

# Summary of Halmos' Naive Set Theory

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# Chapter 1

## Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*,  $\in$ . If  $x \in A$  we say that  $x$  *belongs to*  $A$ ,  $x$  is an *element* of  $A$ , or  $x$  is *contained in*  $A$ . If this does not hold we write  $x \notin A$ .

**Axiom 1.1** (Axiom of Extensionality). *Two sets are equal if and only if they have the same elements.*

**Axiom 1.2** (Axiom of Comprehension, Aussonderungsaxiom). *To every set  $A$  and to every condition  $S(x)$  there corresponds a set  $B$  whose elements are exactly those elements  $x$  of  $A$  for which  $S(x)$  holds.*

**Axiom 1.3** (Axiom of Pairing). *For any two sets, there exists a set that they both belong to.*

**Axiom 1.4** (Union Axiom). *For every set  $A$ , there exists a set that contains all the elements that belong to at least one element of  $A$ .*

**Definition 1.5** (Subset). Let  $A$  and  $B$  be sets. We say that  $A$  is a *subset* of  $B$ , or  $B$  *includes*  $A$ , and write  $A \subseteq B$  or  $B \supseteq A$ , iff every element of  $A$  is an element of  $B$ .

**Axiom 1.6** (Power Set Axiom). *For any set  $A$ , there exists a set that contains all the subsets of  $A$ .*

**Axiom 1.7** (Axiom of Infinity). *There exists a set  $I$  such that:*

- *$I$  has an element that has no elements*
- *for all  $x \in I$ , there exists  $y \in I$  such that the elements of  $y$  are exactly  $x$  and the elements of  $x$ .*

## Chapter 2

# Basic Properties and Operations on Sets

### 2.1 The Subset Relation

**Theorem 2.1.** *For any set  $A$ , we have  $A \subseteq A$ .*

PROOF: Every element of  $A$  is an element of  $A$ .  $\square$

**Theorem 2.2.** *For any sets  $A$ ,  $B$  and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ .*

PROOF: If every element of  $A$  is an element of  $B$ , and every element of  $B$  is an element of  $C$ , then every element of  $A$  is an element of  $C$ .  $\square$

**Theorem 2.3.** *For any sets  $A$  and  $B$ , if  $A \subseteq B$  and  $B \subseteq A$  then  $A = B$ .*

PROOF: If every element of  $A$  is an element of  $B$ , and every element of  $B$  is an element of  $A$ , then  $A$  and  $B$  have the same elements, and therefore are equal by the Axiom of Extensionality.  $\square$

**Definition 2.4** (Proper Subset). Let  $A$  and  $B$  be sets. We say that  $A$  is a *proper* subset of  $B$ , or  $B$  *properly* includes  $A$ , and write  $A \subsetneq B$  or  $B \supsetneq A$ , iff  $A \subseteq B$  and  $A \neq B$ .

### 2.2 Comprehension Notation

**Definition 2.5.** Given a set  $A$  and a condition  $S(x)$ , we write  $\{x \in A : S(x)\}$  for the set whose elements are exactly those elements  $x$  of  $A$  for which  $S(x)$  holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality.  $\square$

**Theorem 2.6.** *There is no set that contains every set.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be a set.

PROVE: There exists a set  $B$  such that  $B \notin A$ .

$\langle 1 \rangle 2$ . LET:  $B = \{x \in A : x \notin x\}$

$\langle 1 \rangle 3$ . If  $B \in A$  then we have  $B \in B$  if and only if  $B \notin B$ .

$\langle 1 \rangle 4$ .  $B \notin A$

□

## 2.3 The Empty Set

**Theorem 2.7.** *There exists a set with no elements.*

PROOF: Immediate from the Axiom of Infinity. □

**Definition 2.8** (Empty Set). The *empty set*  $\emptyset$  is the set with no elements.

**Theorem 2.9.** *For any set  $A$  we have  $\emptyset \subset A$ .*

PROOF: Vacuous. □

## 2.4 Unordered Pairs

**Definition 2.10** ((Unordered) Pair). For any sets  $a$  and  $b$ , the *(unordered) pair*  $\{a, b\}$  is the set whose elements are just  $a$  and  $b$ .

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality. □

**Definition 2.11** (Singleton). For any set  $a$ , the *singleton*  $\{a\}$  is defined to be  $\{a, a\}$ .

## 2.5 Unions

**Definition 2.12** (Union). For any set  $\mathcal{C}$ , the *union* of  $\mathcal{C}$ ,  $\bigcup \mathcal{C}$ , is the set whose elements are the elements of the elements of  $\mathcal{C}$ .

We write  $\bigcup_{X \in \mathcal{A}} t[X]$  for  $\bigcup \{t[X] \mid X \in \mathcal{A}\}$ .

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality. □

**Proposition 2.13.**

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of  $\emptyset$ . □

**Proposition 2.14.** *For any set  $A$ , we have  $\bigcup \{A\} = A$ .*

PROOF: For any  $x$ , we have  $x$  is an element of an element of  $\{A\}$  if and only if  $x$  is an element of  $A$ .  $\square$

**Definition 2.15.** We write  $A \cup B$  for  $\bigcup\{A, B\}$ .

**Proposition 2.16.** For any set  $A$ , we have  $A \cup \emptyset = A$ .

PROOF:  $x \in A \cup \emptyset$  iff  $x \in A$  or  $x \in \emptyset$ , iff  $x \in A$ .  $\square$

**Proposition 2.17** (Idempotence). For any set  $A$ , we have  $A \cup A = A$ .

PROOF:  $x \in A$  or  $x \in A$  is equivalent to  $x \in A$ .  $\square$

**Proposition 2.18.** For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A \cup B = B$ .

PROOF: For any  $x$ , the statement "if  $x \in A$  then  $x \in B$ " is equivalent to " $x \in A$  or  $x \in B$  if and only if  $x \in B$ ".  $\square$

**Proposition 2.19.** For any sets  $a$  and  $b$ , we have  $\{a\} \cup \{b\} = \{a, b\}$ .

PROOF: Immediate from definitions.  $\square$

## 2.6 Intersections

**Definition 2.20** (Intersection). For any sets  $A$  and  $B$ , the *intersection*  $A \cap B$  is defined to be  $\{x \in A : x \in B\}$ .

**Proposition 2.21.** For any set  $A$ , we have  $A \cap \emptyset = \emptyset$ .

PROOF: There is no  $x$  such that  $x \in A$  and  $x \in \emptyset$ .  $\square$

**Proposition 2.22.** For any set  $A$ , we have

$$A \cap A = A.$$

PROOF: We have  $x \in A$  and  $x \in A$  if and only if  $x \in A$ .  $\square$

**Proposition 2.23.** For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A \cap B = A$ .

PROOF: For any  $x$ , the statement "if  $x \in A$  then  $x \in B$ " is equivalent to " $x \in A$  and  $x \in B$  if and only if  $x \in A$ ".  $\square$

**Proposition 2.24.** For any sets  $A$ ,  $B$  and  $C$ , we have  $C \subseteq A$  if and only if  $(A \cap B) \cup C = A \cap (B \cup C)$ .

PROOF: The statement "if  $x \in C$  then  $x \in A$ " is equivalent to the statement " $((x \in A \wedge x \in B) \vee x \in C) \Leftrightarrow (x \in A \wedge (x \in B \vee x \in C))$ ".  $\square$

**Definition 2.25** (Disjoint). Two sets  $A$  and  $B$  are *disjoint* if and only if  $A \cap B = \emptyset$ .

**Definition 2.26** (Pairwise Disjoint). Let  $A$  be a set. We say the elements of  $A$  are *pairwise disjoint* if and only if, for all  $x, y \in A$ , if  $x \cap y \neq \emptyset$  then  $x = y$ .

**Definition 2.27** (Intersection). For any nonempty set  $\mathcal{C}$ , the *intersection* of  $\mathcal{C}$ ,  $\bigcap \mathcal{C}$ , is the set that contains exactly those sets that belong to every element of  $\mathcal{C}$ .

We write  $\bigcap_{X \in \mathcal{A}} t[X]$  for  $\bigcap \{t[X] \mid X \in \mathcal{A}\}$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C}$  be a nonempty set.

$\langle 1 \rangle 2$ . There exists a set  $I$  whose elements are exactly the sets that belong to every element of  $\mathcal{C}$ .

PROOF: Pick  $A \in \mathcal{C}$ , and take  $I = \{x \in A : \forall X \in \mathcal{C}. x \in X\}$ .

$\langle 1 \rangle 3$ . For any sets  $I, J$ , if the elements of  $I$  and  $J$  are exactly the sets that belong to every element of  $\mathcal{C}$  then  $I = J$ .

PROOF: Axiom of Extensionality.

□

## 2.7 Unordered Triples

**Definition 2.28** ((Unordered) Triple). Given sets  $a_1, \dots, a_n$ , define the (*unordered*) *n-tuple*  $\{a_1, \dots, a_n\}$  to be

$$\{a_1, \dots, a_n\} := \{a_1\} \cup \dots \cup \{a_n\} .$$

## 2.8 Relative Complements

**Definition 2.29** (Relative Complement). For any sets  $A$  and  $B$ , the *difference* or *relative complement*  $A - B$  is defined to be

$$A - B := \{x \in A : x \notin B\} .$$

**Proposition 2.30.** For any sets  $A$  and  $E$ , we have  $A \subseteq E$  if and only if

$$E - (E - A) = A$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $E$  be sets.

$\langle 1 \rangle 2$ . If  $A \subseteq E$  then  $E - (E - A) = A$

$\langle 2 \rangle 1$ . ASSUME:  $A \subseteq E$

$\langle 2 \rangle 2$ .  $E - (E - A) \subseteq A$

PROOF: If  $x \in E$  and  $x \notin E - A$  then  $x \in A$ .

$\langle 2 \rangle 3$ .  $A \subseteq E - (E - A)$

PROOF: If  $x \in A$  then  $x \in E$  and  $x \notin E - A$ .

