

Mathematics

Robin Adams

August 15, 2023

Contents

| | | |
|----------|--|-----------|
| 1 | Sets and Classes | 7 |
| 1.1 | Classes | 7 |
| 1.1.1 | Subclasses | 8 |
| 1.1.2 | Constructions of Classes | 8 |
| 1.2 | Sets and the Axiom of Extensionality | 9 |
| 1.3 | The Other Axioms | 9 |
| 1.4 | ZFC Extends Z | 12 |
| 1.5 | Consequences of the Axioms | 13 |
| 1.6 | Transitive Classes | 15 |
| 2 | Relations | 19 |
| 2.1 | Ordered Pairs | 19 |
| 2.2 | Relations | 20 |
| 2.2.1 | Identity Functions | 21 |
| 2.2.2 | Inverses | 21 |
| 2.2.3 | Composition | 21 |
| 2.2.4 | Properties of Relations | 21 |
| 2.3 | n-ary Relations | 23 |
| 2.4 | Well Founded Relations | 23 |
| 3 | Functions | 27 |
| 3.1 | Functions | 27 |
| 3.1.1 | Injective Functions | 29 |
| 3.1.2 | Surjective Functions | 29 |
| 3.1.3 | Bijections | 30 |
| 3.1.4 | Restrictions | 31 |
| 3.1.5 | Images | 31 |
| 3.1.6 | Inverse Images | 34 |
| 3.1.7 | Function Sets | 34 |
| 3.2 | Equinumerosity | 35 |
| 3.3 | Domination | 35 |

| | | |
|-----------|--|------------|
| 4 | Category Theory | 37 |
| 4.1 | Categories | 37 |
| 4.2 | Invertible Morphisms | 38 |
| 5 | Equivalence Relations | 39 |
| 6 | Ordering Relations | 43 |
| 6.1 | Partial Orders | 43 |
| 6.2 | Linear Orders | 48 |
| 6.3 | Well Orderings | 51 |
| 7 | Ordinal Numbers | 55 |
| 7.1 | Ordinals | 55 |
| 7.2 | Natural Numbers | 61 |
| 7.3 | Transitive Closure of a Set | 62 |
| 7.4 | Successors | 72 |
| 7.5 | The Well-Ordering Theorem and Zorn's Lemma | 74 |
| 7.6 | Ordinal Operations | 77 |
| 7.7 | Ordinal Arithmetic | 80 |
| 7.7.1 | Addition | 80 |
| 7.7.2 | Multiplication | 83 |
| 7.7.3 | Exponentiation | 84 |
| 7.8 | Sequences | 87 |
| 7.9 | Strict Supremum | 88 |
| 8 | Cardinal Numbers | 89 |
| 8.1 | Cardinal Numbers | 89 |
| 8.2 | Ordering on Cardinal Numbers | 92 |
| 9 | Natural Numbers | 95 |
| 9.1 | Inductive Sets | 95 |
| 9.2 | Cardinality | 97 |
| 9.3 | Countable Sets | 99 |
| 9.4 | Arithmetic | 99 |
| 9.5 | Sequences | 101 |
| 9.6 | Transitive Closure of a Set | 101 |
| 9.7 | The Veblen Fixed Point Theorem | 102 |
| 9.8 | Cantor Normal Form | 104 |
| 10 | The Cumulative Hierarchy | 105 |
| 11 | Models of Set Theory | 109 |

| | |
|---|------------|
| 12 Infinite Cardinals | 113 |
| 12.1 Arithmetic of Infinite Cardinals | 113 |
| 12.2 Alephs | 115 |
| 12.3 Beths | 115 |
| 12.4 Cofinality | 116 |
| 12.5 Inaccessible Cardinals | 120 |
| 13 Group Theory | 123 |
| 13.1 Groups | 123 |
| 13.2 Abelian Groups | 124 |
| 14 Ring Theory | 125 |
| 14.1 Rings | 125 |
| 14.2 Ordered Rings | 125 |
| 14.3 Integral Domains | 126 |
| 15 Field Theory | 127 |
| 15.1 Fields | 127 |
| 15.2 Ordered Fields | 130 |
| 16 Number Systems | 135 |
| 16.1 The Integers | 135 |
| 16.2 The Rationals | 139 |
| 16.3 The Real Numbers | 142 |
| 17 Real Analysis | 157 |
| 17.1 Step Functions | 157 |
| 17.2 Lebesgue Integration | 161 |
| 18 Complex Analysis | 163 |
| 19 Topology | 167 |
| 19.1 Topological Spaces | 167 |
| 19.2 Continuous Functions | 168 |
| 19.3 Convergence | 168 |
| 19.4 Homotopy | 169 |
| 19.5 Metric Spaces | 169 |
| 20 Ring Theory | 171 |
| 21 Linear Algebra | 173 |
| 21.1 Vector Spaces | 173 |
| 21.2 Subspaces | 174 |
| 21.3 Linear Transformations | 176 |
| 21.4 Normed Spaces | 177 |
| 21.4.1 Functionals | 191 |
| 21.4.2 Contraction Mappings | 191 |

| | |
|-------------------------------------|-----|
| 21.5 Banach Spaces | 191 |
| 21.6 Inner Product Spaces | 202 |

Chapter 1

Sets and Classes

1.1 Classes

Our language is the language of first-order logic with equality over one primitive binary predicate \in . We call all the objects we reason about *sets*. When $a \in b$, we say a is a *member* or *element* of b , or b *contains* a . We write $b \ni a$ for $a \in b$, and $a \notin b$ for $\neg(a \in b)$. We write $\forall x \in a. \phi$ as an abbreviation for $\forall x(x \in a \rightarrow \phi)$, and $\exists x \in a. \phi$ as an abbreviation for $\exists x(x \in a \wedge \phi)$.

We shall speak informally of *classes* as an abbreviation for talking about predicates. A *class* is determined by a unary predicate $\phi[x]$ (possibly with parameters). We write $\{x \mid \phi[x]\}$ or $\{x : \phi[x]\}$ for the class determined by $\phi[x]$. We write ' a is an element of $\{x \mid \phi[x]\}$ ' or ' $a \in \{x \mid \phi[x]\}$ ' for $\phi[a]$.

We write $\{t[x_1, \dots, x_n] \mid P[x_1, \dots, x_n]\}$ for

$$\{y \mid \exists x_1, \dots, x_n (y = t[x_1, \dots, x_n] \wedge P[x_1, \dots, x_n])\} .$$

We say two classes **A** and **B** are *equal*, and write $\mathbf{A} = \mathbf{B}$, iff $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$.

Proposition Schema 1.1.1. *For any class **A**, the following is a theorem.*

$$\mathbf{A} = \mathbf{A}$$

PROOF: We have $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{A})$. \square

Proposition Schema 1.1.2. *For any classes **A** and **B**, the following is a theorem.*

If $\mathbf{A} = \mathbf{B}$ then $\mathbf{B} = \mathbf{A}$.

PROOF: If $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$ then $\forall x(x \in \mathbf{B} \leftrightarrow x \in \mathbf{A})$. \square

Proposition Schema 1.1.3. *For any classes **A**, **B** and **C**, the following is a theorem.*

If $\mathbf{A} = \mathbf{B}$ and $\mathbf{B} = \mathbf{C}$ then $\mathbf{A} = \mathbf{C}$.

PROOF: If $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$ and $\forall x(x \in \mathbf{B} \leftrightarrow x \in \mathbf{C})$ then $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{C})$. \square

1.1.1 Subclasses

Definition 1.1.4 (Subclass). We say a class \mathbf{A} is a *subclass* of \mathbf{B} , or \mathbf{B} is a *superclass* of \mathbf{A} , or \mathbf{B} *includes* \mathbf{A} , and write $\mathbf{A} \subseteq \mathbf{B}$ or $\mathbf{B} \supseteq \mathbf{A}$, iff every element of \mathbf{A} is an element of \mathbf{B} . Otherwise we write $\mathbf{A} \not\subseteq \mathbf{B}$ or $\mathbf{B} \not\supseteq \mathbf{A}$.

We say \mathbf{A} is a *proper* subclass of \mathbf{B} , \mathbf{B} is a *proper* superclass of \mathbf{A} , or \mathbf{B} *properly* includes \mathbf{A} , and write $\mathbf{A} \subsetneq \mathbf{B}$ or $\mathbf{B} \supsetneq \mathbf{A}$, iff $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$.

Proposition Schema 1.1.5. *For any class \mathbf{A} , the following is a theorem.*

$$\mathbf{A} \subseteq \mathbf{A}$$

PROOF: Every element of \mathbf{A} is an element of \mathbf{A} . \square

Proposition Schema 1.1.6. *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem.*

If $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$ then $\mathbf{A} = \mathbf{B}$.

PROOF: If every element of \mathbf{A} is an element of \mathbf{B} , and every element of \mathbf{B} is an element of \mathbf{A} , then \mathbf{A} and \mathbf{B} have exactly the same elements. \square

Proposition Schema 1.1.7. *For any classes \mathbf{A} , \mathbf{B} and \mathbf{C} , the following is a theorem.*

If $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{C}$ then $\mathbf{A} \subseteq \mathbf{C}$.

PROOF: If every element of \mathbf{A} is an element of \mathbf{B} and every element of \mathbf{B} is an element of \mathbf{C} then every element of \mathbf{A} is an element of \mathbf{C} . \square

1.1.2 Constructions of Classes

Definition 1.1.8 (Empty Class). The *empty class* \emptyset is $\{x \mid \perp\}$. Every other class is *nonempty*.

Definition 1.1.9 (Universal Class). The *universal class* \mathbf{V} is $\{x \mid \top\}$.

Definition 1.1.10 (Enumeration). Given objects a_1, \dots, a_n , we define the class $\{a_1, \dots, a_n\}$ to be the class $\{x \mid x = a_1 \vee \dots \vee x = a_n\}$.

Definition 1.1.11 (Intersection). For any classes \mathbf{A} and \mathbf{B} , the *intersection* $\mathbf{A} \cap \mathbf{B}$ is $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$.

Definition 1.1.12 (Union). For any classes \mathbf{A} and \mathbf{B} , the *union* $\mathbf{A} \cup \mathbf{B}$ is $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$.

Definition 1.1.13 (Relative Complement). Let \mathbf{A} and \mathbf{B} be classes. The *relative complement* of \mathbf{B} in \mathbf{A} is the class $\mathbf{A} - \mathbf{B} := \{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

Definition 1.1.14 (Symmetric Difference). For any classes \mathbf{A} and \mathbf{B} , the *symmetric difference* is the class $\mathbf{A} + \mathbf{B} := (\mathbf{A} - \mathbf{B}) \cup (\mathbf{B} - \mathbf{A})$.

Definition 1.1.15 (Pairwise disjoint). Let \mathbf{A} be a class. We say the elements of \mathbf{A} are *pairwise disjoint* iff, for all $x, y \in \mathbf{A}$, if $x \cap y \neq \emptyset$ then $x = y$.

1.2 Sets and the Axiom of Extensionality

Definition 1.2.1 (Axiom of Extensionality). The *Axiom of Extensionality* is the statement: if two sets have exactly the same members, then they are equal.

$$\forall x, y (\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y) \quad .$$

When working in a theory with the Axiom of Extensionality, we may identify a set a with the class $\{x \mid x \in a\}$. Our use of the symbols \in and $=$ is consistent. We say a class \mathbf{A} is a *set* iff there exists a set a such that $a = \mathbf{A}$; that is, $\{x \mid \phi[x]\}$ is a set iff $\exists a \forall x (x \in a \leftrightarrow \phi[x])$. Otherwise, \mathbf{A} is a *proper class*.

Definition 1.2.2 (Subset). If A is a set and $A \subseteq \mathbf{B}$, we say A is a *subset* of \mathbf{B} .

Definition 1.2.3 (Union). The *union* of a class \mathbf{A} is $\{x \mid \exists X \in \mathbf{A}. x \in X\}$. We write $\bigcup_{P(x)} t(x)$ for $\bigcup \{t(x) \mid P(x)\}$.

Definition 1.2.4 (Intersection). The *intersection* of a class \mathbf{A} is $\{x \mid \forall X \in \mathbf{A}. x \in X\}$. We write $\bigcap_{P(x)} t(x)$ for $\bigcap \{t(x) \mid P(x)\}$.

Definition 1.2.5 (Power Class). For any class \mathbf{A} , the *power class* $\mathcal{P}\mathbf{A}$ is $\{X \mid X \subseteq \mathbf{A}\}$.

1.3 The Other Axioms

Definition 1.3.1 (Empty Set Axiom). The *Empty Set Axiom* is the statement: The empty class \emptyset is a set.

$$\exists e \forall x x \notin e$$

Definition 1.3.2 (Pairing Axiom). The *Pairing Axiom* or *Pair Set Axiom* is the statement: for any sets a and b , the class $\{a, b\}$ is a set.

$$\forall a \forall b \exists c \forall x (x \in c \Leftrightarrow x = a \vee x = b)$$

Definition 1.3.3 (Union Axiom). The *Union Axiom* is the statement: for any set A , the class $\bigcup A$ is a set.

$$\forall A \exists B \forall x (x \in B \Leftrightarrow \exists y (y \in A \wedge x \in y))$$

Definition 1.3.4 (Comprehension Axiom Scheme). The *Comprehension Axiom Scheme*, *Subset Axiom Scheme* or *Aussonderungsaxiom Scheme* is the set of sentences of the form, for any class \mathbf{A} : If \mathbf{A} is a subclass of a set then \mathbf{A} is a set.

That is, for any property $P[x, y_1, \dots, y_n]$:

For any sets a_1, \dots, a_n and B , the class $\{x \in B \mid P[x, a_1, \dots, a_n]\}$ is a set.

$$\forall a_1, \dots, a_n, B. \exists C. \forall x (x \in C \Leftrightarrow x \in B \wedge P[x, a_1, \dots, a_n])$$

Definition 1.3.5 (Replacement Axiom Scheme). The *Replacement Axiom Scheme* is the set of sentences of the form, for some property $P[x, y, z_1, \dots, z_n]$:

For any sets a_1, \dots, a_n, B , assume for all $x \in B$ there exists at most one y such that $P[x, y, a_1, \dots, a_n]$. Then $\{y \mid \exists x \in B. P[x, y, a_1, \dots, a_n]\}$ is a set.

$$\forall a_1, \dots, a_n, B (\forall x \in B. \forall y, y' (P[x, y, a_1, \dots, a_n] \wedge P[x, y', a_1, \dots, a_n] \Rightarrow y = y') \Rightarrow \\ \exists C \forall y (y \in C \Leftrightarrow \exists x \in B. P[x, y, a_1, \dots, a_n]))$$

Definition 1.3.6 (Power Set Axiom). The *Power Set Axiom* is the statement: the power class of a set is a set.

$$\forall A \exists B \forall x (x \in B \Leftrightarrow \forall y (y \in x \Rightarrow y \in A))$$

Definition 1.3.7 (Axiom of Infinity). The *Axiom of Infinity* is the statement: there exists a set I such that $\emptyset \in I$ and $\forall x \in I. x \cup \{x\} \in I$.

$$\exists I (\emptyset \in I. \forall x. x \notin e \wedge \forall x \in I. \exists y \in I. \forall z (z \in y \Leftrightarrow z \in x \vee z = x))$$

Definition 1.3.8 (Axiom of Choice). The *Axiom of Choice* is the statement: For any set A of pairwise disjoint, nonempty sets, there exists a set C such that, for all $x \in A$, we have $x \cap C$ has exactly one element.

$$\forall A (\forall x \in A. \exists y y \in x \wedge \\ \forall x, y \in A. \forall z (z \in x \wedge z \in y \Rightarrow x = y) \Rightarrow \\ \exists C. \forall x \in A. \exists y \forall z (z \in x \wedge z \in C \Leftrightarrow z = y))$$

Definition 1.3.9 (Axiom of Regularity). The *Axiom of Regularity* or *Axiom of Foundation* is the statement: for any A , if A has a member, then there exists $m \in A$ such that $m \cap A = \emptyset$.

$$\forall A (\exists x. x \in A \Rightarrow \exists m \in A. \neg \exists x (x \in m \wedge x \in A))$$

Definition 1.3.10 (Skolem Set Theory). *Skolem set theory* (S) is the theory whose axioms are:

- Extensionality
- Empty Set
- Pairing
- Union
- Comprehension
- Power Set

- Regularity

Let SC be the extension of S with the Axiom of Choice.

Definition 1.3.11 (Skolem-Fraenkel Set Theory). *Skolem-Fraenkel set theory* (SF) is the theory whose axioms are:

- Extensionality
- Empty Set
- Union
- Replacement
- Power Set
- Regularity

Let SFC be the extension of SF with the Axiom of Choice.

Definition 1.3.12 (Zermelo Set Theory). *Zermelo set theory* is the theory whose axioms are:

- Extensionality
- Pairing
- Union
- Comprehension
- Power Set
- Infinity
- Regularity

We label theorems with Z when they are provable in Zermelo set theory.

Let ZC be the extension of Z with the Axiom of Choice.

Definition 1.3.13 (Fraenkel-Mostowski Set Theory). *Fraenkel-Mostowski set theory* (FM) is the theory whose axioms are:

- The *Axiom of Extensionality with Urelements*: For any sets x and y , if x is nonempty and x and y have exactly the same elements, then $x = y$.
- Union
- Replacement
- Power Set
- Infinity

- Regularity

We write FMC for the extension of FM with Choice.

Definition 1.3.14 (Zermelo-Fraenkel Set Theory). *Zermelo-Fraenkel set theory* is the theory whose axioms are:

- Extensionality
- Union
- Replacement
- Power Set
- Infinity
- Regularity

We label theorems with ZF when they are provable in Zermelo-Fraenkel set theory.

Let ZFC be the extension of ZF with the Axiom of Choice.

We label a theorem with FOL if it can be proved in first-order logic, i.e. from no axioms.

1.4 ZFC Extends Z

Proposition 1.4.1. *The Axiom of Infinity implies the Empty Set Axiom.*

PROOF: Trivial. \square

Corollary 1.4.1.1. *Z is an extension of S.*

ZF is an extension of SF.

ZC is an extension of SC.

ZFC is an extension of SFC.

Proposition 1.4.2. *The Axiom of Pairing is a theorem of SF.*

PROOF:

$\langle 1 \rangle 1$. LET: a, b be sets.

$\langle 1 \rangle 2$. LET: $P(x, y)$ be the predicate $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$.

$\langle 1 \rangle 3$. For all $x \in \mathcal{P}\mathcal{P}\emptyset$, there exists at most one y such that $P(x, y)$.

$\langle 2 \rangle 1$. LET: $x \in \mathcal{P}\mathcal{P}\emptyset$

$\langle 2 \rangle 2$. LET: y and y' be sets.

$\langle 2 \rangle 3$. ASSUME: $P(x, y)$ and $P(x, y')$

$\langle 2 \rangle 4$. $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$

PROOF: From $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. $(x = \emptyset \wedge y' = a) \vee (x = \mathcal{P}\emptyset \wedge y' = b)$

PROOF: From $\langle 2 \rangle 3$.

⟨2⟩6. $\emptyset \neq \mathcal{P}\emptyset$

PROOF: Since $\emptyset \in \mathcal{P}\emptyset$ and $\emptyset \notin \emptyset$.

⟨2⟩7. $y = y'$

⟨1⟩4. LET: A be the set $\{y \mid \exists x \in \mathcal{P}\mathcal{P}\emptyset. P(x, y)\}$.

⟨1⟩5. $A = \{a, b\}$

□

Proposition Schema 1.4.3. *Every instance of the Comprehension Axiom Scheme is a theorem of SF.*

PROOF:

⟨1⟩1. LET: $P(x)$ be a predicate.

