

Mathematics

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February 9, 2024

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Part I

Category Theory

Chapter 1

Foundations

This is a placeholder — I am not sure what foundation I want to use for this project yet. I will try to work in a way which is foundation-independent. What I do could be formalized in ZFC, ETCS, or some other system. I will assume the usual set theoretic constructions as needed. Sets will be defined up to bijection only.

Chapter 2

Categories

Definition 2.1 (Category). A *category* \mathcal{C} consists of:

- A class $|\mathcal{C}|$ of *objects*. We write $A \in \mathcal{C}$ for $A \in |\mathcal{C}|$.
- For any objects A, B , a set $\mathcal{C}[A, B]$ of *morphisms* from A to B . We write $f : A \rightarrow B$ for $f \in \mathcal{C}[A, B]$.
- For any object A , a morphism $\text{id}_A : A \rightarrow A$, the *identity* morphism on A .
- For any morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, a morphism $g \circ f : A \rightarrow C$, the *composite* of f and g .

such that:

Associativity Given $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Left Unit Law For any morphism $f : A \rightarrow B$, we have $\text{id}_B \circ f = f$.

Right Unit Law For any morphism $f : A \rightarrow B$, we have $f \circ \text{id}_A = f$.

Proposition 2.2. *The identity morphism on an object is unique.*

PROOF: If i and j are identity morphisms on A then $i = i \circ j = j$. \square

Example 2.3 (Category of Sets). The *category of sets* **Set** has objects all sets and morphisms all functions.

Definition 2.4 (Endomorphism). In a category \mathcal{C} , an *endomorphism* on an object A is a morphism $A \rightarrow A$. We write $\text{End}_{\mathcal{C}}(A)$ for $\mathcal{C}[A, A]$.

Definition 2.5 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with the same objects as \mathcal{C} and

$$\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$$

2.1 Preorders

Definition 2.6 (Preorder). A *preorder* on a set A is a relation \leq on A that is reflexive and transitive.

A *preordered set* is a pair (A, \leq) such that \leq is a preorder on A . We usually write A for the preordered set (A, \leq) .

We identify any preordered set A with the category whose objects are the elements of A , with one morphism $a \rightarrow b$ iff $a \leq b$, and no morphism $a \rightarrow b$ otherwise.

Example 2.7. For any ordinal α , let α be the preorder $\{\beta : \beta < \alpha\}$ under \leq .

Definition 2.8 (Discrete Preorder). We identify any set A with the *discrete* preorder $(A, =)$.

2.2 Monomorphisms and Epimorphisms

Definition 2.9 (Monomorphism). In a category, let $f : A \rightarrow B$. Then f is a *monomorphism* or *monic* iff, for every object X and morphism $x, y : X \rightarrow A$, if $fx = fy$ then $x = y$.

Definition 2.10 (Epimorphism). In a category, let $f : A \rightarrow B$. Then f is a *epimorphism* or *epi* iff, for every object X and morphism $x, y : B \rightarrow X$, if $xf = yf$ then $x = y$.

Proposition 2.11. *The composite of two monomorphism is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$ be monic.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$

$\langle 1 \rangle 3$. ASSUME: $g \circ f \circ x = g \circ f \circ y$

$\langle 1 \rangle 4$. $f \circ x = f \circ y$

$\langle 1 \rangle 5$. $x = y$

□

Proposition 2.12. *The composite of two epimorphisms is epi.*

PROOF: Dual. □

Proposition 2.13. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is monic then f is monic.*

PROOF: If $f \circ x = f \circ y$ then $g \circ f \circ x = g \circ f \circ y$ and so $x = y$. □

Proposition 2.14. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is epi then g is epi.*

PROOF: Dual. □

Proposition 2.15. *A function is a monomorphism in **Set** iff it is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is monic then f is injective.

$\langle 2 \rangle 1$. ASSUME: f is monic.

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 4$. LET: $\bar{x}, \bar{y} : 1 \rightarrow A$ be the functions such that $\bar{x}(*) = x$ and $\bar{y}(*) = y$

$\langle 2 \rangle 5$. $f \circ \bar{x} = f \circ \bar{y}$

$\langle 2 \rangle 6$. $\bar{x} = \bar{y}$

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. $x = y$

$\langle 1 \rangle 3$. If f is injective then f is monic.

