

# Mathematics

Robin Adams

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# Chapter 1

## Primitive Terms and Axioms

### 1.1 Primitive Terms

Let there be *sets*.

For any set  $A$ , let there be *elements* of  $A$ . We write  $a \in A$  for:  $a$  is an element of  $A$ .

For any sets  $A$  and  $B$ , let there be a set  $B^A$ , whose elements are called *functions* from  $A$  to  $B$ . We write  $f : A \rightarrow B$  for  $f \in B^A$ .

For any function  $f : A \rightarrow B$  and element  $a \in A$ , let there be an element  $f(a) \in B$ , the *value* of the function  $f$  at the *argument*  $a$ .

### 1.2 Axioms

**Axiom Schema 1.2.1** (Choice). *Let  $P[X, Y, x, y]$  be a formula where  $X$  and  $Y$  are set variables,  $x : \text{El}(X)$  and  $y : \text{El}(Y)$ . Then the following is an axiom.*

*Let  $A$  and  $B$  be sets. Assume that, for all  $a : \text{El}(A)$ , there exists  $b : \text{El}(B)$  such that  $P[A, B, a, b]$ . Then there exists a function  $f : A \rightarrow B$  such that  $\forall a : \text{El}(A). P[A, B, a, f(a)]$ .*

**Axiom 1.2.2** (Extensionality). *Let  $f, g : A \rightarrow B$ . If, for all  $x \in A$ , we have  $f(x) = g(x)$ , then  $f = g$ .*

**Axiom 1.2.3** (Pairing). *For any sets  $A$  and  $B$ , there exists a set  $A \times B$ , the Cartesian product of  $A$  and  $B$ , and functions  $\pi_1 : A \times B \rightarrow A$  and  $\pi_2 : A \times B \rightarrow B$  such that, for all  $a : \text{El}(A)$  and  $b : \text{El}(B)$ , there exists a unique  $(a, b) : \text{El}(A \times B)$  such that  $\pi_1(a, b) = a$  and  $\pi_2(a, b) = b$ .*

**Definition 1.2.4** (Injective). A function  $f : A \rightarrow B$  is *injective* or an *injection* iff, for all  $x, y : \text{El}(A)$ , if  $f(x) = f(y)$  then  $x = y$ .

**Axiom Schema 1.2.5** (Separation). *For every property  $P[X, x]$  where  $X$  is a set variable and  $x : \text{El}(X)$ , the following is an axiom:*

*For every set  $A$ , there exists a set  $S = \{x : \text{El}(A) \mid P[A, x]\}$  and an injection  $i : S \rightarrow A$  such that, for all  $x : \text{El}(A)$ , we have*

$$(\exists y : S. i(y) = x) \Leftrightarrow P[A, x] .$$

**Axiom 1.2.6** (Infinity). *There exists a set  $\mathbb{N}$ , an element  $0 : \text{El}(\mathbb{N})$ , and a function  $s : \mathbb{N} \rightarrow \mathbb{N}$  such that:*

- $\forall n : \text{El}(\mathbb{N}). s(n) \neq 0$
- $\forall m, n : \text{El}(\mathbb{N}). s(m) = s(n) \Rightarrow m = n.$



## Chapter 2

# Sets and Functions

### 2.1 Surjective Functions

**Definition 2.1.1** (Surjective). A function  $f : A \rightarrow B$  is *surjective* iff, for all  $y : \text{El}(B)$ , there exists  $x : \text{El}(A)$  such that  $f(x) = y$ .

### 2.2 Bijections

**Definition 2.2.1** (Bijective). A function  $f : A \rightarrow B$  is *bijective* or a *bijection* iff it is injective and surjective.

Sets  $A$  and  $B$  are *equinumerous*,  $A \approx B$ , iff there exists a bijection between them.

If we prove there exists a set  $X$  such that  $P(X)$ , and that any two sets that satisfy  $P$  are bijective, then we may introduce a constant  $C$  and define "Let  $C$  be the set such that  $P(C)$ ".

### 2.3 Identity Function

**Definition 2.3.1** (Identity). For any set  $A$ , the *identity* function  $\text{id}_A : A \rightarrow A$  is the function defined by  $\text{id}_A(a) = a$ .

### 2.4 Composition

**Definition 2.4.1** (Composite). Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . The *composite*  $g \circ f : A \rightarrow C$  is the function such that, for all  $a \in A$ , we have

$$(g \circ f)(a) = g(f(a)) \text{ .}$$

**Proposition 2.4.2.** *Given functions  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h : C \rightarrow D$ , we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \text{ .}$$

PROOF: Each is the function that maps  $a \in A$  to  $h(g(f(a)))$ .  $\square$

**Proposition 2.4.3.** *The composite of injective functions is injective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be injective.

$\langle 1 \rangle 2$ . LET:  $x, y \in A$  satisfy  $(g \circ f)(x) = (g \circ f)(y)$

$\langle 1 \rangle 3$ .  $g(f(x)) = g(f(y))$

$\langle 1 \rangle 4$ .  $f(x) = f(y)$

$\langle 1 \rangle 5$ .  $x = y$

$\square$

**Proposition 2.4.4.** *For functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , if  $g \circ f$  is injective then  $f$  is injective.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $g \circ f$  is injective.

$\langle 1 \rangle 2$ . LET:  $x, y \in A$

$\langle 1 \rangle 3$ . ASSUME:  $f(x) = f(y)$

$\langle 1 \rangle 4$ .  $g(f(x)) = g(f(y))$

$\langle 1 \rangle 5$ .  $x = y$

$\square$

**Proposition 2.4.5.** *Let  $f : A \rightarrow B$ . Then  $f$  is injective if and only if, for every set  $X$  and functions  $x, y : X \rightarrow A$ , if  $f \circ x = f \circ y$  then  $x = y$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is injective then, for every set  $X$  and functions  $x, y : X \rightarrow A$ , if  $f \circ x = f \circ y$  then  $x = y$ .

$\langle 2 \rangle 1$ . ASSUME:  $f$  is injective.

$\langle 2 \rangle 2$ . LET:  $X$  be a set.

$\langle 2 \rangle 3$ . LET:  $x, y : X \rightarrow A$

$\langle 2 \rangle 4$ . ASSUME:  $f \circ x = f \circ y$

$\langle 2 \rangle 5$ .  $\forall t \in X. x(t) = y(t)$

$\langle 3 \rangle 1$ . LET:  $t \in X$

$\langle 3 \rangle 2$ .  $f(x(t)) = f(y(t))$

PROOF:  $\langle 2 \rangle 4$

$\langle 3 \rangle 3$ .  $x(t) = y(t)$

PROOF:  $\langle 2 \rangle 5$

$\langle 2 \rangle 6$ .  $x = y$

PROOF: Axiom of Extensionality.

$\langle 1 \rangle 2$ . If, for every set  $X$  and functions  $x, y : X \rightarrow A$ , if  $f \circ x = f \circ y$  then  $x = y$ .

PROOF: Take  $X = 1$ .

$\square$

**Proposition 2.4.6.** *The composite of surjective functions is surjective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be surjective.

- $\langle 1 \rangle 2$ . LET:  $c \in C$
- $\langle 1 \rangle 3$ . PICK  $b \in B$  such that  $g(b) = c$ .
- $\langle 1 \rangle 4$ . PICK  $a \in A$  such that  $f(a) = b$ .
- $\langle 1 \rangle 5$ .  $(g \circ f)(a) = c$

□

**Proposition 2.4.7.** *Let  $f : A \rightarrow B$ . Then the following are equivalent.*

- 1.  $f$  is surjective.
- 2. For any set  $X$  and functions  $g, h : B \rightarrow X$ , if  $g \circ f = h \circ f$  then  $g = h$ .
- 3. There exists  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

- $\langle 2 \rangle 1$ . ASSUME:  $f$  is surjective.
- $\langle 2 \rangle 2$ . LET:  $X$  be a set.
- $\langle 2 \rangle 3$ . LET:  $g, h : B \rightarrow X$
- $\langle 2 \rangle 4$ . ASSUME:  $g \circ f = h \circ f$
- $\langle 2 \rangle 5$ . LET:  $b \in B$   
PROVE:  $g(b) = h(b)$
- $\langle 2 \rangle 6$ . PICK  $a \in A$  such that  $f(a) = b$
- $\langle 2 \rangle 7$ .  $g(b) = h(b)$   
PROOF:  $g(b) = g(f(a)) = h(f(a)) = h(b)$

$\langle 1 \rangle 2$ .  $1 \Rightarrow 3$

- $\langle 2 \rangle 1$ . ASSUME:  $f$  is surjective.
- $\langle 2 \rangle 2$ . PICK  $g : B \rightarrow A$  such that, for all  $b \in B$ , we have  $f(g(b)) = b$ .  
PROOF: Axiom of Choice.
- $\langle 2 \rangle 3$ .  $f \circ g = \text{id}_B$ .

$\langle 1 \rangle 3$ .  $3 \Rightarrow 2$

- $\langle 2 \rangle 1$ . LET:  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$
- $\langle 2 \rangle 2$ . LET:  $X$  be a set.
- $\langle 2 \rangle 3$ . LET:  $h, k : B \rightarrow X$
- $\langle 2 \rangle 4$ . ASSUME:  $h \circ f = k \circ f$
- $\langle 2 \rangle 5$ .  $h = k$   
PROOF:  $h = h \circ f \circ g = k \circ f \circ g = k$

$\langle 1 \rangle 4$ .  $2 \Rightarrow 1$

- $\langle 2 \rangle 1$ . ASSUME: 2
- $\langle 2 \rangle 2$ . LET:  $b \in B$
- $\langle 2 \rangle 3$ . LET:  $h : B \rightarrow 2$  be the function that maps everything to 1.
- $\langle 2 \rangle 4$ . LET:  $k : B \rightarrow 2$  be the function that maps  $b$  to 0 and everything else to 1.
- $\langle 2 \rangle 5$ .  $h \neq k$
- $\langle 2 \rangle 6$ .  $h \circ f \neq k \circ f$
- $\langle 2 \rangle 7$ . PICK  $a \in A$  such that  $h(f(a)) \neq k(f(a))$
- $\langle 2 \rangle 8$ .  $f(a) = b$

□

**Proposition 2.4.8.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $g \circ f$  is surjective then  $g$  is surjective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $c \in C$

$\langle 1 \rangle 2$ . There exists  $a \in A$  such that  $g(f(a)) = c$ .

$\langle 1 \rangle 3$ . There exists  $b \in B$  such that  $g(b) = c$ .

□

**Proposition 2.4.9.** *The composite of bijections is a bijection.*

PROOF: Propositions 2.4.3 and 2.4.6. □

**Proposition 2.4.10.** *Let  $f : A \rightarrow B$ . Then  $f$  is bijective if and only if there exists a function  $f^{-1} : B \rightarrow A$ , the inverse of  $f$ , such that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ , in which case the inverse is unique.*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is bijective then there exists  $f^{-1} : B \rightarrow A$  such that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ .

$\langle 2 \rangle 1$ . ASSUME:  $f$  is bijective.

$\langle 2 \rangle 2$ . PICK  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$

PROOF: Proposition 2.4.7.

$\langle 2 \rangle 3$ .  $f \circ g \circ f = f$

$\langle 2 \rangle 4$ .  $g \circ f = \text{id}_A$

PROOF: Proposition 2.4.5.

$\langle 1 \rangle 2$ . If there exists  $f^{-1} : B \rightarrow A$  such that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ , then  $f$  is bijective.

$\langle 2 \rangle 1$ . LET:  $f^{-1} : B \rightarrow A$  satisfy  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$

$\langle 2 \rangle 2$ .  $f$  is injective.

PROOF: If  $f(x) = f(y)$  then  $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$ .

$\langle 2 \rangle 3$ .  $f$  is surjective.

PROOF: Proposition 2.4.7.

$\langle 1 \rangle 3$ . If  $g, h : B \rightarrow A$  satisfy  $f \circ g = \text{id}_B$  and  $g \circ f = \text{id}_A$  and  $f \circ h = \text{id}_B$  and  $h \circ f = \text{id}_A$  then  $g = h$ .

PROOF: We have  $g = g \circ f \circ h = h$ .

□

**Proposition 2.4.11.** *Let  $f : A \rightarrow B$ . Then  $\text{id}_B \circ f = f = f \circ \text{id}_A$ .*

PROOF: Each is the function that maps  $a$  to  $f(a)$ . □

**Proposition 2.4.12.**

$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps  $f$  to  $(\pi_1 \circ f, \pi_2 \circ f)$  is a bijection. □

**Proposition 2.4.13.**

$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function  $\Phi$  such that  $\Phi(f)(c)(b) = f(b, c)$  is a bijection. □

### 2.4.1 The Empty Set

**Theorem 2.4.14.** *There exists a set which has no elements.*

PROOF:

$\langle 1 \rangle 1$ . PICK a set  $A$

PROOF: By the Axiom of Infinity, a set exists.

$\langle 1 \rangle 2$ . LET:  $S = \{x : \text{El}(A) \mid \perp\}$  with injection  $i : S \rightarrow A$

PROOF: Axiom of Separation.

$\langle 1 \rangle 3$ .  $S$  has no elements.

□

**Theorem 2.4.15.** *If  $E$  and  $E'$  have no elements then  $E \approx E'$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $E$  and  $E'$  have no elements.

$\langle 1 \rangle 2$ . PICK a function  $F : E \rightarrow E'$ .

PROOF: Axiom of Choice since vacuously  $\forall x : \text{El}(E) . \exists y : \text{El}(E') . \top$ .

$\langle 1 \rangle 3$ .  $F$  is injective.

PROOF: Vacuously, for all  $x, y : \text{El}(E)$ , if  $F(x) = F(y)$  then  $x = y$ .

$\langle 1 \rangle 4$ .  $F$  is surjective.

PROOF: Vacuously, for all  $y : \text{El}(E')$ , there exists  $x : \text{El}(E)$  such that  $F(x) = y$ .

□

**Definition 2.4.16** (Empty Set). The *empty set*  $\emptyset$  is the set with no elements.

### 2.4.2 The Singleton

**Theorem 2.4.17.** *There exists a set that has exactly one element.*

PROOF:

$\langle 1 \rangle 1$ . PICK a set  $A$  that has an element.

PROOF: By the Axiom of Infinity, there exists a set that has an element.

$\langle 1 \rangle 2$ . PICK  $a : \text{El}(A)$

$\langle 1 \rangle 3$ . PICK a set  $S$  and injection  $i : S \rightarrow A$  such that, for all  $x : \text{El}(A)$ , there exists  $s : \text{El}(S)$  such that  $s = x$  if and only if  $x = a$

$\langle 1 \rangle 4$ .  $S$  has exactly one element.

□

**Theorem 2.4.18.** *If  $A$  and  $B$  both have exactly one element then  $A \approx B$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $B$  both have exactly one element  $a$  and  $b$  respectively.

$\langle 1 \rangle 2$ . LET:  $F : A \rightarrow B$  be the function such that, for all  $x : \text{El}(A)$ , we have  
 $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$ .  $F$  is a bijection.

□

**Definition 2.4.19** (Singleton). Let  $1$  be the set that has exactly one element. Let  $*$  be its element.

### 2.4.3 Subsets

**Definition 2.4.20** (Subset). A *subset* of a set  $A$  consists of a set  $S$  and an injection  $i : S \rightarrow A$ . We write  $(S, i) : \text{Sub}(A)$ .

We say two subsets  $(S, i)$  and  $(T, j)$  are *equal*,  $(S, i) = (T, j)$ , iff there exists a bijection  $\phi : S \approx T$  such that  $j \circ \phi = i$ .

**Proposition 2.4.21.** *For any subset  $(S, i)$  of  $A$  we have  $(S, i) = (S, i)$ .*

PROOF: We have  $\text{id}_S : S \approx S$  and  $i \circ \text{id}_S = i$ .

**Proposition 2.4.22.** *If  $(S, i) = (T, j)$  then  $(T, j) = (S, i)$ .*

PROOF: If  $\phi : S \approx T$  and  $j \circ \phi = i$  then  $\phi^{-1} : T \approx S$  and  $i \circ \phi^{-1} = j$ .  $\square$

**Proposition 2.4.23.** *If  $(R, i) = (S, j)$  and  $(S, j) = (T, k)$  then  $(R, i) = (T, k)$ .*

PROOF: If  $\phi : R \approx S$  and  $j \circ \phi = i$ , and  $\psi : S \approx T$  and  $k \circ \psi = j$ , then  $\psi \circ \phi : R \approx T$  and  $k \circ \psi \circ \phi = i$ .  $\square$

**Definition 2.4.24** (Membership). Given  $(S, i) : \text{Sub}(A)$  and  $a \in A$ , we write  $a \in (S, i)$  for  $\exists s : \text{El}(S) . i(s) = a$ .

