

Mathematics

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Part I

Category Theory

Chapter 1

Foundations

This is a placeholder — I am not sure what foundation I want to use for this project yet. I will try to work in a way which is foundation-independent. What I do could be formalized in ZFC, ETCS, or some other system. I will assume the usual set theoretic constructions as needed. Sets will be defined up to bijection only.

Chapter 2

Categories

Definition 2.1 (Category). A *category* \mathcal{C} consists of:

- A class $|\mathcal{C}|$ of *objects*. We write $A \in \mathcal{C}$ for $A \in |\mathcal{C}|$.
- For any objects A, B , a set $\mathcal{C}[A, B]$ of *morphisms* from A to B . We write $f : A \rightarrow B$ for $f \in \mathcal{C}[A, B]$.
- For any object A , a morphism $\text{id}_A : A \rightarrow A$, the *identity* morphism on A .
- For any morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, a morphism $g \circ f : A \rightarrow C$, the *composite* of f and g .

such that:

Associativity Given $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Left Unit Law For any morphism $f : A \rightarrow B$, we have $\text{id}_B \circ f = f$.

Right Unit Law For any morphism $f : A \rightarrow B$, we have $f \circ \text{id}_A = f$.

Proposition 2.2. *The identity morphism on an object is unique.*

PROOF: If i and j are identity morphisms on A then $i = i \circ j = j$. \square

Example 2.3 (Category of Sets). The *category of sets* **Set** has objects all sets and morphisms all functions.

Definition 2.4 (Endomorphism). In a category \mathcal{C} , an *endomorphism* on an object A is a morphism $A \rightarrow A$. We write $\text{End}_{\mathcal{C}}(A)$ for $\mathcal{C}[A, A]$.

Definition 2.5 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with the same objects as \mathcal{C} and

$$\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$$

2.1 Preorders

Definition 2.6 (Preorder). A *preorder* on a set A is a relation \leq on A that is reflexive and transitive.

A *preordered set* is a pair (A, \leq) such that \leq is a preorder on A . We usually write A for the preordered set (A, \leq) .

We identify any preordered set A with the category whose objects are the elements of A , with one morphism $a \rightarrow b$ iff $a \leq b$, and no morphism $a \rightarrow b$ otherwise.

Example 2.7. For any ordinal α , let α be the preorder $\{\beta : \beta < \alpha\}$ under \leq .

Definition 2.8 (Discrete Preorder). We identify any set A with the *discrete* preorder $(A, =)$.

2.2 Monomorphisms and Epimorphisms

Definition 2.9 (Monomorphism). In a category, let $f : A \rightarrow B$. Then f is a *monomorphism* or *monic* iff, for every object X and morphism $x, y : X \rightarrow A$, if $fx = fy$ then $x = y$.

Definition 2.10 (Epimorphism). In a category, let $f : A \rightarrow B$. Then f is a *epimorphism* or *epi* iff, for every object X and morphism $x, y : B \rightarrow X$, if $xf = yf$ then $x = y$.

Proposition 2.11. *The composite of two monomorphism is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$ be monic.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$

$\langle 1 \rangle 3$. ASSUME: $g \circ f \circ x = g \circ f \circ y$

$\langle 1 \rangle 4$. $f \circ x = f \circ y$

$\langle 1 \rangle 5$. $x = y$

□

Proposition 2.12. *The composite of two epimorphisms is epi.*

PROOF: Dual. □

Proposition 2.13. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is monic then f is monic.*

PROOF: If $f \circ x = f \circ y$ then $g \circ f \circ x = g \circ f \circ y$ and so $x = y$. □

Proposition 2.14. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is epi then g is epi.*

PROOF: Dual. □

Proposition 2.15. *A function is a monomorphism in **Set** iff it is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is monic then f is injective.

$\langle 2 \rangle 1$. ASSUME: f is monic.

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 4$. LET: $\bar{x}, \bar{y} : 1 \rightarrow A$ be the functions such that $\bar{x}(*) = x$ and $\bar{y}(*) = y$

$\langle 2 \rangle 5$. $f \circ \bar{x} = f \circ \bar{y}$

$\langle 2 \rangle 6$. $\bar{x} = \bar{y}$

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. $x = y$

$\langle 1 \rangle 3$. If f is injective then f is monic.

$\langle 2 \rangle 1$. ASSUME: f is injective.

$\langle 2 \rangle 2$. LET: X be a set and $x, y : X \rightarrow A$.

$\langle 2 \rangle 3$. ASSUME: $f \circ x = f \circ y$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $t \in X$

PROVE: $x(t) = y(t)$

$\langle 2 \rangle 5$. $f(x(t)) = f(y(t))$

$\langle 2 \rangle 6$. $x(t) = y(t)$

PROOF: By $\langle 2 \rangle 1$.

□

Proposition 2.16. *A function is an epimorphism in **Set** iff it is surjective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is an epimorphism then f is surjective.

$\langle 2 \rangle 1$. ASSUME: f is an epimorphism.

$\langle 2 \rangle 2$. LET: $b \in B$

$\langle 2 \rangle 3$. LET: $x, y : B \rightarrow 2$ be defined by $x(b) = 1$ and $x(t) = 0$ for all other $t \in B$, $y(t) = 0$ for all $t \in B$.

$\langle 2 \rangle 4$. $x \neq y$

$\langle 2 \rangle 5$. $x \circ f \neq y \circ f$

$\langle 2 \rangle 6$. There exists $a \in A$ such that $f(a) = b$.

$\langle 1 \rangle 3$. If f is surjective then f is an epimorphism.

$\langle 2 \rangle 1$. ASSUME: f is surjective.

$\langle 2 \rangle 2$. LET: $x, y : B \rightarrow X$

$\langle 2 \rangle 3$. ASSUME: $x \circ f = y \circ f$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $b \in B$

PROVE: $x(b) = y(b)$

$\langle 2 \rangle 5$. PICK $a \in A$ such that $f(a) = b$

$\langle 2 \rangle 6$. $x(f(a)) = y(f(a))$

$\langle 2 \rangle 7$. $x(b) = y(b)$

□

Proposition 2.17. *In a preorder, every morphism is monic and epi.*

PROOF: Immediate from definitions. \square

2.3 Sections and Retractions

Definition 2.18 (Section, Retraction). In a category, let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $r \circ s = \text{id}_B$.

Proposition 2.19. *Every identity morphism is a section and retraction of itself.*

PROOF: Immediate from definitions. \square

Proposition 2.20. *Let $r, r' : A \rightarrow B$ and $s : B \rightarrow A$. If r is a retraction of s and r' is a section of s then $r = r'$.*

PROOF:

$$\begin{aligned} r &= r \circ \text{id}_A \\ &= r \circ s \circ r' \\ &= \text{id}_B \circ r' \\ &= r' \end{aligned} \quad \square$$

Proposition 2.21. *Let $r_1 : A \rightarrow B$, $r_2 : B \rightarrow C$, $s_1 : B \rightarrow A$ and $s_2 : C \rightarrow B$. If r_1 is a retraction of s_1 and r_2 is a retraction of s_2 then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= \text{id}_C \end{aligned} \quad \square$$

Proposition 2.22. *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $s : A \rightarrow B$ be a section of $r : B \rightarrow A$.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$ satisfy $sx = sy$.

$\langle 1 \rangle 3$. $rsx = rsy$

$\langle 1 \rangle 4$. $x = y$

\square

Proposition 2.23. *Every retraction is epi.*

PROOF: Dual. \square

Proposition 2.24. *In Set, every epimorphism has a retraction.*

PROOF: By the Axiom of Choice. \square

Example 2.25. It is not true in general that every monomorphism in any category has a section. nor that every epimorphism in any category has a retraction.

In the category **2**, the morphism $0 \leq 1$ is monic and epi but has no retraction or section.

2.4 Isomorphisms

Definition 2.26 (Isomorphism). In a category \mathcal{C} , a morphism $f : A \rightarrow B$ is an *isomorphism*, denoted $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$, the *inverse* of f , such that $f^{-1} \circ f = \text{id}_A$ and $f \circ f^{-1} = \text{id}_B$.

An *automorphism* on an object A is an isomorphism between A and itself. We write $\text{Aut}_{\mathcal{C}}(A)$ for the set of all automorphisms on A .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 2.27. *The inverse of an isomorphism is unique.*

PROOF: Proposition 2.20. \square

Proposition 2.28. *For any object A we have $\text{id}_A : A \cong A$ and $\text{id}_A^{-1} = \text{id}_A$.*

PROOF: Since $\text{id}_A \circ \text{id}_A = \text{id}_A$ by the Unit Laws. \square

Proposition 2.29. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Immediate from definitions. \square

Proposition 2.30. *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF: From Proposition 2.21. \square

Definition 2.31 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

2.5 Initial and Terminal Objects

Definition 2.32 (Initial Object). An object I in a category is *initial* iff, for any object X , there is exactly one morphism $I \rightarrow X$.

Example 2.33. The empty set is the initial object in **Set**.

Definition 2.34 (Terminal Object). An object T in a category is *terminal* iff, for any object X , there is exactly one morphism $X \rightarrow T$.

Example 2.35. Every singleton is terminal in **Set**.

Proposition 2.36. *If I and J are initial in a category, then there exists a unique isomorphism $I \cong J$.*

PROOF:

- $\langle 1 \rangle 1$. LET: i be the unique morphism $I \rightarrow J$.
- $\langle 1 \rangle 2$. LET: i^{-1} be the unique morphism $J \rightarrow I$.
- $\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_J$

PROOF: Since there is only one morphism $J \rightarrow J$.

- $\langle 1 \rangle 4$. $i^{-1} \circ i = \text{id}_I$

PROOF: Since there is only one morphism $I \rightarrow I$.
 \square

Proposition 2.37. *If S and T are terminal in a category, then there exists a unique isomorphism $S \cong T$.*

PROOF: Dual. \square

Chapter 3

Functors

Definition 3.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for any morphism $f : A \rightarrow B : \mathcal{C}$, a morphism $Ff : FA \rightarrow FB : \mathcal{D}$

such that:

- $F\text{id}_A = \text{id}_{FA}$
- $F(g \circ f) = Fg \circ Ff$

Definition 3.2 (Identity Functor). For any category \mathcal{C} , the *identity functor* $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is defined by

$$\begin{aligned} 1_{\mathcal{C}}A &= A \\ 1_{\mathcal{C}}f &= f \end{aligned}$$

Definition 3.3 (Constant Functor). Given categories \mathcal{C} , \mathcal{D} and an object $D \in \mathcal{D}$, the *constant functor* $K^{\mathcal{C}}D : \mathcal{C} \rightarrow \mathcal{D}$ is the functor defined by

$$\begin{aligned} K^{\mathcal{C}}DC &= D \\ K^{\mathcal{C}}Df &= \text{id}_D \end{aligned}$$

3.1 Comma Categories

Definition 3.4 (Comma Category). Let $F : \mathcal{C} \rightarrow \mathcal{E}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors. The *comma category* $F \downarrow G$ is the category with:

- objects all pairs (C, D, f) where $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f : FC \rightarrow GD : \mathcal{E}$

- morphisms $(u, v) : (C, D, f) \rightarrow (C', D', g)$ all pairs $u : C \rightarrow C' : \mathcal{C}$ and $v : D \rightarrow D' : \mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ \downarrow Fu & & \downarrow Gv \\ FC' & \xrightarrow{g} & GD' \end{array}$$

Definition 3.5 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category* over A , denoted \mathcal{C}/A , is the comma category $1_{\mathcal{C}} \downarrow K^1 A$.

Definition 3.6 (Coslice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *coslice category* over A , denoted $\mathcal{C} \backslash A$, is the comma category $K^1 A \downarrow 1_{\mathcal{C}}$.

Definition 3.7 (Pointed Sets). The *category of pointed sets* \mathbf{Set}_* is the coslice category $\mathbf{Set} \backslash 1$.

Part II

Group Theory

Chapter 4

Semigroups

Definition 4.1 (Semigroup). A *semigroup* consists of a set S and an associative binary operation \cdot on S .

Chapter 5

Monoids

Definition 5.1 (Monoid). A *monoid* consists of a semigroup M such that there exists $e \in M$, the *unit*, such that, for all $x \in M$, we have $xe = ex = x$.

We identify a monoid M with the category with one object whose morphisms are the elements of M , with composition given by \cdot .

Proposition 5.2. *The identity in a group is unique.*

PROOF: Proposition 2.2.

Chapter 6

Groups

Definition 6.1 (Group). Let \mathcal{C} be a category with finite products. A *group (object)* in \mathcal{C} consists of an object $G \in \mathcal{C}$ and morphisms

$$m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G$$

such that the following diagrams commute.

$$\begin{array}{ccc} G^3 & \xrightarrow{m \times \text{id}_G} & G^2 \\ \downarrow \text{id}_G \times m & & \downarrow m \\ G^2 & \xrightarrow{m} & G \end{array}$$

$$\begin{array}{ccc} 1 \times G & \xrightarrow{e \times \text{id}_G} & G^2 \\ & \searrow \cong & \downarrow m \\ & & G \end{array} \quad \begin{array}{ccc} G \times 1 & \xrightarrow{\text{id}_G \times e} & G^2 \\ & \searrow \cong & \downarrow m \\ & & G \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{\Delta} & G^2 & \xrightarrow{\text{id}_G \times i} & G^2 \\ \downarrow & & & & \downarrow m \\ 1 & \xrightarrow{e} & G & & G \end{array} \quad \begin{array}{ccccc} G & \xrightarrow{\Delta} & G^2 & \xrightarrow{i \times \text{id}_G} & G^2 \\ \downarrow & & & & \downarrow m \\ 1 & \xrightarrow{e} & G & & G \end{array}$$

Definition 6.2 (Group). We write just 'group' for 'group in **Set**'. Thus, a *group* G consists of a set G and a binary operation $\cdot : G^2 \rightarrow G$ such that \cdot is associative, and there exists $e \in G$, the *identity* element of the group, such that:

- For all $x \in G$ we have $xe = ex = x$
- For all $x \in G$, there exists $x^{-1} \in G$, the *inverse* of x , such that $xx^{-1} = x^{-1}x = e$.

The *order* of a group G , denoted $|G|$, is the number of elements in G if G is finite; otherwise we write $|G| = \infty$.

Proposition 6.3. *The inverse of an element is unique.*

PROOF: If i and j are inverses of x then $i = ixj = j$. \square

Example 6.4. • The *trivial* group is $\{e\}$ under $ee = e$.

- \mathbb{Z} is a group under addition
 - \mathbb{Q} is a group under addition
 - $\mathbb{Q} - \{0\}$ is a group under multiplication
 - \mathbb{R} is a group under addition
 - $\mathbb{R} - \{0\}$ is a group under multiplication
 - \mathbb{C} is a group under addition
 - $\mathbb{C} - \{0\}$ is a group under multiplication
 - $\{-1, 1\}$ is a group under multiplication
 - For any category \mathcal{C} and object $A \in \mathcal{C}$, we have $\text{Aut}_{\mathcal{C}}(A)$ is a group under $gf = f \circ g$.
- For A a set, we call $S_A = \text{Aut}_{\text{Set}}(A)$ the *symmetric group* or *group of permutations* of A .
- For $n \geq 3$, the *dihedral group* D_{2n} consists of the set of rigid motions that map the regular n -gon onto itself under composition.
 - Let $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ under matrix multiplication.
 - The quaternionic group Q_8 is the group

$$\{1, -1, i, -i, j, -j, k, -k\}$$

with multiplication table

1	-1	i	-i	j	-j	k	-k
-1	1	-i	i	-j	j	-k	k
i	-i	-1	1	k	-k	-j	j
-i	i	1	-1	-k	k	j	-j
j	-j	-k	k	-1	1	i	-i
-j	j	k	-k	1	-1	-i	i
k	-k	j	-j	-i	i	-1	1
-k	k	-j	j	i	-i	1	-1

Example 6.5. • The only group of order 1 is the trivial group.

- The only group of order 2 is \mathbb{Z}_2 .

- The only group of order 3 is \mathbb{Z}_3 .
- There are exactly two groups of order 4: \mathbb{Z}_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ under $(a, b)(c, d) = (ac, bd)$.

Proposition 6.6 (Cancellation). *Let G be a group. Let $a, g, h \in G$. If $ag = ah$ or $ga = ha$ then $g = h$.*

PROOF: If $ag = ah$ then $g = a^{-1}ag = a^{-1}ah = h$. Similarly if $ga = ha$. \square

Proposition 6.7. *Let G be a group and $g, h \in G$. Then $(gh)^{-1} = h^{-1}g^{-1}$.*

PROOF: Since $ghh^{-1}g^{-1} = e$. \square

Definition 6.8. Let G be a group. Let $g \in G$. We define $g^n \in G$ for all $n \in \mathbb{Z}$ as follows:

$$\begin{aligned} g^0 &= e \\ g^{n+1} &= g^n g & (n \geq 0) \\ g^{-n} &= (g^{-1})^n & (n > 0) \end{aligned}$$

Proposition 6.9. *Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then*

$$g^{m+n} = g^m g^n .$$

PROOF:

$\langle 1 \rangle 1$. For all $k \in \mathbb{Z}$ we have $g^{k+1} = g^k g$

$\langle 2 \rangle 1$. For all $k \geq 0$ we have $g^{k+1} = g^k g$

PROOF: Immediate from definition.

$\langle 2 \rangle 2$. $g^{-1+1} = g^{-1} g$

PROOF: Both are equal to e .

$\langle 2 \rangle 3$. For all $k > 1$ we have $g^{-k+1} = g^{-k} g$

PROOF:

$$\begin{aligned} g^{-k+1} &= (g^{-1})^{k-1} \\ &= (g^{-1})^{k-1} g^{-1} g \\ &= (g^{-1})^k g \\ &= g^{-k} g \end{aligned}$$

$\langle 1 \rangle 2$. For all $k \in \mathbb{Z}$ we have $g^{k-1} = g^k g^{-1}$

PROOF: Substitute $k = k - 1$ above and multiply by g^{-1} .

$\langle 1 \rangle 3$. $g^{m+0} = g^m g^0$

PROOF: Since $g^m g^0 = g^m e = g^m$.

$\langle 1 \rangle 4$. If $g^{m+n} = g^m g^n$ then $g^{m+n+1} = g^m g^{n+1}$

PROOF:

$$\begin{aligned} g^{m+n+1} &= g^{m+n} g & (\langle 1 \rangle 1) \\ &= g^m g^n g \\ &= g^m g^{n+1} & (\langle 1 \rangle 1) \end{aligned}$$

$\langle 1 \rangle 5$. If $g^{m+n} = g^m g^n$ then $g^{m+n-1} = g^m g^{n-1}$

PROOF:

$$g^{m+n-1} g = g^{m+n} \quad (\langle 1 \rangle 1)$$

$$= g^m g^n$$

$$\therefore g^{m+n-1} = g^m g^n g^{-1}$$

$$= g^m g^{n-1} \quad (\langle 1 \rangle 2)$$

□

Proposition 6.10. Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then

$$(g^m)^n = g^{mn} .$$

PROOF:

$\langle 1 \rangle 1$. $(g^m)^0 = g^0$

PROOF: Both sides are equal to e .

$\langle 1 \rangle 2$. If $(g^m)^n = g^{mn}$ then $(g^m)^{n+1} = g^{m(n+1)}$.

PROOF:

$$(g^m)^{n+1} = (g^m)^n g^m \quad (\text{Proposition 6.9})$$

$$= g^{mn} g^m$$

$$= g^{mn+m} \quad (\text{Proposition 6.9})$$

$\langle 1 \rangle 3$. If $(g^m)^n = g^{mn}$ then $(g^m)^{n-1} = g^{m(n-1)}$.

PROOF:

$$(g^m)^n = g^{mn}$$

$$\therefore (g^m)^{n-1} g^m = g^{mn-m} g^m \quad (\text{Proposition 6.9})$$

$$\therefore (g^m)^{n-1} = g^{mn-m} \quad (\text{Cancellation})$$

□

Definition 6.11 (Commute). Let G be a group and $g, h \in G$. We say g and h *commute* iff $gh = hg$.

Definition 6.12. Let G be a group. Given $g \in G$ and $A \subseteq G$, we define

$$gA = \{ga : a \in A\}, \quad Ag = \{ag : a \in A\} .$$

Given sets $A, B \subseteq G$, we define

$$AB = \{ab : a \in A, b \in B\} .$$

6.1 Order of an Element

Definition 6.13 (Order). Let G be a group. Let $g \in G$. Then g has *finite order* iff there exists a positive integer n such that $g^n = e$. In this case, the *order* of g , denoted $|g|$, is the least positive integer n such that $g^n = e$.

If g does not have finite order, we write $|g| = \infty$.

Proposition 6.14. *Let G be a group. Let $g \in G$ and n be a positive integer. If $g^n = e$ then $|g| \mid n$.*

PROOF:

$\langle 1 \rangle 1$. LET: $n = q|g| + d$ where $0 \leq d < |g|$

PROOF: Division Algorithm.

$\langle 1 \rangle 2$. $g^d = e$

PROOF:

$$\begin{aligned} e &= g^n \\ &= g^{q|g|+d} \\ &= (g^{|g|})^q g^d && \text{(Propositions 6.9, 6.10)} \\ &= e^q g^d \\ &= g^d \end{aligned}$$

$\langle 1 \rangle 3$. $d = 0$

PROOF: By minimality of $|g|$.

$\langle 1 \rangle 4$. $n = q|g|$

□

Corollary 6.14.1. *Let G be a group. Let $g \in G$ have finite order and $n \in \mathbb{Z}$. Then $g^n = e$ if and only if $|g| \mid n$.*

Proposition 6.15. *Let G be a group and $g \in G$. Then $|g| \leq |G|$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: w.l.o.g. G is finite.

$\langle 1 \rangle 2$. PICK i, j with $0 \leq i < j \leq |G|$ such that $g^i = g^j$.

PROOF: Otherwise $g^0, g^1, \dots, g^{|G|}$ would be $|G| + 1$ distinct elements of G .

$\langle 1 \rangle 3$. $g^{j-i} = e$

$\langle 1 \rangle 4$. g has finite order and $|g| \leq |G|$

PROOF: Since $|g| \leq j - i \leq j \leq |G|$.

□

Proposition 6.16. *Let G be a group. Let $g \in G$ have finite order. Let $m \in \mathbb{N}$. Then*

$$|g^m| = \frac{\text{lcm}(m, |g|)}{m} = \frac{|g|}{\text{gcd}(m, |g|)}$$

PROOF: Since for any integer d we have

$$g^{md} = e \Leftrightarrow |g| \mid md \quad (\text{Corollary 6.14.1})$$

$$\Leftrightarrow \text{lcm}(m, |g|) \mid md$$

$$\Leftrightarrow \frac{\text{lcm}(m, |g|)}{m} \mid d$$

□

and so $|g^m| = \frac{\text{lcm}(m, |g|)}{m}$ by Corollary 6.14.1. □

Corollary 6.16.1. *If g has odd order then $|g^2| = |g|$.*

Proposition 6.17. *Let G be a group. Let $g, h \in G$ have finite order. Assume $gh = hg$. Then $|gh|$ has finite order and*

$$|gh| \mid \text{lcm}(|g|, |h|)$$

PROOF: Since $(gh)^{\text{lcm}(|g|, |h|)} = g^{\text{lcm}(|g|, |h|)} h^{\text{lcm}(|g|, |h|)} = e$. \square

Example 6.18. This example shows that we cannot remove the hypothesis that $gh = hg$.

In $\text{GL}_2(\mathbb{R})$, take

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then $|g| = 4$, $|h| = 3$ and $|gh| = \infty$.

Proposition 6.19. *Let G be a group and $g, h \in G$ have finite order. If $gh = hg$ and $\gcd(|g|, |h|) = 1$ then $|gh| = |g||h|$.*

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } N = |gh|$$

$$\langle 1 \rangle 2. g^N = (h^{-1})^N$$

$$\langle 1 \rangle 3. g^{N|g|} = e$$

$$\langle 1 \rangle 4. |g^N| \mid |g|$$

$$\langle 1 \rangle 5. h^{-N|h|} = e$$

$$\langle 1 \rangle 6. |g^N| \mid |h|$$

$$\langle 1 \rangle 7. |g^N| = 1$$

PROOF: Since $\gcd(|g|, |h|) = 1$.

$$\langle 1 \rangle 8. g^N = e$$

$$\langle 1 \rangle 9. |g| \mid N$$

$$\langle 1 \rangle 10. h^{-N} = e$$

$$\langle 1 \rangle 11. |h| \mid N$$

$$\langle 1 \rangle 12. N = |g||h|$$

PROOF: Using Proposition 6.17.

\square

Proposition 6.20. *Let G be a finite group. Assume there is exactly one element $f \in G$ of order 2. Then the product of all the elements of G is f .*

PROOF: Let the elements of G be g_1, g_2, \dots, g_n . Apart from e and f , every element and its inverse are distinct elements of the list. Hence the product of the list is $ef = f$. \square

Proposition 6.21. *Let G be a finite group of order n . Let m be the number of elements of G of order 2. Then $n - m$ is odd.*

PROOF: In the list of all elements that are not of order 2, every element and its inverse are distinct except for e . Hence the list has odd length. \square

Corollary 6.21.1. *If a finite group has even order, then it contains an element of order 2.*

Proposition 6.22. *Let G be a group and $a, g \in G$. Then $|aga^{-1}| = |g|$.*

PROOF: Since

$$\begin{aligned} (aga^{-1})^n = e &\Leftrightarrow ag^na^{-1} = e \\ &\Leftrightarrow g^n = e \end{aligned} \quad \square$$

Proposition 6.23. *Let G be a group and $g, h \in G$. Then $|gh| = |hg|$.*

PROOF: Since $|gh| = |ghgg^{-1}| = |hg|$. \square

Proposition 6.24. *Let G be a group of order n . Let k be relatively prime to n . Then every element in G has the form x^k for some x .*

$\langle 1 \rangle 1$. PICK integers a and b such that $an + bk = 1$.

$\langle 1 \rangle 2$. LET: $g \in G$

$\langle 1 \rangle 3$. $g = (g^b)^k$

PROOF:

$$\begin{aligned} g &= g \cdot (g^n)^{-a} & (g^n = e) \\ &= g^{1-an} \\ &= g^{bk} \end{aligned}$$

\square

6.2 Generators

Definition 6.25 (Generator). Let G be a group and $a \in G$. We say a *generates* the group iff, for all $x \in G$, there exists an integer n such that $x^n = a$.

Example 6.26. $\text{SL}_2(\mathbb{Z})$ is generated by

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

PROOF:

$\langle 1 \rangle 1$. LET: $H = \langle s, t \rangle$

$\langle 1 \rangle 2$. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in H$.

PROOF: It is t^q .

$\langle 1 \rangle 3$. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \in H$.

PROOF:

$$\begin{aligned} st^{-q}s^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \end{aligned}$$

⟨1⟩4.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & qa+b \\ c & qc+d \end{pmatrix}$$

⟨1⟩5.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} a+qb & b \\ c+qd & d \end{pmatrix}$$

⟨1⟩6. For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, if c and d are both nonzero, then there exists $N \in H$ such that the bottom row of MN has one entry the same as M and one entry with smaller absolute value.

PROOF: From ⟨1⟩4 and ⟨1⟩5 taking $q = -1$.

⟨1⟩7. For any $M \in \mathrm{SL}_2(\mathbb{Z})$, there exists $N \in H$ such that MN has a zero on the bottom row.

PROOF: Apply ⟨1⟩6 repeatedly.

⟨1⟩8. Any matrix in $\mathrm{SL}_2(\mathbb{Z})$ with a zero on the bottom row is in H .

⟨2⟩1. $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H$

PROOF: ⟨1⟩2

⟨2⟩2. $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \in H$

PROOF: It is $s^2 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ since $s^2 = -I$.

⟨2⟩3. $\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} \in H$

PROOF: It is $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} s$.

⟨2⟩4. $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \in H$

PROOF: It is $s^2 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s$.

⟨1⟩9. Every matrix in $\mathrm{SL}_2(\mathbb{Z})$ is in H .

□

6.3 p -groups

Definition 6.27 (p -group). Let p be a prime. A p -group is a finite group whose order is a power of p .

Chapter 7

Group Homomorphisms

Definition 7.1 (Homomorphism). Let G and H be groups. A (group) homomorphism $\phi : G \rightarrow H$ is a function such that, for all $x, y \in G$,

$$\phi(xy) = \phi(x)\phi(y) \text{ .}$$

Proposition 7.2. Let G and H be groups with identities e_G and e_H . Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\phi(e_G) = e_H$.

PROOF: Since $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$ and so $\phi(e_G) = e_H$ by Cancellation. \square

Proposition 7.3. Let $\phi : G \rightarrow H$ be a group homomorphism. For all $x \in G$ we have $\phi(x^{-1}) = \phi(x)^{-1}$.

PROOF: Since $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_H$. \square

Proposition 7.4. Let G, H and K be groups. If $\phi : G \rightarrow H$ and $\psi : H \rightarrow K$ are homomorphisms then $\psi \circ \phi : G \rightarrow K$ is a homomorphism.

PROOF: For $x, y \in G$ we have

$$\psi(\phi(xy)) = \psi(\phi(x)\phi(y)) = \psi(\phi(x))\psi(\phi(y)) \text{ .}$$

Proposition 7.5. Let G be a group. Then $\text{id}_G : G \rightarrow G$ is a group homomorphism.

PROOF: For $x, y \in G$ we have $\text{id}_G(xy) = xy = \text{id}_G(x)\text{id}_G(y)$. \square

Proposition 7.6. Let $\phi : G \rightarrow H$ be a group homomorphism. Let $g \in G$ have finite order. Then $|\phi(g)|$ divides $|g|$.

PROOF: Since $\phi(g)^{|g|} = \phi(g^{|g|}) = e$. \square

Definition 7.7 (Category of Groups). Let **Grp** be the category of groups and group homomorphisms.

Example 7.8. There are 49487365402 groups of order 1024 up to isomorphism.

Proposition 7.9. *A group homomorphism $\phi : G \rightarrow H$ is an isomorphism in **Grp** if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: ϕ is bijective.

PROVE: ϕ^{-1} is a group homomorphism.

$\langle 1 \rangle 2$. LET: $h, h' \in H$

$\langle 1 \rangle 3$. $\phi(\phi^{-1}(hh')) = \phi(\phi^{-1}(h)\phi^{-1}(h'))$

PROOF: Both are equal to hh' .

$\langle 1 \rangle 4$. $\phi^{-1}(hh') = \phi^{-1}(h)\phi^{-1}(h')$

□

Corollary 7.9.1.

$$D_6 \cong C_3$$

PROOF: The canonical homomorphism $D_6 \rightarrow C_3$ is bijective. □

Corollary 7.9.2.

$$(\mathbb{R}, +) \cong (\{x \in \mathbb{R} : x > 0\}, \cdot)$$

PROOF: The function that maps x to e^x is a bijective homomorphism. □

Proposition 7.10. *The trivial group is the zero object in **Grp**.*

PROOF: For any group G , the unique function $G \rightarrow \{e\}$ is a group homomorphism, and the only group homomorphism $\{e\} \rightarrow G$ maps e to e_G . □

Proposition 7.11. *For any groups G and H , the set $G \times H$ under $(g, h)(g', h') = (gg', hh')$ is the product of G and H in **Grp**.*

PROOF:

$\langle 1 \rangle 1$. $G \times H$ is a group.

$\langle 2 \rangle 1$. The multiplication is associative.

PROOF: Since $(g_1, h_1)((g_2, h_2)(g_3, h_3)) = ((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1g_2g_3, h_1h_2h_3)$.

$\langle 2 \rangle 2$. (e_G, e_H) is the identity.

PROOF: Since $(g, h)(e_G, e_H) = (e_G, e_H)(g, h) = (g, h)$.

$\langle 2 \rangle 3$. The inverse of (g, h) is (g^{-1}, h^{-1}) .

PROOF: Since $(g, h)(g^{-1}, h^{-1}) = (g^{-1}, h^{-1})(g, h) = (e_G, e_H)$.

$\langle 1 \rangle 2$. $\pi_1 : G \times H \rightarrow G$ is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$. $\pi_2 : G \times H \rightarrow H$ is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 4$. For any group homomorphism $\phi : K \rightarrow G$ and $\psi : K \rightarrow H$, the function $\langle \phi, \psi \rangle : K \rightarrow G \times H$ where $\langle \phi, \psi \rangle(k) = (\phi(k), \psi(k))$ is a group homomorphism.

PROOF:

$$\begin{aligned} \langle \phi, \psi \rangle(kk') &= (\phi(kk'), \psi(kk')) \\ &= (\phi(k)\phi(k'), \psi(k)\psi(k')) \\ &= (\phi(k), \psi(k))(\phi(k'), \psi(k')) \\ &= \langle \phi, \psi \rangle(k)\langle \phi, \psi \rangle(k') \end{aligned}$$

□

7.1 Subgroups

Definition 7.12 (Subgroup). Let (G, \cdot) and $(H, *)$ be groups such that H is a subset of G . Then H is a *subgroup* of G iff the inclusion $i : H \hookrightarrow G$ is a group homomorphism.

Proposition 7.13. *If $(H, *)$ is a subgroup of (G, \cdot) then $*$ is the restriction of \cdot to H .*

PROOF: Given $x, y \in H$ we have

$$x * y = i(x * y) = i(x) \cdot i(y) = x \cdot y . \quad \square$$

Example 7.14. For any group G we have $\{e\}$ is a subgroup of G .

Proposition 7.15. *Let G be a group. Let H be a subset of G . Then H is a subgroup of G iff H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$.*

PROOF:

$\langle 1 \rangle 1$. If H is a subgroup of G then H is nonempty.

PROOF: Since every group has an identity element and so is nonempty.

$\langle 1 \rangle 2$. If H is a subgroup of G then, for all $x, y \in H$, we have $xy^{-1} \in H$.

PROOF: Easy.

$\langle 1 \rangle 3$. If H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$, then H is a subgroup of G .

$\langle 2 \rangle 1$. ASSUME: H is nonempty.

$\langle 2 \rangle 2$. ASSUME: $\forall x, y \in H. xy^{-1} \in H$

$\langle 2 \rangle 3$. $e \in H$

PROOF: Pick $x \in H$. We have $e = xx^{-1} \in H$.

$\langle 2 \rangle 4$. $\forall x \in H. x^{-1} \in H$

PROOF: Given $x \in H$ we have $x^{-1} = ex^{-1} \in H$.

$\langle 2 \rangle 5$. H is closed under the restriction of \cdot

PROOF: Given $x, y \in H$ we have $xy = x(y^{-1})^{-1} \in H$.

$\langle 2 \rangle 6$. H is a group under the restriction of \cdot

PROOF: Associativity is inherited from G and the existence of an identity element and inverses follows from $\langle 2 \rangle 3$ and $\langle 2 \rangle 4$.

$\langle 2 \rangle 7$. The inclusion $H \hookrightarrow G$ is a group homomorphism.

PROOF: For $x, y \in H$ we have $i(xy) = i(x)i(y) = xy$.

\square

Corollary 7.15.1. *The intersection of a set of subgroups of G is a subgroup of G .*

Corollary 7.15.2. *Let $\phi : G \rightarrow H$ be a group homomorphism. Let K be a subgroup of H . Then $\phi^{-1}(K)$ is a subgroup of G .*

PROOF:

$\langle 1 \rangle 1$. $\phi^{-1}(K)$ is nonempty.

