# Mathematics

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# Contents

# Part I

# Category Theory

# Chapter 1

# Foundations

This is a placeholder — I am not sure what foundation I want to use for this project yet. I will try to work in a way which is foundation-independent. What I do could be formalized in ZFC, ETCS, or some other system. I will assume the usual set theoretic constructions as needed. Sets will be defined up to bijection only.

# Chapter 2

# Number Theory

## 2.1 Congruence

**Definition 2.1** (Congruence)**.** Let $a$, $b$, $n$ be integers with $n > 0$. We say $a$ is *congruent* to $b$ *modulo* $n$, and write $a \equiv b \mod n$, iff $n \mid b - a$.

**Proposition 2.2.** *For $n$ a positive integer, congruence modulo $n$ is an equivalence relation.*

Proof:
$\langle 1 \rangle 1$. For any integer $a$ we have $a \equiv a \mod n$.
  Proof: Since $n \mid 0 = a - a$.
$\langle 1 \rangle 2$. If $a \equiv b \mod n$ then $b \equiv a \mod n$.
  Proof: If $n \mid b - a$ then $n \mid a - b = -(b - a)$.
$\langle 1 \rangle 3$. If $a \equiv b \mod n$ and $b \equiv c \mod n$ then $a \equiv c \mod n$.
  Proof: If $n \mid b - a$ and $n \mid c - b$ then $n \mid c - a = (c - b) + (b - a)$.
$\square$

**Definition 2.3.** Let $\mathbb{Z}/n\mathbb{Z}$ be the quotient set of $\mathbb{Z}$ with respect to congruence modulo $n$.

**Proposition 2.4.** *$\mathbb{Z}/n\mathbb{Z}$ has exactly $n$ elements.*

Proof: Every integer is congruent to one of $0, 1, \ldots, n - 1$ by the division algorithm, and no two of them are conguent to one another, since if $0 \leq i < j < n$ then $0 < j - i < n$. $\square$

**Proposition 2.5.** *If $a \equiv a' \mod n$ and $b \equiv b' \mod n$ then $a + b \equiv a' + b' \mod n$.*

Proof: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid (a' + b') - (a + b)$. $\square$

**Proposition 2.6.** *If $a \equiv a' \mod n$ and $b \equiv b' \mod n$ then $ab \equiv a'b' \mod n$.*

Proof: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid a'b' - ab = a'(b' - b) + (a' - a)b$. $\square$

## 2.2 Euler's $\phi$-function

**Definition 2.7.** For $n$ a positive integer, let $(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$.

PROOF: We prove this is well-defined.
$\langle 1\rangle 1$. If $m \equiv m' \mod n$ and $\gcd(m, n) = 1$ then $\gcd(m', n) = 1$.
  $\langle 2\rangle 1$. PICK integers $a$, $b$ such that $am + bn = 1$
  $\langle 2\rangle 2$. PICK an integer $c$ such that $m' - m = cn$
  $\langle 2\rangle 3$. $am' + (b - ac)n = 1$
□

**Definition 2.8.** For $n$ a positive integer, let $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$.

**Proposition 2.9.** *If $n$ is an odd positive integer then $\phi(2n) = \phi(n)$.*

PROOF:
$\langle 1\rangle 1$. LET: $n$ be an odd positive integer.
$\langle 1\rangle 2$. For any integer $m$, if $\gcd(m, n) = 1$ then $\gcd(2m + n, 2n) = 1$
  PROOF: For $p$ a prime, if $p \mid 2m + n$ and $p \mid 2n$ then $p \neq 2$ (since $2m + n$ is odd) so $p \mid n$ and hence $p \mid m$, which is a contradiction.
$\langle 1\rangle 3$. For any integer $r$, if $\gcd(r, 2n) = 1$ then $\gcd(\frac{r+n}{2}, n) = 1$
  PROOF: If $p \mid n$ and $p \mid \frac{r+n}{2}$ then $p \mid r + n$ so $p \mid r$ which is a contradiction.
$\langle 1\rangle 4$. The function that maps $m$ to $2m + n$ is a bijection between $(\mathbb{Z}/n\mathbb{Z})^*$ and $(\mathbb{Z}/2n\mathbb{Z})^*$.
□

# Chapter 3

# Categories

**Definition 3.1** (Category)**.** A *category* $\mathcal{C}$ consists of:

- A class $|\mathcal{C}|$ of *objects*. We write $A \in \mathcal{C}$ for $A \in |\mathcal{C}|$.
- For any objects $A$, $B$, a set $\mathcal{C}[A, B]$ of *morphisms* from $A$ to $B$. We write $f : A \to B$ for $f \in \mathcal{C}[A, B]$.
- For any object $A$, a morphism $\mathrm{id}_A : A \to A$, the *identity* morphism on $A$.
- For any morphisms $f : A \to B$ and $g : B \to C$, a morphism $g \circ f : A \to C$, the *composite* of $f$ and $g$.

such that:

**Associativity** Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have $h \circ (g \circ f) = (h \circ g) \circ f$

**Left Unit Law** For any morphism $f : A \to B$, we have $\mathrm{id}_B \circ f = f$.

**Right Unit Law** For any morphism $f : A \to B$, we have $f \circ \mathrm{id}_A = f$.

**Proposition 3.2.** *The identity morphism on an object is unique.*

PROOF: If $i$ and $j$ are identity morphisms on $A$ then $i = i \circ j = j$. □

**Example 3.3** (Category of Sets)**.** The *category of sets* **Set** has objects all sets and morphisms all functions.

**Definition 3.4** (Endomorphism)**.** In a category $\mathcal{C}$, an *endomorphism* on an object $A$ is a morphism $A \to A$. We write $\mathrm{End}_{\mathcal{C}}(A)$ for $\mathcal{C}[A, A]$.

**Definition 3.5** (Opposite Category)**.** For any category $\mathcal{C}$, the *opposite* category $\mathcal{C}^{\mathrm{op}}$ is the category with the same objects as $\mathcal{C}$ and

$$\mathcal{C}^{\mathrm{op}}[A, B] = \mathcal{C}[B, A]$$

## 3.1 Preorders

**Definition 3.6** (Preorder)**.** A *preorder* on a set $A$ is a relation $\leq$ on $A$ that is reflexive and transitive.

A *preordered set* is a pair $(A, \leq)$ such that $\leq$ is a preorder on $A$. We usually write $A$ for the preordered set $(A, \leq)$.

We identify any preordered set $A$ with the category whose objects are the elements of $A$, with one morphism $a \to b$ iff $a \leq b$, and no morphism $a \to b$ otherwise.

**Example 3.7.** For any ordinal $\alpha$, let $\alpha$ be the preorder $\{\beta : \beta < \alpha\}$ under $\leq$.

**Definition 3.8** (Discrete Preorder)**.** We identify any set $A$ with the *discrete* preorder $(A, =)$.

## 3.2 Monomorphisms and Epimorphisms

**Definition 3.9** (Monomorphism)**.** In a category, let $f : A \to B$. Then $f$ is a *monomorphism* or *monic* iff, for every object $X$ and morphism $x, y : X \to A$, if $fx = fy$ then $x = y$.

**Definition 3.10** (Epimorphism)**.** In a category, let $f : A \to B$. Then $f$ is a *epimorphism* or *epi* iff, for every object $X$ and morphism $x, y : B \to X$, if $xf = yf$ then $x = y$.

