$ PICK k such that $pr = kh$
 $\langle 2 \rangle 4.$ $pqr = e$
 $\langle 2 \rangle 5.$ qr has order p .
 \square

Corollary 7.21.1. *For n an odd integer, any Abelian group of order $2n$ has exactly one element of order 2.*

PROOF: If x and y are distinct elements of order 2 then $\langle x, y \rangle = \{e, x, y, xy\}$ has size 4 and so $4 \mid 2n$ which is a contradiction. \square

Example 7.22. It is not true that, if G is a finite group and $d \mid |G|$, then G has an element of order d . The quaternion group has no element of order 4.

Proposition 7.23. *If G is a finite Abelian group and $d \mid |G|$ then G has a subgroup of size d .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: as induction hypothesis the result is true for all $d' < d$.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $d \neq 1$.

$\langle 1 \rangle 3$. PICK a prime p such that $p \mid d$.

$\langle 1 \rangle 4$. PICK an element $g \in G$ of order p .

$\langle 1 \rangle 5$. $d/p \mid |G/\langle g \rangle|$

$\langle 1 \rangle 6$. PICK a subgroup H of $G/\langle g \rangle$ of size d/p .

$\langle 1 \rangle 7$. $\pi^{-1}(H)$ is a subgroup of G of size d .

□

Proposition 7.24. *Let (G, \cdot) be a group. Let $\circ : G^2 \rightarrow G$ be a group homomorphism such that (G, \circ) is a group. Then \circ and \cdot coincide, and G is Abelian.*

PROOF:

$\langle 1 \rangle 1$. For all $g_1, g_2, h_1, h_2 \in G$ we have

$$(g_1 g_2) \circ (h_1 h_2) = (g_1 \circ h_1)(g_2 \circ h_2)$$

$\langle 1 \rangle 2$. $e \circ e = e$

PROOF:

$$\begin{aligned} e \circ e &= (ee) \circ (ee) \\ &= (e \circ e)(e \circ e) \end{aligned}$$

Hence $e \circ e = e$ by Cancellation.

$\langle 1 \rangle 3$. e is the identity of (G, \circ)

$\langle 1 \rangle 4$. For all $g, h \in G$ we have

$$g \circ h = gh$$

PROOF:

$$\begin{aligned} g \circ h &= (ge) \circ (eh) \\ &= (g \circ e)(e \circ h) \\ &= gh \end{aligned}$$

$\langle 1 \rangle 5$. For all $g, h \in G$ we have $gh = hg$.

PROOF:

$$\begin{aligned} gh &= (e \circ g)(h \circ e) \\ &= (eh) \circ (ge) \\ &= h \circ g \\ &= hg \end{aligned}$$

□

Corollary 7.24.1. *If $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$ is a group object in **Grp** then m is the multiplication of G , $e(*)$ is the identity of G , $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Grp** where $e(*) = e$ and $i(g) = g^{-1}$.*

7.1 The Category of Abelian Groups

Definition 7.25 (Category of Abelian Groups). Let **Ab** be the full subcategory of **Grp** whose objects are the Abelian groups.

Proposition 7.26. *If $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$ is a group object in **Ab** then m is the multiplication of G , $e(*)$ is the identity of G , $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Ab** where $e(*) = e$ and $i(g) = g^{-1}$.*

PROOF: Immediate from Corollary 7.24.1. \square

Definition 7.27 (Direct Sum). Given Abelian groups G and H , we also call the direct product of G and H the *direct sum* and denote it $G \oplus H$.

Proposition 7.28. *Given Abelian groups G and H , the direct sum $G \oplus H$ is the coproduct of G and H in **Ab**.*

PROOF:

$\langle 1 \rangle 1$. LET: $\kappa_1 : G \rightarrow G \oplus H$ be the group homomorphism $\kappa_1(g) = (g, e_H)$.

$\langle 1 \rangle 2$. LET: $\kappa_2 : H \rightarrow G \oplus H$ be the group homomorphism $\kappa_2(h) = (e_G, h)$.