**Proposition 2.4.25.** *If  $a \in (S, i)$  and  $(S, i) = (T, j)$  then  $a \in (T, j)$ .*

PROOF: If  $i(s) = a$  then  $j(\phi(s)) = a$ .  $\square$

## 2.5 Axioms Part Two

**Axiom 2.5.1** (Power Set). *For any set  $A$ , there exists a set  $\mathcal{P}A$ , the power set of  $A$ , and a relation  $\in : A \rightarrow \mathcal{P}A$ , called membership, such that, for any subset  $S$  of  $A$ , there exists a unique  $\overline{S} \in \mathcal{P}A$  such that, for all  $x \in A$ , we have  $x \in \overline{S}$  if and only if  $x \in S$ .*

*We usually write just  $S$  for  $\overline{S}$ .*

**Axiom Schema 2.5.2** (Collection). *Let  $P[X, Y, x]$  be a formula with set variables  $X$  and  $Y$  and an element variable  $x \in X$ . Then the following is an axiom.*

*For any set  $A$ , there exists a set  $B$ , a function  $p : B \rightarrow A$ , a set  $Y$  and a relation  $M : B \rightarrow Y$  such that:*

- $\forall b \in B. P[A, \{y \in Y : bMy\}, p(b)]$
- *For all  $a \in A$ , if  $\exists Y. P[A, Y, a]$ , then there exists  $b \in B$  such that  $a = p(b)$ .*

**Definition 2.5.3** (Universe). Let  $E : U \rightarrow X$  be a relation. Let us say that a set  $A$  is *small* iff there exists  $u \in U$  such that  $A \approx \{x \in X : uEx\}$ .

Then  $(U, X, E)$  form a *universe* if and only if:

- $\mathbb{N}$  is  $U$ -small.
- For any  $U$ -small sets  $A$  and  $B$  and relation  $R : A \rightarrow B$ , the tabulation of  $R$  is  $U$ -small.

- If  $A$  is  $U$ -small then so is  $\mathcal{P}A$
- Let  $f : A \rightarrow B$  be a function. If  $B$  is  $U$ -small and  $f^{-1}(b)$  is  $U$ -small for all  $b \in B$ , then  $A$  is  $U$ -small.
- If  $p : B \twoheadrightarrow A$  is a surjective function such that  $A$  is  $U$ -small, then there exists a  $U$ -small set  $C$ , a surjection  $q : C \twoheadrightarrow A$ , and a function  $f : C \rightarrow B$  such that  $q = pf$ .

**Axiom 2.5.4** (Universe). *There exists a universe.*

Let  $E : U \nrightarrow X$  be a universe. We shall say a set is *small* iff it is  $U$ -small, and *large* otherwise.

## 2.6 Cartesian Product

**Definition 2.6.1** (Cartesian Product). Let  $A$  and  $B$  be sets. The *Cartesian product* of  $A$  and  $B$ ,  $A \times B$ , is the tabulation of the relation  $A \nrightarrow B$  that holds for all  $a \in A$  and  $b \in B$ . The associated functions  $\pi_1 : A \times B \rightarrow A$  and  $\pi_2 : A \times B \rightarrow B$  are called the *projections*.

Given  $a \in A$  and  $b \in B$ , we write  $(a, b)$  for the unique element of  $A \times B$  such that  $\pi_1(a, b) = a$  and  $\pi_2(a, b) = b$ .

## 2.7 Quotient Sets

**Proposition 2.7.1.** *Let  $\sim$  be an equivalence relation on  $X$ . Then there exists a set  $X/\sim$ , the quotient set of  $X$  with respect to  $\sim$ , and a surjective function  $\pi : X \twoheadrightarrow X/\sim$ , the canonical projection, such that, for all  $x, y : \text{El}(X)$ , we have  $x \sim y$  if and only if  $\pi(x) = \pi(y)$ .*

*Further, if  $p : X \twoheadrightarrow Q$  is another quotient with respect to  $\sim$ , then there exists a unique bijection  $\phi : X/\sim \approx Q$  such that  $\phi \circ \pi = p$ .*

## 2.8 Partitions

**Definition 2.8.1** (Partition). A *partition* of a set  $X$  is a set of pairwise disjoint subsets of  $X$  whose union is  $X$ .





## Chapter 3

# Order Theory

### 3.1 Relations

**Definition 3.1.1** (Reflexive). A relation  $R \subseteq A \times A$  is *reflexive* iff, for all  $a \in A$ , we have  $(a, a) \in R$ .

**Definition 3.1.2** (Antisymmetric). A relation  $R \subseteq A \times A$  is *antisymmetric* iff, for all  $a, b \in A$ , if  $(a, b) \in R$  and  $(b, a) \in R$  then  $a = b$ .

**Definition 3.1.3** (Transitive). A relation  $R \subseteq A \times A$  is *transitive* iff, for all  $a, b, c \in A$ , if  $(a, b) \in R$  and  $(b, c) \in R$  then  $(a, c) \in R$ .

**Definition 3.1.4** (Partial Order). A *partial order* on a set  $A$  is a relation on  $A$  that is reflexive, antisymmetric and transitive.

We say  $(A, \leq)$  is a *partially ordered set* or *poset* iff  $\leq$  is a partial order on  $A$ .



## Chapter 4

# Category Theory

### 4.1 Categories

**Definition 4.1.1.** A *category*  $\mathcal{C}$  consists of:

- a set  $\text{Ob}(\mathcal{C})$  of *objects*. We write  $A \in \mathcal{C}$  for  $A \in \text{Ob}(\mathcal{C})$ .
- for any objects  $X$  and  $Y$ , a set  $\mathcal{C}[X, Y]$  of *morphisms* from  $X$  to  $Y$ . We write  $f : X \rightarrow Y$  for  $f \in \mathcal{C}[X, Y]$ .
- for any objects  $X, Y$  and  $Z$ , a function  $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \rightarrow \mathcal{C}[X, Z]$ , called *composition*.

such that:

- Given  $f : X \rightarrow Y, g : Y \rightarrow Z$  and  $h : Z \rightarrow W$ , we have  $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object  $X$ , there exists a morphism  $\text{id}_X : X \rightarrow X$ , the *identity morphism* on  $X$ , such that:
  - for any object  $Y$  and morphism  $f : Y \rightarrow X$  we have  $\text{id}_X \circ f = f$
  - for any object  $Y$  and morphism  $f : X \rightarrow Y$  we have  $f \circ \text{id}_X = f$

We write the composite of morphism  $f_1, \dots, f_n$  as  $f_n \circ \dots \circ f_1$ . This is unambiguous thanks to Associativity.

**Definition 4.1.2.** Let **Set** be the category of small sets and functions.

**Definition 4.1.3.** We identify a poset  $(A, \leq)$  with the category with:

- set of objects  $A$
- for  $a, b \in A$ , the set of homomorphisms is  $\{x \in 1 : a \leq b\}$

**Proposition 4.1.4.** A category is a poset iff, for any two objects, there exists at most one morphism between them.

**Proposition 4.1.5.** *The identity morphism on an object is unique.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C}$  be a category.

$\langle 1 \rangle 2$ . LET:  $A \in \mathcal{C}$

$\langle 1 \rangle 3$ . LET:  $i, j : A \rightarrow A$  be identity morphisms on  $A$ .

$\langle 1 \rangle 4$ .  $i = j$

PROOF:

$$\begin{aligned} i &= i \circ j && (j \text{ is an identity on } A) \\ &= j && (i \text{ is an identity on } A) \end{aligned}$$

□

**Definition 4.1.6.** Given  $f : A \rightarrow B$  and an object  $C$ , define the function  $f^* : \mathcal{C}[B, C] \rightarrow \mathcal{C}[A, C]$  by  $f^*(g) = g \circ f$ .

**Definition 4.1.7.** Given  $f : A \rightarrow B$  and an object  $C$ , define the function  $f_* : \mathcal{C}[C, A] \rightarrow \mathcal{C}[C, B]$  by  $f_*(g) = f \circ g$ .

### 4.1.1 Monomorphisms

**Definition 4.1.8** (Monomorphism). Let  $f : A \rightarrow B$ . Then  $f$  is *monic* or a *monomorphism*,  $f : A \rightarrowtail B$ , iff, for any object  $X$  and functions  $x, y : X \rightarrow A$ , if  $f \circ x = f \circ y$  then  $x = y$ .

### 4.1.2 Epimorphisms

**Definition 4.1.9** (Epimorphism). Let  $f : A \rightarrow B$ . Then  $f$  is *epic* or an *epimorphism*,  $f : A \twoheadrightarrow B$ , iff, for any object  $X$  and functions  $x, y : B \rightarrow X$ , if  $x \circ f = y \circ f$  then  $x = y$ .

### 4.1.3 Sections and Retractions

**Definition 4.1.10** (Section, Retraction). Let  $r : A \rightarrow B$  and  $s : B \rightarrow A$ . Then  $r$  is a *retraction* of  $s$ , and  $s$  is a *section* of  $r$ , iff  $rs = \text{id}_B$ .

**Proposition 4.1.11.** *Let  $f : A \rightarrow B$  and  $r, s : B \rightarrow A$ . If  $r$  is a retraction of  $f$  and  $s$  is a section of  $f$  then  $r = s$ .*

PROOF:

$$\begin{aligned} r &= r \text{id}_B && (\text{Unit Law}) \\ &= rfs && (s \text{ is a section of } f) \\ &= \text{id}_A s && (r \text{ is a retraction of } f) \\ &= s && (\text{Unit Law}) \square \end{aligned}$$

**Proposition 4.1.12.** *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $s : B \rightarrow A$  be a section of  $r : A \rightarrow B$ .

$\langle 1 \rangle 2$ . LET:  $X$  be an object and  $x, y : X \rightarrow B$

$\langle 1 \rangle 3$ . ASSUME:  $s \circ x = s \circ y$

$\langle 1 \rangle 4$ .  $x = y$

PROOF:  $x = r \circ s \circ x = r \circ s \circ y = y$ .

□

**Proposition 4.1.13.** *Every retraction is epic.*

PROOF: Dual. □

#### 4.1.4 Isomorphisms

**Definition 4.1.14** (Isomorphism). A morphism  $f : A \rightarrow B$  is an *isomorphism*,  $f : A \cong B$ , iff there exists a morphism  $f^{-1} : B \rightarrow A$  that is both a retraction and section of  $f$ .

Objects  $A$  and  $B$  are *isomorphic*,  $A \cong B$ , iff there exists an isomorphism between them.

**Proposition 4.1.15.** *The inverse of an isomorphism is unique.*

PROOF: From Proposition 4.1.11. □

**Proposition 4.1.16.** *If  $f : A \cong B$  then  $f^{-1} : B \cong A$  and  $(f^{-1})^{-1} = f$ .*

PROOF: Since  $ff^{-1} = \text{id}_B$  and  $f^{-1}f = \text{id}_A$ . □

Isomorphism.

Define the opposite category.

Slice categories

**Definition 4.1.17.** Let  $\mathcal{C}$  be a category and  $B \in \mathcal{C}$ . The category  $\mathcal{C}_B^B$  of objects *over and under*  $B$  is the category with:

- objects all triples  $(X, u, p)$  such that  $u : B \rightarrow X$  and  $p : X \rightarrow B$
- morphisms  $f : (X, u, p) \rightarrow (Y, u', p')$  all morphisms  $f : X \rightarrow Y$  such that  $fu = u'$  and  $p'f = p$ .

**Proposition 4.1.18.**

$$\mathcal{C}_B^B \cong (\mathcal{C}/B) \backslash \text{id}_B \cong (\mathcal{C} \backslash B) / \text{id}_B$$

$(B, \text{id}_B, \text{id}_B)$  is the zero object in  $\mathcal{C}_B^B$ .

#### 4.1.5 Initial Objects

**Definition 4.1.19** (Initial Object). An object  $I$  is *initial* iff, for any object  $X$ , there exists exactly one morphism  $I \rightarrow X$ .

**Proposition 4.1.20.** *The empty set is initial in **Set**.*

PROOF: For any set  $A$ , the nowhere-defined function is the unique function  $\emptyset \rightarrow A$ .  $\square$

**Proposition 4.1.21.** *If  $I$  and  $I'$  are initial objects, then there exists a unique isomorphism  $I \cong I'$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $i : I \rightarrow I'$  be the unique morphism  $I \rightarrow I'$ .

$\langle 1 \rangle 2$ . LET:  $i^{-1} : I' \rightarrow I$  be the unique morphism  $I' \rightarrow I$ .

$\langle 1 \rangle 3$ .  $ii^{-1} = \text{id}_{I'}$

PROOF: There is only one morphism  $I' \rightarrow I'$ .

$\langle 1 \rangle 4$ .  $i^{-1}i = \text{id}_I$

PROOF: There is only one morphism  $I \rightarrow I$ .

$\square$

### 4.1.6 Terminal Objects

**Definition 4.1.22** (Terminal Object). An object  $T$  is *terminal* iff, for any object  $X$ , there exists exactly one morphism  $X \rightarrow T$ .

**Proposition 4.1.23.** *1 is terminal in Set.*

PROOF: For any set  $A$ , the constant function to  $*$  is the only function  $A \rightarrow 1$ .  $\square$

**Proposition 4.1.24.** *If  $T$  and  $T'$  are terminal objects, then there exists a unique isomorphism  $T \cong T'$ .*

PROOF: Dual to Proposition 4.1.21.  $\square$

### 4.1.7 Zero Objects

**Definition 4.1.25** (Zero Object). An object  $Z$  is a *zero object* iff it is an initial object and a terminal object.

**Definition 4.1.26** (Zero Morphism). Let  $\mathcal{C}$  be a category with a zero object  $Z$ . Let  $A, B \in \mathcal{C}$ . The *zero morphism*  $A \rightarrow B$  is the unique morphism  $A \rightarrow Z \rightarrow B$ .

**Proposition 4.1.27.** *There is no zero object in Set.*

PROOF: Since  $\emptyset \not\approx 1$ .  $\square$

### 4.1.8 Triads

**Definition 4.1.28** (Triad). Let  $\mathcal{C}$  be a category. A *triad* consists of objects  $X, Y, M$  and morphisms  $\alpha : X \rightarrow M, \beta : Y \rightarrow M$ . We call  $M$  the *codomain* of the triad.

### 4.1.9 Cotriads

**Definition 4.1.29** (Cotriad). Let  $\mathcal{C}$  be a category. A *cotriad* consists of objects  $X, Y, W$  and morphisms  $\xi : W \rightarrow X, \eta : W \rightarrow Y$ . We call  $W$  the *domain* of the triad.

## 4.1.10 Pullbacks

**Definition 4.1.30** (Pullback). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

is a *pullback* iff  $\alpha\xi = \beta\eta$  and, for every object  $Z$  and morphism  $f : Z \rightarrow X$  and  $g : Z \rightarrow Y$  such that  $\alpha f = \beta g$ , there exists a unique  $h : Z \rightarrow W$  such that  $\xi h = f$  and  $\eta h = g$ .

In this case we also say that  $\eta$  is the *pullback* of  $\beta$  along  $\alpha$ .

**Proposition 4.1.31.** *If  $\xi : W \rightarrow X$  and  $\eta : W \rightarrow Y$  form a pullback of  $\alpha : X \rightarrow M$  and  $\beta : Y \rightarrow M$ , and  $\xi' : W' \rightarrow X$  and  $\eta' : W' \rightarrow Y$  also form the pullback of  $\alpha$  and  $\beta$ , then there exists a unique isomorphism  $\phi : W \cong W'$  such that  $\eta'\phi = \eta$  and  $\xi'\phi = \xi$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : W \rightarrow W'$  be the unique morphism such that  $\eta'\phi = \eta$  and  $\xi'\phi = \xi$ .

$\langle 1 \rangle 2$ . LET:  $\phi^{-1} : W' \rightarrow W$  be the unique morphism such that  $\eta\phi^{-1} = \eta'$  and  $\xi\phi^{-1} = \xi'$ .

$\langle 1 \rangle 3$ .  $\phi\phi^{-1} = \text{id}_{W'}$

PROOF: Each is the unique  $x : W' \rightarrow W'$  such that  $\eta'x = \eta'$  and  $\xi'x = \xi'$ .

$\langle 1 \rangle 4$ .  $\phi^{-1}\phi = \text{id}_W$

PROOF: Each is the unique  $x : W \rightarrow W$  such that  $\eta x = \eta$  and  $\xi x = \xi$ .

□

**Proposition 4.1.32.** *For any morphism  $h : A \rightarrow B$ , the following diagram is a pullback diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $Z$  be an object.

$\langle 1 \rangle 2$ . LET:  $f : Z \rightarrow B$  and  $g : Z \rightarrow A$  satisfy  $\text{id}_B f = hg$

$\langle 1 \rangle 3$ .  $g : Z \rightarrow A$  is the unique morphism such that  $\text{id}_A g = g$  and  $hg = f$ .