**Proposition 3.11.** *The composite of two monomorphism is monic.*

PROOF:
$\langle 1 \rangle 1.$ LET: $f : A \rightarrowtail B$ and $g : B \rightarrowtail C$ be monic.
$\langle 1 \rangle 2.$ LET: $x, y : X \to A$
$\langle 1 \rangle 3.$ ASSUME: $g \circ f \circ x = g \circ f \circ y$
$\langle 1 \rangle 4.$ $f \circ x = f \circ y$
$\langle 1 \rangle 5.$ $x = y$
□

**Proposition 3.12.** *The composite of two epimorphisms is epi.*

PROOF: Dual. □

**Proposition 3.13.** *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is monic then $f$ is monic.*

PROOF: If $f \circ x = f \circ y$ then $gfx = gfy$ and so $x = y$. □

**Proposition 3.14.** *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is epi then $g$ is epi.*

PROOF: Dual. □

**Proposition 3.15.** *A function is a monomorphism in* **Set** *iff it is injective.*

PROOF:
⟨1⟩1. LET: $f : A \to B$
⟨1⟩2. If $f$ is monic then $f$ is injective.
⟨2⟩1. ASSUME: $f$ is monic.
⟨2⟩2. LET: $x, y \in A$
⟨2⟩3. ASSUME: $f(x) = f(y)$
⟨2⟩4. LET: $\overline{x}, \overline{y} : 1 \to A$ be the functions such that $\overline{x}(*) = x$ and $\overline{y}(*) = y$
⟨2⟩5. $f \circ \overline{x} = f \circ \overline{y}$
⟨2⟩6. $\overline{x} = \overline{y}$
PROOF: By ⟨2⟩1.
⟨2⟩7. $x = y$
⟨1⟩3. If $f$ is injective then $f$ is monic.
⟨2⟩1. ASSUME: $f$ is injective.
⟨2⟩2. LET: $X$ be a set and $x, y : X \to A$.
⟨2⟩3. ASSUME: $f \circ x = f \circ y$
PROVE: $x = y$
⟨2⟩4. LET: $t \in X$
PROVE: $x(t) = y(t)$
⟨2⟩5. $f(x(t)) = f(y(t))$
⟨2⟩6. $x(t) = y(t)$
PROOF: By ⟨2⟩1.
□

**Proposition 3.16.** *A function is an epimorphism in* **Set** *iff it is surjective.*

PROOF:
⟨1⟩1. LET: $f : A \to B$
⟨1⟩2. If $f$ is an epimorphism then $f$ is surjective.
⟨2⟩1. ASSUME: $f$ is an epimorphism.
⟨2⟩2. LET: $b \in B$
⟨2⟩3. LET: $x, y : B \to 2$ be defined by $x(b) = 1$ and $x(t) = 0$ for all other $t \in B$, $y(t) = 0$ for all $t \in B$.
⟨2⟩4. $x \neq y$
⟨2⟩5. $x \circ f \neq y \circ f$
⟨2⟩6. There exists $a \in A$ such that $f(a) = b$.
⟨1⟩3. If $f$ is surjective then $f$ is an epimorphism.
⟨2⟩1. ASSUME: $f$ is surjective.
⟨2⟩2. LET: $x, y : B \to X$
⟨2⟩3. ASSUME: $x \circ f = y \circ f$
PROVE: $x = y$
⟨2⟩4. LET: $b \in B$
PROVE: $x(b) = y(b)$
⟨2⟩5. PICK $a \in A$ such that $f(a) = b$
⟨2⟩6. $x(f(a)) = y(f(a))$
⟨2⟩7. $x(b) = y(b)$
□

**Proposition 3.17.** *In a preorder, every morphism is monic and epi.*

PROOF: Immediate from definitions. □

## 3.3 Sections and Retractions

**Definition 3.18** (Section, Retraction)**.** In a category, let $r : A \to B$ and $s : B \to A$. Then $r$ is a *retraction* of $s$, and $s$ is a *section* of $r$, iff $r \circ s = \mathrm{id}_B$.

**Proposition 3.19.** *Every identity morphism is a section and retraction of itself.*

PROOF: Immediate from definitions. □

**Proposition 3.20.** *Let $r, r' : A \to B$ and $s : B \to A$. If $r$ is a retraction of $s$ and $r'$ is a section of $s$ then $r = r'$.*

PROOF:

$$\begin{aligned} r &= r \circ \mathrm{id}_A \\ &= r \circ s \circ r' \\ &= \mathrm{id}_B \circ r' \\ &= r' \end{aligned}$$

□

**Proposition 3.21.** *Let $r_1 : A \to B$, $r_2 : B \to C$, $s_1 : B \to A$ and $s_2 : C \to B$. If $r_1$ is a retraction of $s_1$ and $r_2$ is a retraction of $s_2$ then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \mathrm{id}_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= \mathrm{id}_C \end{aligned}$$

□

**Proposition 3.22.** *Every section is monic.*

PROOF:
⟨1⟩1. LET: $s : A \to B$ be a section of $r : B \to A$.
⟨1⟩2. LET: $x, y : X \to A$ satisfy $sx = sy$.
⟨1⟩3. $rsx = rsy$
⟨1⟩4. $x = y$
□

**Proposition 3.23.** *Every retraction is epi.*

PROOF: Dual. □

**Proposition 3.24.** *In* **Set***, every epimorphism has a retraction.*

PROOF: By the Axiom of Choice. □

**Example 3.25.** It is not true in general that every monomorphism in any category has a section. nor that every epimorphism in any category has a retraction.

In the category **2**, the morphism $0 \leq 1$ is monic and epi but has no retraction or section.

## 3.4 Isomorphisms

**Definition 3.26** (Isomorphism)**.** In a category $\mathcal{C}$, a morphism $f : A \to B$ is an *isomorphism*, denoted $f : A \cong B$, iff there exists a morphism $f^{-1} : B \to A$, the *inverse* of $f$, such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

An *automorphism* on an object $A$ is an isomorphism between $A$ and itself. We write $\mathrm{Aut}_{\mathcal{C}}(A)$ for the set of all automorphisms on $A$.

Objects $A$ and $B$ are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

**Proposition 3.27.** *The inverse of an isomorphism is unique.*

Proof: Proposition 3.20. ∎

**Proposition 3.28.** *For any object $A$ we have $\mathrm{id}_A : A \cong A$ and $\mathrm{id}_A^{-1} = \mathrm{id}_A$.*

Proof: Since $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$ by the Unit Laws. ∎

**Proposition 3.29.** *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

Proof: Immediate from definitions. ∎

**Proposition 3.30.** *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Proof: From Proposition 3.21. ∎

**Definition 3.31** (Groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

## 3.5 Initial and Terminal Objects

**Definition 3.32** (Initial Object)**.** An object $I$ in a category is *initial* iff, for any object $X$, there is exactly one morphism $I \to X$.

**Example 3.33.** The empty set is the initial object in **Set**.

**Definition 3.34** (Terminal Object)**.** An object $T$ in a category is *terminal* iff, for any object $X$, there is exactly one morphism $X \to T$.

**Example 3.35.** Every singleton is terminal in **Set**.

**Proposition 3.36.** *If $I$ and $J$ are initial in a category, then there exists a unique isomorphism $I \cong J$.*

Proof:
⟨1⟩1. Let: $i$ be the unique morphism $I \to J$.
⟨1⟩2. Let: $i^{-1}$ be the unique morphism $J \to I$.
⟨1⟩3. $i \circ i^{-1} = \mathrm{id}_J$
  Proof: Since there is only one morphism $J \to J$.
⟨1⟩4. $i^{-1} \circ i = \mathrm{id}_I$

PROOF: Since there is only one morphism $I \to I$.
□

**Proposition 3.37.** *If $S$ and $T$ are terminal in a category, then there exists a unique isomorphism $S \cong T$.*

PROOF: Dual. □

# Chapter 4

# Functors

**Definition 4.1** (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for any morphism $f : A \to B : \mathcal{C}$, a morphism $Ff : FA \to FB : \mathcal{D}$

such that:

- $F\mathrm{id}_A = \mathrm{id}_{FA}$
- $F(g \circ f) = Fg \circ Ff$

**Definition 4.2** (Identity Functor)**.** For any category $\mathcal{C}$, the *identity functor* $1_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned} 1_\mathcal{C} A &= A \\ 1_\mathcal{C} f &= f \end{aligned}$$

**Definition 4.3** (Constant Functor)**.** Given categories $\mathcal{C}$, $\mathcal{D}$ and an object $D \in \mathcal{D}$, the *constant functor* $K^\mathcal{C} D : \mathcal{C} \to \mathcal{D}$ is the functor defined by

$$\begin{aligned} K^\mathcal{C} DC &= D \\ K^\mathcal{C} Df &= \mathrm{id}_D \end{aligned}$$

## 4.1 Comma Categories

**Definition 4.4** (Comma Category)**.** Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. The *comma category* $F \downarrow G$ is the category with:

- objects all pairs $(C, D, f)$ where $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f : FC \to GD : \mathcal{E}$

- morphisms $(u, v) : (C, D, f) \to (C', D', g)$ all pairs $u : C \to C' : \mathcal{C}$ and $v : D \to D' : \mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ \downarrow{\scriptstyle Fu} & & \downarrow{\scriptstyle Gv} \\ FC' & \xrightarrow{g} & GD' \end{array}$$

**Definition 4.5** (Slice Category)**.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category* over $A$, denoted $\mathcal{C}/A$, is the comma category $1_{\mathcal{C}} \downarrow K^{\mathbf{1}}A$.