$\langle 2 \rangle 1$. ASSUME: f is injective.

$\langle 2 \rangle 2$. LET: X be a set and $x, y : X \rightarrow A$.

$\langle 2 \rangle 3$. ASSUME: $f \circ x = f \circ y$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $t \in X$

PROVE: $x(t) = y(t)$

$\langle 2 \rangle 5$. $f(x(t)) = f(y(t))$

$\langle 2 \rangle 6$. $x(t) = y(t)$

PROOF: By $\langle 2 \rangle 1$.

□

Proposition 2.16. *A function is an epimorphism in **Set** iff it is surjective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is an epimorphism then f is surjective.

$\langle 2 \rangle 1$. ASSUME: f is an epimorphism.

$\langle 2 \rangle 2$. LET: $b \in B$

$\langle 2 \rangle 3$. LET: $x, y : B \rightarrow 2$ be defined by $x(b) = 1$ and $x(t) = 0$ for all other $t \in B$, $y(t) = 0$ for all $t \in B$.

$\langle 2 \rangle 4$. $x \neq y$

$\langle 2 \rangle 5$. $x \circ f \neq y \circ f$

$\langle 2 \rangle 6$. There exists $a \in A$ such that $f(a) = b$.

$\langle 1 \rangle 3$. If f is surjective then f is an epimorphism.

$\langle 2 \rangle 1$. ASSUME: f is surjective.

$\langle 2 \rangle 2$. LET: $x, y : B \rightarrow X$

$\langle 2 \rangle 3$. ASSUME: $x \circ f = y \circ f$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $b \in B$

PROVE: $x(b) = y(b)$

$\langle 2 \rangle 5$. PICK $a \in A$ such that $f(a) = b$

$\langle 2 \rangle 6$. $x(f(a)) = y(f(a))$

$\langle 2 \rangle 7$. $x(b) = y(b)$

□

Proposition 2.17. *In a preorder, every morphism is monic and epi.*

PROOF: Immediate from definitions. \square

2.3 Sections and Retractions

Definition 2.18 (Section, Retraction). In a category, let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $r \circ s = \text{id}_B$.

Proposition 2.19. *Every identity morphism is a section and retraction of itself.*

PROOF: Immediate from definitions. \square

Proposition 2.20. *Let $r, r' : A \rightarrow B$ and $s : B \rightarrow A$. If r is a retraction of s and r' is a section of s then $r = r'$.*

PROOF:

$$\begin{aligned} r &= r \circ \text{id}_A \\ &= r \circ s \circ r' \\ &= \text{id}_B \circ r' \\ &= r' \end{aligned} \quad \square$$

Proposition 2.21. *Let $r_1 : A \rightarrow B$, $r_2 : B \rightarrow C$, $s_1 : B \rightarrow A$ and $s_2 : C \rightarrow B$. If r_1 is a retraction of s_1 and r_2 is a retraction of s_2 then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= \text{id}_C \end{aligned} \quad \square$$

Proposition 2.22. *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $s : A \rightarrow B$ be a section of $r : B \rightarrow A$.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$ satisfy $sx = sy$.

$\langle 1 \rangle 3$. $rsx = rsy$

$\langle 1 \rangle 4$. $x = y$

\square

Proposition 2.23. *Every retraction is epi.*

PROOF: Dual. \square

Proposition 2.24. *In Set, every epimorphism has a retraction.*

PROOF: By the Axiom of Choice. \square

Example 2.25. It is not true in general that every monomorphism in any category has a section. nor that every epimorphism in any category has a retraction.

In the category **2**, the morphism $0 \leq 1$ is monic and epi but has no retraction or section.

2.4 Isomorphisms

Definition 2.26 (Isomorphism). In a category \mathcal{C} , a morphism $f : A \rightarrow B$ is an *isomorphism*, denoted $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$, the *inverse* of f , such that $f^{-1} \circ f = \text{id}_A$ and $f \circ f^{-1} = \text{id}_B$.