□

**Proposition 4.1.33.** *The pullback of an isomorphism is an isomorphism.*

PROOF:

⟨1⟩1. LET:

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback diagram.

⟨1⟩2. ASSUME:  $\beta$  is an isomorphism.

⟨1⟩3. LET:  $\xi^{-1}$  be the unique morphism  $X \rightarrow W$  such that  $\xi\xi^{-1} = \text{id}_X$  and  $\eta\xi^{-1} = \beta^{-1}\alpha$ .

PROOF: This exists since  $\alpha\text{id}_X = \beta\beta^{-1}\alpha = \alpha$ .

⟨1⟩4.  $\xi^{-1}\xi = \text{id}_W$

PROOF: Each is the unique  $x : W \rightarrow W$  such that  $\xi x = \xi$  and  $\eta x = \eta$ .

□

**Proposition 4.1.34.** *Let  $\beta : (Y, y) \rightarrow (M, m)$  and  $\alpha : (X, x) \rightarrow (M, m)$  in  $\mathcal{C} \setminus A$ . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

*be a pullback in  $\mathcal{C}$ . Let  $w : A \rightarrow W$  be the unique morphism such that  $\xi w = x$  and  $\eta w = y$ . Then  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  is the pullback of  $\beta$  and  $\alpha$  in  $\mathcal{C} \setminus A$ .*

PROOF:

⟨1⟩1. LET:  $(Z, z) \in \mathcal{C} \setminus A$

⟨1⟩2. LET:  $f : (Z, z) \rightarrow (X, x)$  and  $g : (Z, z) \rightarrow (Y, y)$  satisfy  $\alpha f = \beta g$ .

⟨1⟩3. LET:  $h : Z \rightarrow W$  be the unique morphism such that  $\xi h = f$  and  $\eta h = g$ .

⟨1⟩4.  $hz = w$

⟨2⟩1.  $\xi hz = \xi w$

PROOF:

$$\begin{aligned} \xi hz &= fz & (\langle 1 \rangle 3) \\ &= x & (\langle 1 \rangle 2) \\ &= \xi w \end{aligned}$$

⟨2⟩2.  $\eta hz = \eta w$

PROOF: Similar.

⟨1⟩5.  $h : (Z, z) \rightarrow (W, w)$

□

**Proposition 4.1.35.** *Let  $\beta : (Y, y) \rightarrow (M, m)$  and  $\alpha : (X, x) \rightarrow (M, m)$  in*



$\mathcal{C}/A$ . Let

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in  $\mathcal{C}$ . Let  $w = x\xi : W \rightarrow A$ . Then  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  form a pullback of  $\alpha$  and  $\beta$  in  $\mathcal{C}/A$ .

PROOF:

$\langle 1 \rangle 1$ .  $\eta : (W, w) \rightarrow (Y, y)$

PROOF:

$$\begin{aligned} y\eta &= m\beta\eta \\ &= m\alpha\xi \\ &= x\xi \\ &= w \end{aligned}$$

$\langle 1 \rangle 2$ . LET:  $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 3$ . LET:  $f : (Z, z) \rightarrow (X, x)$  and  $g : (Z, z) \rightarrow (Y, y)$  satisfy  $\alpha f = \beta g$ .

$\langle 1 \rangle 4$ . LET:  $h : Z \rightarrow W$  be the unique morphism such that  $\xi h = f$  and  $\eta h = g$ .

$\langle 1 \rangle 5$ .  $h : (Z, z) \rightarrow (W, w)$

PROOF:

$$\begin{aligned} wh &= x\xi h \\ &= xf && (\langle 1 \rangle 4) \\ &= z && (\langle 1 \rangle 3) \end{aligned}$$

□

**Proposition 4.1.36.** In **Set**, let  $\alpha : X \rightarrow M$  and  $\beta : Y \rightarrow M$ . Let  $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$  with inclusion  $i : W \rightarrow X \times Y$ . Let  $\xi = \pi_1 i : W \rightarrow X$  and  $\eta = \pi_2 i : W \rightarrow Y$ . Then  $\xi$  and  $\eta$  form the pullback of  $\alpha$  and  $\beta$ .

PROOF:

$\langle 1 \rangle 1$ .  $\alpha\xi = \beta\eta$

PROOF: For  $w \in W$ , if  $i(w) = (x, y)$  then  $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$ .

$\langle 1 \rangle 2$ . For every set  $Z$  and functions  $f : Z \rightarrow X$ ,  $g : Z \rightarrow Y$  such that  $\alpha f = \beta g$ , there exists a unique  $h : Z \rightarrow W$  such that  $\xi h = f$  and  $\eta h = g$

PROOF: For  $z \in Z$ , let  $h(z)$  be the unique element of  $W$  such that  $i(h(z)) = (f(z), g(z))$ .

□

Pullback lemma

### 4.1.11 Pushouts

**Definition 4.1.37** (Pushout). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array} \quad (4.1)$$

is a *pushout* iff  $\alpha\xi = \beta\eta$  and, for every object  $Z$  and morphism  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  such that  $f\xi = g\eta$ , there exists a unique  $h : M \rightarrow Z$  such that  $h\alpha = f$  and  $h\beta = g$ .

We also say that  $\beta$  is the *pushout* of  $\xi$  along  $\eta$ .

**Proposition 4.1.38.** *If  $\alpha : X \rightarrow M$  and  $\beta : Y \rightarrow M$  form a pushout of  $\xi : W \rightarrow X$  and  $\eta : W \rightarrow Y$ , and  $\alpha' : X \rightarrow M'$  and  $\beta' : Y \rightarrow M'$  also form a pushout of  $\xi$  and  $\eta$ , then there exists a unique isomorphism  $\phi : M \cong M'$  such that  $\phi\alpha = \alpha'$  and  $\phi\beta = \beta'$ .*

PROOF: Dual to Proposition 4.1.31.  $\square$

**Proposition 4.1.39.** *For any morphism  $h : A \rightarrow B$ , the following diagram is a pushout diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF: Dual to Proposition 4.1.32.

**Proposition 4.1.40.** *The diagram (4.1) is a pushout in  $\mathcal{C}$  iff it is a pullback in  $\mathcal{C}^{\text{op}}$ .*

PROOF: Immediate from definitions.  $\square$

**Proposition 4.1.41.** *The pushout of an isomorphism is an isomorphism.*

PROOF: Dual to Proposition 4.1.33.  $\square$

**Proposition 4.1.42.** *Let  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  in  $\mathcal{C} \setminus A$ . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

*be a pushout in  $\mathcal{C}$ . Let  $m := \alpha x : A \rightarrow M$ . Then  $\alpha : (X, x) \rightarrow (M, m)$  and  $\beta : (Y, y) \rightarrow (M, m)$  is the pushout of  $\xi$  and  $\eta$  in  $\mathcal{C} \setminus A$ .*

PROOF: Dual to Proposition 4.1.35.  $\square$

**Proposition 4.1.43.** *Let  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  in  $\mathcal{C}/A$ . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

*be a pushout in  $\mathcal{C}$ . Let  $m : M \rightarrow A$  be the unique morphism such that  $m\alpha = x$  and  $m\beta = y$ . Then  $\alpha : (X, x) \rightarrow (M, m)$  and  $\beta : (Y, y) \rightarrow (M, m)$  is the pushout of  $\xi$  and  $\eta$  in  $\mathcal{C}/A$ .*

PROOF: Dual to Proposition 4.1.34.  $\square$

**Proposition 4.1.44.** *Set has pushouts.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\xi : W \rightarrow X$  and  $\eta : W \rightarrow Y$ .

$\langle 1 \rangle 2$ . LET:  $\sim$  be the equivalence relation on  $X + Y$  generated by  $\xi(w) \sim \eta(w)$  for all  $w \in W$

$\langle 1 \rangle 3$ . LET:  $M = (X + Y)/\sim$  with canonical projection  $\pi : X + Y \twoheadrightarrow M$ .

$\langle 1 \rangle 4$ . LET:  $\alpha = \pi \circ \kappa_1 : X \rightarrow M$

$\langle 1 \rangle 5$ . LET:  $\beta = \pi \circ \kappa_2 : Y \rightarrow M$

$\langle 1 \rangle 6$ . LET:  $Z$  be any set,  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$ .

$\langle 1 \rangle 7$ . ASSUME:  $f\xi = g\eta$

$\langle 1 \rangle 8$ . LET:  $h : X + Y \rightarrow Z$  be the function defined by  $h(x) = f(x)$  and  $h(y) = g(y)$  for  $x \in X$  and  $y \in Y$

$\langle 1 \rangle 9$ .  $h$  respects  $\sim$

PROOF: For  $w \in W$  we have

$$\begin{aligned} h(\xi(w)) &= f(\xi(w)) && (\langle 1 \rangle 8) \\ &= g(\eta(w)) && (\langle 1 \rangle 7) \\ &= h(\eta(w)) && (\langle 1 \rangle 8) \end{aligned}$$

$\langle 1 \rangle 10$ . LET:  $\bar{h} : M \rightarrow Z$  be the induced function.

$\langle 1 \rangle 11$ .  $\bar{h}\alpha = f$

PROOF:

$$\begin{aligned} \bar{h}(\alpha(x)) &= \bar{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x) \end{aligned}$$

$\langle 1 \rangle 12$ .  $\bar{h}\beta = g$

PROOF: Similar.

$\langle 1 \rangle 13$ . For all  $k : M \rightarrow Z$ , if  $k\alpha = f$  and  $k\beta = g$  then  $k = \bar{h}$ .

PROOF:

$$\begin{aligned}
 k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\
 &= f(x) \\
 k(\pi(\kappa_2(y))) &= k(\beta(y)) \\
 &= g(y) \\
 \therefore k \circ \pi &= h \\
 \therefore k &= \bar{h}
 \end{aligned}$$

□

**Definition 4.1.45.** Let  $u : A \rightarrow X$  be an injection. The *pointed set obtained from  $X$  by collapsing  $(A, u)$* , denoted  $X/(A, u)$ , is the pushout

$$\begin{array}{ccc}
 A & \longrightarrow & 1 \\
 \downarrow u & & \downarrow * \\
 X & \longrightarrow & X/(A, u)
 \end{array}$$

**Proposition 4.1.46.** In  $\mathbf{Set}_*$ , any two morphisms  $1 \rightarrow X$  and  $1 \rightarrow Y$  have a pushout.

PROOF: The pushout of  $a : (1, *) \rightarrow (X, x)$  and  $b : (1, *) \rightarrow (Y, y)$  is  $(X+Y/\sim, x)$  where  $\sim$  is the equivalence relation generated by  $x \sim y$ . □

**Definition 4.1.47** (Wedge). The *wedge* of pointed sets  $X$  and  $Y$ ,  $X \vee Y$ , is the pushout of the unique morphism  $1 \rightarrow X$  and  $1 \rightarrow Y$ .

**Definition 4.1.48** (Smash). Let  $X$  and  $Y$  be pointed sets. Let  $\xi : X \vee Y \rightarrow X$  be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc}
 1 & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 Y & \longrightarrow & X \vee Y \\
 & \searrow \xi & \nearrow \\
 & & X
 \end{array}$$

(Note: In the original image, there is an additional arrow labeled  $0$  from  $Y$  to  $X$ .)

Let  $\eta : X \vee Y \rightarrow Y$  be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc}
 1 & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 Y & \longrightarrow & X \vee Y \\
 & \searrow \eta & \nearrow \\
 & & Y
 \end{array}$$

(Note: In the original image, there is an additional arrow labeled  $0$  from  $X$  to  $Y$ .)

Let  $\zeta = \langle \xi, \eta \rangle : X \vee Y \rightarrow X \times Y$ . The *smash* of  $X$  and  $Y$ ,  $X \wedge Y$ , is the result of collapsing  $X \times Y$  with respect to  $\zeta$ .

Pushout lemma

#### 4.1.12 Subcategories

**Definition 4.1.49** (Subcategory). A *subcategory*  $\mathcal{C}'$  of a category  $\mathcal{C}$  consists of:

- a subset  $\text{Ob}(\mathcal{C}')$  of  $\mathcal{C}$
- for all  $A, B \in \text{Ob}(\mathcal{C}')$ , a subset  $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all  $A \in \text{Ob}(\mathcal{C}')$ , we have  $\text{id}_A \in \mathcal{C}'[A, A]$
- for all  $f \in \mathcal{C}'[A, B]$  and  $g \in \mathcal{C}'[B, C]$ , we have  $g \circ f \in \mathcal{C}'[A, C]$ .

It is a *full* subcategory iff, for all  $A, B \in \text{Ob}(\mathcal{C}')$ , we have  $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$ .

#### 4.1.13 Opposite Category

**Definition 4.1.50** (Opposite Category). For any category  $\mathcal{C}$ , the *opposite* category  $\mathcal{C}^{\text{op}}$  is the category with

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- $\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$
- Given  $f \in \mathcal{C}^{\text{op}}[A, B]$  and  $g \in \mathcal{C}^{\text{op}}[B, C]$ , their composite in  $\mathcal{C}^{\text{op}}$  is  $f \circ g$ , where  $\circ$  is composition in  $\mathcal{C}$ .

**Proposition 4.1.51.** *An object is initial in  $\mathcal{C}$  iff it is terminal in  $\mathcal{C}^{\text{op}}$ .*

PROOF: Immediate from definitions.  $\square$

**Proposition 4.1.52.** *An object is terminal in  $\mathcal{C}$  iff it is initial in  $\mathcal{C}^{\text{op}}$ .*

PROOF: Immediate from definitions.  $\square$

**Corollary 4.1.52.1.** *If  $T$  and  $T'$  are terminal objects in  $\mathcal{C}$  then there exists a unique isomorphism  $T \cong T'$ .*

#### 4.1.14 Groupoids

**Definition 4.1.53** (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

### 4.1.15 Concrete Categories

**Definition 4.1.54** (Concrete Category). A *concrete category*  $\mathcal{C}$  consists of:

- a set  $\text{Ob}(\mathcal{C})$  of *objects*
- for any object  $A \in \text{Ob}(\mathcal{C})$ , a set  $|A|$
- for any objects  $A, B \in \text{Ob}(\mathcal{C})$ , a set of functions  $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any  $f \in \mathcal{C}[A, B]$  and  $g \in \mathcal{C}[B, C]$ , we have  $g \circ f \in \mathcal{C}[A, C]$
- for any object  $A$  we have  $\text{id}_{|A|} \in \mathcal{C}[A, A]$ .

### 4.1.16 Power of Categories

**Definition 4.1.55.** Let  $\mathcal{C}$  be a category and  $J$  a set. The category  $\mathcal{C}^J$  is the category with:

- objects all  $J$ -indexed families of objects of  $\mathcal{C}$
- morphisms  $\{X_j\}_{j \in J} \rightarrow \{Y_j\}_{j \in J}$  all families  $\{f_j\}_{j \in J}$  where  $f_j : X_j \rightarrow Y_j$

### 4.1.17 Arrow Category

**Definition 4.1.56** (Arrow Category). Let  $\mathcal{C}$  be a category. The *arrow category*  $\mathcal{C}^\rightarrow$  is the category with:

- objects all triples  $(A, B, f)$  where  $f : A \rightarrow B$  in  $\mathcal{C}$
- morphisms  $(A, B, f) \rightarrow (C, D, g)$  all pairs  $(u : A \rightarrow C, v : B \rightarrow D)$  such that  $vf = gu$ .

### 4.1.18 Slice Category

**Definition 4.1.57** (Slice Category). Let  $\mathcal{C}$  be a category and  $A \in \mathcal{C}$ . The *slice category under A*,  $\mathcal{C} \backslash A$ , is the category with:

- objects all pairs  $(B, f)$  where  $B \in \mathcal{C}$  and  $f : A \rightarrow B$
- morphisms  $(B, f) \rightarrow (C, g)$  are morphisms  $u : B \rightarrow C$  such that  $uf = g$ .

We identify this with the subcategory of  $\mathcal{C}^\rightarrow$  formed by mapping  $(B, f)$  to  $(A, B, f)$  and  $u$  to  $(\text{id}_A, u)$ .

**Proposition 4.1.58.** If  $s : (B, f) \rightarrow (C, g)$  in  $\mathcal{C} \backslash A$ , then any retraction of  $s$  in  $\mathcal{C}$  is a retraction of  $s$  in  $\mathcal{C} \backslash A$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $r : C \rightarrow B$  be a retraction of  $s$  in  $\mathcal{C}$ .

$\langle 1 \rangle 2. rg = f$

PROOF:  $rg = rsf = f$ .