**Definition 4.6** (Coslice Category)**.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *coslice category* over $A$, denoted $\mathcal{C}\backslash A$, is the comma category $K^{\mathbf{1}}A \downarrow 1_{\mathcal{C}}$.

**Definition 4.7** (Pointed Sets)**.** The *category of pointed sets* $\mathbf{Set}_*$ is the coslice category $\mathbf{Set}\backslash 1$.

# Part II

# Group Theory

# Chapter 5

# Groups

**Definition 5.1** (Group)**.** A *group* $G$ consists of a set $G$ and a binary operation $\cdot : G^2 \to G$ such that $\cdot$ is associative, and there exists $e \in G$, the *identity* element of the group, such that:

- For all $x \in G$ we have $xe = ex = x$
- For all $x \in G$, there exists $x^{-1} \in G$, the *inverse* of $x$, such that $xx^{-1} = x^{-1}x = e$.

We identify a group $G$ with the category $G$ with one object and morphisms the elements of $G$, with composition given by $\cdot$.

The *order* of a group $G$, denoted $|G|$, is the number of elements in $G$ if $G$ is finite; otherwise we write $|G| = \infty$.

**Proposition 5.2.** *The identity in a group is unique.*

Proof: Proposition 3.2.

**Proposition 5.3.** *The inverse of an element is unique.*

Proof: If $i$ and $j$ are inverses of $x$ then $i = ixj = j$. □

**Example 5.4.**

- The *trivial* group is $\{e\}$ under $ee = e$.
- $\mathbb{Z}$ is a group under addition
- $\mathbb{Q}$ is a group under addition
- $\mathbb{Q} - \{0\}$ is a group under multiplication
- $\mathbb{R}$ is a group under addition
- $\mathbb{R} - \{0\}$ is a group under multiplication
- $\mathbb{C}$ is a group under addition
- $\mathbb{C} - \{0\}$ is a group under multiplication

- $\{-1, 1\}$ is a group under multiplication
- The set of $2 \times 2$ real matrices with non-zero determinant is a group under matrix multiplication.
- For any positive integer $n$, the set $\mathbb{Z}/n\mathbb{Z}$ of integers modulo $n$ under addition is a group.
- For any category $\mathcal{C}$ and object $A \in \mathcal{C}$, we have $\mathrm{Aut}_{\mathcal{C}}(A)$ is a group under $gf = f \circ g$.

  For $A$ a set, we call $S_A = \mathrm{Aut}_{\mathbf{Set}}(A)$ the *symmetric group* or *group of permutations* of $A$.
- For $n \geq 3$, the *dihedral group* $D_{2n}$ consists of the set of rigid motions that map the regular $n$-gon onto itself under composition.

**Example 5.5.**

- The only group of order 1 is the trivial group.
- The only group of order 2 is $\mathbb{Z}_2$.
- The only group of order 3 is $\mathbb{Z}_3$.
- There are exactly two groups of order 4: $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ under $(a, b)(c, d) = (ac, bd)$.

**Example 5.6.** For any positive integer $n$, the set

$$(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$$

is a group under multiplication.

PROOF:
$\langle 1 \rangle 1$. If $\gcd(m_1, n) = \gcd(m_2, n) = 1$ then $\gcd(m_1 m_2, n) = 1$
  $\langle 2 \rangle 1$. PICK integers $a$, $b$, $c$, $d$ such that $am_1 + bn = cm_2 + dn = 1$
  $\langle 2 \rangle 2$. $acm_1 m_2 + (bcm_2 + d)n =!$
$\langle 1 \rangle 2$. Multiplication is associative.
$\langle 1 \rangle 3$. 1 is the identity element.
$\langle 1 \rangle 4$. Every element has an inverse.
  $\langle 2 \rangle 1$. LET: $a \in (\mathbb{Z}/n\mathbb{Z})^*$
  $\langle 2 \rangle 2$. PICK integers $b$, $c$ such that $ab + cn = 1$
  $\langle 2 \rangle 3$. $ab = 1$ in $(\mathbb{Z}/n\mathbb{Z})^*$
□

**Proposition 5.7** (Cancellation)**.** *Let $G$ be a group. Let $a, g, h \in G$. If $ag = ah$ or $ga = ha$ then $g = h$.*

PROOF: If $ag = ah$ then $g = a^{-1}ag = a^{-1}ah = h$. Similarly if $ga = ha$. □

**Proposition 5.8.** *Let $G$ be a group and $g, h \in G$. Then $(gh)^{-1} = h^{-1}g^{-1}$.*

PROOF: Since $ghh^{-1}g^{-1} = e$. □

**Definition 5.9.** Let $G$ be a group. Let $g \in G$. We define $g^n \in G$ for all $n \in \mathbb{Z}$ as follows:

$$\begin{aligned} g^0 &= e \\ g^{n+1} &= g^n g && (n \geq 0) \\ g^{-n} &= (g^{-1})^n && (n > 0) \end{aligned}$$

**Proposition 5.10.** *Let $G$ be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then*

$$g^{m+n} = g^m g^n \quad .$$

PROOF:
⟨1⟩1. For all $k \in \mathbb{Z}$ we have $g^{k+1} = g^k g$
⟨2⟩1. For all $k \geq 0$ we have $g^{k+1} = g^k g$
PROOF: Immediate from definition.
⟨2⟩2. $g^{-1+1} = g^{-1} g$
PROOF: Both are equal to $e$.
⟨2⟩3. For all $k > 1$ we have $g^{-k+1} = g^{-k} g$
PROOF:

$$\begin{aligned} g^{-k+1} &= (g^{-1})^{k-1} \\ &= (g^{-1})^{k-1} g^{-1} g \\ &= (g^{-1})^k g \\ &= g^{-k} g \end{aligned}$$

⟨1⟩2. For all $k \in \mathbb{Z}$ we have $g^{k-1} = g^k g^{-1}$
PROOF: Substitute $k = k - 1$ above and multiply by $g^{-1}$.
⟨1⟩3. $g^{m+0} = g^m g^0$
PROOF: Since $g^m g^0 = g^m e = g^m$.
⟨1⟩4. If $g^{m+n} = g^m g^n$ then $g^{m+n+1} = g^m g^{n+1}$
PROOF:

$$\begin{aligned} g^{m+n+1} &= g^{m+n} g && (\langle 1 \rangle 1) \\ &= g^m g^n g \\ &= g^m g^{n+1} && (\langle 1 \rangle 1) \end{aligned}$$

⟨1⟩5. If $g^{m+n} = g^m g^n$ then $g^{m+n-1} = g^m g^{n-1}$
PROOF:

$$\begin{aligned} g^{m+n-1} g &= g^{m+n} && (\langle 1 \rangle 1) \\ &= g^m g^n \\ \therefore g^{m+n-1} &= g^m g^n g^{-1} \\ &= g^m g^{n-1} && (\langle 1 \rangle 2) \end{aligned}$$

