Mathematics

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Contents

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*. We write $A : \mathrm{Set}$ for: A is a set.

For any set A, let there be *elements* of A. We write $a : \mathrm{El}\,(A)$ for: a is an element of A.

For any sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B.

For any function $f : A \to B$ and element $a : \mathrm{El}\,(A)$, let there be an element $f(a) : \mathrm{El}\,(B)$, the *value* of the function f at the *argument* a.

1.2 Axioms

Axiom Schema 1.1 (Choice)**.** *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x : \mathrm{El}\,(X)$ and $y : \mathrm{El}\,(Y)$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a : \mathrm{El}\,(A)$, there exists $b : \mathrm{El}\,(B)$ such that $P[A, B, a, b]$. Then there exists a function $f : A \to B$ such that $\forall a : \mathrm{El}\,(A)\,.P[A, B, a, f(a)]$.

Axiom 1.2 (Pairing)**.** *For any sets A and B, there exists a set $A \times B$, the* Cartesian product *of A and B, and functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that, for all $a : \mathrm{El}\,(A)$ and $b : \mathrm{El}\,(B)$, there exists a unique $(a, b) : \mathrm{El}\,(A \times B)$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Definition 1.3 (Injective)**.** A function $f : A \to B$ is *injective* or an *injection* iff, for all $x, y : \mathrm{El}\,(A)$, if $f(x) = f(y)$ then $x = y$.

Axiom Schema 1.4 (Separation)**.** *For every property $P[X, x]$ where X is a set variable and $x : \mathrm{El}\,(X)$, the following is an axiom:*

For every set A, there exists a set $S = \{x : \mathrm{El}(A) \mid P[A, x]\}$ and an injection $i : S \to A$ such that, for all $x : \mathrm{El}(A)$, we have

$$(\exists y : S. i(y) = x) \Leftrightarrow P[A, x] \ .$$

Axiom 1.5 (Infinity)**.** *There exists a set $\mathbb{N}$, an element $0 : \mathrm{El}(\mathbb{N})$, and a function $s : \mathbb{N} \to \mathbb{N}$ such that:*

- $\forall n : \mathrm{El}(\mathbb{N}) . s(n) \neq 0$
- $\forall m, n : \mathrm{El}(\mathbb{N}) . s(m) = s(n) \Rightarrow m = n.$

1.3 Consequences of the Axioms

1.3.1 Definitions

Definition 1.6. Let $f, g : A \to B$. We say f and g are *equal*, $f = g$, iff $\forall x : \mathrm{El}(A) . f(x) = g(x)$.

Definition 1.7 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for all $y : \mathrm{El}(B)$, there exists $x : \mathrm{El}(A)$ such that $f(x) = y$.

Definition 1.8 (Bijective)**.** A function $f : A \to B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$".

1.3.2 The Empty Set

Theorem 1.9. *There exists a set which has no elements.*

Proof:
⟨1⟩1. Pick a set A
Proof: By the Axiom of Infinity, a set exists.
⟨1⟩2. Let: $S = \{x : \mathrm{El}(A) \mid \bot\}$ with injection $i : S \to A$
Proof: Axiom of Separation.
⟨1⟩3. S has no elements.
□

Theorem 1.10. *If E and E' have no elements then $E \approx E'$.*

Proof:
⟨1⟩1. Let: E and E' have no elements.
⟨1⟩2. Pick a function $F : E \to E'$.
Proof: Axiom of Choice since vacuously $\forall x : \mathrm{El}(E) . \exists y : \mathrm{El}(E') . \top$.