⟨1⟩2. LET: A be a set.

⟨1⟩3. LET: $Q(x, y)$ be the predicate $P(x) \wedge y = x$.

⟨1⟩4. For all $x \in A$, there exists at most one y such that $Q(x, y)$.

⟨2⟩1. LET: $x \in A$

⟨2⟩2. LET: y and y' be sets.

⟨2⟩3. ASSUME: $Q(x, y)$ and $Q(x, y')$

⟨2⟩4. $x \in A \wedge P(x) \wedge y = x \wedge y' = x$

PROOF: From ⟨2⟩3.

⟨2⟩5. $y = y'$

PROOF: From ⟨2⟩4.

⟨1⟩5. LET: B be the set $\{y \mid \exists x \in A. Q(x, y)\}$

PROOF: This is a set by an Axiom of Replacement and ⟨1⟩4.

⟨1⟩6. $B = \{y \in A \mid P(y)\}$

PROOF:

$$y \in B \Leftrightarrow \exists x \in A. Q(x, y) \quad (\langle 1 \rangle 5)$$

$$\Leftrightarrow \exists x \in A. (P(x) \wedge y = x) \quad (\langle 1 \rangle 3)$$

$$\Leftrightarrow P(y)$$

□

Corollary 1.4.3.1. *SF is an extension of S.*

SFC is an extension of SC.

ZF is an extension of Z.

ZFC is an extension of ZC.

1.5 Consequences of the Axioms

Proposition 1.5.1 (S). *The union of two sets is a set.*

PROOF: Because $A \cup B = \bigcup \{A, B\}$. □

Proposition Schema 1.5.2 (S). *For any number n , the following is a theorem:*

For any sets a_1, \dots, a_n , the class $\{a_1, \dots, a_n\} = \{x \mid x = a_1 \vee \dots \vee x = a_n\}$ is a set.

PROOF: The case $n = 1$ follows from Pairing since $\{a\} = \{a, a\}$.
 If we have proved the theorem for n we have $\{a_1, \dots, a_n, a_{n+1}\} = \{a_1, \dots, a_n\} \cup \{a_{n+1}\}$. \square

Proposition 1.5.3 (S). *No set is a member of itself.*

PROOF:

- $\langle 1 \rangle 1$. LET: x be any set.
 - $\langle 1 \rangle 2$. PICK $m \in \{x\}$ such that $m \cap \{x\} = \emptyset$.
 PROOF: Axiom of Regularity.
 - $\langle 1 \rangle 3$. $m = x$
 - $\langle 1 \rangle 4$. $x \cap \{x\} = \emptyset$
 - $\langle 1 \rangle 5$. $x \notin x$
- \square

Corollary 1.5.3.1 (S). *The universal class \mathbf{V} is a proper class.*

PROOF: If \mathbf{V} is a set then $\mathbf{V} \in \mathbf{V}$, contradicting the Proposition. \square

Proposition 1.5.4 (S). *There are no sets a and b such that $a \in b$ and $b \in a$.*

PROOF:

- $\langle 1 \rangle 1$. LET: a and b be any sets.
 - $\langle 1 \rangle 2$. PICK $m \in \{a, b\}$ such that $m \cap \{a, b\} = \emptyset$
 - $\langle 1 \rangle 3$. CASE: $m = a$
 PROOF: Then $b \notin a$.
 - $\langle 1 \rangle 4$. CASE: $m = b$
 PROOF: Then $a \notin b$.
- \square

Proposition 1.5.5 (S). *The intersection of a set and a class is a set.*

PROOF: Immediate from Comprehension. \square

Proposition 1.5.6 (S). *The relative complement of a class in a set is a set.*

PROOF: Immediate from Comprehension. \square

Corollary 1.5.6.1 (S). *The symmetric difference of two sets is a set.*

Proposition 1.5.7 (S). *The intersection of a nonempty class is a set.*

PROOF:

- $\langle 1 \rangle 1$. LET: \mathbf{A} be a nonempty class.
 - $\langle 1 \rangle 2$. PICK $B \in \mathbf{A}$
 - $\langle 1 \rangle 3$. $\bigcap \mathbf{A} \subseteq B$
 - $\langle 1 \rangle 4$. $\bigcap \mathbf{A}$ is a set.
- PROOF: By Comprehension.
- \square

Proposition Schema 1.5.8 (FOL). *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem:*

If $\mathbf{A} \subseteq \mathbf{B}$ then $\mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathbf{B}$.

PROOF: Every subset of \mathbf{A} is a subset of \mathbf{B} . \square

Proposition Schema 1.5.9 (FOL). *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem:*

If $\mathbf{A} \subseteq \mathbf{B}$ then $\bigcup \mathbf{A} \subseteq \bigcup \mathbf{B}$.

PROOF: If $x \in X \in \mathbf{A}$ then $x \in X \in \mathbf{B}$. \square

Proposition Schema 1.5.10 (S). *For any class \mathbf{A} , the following is a theorem:*

$$\mathbf{A} = \bigcup \mathcal{P}\mathbf{A}$$

PROOF:

$\langle 1 \rangle 1. \mathbf{A} \subseteq \bigcup \mathcal{P}\mathbf{A}$

PROOF: For all $x \in \mathbf{A}$ we have $x \in \{x\} \in \mathcal{P}\mathbf{A}$.

$\langle 1 \rangle 2. \bigcup \mathcal{P}\mathbf{A} \subseteq \mathbf{A}$

$\langle 2 \rangle 1. \text{ LET: } x \in \bigcup \mathcal{P}\mathbf{A}$

$\langle 2 \rangle 2. \text{ PICK } X \in \mathcal{P}\mathbf{A} \text{ such that } x \in X$

$\langle 2 \rangle 3. X \subseteq \mathbf{A}$

$\langle 2 \rangle 4. x \in \mathbf{A}$

\square

1.6 Transitive Classes

Definition 1.6.1 (Transitive Class). A class \mathbf{A} is a *transitive class* iff whenever $x \in y \in \mathbf{A}$ then $x \in \mathbf{A}$.

Proposition Schema 1.6.2 (FOL). *For any class \mathbf{A} , the following is a theorem:*

The following are equivalent.

1. \mathbf{A} is a transitive class.
2. $\bigcup \mathbf{A} \subseteq \mathbf{A}$
3. Every element of \mathbf{A} is a subset of \mathbf{A} .
4. $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$

PROOF: Immediate from definitions. \square

Proposition Schema 1.6.3 (FOL). *For any class \mathbf{A} , the following is a theorem:*

If \mathbf{A} is a transitive class then $\bigcup \mathbf{A}$ is a transitive class.

PROOF:

⟨1⟩1. ASSUME: \mathbf{A} is a transitive class.
 ⟨1⟩2. LET: $x \in y \in \bigcup \mathbf{A}$
 ⟨1⟩3. $y \in \mathbf{A}$
 PROOF: Since $\bigcup \mathbf{A} \subseteq \mathbf{A}$ by Proposition 1.6.2.
 ⟨1⟩4. $x \in \bigcup \mathbf{A}$
 \square

Proposition Schema 1.6.4 (S). *For any class \mathbf{A} , the following is a theorem:
 We have \mathbf{A} is a transitive class if and only if $\mathcal{P}\mathbf{A}$ is a transitive class.*

PROOF:

⟨1⟩1. If \mathbf{A} is a transitive class then $\mathcal{P}\mathbf{A}$ is a transitive class.
 ⟨2⟩1. ASSUME: \mathbf{A} is a transitive class.
 ⟨2⟩2. $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$
 PROOF: Proposition 1.6.2.
 ⟨2⟩3. $\mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathcal{P}\mathbf{A}$
 PROOF: Proposition 1.5.8.
 ⟨2⟩4. $\mathcal{P}\mathbf{A}$ is a transitive class.
 PROOF: Proposition 1.6.2.
 ⟨1⟩2. If $\mathcal{P}\mathbf{A}$ is a transitive class then \mathbf{A} is a transitive class.
 ⟨2⟩1. ASSUME: $\mathcal{P}\mathbf{A}$ is a transitive class.
 ⟨2⟩2. $\bigcup \mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$
 PROOF: Proposition 1.6.2.
 ⟨2⟩3. $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$
 PROOF: Proposition 1.5.10.
 ⟨2⟩4. \mathbf{A} is a transitive class.
 PROOF: Proposition 1.6.2.
 \square

Proposition Schema 1.6.5 (FOL). *For any class \mathbf{A} , the following is a theorem:*

If every member of \mathbf{A} is a transitive set then $\bigcup \mathbf{A}$ is a transitive class.

PROOF:

⟨1⟩1. ASSUME: Every member of \mathbf{A} is a transitive set.
 ⟨1⟩2. LET: $x \in y \in \bigcup \mathbf{A}$
 ⟨1⟩3. PICK $A \in \mathbf{A}$ such that $y \in A$.
 ⟨1⟩4. $x \in A$
 PROOF: Since A is a transitive set.
 ⟨1⟩5. $x \in \bigcup \mathbf{A}$
 \square

Proposition Schema 1.6.6 (FOL). *For any class \mathbf{A} , the following is a theorem:*

If every member of \mathbf{A} is a transitive set then $\bigcap \mathbf{A}$ is a transitive class.

PROOF:

⟨1⟩1. ASSUME: Every member of \mathbf{A} is a transitive set.

$\langle 1 \rangle 2$. LET: $x \in y \in \bigcap \mathbf{A}$

PROVE: $x \in \bigcap \mathbf{A}$

$\langle 1 \rangle 3$. LET: $A \in \mathbf{A}$

$\langle 1 \rangle 4$. $y \in A$

$\langle 1 \rangle 5$. $x \in A$

PROOF: Since A is a transitive set.

□

Chapter 2

Relations

2.1 Ordered Pairs

Definition 2.1.1 (Ordered Pair). For any sets a and b , the *ordered pair* (a, b) is defined to be $\{\{a\}, \{a, b\}\}$.

Theorem 2.1.2 (S). For any sets a, b, c, d , we have $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

PROOF:

$\langle 1 \rangle 1$. If $(a, b) = (c, d)$ then $a = c$ and $b = d$.

$\langle 2 \rangle 1$. ASSUME: $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$

$\langle 2 \rangle 2$. $\bigcap \{\{a\}, \{a, b\}\} = \bigcap \{\{c\}, \{c, d\}\}$

$\langle 2 \rangle 3$. $\{a\} = \{c\}$

$\langle 2 \rangle 4$. $a = c$

$\langle 2 \rangle 5$. $\bigcup \{\{a\}, \{a, b\}\} = \bigcup \{\{c\}, \{c, d\}\}$

$\langle 2 \rangle 6$. $\{a, b\} = \{c, d\}$

$\langle 2 \rangle 7$. $b = c$ or $b = d$

$\langle 2 \rangle 8$. $a = d$ or $b = d$

$\langle 2 \rangle 9$. If $b = c$ and $a = d$ then $b = d$

PROOF: By $\langle 2 \rangle 4$.

$\langle 2 \rangle 10$. $b = d$

PROOF: From $\langle 2 \rangle 7$, $\langle 2 \rangle 8$, $\langle 2 \rangle 9$.

$\langle 1 \rangle 2$. If $a = c$ and $b = d$ then $(a, b) = (c, d)$.

PROOF: First-order logic.

□

Definition 2.1.3 (Cartesian Product). The *Cartesian product* of classes \mathbf{A} and \mathbf{B} is the class $\mathbf{A} \times \mathbf{B} := \{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$.

Proposition 2.1.4 (S). For any sets A and B , the class $A \times B$ is a set.

PROOF: It is a subset of $\mathcal{PP}(A \cup B)$. □

Proposition Schema 2.1.5 (S). *For any classes \mathbf{A} , \mathbf{B} and \mathbf{C} , the following is a theorem:*

$$\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$$

PROOF:

$$\begin{aligned} (x, y) \in \mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) &\Leftrightarrow x \in \mathbf{A} \wedge (y \in \mathbf{B} \vee y \in \mathbf{C}) \\ &\Leftrightarrow (x \in \mathbf{A} \wedge y \in \mathbf{B}) \vee (x \in \mathbf{A} \wedge y \in \mathbf{C}) \\ &\Leftrightarrow (x, y) \in (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C}) \quad \square \end{aligned}$$

Proposition Schema 2.1.6 (S). *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem:*

If $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ and \mathbf{A} is nonempty then $\mathbf{B} = \mathbf{C}$.

PROOF:

- $\langle 1 \rangle 1$. PICK $a \in \mathbf{A}$
 $\langle 1 \rangle 2$. For all x we have $x \in \mathbf{B}$ iff $x \in \mathbf{C}$.

PROOF:

$$\begin{aligned} x \in \mathbf{B} &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{B} \\ &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{C} \\ &\Leftrightarrow x \in \mathbf{C} \end{aligned}$$

\square

Proposition Schema 2.1.7 (S). *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem:*

$$\mathbf{A} \times \bigcup \mathbf{B} = \{(a, b) \mid \exists Y \in \mathbf{B}. (a \in \mathbf{A} \wedge b \in Y)\}$$

PROOF:

$$\begin{aligned} (x, y) \in \mathbf{A} \times \bigcup \mathbf{B} &\Leftrightarrow x \in \mathbf{A} \wedge \exists Y \in \mathbf{B}. y \in Y \\ &\Leftrightarrow \exists Y \in \mathbf{B} (x \in \mathbf{A} \wedge y \in Y) \quad \square \end{aligned}$$

2.2 Relations

Definition 2.2.1 (Relation). A *relation* is a class of ordered pairs.

A *relation* \mathbf{R} between classes \mathbf{A} and \mathbf{B} is a subclass of $\mathbf{A} \times \mathbf{B}$.

A *(binary) relation on \mathbf{A}* is a relation between \mathbf{A} and \mathbf{A} .

We write $x\mathbf{R}y$ for $(x, y) \in \mathbf{R}$.

Definition 2.2.2 (Domain). The *domain* of a class \mathbf{R} is the class

$$\text{dom } \mathbf{R} := \{x \mid \exists y. x\mathbf{R}y\} .$$

Proposition 2.2.3 (S). *The domain of a set is a set.*

Definition 2.2.4 (Range). The *range* of a class \mathbf{R} is the class

$$\text{ran } \mathbf{R} := \{y \mid \exists x. x\mathbf{R}y\}$$

Proposition 2.2.5 (S). *The range of a set is a set.*

2.2.1 Identity Functions

Definition 2.2.6 (Identity Function). For any class \mathbf{A} , the *identity function* or *diagonal relation* $\text{id}_{\mathbf{A}}$ on \mathbf{A} is

$$\text{id}_{\mathbf{A}} := \{(x, x) \mid x \in \mathbf{A}\} .$$

2.2.2 Inverses

Definition 2.2.7 (Inverse). The *inverse* of a relation \mathbf{R} between \mathbf{A} and \mathbf{B} is the relation \mathbf{R}^{-1} between \mathbf{B} and \mathbf{A} defined by

$$b\mathbf{R}^{-1}a \Leftrightarrow a\mathbf{R}b .$$

Proposition Schema 2.2.8 (S). For any classes \mathbf{A} , \mathbf{B} and \mathbf{R} , the following is a theorem:

If \mathbf{R} is a relation between \mathbf{A} and \mathbf{B} then $(\mathbf{R}^{-1})^{-1} = \mathbf{R}$.

PROOF:

$$\begin{aligned} x(\mathbf{R}^{-1})^{-1}y &\Leftrightarrow y\mathbf{R}^{-1}x \\ &\Leftrightarrow x\mathbf{R}y \end{aligned} \quad \square$$

2.2.3 Composition

Definition 2.2.9 (Composition). Let \mathbf{R} be a relation between \mathbf{A} and \mathbf{B} , and \mathbf{S} be a relation between \mathbf{B} and \mathbf{C} . The *composition* $\mathbf{S} \circ \mathbf{R}$ is the relation between \mathbf{A} and \mathbf{C} defined by

$$a(\mathbf{S} \circ \mathbf{R})c \Leftrightarrow \exists b(a\mathbf{R}b \wedge b\mathbf{S}c) .$$

Proposition Schema 2.2.10 (S). For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{R} and \mathbf{S} , the following is a theorem:

If \mathbf{R} is a relation between \mathbf{A} and \mathbf{B} , and \mathbf{S} is a relation between \mathbf{B} and \mathbf{C} , then

$$(\mathbf{S} \circ \mathbf{R})^{-1} = \mathbf{R}^{-1} \circ \mathbf{S}^{-1} .$$

PROOF:

$$\begin{aligned} z(\mathbf{S} \circ \mathbf{R})^{-1}x &\Leftrightarrow x(\mathbf{S} \circ \mathbf{R})z \\ &\Leftrightarrow \exists y.(x\mathbf{R}y \wedge y\mathbf{S}z) \\ &\Leftrightarrow \exists y.(y\mathbf{R}^{-1}x \wedge z\mathbf{S}^{-1}y) \\ &\Leftrightarrow z(\mathbf{R}^{-1} \circ \mathbf{S}^{-1})x \end{aligned} \quad \square$$

2.2.4 Properties of Relations

Definition 2.2.11 (Reflexive). Let \mathbf{R} be a binary relation on \mathbf{A} . Then \mathbf{R} is *reflexive* on \mathbf{A} iff $\forall x \in \mathbf{A}.(x, x) \in \mathbf{R}$.

Proposition Schema 2.2.12 (S). *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is a reflexive relation on \mathbf{A} then so is \mathbf{R}^{-1} .

PROOF:

$\langle 1 \rangle 1.$ LET: $x \in \mathbf{A}$

$\langle 1 \rangle 2.$ $x\mathbf{R}x$

PROOF: Since \mathbf{R} is reflexive.

$\langle 1 \rangle 3.$ $x\mathbf{R}^{-1}x$

□

Definition 2.2.13 (Irreflexive). A relation \mathbf{R} is *irreflexive* iff there is no x such that $(x, x) \in \mathbf{R}$.

Definition 2.2.14 (Symmetric). A relation \mathbf{R} is *symmetric* iff, whenever $x\mathbf{R}y$, then $y\mathbf{R}x$.

Definition 2.2.15 (Antisymmetric). A relation \mathbf{R} is *antisymmetric* iff, whenever $x\mathbf{R}y$ and $y\mathbf{R}x$, then $x = y$.

Proposition Schema 2.2.16 (S). *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is an antisymmetric relation on \mathbf{A} then so is \mathbf{R}^{-1} .

PROOF:

$\langle 1 \rangle 1.$ ASSUME: $x\mathbf{R}^{-1}y$ and $y\mathbf{R}^{-1}x$

$\langle 1 \rangle 2.$ $y\mathbf{R}x$ and $x\mathbf{R}y$

$\langle 1 \rangle 3.$ $x = y$

PROOF: Since \mathbf{R} is antisymmetric.

□

Definition 2.2.17 (Transitive). A relation \mathbf{R} is *transitive* iff, whenever $x\mathbf{R}y$ and $y\mathbf{R}z$, then $x\mathbf{R}z$.

Proposition Schema 2.2.18 (S). *For any classes \mathbf{A} , \mathbf{B} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is a transitive relation between \mathbf{A} and \mathbf{B} then \mathbf{R}^{-1} is transitive.