□

**Proposition 5.11.** *Let $G$ be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then*

$$(g^m)^n = g^{mn} \quad .$$

PROOF:
⟨1⟩1. $(g^m)^0 = g^0$
PROOF: Both sides are equal to $e$.
⟨1⟩2. If $(g^m)^n = g^{mn}$ then $(g^m)^{n+1} = g^{m(n+1)}$.
PROOF:

$$\begin{aligned}(g^m)^{n+1} &= (g^m)^n g^m && \text{(Proposition 5.10)}\\ &= g^{mn} g^m \\ &= g^{mn+m} && \text{(Proposition 5.10)}\end{aligned}$$

⟨1⟩3. If $(g^m)^n = g^{mn}$ then $(g^m)^{n-1} = g^{m(n-1)}$.
PROOF:

$$\begin{aligned}(g^m)^n &= g^{mn} \\ \therefore (g^m)^{n-1} g^m &= g^{mn-m} g^m && \text{(Proposition 5.10)}\\ \therefore (g^m)^{n-1} &= g^{mn-m} && \text{(Cancellation)}\end{aligned}$$

□

**Definition 5.12** (Commute)**.** Let $G$ be a group and $g, h \in G$. We say $g$ and $h$ *commute* iff $gh = hg$.

## 5.1 Order of an Element

**Definition 5.13** (Order)**.** Let $G$ be a group. Let $g \in G$. Then $g$ has *finite order* iff there exists a positive integer $n$ such that $g^n = e$. In this case, the *order* of $g$, denoted $|g|$, is the least positive integer $n$ such that $g^n = e$.

If $g$ does not have finite order, we write $|g| = \infty$.

**Proposition 5.14.** *Let $G$ be a group. Let $g \in G$ and $n$ be a positive integer. If $g^n = e$ then $|g| \mid n$.*

PROOF:
⟨1⟩1. LET: $n = q|g| + d$ where $0 \leq d < |g|$
PROOF: Division Algorithm.
⟨1⟩2. $g^d = e$
PROOF:

$$\begin{aligned}e &= g^n \\ &= g^{q|g|+d} \\ &= (g^{|g|})^q g^d && \text{(Propositions 5.10, 5.11)}\\ &= e^q g^d \\ &= g^d\end{aligned}$$

⟨1⟩3. $d = 0$
PROOF: By minimality of $|g|$.
⟨1⟩4. $n = q|g|$
□

**Corollary 5.14.1.** *Let $G$ be a group. Let $g \in G$ have finite order and $n \in \mathbb{Z}$. Then $g^n = e$ if and only if $|g| \mid n$.*

**Proposition 5.15.** *Let $G$ be a group and $g \in G$. Then $|g| \leq |G|$.*

PROOF:
$\langle 1 \rangle$1. ASSUME: w.l.o.g. $G$ is finite.
$\langle 1 \rangle$2. PICK $i$, $j$ with $0 \leq i < j \leq |G|$ such that $g^i = g^j$.
PROOF: Otherwise $g^0$, $g^1$, ..., $g^{|G|}$ would be $|G| + 1$ distinct elements of $G$.
$\langle 1 \rangle$3. $g^{j-i} = e$
$\langle 1 \rangle$4. $g$ has finite order and $|g| \leq |G|$
PROOF: Since $|g| \leq j - i \leq j \leq |G|$.
∎

**Proposition 5.16.** *Let $G$ be a group. Let $g \in G$ have finite order. Let $m \in \mathbb{N}$. Then*

$$|g^m| = \frac{\mathrm{lcm}(m, |g|)}{m} = \frac{|g|}{\gcd(m, |g|)}$$

PROOF: Since for any integer $d$ we have

$$\begin{aligned} g^{md} = e &\Leftrightarrow |g| \mid md && \text{(Corollary 5.14.1)} \\ &\Leftrightarrow \mathrm{lcm}(m, |g|) \mid md \\ &\Leftrightarrow \frac{\mathrm{lcm}(m, |g|)}{m} \mid d && \square \end{aligned}$$

and so $|g^m| = \frac{\mathrm{lcm}(m,|g|)}{m}$ by Corollary 5.14.1. ∎

**Corollary 5.16.1.** *If $g$ has odd order then $|g^2| = |g|$.*

**Corollary 5.16.2.** *Let $m$ and $n$ be integers with $n > 0$. The order of $m$ in $\mathbb{Z}/n\mathbb{Z}$ is $\frac{n}{\gcd(m,n)}$.*

PROOF: Since the order of 1 is $n$. ∎

**Proposition 5.17.** *Let $G$ be a group. Let $g, h \in G$ have finite order. Assume $gh = hg$. Then $|gh|$ has finite order and*

$$|gh| \mid \mathrm{lcm}(|g|, |h|)$$

PROOF: Since $(gh)^{\mathrm{lcm}(|g|,|h|)} = g^{\mathrm{lcm}(|g|,|h|)} h^{\mathrm{lcm}(|g|,|h|)} = e$. ∎

**Example 5.18.** This example shows that we cannot remove the hypothesis that $gh = hg$.

In $\mathrm{GL}_2(\mathbb{R})$, take

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} .$$

Then $|g| = 4$, $|h| = 3$ and $|gh| = \infty$.

**Proposition 5.19.** *Let $G$ be a group and $g, h \in G$ have finite order. If $gh = hg$ and $\gcd(|g|, |h|) = 1$ then $|gh| = |g||h|$.*

PROOF:
⟨1⟩1. LET: $N = |gh|$
⟨1⟩2. $g^N = (h^{-1})^N$
⟨1⟩3. $g^{N|g|} = e$
⟨1⟩4. $|g^N| \mid |g|$
⟨1⟩5. $h^{-N|h|} = e$
⟨1⟩6. $|g^N| \mid |h|$
⟨1⟩7. $|g^N| = 1$
PROOF: Since $\gcd(|g|, |h|) = 1$.
⟨1⟩8. $g^N = e$
⟨1⟩9. $|g| \mid N$
⟨1⟩10. $h^{-N} = e$
⟨1⟩11. $|h| \mid N$
⟨1⟩12. $N = |g||h|$
PROOF: Using Proposition 5.17.
□

**Proposition 5.20.** *Let $G$ be a finite group. Assume there is exactly one element $f \in G$ of order 2. Then the product of all the elements of $G$ is $f$.*

PROOF: Let the elements of $G$ be $g_1, g_2, \ldots, g_n$. Apart from $e$ and $f$, every element and its inverse are distinct elements of the list. Hence the product of the list is $ef = f$. □

**Proposition 5.21.** *Let $G$ be a finite group of order $n$. Let $m$ be the number of elements of $G$ of order 2. Then $n - m$ is odd.*

PROOF: In the list of all elements that are not of order 2, every element and its inverse are distinct except for $e$. Hence the list has odd length. □

**Corollary 5.21.1.** *If a finite group has even order, then it contains an element of order 2.*

**Proposition 5.22.** *Let $G$ be a group and $a, g \in G$. Then $|aga^{-1}| = |g|$.*

PROOF: Since

$$\begin{aligned}(aga^{-1})^n = e &\Leftrightarrow ag^n a^{-1} = e \\ &\Leftrightarrow g^n = e\end{aligned}$$

□

**Proposition 5.23.** *Let $G$ be a group and $g, h \in G$. Then $|gh| = |hg|$.*

PROOF: Since $|gh| = |ghgg^{-1}| = |hg|$. □

## 5.2 Generators

**Definition 5.24** (Generator)**.** Let $G$ be a group and $a \in G$. We say *$a$ generates* the group iff, for all $x \in G$, there exists an integer $n$ such that $x^n = a$.