PROOF: Since $e \in \phi^{-1}(K)$.

$\langle 1 \rangle 2$. LET: $x, y \in \phi^{-1}(K)$

- $\langle 1 \rangle 3. \phi(x), \phi(y) \in K$
- $\langle 1 \rangle 4. \phi(x)\phi(y)^{-1} \in K$
- $\langle 1 \rangle 5. \phi(xy^{-1}) \in K$
- $\langle 1 \rangle 6. xy^{-1} \in \phi^{-1}(K)$

□

Corollary 7.15.3. *Let $\phi : G \rightarrow H$ be a group homomorphism. Let K be a subgroup of G . Then $\phi(K)$ is a subgroup of H .*

PROOF:

- $\langle 1 \rangle 1.$ LET: $x, y \in \phi(K)$
- $\langle 1 \rangle 2.$ PICK $a, b \in K$ such that $x = \phi(a)$ and $y = \phi(b)$
- $\langle 1 \rangle 3. xy^{-1} = \phi(ab^{-1})$
- $\langle 1 \rangle 4. xy^{-1} \in \phi(K)$

□

Proposition 7.16. *Let G be a subgroup of \mathbb{Z} . Then there exists $d \geq 0$ such that $G = d\mathbb{Z}$.*

PROOF:

- $\langle 1 \rangle 1.$ ASSUME: w.l.o.g. $G \neq \{0\}$

PROOF: Since $\{0\} = 0\mathbb{Z}$.

- $\langle 1 \rangle 2.$ LET: d be the least positive element of G .

PROVE: $G = d\mathbb{Z}$

PROOF: If $n \in G$ then $-n \in G$ so G must contain a positive element.

- $\langle 1 \rangle 3. G \subseteq d\mathbb{Z}$

- $\langle 2 \rangle 1.$ LET: $n \in G$

- $\langle 2 \rangle 2.$ LET: q and r be the integers such that $n = qd + r$ and $0 \leq r < d$.

- $\langle 2 \rangle 3. r \in G$

PROOF: Since $r = n - qd$.

- $\langle 2 \rangle 4. r = 0$

PROOF: By minimality of d .

- $\langle 2 \rangle 5. n = qd \in d\mathbb{Z}$

- $\langle 1 \rangle 4. d\mathbb{Z} \subseteq G$

□

7.2 Kernel

Definition 7.17 (Kernel). Let $\phi : G \rightarrow H$ be a group homomorphism. The *kernel* of ϕ is

$$\ker \phi = \{g \in G : \phi(g) = e\} .$$

Proposition 7.18. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\ker \phi$ is a subgroup of G .*

PROOF: Corollary 7.15.2. □

Proposition 7.19. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then the inclusion $i : \ker \phi \hookrightarrow G$ is terminal in the category of pairs $(K, \alpha : K \rightarrow G)$ such that $\phi \circ \alpha = 0$.*

PROOF:

$\langle 1 \rangle 1.$ $\phi \circ i = 0$

$\langle 1 \rangle 2.$ For any group K and homomorphism $\alpha : K \rightarrow G$ such that $\phi \circ \alpha = 0$, there exists a unique homomorphism $\beta : K \rightarrow \ker \phi$ such that $i \circ \beta = \alpha$.

□

Proposition 7.20. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then the following are equivalent:*

1. ϕ is monic.
2. $\ker \phi = \{e\}$
3. ϕ is injective.

PROOF:

$\langle 1 \rangle 1.$ $1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: ϕ is monic.

$\langle 2 \rangle 2.$ LET: $i : \ker \phi \hookrightarrow G$, $j : \{e\} \hookrightarrow \ker \phi \hookrightarrow G$ be the inclusions.

$\langle 2 \rangle 3.$ $\phi \circ i = \phi \circ j$

$\langle 2 \rangle 4.$ $i = j$

$\langle 1 \rangle 2.$ $2 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: $\ker \phi = \{e\}$

$\langle 2 \rangle 2.$ LET: $x, y \in G$

$\langle 2 \rangle 3.$ ASSUME: $\phi(x) = \phi(y)$

$\langle 2 \rangle 4.$ $\phi(xy^{-1}) = e$

$\langle 2 \rangle 5.$ $xy^{-1} \in \ker \phi$

$\langle 2 \rangle 6.$ $xy^{-1} = e$

$\langle 2 \rangle 7.$ $x = y$

$\langle 1 \rangle 3.$ $3 \Rightarrow 1$

PROOF: Easy.

□

Proposition 7.21. *A group homomorphism is an epimorphism if and only if it is surjective.*

7.3 Inner Automorphisms

Proposition 7.22. *Let G be a group and $g \in G$. The function $\gamma_g : G \rightarrow G$ defined by $\gamma_g(a) = gag^{-1}$ is an automorphism on G .*

PROOF:

$\langle 1 \rangle 1.$ γ_g is a homomorphism.

PROOF:

$$\begin{aligned}\gamma_g(ab) &= gabg^{-1} \\ &= gag^{-1}gbg^{-1} \\ &= \gamma_g(a)\gamma_g(b)\end{aligned}$$

<1>2. γ_g is injective.

PROOF: By Cancellation.

<1>3. γ_g is surjective.

PROOF: Given $b \in G$, we have $\gamma_g(g^{-1}bg) = b$.

□

Definition 7.23 (Inner Automorphism). Let G be a group. An *inner automorphism* on G is a function of the form $\gamma_g(a) = gag^{-1}$ for some $g \in G$.

We write $\text{Inn}(G)$ for the set of inner automorphisms of G .

Proposition 7.24. Let G be a group. The function $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$ that maps g to γ_g is a group homomorphism.

PROOF: Since $\gamma_{gh}(a) = ghah^{-1}g^{-1} = \gamma_g(\gamma_h(a))$. □

Corollary 7.24.1. $\text{Inn}(G)$ is a subgroup of $\text{Aut}_{\mathbf{Grp}}(G)$.

7.4 Direct Products

Definition 7.25 (Direct Product). The *direct product* of groups G and H is their product in \mathbf{Grp} .

7.5 Free Groups

Proposition 7.26. Let A be a set. Let \mathcal{F}^A be the category whose objects are pairs (G, j) where G is a group and j is a function $A \rightarrow G$, with morphisms $f : (G, j) \rightarrow (H, k)$ the group homomorphisms $f : G \rightarrow H$ such that $f \circ j = k$. Then \mathcal{F}^A has an initial object.

PROOF:

<1>1. LET: $W(A)$ be the set of words in the alphabet whose elements are the elements of A together with $\{a^{-1} : a \in A\}$.

<1>2. LET: $r : W(A) \rightarrow W(A)$ be the function that, given a word w , removes the first pair of letters of the form aa^{-1} or $a^{-1}a$; if there is no such pair, then $r(w) = w$.

<1>3. Let us say that a word w is a *reduced word* iff $r(w) = w$.

<1>4. For any word w of length n , we have $r^{\lceil \frac{n}{2} \rceil}(w)$ is a reduced word.

PROOF: Since we cannot remove more than $n/2$ pairs of letters from w .

<1>5. LET: $R : W(A) \rightarrow W(A)$ be the function $R(w) = r^{\lceil \frac{n}{2} \rceil}(w)$, where n is the length of w .

<1>6. LET: $F(A)$ be the set of reduced words.

<1>7. Define $\cdot : F(A)^2 \rightarrow F(A)$ by $w \cdot w' = R(ww')$

(1)8. \cdot is associative.

PROOF: Both $w_1 \cdot (w_2 \cdot w_3)$ and $(w_1 \cdot w_2) \cdot w_3$ are equal to $R(w_1 w_2 w_3)$.

(1)9. The empty word is the identity element in $F(A)$

(1)10. The inverse of $a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}$ is $a_n^{\mp 1} \dots a_2^{\mp 1} a_1^{\mp 1}$.

(1)11. LET: $j : A \rightarrow F(A)$ be the function that maps a to the word a of length

(1)12. LET: G be any group and $k : A \rightarrow G$ any function.

(1)13. The only morphism $f : (F(A), j) \rightarrow (G, k)$ in \mathcal{F}^A is $f(a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}) = k(a_1)^{\pm 1} k(a_2)^{\pm 1} \dots k(a_n)^{\pm 1}$.

□

Definition 7.27 (Free Group). For any set A , the *free group* on A is the initial object $(F(A), i)$ in \mathcal{F}^A .

Proposition 7.28. $i : A \rightarrow F(A)$ is injective.

PROOF:

(1)1. LET: $x, y \in A$

(1)2. ASSUME: $x \neq y$

PROVE: $i(x) \neq i(y)$

(1)3. LET: $f : A \rightarrow C_2$ be the function that maps x to 0 and all other elements of A to 1.

(1)4. LET: $\phi : F(A) \rightarrow C_2$ be the group homomorphism such that $f = \phi \circ i$.

(1)5. $f(x) \neq f(y)$

(1)6. $\phi(i(x)) \neq \phi(i(y))$

(1)7. $i(x) \neq i(y)$

□

Proposition 7.29.

$$F(0) \cong \{e\}$$

PROOF: For any set A , the unique group homomorphism $\{e\} \rightarrow A$ makes the following diagram commute.

$$\begin{array}{ccc} \{e\} & \longrightarrow & A \\ \uparrow & \nearrow & \\ \emptyset & & \end{array}$$

Proposition 7.30. The free group on 1 is \mathbb{Z} with the injection mapping 0 to 1.

PROOF: Given any group G and function $a : 1 \rightarrow G$, the required unique homomorphism $\phi : \mathbb{Z} \rightarrow G$ is defined by $\phi(n) = a(0)^n$. □

Proposition 7.31. For any sets A and B , we have that $F(A + B)$ is the coproduct of $F(A)$ and $F(B)$ in **Grp**.

$$\begin{array}{ccccc}
& & G & & \\
& f \nearrow & \uparrow k & \nwarrow g & \\
F(A) & \xrightarrow{\kappa_1} & F(A+B) & \xleftarrow{\kappa_2} & F(B) \\
i_A \uparrow & & j \uparrow & & i_B \uparrow \\
A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
\end{array}$$

PROOF:

- (1)1. LET: $i_A : A \rightarrow F(A)$, $i_B : B \rightarrow F(B)$, $j : A+B \rightarrow F(A+B)$ be the canonical injections.
 (1)2. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
 (1)3. LET: G be any group and $f : F(A) \rightarrow G$, $g : F(B) \rightarrow G$ any group homomorphisms.
 (1)4. LET: $h : A+B \rightarrow G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.
 (1)5. LET: $k : F(A+B) \rightarrow G$ be the unique group homomorphism such that $k \circ j = h$.
 (1)6. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
 (1)7. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.
 \square

Definition 7.32 (Subgroup Generated by a Group). Let G be a group and A a subset of G . Let $\phi : F(A) \rightarrow G$ be the unique group homomorphism such that $\phi(a) = a$ for all $a \in A$. The subgroup *generated* by A is

$$\langle A \rangle := \text{im } \phi$$

$$\begin{array}{ccc}
F(A) & \xrightarrow{\phi} & G \\
\uparrow & \nearrow & \\
A & &
\end{array}$$

Proposition 7.33. Let G be a group and A a subset of G . Then $\langle A \rangle$ is the set of all elements of the form $a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$ (where $n \geq 0$) such that $a_1, \dots, a_n \in A$.

PROOF: Immediate from definitions. \square

Corollary 7.33.1. Let G be a group and $g \in G$. Then

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\}.$$

Proposition 7.34. Let G be a group and A a subset of G . Then $\langle A \rangle$ is the intersection of all the subgroups of G that include A .

PROOF: Easy. \square

Definition 7.35 (Finitely Generated). Let G be a group. Then G is *finitely generated* iff there exists a finite subset A of G such that $G = \langle A \rangle$.

Proposition 7.36. *Every subgroup of a finitely generated free group is free.*

PROOF: TODO.

Proposition 7.37. *$F(2)$ includes subgroups isomorphic to the free group on arbitrarily many generators.*

PROOF: TODO

Proposition 7.38.

$$[F(2), F(2)] \cong F(\mathbb{Z})$$

PROOF: TODO

7.6 Normal Subgroups

Definition 7.39 (Normal Subgroup). A subgroup N of G is *normal* iff, for all $g \in G$ and $n \in N$, we have $gng^{-1} \in N$.

Example 7.40. Every subgroup of Q_8 is normal.

Proposition 7.41. *Let G be a group and N a subgroup of G . Then the following are equivalent.*

1. N is normal.
2. $\forall g \in G. gNg^{-1} \subseteq N$
3. $\forall g \in G. gNg^{-1} = N$
4. $\forall g \in G. gN \subseteq Ng$
5. $\forall g \in G. gN = Ng$

PROOF:

$\langle 1 \rangle 1. 1 \Leftrightarrow 2$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: If 2 holds then we have $gNg^{-1} \subseteq N$ and $g^{-1}Ng \subseteq N$ hence $N = gNg^{-1}$.

$\langle 1 \rangle 3. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 4. 2 \Leftrightarrow 4$

PROOF: Easy.

$\langle 1 \rangle 5. 3 \Leftrightarrow 5$

PROOF: Easy.

□

Proposition 7.42. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\ker \phi$ is a normal subgroup of G .*

PROOF: Given $g \in G$ and $n \in \ker \phi$ we have

$$\begin{aligned}\phi(gng^{-1}) &= \phi(g)\phi(n)\phi(g)^{-1} \\ &= \phi(g)\phi(g)^{-1} \\ &= e\end{aligned}$$

and so $gng^{-1} \in \ker \phi$. □

7.7 Quotient Groups

Definition 7.43. Let G be a group. Let \sim be an equivalence relation on G . Then we say that \sim is *compatible* with the group operation on G iff, for all $a, a', g \in G$, if $a \sim a'$ then $ga \sim ga'$ and $ag \sim a'g$.

Proposition 7.44. *Let G be a group. Let \sim be an equivalence relation on G . Then there exists an operation $\cdot : (G/\sim)^2 \rightarrow G/\sim$ such that*

$$\forall a, b \in G. [a][b] = [ab]$$

iff \sim is compatible with the group operation on G . In this case, G/\sim is a group under \cdot and the canonical function $\pi : G \rightarrow G/\sim$ is a group homomorphism, and is universal with respect to group homomorphisms $\phi : G \rightarrow G'$ such that if $a \sim a'$ then $\phi(a) = \phi(a')$.

PROOF: Easy. □

Definition 7.45 (Quotient Group). Let G be a group. Let \sim be an equivalence relation on G that is compatible with the group operation on G . Then G/\sim is the *quotient group* of G by \sim under $[a][b] = [ab]$.

Proposition 7.46. *Let G be a group and H a subgroup of G . Then H is normal if and only if there exists a group K and homomorphism $\phi : G \rightarrow K$ such that $H = \ker \phi$.*

PROOF: One direction is given by Proposition 7.42. For the other direction, take $K = G/H$ and ϕ to be the canonical map $G \rightarrow G/H$. □

Definition 7.47 (Modular Group). The *modular group* $\text{PSL}_2(\mathbb{Z})$ is $\text{SL}_2(\mathbb{Z})/\{I, -I\}$.

Proposition 7.48. $\text{PSL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

PROOF: By Example 6.26.

Proposition 7.49 (Roger Alperin). $\text{PSL}_2(\mathbb{Z})$ is presented by $(x, y | x^2, y^3)$.

PROOF:

⟨1⟩1. LET: $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

⟨1⟩2. LET: $y = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

⟨1⟩3. Define an action of $\text{PSL}_2(\mathbb{Z})$ on $\mathbb{R} - \mathbb{Q}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} r = \frac{ar + b}{cr + d}.$$

⟨2⟩1. Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z})$ and r irrational we have $\frac{ar+b}{cr+d}$ is irrational.

⟨3⟩1. ASSUME: for a contradiction $\frac{ar+b}{cr+d} = \frac{p}{q}$ where p and q are integers with $q > 0$.

⟨3⟩2. $aqr + bq = cpr + dp$

⟨3⟩3. $(aq - cp)r = dp - bq$

⟨3⟩4. $aq = cp = dp - bq = 0$

⟨3⟩5. $adq - cdp = 0$

⟨3⟩6. $cdp - cbq = 0$

⟨3⟩7. $(ad - cb)q = 0$

PROOF: Since $ad - cb = 1$.

⟨3⟩8. $q = 0$

⟨3⟩9. Q.E.D.

PROOF: This contradicts ⟨3⟩1.

⟨2⟩2. $-Ir = r$

PROOF: Since $-Ir = \frac{-r}{-1} = r$.

⟨2⟩3. Given $A, B \in \text{PSL}_2(\mathbb{Z})$ we have $A(Br) = (AB)r$.

PROOF:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} r \right] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{er + f}{gr + h} \\ &= \frac{a \frac{er+f}{gr+h} + b}{c \frac{er+f}{gr+h} + d} \\ &= \frac{a(er + f) + b(gr + h)}{c(er + f) + d(gr + h)} \\ &= \frac{(ae + bg)r + (af + bh)}{(ce + dg)r + (cf + dh)} \\ &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} r \\ &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right] r \end{aligned}$$

⟨1⟩4.

$$yr = 1 - \frac{1}{r}$$

⟨1⟩5.

$$y^{-1}r = \frac{1}{1 - r}$$

PROOF: Since $y^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$
 (1)6.

PROOF: Since $yx = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. $yxr = 1 + r$
 (1)7.

$y^{-1}xr = \frac{r}{1+r}$
 PROOF: Since $y^{-1}x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

(1)8. If $r > -1$ is positive then yxr is positive.

(1)9. If r is positive then $y^{-1}xr$ is positive.

(1)10. If $r < -1$ then $y^{-1}xr$ is positive.

(1)11. If r is negative then yr is positive.

(1)12. If r is negative then $y^{-1}r$ is positive.

(1)13. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)$$

with one or more factors can equal the identity.

PROOF: If the last factor is (yx) , then the product maps numbers in $(-1, 0)$ to positive numbers. If the last factor is $(y^{-1}x)$, then the product maps numbers < -1 to positive numbers.

(1)14. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)y^{\pm 1}$$

with one or more factors can equal the identity.

PROOF: The product maps negative numbers to positive numbers.

(1)15. $\text{PSL}_2(\mathbb{Z})$ is presented by $(x, y|x^2, y^3)$.

□

Corollary 7.49.1. $\text{PSL}_2(\mathbb{Z})$ is the coproduct of C_2 and C_3 in **Grp**.

Theorem 7.50. Every group homomorphism $\phi : G \rightarrow H$ may be decomposed as

$$G \longrightarrow G/\ker \phi \xrightarrow{\cong} \text{im } \phi \longrightarrow H$$

PROOF: Easy. □

Corollary 7.50.1 (First Isomorphism Theorem). Let $\phi : G \rightarrow H$ be a surjective group homomorphism. Then $H \cong G/\ker \phi$.

Proposition 7.51. Let H_1 be a normal subgroup of G_1 and H_2 a normal subgroup of G_2 . Then $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$, and

$$\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}.$$

PROOF: $\pi \times \pi : G_1 \times G_2 \twoheadrightarrow G_1/H_1 \times G_2/H_2$ is a surjective homomorphism with kernel $H_1 \times H_2$. □

Example 7.52.

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

PROOF: Map a real number r to $(\cos r, \sin r)$. The result is a surjective group homomorphism with kernel \mathbb{Z} . \square

Proposition 7.53. *Let H be a normal subgroup of a group G . For every subgroup K of G that includes H , we have H is a normal subgroup of K , and K/H is a subgroup of G/H . The mapping*

$$u : \{\text{subgroups of } G \text{ including } H\} \rightarrow \{\text{subgroups of } G/H\}$$

with $u(K) = K/H$ is a poset isomorphism.

PROOF:

- $\langle 1 \rangle 1$. If K is a subgroup of G that includes H then H is normal in K .
- $\langle 1 \rangle 2$. If K is a subgroup of G that includes H then K/H is a subgroup of G/H .
- $\langle 1 \rangle 3$. If $H \subseteq K_1 \subseteq K_2$ then $K_1/H \subseteq K_2/H$.
- $\langle 1 \rangle 4$. If $K_1/H = K_2/H$ then $K_1 = K_2$
 - $\langle 2 \rangle 1$. ASSUME: $K_1/H = K_2/H$
 - $\langle 2 \rangle 2$. $K_1 \subseteq K_2$
 - $\langle 3 \rangle 1$. LET: $k \in K_1$
 - $\langle 3 \rangle 2$. $kH \in K_2/H$
 - $\langle 3 \rangle 3$. PICK $k' \in K_2$ such that $kH = k'H$
 - $\langle 3 \rangle 4$. $kk'^{-1} \in H$
 - $\langle 3 \rangle 5$. $kk'^{-1} \in K_2$
 - $\langle 3 \rangle 6$. $k \in K_2$
 - $\langle 2 \rangle 3$. $K_2 \subseteq K_1$
- PROOF: Similar.
- $\langle 1 \rangle 5$. For any subgroup L of G/H , there exists a subgroup K of G that includes H such that $L = K/H$.
 - $\langle 2 \rangle 1$. LET: L be a subgroup of G/H .
 - $\langle 2 \rangle 2$. LET: $K = \{k \in G : kH \in L\}$
 - $\langle 2 \rangle 3$. K is a subgroup of G .
 - PROOF: Given $k, k' \in K$ we have $kH, k'H \in L$ hence $kk'^{-1}H \in L$ and so $kk'^{-1} \in K$.
 - $\langle 2 \rangle 4$. $H \subseteq K$
 - PROOF: For all $h \in H$ we have $hH = H \in L$.
 - $\langle 2 \rangle 5$. $L = K/H$
 - PROOF: By definition.

\square

Proposition 7.54 (Third Isomorphism Theorem). *Let H be a normal subgroup of a group G . Let N be a subgroup of G that includes H . Then N/H is normal in G/H if and only if N is normal in G , in which case*

$$\frac{G/H}{N/H} \cong \frac{G}{N}$$

PROOF:

- ⟨1⟩1. If N/H is normal in G/H then N is normal in G .
 - ⟨2⟩1. ASSUME: N/H is normal in G/H .
 - ⟨2⟩2. LET: $g \in G$ and $n \in N$.
 - ⟨2⟩3. $gng^{-1}H \in N/H$
 - ⟨2⟩4. PICK $n' \in N$ such that $gng^{-1}H = n'H$
 - ⟨2⟩5. $gng^{-1}n'^{-1} \in H$
 - ⟨2⟩6. $gng^{-1}n'^{-1} \in N$
 - ⟨2⟩7. $gng^{-1} \in N$
- ⟨1⟩2. If N is normal in G then N/H is normal in G/H and $(G/H)/(N/H) \cong G/N$.
 - ⟨2⟩1. ASSUME: N is normal in G .
 - ⟨2⟩2. LET: $\phi : G/H \rightarrow G/N$ be the homomorphism $\phi(gH) = gN$
 - ⟨3⟩1. If $gH = g'H$ then $gN = g'N$
 PROOF: If $gg'^{-1} \in H$ then $gg'^{-1} \in N$.
 - ⟨3⟩2. $\phi((gH)(g'H)) = \phi(gH)\phi(g'H)$
 PROOF: Both are $gg'N$.
 - ⟨2⟩3. ϕ is surjective.
 - ⟨2⟩4. $\ker \phi = N/H$
 - ⟨2⟩5. $(G/H)/(N/H) \cong G/N$
 PROOF: By the First Isomorphism Theorem.

□

Proposition 7.55 (Second Isomorphism Theorem). *Let H and K be subgroups of a group G . Assume that H is normal in G . Then:*

- 1. HK is a subgroup of G , and H is normal in HK .
- 2. $H \cap K$ is normal in K , and

$$\frac{HK}{H} \cong \frac{K}{H \cap K}.$$

PROOF:

- ⟨1⟩1. HK is a subgroup of G .
 PROOF: Since $hkh'k' = hh'(h'^{-1}kh')k' \in HK$.
- ⟨1⟩2. H is normal in HK .
- ⟨1⟩3. $H \cap K$ is normal in K and $HK/H \cong K/(H \cap K)$
 PROOF: The function that maps k to kH is a surjective homomorphism $K \twoheadrightarrow HK/H$ with kernel $H \cap K$. Surjectivity follows because $hkh = hkh^{-1}H$.

□

See also Proposition 7.70 for a result that holds even if H is not normal.

7.8 Cosets

Proposition 7.56. *Let G be a group. Let \sim be an equivalence relation on G such that, for all $a, b, g \in G$, if $a \sim b$ then $ga \sim gb$. Let $H = \{h \in G : h \sim e\}$.*

Then H is a subgroup of G and, for all $a, b \in G$, we have

$$a \sim b \Leftrightarrow a^{-1}b \in H \Leftrightarrow aH = bH .$$

PROOF:

$\langle 1 \rangle 1.$ $e \in H$

$\langle 1 \rangle 2.$ For all $x, y \in H$ we have $xy^{-1} \in H$.

$\langle 2 \rangle 1.$ ASSUME: $x \sim e$ and $y \sim e$.

$\langle 2 \rangle 2.$ $e \sim y^{-1}$

PROOF: Since $yy^{-1} \sim ey^{-1}$.

$\langle 2 \rangle 3.$ $xy^{-1} \sim e$

PROOF: Since $xy^{-1} \sim ey^{-1} \sim e$.

$\langle 1 \rangle 3.$ If $a \sim b$ then $a^{-1}b \in H$.

PROOF: If $a \sim b$ then $a^{-1}b \sim a^{-1}a = e$.

$\langle 1 \rangle 4.$ If $a^{-1}b \in H$ then $aH = bH$.

$\langle 2 \rangle 1.$ ASSUME: $a^{-1}b \in H$

$\langle 2 \rangle 2.$ $bH \subseteq aH$

PROOF: For any $h \in H$ we have $bh = aa^{-1}bh \in aH$.

$\langle 2 \rangle 3.$ $aH \subseteq bH$

PROOF: Similar since $b^{-1}a \in H$.

$\langle 1 \rangle 5.$ If $aH = bH$ then $a \sim b$.

$\langle 2 \rangle 1.$ ASSUME: $aH = bH$

$\langle 2 \rangle 2.$ PICK $h \in H$ such that $a = bh$.

$\langle 2 \rangle 3.$ $b^{-1}a = h$

$\langle 2 \rangle 4.$ $b^{-1}a \in H$

$\langle 2 \rangle 5.$ $b^{-1}a \sim e$

$\langle 2 \rangle 6.$ $a \sim b$

PROOF: $a = bb^{-1}a \sim be = b$.

□

Definition 7.57 (Coset). Let G be a group and H a subgroup of G . A *left coset* of H is a set of the form aH for $a \in G$. A *right coset* of H is a set of the form Ha for some $a \in G$.

We write G/H for the set of all left cosets of H , and $G \backslash H$ for the set of all right cosets of H .

Proposition 7.58.

$$G/H \cong G \backslash H$$

PROOF: The function that maps aH to Ha^{-1} is a bijection. □

Proposition 7.59. Let G be a group and H a subgroup of G . Define \sim_H on G by: $a \sim b$ iff $a^{-1}b \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations \sim on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ga \sim gb$. The equivalence class of a is aH .

PROOF:

$\langle 1 \rangle 1.$ For any subgroup H , we have \sim_H is an equivalence relation on G .

⟨2⟩1. \sim is reflexive.

PROOF: For any $a \in G$ we have $a^{-1}a = e \in H$.

⟨2⟩2. \sim is symmetric.

PROOF: If $a^{-1}b \in H$ then $b^{-1}a \in H$.

⟨2⟩3. \sim is transitive.

PROOF: If $a^{-1}b \in H$ and $b^{-1}c \in H$ then $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$.

⟨1⟩2. If $a \sim_H b$ then $ga \sim_H gb$.

PROOF: If $a^{-1}b \in H$ then $(ga)^{-1}(gb) = a^{-1}g^{-1}gb = a^{-1}b \in H$.

⟨1⟩3. For any equivalence relation \sim on G such that, whenever $a \sim b$, then $ga \sim gb$, there exists a subgroup H such that $\sim = \sim_H$.

PROOF: Proposition 7.56.

⟨1⟩4. The \sim_H -equivalence class of a is aH .

PROOF:

$$\begin{aligned} a \sim b &\Leftrightarrow a^{-1}b \in H \\ &\Leftrightarrow \exists h \in H. a^{-1}b = h \\ &\Leftrightarrow \exists h \in H. b = ah \\ &\Leftrightarrow b \in aH \end{aligned}$$

□

Proposition 7.60. Let G be a group and H a subgroup of G . Define \sim_H on G by: $a \sim b$ iff $ab^{-1} \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations \sim on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ag \sim bg$. The equivalence class of a is Ha .

PROOF: Similar. □

Proposition 7.61. Let G be a group and H be a subgroup of G . Define \sim_L and \sim_R on G by:

$$a \sim_L b \Leftrightarrow a^{-1}b \in H, \quad a \sim_R b \Leftrightarrow ab^{-1} \in H.$$

Then $\sim_L = \sim_R$ if and only if H is normal.

PROOF:

⟨1⟩1. If $\sim_L = \sim_R$ then H is normal.

⟨2⟩1. ASSUME: $\sim_L = \sim_R$

⟨2⟩2. LET: $h \in H$ and $g \in G$

⟨2⟩3. $g \sim_L gh^{-1}$

⟨2⟩4. $g \sim_R gh^{-1}h$

⟨2⟩5. $ghg^{-1} \in H$

⟨1⟩2. If H is normal then $\sim_L = \sim_R$.

⟨2⟩1. ASSUME: H is normal.

⟨2⟩2. If $a \sim_L b$ then $a \sim_R b$.

⟨3⟩1. ASSUME: $a \sim_L b$

⟨3⟩2. $a^{-1}b \in H$

⟨3⟩3. $aa^{-1}ba^{-1} \in H$

⟨3⟩4. $ba^{-1} \in H$

- $\langle 3 \rangle 5. a \sim_R b$
 $\langle 2 \rangle 3. \text{ If } a \sim_R b \text{ then } a \sim_L b.$

PROOF: Similar.

□

Corollary 7.61.1. *Let G be a group and H be a normal subgroup of G . Define \sim on G by $a \sim b$ iff $a^{-1}b \in H$. Then G/\sim is a group under $[a][b] = [ab]$.*

Definition 7.62 (Quotient Group). Let G be a group and H be a normal subgroup of G . The *quotient group* G/H is G/\sim where $a \sim b$ iff $a^{-1}b \in H$, under $[a][b] = [ab]$ or $(aH)(bH) = abH$.

Corollary 7.62.1. *Let H be a normal subgroup of a group G . For every group homomorphism $\phi : G \rightarrow G'$ such that $H \subseteq \ker \phi$, there exists a unique group homomorphism $\bar{\phi} : G/H \rightarrow G'$ such that the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ & \searrow \pi & \nearrow \bar{\phi} \\ & G/H & \end{array}$$

Proposition 7.63. $\mathbb{Z}/n\mathbb{Z}$ has exactly n elements.

PROOF: Every integer is congruent to one of $0, 1, \dots, n-1$ by the division algorithm, and no two of them are congruent to one another, since if $0 \leq i < j < n$ then $0 < j - i < n$. □

Proposition 7.64. *Let m and n be integers with $n > 0$. The order of m in $\mathbb{Z}/n\mathbb{Z}$ is $\frac{n}{\gcd(m,n)}$.*

PROOF: By Proposition 6.16 since the order of 1 is n . □

Proposition 7.65. *The integer m generates $\mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(m, n) = 1$.*

PROOF: By Proposition 7.64. □

Corollary 7.65.1. *If p is prime then every non-zero element in $\mathbb{Z}/p\mathbb{Z}$ is a generator.*

Proposition 7.66.

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$$

PROOF: Every permutation of $\{(1, 0), (0, 1), (1, 1)\}$ gives an automorphism of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

Example 7.67. Not all monomorphisms split in \mathbf{Grp} .

Define $\phi : \mathbb{Z}/3\mathbb{Z} \rightarrow S_3$ by

$$\phi(0) = \text{id}_3, \quad \phi(1) = (1 \ 3 \ 2), \quad \phi(2) = (1 \ 2 \ 3) .$$

Then ϕ is monic but has no retraction.

For if $r : S_3 \rightarrow \mathbb{Z}/3\mathbb{Z}$ is a retraction, then we would have

$$r(1\ 2) + r(2\ 3) = 1, \quad r(2\ 3) + r(1\ 2) = 2$$

which is impossible.

Proposition 7.68. *Let G be a group, H a subgroup of G , and $g \in G$. The function that maps h to gh is a bijection $H \cong gH$.*

PROOF: By Cancellation. \square

Proposition 7.69. *Let G be a group, H a subgroup of G , and $g \in G$. The function that maps h to hg is a bijection $H \cong Hg$.*

PROOF: By Cancellation. \square

Proposition 7.70. *Let H and K be finite subgroups of a group G . Then*

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

PROOF:

$\langle 1 \rangle 1$. LET: $f : \{hK : h \in H\} \rightarrow H/(H \cap K)$ be the function $f(hK) = h(H \cap K)$

PROOF: This is well-defined because if $hK = h'K$ then $h^{-1}h' \in H \cap K$ so $h(H \cap K) = h'(H \cap K)$.

$\langle 1 \rangle 2$. f is injective.

PROOF: If $h(H \cap K) = h'(H \cap K)$ then $hK = h'K$.

$\langle 1 \rangle 3$. f is surjective.

PROOF: Clear.

$\langle 1 \rangle 4$.

$$\frac{|HK|}{|K|} = \frac{|H|}{|H \cap K|}$$

\square

7.9 Congruence

Definition 7.71 (Congruence). Given integers a, b, n with n positive, we say a is *congruent to b modulo n* , and write $a \equiv b \pmod{n}$, iff $a + n\mathbb{Z} = b + n\mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$.

Proposition 7.72. *Given integers a, b, n with n positive, we have $a \equiv b \pmod{n}$ iff $n \mid a - b$.*

PROOF: By Proposition 7.56. \square

Proposition 7.73. *If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then $a + b \equiv a' + b' \pmod{n}$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid (a' + b') - (a + b)$. \square

Proposition 7.74. *If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then $ab \equiv a'b' \pmod{n}$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid a'b' - ab = a'(b' - b) + (a' - a)b$. \square

7.10 Cyclic Groups

Definition 7.75 (Cyclic Group). The *cyclic* groups are \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$ for positive integers n .

Proposition 7.76. *If m and n are positive integers with $\gcd(m, n) = 1$ then $C_{mn} \cong C_m \times C_n$.*

PROOF: The function that maps x to $(x \bmod m, x \bmod n)$ is an isomorphism. \square

Proposition 7.77. *Let G be a group and $g \in G$. Then $\langle g \rangle$ is cyclic.*

PROOF: If g has finite order then $\langle g \rangle \cong C_{|g|}$, otherwise $\langle g \rangle \cong \mathbb{Z}$. \square

Proposition 7.78. *Every finitely generated subgroup of \mathbb{Q} is cyclic.*

PROOF:

$\langle 1 \rangle$ 1. LET: $G = \langle a_1/b, \dots, a_n/b \rangle$ where a_1, \dots, a_n, b are integers with $b > 0$

$\langle 1 \rangle$ 2. LET: $a = \gcd(a_1, \dots, a_n)$

$\langle 1 \rangle$ 3. $G = \langle a/b \rangle$

\square

Corollary 7.78.1. \mathbb{Q} is not finitely generated.