$\langle 1 \rangle 3$. Given group homomorphism $\phi : G \rightarrow K$ and $\psi : H \rightarrow K$, define $[\phi, \psi] : G \oplus H \rightarrow K$ by $[\phi, \psi](g, h) = \phi(g) + \psi(h)$.

$\langle 1 \rangle 4$. $[\phi, \psi]$ is a group homomorphism.

PROOF:

$$\begin{aligned} [\phi, \psi]((g, h) + (g', h')) &= [\phi, \psi](g + g', h + h') \\ &= \phi(g + g') + \psi(h + h') \\ &= \phi(g) + \phi(g') + \psi(h) + \psi(h') \\ &= \phi(g) + \psi(h) + \phi(g') + \psi(h') \\ &= [\phi, \psi](g, h) + [\phi, \psi](g', h') \end{aligned}$$

$\langle 1 \rangle 5$. $[\phi, \psi] \circ \kappa_1 = \phi$

PROOF:

$$\begin{aligned} [\phi, \psi](\kappa_1(g)) &= [\phi, \psi](g, e_H) \\ &= \phi(g) + \psi(e_H) \\ &= \phi(g) + e_K \\ &= \phi(g) \end{aligned}$$

$\langle 1 \rangle 6$. $[\phi, \psi] \circ \kappa_2 = \psi$

PROOF: Similar.

$\langle 1 \rangle 7$. If $f : G \oplus H \rightarrow K$ is a group homomorphism with $f \circ \kappa_1 = \phi$ and $f \circ \kappa_2 = \psi$ then $f = [\phi, \psi]$.

PROOF:

$$\begin{aligned} f(g, h) &= f((g, e_H) + (e_G, h)) \\ &= f(\kappa_1(g)) + f(\kappa_2(h)) \\ &= \phi(g) + \psi(h) \end{aligned}$$

□

Theorem 7.29. *Every finitely generated Abelian group is a direct sum of cyclic groups.*

PROOF: TODO □

7.2 Free Abelian Groups

Proposition 7.30. *Let A be a set. Let \mathcal{F}^A be the category whose objects are pairs (G, j) where G is an Abelian group and j is a function $A \rightarrow G$, with morphisms $f : (G, j) \rightarrow (H, k)$ the group homomorphisms $f : G \rightarrow H$ such that $f \circ j = k$. Then \mathcal{F}^A has an initial object.*

PROOF:

- ⟨1⟩1. LET: $\mathbb{Z}^{\oplus A}$ be the subgroup of \mathbb{Z}^A consisting of all functions $\alpha : A \rightarrow \mathbb{Z}$ such that $\alpha(a) = 0$ for only finitely many $a \in A$.
- ⟨1⟩2. LET: $i : A \rightarrow \mathbb{Z}^{\oplus A}$ be the function such that $i(a)(b) = 1$ if $a = b$ and 0 if $a \neq b$.
- ⟨1⟩3. LET: G be any Abelian group and $j : A \rightarrow G$ any function.
- ⟨1⟩4. The unique homomorphism $\phi : \mathbb{Z}^{\oplus A} \rightarrow G$ required is defined by $\phi(\alpha) = \sum_{a \in A} \alpha(a)j(a)$

□

Definition 7.31 (Free Abelian Group). For any set A , the *free Abelian group* on A is the initial object $(F^{ab}(A), i)$ in \mathcal{F}^A .