$\langle 1 \rangle 3$. F is injective.
PROOF: Vacuously, for all $x, y : \mathrm{El}(E)$, if $F(x) = F(y)$ then $x = y$.
$\langle 1 \rangle 4$. F is surjective.
PROOF: Vacuously, for all $y : \mathrm{El}(E)$, there exists $x : \mathrm{El}(E)$ such that $F(x) = y$.
□

Definition 1.11 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

1.3.3 The Singleton

Theorem 1.12. *There exists a set that has exactly one element.*

PROOF:
$\langle 1 \rangle 1$. PICK a set A that has an element.
PROOF: By the Axiom of Infinity, there exists a set that has an element.
$\langle 1 \rangle 2$. PICK $a : \mathrm{El}(A)$
$\langle 1 \rangle 3$. LET: $R : A \looparrowright A$ be the relation such that, for all $x, y : \mathrm{El}(A)$, we have xRy if and only if $x = y = a$.
PROOF: By the Axiom of Comprehension.
$\langle 1 \rangle 4$. LET: $|R|$ be the tabulation of R with projections $p, q : |R| \to A$.
PROVE: $|R|$ has exactly one element.
PROOF: By the Axiom of Tabulations.
$\langle 1 \rangle 5$. LET: $r : \mathrm{El}(|R|)$ be the element such that $p(r) = q(r) = a$
PROOF: Since aRa by $\langle 1 \rangle 3$.
$\langle 1 \rangle 6$. LET: $s : \mathrm{El}(|R|)$
PROVE: $s = r$
$\langle 1 \rangle 7$. $p(s)Rq(s)$
PROOF: By the Axiom of Tabulations.
$\langle 1 \rangle 8$. $p(s) = q(s) = a$
PROOF: By $\langle 1 \rangle 3$.
$\langle 1 \rangle 9$. $p(s) = p(r)$ and $q(s) = q(r)$
PROOF: By $\langle 1 \rangle 5$.
$\langle 1 \rangle 10$. $s = r$
PROOF: By the Axiom of Tabulations.
□

Theorem 1.13. *If A and B both have exactly one element then $A \approx B$.*

PROOF:
$\langle 1 \rangle 1$. LET: A and B both have exactly one element.
$\langle 1 \rangle 2$. LET: $F : A \looparrowright B$ be the relation such that, for all $x : \mathrm{El}(A)$ and $y : \mathrm{El}(B)$, we have xFy.
$\langle 1 \rangle 3$. F is a function.
PROOF: If xFy and xFy' then $y = y'$ because B has only one element.
$\langle 1 \rangle 4$. F is injective.
PROOF: If $F(x) = F(x')$ then $x = x'$ because A has only one element.

⟨1⟩5. F is surjective.
⟨2⟩1. LET: $y : \mathrm{El}(B)$
⟨2⟩2. LET: x be the element of A.
⟨2⟩3. $F(x) = y$
□

Definition 1.14 (Singleton)**.** Let 1 be the set that has exactly one element. Let $*$ be its element.

1.3.4 Subsets

Definition 1.15 (Subset)**.** A *subset* of a set A is a relation $1 \nrightarrow S$.

Given $S : 1 \nrightarrow S$ and $a : \mathrm{El}(A)$, we write $a \in S$ for $*Sa$.

Theorem Schema 1.16. *For any property $P[X, x]$ where X is a set variable and $x : \mathrm{El}(X)$, the following is a theorem:*

For any set A, there exists a set B and injection $i : B \to A$ such that, for all $x : \mathrm{El}(A)$, we have $P[A, x]$ if and only if there exists $b : \mathrm{El}(B)$ such that $i(b) = x$.

PROOF:
⟨1⟩1. LET: $S : 1 \nrightarrow A$ be the relation such that, for all $e : \mathrm{El}(1)$ and $a : \mathrm{El}(A)$, we have eSa if and only if $P[A, a]$.
PROOF: Axiom of Comprehension.
⟨1⟩2. LET: B be the tabulation of S with projections $p : B \to 1$ and $i : B \to A$.
PROOF: Axiom of Tabulations.
⟨1⟩3. i is injective.
⟨2⟩1. LET: $r, s : \mathrm{El}(B)$
⟨2⟩2. ASSUME: $i(r) = i(s)$
⟨2⟩3. $p(r) = p(s)$
PROOF: Since 1 has only one element.
⟨2⟩4. $r = s$
PROOF: Axiom of Tabulations.
⟨1⟩4. For all $x : \mathrm{El}(A)$, we have $P[A, x]$ if and only if there exists $b : \mathrm{El}(B)$ such that $i(b) = x$.
⟨2⟩1. LET: $x : \mathrm{El}(A)$
⟨2⟩2. If $P[A, x]$ then there exists $b : \mathrm{El}(B)$ such that $i(b) = x$
⟨3⟩1. ASSUME: $P[A, x]$
⟨3⟩2. $*Sx$
PROOF: ⟨1⟩1
⟨3⟩3. There exists $b : \mathrm{El}(B)$ such that $p(b) = *$ and $i(b) = x$
PROOF: Axiom of Tabulations.
⟨2⟩3. For all $b : \mathrm{El}(B)$ we have $P[A, i(b)]$
⟨3⟩1. LET: $b : \mathrm{El}(B)$
⟨3⟩2. $p(b)Si(b)$
PROOF: Axiom of Tabulations.
⟨3⟩3. $P[A, i(b)]$