PROOF:

$\langle 1 \rangle 1.$ ASSUME: $(x, y), (y, z) \in \mathbf{R}^{-1}$

$\langle 1 \rangle 2.$ $(y, x), (z, y) \in \mathbf{R}$

$\langle 1 \rangle 3.$ $(z, x) \in \mathbf{R}$

$\langle 1 \rangle 4.$ $(x, z) \in \mathbf{R}^{-1}$

□

Proposition 2.2.19 (S). *For any relation R on a set A , there exists a smallest transitive relation on A that includes R .*

PROOF: The relation is $\bigcap \{S \in \mathcal{P}A^2 \mid R \subseteq S, S \text{ is transitive}\}$. □

Definition 2.2.20 (Transitive Closure). For any relation R on a set A , the *transitive closure* of R is the smallest transitive relation that includes R .

Definition 2.2.21 (Minimal). Let \mathbf{R} be a relation on \mathbf{A} . An element $m \in \mathbf{A}$ is *minimal* iff there is no $x \in \mathbf{A}$ such that $x\mathbf{R}m$.

Definition 2.2.22 (Maximal). Let \mathbf{R} be a relation on \mathbf{A} . An element $m \in \mathbf{A}$ is *maximal* iff there is no $x \in \mathbf{A}$ such that $m\mathbf{R}x$.

2.3 n-ary Relations

Definition Schema 2.3.1. For any sets a_1, \dots, a_n , define the *ordered n -tuple* (a_1, \dots, a_n) by

$$(a_1) := a_1$$

$$(a_1, \dots, a_n, a_{n+1}) = ((a_1, \dots, a_n), a_{n+1})$$

Definition Schema 2.3.2. An *n -ary relation on \mathbf{A}* is a class of ordered n -tuples all of whose components are in \mathbf{A} .

2.4 Well Founded Relations

Definition 2.4.1 (Well Founded). A relation \mathbf{R} is *well founded* iff:

- Every nonempty set has an \mathbf{R} -minimal element.
- For every set x , there exists a set u such that $x \subseteq u$ and, for all w, y , if $y \in u$ and $w\mathbf{R}y$ then $w \in u$.

Proposition Schema 2.4.2 (S). *For any class \mathbf{R} , the following is a theorem:*
Assume \mathbf{R} is a well founded relation. For any set a , the class $\{x \mid x\mathbf{R}a\}$ is a set.

PROOF:

$\langle 1 \rangle 1$. PICK a set u such that $\{a\} \subseteq u$ and, for all w, y , if $y \in u$ and $w\mathbf{R}y$ then $w \in u$.

$\langle 1 \rangle 2$. $\{x \mid x\mathbf{R}a\} \subseteq u$

□

Proposition Schema 2.4.3 (S). *For any classes \mathbf{A} , \mathbf{B} and \mathbf{R} , the following is a theorem:*

Assume \mathbf{R} is a well founded relation on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$ is nonempty. Then \mathbf{B} has an \mathbf{R} -minimal element.

PROOF:

$\langle 1 \rangle 1$. PICK $b \in \mathbf{B}$

$\langle 1 \rangle 2$. LET: $S = \{x \in \mathbf{B} \mid x\mathbf{R}b\}$

$\langle 1 \rangle 3$. CASE: $S = \emptyset$

PROOF: In this case b is an \mathbf{R} -minimal element of \mathbf{B} .

$\langle 1 \rangle 4$. CASE: $S \neq \emptyset$

PROOF: In this cases S has an \mathbf{R} -minimal element, which is an \mathbf{R} -minimal element of \mathbf{B} .

□

Proposition Schema 2.4.4 (FOL). *For any classes \mathbf{R} and \mathbf{S} , the following is a theorem:*

Assume \mathbf{R} is a relation. If \mathbf{S} is a well-founded relation and $\mathbf{R} \subseteq \mathbf{S}$ then \mathbf{R} is a well founded relation.

PROOF: Immediate from definitions. □

Theorem Schema 2.4.5 (Transfinite Induction Principle (S)). *For any classes \mathbf{A} , \mathbf{B} and \mathbf{R} , the following is a theorem:*

Assume \mathbf{R} is a well founded relation on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Assume that, for all $t \in \mathbf{A}$,

$$\{x \in \mathbf{A} \mid x\mathbf{R}t\} \subseteq \mathbf{B} \Rightarrow t \in \mathbf{B} .$$

Then $\mathbf{B} = \mathbf{A}$.

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $\mathbf{B} \neq \mathbf{A}$

$\langle 1 \rangle 2$. PICK an \mathbf{R} -minimal element m of $\mathbf{A} - \mathbf{B}$.

PROOF: Proposition 2.4.3.

$\langle 1 \rangle 3$. $\{x \in \mathbf{A} \mid x\mathbf{R}m\} \subseteq \mathbf{B}$

PROOF: By minimality of m .

$\langle 1 \rangle 4$. $m \in \mathbf{B}$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

□

Theorem 2.4.6 (S). *The transitive closure of a well founded relation on a set is well founded.*

PROOF:

$\langle 1 \rangle 1$. LET: R be a well founded relation on the set A .

$\langle 1 \rangle 2$. LET: R^t be the transitive closure of R .

$\langle 1 \rangle 3$. For any $x, y \in A$, if $xR^t y$ then there exists $z \in A$ such that zRy .

PROOF: $\{(x, y) \in A^2 \mid \exists z \in A. zRy\}$ is a transitive relation on A that includes R .

$\langle 1 \rangle 4$. LET: B be a nonempty subset of A .

$\langle 1 \rangle 5$. PICK an R -minimal element b of B .

$\langle 1 \rangle 6$. b is R^t -minimal in B .

PROOF: If there exists x such that $xR^t b$ then there exists z such that zRb by

$\langle 1 \rangle 3$.

□

Definition 2.4.7 (Initial Segment). Let \mathbf{R} be a relation on \mathbf{A} and $a \in \mathbf{A}$. The *initial segment* up to a is

$$\text{seg } a := \{x \in \mathbf{A} \mid x\mathbf{R}a\} .$$

Theorem Schema 2.4.8 (Transfinite Recursion Theorem Schema (ZF)). *For any classes \mathbf{A} , \mathbf{R} and any property $G[x, y, z]$, there exists a class \mathbf{F} such that, for any class \mathbf{F}' the following is a theorem:*

Assume that \mathbf{R} is a well-founded relation on \mathbf{A} . Assume that, for any f and t , there exists a unique z such that $G[f, t, z]$. Then $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{V}$ such that, for all $t \in \mathbf{A}$, we have $\mathbf{F} \upharpoonright \text{seg } t$ is a set and

$$G[\mathbf{F} \upharpoonright \text{seg } t, t, \mathbf{F}(t)] .$$

If $\mathbf{F}' : \mathbf{A} \rightarrow \mathbf{V}$ satisfies that, for all $t \in \mathbf{A}$, we have $\mathbf{F}' \upharpoonright \text{seg } t$ is a set and $G[\mathbf{F}' \upharpoonright \text{seg } t, t, \mathbf{F}'(t)]$, then $\mathbf{F}' = \mathbf{F}$.

PROOF:

- $\langle 1 \rangle 1$. For B a subset of \mathbf{A} , let us say a function $v : B \rightarrow \mathbf{V}$ is *acceptable* iff, for all $x \in B$, we have $\text{seg } x \subseteq B$ and $G[v \upharpoonright \text{seg } x, x, v(x)]$
 - $\langle 1 \rangle 2$. LET: \mathbf{K} be the class of all acceptable functions.
 - $\langle 1 \rangle 3$. LET: $\mathbf{F} = \bigcup \mathbf{K}$
 - $\langle 1 \rangle 4$. For all $B, C \subseteq \mathbf{A}$, given $v_1 : B \rightarrow \mathbf{V}$ and $v_2 : C \rightarrow \mathbf{V}$ acceptable and $x \in B \cap C$, we have $v_1(x) = v_2(x)$
 - $\langle 2 \rangle 1$. ASSUME: as transfinite induction hypothesis $\forall y \mathbf{R} x. y \in B \cap C \Rightarrow v_1(y) = v_2(y)$
 - $\langle 2 \rangle 2$. $v_1 \upharpoonright \text{seg } x = v_2 \upharpoonright \text{seg } x$
 - $\langle 2 \rangle 3$. $G[v_1 \upharpoonright \text{seg } x, x, v_1(x)]$
 - $\langle 2 \rangle 4$. $G[v_2 \upharpoonright \text{seg } x, x, v_2(x)]$
 - $\langle 2 \rangle 5$. $v_1(x) = v_2(x)$
 - $\langle 1 \rangle 5$. \mathbf{F} is a function.
 - $\langle 2 \rangle 1$. ASSUME: $(x, y), (x, z) \in \mathbf{F}$
 - $\langle 2 \rangle 2$. PICK acceptable $v_1 : B \rightarrow \mathbf{V}$ and $v_2 : C \rightarrow \mathbf{V}$ such that $v_1(x) = y$ and $v_2(x) = z$
 - $\langle 2 \rangle 3$. $y = z$
- PROOF: By $\langle 1 \rangle 4$.
- $\langle 1 \rangle 6$. For all $t \in \text{dom } \mathbf{F}$, we have $\mathbf{F} \upharpoonright \text{seg } t$ is a set and $G[\mathbf{F} \upharpoonright \text{seg } t, t, \mathbf{F}(t)]$
 - $\langle 2 \rangle 1$. LET: $t \in \text{dom } \mathbf{F}$
 - $\langle 2 \rangle 2$. PICK an acceptable $v : A \rightarrow \mathbf{V}$ such that $t \in A$
 - $\langle 2 \rangle 3$. For all $y \mathbf{R} x$ we have $v(y) = \mathbf{F}(y)$
 - $\langle 2 \rangle 4$. $\mathbf{F} \upharpoonright \text{seg } x = v \upharpoonright \text{seg } x$
 - $\langle 2 \rangle 5$. $G[v \upharpoonright \text{seg } x, x, v(x)]$
 - $\langle 2 \rangle 6$. $G[\mathbf{F} \upharpoonright \text{seg } x, x, \mathbf{F}(x)]$
 - $\langle 1 \rangle 7$. $\text{dom } \mathbf{F} = \mathbf{A}$
 - $\langle 2 \rangle 1$. LET: $x \in \mathbf{A}$
 - $\langle 2 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall y \mathbf{R} x. y \in \mathbf{A}$
 - $\langle 2 \rangle 3$. ASSUME: for a contradiction $x \notin \text{dom } \mathbf{F}$

$\langle 2 \rangle 4.$ $\mathbf{F} \upharpoonright \text{seg } x$ is a set

PROOF: Axiom of Replacement, Proposition 2.4.2.

$\langle 2 \rangle 5.$ $\mathbf{F} \upharpoonright \text{seg } x$ is acceptable

$\langle 2 \rangle 6.$ LET: y be the unique object such that $G[\mathbf{F} \upharpoonright \text{seg } x, x, y]$

$\langle 2 \rangle 7.$ $\mathbf{F} \upharpoonright \text{seg } x \cup \{(x, y)\}$ is acceptable

$\langle 2 \rangle 8.$ $x \in \text{dom } \mathbf{F}$

$\langle 2 \rangle 9.$ Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 8.$ If $\mathbf{F}' : \mathbf{A} \rightarrow \mathbf{V}$ satisfies the theorem, then $\mathbf{F}' = \mathbf{F}$.

$\langle 2 \rangle 1.$ LET: $x \in \mathbf{A}$

PROVE: $\mathbf{F}'(x) = \mathbf{F}(x)$

$\langle 2 \rangle 2.$ ASSUME: as transfinite induction hypothesis $\forall y \mathbf{R}x. \mathbf{F}'(y) = \mathbf{F}(y)$

$\langle 2 \rangle 3.$ $\mathbf{F} \upharpoonright x = \mathbf{F}' \upharpoonright x$

$\langle 2 \rangle 4.$ $G[\mathbf{F} \upharpoonright x, x, \mathbf{F}(x)]$

$\langle 2 \rangle 5.$ $G[\mathbf{F}' \upharpoonright x, x, \mathbf{F}'(x)]$

$\langle 2 \rangle 6.$ $\mathbf{F}(x) = \mathbf{F}'(x)$

□

Chapter 3

Functions

3.1 Functions

Definition 3.1.1 (Function). A *function* from \mathbf{A} to \mathbf{B} is a relation \mathbf{F} between \mathbf{A} and \mathbf{B} such that, for all $x \in \mathbf{A}$, there is only one y such that $x\mathbf{F}y$. We denote this y by $\mathbf{F}(x)$.

A *binary operation* on a class \mathbf{A} is a function $\mathbf{A}^2 \rightarrow \mathbf{A}$.

Definition 3.1.2 (Closed). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{A}$ be a function and $\mathbf{B} \subseteq \mathbf{A}$. Then \mathbf{B} is *closed* under \mathbf{F} iff $\forall x \in \mathbf{B}. \mathbf{F}(x) \in \mathbf{B}$.

Proposition 3.1.3 (S). *For any class \mathbf{A} , the following is a theorem:*

$$\text{id}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}$$

PROOF: For all $x \in \mathbf{A}$, the only y such that $(x, y) \in \text{id}_{\mathbf{A}}$ is $y = x$. \square

Proposition Schema 3.1.4 (S). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{F} and \mathbf{G} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{G} : \mathbf{B} \rightarrow \mathbf{C}$. Then $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \rightarrow \mathbf{C}$ and, for all $x \in \mathbf{A}$, we have

$$(\mathbf{G} \circ \mathbf{F})(x) = \mathbf{G}(\mathbf{F}(x)) .$$

PROOF:

$\langle 1 \rangle 1. \forall x \in \mathbf{A}. (x, \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G} \circ \mathbf{F}$

PROOF: Because $(x, \mathbf{F}(x)) \in \mathbf{F}$ and $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$.

$\langle 1 \rangle 2. \text{ If } (x, z) \in \mathbf{F} \circ \mathbf{G} \text{ then } z = \mathbf{G}(\mathbf{F}(x))$

$\langle 2 \rangle 1. \text{ PICK } y \in \mathbf{B} \text{ such that } x\mathbf{F}y \text{ and } y\mathbf{G}z$

$\langle 2 \rangle 2. y = \mathbf{F}(x)$

$\langle 2 \rangle 3. z = \mathbf{G}(y)$

$\langle 2 \rangle 4. z = \mathbf{G}(\mathbf{F}(x))$

\square

Proposition 3.1.5 (SC). *For any set A there exists a function $F : \mathcal{P}A - \{\emptyset\} \rightarrow A$ (a choice function for A) such that, for every nonempty $B \subseteq A$, we have $F(B) \in B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A be a set.

$\langle 1 \rangle 2$. LET: $\mathcal{A} = \{\{B\} \times B \mid B \in \mathcal{P}A - \{\emptyset\}\}$

$\langle 1 \rangle 3$. Every member of \mathcal{A} is nonempty.

$\langle 1 \rangle 4$. Any two distinct members of \mathcal{A} are disjoint.

$\langle 1 \rangle 5$. PICK a set C such that, for all $X \in \mathcal{A}$, we have $C \cap X$ is a singleton.

PROOF: Axiom of Choice.

$\langle 1 \rangle 6$. LET: $F = C \cap \bigcup \mathcal{A}$

$\langle 1 \rangle 7$. $F : \mathcal{P}A - \{\emptyset\} \rightarrow A$

$\langle 2 \rangle 1$. F is a function.

$\langle 3 \rangle 1$. LET: $(B, b), (B, b') \in F$

$\langle 3 \rangle 2$. $(B, b), (B, b') \in \{B\} \times B$

PROOF: Since $(B, b), (B, b') \in \bigcup \mathcal{A}$.

$\langle 3 \rangle 3$. $(B, b), (B, b') \in C \cap (\{B\} \times B)$

$\langle 3 \rangle 4$. $(B, b) = (B, b')$

PROOF: From $\langle 1 \rangle 5$.

$\langle 3 \rangle 5$. $b = b'$

$\langle 2 \rangle 2$. $\text{dom } F = \mathcal{P}A - \{\emptyset\}$

PROOF:

$$B \in \text{dom } F$$

$$\Leftrightarrow \exists b. (B, b) \in F$$

$$\Leftrightarrow \exists b. ((B, b) \in \bigcup \mathcal{A} \wedge (B, b) \in C)$$

$$\Leftrightarrow \exists b. \exists B' \in \mathcal{P}A - \{\emptyset\}. ((B, b) \in \{B'\} \times B' \wedge (B, b) \in C)$$

$$\Leftrightarrow B \in \mathcal{P}A - \{\emptyset\} \wedge \exists b \in B. (B, b) \in C$$

$$\Leftrightarrow B \in \mathcal{P}A - \{\emptyset\} \quad (\langle 1 \rangle 5)$$

$\langle 2 \rangle 3$. $\text{ran } F \subseteq A$

$\langle 1 \rangle 8$. For every nonempty $B \subseteq A$ we have $F(B) \in B$

□

Proposition 3.1.6 (SC). *For any relation R between A and B , there exists a function $H : A \rightarrow B$ such that $H \subseteq R$ (i.e. $\forall x \in A. xRH(x)$).*

PROOF:

$\langle 1 \rangle 1$. LET: R be a relation between A and B .

$\langle 1 \rangle 2$. PICK a choice function G for B .

$\langle 1 \rangle 3$. Define $H : A \rightarrow B$ by $H(x) = G(\{y \mid xRy\})$

$\langle 1 \rangle 4$. $H \subseteq R$

□

3.1.1 Injective Functions

Definition 3.1.7 (Injective). A function $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ is *one-to-one*, *injective* or an *injection*, $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$, iff, for all $x, y \in \mathbf{A}$, if $\mathbf{F}(x) = \mathbf{F}(y)$, then $x = y$.

Proposition 3.1.8 (S). For any class \mathbf{A} , the following is a theorem:

$\text{id}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}$ is injective.

PROOF: If $\text{id}_{\mathbf{A}}(x) = \text{id}_{\mathbf{A}}(y)$ then immediately $x = y$. \square

Proposition Schema 3.1.9 (S). For any classes $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{F}, \mathbf{G}$, the following is a theorem:

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{G} : \mathbf{B} \rightarrow \mathbf{C}$. Then $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \rightarrow \mathbf{C}$.

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \mathbf{A}$

$\langle 1 \rangle 2$. ASSUME: $(\mathbf{G} \circ \mathbf{F})(x) = (\mathbf{G} \circ \mathbf{F})(y)$

$\langle 1 \rangle 3$. $\mathbf{G}(\mathbf{F}(x)) = \mathbf{G}(\mathbf{F}(y))$

$\langle 1 \rangle 4$. $\mathbf{F}(x) = \mathbf{F}(y)$

PROOF: Since \mathbf{G} is injective.

$\langle 1 \rangle 5$. $x = y$

PROOF: Since \mathbf{F} is injective.

\square

Proposition 3.1.10 (S). Let $F : A \rightarrow B$ where A is nonempty. There exists $G : B \rightarrow A$ (a left inverse) such that $G \circ F = \text{id}_A$ if and only if F is one-to-one.

PROOF:

$\langle 1 \rangle 1$. If there exists $G : B \rightarrow A$ such that $G \circ F = \text{id}_A$ then F is one-to-one.

$\langle 2 \rangle 1$. ASSUME: $G : B \rightarrow A$ and $G \circ F = I_A$

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: $F(x) = F(y)$

$\langle 2 \rangle 4$. $x = y$

PROOF: $x = G(F(x)) = G(F(y)) = y$

$\langle 1 \rangle 2$. If F is one-to-one then there exists $G : B \rightarrow A$ such that $G \circ F = I_A$.

$\langle 2 \rangle 1$. ASSUME: F is one-to-one.

$\langle 2 \rangle 2$. PICK $a \in A$

$\langle 2 \rangle 3$. LET: $G : B \rightarrow A$ be the function defined by: $G(b)$ is the (unique) $x \in A$ such that $F(x) = b$ if there exists such an x , $G(b) = a$ otherwise.