$\langle 1 \rangle 3$ . If  $E - (E - A) = A$  then  $A \subseteq E$ .

PROOF: Since  $E - (E - A) \subseteq E$ .

□

**Proposition 2.31.** *For any set  $E$  we have*

$$E - \emptyset = E$$

PROOF:  $x \in E$  if and only if  $x \in E$  and  $x \notin \emptyset$ . □

**Proposition 2.32.** *For any set  $E$  we have*

$$E - E = \emptyset .$$

PROOF: There is no  $x$  such that  $x \in E$  and  $x \notin E$ . □

**Proposition 2.33.** *For any sets  $A$  and  $E$ , we have*

$$A \cap (E - A) = \emptyset .$$

PROOF: There is no  $x$  such that  $x \in A$  and  $x \in E - A$ . □

**Proposition 2.34.** *Let  $A$  and  $E$  be sets. Then  $A \subseteq E$  if and only if*

$$A \cup (E - A) = E .$$

PROOF:

⟨1⟩1. LET:  $A$  and  $E$  be sets.

⟨1⟩2. If  $A \subseteq E$  then  $A \cup (E - A) = E$ .

⟨2⟩1. ASSUME:  $A \subseteq E$

⟨2⟩2.  $A \cup (E - A) \subseteq E$

PROOF: If  $x \in A$  or  $x \in E - A$  then  $x \in E$ .

⟨2⟩3.  $E \subseteq A \cup (E - A)$

PROOF: If  $x \in E$  then either  $x \in A$  or  $x \notin A$ . In the latter case,  $x \in E - A$ .

⟨1⟩3. If  $A \cup (E - A) = E$  then  $A \subseteq E$

PROOF: Since  $A \subseteq A \cup (E - A)$ .

□

**Proposition 2.35.** *Let  $A$ ,  $B$  and  $E$  be sets. Then:*

1. *If  $A \subseteq B$  then  $E - B \subseteq E - A$ .*

2. *If  $A \subseteq E$  and  $E - B \subseteq E - A$  then  $A \subseteq B$ .*

PROOF:

⟨1⟩1. LET:  $A$ ,  $B$  and  $E$  be sets.

⟨1⟩2. If  $A \subseteq B$  then  $E - B \subseteq E - A$ .

PROOF: If  $A \subseteq B$ ,  $x \in E$  and  $x \notin B$ , then we have  $x \in E$  and  $x \notin A$ .

⟨1⟩3. If  $A \subseteq E$  and  $E - B \subseteq E - A$  then  $A \subseteq B$ .

⟨2⟩1. ASSUME:  $A \subseteq E$

⟨2⟩2. ASSUME:  $E - B \subseteq E - A$

⟨2⟩3. LET:  $x \in A$

⟨2⟩4.  $x \in E$



⟨2⟩5.  $x \notin E - A$

⟨2⟩6.  $x \notin E - B$

⟨2⟩7.  $x \in B$

□

**Example 2.36.** We cannot remove the hypothesis  $A \subseteq E$  in item 2 above. Let  $E = \emptyset$ ,  $A = \{\emptyset\}$  and  $B = \emptyset$ . Then  $E - B = E - A = \emptyset$  but  $A \not\subseteq B$ .

**Proposition 2.37** (De Morgan's Law). *For any sets  $A$ ,  $B$  and  $E$ , we have  $E - (A \cup B) = (E - A) \cap (E - B)$ .*

PROOF:  $(x \in E \wedge \neg(x \in A \vee x \in B)) \Leftrightarrow (x \in E \wedge x \notin A \wedge x \in E \wedge x \notin B)$ . □

**Proposition 2.38** (De Morgan's Law). *For any sets  $A$ ,  $B$  and  $E$ , we have  $E - (A \cap B) = (E - A) \cup (E - B)$ .*

PROOF:  $(x \in E \vee \neg(x \in A \wedge x \in B)) \Leftrightarrow (x \in E \wedge x \notin A) \vee (x \in E \wedge x \notin B)$ . □

**Proposition 2.39.** *For any sets  $A$ ,  $B$  and  $E$ , if  $A \subseteq E$  then*

$$A - B = A \cap (E - B) .$$

PROOF: If  $A \subseteq E$  then we have  $(x \in A \wedge x \notin B) \Leftrightarrow (x \in A \wedge x \in E \wedge x \notin B)$ . □

**Proposition 2.40.** *For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A - B = \emptyset$ .*

PROOF: Both are equivalent to the statement that there is no  $x$  such that  $x \in A$  and  $x \notin B$ . □

**Proposition 2.41.** *For any sets  $A$  and  $B$ , we have*

$$A - (A - B) = A \cap B .$$

PROOF:  $(x \in A \wedge \neg(x \in A \wedge x \notin B)) \Leftrightarrow x \in A \wedge x \in B$ . □

**Proposition 2.42.** *For any sets  $A$ ,  $B$  and  $C$ , we have*

$$A \cap (B - C) = (A \cap B) - (A \cap C) .$$

PROOF:  $(x \in A \wedge x \in B \wedge x \notin C) \Leftrightarrow (x \in A \wedge x \in B \wedge \neg(x \in A \wedge x \in C))$ . □

**Proposition 2.43.** *For any sets  $A$ ,  $B$ ,  $C$  and  $E$ , if  $(A \cap B) - C \subseteq E$  then we have*

$$A \cap B \subseteq (A \cap C) \cup (B \cap (E - C)) .$$

PROOF:

⟨1⟩1. LET:  $x \in A \cap B$

PROVE:  $x \in (A \cap C) \cup (B \cap (E - C))$

⟨1⟩2. CASE:  $x \in C$

PROOF: Then  $x \in A \cap C$ .

⟨1⟩3. CASE:  $x \notin C$

PROOF: Then  $x \in E$  and so  $x \in B \cap (E - C)$ .  
 $\square$

**Proposition 2.44.** *For any sets  $A, B, C$  and  $E$ , we have*

$$(A \cup C) \cap (B \cup (E - C)) \subseteq A \cup B .$$

PROOF: The statement  $(x \in A \vee x \in C) \wedge (x \in B \vee (x \in E \wedge x \notin C))$  implies  $x \in A \vee x \in B$ .  $\square$

**Proposition 2.45** (De Morgan's Law). *Let  $E$  be a set and  $\mathcal{C}$  a nonempty set. Then*

$$E - \bigcup \mathcal{C} = \bigcap_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy.  $\square$

**Proposition 2.46** (De Morgan's Law). *Let  $E$  be a set and  $\mathcal{C}$  a nonempty set. Then*

$$E - \bigcap \mathcal{C} = \bigcup_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy.  $\square$

## 2.9 Symmetric Difference

**Definition 2.47** (Symmetric Difference). For any sets  $A$  and  $B$ , the *symmetric difference*  $A + B$  is defined to be

$$A + B := (A - B) \cup (B - A) .$$

**Proposition 2.48.** *For any sets  $A$  and  $B$ , we have*

$$A + B = B + A$$

PROOF: From the commutativity of union.  $\square$

**Proposition 2.49.** *For any sets  $A, B$  and  $C$ , we have*

$$A + (B + C) = (A + B) + C .$$

PROOF: Each is the set of all  $x$  that belong to either exactly one or all three of  $A, B$  and  $C$ .  $\square$

**Proposition 2.50.** *For any set  $A$ , we have*

$$A + \emptyset = A .$$

PROOF:

$$\begin{aligned} A + \emptyset &= (A - \emptyset) \cup (\emptyset - A) \\ &= A \cup \emptyset \\ &= A \end{aligned}$$

$\square$

**Proposition 2.51.** *For any set  $A$  we have*

$$A + A = \emptyset .$$

PROOF:

$$\begin{aligned} A + A &= (A - A) \cup (A - A) \\ &= \emptyset \cup \emptyset \\ &= \emptyset \end{aligned}$$

□

## 2.10 Power Sets

**Definition 2.52** (Power Set). For any set  $A$ , the *power set* of  $A$ ,  $\mathcal{P}A$ , is the set whose elements are exactly the subsets of  $A$ .

PROOF: This exists by the Power Set Axiom and Axiom of Comprehension, and is unique by the Axiom of Extensionality. □

**Proposition 2.53.**

$$\mathcal{P}\emptyset = \{\emptyset\}$$

PROOF: The only subset of  $\emptyset$  is  $\emptyset$ . □

**Proposition 2.54.** *For any set  $a$ , we have*

$$\mathcal{P}\{a\} = \{\emptyset, \{a\}\} .$$

PROOF: The only subsets of  $\{a\}$  are  $\emptyset$  and  $\{a\}$ . □

**Proposition 2.55.** *For any sets  $a$  and  $b$ , we have*

$$\mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} .$$

PROOF: The only subsets of  $\{a, b\}$  are  $\emptyset$ ,  $\{a\}$ ,  $\{b\}$  and  $\{a, b\}$ . □

**Proposition 2.56.** *For any nonempty set  $\mathcal{C}$  we have*

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}X = \mathcal{P}\left(\bigcap \mathcal{C}\right) .$$

PROOF:

$$\begin{aligned} x \in \bigcup_{X \in \mathcal{C}} \mathcal{P}X &\Leftrightarrow \forall X \in \mathcal{C}. x \subseteq X \\ &\Leftrightarrow \forall X \in \mathcal{C}. \forall y \in x. y \in X \\ &\Leftrightarrow \forall y \in x. \forall X \in \mathcal{C}. y \in X \\ &\Leftrightarrow x \subseteq \bigcap \mathcal{C} \end{aligned}$$

□

**Proposition 2.57.** *For any set  $\mathcal{C}$  we have*

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}X \subseteq \mathcal{P}\bigcup \mathcal{C} .$$

PROOF: If there exists  $X \in \mathcal{C}$  such that  $x \subseteq X$  then  $x \subseteq \bigcup \mathcal{C}$ .  $\square$

**Proposition 2.58.** *For any set  $E$ , we have*

$$\bigcap \mathcal{P}E = \emptyset .$$

PROOF: Since  $\emptyset \in \mathcal{P}E$ .  $\square$

**Proposition 2.59.** *For any sets  $E$  and  $F$ , if  $E \subseteq F$  then  $\mathcal{P}E \subseteq \mathcal{P}F$ .*

PROOF: If  $E \subseteq F$  and  $X \subseteq E$  then  $X \subseteq F$ .  $\square$

## Chapter 3

# Relations and Functions

### 3.1 Ordered Pairs

**Definition 3.1** (Ordered Pair). For any sets  $a$  and  $b$ , the *ordered pair*  $(a, b)$  is defined by

$$(a, b) := \{\{a\}, \{a, b\}\} .$$

**Proposition 3.2.** For any sets  $a, b, x$  and  $y$ , if  $(a, b) = (x, y)$  then  $a = x$  and  $b = y$ .

PROOF:

⟨1⟩1. LET:  $a, b, x$  and  $y$  be sets.