PROOF: $\langle 1 \rangle 1$

$\square$

1.4 Composition

Definition 1.17 (Composite)**.** Let $\phi : A \looparrowright B$ and $\psi : B \looparrowright C$. The *composite* $\psi \circ \phi : A \looparrowright C$ is the relation such that $a(\psi \circ \phi)c$ iff there exists b such that $a\phi b$ and $b\psi c$.

Definition 1.18 (Identity)**.** For any set A, the *identity* function $\mathrm{id}_A : A \to A$ is the function defined by $\mathrm{id}_A(a) = a$.

Theorem 1.19. *Composition of relations is associative, and the identity function is an identity for composition. The composite of functions is a function. The composite of injective functions is injective. The composite of surjective functions is surjective. The composite of bijections is a bijection. A function $f : A \to B$ is a bijection iff there exists a function $f^{-1} : B \to A$ such that $f^{-1}f = \mathrm{id}_A$ and $ff^{-1} = \mathrm{id}_B$, in which case f^{-1} is unique.*

1.5 Axioms Part Two

Axiom 1.20 (Power Set)**.** *For any set A, there exists a set $\mathcal{P}A$, the* power set *of A, and a relation $\in : A \looparrowright \mathcal{P}A$, called* membership*, such that, for any subset S of A, there exists a unique $\overline{S} \in \mathcal{P}A$ such that, for all $x \in A$, we have $x \in \overline{S}$ if and only if $x \in S$.*

We usually write just S for $\overline{S}$.

Axiom Schema 1.21 (Collection)**.** *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A, there exists a set B, a function $p : B \to A$, a set Y and a relation $M : B \looparrowright Y$ such that:

- $\forall b \in B. P[A, \{y \in Y : bMy\}, p(b)]$
- *For all $a \in A$, if $\exists Y. P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 1.22 (Universe)**.** Let $E : U \looparrowright X$ be a relation. Let us say that a set A is *small* iff there exists $u \in U$ such that $A \approx \{x \in X : uEx\}$.

Then (U, X, E) form a *universe* if and only if:

- $\mathbb{N}$ is U-small.
- For any U-small sets A and B and relation $R : A \looparrowright B$, the tabulation of R is U-small.
- If A is U-small then so is $\mathcal{P}A$
- Let $f : A \to B$ be a function. If B is U-small and $f^{-1}(b)$ is U-small for all $b \in B$, then A is U-small.

- If $p : B \twoheadrightarrow A$ is a surjective function such that A is U-small, then there exists a U-small set C, a surjection $q : C \twoheadrightarrow A$, and a function $f : C \to B$ such that $q = pf$.

Axiom 1.23 (Universe)**.** *There exists a universe.*

Let $E : U \looparrowright X$ be a universe. We shall say a set is *small* iff it is U-small, and *large* otherwise.

1.6 Cartesian Product

Definition 1.24 (Cartesian Product)**.** Let A and B be sets. The *Cartesian product* of A and B, $A \times B$, is the tabulation of the relation $A \looparrowright B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

1.7 Quotient Sets

Proposition 1.25. *Let $\sim$ be an equivalence relation on X. Then there exists a set $X/\sim$, the* quotient set *of X with respect to $\sim$, and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the* canonical projection*, such that, for all $x, y : \mathrm{El}(X)$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to $\sim$, then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