An *automorphism* on an object A is an isomorphism between A and itself. We write $\text{Aut}_{\mathcal{C}}(A)$ for the set of all automorphisms on A .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 2.27. *The inverse of an isomorphism is unique.*

PROOF: Proposition 2.20. \square

Proposition 2.28. *For any object A we have $\text{id}_A : A \cong A$ and $\text{id}_A^{-1} = \text{id}_A$.*

PROOF: Since $\text{id}_A \circ \text{id}_A = \text{id}_A$ by the Unit Laws. \square

Proposition 2.29. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Immediate from definitions. \square

Proposition 2.30. *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF: From Proposition 2.21. \square

Definition 2.31 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

2.5 Initial and Terminal Objects

Definition 2.32 (Initial Object). An object I in a category is *initial* iff, for any object X , there is exactly one morphism $I \rightarrow X$.

Example 2.33. The empty set is the initial object in **Set**.

Definition 2.34 (Terminal Object). An object T in a category is *terminal* iff, for any object X , there is exactly one morphism $X \rightarrow T$.

Example 2.35. Every singleton is terminal in **Set**.

Proposition 2.36. *If I and J are initial in a category, then there exists a unique isomorphism $I \cong J$.*

PROOF:

- $\langle 1 \rangle 1$. LET: i be the unique morphism $I \rightarrow J$.
- $\langle 1 \rangle 2$. LET: i^{-1} be the unique morphism $J \rightarrow I$.
- $\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_J$

PROOF: Since there is only one morphism $J \rightarrow J$.

- $\langle 1 \rangle 4$. $i^{-1} \circ i = \text{id}_I$

PROOF: Since there is only one morphism $I \rightarrow I$.
 \square

Proposition 2.37. *If S and T are terminal in a category, then there exists a unique isomorphism $S \cong T$.*

PROOF: Dual. \square

Chapter 3

Functors

Definition 3.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for any morphism $f : A \rightarrow B : \mathcal{C}$, a morphism $Ff : FA \rightarrow FB : \mathcal{D}$

such that:

- $F\text{id}_A = \text{id}_{FA}$
- $F(g \circ f) = Fg \circ Ff$

Definition 3.2 (Identity Functor). For any category \mathcal{C} , the *identity functor* $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is defined by

$$\begin{aligned} 1_{\mathcal{C}}A &= A \\ 1_{\mathcal{C}}f &= f \end{aligned}$$

Definition 3.3 (Constant Functor). Given categories \mathcal{C} , \mathcal{D} and an object $D \in \mathcal{D}$, the *constant functor* $K^{\mathcal{C}}D : \mathcal{C} \rightarrow \mathcal{D}$ is the functor defined by

$$\begin{aligned} K^{\mathcal{C}}DC &= D \\ K^{\mathcal{C}}Df &= \text{id}_D \end{aligned}$$

3.1 Comma Categories

Definition 3.4 (Comma Category). Let $F : \mathcal{C} \rightarrow \mathcal{E}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors. The *comma category* $F \downarrow G$ is the category with:

- objects all pairs (C, D, f) where $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f : FC \rightarrow GD : \mathcal{E}$

- morphisms $(u, v) : (C, D, f) \rightarrow (C', D', g)$ all pairs $u : C \rightarrow C' : \mathcal{C}$ and $v : D \rightarrow D' : \mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ \downarrow Fu & & \downarrow Gv \\ FC' & \xrightarrow{g} & GD' \end{array}$$

Definition 3.5 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category* over A , denoted \mathcal{C}/A , is the comma category $1_{\mathcal{C}} \downarrow K^1 A$.

Definition 3.6 (Coslice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *coslice category* over A , denoted $\mathcal{C} \backslash A$, is the comma category $K^1 A \downarrow 1_{\mathcal{C}}$.

Definition 3.7 (Pointed Sets). The *category of pointed sets* \mathbf{Set}_* is the coslice category $\mathbf{Set} \backslash 1$.

Part II

Group Theory

Chapter 4

Groups

Definition 4.1 (Group). A *group* G consists of a set G and a binary operation $\cdot : G^2 \rightarrow G$ such that \cdot is associative, and there exists $e \in G$, the *identity* element of the group, such that:

- For all $x \in G$ we have $xe = ex = x$
- For all $x \in G$, there exists $x^{-1} \in G$, the *inverse* of x , such that $xx^{-1} = x^{-1}x = e$.

We identify a group G with the category G with one object and morphisms the elements of G , with composition given by \cdot .