⟨1⟩2. ASSUME:  $(a, b) = (x, y)$

⟨1⟩3.  $a = x$

PROOF:  $\{a\} = \bigcap(a, b) = \bigcap(x, y) = \{x\}$ .

⟨1⟩4.  $\{a, b\} = \{x, y\}$

⟨1⟩5. CASE:  $a = b$

⟨2⟩1.  $x = y$

PROOF: Since  $\{x, y\} = \{a, b\}$  is a singleton.

⟨2⟩2.  $b = y$

PROOF:  $b = a = x = y$

⟨1⟩6. CASE:  $a \neq b$

⟨2⟩1.  $x \neq y$

PROOF: Since  $\{x, y\} = \{a, b\}$  is not a singleton.

⟨2⟩2.  $b = y$

PROOF:  $\{b\} = \{a, b\} - \{a\} = \{x, y\} - \{x\} = \{y\}$ .

□

**Definition 3.3** (Cartesian Product). For any sets  $A$  and  $B$ , the *Cartesian product*  $A \times B$  is

$$A \times B := \{p \in \mathcal{PP}(A \cup B) : \exists a \in A. \exists b \in B. p = (a, b)\} .$$

**Proposition 3.4.** For any sets  $A$ ,  $B$  and  $X$ , we have

$$(A - B) \times X = (A \times X) - (B \times X) .$$

PROOF: Easy.  $\square$

**Proposition 3.5.** For any sets  $A$  and  $B$ , we have  $A \times B = \emptyset$  if and only if  $A = \emptyset$  or  $B = \emptyset$ .

PROOF: Easy.  $\square$

**Proposition 3.6.** For any sets  $A$ ,  $B$ ,  $X$  and  $Y$ , if  $A \subseteq X$  and  $B \subseteq Y$  then  $A \times B \subseteq X \times Y$ . The converse holds assuming  $A \neq \emptyset$  and  $B \neq \emptyset$ .

PROOF: Easy.  $\square$

## 3.2 Relations

**Definition 3.7** (Relation). A *relation* is a set of ordered pairs.

If  $R$  is a relation, we write  $xRy$  for  $(x, y) \in R$ .

Given sets  $X$  and  $Y$ , a relation *between*  $X$  and  $Y$  is a subset of  $X \times Y$ .

Given a set  $X$ , a relation *on*  $X$  is a relation between  $X$  and  $X$ .

**Definition 3.8** (Domain). The *domain* of a relation  $R$  is the set

$$\text{dom } R := \left\{ x \in \bigcup \bigcup R : \exists y. (x, y) \in R \right\} .$$

**Definition 3.9** (Range). The *range* of a relation  $R$  is the set

$$\text{ran } R := \left\{ y \in \bigcup \bigcup R : \exists x. (x, y) \in R \right\} .$$

**Definition 3.10** (Reflexive). Let  $R$  be a relation on  $X$ . Then  $R$  is *reflexive* iff, for all  $x \in X$ , we have  $xRx$ .

**Definition 3.11** (Symmetric). Let  $R$  be a relation on  $X$ . Then  $R$  is *symmetric* iff, whenever  $xRy$ , then  $yRx$ .

**Definition 3.12** (Antisymmetric). A relation  $R$  is *antisymmetric* iff, whenever  $xRy$  and  $yRx$ , then  $x = y$ .

**Definition 3.13** (Transitive). Let  $R$  be a relation on  $X$ . Then  $R$  is *transitive* iff, whenever  $xRy$  and  $yRz$ , then  $xRz$ .

## 3.3 Equivalence Relations

**Definition 3.14** (Equivalence Relation). Let  $R$  be a relation on  $X$ . Then  $R$  is an *equivalence relation* iff it is reflexive, symmetric and transitive.

**Definition 3.15** (Partition). Let  $X$  be a set. A *partition* of  $X$  is a pairwise disjoint set of nonempty subsets of  $X$  whose union is  $X$ .

**Definition 3.16** (Equivalence Class). Let  $R$  be an equivalence relation on  $X$ . Let  $x \in X$ . The *equivalence class* of  $x$  with respect to  $R$  is

$$x/R := \{y \in X : xRy\} .$$

We write  $X/R$  for the set of all equivalence classes with respect to  $R$ .

**Definition 3.17** (Induced). Let  $P$  be a partition of  $X$ . The relation *induced* by  $P$  is  $X/P$  where  $x(X/P)y$  iff there exists  $X \in P$  such that  $x \in X$  and  $y \in X$ .

**Theorem 3.18.** *Let  $R$  be an equivalence relation on  $X$ . Then  $X/R$  is a partition of  $X$  that induces the relation  $R$ .*

PROOF: Easy.  $\square$

**Theorem 3.19.** *Let  $P$  be a partition of  $X$ . Then  $X/P$  is an equivalence relation on  $X$ , and  $P = X/(X/P)$ .*

PROOF: Easy.  $\square$

**Definition 3.20** (Composition). Let  $R$  be a relation between  $X$  and  $Y$ , and  $S$  a relation between  $Y$  and  $Z$ . The *composite* or *relative product*  $S \circ R = SR$  is the relation between  $X$  and  $Z$  defined by

$$x(S \circ R)z \Leftrightarrow \exists y \in Y (xRy \wedge ySz) .$$

**Proposition 3.21.** *Let  $R$  be a relation between  $X$  and  $Y$ ,  $S$  a relation between  $Y$  and  $Z$ , and  $T$  a relation between  $Z$  and  $W$ . Then*

$$T(SR) = (TS)R .$$

PROOF: Easy.  $\square$

**Example 3.22.** Composition of relations is not commutative in general. Let  $X = \{a, b\}$  where  $a \neq b$ . Let  $R = \{(a, a), (b, a)\}$  and  $S = \{(a, b), (b, b)\}$ . Then  $SR = S$  but  $RS = R \neq S$ .

**Proposition 3.23.** *A relation  $R$  is transitive if and only if  $RR \subseteq R$ .*

PROOF: Easy.  $\square$

**Definition 3.24** (Inverse). Let  $R$  be a relation between  $X$  and  $Y$ . The *inverse* or *converse*  $R^{-1}$  is the relation between  $Y$  and  $X$  defined by

$$yR^{-1}x \Leftrightarrow xRy .$$

**Proposition 3.25.** *For any relation  $R$ , we have*

$$\text{dom } R^{-1} = \text{ran } R .$$

PROOF: Easy.  $\square$

**Proposition 3.26.** *For any relation  $R$ , we have*

$$\text{ran } R^{-1} = \text{dom } R .$$

PROOF: Easy.  $\square$

**Proposition 3.27.** *Let  $R$  be a relation between  $X$  and  $Y$ , and  $S$  a relation between  $Y$  and  $Z$ . Then*

$$(SR)^{-1} = R^{-1}S^{-1} .$$

PROOF: Easy.  $\square$

**Proposition 3.28.** *A relation  $R$  is symmetric if and only if  $R \subseteq R^{-1}$ .*

PROOF: Easy.  $\square$

**Definition 3.29** (Identity Relation). For any set  $X$ , the *identity relation*  $I_X$  on  $X$  is

$$I_X = \{(x, x) : x \in X\} .$$

**Proposition 3.30.** *Let  $R$  be a relation between  $X$  and  $Y$ . Then*

$$I_Y R = R I_X = R .$$

PROOF: Easy.  $\square$

**Proposition 3.31.** *A relation  $R$  on a set  $X$  is reflexive if and only if  $I_X \subseteq R$ .*

PROOF: Easy.  $\square$

## 3.4 Functions

**Definition 3.32** (Function). Let  $X$  and  $Y$  be sets. A *function*, *map*, *mapping*, *transformation* or *operator*  $f$  from  $X$  to  $Y$ ,  $f : X \rightarrow Y$ , is a relation  $f$  between  $X$  and  $Y$  such that, for all  $x \in X$ , there exists a unique  $f(x) \in Y$ , called the *value* of  $f$  at the *argument*  $x$ , such that  $(x, f(x)) \in f$ .

**Definition 3.33** (Onto). Let  $f : X \rightarrow Y$ . We say  $f$  maps  $X$  *onto*  $Y$  iff  $\text{ran } f = Y$ .

**Definition 3.34** (Image). Let  $f : X \rightarrow Y$  and  $A \subseteq X$ . The *image* of  $A$  under  $f$  is

$$f(A) := \{f(x) : x \in A\} .$$

**Definition 3.35** (Inclusion Map). Let  $Y$  be a set and  $X \subseteq Y$ . Then the *inclusion map*  $i : X \hookrightarrow Y$  is the function defined by  $i(x) = x$  for all  $x \in X$ .

**Proposition 3.36.** *For any set  $X$ , the identity relation  $I_X$  is a function  $X \rightarrow X$ .*



PROOF: Easy.  $\square$

**Definition 3.37** (Restriction). Let  $f : Y \rightarrow Z$  and  $X \subseteq Y$ . The *restriction* of  $f$  to  $X$  is the function  $f \upharpoonright X : X \rightarrow Z$  defined by

$$(f \upharpoonright X)(x) = f(x) \quad (x \in X) .$$

Given sets  $X, Y$  and  $Z$  with  $X \subseteq Y$ , if  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$ , we say  $g$  is an *extension* of  $f$  to  $Y$  iff  $f = g \upharpoonright X$ .