Chapter 2

Category Theory

2.1 Categories

Definition 2.1. A *category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}(\mathcal{C})$ of *objects*
- for any objects X and Y, a set $\mathrm{Mor}(X, Y)$ of *morphisms* from X to Y. We write $f : X \to Y$ for $f : \mathrm{El}\,(\mathrm{Mor}(X, Y))$.
- for any objects X, Y and Z, a function $\circ : \mathrm{Mor}(Y, Z) \times \mathrm{Mor}(X, Y) \to \mathrm{Mor}(X, Z)$, called *composition*.

such that:

- Given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X, there exists a morphism $\mathrm{id}_X : X \to X$, the *identity morphism* on X, such that:
  - for any object Y and morphism $f : Y \to X$ we have $\mathrm{id}_X \circ f = f$
  - for any object Y and morphism $f : X \to Y$ we have $f \circ \mathrm{id}_X = f$

Chapter 3

Topology

3.1 Topological Spaces

Definition 3.1 (Topological Space)**.** Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say $(X, \mathcal{O})$ is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call $\mathcal{O}$ the *topology* of the toplogical space, and call its elements *open* sets. We shall often write X for the topological space $(X, \mathcal{O})$.

Definition 3.2 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 3.3. *A set B is open if and only if $X - B$ is closed.*

Proposition 3.4. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that $\mathcal{C}$ is the set of closed sets if and only if:*

- *For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$*
- *For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*
- $\emptyset \in \mathcal{C}$

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Definition 3.5 (Neighbourhood)**.** Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x, and x is an *interior* point of U, iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 3.6. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 3.7. *Let X be a set and $\mathcal{N} : X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that, for all $x \in X$, we have $\mathcal{N}_x$ is the set of neighbourhoods of x, if and only if:*

- *For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$*
- *For all $x \in X$ we have $X \in \mathcal{N}_x$*
- *For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$*
- *For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$*
- *For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M.M \in \mathcal{N}_y$.*

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{U : \forall x \in U.U \in \mathcal{N}_x\}$.

Definition 3.8 (Exterior Point)**.** Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x.

Definition 3.9 (Boundary Point)**.** Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B.

Definition 3.10 (Interior)**.** Let X be a topological space and $B \subseteq X$. The *interior* of B, B°, is the set of all interior points of B.

Proposition 3.11. *The interior of B is the union of all the open sets included in B.*

Definition 3.12 (Closure)**.** Let X be a topological space and $B \subseteq X$. The *closure* of B, $\overline{B}$, is the set of all points that are not exterior points of B.

Proposition 3.13. *The closure of B is the intersection of all the closed sets that include B.*

Proposition 3.14. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 3.15 (Kuratowski Closure Axioms)**.** *Let X be a set and $\overline{} : \mathcal{P}X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ such that, for all $B \subseteq X$, $\overline{B}$ is the closure of B, if and only if:*

- $\overline{\emptyset} = \emptyset$
- *For all $A \subseteq X$ we have $A \subseteq \overline{A}$*
- *For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$*
- *For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

In this case, $\mathcal{O}$ is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

3.1.1 Subspaces

Definition 3.16 (Subspace)**.** Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

Example 3.17. The *unit sphere* S^2 is $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a subspace of $\mathbb{R}^3$.

3.1.2 Topological Disjoint Union

Definition 3.18. Let X and Y be topological spaces. The *disjoint union* is $X + Y$ where $U \subseteq X + Y$ is open if and only if ${\kappa_1}^{-1}(U)$ is open in X and ${\kappa_2}^{-1}(U)$ is open in Y.

3.1.3 Product Topology

Definition 3.19. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the set of all subsets $W \subseteq X \times Y$ such that, for all $(x, y) \in W$, there exist neighbourhoods U of x in X and V of y in Y such that $U \times V \subseteq W$.

3.1.4 Bases

Definition 3.20 (Basis)**.** Let X be a topological space. A *basis* for the topology on X is a set of open sets $\mathcal{B}$ such that every open set is the union of a subset of $\mathcal{B}$.

3.1.5 Subbases

Definition 3.21 (Subbasis)**.** Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of $\mathcal{S}$.

3.2 Continuous Functions

Definition 3.22 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* iff, for every open set V in Y, the inverse image $f^{-1}(V)$ is open in X.