The *order* of a group G , denoted $|G|$, is the number of elements in G if G is finite; otherwise we write $|G| = \infty$.

Proposition 4.2. *The identity in a group is unique.*

PROOF: Proposition 2.2.

Proposition 4.3. *The inverse of an element is unique.*

PROOF: If i and j are inverses of x then $i = ixj = j$. \square

Example 4.4. • The *trivial* group is $\{e\}$ under $ee = e$.

- \mathbb{Z} is a group under addition
- \mathbb{Q} is a group under addition
- $\mathbb{Q} - \{0\}$ is a group under multiplication
- \mathbb{R} is a group under addition
- $\mathbb{R} - \{0\}$ is a group under multiplication
- \mathbb{C} is a group under addition
- $\mathbb{C} - \{0\}$ is a group under multiplication

- $\{-1, 1\}$ is a group under multiplication
- The set of 2×2 real matrices with non-zero determinant is a group under matrix multiplication.

Proposition 4.5 (Cancellation). *Let G be a group. Let $a, g, h \in G$. If $ag = ah$ or $ga = ha$ then $g = h$.*

PROOF: If $ag = ah$ then $g = a^{-1}ag = a^{-1}ah = h$. Similarly if $ga = ha$. \square

Definition 4.6. Let G be a group. Let $g \in G$. We define $g^n \in G$ for all $n \in \mathbb{Z}$ as follows:

$$\begin{aligned} g^0 &= e \\ g^{n+1} &= g^n g & (n \geq 0) \\ g^{-n} &= (g^{-1})^n & (n > 0) \end{aligned}$$

Proposition 4.7. *Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then*

$$g^{m+n} = g^m g^n .$$

PROOF:

$\langle 1 \rangle 1$. For all $k \in \mathbb{Z}$ we have $g^{k+1} = g^k g$

$\langle 2 \rangle 1$. For all $k \geq 0$ we have $g^{k+1} = g^k g$

PROOF: Immediate from definition.

$\langle 2 \rangle 2$. $g^{-1+1} = g^{-1} g$

PROOF: Both are equal to e .

$\langle 2 \rangle 3$. For all $k > 1$ we have $g^{-k+1} = g^{-k} g$

PROOF:

$$\begin{aligned} g^{-k+1} &= (g^{-1})^{k-1} \\ &= (g^{-1})^{k-1} g^{-1} g \\ &= (g^{-1})^k g \\ &= g^{-k} g \end{aligned}$$

$\langle 1 \rangle 2$. For all $k \in \mathbb{Z}$ we have $g^{k-1} = g^k g^{-1}$

PROOF: Substitute $k = k - 1$ above and multiply by g^{-1} .

$\langle 1 \rangle 3$. $g^{m+0} = g^m g^0$

PROOF: Since $g^m g^0 = g^m e = g^m$.

$\langle 1 \rangle 4$. If $g^{m+n} = g^m g^n$ then $g^{m+n+1} = g^m g^{n+1}$

PROOF:

$$\begin{aligned} g^{m+n+1} &= g^{m+n} g & (\langle 1 \rangle 1) \\ &= g^m g^n g \\ &= g^m g^{n+1} & (\langle 1 \rangle 1) \end{aligned}$$

$\langle 1 \rangle 5$. If $g^{m+n} = g^m g^n$ then $g^{m+n-1} = g^m g^{n-1}$

PROOF:

$$\begin{aligned}
 g^{m+n-1}g &= g^{m+n} && (\langle 1 \rangle 1) \\
 &= g^m g^n \\
 \therefore g^{m+n-1} &= g^m g^n g^{-1} \\
 &= g^m g^{n-1} && (\langle 1 \rangle 2)
 \end{aligned}$$

□

Proposition 4.8. *Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then*

$$(g^m)^n = g^{mn}.$$

PROOF:

$$\langle 1 \rangle 1. (g^m)^0 = g^0$$

PROOF: Both sides are equal to e .

$$\langle 1 \rangle 2. \text{ If } (g^m)^n = g^{mn} \text{ then } (g^m)^{n+1} = g^{m(n+1)}.$$