7.11 Commutator Subgroup

Definition 7.79 (Commutator Subgroup). Let G be a group. The *commutator subgroup* $[G, G]$ is the subgroup generated by the elements of the form $aba^{-1}b^{-1}$.

Proposition 7.80. *The commutator subgroup is normal.*

PROOF: Since

$$\begin{aligned} & ga_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_nb_na_n^{-1}b_n^{-1}g^{-1} \\ &= (ga_1g^{-1})(gb_1g^{-1})(ga_1g^{-1})^{-1}(gb_1g^{-1})^{-1} \cdots (ga_ng^{-1})(gb_ng^{-1})(ga_ng^{-1})^{-1}(gb_ng^{-1})^{-1}. \end{aligned} \quad \square$$

7.12 Presentations

Definition 7.81 (Presentation). A *presentation* of a group G is a pair (A, R) where A is a set and $R \subseteq F(A)$ is a set of words such that

$$G \cong F(A)/N(R)$$

where $N(R)$ is the smallest normal subgroup of $F(A)$ that includes R .

Example 7.82. • The free group on a set A is presented by (A, \emptyset) .

- S_3 is presented by $(x, y | x^2, y^3, xyxy)$.
- $(a, b | a^2, b^2, (ab)^n)$ is a presentation of D_{2n} .

- $(x, y \mid x^2y^{-2}, y^4, xyx^{-1}y)$ is a presentation of Q_8 .

Proposition 7.83 (Word Problem). *Let (A, R) be a presentation of the group G . Let $w_1, w_2 \in F(A)$ be two words. Then it is undecidable in general if $w_1N(R) = w_2N(R)$ in G .*

Definition 7.84 (Finitely Presented). A group is *finitely presented* iff it has a presentation (A, R) where both A and R are finite.

Proposition 7.85. *Let $(A|R)$ be a presentation of G and $(A'|R')$ a presentation of H . Assume w.l.o.g. A and A' are disjoint. Then the group $G * G'$ presented by $(A \cup A' | R \cup R')$ is the coproduct of G and G' in \mathbf{Grp} .*

$$\begin{array}{ccccc}
 A & \longrightarrow & A \cup A' & \longleftarrow & A' \\
 \downarrow & & \downarrow & & \downarrow \\
 F(A) & \longrightarrow & F(A \cup A') & \longleftarrow & F(A') \\
 \downarrow & & \downarrow & & \downarrow \\
 G & \xrightarrow{\kappa_1} & G * G' & \xleftarrow{\kappa_2} & G'
 \end{array}$$

PROOF:

- $\langle 1 \rangle 1$. LET: $\kappa_1 : G \rightarrow G * G'$ and $\kappa_2 : G' \rightarrow G * G'$ be the unique homomorphisms that make the diagram above commute.
- $\langle 1 \rangle 2$. LET: $\phi : G \rightarrow H$ and $\psi : G' \rightarrow H$ be any homomorphisms.
- $\langle 1 \rangle 3$. LET: $[\phi, \psi] : F(A \cup A') \rightarrow H$ be the unique homomorphism such that ...
- $\langle 1 \rangle 4$. $R \cup R' \subseteq \ker[\phi, \psi]$
- $\langle 1 \rangle 5$. $[\phi, \psi]$ factors uniquely through the morphism $F(A \cup A') \rightarrow G * G'$

□

7.13 Index of a Subgroup

Definition 7.86 (Index). Let G be a group and H a subgroup of G . The *index* of H in G , denoted $|G : H|$, is the number of left cosets of H in G if this is finite, otherwise ∞ .

Theorem 7.87 (Lagrange's Theorem). *Let G be a finite group and H a subgroup of G . Then*

$$|G| = |G : H||H| .$$

PROOF: G/H is a partition of G into $|G : H|$ subsets, each of size $|H|$. □

Corollary 7.87.1. *For p a prime number, the only group of order p is C_p .*

PROOF: Let G be a group of order p and $g \in G$ with $g \neq e$. Then $|\langle g \rangle|$ divides p and is not 1, hence is p , that is, $G = \langle g \rangle$. □

Theorem 7.88 (Cauchy's Theorem). *Let G be a finite group. If p is prime and $p \mid |G|$ then the number of cyclic subgroups of order p is congruent to 1 modulo p . In particular, there exists an element of order p .*

PROOF:

$\langle 1 \rangle 1$. LET: $S = \{(a_1, a_2, \dots, a_p) \in G^p : a_1 a_2 \cdots a_p = e\}$

$\langle 1 \rangle 2$. $|S| = |G|^{p-1}$

PROOF: Given any $a_1, \dots, a_{p-1} \in G$, there exists a unique a_p such that $(a_1, \dots, a_p) \in S$, namely $a_p = (a_1 \cdots a_{p-1})^{-1}$.

$\langle 1 \rangle 3$. $p \mid |S|$

$\langle 1 \rangle 4$. Define an action of $\mathbb{Z}/p\mathbb{Z}$ on S by

$$m \cdot (a_1, \dots, a_p) = (a_m, a_{m+1}, \dots, a_p, a_1, a_2, \dots, a_{m-1}) .$$

PROOF: If $(a_1, \dots, a_p) \in S$ then $(a_2, a_3, \dots, a_p, a_1) \in S$ since $a_1 = (a_2 \cdots a_p)^{-1}$.

$\langle 1 \rangle 5$. LET: Z be the set of fixed points of this action.

$\langle 1 \rangle 6$. $|Z| \equiv 0 \pmod{p}$

PROOF: Corollary 9.18.1, $\langle 1 \rangle 3$.

$\langle 1 \rangle 7$. $Z = \{(a, a, \dots, a) : a^p = e\}$

$\langle 1 \rangle 8$. $Z \neq \emptyset$

PROOF: Since $(e, e, \dots, e) \in Z$.

$\langle 1 \rangle 9$. An element a has order p iff $(a, a, \dots, a) \in Z$ and $a \neq e$.

$\langle 1 \rangle 10$. LET: N be the number of cyclic subgroups of order p .

$\langle 1 \rangle 11$. The number of elements of order p is $N(p-1)$

$\langle 1 \rangle 12$. $|Z| = N(p-1) + 1$

$\langle 1 \rangle 13$. $-N + 1 \equiv 0 \pmod{p}$

PROOF: From $\langle 1 \rangle 6$.

$\langle 1 \rangle 14$. $N \equiv 1 \pmod{p}$

□

Proposition 7.89. *Let G be a group. Let K be a subgroup of G and H a subgroup of K . If $|G : H|$, $|G : K|$ and $|K : H|$ are all finite then*

$$|G : H| = |G : K| |K : H| .$$

PROOF:

$\langle 1 \rangle 1$. LET: $G/K = \{g_1 K, g_2 K, \dots, g_m K\}$

$\langle 1 \rangle 2$. LET: $K/H = \{k_1 H, k_2 H, \dots, k_n H\}$

$\langle 1 \rangle 3$. $G/H = \{g_i k_j H : 1 \leq i \leq m, 1 \leq j \leq n\}$

$\langle 2 \rangle 1$. LET: $g \in G$

$\langle 2 \rangle 2$. PICK i such that $gK = g_i K$

$\langle 2 \rangle 3$. $g^{-1} g_i \in K$

$\langle 2 \rangle 4$. PICK j such that $g^{-1} g_i H = k_j H$

$\langle 2 \rangle 5$. $g^{-1} g_i k_j \in H$

$\langle 2 \rangle 6$. $gH = g_i k_j H$

$\langle 1 \rangle 4$. If $g_i k_j H = g_{i'} k_{j'} H$ then $i = i'$ and $j = j'$.

$\langle 2 \rangle 1$. ASSUME: $g_i k_j H = g_{i'} k_{j'} H$

$\langle 2 \rangle 2$. $g_i K = g_{i'} K$

$\langle 2 \rangle 3$. $i = i'$

- $\langle 2 \rangle 4. k_j H = k_{j'} H$
 $\langle 2 \rangle 5. j = j'$

□

7.14 Cokernels

Proposition 7.90. *Let $\phi : G \rightarrow H$ be a homomorphism between groups. Then there exists a group K and homomorphism $\pi : H \rightarrow K$ that is initial with respect to all homomorphism $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.*

PROOF:

- $\langle 1 \rangle 1.$ LET: N be the intersection of all the normal subgroups of H that include $\text{im } \phi$.
 $\langle 1 \rangle 2.$ LET: $K = H/N$ and π be the canonical homomorphism.
 $\langle 1 \rangle 3.$ LET: $\pi \circ \phi = 0$
 $\langle 1 \rangle 4.$ LET: $\alpha : H \rightarrow L$ satisfy $\alpha \circ \phi = 0$
 $\langle 1 \rangle 5.$ $\text{im } \phi \subseteq \ker \alpha$
 $\langle 1 \rangle 6.$ $N \subseteq \ker \alpha$
 $\langle 1 \rangle 7.$ There exists a unique $\bar{\alpha} : H/\text{im } \phi \rightarrow L$ such that $\bar{\alpha} \circ \pi = \alpha$

□

Definition 7.91 (Cokernel). For any homomorphism $\phi : G \rightarrow H$ in **Grp**, the *cokernel* of ϕ is the group $\text{coker } \phi$ and homomorphism $\pi : H \rightarrow \text{coker } \phi$ that is initial among homomorphisms $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.

Example 7.92. It is not true that a homomorphism with trivial cokernel is epi. The inclusion $\langle (1 \ 2) \rangle \hookrightarrow S_3$ has trivial cokernel but is not epi.

7.15 Cayley Graphs

Definition 7.93 (Cayley Graph). Let G be a finitely generated group. Let A be a finite set of generators for G . The *Cayley graph* of G with respect to A is the directed graph whose vertices are the elements of G , with an edge $g_1 \rightarrow g_2$ labelled by $a \in A$ iff $g_2 = g_1 a$.

Proposition 7.94. *G is the free group on A iff the Cayley graph with respect to A is a tree.*

PROOF: Both are equivalent to saying that the product of two different strings of elements of A and/or their inverses are not equal. □

7.16 Characteristic Subgroups

Definition 7.95 (Characteristic Subgroup). Let G be a group. Let H be a subgroup of G . Then H is a *characteristic* subgroup of G iff, for every automorphism ϕ of G , we have $\phi(H) \subseteq H$.

Proposition 7.96. *Characteristic subgroups are normal.*

PROOF: Take ϕ to be conjugation with respect to an arbitrary element. \square

Proposition 7.97. *Let G be a group. Let K be a normal subgroup of G and H a characteristic subgroup of K . Then H is normal in G .*

PROOF: For any $a \in G$ we have conjugation by a is an automorphism on K , hence H is closed under it. \square

Proposition 7.98. *Let G be a group. Let H be a subgroup of G . Suppose there is no other subgroup of G isomorphic to H . Then H is characteristic, hence normal.*

PROOF: For any automorphism ϕ on G , we have $\phi(H)$ is isomorphic to H , hence $\phi(H) = H$. \square

Proposition 7.99. *Let G be a finite group. Let K be a normal subgroup of G . Assume $|K|$ and $|G/K|$ are relatively prime. Then K is characteristic.*

PROOF:

$\langle 1 \rangle 1$. LET: K' be a subgroup of G isomorphic to K .

PROVE: $K' = K$

$\langle 1 \rangle 2$. $|K'/(K \cap K')|$ divides both $|K'| = |K|$ and $|G/K|$

$\langle 1 \rangle 3$. $|K'/(K \cap K')| = 1$

$\langle 1 \rangle 4$. $K' = K \cap K'$

$\langle 1 \rangle 5$. $K' = K$

\square

7.17 Simple Groups

Definition 7.100 (Simple Group). A group G is *simple* iff its only normal subgroups are $\{e\}$ and G .

Proposition 7.101. *Let G be a group. Then G is simple if and only if the only homomorphic images of G are 1 and G .*

PROOF: Both are equivalent to saying that, for any surjective homomorphism $\phi : G \rightarrow G'$, either ϕ has kernel $\{e\}$ (in which case it is an isomorphism) or ϕ has kernel G (in which case $G' = 1$.) \square

7.18 Sylow Subgroups

Definition 7.102 (Sylow Subgroup). Let p be a prime number. Let G be a finite group. A *p-Sylow subgroup* of G is a subgroup of order p^r , where r is the largest integer such that p^r divides $|G|$.

Chapter 8

Abelian Groups

Definition 8.1 (Abelian Group). A group is *Abelian* iff any two elements commute.

In an Abelian group G , we often denote the group operation by $+$, the identity element by 0 and the inverse of an element g by $-g$. We write ng for g^n ($g \in G, n \in \mathbb{Z}$).

Example 8.2. Every group of order ≤ 4 is Abelian.

Example 8.3. For any positive integer n , we have $\mathbb{Z}/n\mathbb{Z}$ is an Abelian group under addition.

Example 8.4. S_n is not Abelian for $n \geq 3$. If $x = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ then $xy = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $yx = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$.

Example 8.5. There are 42 Abelian groups of order 1024 up to isomorphism.

Proposition 8.6. Let G be a group. If $g^2 = e$ for all $g \in G$ then G is Abelian.

PROOF: For any $g, h \in G$ we have

$$\begin{aligned} ghgh &= e \\ \therefore hgh &= g && \text{(multiplying on the left by } g) \\ \therefore hg &= gh && \text{(multiplying on the right by } h) \square \end{aligned}$$

Proposition 8.7. Let G be a group. Then G is Abelian if and only if the function that maps g to g^{-1} is a group homomorphism.

PROOF:

(1)1. If G is Abelian then the function that maps g to g^{-1} is a group homomorphism.

PROOF: Since $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$.

(1)2. If the function that maps g to g^{-1} is a group homomorphism then G is Abelian.

PROOF: Since $gh = (g^{-1})^{-1}(h^{-1})^{-1} = (g^{-1}h^{-1})^{-1} = hg$.
 \square

Proposition 8.8. *Let G be a group. Then G is Abelian if and only if the function that maps g to g^2 is a group homomorphism.*

PROOF:

$\langle 1 \rangle 1$. If G is Abelian then the function that maps g to g^2 is a group homomorphism.

PROOF: Since $(gh)^2 = g^2h^2$.

$\langle 1 \rangle 2$. If the function that maps g to g^2 is a group homomorphism then G is Abelian.

PROOF: Since we have $(gh)^2 = ghgh = g^2h^2$ and so $hg = gh$.

\square

Proposition 8.9. *Let G be a group. Then G is Abelian if and only if the homomorphism $\gamma : G \rightarrow \text{Aut}_{\text{Grp}}(G)$ is the trivial homomorphism.*

PROOF:

$\langle 1 \rangle 1$. If G is Abelian then γ is trivial.

PROOF: Since $\gamma_g(a) = gag^{-1} = a$.

$\langle 1 \rangle 2$. If γ is trivial then G is Abelian.

PROOF: If $\gamma_g(a) = gag^{-1} = a$ for all g and a then $ga = ag$ for all g, a .

\square

Proposition 8.10. *Let G be an Abelian group. Let $g, h \in G$. If g has maximal finite order in G , and h has finite order, then $|h| \mid |g|$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $|h| \nmid |g|$.

$\langle 1 \rangle 2$. PICK a prime p such that $|g| = p^m r$, $|h| = p^n s$ where $p \nmid r$, $p \nmid s$ and $m < n$.

$\langle 1 \rangle 3$. $|g^{p^m} h^s| = p^n r$

PROOF: Proposition 6.19.

$\langle 1 \rangle 4$. $|g| < |g^{p^m} h^s|$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts the maximality of $|g|$.

\square

Proposition 8.11. *Given a set A and an Abelian group H , the set H^A is an Abelian group under*

$$(\phi + \psi)(a) = \phi(a) + \psi(a) \quad (\phi, \psi \in H^A, a \in A) .$$

PROOF:

$\langle 1 \rangle 1$. $\phi + (\psi + \chi) = (\phi + \psi) + \chi$

$\langle 1 \rangle 2$. $\phi + \psi = \psi + \phi$

$\langle 1 \rangle 3$. LET: $0 : A \rightarrow H$ be the function $0(a) = 0$.

$\langle 1 \rangle 4$. $\phi + 0 = 0 + \phi = \phi$

$\langle 1 \rangle 5$. Given $\phi : A \rightarrow H$, define $-\phi : A \rightarrow H$ by $(-\phi)(a) = -(\phi(a))$.

$\langle 1 \rangle 6$. $\phi + (-\phi) = (-\phi) + \phi = 0$

□

Proposition 8.12. *Given a group G and an Abelian group H , the set $\mathbf{Grp}[G, H]$ is a subgroup of H^G .*

PROOF:

$\langle 1 \rangle 1$. Given $\phi, \psi : G \rightarrow H$ group homomorphisms, we have $\phi - \psi$ is a group homomorphism.

PROOF:

$$\begin{aligned} (\phi - \psi)(g + g') &= \phi(g + g') - \psi(g + g') \\ &= \phi(g) + \phi(g') - \psi(g) - \psi(g') \\ &= \phi(g) - \psi(g) + \phi(g') - \psi(g') \\ &= (\phi - \psi)(g) + (\phi - \psi)(g') \end{aligned}$$

□

Proposition 8.13. *Let G be a group. The following are equivalent.*

1. $\text{Inn}(G)$ is cyclic.
2. $\text{Inn}(G)$ is trivial.
3. G is Abelian.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: $\text{Inn}(G) = \langle \gamma_g \rangle$

$\langle 2 \rangle 2$. g commutes with every element of G

$\langle 3 \rangle 1$. LET: $x \in G$

$\langle 3 \rangle 2$. PICK $n \in \mathbb{Z}$ such that $\gamma_x = \gamma_g^n$

$\langle 3 \rangle 3$. $\forall y \in G. xyx^{-1} = g^n yg^{-n}$

$\langle 3 \rangle 4$. $xgx^{-1} = g$

$\langle 2 \rangle 3$. $\gamma_g = \text{id}_G$

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

$\langle 2 \rangle 1$. ASSUME: $\forall g \in G. \gamma_g = \text{id}_G$

$\langle 2 \rangle 2$. LET: $x, y \in G$

$\langle 2 \rangle 3$. $\gamma_x(y) = y$

$\langle 2 \rangle 4$. $xyx^{-1} = y$

$\langle 2 \rangle 5$. $xy = yx$

$\langle 1 \rangle 3$. $3 \Rightarrow 2$

PROOF: If $xy = yx$ for all x, y then $\gamma_x(y) = y$ for all x, y .

$\langle 1 \rangle 4$. $2 \Rightarrow 1$

PROOF: Easy.

□

Corollary 8.13.1. *If $\text{Aut}_{\mathbf{Grp}}(G)$ is cyclic then G is Abelian.*

Proposition 8.14. *Every subgroup of an Abelian group is normal.*

PROOF: Let G be an Abelian group and N a subgroup of G . Given $g \in G$ and $n \in N$ we have $gng^{-1} = n \in N$. \square

Proposition 8.15. *For any group G , the group $G/[G, G]$ is Abelian.*

PROOF: For any $g, h \in G$ we have

$$gh(hg)^{-1} \in [G, G]$$

$$\therefore gh[G, G] = hg[G, G] \quad \square$$

Proposition 8.16. *Let G be a finite Abelian group. Let p be a prime divisor of $|G|$. Then G has an element of order p .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: as induction hypothesis the result holds for all groups smaller than G .

$\langle 1 \rangle 2$. PICK $g \in G - \{0\}$.

$\langle 1 \rangle 3$. PICK an element $h \in \langle g \rangle$ with prime order q .

$\langle 1 \rangle 4$. CASE: $q = p$

PROOF: h is the required element.

$\langle 1 \rangle 5$. CASE: $q \neq p$

$\langle 2 \rangle 1$. PICK $r \in G$ such that $r + \langle h \rangle$ has order p in $G/\langle h \rangle$.

PROOF: By induction hypothesis since $|G/\langle h \rangle| = |G|/q$.

$\langle 2 \rangle 2$. $pr \in \langle h \rangle$

$\langle 2 \rangle 3$. PICK k such that $pr = kh$

$\langle 2 \rangle 4$. $pqr = e$

$\langle 2 \rangle 5$. qr has order p .

\square

Corollary 8.16.1. *For n an odd integer, any Abelian group of order $2n$ has exactly one element of order 2.*

PROOF: If x and y are distinct elements of order 2 then $\langle x, y \rangle = \{e, x, y, xy\}$ has size 4 and so $4 \mid 2n$ which is a contradiction. \square

Example 8.17. It is not true that, if G is a finite group and $d \mid |G|$, then G has an element of order d . The quaternionic group has no element of order 4.

Proposition 8.18. *If G is a finite Abelian group and $d \mid |G|$ then G has a subgroup of size d .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: as induction hypothesis the result is true for all $d' < d$.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $d \neq 1$.

$\langle 1 \rangle 3$. PICK a prime p such that $p \mid d$.

$\langle 1 \rangle 4$. PICK an element $g \in G$ of order p .

$\langle 1 \rangle 5$. $d/p \mid |G/\langle g \rangle|$

$\langle 1 \rangle 6$. PICK a subgroup H of $G/\langle g \rangle$ of size d/p .

$\langle 1 \rangle 7$. $\pi^{-1}(H)$ is a subgroup of G of size d .

\square

Proposition 8.19. *Let (G, \cdot) be a group. Let $\circ : G^2 \rightarrow G$ be a group homomorphism such that (G, \circ) is a group. Then \circ and \cdot coincide, and G is Abelian.*

PROOF:

$\langle 1 \rangle 1$. For all $g_1, g_2, h_1, h_2 \in G$ we have

$$(g_1 g_2) \circ (h_1 h_2) = (g_1 \circ h_1)(g_2 \circ h_2)$$

$\langle 1 \rangle 2$. $e \circ e = e$

PROOF:

$$\begin{aligned} e \circ e &= (ee) \circ (ee) \\ &= (e \circ e)(e \circ e) \end{aligned}$$

Hence $e \circ e = e$ by Cancellation.

$\langle 1 \rangle 3$. e is the identity of (G, \circ)

$\langle 1 \rangle 4$. For all $g, h \in G$ we have

$$g \circ h = gh$$

PROOF:

$$\begin{aligned} g \circ h &= (ge) \circ (eh) \\ &= (g \circ e)(e \circ h) \\ &= gh \end{aligned}$$

$\langle 1 \rangle 5$. For all $g, h \in G$ we have $gh = hg$.

PROOF:

$$\begin{aligned} gh &= (e \circ g)(h \circ e) \\ &= (eh) \circ (ge) \\ &= h \circ g \\ &= hg \end{aligned}$$

□

Corollary 8.19.1. *If $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$ is a group object in **Grp** then m is the multiplication of G , $e(*)$ is the identity of G , $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Grp** where $e(*) = e$ and $i(g) = g^{-1}$.*

Proposition 8.20. *Let G be a group. If every element of G has order ≤ 2 then G is Abelian.*

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in G$

PROVE: $xy = yx$

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $x \neq e \neq y$.

$\langle 1 \rangle 3$. $x^2 = e = y^2$

$\langle 1 \rangle 4$. $x^{-1} = x$ and $y^{-1} = y$.

$\langle 1 \rangle 5$. CASE: $xy = e$

PROOF: Then $y = x^{-1}$ and so $xy = yx = e$.

$\langle 1 \rangle 6$. CASE: $xy \neq e$

$\langle 2 \rangle 1$. $(xy)^2 = e$

$\langle 2 \rangle 2$. $xyxy = e$

$$\langle 2 \rangle 3. \quad xy = y^{-1}x^{-1}$$

$$\langle 2 \rangle 4. \quad xy = yx$$

□

8.1 The Category of Abelian Groups

Definition 8.21 (Category of Abelian Groups). Let **Ab** be the full subcategory of **Grp** whose objects are the Abelian groups.

Proposition 8.22. *If $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$ is a group object in **Ab** then m is the multiplication of G , $e(*)$ is the identity of G , $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Ab** where $e(*) = e$ and $i(g) = g^{-1}$.*

PROOF: Immediate from Corollary 8.19.1. □

Definition 8.23 (Direct Sum). Given Abelian groups G and H , we also call the direct product of G and H the *direct sum* and denote it $G \oplus H$.

Proposition 8.24. *Given Abelian groups G and H , the direct sum $G \oplus H$ is the coproduct of G and H in **Ab**.*

PROOF:

$\langle 1 \rangle 1.$ LET: $\kappa_1 : G \rightarrow G \oplus H$ be the group homomorphism $\kappa_1(g) = (g, e_H)$.

$\langle 1 \rangle 2.$ LET: $\kappa_2 : H \rightarrow G \oplus H$ be the group homomorphism $\kappa_2(h) = (e_G, h)$.

$\langle 1 \rangle 3.$ Given group homomorphism $\phi : G \rightarrow K$ and $\psi : H \rightarrow K$, define $[\phi, \psi] :$

$$G \oplus H \rightarrow K \text{ by } [\phi, \psi](g, h) = \phi(g) + \psi(h).$$

$\langle 1 \rangle 4.$ $[\phi, \psi]$ is a group homomorphism.

PROOF:

$$\begin{aligned} [\phi, \psi]((g, h) + (g', h')) &= [\phi, \psi](g + g', h + h') \\ &= \phi(g + g') + \psi(h + h') \\ &= \phi(g) + \phi(g') + \psi(h) + \psi(h') \\ &= \phi(g) + \psi(h) + \phi(g') + \psi(h') \\ &= [\phi, \psi](g, h) + [\phi, \psi](g', h') \end{aligned}$$

$\langle 1 \rangle 5.$ $[\phi, \psi] \circ \kappa_1 = \phi$

PROOF:

$$\begin{aligned} [\phi, \psi](\kappa_1(g)) &= [\phi, \psi](g, e_H) \\ &= \phi(g) + \psi(e_H) \\ &= \phi(g) + e_K \\ &= \phi(g) \end{aligned}$$

$\langle 1 \rangle 6.$ $[\phi, \psi] \circ \kappa_2 = \psi$

PROOF: Similar.

$\langle 1 \rangle 7.$ If $f : G \oplus H \rightarrow K$ is a group homomorphism with $f \circ \kappa_1 = \phi$ and $f \circ \kappa_2 = \psi$ then $f = [\phi, \psi]$.

PROOF:

$$\begin{aligned} f(g, h) &= f((g, e_H) + (e_G, h)) \\ &= f(\kappa_1(g)) + f(\kappa_2(h)) \\ &= \phi(g) + \psi(h) \end{aligned}$$

□

Theorem 8.25. *Every finitely generated Abelian group is a direct sum of cyclic groups.*

PROOF: TODO □

8.2 Free Abelian Groups

Proposition 8.26. *Let A be a set. Let \mathcal{F}^A be the category whose objects are pairs (G, j) where G is an Abelian group and j is a function $A \rightarrow G$, with morphisms $f : (G, j) \rightarrow (H, k)$ the group homomorphisms $f : G \rightarrow H$ such that $f \circ j = k$. Then \mathcal{F}^A has an initial object.*

PROOF:

- ⟨1⟩1. LET: $\mathbb{Z}^{\oplus A}$ be the subgroup of \mathbb{Z}^A consisting of all functions $\alpha : A \rightarrow \mathbb{Z}$ such that $\alpha(a) = 0$ for only finitely many $a \in A$.
- ⟨1⟩2. LET: $i : A \rightarrow \mathbb{Z}^{\oplus A}$ be the function such that $i(a)(b) = 1$ if $a = b$ and 0 if $a \neq b$.
- ⟨1⟩3. LET: G be any Abelian group and $j : A \rightarrow G$ any function.
- ⟨1⟩4. The unique homomorphism $\phi : \mathbb{Z}^{\oplus A} \rightarrow G$ required is defined by $\phi(\alpha) = \sum_{a \in A} \alpha(a)j(a)$

□

Definition 8.27 (Free Abelian Group). For any set A , the *free Abelian group* on A is the initial object $(F^{ab}(A), i)$ in \mathcal{F}^A .

Proposition 8.28. *For any sets A and B , we have that $F^{ab}(A + B)$ is the coproduct of $F^{ab}(A)$ and $F^{ab}(B)$ in **Grp**.*

$$\begin{array}{ccccc} & & G & & \\ & \nearrow f & \uparrow k & \nwarrow g & \\ F^{ab}(A) & \xrightarrow{\kappa_1} & F^{ab}(A+B) & \xleftarrow{\kappa_2} & F^{ab}(B) \\ \uparrow i_A & & \uparrow j & & \uparrow i_B \\ A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B \end{array}$$

PROOF:

- ⟨1⟩1. LET: $i_A : A \rightarrow F^{ab}(A)$, $i_B : B \rightarrow F^{ab}(B)$, $j : A + B \rightarrow F^{ab}(A + B)$ be the canonical injections.
- ⟨1⟩2. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.

- (1)3. LET: G be any group and $f : F^{\text{ab}}(A) \rightarrow G$, $g : F^{\text{ab}}(B) \rightarrow G$ any group homomorphisms.
 (1)4. LET: $h : A + B \rightarrow G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.
 (1)5. LET: $k : F^{\text{ab}}(A + B) \rightarrow G$ be the unique group homomorphism such that $k \circ j = h$.
 (1)6. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
 (1)7. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.
 \square

Proposition 8.29. *For A and B finite sets, if $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$ then $A \cong B$.*

PROOF:

- (1)1. For any set C , define \sim on $F^{\text{ab}}(C)$ by: $f \sim f'$ iff there exists $g \in F^{\text{ab}}(C)$ such that $f - f' = 2g$.
 (1)2. For any set C , \sim is an equivalence relation on $F^{\text{ab}}(C)$.
 (1)3. For any set C , we have $F^{\text{ab}}(C) / \sim$ is finite if and only if C is finite, in which case $|F^{\text{ab}}(C) / \sim| = 2^{|C|}$.

PROOF: There is a bijection between $F^{\text{ab}}(C) / \sim$ and the finite subsets of C , which maps f to $\{c \in C : f(c) \text{ is odd}\}$.

- (1)4. If $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$ then $A \cong B$.

PROOF: If $|F^{\text{ab}}(A) / \sim| = |F^{\text{ab}}(B) / \sim|$ then $2^{|A|} = 2^{|B|}$ and so $|A| = |B|$.

\square

Proposition 8.30. *Let G be an Abelian group. Then G is finitely generated if and only if there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n .*

PROOF:

- (1)1. If G is finitely generated then there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n .

PROOF: Let $G = \langle a_1, \dots, a_n \rangle$. Define $\phi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$ by $\phi(i_1, \dots, i_n) = i_1 \cdot a_1 + \dots + i_n \cdot a_n$.

- (1)2. If there exists a surjective homomorphism $\phi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n then G is finitely generated.

PROOF: G is generated by $\phi(1, 0, \dots, 0)$, $\phi(0, 1, 0, \dots, 0)$, \dots , $\phi(0, \dots, 0, 1)$.

\square

Proposition 8.31. *Let A be a set. Let $i : A \hookrightarrow F(A)$ be the free group on A . Then $\pi \circ i : A \rightarrow F(A) / [F(A), F(A)]$ is the free Abelian group on A .*

$$\begin{array}{ccc}
 & F(A) / [F(A), F(A)] & \\
 & \uparrow \pi & \searrow h \\
 F(A) & \xrightarrow{g} & G \\
 & \uparrow i & \nearrow f \\
 & A &
 \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: G be an Abelian group and $f : A \rightarrow G$ a function.

$\langle 1 \rangle 2$. LET: $g : F(A) \rightarrow G$ be the unique group homomorphism such that $g \circ i = f$.

$\langle 1 \rangle 3$. $[F(A), F(A)] \subseteq \ker g$

PROOF: For all $x, y \in F(A)$ we have $g(xy x^{-1} y^{-1}) = g(x) + g(y) - g(x) - g(y) = 0$.

$\langle 1 \rangle 4$. LET: $h : F(A)/[F(A), F(A)]$ be the unique group homomorphism such that $h \circ \pi = g$.

$\langle 1 \rangle 5$. h is the unique group homomorphism such that $h \circ \pi \circ i = f$.

□

Corollary 8.31.1. *Let A and B be sets. Let $F(A)$ and $F(B)$ be the free groups on A and B respectively. If $F(A) \cong F(B)$ then $A \cong B$.*

PROOF: Proposition 8.29. □

8.3 Cokernels

Proposition 8.32. *Let $\phi : G \rightarrow H$ be a homomorphism between Abelian groups. Then there exists an Abelian group K and homomorphism $\pi : H \rightarrow K$ that is initial with respect to all homomorphism $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.*

PROOF:

$\langle 1 \rangle 1$. LET: $K = H/\text{im } \phi$ and π be the canonical homomorphism.

$\langle 1 \rangle 2$. LET: $\pi \circ \phi = 0$

$\langle 1 \rangle 3$. LET: $\alpha : H \rightarrow L$ satisfy $\alpha \circ \phi = 0$

$\langle 1 \rangle 4$. $\text{im } \phi \subseteq \ker \alpha$

$\langle 1 \rangle 5$. There exists a unique $\bar{\alpha} : H/\text{im } \phi \rightarrow L$ such that $\bar{\alpha} \circ \pi = \alpha$

□

Definition 8.33 (Cokernel). For any homomorphism $\phi : G \rightarrow H$ in **Ab**, the *cokernel* of ϕ is the Abelian group $\text{coker } \phi$ and homomorphism $\pi : H \rightarrow \text{coker } \phi$ that is initial among homomorphisms $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.

Proposition 8.34. $\pi : H \rightarrow \text{coker } \phi$ is initial among functions $f : H \rightarrow X$ such that, for all $x, y \in H$, if $x + \text{im } \phi = y + \text{im } \phi$ then $f(x) = f(y)$.

PROOF: Easy. □

Proposition 8.35. *Let $\phi : G \rightarrow H$ be a homomorphism of Abelian groups. Then the following are equivalent.*

- ϕ is an epimorphism.
- $\text{coker } \phi$ is trivial.
- ϕ is surjective.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: ϕ is epi.

$\langle 2 \rangle 2.$ LET: $\pi : H \rightarrow \text{coker } \phi$ be the canonical homomorphism.

$\langle 2 \rangle 3.$ $\pi \circ \phi = 0 \circ \phi$

$\langle 2 \rangle 4.$ $\pi = 0$

$\langle 2 \rangle 5.$ $\text{coker } \phi = \text{im } \pi$ is trivial.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: If $\text{coker } \phi = H / \text{im } \phi$ is trivial then $\text{im } \phi = H$.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: If it is surjective then it is epi in **Set**.

□

Chapter 9

Group Actions

9.1 Group Actions

Definition 9.1 (Action). Let G be a group. Let A be an object of a category \mathcal{C} . A (left) action of G on A is a group homomorphism $G \rightarrow \text{Aut}_{\mathcal{C}}(A)$.

It is *faithful* or *effective* iff it is injective.

Proposition 9.2. Let A be a set. An action of the group G on the set A is given by a function $\cdot : G \times A \rightarrow A$ such that

- $\forall a \in A. ea = a$
- $\forall g, h \in G. \forall a \in A. (gh)a = g(ha)$

PROOF: Just unfolding definitions. \square

Example 9.3. Left multiplication defines a faithful action of any group on its own underlying set.

In fact, for any subgroup H of a group G , left multiplication defines an action of G on G/H .

Corollary 9.3.1 (Cayley's Theorem). Every group G is a subgroup of a symmetric group, namely $\text{Aut}_{\text{Set}}(G)$.