$\langle 1 \rangle 3. r : (C, g) \rightarrow (B, f)$  in  $\mathcal{C} \backslash A$

$\langle 1 \rangle 4. rs = \text{id}_{(B, f)}$

PROOF: Because composition is inherited from  $\mathcal{C}$ .

□

**Proposition 4.1.59.**  $\text{id}_A$  is the initial object in  $\mathcal{C} \backslash A$ .

PROOF: For any  $(B, f) \in \mathcal{C} \backslash A$ , we have  $f$  is the only morphism  $A \rightarrow B$  such that  $f \text{id}_A = f$ . □

**Proposition 4.1.60.** If  $A$  is terminal in  $\mathcal{C}$  then  $\text{id}_A$  is the zero object in  $\mathcal{C} \backslash A$ .

PROOF: For any  $(B, f) \in \mathcal{C} \backslash A$ , the unique morphism  $! : B \rightarrow A$  is the unique morphism such that  $!f = \text{id}_A$ . □

**Definition 4.1.61** (Pointed Sets). The category of pointed sets is **Set** $\backslash 1$ .

**Definition 4.1.62.** Let  $\mathcal{C}$  be a category and  $A \in \mathcal{C}$ . The slice category over  $A$ ,  $\mathcal{C}/A$ , is the category with:

- objects all pairs  $(B, f)$  with  $f : B \rightarrow A$
- morphisms  $u : (B, f) \rightarrow (C, g)$  all morphisms  $u : B \rightarrow C$  such that  $gu = f$ .

**Proposition 4.1.63.** Let  $u : (B, f) \rightarrow (C, g) : \mathcal{C}/A$ . Any section of  $u$  in  $\mathcal{C}$  is a section of  $u$  in  $\mathcal{C}/A$ .

PROOF: Dual to Proposition 4.1.58. □

**Proposition 4.1.64.**  $\text{id}_A$  is terminal in  $\mathcal{C}/A$ .

PROOF: Dual to Proposition 4.1.59. □

**Proposition 4.1.65.** If  $A$  is initial in  $\mathcal{C}$  then  $\text{id}_A$  is the zero object in  $\mathcal{C}/A$ .

PROOF: Dual to Proposition 4.1.60. □

**Definition 4.1.66.** Let  $A \in \mathcal{C}$ . The category of objects over and under  $A$ , written  $\mathcal{C}_A^A$ , is the category with:

- objects all triples  $(X, u, p)$  where  $u : A \rightarrow X$ ,  $p : X \rightarrow A$  and  $pu = \text{id}_A$
- morphism  $f : (X, u, p) \rightarrow (Y, v, q)$  all morphisms  $f : X \rightarrow Y$  such that  $fu = v$  and  $qf = p$

**Proposition 4.1.67.**  $(A, \text{id}_A, \text{id}_A)$  is the zero object in  $\mathcal{C}_A^A$ .

PROOF: For any object  $(X, u, p)$ , we have  $p$  is the unique morphism  $(X, u, p) \rightarrow (A, \text{id}_A, \text{id}_A)$ , and  $u$  is the unique morphism  $(A, \text{id}_A, \text{id}_A) \rightarrow (X, u, p)$ . □

**Definition 4.1.68** (Fibre Collapsing). Let  $B$  be a set. Let  $u : (A, a) \rightarrow (X, x)$  in  $\mathbf{Set}/B$ . Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow u & & \downarrow j \\ X & \xrightarrow{i} & C \end{array}$$

Let  $c : C \rightarrow B$  be the unique morphism such that  $cj = \text{id}_B$  and  $ci = x$ . Then  $(C, j, c) \in \mathbf{Set}_B^B$  is called the set over and under  $B$  obtained from  $X$  by *fibre collapsing* with respect to  $u$ . If  $(A, u)$  is a subset of  $X$ , we denote this set over and under  $B$  by  $X/_B(A, u)$ .

**Definition 4.1.69** (Fibre Wedge). Let  $B$  be a small set. Let  $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$ . The *fibre wedge* of  $X$  and  $Y$  is the pushout of  $u_X$  and  $u_Y$ :

$$\begin{array}{ccc} B & \xrightarrow{u_X} & X \\ \downarrow u_Y & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \end{array}$$

**Definition 4.1.70** (Fibre Smash). Let  $X, Y \in \mathbf{Set}_B^B$ . Let  $\xi : X \vee_B Y \rightarrow X$  be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow 0 & \nearrow \xi \\ & & X \end{array}$$

Let  $\eta : X \vee_B Y \rightarrow Y$  be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \eta & \nearrow 0 \\ & & Y \end{array}$$

Let  $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \rightarrow X \times Y$ . The *fibre smash* of  $X$  and  $Y$ ,  $X \wedge_B Y$ , is the result of collapsing  $X \times Y$  with respect to  $\zeta$ .

**Proposition 4.1.71.** *Set has products and coproducts.*



**Proposition 4.1.72.** *Let  $\mathcal{C}$  be a category. Let  $\{X_\alpha\}_{\alpha \in I}$  be a family of objects in  $\mathcal{C}$  and  $Z \in \mathcal{C}$ . Let  $\coprod_{\alpha \in I} X_\alpha$  be the coproduct of  $\{X_\alpha\}_{\alpha \in I}$ . Then*

$$\mathcal{C}[\coprod_{\alpha \in I} X_\alpha, Z] \approx \prod_{\alpha \in I} \mathcal{C}[X_\alpha, Z] .$$

**Proposition 4.1.73.** *Let  $\mathcal{C}$  be a category. Let  $\{X_\alpha\}_{\alpha \in I}$  be a family of objects in  $\mathcal{C}$  and  $Z \in \mathcal{C}$ . Let  $\prod_{\alpha \in I} X_\alpha$  be the product of  $\{X_\alpha\}_{\alpha \in I}$ . Then*

$$\mathcal{C}[Z, \prod_{\alpha \in I} X_\alpha] \approx \prod_{\alpha \in I} \mathcal{C}[Z, X_\alpha] .$$

**Proposition 4.1.74.** *A product in  $\mathcal{C}$  constitutes a product in  $\mathcal{C}/A$ .*

**Proposition 4.1.75.** *A coproduct in  $\mathcal{C}$  constitutes a product in  $\mathcal{C}/A$ .*

## 4.2 Functors

**Definition 4.2.1** (Functor). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of:

- a function  $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism  $f : A \rightarrow B$  in  $\mathcal{C}$ , a morphism  $Ff : FA \rightarrow FB$  in  $\mathcal{D}$

such that:

- for all  $A : \text{El}(\text{Ob}(\mathcal{C}))$  we have  $F\text{id}_A = \text{id}_{FA}$
- for any morphism  $f : A \rightarrow B$  and  $g : B \rightarrow C$  in  $\mathcal{C}$ , we have  $F(g \circ f) = Fg \circ Ff$

**Proposition 4.2.2.** *Functors preserve isomorphisms.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

$\langle 1 \rangle 2$ . LET:  $f : A \cong B$  in  $\mathcal{C}$

$\langle 1 \rangle 3$ .  $Ff^{-1} \circ Ff = \text{id}_{FA}$

PROOF:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\text{id}_A \\ &= \text{id}_{FA} \end{aligned}$$

$\langle 1 \rangle 4$ .  $Ff \circ Ff^{-1} = \text{id}_{FB}$

PROOF:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

**Definition 4.2.3** (Identity Functor). For any category  $\mathcal{C}$ , the *identity* functor on  $\mathcal{C}$  is the functor  $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  defined by

$$\begin{aligned} I_{\mathcal{C}}A &:= A & (A \in \mathcal{C}) \\ I_{\mathcal{C}}f &:= f & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

**Proposition 4.2.4.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$ . If  $r : A \rightarrow B$  is a retraction of  $s : B \rightarrow A$  in  $\mathcal{C}$  then  $Fr$  is a retraction of  $Fs$ .

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned} \quad \square$$

**Corollary 4.2.4.1.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$ . If  $\phi : A \cong B$  is an isomorphism in  $\mathcal{C}$  then  $F\phi : FA \cong FB$  is an isomorphism in  $\mathcal{D}$  with  $(F\phi)^{-1} = F\phi^{-1}$ .

**Definition 4.2.5** (Composition of Functors). Given functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$ , the *composite* functor  $GF : \mathcal{C} \rightarrow \mathcal{E}$  is defined by

$$\begin{aligned} (GF)A &= G(FA) & (A \in \mathcal{C}) \\ (GF)f &= G(Ff) & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

**Definition 4.2.6** (Category of Categories). Let **Cat** be the category of small categories and functors.

**Definition 4.2.7** (Isomorphism of Categories). Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Then  $F$  is an *isomorphism of categories* iff there exists a functor  $F^{-1} : \mathcal{D} \rightarrow \mathcal{C}$ , the *inverse* of  $F$ , such that  $FF^{-1} = I_{\mathcal{D}}$  and  $F^{-1}F = I_{\mathcal{C}}$ .

Categories  $\mathcal{C}$  and  $\mathcal{D}$  are *isomorphic*,  $\mathcal{C} \cong \mathcal{D}$ , iff there exists an isomorphism between them.

**Proposition 4.2.8.** If  $A$  is initial in  $\mathcal{C}$  then  $\mathcal{C} \setminus A \cong \mathcal{C}$ .

PROOF:

$\langle 1 \rangle 1$ . Define  $F : \mathcal{C} \setminus A \rightarrow \mathcal{C}$  by

$$F(B, f) = B$$

$$F(u : (B, f) \rightarrow (C, g)) = u$$

$\langle 1 \rangle 2$ . Define  $G : \mathcal{C} \rightarrow \mathcal{C} \setminus A$  by

$$GB = (B, !_B)$$

where  $!_B$  is the unique morphism  $A \rightarrow B$

$$G(u : B \rightarrow C) = u : (B, !_B) \rightarrow (C, !_C)$$

$\langle 1 \rangle 3$ .  $FG = \text{id}_{\mathcal{C}}$

$\langle 1 \rangle 4$ .  $GF = \text{id}_{\mathcal{C} \setminus A}$

PROOF: Since  $GF(B, f) = (B, !_B) = (B, f)$  because the morphism  $A \rightarrow B$  is unique.

$\square$

**Proposition 4.2.9.** If  $A$  is terminal in  $\mathcal{C}$  then  $\mathcal{C}/A \cong \mathcal{C}$ .

PROOF: Dual.  $\square$

**Proposition 4.2.10.**

$$\mathcal{C}_A^A \cong (\mathcal{C}/A) \backslash (A, \text{id}_A) \cong (\mathcal{C} \backslash A) / (A, \text{id}_A)$$

PROOF:

$\langle 1 \rangle 1$ . Define a functor  $F : \mathcal{C}_A^A \rightarrow (\mathcal{C}/A) \backslash (A, \text{id}_A)$ .

$\langle 2 \rangle 1$ . Given  $A \xrightarrow{u} X \xrightarrow{p} A$  in  $\mathcal{C}_A^A$ , let  $F(X, u, p) = ((X, p), u)$

$\langle 2 \rangle 2$ . Given  $f : (A \xrightarrow{u} X \xrightarrow{p} A) \rightarrow (A \xrightarrow{v} Y \xrightarrow{q} A)$ , let  $Ff = f$ .

$\langle 1 \rangle 2$ . Define a functor  $G : (\mathcal{C}/A) \backslash (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$ .

$\langle 1 \rangle 3$ . Define a functor  $H : \mathcal{C}_A^A \rightarrow (\mathcal{C} \backslash A) / (A, \text{id}_A)$ .

$\langle 1 \rangle 4$ . Define a functor  $K : (\mathcal{C} \backslash A) / (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$ .

$\square$

**Definition 4.2.11** (Forgetful Functor). For any concrete category  $\mathcal{C}$ , define the *forgetful* functor  $U : \mathcal{C} \rightarrow \mathbf{Set}$  by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

**Definition 4.2.12** (Switching Functor). For any category  $\mathcal{C}$ , define the *switching* functor  $T : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$  by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

**Definition 4.2.13** (Reduction). Let  $\Phi : \mathbf{Set} \rightarrow \mathbf{Set}$  be a functor. The *reduction* of  $\Phi$  is the functor  $\Phi^* : \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  defined by:  $\Phi^*(X, a)$  is the collapse of  $\Phi(X)$  with respect to  $\Phi(a) : \Phi(1) \rightarrow \Phi(X)$ .

**Definition 4.2.14.** Extend the wedge  $\vee$  to a functor  $\mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  by defining, given  $f : X \rightarrow X'$  and  $g : Y \rightarrow Y'$ , there  $f \vee g$  is the unique morphism that makes the following diagram commute.

$$\begin{array}{ccccc} 1 & \longrightarrow & X & & \\ \downarrow & & \downarrow & \searrow f & \\ Y & \longrightarrow & X \vee Y & & X' \\ & \searrow g & \downarrow f \vee g & \downarrow & \\ & & Y' & \longrightarrow & X' \vee Y' \end{array}$$

**Definition 4.2.15.** Extend smash to a functor  $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  as follows. Given  $f : X \rightarrow X'$  and  $g : Y \rightarrow Y'$ , let  $f \wedge g : X \wedge Y \rightarrow X' \wedge Y'$  be the

unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc}
 X \vee Y & \xrightarrow{\quad} & 1 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 X \times Y & \xrightarrow{\quad} & X \wedge Y & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & X' \vee Y' & \xrightarrow{\quad} & 1 \\
 f \times g \swarrow & & \downarrow & & \downarrow \\
 & & X' \times Y' & \xrightarrow{\quad} & X' \wedge Y'
 \end{array}$$

**Definition 4.2.16** (Reduction). Let  $B$  be a small set. Let  $\Phi_B : \mathbf{Set}/B \rightarrow \mathbf{Set}/B$  be a functor. The *reduction* of  $\Phi_B$  is the functor  $\Phi_B^B : \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  defined as follows.

For  $(X, u : B \rightarrow X, p : X \rightarrow B) \in \mathbf{Set}_B^B$ , let  $\Phi_B^B(X)$  be the set over and under  $B$  obtained from  $\Phi_B(X)$  by collapsing with respect to  $\Phi_B(u) : \Phi_B(B) \rightarrow \Phi_B(X)$ .

**Definition 4.2.17.** Extend  $\vee_B$  to a functor  $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ .

**Definition 4.2.18.** Extend  $\wedge_B$  to a functor  $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ .

**Definition 4.2.19** (Faithful). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *faithful* iff, for any objects  $A, B \in \mathcal{C}$  and morphisms  $f, g : A \rightarrow B : \mathcal{C}$ , if  $Ff = Fg$  then  $f = g$ .

**Definition 4.2.20** (Full). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *full* iff, for any objects  $A, B \in \mathcal{C}$  and morphism  $g : FA \rightarrow FB : \mathcal{D}$ , there exists  $f : A \rightarrow B : \mathcal{C}$  such that  $Ff = g$ .

**Definition 4.2.21** (Fully Faithful). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *fully faithful* iff it is full and faithful.

**Definition 4.2.22** (Full Embedding). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a *full embedding* iff it is fully faithful and injective on objects.

### 4.3 Natural Transformations

**Definition 4.3.1** (Natural Transformation). Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$ . A *natural transformation*  $\tau : F \Rightarrow G$  is a family of morphisms  $\{\tau_X : FX \rightarrow GX\}_{X \in \mathcal{C}}$  such that, for every morphism  $f : X \rightarrow Y : \mathcal{C}$ , we have  $Gf \circ \tau_X = \tau_Y \circ Ff$ .

$$\begin{array}{ccc}
 FX & \xrightarrow{Ff} & FY \\
 \tau_X \downarrow & & \downarrow \tau_Y \\
 GX & \xrightarrow{Gf} & GY
 \end{array}$$

**Definition 4.3.2** (Natural Isomorphism). A natural transformation  $\tau : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$  is a *natural isomorphism*,  $\tau : F \cong G$ , iff for all  $X \in \mathcal{C}$ ,  $\tau_X$  is an isomorphism  $FX \cong GX$ .

Functors  $F$  and  $G$  are *naturally isomorphic*,  $F \cong G$ , iff there exists a natural isomorphism between them.

**Definition 4.3.3** (Inverse). Let  $\tau : F \cong G$ . The *inverse* natural isomorphism  $\tau^{-1} : G \cong F$  is defined by  $(\tau^{-1})_X = \tau_X^{-1}$ .

## 4.4 Bifunctors

**Definition 4.4.1** (Commutative). A bifunctor  $\square : \mathcal{C}^2 \rightarrow \mathcal{C}$  is *commutative* iff  $\square \cong \square \circ T$ , where  $T : \mathcal{C}^2 \rightarrow \mathcal{C}^2$  is the swap functor.