Proposition 7.32. *For any sets A and B , we have that $F^{ab}(A + B)$ is the coproduct of $F^{ab}(A)$ and $F^{ab}(B)$ in **Grp**.*

$$\begin{array}{ccccc}
 & & G & & \\
 & f \nearrow & \uparrow k & \nwarrow g & \\
 F^{ab}(A) & \xrightarrow{\kappa_1} & F^{ab}(A+B) & \xleftarrow{\kappa_2} & F^{ab}(B) \\
 i_A \uparrow & & j \uparrow & & i_B \uparrow \\
 A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
 \end{array}$$

PROOF:

- ⟨1⟩1. LET: $i_A : A \rightarrow F^{ab}(A)$, $i_B : B \rightarrow F^{ab}(B)$, $j : A + B \rightarrow F^{ab}(A + B)$ be the canonical injections.
- ⟨1⟩2. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
- ⟨1⟩3. LET: G be any group and $f : F^{ab}(A) \rightarrow G$, $g : F^{ab}(B) \rightarrow G$ any group homomorphisms.
- ⟨1⟩4. LET: $h : A + B \rightarrow G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.

- (1)5. LET: $k : F^{\text{ab}}(A + B) \rightarrow G$ be the unique group homomorphism such that $k \circ j = h$.
- (1)6. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
- (1)7. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.
 \square

Proposition 7.33. *For A and B finite sets, if $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$ then $A \cong B$.*

PROOF:

- (1)1. For any set C , define \sim on $F^{\text{ab}}(C)$ by: $f \sim f'$ iff there exists $g \in F^{\text{ab}}(C)$ such that $f - f' = 2g$.
- (1)2. For any set C , \sim is an equivalence relation on $F^{\text{ab}}(C)$.
- (1)3. For any set C , we have $F^{\text{ab}}(C) / \sim$ is finite if and only if C is finite, in which case $|F^{\text{ab}}(C) / \sim| = 2^{|C|}$.
- PROOF: There is a bijection between $F^{\text{ab}}(C) / \sim$ and the finite subsets of C , which maps f to $\{c \in C : f(c) \text{ is odd}\}$.
- (1)4. If $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$ then $A \cong B$.
- PROOF: If $|F^{\text{ab}}(A) / \sim| = |F^{\text{ab}}(B) / \sim|$ then $2^{|A|} = 2^{|B|}$ and so $|A| = |B|$.
 \square

Proposition 7.34. *Let G be an Abelian group. Then G is finitely generated if and only if there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n .*

PROOF:

- (1)1. If G is finitely generated then there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n .
- PROOF: Let $G = \langle a_1, \dots, a_n \rangle$. Define $\phi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$ by $\phi(i_1, \dots, i_n) = i_1 \cdot a_1 + \dots + i_n \cdot a_n$.
- (1)2. If there exists a surjective homomorphism $\phi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n then G is finitely generated.
- PROOF: G is generated by $\phi(1, 0, \dots, 0), \phi(0, 1, 0, \dots, 0), \dots, \phi(0, \dots, 0, 1)$.
 \square

Proposition 7.35. *Let A be a set. Let $i : A \hookrightarrow F(A)$ be the free group on A . Then $\pi \circ i : A \rightarrow F(A)/[F(A), F(A)]$ is the free Abelian group on A .*

$$\begin{array}{ccc}
 & F(A)/[F(A), F(A)] & \\
 \uparrow \pi & \searrow h & \\
 F(A) & \xrightarrow{g} & G \\
 \uparrow i & \nearrow f & \\
 A & &
 \end{array}$$

PROOF:

- (1)1. LET: G be an Abelian group and $f : A \rightarrow G$ a function.

- $\langle 1 \rangle 2$. LET: $g : F(A) \rightarrow G$ be the unique group homomorphism such that $g \circ i = f$.
- $\langle 1 \rangle 3$. $[F(A), F(A)] \subseteq \ker g$
 PROOF: For all $x, y \in F(A)$ we have $g(xy x^{-1} y^{-1}) = g(x) + g(y) - g(x) - g(y) = 0$.
- $\langle 1 \rangle 4$. LET: $h : F(A)/[F(A), F(A)]$ be the unique group homomorphism such that $h \circ \pi = g$.
- $\langle 1 \rangle 5$. h is the unique group homomorphism such that $h \circ \pi \circ i = f$.
 \square