$\langle 2 \rangle 4$. For all $x \in A$ we have $G(F(x)) = x$.

\square

3.1.2 Surjective Functions

Definition 3.1.11 (Surjective). Let $F : A \rightarrow B$. We say that F is *surjective*, or maps A *onto* B , and write $F : A \rightarrow B$, iff for all $y \in B$ there exists $x \in A$ such that $F(x) = y$.

Proposition Schema 3.1.12 (S). For any class \mathbf{A} , the following is a theorem:

$\text{id}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}$ is surjective.

PROOF: For any $y \in \mathbf{A}$ we have $\text{id}_{\mathbf{A}}(y) = y$. \square

Proposition Schema 3.1.13 (S). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{F} and \mathbf{G} , the following is a theorem:*

If $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{G} : \mathbf{B} \rightarrow \mathbf{C}$, then $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \rightarrow \mathbf{C}$.

PROOF:

- $\langle 1 \rangle 1$. LET: $c \in \mathbf{C}$
- $\langle 1 \rangle 2$. PICK $b \in \mathbf{B}$ such that $\mathbf{G}(b) = c$.
- $\langle 1 \rangle 3$. PICK $a \in \mathbf{A}$ such that $\mathbf{F}(a) = b$.
- $\langle 1 \rangle 4$. $(\mathbf{G} \circ \mathbf{F})(a) = c$

\square

Proposition 3.1.14 (SC). *Let $F : A \rightarrow B$. There exists $H : B \rightarrow A$ (a right inverse) such that $F \circ H = \text{id}_B$ if and only if F maps A onto B .*

PROOF:

- $\langle 1 \rangle 1$. If F has a right inverse then F is surjective.
 - $\langle 2 \rangle 1$. ASSUME: F has a right inverse $H : B \rightarrow A$.
 - $\langle 2 \rangle 2$. LET: $y \in B$
 - $\langle 2 \rangle 3$. $F(H(y)) = y$
 - $\langle 2 \rangle 4$. There exists $x \in A$ such that $F(x) = y$
- $\langle 1 \rangle 2$. If F is surjective then F has a right inverse.
 - $\langle 2 \rangle 1$. ASSUME: F is surjective.
 - $\langle 2 \rangle 2$. PICK a function $H : B \rightarrow A$ such that $H \subseteq F^{-1}$

PROOF: Proposition 3.1.6.

- $\langle 2 \rangle 3$. $F \circ H = \text{id}_B$
 - $\langle 3 \rangle 1$. LET: $y \in B$
 - $\langle 3 \rangle 2$. $(y, H(y)) \in F^{-1}$
 - $\langle 3 \rangle 3$. $F(H(y)) = y$

\square

3.1.3 Bijections

Definition 3.1.15 (Bijection). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Then \mathbf{F} is *bijective* or a *bijection*, $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$, iff it is injective and surjective.

Proposition Schema 3.1.16 (S). *For any class \mathbf{A} , the following is a theorem:*
The identity function $\text{id}_{\mathbf{A}} : \mathbf{A} \approx \mathbf{A}$ is a bijection.

PROOF: Proposition 3.1.8 and 3.1.12. \square

Proposition Schema 3.1.17 (S). *For any classes \mathbf{A} , \mathbf{B} and \mathbf{F} , the following is a theorem:*

If $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$ then $\mathbf{F}^{-1} : \mathbf{B} \approx \mathbf{A}$.

PROOF:

- $\langle 1 \rangle 1$. $\mathbf{F}^{-1} : \mathbf{B} \rightarrow \mathbf{A}$
- $\langle 2 \rangle 1$. LET: $b \in \mathbf{B}$

⟨2⟩2. PICK $a \in \mathbf{A}$ such that $\mathbf{F}(a) = b$.

PROOF: Since \mathbf{F} is surjective.

⟨2⟩3. $(b, a) \in \mathbf{F}^{-1}$

⟨2⟩4. If $(b, a') \in \mathbf{F}^{-1}$ then $a' = a$.

⟨3⟩1. LET: $a' \in \mathbf{A}$ such that $(b, a') \in \mathbf{F}^{-1}$

⟨3⟩2. $\mathbf{F}(a') = \mathbf{F}(a)$

⟨3⟩3. $a' = a$

PROOF: Since \mathbf{F} is injective.

⟨1⟩2. \mathbf{F}^{-1} is injective.

⟨2⟩1. LET: $x, y \in \mathbf{B}$

⟨2⟩2. ASSUME: $\mathbf{F}^{-1}(x) = \mathbf{F}^{-1}(y)$

⟨2⟩3. $x = y$

PROOF: $x = \mathbf{F}(\mathbf{F}^{-1}(x)) = \mathbf{F}(\mathbf{F}^{-1}(y)) = y$.

⟨1⟩3. \mathbf{F}^{-1} is surjective.

PROOF: For all $a \in \mathbf{A}$ we have $\mathbf{F}^{-1}(\mathbf{F}(a)) = a$.

□

Proposition Schema 3.1.18 (S). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{F} and \mathbf{G} , the following is a theorem:*

If $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$ and $\mathbf{G} : \mathbf{B} \approx \mathbf{C}$ then $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \approx \mathbf{C}$.

PROOF: Propositions 3.1.9 and 3.1.13. □

3.1.4 Restrictions

Definition 3.1.19 (Restriction). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Let $\mathbf{C} \subseteq \mathbf{A}$. The *restriction* of \mathbf{F} to \mathbf{C} , denoted $\mathbf{F} \upharpoonright \mathbf{C}$, is the function

$$\begin{aligned} \mathbf{F} \upharpoonright \mathbf{C} : \mathbf{C} &\rightarrow \mathbf{B} \\ (\mathbf{F} \upharpoonright \mathbf{C})(x) &= \mathbf{F}(x) \quad (x \in \mathbf{C}) \end{aligned}$$

3.1.5 Images

Definition 3.1.20 (Image). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C} \subseteq \mathbf{A}$. The *image* of \mathbf{C} under \mathbf{F} is the class

$$\mathbf{F}(\mathbf{C}) := \{\mathbf{F}(x) \mid x \in \mathbf{C}\}.$$

Proposition Schema 3.1.21 (S). *For any classes \mathbf{F} , \mathbf{A} and \mathbf{B} , the following is a theorem.*

If $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$, then for any subset $S \subseteq \mathbf{A}$, the class $\mathbf{F}(S)$ is a set.

PROOF: By an Axiom of Replacement. □

Proposition Schema 3.1.22 (S). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{F} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C} \subseteq \mathcal{P}\mathbf{A}$. Then

$$\mathbf{F}\left(\bigcup \mathbf{C}\right) = \{y \mid \exists X \in \mathbf{C}. y \in \mathbf{F}(X)\}$$

PROOF:

$$\begin{aligned}
 y \in \mathbf{F}\left(\bigcup \mathbf{C}\right) &\Leftrightarrow \exists x \in \bigcup \mathbf{C}. y = \mathbf{F}(x) \\
 &\Leftrightarrow \exists x. \exists X. X \in \mathbf{C} \wedge x \in X \wedge y = \mathbf{F}(x) \\
 &\Leftrightarrow \exists X \in \mathbf{C}. y \in \mathbf{F}(X) \quad \square
 \end{aligned}$$

Proposition Schema 3.1.23 (S). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} and \mathbf{F} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$. Then

$$\mathbf{F}(\mathbf{C} \cup \mathbf{D}) = \mathbf{F}(\mathbf{C}) \cup \mathbf{F}(\mathbf{D}) .$$

PROOF:

$$\begin{aligned}
 y \in \mathbf{F}(\mathbf{C} \cup \mathbf{D}) &\Leftrightarrow \exists x \in \mathbf{C} \cup \mathbf{D}. y = \mathbf{F}(x) \\
 &\Leftrightarrow \exists x \in \mathbf{C}. y = \mathbf{F}(x) \vee \exists x \in \mathbf{D}. y = \mathbf{F}(x) \\
 &\Leftrightarrow y \in \mathbf{F}(\mathbf{C}) \cup \mathbf{F}(\mathbf{D}) \quad \square
 \end{aligned}$$

Proposition 3.1.24 (S). *For any classes \mathbf{F} , \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$. Then

$$\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B}) .$$

Equality holds if \mathbf{F} is injective.

PROOF:

- $\langle 1 \rangle 1. \mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
- $\langle 2 \rangle 1. \text{ LET: } y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$
- $\langle 2 \rangle 2. \text{ PICK } x \in \mathbf{A} \cap \mathbf{B} \text{ such that } y = \mathbf{F}(x)$
- $\langle 2 \rangle 3. y \in \mathbf{F}(\mathbf{A})$
- PROOF: Since $x \in \mathbf{A}$.
- $\langle 2 \rangle 4. y \in \mathbf{F}(\mathbf{B})$
- PROOF: Since $x \in \mathbf{B}$.
- $\langle 1 \rangle 2. \text{ If } \mathbf{F} \text{ is injective then } \mathbf{F}(\mathbf{A} \cap \mathbf{B}) = \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B}).$
- $\langle 2 \rangle 1. \text{ ASSUME: } \mathbf{F} \text{ is injective.}$
- $\langle 2 \rangle 2. \text{ LET: } y \in \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
- $\langle 2 \rangle 3. \text{ PICK } x \in \mathbf{A} \text{ such that } y = \mathbf{F}(x)$
- $\langle 2 \rangle 4. \text{ PICK } x' \in \mathbf{B} \text{ such that } y = \mathbf{F}(x')$
- $\langle 2 \rangle 5. x = x'$
- PROOF: $\langle 2 \rangle 1$
- $\langle 2 \rangle 6. x \in \mathbf{A} \cap \mathbf{B}$
- $\langle 2 \rangle 7. y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$

□

Proposition Schema 3.1.25 (S). *For any classes \mathbf{F} , \mathbf{A} , \mathbf{B} , and \mathbf{C} , the following is a theorem:*

Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C} \subseteq \mathcal{P}\mathbf{A}$. Then

$$\mathbf{F}\left(\bigcap \mathbf{C}\right) \subseteq \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\} .$$

Equality holds if \mathbf{F} is injective and \mathbf{A} is nonempty.

PROOF:

- $\langle 1 \rangle 1.$ $\mathbf{F}(\bigcap \mathbf{A}) \subseteq \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
 - $\langle 2 \rangle 1.$ LET: $y \in \mathbf{F}(\bigcap \mathbf{A})$
 - $\langle 2 \rangle 2.$ PICK $x \in \bigcap \mathbf{A}$ such that $y = \mathbf{F}(x)$
 - $\langle 2 \rangle 3.$ LET: $X \in \mathbf{A}$
 - $\langle 2 \rangle 4.$ $x \in X$
 - $\langle 2 \rangle 5.$ $y \in \mathbf{F}(X)$
- $\langle 1 \rangle 2.$ If \mathbf{F} is injective then $\mathbf{F}(\bigcap \mathbf{A}) = \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
 - $\langle 2 \rangle 1.$ ASSUME: \mathbf{F} is injective.
 - $\langle 2 \rangle 2.$ ASSUME: \mathbf{A} is nonempty.
 - $\langle 2 \rangle 3.$ LET: $y \in \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
 - $\langle 2 \rangle 4.$ PICK $X_0 \in \mathbf{A}$
 - $\langle 2 \rangle 5.$ PICK $x \in X_0$ such that $(x, y) \in \mathbf{F}$
 - $\langle 2 \rangle 6.$ $x \in \bigcap \mathbf{A}$
 - $\langle 3 \rangle 1.$ LET: $X \in \mathbf{A}$
 - $\langle 3 \rangle 2.$ PICK $x' \in X$ such that $(x', y) \in \mathbf{F}$.
 - $\langle 3 \rangle 3.$ $x = x'$
 - PROOF: $\langle 2 \rangle 1$
 - $\langle 3 \rangle 4.$ $x \in X$
 - $\langle 2 \rangle 7.$ $y \in \mathbf{F}(\bigcap \mathbf{A})$

□

Proposition 3.1.26 (S). *For any classes $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ and \mathbf{F} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$. Then

$$\mathbf{F}(\mathbf{C}) - \mathbf{F}(\mathbf{D}) \subseteq \mathbf{F}(\mathbf{C} - \mathbf{D}) .$$

Equality holds if \mathbf{F} is injective.

PROOF:

- $\langle 1 \rangle 1.$ $\mathbf{F}(\mathbf{C}) - \mathbf{F}(\mathbf{D}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
 - $\langle 2 \rangle 1.$ LET: $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
 - $\langle 2 \rangle 2.$ PICK $x \in \mathbf{A}$ such that $y = \mathbf{F}(x)$
 - $\langle 2 \rangle 3.$ $x \notin \mathbf{B}$
 - $\langle 2 \rangle 4.$ $x \in \mathbf{A} - \mathbf{B}$
 - $\langle 2 \rangle 5.$ $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
- $\langle 1 \rangle 2.$ If \mathbf{F} is injective then $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) = \mathbf{F}(\mathbf{A} - \mathbf{B})$
 - $\langle 2 \rangle 1.$ ASSUME: \mathbf{F} is injective.
 - $\langle 2 \rangle 2.$ LET: $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
 - $\langle 2 \rangle 3.$ PICK $x \in \mathbf{A} - \mathbf{B}$ such that $y = \mathbf{F}(x)$
 - $\langle 2 \rangle 4.$ $y \in \mathbf{F}(\mathbf{A})$
 - $\langle 2 \rangle 5.$ $y \notin \mathbf{F}(\mathbf{B})$

- ⟨3⟩1. ASSUME: for a contradiction $y \in \mathbf{F}(\mathbf{B})$
 ⟨3⟩2. PICK $x' \in \mathbf{B}$ such that $y = \mathbf{F}(x')$
 ⟨3⟩3. $x = x'$
 PROOF: ⟨2⟩1
 ⟨3⟩4. $x \in \mathbf{B}$
 ⟨3⟩5. Q.E.D.
 PROOF: This contradicts ⟨2⟩3.

□

3.1.6 Inverse Images

Definition 3.1.27 (Inverse Image). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C} \subseteq \mathbf{B}$. Then the *inverse image* of \mathbf{C} under \mathbf{F} is

$$\mathbf{F}^{-1}(\mathbf{C}) = \{x \in \mathbf{A} \mid \mathbf{F}(x) \in \mathbf{C}\} .$$

Proposition Schema 3.1.28 (S). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{F} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C} \subseteq \mathcal{PB}$. Then

$$\mathbf{F}^{-1}\left(\bigcap \mathbf{C}\right) = \bigcap \{\mathbf{F}^{-1}(X) \mid X \in \mathbf{C}\} .$$

PROOF:

$$\begin{aligned}
 x \in \mathbf{F}^{-1}\left(\bigcap \mathbf{C}\right) &\Leftrightarrow \mathbf{F}(x) \in \bigcap \mathbf{C} \\
 &\Leftrightarrow \forall X \in \mathbf{C}. \mathbf{F}(x) \in X \\
 &\Leftrightarrow \forall X \in \mathbf{C}. x \in \mathbf{F}^{-1}(X) \quad \square
 \end{aligned}$$

Proposition Schema 3.1.29 (S). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} and \mathbf{F} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C}, \mathbf{D} \subseteq \mathbf{B}$. Then

$$\mathbf{F}^{-1}(\mathbf{C} - \mathbf{D}) = \mathbf{F}^{-1}(\mathbf{C}) - \mathbf{F}^{-1}(\mathbf{D}) .$$

PROOF:

$$\begin{aligned}
 x \in \mathbf{F}^{-1}(\mathbf{C} - \mathbf{D}) &\Leftrightarrow \mathbf{F}(x) \in \mathbf{C} - \mathbf{D} \\
 &\Leftrightarrow \mathbf{F}(x) \in \mathbf{C} \wedge \mathbf{F}(x) \notin \mathbf{D} \\
 &\Leftrightarrow x \in \mathbf{F}^{-1}(\mathbf{C}) \wedge x \notin \mathbf{F}^{-1}(\mathbf{D}) \\
 &\Leftrightarrow x \in \mathbf{F}^{-1}(\mathbf{C}) - \mathbf{F}^{-1}(\mathbf{D}) \quad \square
 \end{aligned}$$

3.1.7 Function Sets

Proposition 3.1.30 (SF). *For any classes \mathbf{B} and \mathbf{F} , the following is a theorem:*

Let A be a set. If $\mathbf{F} : A \rightarrow \mathbf{B}$ then \mathbf{F} is a set.

PROOF: By an Axiom of Replacement, we have $R = \{\mathbf{F}(x) \mid x \in A\}$ is a set. Hence \mathbf{F} is a set since $\mathbf{F} \subseteq A \times R$. □

Definition 3.1.31 (Dependent Product Class). Let I be a set and let $\mathbf{H}(i)$ be a class for all $i \in I$. We write $\prod_{i \in I} \mathbf{H}(i)$ for the class of all functions $f : I \rightarrow \bigcup_{i \in I} \mathbf{H}(i)$ such that $\forall i \in I. f(i) \in \mathbf{H}(i)$.

We write \mathbf{B}^I for $\prod_{i \in I} \mathbf{B}$ where \mathbf{B} does not depend on I .

Proposition Schema 3.1.32 (SF). *Let I be a set. Let $H(i)$ be a set for every $i \in I$. Then $\prod_{i \in I} H(i)$ is a set.*

PROOF:

$\langle 1 \rangle 1$. $\{\mathbf{H}(i) \mid i \in I\}$ is a set.

PROOF: By an Axiom of Replacement.

$\langle 1 \rangle 2$. $\bigcup_{i \in I} \mathbf{H}(i)$ is a set.

$\langle 1 \rangle 3$. $\prod_{i \in I} \mathbf{H}(i)$ is a set.

PROOF: It is a subset of $\mathcal{P}(I \times \bigcup_{i \in I} \mathbf{H}(i))$.

□

Proposition 3.1.33 (SFC). *Let I be a set. Let $H(i)$ be a set for all $i \in I$. If $\forall i \in I. H(i) \neq \emptyset$ then $\prod_{i \in I} H(i) \neq \emptyset$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: $\forall i \in I. H(i) \neq \emptyset$

$\langle 1 \rangle 2$. LET: $R = \{(i, x) \mid i \in I, x \in H(i)\}$

$\langle 1 \rangle 3$. PICK a function $f : I \rightarrow \bigcup_{i \in I} H(i)$ such that $f \subseteq R$

PROOF: Proposition 3.1.6.

$\langle 1 \rangle 4$. $f \in \prod_{i \in I} H(i)$

□

3.2 Equinumerosity

Definition 3.2.1 (Equinumerous). Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between A and B .

3.3 Domination

Definition 3.3.1 (Dominate). A set A is *dominated* by a set B , $A \preceq B$, iff there exists an injection $A \rightarrow B$.

Proposition 3.3.2 (SC). *Given sets A and B , if $A \neq \emptyset$ or $B = \emptyset$, then we have $A \preceq B$ iff there exists a surjective function $B \rightarrow A$.*

PROOF:

$\langle 1 \rangle 1$. If $A \preceq B$ and $A \neq \emptyset$ then there exists a surjective function $B \rightarrow A$.

$\langle 2 \rangle 1$. ASSUME: $f : A \rightarrow B$ be injective.

$\langle 2 \rangle 2$. PICK $a \in A$

$\langle 2 \rangle 3$. LET: $g : B \rightarrow A$ be the function defined by $g(b) = f^{-1}(b)$ if $b \in \text{ran } f$,
and $g(b) = a$ otherwise.