**Proposition 5.25.** *The integer $m$ generates $\mathbb{Z}/n\mathbb{Z}$ if and only if* $\gcd(m, n) = 1$.

PROOF: By Corollary 5.16.2. □

**Corollary 5.25.1.** *If $p$ is prime then every non-zero element in $\mathbb{Z}/p\mathbb{Z}$ is a generator.*

# Chapter 6

# Group Homomorphisms

**Definition 6.1** (Homomorphism)**.** Let $G$ and $H$ be groups. A *(group) homomorphism* $\phi : G \to H$ is a function such that, for all $x, y \in G$,

$$\phi(xy) = \phi(x)\phi(y) \quad .$$

**Proposition 6.2.** *Let $G$ and $H$ be groups with identities $e_G$ and $e_H$. Let $\phi : G \to H$ be a group homomorphism. Then $\phi(e_G) = e_H$.*

Proof: Since $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$ and so $\phi(e_G) = e_H$ by Cancellation. □

**Proposition 6.3.** *Let $\phi : G \to H$ be a group homomorphism. For all $x \in G$ we have $\phi(x^{-1}) = \phi(x)^{-1}$.*

Proof: Since $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_H$. □

**Proposition 6.4.** *Let $G$, $H$ and $K$ be groups. If $\phi : G \to H$ and $\psi : H \to K$ are homomorphisms then $\psi \circ \phi : G \to K$ is a homomorphism.*

Proof: For $x, y \in G$ we have

$$\psi(\phi(xy)) = \psi(\phi(x)\phi(y)) = \psi(\phi(x))\psi(\phi(y)) \quad .$$

**Proposition 6.5.** *Let $G$ be a group. Then $\mathrm{id}_G : G \to G$ is a group homomorphism.*

Proof: For $x, y \in G$ we have $\mathrm{id}_G(xy) = xy = \mathrm{id}_G(x)\mathrm{id}_G(y)$. □

**Proposition 6.6.** *Let $\phi : G \to H$ be a group homomorphism. Let $g \in G$ have finite order. Then $|\phi(g)|$ divides $|g|$.*

Proof: Since $\phi(g)^{|g|} = \phi(g^{|g|}) = e$. □

**Definition 6.7** (Category of Groups)**.** Let **Grp** be the category of groups and group homomorphisms.

**Proposition 6.8.** *A group homomorphism $\phi : G \to H$ is an isomorphism in* **Grp** *if and only if it is bijective.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $\phi$ is bijective.
PROVE: $\phi^{-1}$ is a group homomorphism.
$\langle 1 \rangle 2$. LET: $h, h' \in H$
$\langle 1 \rangle 3$. $\phi(\phi^{-1}(hh')) = \phi(\phi^{-1}(h)\phi^{-1}(h'))$
PROOF: Both are equal to $hh'$.
$\langle 1 \rangle 4$. $\phi^{-1}(hh') = \phi^{-1}(h)\phi^{-1}(h')$
□

**Corollary 6.8.1.**

$$D_6 \cong C_3$$

PROOF: The canonical homomorphism $D_6 \to C_3$ is bijective. □

**Corollary 6.8.2.**

$$(\mathbb{R}, +) \cong (\{x \in \mathbb{R} : x > 0\}, \cdot)$$

PROOF: The function that maps $x$ to $e^x$ is a bijective homomorphism. □

**Proposition 6.9.** *The trivial group is the zero object in* **Grp**.

PROOF: For any group $G$, the unique function $G \to \{e\}$ is a group homomorphism, and the only group homomorphism $\{e\} \to G$ maps $e$ to $e_G$. □

**Proposition 6.10.** *For any groups $G$ and $H$, the set $G \times H$ under $(g, h)(g', h') = (gg', hh')$ is the product of $G$ and $H$ in* **Grp**.

PROOF:
$\langle 1 \rangle 1$. $G \times H$ is a group.
$\langle 2 \rangle 1$. The multiplication is associative.
PROOF: Since $(g_1, h_1)((g_2, h_2)(g_3, h_3)) = ((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1 g_2 g_3, h_1 h_2 h_3)$.
$\langle 2 \rangle 2$. $(e_G, e_H)$ is the identity.
PROOF: Since $(g, h)(e_G, e_H) = (e_G, e_H)(g, h) = (g, h)$.
$\langle 2 \rangle 3$. The inverse of $(g, h)$ is $(g^{-1}, h^{-1})$.
PROOF: Since $(g, h)(g^{-1}, h^{-1}) = (g^{-1}, h^{-1})(g, h) = (e_G, e_H)$.
$\langle 1 \rangle 2$. $\pi_1 : G \times H \to G$ is a group homomorphism.
PROOF: Immediate from definitions.
$\langle 1 \rangle 3$. $\pi_2 : G \times H \to H$ is a group homomorphism.
PROOF: Immediate from definitions.
$\langle 1 \rangle 4$. For any group homomorphism $\phi : K \to G$ and $\psi : K \to H$, the function $\langle \phi, \psi \rangle : K \to G \times H$ where $\langle \phi, \psi \rangle(k) = (\phi(k), \psi(k))$ is a group homomorphism.
PROOF:

$$\begin{aligned}
\langle \phi, \psi \rangle(kk') &= (\phi(kk'), \psi(kk')) \\
&= (\phi(k)\phi(k'), \psi(k)\psi(k')) \\
&= (\phi(k), \psi(k))(\phi(k'), \psi(k')) \\
&= \langle \phi, \psi \rangle(k)\langle \phi, \psi \rangle(k')
\end{aligned}$$

□

**Proposition 6.11.**

$$\mathrm{Aut}_{\mathbf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$$

PROOF: Every permutation of $\{(1,0),(0,1),(1,1)\}$ gives an automorphism of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

**Proposition 6.12.**

$$|\mathrm{Aut}_{\mathbf{Grp}}(C_n)| = \phi(n)$$

PROOF: An automorphism $\alpha$ is determined by $\alpha(1)$ which is any element of order $n$, and $g$ has order $n$ iff $\gcd(g,n) = 1$. □

**Example 6.13.**

$$\mathrm{Aut}_{\mathbf{Grp}}(\mathbb{Z}) \cong C_2$$

PROOF: The only automorphisms are the identity and multiplication by -1. □

## 6.1 Subgroups

**Definition 6.14** (Subgroup)**.** Let $(G,\cdot)$ and $(H,*)$ be groups such that $H$ is a subset of $G$. Then $H$ is a *subgroup* of $G$ iff the inclusion $i : H \hookrightarrow G$ is a group homomorphism.

**Proposition 6.15.** *If $(H,*)$ is a subgroup of $(G,\cdot)$ then $*$ is the restriction of $\cdot$ to $H$.*

PROOF: Given $x, y \in H$ we have

$$x * y = i(x * y) = i(x) \cdot i(y) = x \cdot y \ . \qquad \square$$

**Example 6.16.** For any group $G$ we have $\{e\}$ is a subgroup of $G$.

**Proposition 6.17.** *Let $G$ be a group. Let $H$ be a subset of $G$. Then $H$ is a subgroup of $G$ iff $H$ is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$.*

PROOF:

$\langle 1\rangle$1. If $H$ is a subgroup of $G$ then $H$ is nonempty.
  PROOF: Since every group has an identity element and so is nonempty.
$\langle 1\rangle$2. If $H$ is a subgroup of $G$ then, for all $x, y \in H$, we have $xy^{-1} \in H$.
  PROOF: Easy.
$\langle 1\rangle$3. If $H$ is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$, then $H$ is a subgroup of $G$.
  $\langle 2\rangle$1. ASSUME: $H$ is nonempty.
  $\langle 2\rangle$2. ASSUME: $\forall x, y \in H. xy^{-1} \in H$
  $\langle 2\rangle$3. $e \in H$
    PROOF: Pick $x \in H$. We have $e = xx^{-1} \in H$.
  $\langle 2\rangle$4. $\forall x \in H. x^{-1} \in H$
    PROOF: Given $x \in H$ we have $x^{-1} = ex^{-1} \in H$.

⟨2⟩5. $H$ is closed under the restriction of $\cdot$

PROOF: Given $x, y \in H$ we have $xy = x(y^{-1})^{-1} \in H$.

⟨2⟩6. $H$ is a group under the restriction of $\cdot$

PROOF: Associativity is inherited from $G$ and the existence of an identity element and inverses follows from ⟨2⟩3 and ⟨2⟩4.

⟨2⟩7. The inclusion $H \hookrightarrow G$ is a group homomorphism.

PROOF: For $x, y \in H$ we have $i(xy) = i(x)i(y) = xy$.