Corollary 7.35.1. *Let A and B be sets. Let $F(A)$ and $F(B)$ be the free groups on A and B respectively. If $F(A) \cong F(B)$ then $A \cong B$.*

PROOF: Proposition 7.33. \square

7.3 Cokernels

Proposition 7.36. *Let $\phi : G \rightarrow H$ be a homomorphism between Abelian groups. Then there exists an Abelian group K and homomorphism $\pi : H \rightarrow K$ that is initial with respect to all homomorphism $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.*

PROOF:

- $\langle 1 \rangle 1$. LET: $K = H/\text{im } \phi$ and π be the canonical homomorphism.
- $\langle 1 \rangle 2$. LET: $\pi \circ \phi = 0$
- $\langle 1 \rangle 3$. LET: $\alpha : H \rightarrow L$ satisfy $\alpha \circ \phi = 0$
- $\langle 1 \rangle 4$. $\text{im } \phi \subseteq \ker \alpha$
- $\langle 1 \rangle 5$. There exists a unique $\bar{\alpha} : H/\text{im } \phi \rightarrow L$ such that $\bar{\alpha} \circ \pi = \alpha$
 \square

Definition 7.37 (Cokernel). For any homomorphism $\phi : G \rightarrow H$ in **Ab**, the *cokernel* of ϕ is the Abelian group $\text{coker } \phi$ and homomorphism $\pi : H \rightarrow \text{coker } \phi$ that is initial among homomorphisms $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.

Proposition 7.38. $\pi : H \rightarrow \text{coker } \phi$ is initial among functions $f : H \rightarrow X$ such that, for all $x, y \in H$, if $x + \text{im } \phi = y + \text{im } \phi$ then $f(x) = f(y)$.

PROOF: Easy. \square

Proposition 7.39. *Let $\phi : G \rightarrow H$ be a homomorphism of Abelian groups. Then the following are equivalent.*

- ϕ is an epimorphism.
- $\text{coker } \phi$ is trivial.
- ϕ is surjective.

PROOF:

- $\langle 1 \rangle 1$. $1 \Rightarrow 2$

- $\langle 2 \rangle 1$. ASSUME: ϕ is epi.
- $\langle 2 \rangle 2$. LET: $\pi : H \rightarrow \text{coker } \phi$ be the canonical homomorphism.
- $\langle 2 \rangle 3$. $\pi \circ \phi = 0 \circ \phi$
- $\langle 2 \rangle 4$. $\pi = 0$
- $\langle 2 \rangle 5$. $\text{coker } \phi = \text{im } \pi$ is trivial.
- $\langle 1 \rangle 2$. $2 \Rightarrow 3$
PROOF: If $\text{coker } \phi = H / \text{im } \phi$ is trivial then $\text{im } \phi = H$.
- $\langle 1 \rangle 3$. $3 \Rightarrow 1$
PROOF: If it is surjective then it is epi in **Set**.

□

Chapter 8

Group Actions

8.1 Group Actions

Definition 8.1 (Action). Let G be a group. Let A be an object of a category \mathcal{C} . A (left) action of G on A is a group homomorphism $G \rightarrow \text{Aut}_{\mathcal{C}}(A)$.

It is *faithful* or *effective* iff it is injective.

Proposition 8.2. Let A be a set. An action of the group G on the set A is given by a function $\cdot : G \times A \rightarrow A$ such that

- $\forall a \in A. ea = a$
- $\forall g, h \in G. \forall a \in A. (gh)a = g(ha)$

PROOF: Just unfolding definitions. \square

Example 8.3. Left multiplication defines a faithful action of any group on its own underlying set.

In fact, for any subgroup H of a group G , left multiplication defines an action of G on G/H .

Corollary 8.3.1 (Cayley's Theorem). Every group G is a subgroup of a symmetric group, namely $\text{Aut}_{\text{Set}}(G)$.

Example 8.4. Conjugation $g * h = ghg^{-1}$ is an action of any group on its own underlying set.