Proposition 3.23.

1. *id_X is continuous*
2. *The composite of two continuous functions is continuous.*
3. *If $f : X \to Y$ is continuous and $X_0 \subseteq X$ then $f{\restriction}X_0 : X_0 \to Y$ is continuous.*
4. *If $f : X + Y \to Z$, then f is continuous iff $f \circ \kappa_1 : X \to Z$ and $f \circ \kappa_2 : Y \to Z$ are continuous.*
5. *If $f : Z \to X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.*

Definition 3.24 (Homeomorphism)**.** Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Definition 3.25 (Retraction)**.** Let X be a topological space and A a subspace of X. A continuous function $\rho : X \to A$ is a *retraction* iff $\rho \upharpoonright A = \mathrm{id}_A$. We say A is a *retract* of X iff there exists a retraction.

3.3 Convergence

Definition 3.26 (Convergence)**.** Let X be a topological space. Let (x_n) be a sequence in X. A point $a : \mathrm{El}\,(X)$ is a *limit* of the sequence iff, for every neighbourhood U of a, there exists n_0 such that $\forall n \geqslant n_0 . x_n \in U$.

3.4 Connected Spaces

Definition 3.27 (Connected)**.** A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 3.28. *The continuous image of a connected space is connected.*

Proposition 3.29. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are connected, then X is connected.*

Proposition 3.30. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 3.31 (Path-connected)**.** A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \to X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 3.32. *The continuous image of a path connected space is path connected.*

Proposition 3.33. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are path connected, then X is path connected.*

Proposition 3.34. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

3.5 Hausdorff Spaces

Definition 3.35 (Hausdorff)**.** A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 3.36. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 3.37.

1. *Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

Proposition 3.38. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \to B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X, then $f = g$.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.
⟨1⟩2. PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.
⟨1⟩3. PICK $x \in f^{-1}(U) \cap g^{-1}(V)$
⟨1⟩4. $f(x) = g(x) \in U \cap V$
⟨1⟩5. Q.E.D.
PROOF: This is a contradiction.
□

Proposition 3.39. *Let X and Y be metric spaces. Let $f : X \to Y$ be uniformly continuous. Let $\hat{X}$ and $\hat{Y}$ be the completions of X and Y. Then f extends uniquely to a continuous map $\hat{X} \to \hat{Y}$.*

PROOF: The extension maps $\lim_{n\to\infty} x_n$ to $\lim_{n\to\infty} f(x_n)$. □

3.6 Compactness

Definition 3.40 (Compact)**.** A topological space is *compact* iff every open cover has a finite subcover.

Proposition 3.41. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P. Then $X \in P$.*

PROOF:
⟨1⟩1. P is an open cover of X
⟨1⟩2. PICK a finite subcover $U_1, \ldots, U_n \in P$
⟨1⟩3. $X = U_1 \cup \cdots \cup U_n \in P$
□

Corollary 3.41.1. *Let f be a compact space and $f : X \to \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. □

Proposition 3.42. *The continuous image of a compact space is compact.*

Proposition 3.43. *A closed subspace of a compact space is compact.*

Proposition 3.44. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. *X and Y are compact.*

2. *$X + Y$ is compact.*

3. *$X \times Y$ is compact.*

Proposition 3.45. *A compact subspace of a Hausdorff space is closed.*

Proposition 3.46. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

3.7 Quotient Spaces

Definition 3.47 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. The *quotient topology* on $X/\sim$ is defined by: $U : \mathrm{El}\,(\mathcal{P}X)$ is open in $X/\sim$ if and only if $\pi^{-1}(U)$ is open in X.

Proposition 3.48. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $f : X/\sim \to Y$. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proposition 3.49. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $\phi : Y \to X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \to X$ such that $\pi \circ \Phi = \phi \restriction U$. Then ϕ is continuous.

Proposition 3.50. *A quotient of a connected space is connected.*

Proposition 3.51. *A quotient of a path connected space is path connected.*

Proposition 3.52. *Let X be a topological space and $\sim$ an equivalence relation on X. If $X/\sim$ is Hausdorff then every equivalence class of $\sim$ is closed in X.*

Definition 3.53. Let X be a topological space and $A_1, \ldots, A_r \subseteq X$. Then $X/A_1, \ldots, A_r$ is the quotient space of X with respect to $\sim$ where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 3.54 (Cone)**.** Let X be a topological space. The *cone over* X is the space $(X \times [0, 1])/(X \times \{1\})$.