**Definition 3.38** (Projection). Given sets  $X$  and  $Y$ , the *projection* maps  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  are defined by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y \quad (x \in X, y \in Y) .$$

**Definition 3.39** (Canonical Map). Let  $X$  be a set and  $R$  an equivalence relation on  $X$ . The *canonical map*  $\pi : X \rightarrow X/R$  is the map defined by  $\pi(x) = x/R$ .

**Definition 3.40** (One-to-One). A function  $f : X \rightarrow Y$  is *one-to-one*, or a *one-to-one correspondence*, iff, for all  $x, y \in X$ , if  $f(x) = f(y)$  then  $x = y$ .

**Proposition 3.41.** *Let  $f : X \rightarrow Y$ . Then the following are equivalent:*

1.  $f$  is one-to-one.
2. For all  $A, B \subseteq X$ , we have  $f(A \cap B) = f(A) \cap f(B)$ .
3. For all  $A \subseteq X$ , we have  $f(X - A) \subseteq Y - f(A)$ .

PROOF: Easy.  $\square$

**Proposition 3.42.** *Let  $f : X \rightarrow Y$ . Then  $f$  maps  $X$  onto  $Y$  if and only if, for all  $A \subseteq X$ , we have  $Y - f(A) \subseteq f(X - A)$ .*

PROOF: Easy.  $\square$

## 3.5 Families

**Definition 3.43** (Family). Let  $I$  and  $X$  be sets. A *family* of elements of  $X$  indexed by  $I$  is a function  $a : I \rightarrow X$ . We write  $a_i$  for  $a(i)$ , and  $\{a_i\}_{i \in I}$  for  $a$ .

**Proposition 3.44** (Generalized Associative Law for Unions). *Let  $\{I_j\}_{j \in J}$  be a family of sets. Let  $K = \bigcup_{j \in J} I_j$ . Let  $\{A_k\}_{k \in K}$  be a family of sets indexed by  $K$ . Then*

$$\bigcup_{k \in K} A_k = \bigcup_{j \in J} \bigcup_{i \in I_j} A_i .$$

PROOF: Easy.  $\square$

**Proposition 3.45** (Generalized Commutative Law for Unions). *Let  $\{I_j\}_{j \in J}$  be a family of sets. Let  $f : J \rightarrow J$  be a one-to-one correspondence from  $J$  onto  $J$ . Then*

$$\bigcup_{j \in J} I_j = \bigcup_{j \in J} I_{f(j)} .$$

PROOF: Easy.  $\square$

**Proposition 3.46** (Generalized Associative Law for Intersections). *Let  $\{I_j\}_{j \in J}$  be a nonempty family of nonempty sets. Let  $K = \bigcup_{j \in J} I_j$ . Let  $\{A_k\}_{k \in K}$  be a family of sets indexed by  $K$ . Then*

$$\bigcap_{k \in K} A_k = \bigcap_{j \in J} \bigcap_{i \in I_j} A_i .$$

PROOF: Easy.  $\square$

**Proposition 3.47** (Generalized Commutative Law for Intersections). *Let  $\{I_j\}_{j \in J}$  be a nonempty family of sets. Let  $f : J \rightarrow J$  be a one-to-one correspondence from  $J$  onto  $J$ . Then*

$$\bigcap_{j \in J} I_j = \bigcap_{j \in J} I_{f(j)} .$$

PROOF: Easy.  $\square$

**Proposition 3.48.** *Let  $B$  be a set and  $\{A_i\}_{i \in I}$  a family of sets. Then*

$$B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i)$$

PROOF: Easy.  $\square$

**Proposition 3.49.** *Let  $B$  be a set and  $\{A_i\}_{i \in I}$  a nonempty family of sets. Then*

$$B \cup \bigcap_{i \in I} A_i = \bigcap_{i \in I} (B \cup A_i)$$

PROOF: Easy.  $\square$

**Definition 3.50** (Cartesian Product of a Family of Sets). Let  $\{A_i\}_{i \in I}$  be a family of sets. The *Cartesian product*  $\times_{i \in I} A_i$  is the set of all families  $\{a_i\}_{i \in I}$  such that  $\forall i \in I. a_i \in A_i$ .

We write  $A^I$  for  $\times_{i \in I} A_i$ .

**Definition 3.51** (Projection). Let  $\{A_i\}_{i \in I}$  be a family of sets and  $i \in I$ . The *projection function*  $\pi_i : \times_{i \in I} A_i \rightarrow A_i$  is defined by  $\pi_i(a) = a_i$ .

**Proposition 3.52.** *Let  $\{A_i\}_{i \in I}$  and  $\{B_j\}_{j \in J}$  be families of sets. Then*

$$\left( \bigcup_{i \in I} A_i \right) \times \left( \bigcup_{j \in J} B_j \right) = \bigcup_{i \in I} \bigcup_{j \in J} (A_i \times B_j) .$$

PROOF: Easy.  $\square$

**Proposition 3.53.** Let  $\{A_i\}_{i \in I}$  and  $\{B_j\}_{j \in J}$  be nonempty families of sets. Then

$$\left( \bigcap_{i \in I} A_i \right) \times \left( \bigcap_{j \in J} B_j \right) = \bigcap_{i \in I} \bigcap_{j \in J} (A_i \times B_j) .$$

PROOF: Easy.  $\square$

**Proposition 3.54.** Let  $f : X \rightarrow Y$ . Let  $\{A_i\}_{i \in I}$  be a family of subsets of  $X$ . Then

$$f \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} f(A_i) .$$

PROOF: Easy.  $\square$

**Example 3.55.** It is not true in general that, if  $f : X \rightarrow Y$  and  $\{A_i\}_{i \in I}$  is a nonempty family of subsets of  $X$ , then  $f(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f(A_i)$ .

Take  $X = \{a, b\}$  and  $Y = \{c\}$  where  $a \neq b$ . Take  $I = \{i, j\}$  with  $i \neq j$ . Let  $A_i = \{a\}$  and  $A_j = \{b\}$ . Let  $f$  be the unique function  $X \rightarrow Y$ . Then  $f(\bigcap_{i \in I} A_i) = f(\emptyset) = \emptyset$  but  $\bigcap_{i \in I} f(A_i) = \{c\}$ .

## 3.6 Inverses and Composites

**Definition 3.56** (Inverse). Given a function  $f : X \rightarrow Y$ , the *inverse* of  $f$  is the function  $f^{-1} : \mathcal{P}Y \rightarrow \mathcal{P}X$  defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\} .$$

We call  $f^{-1}(B)$  the *inverse image* of  $B$  under  $f$ .

**Proposition 3.57.** Let  $f : X \rightarrow Y$ . Then  $f$  maps  $X$  onto  $Y$  if and only if the inverse image of any nonempty subset of  $Y$  is nonempty.

PROOF: Easy.  $\square$

**Proposition 3.58.** Let  $f : X \rightarrow Y$ . Then  $f$  is one-to-one if and only if the inverse image of any singleton subset of  $Y$  is a singleton.

PROOF: Easy.  $\square$

**Proposition 3.59.** Let  $f : X \rightarrow Y$ . Let  $B \subseteq Y$ . Then

$$f(f^{-1}(B)) \subseteq B .$$

PROOF: Easy.  $\square$

**Proposition 3.60.** Let  $f : X \rightarrow Y$ . Let  $A \subseteq X$ . Then

$$A \subseteq f^{-1}(f(A)) .$$

Equality holds if  $f$  is one-to-one.

PROOF: Easy.  $\square$

**Proposition 3.61.** *Let  $f : X \rightarrow Y$ . Let  $\{B_i\}_{i \in I}$  be a family of subsets of  $Y$ . Then*

$$f^{-1} \left( \bigcup_{i \in I} B_i \right) = \bigcup_{i \in I} f^{-1}(B_i) .$$

PROOF: Easy.  $\square$

**Proposition 3.62.** *Let  $f : X \rightarrow Y$ . Let  $\{B_i\}_{i \in I}$  be a nonempty family of subsets of  $Y$ . Then*

$$f^{-1} \left( \bigcap_{i \in I} B_i \right) = \bigcap_{i \in I} f^{-1}(B_i) .$$

PROOF: Easy.  $\square$

**Proposition 3.63.** *Let  $f : X \rightarrow Y$  and  $B \subseteq Y$ . Then  $f^{-1}(Y - B) = X - f^{-1}(B)$ .*

PROOF: Easy.  $\square$

**Proposition 3.64.** *Let  $f : X \rightarrow Y$  be one-to-one. Then the inverse of  $f$  as a relation,  $f^{-1}$ , is a function  $f^{-1} : \text{ran } f \rightarrow X$ , and for all  $y \in \text{ran } f$ , we have  $f^{-1}(y)$  is the unique  $x$  such that  $f(x) = y$ .*

PROOF: Easy.  $\square$

**Proposition 3.65.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . Then  $gf : X \rightarrow Z$  and, for all  $x \in X$ , we have*

$$(g \circ f)(x) = g(f(x)) .$$

PROOF: Easy.  $\square$

**Example 3.66.** Example 3.22 shows that function composition is not commutative in general.

**Proposition 3.67.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . Then*

$$(gf)^{-1} = f^{-1}g^{-1} : \mathcal{P}Z \rightarrow \mathcal{P}X .$$

PROOF: Easy.  $\square$

**Proposition 3.68.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$ . If  $gf = I_X$  then  $f$  is one-to-one and  $g$  maps  $Y$  onto  $X$ .*

PROOF: Easy.  $\square$

## Chapter 4

# Order

**Definition 4.1** (Partial Order). A *partial order* on a set  $X$  is a relation on  $X$  that is reflexive, antisymmetric and transitive.

**Definition 4.2** (Total Order). A partial order  $\leq$  on a set  $X$  is a *total order*, *simple order* or *linear order* iff, for all  $x, y \in X$ , either  $x \leq y$  or  $y \leq x$ .