□

## 6.2 Inner Automorphisms

**Proposition 6.18.** *Let $G$ be a group and $g \in G$. The function $\gamma_g : G \to G$ defined by $\gamma_g(a) = gag^{-1}$ is an automorphism on $G$.*

PROOF:

⟨1⟩1. $\gamma_g$ is a homomorphism.

PROOF:

$$\begin{aligned}\gamma_g(ab) &= gabg^{-1} \\ &= gag^{-1}gbg^{-1} \\ &= \gamma_g(a)\gamma_g(b)\end{aligned}$$

⟨1⟩2. $\gamma_g$ is injective.

PROOF: By Cancellation.

⟨1⟩3. $\gamma_g$ is surjective.

PROOF: Given $b \in G$, we have $\gamma_g(g^{-1}bg) = b$.

□

**Definition 6.19** (Inner Automorphism)**.** Let $G$ be a group. An *inner automorphism* on $G$ is a function of the form $\gamma_g(a) = gag^{-1}$ for some $g \in G$.

**Proposition 6.20.** *Let $G$ be a group. The function $\gamma : G \to \mathrm{Aut}_{\mathbf{Grp}}(G)$ that maps $g$ to $\gamma_g$ is a group homomorphism.*

PROOF: Since $\gamma_{gh}(a) = ghah^{-1}g^{-1} = \gamma_g(\gamma_h(a))$. □

## 6.3 Direct Products

**Definition 6.21** (Direct Product)**.** The *direct product* of groups $G$ and $H$ is their product in **Grp**.

**Proposition 6.22.** *If $m$ and $n$ are positive integers with $\gcd(m, n) = 1$ then $C_{mn} \cong C_m \times C_n$.*

PROOF: The function that maps $x$ to $(x \mod m, x \mod n)$ is an isomorphism.

□

**Definition 6.23** (Cyclic Group)**.** The *cyclic* groups are $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ for positive integers $n$.

## 6.4 Free Groups

**Proposition 6.24.** *Let $A$ be a set. Let $\mathcal{F}^A$ be the category whose objects are pairs $(G, j)$ where $G$ is a group and $j$ is a function $A \to G$, with morphisms $f : (G, j) \to (H, k)$ the group homomorphisms $f : G \to H$ such that $f \circ j = k$. Then $\mathcal{F}^A$ has an initial object.*

PROOF:
⟨1⟩1. LET: $W(A)$ be the set of words in the alphabet whose elements are the elements of $A$ together with $\{a^{-1} : a \in A\}$.
⟨1⟩2. LET: $r : W(A) \to W(A)$ be the function that, given a word $w$, removes the first pair of letters of the form $aa^{-1}$ or $a^{-1}a$; if there is no such pair, then $r(w) = w$.
⟨1⟩3. Let us say that a word $w$ is a *reduced word* iff $r(w) = w$.
⟨1⟩4. For any word $w$ of length $n$, we have $r^{\ulcorner \frac{n}{2} \urcorner}(w)$ is a reduced word.
PROOF: Since we cannot remove more than $n/2$ pairs of letters from $w$.
⟨1⟩5. LET: $R : W(A) \to W(A)$ be the function $R(w) = r^{\ulcorner \frac{n}{2} \urcorner}(w)$, where $n$ is the length of $w$.
⟨1⟩6. LET: $F(A)$ be the set of reduced words.
⟨1⟩7. Define $\cdot : F(A)^2 \to F(A)$ by $w \cdot w' = R(ww')$
⟨1⟩8. $\cdot$ is associative.
PROOF: Both $w_1 \cdot (w_2 \cdot w_3)$ and $(w_1 \cdot w_2) \cdot w_3$ are equal to $R(w_1 w_2 w_3)$.
⟨1⟩9. The empty word is the identity element in $F(A)$
⟨1⟩10. The inverse of $a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$ is $a_n^{\mp 1} \cdots a_2^{\mp 1} a_1^{\mp 1}$.
⟨1⟩11. LET: $j : A \to F(A)$ be the function that maps $a$ to the word $a$ of length .
⟨1⟩12. LET: $G$ be any group and $k : A \to G$ any function.
⟨1⟩13. The only morphism $f : (F(A), j) \to (G, k)$ in $\mathcal{F}^A$ is $f(a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}) = k(a_1)^{\pm 1} k(a_2)^{\pm 1} \cdots k(a_n)^{\pm 1}$.
□

**Definition 6.25** (Free Group)**.** For any set $A$, the *free group* on $A$ is the initial object $(F(A), i)$ in $\mathcal{F}^A$.

**Proposition 6.26.** *$i : A \to F(A)$ is injective.*

PROOF:
⟨1⟩1. LET: $x, y \in A$
⟨1⟩2. ASSUME: $x \neq y$
PROVE: $i(x) \neq i(y)$
⟨1⟩3. LET: $f : A \to C_2$ be the function that maps $x$ to 0 and all other elements of $A$ to 1.
⟨1⟩4. LET: $\phi : F(A) \to C_2$ be the group homomorphism such that $f = \phi \circ i$.
⟨1⟩5. $f(x) \neq f(y)$
⟨1⟩6. $\phi(i(x)) \neq \phi(i(y))$
⟨1⟩7. $i(x) \neq i(y)$
□

**Proposition 6.27.**

$$F(0) \cong \{e\}$$

PROOF: For any set $A$, the unique group homomorphism $\{e\} \to A$ makes the following diagram commute.

$$\begin{tikzcd}
\{e\} \arrow[r] & A \\
\emptyset \arrow[u] \arrow[ur] &
\end{tikzcd}$$

**Proposition 6.28.** *The free group on 1 is $\mathbb{Z}$ with the injection mapping 0 to* 1*.*

PROOF: Given any group $G$ and function $a : 1 \to G$, the required unique homomorphism $\phi : \mathbb{Z} \to G$ is defined by $\phi(n) = a(0)^n$. □

**Proposition 6.29.** *For any sets $A$ and $B$, we have that $F(A + B)$ is the coproduct of $F(A)$ and $F(B)$ in* **Grp***.*

$$\begin{tikzcd}
 & G & \\
F(A) \arrow[ur, "f"] \arrow[r, "\kappa_1"] & F(A+B) \arrow[u, "k"] & F(B) \arrow[ul, "g"'] \arrow[l, "\kappa_2"] \\
A \arrow[u, "i_A"] \arrow[r, "k_1"] & A+B \arrow[u, "j"] & B \arrow[u, "i_B"'] \arrow[l, "k_2"]
\end{tikzcd}$$

PROOF:

⟨1⟩1. LET: $i_A : A \to F(A)$, $i_B : B \to F(B)$, $j : A + B \to F(A + B)$ be the canonical injections.

⟨1⟩2. LET: $\kappa_1$, $\kappa_2$ be the unique group homomorphisms that make the diagram above commute.

⟨1⟩3. LET: $G$ be any group and $f : F(A) \to G$, $g : F(B) \to G$ any group homomorphisms.

⟨1⟩4. LET: $h : A + B \to G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.

⟨1⟩5. LET: $k : F(A + B) \to G$ be the unique group homomorphism such that $k \circ j = h$.

⟨1⟩6. $k$ is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.

⟨1⟩7. $k$ is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.

□

# Chapter 7

# Abelian Groups

**Definition 7.1** (Abelian Group)**.** A group is *Abelian* iff any two elements commute.

In an Abelian group $G$, we often denote the group operation by $+$, the identity element by $0$ and the inverse of an element $g$ by $-g$. We write $ng$ for $g^n$ ($g \in G$, $n \in \mathbb{Z}$).

**Example 7.2.** Every group of order $\leq 4$ is Abelian.

**Example 7.3.** For any positive integer $n$, we have $\mathbb{Z}/n\mathbb{Z}$ is an Abelian group under addition.

**Example 7.4.** $S_n$ is not Abelian for $n \geq 3$. If $x = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}$ then $xy = \begin{pmatrix} 2 & 3 \end{pmatrix}$ and $yx = \begin{pmatrix} 1 & 3 \end{pmatrix}$.