Definition 3.55 (Suspension)**.** Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 3.56 (Wedge Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 3.57 (Smash Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 3.58. $D^n/S^{n-1} \cong S^n$

PROOF:
⟨1⟩1. LET: $\phi : D^n/S^{n-1} \to S^n$ be the function induced by the map $D^n \to S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.
⟨1⟩2. ϕ is a bijection.
⟨1⟩3. ϕ is a homeomorphism.
PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.
□

3.8 Gluing

Definition 3.59 (Gluing)**.** Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \to Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X+Y)/\sim$, where $\sim$ is the equivalence relation generated by $x \sim \phi(x)$ for all $x : \mathrm{El}(X)$.

Proposition 3.60. *Y is a subspace of $Y \cup_\phi X$.*

Definition 3.61. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0,1])/\alpha$ is the quotient space of $X \times [0,1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x : \mathrm{El}(X)$.

Definition 3.62 (Möbius Strip)**.** The *Möbius strip* is $([-1,1] \times [0,1])/\alpha$ where $\alpha(x) = -x$.

Definition 3.63 (Klein Bottle)**.** The *Klein bottle* is $(S^1 \times [0,1])/\alpha$ where $\alpha(z) = \overline{z}$.

Proposition 3.64. *Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\mathrm{id}_{\partial M}} M \cong K$.*

PROOF:
⟨1⟩1. LET: $f : ([-1,1] \times [0,1]) + ([-1,1] \times [0,1]) \to S^1 \times [0,1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{\pi i\theta/2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-\pi i\theta/2}, t)$.
⟨1⟩2. f induces a bijection $M \cup_{\mathrm{id}_{\partial M}} M \approx K$
⟨1⟩3. f is a homeomorphism.
□

3.9 Metric Spaces

Definition 3.65 (Metric Space)**.** Let X be a set and $d : X^2 \to \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geqslant 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$

- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leqslant d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d). We often write X for the metric space (X, d).

Definition 3.66 (Topology of a Metric Space)**.** Let (X, d) be a metric space. The topology *induced* by the metric d is defined by: for $V \subseteq X$, we have V is open if and only if, for all $x \in V$, there exists $\epsilon > 0$ such that $\{y \in X : d(x, y) < \epsilon\} \subseteq V$.

Definition 3.67 (Metrizable)**.** A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 3.68. *Every metrizable space is Hausdorff.*

3.10 Complete Metric Spaces

Definition 3.69 (Complete)**.** A metric space is *complete* iff every Cauchy sequence converges.

Example 3.70. $\mathbb{R}$ is complete.

Proposition 3.71. *The product of two complete metric spaces is complete.*

Proposition 3.72. *Every compact metric space is complete.*

Proposition 3.73. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 3.74 (Completion)**.** Let X be a metric space. A *completion* of X is a complete metric space $\hat{X}$ and injection $i : X \rightarrowtail \hat{X}$ such that:

- The metric on X is the restriction of the metric on $\hat{X}$
- X is dense in $\hat{X}$.

Proposition 3.75. *Let $i_1 : X \to Y_1$ and $i_2 : X \to Y_2$ be completions of X. Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

Proof: Define $\phi(\lim_{n\to\infty} i_1(x_n)) = \lim_{n\to\infty} i_2(x_n)$. □

Theorem 3.76. *Every metric space has a completion.*

Proof: Let $\hat{X}$ be the set of Cauchy sequences in X quotiented by $\sim$ where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \to 0$. □

Chapter 4

Homotopy Theory

4.1 Homotopies

Definition 4.1 (Homotopy)**.** Let X and Y be topological spaces. Let $f, g : X \to Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0,1] \to Y$ such that

- $\forall x : \mathrm{El}\,(X)\,.h(x,0) = f(x)$
- $\forall x : \mathrm{El}\,(X)\,.h(x,1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \to Y$.