**Proposition 4.4.2.**  $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is commutative.

PROOF: Since the pushout of  $f$  and  $g$  is the pushout of  $g$  and  $f$ .  $\square$

**Proposition 4.4.3.**  $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is commutative.

PROOF: In the diagram defining  $X \wedge Y$ , construct the isomorphism between the version with  $X$  and  $Y$  and the version with  $Y$  with  $X$  for every object.  $\square$

**Proposition 4.4.4.**  $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is commutative.

**Proposition 4.4.5.**  $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is commutative.

**Definition 4.4.6** (Associative). A bifunctor  $\square$  is *associative* iff  $\square \circ (\square \times \text{id}) \cong \square \circ (\text{id} \times \square)$ .

**Proposition 4.4.7.**  $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is associative.

PROOF: Since  $X \vee (Y \vee Z)$  and  $(X \vee Y) \vee Z$  are both the pushout of the unique morphisms  $1 \rightarrow X$ ,  $1 \rightarrow Y$  and  $1 \rightarrow Z$ .  $\square$

**Proposition 4.4.8.**  $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is associative.

PROOF: Draw isomorphisms between the diagrams for  $X \wedge (Y \wedge Z)$  and  $(X \wedge Y) \wedge Z$ .  $\square$

Product and coproduct are commutative and associative.

**Proposition 4.4.9.**  $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is associative.

**Proposition 4.4.10.**  $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is associative.

**Proposition 4.4.11.** Let  $\mathcal{C}$  be a category with binary coproducts. Let  $\square : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  be a bifunctor. Then  $\square$  distributes over  $+$  iff the canonical morphism

$$(X \square Z) + (Y \square Z) \rightarrow (X + Y) \square Z$$

is an isomorphism for all  $X, Y, Z$ .

**Proposition 4.4.12.** *In a category with binary products and binary coproducts, then  $\times$  distributes over  $+$ .*

**Proposition 4.4.13.** *In  $\mathbf{Set}/*$ , we have  $\times$  does not distribute over  $\vee$ .*

**Proposition 4.4.14.** *In  $\mathbf{Set}/*$ , we have  $\wedge$  distributes over  $\vee$ .*

**Proposition 4.4.15.** *In  $\mathbf{Set}/B$ , we have  $\times_B$  distributes over  $+_B$ .*

**Proposition 4.4.16.** *In  $\mathbf{Set}/B^B$ , we have  $\wedge_B$  distributes over  $\vee_B$ .*

## 4.5 Functor Categories

**Definition 4.5.1** (Functor Category). Given categories  $\mathcal{C}$  and  $\mathcal{D}$ , define the *functor category*  $\mathcal{C}^{\mathcal{D}}$  to be the category with objects the functors from  $\mathcal{D}$  to  $\mathcal{C}$  and morphisms the natural transformations.

**Definition 4.5.2** (Yoneda Embedding). Let  $\mathcal{C}$  be a category. The *Yoneda embedding*  $Y : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$  is the functor that maps an object  $A$  to  $\mathcal{C}[-, A]$  and morphisms similarly.

**Theorem 4.5.3** (Yoneda Lemma). *Let  $\mathcal{C}$  be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\text{op}}}[\mathcal{C}[-, X], F] \cong FX$$

*that maps  $\tau : \mathcal{C}[-, X] \Rightarrow F$  to  $\tau_X(\text{id}_X)$ .*

PROOF:

$\langle 1 \rangle 1.$   $\phi$  is natural in  $X$ .

PROOF:

$\langle 2 \rangle 1.$  LET:  $f : X \rightarrow Y : \mathcal{C}$

$\langle 2 \rangle 2.$  LET:  $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3.$   $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$

PROOF:

$$\begin{aligned} \phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\text{id}_Y \circ f) \\ &= \tau_Y(f) \\ &= \tau_Y(f \circ \text{id}_X) \\ &= Ff(\tau_X(\text{id}_X)) && (\tau \text{ natural}) \\ &= Ff(\phi(\tau)) \end{aligned}$$

$\langle 1 \rangle 2.$   $\phi$  is natural in  $F$ .

$\langle 2 \rangle 1.$  LET:  $\alpha : F \Rightarrow G : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$

$\langle 2 \rangle 2.$  LET:  $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3.$   $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$

PROOF:  $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\text{id}_X)) = \alpha_X(\phi(\tau))$

$\langle 1 \rangle 3.$  Each  $\phi_{XF}$  is injective.

$\langle 2 \rangle 1.$  LET:  $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 2.$  ASSUME:  $\phi(\sigma) = \phi(\tau)$

$\langle 2 \rangle 3$ . LET:  $f : Y \rightarrow X$

$\langle 2 \rangle 4$ .  $\sigma_Y(f) = \tau_Y(f)$

PROOF:

$$\begin{aligned}
 \sigma_Y(f) &= \sigma_Y(\text{id}_X \circ f) \\
 &= Ff(\sigma_X(\text{id}_X)) && (\sigma \text{ is natural}) \\
 &= Ff(\tau_X(\text{id}_X)) && (\langle 2 \rangle 2) \\
 &= \tau_Y(\text{id}_X \circ f) && (\tau \text{ is natural}) \\
 &= \tau_Y(f)
 \end{aligned}$$

$\langle 1 \rangle 4$ . Each  $\phi_{XF}$  is surjective.

$\langle 2 \rangle 1$ . LET:  $X \in \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$

$\langle 2 \rangle 2$ . LET:  $a \in FX$

$\langle 2 \rangle 3$ . LET:  $\tau : \mathcal{C}[-, X] \Rightarrow F$  be given by  $\tau_Y(g) = Fg(a)$  for  $g : Y \rightarrow X$

$\langle 2 \rangle 4$ .  $\tau$  is natural.

$\langle 3 \rangle 1$ . LET:  $h : Y \rightarrow Z : \mathcal{C}$

PROVE:  $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \text{id}_X]$

$\langle 3 \rangle 2$ . LET:  $g : Z \rightarrow X$

$\langle 3 \rangle 3$ .  $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$

PROOF:

$$\begin{aligned}
 \tau_Y(g \circ h) &= F(g \circ h)(a) \\
 &= Fh(Fg(a)) \\
 &= Fh(\tau_Z(g))
 \end{aligned}$$

$\langle 2 \rangle 5$ .  $\phi(\tau) = a$

PROOF:

$$\begin{aligned}
 \phi_X(\tau) &= \tau_X(\text{id}_X) \\
 &= F\text{id}_X(a) \\
 &= a
 \end{aligned}$$

□

**Corollary 4.5.3.1.** *The Yoneda embedding is fully faithful.*

**Corollary 4.5.3.2.** *Given objects  $A$  and  $B$  in  $\mathcal{C}$ , we have  $A \cong B$  if and only if  $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$ .*





## Chapter 5

# Monoid Theory

**Definition 5.0.1** (Monoid). A *monoid* is a category with one object.

**Definition 5.0.2.** Let  $\mathcal{C}$  be a category and  $X \in \mathcal{C}$ . The monoid  $\text{End}_{\mathcal{C}}(X)$  is the set of all morphisms  $X \rightarrow X$  under composition.

**Proposition 5.0.3.** *For any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $X \in \mathcal{C}$ , we have that  $F : \text{End}_{\mathcal{C}}(X) \rightarrow \text{End}_{\mathcal{D}}(FX)$  is a monoid homomorphism.*

PROOF: Since  $F\text{id}_X = \text{id}_{FX}$  and  $F(g \circ f) = Fg \circ Ff$ .  $\square$



## Chapter 6

# Group Theory

**Definition 6.0.1.** Let  $\mathbf{Grp}$  be the category of small groups and group homomorphisms.

**Definition 6.0.2.** We identify any group  $G$  with the category with one object whose morphisms are the elements of  $G$  with composition given by the multiplication in  $G$ .

**Proposition 6.0.3.** *The trivial group is a zero object in  $\mathbf{Grp}$ .*

PROOF: Easy.  $\square$

The zero morphism  $G \rightarrow H$  maps every element in  $G$  to  $e$ .

**Definition 6.0.4.** Let  $\mathcal{C}$  be a category and  $X \in \mathcal{C}$ . We write  $\text{Aut}_{\mathcal{C}}(X)$  for the set of all isomorphisms  $X \cong X$  under composition.

**Proposition 6.0.5.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor and  $X \in \mathcal{C}$ . Then  $F : \text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(FX)$  is a group homomorphism.*

PROOF: Since  $F\text{id}_X = \text{id}_{FX}$ ,  $F(g \circ f) = Fg \circ Ff$ , and  $Ff^{-1} = (Ff)^{-1}$ .  $\square$

**Proposition 6.0.6.**  $\mathbf{Grp}$  has products.

**Definition 6.0.7** (Free Product). The product of a family of groups in  $\mathbf{Grp}$  is called the *free product*.

**Proposition 6.0.8.**  $\mathbf{Ab}$  has products given by direct sums.



## Chapter 7

# Ring Theory

**Definition 7.0.1.** Let **Ring** be the concrete category of rings and ring homomorphisms.

**Definition 7.0.2** (Spectrum). Let  $R$  be a commutative ring. The *spectrum* of  $R$ ,  $\text{spec } R$ , is the set of all prime ideals of  $R$ .

**Definition 7.0.3** (Zariski Topology). Let  $R$  be a commutative ring. The *Zariski topology* on  $\text{spec } R$  is the topology where the closed sets are the sets of the form

$$VE := \{p \in \text{spec } R : E \subseteq p\}$$

for any  $E \in \mathcal{P}R$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$

$\langle 1 \rangle 2$ . For all  $\mathcal{A} \subseteq \mathcal{C}$  we have  $\bigcap \mathcal{A} \in \mathcal{C}$

$\langle 2 \rangle 1$ . LET:  $\mathcal{A} \subseteq \mathcal{C}$

$\langle 2 \rangle 2$ . LET:  $E = \bigcup \{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$

PROVE:  $VE = \bigcap \mathcal{A}$

$\langle 2 \rangle 3$ . For all  $p \in \text{spec } R$ , if  $E \subseteq p$  then  $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 1$ . LET:  $p \in \text{spec } R$

$\langle 3 \rangle 2$ . ASSUME:  $E \subseteq p$

$\langle 3 \rangle 3$ . LET:  $E' \in \mathcal{P}R$  with  $VE' \in \mathcal{A}$

$\langle 3 \rangle 4$ .  $E' \subseteq E$

$\langle 3 \rangle 5$ .  $E' \subseteq p$

$\langle 3 \rangle 6$ .  $p \in VE'$

$\langle 2 \rangle 4$ . For all  $p \in \text{spec } R$ , if  $p \in \bigcap \mathcal{A}$  then  $E \subseteq p$

$\langle 3 \rangle 1$ . LET:  $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 2$ . For all  $E' \in \mathcal{P}R$  with  $VE' \in \mathcal{A}$  we have  $E' \subseteq p$

$\langle 3 \rangle 3$ .  $E \subseteq p$

$\langle 1 \rangle 3$ . For all  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .

PROOF: Since  $VE \cup VE' = V(E \cap E')$

$\langle 1 \rangle 4. \emptyset \in \mathcal{C}$

$\langle 2 \rangle 1. VR = \emptyset$

PROOF: If  $p \in VR$  then  $R \subseteq p$  contradicting the fact that  $p$  is a prime ideal.

□

**Definition 7.0.4.** For any ring  $R$ , let  $R - \mathbf{Mod}$  be the category of small  $R$ -modules and  $R$ -module homomorphisms.

**Proposition 7.0.5.**  $R - \mathbf{Mod}$  has products and coproducts.

## Chapter 8

# Field Theory

**Proposition 8.0.1.** *Field does not have binary products.*

PROOF: There cannot be a field  $K$  with field homomorphisms  $K \rightarrow \mathbb{Z}_2$  and  $K \rightarrow \mathbb{Z}_3$ , because its characteristic would be both 2 and 3.  $\square$





## Chapter 9

# Linear Algebra

**Definition 9.0.1.** For any field  $K$ , we write  $\mathbf{Vect}_K$  for  $K\text{-Mod}$ .

Dual space functor  $\mathbf{Vect}_K^{\text{op}} \rightarrow \mathbf{Vect}_K$ .



# Chapter 10

## Topology

### 10.1 Topological Spaces

**Definition 10.1.1** (Topological Space). Let  $X$  be a set and  $\mathcal{O} \subseteq \mathcal{P}X$ . Then we say  $(X, \mathcal{O})$  is a *topological space* iff:

- For any  $\mathcal{U} \subseteq \mathcal{O}$  we have  $\bigcup \mathcal{U} \in \mathcal{O}$ .
- For any  $U, V \in \mathcal{O}$  we have  $U \cap V \in \mathcal{O}$ .
- $X \in \mathcal{O}$

We call  $\mathcal{O}$  the *topology* of the topological space, and call its elements *open* sets. We shall often write  $X$  for the topological space  $(X, \mathcal{O})$ .

**Definition 10.1.2** (Discrete Topology). For any set  $X$ , the power set  $\mathcal{P}X$  is called the *discrete* topology on  $X$ .

**Proposition 10.1.3.** *For any set  $X$ , the discrete topology on  $X$  is a topology on  $X$ .*

**Definition 10.1.4** (Indiscrete Topology). For any set  $X$ , the *indiscrete* or *trivial* topology on  $X$  is  $\{\emptyset, X\}$ .

**Proposition 10.1.5.** *For any set  $X$ , the indiscrete topology on  $X$  is a topology on  $X$ .*

**Definition 10.1.6** (Cofinite Topology). For any set  $X$ , the *cofinite* topology is  $\{X - U : U \subseteq X \text{ is finite}\}$ .

**Definition 10.1.7** (Cocountable Topology). For any set  $X$ , the *cocountable* topology is  $\{X - U : U \subseteq X \text{ is countable}\}$ .

**Definition 10.1.8** (Sierpiński Two-Point Space). The *Sierpiński two-point space* is  $\{0, 1\}$  under the topology  $\{\emptyset, \{1\}, \{0, 1\}\}$ .

**Definition 10.1.9** (Closed Set). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is *closed* iff  $X - A$  is open.

**Proposition 10.1.10.** *A set  $B$  is open if and only if  $X - B$  is closed.*

**Proposition 10.1.11.** *Let  $X$  be a set and  $\mathcal{C} \subseteq \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  on  $X$  such that  $\mathcal{C}$  is the set of closed sets if and only if:*

- For any  $\mathcal{D} \subseteq \mathcal{C}$  we have  $\bigcap \mathcal{D} \in \mathcal{C}$
- For any  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .
- $\emptyset \in \mathcal{C}$

*In this case,  $\mathcal{O}$  is unique and is given by  $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$ .*

**Theorem 10.1.12.** *There are infinitely many primes.*

Furstenberg's proof:

PROOF:

$\langle 1 \rangle 1$ . For  $a \in \mathbb{Z} - \{0\}$  and  $b \in \mathbb{Z}$ ,

LET:  $S(a, b) := \{an + b : n \in \mathbb{N}\}$

$\langle 1 \rangle 2$ . LET:  $\mathcal{T}$  be the topology generated by the basis  $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$

$\langle 2 \rangle 1$ . For every  $n \in \mathbb{Z}$ , there exist  $a, b$  such that  $n \in S(a, b)$ .

PROOF:  $n \in S(n, 0)$

$\langle 2 \rangle 2$ . If  $n \in S(a_1, b_1) \cap S(a_2, b_2)$  then there exist  $a_3, b_3$  such that  $n \in S(a_3, b_3) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 1$ . LET:  $d = \text{lcm}(a_1, a_2)$

PROVE:  $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 2$ . LET:  $d = a_1k = a_2l$

$\langle 3 \rangle 3$ . LET:  $n = a_1c + b_1 = a_2d + b_2$

$\langle 3 \rangle 4$ . LET:  $z \in \mathbb{Z}$

PROVE:  $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 5$ .  $dz + n \in S(a_1, b_1)$

PROOF:

$$\begin{aligned} dz + n &= a_1kz + a_1c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

$\langle 3 \rangle 6$ .  $dz + n \in S(a_2, b_2)$

PROOF: Similar.

$\langle 1 \rangle 3$ . For all  $a \in \mathbb{Z} - \{0\}$  and  $b \in \mathbb{Z}$  we have  $S(a, b)$  is closed.

$\langle 2 \rangle 1$ . LET:  $a \in \mathbb{Z} - \{0\}$  and  $b \in \mathbb{Z}$

$\langle 2 \rangle 2$ . LET:  $n \in \mathbb{Z} - S(a, b)$

$\langle 2 \rangle 3$ .  $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$

$\langle 3 \rangle 1$ . LET:  $x \in S(a, n)$

$\langle 3 \rangle 2$ . ASSUME: for a contradiction  $x \in S(a, b)$

$\langle 3 \rangle 3$ . PICK  $m$  such that  $x = am + b$

$\langle 3 \rangle 4$ . PICK  $l$  such that  $x = al + n$

$\langle 3 \rangle 5$ .  $n = a(m - l) + b$

⟨3⟩6.  $n \in S(a, b)$

⟨3⟩7. Q.E.D.