PROOF:

$$\begin{aligned}
 (g^m)^{n+1} &= (g^m)^n g^m && (\text{Proposition 4.7}) \\
 &= g^{mn} g^m \\
 &= g^{mn+m} && (\text{Proposition 4.7})
 \end{aligned}$$

$$\langle 1 \rangle 3. \text{ If } (g^m)^n = g^{mn} \text{ then } (g^m)^{n-1} = g^{m(n-1)}.$$

PROOF:

$$\begin{aligned}
 (g^m)^n &= g^{mn} \\
 \therefore (g^m)^{n-1} g^m &= g^{mn-m} g^m && (\text{Proposition 4.7}) \\
 \therefore (g^m)^{n-1} &= g^{mn-m} && (\text{Cancellation})
 \end{aligned}$$

□

Definition 4.9 (Commute). Let G be a group and $g, h \in G$. We say g and h *commute* iff $gh = hg$.

4.1 Order of an Element

Definition 4.10 (Order). Let G be a group. Let $g \in G$. Then g has *finite order* iff there exists a positive integer n such that $g^n = e$. In this case, the *order* of g , denoted $|g|$, is the least positive integer n such that $g^n = e$.

If g does not have finite order, we write $|g| = \infty$.

Proposition 4.11. *Let G be a group. Let $g \in G$ and n be a positive integer. If $g^n = e$ then $|g| \mid n$.*

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } n = q|g| + d \text{ where } 0 \leq d < |g|$$

PROOF: Division Algorithm.

$$\langle 1 \rangle 2. g^d = e$$

PROOF:

$$\begin{aligned}
 e &= g^n \\
 &= g^{q|g|+d} \\
 &= (g^{|g|})^q g^d && \text{(Propositions 4.7, 4.8)} \\
 &= e^q g^d \\
 &= g^d
 \end{aligned}$$

$\langle 1 \rangle 3. d = 0$

PROOF: By minimality of $|g|$.

$\langle 1 \rangle 4. n = q|g|$

□

Corollary 4.11.1. *Let G be a group. Let $g \in G$ have finite order and $n \in \mathbb{Z}$. Then $g^n = e$ if and only if $|g| \mid n$.*

Proposition 4.12. *Let G be a group and $g \in G$. Then $|g| \leq |G|$.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: w.l.o.g. G is finite.

$\langle 1 \rangle 2.$ PICK i, j with $0 \leq i < j \leq |G|$ such that $g^i = g^j$.

PROOF: Otherwise $g^0, g^1, \dots, g^{|G|}$ would be $|G| + 1$ distinct elements of G .

$\langle 1 \rangle 3. g^{j-i} = e$

$\langle 1 \rangle 4. g$ has finite order and $|g| \leq |G|$

PROOF: Since $|g| \leq j - i \leq j \leq |G|$.

□

Proposition 4.13. *Let G be a group. Let $g \in G$ have finite order. Let $m \in \mathbb{N}$. Then*

$$|g^m| = \frac{\text{lcm}(m, |g|)}{m} = \frac{|g|}{\text{gcd}(m, |g|)}$$

PROOF: Since for any integer d we have

$$g^{md} = e \Leftrightarrow |g| \mid md \quad \text{(Corollary 4.11.1)}$$

$$\Leftrightarrow \text{lcm}(m, |g|) \mid md$$

$$\Leftrightarrow \frac{\text{lcm}(m, |g|)}{m} \mid d \quad \square$$

and so $|g^m| = \frac{\text{lcm}(m, |g|)}{m}$ by Corollary 4.11.1. □

Proposition 4.14. *Let G be a group. Let $g, h \in G$ have finite order. Assume $gh = hg$. Then $|gh|$ has finite order and*

$$|gh| \mid \text{lcm}(|g|, |h|)$$

PROOF: Since $(gh)^{\text{lcm}(|g|, |h|)} = g^{\text{lcm}(|g|, |h|)} h^{\text{lcm}(|g|, |h|)} = e$. □

Chapter 5

Abelian Groups

Definition 5.1 (Abelian Group). A group is *Abelian* iff any two elements commute.

In an Abelian group G , we often denote the group operation by $+$, the identity element by 0 and the inverse of an element g by $-g$. We write ng for g^n ($g \in G, n \in \mathbb{Z}$).

Part III

Linear Algebra

Definition 5.2. Let $\text{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ real matrices.