Definition 8.5 (Transitive). An action of a group G on a set A is *transitive* iff, for all $a, b \in A$, there exists $g \in G$ such that $ga = b$.

Example 8.6. Left multiplication of a group G is a transitive action of G on G .

Definition 8.7 (Orbit). Given an action of a group G on a set A and $a \in A$, the *orbit* of a is

$$\text{O}_G(a) := \{ga : g \in G\} .$$

Proposition 8.8. *Given an action of a group G on a set A , the orbits form a partition of A .*

PROOF:

$\langle 1 \rangle 1$. Every element of A is in some orbit.

PROOF: Since $a \in O_G(a)$.

$\langle 1 \rangle 2$. Distinct orbits are disjoint.

$\langle 2 \rangle 1$. LET: $a \in O_G(b) \cap O_G(c)$

$\langle 2 \rangle 2$. PICK $g, h \in G$ such that $a = gb = hc$.

$\langle 2 \rangle 3$. $O_G(b) \subseteq O_G(c)$

PROOF: For all $k \in G$ we have $kb = kg^{-1}hc$.

$\langle 2 \rangle 4$. $O_G(c) \subseteq O_G(b)$

PROOF: Similar.

□

Proposition 8.9. *Given an action of a group G on a set A and $a \in A$, the action is transitive on $O_G(a)$.*

PROOF:

$\langle 1 \rangle 1$. The restriction of the action is an action on $O_G(a)$.

PROOF: Since $g(ha) = (gh)a$, the action maps $O_G(a)$ to itself.

$\langle 1 \rangle 2$. The restricted action is transitive.

PROOF: Given $ga, ha \in O_G(a)$, we have $ha = (hg^{-1})(ga)$.

□

Definition 8.10 (Stabilizer Subgroup). Given an action of a group G on a set A and $a \in A$, the *stabilizer subgroup* of a is

$$\text{Stab}_G(a) := \{g \in G : ga = a\} .$$

Proposition 8.11. *Stabilizer subgroups are subgroups.*

PROOF: If $g, h \in \text{Stab}_G(a)$ then $gh^{-1}a = a$ so $gh^{-1} \in \text{Stab}_G(a)$. □

Proposition 8.12. *Let G act on a set A . Let $a \in A$ and $g \in G$. Then*

$$\text{Stab}_G(ga) = g\text{Stab}_G(a)g^{-1} .$$

PROOF:

$$h \in \text{Stab}_G(ga) \Leftrightarrow hga = ga$$

$$\Leftrightarrow g^{-1}hga = a$$

$$\Leftrightarrow g^{-1}hg \in \text{Stab}_G(a)$$

$$\Leftrightarrow h \in g\text{Stab}_G(a)g^{-1}$$

□

Corollary 8.12.1. *Let G be an action on a set A and $a \in A$. If $\text{Stab}_G(a)$ is normal in G , then for any $b \in O_G(a)$ we have $\text{Stab}_G(a) = \text{Stab}_G(b)$.*

Definition 8.13 (Free). An action of a group G on a set A is *free* iff, whenever $ga = a$, then $g = e$.

Example 8.14. The action of left multiplication is free.

Proposition 8.15. *Let G be a group. Let H be a subgroup of G of finite index n . Then H includes a subgroup K that is normal in G and such that $|G : K|$ divides $\gcd(|G|, n!)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\sigma : G \rightarrow \text{Aut}_{\text{Set}}(G/H)$ be the action of left multiplication.

$\langle 1 \rangle 2$. LET: $K = \ker \sigma$

$\langle 1 \rangle 3$. $K \subseteq H$

$\langle 2 \rangle 1$. LET: $g \in K$

$\langle 2 \rangle 2$. $\sigma(g)(H) = H$

$\langle 2 \rangle 3$. $gH = H$

$\langle 2 \rangle 4$. $g \in H$

$\langle 1 \rangle 4$. K is normal in G .

PROOF: Proposition 6.43.

$\langle 1 \rangle 5$. $|G : K| \mid |G|$

PROOF: Lagrange's Theorem.