- $\langle 2 \rangle 4$. g is surjective.
- $\langle 1 \rangle 2$. If there exists a surjective function $B \rightarrow A$ then $A \preceq B$.
- $\langle 2 \rangle 1$. ASSUME: there exists a surjective function $g : B \rightarrow A$
- $\langle 2 \rangle 2$. $\forall a \in A. \exists b \in B. g(b) = a$
- $\langle 2 \rangle 3$. Choose a function $f : A \rightarrow B$ such that $\forall a \in A. g(f(a)) = a$
- $\langle 2 \rangle 4$. f is injective.

□

Chapter 4

Category Theory

4.1 Categories

Definition 4.1.1 (Category). A *category* \mathbf{C} consists of:

- a class of *objects*;
- for any objects X, Y , a set $\mathbf{C}(X, Y)$ whose elements are called *morphisms*. We write $f : X \rightarrow Y$ for $f \in \mathbf{C}(X, Y)$.
- for any morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, a morphism $gf : X \rightarrow Z$

such that:

- For all $f : X \rightarrow Y, g : Y \rightarrow Z, h : Z \rightarrow W$, we have $h(gf) = (hg)f$.
- For every object X , there exists a morphism $\text{id}_X : X \rightarrow X$ such that:
 - for any object Y and morphism $f : X \rightarrow Y$ we have $f\text{id}_X = f$.
 - for any object Y and morphism $f : Y \rightarrow X$ we have $\text{id}_X f = f$.

Definition 4.1.2 (Category of Sets). Let Set be the category of sets and functions.

Definition 4.1.3 (Category of Pointed Sets). The *category of pointed sets* Set_* has:

- objects all pairs (A, a) where A is a set and $a \in A$;
- morphisms $f : (A, a) \rightarrow (B, b)$ all functions $f : A \rightarrow B$ such that $f(a) = b$.

Definition 4.1.4 (Opposite Category). For any category \mathbf{C} , the *opposite category* \mathbf{C}^{op} is the category with the same objects as \mathbf{C} and $\mathbf{C}^{\text{op}}(X, Y) = \mathbf{C}(Y, X)$.

4.2 Invertible Morphisms

Definition 4.2.1 (Left Inverse, Right Inverse). In any category, let $f : A \rightarrow B$ and $g : B \rightarrow A$. Then g is a *left inverse* of f , and f is a *right inverse* of f , iff $gf = \text{id}_A$.

Proposition 4.2.2 (FOL). Let $f : A \rightarrow B$ and $g, h : B \rightarrow A$. If g is a left inverse to f and h is a right inverse to f then $g = h$.

PROOF: Since $g = g\text{id}_B = gfh = \text{id}_Ah = h$. \square

Definition 4.2.3 (Isomorphism). A morphism $f : A \rightarrow B$ is an *isomorphism*, $f : A \cong B$, iff it has both a left and a right inverse. In this case, its unique inverse is denoted f^{-1} .

Two objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 4.2.4 (S). A function is an isomorphism in Set iff it is a bijection.

Proposition 4.2.5 (S). Let $f : A \rightarrow B$ in the category \mathbf{C} . Then the following are equivalent.

1. f is an isomorphism.
2. For every object X , the function $\mathbf{C}(X, f) : \mathbf{C}(X, A) \rightarrow \mathbf{C}(X, B)$ is a bijection.
3. For every object X , the function $\mathbf{C}(f, X) : \mathbf{C}(B, X) \rightarrow \mathbf{C}(A, X)$ is a bijection.

Chapter 5

Equivalence Relations

Definition 5.0.1 (Equivalence Relation). An *equivalence relation* on a class \mathbf{A} is a binary relation on \mathbf{A} that is reflexive, symmetric and transitive.

Proposition 5.0.2 (Z). *Equinumerosity is an equivalence relation on the class of all sets.*

PROOF: Propositions 3.1.16, 3.1.17, 3.1.18. \square

Definition 5.0.3 (Respects). Let \mathbf{R} be an equivalence relation on \mathbf{A} and $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Then \mathbf{F} *respects* \mathbf{A} iff, whenever $(x, y) \in \mathbf{R}$, then $\mathbf{F}(x) = \mathbf{F}(y)$.

Definition 5.0.4 (Equivalence Class). Let \mathbf{R} be an equivalence relation on \mathbf{A} and $a \in \mathbf{A}$. The *equivalence class* of a modulo \mathbf{R} is

$$[a]_{\mathbf{R}} := \{x \mid a\mathbf{R}x\} .$$

Proposition Schema 5.0.5 (Z). *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem.*

Assume \mathbf{R} be an equivalence relation on \mathbf{A} . Let $a, b \in \mathbf{A}$. Then $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ if and only if $a\mathbf{R}b$.

PROOF:

$\langle 1 \rangle 1$. If $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ then $a\mathbf{R}b$.

$\langle 2 \rangle 1$. ASSUME: $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$

$\langle 2 \rangle 2$. $b\mathbf{R}b$

PROOF: Reflexivity

$\langle 2 \rangle 3$. $b \in [b]_{\mathbf{R}}$

$\langle 2 \rangle 4$. $b \in [a]_{\mathbf{R}}$

$\langle 2 \rangle 5$. $a\mathbf{R}b$

$\langle 1 \rangle 2$. If $a\mathbf{R}b$ then $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$.

$\langle 2 \rangle 1$. For all $x, y \in \mathbf{A}$, if $x\mathbf{R}y$ then $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$

$\langle 3 \rangle 1$. LET: $x, y \in \mathbf{A}$

$\langle 3 \rangle 2$. ASSUME: $x\mathbf{R}y$

$\langle 3 \rangle 3$. LET: $t \in [y]_{\mathbf{R}}$
 $\langle 3 \rangle 4$. $y \mathbf{R} t$
 $\langle 3 \rangle 5$. $x \mathbf{R} t$
 PROOF: Transitivity, $\langle 3 \rangle 2$, $\langle 3 \rangle 4$.
 $\langle 3 \rangle 6$. $t \in [x]_{\mathbf{R}}$
 $\langle 2 \rangle 2$. ASSUME: $a \mathbf{R} b$
 $\langle 2 \rangle 3$. $[b]_{\mathbf{R}} \subseteq [a]_{\mathbf{R}}$
 PROOF: $\langle 2 \rangle 1$, $\langle 2 \rangle 2$.
 $\langle 2 \rangle 4$. $b \mathbf{R} a$
 PROOF: Symmetry, $\langle 2 \rangle 2$.
 $\langle 2 \rangle 5$. $[a]_{\mathbf{R}} \subseteq [b]_{\mathbf{R}}$
 PROOF: $\langle 2 \rangle 1$, $\langle 2 \rangle 4$.
 $\langle 2 \rangle 6$. $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$
 PROOF: $\langle 2 \rangle 3$, $\langle 2 \rangle 5$.

□

Definition 5.0.6 (Partition). A *partition* Π of a set A is a set of nonempty subsets of A that is disjoint and exhaustive, i.e.

1. no two different sets in Π have any common elements, and
2. each element of A is in some set in Π .

Definition 5.0.7. Let R be an equivalence relation on a set A . The *quotient set* A/R is the set of all equivalence classes.

Theorem 5.0.8 (Z). Let A be a set and \mathbf{B} a class. Let R be an equivalence relation on A and $F : A \rightarrow \mathbf{B}$. Then F respects R if and only if there exists $\hat{F} : A/R \rightarrow \mathbf{B}$ such that

$$\forall a \in A. \hat{F}([a]_R) = F(a) .$$

In this case, \hat{F} is unique.

PROOF:

$\langle 1 \rangle 1$. If F respects R then there exists $\hat{F} : A/R \rightarrow \mathbf{B}$ such that $\forall a \in A. \hat{F}([a]_R) = F(a)$.
 $\langle 2 \rangle 1$. ASSUME: F respects R .
 $\langle 2 \rangle 2$. LET: $\hat{F} = \{([a]_R, F(a)) \mid a \in A\}$
 $\langle 2 \rangle 3$. \hat{F} is a function.
 $\langle 3 \rangle 1$. ASSUME: $a, a' \in A$ and $[a]_R = [a']_R$
 PROVE: $F(a) = F(a')$
 $\langle 3 \rangle 2$. $(a, a') \in R$
 PROOF: Proposition 5.0.5.
 $\langle 3 \rangle 3$. $F(a) = F(a')$
 PROOF: $\langle 2 \rangle 1$
 $\langle 2 \rangle 4$. $\text{dom } \hat{F} = A/R$
 $\langle 2 \rangle 5$. $\text{ran } \hat{F} \subseteq \mathbf{B}$

- $\langle 2 \rangle 6. \forall a \in A. \hat{F}([a]_R) = F(a)$
 $\langle 1 \rangle 2.$ If there exists $\hat{F} : A/R \rightarrow \mathbf{B}$ such that $\forall a \in A. \hat{F}([a]_R) = F(a)$ then F respects R .
 $\langle 2 \rangle 1.$ ASSUME: $\hat{F} : A/R \rightarrow \mathbf{B}$ and $\forall a \in A. \hat{F}([a]_R) = F(a)$
 $\langle 2 \rangle 2.$ LET: $a, a' \in A$
 $\langle 2 \rangle 3.$ ASSUME: $(a, a') \in R$
 $\langle 2 \rangle 4.$ $[a]_R = [a']_R$
 PROOF: Proposition 5.0.5.
 $\langle 2 \rangle 5.$ $F(a) = F(a')$
 PROOF: $\langle 2 \rangle 1$
 $\langle 1 \rangle 3.$ If $G, H : A/R \rightarrow \mathbf{B}$ and $\forall a \in A. G([a]_R) = H([a]_R)$ then $G = H$.
 \square

Proposition 5.0.9 (Z). *Let R be an equivalence relation on a set A . Then A/R is a partition of A .*

PROOF:

- $\langle 1 \rangle 1.$ Every member of A/R is nonempty.
 PROOF: Since $a \in [a]_R$ by reflexivity.
 $\langle 1 \rangle 2.$ No two different sets in A/R have any common elements.
 $\langle 2 \rangle 1.$ LET: $[a]_R, [b]_R \in A/R$
 $\langle 2 \rangle 2.$ LET: $c \in [a]_R \cap [b]_R$
 PROVE: $[a]_R = [b]_R$
 $\langle 2 \rangle 3.$ aRc
 PROOF: $\langle 2 \rangle 2$
 $\langle 2 \rangle 4.$ bRc
 PROOF: $\langle 2 \rangle 2$
 $\langle 2 \rangle 5.$ cRb
 PROOF: Symmetry, $\langle 2 \rangle 4$
 $\langle 2 \rangle 6.$ aRb
 PROOF: Transitivity, $\langle 2 \rangle 3, \langle 2 \rangle 5$
 $\langle 2 \rangle 7.$ $[a]_R = [b]_R$
 PROOF: Proposition 5.0.5, $\langle 2 \rangle 6$
 $\langle 1 \rangle 3.$ Each element of A is in some set in A/R .
 PROOF: Since $a \in [a]_R$ by reflexivity.
 \square

Proposition 5.0.10 (Z). *For any partition P of a set A , there exists a unique equivalence relation R on A such that $A/R = P$, namely xRy iff $\exists X \in P(x \in X \wedge y \in X)$.*

PROOF: Easy. \square

Definition 5.0.11 (Natural Map). Let A be a set and R an equivalence relation on A . The *natural map* $A \rightarrow A/R$ is the function that maps $a \in A$ to $[a]_R$.

Chapter 6

Ordering Relations

6.1 Partial Orders

Definition 6.1.1 (Partial Ordering). Let \mathbf{A} be a class. A *partial ordering* on \mathbf{A} is a relation \mathbf{R} on \mathbf{A} that is reflexive, antisymmetric and transitive.

We often write \leq for a partial ordering, and then write $x < y$ for $x \leq y \wedge x \neq y$.

Proposition Schema 6.1.2 (Z). *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is a partial order on \mathbf{A} then so is \mathbf{R}^{-1} .

PROOF:

$\langle 1 \rangle 1.$ \mathbf{R}^{-1} is reflexive.

PROOF: Proposition 2.2.12.

$\langle 1 \rangle 2.$ \mathbf{R}^{-1} is antisymmetric.

PROOF: Proposition 2.2.16.

$\langle 1 \rangle 3.$ \mathbf{R}^{-1} is transitive.

$\langle 2 \rangle 1.$ ASSUME: $x\mathbf{R}^{-1}y$ and $y\mathbf{R}^{-1}z$

$\langle 2 \rangle 2.$ $y\mathbf{R}x$ and $z\mathbf{R}y$

$\langle 2 \rangle 3.$ $z\mathbf{R}x$

PROOF: Since \mathbf{R} is transitive.

$\langle 2 \rangle 4.$ $x\mathbf{R}^{-1}z$

□

Proposition Schema 6.1.3 (Z). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{F} and \mathbf{R} , the following is a theorem:*

Assume \mathbf{R} is a partial order on \mathbf{B} and $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ is injective. Define \mathbf{S} on \mathbf{A} by $x\mathbf{S}y$ iff $\mathbf{F}(x)\mathbf{R}\mathbf{F}(y)$. Then \mathbf{S} is a partial order on \mathbf{A} .

PROOF:

$\langle 1 \rangle 1.$ \mathbf{S} is reflexive.

PROOF: For any $x \in \mathbf{A}$ we have $\mathbf{F}(x)\mathbf{R}\mathbf{F}(x)$.

- ⟨1⟩2. **S** is antisymmetric.
 ⟨2⟩1. LET: $x, y \in \mathbf{A}$
 ⟨2⟩2. ASSUME: $x\mathbf{S}y$ and $y\mathbf{S}x$
 ⟨2⟩3. $\mathbf{F}(x)\mathbf{R}\mathbf{F}(y)$ and $\mathbf{F}(y)\mathbf{R}\mathbf{F}(x)$
 ⟨2⟩4. $\mathbf{F}(x) = \mathbf{F}(y)$
 PROOF: **R** is antisymmetric.
 ⟨2⟩5. $x = y$
 ⟨1⟩3. **S** is transitive.
 □

Corollary Schema 6.1.3.1 (Z). *For any classes **A**, **B** and **R**, the following is a theorem:*

*Assume **R** be a partial order on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Then $\mathbf{R} \cap \mathbf{B}^2$ is a partial order on **B**.*

Definition 6.1.4 (Partially Ordered Set). A *partially ordered set* or *poset* is a pair (A, \leq) where A is a set and \leq is a partial ordering on A . We often write just A for (A, \leq) .

If (A, \leq) is a poset and $B \subseteq A$ we write just B for the poset $(B, \leq \cap B^2)$.

Definition 6.1.5 (Strictly Monotone). Let $(A, <_A)$ and $(B, <_B)$ be posets. A function $f : A \rightarrow B$ is *strictly monotone* iff, whenever $x <_A y$, then $f(x) <_B f(y)$.

Definition 6.1.6 (Least). Let \leq be a partial order on **A**. An element $m \in \mathbf{A}$ is *least* iff for all $x \in \mathbf{A}$ we have $m \leq x$.

Proposition 6.1.7 (Z). *A partial order has at most one least element.*

PROOF: If m and m' are least then $m \leq m'$ and $m' \leq m$, so $m = m'$. □

Definition 6.1.8 (Greatest). Let \leq be a partial order on **A**. An element $m \in \mathbf{A}$ is *greatest* iff for all $x \in \mathbf{A}$ we have $x \leq m$.

Proposition 6.1.9 (Z). *A poset has at most one greatest element.*

PROOF: If m and m' are greatest then $m \leq m'$ and $m' \leq m$, so $m = m'$. □

Definition 6.1.10 (Upper Bound). Let \leq be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Let $u \in \mathbf{A}$. Then u is an *upper bound* for **B** iff $\forall x \in \mathbf{B}. x \leq u$.

Definition 6.1.11 (Lower Bound). Let \leq be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Let $l \in \mathbf{A}$. Then l is a *lower bound* for **B** iff $\forall x \in \mathbf{B}. l \leq x$.

Definition 6.1.12 (Bounded Above). Let \leq be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Then **B** is *bounded above* iff it has an upper bound.

Definition 6.1.13 (Bounded Below). Let \leq be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Then **B** is *bounded below* iff it has a lower bound.

Definition 6.1.14 (Least Upper Bound). Let \leq be a partial ordering on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Let $s \in \mathbf{A}$. Then s is the *least upper bound* or *supremum* of \mathbf{B} iff s is an upper bound for \mathbf{B} and, for every upper bound u for \mathbf{B} , we have $s \leq u$.

Definition 6.1.15 (Greatest Lower Bound). Let \leq be a partial ordering on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Let $i \in \mathbf{A}$. Then i is the *greatest lower bound* or *infimum* of \mathbf{B} iff i is a lower bound for \mathbf{B} and, for every lower bound l for \mathbf{B} , we have $i \leq l$.

Definition 6.1.16 (Complete). A partial order is *complete* iff every nonempty subset bounded above has a supremum, and every nonempty subset bounded below has an infimum.

Definition 6.1.17 (Order Isomorphism). Let A and B be posets. An *order isomorphism* between A and B , $f : A \cong B$, is a bijection $f : A \approx B$ such that f and f^{-1} are monotone.

Theorem 6.1.18 (Knaster Fixed-Point Theorem (Z)). *Let A be a complete poset with a greatest and least element. Let $\phi : A \rightarrow A$ be monotone. Then there exists $a \in A$ such that $\phi(a) = a$.*

PROOF:

$\langle 1 \rangle 1$. LET: $B = \{x \in A \mid x \leq \phi(x)\}$

$\langle 1 \rangle 2$. LET: $a = \sup B$

PROOF: B is nonempty because the least element of A is in B , and it is bounded above by the greatest element of A .

$\langle 1 \rangle 3$. For all $b \in B$ we have $b \leq \phi(a)$

$\langle 2 \rangle 1$. LET: $b \in B$

$\langle 2 \rangle 2$. $b \leq \phi(b)$

$\langle 2 \rangle 3$. $b \leq a$

$\langle 2 \rangle 4$. $\phi(b) \leq \phi(a)$

$\langle 2 \rangle 5$. $b \leq \phi(a)$

$\langle 1 \rangle 4$. $a \leq \phi(a)$

$\langle 1 \rangle 5$. $\phi(a) \leq \phi(\phi(a))$

$\langle 1 \rangle 6$. $\phi(a) \in B$

$\langle 1 \rangle 7$. $\phi(a) \leq a$

$\langle 1 \rangle 8$. $\phi(a) = a$

□

Definition 6.1.19 (Dense). Let \leq be a partial order on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Then \mathbf{B} is *dense* iff, for all $x, y \in \mathbf{A}$, if $x < y$ then there exists $z \in \mathbf{B}$ such that $x < z < y$.