Proposition 4.2. *Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.*

4.2 Homotopy Equivalence

Definition 4.3 (Homotopy Equivalence)**.** Let X and Y be topological spaces. A *homotopy equivalence* between X and Y, $f : X \simeq Y$, is a continuous function $f : X \to Y$ such that there exists a continuous function $g : Y \to X$, the *homotopy inverse* to f, such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Definition 4.4 (Contractible)**.** A topological space X is *contractible* iff $X \simeq 1$.

Example 4.5. $\mathbb{R}^n$ is contractible.

Example 4.6. D^n is contractible.

Definition 4.7 (Deformation Retract)**.** Let X be a topological space and A a subspace of X. A retraction $\rho : X \to A$ is a *deformation retraction* iff $i \circ \rho \simeq \mathrm{id}_X$, where i is the inclusion $A \rightarrowtail X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 4.8 (Strong Deformation Retract)**.** Let X be a topological space and A a subspace of X. A *strong deformation retraction* $\rho : X \to A$ is a continuous function such that there exists a homotopy $h : X \times [0,1] \to X$ between $i \circ \rho$ and id_X such that, for all $a : \mathrm{El}\,(X)$ and $t : \mathrm{El}\,([0,1])$, we have $h(a,t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 4.9. $\{0\}$ is a strong deformation retract of $\mathbb{R}^n$ and of D^n.

Example 4.10. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 4.11. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 4.12. For any topological space X, the singleton consisting of the vertex is a strong deformation retract of the cone over X.

Chapter 5

Topological Groups

Definition 5.1 (Topological Group)**.** A *topological group* is a group G with a topology such that the function $G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.

Example 5.2. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 5.3. *Any subgroup of a topological group is a topological group under the subspace topology.*

Definition 5.4 (Homogeneous Space)**.** A *homogeneous space* is a topological space of the form G/H, where G is a topological group and H is a normal subgroup of G, under the quotient topology.

Proposition 5.5. *Let G be a topological group and H a normal subgroup of G. Then G/H is Hausdorff if and only if H is closed.*

Proof: See Bourbaki, N., General Topology. III.12 □

5.1 Continuous Actions

Definition 5.6 (Continuous Action)**.** Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \to X$ such that:

- $\forall x : \mathrm{El}\,(X)\,.ex = x$
- $\forall g, h : \mathrm{El}\,(G)\,.\forall x : \mathrm{El}\,(X)\,.g(hx) = (gh)x$

A *G-space* consists of a topological space X and a continuous action of G on X.

Definition 5.7 (Orbit)**.** Let X be a G-space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 5.8. *Define an action of $SO(2)$ on S^2 by $g(x_1, x_2, x_3) = (g(x_1, x_2), x_3)$. Then $S^2/SO(2) \cong [-1, 1]$.*

PROOF:
⟨1⟩1. LET: $f_3 : S^2/SO(2) \to [-1, 1]$ be the function induced by $\pi_3 : S^2 \to [-1, 1]$
⟨1⟩2. f_3 is bijective.
⟨1⟩3. $S^2/SO(2)$ is compact.
PROOF: It is the continuous image of S^2 which is compact.
⟨1⟩4. $[-1, 1]$ is Hausdorff.
⟨1⟩5. f_3 is a homeomorphism.
□

Definition 5.9 (Stabilizer)**.** Let X be a G-space and $x \in X$. The *stabilizer* of x is $G_x := \{g : \mathrm{El}(G) \mid gx = x\}$.

Proposition 5.10. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx.*

PROOF:
⟨1⟩1. If $gG_x = hG_x$ then $gx = hx$.
⟨2⟩1. ASSUME: $gG_x = hG_x$
⟨2⟩2. $g^{-1}h \in G_x$
⟨2⟩3. $g^{-1}hx = x$
⟨2⟩4. $gx = hx$
⟨1⟩2. If $gx = hx$ then $gG_x = hG_x$.
PROOF: Similar.
⟨1⟩3. The function is continuous.
PROOF: Proposition 2.48.
□

Chapter 6

Topological Vector Spaces

Definition 6.1 (Topological Vector Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Substraction is a continuous function $E^2 \to E$
- Multiplication is a continuous function $K \times E \to E$

Proposition 6.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. $\square$

Theorem 6.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 $\square$

Proposition 6.4. *Let E be a topological vector space and E_0 a subspace of E. Then $\overline{E_0}$ is a subspace of E.*

Definition 6.5. Let E be a topological vector space. The topological space *associated* with E is $E/\overline{\{0\}}$.