# Chapter 5

## Natural Numbers

### 5.1 Natural Numbers

**Definition 5.1** (Successor). The *successor* of a set  $x$ ,  $x^+$ , is defined by

$$x^+ := x \cup \{x\} .$$

**Definition 5.2.** We define

$$\begin{aligned} 0 &= \emptyset \\ 1 &= 0^+ \\ 2 &= 1^+ \end{aligned}$$

etc.

**Definition 5.3** (Characteristic Function). Let  $X$  be a set and  $A \subseteq X$ . The *characteristic function* of  $A$  is the function  $\chi_A : X \rightarrow 2$  defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

**Theorem 5.4.** Let  $X$  be a set. The function  $\chi : \mathcal{P}X \rightarrow 2^X$  that maps a subset  $A$  of  $X$  to  $\chi_A$  is a one-to-one correspondence.

PROOF: Easy.  $\square$

**Definition 5.5.** The set  $\omega$  of *natural numbers* is the set such that:

- $0 \in \omega$
- For all  $n \in \omega$  we have  $n^+ \in \omega$
- For any set  $X$ , if  $0 \in X$  and  $\forall n \in X. n^+ \in X$  then  $\omega \subseteq X$

PROOF: To show this exists, pick a set  $A$  such that  $0 \in A$  and  $\forall n \in A. n^+ \in A$  (by the Axiom of Infinity), and let  $\omega = \bigcap \{X \in \mathcal{P}A : 0 \in X \wedge \forall n \in X. n^+ \in X\}$ .  
 $\square$

**Definition 5.6** (Sequence). A *finite sequence* is a family whose index set is a natural number. An *infinite sequence* is a family whose index set is  $\omega$ .

Given a finite sequence of sets  $\{A_i\}_{i \in n^+}$ , we write  $\bigcup_{i=0}^n A_i$  for  $\bigcup_{i \in n^+} A_i$ . Given an infinite sequence of sets  $\{A_i\}_{i \in \omega}$ , we write  $\bigcup_{i=0}^{\infty} A_i$  for  $\bigcup_{i \in \omega} A_i$ .

We make similar definitions for  $\bigcap$  and  $\times$ .

**Proposition 5.7.** For any natural numbers  $m$  and  $n$ , if  $m \in n$  then  $m^+ \in n^+$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the property  $\forall m \in n. m^+ \in n^+$

$\langle 1 \rangle 2$ .  $P(0)$

PROOF: Vacuous.

$\langle 1 \rangle 3$ . For any natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$

$\langle 2 \rangle 3$ . LET:  $m \in n^+$

$\langle 2 \rangle 4$ .  $m \in n$  or  $m = n$

$\langle 2 \rangle 5$ .  $m^+ \in n^+$  or  $m^+ = n^+$

PROOF:  $\langle 2 \rangle 2$

$\langle 2 \rangle 6$ . CASE:  $m^+ \in n^{++}$

$\square$

**Theorem 5.8** (Principle of Mathematical Induction). For any subset  $S$  of  $\omega$ , if  $0 \in S$  and  $\forall n \in S. n^+ \in S$ , then  $S = \omega$ .

PROOF: From the definition of  $\omega$ .  $\square$

**Proposition 5.9.**

$$\forall n \in \omega. \forall x \in n. n \not\subseteq x$$

PROOF:

$\langle 1 \rangle 1$ .  $\forall x \in 0. 0 \not\subseteq x$

PROOF: Vacuous.

$\langle 1 \rangle 2$ . For any natural number  $n$ , if  $\forall x \in n. n \not\subseteq x$  then  $\forall x \in n^+. n^+ \not\subseteq x$ .

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $\forall x \in n. n \not\subseteq x$

$\langle 2 \rangle 3$ . LET:  $x \in n^+$

$\langle 2 \rangle 4$ . ASSUME: for a contradiction  $n^+ \subseteq x$

$\langle 2 \rangle 5$ .  $x \in n$  or  $x = n$

$\langle 2 \rangle 6$ . CASE:  $x \in n$

PROOF: Then we have  $n \subseteq n^+ \subseteq x$  contradicting  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 7$ . CASE:  $x = n$

PROOF: Then we have  $n \in n^+ \subseteq x = n$  and  $n \subseteq n$  contradicting  $\langle 2 \rangle 2$ .

$\square$

**Corollary 5.9.1.** *For any natural number  $n$  we have  $n \notin n$ .*

**Corollary 5.9.2.** *For any natural number  $n$  we have  $n \neq n^+$ .*

**Definition 5.10** (Transitive Set). A set  $E$  is a *transitive set* iff, whenever  $x \in y \in E$ , then  $x \in E$ .

**Proposition 5.11.** *Every natural number is a transitive set.*

PROOF:

$\langle 1 \rangle 1$ . 0 is a transitive set.

PROOF: Vacuously, if  $x \in y \in 0$  then  $x \in 0$ .

$\langle 1 \rangle 2$ . For any natural number  $n$ , if  $n$  is a transitive set, then  $n^+$  is a transitive set.

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $n$  is a transitive set.

$\langle 2 \rangle 3$ . LET:  $x \in y \in n^+$

$\langle 2 \rangle 4$ .  $y \in n$  or  $y = n$

$\langle 2 \rangle 5$ . CASE:  $y \in n$

$\langle 3 \rangle 1$ .  $x \in n$

PROOF:  $\langle 2 \rangle 2$ ,  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 5$ .

$\langle 3 \rangle 2$ .  $x \in n^+$

$\langle 2 \rangle 6$ . CASE:  $y = n$

$\langle 3 \rangle 1$ .  $x \in n$

PROOF:  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 6$

$\langle 3 \rangle 2$ .  $x \in n^+$

□

**Proposition 5.12.** *For any natural numbers  $m$  and  $n$ , if  $m^+ = n^+$  then  $m = n$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $m$  and  $n$  be natural numbers.

$\langle 1 \rangle 2$ . ASSUME:  $m^+ = n^+$

$\langle 1 \rangle 3$ .  $m \in m^+ = n^+$

$\langle 1 \rangle 4$ .  $m \in n$  or  $m = n$

$\langle 1 \rangle 5$ .  $n \in n^+ = m^+$

$\langle 1 \rangle 6$ .  $n \in m$  or  $n = m$

$\langle 1 \rangle 7$ . We cannot have  $m \in n$  and  $n \in m$

$\langle 2 \rangle 1$ . ASSUME: for a contradiction  $m \in n$  and  $n \in m$

$\langle 2 \rangle 2$ .  $m \in m$

PROOF: Since  $m$  is a transitive set (Proposition 5.11).

$\langle 2 \rangle 3$ . Q.E.D.

PROOF: This contradicts Proposition 5.9.

$\langle 1 \rangle 8$ .  $m = n$

□

**Theorem 5.13** (Recursion Theorem). *Let  $X$  be a set. Let  $a \in X$ . Let  $f : X \rightarrow X$ . There exists a function  $u : \omega \rightarrow X$  such that  $u(0) = a$  and, for all  $n \in \omega$ , we have  $u(n^+) = f(u(n))$ .*



PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C} = \{A \in \mathcal{P}(\omega \times X) : (0, a) \in A \wedge \forall n \in \omega. \forall x \in X. (n, x) \in A \Rightarrow (n^+, f(x)) \in A\}$

$\langle 1 \rangle 2$ .  $\mathcal{C} \neq \emptyset$

PROOF:  $\omega \times X \in \mathcal{C}$

$\langle 1 \rangle 3$ . LET:  $u = \bigcap \mathcal{C}$

$\langle 1 \rangle 4$ .  $u \in \mathcal{C}$

$\langle 1 \rangle 5$ .  $u$  is a function.

$\langle 2 \rangle 1$ . LET:  $P(n)$  be the property:  $\forall x, y \in X. (n, x) \in u \wedge (n, y) \in u \Rightarrow x = y$

$\langle 2 \rangle 2$ .  $P(0)$

$\langle 3 \rangle 1$ .  $\forall x \in X. (0, x) \in u \Rightarrow x = a$

PROOF: If  $(0, x) \in u$  and  $x \neq a$  then  $u - \{(0, x)\} \in \mathcal{C}$  and so  $u - \{(0, x)\} \subseteq u$ , which is impossible.

$\langle 2 \rangle 3$ . For every natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .

$\langle 3 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 3 \rangle 2$ . ASSUME:  $P(n)$

$\langle 3 \rangle 3$ . LET:  $x, y \in X$

$\langle 3 \rangle 4$ . ASSUME:  $(n^+, x), (n^+, y) \in u$

$\langle 3 \rangle 5$ . PICK  $x', y' \in X$  such that  $(n, x') \in u$ ,  $(n, y') \in u$  and  $f(x') = x$  and  $f(y') = y$

PROOF: If no such  $x'$  exists then  $u - \{(n^+, x)\} \in \mathcal{C}$  and so  $u - \{(n^+, x)\} \subseteq u$  which is impossible. Similarly for  $y'$ .

$\langle 3 \rangle 6$ .  $x' = y'$

PROOF:  $\langle 3 \rangle 2$

$\langle 3 \rangle 7$ .  $x = y$

□

**Proposition 5.14.** *For any natural number  $n$ , either  $n = 0$  or there exists a natural number  $m$  such that  $n = m^+$ .*

PROOF: Easy induction on  $n$ . □

**Proposition 5.15.**  *$\omega$  is a transitive set.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the property  $\forall x \in n. x \in \omega$

$\langle 1 \rangle 2$ .  $P(0)$

PROOF: Vacuous.

$\langle 1 \rangle 3$ . For any natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$

$\langle 2 \rangle 3$ . LET:  $x \in n^+$

$\langle 2 \rangle 4$ .  $x \in n$  or  $x = n$

$\langle 2 \rangle 5$ . CASE:  $x \in n$

PROOF: Then  $x \in \omega$  by  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 6$ . CASE:  $x = n$

PROOF: Then  $x \in \omega$  by  $\langle 2 \rangle 1$ .