PROOF: This contradicts ⟨2⟩2.

⟨1⟩4.

$$\mathbb{Z} - \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$$

PROOF: Since every integer except 1 and  $-1$  is divisible by a prime.

⟨1⟩5. No nonempty finite set is open.

⟨2⟩1. LET:  $U$  be a nonempty open set

⟨2⟩2. PICK  $n \in U$

⟨2⟩3. There exist  $a, b$  such that  $n \in S(a, b) \subseteq U$

⟨2⟩4.  $U$  is infinite.

⟨1⟩6.  $\mathbb{Z} - \{1, -1\}$  is not closed.

⟨1⟩7.  $\bigcup_{p \text{ prime}} S(p, 0)$  is not closed.

⟨1⟩8. The union of finitely many closed sets is closed.

⟨1⟩9. There are infinitely many primes.

□

**Definition 10.1.13** (Neighbourhood). Let  $X$  be a topological space,  $Sx \in X$  and  $U \subseteq X$ . Then  $U$  is a *neighbourhood* of  $x$ , and  $x$  is an *interior* point of  $U$ , iff there exists an open set  $V$  such that  $x \in V \subseteq U$ .

**Proposition 10.1.14.** A set  $B$  is open if and only if it is a neighbourhood of each of its points.

**Proposition 10.1.15.** Let  $X$  be a set and  $\mathcal{N} : X \rightarrow \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  on  $X$  such that, for all  $x \in X$ , we have  $\mathcal{N}_x$  is the set of neighbourhoods of  $x$ , if and only if:

- For all  $x \in X$  and  $N \in \mathcal{N}_x$  we have  $x \in N$
- For all  $x \in X$  we have  $X \in \mathcal{N}_x$
- For all  $x \in X$ ,  $N \in \mathcal{N}_x$  and  $V \subseteq \mathcal{P}X$ , if  $N \subseteq V$  then  $V \in \mathcal{N}_x$
- For all  $x \in X$  and  $M, N \in \mathcal{N}_x$  we have  $M \cap N \in \mathcal{N}_x$
- For all  $x \in X$  and  $N \in \mathcal{N}_x$ , there exists  $M \in \mathcal{N}_x$  such that  $M \subseteq N$  and  $\forall y \in M. M \in \mathcal{N}_y$ .

In this case,  $\mathcal{O}$  is unique and is given by  $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$ .

**Definition 10.1.16** (Exterior Point). Let  $X$  be a topological space,  $x \in X$  and  $B \subseteq X$ . Then  $x$  is an *exterior point* of  $B$  iff  $B - X$  is a neighbourhood of  $x$ .

**Definition 10.1.17** (Boundary Point). Let  $X$  be a topological space,  $x \in X$  and  $B \subseteq X$ . Then  $x$  is a *boundary point* of  $B$  iff it is neither an interior point nor an exterior point of  $B$ .

**Definition 10.1.18** (Interior). Let  $X$  be a topological space and  $B \subseteq X$ . The *interior* of  $B$ ,  $B^\circ$ , is the set of all interior points of  $B$ .

**Proposition 10.1.19.** *The interior of  $B$  is the union of all the open sets included in  $B$ .*

**Definition 10.1.20** (Closure). Let  $X$  be a topological space and  $B \subseteq X$ . The closure of  $B$ ,  $\overline{B}$ , is the set of all points that are not exterior points of  $B$ .

**Proposition 10.1.21.** *The closure of  $B$  is the intersection of all the closed sets that include  $B$ .*

**Proposition 10.1.22.** *A set  $B$  is open iff  $X - B = \overline{X - B}$ .*

**Proposition 10.1.23** (Kuratowski Closure Axioms). *Let  $X$  be a set and  $- : \mathcal{P}X \rightarrow \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  such that, for all  $B \subseteq X$ ,  $\overline{B}$  is the closure of  $B$ , if and only if:*

- $\overline{\emptyset} = \emptyset$
- For all  $A \subseteq X$  we have  $A \subseteq \overline{A}$
- For all  $A \subseteq X$  we have  $\overline{\overline{A}} = \overline{A}$
- For all  $A, B \subseteq X$  we have  $\overline{A \cup B} = \overline{A} \cup \overline{B}$

*In this case,  $\mathcal{O}$  is unique and is defined by  $\mathcal{O} = \{U : X - U = \overline{X - U}\}$ .*

**Definition 10.1.24** (Finer, Coarser). Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the set  $X$ . Then  $\mathcal{T}$  is *coarser*, *smaller* or *weaker* than  $\mathcal{T}'$ , or  $\mathcal{T}'$  is *finer*, *larger* or *weaker* than  $\mathcal{T}$ , iff  $\mathcal{T} \subseteq \mathcal{T}'$ .

### 10.1.1 Subspaces

**Definition 10.1.25** (Subspace). Let  $X$  be a topological space and  $X_0 \subseteq X$ . The *subspace topology* on  $X_0$  is  $\{U \cap X_0 : U \text{ is open in } X\}$ .

**Example 10.1.26.** The *unit sphere*  $S^2$  is  $\{x \in \mathbb{R}^3 : \|x\| = 1\}$  as a subspace of  $\mathbb{R}^3$ .

**Theorem 10.1.27.** *Let  $X$  be a topological space and  $(Y, i)$  a subset of  $X$ . Then the subspace topology on  $Y$  is the unique topology such that, for every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f : Z \rightarrow X$  is continuous.*

PROOF:

- ⟨1⟩1. If we give  $Y$  the subspace topology then, for every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f$  is continuous.
- ⟨2⟩1. Given  $Y$  the subspace topology.
- ⟨2⟩2. LET:  $Z$  be a topological space.
- ⟨2⟩3. LET:  $f : Z \rightarrow Y$
- ⟨2⟩4. If  $f$  is continuous then  $i \circ f$  is continuous.

PROOF: Since  $i$  is continuous.

- ⟨2⟩5. If  $i \circ f$  is continuous then  $f$  is continuous.
- ⟨3⟩1. ASSUME:  $i \circ f$  is continuous.
- ⟨3⟩2. LET:  $U$  be open in  $Y$ .
- ⟨3⟩3.  $f^{-1}(i^{-1}(i(U)))$  is open in  $Z$ .
- ⟨3⟩4.  $f^{-1}(U)$  is open in  $Z$ .
- ⟨1⟩2. If, for every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f$  is continuous.
- ⟨2⟩1. ASSUME: For every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f$  is continuous.
- ⟨2⟩2.  $i$  is continuous.
- ⟨2⟩3. For every open set  $U$  in  $X$ , we have  $i^{-1}(X)$  is open in  $Y$ .
- ⟨2⟩4. LET:  $Z$  be the set  $Y$  under the subspace topology and  $f : Z \rightarrow Y$  the identity function.
- ⟨2⟩5.  $i \circ f$  is continuous.
- ⟨2⟩6.  $f$  is continuous.
- ⟨2⟩7. Every set open in  $Y$  is open in  $Z$ .

□

### 10.1.2 Topological Disjoint Union

**Definition 10.1.28.** Let  $X$  and  $Y$  be topological spaces. The *disjoint union* is  $X + Y$  where  $U \subseteq X + Y$  is open if and only if  $\kappa_1^{-1}(U)$  is open in  $X$  and  $\kappa_2^{-1}(U)$  is open in  $Y$ .

### 10.1.3 Product Topology

**Definition 10.1.29** (Product Topology). Let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be a family of topological spaces. The *product topology* on  $\prod_{\lambda \in \Lambda} X_\lambda$  is the coarsest topology such that every projection onto  $X_\lambda$  is continuous.

### 10.1.4 Bases

**Definition 10.1.30** (Basis). Let  $X$  be a topological space. A *basis* for the topology on  $X$  is a set of open sets  $\mathcal{B}$  such that every open set is the union of a subset of  $\mathcal{B}$ . The elements of  $\mathcal{B}$  are called *basic open neighbourhoods* of their elements.

**Definition 10.1.31** (Order Topology). Let  $X$  be a linearly ordered set. The *order topology* on  $X$  is the topology generated by the open interval  $(a, b)$  as well as the open rays  $(a, +\infty)$  and  $(-\infty, b)$  for  $a, b \in X$ .

**Definition 10.1.32** (Lower Limit Topology). The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology generated by the basis consisting of all half-open intervals  $[a, b)$ .

**Proposition 10.1.33.** Let  $X$  be a topological space and  $\mathcal{B} \subseteq \mathcal{P}X$ . Then  $\mathcal{B}$  is a basis for a topology on  $X$  if and only if:

$$1. \bigcup \mathcal{B} = X$$

2. For all  $A, B \in \mathcal{B}$  and  $x \in A \cap B$ , there exists  $C \in \mathcal{B}$  such that  $x \in C \subseteq A \cap B$ .

In this case, the topology is unique and is the set of all unions of subsets of  $\mathcal{B}$ . We call it the topology generated by  $\mathcal{B}$ .

### 10.1.5 Subbases

**Definition 10.1.34** (Subbasis). Let  $X$  be a topological space. A *subbasis* for the topology on  $X$  is a subset  $\mathcal{S} \subseteq \mathcal{P}X$  such that every open set is a union of finite intersections of  $\mathcal{S}$ .

**Definition 10.1.35** (Space with Basepoint). A *space with basepoint* is a pair  $(X, x)$  where  $X$  is a topological space and  $x : \text{El}(X)$ .

### 10.1.6 Countability Axioms

**Definition 10.1.36** (Neighbourhood Basis). Let  $X$  be a topological space and  $x_0 : \text{El}(X)$ . A *neighbourhood basis* of  $x_0$  is a set  $\mathcal{U}$  of neighbourhoods of  $x_0$  such that every neighbourhood of  $x_0$  includes an element of  $\mathcal{U}$ .

**Definition 10.1.37** (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

**Definition 10.1.38** (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

$\mathbb{R}^n$  is second countable.

An uncountable discrete space is first countable but not second countable.

**Proposition 10.1.39.** Let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be a family of topological spaces such that no  $X_\lambda$  is indiscrete. If  $\Lambda$  is uncountable, then  $\prod_{\lambda \in \Lambda} X_\lambda$  is not first countable.

PROOF:

$\langle 1 \rangle 1$ . For all  $\lambda : \text{El}(\Lambda)$ , PICK  $U_\lambda$  open in  $X_\lambda$  such that  $\emptyset \neq U_\lambda \neq X_\lambda$ .

$\langle 1 \rangle 2$ . For all  $\lambda : \text{El}(\Lambda)$ , PICK  $x_\lambda \in U_\lambda$ .

$\langle 1 \rangle 3$ . ASSUME: for a contradiction  $B$  is a countable neighbourhood basis for  $(x_\lambda)_{\lambda \in \Lambda}$ .

$\langle 1 \rangle 4$ . PICK  $\lambda \in \Lambda$  such that, for all  $U \in B$ , we have  $\pi_\lambda(U) = X_\lambda$

$\langle 1 \rangle 5$ . There is no  $U \in \lambda$  such that  $U \subseteq \pi_\lambda^{-1}(U_\lambda)$

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This is a contradiction.

□



## 10.2 Continuous Functions

**Definition 10.2.1** (Continuous). Let  $X$  and  $Y$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous* iff, for every open set  $V$  in  $Y$ , the inverse image  $f^{-1}(V)$  is open in  $X$ .

**Proposition 10.2.2.** 1.  $\text{id}_X$  is continuous

2. The composite of two continuous functions is continuous.

3. If  $f : X \rightarrow Y$  is continuous and  $X_0 \subseteq X$  then  $f|_{X_0} : X_0 \rightarrow Y$  is continuous.

4. If  $f : X + Y \rightarrow Z$ , then  $f$  is continuous iff  $f \circ \kappa_1 : X \rightarrow Z$  and  $f \circ \kappa_2 : Y \rightarrow Z$  are continuous.

5. If  $f : Z \rightarrow X \times Y$ , then  $f$  is continuous iff  $\pi_1 \circ f$  and  $\pi_2 \circ f$  are continuous.

**Proposition 10.2.3.** Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous then, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{B}$  is open in  $Y$ .

$\langle 1 \rangle 2$ . If, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ , then  $f$  is continuous.

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

$\langle 2 \rangle 2$ . LET:  $U$  be open in  $Y$ .

$\langle 2 \rangle 3$ . LET:  $x \in f^{-1}(U)$

$\langle 2 \rangle 4$ . PICK  $B \in \mathcal{B}$  such that  $f(x) \in B \subseteq U$ .

$\langle 2 \rangle 5$ .  $x \in f^{-1}(B) \subseteq f^{-1}(U)$

□

**Definition 10.2.4** (Homeomorphism). Let  $X$  and  $Y$  be topological spaces. A *homeomorphism* between  $X$  and  $Y$  is a bijection  $f : X \approx Y$  such that  $f$  and  $f^{-1}$  are continuous.

**Definition 10.2.5** (Retraction). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A continuous function  $\rho : X \rightarrow A$  is a *retraction* iff  $\rho|_A = \text{id}_A$ . We say  $A$  is a *retract* of  $X$  iff there exists a retraction.

**Definition 10.2.6.** Let **Top** be the category of small topological spaces and continuous functions.

**Proposition 10.2.7.**  $\emptyset$  is initial in **Top**.

**Proposition 10.2.8.**  $1$  is terminal in **Top**.

Forgetful functor **Top**  $\rightarrow$  **Set**.

Basepoint preserving continuous functor.

**Proposition 10.2.9.** *Let  $(X, \mathcal{T})$  be a topological space. Let  $S$  be the Sierpiński two-point space. Define  $\Phi : \mathcal{T} \rightarrow \mathbf{Top}[X, S]$  by  $\Phi(U)(x) = 1$  iff  $x \in U$ . Then  $\Phi$  is a bijection.*

PROOF:

$\langle 1 \rangle 1$ . For all  $U \in \mathcal{T}$  we have  $\Phi(U)$  is continuous.

$\langle 2 \rangle 1$ . LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 2$ .  $\Phi(U)(\{1\})$  is open.

PROOF: Since  $\Phi(U)(\{1\}) = U$ .

$\langle 1 \rangle 2$ .  $\Phi$  is injective.

PROOF: If  $\Phi(U) = \Phi(V)$  then we have  $\forall x (x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$ .

$\langle 1 \rangle 3$ .  $\Phi$  is surjective.

PROOF: Given  $f : X \rightarrow S$  continuous we have  $\Phi(f^{-1}(1)) = f$ .

□

### 10.2.1 Paths

**Definition 10.2.10** (Path). A *path* in a topological space  $X$  is a continuous function  $[0, 1] \rightarrow X$ .

### 10.2.2 Loops

**Definition 10.2.11** (Loop). A *loop* in a topological space  $X$  is a path  $\alpha : [0, 1] \rightarrow X$  such that  $\alpha(0) = \alpha(1)$ .

## 10.3 Convergence

**Definition 10.3.1** (Convergence). Let  $X$  be a topological space. Let  $(x_n)$  be a sequence in  $X$ . A point  $a \in \text{El}(X)$  is a *limit* of the sequence iff, for every neighbourhood  $U$  of  $a$ , there exists  $n_0$  such that  $\forall n \geq n_0, x_n \in U$ .

Convergence in a product space is pointwise convergence.

If  $f : X \rightarrow Y$  is continuous and  $x_n \rightarrow l$  in  $X$  then  $f(x_n) \rightarrow f(l)$  in  $Y$ .

**Example 10.3.2.** The converse does not hold.

Let  $X$  be the set of all continuous functions  $[0, 1] \rightarrow [-1, 1]$  under the product topology. Let  $i : X \rightarrow L^2([0, 1])$  be the inclusion.

If  $f_n \rightarrow f$  then  $i(f_n) \rightarrow i(f)$  — Lebesgue convergence theorem.

We prove that  $i$  is not continuous.

Assume for a contradiction  $i$  is continuous. Choose a neighbourhood  $K$  of 0 in  $X$  such that  $\forall \phi \in K, \int \phi^2 < 1/2$ . Let  $K = \prod_{\lambda \in [0, 1]} U_\lambda$  where  $U_\lambda = [-1, 1]$  except for  $\lambda = \lambda_1, \dots, \lambda_n$ . Let  $\phi$  be the function that is 0 at  $\lambda_1, \dots, \lambda_n$  and 1 everywhere else. Then  $\phi \in K$  but  $\int \phi^2 = 1$ .

**Proposition 10.3.3.** *The converse does hold for first countable spaces. If  $f : X \rightarrow Y$  where  $X$  is first countable, and  $Y$  is a topological space, and whenever  $x_n \rightarrow x$  then  $f(x_n) \rightarrow f(x)$ , then  $f$  is continuous.*

## 10.4 Subspaces

**Definition 10.4.1** (Subspace). Let  $X$  be a topological space,  $Y$  a set, and  $f : Y \rightarrow X$ . The *subspace topology* on  $Y$  induced by  $f$  is  $\mathcal{T} = \{f^{-1}(U) : U \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ .

PROOF: Since  $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$ .

$\langle 1 \rangle 2$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .

PROOF: Since  $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$ .

$\langle 1 \rangle 3$ .  $Y \in \mathcal{T}$ .

PROOF: Since  $Y = f^{-1}(X)$ .

□

**Proposition 10.4.2.** Let  $X$  be a topological space,  $Y$  a set and  $f : Y \rightarrow X$  a function. Then the subspace topology on  $Y$  is the coarsest topology such that  $f$  is continuous.

PROOF: Immediate from definition. □

## 10.5 Embedding

**Definition 10.5.1** (Embedding). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *embedding* iff  $f$  is injective and the topology on  $X$  is the subspace induced by  $f$ .

## 10.6 Quotient Spaces

**Definition 10.6.1** (Quotient Topology). Let  $X$  be a topological space,  $S$  a set, and  $\pi : X \rightarrow S$  be a surjection. The *quotient topology* on  $S$  induced by  $\pi$  is  $\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ .

PROOF: Since  $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}$ .

$\langle 1 \rangle 2$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .

PROOF: Since  $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$ .

$\langle 1 \rangle 3$ .  $X \in \mathcal{T}$ .

PROOF: Since  $X = \pi^{-1}(Y)$ .

□

## 10.7 Connected Spaces

**Definition 10.7.1** (Connected). A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

**Proposition 10.7.2.** *The continuous image of a connected space is connected.*

**Proposition 10.7.3.** *Let  $X$  be a topological space and  $A, B \subseteq X$ . If  $X = A \cup B$ ,  $A \cap B \neq \emptyset$ , and  $A$  and  $B$  are connected, then  $X$  is connected.*

**Proposition 10.7.4.** *If  $X$  and  $Y$  are nonempty topological spaces, then  $X \times Y$  is connected if and only if  $X$  and  $Y$  are connected.*

**Definition 10.7.5** (Path-connected). A topological space  $X$  is *path-connected* iff, for any points  $a, b \in X$ , there exists a continuous function  $\alpha : [0, 1] \rightarrow X$ , called a *path*, such that  $\alpha(0) = a$  and  $\alpha(1) = b$ .

**Proposition 10.7.6.** *The continuous image of a path connected space is path connected.*

**Proposition 10.7.7.** *Let  $X$  be a topological space and  $A, B \subseteq X$ . If  $X = A \cup B$ ,  $A \cap B \neq \emptyset$ , and  $A$  and  $B$  are path connected, then  $X$  is path connected.*

**Proposition 10.7.8.** *If  $X$  and  $Y$  are nonempty topological spaces, then  $X \times Y$  is path connected if and only if  $X$  and  $Y$  are path connected.*

## 10.8 Hausdorff Spaces

**Definition 10.8.1** (Hausdorff). A topological space is a *Hausdorff* space or a  $T_2$  space iff any two distinct points have disjoint neighbourhoods.

**Proposition 10.8.2.** *In a Hausdorff space, a sequence has at most one limit.*

**Proposition 10.8.3.** 1. *Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

**Proposition 10.8.4.** *Let  $A$  be a topological space and  $B$  a Hausdorff space. Let  $f, g : A \rightarrow B$  be continuous. Let  $X \subseteq A$  be dense. If  $f$  and  $g$  agree on  $X$ , then  $f = g$ .*

PROOF:

<1>1. ASSUME: for a contradiction  $a \in A$  and  $f(a) \neq g(a)$ .

<1>2. PICK disjoint neighbourhoods  $U$  and  $V$  of  $f(a)$  and  $g(a)$  respectively.

<1>3. PICK  $x \in f^{-1}(U) \cap g^{-1}(V)$

<1>4.  $f(x) = g(x) \in U \cap V$

<1>5. Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 10.8.5.** *Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$  be uniformly continuous. Let  $\hat{X}$  and  $\hat{Y}$  be the completions of  $X$  and  $Y$ . Then  $f$  extends uniquely to a continuous map  $\hat{X} \rightarrow \hat{Y}$ .*

PROOF: The extension maps  $\lim_{n \rightarrow \infty} x_n$  to  $\lim_{n \rightarrow \infty} f(x_n)$ . □

## 10.9 Separable Spaces

**Definition 10.9.1** (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

## 10.10 Sequential Compactness

**Definition 10.10.1** (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

## 10.11 Compactness

**Definition 10.11.1** (Compact). A topological space is *compact* iff every open cover has a finite subcover.

**Proposition 10.11.2.** *Let  $X$  be a compact topological space. Let  $P$  be a set of open sets such that, for all  $U, V \in P$ , we have  $U \cup V \in P$ . Assume that every point has an open neighbourhood in  $P$ . Then  $X \in P$ .*

PROOF:

- ⟨1⟩1.  $P$  is an open cover of  $X$
- ⟨1⟩2. PICK a finite subcover  $U_1, \dots, U_n \in P$
- ⟨1⟩3.  $X = U_1 \cup \dots \cup U_n \in P$

□

**Corollary 10.11.2.1.** *Let  $f$  be a compact space and  $f : X \rightarrow \mathbb{R}$  be locally bounded. Then  $f$  is bounded.*

PROOF: Take  $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$ . □

**Proposition 10.11.3.** *The continuous image of a compact space is compact.*

**Proposition 10.11.4.** *A closed subspace of a compact space is compact.*

**Proposition 10.11.5.** *Let  $X$  and  $Y$  be nonempty spaces. Then the following are equivalent.*

1.  $X$  and  $Y$  are compact.

2.  $X + Y$  is compact.

3.  $X \times Y$  is compact.

**Proposition 10.11.6.** *A compact subspace of a Hausdorff space is closed.*

**Proposition 10.11.7.** *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

**Proposition 10.11.8.** *A first countable compact space is sequentially compact.*

## 10.12 Quotient Spaces

**Definition 10.12.1** (Quotient Space). Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . The *quotient topology* on  $X/\sim$  is defined by:  $U : \text{El}(\mathcal{P}X)$  is open in  $X/\sim$  if and only if  $\pi^{-1}(U)$  is open in  $X$ .

**Proposition 10.12.2.** *Let  $X$  and  $Y$  be topological spaces. Let  $\sim$  be an equivalence relation on  $X$ . Let  $f : X/\sim \rightarrow Y$ . Then  $f$  is continuous if and only if  $f \circ \pi$  is continuous.*

**Proposition 10.12.3.** *Let  $X$  and  $Y$  be topological spaces. Let  $\sim$  be an equivalence relation on  $X$ . Let  $\phi : Y \rightarrow X/\sim$ .*

*Assume that, for all  $y \in Y$ , there exists a neighbourhood  $U$  of  $y$  and a continuous function  $\Phi : U \rightarrow X$  such that  $\pi \circ \Phi = \phi|_U$ . Then  $\phi$  is continuous.*

**Proposition 10.12.4.** *A quotient of a connected space is connected.*

**Proposition 10.12.5.** *A quotient of a path connected space is path connected.*

**Proposition 10.12.6.** *Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . If  $X/\sim$  is Hausdorff then every equivalence class of  $\sim$  is closed in  $X$ .*

**Definition 10.12.7.** Let  $X$  be a topological space and  $A_1, \dots, A_r \subseteq X$ . Then  $X/A_1, \dots, A_r$  is the quotient space of  $X$  with respect to  $\sim$  where  $x \sim y$  iff  $x = y$  or  $\exists i(x \in A_i \wedge y \in A_i)$ .

**Definition 10.12.8** (Cone). Let  $X$  be a topological space. The *cone over  $X$*  is the space  $(X \times [0, 1])/(X \times \{1\})$ .

**Definition 10.12.9** (Suspension). Let  $X$  be a topological space. The *suspension* of  $X$  is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

**Definition 10.12.10** (Wedge Product). Let  $x_0 \in X$  and  $y_0 \in Y$ . The *wedge product*  $X \vee Y$  is  $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$  as a subspace of  $X \times Y$ .

**Definition 10.12.11** (Smash Product). Let  $x_0 \in X$  and  $y_0 \in Y$ . The *smash product*  $X \wedge Y$  is  $(X \times Y)/(X \vee Y)$ .

**Example 10.12.12.**  $D^n/S^{n-1} \cong S^n$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : D^n/S^{n-1} \rightarrow S^n$  be the function induced by the map  $D^n \rightarrow S^n$  that maps the radii of  $D^n$  onto the meridians of  $S^n$  from the north to the south pole.

$\langle 1 \rangle 2$ .  $\phi$  is a bijection.

$\langle 1 \rangle 3$ .  $\phi$  is a homeomorphism.

PROOF: Since  $D^n/S^{n-1}$  is compact and  $S^n$  is Hausdorff.

□

## 10.13 Gluing

**Definition 10.13.1** (Gluing). Let  $X$  and  $Y$  be topological spaces,  $X_0 \subseteq X$  and  $\phi : X_0 \rightarrow Y$  a continuous map. Then  $Y \cup_\phi X$  is the quotient space  $(X + Y)/\sim$ , where  $\sim$  is the equivalence relation generated by  $x \sim \phi(x)$  for all  $x : \text{El}(X)$ .

**Proposition 10.13.2.**  $Y$  is a subspace of  $Y \cup_\phi X$ .

**Definition 10.13.3.** Let  $X$  be a topological space and  $\alpha : X \cong X$  a homeomorphism. Then  $(X \times [0, 1])/\alpha$  is the quotient space of  $X \times [0, 1]$  by the equivalence relation generated by  $(x, 0) \sim (\alpha(x), 1)$  for all  $x : \text{El}(X)$ .

**Definition 10.13.4** (Möbius Strip). The *Möbius strip* is  $([-1, 1] \times [0, 1])/\alpha$  where  $\alpha(x) = -x$ .

**Definition 10.13.5** (Klein Bottle). The *Klein bottle* is  $(S^1 \times [0, 1])/\alpha$  where  $\alpha(z) = \bar{z}$ .

**Proposition 10.13.6.** Let  $M$  be the Möbius strip and  $K$  the Klein bottle. Then  $M \cup_{\text{id}_M} M \cong K$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$  be the function that maps  $\kappa_1(\theta, t)$  to  $(e^{\pi i \theta/2}, t)$  and  $\kappa_2(\theta, t)$  to  $(-e^{-\pi i \theta/2}, t)$ .

$\langle 1 \rangle 2$ .  $f$  induces a bijection  $M \cup_{\text{id}_M} M \approx K$

$\langle 1 \rangle 3$ .  $f$  is a homeomorphism.

□

## 10.14 Metric Spaces

**Definition 10.14.1** (Metric Space). Let  $X$  be a set and  $d : X^2 \rightarrow \mathbb{R}$ . We say  $(X, d)$  is a *metric space* iff:

- For all  $x, y \in X$  we have  $d(x, y) \geq 0$
- For all  $x, y \in X$  we have  $d(x, y) = 0$  iff  $x = y$
- For all  $x, y \in X$  we have  $d(x, y) = d(y, x)$

- (*Triangle Inequality*) For all  $x, y, z \in X$  we have  $d(x, z) \leq d(x, y) + d(y, z)$

We call  $d$  the *metric* of the metric space  $(X, d)$ . We often write  $X$  for the metric space  $(X, d)$ .

**Definition 10.14.2** (Ball). Let  $X$  be a metric space. Let  $x \in X$  and  $r > 0$ . The *ball* with *centre*  $x$  and *radius*  $r$  is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} .$$

**Definition 10.14.3** (Metric Topology). Let  $(X, d)$  be a metric space. The *metric topology* on  $X$  is the topology generated by the basis consisting of the balls.

**Definition 10.14.4** (Metrizable). A topological space is *metrizable* iff there exists a metric that induces its topology.

**Proposition 10.14.5.** *Every metrizable space is Hausdorff.*

Every metrizable space is first countable.

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

## 10.15 Complete Metric Spaces

**Definition 10.15.1** (Complete). A metric space is *complete* iff every Cauchy sequence converges.

**Example 10.15.2.**  $\mathbb{R}$  is complete.

**Proposition 10.15.3.** *The product of two complete metric spaces is complete.*

**Proposition 10.15.4.** *Every compact metric space is complete.*

**Proposition 10.15.5.** *Let  $X$  be a complete metric space and  $A \subseteq X$ . Then  $A$  is complete if and only if  $A$  is closed.*

**Definition 10.15.6** (Completion). Let  $X$  be a metric space. A *completion* of  $X$  is a complete metric space  $\hat{X}$  and injection  $i : X \rightarrow \hat{X}$  such that:

- The metric on  $X$  is the restriction of the metric on  $\hat{X}$
- $X$  is dense in  $\hat{X}$ .

**Proposition 10.15.7.** *Let  $i_1 : X \rightarrow Y_1$  and  $i_2 : X \rightarrow Y_2$  be completions of  $X$ . Then there exists a unique isometry  $\phi : Y_1 \cong Y_2$  such that  $\phi \circ i_1 = i_2$ .*

PROOF: Define  $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$ .  $\square$

**Theorem 10.15.8.** *Every metric space has a completion.*

PROOF: Let  $\hat{X}$  be the set of Cauchy sequences in  $X$  quotiented by  $\sim$  where  $(x_n) \sim (y_n)$  if and only if  $d(x_n, y_n) \rightarrow 0$ .  $\square$



## 10.16 Manifolds

**Definition 10.16.1** (Manifold). An *n-dimensional manifold* is a second countable Hausdorff space locally homeomorphic to  $\mathbb{R}^n$ .



# Chapter 11

## Homotopy Theory

### 11.1 Homotopies

**Definition 11.1.1** (Homotopy). Let  $X$  and  $Y$  be topological spaces. Let  $f, g : X \rightarrow Y$  be continuous. A *homotopy* between  $f$  and  $g$  is a continuous function  $h : X \times [0, 1] \rightarrow Y$  such that

- $\forall x : \text{El}(X) . h(x, 0) = f(x)$
- $\forall x : \text{El}(X) . h(x, 1) = g(x)$

We say  $f$  and  $g$  are *homotopic*,  $f \simeq g$ , iff there exists a homotopy between them.

Let  $[X, Y]$  be the set of all homotopy classes of functions  $X \rightarrow Y$ .

**Proposition 11.1.2.** Let  $f, f' : X \rightarrow Y$  and  $g, g' : Y \rightarrow Z$  be continuous. If  $f \simeq f'$  and  $g \simeq g'$  then  $g \circ f \simeq g' \circ f'$ .

**Definition 11.1.3.** Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

**Definition 11.1.4.** A functor  $F : \mathbf{Top} \rightarrow \mathcal{C}$  is *homotopy invariant* iff, for any topological spaces  $X, Y$  and continuous functions  $f, g : X \rightarrow Y$ , if  $f \simeq g$  then  $Hf = Hg$ .

Basepoint-preserving homotopy.

### 11.2 Homotopy Equivalence

**Definition 11.2.1** (Homotopy Equivalence). Let  $X$  and  $Y$  be topological spaces. A *homotopy equivalence* between  $X$  and  $Y$ ,  $f : X \simeq Y$ , is a continuous function  $f : X \rightarrow Y$  such that there exists a continuous function  $g : Y \rightarrow X$ , the *homotopy inverse* to  $f$ , such that  $g \circ f \simeq \text{id}_X$  and  $f \circ g \simeq \text{id}_Y$ .

**Definition 11.2.2** (Contractible). A topological space  $X$  is *contractible* iff  $X \simeq 1$ .

**Example 11.2.3.**  $\mathbb{R}^n$  is contractible.

**Example 11.2.4.**  $D^n$  is contractible.

**Definition 11.2.5** (Deformation Retract). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A retraction  $\rho : X \rightarrow A$  is a *deformation retraction* iff  $i \circ \rho \simeq \text{id}_X$ , where  $i$  is the inclusion  $A \hookrightarrow X$ . We say  $A$  is a *deformation retract* of  $X$  iff there exists a deformation retraction.

**Definition 11.2.6** (Strong Deformation Retract). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A *strong deformation retraction*  $\rho : X \rightarrow A$  is a continuous function such that there exists a homotopy  $h : X \times [0, 1] \rightarrow X$  between  $i \circ \rho$  and  $\text{id}_X$  such that, for all  $a : \text{El}(X)$  and  $t : \text{El}([0, 1])$ , we have  $h(a, t) = a$ .

We say  $A$  is a *strong deformation retract* of  $X$  iff a strong deformation retraction exists.

**Example 11.2.7.**  $\{0\}$  is a strong deformation retract of  $\mathbb{R}^n$  and of  $D^n$ .

**Example 11.2.8.**  $S^1$  is a strong deformation retract of the torus  $S^1 \times D^2$ .

**Example 11.2.9.**  $S^{n-1}$  is a strong deformation retract of  $D^n - \{0\}$ .

**Example 11.2.10.** For any topological space  $X$ , the singleton consisting of the vertex is a strong deformation retract of the cone over  $X$ .

## Chapter 12

# Simplicial Complexes

**Definition 12.0.1** (Simplex). A  $k$ -dimensional simplex or  $k$ -simplex in  $\mathbb{R}^n$  is the convex hull  $s(x_0, \dots, x_k)$  of  $k + 1$  points in general position.

**Definition 12.0.2** (Face). A *sub-simplex* or *face* of  $s(x_0, \dots, x_k)$  is the convex hull of a subset of  $\{x_0, \dots, x_k\}$ .

**Definition 12.0.3** (Simplicial Complex). A *simplicial complex* in  $\mathbb{R}^n$  is a set  $K$  of simplices such that:

- for every simplex  $s$  in  $K$ , every face of  $s$  is in  $K$ .
- The intersection of two simplices  $s_1, s_2 \in K$  is either empty or is a face of both  $s_1$  and  $s_2$ .
- $K$  is locally finite, i.e. every point of  $\mathbb{R}^n$  has a neighbourhood that only intersects finitely many elements of  $K$ .

The topological space *underlying*  $K$  is  $|K| = \bigcup K$  as a subspace of  $\mathbb{R}^n$ .

### 12.1 Cell Decompositions

**Definition 12.1.1** ( $n$ -cell). An  $n$ -cell is a topological space homeomorphic to  $\mathbb{R}^n$ .

**Definition 12.1.2** (Cell Decomposition). Let  $X$  be a topological space. A *cell decomposition* of  $X$  is a partition of  $X$  into subspaces that are  $n$ -cells.

**Definition 12.1.3** ( $n$ -skeleton). Given a cell decomposition of  $X$ , the  $n$ -skeleton  $X^n$  is the union of all the cells of dimension  $\leq n$ .

### 12.2 CW-complexes

**Definition 12.2.1** (CW-Complex). A *CW-complex* consists of a topological space  $X$  and a cell decomposition  $\mathcal{E}$  of  $X$  such that:

1. *Characteristic Maps* For every  $n$ -cell  $e \in \mathcal{E}$ , there exists a continuous map  $\Phi_e : D^n \rightarrow X$  such that  $\Phi_e((D^n)^\circ) = e$ , the corestriction  $\Phi_e : (D^n)^\circ \approx e$  is a homeomorphism, and  $\Phi_e(S^n)$  is the union of all the cells in  $\mathcal{E}$  of dimension  $< n$ .
2. *Closure Finiteness* For all  $e \in \mathcal{E}$ , we have  $\bar{e}$  intersects only finitely many other cells in  $\mathcal{E}$ .
3. *Weak Topology* Given  $A \subseteq X$ , we have  $A$  is closed iff for all  $e \in \mathcal{E}$ ,  $A \cap \bar{e}$  is closed.

**Proposition 12.2.2.** *If a cell decomposition  $\mathcal{E}$  satisfies the Characteristic Maps axiom, then for every  $n$ -cell  $e \in \mathcal{E}$  we have  $\bar{e} = \Phi_e(D^n)$ . Therefore  $\bar{e}$  is compact and  $\bar{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$ .*

PROOF:

$\langle 1 \rangle 1.$   $e \subseteq \Phi_e(D^n) \subseteq \bar{e}$

PROOF:

$$\begin{aligned}
 e &= \Phi_e((D^n)^\circ) \\
 &\subseteq \Phi_e(D^n) \\
 &= \Phi_e(\overline{(D^n)^\circ}) \\
 &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
 &= \bar{e}
 \end{aligned}$$

$\langle 1 \rangle 2.$   $\Phi_e(D^n)$  is compact.

PROOF: Because  $D^n$  is compact.

$\langle 1 \rangle 3.$   $\Phi_e(D^n)$  is closed.

$\langle 1 \rangle 4.$   $\Phi_e(D^n) = \bar{e}$

□

## Chapter 13

# Topological Groups

**Definition 13.0.1** (Topological Group). A *topological group* is a group  $G$  with a topology such that the function  $G^2 \rightarrow G$  that maps  $(x, y)$  to  $xy^{-1}$  is continuous.

**Example 13.0.2.**  $GL(n, \mathbb{R})$  and  $GL(n, \mathbb{C})$  are topological groups.

**Proposition 13.0.3.** Any subgroup of a topological group is a topological group under the subspace topology.

**Definition 13.0.4** (Homogeneous Space). A *homogeneous space* is a topological space of the form  $G/H$ , where  $G$  is a topological group and  $H$  is a normal subgroup of  $G$ , under the quotient topology.

**Proposition 13.0.5.** Let  $G$  be a topological group and  $H$  a normal subgroup of  $G$ . Then  $G/H$  is Hausdorff if and only if  $H$  is closed.

PROOF: See Bourbaki, N., General Topology. III.12  $\square$

### 13.1 Continuous Actions

**Definition 13.1.1** (Continuous Action). Let  $G$  be a topological group and  $X$  a topological space. A *continuous action* of  $G$  on  $X$  is a continuous function  $\cdot : G \times X \rightarrow X$  such that:

- $\forall x : \text{El}(X) . ex = x$
- $\forall g, h : \text{El}(G) . \forall x : \text{El}(X) . g(hx) = (gh)x$

A  $G$ -space consists of a topological space  $X$  and a continuous action of  $G$  on  $X$ .

**Definition 13.1.2** (Orbit). Let  $X$  be a  $G$ -space and  $x \in X$ . The *orbit* of  $x$  is  $\{gx : g \in G\}$ .

The *orbit space*  $X/G$  is the set of all orbits under the quotient topology.

**Proposition 13.1.3.** *Define an action of  $SO(2)$  on  $S^2$  by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

*Then  $S^2/SO(2) \cong [-1, 1]$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f_3 : S^2/SO(2) \rightarrow [-1, 1]$  be the function induced by  $\pi_3 : S^2 \rightarrow [-1, 1]$

$\langle 1 \rangle 2$ .  $f_3$  is bijective.

$\langle 1 \rangle 3$ .  $S^2/SO(2)$  is compact.

PROOF: It is the continuous image of  $S^2$  which is compact.

$\langle 1 \rangle 4$ .  $[-1, 1]$  is Hausdorff.

$\langle 1 \rangle 5$ .  $f_3$  is a homeomorphism.

□

**Definition 13.1.4** (Stabilizer). Let  $X$  be a  $G$ -space and  $x \in X$ . The *stabilizer* of  $x$  is  $G_x := \{g \in G \mid gx = x\}$ .

**Proposition 13.1.5.** *The function that maps  $gG_x$  to  $gx$  is a continuous bijection from  $G/G_x$  to  $Gx$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $gG_x = hG_x$  then  $gx = hx$ .

$\langle 2 \rangle 1$ . ASSUME:  $gG_x = hG_x$

$\langle 2 \rangle 2$ .  $g^{-1}h \in G_x$

$\langle 2 \rangle 3$ .  $g^{-1}hx = x$

$\langle 2 \rangle 4$ .  $gx = hx$

$\langle 1 \rangle 2$ . If  $gx = hx$  then  $gG_x = hG_x$ .

PROOF: Similar.

$\langle 1 \rangle 3$ . The function is continuous.

PROOF: Proposition 10.12.2.

□



## Chapter 14

# Topological Vector Spaces

**Definition 14.0.1** (Topological Vector Space). Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . A *topological vector space* over  $K$  consists of a vector space  $E$  over  $K$  and a topology on  $E$  such that:

- Subtraction is a continuous function  $E^2 \rightarrow E$
- Multiplication is a continuous function  $K \times E \rightarrow E$

**Proposition 14.0.2.** *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition.  $\square$

**Theorem 14.0.3.** *The usual topology on a finite dimensional vector space over  $K$  is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18  $\square$

**Proposition 14.0.4.** *Let  $E$  be a topological vector space and  $E_0$  a subspace of  $E$ . Then  $\overline{E_0}$  is a subspace of  $E$ .*

**Definition 14.0.5.** Let  $E$  be a topological vector space. The topological space associated with  $E$  is  $E/\overline{\{0\}}$ .

### 14.1 Cauchy Sequences

**Definition 14.1.1** (Cauchy Sequence). Let  $E$  be a topological vector space. A sequence  $(x_n)$  in  $E$  is a *Cauchy sequence* iff, for every neighbourhood  $U$  of 0, there exists  $n_0$  such that  $\forall m, n \geq n_0, x_n - x_m \in U$ .

**Definition 14.1.2** (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

## 14.2 Seminorms

**Definition 14.2.1** (Seminorm). Let  $E$  be a vector space over  $K$ . A *seminorm* on  $E$  is a function  $\| \cdot \| : E \rightarrow \mathbb{R}$  such that:

1.  $\forall x : \text{El}(E) . \|x\| \geq 0$
2.  $\forall \alpha : \text{El}(K) . \forall x : \text{El}(E) . \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality*  $\forall x, y : \text{El}(E) . \|x + y\| \leq \|x\| + \|y\|$

**Example 14.2.2.** The function that maps  $(x_1, \dots, x_n)$  to  $|x_i|$  is a seminorm on  $\mathbb{R}^n$ .

**Definition 14.2.3.** Let  $E$  be a vector space over  $K$ . Let  $\Lambda$  be a set of seminorms on  $E$ . The topology *generated* by  $\Lambda$  is the topology generated by the subbasis consisting of all sets of the form  $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$  for  $\epsilon > 0$ ,  $\lambda \in \Lambda$  and  $x : \text{El}(E)$ .

**Proposition 14.2.4.**  $E$  is a topological vector space under this topology. It is Hausdorff iff, for all  $x : \text{El}(E)$ , if  $\forall \lambda \in \Lambda . \lambda(x) = 0$  then  $x = 0$ .

## 14.3 Fréchet Spaces

**Definition 14.3.1** (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

**Proposition 14.3.2.** Let  $E$  be a pre-Fréchet space whose topology is generated by the family of seminorms  $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$ . Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

**Definition 14.3.3** (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

## 14.4 Normed Spaces

**Definition 14.4.1** (Normed Space). Let  $E$  be a vector space over  $K$ . A *norm* on  $E$  is a function  $\| \cdot \| : E \rightarrow \mathbb{R}$  is a seminorm such that,  $\forall x \in E . \|x\| = 0 \Leftrightarrow x = 0$ .

A *normed space* consists of a vector space with a norm.

**Proposition 14.4.2.** If  $E$  is a normed space then  $d(x, y) = \|x - y\|$  is a metric on  $E$  that makes  $E$  into a topological vector space. The two definitions of Cauchy sequence agree on  $E$ .

**Definition 14.4.3** ( $p$ -norm). For any  $p \geq 1$ , the  $p$ -norm on  $\mathbb{R}^n$  is defined by

$$\|\vec{x}\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1$ . For all  $\vec{x} \in \mathbb{R}^n$  we have  $\|\vec{x}\|_p \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$ . For all  $\alpha \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^n$  we have  $\|\alpha \vec{x}\|_p = |\alpha| \|\vec{x}\|_p$

PROOF:

$$\begin{aligned} \|\alpha(x_1, \dots, x_n)\| &= \|(\alpha x_1, \dots, \alpha x_n)\| \\ &= \left( \sum_{i=1}^n (\alpha x_i)^p \right)^{\frac{1}{p}} \\ &= \left( |\alpha|^p \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|\vec{x}\|_p \end{aligned}$$

$\langle 1 \rangle 3$ . The triangle inequality holds.

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\ &= \sum_{i=1}^n |x_i + y_i| |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^n (|x_i| + |y_i|) |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} \\ &\leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} \quad (\text{Hölder's Inequality}) \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p) \|\vec{x} + \vec{y}\|_p^{p-1} \end{aligned}$$

Assuming w.l.o.g.  $\|\vec{x} + \vec{y}\|_p^{p-1} \neq 0$  (using ??) we have  $\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$ .

$\langle 1 \rangle 4$ . For any  $\vec{x} \in \mathbb{R}^n$ , we have  $\|\vec{x}\| = 0$  iff  $\vec{x} = \vec{0}$ .

PROOF:  $\sum_{i=1}^n x_i^p = 0$  iff  $x_1 = \dots = x_n = 0$ .

□

**Definition 14.4.4** (Sup-norm). The *sup-norm* on  $\mathbb{R}^n$  is defined by

$$\|(x_1, \dots, x_n)\|_\infty := \max(|x_1|, \dots, |x_n|) .$$

**Proposition 14.4.5.** *The 2-norm on  $\mathbb{R}^n$  induces the standard metric.*

PROOF: Immediate from definitions.  $\square$

**Definition 14.4.6.** For  $p \geq 1$ , the normed space  $l_p$  is the set of all sequences  $(x_n)$  in  $\mathbb{R}$  such that  $\sum_{n=1}^\infty x_n^p$  converges, under

$$\|(x_n)\|_p := \left( \sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} .$$

**Proposition 14.4.7.** *The spaces  $l_p$  for  $p \geq 1$  are all homeomorphic.*

PROOF: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. <http://dx.doi.org/10.1007/BF01075865>.

**Definition 14.4.8.** Let  $l_\infty$  be the set of all bounded sequences in  $\mathbb{R}$  under

$$\|(x_n)\| := \sup_n |x_n|$$

**Proposition 14.4.9.** *For all  $p \geq 1$  we have  $l_p$  is not homeomorphic to  $l_\infty$ .*

**Proposition 14.4.10.** *Let  $\| \cdot \|$  be a seminorm on the vector space  $E$ . Then  $\| \cdot \|$  defines a norm on  $E/\{0\}$ .*

**Proposition 14.4.11.** *Let  $E$  and  $F$  be normed spaces. Any continuous linear map  $E \rightarrow F$  is uniformly continuous.*

**Definition 14.4.12.** For  $p \geq 1$ . let  $\mathcal{L}^p(\mathbb{R}^n)$  be the vector space of all Lebesgue-measurable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $|f|^p$  is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on  $\mathcal{L}^p(\mathbb{R}^n)$ . Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\{0\} .$$

## 14.5 Inner Product Spaces

**Proposition 14.5.1.** *If  $E$  is an inner product space then  $\|x\| = \sqrt{\langle x, x \rangle}$  is a norm on  $E$ .*

## 14.6 Banach Spaces

**Definition 14.6.1** (Banach Space). A *Banach space* is a complete normed space.

**Example 14.6.2.** For any topological space  $X$ , the set  $C(X)$  of bounded continuous functions  $X \rightarrow \mathbb{R}$  is a Banach space under  $\|f\| = \sup_{x \in X} |f(x)|$ .

**Proposition 14.6.3.** *The completion of a normed space is a Banach space.*

**Proposition 14.6.4.** *Let  $E$  and  $F$  be normed spaces. Let  $f : E \rightarrow F$  be a continuous linear map. Then the extension to the completions  $\hat{E} \rightarrow \hat{F}$  is linear.*

**Proposition 14.6.5.**  $L^p(\mathbb{R}^n)$  is a Banach space.

**Proposition 14.6.6.**  $C(\mathbb{R})$  is first countable but not second countable.

PROOF: For every sequence of 0s and 1s  $s = (s_n)$ , let  $f_s$  be a continuous bounded function whose value at  $n$  is  $s_n$ . Then the set of all  $f_s$  is an uncountable discrete set in  $C(\mathbb{R})$ . Hence  $C(\mathbb{R})$  is not second countable.

It is first countable because it is metrizable.  $\square$

## 14.7 Hilbert Spaces

**Definition 14.7.1** (Hilbert Space). A *Hilbert space* is a complete inner product space.

**Example 14.7.2.** The set of *square-integrable functions* is the set of Lebesgue integrable functions  $[-\pi, \pi] \rightarrow \mathbb{R}$  quotiented by:  $f \sim g$  iff  $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$  has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

**Proposition 14.7.3.** *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

## 14.8 Locally Convex Spaces

**Definition 14.8.1** (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

**Proposition 14.8.2.** *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18.  $\square$

**Proposition 14.8.3.** *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18.  $\square$

**Example 14.8.4.** Let  $E$  be an infinite dimensional Hilbert space. Let  $E'$  be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map  $E \rightarrow \mathbb{R}$  is continuous as a map  $E' \rightarrow \mathbb{R}$ . Then  $E$  is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

**Definition 14.8.5** (Thom Space). Let  $E$  be a vector bundle with a Riemannian metric,  $DE = \{x : \text{El}(E) \mid \|x\| \leq 1\}$  its disc bundle and  $SE := \{v : \text{El}(E) \mid \|v\| = 1\}$  its sphere bundle. The *Thom space* of  $E$  is the quotient space  $DE/SE$ .