Proposition 6.1.20 (Z). *Let A be a complete poset with no least element. Let $B \subseteq A$ be dense. Let $\theta : A \rightarrow A$ be a monotone map that is the identity on B . Then $\theta = \text{id}_A$.*

PROOF:

$\langle 1 \rangle 1$. LET: $a \in A$

PROVE: $\theta(a) = a$

- ⟨1⟩2. LET: $S(a) = \{b \in B \mid b < a\}$
- ⟨1⟩3. $S(a)$ is nonempty and bounded above.
 - ⟨2⟩1. $S(a)$ is nonempty.
 - ⟨3⟩1. PICK $a_1 < a$
 - PROOF: Since a is not least.
 - ⟨3⟩2. There exists $b \in B$ such that $a_1 < b < a$.
 - ⟨2⟩2. $S(a)$ is bounded above by a .
- ⟨1⟩4. $\sup S(a) \leq a$
- ⟨1⟩5. $\sup S(a) = a$
 - ⟨2⟩1. ASSUME: for a contradiction $\sup S(a) < a$
 - ⟨2⟩2. PICK $b \in B$ such that $\sup S(a) < b < a$
 - ⟨2⟩3. $b \in S(a)$
 - ⟨2⟩4. Q.E.D.
- PROOF: This contradicts the fact that $\sup S(a) < b$.
- ⟨1⟩6. For all $b \in S(a)$ we have $b \leq \theta(a)$
 - ⟨2⟩1. LET: $b \in S(a)$
 - ⟨2⟩2. $b < a$
 - ⟨2⟩3. $\theta(b) \leq \theta(a)$
 - ⟨2⟩4. $b \leq \theta(a)$
 - PROOF: $\theta(b) = b$
- ⟨1⟩7. $a \leq \theta(a)$
 - PROOF: Since $a = \sup S(a)$ and $\theta(a)$ is an upper bound for $S(a)$.
- ⟨1⟩8. $a \not\leq \theta(a)$
 - ⟨2⟩1. ASSUME: for a contradiction $a < \theta(a)$.
 - ⟨2⟩2. PICK $b \in B$ such that $a < b < \theta(a)$
 - ⟨2⟩3. $\theta(a) \leq \theta(b) = b$
 - ⟨2⟩4. Q.E.D.
 - PROOF: This contradicts the fact that $b < \theta(a)$.
- ⟨1⟩9. $\theta(a) = a$

□

Theorem 6.1.21 (Z). *Let A and P be complete posets with no least or greatest element. Let B be dense in A and Q be dense in P . Every order isomorphism $\phi : B \cong Q$ extends uniquely to an order isomorphism $A \cong P$.*

PROOF:

- ⟨1⟩1. For $a \in A$, let $S(a) = \{b \in B \mid b < a\}$.
- ⟨1⟩2. Define $\bar{\phi} : A \rightarrow P$ by $\bar{\phi}(a) = \sup \phi(S(a))$.
 - ⟨2⟩1. $\phi(S(a))$ is nonempty.
 - ⟨3⟩1. PICK $a_1 < a$
 - PROOF: Since a is not least.
 - ⟨3⟩2. PICK $b \in B$ such that $a_1 < b < a$.
 - ⟨3⟩3. $\phi(b) \in \phi(S(a))$
 - ⟨2⟩2. $\phi(S(a))$ is bounded above.
 - ⟨3⟩1. PICK $a_2 > a$
 - PROOF: Since a is not greatest.
 - ⟨3⟩2. PICK $b \in B$ such that $a < b < a_2$

- (3)3. $\phi(b)$ is an upper bound for $\phi(S(a))$.
 (1)3. $\bar{\phi}$ is monotone.
 PROOF: If $a \leq a'$ then $S(a) \subseteq S(a')$ and so $\bar{\phi}(a) \leq \bar{\phi}(a')$.
 (1)4. $\bar{\phi}$ extends ϕ .
 (2)1. LET: $b \in B$
 PROVE: $\phi(b) = \sup \phi(S(b))$
 (2)2. $\phi(b)$ is an upper bound for $\phi(S(b))$
 (2)3. LET: u be any upper bound for $\phi(S(b))$
 PROVE: $\phi(b) \leq u$
 (2)4. ASSUME: for a contradiction $u < \phi(b)$
 (2)5. PICK $q \in Q$ such that $u < q < \phi(b)$
 (2)6. PICK $b' \in B$ such that $\phi(b') = q$
 (2)7. $b' < b$
 (2)8. $b' \in S(b)$
 (2)9. $q = \phi(b') \leq u$
 (2)10. Q.E.D.
 PROOF: This is a contradiction.
 (1)5. LET: $\bar{\psi} = \phi^{-1}$
 (1)6. LET: $\bar{\psi} : P \rightarrow A$ be the function $\bar{\psi}(p) = \sup\{\psi(q) \mid q \in Q, q < p\}$
 (1)7. $\bar{\psi}$ is monotone and extends ψ
 PROOF: Similar.
 (1)8. $\bar{\psi} \circ \bar{\phi} : A \rightarrow A$ is monotone and the identity on B .
 (1)9. $\bar{\psi} \circ \bar{\phi} = \text{id}_A$
 PROOF: Proposition 6.1.20.
 (1)10. $\bar{\phi} \circ \bar{\psi} = \text{id}_B$
 PROOF: Proposition 6.1.20.
 (1)11. If $\phi^* : A \cong P$ is any order isomorphism that extends ϕ then $\phi^* = \bar{\phi}$.
 (2)1. LET: $a \in A$
 PROVE: $\phi^*(a) = \sup \phi(S(a))$
 (2)2. $\phi^*(a)$ is an upper bound for $\phi(S(a))$
 (2)3. LET: u be any upper bound for $\phi(S(a))$
 PROVE: $\phi^*(a) \leq u$
 (2)4. ASSUME: for a contradiction $u < \phi^*(a)$
 (2)5. PICK $q \in Q$ such that $u < q < \phi^*(a)$
 (2)6. PICK $b \in B$ such that $q = \phi(b)$
 (2)7. $b < a$
 (2)8. $b \in S(a)$
 (2)9. $q = \phi(b) \leq u$
 (2)10. Q.E.D.
 PROOF: This is a contradiction.

□

Definition 6.1.22 (Initial Segment). Let \leq be a partial order on \mathbf{A} and $t \in A$. The *initial segment* up to t is the class

$$\text{seg } t := \{x \in \mathbf{A} \mid x < t\} .$$

Definition 6.1.23 (Lexicographic Ordering). Let \mathbf{R} be a partial order on \mathbf{A} and \mathbf{S} a partial order on \mathbf{B} . The *lexicographic ordering* \leq on $\mathbf{A} \times \mathbf{B}$ is defined by:

$$(a, b) \leq (a', b') \Leftrightarrow (a\mathbf{R}a' \wedge a \neq a') \vee (a = a' \wedge b\mathbf{S}b') .$$

Proposition Schema 6.1.24 (Z). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{R} and \mathbf{S} , the following is a theorem:*

If \mathbf{R} is a partial order on \mathbf{A} and \mathbf{S} is a partial order on \mathbf{B} then the lexicographic ordering on $\mathbf{A} \times \mathbf{B}$ is a partial order.

PROOF:

$\langle 1 \rangle 1$. LET: \leq be the lexicographic ordering on $\mathbf{A} \times \mathbf{B}$

$\langle 1 \rangle 2$. \leq is reflexive.

PROOF: For any $a \in \mathbf{A}$ and $b \in \mathbf{B}$ we have $a = a$ and $b\mathbf{S}b$, so $(a, b) \leq (a, b)$.

$\langle 1 \rangle 3$. \leq is antisymmetric.

$\langle 2 \rangle 1$. ASSUME: $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$

$\langle 2 \rangle 2$. $(a\mathbf{R}a' \wedge a \neq a') \vee (a = a' \wedge b\mathbf{S}b')$

$\langle 2 \rangle 3$. $(a'\mathbf{R}a \wedge a' \neq a) \vee (a' = a \wedge b\mathbf{S}b')$

$\langle 2 \rangle 4$. CASE: $a = a'$

PROOF: Then $b\mathbf{S}b'$ and $b'\mathbf{S}b$ hence $b = b'$ and $(a, b) = (a', b')$.

$\langle 2 \rangle 5$. CASE: $a \neq a'$

PROOF: Then $a\mathbf{R}a'$ and $a'\mathbf{R}a$ hence $a = a'$ which is a contradiction.

$\langle 1 \rangle 4$. \leq is transitive.

$\langle 2 \rangle 1$. ASSUME: $(a_1, b_1) \leq (a_2, b_2) \leq (a_3, b_3)$

$\langle 2 \rangle 2$. $(a_1\mathbf{R}a_2 \wedge a_1 \neq a_2) \vee (a_1 = a_2 \wedge b_1\mathbf{S}b_2)$

$\langle 2 \rangle 3$. $(a_2\mathbf{R}a_3 \wedge a_2 \neq a_3) \vee (a_2 = a_3 \wedge b_2\mathbf{S}b_3)$

$\langle 2 \rangle 4$. CASE: $a_1\mathbf{R}a_2, a_1 \neq a_2, a_2\mathbf{R}a_3, a_2 \neq a_3$

$\langle 3 \rangle 1$. $a_1\mathbf{R}a_3$

PROOF: Since \mathbf{R} is transitive.

$\langle 3 \rangle 2$. $a_1 \neq a_3$

PROOF: If $a_1 = a_3$ then $a_1\mathbf{R}a_2$ and $a_2\mathbf{R}a_1$ so $a_1 = a_2$ which is a contradiction.

$\langle 2 \rangle 5$. CASE: $a_1\mathbf{R}a_2, a_1 \neq a_2, a_2 = a_3, b_2\mathbf{S}b_3$

PROOF: Then $a_1\mathbf{R}a_3$ and $a_1 \neq a_3$.

$\langle 2 \rangle 6$. CASE: $a_1 = a_2, b_1\mathbf{S}b_2, a_2\mathbf{R}a_3, a_2 \neq a_3$

PROOF: Then $a_1\mathbf{R}a_3$ and $a_1 \neq a_3$.

$\langle 2 \rangle 7$. CASE: $a_1 = a_2, b_1\mathbf{S}b_2, a_2 = a_3, b_2\mathbf{S}b_3$

PROOF: Then $a_1 = a_3$ and $b_1\mathbf{S}b_3$.

□

6.2 Linear Orders

Definition 6.2.1 (Linear Ordering). Let \mathbf{A} be a class. A *linear ordering* or *total ordering* on \mathbf{A} is a partial ordering \leq on \mathbf{A} that is *total*, i.e.

$$\forall x, y \in \mathbf{A}. x \leq y \vee y \leq x$$

We often use the symbol $<$ for a linear ordering, and then write $x < y$ for $(x, y) \in <$.

Proposition Schema 6.2.2 (Trichotomy (Z)). *For any classes \mathbf{A} and \leq , the following is a theorem:*

Assume \leq be a linear ordering on \mathbf{A} . For any $x, y \in \mathbf{A}$, exactly one of $x < y$, $x = y$, $y < x$ holds.

PROOF: Immediate from definitions. \square

Proposition Schema 6.2.3 (Z). *For any classes \mathbf{A} and $<$, the following is a theorem:*

Let $<$ be a transitive relation on \mathbf{A} that satisfies trichotomy. Define \leq on \mathbf{A} by $x \leq y$ iff $x < y$ or $x = y$. Then \leq is a linear ordering on \mathbf{A} and $x < y$ iff $x \leq y$ and $x \neq y$.

PROOF:

$\langle 1 \rangle 1.$ \leq is reflexive.

PROOF: By definition we have $\forall x \in \mathbf{A}. x \leq x$.

$\langle 1 \rangle 2.$ \leq is antisymmetric.

$\langle 2 \rangle 1.$ ASSUME: $x \leq y$ and $y \leq x$

$\langle 2 \rangle 2.$ $x < y$ or $x = y$

$\langle 2 \rangle 3.$ $y < x$ or $y = x$

$\langle 2 \rangle 4.$ We cannot have $x < y$ and $y < x$

PROOF: Trichotomy.

$\langle 2 \rangle 5.$ $x = y$

$\langle 1 \rangle 3.$ \leq is transitive.

$\langle 2 \rangle 1.$ ASSUME: $x \leq y$ and $y \leq z$

$\langle 2 \rangle 2.$ $x < y$ or $x = y$

$\langle 2 \rangle 3.$ $y < z$ or $y = z$

$\langle 2 \rangle 4.$ CASE: $x < y$ and $y < z$

PROOF: Then $x < z$ by transitivity, so $x \leq z$.

$\langle 2 \rangle 5.$ CASE: $x = y$

PROOF: Then we have $y \leq z$ and so $x \leq z$.

$\langle 2 \rangle 6.$ CASE: $y = z$

PROOF: Then we have $x \leq y$ and so $x \leq z$.

$\langle 1 \rangle 4.$ \leq is total.

PROOF: Immediate from trichotomy.

\square

Proposition Schema 6.2.4 (Z). *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is a linear ordering on \mathbf{A} then \mathbf{R}^{-1} is also a linear ordering on \mathbf{A} .

PROOF:

$\langle 1 \rangle 1.$ \mathbf{R}^{-1} is a partial order on \mathbf{A} .

PROOF: Proposition 6.1.2.

$\langle 1 \rangle 2.$ \mathbf{R}^{-1} is total.

- $\langle 2 \rangle 1.$ LET: $x, y \in \mathbf{A}$
- $\langle 2 \rangle 2.$ $x\mathbf{R}y$ or $y\mathbf{R}x$.
- $\langle 2 \rangle 3.$ $y\mathbf{R}^{-1}x$ or $x\mathbf{R}^{-1}y$.

□

Proposition Schema 6.2.5 (Z). *For any classes $\mathbf{A}, \mathbf{B}, \mathbf{F}, \mathbf{R}, \mathbf{S}$, the following is a theorem:*

Assume \mathbf{R} is a linear order on \mathbf{A} , \mathbf{S} is a partial order on \mathbf{B} , and $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. If \mathbf{F} is strictly monotone then it is injective.

PROOF:

- $\langle 1 \rangle 1.$ LET: $x, y \in \mathbf{A}$
- $\langle 1 \rangle 2.$ ASSUME: $x \neq y$
PROVE: $\mathbf{F}(x) \neq \mathbf{F}(y)$
- $\langle 1 \rangle 3.$ ASSUME: w.l.o.g. $x\mathbf{R}y$
PROOF: \mathbf{R} is total.
- $\langle 1 \rangle 4.$ $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$ and $\mathbf{F}(x) \neq \mathbf{F}(y)$
PROOF: \mathbf{F} is strictly monotone.

□

Proposition Schema 6.2.6 (Z). *For any classes \mathbf{A}, \mathbf{B} , \leq , \preceq and \mathbf{F} , the following is a theorem:*

Assume \leq is a linear order on \mathbf{A} and \preceq is a linear order on \mathbf{B} . Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and \mathbf{F} is strictly monotone. For all $x, y \in \mathbf{A}$, if $\mathbf{F}(x) \prec \mathbf{F}(y)$ then $x < y$.

PROOF:

- $\langle 1 \rangle 1.$ $\mathbf{F}(x) \neq \mathbf{F}(y)$ and $\mathbf{F}(y) \not\prec \mathbf{F}(x)$
PROOF: Trichotomy.
- $\langle 1 \rangle 2.$ $x \neq y$ and $y \not\prec x$
PROOF: \mathbf{F} is strictly monotone.
- $\langle 1 \rangle 3.$ $x < y$
PROOF: Trichotomy.

□

Corollary Schema 6.2.6.1 (Z). *For any classes \mathbf{A}, \mathbf{B} , \leq , \preceq and \mathbf{F} , the following is a theorem:*

Assume \leq is a linear order on \mathbf{A} and \preceq is a linear order on \mathbf{B} . Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and \mathbf{F} is strictly monotone. Then \mathbf{F} is an order isomorphism.

Proposition Schema 6.2.7 (Z). *For any classes $\mathbf{A}, \mathbf{B}, \mathbf{F}$ and \mathbf{S} , the following is a theorem:*

Assume \mathbf{S} is a linear order on \mathbf{B} and $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Define \mathbf{R} on \mathbf{A} by $x\mathbf{R}y$ if and only if $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$. Then \mathbf{R} is a linear order on \mathbf{A} .

PROOF:

- $\langle 1 \rangle 1.$ \mathbf{R} is a partial order on \mathbf{A} .
PROOF: Proposition 6.1.3.

$\langle 1 \rangle 2$. \mathbf{R} is total.

PROOF: For all $x, y \in \mathbf{A}$ we have $\mathbf{F}(x)\mathbf{SF}(y)$ or $\mathbf{F}(y)\mathbf{SF}(x)$.

□

Corollary Schema 6.2.7.1 (Z). *For any classes \mathbf{A} , \mathbf{B} and \mathbf{R} , the following is a theorem:*

Assume \mathbf{R} be a linear order on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Then $\mathbf{R} \cap \mathbf{B}^2$ is a linear order on \mathbf{B} .

Proposition Schema 6.2.8 (Z). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{R} and \mathbf{S} , the following is a theorem:*

Assume \mathbf{R} is a linear order on \mathbf{A} and \mathbf{S} is a linear order on \mathbf{B} . Then the lexicographic ordering is a linear order on $\mathbf{A} \times \mathbf{B}$.

PROOF:

$\langle 1 \rangle 1$. LET: \leq be the lexicographic order on $\mathbf{A} \times \mathbf{B}$

$\langle 1 \rangle 2$. \leq is a partial order.

PROOF: Proposition 6.1.24.

$\langle 1 \rangle 3$. \leq is total.

$\langle 2 \rangle 1$. LET: $a, a' \in \mathbf{A}$ and $b, b' \in \mathbf{B}$

$\langle 2 \rangle 2$. CASE: $a\mathbf{R}a'$ and $a \neq a'$

PROOF: Then $(a, b) \leq (a', b')$.

$\langle 2 \rangle 3$. CASE: $a = a'$

PROOF: We have $b\mathbf{S}b'$ or $b'\mathbf{S}b$, so $(a, b) \leq (a', b')$ or $(a', b') \leq (a, b)$.

$\langle 2 \rangle 4$. CASE: $a'\mathbf{R}a$ and $a \neq a'$

PROOF: Then $(a', b') \leq (a, b)$.

□

6.3 Well Orderings

Definition 6.3.1 (Well Ordering). A *well ordering* on a class \mathbf{A} is a well-founded linear ordering on \mathbf{A} .

Proposition 6.3.2 (Z). *Let S be a well ordering of the set B and $f : A \rightarrow B$ a function. Define R on A by xRy if and only if $F(x)SF(y)$. Then R well orders A .*

PROOF:

$\langle 1 \rangle 1$. R linearly orders A .

PROOF: Proposition 6.2.7.

$\langle 1 \rangle 2$. Every nonempty subset of A has a least element.

$\langle 2 \rangle 1$. LET: C be a nonempty subset of A .

$\langle 2 \rangle 2$. LET: y be the least element of $f(C)$.

$\langle 2 \rangle 3$. PICK $x \in C$ such that $f(x) = y$.

$\langle 2 \rangle 4$. x is least in C .

□

Proposition Schema 6.3.3 (Z). *For any classes \mathbf{A} , \mathbf{B} and \mathbf{R} , the following is a theorem:*

Assume \mathbf{R} well orders \mathbf{B} and $\mathbf{A} \subseteq \mathbf{B}$. Then $\mathbf{R} \cap \mathbf{A}^2$ well orders \mathbf{A} .

PROOF:

$\langle 1 \rangle 1$. LET: $\mathbf{R}' = \mathbf{R} \cap \mathbf{A}^2$

$\langle 1 \rangle 2$. \mathbf{R}' linearly orders \mathbf{A} .

PROOF: Corollary 6.2.7.1.

$\langle 1 \rangle 3$. \mathbf{R}' is well founded.

PROOF: Proposition 2.4.4.

□

Proposition Schema 6.3.4 (ZFC). *For any classes \mathbf{A} , \mathbf{B} , \mathbf{F} and \mathbf{S} , the following is a theorem:*

Assume \mathbf{S} well orders \mathbf{B} and $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Define \mathbf{R} on \mathbf{A} by $x\mathbf{R}y$ if and only if $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$. Then \mathbf{R} well orders \mathbf{A} .

PROOF:

$\langle 1 \rangle 1$. \mathbf{R} linearly orders \mathbf{A} .

PROOF: Proposition 6.2.7.

$\langle 1 \rangle 2$. For all $t \in \mathbf{A}$ we have $\{x \in \mathbf{A} \mid x\mathbf{R}t \wedge x \neq t\}$ is a set.