□

**Proposition 5.16.** *For any natural number  $n$  and any nonempty subset  $E \subseteq n$ , there exists  $k \in E$  such that  $\forall m \in E. k = m \vee k \in m$ .*

PROOF:

⟨1⟩1. LET:  $P(n)$  be the property: for any nonempty subset  $E \subseteq n$ , there exists  $k \in E$  such that  $\forall m \in E. k = m \vee k \in m$

⟨1⟩2.  $P(0)$

PROOF: Vacuous as there is no nonempty subset of 0.

⟨1⟩3. For any natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .

⟨2⟩1. LET:  $n$  be a natural number.

⟨2⟩2. ASSUME:  $P(n)$

⟨2⟩3. LET:  $E$  be a nonempty subset of  $n^+$

⟨2⟩4. CASE:  $E - \{n\} = \emptyset$

PROOF: Then  $E = \{n\}$  so take  $k = n$ .

⟨2⟩5. CASE:  $E - \{n\} \neq \emptyset$

⟨3⟩1. PICK  $k \in E - \{n\}$  such that  $\forall m \in E - \{n\}. k = m \vee k \in m$

PROOF: By ⟨2⟩2.

⟨3⟩2.  $\forall m \in E. k = m \vee k \in m$

PROOF: Since  $k \in n$ .

□

## 5.2 Arithmetic

**Definition 5.17** (Addition). Define *addition*  $+$  on  $\omega$  by recursion thus:

$$\begin{aligned} m + 0 &= m \\ m + n^+ &= (m + n)^+ \end{aligned}$$

**Proposition 5.18.** *For all  $m, n, p \in \omega$  we have*

$$m + (n + p) = (m + n) + p .$$

PROOF:

⟨1⟩1. LET:  $P(p)$  be the property  $\forall m, n \in \omega. m + (n + p) = (m + n) + p$

⟨1⟩2.  $P(0)$

PROOF:  $m + (n + 0) = m + n = (m + n) + 0$ .

⟨1⟩3.  $\forall p \in \omega. P(p) \Rightarrow P(p^+)$

⟨2⟩1. LET:  $p \in \omega$

⟨2⟩2. ASSUME:  $P(p)$

⟨2⟩3. LET:  $m, n \in \omega$

⟨2⟩4.  $m + (n + p^+) = (m + n) + p^+$

PROOF:

$$\begin{aligned}
m + (n + p^+) &= m + (n + p)^+ \\
&= (m + (n + p))^+ \\
&= ((m + n) + p)^+ \\
&= (m + n) + p^+
\end{aligned}$$

□

**Proposition 5.19.** *For all  $m, n \in \omega$ , we have*

$$m + n = n + m .$$

PROOF:

⟨1⟩1. LET:  $P(m)$  be the property  $\forall n \in \omega. m + n = n + m$

⟨1⟩2.  $P(0)$

⟨2⟩1. LET:  $Q(n)$  be the property  $0 + n = n + 0$

⟨2⟩2.  $Q(0)$

PROOF: Trivial.

⟨2⟩3.  $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

⟨3⟩1. LET:  $n \in \omega$

⟨3⟩2. ASSUME:  $Q(n)$

⟨3⟩3.  $0 + n^+ = n^+ + 0$

PROOF:

$$\begin{aligned}
0 + n^+ &= (0 + n)^+ \\
&= (n + 0)^+ && (\langle 3 \rangle 2) \\
&= n^+ \\
&= n^+ + 0
\end{aligned}$$

⟨1⟩3.  $\forall m \in \omega. P(m) \Rightarrow P(m^+)$

⟨2⟩1. LET:  $m \in \omega$

⟨2⟩2. ASSUME:  $P(m)$

⟨2⟩3. LET:  $Q(n)$  be the property  $m^+ + n = n + m^+$

⟨2⟩4.  $Q(0)$

PROOF: ⟨1⟩2

⟨2⟩5.  $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

⟨3⟩1. LET:  $n \in \omega$

⟨3⟩2. ASSUME:  $Q(n)$

⟨3⟩3.  $Q(n^+)$

PROOF:

$$\begin{aligned}
m^+ + n^+ &= (m^+ + n)^+ \\
&= (n + m^+)^+ & (\langle 3 \rangle 2) \\
&= (n + m)^{++} \\
&= (m + n)^{++} & (\langle 2 \rangle 2) \\
&= (m + n^+)^+ \\
&= (n^+ + m)^+ & (\langle 2 \rangle 2) \\
&= n^+ + m^+
\end{aligned}$$

□

**Definition 5.20** (Multiplication). Define *multiplication*  $\cdot$  on  $\omega$  by

$$\begin{aligned}
m0 &= 0 \\
mn^+ &= mn + m
\end{aligned}$$

**Proposition 5.21.** For all  $m, n, p \in \omega$ , we have

$$m(n + p) = mn + mp .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(p)$  be the statement  $\forall m, n \in \omega. m(n + p) = mn + mp$

$\langle 1 \rangle 2$ .  $P(0)$

PROOF:

$$\begin{aligned}
m(n + 0) &= mn \\
&= mn + 0 \\
&= mn + m0
\end{aligned}$$

$\langle 1 \rangle 3$ .  $\forall p \in \omega. P(p) \Rightarrow P(p^+)$

$\langle 2 \rangle 1$ . LET:  $p \in \omega$

$\langle 2 \rangle 2$ . ASSUME:  $P(p)$

$\langle 2 \rangle 3$ . LET:  $m, n \in \omega$

$\langle 2 \rangle 4$ .  $m(n + p^+) = mn + mp^+$

PROOF:

$$\begin{aligned}
m(n + p^+) &= m(n + p)^+ \\
&= m(n + p) + m \\
&= (mn + mp) + m & (\langle 2 \rangle 2) \\
&= mn + (mp + m) & (\text{Proposition 5.18}) \\
&= mn + mp^+
\end{aligned}$$

□

**Proposition 5.22.** For all  $m, n, p \in \omega$  we have

$$m(np) = (mn)p .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(p)$  be the statement  $\forall m, n \in \omega. m(np) = (mn)p$

$\langle 1 \rangle 2$ .  $P(0)$

PROOF:

$$\begin{aligned} m(n0) &= m0 \\ &= 0 \\ &= (mn)0 \end{aligned}$$

$\langle 1 \rangle 3$ .  $\forall p \in \omega. P(p) \Rightarrow P(p^+)$

$\langle 2 \rangle 1$ . LET:  $p \in \omega$

$\langle 2 \rangle 2$ . ASSUME:  $P(p)$

$\langle 2 \rangle 3$ . LET:  $m, n \in \omega$

$\langle 2 \rangle 4$ .  $m(np^+) = (mn)p^+$

PROOF:

$$\begin{aligned} m(np^+) &= m(np + n) \\ &= m(np) + mn && \text{(Proposition 5.21)} \\ &= (mn)p + mn && (\langle 2 \rangle 2) \\ &= (mn)p^+ \end{aligned}$$

□

**Proposition 5.23.** *For all  $m, n \in \omega$ , we have*

$$mn = nm \ .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(m)$  be the statement  $\forall n \in \omega. mn = nm$

$\langle 1 \rangle 2$ .  $P(0)$

$\langle 2 \rangle 1$ . LET:  $Q(n)$  be the statement  $0n = n0$

$\langle 2 \rangle 2$ .  $Q(0)$

PROOF: Trivial.

$\langle 2 \rangle 3$ .  $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$ . LET:  $n \in \omega$

$\langle 3 \rangle 2$ . ASSUME:  $Q(n)$

$\langle 3 \rangle 3$ .  $Q(n^+)$

PROOF:

$$\begin{aligned} 0n^+ &= 0n + 0 \\ &= 0n \\ &= n0 && (\langle 3 \rangle 2) \\ &= 0 \\ &= n^+0 \end{aligned}$$

$\langle 1 \rangle 3$ .  $\forall m \in \omega. P(m) \Rightarrow P(m^+)$

$\langle 2 \rangle 1$ . LET:  $m \in \omega$

$\langle 2 \rangle 2$ . ASSUME:  $P(m)$

$\langle 2 \rangle 3$ . LET:  $Q(n)$  be the statement  $m^+n = nm^+$

$\langle 2 \rangle 4$ .  $Q(0)$

PROOF:  $\langle 1 \rangle 2$

$\langle 2 \rangle 5. \forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1. \text{ LET: } n \in \omega$

$\langle 3 \rangle 2. \text{ ASSUME: } Q(n)$

$\langle 3 \rangle 3. Q(n^+)$

PROOF:

$$\begin{aligned}
m^+n^+ &= m^+n + m^+ \\
&= (m^+n + m)^+ \\
&= (nm^+ + m)^+ & (\langle 3 \rangle 2) \\
&= (nm + n + m)^+ \\
&= (mn + m + n)^+ & (\langle 2 \rangle 2, \text{ Proposition 5.18, Proposition 5.19}) \\
&= (mn^+ + n)^+ \\
&= (n^+m + n)^+ & (\langle 2 \rangle 2) \\
&= n^+m + n^+ \\
&= n^+m^+
\end{aligned}$$

□

**Definition 5.24** (Exponentiation). Define *exponentiation* on  $\omega$  by recursion:

$$\begin{aligned}
m^0 &= 1 \\
m^{n^+} &= m^n m
\end{aligned}$$

**Proposition 5.25.** For all  $m, n, p \in \omega$  we have

$$m^{n+p} = m^n m^p .$$

PROOF:

$\langle 1 \rangle 1. m^{n+0} = m^n m^0$

PROOF:

$$\begin{aligned}
m^{n+0} &= m^n \\
&= m^n 1 \\
&= m^n m^0
\end{aligned}$$

$\langle 1 \rangle 2. \text{ If } m^{n+p} = m^n m^p \text{ then } m^{n+p^+} = m^n m^{p^+}$

PROOF:

$$\begin{aligned}
m^{n+p^+} &= m^{n+p} m \\
&= m^n m^p m \\
&= m^n m^{p^+}
\end{aligned}$$

□

**Proposition 5.26.** For all  $m, n, p \in \omega$  we have

$$(m^n)^p = m^{np} .$$

PROOF:

⟨1⟩1.  $(m^n)^0 = m^{n0}$

PROOF: Both are equal to 1.