**Proposition 7.5.** *Let $G$ be a group. If $g^2 = e$ for all $g \in G$ then $G$ is Abelian.*

Proof: For any $g, h \in G$ we have

$$\begin{aligned} ghgh &= e \\ \therefore hgh &= g && \text{(multiplying on the left by } g\text{)} \\ \therefore hg &= gh && \text{(multiplying on the right by } h\text{)} \end{aligned}$$ □

**Proposition 7.6.** *Let $G$ be a group. Then $G$ is Abelian if and only if the function that maps $g$ to $g^{-1}$ is a group homomorphism.*

Proof:

⟨1⟩1. If $G$ is Abelian then the function that maps $g$ to $g^{-1}$ is a group homomorphism.

Proof: Since $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$.

⟨1⟩2. If the function that maps $g$ to $g^{-1}$ is a group homomorphism then $G$ is Abelian.

Proof: Since $gh = (g^{-1})^{-1}(h^{-1})^{-1} = (g^{-1}h^{-1})^{-1} = hg$.

□

**Proposition 7.7.** *Let $G$ be a group. Then $G$ is Abelian if and only if the function that maps $g$ to $g^2$ is a group homomorphism.*

PROOF:
$\langle 1\rangle 1$. If $G$ is Abelian then the function that maps $g$ to $g^2$ is a group homomorphism.
PROOF: Since $(gh)^2 = g^2h^2$.
$\langle 1\rangle 2$. If the function that maps $g$ to $g^2$ is a group homomorphism then $G$ is Abelian.
PROOF: Since we have $(gh)^2 = ghgh = g^2h^2$ and so $hg = gh$.
□

**Proposition 7.8.** *Let $G$ be a group. Then $G$ is Abelian if and only if the homomorphism $\gamma : G \to \mathrm{Aut}_{\mathbf{Grp}}(G)$ is the trivial homomorphism.*

PROOF:
$\langle 1\rangle 1$. If $G$ is Abelian then $\gamma$ is trivial.
PROOF: Since $\gamma_g(a) = gag^{-1} = a$.
$\langle 1\rangle 2$. If $\gamma$ is trivial then $G$ is Abelian.
PROOF: If $\gamma_g(a) = gag^{-1} = a$ for all $g$ and $a$ then $ga = ag$ for all $g$, $a$.
□

**Proposition 7.9.** *Let $G$ be an Abelian group. Let $g, h \in G$. If $g$ has maximal finite order in $G$, and $h$ has finite order, then $|h| \mid |g|$.*

PROOF:
$\langle 1\rangle 1$. ASSUME: for a contradiction $|h| \nmid |g|$.
$\langle 1\rangle 2$. PICK a prime $p$ such that $|g| = p^m r$, $|h| = p^n s$ where $p \nmid r$, $p \nmid s$ and $m < n$.
$\langle 1\rangle 3$. $|g^{p^m} h^s| = p^n r$
PROOF: Proposition 5.19.
$\langle 1\rangle 4$. $|g| < |g^{p^m} h^s|$
$\langle 1\rangle 5$. Q.E.D.
PROOF: This contradicts the maximality of $|g|$.
□

**Proposition 7.10.** *If $p$ is prime then $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.*

PROOF:
$\langle 1\rangle 1$. LET: $g$ be an element of maximal order in $(\mathbb{Z}/p\mathbb{Z})^*$.
$\langle 1\rangle 2$. For all $h \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $h^{|g|} = 1$.
PROOF: Proposition 7.9.
$\langle 1\rangle 3$. There are at most $|g|$ elements $x$ such that $x^{|g|} = 1$ in $\mathbb{Z}/p\mathbb{Z}$
$\langle 1\rangle 4$. $p - 1 \leq |g|$
$\langle 1\rangle 5$. $|g| = p - 1$
$\langle 1\rangle 6$. $g$ generates $(\mathbb{Z}/p\mathbb{Z})^*$.
□

**Example 7.11.** $(\mathbb{Z}/12\mathbb{Z})^*$ is not cyclic. Its elements are 1, 5, 7 and 11 with orders 1, 2, 2 and 2.

**Theorem 7.12** (Wilson's Theorem)**.** *A positive integer $p$ is prime if and only if $(p-1)! \equiv 1(\mod p)$.*

⟨1⟩1. If $p$ is prime then $(p-1)! \equiv 1(\mod p)$.
  ⟨2⟩1. Assume: $p$ is prime.
  ⟨2⟩2. $(p-1)!$ is the product of all the elements of $(\mathbb{Z}/p\mathbb{Z})^*$
  ⟨2⟩3. The only element of $(\mathbb{Z}/p\mathbb{Z})^*$ with order 2 is $-1$.
  ⟨2⟩4. $(p-1)! \equiv -1(\mod p)$
    Proof: Proposition 5.20.
⟨1⟩2. If $(p-1)! \equiv -1(\mod p)$ then $p$ is prime.
  ⟨2⟩1. Assume: (
    $(p-1)! \equiv -1(\mod p)$)
  ⟨2⟩2. Let: $d$ be a proper divisor of $p$.
    Prove: $d = 1$
  ⟨2⟩3. $d \mid (p-1)!$
  ⟨2⟩4. $d \mid 1$
    Proof: Since $d \mid p \mid (p-1)! + 1$.
  ⟨2⟩5. $d = 1$
□

**Proposition 7.13.** *If $p$ and $q$ are distinct odd primes then $(\mathbb{Z}/pq\mathbb{Z})^*$ is not cyclic.*

Proof:
⟨1⟩1. $|(\mathbb{Z}/pq\mathbb{Z})^*| = (p-1)(q-1)$
⟨1⟩2. Let: $g \in (\mathbb{Z}/pq\mathbb{Z})^*$
  Prove: $g$ does not have order $(p-1)(q-1)$
⟨1⟩3. $g^{(p-1)(q-1)/2} \equiv 1(\mod p)$
⟨1⟩4. $g^{(p-1)(q-1)/2} \equiv 1(\mod q)$
⟨1⟩5. $pq \mid g^{(p-1)(q-1)/2} - 1$
⟨1⟩6. $g^{(p-1)(q-1)/2} \equiv 1(\mod pq)$
⟨1⟩7. $|g| \mid (p-1)(q-1)/2$
□

**Proposition 7.14.** *For any prime $p$, we have $\mathrm{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$.*

Proof:
⟨1⟩1. Let: $\phi : \mathrm{Aut}_{\mathbf{Grp}}(C_p) \to (\mathbb{Z}/p\mathbb{Z})^*$ be the function $\phi(\alpha) = \alpha(1)$.
  Proof: $\alpha(1)$ has order $p$ in $C_p$ and so is coprime with $p$.
⟨1⟩2. $\phi$ is a homomorphism.
  Proof: $\phi(\alpha \circ \beta) = \alpha(\beta(1)) = \alpha(\beta(1)1) = \beta(1)\alpha(1) = \phi(\alpha)\phi(\beta)$
⟨1⟩3. $\phi$ is injective.
  Proof: If $\phi(\alpha) = \phi(\beta)$ then for any $n$ we have $\alpha(n) = n\alpha(1) = n\phi(\alpha) = n\phi(\beta) = n\beta(1) = \beta(n)$.
⟨1⟩4. $\phi$ is surjective.
  Proof: For any $r \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $r = \phi(\alpha)$ where $\alpha(n) = nr \mod p$.
⟨1⟩5. $(\mathbb{Z}/p\mathbb{Z})^* \cong C_{p-1}$
□

**Proposition 7.15.** *Given a set $A$ and an Abelian group $H$, the set $H^A$ is an Abelian group under*

$$(\phi + \psi)(a) = \phi(a) + \psi(a) \qquad (\phi, \psi \in H^A, a \in A) \ .$$

PROOF:
⟨1⟩1. $\phi + (\psi + \chi) = (\phi + \psi) + \chi$
⟨1⟩2. $\phi + \psi = \psi + \phi$
⟨1⟩3. LET: $0 : A \to H$ be the function $0(a) = 0$.
⟨1⟩4. $\phi + 0 = 0 + \phi = \phi$
⟨1⟩5. Given $\phi : A \to H$, define $-\phi : A \to H$ by $(-\phi)(a) = -(\phi(a))$.
⟨1⟩6. $\phi + (-\phi) = (-\phi) + \phi = 0$
□

**Proposition 7.16.** *Given a group $G$ and an Abelian group $H$, the set* $\mathbf{Grp}[G, H]$ *is a subgroup of $H^G$.*

PROOF:
⟨1⟩1. Given $\phi, \psi : G \to H$ group homomorphisms, we have $\phi - \psi$ is a group homomorphism.
PROOF:

$$\begin{aligned}(\phi - \psi)(g + g') &= \phi(g + g') - \psi(g + g') \\ &= \phi(g) + \phi(g') - \psi(g) - \psi(g') \\ &= \phi(g) - \psi(g) + \phi(g') - \psi(g') \\ &= (\phi - \psi)(g) + (\phi - \psi)(g')\end{aligned}$$

□

## 7.1 The Category of Abelian Groups

**Definition 7.17** (Category of Abelian Groups)**.** Let **Ab** be the full subcategory of **Grp** whose objects are the Abelian groups.