$\langle 2 \rangle 1$. LET: $t \in \mathbf{A}$

$\langle 2 \rangle 2$. LET: $S = \{y \in \mathbf{B} \mid y\mathbf{S}\mathbf{F}(t) \wedge y \neq \mathbf{F}(t)\}$

$\langle 2 \rangle 3$. LET: $P(x, y)$ be the property $\mathbf{F}(y) = x$

$\langle 2 \rangle 4$. For all $x \in S$ there exists at most one y such that $P(x, y)$

PROOF: \mathbf{F} is injective.

$\langle 2 \rangle 5$. LET: $T = \{y \mid \exists x \in S. P(x, y)\}$

PROOF: Axiom of Replacement.

$\langle 2 \rangle 6$. $T = \{x \in \mathbf{A} \mid x\mathbf{R}t \wedge x \neq t\}$

$\langle 1 \rangle 3$. Every nonempty subset of \mathbf{A} has a least element.

$\langle 2 \rangle 1$. LET: S be a nonempty subset of \mathbf{A} .

$\langle 2 \rangle 2$. $\mathbf{F}(S)$ is a nonempty subset of \mathbf{B}

PROOF: Axiom of Replacement.

$\langle 2 \rangle 3$. LET: y be the least element of $\mathbf{F}(S)$.

$\langle 2 \rangle 4$. PICK $x \in S$ such that $\mathbf{F}(x) = y$.

$\langle 2 \rangle 5$. x is least in S .

□

Proposition 6.3.5 (Z). *For any well ordered sets A and B , the lexicographic order well orders $A \times B$.*

PROOF:

$\langle 1 \rangle 1$. $A \times B$ is linearly ordered.

PROOF: Proposition 6.2.8.

$\langle 1 \rangle 2$. Every nonempty subset of $A \times B$ has a least element.

$\langle 2 \rangle 1$. LET: S be a nonempty subset of $A \times B$.

$\langle 2 \rangle 2$. LET: a be the least element of $\{x \in A \mid \exists y \in B. (x, y) \in S\}$.

$\langle 2 \rangle 3$. LET: b be the least element of $\{y \in B \mid (a, y) \in S\}$.

⟨2⟩4. (a, b) is least in S .
 \square

Definition 6.3.6 (End Extension). Let A and B be well ordered sets. Then B is an *end extension* of A iff $A \subseteq B$ and:

- Whenever $x, y \in A$ then $x \leq_A y$ iff $x \leq_B y$.
- Whenever $x \in A$ and $y \in B - A$ then $x < y$.

Theorem 6.3.7 (Z). Let \leq be a linear ordering on A . Assume that, for any $B \subseteq A$ such that $\forall t \in A. \text{seg } t \subseteq B \Rightarrow t \in B$, we have $B = A$. Then \leq is a well ordering on A .

PROOF:

- ⟨1⟩1. LET: $C \subseteq A$ be nonempty.
 ⟨1⟩2. LET: $B = \{t \in A \mid \forall x \in C. t < x\}$
 ⟨1⟩3. $B \cap C = \emptyset$
 ⟨1⟩4. $B \neq A$
 ⟨1⟩5. PICK $t \in A$ such that $\text{seg } t \subseteq B$ and $t \notin B$
 ⟨1⟩6. t is least in C .
 \square

Proposition Schema 6.3.8 (Z). For any classes $\mathbf{A}, \mathbf{B}, \mathbf{F}, \mathbf{G}, \leq$ and \preceq , the following is a theorem:

Assume \leq well orders \mathbf{A} and \preceq well orders \mathbf{B} . Assume \mathbf{F} and \mathbf{G} are order isomorphisms between \mathbf{A} and \mathbf{B} . Then $\mathbf{F} = \mathbf{G}$.

PROOF:

- ⟨1⟩1. For all $x \in \mathbf{A}$, if $\forall t < x. \mathbf{F}(t) = \mathbf{G}(t)$, then $\mathbf{F}(x) = \mathbf{G}(x)$
 ⟨2⟩1. LET: $x \in \mathbf{A}$
 ⟨2⟩2. ASSUME: $\forall t < x. \mathbf{F}(t) = \mathbf{G}(t)$
 ⟨2⟩3. $\mathbf{F}(\text{seg } x) = \mathbf{G}(\text{seg } x)$
 ⟨2⟩4. $\mathbf{F}(x)$ is the least element of $\mathbf{B} - \mathbf{F}(\text{seg } x)$
 ⟨2⟩5. $\mathbf{G}(x)$ is the least element of $\mathbf{B} - \mathbf{G}(\text{seg } x)$
 ⟨2⟩6. $\mathbf{F}(x) = \mathbf{G}(x)$
 ⟨1⟩2. $\forall x \in \mathbf{A}. \mathbf{F}(x) = \mathbf{G}(x)$

PROOF: Transfinite induction.

\square

Theorem 6.3.9 (ZFC). Let A and B be well ordered sets. Then one of the following holds: $A \cong B$; there exists $b \in B$ such that $A \cong \text{seg } b$; there exists $a \in A$ such that $\text{seg } a \cong B$.

PROOF:

- ⟨1⟩1. PICK e that is not in A or B .
 ⟨1⟩2. LET: $F : A \rightarrow B \cup \{e\}$ be the function defined by transfinite recursion thus:

$$F(t) = \begin{cases} \text{the least element of } B - F(\text{seg } t) & \text{if } B - F(\text{seg } t) \neq \emptyset \\ e & \text{if } B - F(\text{seg } t) = \emptyset \end{cases}$$

$\langle 2 \rangle 1$. LET: t be least such that $F(t) = e$

$\langle 2 \rangle 1$. LET: t be least such that $F(t) = e$

$$\langle 2 \rangle 2. \quad F \upharpoonright \text{seg } t : \text{seg } t \cong B$$

PROOF: We have $F : A \cong B$

PROOF: We have $F : A \cong B$

$\langle 2 \rangle 1$. LET: b be the least element of $B - \text{ran } F$

$\langle 2 \rangle 1$. LET: b be the least element of $B - \text{ran } F$

$\langle 2 \rangle 2$. $F : A \cong \text{seg } b$

1

Chapter 7

Ordinal Numbers

7.1 Ordinals

Definition 7.1.1 (Ordinal). A set α is an *ordinal (number)* iff:

- α is a transitive set
- $\{(x, y) \in \alpha^2 \mid x \in y\}$ is a strict well ordering of α .

Given ordinals α, β , we write $\alpha < \beta$ iff $\alpha \in \beta$, and $\alpha \leq \beta$ iff $\alpha \in \beta$ or $\alpha = \beta$.

Let **On** be the class of ordinals.

Proposition 7.1.2 (S). *Let α be a set. Then α is an ordinal if and only if α is a transitive set and, for all $x, y \in \alpha$, we have $x \in y$ or $x = y$ or $y \in x$.*

PROOF:

$\langle 1 \rangle 1$. LET: α be a set.

$\langle 1 \rangle 2$. If α is an ordinal then α is a transitive set and, for all $x, y \in \alpha$, we have $x \in y$ or $x = y$ or $y \in x$.

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$. If α is a transitive set and $\forall x, y \in \alpha (x \in y \vee x = y \vee y \in x)$, then α is an ordinal.

$\langle 2 \rangle 1$. ASSUME: α is a transitive set.

$\langle 2 \rangle 2$. ASSUME: $\forall x, y \in \alpha (x \in y \vee x = y \vee y \in x)$

$\langle 2 \rangle 3$. LET: $E = \{(x, y) \in \alpha^2 \mid x \in y\}$

$\langle 2 \rangle 4$. $\forall x, y, z \in \alpha. xEy \wedge yEz \Rightarrow xEz$

$\langle 3 \rangle 1$. LET: $x, y, z \in \alpha$

$\langle 3 \rangle 2$. ASSUME: xEy and yEz

$\langle 3 \rangle 3$. $x \in z$ or $x = z$ or $z \in x$

$\langle 3 \rangle 4$. $x \neq z$

PROOF: We cannot have $x \in y \in x$ by the Axiom of Foundation.

$\langle 3 \rangle 5$. $z \notin x$

PROOF: We cannot have $x \in y \in z \in x$ by the Axiom of Foundation.

- $\langle 3 \rangle 6. x \in z$
- $\langle 2 \rangle 5. \text{ There is no } x \in \alpha \text{ such that } xEx.$
PROOF: Axiom of Foundation.
- $\langle 2 \rangle 6. \text{ Every nonempty subset of } \alpha \text{ has an } E\text{-least element.}$
PROOF: Axiom of Foundation.

□

Proposition 7.1.3 (S without Foundation). ***On** is a transitive class. That is, if β is an ordinal and $\alpha \in \beta$ then α is an ordinal.*

PROOF:

- $\langle 1 \rangle 1. \text{ LET: } \beta \text{ be an ordinal.}$
- $\langle 1 \rangle 2. \text{ LET: } \alpha \in \beta$
- $\langle 1 \rangle 3. \alpha \text{ is a transitive set.}$
 - $\langle 2 \rangle 1. \text{ LET: } x \in y \in \alpha$
 - $\langle 2 \rangle 2. y \in \beta$
PROOF: β is a transitive set.
 - $\langle 2 \rangle 3. x \in \beta$
PROOF: β is a transitive set.
 - $\langle 2 \rangle 4. x \in \alpha \text{ or } x = \alpha \text{ or } \alpha \in x$
 - $\langle 2 \rangle 5. x \neq \alpha$
PROOF: We cannot have $x \in y \in x$ since β is well-ordered by \in .
 - $\langle 2 \rangle 6. \alpha \notin x$
PROOF: We cannot have $x \in y \in \alpha \in x$ since β is well-ordered by \in .
 - $\langle 2 \rangle 7. x \in \alpha$
- $\langle 1 \rangle 4. \alpha \text{ is well-ordered by } \in$
PROOF: This holds because $\alpha \subseteq \beta$ since β is a transitive set.

□

Proposition 7.1.4 (S without Foundation). *For ordinals α and β , we have $\alpha \subseteq \beta$ if and only if $\alpha \leq \beta$.*

PROOF:

- $\langle 1 \rangle 1. \text{ If } \alpha \leq \beta \text{ then } \alpha \subseteq \beta.$
PROOF: Since β is a transitive set.
- $\langle 1 \rangle 2. \text{ If } \alpha \subseteq \beta \text{ then } \alpha \leq \beta.$
 - $\langle 2 \rangle 1. \text{ ASSUME: } \alpha \subsetneq \beta$
PROVE: $\alpha \in \beta$
 - $\langle 2 \rangle 2. \text{ LET: } u \text{ be the } \in\text{-least element of } \beta - \alpha$
 - $\langle 2 \rangle 3. u \in \beta$
 - $\langle 2 \rangle 4. u \notin \alpha$
 - $\langle 2 \rangle 5. \text{ For all } v \in \beta, \text{ if } v \notin \alpha \text{ then } v \notin u$
PROOF: From $\langle 2 \rangle 2$
 - $\langle 2 \rangle 6. \text{ For all } v \in u \text{ we have } v \in \alpha$
 - $\langle 2 \rangle 7. \text{ For all } w \in \alpha \text{ we have } w \in u$
 - $\langle 3 \rangle 1. \text{ LET: } w \in \alpha$
 - $\langle 3 \rangle 2. w \in \beta$

PROOF: $\langle 2 \rangle 1$

$\langle 3 \rangle 3$. $w \in u$ or $w = u$ or $u \in w$

PROOF: β is an ordinal, $\langle 2 \rangle 3$, $\langle 3 \rangle 2$

$\langle 3 \rangle 4$. $w \neq u$

PROOF: $\langle 2 \rangle 4$, $\langle 3 \rangle 1$

$\langle 3 \rangle 5$. $u \notin w$

PROOF: $\langle 2 \rangle 4$, $\langle 3 \rangle 1$, α is a transitive set.

$\langle 3 \rangle 6$. $w \in u$

PROOF: $\langle 3 \rangle 3$, $\langle 3 \rangle 4$, $\langle 3 \rangle 5$

$\langle 2 \rangle 8$. $u = \alpha$

PROOF: $\langle 2 \rangle 6$, $\langle 2 \rangle 7$, Axiom of Extensionality.

$\langle 2 \rangle 9$. $\alpha \in \beta$

PROOF: $\langle 2 \rangle 3$, $\langle 2 \rangle 8$

□

Proposition 7.1.5 (S). *The intersection of any nonempty set of ordinals is an ordinal.*

PROOF:

$\langle 1 \rangle 1$. LET: x be a nonempty set of ordinals.

$\langle 1 \rangle 2$. $\bigcap x$ is a transitive set.

PROOF: Proposition 1.6.6.

$\langle 1 \rangle 3$. $\forall y, z \in \bigcap x. y \in z \vee y = z \vee z \in y$

$\langle 2 \rangle 1$. PICK $\alpha \in x$

$\langle 2 \rangle 2$. $y, z \in \alpha$

$\langle 2 \rangle 3$. $y \in z$ or $y = z$ or $z \in y$

□

Theorem 7.1.6 (S). *For any ordinals α and β , we have $\alpha < \beta$ or $\alpha = \beta$ or $\beta < \alpha$.*

PROOF:

$\langle 1 \rangle 1$. LET: α and β be ordinals.

$\langle 1 \rangle 2$. CASE: $\alpha \cap \beta = \alpha$ or $\alpha \cap \beta = \beta$

$\langle 2 \rangle 1$. ASSUME: for a contradiction $\alpha \cap \beta \subsetneq \alpha$ and $\alpha \cap \beta \subsetneq \beta$

$\langle 2 \rangle 2$. $\alpha \cap \beta \in \alpha$ and $\alpha \cap \beta \in \beta$

PROOF: Propositions 7.1.4, 7.1.5.

$\langle 2 \rangle 3$. $\alpha \cap \beta \in \alpha \cap \beta$

$\langle 2 \rangle 4$. Q.E.D.

PROOF: This contradicts the Axiom of Foundation.

$\langle 1 \rangle 3$. $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

$\langle 1 \rangle 4$. $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$

PROOF: Proposition 7.1.4.

□

Corollary 7.1.6.1 (Burali-Forti Paradox (S)). *The class \mathbf{On} is a proper class.*

PROOF: If \mathbf{On} is a set then it is an ordinal by Propositions 7.1.3 and 7.3.8, hence $\mathbf{On} \in \mathbf{On}$ contradicting the Axiom of Foundation. □

Theorem 7.1.7 (S). *The class **On** is well ordered by \leq .*

PROOF:

$\langle 1 \rangle 1$. For every ordinal α we have $\alpha \leq \alpha$.

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. For any ordinals α , β and γ , if $\alpha \leq \beta \leq \gamma$ then $\alpha \leq \gamma$.

PROOF: Since γ is a transitive set.

$\langle 1 \rangle 3$. For any ordinals α and β , if $\alpha \leq \beta \leq \alpha$ then $\alpha = \beta$.

PROOF: If $\alpha \neq \beta$ then $\alpha \in \beta \in \alpha$ contradicting the Axiom of Foundation.

$\langle 1 \rangle 4$. For any ordinals α and β , either $\alpha \leq \beta$ or $\beta \leq \alpha$.

PROOF: Theorem 7.1.6.

$\langle 1 \rangle 5$. Any set of ordinals x has a least element.

PROOF: Axiom of Foundation.

□

Proposition 7.1.8 (S). *For any nonempty set of ordinals x , we have $\bigcap x$ is the infimum of x in **On**.*

PROOF: It is an ordinal by Proposition 7.1.5. The fact it is the infimum is immediate from definitions. □

Proposition 7.1.9 (S). *For any nonempty set of ordinals x , we have $\bigcup x$ is the supremum of x in **On**.*

PROOF:

$\langle 1 \rangle 1$. $\bigcup x$ is an ordinal.

$\langle 2 \rangle 1$. $\bigcup x$ is a transitive set.

PROOF: Proposition 1.6.3.

$\langle 2 \rangle 2$. For all $y, z \in \bigcup x$ we have $y \in z$ or $y = z$ or $z \in y$

$\langle 3 \rangle 1$. PICK $\alpha, \beta \in x$ such that $y \in \alpha$ and $z \in \beta$.

$\langle 3 \rangle 2$. ASSUME: w.l.o.g. $\alpha \leq \beta$.

$\langle 3 \rangle 3$. $y, z \in \beta$

$\langle 3 \rangle 4$. $y \in z$ or $y = z$ or $z \in y$.

□

Proposition 7.1.10 (S). *For any set α , we have α is an ordinal if and only if α is a transitive set of transitive sets.*

PROOF:

$\langle 1 \rangle 1$. If α is an ordinal then α is a transitive set of transitive sets.

PROOF: Proposition 7.1.3.

$\langle 1 \rangle 2$. If α is a transitive set of transitive sets then α is an ordinal.

$\langle 2 \rangle 1$. ASSUME: α is a transitive set of transitive sets.

$\langle 2 \rangle 2$. ASSUME: for a contradiction there exist $x, y \in \alpha$ such that $x \notin y$ and $y \notin x$

$\langle 2 \rangle 3$. $S = \{x \in \alpha \mid \exists y \in \alpha. x \notin y \wedge x \neq y \wedge y \notin x\}$ is nonempty

$\langle 2 \rangle 4$. PICK $x \in S$ such that $x \cap S = \emptyset$

PROOF: Axiom of Foundation.

$\langle 2 \rangle 5$. $T = \{y \in \alpha \mid x \notin y \wedge x \neq y \wedge y \notin x\}$ is nonempty.

$\langle 2 \rangle 6$. PICK $y \in T$ such that $y \cap T = \emptyset$

PROOF: Axiom of Foundation.

$\langle 2 \rangle 7$. $x \subseteq y$

$\langle 3 \rangle 1$. LET: $z \in x$

$\langle 3 \rangle 2$. $z \notin S$

PROOF: $\langle 2 \rangle 4$

$\langle 3 \rangle 3$. $z \in y$ or $z = y$ or $y \in z$

$\langle 3 \rangle 4$. $z \neq y$

PROOF: Since $y \notin x$ ($\langle 2 \rangle 6$).

$\langle 3 \rangle 5$. $y \notin z$

PROOF: Since x is transitive and $y \notin x$.

$\langle 3 \rangle 6$. $z \in y$

$\langle 2 \rangle 8$. $y \subseteq x$

$\langle 3 \rangle 1$. LET: $z \in y$

$\langle 3 \rangle 2$. $z \notin T$

PROOF: $\langle 2 \rangle 6$

$\langle 3 \rangle 3$. $z \in x$ or $z = x$ or $x \in z$

$\langle 3 \rangle 4$. $z \neq x$

PROOF: Since $x \notin y$

$\langle 3 \rangle 5$. $x \notin z$

PROOF: Since y is a transitive set and $x \notin y$.

$\langle 3 \rangle 6$. $z \in x$

$\langle 2 \rangle 9$. $x = y$

PROOF: Axiom of Extensionality.

$\langle 2 \rangle 10$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 6$.

□

Proposition 7.1.11 (S). \emptyset is an ordinal.

PROOF: Vacuous. □

Definition 7.1.12.

$$0 := \emptyset$$

Definition 7.1.13 (Successor). For any ordinal α , the *successor* of α is

$$\alpha^+ := \alpha \cup \{\alpha\} .$$

Proposition 7.1.14 (S). For any ordinal α , the successor α^+ is an ordinal.

PROOF:

$\langle 1 \rangle 1$. LET: α be an ordinal.

$\langle 1 \rangle 2$. α^+ is a transitive set.

$\langle 2 \rangle 1$. LET: $x \in y \in \alpha^+$

$\langle 2 \rangle 2$. $y \in \alpha$ or $y = \alpha$

$\langle 2 \rangle 3$. $x \in \alpha$

PROOF: If $y \in \alpha$ then this follows because α is a transitive set. If $y = \alpha$ this follows immediately.