⟨1⟩2. If  $(m^n)^p = m^{np}$  then  $(m^n)^{p^+} = m^{np^+}$

PROOF:

$$\begin{aligned} (m^n)^{p^+} &= (m^n)^p m^n \\ &= m^{np} m^n \\ &= m^{np+n} && \text{(Proposition 5.25)} \\ &= m^{np^+} \end{aligned}$$

□

**Proposition 5.27.** *For any natural numbers  $m$  and  $n$ , either  $m \in n$  or  $m = n$  or  $n \in m$ .*

PROOF:

⟨1⟩1. LET:  $P(n)$  be the property: for all  $m \in \omega$ , either  $m \in n$  or  $m = n$  or  $n \in m$

⟨1⟩2.  $P(0)$

⟨2⟩1. LET:  $Q(m)$  be the property: either  $m = 0$  or  $0 \in m$

⟨2⟩2.  $Q(0)$

PROOF: Since  $0 = 0$ .

⟨2⟩3. For all  $m \in \omega$ , if  $Q(m)$  then  $Q(m^+)$

PROOF: If  $m = 0$  or  $0 \in m$  then  $0 \in m^+$ .

⟨1⟩3. For any natural number  $n$ , if  $P(n)$  then  $P(n^+)$

⟨2⟩1. LET:  $n$  be a natural number.

⟨2⟩2. ASSUME:  $P(n)$

⟨2⟩3. LET:  $m \in \omega$

⟨2⟩4.  $m \in n$  or  $m = n$  or  $n \in m$

PROOF: ⟨2⟩2

⟨2⟩5. CASE:  $m \in n$  or  $m = n$

PROOF: Then  $m \in n^+$ .

⟨2⟩6. CASE:  $n \in m$

⟨3⟩1. PICK  $p$  such that  $m = p^+$

⟨3⟩2.  $n \in p$  or  $n = p$

⟨3⟩3. CASE:  $n \in p$

PROOF: Then  $n^+ \in p^+ = m$  by Proposition 5.7.

⟨3⟩4. CASE:  $n = p$

PROOF: Then  $m = n^+$ .

□

**Corollary 5.27.1** (Trichotomy). *For any natural numbers  $m$  and  $n$ , exactly one of  $m \in n$ ,  $m = n$ ,  $n \in m$  holds.*

PROOF:

⟨1⟩1. We never have  $m \in n$  and  $m = n$ .

PROOF: By Corollary 5.9.1.

⟨1⟩2. We never have  $m \in n$  and  $n \in m$ .

PROOF: Since  $m$  is a transitive set this would imply  $m \in m$  contradicting Corollary 5.9.1.

$\langle 1 \rangle 3$ . We never have  $m = n$  and  $n \in m$ .

PROOF: By Corollary 5.9.1.

□

**Proposition 5.28.** *For any natural numbers  $m$  and  $n$ , we have  $m \in n$  if and only if  $m \subsetneq n$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $m$  and  $n$  be natural numbers.

$\langle 1 \rangle 2$ . If  $m \in n$  then  $m \subsetneq n$ .

PROOF: Since  $n$  is a transitive set, and  $m \neq n$  by Corollary 5.9.1.

$\langle 1 \rangle 3$ . If  $m \subsetneq n$  then  $m \in n$ .

$\langle 2 \rangle 1$ . ASSUME:  $m \subsetneq n$

$\langle 2 \rangle 2$ .  $n \notin m$

PROOF: Proposition 5.9.

$\langle 2 \rangle 3$ .  $m \neq n$

$\langle 2 \rangle 4$ .  $m \in n$

PROOF: Trichotomy.

□

**Definition 5.29.** Given natural numbers  $m$  and  $n$ , we write  $m < n$  iff  $m \in n$ .

We write  $m \leq n$  iff  $m < n \vee m = n$ .

**Proposition 5.30.** *For natural numbers  $m$  and  $n$ , if  $m \leq n$  and  $n \leq m$  then  $m = n$ .*

PROOF: We cannot have  $m < n$  and  $n < m$  by trichotomy. □

**Proposition 5.31.** *For natural numbers  $m$ ,  $n$  and  $k$ , if  $m < n$  then  $m + k < n + k$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $m, n \in \omega$

$\langle 1 \rangle 2$ . ASSUME:  $m < n$

$\langle 1 \rangle 3$ .  $m + 0 < n + 0$

$\langle 1 \rangle 4$ .  $\forall k \in \omega. m + k < n + k \Rightarrow m + k^+ < n + k^+$

PROOF: By Proposition 5.7.

□

**Proposition 5.32.** *For natural numbers  $m$ ,  $n$  and  $k$ , if  $m < n$  and  $k \neq 0$  then  $mk < nk$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $m, n \in \omega$

$\langle 1 \rangle 2$ . ASSUME:  $m < n$

$\langle 1 \rangle 3$ .  $m1 < n1$

$\langle 1 \rangle 4$ . For all  $k \in \omega$ , if  $k \neq 0$  and  $mk < nk$  then  $m(k + 1) < n(k + 1)$



PROOF:

$$\begin{aligned}
m(k+1) &= mk + m \\
&< mk + n && \text{(Proposition 5.31)} \\
&< nk + n && \text{(Proposition 5.31)} \\
&= n(k+1)
\end{aligned}$$

□

**Proposition 5.33.** *For any nonempty set of natural numbers  $E$ , there exists  $k \in E$  such that  $\forall m \in E. k \leq m$ .*

PROOF:

- ⟨1⟩1. LET:  $E \subseteq \omega$
- ⟨1⟩2. ASSUME: there is no  $k \in E$  such that  $\forall m \in E. k \leq m$ .  
PROVE:  $E = \emptyset$
- ⟨1⟩3.  $\forall n \in \omega. n \notin E$ 
  - ⟨2⟩1. LET:  $P(n)$  be the property:  $\forall m < n. m \notin E$
  - ⟨2⟩2.  $P(0)$   
PROOF: Vacuous.
  - ⟨2⟩3.  $\forall n \in \omega. P(n) \Rightarrow P(n+1)$ 
    - ⟨3⟩1. LET:  $n \in \omega$
    - ⟨3⟩2. ASSUME:  $\forall m < n. m \notin E$
    - ⟨3⟩3.  $n \notin E$   
PROOF: From ⟨1⟩2.
    - ⟨3⟩4.  $\forall m < n+1. m \notin E$

□

**Definition 5.34** (Equivalent). Sets  $E$  and  $F$  are *equivalent*,  $E \sim F$ , iff there exists a one-to-one correspondence between them.

**Proposition 5.35.** *For any set  $X$ , equivalence is an equivalence relation on  $\mathcal{P}X$ .*

PROOF: Easy.

**Proposition 5.36.** *Let  $n$  be a natural number. Let  $X$  be a proper subset of  $n$ . Then there exists  $m < n$  such that  $X \sim m$ .*

PROOF:

- ⟨1⟩1. LET:  $P(n)$  be the property: for every proper subset  $X \subsetneq n$ , there exists  $m < n$  such that  $X \sim m$ .
- ⟨1⟩2.  $P(0)$   
PROOF: Vacuous.
- ⟨1⟩3.  $\forall n \in \omega. P(n) \Rightarrow P(n+1)$ 
  - ⟨2⟩1. LET:  $n \in \omega$
  - ⟨2⟩2. ASSUME:  $P(n)$
  - ⟨2⟩3. LET:  $X$  be a proper subset of  $n+1$
  - ⟨2⟩4. CASE:  $X - \{n\} = n$

PROOF: Then  $X = n$  so  $X \sim n < n + 1$ .

$\langle 2 \rangle 5$ . CASE:  $X - \{n\} \subsetneq n$

$\langle 3 \rangle 1$ . PICK  $m < n$  such that  $X - \{n\} \sim m$

$\langle 3 \rangle 2$ .  $X \sim m$  or  $X \sim m + 1$

PROOF: If  $n \in X$  then  $X \sim m + 1$ . If  $n \notin X$  then  $X \sim m$ .

□

**Proposition 5.37.** *For every natural number  $n$ , we have  $n$  is not equivalent to a proper subset of  $n$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the property: every one-to-one function  $n \rightarrow n$  is onto.

$\langle 1 \rangle 2$ .  $P(0)$

PROOF: The only function  $0 \rightarrow 0$  is  $\emptyset$ .

$\langle 1 \rangle 3$ .  $\forall n \in \omega. P(n) \Rightarrow P(n + 1)$

$\langle 2 \rangle 1$ . LET:  $n \in \omega$

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$

$\langle 2 \rangle 3$ . ASSUME:  $f : n + 1 \rightarrow n + 1$  is one-to-one.

$\langle 2 \rangle 4$ . LET:  $g : n \rightarrow n$  be the function

$$g(k) = \begin{cases} f(k) & \text{if } f(k) < n \\ f(n) & \text{if } f(k) = n \end{cases}$$

PROOF: If  $k < n$  and  $f(k) = n$  then  $f(n) < n$  since  $f$  is one-to-one.

$\langle 2 \rangle 5$ .  $g$  is one-to-one.

$\langle 3 \rangle 1$ . LET:  $k, l < n$

$\langle 3 \rangle 2$ . ASSUME:  $g(k) = g(l)$

$\langle 3 \rangle 3$ . CASE:  $f(k) < n$  and  $f(l) < n$

PROOF: Then  $f(k) = g(k) = g(l) = f(l)$  so  $k = l$  since  $f$  is one-to-one.

$\langle 3 \rangle 4$ . CASE:  $f(k) < n$  and  $f(l) = n$

PROOF: Then  $f(k) = g(k) = g(l) = f(n)$  contradicting the fact that  $f$  is one-to-one.

$\langle 3 \rangle 5$ . CASE:  $f(k) = n$  and  $f(l) < n$

PROOF: Similar.

$\langle 3 \rangle 6$ . CASE:  $f(k) = n$  and  $f(l) = n$

PROOF: Then  $k = l$  since  $f$  is one-to-one.

$\langle 2 \rangle 6$ .  $g$  maps  $n$  onto  $n$ .

PROOF:  $\langle 2 \rangle 2$

$\langle 2 \rangle 7$ .  $f$  maps  $n + 1$  onto  $n + 1$ .

$\langle 3 \rangle 1$ . LET:  $l < n + 1$

$\langle 3 \rangle 2$ . CASE:  $l < n$

$\langle 4 \rangle 1$ . PICK  $k < n$  such that  $g(k) = l$

$\langle 4 \rangle 2$ .  $f(k) = l$  or  $f(n) = l$

$\langle 3 \rangle 3$ . CASE:  $l = n$

$\langle 4 \rangle 1$ . CASE:  $f(n) = n$

PROOF: Then  $l \in \text{ran } f$  as required.

$\langle 4 \rangle 2$ . CASE:  $f(n) < n$

$\langle 5 \rangle 1$ . PICK  $k < n$  such that  $g(k) = f(n)$

□  $\langle 5 \rangle 2. f(k) = n$

**Corollary 5.37.1.** *Equivalent natural numbers are equal.*

## 5.3 Finite Sets

**Definition 5.38** (Finite). A set is *finite* iff it is equivalent to a natural number; otherwise, it is *infinite*.

**Proposition 5.39.** *No finite set is equivalent to one of its proper subsets.*

PROOF: From Proposition 5.37. □

**Proposition 5.40.**  $\omega$  is infinite.

PROOF: Since the function that maps  $n$  to  $n + 1$  is a one-to-one correspondence between  $\omega$  and  $\omega - \{0\}$ . □

**Proposition 5.41.** *Every subset of a finite set is finite.*

PROOF: Proposition 5.36. □

**Definition 5.42** (Number of Elements). For any finite set  $E$ , the *number of elements* in  $E$ ,  $\sharp(E)$ , is the unique natural number such that  $E \sim \sharp(E)$ .

**Proposition 5.43.** *Let  $E$  and  $F$  be finite sets. If  $E \subseteq F$  then  $\sharp(E) \leq \sharp(F)$ .*

PROOF: Proposition 5.36. □

**Proposition 5.44.** *Let  $E$  and  $F$  be disjoint finite sets. Then  $E \cup F$  is finite and  $\sharp(E \cup F) = \sharp(E) + \sharp(F)$ .*

PROOF:

$\langle 1 \rangle 1.$  LET:  $P(n)$  be the statement:  $n \in \omega$  and for any  $m \in \omega$ , if  $E \sim m$ ,  $F \sim n$  and  $E \cap F = \emptyset$ , then  $E \cup F \sim m + n$

$\langle 1 \rangle 2. P(0)$

$\langle 2 \rangle 1.$  LET:  $m \in \omega$

$\langle 2 \rangle 2.$  LET:  $E \sim m$  and  $F \sim 0$

$\langle 2 \rangle 3.$   $F = \emptyset$

$\langle 2 \rangle 4.$   $E \cup F = E \sim m = m + 0$

$\langle 1 \rangle 3. \forall n \in \omega. P(n) \Rightarrow P(n + 1)$

$\langle 2 \rangle 1.$  LET:  $n \in \omega$

$\langle 2 \rangle 2.$  ASSUME:  $P(n)$

$\langle 2 \rangle 3.$  LET:  $m \in \omega$

$\langle 2 \rangle 4.$  LET:  $E \sim m$  and  $F \sim n + 1$

$\langle 2 \rangle 5.$  ASSUME:  $E \cap F = \emptyset$

$\langle 2 \rangle 6.$  PICK  $f \in F$

$\langle 2 \rangle 7.$   $F - \{f\} \sim n$

$\langle 2 \rangle 8.$   $E \cap (F - \{f\}) = \emptyset$

$\langle 2 \rangle 9. E \cup (F - \{f\}) \sim m + n$

PROOF:  $\langle 2 \rangle 2$

$\langle 2 \rangle 10. E \cup F \sim m + n + 1$

□

**Corollary 5.44.1.** *The union of two finite sets is finite.*

PROOF: Since, if  $E$  and  $F$  are finite, then  $E \cup F = (E - F) \cup (E \cap F) \cup (F - E)$  and these are finite and disjoint. □

**Proposition 5.45.** *If  $E$  and  $F$  are finite sets then  $E \times F$  is finite and  $\sharp(E \times F) = \sharp(E)\sharp(F)$ .*

PROOF:

$\langle 1 \rangle 1.$  LET:  $P(n)$  be the statement:  $n \in \omega$  and for all  $m \in \omega$ , if  $E \sim m$  and  $F \sim n$  then  $E \times F \sim mn$

$\langle 1 \rangle 2. P(0)$

PROOF: If  $F \sim 0$  then  $F = \emptyset$  so  $E \times F = \emptyset \sim 0$ .

$\langle 1 \rangle 3. \forall n \in \omega. P(n) \Rightarrow P(n + 1)$

$\langle 2 \rangle 1.$  LET:  $n \in \omega$

$\langle 2 \rangle 2.$  ASSUME:  $P(n)$

$\langle 2 \rangle 3.$  LET:  $m \in \omega$

$\langle 2 \rangle 4.$  ASSUME:  $E \sim m$  and  $F \sim n + 1$

$\langle 2 \rangle 5.$  PICK  $f \in F$

$\langle 2 \rangle 6. F - \{f\} \sim n$

$\langle 2 \rangle 7. E \times (F - \{f\}) \sim mn$

$\langle 2 \rangle 8. E \times F = (E \times (F - \{f\})) \cup (E \times \{f\})$

$\langle 2 \rangle 9. E \times \{f\} \sim m$

$\langle 2 \rangle 10. E \times F \sim mn + m$

PROOF: Proposition 5.44.

□

**Proposition 5.46.** *For any finite sets  $E$  and  $F$ , we have  $E^F$  is finite and  $\sharp(E^F) = \sharp(E)^{\sharp(F)}$ .*

PROOF:

$\langle 1 \rangle 1.$  LET:  $P(n)$  be the property:  $n \in \omega$  and for all  $m \in \omega$ , if  $E \sim m$  and  $F \sim n$  then  $E^F \sim m^n$

$\langle 1 \rangle 2. P(0)$

PROOF: Since  $E^\emptyset = \{\emptyset\} \sim 1$

$\langle 1 \rangle 3. \forall n \in \omega. P(n) \Rightarrow P(n + 1)$

$\langle 2 \rangle 1.$  LET:  $n \in \omega$

$\langle 2 \rangle 2.$  ASSUME:  $P(n)$

$\langle 2 \rangle 3.$  LET:  $m \in \omega$

$\langle 2 \rangle 4.$  LET:  $E \sim m$  and  $F \sim n + 1$

$\langle 2 \rangle 5.$  PICK  $f \in F$

$\langle 2 \rangle 6. F - \{f\} \sim n$

$\langle 2 \rangle 7.$  LET:  $\phi : E^F \rightarrow E^{F - \{f\}} \times E$  be the function  $\phi(g) = (g \upharpoonright (F - \{f\}), g(f))$

- $\langle 2 \rangle 8.$   $\phi$  is a one-to-one correspondence  
 $\langle 2 \rangle 9.$   $\sharp(E^F) = m^{n+1}$

PROOF:

$$\begin{aligned}
 \sharp(E^F) &= \sharp(E^{F-\{f\}} \times E) \\
 &= \sharp(E^{F-\{f\}}) \sharp(E) && \text{(Proposition 5.45)} \\
 &= m^n m && (\langle 2 \rangle 2, \langle 2 \rangle 4) \\
 &= m^{n+1}
 \end{aligned}$$

□

**Corollary 5.46.1.** *If  $E$  is finite then  $\mathcal{P}E$  is finite and  $\sharp(\mathcal{P}E) = 2^{\sharp(E)}$ .*

**Proposition 5.47.** *The union of a finite set of finite sets is finite.*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $P(n)$  be the property: for any set  $E$ , if  $E \sim n$  and every element of  $E$  is finite, then  $\bigcup E$  is finite.

- $\langle 1 \rangle 2.$   $P(0)$

PROOF: Since  $\bigcup \emptyset = \emptyset$  is finite.

- $\langle 1 \rangle 3.$   $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

- $\langle 2 \rangle 1.$  LET:  $n$  be a natural number.

- $\langle 2 \rangle 2.$  ASSUME:  $P(n)$

- $\langle 2 \rangle 3.$  LET:  $E \sim n+1$

- $\langle 2 \rangle 4.$  PICK  $X \in E$

- $\langle 2 \rangle 5.$   $E - \{X\} \sim n$

- $\langle 2 \rangle 6.$   $\bigcup(E - \{X\})$  is finite.

PROOF:  $\langle 2 \rangle 2$

- $\langle 2 \rangle 7.$   $\bigcup E = \bigcup(E - \{X\}) \cup X$

- $\langle 2 \rangle 8.$   $\bigcup E$  is finite.

PROOF: Corollary 5.44.1.

□

**Proposition 5.48.** *Every nonempty finite set of natural numbers has a greatest element.*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $P(n)$  be the property: for every  $E \subseteq \mathbb{N}$ , if  $E \sim n$  then  $E$  has a greatest element.

- $\langle 1 \rangle 2.$   $P(1)$

PROOF: Since  $k$  is the greatest element of  $\{k\}$ .

- $\langle 1 \rangle 3.$   $\forall n \geq 1. P(n) \Rightarrow P(n+1)$

- $\langle 2 \rangle 1.$  LET:  $n \geq 1$

- $\langle 2 \rangle 2.$  ASSUME:  $P(n)$

- $\langle 2 \rangle 3.$  ASSUME:  $E \subseteq \omega$  and  $E \sim n+1$

- $\langle 2 \rangle 4.$  PICK  $k \in E$

- $\langle 2 \rangle 5.$  LET:  $l$  be the greatest element of  $E - \{k\}$

- $\langle 2 \rangle 6.$  Either  $k$  or  $l$  is greatest in  $E$ .

□