6.1 Cauchy Sequences

Definition 6.6 (Cauchy Sequence)**.** Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geqslant n_0 . x_n - x_m \in U$.

Definition 6.7 (Complete Topological Vector Space)**.** A topological vector space is *complete* iff every Cauchy sequence converges.

6.2 Seminorms

Definition 6.8 (Seminorm)**.** Let E be a vector space over K. A *seminorm* on E is a function $\| \| : E \to \mathbb{R}$ such that:

1. $\forall x : \mathrm{El}\,(E)\,.\|x\| \geqslant 0$
2. $\forall \alpha : \mathrm{El}\,(K)\,.\forall x : \mathrm{El}\,(E)\,.\|\alpha x\| = |\alpha|\|x\|$
3. *Triangle Inequality* $\forall x, y : \mathrm{El}\,(E)\,.\|x + y\| \leqslant \|x\| + \|y\|$

Example 6.9. The function that maps $(x_1, \ldots, x_n)$ to $|x_i|$ is a seminorm on $\mathbb{R}^n$.

Definition 6.10. Let E be a vector space over K. Let Λ be a set of seminorms on E. The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x : \mathrm{El}\,(E)$.

Proposition 6.11. *E is a topological vector space under this topology. It is Hausdorff iff, for all $x : \mathrm{El}\,(E)$, if $\forall \lambda \in \Lambda.\lambda(x) = 0$ then $x = 0$.*

6.3 Fréchet Spaces

Definition 6.12 (Pre-Fréchet Space)**.** A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 6.13. *Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \|_n : n \in \mathbb{Z}^+\}$. Then*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 6.14 (Fréchet Space)**.** A *Fréchet space* is a complete pre-Fréchet space.

6.4 Normed Spaces

Definition 6.15 (Normed Space)**.** Let E be a vector space over K. A *norm* on E is a function $\| \| : E \to \mathbb{R}$ is a seminorm such that, $\forall x \in E.\|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 6.16. *If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E.*

Proposition 6.17. *Let $\| \ \|$ be a seminorm on the vector space E. Then $\| \ \|$ defines a norm on $E/\overline{\{0\}}$.*

Proposition 6.18. *Let E and F be normed spaces. Any continuous linear map $E \to F$ is uniformly continuous.*

Definition 6.19. For $p \geqslant 1$. let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt{p} \int_{\mathbb{R}^n} |f(x)|^p dx$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\overline{\{0\}} \ .$$

6.5 Inner Product Spaces

Proposition 6.20. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E.*

6.6 Banach Spaces

Definition 6.21 (Banach Space)**.** A *Banach space* is a complete normed space.

Example 6.22. For any topological space X, the set $C(X)$ of bounded continuous functions $X \to \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 6.23. *The completion of a normed space is a Banach space.*

Proposition 6.24. *Let E and F be normed spaces. Let $f : E \to F$ be a continuous linear map. Then the extension to the completions $\hat{E} \to \hat{F}$ is linear.*

Proposition 6.25. *$L^p(\mathbb{R}^n)$ is a Banach space.*

6.7 Hilbert Spaces

Definition 6.26 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Example 6.27. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \to \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi} \pi f(x) g(x) dx \ .$$

Proposition 6.28. *The completion of an inner product space is a Hilbert space.*

6.8 Locally Convex Spaces

Definition 6.29 (Locally Convex Space)**.** A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 6.30. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Proposition 6.31. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Example 6.32. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \to \mathbb{R}$ is continuous as a map $E' \to \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 6.33 (Thom Space)**.** Let E be a vector bundle with a Riemannian metric, $DE = \{x : \mathrm{El}\,(E) \mid \|x\| \leqslant 1\}$ its disc bundle and $SE := \{v : \mathrm{El}\,(E) \mid \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE.