

Mathematics

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Chapter 1

Primitive Notions and Axioms

Let there be *sets*.

Given sets A and B , let there be *functions* from A to B . We write $f : A \rightarrow B$ for ' f is a function from A to B '. We call A the *domain* of f , and B the *codomain*.

Given sets A , B and C , and functions $f : A \rightarrow B$ and $g : B \rightarrow C$, let there be a function $gf = g \circ f : A \rightarrow C$, the *composite* of f and g .

Axiom 1.1 (Associativity). *For any functions $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f .$$

Axiom 1.2 (Identity). *For any set A , there exists a function $\text{id}_A : A \rightarrow A$, called an identity function on A , such that:*

- *for every set B and function $f : A \rightarrow B$, we have $f \circ \text{id}_A = f$;*
- *for every set B and function $f : B \rightarrow A$, we have $\text{id}_A \circ f = f$.*

Proposition 1.3. *The identity function on a set is unique.*

PROOF: If $i, j : A \rightarrow A$ are identity functions on A then we have $i = i \circ j = j$. \square

Definition 1.4 (Isomorphism). A function $i : A \rightarrow B$ is an *isomorphism*, $i : A \cong B$, iff there exists a function $i^{-1} : B \rightarrow A$, the *inverse* of i , such that $i^{-1} \circ i = \text{id}_A$ and $i \circ i^{-1} = \text{id}_B$.

Axiom 1.5 (Terminal Set). *There exists a set 1 such that, for any set A , there exists a unique function $A \rightarrow 1$.*

Proposition 1.6. *The terminal set is unique up to unique isomorphism.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B be terminal sets.

$\langle 1 \rangle 2$. LET: i be the unique function $A \rightarrow B$.

$\langle 1 \rangle 3$. LET: i^{-1} be the unique function $B \rightarrow A$.

$\langle 1 \rangle 4$. $i \circ i^{-1} = \text{id}_B$

PROOF: Since there is only one function $B \rightarrow B$.

$\langle 1 \rangle 5$. $i^{-1} \circ i = \text{id}_A$

PROOF: Since there is only one function $A \rightarrow A$.

□

Definition 1.7 (Element). For any set A , an *element* of A is a function $1 \rightarrow A$.

We write $a \in A$ for $a : 1 \rightarrow A$. Given $f : A \rightarrow B$ and $a \in A$, we write $f(a)$ for $f \circ a$.

Axiom 1.8 (Extensionality). Let A and B be sets. Let $f, g : A \rightarrow B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.

Axiom 1.9 (Empty Set). There exists a set with no elements.

Axiom 1.10 (Products). Let A and B be sets. There exists a set $A \times B$ and functions $\pi_1 : A \times B \rightarrow A$, $\pi_2 : A \times B \rightarrow B$, the projections, such that, for every set X and functions $f : X \rightarrow A$, $g : X \rightarrow B$, there exists a unique function $\langle f, g \rangle : X \rightarrow A \times B$ such that

$$\pi_1 \circ \langle f, g \rangle = f, \quad \pi_2 \circ \langle f, g \rangle = g \quad .$$

Proposition 1.11. If $\pi_1 : P \rightarrow A$ and $\pi_2 : P \rightarrow B$ form a product of A and B , and $p_1 : Q \rightarrow A$ and $p_2 : Q \rightarrow B$ form a product of A and B , then there exists a unique isomorphism $i : P \cong Q$ such that $p_1 \circ i = \pi_1$ and $p_2 \circ i = \pi_2$.

PROOF:

$\langle 1 \rangle 1$. LET: $i : P \rightarrow Q$ be the unique function such that $p_1 \circ i = \pi_1$ and $p_2 \circ i = \pi_2$.

$\langle 1 \rangle 2$. LET: $i^{-1} : Q \rightarrow P$ be the unique function such that $\pi_1 \circ i^{-1} = p_1$ and $\pi_2 \circ i^{-1} = p_2$

$\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_Q$

PROOF: Each is the unique $x : Q \rightarrow Q$ such that $p_1 \circ x = p_1$ and $p_2 \circ x = p_2$.

$\langle 1 \rangle 4$. $i^{-1} \circ i = \text{id}_P$

PROOF: Each is the unique $x : P \rightarrow P$ such that $\pi_1 \circ x = \pi_1$ and $\pi_2 \circ x = \pi_2$.

□

Definition 1.12. Given functions $f : A \rightarrow B$ and $g : C \rightarrow D$, define $f \times g : A \times C \rightarrow B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle \quad .$$

Axiom 1.13 (Function Sets). Let A and B be sets. There exists a set A^B and function $\epsilon : A^B \times B \rightarrow A$ such that, for any set X and function $f : X \times B \rightarrow A$, there exists a unique function $\lambda f : X \rightarrow A^B$ such that

$$f = \epsilon \circ \langle \lambda f \circ \pi_1, \pi_2 \rangle \quad .$$

Definition 1.14 (Inverse Image). Let A , X and Y be sets. Let $f : X \rightarrow Y$, $a \in Y$ and $j : A \rightarrow X$. Then j is the *inverse image* of a under f if and only if:

- $f \circ j = a \circ !_A$
- for every set I and function $q : I \rightarrow X$ such that $f \circ q = a \circ !_I$, there exists a unique $\bar{q} : I \rightarrow A$ such that $q = j \circ \bar{q}$.

Axiom 1.15 (Inverse Images). For any sets X and Y , function $f : X \rightarrow Y$ and element $a \in Y$, there exists a set $f^{-1}(a)$ and function $j : f^{-1}(a) \rightarrow X$ such that j is the inverse image of a under f .

Definition 1.16 (Injective). A function $f : A \rightarrow B$ is *injective* iff, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

Definition 1.17 (Surjective). A function $f : A \rightarrow B$ is *surjective* iff, for every set X and functions $x, y : B \rightarrow X$, if $x \circ f = y \circ f$ then $x = y$.

Axiom 1.18 (Subset Classifier). There exists a set 2 and function $\top : 1 \rightarrow 2$ such that, for every injective function $f : A \rightarrow X$, there exists a unique function $\chi : X \rightarrow 2$ such that f is the inverse image of \top under χ .

Axiom 1.19 (Natural Numbers). There exists a set \mathbb{N} , an element $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that, for every set X , element $a \in X$ and function $r : X \rightarrow X$, there exists a unique function $x : \mathbb{N} \rightarrow X$ such that $x \circ 0 = a$ and $x \circ s = r \circ x$.

Axiom 1.20 (Choice). For every surjective function $r : X \rightarrow Y$, there exists $s : Y \rightarrow X$ such that $r \circ s$ is an identity function on Y .