$\langle 1 \rangle 6$. $|G : K| \mid n!$

PROOF: Since G/K is a subgroup of $\text{Aut}_{\text{Set}}(G/H)$.

□

Corollary 8.15.1. *Let G be a finite group. Let H be a subgroup of G of index p where p is the smallest prime that divides $|G|$. Then H is normal in G .*

PROOF:

$\langle 1 \rangle 1$. PICK a subgroup K of H normal in G such that $|G : K|$ divides $\gcd(|G|, p!)$.

$\langle 1 \rangle 2$. $|G : K|$ divides p .

$\langle 1 \rangle 3$. $|G : H| |H : K|$ divides p .

$\langle 1 \rangle 4$. $|H : K| = 1$

$\langle 1 \rangle 5$. $H = K$

$\langle 1 \rangle 6$. H is normal.

□

Corollary 8.15.2. *Any subgroup of index 2 is normal.*

Proposition 8.16. *Let G be a group with finite set of generators A . Then left multiplication defines a free action of G on its Cayley graph.*

PROOF: Easy since if $g_2 = g_1 a$ then $hg_2 = hg_1 a$. □

Corollary 8.16.1. *A free group acts freely on a tree.*

Theorem 8.17. *If a group G acts freely on a tree then G is free.*

Corollary 8.17.1. *Every subgroup of the free group on a finite set is free.*

PROOF: If H is a subgroup of $F(A)$ then left multiplication defines a free action of H on the Cayley graph of $F(A)$, which is a tree. □

8.2 Category of G -Sets

Definition 8.18. Given a group G , let $G - \mathbf{Set}$ be the category with:

- objects all pairs (A, ρ) such that A is a set and $\rho : G \times A \rightarrow A$ is an action of G on A ;
- morphisms $f : (A, \rho) \rightarrow (B, \sigma)$ are functions $f : A \rightarrow B$ that are $(G-)$ equivariant, i.e.

$$\forall g \in G. \forall a \in A. f(\rho(g, a)) = \sigma(g, f(a)) .$$

Proposition 8.19. *A G -equivariant function $f : A \rightarrow B$ is an isomorphism in $G - \mathbf{Set}$ if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ be G -equivariant and bijective.

PROVE: f^{-1} is G -equivariant.

$\langle 1 \rangle 2$. LET: $g \in G$ and $b \in B$

$\langle 1 \rangle 3$. $f^{-1}(gb) = gf^{-1}(b)$

PROOF:

$$\begin{aligned} f(f^{-1}(gb)) &= gb \\ &= gf(f^{-1}(b)) \\ &= f(gf^{-1}(b)) \end{aligned}$$

□

Proposition 8.20. *Let G be a group and A a transitive G -set. Let $a \in A$. Then A is isomorphic to $G/\text{Stab}_G(a)$ under left multiplication.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : G/\text{Stab}_G(a) \rightarrow A$ be the function $f(g\text{Stab}_G(a)) = ga$.

$\langle 2 \rangle 1$. ASSUME: $g\text{Stab}_G(a) = h\text{Stab}_G(a)$

PROVE: $ga = ha$

$\langle 2 \rangle 2$. $g^{-1}h \in \text{Stab}_G(a)$

$\langle 2 \rangle 3$. $g^{-1}ha = a$

$\langle 2 \rangle 4$. $ha = ga$

$\langle 1 \rangle 2$. f is G -equivariant.

PROOF: Since $f(gh\text{Stab}_G(a)) = gha = gf(h\text{Stab}_G(a))$.

$\langle 1 \rangle 3$. f is injective.

PROOF: If $ga = ha$ then $g^{-1}h \in \text{Stab}_G(a)$ so $g\text{Stab}_G(a) = h\text{Stab}_G(a)$.

$\langle 1 \rangle 4$. f is surjective.

PROOF: Since for all $b \in A$ there exists $g \in G$ such that $ga = b$.

□

Corollary 8.20.1. *If O is an orbit of the action of a finite group G on a set A , then O is finite and $|O|$ divides $|G|$.*

Corollary 8.20.2. *Let H be a subgroup of G and $g \in G$. Then*

$$G/H \cong G/(gHg^{-1})$$

in $G - \mathbf{Set}$.

PROOF: Taking $A = G/H$ and $a = gH$. \square

Proposition 8.21. *Given a family of G -sets $\{A_i\}_{i \in I}$, we have $\prod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g\{a_i\}_{i \in I} = \{ga_i\}_{i \in I}.$$

PROOF: Easy. \square

Proposition 8.22. *Given a family of G -sets $\{A_i\}_{i \in I}$, we have $\coprod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g(i, a_i) = (i, ga_i).$$

PROOF: Easy. \square

Proposition 8.23. *Every finite G -set is a coproduct of G -sets of the form G/H .*

PROOF: If $O(a_1), \dots, O(a_n)$ are the orbits of the G -set A , then G is the coproduct of $G/\text{Stab}_G(a_1), \dots, G/\text{Stab}_G(a_n)$. \square

Proposition 8.24. *For any group G we have $G \cong \text{Aut}_{G-\mathbf{Set}}(G)$ (considering G as a G -set under left multiplication).*

PROOF:

$\langle 1 \rangle 1$. Define $\phi : G \rightarrow \text{Aut}_{G-\mathbf{Set}}(G)$ by $\phi(g)(g') = g'g^{-1}$.

$\langle 2 \rangle 1$. LET: $g \in G$

PROVE: $\lambda g' \in G.g'g^{-1}$ is an automorphism of G in $G - \mathbf{Set}$.

$\langle 2 \rangle 2$. $\phi(g)$ is G -equivariant.

PROOF: Since $\phi(g)(h_1h_2) = h_1h_2g^{-1} = h_1\phi(g)(h_2)$.

$\langle 2 \rangle 3$. $\phi(g)$ is injective.

PROOF: By Cancellation.

$\langle 2 \rangle 4$. $\phi(g)$ is surjective.

PROOF: For any $h \in G$ we have $h = \phi(g)(hg)$.

$\langle 1 \rangle 2$. ϕ is a group homomorphism.

PROOF: $\phi(g_1g_2)(h) = hg_2^{-1}g_1^{-1} = \phi(g_1)(\phi(g_2)(h))$.

$\langle 1 \rangle 3$. ϕ is injective.

PROOF: If $\phi(g) = \phi(g')$ then $g = \phi(g)(e) = \phi(g')(e) = g'$.

$\langle 1 \rangle 4$. ϕ is surjective.

$\langle 2 \rangle 1$. LET: $\sigma \in \text{Aut}_{G-\mathbf{Set}}(G)$

$\langle 2 \rangle 2$. LET: $g = \sigma(e)$

PROVE: $\sigma = \phi(g^{-1})$

$\langle 2 \rangle 3$. $\sigma(h) = hg$

PROOF: $\sigma(h) = \sigma(hg) = h\sigma(e) = hg$.

\square

Part III

Ring Theory

Chapter 9

Rings

9.1 Rngs

Definition 9.1 (Ring). A *rng* consists of a set R and binary operations $+, \cdot : R^2 \rightarrow R$ such that:

- $(R, +)$ is an Abelian group
- \cdot is associative.
- The *distributive properties* hold: for all $r, s, t \in R$ we have

$$(r + s)t = rt + st, \quad r(s + t) = rs + rt .$$

Example 9.2. • The *zero rng* is $\{0\}$.

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} are rngs.
- $2\mathbb{Z}$ is a rng.

Proposition 9.3. *In any rng R we have*

$$\forall x \in R. x0 = 0x = 0 .$$

PROOF:

$$\begin{aligned} x0 &= x(0 + 0) \\ &= x0 + x0 \end{aligned}$$

and so $x0 = 0$ by Cancellation. Similarly $0x = 0$. \square

9.2 Rings

Definition 9.4 (Ring). A *ring* R is a rng such that there exists $1 \in R$, the *multiplicative identity*, such that

$$\forall x \in R. x1 = 1x = x .$$

Example 9.5. • The zero rng is a ring with $1 = 0$.

- \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are rngs.
- $2\mathbb{Z}$ is not a ring.

Proposition 9.6. *In any ring we have $(-1)x = -x$.*

PROOF: Since

$$\begin{aligned}x + (-1)x &= 1x + (-1)x \\&= (1 + (-1))x \\&= 0x \\&= 0\end{aligned}$$

□

Part IV

Linear Algebra

Definition 9.7. Let $\text{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ real matrices.

$\text{GL}_n(\mathbb{R})$ acts on \mathbb{R}^n by matrix multiplication.

Definition 9.8. Let $\text{GL}_n(\mathbb{C})$ be the group of invertible $n \times n$ complex matrices.

$\text{GL}_n(\mathbb{C})$ acts on \mathbb{C}^n by matrix multiplication.

Definition 9.9. Let $\text{SL}_n(\mathbb{R}) = \{M \in \text{GL}_n(\mathbb{R}) : \det M = 1\}$.

Proposition 9.10. $\text{SL}_n(\mathbb{R})$ is a normal subgroup of $\text{GL}_n(\mathbb{R})$.

PROOF: If $\det M = 1$ then $\det(AMA^{-1}) = (\det A)(\det M)(\det A)^{-1} = 1$. \square

Proposition 9.11.

$$\text{GL}_n(\mathbb{R})/\text{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$$

Definition 9.12. Let $\text{SL}_n(\mathbb{C}) = \{M \in \text{GL}_n(\mathbb{C}) : \det M = 1\}$.

Definition 9.13. Let $\text{O}_n(\mathbb{R}) = \{M \in \text{GL}_n(\mathbb{R}) : MM^T = M^T M = I_n\}$.

Proposition 9.14. The action of $\text{O}_n(\mathbb{R})$ on \mathbb{R}^n preserves lengths and angles.

Definition 9.15. Let $\text{SO}_n(\mathbb{R}) = \{M \in \text{O}_n(\mathbb{R}) : \det M = 1\}$.

Definition 9.16. Let $\text{U}_n(\mathbb{C}) = \{M \in \text{GL}_n(\mathbb{C}) : MM^\dagger = M^\dagger M = I_n\}$.

Definition 9.17. Let $\text{SU}_n(\mathbb{C}) = \{M \in \text{U}_n(\mathbb{C}) : \det M = 1\}$.

Proposition 9.18. Every matrix in $\text{SU}_2(\mathbb{C})$ can be written in the form

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 + c^2 + d^2 = 1$.

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SU}_2(\mathbb{C})$$

$$\langle 1 \rangle 2. M^{-1} = M^\dagger$$

$$\langle 1 \rangle 3. \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

$$\langle 1 \rangle 4. \text{ LET: } \alpha = a + bi \text{ and } \beta = c + di.$$

$$\langle 1 \rangle 5. \delta = \bar{\alpha} = a - bi$$

$$\langle 1 \rangle 6. \gamma = -\bar{\beta} = -c + di$$

$$\langle 1 \rangle 7. \det M = a^2 + b^2 + c^2 + d^2 = 1$$

\square

Corollary 9.18.1. $\text{SU}_2(\mathbb{C})$ is simply connected.

Corollary 9.18.2.

$$\text{SO}_3(\mathbb{R}) \cong \text{SU}_2(\mathbb{C})/\{I, -I\}$$

PROOF: The function that maps $\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$ to $\begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(ac + bd) \\ 2(ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(cd - ab) \\ 2(bd - ac) & 2(ab + cd) & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$

is a surjective homomorphism with kernel $\{I, -I\}$. \square

Corollary 9.18.3. The fundamental group of $\text{SO}_3(\mathbb{R})$ is C_2 .