**Definition 7.18** (Direct Sum)**.** Given Abelian groups $G$ and $H$, we also call the direct product of $G$ and $H$ the *direct sum* and denote it $G \oplus H$.

**Proposition 7.19.** *Given Abelian groups $G$ and $H$, the direct sum $G \oplus H$ is the coproduct of $G$ and $H$ in* **Ab**.

PROOF:
⟨1⟩1. LET: $\kappa_1 : G \to G \oplus H$ be the group homomorphism $\kappa_1(g) = (g, e_H)$.
⟨1⟩2. LET: $\kappa_2 : H \to G \oplus H$ be the group homomorphism $\kappa_2(h) = (e_G, h)$.
⟨1⟩3. Given group homomorphism $\phi : G \to K$ and $\psi : H \to K$, define $[\phi, \psi] : G \oplus H \to K$ by $[\phi, \psi](g, h) = \phi(g) + \psi(h)$.
⟨1⟩4. $[\phi, \psi]$ is a group homomorphism.

PROOF:

$$\begin{aligned}[\phi, \psi]((g, h) + (g', h')) &= [\phi, \psi](g + g', h + h') \\ &= \phi(g + g') + \psi(h + h') \\ &= \phi(g) + \phi(g') + \psi(h) + \psi(h') \\ &= \phi(g) + \psi(h) + \phi(g') + \psi(h') \\ &= [\phi, \psi](g, h) + [\phi, \psi](g', h')\end{aligned}$$

$\langle 1 \rangle 5.$ $[\phi, \psi] \circ \kappa_1 = \phi$

PROOF:

$$\begin{aligned}[\phi, \psi](\kappa_1(g)) &= [\phi, \psi](g, e_h) \\ &= \phi(g) + \psi(e_H) \\ &= \phi(g) + e_K \\ &= \phi(g)\end{aligned}$$

$\langle 1 \rangle 6.$ $[\phi, \psi] \circ \kappa_2 = \psi$

PROOF: Similar.

$\langle 1 \rangle 7.$ If $f : G \oplus H \to K$ is a group homomorphism with $f \circ \kappa_1 = \phi$ and $f \circ \kappa_2 = \psi$ then $f = [\phi, \psi]$.

PROOF:

$$\begin{aligned}f(g, h) &= f((g, e_H) + (e_G, h)) \\ &= f(\kappa_1(g)) + f(\kappa_2(h)) \\ &= \phi(g) + \psi(h)\end{aligned}$$

□

## 7.2 Free Abelian Groups

**Proposition 7.20.** *Let $A$ be a set. Let $\mathcal{F}^A$ be the category whose objects are pairs $(G, j)$ where $G$ is an Abelian group and $j$ is a function $A \to G$, with morphisms $f : (G, j) \to (H, k)$ the group homomorphisms $f : G \to H$ such that $f \circ j = k$. Then $\mathcal{F}^A$ has an initial object.*

PROOF:

$\langle 1 \rangle 1.$ LET: $\mathbb{Z}^{\oplus A}$ be the subgroup of $\mathbb{Z}^A$ consisting of all functions $\alpha : A \to \mathbb{Z}$ such that $\alpha(a) = 0$ for only finitely many $a \in A$.

$\langle 1 \rangle 2.$ LET: $i : A \to \mathbb{Z}^{\oplus A}$ be the function such that $i(a)(b) = 1$ if $a = b$ and 0 if $a \neq b$.

$\langle 1 \rangle 3.$ LET: $G$ be any Abelian group and $j : A \to G$ any function.

$\langle 1 \rangle 4.$ The unique homomorphism $\phi : \mathbb{Z}^{\oplus A} \to G$ required is defined by $\phi(\alpha) = \sum_{a \in A} \alpha(a) j(a)$

□

**Definition 7.21** (Free Abelian Group)**.** For any set $A$, the *free Abelian group* on $A$ is the initial object $(F^{ab}(A), i)$ in $\mathcal{F}^A$.

**Proposition 7.22.** *For any sets $A$ and $B$, we have that $F^{ab}(A + B)$ is the coproduct of $F^{ab}(A)$ and $F^{ab}(B)$ in* **Grp**.

$$\begin{tikzcd}
& G & \\
F^{ab}(A) \arrow[ur, "f"] \arrow[r, "\kappa_1"] & F^{ab}(A+B) \arrow[u, "k"] & F^{ab}(B) \arrow[ul, "g"'] \arrow[l, "\kappa_2"] \\
A \arrow[u, "i_A"] \arrow[r, "k_1"] & A+B \arrow[u, "j"] & B \arrow[u, "i_B"] \arrow[l, "k_2"]
\end{tikzcd}$$

PROOF:

⟨1⟩1. LET: $i_A : A \to F^{ab}(A)$, $i_B : B \to F^{ab}(B)$, $j : A+B \to F^{ab}(A+B)$ be the canonical injections.

⟨1⟩2. LET: $\kappa_1$, $\kappa_2$ be the unique group homomorphisms that make the diagram above commute.

⟨1⟩3. LET: $G$ be any group and $f : F^{ab}(A) \to G$, $g : F^{ab}(B) \to G$ any group homomorphisms.

⟨1⟩4. LET: $h : A+B \to G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.

⟨1⟩5. LET: $k : F^{ab}(A+B) \to G$ be the unique group homomorphism such that $k \circ j = h$.

⟨1⟩6. $k$ is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.

⟨1⟩7. $k$ is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.

$\square$

**Proposition 7.23.** *For $A$ and $B$ finite sets, if $F^{\mathrm{ab}}(A) \cong F^{\mathrm{ab}}(B)$ then $A \cong B$.*

PROOF:

⟨1⟩1. For any set $C$, define $\sim$ on $F^{\mathrm{ab}}(C)$ by: $f \sim f'$ iff there exists $g \in F^{\mathrm{ab}}(C)$ such that $f - f' = 2g$.

⟨1⟩2. For any set $C$, $\sim$ is an equivalence relation on $F^{\mathrm{ab}}(C)$.

⟨1⟩3. For any set $C$, we have $F^{\mathrm{ab}}(C)/\sim$ is finite if and only if $C$ is finite, in which case $|F^{\mathrm{ab}}(C)/\sim| = 2^{|C|}$.

PROOF: There is a bijection between $F^{\mathrm{ab}}(C)/\sim$ and the finite subsets of $C$, which maps $f$ to $\{c \in C : f(c) \text{ is odd}\}$.

⟨1⟩4. If $F^{\mathrm{ab}}(A) \cong F^{\mathrm{ab}}(B)$ then $A \cong B$.

PROOF: If $|F^{\mathrm{ab}}(A)/\sim| = |F^{\mathrm{ab}}(B)/\sim|$ then $2^{|A|} = 2^{|B|}$ and so $|A| = |B|$.

$\square$

# Part III

# Linear Algebra

**Definition 7.24.** Let $\mathrm{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ real matrices.