Example 9.4. Conjugation $g * h = ghg^{-1}$ is an action of any group on its own underlying set.

Definition 9.5 (Transitive). An action of a group G on a set A is *transitive* iff, for all $a, b \in A$, there exists $g \in G$ such that $ga = b$.

Example 9.6. Left multiplication of a group G is a transitive action of G on G .

Definition 9.7 (Orbit). Given an action of a group G on a set A and $a \in A$, the *orbit* of a is

$$\text{O}_G(a) := \{ga : g \in G\} .$$

Proposition 9.8. *Given an action of a group G on a set A , the orbits form a partition of A .*

PROOF:

$\langle 1 \rangle 1$. Every element of A is in some orbit.

PROOF: Since $a \in O_G(a)$.

$\langle 1 \rangle 2$. Distinct orbits are disjoint.

$\langle 2 \rangle 1$. LET: $a \in O_G(b) \cap O_G(c)$

$\langle 2 \rangle 2$. PICK $g, h \in G$ such that $a = gb = hc$.

$\langle 2 \rangle 3$. $O_G(b) \subseteq O_G(c)$

PROOF: For all $k \in G$ we have $kb = kg^{-1}hc$.

$\langle 2 \rangle 4$. $O_G(c) \subseteq O_G(b)$

PROOF: Similar.

□

Proposition 9.9. *Given an action of a group G on a set A and $a \in A$, the action is transitive on $O_G(a)$.*

PROOF:

$\langle 1 \rangle 1$. The restriction of the action is an action on $O_G(a)$.

PROOF: Since $g(ha) = (gh)a$, the action maps $O_G(a)$ to itself.

$\langle 1 \rangle 2$. The restricted action is transitive.

PROOF: Given $ga, ha \in O_G(a)$, we have $ha = (hg^{-1})(ga)$.

□

Definition 9.10 (Stabilizer Subgroup). Given an action of a group G on a set A and $a \in A$, the *stabilizer subgroup* of a is

$$\text{Stab}_G(a) := \{g \in G : ga = a\} .$$

Proposition 9.11. *Stabilizer subgroups are subgroups.*

PROOF: If $g, h \in \text{Stab}_G(a)$ then $gh^{-1}a = a$ so $gh^{-1} \in \text{Stab}_G(a)$. □

Proposition 9.12. *Let G act on a set A . Let $a \in A$ and $g \in G$. Then*

$$\text{Stab}_G(ga) = g\text{Stab}_G(a)g^{-1} .$$

PROOF:

$$h \in \text{Stab}_G(ga) \Leftrightarrow hga = ga$$

$$\Leftrightarrow g^{-1}hga = a$$

$$\Leftrightarrow g^{-1}hg \in \text{Stab}_G(a)$$

$$\Leftrightarrow h \in g\text{Stab}_G(a)g^{-1}$$

□

Corollary 9.12.1. *Let G be an action on a set A and $a \in A$. If $\text{Stab}_G(a)$ is normal in G , then for any $b \in O_G(a)$ we have $\text{Stab}_G(a) = \text{Stab}_G(b)$.*

Definition 9.13 (Free). An action of a group G on a set A is *free* iff, whenever $ga = a$, then $g = e$.

Example 9.14. The action of left multiplication is free.

Proposition 9.15. *Let G be a group. Let H be a subgroup of G of finite index n . Then H includes a subgroup K that is normal in G and such that $|G : K|$ divides $\gcd(|G|, n!)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\sigma : G \rightarrow \text{Aut}_{\text{Set}}(G/H)$ be the action of left multiplication.

$\langle 1 \rangle 2$. LET: $K = \ker \sigma$

$\langle 1 \rangle 3$. $K \subseteq H$

$\langle 2 \rangle 1$. LET: $g \in K$

$\langle 2 \rangle 2$. $\sigma(g)(H) = H$

$\langle 2 \rangle 3$. $gH = H$

$\langle 2 \rangle 4$. $g \in H$

$\langle 1 \rangle 4$. K is normal in G .

PROOF: Proposition 7.42.

$\langle 1 \rangle 5$. $|G : K| \mid |G|$

PROOF: Lagrange's Theorem.

$\langle 1 \rangle 6$. $|G : K| \mid n!$

PROOF: Since G/K is a subgroup of $\text{Aut}_{\text{Set}}(G/H)$.

□

Corollary 9.15.1. *Let G be a finite group. Let H be a subgroup of G of index p where p is the smallest prime that divides $|G|$. Then H is normal in G .*

PROOF:

$\langle 1 \rangle 1$. PICK a subgroup K of H normal in G such that $|G : K|$ divides $\gcd(|G|, p!)$.

$\langle 1 \rangle 2$. $|G : K|$ divides p .

$\langle 1 \rangle 3$. $|G : H| |H : K|$ divides p .

$\langle 1 \rangle 4$. $|H : K| = 1$

$\langle 1 \rangle 5$. $H = K$

$\langle 1 \rangle 6$. H is normal.

□

Corollary 9.15.2. *Any subgroup of index 2 is normal.*

Proposition 9.16. *Let G be a group with finite set of generators A . Then left multiplication defines a free action of G on its Cayley graph.*

PROOF: Easy since if $g_2 = g_1 a$ then $hg_2 = hg_1 a$. □

Corollary 9.16.1. *A free group acts freely on a tree.*

Theorem 9.17. *If a group G acts freely on a tree then G is free.*

Corollary 9.17.1. *Every subgroup of the free group on a finite set is free.*

PROOF: If H is a subgroup of $F(A)$ then left multiplication defines a free action of H on the Cayley graph of $F(A)$, which is a tree. □

Proposition 9.18. *Let S be a finite set. Let G be a group acting on S . Let Z be the set of fixed points of the action:*

$$Z = \{a \in S : \forall g \in G. ga = a\} .$$

Let A be a set of representatives for the nontrivial orbits of the action. Then

$$|S| = |Z| + \sum_{a \in A} [G : \text{Stab}_G(a)] .$$

PROOF: Immediate from the fact that the orbits partition S . \square

Corollary 9.18.1. *Let p be a prime. Let S be a finite set. Let G be a p -group acting on S . Let Z be the set of fixed points of the action. Then $|Z| \equiv |S| \pmod{p}$.*

Corollary 9.18.2. *Let p be a prime. Let S be a finite set. Let G be a p -group acting on S . If p does not divide $|S|$ then the action has a fixed point.*

9.2 Category of G -Sets

Definition 9.19. Given a group G , let $G - \mathbf{Set}$ be the category with:

- objects all pairs (A, ρ) such that A is a set and $\rho : G \times A \rightarrow A$ is an action of G on A ;
- morphisms $f : (A, \rho) \rightarrow (B, \sigma)$ are functions $f : A \rightarrow B$ that are $(G-)$ equivariant, i.e.

$$\forall g \in G. \forall a \in A. f(\rho(g, a)) = \sigma(g, f(a)) .$$

Proposition 9.20. *A G -equivariant function $f : A \rightarrow B$ is an isomorphism in $G - \mathbf{Set}$ if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ be G -equivariant and bijective.

PROVE: f^{-1} is G -equivariant.

$\langle 1 \rangle 2$. LET: $g \in G$ and $b \in B$

$\langle 1 \rangle 3$. $f^{-1}(gb) = gf^{-1}(b)$

PROOF:

$$\begin{aligned} f(f^{-1}(gb)) &= gb \\ &= gf(f^{-1}(b)) \\ &= f(gf^{-1}(b)) \end{aligned}$$

\square

Proposition 9.21. *Let G be a group and A a transitive G -set. Let $a \in A$. Then A is isomorphic to $G/\text{Stab}_G(a)$ under left multiplication.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : G/\text{Stab}_G(a) \rightarrow A$ be the function $f(g\text{Stab}_G(a)) = ga$.

$\langle 2 \rangle 1$. ASSUME: $g\text{Stab}_G(a) = h\text{Stab}_G(a)$

PROVE: $ga = ha$

$\langle 2 \rangle 2$. $g^{-1}h \in \text{Stab}_G(a)$

$\langle 2 \rangle 3$. $g^{-1}ha = a$

$\langle 2 \rangle 4$. $ha = ga$

$\langle 1 \rangle 2$. f is G -equivariant.

PROOF: Since $f(gh\text{Stab}_G(a)) = gha = gf(h\text{Stab}_G(a))$.

$\langle 1 \rangle 3$. f is injective.

PROOF: If $ga = ha$ then $g^{-1}h \in \text{Stab}_G(a)$ so $g\text{Stab}_G(a) = h\text{Stab}_G(a)$.

$\langle 1 \rangle 4$. f is surjective.

PROOF: Since for all $b \in A$ there exists $g \in G$ such that $ga = b$.

□

Corollary 9.21.1. *If O is an orbit of the action of a finite group G on a set A , then O is finite and $|O|$ divides $|G|$.*

Corollary 9.21.2. *Let H be a subgroup of G and $g \in G$. Then*

$$G/H \cong G/(gHg^{-1})$$

in $G - \mathbf{Set}$.

PROOF: Taking $A = G/H$ and $a = gH$. □

Proposition 9.22. *Given a family of G -sets $\{A_i\}_{i \in I}$, we have $\prod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g\{a_i\}_{i \in I} = \{ga_i\}_{i \in I}.$$

PROOF: Easy. □

Proposition 9.23. *Given a family of G -sets $\{A_i\}_{i \in I}$, we have $\coprod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g(i, a_i) = (i, ga_i).$$

PROOF: Easy. □

Proposition 9.24. *Every finite G -set is a coproduct of G -sets of the form G/H .*

PROOF: If $O(a_1), \dots, O(a_n)$ are the orbits of the G -set A , then A is the coproduct of $G/\text{Stab}_G(a_1), \dots, G/\text{Stab}_G(a_n)$. □

Proposition 9.25. *For any group G we have $G \cong \text{Aut}_{G - \mathbf{Set}}(G)$ (considering G as a G -set under left multiplication).*

PROOF:

$\langle 1 \rangle 1$. Define $\phi : G \rightarrow \text{Aut}_{G - \mathbf{Set}}(G)$ by $\phi(g)(g') = g'g^{-1}$.

- ⟨2⟩1. LET: $g \in G$
 PROVE: $\lambda_{g'} \in G.g'g^{-1}$ is an automorphism of G in $G - \mathbf{Set}$.
 ⟨2⟩2. $\phi(g)$ is G -equivariant.
 PROOF: Since $\phi(g)(h_1h_2) = h_1h_2g^{-1} = h_1\phi(g)(h_2)$.
 ⟨2⟩3. $\phi(g)$ is injective.
 PROOF: By Cancellation.
 ⟨2⟩4. $\phi(g)$ is surjective.
 PROOF: For any $h \in G$ we have $h = \phi(g)(hg)$.
 ⟨1⟩2. ϕ is a group homomorphism.
 PROOF: $\phi(g_1g_2)(h) = hg_2^{-1}g_1^{-1} = \phi(g_1)(\phi(g_2)(h))$.
 ⟨1⟩3. ϕ is injective.
 PROOF: If $\phi(g) = \phi(g')$ then $g = \phi(g)(e) = \phi(g')(e) = g'$.
 ⟨1⟩4. ϕ is surjective.
 ⟨2⟩1. LET: $\sigma \in \text{Aut}_{G-\mathbf{Set}}(G)$
 ⟨2⟩2. LET: $g = \sigma(e)$
 PROVE: $\sigma = \phi(g^{-1})$
 ⟨2⟩3. $\sigma(h) = hg$
 PROOF: $\sigma(h) = \sigma(hg) = h\sigma(e) = hg$.
 □

9.3 Center

Definition 9.26 (Center). The *center* of a group G , $Z(G)$, is the kernel of the conjugation action $\sigma : G \rightarrow S_G$.

Proposition 9.27. *The center of a group G is*

$$Z(G) = \{g \in G : \forall a \in G. ag = ga\} .$$

PROOF: Immediate from definitions. □

Lemma 9.28. *Let G be a finite group. Assume $G/Z(G)$ is cyclic. Then G is Abelian and so $G/Z(G)$ is trivial.*

PROOF:

- ⟨1⟩1. PICK $g \in G$ such that $gZ(G)$ generates $G/Z(G)$.
 ⟨1⟩2. LET: $a, b \in G$
 ⟨1⟩3. PICK $r, s \in \mathbb{Z}$ such that $aZ(G) = g^rZ(G)$ and $bZ(G) = g^sZ(G)$
 ⟨1⟩4. LET: $z = g^{-r}a \in Z(G)$ and $w = g^{-s}b \in Z(G)$
 ⟨1⟩5. $a = g^rz$ and $b = g^sw$
 ⟨1⟩6. $ab = ba$

PROOF:

$$\begin{aligned}
 ab &= g^rzg^sw \\
 &= g^{r+s}zw \\
 &= g^swg^rz \\
 &= ba
 \end{aligned}$$

□

Proposition 9.29. *Let G be a group. Let N be a subgroup of $Z(G)$. Then N is normal in G .*

PROOF: For all $n \in N$ and $g \in G$ we have $gng^{-1} = ngg^{-1} = n \in N$ since $n \in Z(G)$. □

Proposition 9.30. *For any group G we have $G/Z(G) \cong \text{Inn}(G)$.*

PROOF: The homomorphism $g \mapsto \gamma_g$ is a surjective homomorphism with kernel $Z(G)$. □

Proposition 9.31. *Let p and q be prime integers. Let G be a group of order pq . Then either G is Abelian or the center of G is trivial.*

PROOF: Otherwise we would have $|Z(G)| = p$ say and so $|\text{Inn}(G)| = q$, meaning $\text{Inn}(G)$ is cyclic, hence trivial, which is a contradiction. □

Theorem 9.32 (First Sylow Theorem). *Let p be a prime and $k \in \mathbb{N}$. Let G be a finite group. If p^k divides $|G|$ then G has a subgroup of order p^k .*

PROOF:

⟨1⟩1. ASSUME: as induction hypothesis the statement is true for all groups smaller than G .

⟨1⟩2. ASSUME: w.l.o.g. $k \neq 0$ and $|G| \neq p$

⟨1⟩3. CASE: There exists a proper subgroup H of G such that p does not divide $[G : H]$.

PROOF: Then H has a subgroup of order p^k by induction hypothesis ⟨1⟩1.

⟨1⟩4. CASE: For every proper subgroup H of G we have p divides $[G : H]$.

⟨2⟩1. p divides $|Z(G)|$.

PROOF: By the Class Formula.

⟨2⟩2. PICK $a \in Z(G)$ that has order p .

PROOF: Cauchy's Theorem.

⟨2⟩3. LET: $N = \langle a \rangle$

⟨2⟩4. N is normal.

PROOF: Proposition 9.29.

⟨2⟩5. p^{k-1} divides $|G/N|$.

⟨2⟩6. PICK a subgroup Q of G/N of order p^{k-1} .

PROOF: Induction hypothesis ⟨1⟩1.

⟨2⟩7. LET: $P = \pi^{-1}(Q)$

⟨2⟩8. $|P| = p^k$

□

Proposition 9.33. *The only non-trivial simple finite Abelian groups are $\mathbb{Z}/p\mathbb{Z}$ for p a prime.*

PROOF:

⟨1⟩1. LET: G be a non-trivial simple finite Abelian group.

⟨1⟩2. PICK a prime p that divides $|G|$.

⟨1⟩3. PICK an element $a \in G$ of order p .

PROOF: Cauchy's Theorem.

⟨1⟩4. $\langle a \rangle = G$

□

Theorem 9.34 (Second Sylow Theorem). *Let G be a finite group. Let p be a prime. Let P be a p -Sylow subgroup of G . Let H be a subgroup of G that is a p -group. Then H is a subgroup of a conjugate of P .*

PROOF:

⟨1⟩1. PICK a fixed point gP for the action of H on the set of left cosets of P by left multiplication.

PROOF: Corollary 9.18.2.

⟨1⟩2. For all $h \in H$ we have $hgP = gP$

⟨1⟩3. $H \subseteq gPg^{-1}$

□

9.4 Centralizer

Definition 9.35 (Centralizer). Let G be a group. Let $a \in G$. The *centralizer* or *normalizer* of a , denoted $Z_G(a)$, is the stabilizer of a under the action of conjugation.

Proposition 9.36.

$$Z_G(a) = \{g \in G : ga = ag\}$$

PROOF: Immediate from definitions. □

9.5 Conjugacy Class

Definition 9.37 (Conjugacy Class). Let G be a group. Let $a \in G$. The *conjugacy class* of a , denoted $[a]$, is the orbit of a under the action of conjugation.

Proposition 9.38 (Class Formula). *Let G be a finite group. Let A be a set of representatives of the non-trivial conjugacy classes. Then*

$$|G| = |Z(G)| + \sum_{a \in A} [G : Z(a)] .$$

PROOF: Proposition 9.18. □

Corollary 9.38.1. *Let p be a prime. Let G be a p -group and H a nontrivial normal subgroup of G . Then $H \cap Z(G) \neq \{e\}$.*

PROOF: Let A be a set of representatives of the non-trivial conjugacy classes. Let $A \cap H = \{a_1, \dots, a_n\}$. Then

$$|H| = |H \cap Z(G)| + \sum_{i=1}^n [G : Z(a_i)] .$$

Since $p \mid |H|$ and $p \mid [G : Z(a_i)]$ for all i , we have $p \mid |H \cap Z(G)|$. \square

Corollary 9.38.2. *Let p be a prime. Every p -group has a non-trivial center.*

Corollary 9.38.3. *Let p be a prime. Every group G of order p^2 is Abelian.*

PROOF: By Proposition 9.31. \square

Proposition 9.39. *Let p be a prime and r a non-negative integer. Let G be a group of order p^r . Then, for $k = 0, 1, \dots, r$, we have G has a normal subgroup of order p^k .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: as induction hypothesis the result holds for $r' < r$.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $k > 0$

PROOF: Since $\{e\}$ is a normal subgroup of order p^0 .

$\langle 1 \rangle 3$. PICK a subgroup N of $Z(G)$ of order p .

$\langle 2 \rangle 1$. $p \mid |Z(G)|$

PROOF: From Corollary 9.38.2.

$\langle 2 \rangle 2$. $Z(G)$ has a subgroup of order p .

PROOF: Cauchy's Theorem.

$\langle 1 \rangle 4$. N is normal.

PROOF: Proposition 9.29.

$\langle 1 \rangle 5$. PICK a normal subgroup M of G/N of order p^{k-1} .

PROOF: From the induction hypothesis $\langle 1 \rangle 1$.

$\langle 1 \rangle 6$. $\pi^{-1}(M)$ is a normal subgroup of G of order p^k .

\square

Example 9.40. The only non-Abelian group of order 6 is S_3 .

PROOF:

$\langle 1 \rangle 1$. LET: G be a non-Abelian group of order 6.

$\langle 1 \rangle 2$. $Z(G) = \{e\}$

PROOF: Otherwise $Z(G)$ has order 2 or 3 and is cyclic, contradicting Lemma 9.28.

$\langle 1 \rangle 3$. G has three conjugacy classes: $Z(G)$, a class of size 2 and a class of size 3.

PROOF: By the Class Formula since the only way to make 6 using non-trivial factors of 6 is $2 + 3$.

$\langle 1 \rangle 4$. PICK an element $y \in G$ of order 3.

PROOF: It cannot be that every element is of order ≤ 2 by Proposition 8.20.

$\langle 1 \rangle 5$. $\langle y \rangle$ is normal in G .

PROOF: Since it has index 2.

$\langle 1 \rangle 6$. The conjugacy class y is $\{y, y^2\}$.

PROOF: Since $\langle y \rangle$ must be a union of conjugacy classes.

$\langle 1 \rangle 7$. The conjugacy class of size 2 is $\{y, y^2\}$.

PROOF: Since y^2 has order 3 and so its conjugacy class is of size 2 similarly, and there is only one conjugacy class of size 2.

⟨1⟩8. PICK $x \in G$ such that $yx = xy^2$.

PROOF: y^2 is conjugate to y so there exists x such that $x^{-1}yx = y^2$.

⟨1⟩9. x has order 2.

PROOF: x is not in the conjugacy class of size 2 so its order cannot be 3.

⟨1⟩10. x and y generate G .

PROOF: Since e, y, y^2, x, xy, xy^2 are all distinct.

⟨1⟩11. $G \cong S_3$

PROOF: We now know the entire multiplication table of G .

□

Proposition 9.41. *Let G be a finite group. Let H be a subgroup of G of order 2. Let $a \in H$. Let $[a]_H$ be the conjugacy class of a in H , and $[a]_G$ the conjugacy class of a in G . If $Z_G(a) \subseteq H$ then $[a]_H$ is half the size of $[a]_G$; otherwise, $[a]_H = [a]_G$.*

PROOF:

⟨1⟩1. H is normal in G .

PROOF: Corollary 9.15.2.

⟨1⟩2. $HZ_G(a)$ is a subgroup of G .

⟨1⟩3. H is normal in $HZ_G(a)$.

⟨1⟩4. $H \cap Z_G(a)$ is normal in $Z_G(a)$.

⟨1⟩5.

$$\frac{HZ_G(a)}{H} \cong \frac{Z_G(a)}{H \cap Z_G(a)}$$

⟨1⟩6. If $Z_G(a) \subseteq H$ then $|[a]_H| = |[a]_G|/2$.

PROOF: In this case we have $Z_H(a) = Z_G(a)$ and so $|[a]_H| = |H|/|Z_H(a)| = (|G|/2)/|Z_G(a)| = |[a]_G|/2$.

⟨1⟩7. If $Z_G(a) \not\subseteq H$ then $[a]_H = [a]_G$.

PROOF:

⟨2⟩1. PICK $b \in Z_G(a) - H$

⟨2⟩2. $Hb^{-1} = G - H$

⟨2⟩3. $G = HZ_G(a)$

PROOF: For $x \in H$ we have $x = xe$ and for $x \notin H$ we have $x \in Hb^{-1}$ hence $xb \in H$ and $x = (xb)b$.

⟨2⟩4. $|[a]_H| = |[a]_G|$

PROOF:

$$\begin{aligned}
 |[a]_H| &= \frac{|H|}{|Z_H(a)|} \\
 &= \frac{|H|}{|H \cap Z_G(a)|} \\
 &= \frac{|Z_G(a)||H|}{|Z_G(a)||H \cap Z_G(a)|} \\
 &= \frac{|HZ_G(a)|}{|Z_G(a)|} \\
 &= \frac{|G|}{|Z_G(a)|} \\
 &= |[a]_G|
 \end{aligned}$$

□

9.6 Conjugation on Sets

Definition 9.42 (Conjugation). Let G be a group. Define an action of G on $\mathcal{P}G$ called *conjugation* that takes g and A to

$$gAg^{-1} = \{gag^{-1} : a \in A\} .$$

Proposition 9.43. *The conjugate of a subgroup is a subgroup.*

PROOF: Let H be a subgroup of G . Given $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$, we have

$$(gh_1g^{-1})(gh_2g^{-1})^{-1} = gh_1h_2^{-1}g^{-1} \in gHg^{-1} . \quad \square$$

Definition 9.44 (Normalizer). Let G be a group and $A \subseteq G$. The *normalizer* of A , denoted $N_G(A)$, is its stabilizer under conjugation.

Proposition 9.45. *Let G be a group, $g \in G$ and A a finite subset of G . If $gAg^{-1} \subseteq A$ then $gAg^{-1} = A$ and so $g \in N_G(A)$.*

PROOF: Conjugation by g is an injection from A into A , hence a bijection. □

Proposition 9.46. *Let G be a group and H a subgroup of G . Then $N_G(H)$ is the largest subgroup of G that includes H such that H is normal in $N_G(H)$.*

PROOF:

⟨1⟩1. $N_G(H)$ is a subgroup of G .

PROOF: If $a, b \in N_G(H)$ then $ab^{-1}Hba^{-1} = aHa^{-1} = H$ so $ab^{-1} \in N_G(H)$.

⟨1⟩2. $H \subseteq N_G(H)$

PROOF: Easy.

⟨1⟩3. H is normal in $N_G(H)$.

PROOF: If $a \in N_G(H)$ then $aHa^{-1} = H$ by definition.

⟨1⟩4. For any subgroup K of G , if $H \subseteq K$ and H is normal in K then $K \subseteq N_G(H)$.

PROOF: H is normal in K means that, for all $a \in K$, we have $aHa^{-1} = H$ and so $a \in N_G(H)$.

□

Corollary 9.46.1. *Let G be a group and H a subgroup of G . Then H is normal if and only if $H = N_G(H)$.*

Proposition 9.47. *Let G be a group and H a subgroup of G . If $[G : N_G(H)]$ is finite, then it is the number of subgroups conjugate to H .*

PROOF: By the Orbit-Stabilizer Theorem. □

Corollary 9.47.1. *Let G be a group and H a subgroup of G . If $[G : H]$ is finite, then the number of subgroups conjugate to H is finite and divides $[G : H]$.*

Lemma 9.48. *Let H be a p -group that is a subgroup of a finite group G . Then*

$$[N_G(H) : H] \equiv [G : H] \pmod{p} .$$

PROOF:

⟨1⟩1. ASSUME: w.l.o.g. H is non-trivial.

⟨1⟩2. gH is a fixed point of the action of H on the set of left cosets of H by left multiplication if and only if $g \in N_G(H)$.

PROOF:

$$gH \text{ is a fixed point} \Leftrightarrow \forall h \in H. hgH = gH$$

$$\Leftrightarrow H \subseteq gHg^{-1}$$

$$\Leftrightarrow H = gHg^{-1} \quad (|gHg^{-1}| = |H|)$$

$$\Leftrightarrow g \in N_G(H)$$

⟨1⟩3. The number of fixed points in $[N_G(H) : H]$.

⟨1⟩4. Q.E.D.

PROOF: Corollary 9.18.1.

□

Proposition 9.49. *Let H be a p -subgroup of a finite group G that is not a p -Sylow subgroup. Then there exists a p -subgroup H' of G such that H is a normal subgroup of H' and $[H' : H] = p$.*

PROOF:

⟨1⟩1. p divides $[N_G(H) : H]$.

PROOF: Lemma 9.48.

⟨1⟩2. PICK $gH \in N_G(H)/H$ of order p .

PROOF: Cauchy's Theorem.

⟨1⟩3. LET: $H' = \pi^{-1}(\langle gH \rangle)$

⟨1⟩4. H is a normal subgroup of H' .

⟨1⟩5. $[H' : H] = p$

□

Corollary 9.49.1. *No p -group of order $\geq p^2$ is simple.*

Lemma 9.50. *Let p be a prime. Let G be a finite group. Let P be a p -Sylow subgroup of G . Every p -subgroup of $N_G(P)$ is a subgroup of P .*

PROOF:

$\langle 1 \rangle 1$. LET: H be a p -subgroup of $N_G(P)$.

$\langle 1 \rangle 2$. P is normal in $N_G(P)$.

PROOF: Proposition 9.46.

$\langle 1 \rangle 3$. PH is a subgroup of $N_G(P)$.

PROOF: Second Isomorphism Theorem.

$\langle 1 \rangle 4$. $|PH/P| = |H/(P \cap H)|$

PROOF: Second Isomorphism Theorem.

$\langle 1 \rangle 5$. PH is a p -group.

$\langle 2 \rangle 1$. ASSUME: for a contradiction q is prime, $q \mid |PH|$ and $q \neq p$

$\langle 2 \rangle 2$. $q \mid |PH/P|$

$\langle 2 \rangle 3$. $q \mid |H/(P \cap H)|$

$\langle 2 \rangle 4$. $q \mid |H|$

$\langle 2 \rangle 5$. Q.E.D.

PROOF: This contradicts the fact that H is a p -group, $\langle 1 \rangle 1$.

$\langle 1 \rangle 6$. $PH = P$

PROOF: By maximality of P .

$\langle 1 \rangle 7$. $H \subseteq P$

□

Lemma 9.51. *Let p be a prime. Let G be a finite group. Let P be a p -Sylow subgroup of G . Let P act by conjugation on the set of p -Sylow subgroups of G . Then P is the unique fixed point of this action.*

PROOF:

$\langle 1 \rangle 1$. P is a fixed point of this action.

PROOF: For any $x \in P$ we have $xPx^{-1} = P$.

$\langle 1 \rangle 2$. If Q is any fixed point of the action then $Q = P$.

$\langle 2 \rangle 1$. LET: Q be a fixed point of the action.

$\langle 2 \rangle 2$. For all $x \in P$ we have $xQx^{-1} = Q$.

$\langle 2 \rangle 3$. $P \subseteq N_G(Q)$

$\langle 2 \rangle 4$. $P \subseteq Q$

PROOF: Lemma 9.50.

$\langle 2 \rangle 5$. $P = Q$

PROOF: Since $|P| = |Q|$.

□

Theorem 9.52 (Third Sylow Theorem). *Let p be a prime. Let G be a finite group of order $p^r m$ where p does not divide m . Then the number of p -Sylow subgroups of G divides m and is congruent to 1 modulo p .*

PROOF:

$\langle 1 \rangle 1$. LET: N_p be the number of p -Sylow subgroups of G .

$\langle 1 \rangle 2$. PICK a p -Sylow subgroup P .

PROOF: One exists by the First Sylow Theorem.

$\langle 1 \rangle 3$. The p -Sylow subgroups of G are exactly the conjugates of P .

PROOF: Second Sylow Theorem

$\langle 1 \rangle 4$. $m = N_p[N_G(P) : P]$

PROOF: Since $N_p = [G : N_G(P)]$ by Proposition 9.47.

$\langle 1 \rangle 5$. N_p divides m .

$\langle 1 \rangle 6$. $mN_p \equiv m \pmod{p}$

$\langle 2 \rangle 1$. $m \equiv [N_G(P) : P] \pmod{p}$

PROOF: Lemma 9.48.

$\langle 2 \rangle 2$. $mN_p \equiv m \pmod{p}$

PROOF: By $\langle 1 \rangle 4$.

$\langle 1 \rangle 7$. $N_p \equiv 1 \pmod{p}$

□

PROOF:

$\langle 1 \rangle 1$. LET: N_p be the number of p -Sylow subgroups of G .

$\langle 1 \rangle 2$. PICK a p -Sylow subgroup P of G .

PROOF: First Sylow Theorem.

$\langle 1 \rangle 3$. N_p is the number of conjugates of P .

PROOF: Second Sylow Theorem.

$\langle 1 \rangle 4$. $N_p \mid m$

PROOF: Corollary 9.47.1.

$\langle 1 \rangle 5$. P acts on the set of conjugates of P with one fixed point.

PROOF: Lemma 9.51.

$\langle 1 \rangle 6$. $N_p \equiv 1 \pmod{p}$

PROOF: Corollary 9.18.1.

□

Corollary 9.52.1. *Let G be a finite group. Let p be a prime number. If $|G| = mp^r$ where $1 < m < p$ then G is not simple.*

PROOF: There must be 1 p -Sylow subgroup, which has order p^r and is normal.

□

Corollary 9.52.2. *Let p and q be prime numbers with $p < q$ and $q \not\equiv 1 \pmod{p}$. Then the only group of order pq is the cyclic group.*

PROOF:

$\langle 1 \rangle 1$. LET: G be a group of order pq .

$\langle 1 \rangle 2$. LET: H be the unique p -Sylow subgroup of G .

PROOF: By the Third Sylow Theorem, the number of p -Sylow subgroups divides q and is congruent to 1 modulo p , so must be 1.

$\langle 1 \rangle 3$. LET: $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(H)$ be the action of conjugation.

$\langle 1 \rangle 4$. γ is the zero map.

PROOF: $|\text{im } \gamma|$ must divide both $|G| = pq$ and $|\text{Aut}_{\mathbf{Grp}}(H)| = p - 1$ and so must be 1.

$\langle 1 \rangle 5$. $H \subseteq Z(G)$

$\langle 1 \rangle 6$. G is Abelian.

$\langle 2 \rangle 1$. G/H is cyclic.

PROOF: Since its order is q .

$\langle 2 \rangle 2$. $G/Z(G)$ is cyclic.

PROOF: It is a quotient of G/H by the Third Isomorphism Theorem.

$\langle 2 \rangle 3$. Q.E.D.

PROOF: Lemma 9.28.

$\langle 1 \rangle 7$. G is cyclic.

$\langle 2 \rangle 1$. PICK an element g of order p .

PROOF: Cauchy's Theorem.

$\langle 2 \rangle 2$. PICK an element h of order q .

PROOF: Cauchy's Theorem.

$\langle 2 \rangle 3$. $|gh| = pq$

PROOF: Proposition 6.19.

□

Definition 9.53 (Centralizer). Let G be a group and $A \subseteq G$. The *centralizer* of A is

$$Z_G(A) := \{g \in G : \forall a \in A. gag^{-1} = a\} .$$

Proposition 9.54. Let H and K be subgroups of G with $H \subseteq N_G(K)$. Then the function $\gamma : H \rightarrow \text{Aut}_{\mathbf{Grp}}(K)$ defined by conjugation

$$\gamma_h(k) = hkh^{-1}$$

is a homomorphism of groups with $\ker \gamma = H \cap Z_G(K)$.

PROOF:

$\langle 1 \rangle 1$. For all $g, h \in H$ we have $\gamma_{gh} = \gamma_g \circ \gamma_h$.

PROOF: Since $\gamma_{gh}(k) = \gamma_g(\gamma_h(k)) = ghkh^{-1}g^{-1}$.

$\langle 1 \rangle 2$. For all $h \in H$ we have $\gamma_h = \text{id}_K$ iff $h \in Z_G(K)$.

PROOF: Both are equivalent to $\forall k \in K. hkh^{-1} = k$, i.e. $\forall k \in K. hk = kh$.

□

Part III

Ring Theory

Chapter 10

Rngs

Definition 10.1 (Ring). A *rng* consists of a set R and binary operations $+, \cdot : R^2 \rightarrow R$ such that:

- $(R, +)$ is an Abelian group
- \cdot is associative.
- The *distributive properties* hold: for all $r, s, t \in R$ we have

$$(r + s)t = rt + st, \quad r(s + t) = rs + rt .$$

Example 10.2. • The *zero rng* is $\{0\}$.

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} are rngs.
- $2\mathbb{Z}$ is a rng.
- Given a rng R and natural number n , then the set $\mathfrak{gl}_n(R)$ of all $n \times n$ matrices with entries in R is a rng under matrix addition and matrix multiplication.
- For any set S , the power set $\mathcal{P}S$ is a rng under $A + B = (A \cup B) - (A \cap B)$ and $AB = A \cap B$.
- Given a rng R and a set S , then R^S is a rng under $(f + g)(s) = f(s) + g(s)$ and $(fg)(s) = f(s)g(s)$ for all $f, g \in R^S$ and $s \in S$.
- The set $\mathfrak{sl}_n(\mathbb{R}) = \{M \in \mathfrak{gl}_n(\mathbb{R}) : \text{tr } M = 0\}$ is a rng.
- The set $\mathfrak{sl}_n(\mathbb{C}) = \{M \in \mathfrak{gl}_n(\mathbb{C}) : \text{tr } M = 0\}$ is a rng.
- $\mathbb{Z}/n\mathbb{Z}$ is a rng.

- The ring \mathbb{H} of *quaternions* is \mathbb{R}^4 under the following operations, where we write (a, b, c, d) as $a + bi + cj + dk$:

$$\begin{aligned}
 (a + bi + cj + dk) + (a' + b'i + c'j + d'k) &= (a + a') + (b + b')i \\
 &\quad + (c + c')j + (d + d')k \\
 (a + bi + cj + dk)(a' + b'i + c'j + d'k) &= (aa' - bb' - cc' - dd') \\
 &\quad + (ab' + ba' + cd' - dc')i \\
 &\quad + (ac' - bd' + ca' + db')j \\
 &\quad + (ad' + bc' - cb' + da')k
 \end{aligned}$$

- For any Abelian group G , the set $\text{End}_{\mathbf{Ab}}(G)$ is a ring under pointwise addition and composition.

Proposition 10.3. *In any rng R we have*

$$\forall x \in R. x0 = 0x = 0 \quad .$$

PROOF:

$$\begin{aligned}
 x0 &= x(0 + 0) \\
 &= x0 + x0
 \end{aligned}$$

and so $x0 = 0$ by Cancellation. Similarly $0x = 0$. \square

Definition 10.4 (Zero Divisor). Let R be a rng and $a \in R$.

Then a is a *left-zero-divisor* iff there exists $b \in R - \{0\}$ such that $ab = 0$.

The element a is a *right-zero-divisor* iff there exists $b \in R - \{0\}$ such that $ba = 0$.

Example 10.5. 0 is a left- and right-zero-divisor in every non-zero rng.

The zero rng is the only ring with no zero-divisors.

Proposition 10.6. *Let R be a rng and $a \in R$. Then a is not a left-zero-divisor if and only if left multiplication by a is an injective function $R \rightarrow R$.*

PROOF:

$\langle 1 \rangle 1$. If a is not a left-zero-divisor then left multiplication by a is injective.

$\langle 2 \rangle 1$. ASSUME: a is not a left-zero-divisor.

$\langle 2 \rangle 2$. LET: $ab = ac$

$\langle 2 \rangle 3$. $a(b - c) = 0$

$\langle 2 \rangle 4$. $b - c = 0$

$\langle 2 \rangle 5$. $b = c$

$\langle 1 \rangle 2$. If a is a left-zero-divisor then left multiplication by a is not injective.

$\langle 2 \rangle 1$. PICK $b \neq 0$ such that $ab = 0$.

$\langle 2 \rangle 2$. $ab = a0$ but $b \neq 0$

\square

10.1 Commutative Rings

Definition 10.7 (Commutative). A rng R is *commutative* iff $\forall x, y \in R. xy = yx$.

Example 10.8. • The zero rng is commutative.

- \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are commutative.
- $2\mathbb{Z}$ is commutative.
- $\mathfrak{gl}_2(\mathbb{R})$ is not commutative.
- For any set S , the rng $\mathcal{P}S$ is commutative.
- If R is commutative then R^S is commutative.

10.2 Rng Homomorphisms

Definition 10.9. Let R and S be rngs. A *rng homomorphism* $\phi : R \rightarrow S$ is a function such that, for all $x, y \in R$, we have

$$\begin{aligned}\phi(x + y) &= \phi(x) + \phi(y) \\ \phi(xy) &= \phi(x)\phi(y)\end{aligned}$$

Let **Rng** be the category of rngs and rng homomorphisms.

10.3 Quaternions

Definition 10.10 (Norm). The *norm* of a quaternion is defined by

$$N(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2 \ .$$

Chapter 11

Rings

Definition 11.1 (Ring). A *ring* R is a rng such that there exists $1 \in R$, the *multiplicative identity*, such that

$$\forall x \in R. x1 = 1x = x \text{ .}$$

Example 11.2. • The zero rng is a ring with $1 = 0$.

- \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are rngs.
- $2\mathbb{Z}$ is not a ring.
- If R is a ring then $\mathfrak{gl}_n(R)$ is a ring.
- For any set S , the rng $\mathcal{P}S$ is a ring with $1 = S$.
- If R is a ring then R^S is a ring.
- $\mathfrak{sl}_n(\mathbb{R})$ is not a ring for $n > 0$.
- $\mathfrak{sl}_n(\mathbb{C})$ is not a ring for $n > 0$.
- $\mathfrak{so}_n(\mathbb{R}) = \{M \in \mathfrak{sl}_n(\mathbb{R}) : M + M^T = 0\}$ is not a ring.
- $\mathbb{Z}/n\mathbb{Z}$ is a ring.

Proposition 11.3. *In any ring R , if $0 = 1$ then R is the zero ring.*

PROOF: For any $x \in R$ we have $x = 1x = 0x = 0$. \square

Proposition 11.4. *In any ring we have $(-1)x = -x$.*

PROOF: Since

$$\begin{aligned} x + (-1)x &= 1x + (-1)x \\ &= (1 + (-1))x \\ &= 0x \\ &= 0 \end{aligned}$$

\square

11.1 Units

Definition 11.5 (Left-Unit, Right-Unit). Let R be a ring and $a \in R$. Then a is a *left-unit* iff there exists $b \in R$ such that $ab = 1$. The element a is a *right-unit* iff there exists $b \in R$ such that $ba = 1$.

An element is a *unit* iff it is a left-unit and a right-unit.

Proposition 11.6. *Let R be a ring and $a \in R$. Then a is a left-unit iff left multiplication by a is a surjective function $R \rightarrow R$.*

PROOF:

$\langle 1 \rangle 1$. If a is a left-unit then left multiplication by a is surjective.

$\langle 2 \rangle 1$. PICK $b \in R$ such that $ab = 1$.

$\langle 2 \rangle 2$. For all $c \in R$ we have $c = a(bc)$.

$\langle 1 \rangle 2$. If left multiplication by a is surjective then a is a left-unit.

PROOF: Immediate.

□

Proposition 11.7. *Let R be a ring and $a \in R$. Then a is a right-unit iff right multiplication by a is a surjective function $R \rightarrow R$.*

PROOF: Similar. □

Proposition 11.8. *No left-unit is a right-zero-divisor.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $ab = 1$ and $ca = 0$ where $c \neq 0$.

$\langle 1 \rangle 2$. $c = 0$

PROOF:

$$0 = 0b$$

$$= cab$$

$$= c1$$

$$= c$$

$\langle 1 \rangle 3$. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 11.9. *No right-unit is a left-zero-divisor.*

PROOF: Similar. □

Proposition 11.10. *The inverse of a unit is unique.*

PROOF: If $ba = 1$ and $ac = 1$ then $b = bac = c$. □

Proposition 11.11. *The units of a ring form a group under multiplication.*

PROOF:

$\langle 1 \rangle 1$. If a and b are units then ab is a unit.

PROOF: We have $b^{-1}a^{-1}ab = 1$ and $abb^{-1}a^{-1} = 1$.

⟨1⟩2. 1 is a unit.

PROOF: Since $1 \cdot 1 = 1$.

⟨1⟩3. If a is a unit then its inverse is a unit.

PROOF: Immediate from definitions.

□

Definition 11.12 (Group of Units). For any ring R , we write R^* for the group of the units of R under multiplication.

Example 11.13. The quaternionic group is a subgroup of \mathbb{H}^* .

Example 11.14. The norm is a group homomorphism $\mathbb{H}^* \rightarrow \mathbb{R}^+$ where \mathbb{R}^+ is the group of positive real numbers under multiplication with kernel isomorphic to $\text{SU}_2(\mathbb{C})$. The isomorphism maps a quaternion $a + bi + cj + dk$ to

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

Theorem 11.15 (Fermat's Little Theorem). *Let p be a prime number and a any integer. Then $a^p \equiv a \pmod{p}$.*

PROOF: If $p \mid a$ then $a^p \equiv a \equiv 0 \pmod{p}$. Otherwise, we have $a^{p-1} \equiv 1 \pmod{p}$ by applying Lagrange's Theorem to $(\mathbb{Z}/p\mathbb{Z})^*$. □

Example 11.16. It is not true that, if $n \mid |G|$, then G has a subgroup of order n . The group A_4 has order 12 but no subgroup of order 6.

Proposition 11.17. *If p is prime then $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.*

PROOF:

⟨1⟩1. LET: g be an element of maximal order in $(\mathbb{Z}/p\mathbb{Z})^*$.

⟨1⟩2. For all $h \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $h^{|g|} = 1$.

PROOF: Proposition 8.10.

⟨1⟩3. There are at most $|g|$ elements x such that $x^{|g|} = 1$ in $\mathbb{Z}/p\mathbb{Z}$

⟨1⟩4. $p - 1 \leq |g|$

⟨1⟩5. $|g| = p - 1$

⟨1⟩6. g generates $(\mathbb{Z}/p\mathbb{Z})^*$.

□

Example 11.18. $(\mathbb{Z}/12\mathbb{Z})^*$ is not cyclic. Its elements are 1, 5, 7 and 11 with orders 1, 2, 2 and 2.

Theorem 11.19 (Wilson's Theorem). *A positive integer p is prime if and only if $(p - 1)! \equiv -1 \pmod{p}$.*

⟨1⟩1. If p is prime then $(p - 1)! \equiv -1 \pmod{p}$.

⟨2⟩1. ASSUME: p is prime.

⟨2⟩2. $(p - 1)!$ is the product of all the elements of $(\mathbb{Z}/p\mathbb{Z})^*$

⟨2⟩3. The only element of $(\mathbb{Z}/p\mathbb{Z})^*$ with order 2 is -1 .

⟨2⟩4. $(p - 1)! \equiv -1 \pmod{p}$

PROOF: Proposition 6.20.

$\langle 1 \rangle 2$. If $(p-1)! \equiv -1 \pmod{p}$ then p is prime.

$\langle 2 \rangle 1$. ASSUME: $(p-1)! \equiv -1 \pmod{p}$

$\langle 2 \rangle 2$. LET: d be a proper divisor of p .

PROVE: $d = 1$

$\langle 2 \rangle 3$. $d \mid (p-1)!$

$\langle 2 \rangle 4$. $d \mid 1$

PROOF: Since $d \mid p \mid (p-1)! + 1$.

$\langle 2 \rangle 5$. $d = 1$

□

Proposition 11.20. *If p and q are distinct odd primes then $(\mathbb{Z}/pq\mathbb{Z})^*$ is not cyclic.*

PROOF:

$\langle 1 \rangle 1$. $|(\mathbb{Z}/pq\mathbb{Z})^*| = (p-1)(q-1)$

$\langle 1 \rangle 2$. LET: $g \in (\mathbb{Z}/pq\mathbb{Z})^*$

PROVE: g does not have order $(p-1)(q-1)$

$\langle 1 \rangle 3$. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{p}$

$\langle 1 \rangle 4$. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{q}$

$\langle 1 \rangle 5$. $pq \mid g^{(p-1)(q-1)/2} - 1$

$\langle 1 \rangle 6$. $g^{(p-1)(q-1)/2} \equiv 1 \pmod{pq}$

$\langle 1 \rangle 7$. $|g| \mid (p-1)(q-1)/2$

□

Proposition 11.21. *For any prime p , we have $\text{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : \text{Aut}_{\mathbf{Grp}}(C_p) \rightarrow (\mathbb{Z}/p\mathbb{Z})^*$ be the function $\phi(\alpha) = \alpha(1)$.

PROOF: $\alpha(1)$ has order p in C_p and so is coprime with p .

$\langle 1 \rangle 2$. ϕ is a homomorphism.

PROOF: $\phi(\alpha \circ \beta) = \alpha(\beta(1)) = \alpha(\beta(1)1) = \beta(1)\alpha(1) = \phi(\alpha)\phi(\beta)$

$\langle 1 \rangle 3$. ϕ is injective.

PROOF: If $\phi(\alpha) = \phi(\beta)$ then for any n we have $\alpha(n) = n\alpha(1) = n\phi(\alpha) = n\phi(\beta) = n\beta(1) = \beta(n)$.

$\langle 1 \rangle 4$. ϕ is surjective.

PROOF: For any $r \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $r = \phi(\alpha)$ where $\alpha(n) = nr \pmod{p}$.

$\langle 1 \rangle 5$. $(\mathbb{Z}/p\mathbb{Z})^* \cong C_{p-1}$

□

11.2 Euler's ϕ -function

Proposition 11.22. *For n a positive integer, we have $(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$.*

PROOF:

$$\begin{aligned} m \in (\mathbb{Z}/n\mathbb{Z})^* &\Leftrightarrow \exists a.am \equiv 1 \pmod{n} \\ &\Leftrightarrow \exists a, b.am + bn = 1 \\ &\Leftrightarrow \gcd(m, n) = 1 \quad \square \end{aligned}$$

Definition 11.23 (Euler's Totient Function). For n a positive integer, let $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$.

Proposition 11.24. *If n is an odd positive integer then $\phi(2n) = \phi(n)$.*

PROOF:

(1)1. LET: n be an odd positive integer.

(1)2. For any integer m , if $\gcd(m, n) = 1$ then $\gcd(2m + n, 2n) = 1$

PROOF: For p a prime, if $p \mid 2m + n$ and $p \mid 2n$ then $p \neq 2$ (since $2m + n$ is odd) so $p \mid n$ and hence $p \mid m$, which is a contradiction.

(1)3. For any integer r , if $\gcd(r, 2n) = 1$ then $\gcd(\frac{r+n}{2}, n) = 1$

PROOF: If $p \mid n$ and $p \mid \frac{r+n}{2}$ then $p \mid r + n$ so $p \mid r$ which is a contradiction.

(1)4. The function that maps m to $2m + n$ is a bijection between $(\mathbb{Z}/n\mathbb{Z})^*$ and $(\mathbb{Z}/2n\mathbb{Z})^*$.

□

Theorem 11.25. *For any positive integer n we have*

$$\sum_{m>0, m|n} \phi(m) = n.$$

PROOF:

(1)1. Define $\chi : \{0, 1, \dots, n-1\} \rightarrow \{(m, d) : m > 0, m \mid n, d \text{ generates } \langle n/m \rangle\}$
by: $\chi(x) = (\gcd(x, n), x)$.

(1)2. χ is injective.

(1)3. χ is surjective.

PROOF: Given (m, d) such that d generates $\langle n/m \rangle$ we have $\chi(d) = (m, d)$.

(1)4. $n = \sum_{m>0, m|n} \phi(m)$

PROOF: Since $\langle n/m \rangle \cong C_m$ and so has $\phi(m)$ generators.

□

Proposition 11.26. *For any positive integers a and n , we have $n \mid \phi(a^n - 1)$.*

PROOF: Since the order of a is n in $(\mathbb{Z}/(a^n - 1)\mathbb{Z})^*$. □

Theorem 11.27 (Euler's Theorem). *For any coprime integers a and n we have $a^{\phi(n)} \equiv a \pmod{n}$.*

PROOF: Immediate from Lagrange's Theorem. □

Proposition 11.28.

$$|\text{Aut}_{\mathbf{Grp}}(C_n)| = \phi(n)$$

PROOF: An automorphism α is determined by $\alpha(1)$ which is any element of order n , and g has order n iff $\gcd(g, n) = 1$. □

Example 11.29.

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}) \cong C_2$$

PROOF: The only automorphisms are the identity and multiplication by -1. \square

11.3 Nilpotent Elements

Definition 11.30 (Nilpotent). Let R be a ring and $a \in R$. Then a is *nilpotent* iff there exists n such that $a^n = 0$.

Proposition 11.31. *Let R be a ring and $a, b \in R$. If a and b are nilpotent and $ab = ba$ then $a + b$ is nilpotent.*

PROOF:

$\langle 1 \rangle 1$. PICK m and n such that $a^m = b^n = 0$.

$\langle 1 \rangle 2$. $(a + b)^{m+n} = 0$

PROOF: Since $(a + b)^{m+n} = \sum_k \binom{m+n}{k} a^k b^{m+n-k}$ and every term in this sum is 0 since, for every k , either $k \geq m$ or $m + n - k \geq n$.

\square

Proposition 11.32. *m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$ if and only if m is divisible by all the prime factors of n .*

PROOF:

$\langle 1 \rangle 1$. If m is nilpotent then m is divisible by all the prime factors of n .

$\langle 2 \rangle 1$. ASSUME: $m^a \equiv 0 \pmod{n}$

$\langle 2 \rangle 2$. For every prime p , if $p \mid n$ then $p \mid m^a$.

$\langle 2 \rangle 3$. For every prime p , if $p \mid n$ then $p \mid m$.

$\langle 1 \rangle 2$. If m is divisible by all the prime factors of n then m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

$\langle 2 \rangle 1$. ASSUME: m is divisible by all the prime factors of n .

$\langle 2 \rangle 2$. LET: a be the largest number such that $p^a \mid n$ for some prime p .

$\langle 2 \rangle 3$. For every prime p that divides n we have $p^a \mid m^a$

$\langle 2 \rangle 4$. $n \mid m^a$

$\langle 2 \rangle 5$. $m^a \equiv 0 \pmod{n}$

$\langle 2 \rangle 6$. m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

\square

Chapter 12

Ring Homomorphisms

Definition 12.1 (Ring Homomorphism). Let R and S be rings. A *ring homomorphism* $\phi : R \rightarrow S$ is a rng homomorphism such that $\phi(1) = 1$.

Proposition 12.2. *The zero-ring is terminal in **Ring**.*

PROOF: Easy. \square

Proposition 12.3. *The ring \mathbb{Z} is initial in **Ring**.*

PROOF: Easy. \square

Proposition 12.4. *Let R and S be rings and $\phi : R \rightarrow S$ be a rng homomorphism. If ϕ is surjective, then ϕ is a ring homomorphism.*

PROOF:

$\langle 1 \rangle 1$. PICK $a \in R$ such that $\phi(a) = 1$

$\langle 1 \rangle 2$. $\phi(1) = 1$

PROOF:

$$\phi(1) = \phi(1)\phi(a)$$

$$= \phi(1a)$$

$$= \phi(a)$$

$$= 1$$

\square

Example 12.5. For any set S we have $\mathcal{P}S \cong (\mathbb{Z}/2\mathbb{Z})^S$ in **Ring** with the isomorphism

$$\begin{aligned} \phi : \mathcal{P}S &\cong (\mathbb{Z}/2\mathbb{Z})^S \\ \phi(A)(s) &= \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases} \end{aligned}$$

Example 12.6. The function $\mathbb{H} \rightarrow \mathfrak{gl}_4(\mathbb{R})$ that maps $a + bi + cj + dk$ to

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

is a monomorphism in **Ring**, as is the function $\mathbb{H} \rightarrow \mathfrak{sl}_2(\mathbb{C})$ that maps $a + bi + cj + dk$ to

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

Proposition 12.7. *Ring homomorphisms preserve units.*

PROOF: If $uv = 1$ then $\phi(u)\phi(v) = 1$. \square

Proposition 12.8. *Let $\phi : R \rightarrow S$ be a ring homomorphism. Then the following are equivalent.*

1. ϕ is a monomorphism.
2. $\ker \phi = \{0\}$
3. ϕ is injective.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: ϕ is a monomorphism.

$\langle 2 \rangle 2.$ LET: $r \in \ker \phi$

$\langle 2 \rangle 3.$ LET: $\text{ev}_r : \mathbb{Z}[x] \rightarrow R$ be the unique ring homomorphism such that $\text{ev}_r(x) = r$.

$\langle 2 \rangle 4.$ LET: $\text{ev}_0 : \mathbb{Z}[x] \rightarrow R$ be the unique ring homomorphism such that $\text{ev}_0(x) = 0$.

$\langle 2 \rangle 5.$ $\phi \circ \text{ev}_r = \phi \circ \text{ev}_0$

$\langle 2 \rangle 6.$ $\text{ev}_r = \text{ev}_0$

$\langle 2 \rangle 7.$ $r = 0$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: Proposition 7.20.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: Easy.

\square

Example 12.9. It is not true that every epimorphism in **Ring** is surjective. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism but not surjective.

The same example shows that a ring homomorphism may be a monomorphism and an epimorphism but not be an isomorphism.

Example 12.10.

$$\text{End}_{\mathbf{Ab}}(\mathbb{Z}) \cong \mathbb{Z}$$

The isomorphism maps any group endomorphism $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ to $\phi(1)$.

Example 12.11. The group of units of $\text{End}_{\mathbf{Ab}}(G)$ is $\text{Aut}_{\mathbf{Ab}}(G)$.

Example 12.12. Let R be a ring. Then the function $\lambda : R \rightarrow \text{End}_{\mathbf{Ab}}(R)$ defined by

$$\lambda(a)(b) = ab$$

is a ring monomorphism.

PROOF: Easy. \square

12.1 Products

Proposition 12.13. *Let R and S be rings. Then $R \times S$ is a ring under componentwise addition and multiplication, and this ring is the product of R and S in **Ring**.*

PROOF: Easy. \square

Chapter 13

Subrings

Definition 13.1 (Subring). Let S be a ring. A *subring* of S is a ring R such that R is a subset of S and the inclusion $R \hookrightarrow S$ is a ring homomorphism.

Proposition 13.2. *Let R and S be rings. Then R is a subring of S if and only if R is a subset of S , the unit 1 of S is an element of R , and the operations of R are the restrictions of the operations of S to R .*

PROOF: Easy. \square

Corollary 13.2.1. *The zero ring is not a subring of any non-zero ring.*

Proposition 13.3. *Let $\phi : R \rightarrow S$ be a ring homomorphism. Then $\phi(R)$ is a subring of S .*

PROOF: Easy. \square

13.1 Centralizer

Definition 13.4 (Centralizer). Let R be a ring and $a \in R$. The *centralizer* of a is $\{r \in R : ar = ra\}$.

Proposition 13.5. *The centralizer of a is a subring of R .*

PROOF: Easy. \square

13.2 Center

Definition 13.6 (Center). The *center* of a ring R is $\{x \in R : \forall y \in R. xy = yx\}$.

Proposition 13.7. *The center of a ring is a subring.*

PROOF: Easy. \square

Proposition 13.8. *Let R be a ring. The center of $\text{End}_{\mathbf{Ab}}(R)$ is isomorphic to the center of R .*

PROOF:

$\langle 1 \rangle 1$. LET: $\lambda : R \rightarrow \text{End}_{\mathbf{Ab}}(R)$ be left multiplication.

$\langle 1 \rangle 2$. λ maps $Z(R)$ to $Z(\text{End}_{\mathbf{Ab}}(R))$.

$\langle 2 \rangle 1$. LET: $a \in Z(R)$

$\langle 2 \rangle 2$. LET: $\phi \in \text{End}_{\mathbf{Ab}}(R)$

PROVE: $\lambda(a) \circ \phi = \phi \circ \lambda(a)$

$\langle 2 \rangle 3$. LET: $x \in R$

$\langle 2 \rangle 4$. $a + \phi(x) = \phi(a + x)$

$\langle 1 \rangle 3$. $\lambda(Z(R)) = Z(\text{End}_{\mathbf{Ab}}(R))$

$\langle 2 \rangle 1$. LET: $\phi \in Z(\text{End}_{\mathbf{Ab}}(R))$

$\langle 2 \rangle 2$. For all $r \in R$,

LET: $\mu_r \in \text{End}_{\mathbf{Ab}}(R)$ be right multiplication by r .

$\langle 2 \rangle 3$. For all $r \in R$ we have $\phi \circ \mu_r = \mu_r \circ \phi$.

$\langle 2 \rangle 4$. For all $r, x \in R$ we have $\phi(xr) = \phi(x)r$

$\langle 2 \rangle 5$. For all $r \in R$ we have $\phi(r) = \phi(1)r$

$\langle 2 \rangle 6$. $\phi = \lambda(\phi(1))$

□

Corollary 13.8.1. *If R is a commutative ring then R is isomorphic to the center of $\text{End}_{\mathbf{Ab}}(R)$.*

Example 13.9. For n a positive integer we have $\mathbb{Z}/n\mathbb{Z} \cong \text{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$.

Since, for any $\phi \in \text{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$ we have $\phi(m) = m\phi(1)$ and so the whole of $\text{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$ is the image of λ .

Chapter 14

Monoid Rings

Definition 14.1 (Monoid Ring). Let R be a ring and M a monoid. Define $R[M]$ to be the ring whose elements are the families $\{a_m\}_{m \in M}$ such that $a_m = 0$ for all but finitely many $m \in M$, written

$$\sum_{m \in M} a_m m \text{ ,}$$

under

$$\begin{aligned} \sum_m a_m m + \sum_m b_m m &= \sum_m (a_m + b_m) m \\ \left(\sum_m a_m m \right) \left(\sum_m b_m m \right) &= \sum_{m \in M} \sum_{m_1 m_2 = m} a_{m_1} b_{m_2} m \end{aligned}$$

Example 14.2. Ring homomorphisms do not necessarily preserve zero-divisors. The canonical homomorphism $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z}$ maps the non-zero-divisor 2 to a zero-divisor.

14.1 Polynomials

Definition 14.3 (Polynomial). Let R be a ring. The ring of *polynomials* $R[x]$ is $R[\mathbb{N}]$. We write

$$\sum_n a_n x^n \text{ for } \sum_n a_n n \text{ .}$$

Concretely, a *polynomial* in R is a sequence (a_n) in R such that there exists N such that $\forall n \geq N. a_n = 0$. We write the polynomial as

$$\sum_{n=0}^{N-1} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1} \text{ .}$$

We write $R[x]$ for the set of all polynomials in R .

Define addition and multiplication on $R[x]$ by

$$\begin{aligned}\sum_n a_n x^n + \sum_n b_n x^n &= \sum_n (a_n + b_n) x^n \\ \left(\sum_n a_n x^n \right) \left(\sum_n b_n x^n \right) &= \sum_n \sum_{i+j=n} a_i b_j x^n\end{aligned}$$

A *constant* is a polynomial of the form $a + 0x + 0x^2 + \cdots$ for some $a \in R$. We write $R[x_1, \dots, x_n]$ for $R[x_1][x_2] \cdots [x_n]$.

Proposition 14.4. *For any ring R , the set of polynomials $R[x]$ is a ring.*

PROOF: Easy. \square

Definition 14.5 (Degree). The *degree* of a polynomial $\sum_n a_n x^n$ is the largest integer d such that $a_d \neq 0$. We take the degree of the zero polynomial to be $-\infty$.

Proposition 14.6. *Let R be a ring and $f, g \in R[x]$ be nonzero polynomials. Then*

$$\deg(f + g) \leq \max(\deg f, \deg g) .$$

PROOF: If $a_n + b_n \neq 0$ then $a_n \neq 0$ or $b_n \neq 0$. \square

Proposition 14.7. *The function $i : n \rightarrow \mathbb{Z}[x_1, \dots, x_n]$ that maps k to x_k is initial in the category with:*

- *objects all pairs $j : n \rightarrow R$ where R is a commutative ring and j a function*
- *morphisms $\phi : (j_1, R_1) \rightarrow (j_2, R_2)$ are ring homomorphisms $\phi : R_1 \rightarrow R_2$ such that $\phi \circ j_1 = j_2$.*

PROOF: The unique morphism $(i, \mathbb{Z}[x_1, \dots, x_n]) \rightarrow (j, R)$ maps a polynomial p to $p(j(0), j(1), \dots, j(n-1))$. \square

Proposition 14.8. *Let $\alpha : R \rightarrow S$ be a ring homomorphism. Let $s \in S$ commute with $\alpha(r)$ for all $r \in R$. Then there exists a unique ring homomorphism $\bar{\alpha} : R[x] \rightarrow S$ such that $\bar{\alpha}(x) = s$ and the following diagram commutes:*

$$\begin{array}{ccc} R[x] & \xrightarrow{\bar{\alpha}} & S \\ \uparrow & \nearrow \alpha & \\ R & & \end{array}$$

PROOF: The map $\bar{\alpha}$ is given by

$$\bar{\alpha}(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = \alpha(a_0) + \alpha(a_1)s + \alpha(a_2)s^2 + \cdots + \alpha(a_n)s^n .$$

\square

Definition 14.9. Let R be a commutative ring. Given a polynomial $p \in R[x]$, the *polynomial function* $p : R \rightarrow R$ is the function given by: $p(r) = \alpha_r(p)$, where $\alpha_r : R[x] \rightarrow R$ is the unique ring homomorphism such that the following diagram commutes.

$$\begin{array}{ccc} R[x] & \xrightarrow{\alpha_r} & R \\ \uparrow x & \nearrow r & \\ 1 & & \end{array}$$

Proposition 14.10. $\mathbb{Z}[x, y]$ is the coproduct of $\mathbb{Z}[x]$ and $\mathbb{Z}[y]$ in the category of commutative rings.

PROOF: Given ring homomorphisms $f : \mathbb{Z}[x] \rightarrow R$ and $g : \mathbb{Z}[y] \rightarrow R$, the required morphism $\mathbb{Z}[x, y] \rightarrow R$ maps $p(x, y)$ to $p(f(x), g(y))$. \square

Example 14.11. $\mathbb{Z}[x, y]$ is not the coproduct of $\mathbb{Z}[x]$ and $\mathbb{Z}[y]$ in **Ring**. Given $f : \mathbb{Z}[x] \rightarrow R$ and $g : \mathbb{Z}[y] \rightarrow R$ with $f(x) \neq g(y)$, the mediating morphism $\mathbb{Z}[x, y] \rightarrow R$ cannot exist since it must map xy to both $f(x)g(y)$ and $g(y)f(x)$. \square

Definition 14.12. A polynomial is *monic* iff its last non-zero coefficient is 1.

Proposition 14.13. A monic polynomial is not a left- or right-zero-divisor.

PROOF: Easy. \square

Proposition 14.14. Let R be a ring. Let $f, g \in R[x]$ with f monic. Then there exist unique polynomials $q, r \in R[x]$ with $\deg r < \deg f$ such that

$$g = qf + r .$$

PROOF:

$\langle 1 \rangle 1$. LET: $d = \deg f$

$\langle 1 \rangle 2$. For all $a \in R$ and $n > d$, there exists $h \in R[x]$ with $\deg h < n$ such that

$$ax^n = ax^{n-d}f + h .$$

PROOF: Take $h = ax^n - ax^{n-d}f$.

$\langle 1 \rangle 3$. For all $a \in R$ and $n > d$, there exists $q, h \in R[x]$ with $\deg h \leq d$ such that

$$ax^n = qf + h .$$

PROOF: Repeating $\langle 1 \rangle 2$ by induction.

$\langle 1 \rangle 4$. LET: $g = \sum_{i=0}^n a_i x^i$

$\langle 1 \rangle 5$. For $i > d$, PICK $q_i h_i \in R[x]$ with $\deg h < \deg f$ such that $a_i x^i = q_i f + h_i$

$\langle 1 \rangle 6$. $g = (\sum_{i=d+1}^n q_i) f + \sum_{i=d+1}^n h_i$

$\langle 1 \rangle 7$. q and r are unique.

PROOF: If $q_1 f + r_1 = q_2 f + r_2$ then $r_1 - r_2 = (q_2 - q_1)f$ and so $r_1 - r_2 = (q_2 - q_1)f = 0$ since $\deg(r_1 - r_2) < \deg f$.

\square

14.2 Laurent Polynomials

Definition 14.15 (Laurent Polynomial). Let R be a ring. The ring of *Laurent polynomials* is the group ring $R[\mathbb{Z}]$. We write $\sum_{n \in \mathbb{Z}} a_n x^n$ for $\sum_n a_n n$.

14.3 Power Series

Definition 14.16 (Power Series). Let R be a ring. A *power series* in R is a sequence (a_n) in R . We write the power series as

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

We write $R[[x]]$ for the set of all power series in R .

Define addition and multiplication on $R[[x]]$ by

$$\begin{aligned} \sum_n a_n x^n + \sum_n b_n x^n &= \sum_n (a_n + b_n) x^n \\ \left(\sum_n a_n x^n \right) \left(\sum_n b_n x^n \right) &= \sum_n \sum_{i+j=n} a_i b_j x^n \end{aligned}$$

Proposition 14.17. *For any ring R , the set of power series $R[[x]]$ is a ring.*

PROOF: Easy. \square

Proposition 14.18. *A power series $\sum_n a_n x^n$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R .*

PROOF:

$\langle 1 \rangle 1$. If $\sum_n a_n x^n$ is a unit then a_0 is a unit.

$\langle 2 \rangle 1$. LET: $\sum_n b_n x^n$ be the inverse of $\sum_n a_n x^n$.

$\langle 2 \rangle 2$. $a_0 b_0 = b_0 a_0 = 1$

$\langle 1 \rangle 2$. If a_0 is a unit then $\sum_n a_n x^n$ is a unit.

PROOF: Define the sequence (b_n) in R by

$$b_n = -a_0^{-1} \sum_{i=1}^n a_i b_{n-i}$$

Then $\sum_n b_n x^n$ is the inverse of $\sum_n a_n x^n$.

\square

Chapter 15

Ideals

Definition 15.1 (Left-Ideal). Let R be a ring.

A subgroup I of R is a *left-ideal* iff, for all $r \in R$, we have $rI \subseteq I$.

A subgroup I of R is a *right-ideal* iff, for all $r \in R$, we have $Ir \subseteq I$.

A subgroup I of R is a *(two-sided) ideal* iff it is a left-ideal and a right-ideal.

Example 15.2. Let R be a ring and $a \in R$. Then Ra is a left-ideal and aR is a right-ideal.

In particular, $\{0\}$ is always a two-sided ideal.

Example 15.3. Let S be a set and $T \subseteq S$. Then $\{X \in \mathcal{P}S : X \subseteq T\}$ is an ideal in $\mathcal{P}S$.

Proposition 15.4. Let S be a finite set. Then every ideal in $\mathcal{P}S$ is of the form $\{X \in \mathcal{P}S : X \subseteq T\}$ for some $T \subseteq S$.

PROOF:

$\langle 1 \rangle 1$. LET: I be an ideal in $\mathcal{P}S$.

$\langle 1 \rangle 2$. LET: $T = \bigcup I$

$\langle 1 \rangle 3$. For all $i \in T$ we have $\{i\} \in I$.

$\langle 2 \rangle 1$. LET: $i \in T$

$\langle 2 \rangle 2$. PICK $X \in I$ such that $i \in X$

$\langle 2 \rangle 3$. $\{i\} = \{i\} \cap X \in I$

$\langle 1 \rangle 4$. For all $X \subseteq T$ we have $X \in I$.

PROOF: If $X = \{x_1, \dots, x_n\}$ then $X = \{x_1\} + \dots + \{x_n\} \in I$.

□

Example 15.5. If S is an infinite set, then there is always an ideal in $\mathcal{P}S$ that is not of the form $\{X \in \mathcal{P}S : X \subseteq T\}$ for some $T \subseteq S$, namely the set of all finite subsets of S .

Proposition 15.6. Let $\phi : R \twoheadrightarrow S$ be a surjective ring homomorphism. Let J be an ideal in R . Then $\phi(J)$ is an ideal in S .

PROOF:

- $\langle 1 \rangle 1$. LET: $j \in J$ and $s \in S$
 PROVE: $s\phi(j), \phi(j)s \in \phi(J)$
 $\langle 1 \rangle 2$. PICK $r \in R$ such that $\phi(r) = s$
 $\langle 1 \rangle 3$. $rj, jr \in J$
 $\langle 1 \rangle 4$. $s\phi(j), \phi(j)s \in \phi(J)$
 \square

Example 15.7. We cannot remove the hypothesis that ϕ is surjective.

Let $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion. Then $i(2\mathbb{Z}) = 2\mathbb{Z}$ is not an ideal in \mathbb{Q} .

Proposition 15.8. Let $\phi : R \rightarrow S$ be a ring homomorphism and I a (left-, right-)ideal in S . Then $\phi^{-1}I$ is a (left-, right-)ideal in R .

PROOF: Easy. \square

Corollary 15.8.1. Let $\phi : R \rightarrow S$ be a ring homomorphism. Then $\ker \phi$ is an ideal in R .

Definition 15.9 (Quotient Ring). Let I be an ideal in R . The *quotient ring* R/I is the quotient group R/I under

$$(a + I)(b + I) = ab + I.$$

This is well-defined as, if $a + I = a' + I$ and $b + I = b' + I$ then

$$\begin{aligned}
 a - a' &\in I \\
 b - b' &\in I \\
 \therefore ab - a'b &\in I \\
 a'b - a'b' &\in I \\
 \therefore ab - a'b' &\in I
 \end{aligned}$$

Proposition 15.10. Let I be an ideal in R . Then the canonical group homomorphism $\pi : R \rightarrow R/I$ is a ring homomorphism.

PROOF: By construction. \square

Proposition 15.11. Let I be an ideal in a ring R . For every ring homomorphism $\phi : R \rightarrow S$ such that $I \subseteq \ker \phi$, there exists a unique ring homomorphism $\bar{\phi} : R/I \rightarrow S$ such that the following diagram commutes.

$$\begin{array}{ccc}
 R & \xrightarrow{\phi} & S \\
 \searrow \pi & & \nearrow \bar{\phi} \\
 & R/I &
 \end{array}$$

PROOF: Easy. \square

Corollary 15.11.1. Every ring homomorphism $\phi : R \rightarrow S$ decomposes as follows.

$$\begin{array}{ccccc}
 & & \phi & & \\
 & \searrow & \text{---} & \nearrow & \\
 R & \twoheadrightarrow & R/\ker \phi & \xrightarrow{\cong} & \text{im } \phi & \hookrightarrow & S
 \end{array}$$

Corollary 15.11.2 (First Isomorphism Theorem). *Let $\phi : R \rightarrow S$ be a surjective ring homomorphism. Then*

$$S \cong R/\ker \phi .$$

Theorem 15.12 (Third Isomorphism Theorem). *Let I and J be ideals in R with $I \subseteq J$. Then J/I is an ideal in R/I , and*

$$\frac{R/I}{J/I} \cong R/J$$

PROOF: Since the function $R/I \rightarrow R/J$ that maps $r + I$ to $r + J$ is a surjective ring homomorphism with kernel J/I . \square

Corollary 15.12.1. *Let $\phi : R \rightarrow S$ be a surjective ring homomorphism. Let J be an ideal in R . Then*

$$\frac{S}{\phi(J)} \cong \frac{R}{\ker \phi + J}$$

Proposition 15.13. *Let R be a ring and J an ideal in $\mathfrak{gl}_n(R)$. Let $A \in \mathfrak{gl}_n(R)$. Then $A \in J$ if and only if the matrices obtained by placing any entry of A in any position and zeros elsewhere all belong to J .*

PROOF: Each such matrix can be obtained by pre- and post-multiplying A by matrices which have a single 1 and 0s elsewhere. Conversely, A is a sum of such matrices. \square

Corollary 15.13.1. *Let R be a ring. Let J be an ideal in $\mathfrak{gl}_n(R)$. Let I be the set of all entries of elements of J . Then I is an ideal in R , and J is the set of all matrices whose entries are in I .*

Proposition 15.14. *Let R be a ring. Let $\{I_\alpha\}_{\alpha \in A}$ be a family of ideals in R . Let*

$$\sum_{\alpha \in A} I_\alpha = \left\{ \sum_{\alpha \in A} r_\alpha : \forall \alpha, r_\alpha \in I_\alpha, r_\alpha = 0 \text{ for all but finitely many } \alpha \in A \right\} .$$

Then $\sum_{\alpha \in A} I_\alpha$ is an ideal, and is the smallest ideal that includes every I_α .

PROOF: Easy. \square

Proposition 15.15. *The intersection of a set of ideals is an ideal.*

PROOF: Easy. \square

15.1 Characteristic

Definition 15.16 (Characteristic). The *characteristic* of a ring R is the non-negative integer n such that $n\mathbb{Z}$ is the kernel of the unique ring homomorphism $\mathbb{Z} \rightarrow R$.

Proposition 15.17. *Let R be a ring. If the unit 1 has finite order in R , then its order is the characteristic of R ; otherwise, the characteristic of R is 0.*

PROOF: Easy. \square

Example 15.18. The zero ring is the only ring with characteristic 1.

15.2 Nilradical

Definition 15.19 (Nilradical). Let R be a commutative ring. The *nilradical* of R is the set of all nilpotent elements.

Proposition 15.20. *Let R be a commutative ring. The nilradical of R is an ideal in R .*

PROOF: If $a^n = 0$ then for any b we have $(ba)^n = 0$. \square

Example 15.21. We cannot remove the assumption that R is commutative. In $\mathfrak{gl}_2(\mathbb{R})$ we have that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not.

15.3 Principal Ideals

Definition 15.22 (Principal Ideal). Let R be a commutative ring and $a \in R$. The *principal ideal* generated by a is $(a) = Ra = aR$.

Example 15.23. $\{0\} = (0)$ and $R = \{1\}$ are principal ideals.

Definition 15.24. Let R be a commutative ring and $\{a_\alpha\}_{\alpha \in A}$ be a family of elements of R . The *ideal generated by the elements a_α* is

$$(a_\alpha)_{\alpha \in A} := \sum_{\alpha \in A} (a_\alpha) \ .$$

An ideal is *finitely generated* iff it is generated by a finite family of elements.

Definition 15.25. Let R be a commutative ring and I, J be ideals in R . Then IJ is the ideal generated by $\{ij\}_{i \in I, j \in J}$.

Proposition 15.26.

$$IJ \subseteq I \cap J$$

PROOF: Easy. \square

Proposition 15.27. *Let R be a commutative ring. Let I and J be ideals in R . If $I + J = R$ then $IJ = I \cap J$.*

PROOF:

$\langle 1 \rangle 1$. LET: $r \in I \cap J$

$\langle 1 \rangle 2$. PICK $i \in I$ and $j \in J$ such that $i + j = 1$.

$\langle 1 \rangle 3$. $ri, rj \in IJ$

$\langle 1 \rangle 4$. $r = ri + rj \in IJ$

\square

Proposition 15.28. *Let R be a commutative ring. Let $f \in R[x]$ be a monic polynomial of degree d . Then the function*

$$\phi : R[x] \rightarrow R^{\oplus d}$$

that sends a polynomial g to the remainder of the division of g by f induces an isomorphism of Abelian groups

$$\frac{R[x]}{(f(x))} \cong R^{\oplus d}.$$

PROOF: It is clearly a group homomorphism; it is surjective since it maps any polynomial of degree $< d$ to itself, and its kernel is $(f(x))$ since these are the polynomials with remainder 0. \square

Corollary 15.28.1. *Let R be a commutative ring and $a \in R$. Then we have*

$$\frac{R[x]}{(x - a)} \cong R$$

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : R[x] \rightarrow R$ be evaluation at a .

$\langle 1 \rangle 2$. $\phi(g)$ is the remainder when dividing g by $x - a$.

PROOF: If $g = (x - a)q + r$ then $g(a) = (a - a)q(a) + r = r$.

$\langle 1 \rangle 3$. ϕ induces a group isomorphism $R[x]/(x - a) \cong R$

PROOF: By the theorem.

$\langle 1 \rangle 4$. This isomorphism is a ring isomorphism.

PROOF: Since evaluation at a is a ring homomorphism.

\square

Example 15.29. We have

$$\frac{\mathbb{R}[x]}{(x^2 + 1)} \cong \mathbb{C}$$

as rings.

15.4 Maximal Ideals

Definition 15.30 (Maximal Ideal). Let R be a ring and I an ideal in R . Then I is a *maximal ideal* iff $I \neq R$ and, whenever J is an ideal with $I \subseteq J$, then either $I = J$ or $J = R$.

Chapter 16

Integral Domains

Definition 16.1 (Integral Domain). An *integral domain* is a non-trivial commutative ring with no nonzero zero-divisors.

Example 16.2. \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are integral domains.

Proposition 16.3. $\mathbb{Z}/n\mathbb{Z}$ is an integral domain if and only if n is prime.

PROOF:

$$\begin{aligned} n \text{ is prime} &\Leftrightarrow \forall a, b \in \mathbb{Z} (n \mid ab \Rightarrow n \mid a \vee n \mid b) \\ &\Leftrightarrow \forall a, b \in \mathbb{Z}/n\mathbb{Z} (ab \cong 0(\bmod n) \Rightarrow a \cong 0(\bmod n) \vee b \cong 0(\bmod n)) \\ &\Leftrightarrow \mathbb{Z}/n\mathbb{Z} \text{ is an integral domain} \quad \square \end{aligned}$$

Proposition 16.4. In an integral domain, if $x^2 = 1$ then $x = \pm 1$.

PROOF: We have $x^2 - 1 = (x - 1)(x + 1) = 0$ so $x - 1 = 0$ or $x + 1 = 0$. \square

Proposition 16.5. Let R be an integral domain and $f, g \in R[x]$. Then

$$\deg(fg) = \deg f + \deg g$$

PROOF:

$\langle 1 \rangle 1$. LET: $f = \sum_n a_n x^n$ and $g = \sum_n b_n x^n$.

$\langle 1 \rangle 2$. LET: $d = \deg f$ and $e = \deg g$.

$\langle 1 \rangle 3$. The $d + e$ th term of fg is

$$a_d b_e x^{d+e}$$

which is non-zero.

$\langle 1 \rangle 4$. For $n > d + e$ the n th term of fg is 0.

\square

Corollary 16.5.1. Let R be a ring. Then $R[x]$ is an integral domain if and only if R is an integral domain.

Proposition 16.6. Let R be a ring. Then $R[[x]]$ is an integral domain if and only if R is an integral domain.

PROOF:

$\langle 1 \rangle 1$. If $R[[x]]$ is an integral domain then R is an integral domain.

PROOF: Easy.

$\langle 1 \rangle 2$. If R is an integral domain then $R[[x]]$ is an integral domain.

$\langle 2 \rangle 1$. ASSUME: R is an integral domain.

$\langle 2 \rangle 2$. LET: $(\sum_n a_n x^n)(\sum_n b_n x^n) = 0$

$\langle 2 \rangle 3$. $a_0 b_0 = 0$

$\langle 2 \rangle 4$. $a_0 = 0$ or $b_0 = 0$

$\langle 2 \rangle 5$. ASSUME: w.l.o.g. $b_0 \neq 0$

PROVE: For all n we have $a_n = 0$

$\langle 2 \rangle 6$. ASSUME: as induction hypothesis $a_0 = a_1 = \cdots = a_{n-1} = 0$

$\langle 2 \rangle 7$. $\sum_{i=0}^n a_i b_{n-i} = 0$

$\langle 2 \rangle 8$. $a_n b_0 = 0$

$\langle 2 \rangle 9$. $a_n = 0$

□

Proposition 16.7. *Let R be a ring and S an integral domain. Every ring homomorphism $\phi : R \rightarrow S$ is a ring homomorphism.*

PROOF:

$$\begin{aligned}\phi(1) &= \phi(1 \cdot 1) \\ &= \phi(1)\phi(1)\end{aligned}$$

and so $\phi(1) = 1$ by Cancellation. □

Proposition 16.8. *The characteristic of an integral domain is either 0 or a prime number.*

PROOF:

$\langle 1 \rangle 1$. LET: D be an integral domain.

$\langle 1 \rangle 2$. LET: n be the characteristic of D

$\langle 1 \rangle 3$. ASSUME: $n \neq 0$

$\langle 1 \rangle 4$. ASSUME: $n = ab$

$\langle 1 \rangle 5$. $ab = 0$ in D

$\langle 1 \rangle 6$. $a = 0$ or $b = 0$ in D

$\langle 1 \rangle 7$. $n \mid a$ or $n \mid b$

$\langle 1 \rangle 8$. One of a, b is 1 and the other is n .

□

16.1 Prime Ideals

Definition 16.9 (Prime Ideal). Let I be an ideal in a commutative ring R . Then I is a *prime ideal* iff R/I is an integral domain.

Example 16.10. Let R be a commutative ring and $a \in R$. Then $(x - a)$ is a prime ideal in R iff R is an integral domain.

Proposition 16.11. *Let R be a commutative ring and I a proper ideal in R . Then I is prime iff, whenever $ab \in I$, then $a \in I$ or $b \in I$.*

PROOF: The condition is the same as saying that, if $(a + I)(b + I) = I$, then $a + I = I$ or $b + I = I$. \square

Definition 16.12 (Spectrum). The *spectrum* of a commutative ring R , $\text{Spec } R$, is the set of prime ideals.

Proposition 16.13. Let $\phi : R \rightarrow S$ be a ring homomorphism. If I is a prime ideal in S then $\phi^{-1}(I)$ is a prime ideal in R .

PROOF: If $ab \in \phi^{-1}(I)$ then $\phi(a)\phi(b) \in I$ so either $\phi(a) \in I$ or $\phi(b) \in I$, i.e. either $a \in \phi^{-1}(I)$ or $b \in \phi^{-1}(I)$. \square

Proposition 16.14. Let R be a commutative ring. Suppose there exists a prime ideal P in R such that the only zero-divisor in P is 0. Then R is an integral domain.

PROOF:

$\langle 1 \rangle$ 1. ASSUME: $ab = 0$ in R

$\langle 1 \rangle$ 2. $ab \in P$

$\langle 1 \rangle$ 3. $a \in P$ or $b \in P$

$\langle 1 \rangle$ 4. $a = 0$ or $b = 0$

\square

Proposition 16.15. Let R be a commutative ring. The nilradical of R is included in every prime ideal of R .

PROOF: Let P be a prime ideal. If $a^n = 0$ then $a^n \in P$ hence $a \in P$. \square

Definition 16.16 (Krull Dimension). The (*Krull*) *dimension* of a commutative ring R is the length of the longest chain of prime ideals in R .

Example 16.17. $\mathbb{Z}[x]$ has Krull dimension 2.

Chapter 17

Unique Factorization Domains

Example 17.1. \mathbb{Z} is a UFD.

Chapter 18

Noetherian Rings

Definition 18.1 (Noetherian Ring). A commutative ring is *Noetherian* iff every ideal is finitely generated.

Proposition 18.2. *The homomorphic image of a Noetherian ring is Noetherian.*

PROOF:

⟨1⟩1. LET: R be a Noetherian ring, S be a commutative ring, and $\phi : R \rightarrow S$ a surjective ring homomorphism.

⟨1⟩2. LET: I be an ideal in S .

⟨1⟩3. LET: $\phi^{-1}(I) = (a_1, \dots, a_n)$

⟨1⟩4. $I = (\phi(a_1), \dots, \phi(a_n))$

□

Chapter 19

Principal Ideal Domains

Definition 19.1 (Principal Ideal Domain). A commutative ring is a *principal ideal domain* (PID) iff every ideal is principal.

Example 19.2. \mathbb{Z} is a PID by Proposition 7.16.

Example 19.3. $\mathbb{Z}[x]$ is not a PID. The ideal $(2, x)$ is not principal.

Proposition 19.4. *Every PID is Noetherian.*

PROOF: Trivial. \square

Proposition 19.5. *Every nonzero prime ideal in a PID is maximal.*

PROOF:

$\langle 1 \rangle$ 1. LET: R be a PID.

$\langle 1 \rangle$ 2. LET: I be a nonzero prime ideal in R .

$\langle 1 \rangle$ 3. PICK $a \in R$ such that $I = (a)$.

$\langle 1 \rangle$ 4. LET: J be an ideal such that $I \subseteq J$

$\langle 1 \rangle$ 5. PICK $b \in R$ such that $J = (b)$.

$\langle 1 \rangle$ 6. PICK $t \in R$ such that $a = bt$.

$\langle 1 \rangle$ 7. $b \in I$ or $t \in I$

$\langle 1 \rangle$ 8. CASE: $b \in I$

PROOF: Then $J \subseteq I$ so $I = J$.

$\langle 1 \rangle$ 9. CASE: $t \in I$

$\langle 2 \rangle$ 1. PICK $s \in R$ such that $t = as$.

$\langle 2 \rangle$ 2. $a = ast$

$\langle 2 \rangle$ 3. $st = 1$

PROOF: Since R is an integral domain.

$\langle 2 \rangle$ 4. $1 \in I$

$\langle 2 \rangle$ 5. $I = R$

\square

Corollary 19.5.1. *Any PID has Krull dimension 1.*

Chapter 20

Euclidean Domains

Example 20.1. \mathbb{Z} is a Euclidean domain.

Chapter 21

Division Rings

Definition 21.1 (Division Ring). A *division ring* is a ring in which every nonzero element is a two-sided unit.

Example 21.2. The quaternions form a division ring, with the inverse of a non-zero element $a + bi + cj + dk$ being

$$\frac{1}{a^2 + b^2 + c^2 + d^2}(a - bi - cj - dk) .$$

Example 21.3. For any ring R , the ring of polynomials $R[x]$ is not a division ring, since x has no inverse.

Proposition 21.4. *Every centralizer in a division ring is a division ring.*

PROOF: If $ar = ra$ then $ra^{-1} = a^{-1}r$. \square

Proposition 21.5. *A non-trivial ring R is a division ring if and only if its only left-ideals and right-ideals are $\{0\}$ and R .*

PROOF:

$\langle 1 \rangle 1$. If R is a division ring then the only left-ideals and right-ideals are $\{0\}$ and R .

$\langle 2 \rangle 1$. ASSUME: R is a division ring.

$\langle 2 \rangle 2$. The only left-ideals are $\{0\}$ and R .

$\langle 3 \rangle 1$. LET: I be a left-ideal that is not $\{0\}$.

PROVE: $I = R$

$\langle 3 \rangle 2$. PICK $a \in I - \{0\}$

$\langle 3 \rangle 3$. PICK a left inverse b for a

$\langle 3 \rangle 4$. $1 \in I$

PROOF: Since $1 = ba$.

$\langle 3 \rangle 5$. $I = R$

PROOF: For any $r \in R$ we have $r = r1 \in I$.

$\langle 2 \rangle 3$. The only right-ideals are $\{0\}$ and R .

PROOF: Similar.

$\langle 1 \rangle 2$. If the only left-ideals and right-ideals are $\{0\}$ and R then R is a division ring.

□

Proposition 21.6. *Let K be a division ring and R a non-trivial ring. Every ring homomorphism $K \rightarrow R$ is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : K \rightarrow R$ be a ring homomorphism.

PROVE: $\ker \phi = \{0\}$

$\langle 1 \rangle 2$. LET: $x \in \ker \phi$

$\langle 1 \rangle 3$. ASSUME: for a contradiction $x \neq 0$.

$\langle 1 \rangle 4$. $\phi(xx^{-1}) = 1$

$\langle 1 \rangle 5$. $0 = 1$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts the assumption that R is non-trivial.

□

Chapter 22

Simple Rings

Definition 22.1 (Simple Ring). A non-trivial ring is *R simple* iff its only two-sided ideals are $\{0\}$ and R .

Example 22.2. For any simple ring R we have $\mathfrak{gl}_n(R)$ is simple, by Corollary 15.13.1.

Proposition 22.3. *Let R be a ring and I an ideal in R . Then I is maximal iff R/I is simple.*

PROOF:

$$\begin{aligned} R/I \text{ is simple} &\Leftrightarrow \text{the only ideals in } R/I \text{ are } \{I\} \text{ and } R/I \\ &\Leftrightarrow \text{the only ideals in } R \text{ that include } I \text{ are } I \text{ and } R \\ &\Leftrightarrow I \text{ is maximal} \end{aligned}$$

□

Chapter 23

Reduced Rings

Definition 23.1 (Reduced Ring). A ring is *reduced* iff it has no non-zero nilpotent elements.

Proposition 23.2. *Let R be a commutative ring. Let N be its nilradical. Then R/N is reduced.*

PROOF:

- $\langle 1 \rangle 1.$ LET: $r + N$ be nilpotent.
- $\langle 1 \rangle 2.$ PICK n such that $(r + N)^n = N$
- $\langle 1 \rangle 3.$ $r^n \in N$
- $\langle 1 \rangle 4.$ PICK k such that $(r^n)^k = 0$
- $\langle 1 \rangle 5.$ $r^{nk} = 0$
- $\langle 1 \rangle 6.$ $r \in N$
- $\langle 1 \rangle 7.$ $r + N = N$

□

Proposition 23.3. *Let R be a commutative ring. Let I and J be ideals in R . If R/IJ is reduced then $IJ = I \cap J$.*

PROOF:

- $\langle 1 \rangle 1.$ LET: $r \in I \cap J$
PROVE: $r \in IJ$
- $\langle 1 \rangle 2.$ $r^2 \in IJ$
- $\langle 1 \rangle 3.$ $(r + IJ)^2 = IJ$
- $\langle 1 \rangle 4.$ $r + IJ = IJ$

PROOF: Since R/IJ is reduced.

- $\langle 1 \rangle 5.$ $r \in IJ$

□

Chapter 24

Boolean Rings

Definition 24.1 (Boolean). A ring is *Boolean* iff $a^2 = a$ for every element a .

Example 24.2. For any set S , the ring $\mathcal{P}S$ is Boolean.

Proposition 24.3. *Every non-trivial Boolean ring has characteristic 2.*

PROOF: We have $4 = 2$ and so $2 = 0$. \square

Proposition 24.4. *Every Boolean ring is commutative.*

PROOF:

$$\begin{aligned}(a+b)^2 &= a+b \\ \therefore a^2 + ab + ba + b^2 &= a+b \\ \therefore a + ab + ba + b &= a+b \\ \therefore ab + ba &= 0 \\ \therefore ab &= -ba \\ &= ba \quad (\text{Proposition 24.3})\end{aligned}$$

Example 24.5. The only Boolean integral domain is $\mathbb{Z}/2\mathbb{Z}$. For, if D is a Boolean integral domain and $x \in D$, we have $x^2 = x$, so $x^2 - x = x(x-1) = 0$ and so $x = 0$ or $x = 1$, i.e. $D = \{0, 1\}$.

Proposition 24.6. *Every Boolean ring has Krull dimension 0.*

PROOF:

$\langle 1 \rangle 1$. LET: R be a Boolean ring.

$\langle 1 \rangle 2$. LET: I be a prime ideal in R .

PROVE: I is maximal.

$\langle 1 \rangle 3$. LET: J be an ideal with $I \subsetneq J$

$\langle 1 \rangle 4$. PICK $a \in J$ with $a \notin I$

$\langle 1 \rangle 5$. $a^2 - a = 0 \in I$

$\langle 1 \rangle 6$. $a(a-1) \in I$

$$\langle 1 \rangle 7. \ a - 1 \in I$$

$$\langle 1 \rangle 8. \ a - 1 \in J$$

$$\langle 1 \rangle 9. \ 1 \in J$$

$$\langle 1 \rangle 10. \ J = R$$

□

Chapter 25

Modules

Definition 25.1 (Left Module). Let R be a ring and M an Abelian group. A *left-action* of R on M is a ring homomorphism

$$R \rightarrow \text{End}_{\mathbf{Ab}}(M) \quad .$$

A *left R -module* consists of an Abelian group M and a left-action of R on M .

Proposition 25.2. *Let R be a ring and M an Abelian group. Let $\cdot : R \times M \rightarrow M$. Then \cdot defines a left-action of R on M if and only if, for all $r, s \in R$ and $m, n \in M$:*

- $r(m + n) = rm + rn$
- $(r + s)m = rm + sm$
- $(rs)m = r(sm)$
- $1m = m$

PROOF: Immediate from definitions. \square

Proposition 25.3. *In any R -module M we have $0m = 0$ for all $m \in M$.*

PROOF: Since $0m = (0 + 0)m = 0m + 0m$ and so $0m = 0$ by cancellation in M . \square

Proposition 25.4. *In any R -module M we have $(-1)m = -m$ for all $m \in M$.*

PROOF: Since $m + (-1)m = 1m + (-1)m = (1 + (-1))m = 0m = 0$. \square

Proposition 25.5. *Every Abelian group is a \mathbb{Z} -module in exactly one way.*

PROOF: Since \mathbb{Z} is initial in **Ring**. \square

Definition 25.6 (Right Module). Let R be a ring. A *right R -module* consists of an Abelian group M and a function $\cdot : M \times R \rightarrow M$ such that, for all $r, s \in R$ and $m, n \in M$:

- $(m + n)r = mr + nr$
- $m(r + s) = mr + ms$
- $m(rs) = (mr)s$
- $m1 = m$

25.1 Homomorphisms

Definition 25.7 (Homomorphism of Left-Modules). Let R be a ring. Let M and N be left- R -modules. A *homomorphism of left- R -modules* $\phi : M \rightarrow N$ is a group homomorphism such that, for all $r \in R$ and $m \in M$, we have $\phi(rm) = r\phi(m)$.

Let $R - \mathbf{Mod}$ be the category of left- R -modules and left- R -module homomorphisms.

Example 25.8.

$$\mathbb{Z} - \mathbf{Mod} \cong \mathbf{Ab}$$

Example 25.9. The trivial group 0 is the zero object in $R - \mathbf{Mod}$.

Proposition 25.10. *Every bijective R -module homomorphism is an isomorphism.*

PROOF: Easy. \square

Proposition 25.11. *Let R be a ring. Let M be an R -module. Then*

$$M \cong R - \mathbf{Mod}[R, M]$$

as R -modules.

PROOF: The isomorphism maps m to the function $\lambda r.rm$. Its inverse maps an R -module homomorphism α to $\alpha(1)$. \square

Proposition 25.12. *Let R be a commutative ring. Let M be an R -module. Then there is a bijection between the set of $R[x]$ -module structures on M that extend the given R -module structure and $\text{End}_{R - \mathbf{Mod}}(M)$.*

PROOF:

- (1)1. LET: $\alpha : R \rightarrow \text{End}_{\mathbf{Ab}}(M)$ be the given R -module structure on M .
- (1)2. An $R[x]$ -module structure on M that extends α is a ring homomorphism $\beta : R[x] \rightarrow \text{End}_{\mathbf{Ab}}(M)$ such that $\beta \circ i = \alpha$, where i is the inclusion $R \rightarrow R[x]$.
- (1)3. There is a bijection between the $R[x]$ -module structures on M that extend α and the elements $s \in \text{End}_{\mathbf{Ab}}(M)$ that commute with $\alpha(r)$ for all $r \in R$.

PROOF: By the universal property for polynomials.

- (1)4. There is a bijection between the $R[x]$ -module structures on M that extend α and the R -module homomorphisms $(M, \alpha) \rightarrow (M, \alpha)$.

□

Proposition 25.13. *Let R be a commutative ring. Let M and N be R -modules. Then $R - \mathbf{Mod}[M, N]$ is an R -module under*

$$\begin{aligned}(\phi + \psi)(m) &= \phi(m) + \psi(m) \\ (r\phi)(m) &= r\phi(m)\end{aligned}$$

PROOF: Easy. □

Proposition 25.14. *Let R be an integral domain. Let I be a nonzero principal ideal of R . Then $I \cong R$ in $R - \mathbf{Mod}$.*

PROOF:

⟨1⟩1. PICK $a \in R$ such that $I = (a)$.

⟨1⟩2. LET: $\phi : R \rightarrow I$ be the map $\phi(r) = ra$.

⟨1⟩3. ϕ is an R -module homomorphism.

PROOF: Since $(r + s)a = ra + sa$ and $(rs)a = r(sa)$.

⟨1⟩4. ϕ is surjective.

⟨1⟩5. ϕ is injective.

PROOF: If $ra = sa$ then $(r - s)a = 0$ so $r - s = 0$ and $r = s$.

⟨1⟩6. $\phi : R \cong I$

□

25.2 Submodules

Definition 25.15 (Submodule). Let M be a left- R -module and $N \subseteq M$. Then N is a *submodule* of M iff N is a subgroup of M and $\forall r \in R, \forall n \in N, rn \in N$.

Proposition 25.16. *Let R be a ring and $I \subseteq R$. Then I is a left-ideal in R iff I is a submodule of R as an R -module.*

PROOF: Immediate from definitions. □

Proposition 25.17. *Let R be a ring. Let M and N be left- R -modules and $\phi : M \rightarrow N$ an R -module homomorphism. Then $\ker \phi$ is a submodule of M and $\text{im } \phi$ is a submodule of N .*

PROOF: Easy. □

Proposition 25.18. *Let R be a commutative ring. Let M be a left- R -module. Let $r \in R$. Then $rM = \{rm : m \in M\}$ is a submodule of M .*

PROOF: Easy. □

Proposition 25.19. *Let R be a ring. Let M be a left- R -module. Let I be a left-ideal in R . Then $IM = \{rm : r \in I, m \in M\}$ is a submodule of M .*

PROOF:

- ⟨1⟩1. IM is a subgroup of M .
 ⟨2⟩1. LET: $r, s \in I$ and $m, n \in M$.
 PROVE: $rm + sn \in IM$
 ⟨2⟩2. $rm + sn = r(m - n) + (s - r)n$
 ⟨1⟩2. For all $r \in R$ and $x \in IM$ we have $rx \in IM$.
 \square

25.3 Quotient Modules

Definition 25.20 (Quotient Module). Let R be a ring. Let M be a left- R -module. Let N be a submodule of M . Then the *quotient module* M/N is the quotient group M/N under

$$r(m + N) = rm + N \ .$$

Proposition 25.21. Let R be a ring. Let M and P be left- R -modules. Let N be a submodule of M . Let $\phi : M \rightarrow P$ be an R -module homomorphism. If $N \subseteq \ker \phi$, then there exists a unique R -module homomorphism $\bar{\phi} : M/N \rightarrow P$ such that the following diagram commutes.

$$\begin{array}{ccc}
 M & \xrightarrow{\phi} & P \\
 & \searrow & \nearrow \bar{\phi} \\
 & M/N &
 \end{array}$$

PROOF: Easy. \square

Theorem 25.22. Every R -module homomorphism $\phi : M \rightarrow M'$ may be decomposed as:

$$M \longrightarrow M/\ker \phi \xrightarrow{\cong} \text{im } \phi \longrightarrow M'$$

PROOF: Easy. \square

Corollary 25.22.1 (First Isomorphism Theorem). Let $\phi : M \rightarrow M'$ be a surjective R -module homomorphism. Then

$$M' \cong \frac{M}{\ker \phi} \ .$$

Proposition 25.23 (Second Isomorphism Theorem). Let R be a ring. Let M be a left- R -module. Let N and P be submodules of M . Then $N + P$ is a submodule of M , $N \cap P$ is a submodule of P , and

$$\frac{N + P}{N} \cong \frac{P}{N \cap P}$$

PROOF: The function that maps P to $p + N$ is a surjective homomorphism $P \rightarrow (N + P)/N$ with kernel $N \cap P$. \square

Proposition 25.24 (Third Isomorphism Theorem). *Let R be a ring. Let M be a left- R -module. Let N be a submodule of M and P a submodule of N . Then N/P is a submodule of M/P and*

$$\frac{M/P}{N/P} \cong \frac{M}{N}$$

PROOF: The canonical map $M \rightarrow M/N$ induces a surjective homomorphism $M/P \rightarrow M/N$ which has kernel N/P . \square

Proposition 25.25. *Let R be a ring. Let M be a left- R -module. The sum and intersection of a family of submodules of M are submodules of M .*

PROOF: Easy. \square

25.4 Products

Proposition 25.26. $R - \mathbf{Mod}$ has products.

PROOF: Given a family $\{M_\alpha\}_{\alpha \in A}$ of left- R -modules, we make $\prod_{\alpha \in A} M_\alpha$ into a left- R -module by

$$(f + g)(\alpha) = f(\alpha) + g(\alpha)$$

$$(rf)(\alpha) = rf(\alpha)$$

\square

25.5 Coproducts

Proposition 25.27. $R - \mathbf{Mod}$ has coproducts.

PROOF: Given a family $\{M_\alpha\}_{\alpha \in A}$ of left- R -modules, take $\bigoplus_{\alpha \in A} M_\alpha$ to be $\{f \in \prod_{\alpha \in A} M_\alpha : f(\alpha) = 0 \text{ for all but finitely many } \alpha \in A\}$. \square

25.6 Direct Sum

Definition 25.28 (Direct Sum). Let R be a ring. Let M and N be left- R -modules. Then the direct sum $M \oplus N$ is an R -module under

$$r(m, n) = (rm, rn) .$$

Proposition 25.29. $M \oplus N$ is the biproduct of M and N in $R - \mathbf{Mod}$.

PROOF: Easy. \square

Example 25.30. Infinite products and coproducts are in general different. We have $\mathbb{Z}^{\mathbb{N}} \not\cong \mathbb{Z}^{\oplus \mathbb{N}}$ since $\mathbb{Z}^{\mathbb{N}}$ is uncountable but $\mathbb{Z}^{\oplus \mathbb{N}}$ is countable.

25.7 Kernels and Cokernels

Proposition 25.31. *Let R be a ring. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then $\ker \phi \hookrightarrow M$ is terminal in the category of left- R -module homomorphisms $\alpha : P \rightarrow M$ such that $\phi \circ \alpha = 0$.*

PROOF: Easy. \square

Proposition 25.32. *Let R be a ring. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then $N \twoheadrightarrow \operatorname{coker} \phi$ is initial in the category of left- R -module homomorphisms $\alpha : N \rightarrow P$ such that $\alpha \circ \phi = 0$.*

PROOF: Easy. \square

Proposition 25.33. *Let R be a ring. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then the following are equivalent.*

- ϕ is a monomorphism.
- $\ker \phi$ is trivial.
- ϕ is injective.

PROOF: Easy. \square

Proposition 25.34. *Let R be a ring. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then the following are equivalent.*

- ϕ is an epimorphism.
- $\operatorname{coker} \phi$ is trivial.
- ϕ is surjective.

PROOF: Easy. \square

Proposition 25.35. *Every monomorphism in $R - \mathbf{Mod}$ is the kernel of some homomorphism.*

PROOF: If $\phi : M \rightarrow N$ is a monomorphism then it is the kernel of $N \twoheadrightarrow N/\operatorname{im} \phi$. \square

Proposition 25.36. *Every epimorphism in $R - \mathbf{Mod}$ is the cokernel of some homomorphism.*

PROOF: If $\phi : M \rightarrow N$ is epi then it is the cokernel of $\ker \phi \hookrightarrow M$. \square

Example 25.37. Monomorphisms do not split in $R - \mathbf{Mod}$. Multiplication by 2 is a monomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$ but has no left inverse.

Example 25.38. Epimorphisms do not split in $R - \mathbf{Mod}$. The canonical map $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ is an epimorphism without a right inverse.

25.8 Free Modules

Proposition 25.39. *Let R be a ring and A a set. Then there exists a left- R -module $F^R(A)$ and function $j : A \rightarrow F^R(A)$ such that, for any left- R -module M and function $f : A \rightarrow M$, there exists a unique left- R -module homomorphism $\bar{f} : F^R(A) \rightarrow M$ such that the following diagram commutes.*

$$\begin{array}{ccc} F^R(A) & \xrightarrow{\bar{f}} & M \\ j \uparrow & \nearrow f & \\ A & & \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: $R^{\oplus A} = \{\alpha : A \rightarrow R : \alpha(a) = 0 \text{ for all but finitely many } a \in A\}$
under the operations

$$\begin{aligned} (\alpha + \beta)(a) &= \alpha(a) + \beta(a) \\ (r\alpha)(a) &= r\alpha(a) \end{aligned}$$

$\langle 1 \rangle 2$. $R^{\oplus A}$ is a left- R -module.

$\langle 1 \rangle 3$. LET: $j : A \rightarrow R^{\oplus A}$ be the function

$$j(a)(a') = \begin{cases} 1 & \text{if } a = a' \\ 0 & \text{if } a \neq a' \end{cases}$$

$\langle 1 \rangle 4$. LET: M be any left- R -module.

$\langle 1 \rangle 5$. LET: $f : A \rightarrow M$ be a function.

$\langle 1 \rangle 6$. LET: $\bar{f} : R^{\oplus A} \rightarrow M$ be the function

$$\bar{f}(\alpha) = \sum_{a \in A, \alpha(a) \neq 0} \alpha(a)f(a)$$

$\langle 1 \rangle 7$. \bar{f} is a left- R -module homomorphism.

$\langle 1 \rangle 8$. $\bar{f} \circ j = f$

$\langle 1 \rangle 9$. \bar{f} is unique.

Definition 25.40. We call $j : A \rightarrow F^R(A)$ the *free* left- R -module over A .

Proposition 25.41. j is injective.

PROOF: By the proof of the previous proposition. \square

Proposition 25.42. *Let R be a ring. Let F be a non-zero free left- R -module. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then ϕ is onto if and only if, for every left- R -module homomorphism $\alpha : F \rightarrow N$, there exists a left- R -module homomorphism $\beta : F \rightarrow M$ such that the diagram below commutes.*

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \beta \uparrow & \nearrow \alpha & \\ F & & \end{array}$$

PROOF:

- (1)1. LET: F be the free left- R -module over A with injection $j : A \rightarrow F$.
 (1)2. If ϕ is onto then, for every homomorphism $\alpha : F \rightarrow N$, there exists a homomorphism $\beta : F \rightarrow M$ such that $\phi \circ \beta = \alpha$.
 (2)1. ASSUME: ϕ is onto.
 (2)2. LET: $\alpha : F \rightarrow N$ be a homomorphism.
 (2)3. For $a \in A$, PICK $f(a) \in M$ such that $\phi(f(a)) = \alpha(j(a))$
 (2)4. LET: $\beta : F \rightarrow M$ be the unique homomorphism such that $\beta \circ j = f$
 (2)5. $\phi \circ \beta = \alpha$
 PROOF: Each is the unique homomorphism such that $\alpha \circ j = \phi \circ f$.

□

$$\begin{array}{ccccc}
 & & M & \xrightarrow{\phi} & N \\
 & f \nearrow & \uparrow \beta & \nearrow \alpha & \\
 A & \xrightarrow{j} & F & &
 \end{array}$$

- (1)3. If, for every homomorphism $\alpha : F \rightarrow N$, there exists a homomorphism $\beta : F \rightarrow M$ such that $\phi \circ \beta = \alpha$, then ϕ is onto.
 (2)1. ASSUME: For every homomorphism $\alpha : F \rightarrow N$ there exists a homomorphism $\beta : F \rightarrow M$ such that $\phi \circ \beta = \alpha$.
 (2)2. LET: $n \in N$
 (2)3. LET: $\alpha : F \rightarrow N$ be the unique homomorphism such that, for all $a \in A$, we have $\alpha(j(a)) = n$
 (2)4. PICK a homomorphism $\beta : F \rightarrow M$ such that $\phi \circ \beta = \alpha$
 (2)5. PICK $a \in A$
 (2)6. $\phi(\beta(j(a))) = n$

□

25.9 Generators

Definition 25.43 (Submodule Generated by a Set). Let R be a ring. Let M be a left- R -module. Let A be a subset of M . Let $\phi_A : F^R(A) \rightarrow M$ be the unique left- R -module homomorphism such that the following diagram commutes.

$$\begin{array}{ccc}
 F^R(A) & \xrightarrow{\phi_A} & M \\
 \uparrow & \nearrow & \\
 A & &
 \end{array}$$

The submodule of M generated by A , denoted $\langle A \rangle$, is defined to be $\text{im } \phi_A$.

Definition 25.44 (Finitely Generated). Let R be a ring. Let M be a left- R -module. Then M is *finitely generated* iff there exists a finite set $A \subseteq M$ such that $M = \langle A \rangle$.

Example 25.45. A submodule of a finitely generated module is not necessarily finitely generated.

Let $R = \mathbb{Z}[x_1, x_2, \dots]$. Then R is finitely generated as an R -module, but (x_1, x_2, \dots) is not.

Proposition 25.46. *The homomorphic image of a finitely generated module is finitely generated.*

PROOF: Easy. \square

Proposition 25.47. *Let R be a ring. Let M be a left- R -module. Let N be a submodule of M . If N and M/N are finitely generated then M is finitely generated.*

PROOF:

$\langle 1 \rangle 1$. PICK a_1, \dots, a_n that generate N .

$\langle 1 \rangle 2$. PICK b_1, \dots, b_m such that $b_1 + N, \dots, b_m + N$ generate M/N .

PROVE: $a_1, \dots, a_n, b_1, \dots, b_m$ generate M .

$\langle 1 \rangle 3$. LET: $m \in M$

$\langle 1 \rangle 4$. PICK $r_1, \dots, r_m \in R$ such that $m + N = r_1 b_1 + \dots + r_m b_m + N$

$\langle 1 \rangle 5$. $m - r_1 b_1 - \dots - r_m b_m \in N$

$\langle 1 \rangle 6$. PICK $s_1, \dots, s_n \in R$ such that $m - r_1 b_1 - \dots - r_m b_m = s_1 a_1 + \dots + s_n a_n$

$\langle 1 \rangle 7$. $m = r_1 b_1 + \dots + r_m b_m + s_1 a_1 + \dots + s_n a_n$

\square

25.10 Projections

Definition 25.48 (Projection). Let R be a ring. Let M be a left- R -module. Let $p : M \rightarrow M$ be a left- R -module homomorphism. Then p is a *projection* iff $p^2 = p$.

Proposition 25.49. *Let R be a ring. Let M be a left- R -module. Let $p : M \rightarrow M$ be a projection. Then*

$$M \cong \ker p \oplus \operatorname{im} p .$$

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : M \rightarrow \ker p \oplus \operatorname{im} p$ be the map $\phi(m) = (m - p(m), p(m))$

$\langle 1 \rangle 2$. ϕ is a left- R -module homomorphism.

$\langle 1 \rangle 3$. ϕ is injective.

$\langle 1 \rangle 4$. ϕ is surjective.

\square

25.11 Pullbacks

Proposition 25.50. *$R - \operatorname{Mod}$ has pullbacks.*

PROOF:

$\langle 1 \rangle 1$. LET: $\mu : M \rightarrow Z, \nu : N \rightarrow Z$ be left- R -module homomorphisms.

$\langle 1 \rangle 2$. LET: $M \times_Z N = \{(m, n) \in M \times N : \mu(m) = \nu(n)\}$ under

$$(m, n) + (m', n') = (m + m', n + n')$$

$$r(m, n) = (rm, rn)$$

$\langle 1 \rangle 3$. $M \times_Z N$ is the pullback of M and N .

\square

25.12 Pushouts

Proposition 25.51. $R - \mathbf{Mod}$ has pushouts.

PROOF:

$\langle 1 \rangle 1$. LET: $\mu : A \rightarrow M$ and $\nu : A \rightarrow N$ be left- R -module homomorphisms.

Chapter 26

Cyclic Modules

Definition 26.1 (Cyclic Module). Let R be a ring. Let M be a left- R -module. Then M is *cyclic* iff there exists $m \in M$ such that $M = \langle m \rangle$.

Proposition 26.2. *Let R be a ring. Let M be a left- R -module. Then M is cyclic if and only if there exists a left-ideal I in R such that $M \cong R/I$.*

PROOF:

$\langle 1 \rangle 1$. If M is cyclic then there exists a left-ideal I in R such that $M \cong R/I$.

$\langle 2 \rangle 1$. ASSUME: M is cyclic.

$\langle 2 \rangle 2$. PICK $m \in M$ such that $M = \langle m \rangle$

$\langle 2 \rangle 3$. LET: $\phi : R \rightarrow M$ be the left- R -module homomorphism $\phi(r) = rm$.

$\langle 2 \rangle 4$. ϕ is surjective.

$\langle 2 \rangle 5$. $M \cong R/\ker \phi$

$\langle 1 \rangle 2$. For every left-ideal I in R , we have that R/I is cyclic.

PROOF: R/I is generated by $1 + I$.

□

Proposition 26.3. *A quotient of a cyclic module is cyclic.*

PROOF: If M is generated by m then M/N is generated by $m + N$. □

Proposition 26.4. *Let R be a ring. For any left-ideal I in R and any left- R -module N , we have*

$$R - \mathbf{Mod}[R/I, N] \cong \{n \in N : \forall a \in I. an = 0\} .$$

PROOF:

$\langle 1 \rangle 1$. LET: $\Phi : R - \mathbf{Mod}[R/I, N] \rightarrow \{n \in N : \forall a \in I. an = 0\}$ be the function

$$\Phi(\alpha) = \alpha(1 + I)$$

PROOF: For all $a \in I$ we have $a\alpha(1 + I) = \alpha(a + I) = \alpha(I) = 0$.

$\langle 1 \rangle 2$. Φ is injective.

PROOF: If $\alpha(1 + I) = \beta(1 + I)$ then $\alpha(r + I) = r\alpha(1 + I) = r\beta(1 + I) = \beta(r + I)$ for all $r \in R$, hence $\alpha = \beta$.

$\langle 1 \rangle 3$. Φ is surjective.

PROOF: Given $n \in N$ such that $\forall a \in I. an = 0$, define $\alpha : R/I \rightarrow N$ by $\alpha(r + I) = rn$.

$\langle 1 \rangle 4$. If R is commutative then Φ is an R -module homomorphism.

□

Corollary 26.4.1. *For all $a, b \in \mathbb{Z}$ we have $\mathbf{Ab}[\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}] \cong \mathbb{Z}/\gcd(a, b)\mathbb{Z}$.*

PROOF:

$$\begin{aligned} \mathbf{Ab}[\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}] &\cong \mathbb{Z} - \mathbf{Mod}[\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}] \\ &\cong \{n \in \mathbb{Z}/b\mathbb{Z} : \forall x \in a\mathbb{Z}. xn \cong 0(\text{mod } b)\} \\ &\cong \{n \in \mathbb{Z}/b\mathbb{Z} : \forall x \in \mathbb{Z}. b \mid xan\} \\ &= \{n \in \mathbb{Z}/b\mathbb{Z} : b \mid an\} \end{aligned}$$

Chapter 27

Simple Modules

Definition 27.1 (Simple Module). Let R be a ring. An R -module M is *simple* or *irreducible* iff its only submodules are $\{0\}$ and M .

Proposition 27.2 (Schur's Lemma). Let R be a ring. Let M and N be simple R -modules. Let $\phi : M \rightarrow N$ be an R -module homomorphism. Then either $\phi = 0$ or ϕ is an isomorphism.

PROOF:

$\langle 1 \rangle 1$. ASSUME: $\phi \neq 0$

$\langle 1 \rangle 2$. $\ker \phi = 0$

PROOF: Since $\ker \phi$ is a submodule of M that is not M .

$\langle 1 \rangle 3$. $\operatorname{im} \phi = N$

PROOF: Since $\operatorname{im} \phi$ is a submodule of N that is not $\{0\}$.

□

Proposition 27.3. Every simple module is cyclic.

PROOF:

$\langle 1 \rangle 1$. LET: M be a simple module.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $M \neq \{0\}$

PROOF: $\{0\} = \langle 0 \rangle$ is cyclic.

$\langle 1 \rangle 3$. PICK $m \in M$ with $m \neq 0$

$\langle 1 \rangle 4$. $\langle m \rangle = M$

PROOF: Since $\langle m \rangle$ is a submodule of M that is not $\{0\}$.

□

Chapter 28

Noetherian Modules

Definition 28.1 (Noetherian Module). Let R be a ring. A left- R -module is *Noetherian* iff every submodule is finitely generated.

Proposition 28.2. *Let R be a ring. Let M be a left- R -module and N a submodule of M . Then M is Noetherian if and only if N and M/N are Noetherian.*

PROOF:

$\langle 1 \rangle 1$. If M is Noetherian then N is Noetherian.

PROOF: Every submodule of N is a submodule of M , hence finitely generated.

$\langle 1 \rangle 2$. If M is Noetherian then M/N is Noetherian.

$\langle 2 \rangle 1$. ASSUME: M is Noetherian.

$\langle 2 \rangle 2$. LET: $\pi : M \rightarrow M/N$ be the canonical epimorphism.

$\langle 2 \rangle 3$. LET: P be a submodule of M/N .

$\langle 2 \rangle 4$. PICK $a_1, \dots, a_n \in M$ that generate $\pi^{-1}(P)$.

$\langle 2 \rangle 5$. $a_1 + N, \dots, a_n + N$ generate P .

$\langle 1 \rangle 3$. If N and M/N are Noetherian then M is Noetherian.

$\langle 2 \rangle 1$. ASSUME: N and M/N are Noetherian.

$\langle 2 \rangle 2$. LET: P be a submodule of M .

$\langle 2 \rangle 3$. PICK $a_1, \dots, a_m \in P$ such that $a_1 + N, \dots, a_m + N$ generate $\pi(P)$.

$\langle 2 \rangle 4$. PICK $b_1, \dots, b_n \in M$ that generated $P \cap N$.

PROVE: $a_1, \dots, a_m, b_1, \dots, b_n$ generate P .

$\langle 2 \rangle 5$. LET: $p \in P$

$\langle 2 \rangle 6$. PICK $r_1, \dots, r_m \in R$ such that $p + N = r_1 a_1 + \dots + r_m a_m + N$

$\langle 2 \rangle 7$. $p - r_1 a_1 - \dots - r_m a_m \in P \cap N$

$\langle 2 \rangle 8$. PICK $s_1, \dots, s_n \in R$ such that $p - r_1 a_1 - \dots - r_m a_m = s_1 b_1 + \dots + s_n b_n$

$\langle 2 \rangle 9$. $p = r_1 a_1 + \dots + r_m a_m + s_1 b_1 + \dots + s_n b_n$

□

Corollary 28.2.1. *If R is a Noetherian ring then $R^{\oplus n}$ is a Noetherian left- R -module.*

PROOF: The proof is by induction on n . The case $n = 1$ is immediate.

The induction step holds since $R^{\oplus(n+1)}/R^{\oplus n} \cong R$. □

Corollary 28.2.2. *If R is a Noetherian ring and M is a finitely generated left- R -module then M is Noetherian.*

PROOF: There is a surjective homomorphism $R^{\oplus n} \twoheadrightarrow M$ for some n , so M is a quotient of $R^{\oplus n}$. \square

Chapter 29

Algebras

Definition 29.1 (Algebra). Let R be a commutative ring. An R -algebra consists of a ring S and a ring homomorphism $\alpha : R \rightarrow S$ such that $\alpha(R)$ is included in the center of S . We write rs for $\alpha(r)s$.

Proposition 29.2. Let R be a commutative ring and S a ring. Let $\cdot : R \times S \rightarrow S$. Then there exists $\alpha : R \rightarrow S$ that makes S into an R -algebra such that

$$rs = \alpha(r)s \quad (r \in R, s \in S)$$

iff S is an R -module under \cdot and, for all $r_1, r_2 \in R$ and $s_1, s_2 \in S$,

$$(r_1 s_1)(r_2 s_2) = (r_1 r_2)(s_1 s_2) .$$

PROOF: Immediate from definitions. \square

Example 29.3. Let R be a commutative ring. Then R is an R -algebra under multiplication.

Example 29.4. Let R be a commutative ring and I an ideal in R . Then R/I is an R -algebra.

Example 29.5. Let R be a commutative ring and M an R -module. Then $\text{End}_{R\text{-Mod}}(M)$ is an R -algebra under composition.

Example 29.6. Let R be a commutative ring. Then $\mathfrak{gl}_n(R)$ is an R -algebra under matrix multiplication.

Definition 29.7 (Algebra Homomorphism). Let R be a commutative ring. Let S and T be R -algebras. An R -algebra homomorphism $\phi : S \rightarrow T$ is a ring homomorphism such that, for all $r \in R$ and $s \in S$, we have $\phi(rs) = r\phi(s)$.

Let $R\text{-Alg}$ be the category of R -algebras and R -algebra homomorphisms.

Example 29.8.

$$\mathbb{Z}\text{-Alg} \cong \mathbf{Ring}$$

Example 29.9. Let R be a commutative ring. Then $R[x_1, \dots, x_n]$, and any quotient ring of $R[x_1, \dots, x_n]$, is a commutative R -algebra.

Example 29.10. R is the initial object in $R\text{-Alg}$.

29.1 Rees Algebra

Definition 29.11 (Rees Algebra). Let R be a commutative ring. Let I be an ideal in R . The *Rees algebra* is the direct sum

$$\text{Rees}_R(I) = \bigoplus_{j \geq 0} I^j$$

under the multiplication

$$\begin{aligned} (r_0, r_1, r_2, r_3, \dots)(s_0, s_1, s_2, \dots) &= (r_0s_0, r_1s_0 + r_0s_1, r_2s_0 + r_1s_1 + r_0s_2, \dots) \\ r(r_0, r_1, r_2, \dots) &= (rr_0, rr_1, rr_2, \dots) \end{aligned}$$

Proposition 29.12. *Let R be a commutative ring. Let $a \in R$ be a non-zero-divisor. Then $R[x]$ is the Rees algebra of (a) .*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : R[x] \rightarrow \text{Rees}_R((a))$ be the function $\phi(r_0 + r_1x + r_2x^2 + \dots) = (r_0, r_1a, r_2a^2, \dots)$.

$\langle 1 \rangle 2$. ϕ is an R -algebra homomorphism.

$\langle 1 \rangle 3$. ϕ is injective.

$\langle 2 \rangle 1$. LET: $\phi(r_0 + r_1x + r_2x^2 + \dots) = \phi(s_0 + s_1x + s_2x^2 + \dots)$

$\langle 2 \rangle 2$. For all n we have $r_na^n = s_na^n$

$\langle 2 \rangle 3$. $(r_n - s_n)a^n = 0$

$\langle 2 \rangle 4$. $r_n - s_n = 0$

PROOF: Since a is not a zero-divisor.

$\langle 2 \rangle 5$. $r_n = s_n$

$\langle 1 \rangle 4$. ϕ is surjective.

□

Proposition 29.13. *Let R be a commutative ring. Let $a \in R$ be a non-zero-divisor. Let I be an ideal of R . Then $\text{Rees}_R(I) \cong \text{Rees}_R(aI)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : \text{Rees}_R(I) \rightarrow \text{Rees}_R(aI)$ be the function $\phi(r_0, r_1, r_2, \dots) = (r_0, ar_1, a^2r_2, \dots)$.

$\langle 1 \rangle 2$. ϕ is an R -algebra homomorphism.

$\langle 1 \rangle 3$. ϕ is injective.

$\langle 1 \rangle 4$. ϕ is surjective.

□

29.2 Free Algebras

Proposition 29.14. *Let R be a ring. Then $R[x_1, \dots, x_n]$ is the free commutative R -algebra on $\{1, \dots, n\}$.*

PROOF: Easy. □

Proposition 29.15. *Let R be a ring and A a set. Let A^* be the free monoid on A . Then the monoid ring $R[A^*]$ is the free R -algebra on A .*

PROOF: Easy. \square

Proposition 29.16. *Let R be a commutative ring and S a commutative R -algebra. Then S is finitely generated as an R -algebra if and only if S is finitely generated as a commutative R -algebra.*

PROOF: Since a subalgebra of a commutative subalgebra is commutative, so the smallest algebra that contains $\{a_1, \dots, a_n\}$ is the smallest commutative subalgebra that contains $\{a_1, \dots, a_n\}$. \square

Chapter 30

Algebras of Finite Type

Definition 30.1 (Algebra of Finite Type). Let R be a ring. Let S be an R -algebra. Then R is of *finite type* iff S is a finitely generated R -algebra.

Proposition 30.2. *Let R be a Noetherian ring. Let S be a finite-type R -algebra. Then S is a Noetherian ring.*

Chapter 31

Finite Algebras

Definition 31.1 (Finite Algebra). Let R be a ring. Let S be an R -algebra. Then S is a *finite* R -algebra iff it is a finitely generated left- R -module.

Proposition 31.2. *Let R be a ring. Every finite R -algebra is of finite type.*

PROOF: If S is generated by a_1, \dots, a_n as an R -module, then it is generated by a_1, \dots, a_n as an R -algebra. \square

Example 31.3. The converse does not hold. $R[x]$ is of finite type but is not finite.

Chapter 32

Division Algebras

Definition 32.1 (Division Algebra). Let R be a commutative ring. A *division R -algebra* is an R -algebra that is a division ring.

Example 32.2. Let R be a commutative ring. Let M be a simple R -algebra. Then $\text{End}_{R\text{-Mod}}(M)$ is a division algebra. For if $\phi \circ \psi = 0$ then ϕ and ψ cannot both be isomorphisms, hence $\phi = 0$ or $\psi = 0$ by Schur's Lemma.

Chapter 33

Chain Complexes

Definition 33.1 (Chain Complex). Let R be a ring. A *chain complex of left- R -modules* $M_\bullet = (M_\bullet, d_\bullet)$ consists of a family of left- R -modules $\{M_i\}_{i \in \mathbb{Z}}$ and a family of left- R -module homomorphisms $\{d_i : M_i \rightarrow M_{i-1}\}_{i \in \mathbb{Z}}$ such that, for all i ,

$$d_i \circ d_{i+1} = 0 \ .$$

We call each d_i a *differential* and the family $\{d_i\}_i$ the *boundary* of the chain complex.

Definition 33.2 (Exact). A chain complex M_\bullet is *exact* at M_i iff $\text{im } d_{i+1} = \ker d_i$.

It is *exact* or an *exact sequence* iff it is exact at M_i for all i .

Proposition 33.3. A complex

$$\cdots \rightarrow 0 \rightarrow L \xrightarrow{\alpha} M \rightarrow \cdots$$

is exact at L iff α is a monomorphism.

PROOF: Since both are equivalent to $\ker \alpha = 0$. \square

Proposition 33.4. A complex

$$\cdots \rightarrow M \xrightarrow{\beta} N \rightarrow 0 \rightarrow \cdots$$

is exact at N iff β is an epimorphism.

PROOF: Since both are equivalent to $\text{im } \beta = N$. \square

Definition 33.5 (Short Exact Sequence). A *short exact sequence* is an exact complex of the form

$$0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0 \ .$$

Proposition 33.6 (Four-Lemma). *If*

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & D_1 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta \\ A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{h_2} & D_2 \end{array}$$

is a commutative diagram of left- R -modules with exact rows, α is an epimorphism, and β and δ are monomorphisms, then γ is an monomorphism.

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in C_1$

$\langle 1 \rangle 2$. ASSUME: $\gamma(x) = \gamma(y)$

$\langle 1 \rangle 3$. $\delta(h_1(x)) = \delta(h_1(y))$

$\langle 1 \rangle 4$. $h_1(x) = h_1(y)$

PROOF: δ is injective.

$\langle 1 \rangle 5$. $x - y \in \ker h_1$

$\langle 1 \rangle 6$. $x - y \in \operatorname{im} g_1$

$\langle 1 \rangle 7$. PICK $b \in B_1$ such that $g_1(b) = x - y$.

$\langle 1 \rangle 8$. $g_2(\beta(b)) = 0$

PROOF: $g_2(\beta(b)) = \gamma(g_1(b)) = \gamma(x - y) = 0$

$\langle 1 \rangle 9$. $\beta(b) \in \ker g_2$

$\langle 1 \rangle 10$. $\beta(b) \in \operatorname{im} f_2$

$\langle 1 \rangle 11$. PICK $a' \in A_2$ such that $f_2(a') = \beta(b)$

$\langle 1 \rangle 12$. PICK $a \in A_1$ such that $\alpha(a) = a'$

PROOF: α is surjective.

$\langle 1 \rangle 13$. $\beta(f_1(a)) = \beta(b)$

$\langle 1 \rangle 14$. $f_1(a) = b$

PROOF: β is injective.

$\langle 1 \rangle 15$. $0 = g_1(b)$

PROOF: Since $g_1(b) = g_1(f_1(a)) = 0$.

$\langle 1 \rangle 16$. $x = y$

PROOF: $\langle 1 \rangle 7$

□

Proposition 33.7 (Four-Lemma). *If*

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & D_1 \\ \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{h_2} & D_2 \end{array}$$

is a commutative diagram of left- R -modules with exact rows, β and δ are epimorphisms, and ϵ is a monomorphism, then γ is an epimorphism.

PROOF:

$\langle 1 \rangle 1$. LET: $b_2 \in B_2$

$\langle 1 \rangle 2$. PICK $c_1 \in C_1$ such that $\delta(c_1) = g_2(b_2)$

PROOF: δ is surjective.

$\langle 1 \rangle 3$. $\epsilon(h_1(c_1)) = 0$

$\langle 1 \rangle 4$. $h_1(c_1) = 0$

PROOF: ϵ is injective.

$\langle 1 \rangle 5$. $c_1 \in \ker h_1$

$\langle 1 \rangle 6$. $c_1 \in \operatorname{im} g_1$

$\langle 1 \rangle 7$. PICK $b_1 \in B_1$ such that $g_1(b_1) = c_1$

$\langle 1 \rangle 8$. $g_2(\gamma(b_1)) = g_2(b_2)$

$\langle 1 \rangle 9$. $\gamma(b_1) - b_2 \in \ker g_2$

$\langle 1 \rangle 10$. $\gamma(b_1) - b_2 \in \operatorname{im} f_2$

$\langle 1 \rangle 11$. PICK $a_2 \in A_2$ such that $f_2(a_2) = \gamma(b_1) - b_2$.

$\langle 1 \rangle 12$. PICK $a_1 \in A_1$ such that $\beta(a_1) = a_2$.

PROOF: β is surjective.

$\langle 1 \rangle 13$. $\gamma(f_1(a_1)) = \gamma(b_1) - b_2$

$\langle 1 \rangle 14$. $b_2 = \gamma(b_1 - f_1(a_1))$

□

Theorem 33.8 (Snake Lemma). *Suppose we have R -modules and homomorphisms*

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 \longrightarrow 0 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu \\ 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 \longrightarrow 0 \end{array}$$

such that the diagram commutes and the two rows are short exact sequences. Then there exists a homomorphism $\delta : \ker \nu \rightarrow \operatorname{coker} \lambda$ such that the following is an exact sequence.

$$0 \rightarrow \ker \lambda \xrightarrow{\alpha_1} \ker \mu \xrightarrow{\beta_1} \ker \nu \xrightarrow{\delta} \operatorname{coker} \lambda \xrightarrow{\alpha_0} \operatorname{coker} \mu \xrightarrow{\beta_0} \operatorname{coker} \nu \rightarrow 0 .$$

PROOF:

$\langle 1 \rangle 1$. Define $\delta : \ker \nu \rightarrow \operatorname{coker} \lambda$.

$\langle 2 \rangle 1$. LET: $a \in \ker \nu$

$\langle 2 \rangle 2$. PICK $c \in M_1$ such that $\beta_1(c) = a$.

PROOF: Since β_1 is surjective.

$\langle 2 \rangle 3$. LET: $d = \mu(c)$

$\langle 2 \rangle 4$. $d \in \ker \beta_0 = \operatorname{im} \alpha_0$

PROOF: Since $\beta_0(d) = \beta_0(\mu(c)) = \nu(a) = 0$.

$\langle 2 \rangle 5$. LET: $e \in L_0$ be the element such that $\alpha_0(e) = d$.

$\langle 2 \rangle 6$. LET: $\delta(a) = e + \operatorname{im} \lambda$

$\langle 1 \rangle 2$. δ is a left- R -module homomorphism.

$\langle 2 \rangle 1$. For $a, a' \in \ker \nu$ we have $\delta(a + a') = \delta(a) + \delta(a')$.

$\langle 3 \rangle 1$. LET: $a, a' \in \ker \nu$

$\langle 3 \rangle 2$. LET: $c, c', c'' \in M_1$ and $e, e', e'' \in L_0$ be the elements such that

$$\beta_1(c) = a$$

$$\beta_1(c') = a'$$

$$\beta_1(c'') = a + a'$$

$$\alpha_0(e) = \mu(c)$$

$$\alpha_0(e') = \mu(c')$$

$$\alpha_0(e'') = \mu(c'')$$

$$\delta(a) = e + \text{im } \lambda$$

$$\delta(a') = e' + \text{im } \lambda$$

$$\delta(a + a') = e'' + \text{im } \lambda$$

$\langle 3 \rangle 3$. $c'' - c - c' \in \ker \beta_1 = \text{im } \alpha_1$

$\langle 3 \rangle 4$. PICK $g \in L_1$ such that $\alpha_1(g) = c'' - c - c'$.

$\langle 3 \rangle 5$. $\alpha_0(\lambda(g)) = \alpha_0(e'' - e - e')$

$\langle 3 \rangle 6$. $\lambda(g) = e'' - e - e'$

$\langle 3 \rangle 7$. $e'' - e - e' \in \text{im } \lambda$

$\langle 3 \rangle 8$. $e'' + \text{im } \lambda = e + e' + \text{im } \lambda$

$\langle 3 \rangle 9$. $\delta(a + a') = \delta(a) + \delta(a')$

$\langle 2 \rangle 2$. For $r \in R$ and $a \in \ker \nu$ we have $\delta(ra) = r\delta(a)$.

$\langle 3 \rangle 1$. LET: $r \in R$ and $a \in \ker \nu$

$\langle 3 \rangle 2$. LET: $c, c' \in M_1$ and $e, e' \in L_0$ be the elements such that

$$\beta_1(c) = a$$

$$\beta_1(c') = ra$$

$$\alpha_0(e) = \mu(c)$$

$$\alpha_0(e') = \mu(c')$$

$$\delta(a) = e + \text{im } \lambda$$

$$\delta(ra) = e' + \text{im } \lambda$$

$\langle 3 \rangle 3$. $rc - c' \in \ker \beta_1 = \text{im } \alpha_1$

$\langle 3 \rangle 4$. PICK $g \in L_1$ such that $\alpha_1(g) = rc - c'$.

$\langle 3 \rangle 5$. $\alpha_0(\lambda(g)) = \alpha_0(re - e')$

$\langle 3 \rangle 6$. $\lambda(g) = re - e'$

$\langle 3 \rangle 7$. $re - e' \in \text{im } \lambda$

$\langle 3 \rangle 8$. $re + \text{im } \lambda = e' + \text{im } \lambda$

$\langle 3 \rangle 9$. $r\delta(a) = \delta(ra)$

$\langle 1 \rangle 3$. The sequence is exact at $\ker \lambda$.

PROOF: Since α_1 is injective.

$\langle 1 \rangle 4$. The sequence is exact at $\ker \mu$.

PROOF: Since $\text{im } \alpha_1 = \ker \beta_1$.

$\langle 1 \rangle 5$. The sequence is exact at $\ker \nu$, i.e.

$$\beta a_1(\ker \mu) = \ker \delta.$$

$\langle 2 \rangle 1$. LET: $a \in \ker \nu$

$\langle 2 \rangle 2$. LET: $c \in M_1$ and $e \in L_0$ be the elements such that $\beta_1(c) = a$, $\alpha_0(e) = \mu(c)$, and $\delta(a) = e + \text{im } \lambda$.

- (2)3. If $\delta(a) = \text{im } \lambda$ then $a \in \beta_1(\ker \mu)$
 (3)1. ASSUME: $\delta(a) = \text{im } \lambda$
 (3)2. $e \in \text{im } \lambda$
 (3)3. PICK $g \in L_1$ such that $\lambda(g) = e$
 (3)4. $\mu(\alpha_1(g)) = \mu(c)$
 (3)5. $c - \alpha_1(g) \in \ker \mu$
 (3)6. $a = \beta_1(c - \alpha_1(g))$
 (2)4. If $a \in \beta_1(\ker \mu)$ then $\delta(a) = \text{im } \lambda$
 (3)1. ASSUME: $c' \in \ker \mu$ and $a = \beta_1(c')$
 (3)2. $c - c' \in \ker \beta_1 = \text{im } \alpha_1$
 (3)3. PICK $g \in L_1$ such that $\alpha_1(g) = c - c'$
 (3)4. $\alpha_0(\lambda(g)) = \mu(c) - \mu(c') = \alpha_0(e) - 0 = \alpha_0(e)$
 (3)5. $\lambda(g) = e$
 (3)6. $e \in \text{im } \lambda$
 (3)7. $\delta(a) = \text{im } \lambda$
 (1)6. The sequence is exact at $\text{coker } \lambda$.
 (2)1. LET: $e \in L_0$
 PROVE: $e + \text{im } \lambda \in \text{im } \delta$ iff $\alpha_0(e) \in \text{im } \mu$.
 (2)2. For all $a \in \ker \nu$, if $\delta(a) = e + \text{im } \lambda$ then $\alpha_0(e) \in \text{im } \mu$
 PROOF: From (1)1 and the fact that α_0 is injective hence e is unique given a .
 (2)3. For all $e \in L_0$, if $\alpha_0(e) \in \text{im } \mu$ then $e + \text{im } \lambda \in \text{im } \delta$.
 (3)1. LET: $e \in L_0$
 (3)2. ASSUME: $\alpha_0(e) \in \text{im } \mu$
 (3)3. PICK $c \in M_1$ such that $\mu(c) = \alpha_0(e)$.
 PROVE: $e + \text{im } \lambda = \delta(\beta_1(c))$
 (3)4. PICK $c' \in M_1$ and $e' \in L_0$ such that $\beta_1(c') = \beta_1(c)$, $\alpha_0(e') = \mu(c')$
 and $\delta(\beta_1(c)) = e' + \text{im } \lambda$
 (3)5. $c - c' \in \ker \beta_1 = \text{im } \alpha_1$
 (3)6. PICK $g \in L_1$ such that $\alpha_1(g) = c - c'$.
 (3)7. $\alpha_0(\lambda(g)) = \alpha_0(e - e')$
 (3)8. $\lambda(g) = e - e'$
 (3)9. $e + \text{im } \lambda = e' + \text{im } \lambda = \delta(\beta_1(c))$
 (1)7. The sequence is exact at $\text{coker } \mu$.
 PROOF: Since $\text{im } \alpha_0 = \ker \beta_0$.
 (1)8. The sequence is exact at $\text{coker } \nu$.
 PROOF: Since β_0 is surjective.

□

Corollary 33.8.1. *Suppose we have R -modules and homomorphisms*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 \longrightarrow 0 \\
 & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu \\
 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 \longrightarrow 0
 \end{array}$$

such that the diagram commutes and the two rows are short exact sequences.

Suppose μ is surjective and ν is injective. Then λ is surjective and ν is an isomorphism.

PROOF: We have $\ker \nu = \operatorname{coker} \mu = 0$ and so $0 \xrightarrow{\delta} \operatorname{coker} \lambda \xrightarrow{\alpha_0} 0$ is an exact sequence, hence $\operatorname{coker} \lambda = 0$ and so λ is surjective.

Since $\operatorname{coker} \mu = 0$ we have $0 \rightarrow \operatorname{coker} \nu \rightarrow 0$ is an exact sequence and so $\operatorname{coker} \nu = 0$, hence ν is surjective, hence ν is an isomorphism. \square

Proposition 33.9 (Short Five-Lemma). *Suppose we have R -modules and homomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 & \longrightarrow & 0 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & & \\ 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 & \longrightarrow & 0 \end{array}$$

such that the diagram commutes and the two rows are short exact sequences. If λ and ν are isomorphisms then μ is an isomorphism.

PROOF:

$\langle 1 \rangle 1$. There exists a homomorphism $\delta : 0 \rightarrow L_0$ such that the following is an exact sequence.

$$0 \rightarrow 0 \rightarrow \ker \mu \rightarrow 0 \xrightarrow{\delta} L_0 \xrightarrow{\alpha_0} \operatorname{coker} \mu \xrightarrow{\beta_0} N_0 \rightarrow 0.$$

PROOF: Snake Lemma

$\langle 1 \rangle 2$. $\ker \mu = 0$

$\langle 1 \rangle 3$. $\operatorname{coker} \mu = M_0$

\square

Proposition 33.10. *If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is an exact sequence and L and N are Noetherian then M is Noetherian.*

PROOF:

$\langle 1 \rangle 1$. LET: P be a submodule of M .

$\langle 1 \rangle 2$. PICK a_1, \dots, a_m generate $\alpha^{-1}(P)$.

$\langle 1 \rangle 3$. PICK c_1, \dots, c_n that generate $\beta(P)$.

$\langle 1 \rangle 4$. For $i = 1, \dots, n$, PICK b_i such that $\beta(b_i) = c_i$.

PROVE: $\alpha(a_1), \dots, \alpha(a_m), b_1, \dots, b_n$ generate P .

$\langle 1 \rangle 5$. LET: $p \in P$

$\langle 1 \rangle 6$. PICK $r_1, \dots, r_n \in R$ such that $r_1 c_1 + \dots + r_n c_n = \beta(p)$

$\langle 1 \rangle 7$. $r_1 b_1 + \dots + r_n b_n - p \in \ker \beta = \operatorname{im} \alpha$

$\langle 1 \rangle 8$. PICK $s_1, \dots, s_m \in R$ such that $\alpha(s_1 a_1 + \dots + s_m a_m) = r_1 b_1 + \dots + r_n b_n - p$.

$\langle 1 \rangle 9$. $p = s_1 \alpha(a_1) + \dots + s_m \alpha(a_m) + r_1 b_1 + \dots + r_n b_n$

\square

Proposition 33.11. *Let R be a ring. Let*

$$0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} P \rightarrow 0$$

be a short exact sequence of left- R -modules. Let L be an R -module. Then the following is an exact sequence:

$$0 \rightarrow R\text{-}\mathbf{Mod}[P, L] \xrightarrow{R\text{-}\mathbf{Mod}[\beta, \text{id}_L]} R\text{-}\mathbf{Mod}[N, L] \xrightarrow{R\text{-}\mathbf{Mod}[\alpha, \text{id}_L]} R\text{-}\mathbf{Mod}[M, L] .$$

PROOF:

$\langle 1 \rangle 1.$ $R\text{-}\mathbf{Mod}[\beta, \text{id}_L]$ is injective.

PROOF: Since β is epi.

$\langle 1 \rangle 2.$ $\text{im } R\text{-}\mathbf{Mod}[\beta, \text{id}_L] = \ker R\text{-}\mathbf{Mod}[\alpha, \text{id}_L]$

$\langle 2 \rangle 1.$ $\text{im } R\text{-}\mathbf{Mod}[\beta, \text{id}_L] \subseteq \ker R\text{-}\mathbf{Mod}[\alpha, \text{id}_L]$

PROOF: For any $\gamma \in R\text{-}\mathbf{Mod}[P, L]$ we have $\gamma \circ \beta \circ \alpha = 0$ because $\beta \circ \alpha = 0$.

$\langle 2 \rangle 2.$ $\ker R\text{-}\mathbf{Mod}[\alpha, \text{id}_L] \subseteq \text{im } R\text{-}\mathbf{Mod}[\beta, \text{id}_L]$

$\langle 3 \rangle 1.$ LET: $\gamma \in \ker R\text{-}\mathbf{Mod}[\alpha, \text{id}_L]$

$\langle 3 \rangle 2.$ $\gamma \circ \alpha = 0$

$\langle 3 \rangle 3.$ PICK $\delta : P \rightarrow L$ by: for all $p \in P$, we have $\delta(p) = \gamma(n)$ where $n \in N$ is an element such that $\beta(n) = p$.

PROVE: $\delta \circ \beta = \gamma$

$\langle 3 \rangle 4.$ LET: $n \in N$

PROVE: $\delta(\beta(n)) = \gamma(n)$

$\langle 3 \rangle 5.$ PICK $n' \in N$ such that $\delta(\beta(n)) = \gamma(n')$ and $\beta(n') = \beta(n)$

$\langle 3 \rangle 6.$ $n - n' \in \ker \beta = \text{im } \alpha$

$\langle 3 \rangle 7.$ PICK $m \in M$ such that $\alpha(m) = n - n'$

$\langle 3 \rangle 8.$ $0 = \gamma(\alpha(m)) = \gamma(n) - \gamma(n')$

$\langle 3 \rangle 9.$ $\gamma(n) = \gamma(n') = \delta(\beta(n))$

□

Theorem 33.12 (Nine-Lemma). *Let the following be a commuting diagram of left- R -modules.*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \longrightarrow 0 \\
 & & \downarrow \alpha_1 & & \downarrow \beta_1 & & \downarrow \gamma_1 \\
 0 & \longrightarrow & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \longrightarrow 0 \\
 & & \downarrow \alpha_0 & & \downarrow \beta_0 & & \downarrow \gamma_0 \\
 0 & \longrightarrow & L_0 & \xrightarrow{f_0} & M_0 & \xrightarrow{g_0} & N_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

If the rows are exact and the two rightmost columns are exact then the left column is exact.

PROOF:

$\langle 1 \rangle 1.$ (L_2, f_2) is the kernel of g_2 , (L_1, f_1) is the kernel of g_1 and (L_0, f_0) is the kernel of g_0 .

$\langle 1 \rangle 2$. 0 is the cokernel of g_2, g_1 and g_0 .

$\langle 1 \rangle 3$. PICK a homomorphism $\delta : L_0 \rightarrow 0$ such that the following is an exact sequence:

$$L_2 \xrightarrow{\beta_1 \upharpoonright L_2} L_1 \xrightarrow{\beta_0 \upharpoonright L_1} L_0 \xrightarrow{\delta} 0 \rightarrow 0 \rightarrow 0$$

PROOF: Snake Lemma.

$\langle 1 \rangle 4$. $\beta_1 \upharpoonright L_2 = \alpha_1$

$\langle 1 \rangle 5$. $\beta_0 \upharpoonright L_1 = \alpha_0$

$\langle 1 \rangle 6$. The following is an exact sequence:

$$0 \rightarrow L_2 \xrightarrow{\alpha_1} L_1 \xrightarrow{\alpha_0} L_0 \rightarrow 0$$

□

Theorem 33.13. *Let the following be a commuting diagram of left- R -modules.*

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & L_{i+1} & \longrightarrow & M_{i+1} & \longrightarrow & N_{i+1} \longrightarrow 0 \\
 & & \downarrow \alpha_{i+1} & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L_i & \longrightarrow & M_i & \longrightarrow & N_i \longrightarrow 0 \\
 & & \downarrow \alpha_i & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L_{i-1} & \xrightarrow{f_{i-1}} & M_{i-1} & \longrightarrow & N_{i-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Assume the central column is a complex and every row is an exact complex. Then the left and right columns are complexes. Further, if any two of the columns are exact, then so is the third.

PROOF:

$\langle 1 \rangle 1$. The left column is a complex.

$\langle 2 \rangle 1$. LET: $x \in L_{i+1}$

$\langle 2 \rangle 2$. $f_{i-1}(\alpha_i(\alpha_{i+1}(x))) = 0$

$\langle 2 \rangle 3$. $\alpha_i(\alpha_{i+1}(x)) = 0$

PROOF: f_{i-1} is injective.

$\langle 1 \rangle 2$. The right column is a complex.

$\langle 2 \rangle 1$. LET: $x \in N_{i+1}$

$\langle 2 \rangle 2$. PICK $y \in N_{i+1}$ such that $g_{i+1}(y) = x$

$\langle 2 \rangle 3$. $\gamma_i(\gamma_{i+1}(x)) = 0$

PROOF:

$$\begin{aligned}
 \gamma_i(\gamma_{i+1}(x)) &= \gamma_i(\gamma_{i+1}(g_{i+1}(y))) \\
 &= g_{i-1}(\beta_i(\beta_{i+1}(y))) \\
 &= g_{i-1}(0) \\
 &= 0
 \end{aligned}$$

(1)3. If the left and center columns are exact then the right column is exact.

- (2)1. LET: $n_i \in \ker \gamma_{i-1}$
PROVE: $n_i \in \operatorname{im} \gamma_i$
- (2)2. PICK $m_i \in M_i$ such that $g_i(m_i) = n_i$
- (2)3. $g_{i-1}(\beta_i(m_i)) = 0$
- (2)4. $\beta_i(m_i) \in \ker g_{i-1} = \operatorname{im} f_{i-1}$
- (2)5. PICK $l_{i-1} \in L_{i-1}$ such that $f_{i-1}(l_{i-1}) = \beta_i(m_i)$
- (2)6. $\beta_{i-1}(f_{i-1}(l_{i-1})) = 0$
- (2)7. $f_{i-2}(\alpha_{i-1}(l_{i-1})) = 0$
- (2)8. $\alpha_{i-1}(l_{i-1}) = 0$
- (2)9. $l_{i-1} \in \ker \alpha_{i-1} = \operatorname{im} \alpha_i$
- (2)10. PICK $l_i \in L_i$ such that $\alpha_i(l_i) = l_{i-1}$
- (2)11. $\beta_i(f_i(l_i)) = \beta_i(m_i)$
- (2)12. $f_i(l_i) - m_i \in \ker \beta_i = \operatorname{im} \beta_{i+1}$
- (2)13. PICK $m_{i+1} \in M_{i+1}$ such that $\beta_{i+1}(m_{i+1}) = f_i(l_i) - m_i$
- (2)14. $\gamma_{i+1}(-g_{i+1}(m_{i+1})) = n_i$

(1)4. If the left and right columns are exact then the center column is exact.

- (2)1. LET: $x \in \ker \beta_i$
PROVE: $x \in \operatorname{im} \beta_{i+1}$
- (2)2. $g_{i-1}(\beta_i(x)) = 0$
- (2)3. $\gamma_i(g_i(x)) = 0$
- (2)4. $g_i(x) \in \ker \gamma_i = \operatorname{im} \gamma_{i+1}$
- (2)5. PICK $n_{i+1} \in N_{i+1}$ such that $\gamma_{i+1}(n_{i+1}) = g_i(x)$
- (2)6. PICK $m_{i+1} \in M_{i+1}$ such that $g_{i+1}(m_{i+1}) = n_{i+1}$
- (2)7. $g_i(\beta_{i+1}(m_{i+1})) = g_i(x)$
- (2)8. $\beta_{i+1}(m_{i+1}) - x \in \ker g_i = \operatorname{im} f_i$
- (2)9. PICK $l_i \in L_i$ such that $f_i(l_i) = \beta_{i+1}(m_{i+1}) - x$
- (2)10. $\beta_i(f_i(l_i)) = 0$
- (2)11. $f_{i-1}(\alpha_i(l_i)) = 0$
- (2)12. $\alpha_i(l_i) = 0$
- (2)13. $l_i \in \ker \alpha_i = \operatorname{im} \alpha_{i+1}$
- (2)14. PICK $l_{i+1} \in L_{i+1}$ such that $\alpha_{i+1}(l_{i+1}) = l_i$
- (2)15. $\beta_{i+1}(f_{i+1}(l_{i+1})) = \beta_{i+1}(m_{i+1}) - x$
- (2)16. $x = \beta_{i+1}(m_{i+1} - f_{i+1}(l_{i+1}))$

(1)5. If the center and right columns are exact then the left column is exact.

- (2)1. LET: $l_i \in \ker \alpha_i$
PROVE: $l_i \in \operatorname{im} \alpha_{i+1}$
- (2)2. $\beta_i(f_i(l_i)) = 0$
- (2)3. $f_i(l_i) \in \ker \beta_i = \operatorname{im} \beta_{i+1}$
- (2)4. PICK $m_{i+1} \in M_{i+1}$ such that $\beta_{i+1}(m_{i+1}) = f_i(l_i)$
- (2)5. $\gamma_{i+1}(g_{i+1}(m_{i+1})) = 0$
- (2)6. $g_{i+1}(m_{i+1}) \in \ker \gamma_{i+1} = \operatorname{im} \gamma_{i+2}$
- (2)7. PICK $n_{i+2} \in N_{i+2}$ such that $\gamma_{i+2}(n_{i+2}) = g_{i+1}(m_{i+1})$
- (2)8. PICK $m_{i+2} \in M_{i+2}$ such that $g_{i+2}(m_{i+2}) = n_{i+2}$
- (2)9. $g_{i+1}(\beta_{i+2}(n_{i+2})) = g_{i+1}(m_{i+1})$
- (2)10. $\beta_{i+2}(n_{i+2}) - m_{i+1} \in \ker g_{i+1} = \operatorname{im} f_{i+1}$

- (2)11. PICK $l_{i+1} \in L_{i+1}$ such that $f_{i+1}(l_{i+1}) = \beta_{i+2}(n_{i+2}) - m_{i+1}$
 (2)12. $f_i(\alpha_{i+1}(l_{i+1})) = -f_i(l_i)$
 (2)13. $l_i = \alpha_{i+1}(-l_{i+1})$

□

Corollary 33.13.1 (Nine-Lemma). *Let the following be a commuting diagram of left- R -modules.*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \longrightarrow 0 \\
 & & \downarrow \alpha_1 & & \downarrow \beta_1 & & \downarrow \gamma_1 \\
 0 & \longrightarrow & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \longrightarrow 0 \\
 & & \downarrow \alpha_0 & & \downarrow \beta_0 & & \downarrow \gamma_0 \\
 0 & \longrightarrow & L_0 & \xrightarrow{f_0} & M_0 & \xrightarrow{g_0} & N_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

If the rows are exact and the two leftmost columns are exact then the right column is exact.

Proposition 33.14. *Let the following be a commuting diagram of left- R -modules.*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \longrightarrow 0 \\
 & & \downarrow \alpha_1 & & \downarrow \beta_1 & & \downarrow \gamma_1 \\
 0 & \longrightarrow & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \longrightarrow 0 \\
 & & \downarrow \alpha_0 & & \downarrow \beta_0 & & \downarrow \gamma_0 \\
 0 & \longrightarrow & L_0 & \xrightarrow{f_0} & M_0 & \xrightarrow{g_0} & N_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

If the rows are exact and the left and right columns are exact then β_1 is monic.

PROOF: By the Snake Lemma, the following is an exact sequence

$$0 \rightarrow \ker \alpha_1 \rightarrow \ker \beta_1 \rightarrow \ker \gamma_1$$

But $\ker \alpha_1 = \ker \gamma_1 = 0$ so $\ker \beta_1 = 0$. □

Proposition 33.15. *Let the following be a commuting diagram of left- R -modules.*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \longrightarrow 0 \\
 & & \downarrow \alpha_1 & & \downarrow \beta_1 & & \downarrow \gamma_1 \\
 0 & \longrightarrow & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \longrightarrow 0 \\
 & & \downarrow \alpha_0 & & \downarrow \beta_0 & & \downarrow \gamma_0 \\
 0 & \longrightarrow & L_0 & \xrightarrow{f_0} & M_0 & \xrightarrow{g_0} & N_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

If the rows are exact and the left and right columns are exact then β_0 is epi.

PROOF: Similar. \square

Proposition 33.16. *Let the following be a commuting diagram of left- R -modules.*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \longrightarrow 0 \\
 & & \downarrow \alpha_1 & & \downarrow \beta_1 & & \downarrow \gamma_1 \\
 0 & \longrightarrow & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \longrightarrow 0 \\
 & & \downarrow \alpha_0 & & \downarrow \beta_0 & & \downarrow \gamma_0 \\
 0 & \longrightarrow & L_0 & \xrightarrow{f_0} & M_0 & \xrightarrow{g_0} & N_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

If the rows are exact, the left and right columns are exact, and the central column is a complex, then the central column is exact.

PROOF:

$\langle 1 \rangle 1.$ LET: $x \in \ker \beta_0$

PROVE: $x \in \operatorname{im} \beta_1$

$\langle 1 \rangle 2.$ $\gamma_0(g_1(x)) = 0$

$\langle 1 \rangle 3.$ $g_1(x) \in \ker \gamma_0 = \operatorname{im} \gamma_1$

$\langle 1 \rangle 4.$ PICK $n_2 \in N_2$ such that $\gamma_1(n_2) = g_1(x)$

$\langle 1 \rangle 5.$ PICK $m_2 \in M_2$ such that $g_2(m_2) = n_2$

$\langle 1 \rangle 6.$ $g_1(\beta_1(m_2)) = g_1(x)$

$\langle 1 \rangle 7.$ $\beta_1(m_2) - x \in \ker g_1 = \operatorname{im} f_1$

$\langle 1 \rangle 8.$ PICK $l_1 \in L_1$ such that $f_1(l_1) = \beta_1(m_2) - x$.

- ⟨1⟩9. $f_0(\alpha_0(l_1)) = 0$
 ⟨1⟩10. $\alpha_0(l_1) = 0$
 ⟨1⟩11. $l_1 \in \ker \alpha_0 = \operatorname{im} \alpha_1$
 ⟨1⟩12. PICK $l_2 \in L_2$ such that $\alpha_1(l_2) = l_1$.
 ⟨1⟩13. $\beta_1(f_2(l_2)) = \beta_1(m_2) - x$
 ⟨1⟩14. $x = \beta_1(m_2 - f_2(l_2))$
 \square

Example 33.17. We cannot remove the hypothesis that the central column is a complex. Consider the situation

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xlongequal{\quad} & \mathbb{Z} \longrightarrow 0 \\
 & & \downarrow & & \downarrow \Delta & & \parallel \\
 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\kappa_1} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\pi_2} & \mathbb{Z} \longrightarrow 0 \\
 & & \parallel & & \downarrow \pi_1 & & \downarrow \\
 0 & \longrightarrow & \mathbb{Z} & \xlongequal{\quad} & \mathbb{Z} & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

This diagram commutes, the rows are exact, the left and right columns are exact, but the central column is not a complex and $\operatorname{im} \Delta \neq \ker \pi_1$.

33.1 Split Exact Sequences

Definition 33.18 (Split Sequence). Let $0 \rightarrow M_1 \xrightarrow{\alpha} N \xrightarrow{\beta} M_2 \rightarrow 0$ be a short exact sequence. Then this sequence *splits* iff there exists an isomorphism

$$\phi : N \cong M_1 \oplus M_2$$

such that $\phi \circ \alpha = \kappa_1 : M_1 \rightarrow M_1 \oplus M_2$ and $\beta \circ \phi^{-1} = \pi_2 : M_1 \oplus M_2 \rightarrow M_2$.

Proposition 33.19. Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then ϕ has a left-inverse if and only if the sequence

$$0 \rightarrow M \xrightarrow{\phi} N \rightarrow \operatorname{coker} \phi \rightarrow 0$$

splits.

PROOF:

- ⟨1⟩1. If ϕ has a left-inverse then the sequence splits.
 ⟨2⟩1. ASSUME: ϕ has a left-inverse $\psi : N \rightarrow M$.
 ⟨2⟩2. Define $i : N \rightarrow M \oplus \operatorname{coker} \phi$ by $i(n) = (\psi(n), n + \operatorname{im} \phi)$.

$\langle 2 \rangle 3$. Define $i^{-1} : M \oplus \text{coker } \phi$ by $i^{-1}(m, x + \text{im } \phi) = \phi(m) + x - \phi(\psi(x))$.

$\langle 2 \rangle 4$. $i \circ i^{-1} = \text{id}_{M \oplus \text{coker } \phi}$

PROOF:

$$\begin{aligned} \psi(\phi(m) + x - \phi(\psi(x))) &= m + \psi(x) - \psi(x) \\ &= m \end{aligned}$$

$\langle 2 \rangle 5$. $i^{-1} \circ i = \text{id}_N$

PROOF:

$$\begin{aligned} i^{-1}(\psi(n), n + \text{im } \phi) &= \phi(\psi(n)) + n - \phi(\psi(n)) \\ &= n \end{aligned}$$

$\langle 2 \rangle 6$. $i \circ \phi = \kappa_1 : M \rightarrow M \oplus \text{coker } \phi$

PROOF:

$$\begin{aligned} i(\phi(m)) &= (\psi(\phi(m)), \phi(m) + \text{im } \phi) \\ &= (m, \text{im } \phi) \end{aligned}$$

$\langle 2 \rangle 7$. $\pi \circ i^{-1} = \pi_2 : M \oplus \text{coker } \phi \rightarrow \text{coker } \phi$

PROOF:

$$\begin{aligned} i^{-1}(\psi(n), n + \text{im } \phi) + \text{im } \phi &= \phi(\psi(n)) + n - \phi(\psi(n)) + \text{im } \phi \\ &= n + \text{im } \phi \end{aligned}$$

$\langle 1 \rangle 2$. If the sequence splits then ϕ has a left-inverse.

PROOF: Since $\kappa_1 : M \rightarrow M \oplus \text{coker } \phi$ has left inverse π_1 .

□

Proposition 33.20. *Let $\phi : M \rightarrow N$ be a left- R -module homomorphism. Then ϕ has a right-inverse if and only if the sequence*

$$0 \rightarrow \ker \phi \rightarrow M \xrightarrow{\phi} N \rightarrow 0$$

splits.

PROOF:

$\langle 1 \rangle 1$. If ϕ has a right-inverse then the sequence splits.

$\langle 2 \rangle 1$. LET: $\psi : N \rightarrow M$ be a right inverse to ϕ .

$\langle 2 \rangle 2$. LET: $i : M \rightarrow \ker \phi \oplus N$ be the function $i(m) = (m - \psi(\phi(m)), \phi(m))$.

PROOF: $m - \psi(\phi(m)) \in \ker \phi$ since $\phi(m - \psi(\phi(m))) = \phi(m) - \phi(m) = 0$.

$\langle 2 \rangle 3$. LET: $i^{-1} : \ker \phi \oplus N \rightarrow M$ be the function $i^{-1}(x, n) = x + \psi(n)$.

$\langle 2 \rangle 4$. $i \circ i^{-1} = \text{id}_{\ker \phi \oplus N}$

PROOF:

$$\begin{aligned} i(i^{-1}(x, n)) &= i(x + \psi(n)) \\ &= (x + \psi(n) - \psi(\phi(x)) - \psi(\phi(\psi(n))), \phi(x) + \phi(\psi(n))) \\ &= (x + \psi(n) - \psi(n), n) \\ &= (x, n) \end{aligned}$$

$\langle 2 \rangle 5$. $i^{-1} \circ i = \text{id}_M$

PROOF:

$$\begin{aligned} i^{-1}(i(m)) &= m - \psi(\phi(m)) + \psi(\phi(m)) \\ &= m \end{aligned}$$

$\langle 2 \rangle 6$. $i \circ \iota = \kappa_1$

PROOF: For $m \in \ker \phi$ we have $i(m) = (m - \psi(\phi(m)), \phi(m)) = (m, 0)$.

$\langle 2 \rangle 7$. $\phi \circ i^{-1} = \pi_2$

PROOF:

$$\begin{aligned}\phi(i^{-1}(x, n)) &= \phi(x) + \phi(\psi(n)) \\ &= 0 + n \\ &= n\end{aligned}$$

$\langle 1 \rangle 2$. If the sequence splits then ϕ has a right-inverse.

PROOF: Since $\kappa_2 : N \rightarrow M \oplus N$ is a right-inverse to π_2 .

□

Proposition 33.21. *Let*

$$0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} F \rightarrow 0$$

be a short exact sequence where F is free. Then the sequence splits.

PROOF:

$\langle 1 \rangle 1$. LET: $F = R^{\oplus A}$

$\langle 1 \rangle 2$. PICK $\gamma : F \rightarrow N$ such that $\text{id}_F = \beta \circ \gamma$

$\langle 1 \rangle 3$. LET: $i : M \oplus F \rightarrow N$ be the homomorphism $i(m, f) = \alpha(m) + \gamma(f)$

$\langle 1 \rangle 4$. i is injective.

$\langle 2 \rangle 1$. ASSUME: $i(m, f) = i(m', f')$

$\langle 2 \rangle 2$. $\alpha(m) + \gamma(f) = \alpha(m') + \gamma(f')$

$\langle 2 \rangle 3$. $\alpha(m - m') = \gamma(f - f')$

$\langle 2 \rangle 4$. $f - f' = 0$

PROOF: Applying β to both sides of $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. $f = f'$

$\langle 2 \rangle 6$. $\alpha(m - m') = 0$

$\langle 2 \rangle 7$. $m = m'$

PROOF: Since α is injective.

$\langle 1 \rangle 5$. i is surjective.

$\langle 2 \rangle 1$. LET: $n \in N$

$\langle 2 \rangle 2$. $n - \gamma(\beta(n)) \in \ker \beta = \text{im } \alpha$

$\langle 2 \rangle 3$. PICK $m \in M$ such that $\alpha(m) = n - \gamma(\beta(n))$

$\langle 2 \rangle 4$. $n = i(m, \beta(n))$

$\langle 1 \rangle 6$. $\alpha = i \circ \kappa_1$

$\langle 1 \rangle 7$. $\beta \circ i = \pi_2$

□

Chapter 34

Homology

Definition 34.1 (Homology). Let (M_\bullet, d_\bullet) be a chain complex. The *i*th homology of the complex is the R -module

$$H_i(M_\bullet) := \frac{\ker d_i}{\operatorname{im} d_{i+1}} .$$

Proposition 34.2. *Consider the complex*

$$0 \rightarrow M_1 \xrightarrow{\phi} M_0 \rightarrow 0 .$$

The 1st homology is $\ker \phi$, and the 0th homology is $\operatorname{coker} \phi$.

Part IV

Field Theory

Chapter 35

Fields

Definition 35.1 (Field). A *field* is a non-trivial commutative division ring.

Example 35.2. \mathbb{Q} , \mathbb{R} and \mathbb{C} are fields.

Proposition 35.3. *Every field is an integral domain.*

PROOF: By Propositions 11.8 and 11.9. \square

Example 35.4. The converse does not hold: \mathbb{Z} is an integral domain but not a field.

Proposition 35.5. *Every finite integral domain is a field.*

PROOF: In a finite integral domain, multiplication by any non-zero element is injective, hence surjective. \square

Corollary 35.5.1. *For any positive integer n , the following are equivalent:*

- n is prime.
- $\mathbb{Z}/n\mathbb{Z}$ is an integral domain.
- $\mathbb{Z}/n\mathbb{Z}$ is a field.

Theorem 35.6 (Wedderburn's Little Theorem). *Every finite division ring is a field.*

Proposition 35.7. *Every subring of a field is an integral domain.*

PROOF: Easy. \square

Proposition 35.8. *The center of a division ring is a field.*

PROOF:

$\langle 1 \rangle$ 1. LET: R be a division ring.

$\langle 1 \rangle$ 2. LET: Z be the center of R .

$\langle 1 \rangle$ 3. Z is non-trivial.

PROOF: Since $1 \in Z$.

$\langle 1 \rangle 4$. Z is commutative.

$\langle 1 \rangle 5$. Z is a division ring.

$\langle 2 \rangle 1$. LET: $a \in Z$

$\langle 2 \rangle 2$. $a^{-1} \in Z$

$\langle 3 \rangle 1$. LET: $x \in R$

$\langle 3 \rangle 2$. $ax = xa$

$\langle 3 \rangle 3$. $xa^{-1} = a^{-1}x$

□

Definition 35.9. For any prime p and positive integer r , define a multiplication on $(\mathbb{Z}/p\mathbb{Z})^r$ that makes this group into a field by:

Proposition 35.10. *A commutative ring is a field if and only if it is simple.*

PROOF: Proposition 21.5. □

Corollary 35.10.1. *Every field has Krull dimension 0.*

Proposition 35.11. *Let K be a field. Then $K[x]$ is a PID, and every non-zero ideal in $K[x]$ is generated by a unique monic polynomial.*

PROOF:

$\langle 1 \rangle 1$. LET: I be a non-zero ideal in $K[x]$

$\langle 1 \rangle 2$. PICK a monic polynomial $f \in K[x]$ of minimal degree.

PROVE: $I = (f)$

$\langle 1 \rangle 3$. LET: $g \in I$

$\langle 1 \rangle 4$. PICK polynomials q, r with $\deg r < \deg f$ such that $g = qf + r$

$\langle 1 \rangle 5$. $r \in I$

$\langle 1 \rangle 6$. $r = 0$

$\langle 1 \rangle 7$. $g \in (f)$

□

Proposition 35.12. *Let R be a commutative ring and I an ideal in R . Then I is maximal iff R/I is a field.*

PROOF: From Proposition 22.3. □

Example 35.13. Let R be a commutative ring and $a \in R$. Then $(x - a)$ is a maximal ideal in $R[x]$ iff R is a field, since $R[x]/(x - a) \cong R$.

Example 35.14. The ideal $(2, x)$ is a maximal ideal in $\mathbb{Z}[x]$, since $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}/2\mathbb{Z}$.

Proposition 35.15. *Every maximal ideal in a commutative ring is a prime ideal.*

PROOF: Since every field is an integral domain. □

Proposition 35.16. *Let R be a commutative ring and I an ideal in R . If I is a prime ideal and R/I is finite then I is a maximal ideal.*

PROOF: Since every finite integral domain is a field. \square

Proposition 35.17. *Let R be a commutative ring and I a proper ideal in R . Then I is maximal iff, whenever J is an ideal and $I \subseteq J$, then $I = J$ or $J = R$.*

Example 35.18. The inverse image of a maximal ideal under a homomorphism is not necessarily maximal.

Let $i : \mathbb{Z}[x] \rightarrow \mathbb{Q}[x]$ be the inclusion. Then (x) is maximal in $\mathbb{Q}[x]$ but its inverse image (x) is not maximal in $\mathbb{Z}[x]$.

Definition 35.19 (Maximal Spectrum). Let R be a commutative ring. The *maximal spectrum* of R is the set of all maximal ideals in R .

Proposition 35.20. *Let K be a field. The Krull dimension of $K[x_1, \dots, x_n]$ is n .*

Theorem 35.21 (Hilbert's Nullstellensatz). *Let K be a field and L a subfield of K . If K is an L -algebra of finite type, then K is a finite L -algebra.*

Proposition 35.22. *Let K be a subfield of L . Then L is a K -algebra under multiplication.*

PROOF: Easy. \square

Chapter 36

Algebraically Closed Fields

Definition 36.1 (Algebraically Closed). A field K is *algebraically closed* iff, for every $f \in K[x]$ that is not constant, there exists $r \in K$ such that $f(r) = 0$.

Theorem 36.2. \mathbb{C} is algebraically closed.

Proposition 36.3. Let K be an algebraically closed field. Let I be an ideal in $K[x]$. Then I is maximal if and only if $I = (x - c)$ for some $c \in K$.

PROOF:

$\langle 1 \rangle 1$. If I is maximal then there exists $c \in K$ such that $I = (x - c)$.

$\langle 2 \rangle 1$. ASSUME: I is maximal.

$\langle 2 \rangle 2$. PICK f monic of minimal degree such that $f \in I$.

$\langle 2 \rangle 3$. f is not constant.

PROOF: Otherwise $f = 1$ and $I = K[x]$.

$\langle 2 \rangle 4$. PICK $c \in K$ such that $f(c) = 0$

$\langle 2 \rangle 5$. $x - c \mid f$

$\langle 2 \rangle 6$. $I \subseteq (x - c)$

$\langle 2 \rangle 7$. $I = (x - c)$

$\langle 1 \rangle 2$. For all $c \in K$ we have $(x - c)$ is maximal.

PROOF: Example 35.13.

□

Part V

Linear Algebra

Chapter 37

Vector Spaces

Definition 37.1 (Vector Space). Let K be a field. A K -vector space is a K -module. A *linear map* is a homomorphism of K -modules. We write $K - \mathbf{Vect}$ for $K - \mathbf{Mod}$.

Definition 37.2. Let $\mathrm{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ real matrices. $\mathrm{GL}_n(\mathbb{R})$ acts on \mathbb{R}^n by matrix multiplication.

Definition 37.3. Let $\mathrm{GL}_n(\mathbb{C})$ be the group of invertible $n \times n$ complex matrices. $\mathrm{GL}_n(\mathbb{C})$ acts on \mathbb{C}^n by matrix multiplication.

Definition 37.4. Let $\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : \det M = 1\}$.

Proposition 37.5. $\mathrm{SL}_n(\mathbb{R})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$.

PROOF: If $\det M = 1$ then $\det(AMA^{-1}) = (\det A)(\det M)(\det A)^{-1} = 1$. \square

Proposition 37.6.

$$\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$$

Definition 37.7. Let $\mathrm{SL}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : \det M = 1\}$.

Definition 37.8. Let $\mathrm{O}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : MM^T = M^T M = I_n\}$.

Proposition 37.9. The action of $\mathrm{O}_n(\mathbb{R})$ on \mathbb{R}^n preserves lengths and angles.

Definition 37.10. Let $\mathrm{SO}_n(\mathbb{R}) = \{M \in \mathrm{O}_n(\mathbb{R}) : \det M = 1\}$.

Definition 37.11. Let $\mathrm{U}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : MM^\dagger = M^\dagger M = I_n\}$.

Definition 37.12. Let $\mathrm{SU}_n(\mathbb{C}) = \{M \in \mathrm{U}_n(\mathbb{C}) : \det M = 1\}$.

Proposition 37.13. Every matrix in $\mathrm{SU}_2(\mathbb{C})$ can be written in the form

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 + c^2 + d^2 = 1$.

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SU}_2(\mathbb{C})$$

$$\langle 1 \rangle 2. M^{-1} = M^\dagger$$

$$\langle 1 \rangle 3. \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

$$\langle 1 \rangle 4. \text{ LET: } \alpha = a + bi \text{ and } \beta = c + di.$$

$$\langle 1 \rangle 5. \delta = \bar{\alpha} = a - bi$$

$$\langle 1 \rangle 6. \gamma = -\bar{\beta} = -c + di$$

$$\langle 1 \rangle 7. \det M = a^2 + b^2 + c^2 + d^2 = 1$$

□

Corollary 37.13.1. $\text{SU}_2(\mathbb{C})$ is simply connected.

Corollary 37.13.2.

$$\text{SO}_3(\mathbb{R}) \cong \text{SU}_2(\mathbb{C}) / \{I, -I\}$$

$$\text{PROOF: The function that maps } \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} \text{ to } \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(ad + bc) \\ 2(ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(bd - ac) \\ 2(bd - ac) & 2(ab + cd) & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$$

is a surjective homomorphism with kernel $\{I, -I\}$. □

Corollary 37.13.3. The fundamental group of $\text{SO}_3(\mathbb{R})$ is C_2 .