$\langle 2 \rangle 4$. $x \in \alpha^+$

$\langle 1 \rangle 3$. For any $x, y \in \alpha^+$, we have $x \in y$ or $x = y$ or $y \in X$.

$\langle 2 \rangle 1$. LET: $x, y \in \alpha^+$

$\langle 2 \rangle 2$. CASE: $x \in \alpha$ and $y \in \alpha$

PROOF: Then $x \in y$ or $x = y$ or $y \in x$ because α is an ordinal.

$\langle 2 \rangle 3$. CASE: $x \in \alpha$ and $y = \alpha$

PROOF: Then $x \in y$.

$\langle 2 \rangle 4$. CASE: $x = \alpha$ and $y \in \alpha$

PROOF: Then $y \in x$.

$\langle 2 \rangle 5$. CASE: $x = \alpha$ and $y = \alpha$

PROOF: Then $x = y$.

□

Proposition 7.1.15 (S). *There are no ordinals α and β such that $\alpha < \beta < \alpha^+$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $\alpha \in \beta \in \alpha^+$

$\langle 1 \rangle 2$. $\beta \in \alpha$ or $\beta = \alpha$.

$\langle 1 \rangle 3$. CASE: $\beta \in \alpha$

PROOF: We have $\alpha \in \beta \in \alpha$ contradicting the Axiom of Foundation.

$\langle 1 \rangle 4$. CASE: $\beta = \alpha$

PROOF: We have $\alpha \in \alpha$ contradicting the Axiom of Foundation.

□

Definition 7.1.16 (Successor Ordinal). An ordinal α is a *successor ordinal* iff there exists an ordinal β such that $\alpha = \beta^+$.

Definition 7.1.17 (Limit Ordinal). An ordinal is a *limit ordinal* iff it is neither 0 nor a successor ordinal.

Proposition 7.1.18 (S). *The relation $\{(\alpha, \beta) \in \mathbf{On}^2 \mid \alpha < \beta\}$ is well founded.*

PROOF:

$\langle 1 \rangle 1$. Any nonempty set has an \in -minimal element.

PROOF: Axiom of Foundation.

$\langle 1 \rangle 2$. For any set x , there exists a set u such that $x \in u$ and for all ordinals α, β , if $\beta \in u$ and $\alpha < \beta$ then $\alpha \in u$.

$\langle 2 \rangle 1$. LET: x be a set.

$\langle 2 \rangle 2$. LET: $u = \{x\}$ if x is not an ordinal, x^+ if x is an ordinal.

$\langle 2 \rangle 3$. $x \in u$

$\langle 2 \rangle 4$. LET: α, β be ordinals.

$\langle 2 \rangle 5$. ASSUME: $\beta \in u$

$\langle 2 \rangle 6$. ASSUME: $\alpha < \beta$

$\langle 2 \rangle 7$. x is an ordinal

$\langle 2 \rangle 8$. $\alpha < \beta \leq x$

$\langle 2 \rangle 9$. $\alpha \in x^+ = u$

□

7.2 Natural Numbers

Definition 7.2.1 (Natural Number). An ordinal n is a *natural number* iff either it is 0, or it is a successor ordinal and every element of n is either 0 or a successor ordinal.

We write ω or \mathbb{N} for the class of natural numbers.

Theorem Schema 7.2.2 (Mathematical Induction (S)). *For any predicate $P(x)$, the following is a theorem:*

Assume $P(0)$. Assume that, for any ordinal α , if $P(\alpha)$ then $P(\alpha^+)$. Then, for every natural number n , we have $P(n)$.

PROOF:

$\langle 1 \rangle 1$. ASSUME: n is a natural number such that $\neg P(n)$

$\langle 1 \rangle 2$. LET: $y = \{\gamma \in n^+ \mid \neg P(\gamma)\}$

$\langle 1 \rangle 3$. $y \neq \emptyset$

$\langle 1 \rangle 4$. PICK $\gamma_0 \in y$ such that $y \cap \gamma_0 = \emptyset$

$\langle 1 \rangle 5$. CASE: $\gamma_0 = 0$

PROOF: Then $\neg P(0)$

$\langle 1 \rangle 6$. CASE: $\gamma_0 \neq 0$

$\langle 2 \rangle 1$. PICK γ_1 such that $\gamma_0 = \gamma_1^+$

PROOF: Either $\gamma_0 = n$ or $\gamma_0 \in n$, and in either case γ_0 is a successor because n is a natural number.

$\langle 2 \rangle 2$. $\gamma_1 \notin y$

PROOF: From $\langle 1 \rangle 4$ since $\gamma_1 \in \gamma_0$.

$\langle 2 \rangle 3$. $\gamma_1 \in n^+$

PROOF: Since $\gamma_1 \in \gamma_0 \in n^+$ and n^+ is a transitive set.

$\langle 2 \rangle 4$. $P(\gamma_1)$ and $\neg P(\gamma_1^+)$

PROOF: $\langle 1 \rangle 2$, $\langle 2 \rangle 2$, $\langle 2 \rangle 3$

□

Theorem 7.2.3 (Z). *The class of natural numbers ω is a set.*

PROOF:

$\langle 1 \rangle 1$. PICK a set I such that $\emptyset \in I$ and $\forall x \in I. x \cup \{x\} \in I$.

PROOF: Axiom of Infinity.

$\langle 1 \rangle 2$. Every natural number is in I .

PROOF: By induction.

$\langle 1 \rangle 3$. Q.E.D.

PROOF: By an Axiom of Comprehension.

□

Theorem 7.2.4 (Z). *ω is a limit ordinal.*

PROOF:

$\langle 1 \rangle 1$. ω is a transitive set.

$\langle 2 \rangle 1$. LET: $m \in n \in \omega$

PROVE: m is a natural number.

- ⟨2⟩2. ASSUME: $m \neq 0$
- ⟨2⟩3. m is a successor ordinal.
PROOF: Since $m \in n$ and n is a natural number.
- ⟨2⟩4. Every element of m is either 0 or a successor ordinal.
 - ⟨3⟩1. LET: $x \in m$
 - ⟨3⟩2. $x \in n$
PROOF: Since n is a transitive set.
 - ⟨3⟩3. $x = 0$ or x is a successor ordinal.
- ⟨1⟩2. For all $m, n \in \omega$, either $m \in n$ or $m = n$ or $n \in m$.
PROOF: Theorem 7.1.6.
- ⟨1⟩3. $\omega \neq 0$
PROOF: Since $0 \in \omega$.
- ⟨1⟩4. ω is not a successor ordinal.
 - ⟨2⟩1. ASSUME: for a contradiction $\omega = n^+$
 - ⟨2⟩2. n is a natural number.
 - ⟨2⟩3. ω is a natural number.
 - ⟨2⟩4. $\omega \in \omega$
 - ⟨2⟩5. Q.E.D.

PROOF: This contradicts the Axiom of Foundation.

□

7.3 Transitive Closure of a Set

Definition 7.3.1 (Transitive Closure). Let x be a set. The *transitive closure* of x is the class

$$\text{TC}(x) = \bigcap \{y \mid y \text{ is a transitive set} \wedge x \in y\} .$$

Theorem 7.3.2 (ZF). For any set x , the transitive closure $\text{TC}(x)$ is a set.

PROOF:

- ⟨1⟩1. LET: $E(n, y, x)$ be the predicate: There exists a function $f : n^+ \rightarrow \mathbf{V}$ such that $f(0) = y$ and $f(n) = x$ and $\forall m \in n. f(m) \in f(m^+)$
- ⟨1⟩2. $\forall x, y. E(0, y, x) \Leftrightarrow y = x$
- ⟨1⟩3. LET: x be a set.
- ⟨1⟩4. LET: $\mathbf{W} = \{y \mid \exists n \in \mathbb{N}. E(n, y, x)\}$
- ⟨1⟩5. \mathbf{W} is a set.
 - ⟨2⟩1. For $n \in \mathbb{N}$,
LET: $\mathbf{W}_n = \{y \mid E(n, y, x)\}$
 - ⟨2⟩2. For all $n \in \mathbb{N}$, \mathbf{W}_n is a set.
 - ⟨3⟩1. \mathbf{W}_0 is a set.
PROOF: $\mathbf{W}_0 = \{x\}$.
 - ⟨3⟩2. For all $n \in \mathbb{N}$, if \mathbf{W}_n is a set then \mathbf{W}_{n^+} is a set.
 - ⟨4⟩1. $\mathbf{W}_{n^+} = \bigcup \mathbf{W}_n$
 - ⟨5⟩1. $\mathbf{W}_{n^+} \subseteq \bigcup \mathbf{W}_n$
 - ⟨6⟩1. LET: $y \in \mathbf{W}_{n^+}$

$\langle 4 \rangle 1$. LET: $n \in \mathbb{N}$

- ⟨4⟩2. ASSUME: $\forall y. E(n, y, x) \Rightarrow y \in w'$
- ⟨4⟩3. LET: y be a set.
- ⟨4⟩4. ASSUME: $E(n^+, y, x)$
- ⟨4⟩5. PICK $f : n^{++} \rightarrow \mathbf{V}$ such that $f(0) = y$ and $f(n^+) = x$ and $\forall m \in n^+. f(m) \in f(m^+)$
- ⟨4⟩6. LET: $g : n^+ \rightarrow \mathbf{V}$ be the function $g(m) = f(m^+)$
- ⟨4⟩7. $g(0) = f(0^+)$ and $g(n) = x$ and $\forall m \in n. g(m) \in g(m^+)$
- ⟨4⟩8. $E(n, f(0^+), x)$
- ⟨4⟩9. $f(0^+) \in w'$
- PROOF: ⟨4⟩5
- ⟨4⟩10. $y \in f(0^+)$
- PROOF: $y = f(0) \in f(0^+)$ by ⟨4⟩5.
- ⟨4⟩11. $y \in w'$
- PROOF: From ⟨3⟩2 and ⟨4⟩10 since w' is transitive (⟨2⟩1).
- ⟨2⟩3. $\mathbf{W} \subseteq w'$

□

Proposition 7.3.3. *The transitive closure of an ordinal α is α^+ .*

PROOF:

- ⟨1⟩1. α^+ is a transitive set that contains α .
- ⟨1⟩2. For any transitive set x , if $\alpha \in x$ then $\alpha^+ \subseteq x$.

□

Proposition Schema 7.3.4 (ZF). *For any class \mathbf{A} , the following is a theorem:
The relation $\{(x, y) \in \mathbf{A}^2 \mid x \in y\}$ is well founded.*

PROOF:

- ⟨1⟩1. Every nonempty set has an \in -minimal element.
PROOF: This is the Axiom of Foundation.
- ⟨1⟩2. For any set x , there exists u such that $x \subseteq u$ and for all w, y , if $y \in u$ and $w, y \in \mathbf{A}$ and $w \in y$ then $w \in u$
PROOF: Take $u = \text{TC}(x)$.

□

Proposition 7.3.5 (Z). *For any ordinal numbers α and β , if $\alpha \cong \beta$ then $\alpha = \beta$.*

PROOF:

- ⟨1⟩1. LET: $f : \alpha \cong \beta$
- ⟨1⟩2. For all $x \in \alpha$, if $\forall t < x. f(t) = t$ then $f(x) = x$
 - ⟨2⟩1. $f(x) \subseteq x$
 - ⟨3⟩1. LET: $y \in f(x)$
 - ⟨3⟩2. $y \in \beta$
 - ⟨3⟩3. PICK $t \in \alpha$ such that $f(t) = y$
PROOF: f is surjective.
 - ⟨3⟩4. $f(t) \in f(x)$
 - ⟨3⟩5. $t \in x$
PROOF: Since f is an order isomorphism.

⟨3⟩6. $f(t) = t$

PROOF: Induction hypothesis.

⟨3⟩7. $y = t$

⟨3⟩8. $y \in x$

⟨2⟩2. $x \subseteq f(x)$

⟨3⟩1. LET: $t \in x$

⟨3⟩2. $f(t) \in f(x)$

⟨3⟩3. $f(t) = t$

⟨3⟩4. $t \in f(x)$

⟨1⟩3. $\forall x \in \alpha. f(x) = x$

PROOF: Transfinite induction.

⟨1⟩4. $\alpha = \beta$

PROOF: Since $\beta = \{f(t) \mid t \in \alpha\} = \{t \mid t \in \alpha\} = \alpha$.

□

Theorem 7.3.6 (ZFC). *Every well-ordered set is isomorphic to a unique ordinal.*

PROOF:

⟨1⟩1. For any well-ordered set A , there exists an ordinal α such that $A \cong \alpha$.

⟨2⟩1. LET: A be a well-ordered set.

⟨2⟩2. Define the function E on A by transfinite recursion thus:

$$E(t) = \{E(x) \mid x < t\} \quad (t \in A) .$$

⟨2⟩3. LET: $\alpha = \{E(x) \mid x \in A\}$

⟨2⟩4. α is an ordinal.

⟨3⟩1. α is a transitive set.

⟨4⟩1. LET: $x \in y \in \alpha$

⟨4⟩2. PICK $t \in A$ such that $y = E(t)$

⟨4⟩3. $x \in E(t) = \{E(s) \mid s < t\}$

⟨4⟩4. PICK $s < t$ such that $x = E(s)$

⟨4⟩5. $x \in \alpha$

⟨3⟩2. α is well-ordered by \in .

⟨4⟩1. LET: $< = \{(x, y) \in \alpha \mid x \in y\}$

⟨4⟩2. $<$ is transitive.

⟨5⟩1. LET: $x, y, z \in \alpha$ with $x \in y \in z$

⟨5⟩2. PICK $t \in A$ such that $z = E(t)$

⟨5⟩3. PICK $s \in A$ such that $s < t$ and $y = E(s)$

⟨5⟩4. PICK $r \in A$ such that $r < s$ and $x = E(r)$

⟨5⟩5. $r < t$

⟨5⟩6. $x \in z$

⟨4⟩3. $<$ satisfies trichotomy.

⟨5⟩1. LET: $x, y \in \alpha$

⟨5⟩2. PICK $s, t \in A$ such that $E(s) = x$ and $E(t) = y$

⟨5⟩3. Exactly one of $s < t$, $s = t$, $t < s$ holds.

⟨5⟩4. CASE: $s < t$

⟨6⟩1. $x \in y$

⟨6⟩2. $x \neq y$ and $y \notin x$

PROOF: Axiom of Regularity.

$\langle 5 \rangle 5$. CASE: $s = t$

$\langle 6 \rangle 1$. $x = y$

$\langle 6 \rangle 2$. $x \notin y$ and $y \notin x$

PROOF: Axiom of Regularity.

$\langle 5 \rangle 6$. CASE: $t < s$

PROOF: Similar to $\langle 5 \rangle 4$.

$\langle 4 \rangle 4$. \leq is a linear order on α .

PROOF: Proposition 6.2.3.

$\langle 4 \rangle 5$. Every nonempty subset of α has a least element.

$\langle 5 \rangle 1$. LET: S be a nonempty subset of α

$\langle 5 \rangle 2$. LET: $T = \{x \in A \mid E(x) \in S\}$

$\langle 5 \rangle 3$. LET: t be the least element of T .

PROVE: $E(t)$ is least in S

$\langle 5 \rangle 4$. LET: $y \in S$

$\langle 5 \rangle 5$. PICK $s \in T$ such that $E(s) = y$

$\langle 5 \rangle 6$. $t \leq s$

$\langle 5 \rangle 7$. $x \leq y$

$\langle 2 \rangle 5$. E is surjective.

PROOF: By definition of α .

$\langle 2 \rangle 6$. E is strictly monotone.

PROOF: If $s < t$ then $E(s) \in E(t)$ by definition of $E(t)$.

$\langle 2 \rangle 7$. Q.E.D.

PROOF: Corollary 6.2.6.1.

$\langle 1 \rangle 2$. For any ordinals α and β , if $\alpha \cong \beta$ then $\alpha = \beta$.

PROOF: Proposition 7.3.5.

□

Proposition 7.3.7 (Z). *The class \mathbf{On} is a transitive class. That is, every element of an ordinal is an ordinal.*

PROOF:

$\langle 1 \rangle 1$. LET: α be an ordinal.

$\langle 1 \rangle 2$. LET: $\beta \in \alpha$

$\langle 1 \rangle 3$. β is a transitive set.

$\langle 2 \rangle 1$. LET: $x \in y \in \beta$

$\langle 2 \rangle 2$. $y \in \alpha$

PROOF: α is transitive.

$\langle 2 \rangle 3$. $x \in \alpha$

PROOF: α is transitive.

$\langle 2 \rangle 4$. $x \in \beta$

PROOF: Since $\{(x, y) \in \alpha^2 \mid x \in y\}$ is transitive.

$\langle 1 \rangle 4$. β is well ordered by \in .

PROOF: By Proposition 6.3.3.

□

Proposition 7.3.8 (ZFC). *Given two ordinal numbers α, β , exactly one of $\alpha \in \beta$, $\alpha = \beta$, $\beta \in \alpha$ holds.*

PROOF:

$\langle 1 \rangle 1$. At most one holds.

PROOF: Since every ordinal is a transitive set and we never have $\alpha \in \alpha$.

$\langle 1 \rangle 2$. At least one holds.

$\langle 2 \rangle 1$. Either $\alpha \cong \beta$ or $\exists t \in \beta. \alpha \cong \text{seg } t$ or $\exists t \in \alpha. \text{seg } t \cong \beta$.

$\langle 2 \rangle 2$. CASE: $\alpha \cong \beta$

PROOF: Then $\alpha = \beta$ by Proposition 7.3.5.

$\langle 2 \rangle 3$. CASE: There exists $t \in \beta$ such that $\alpha \cong \text{seg } t$

$\langle 3 \rangle 1$. t is an ordinal number.

PROOF: Proposition 7.3.7.

$\langle 3 \rangle 2$. $t = \text{seg } t$

$\langle 4 \rangle 1$. $t \subseteq \text{seg } t$

$\langle 5 \rangle 1$. LET: $s \in t$

$\langle 5 \rangle 2$. $s \in \beta$

PROOF: β is a transitive set.

$\langle 5 \rangle 3$. $s \in \text{seg } t$

$\langle 4 \rangle 2$. $\text{seg } t \subseteq t$

PROOF: Immediate from definitions.

$\langle 3 \rangle 3$. $\alpha = t$

PROOF: Proposition 7.3.5.

$\langle 3 \rangle 4$. $\alpha \in \beta$

$\langle 2 \rangle 4$. CASE: There exists $t \in \alpha$ such that $\text{seg } t \cong \beta$

PROOF: $\beta \in \alpha$ similarly.

□

Proposition 7.3.9 (Z). *Any nonempty set S of ordinal numbers has a least element.*

PROOF:

$\langle 1 \rangle 1$. PICK $\beta \in S$

$\langle 1 \rangle 2$. CASE: $\beta \cap S = \emptyset$

PROOF: Then β is least in S .

$\langle 1 \rangle 3$. CASE: $\beta \cap S \neq \emptyset$

PROOF: The least element of $\beta \cap S$ is least in S .

□

Theorem 7.3.10 (ZFC). *The class **On** is well ordered by \in .*

PROOF:

$\langle 1 \rangle 1$. LET: $\mathbf{E} = \{(x, y) \in \mathbf{On}^2 \mid x \in y\}$

$\langle 1 \rangle 2$. \mathbf{E} is transitive.

PROOF: If $\alpha \in \beta \in \gamma$ then $\alpha \in \gamma$ because every ordinal is a transitive set.

$\langle 1 \rangle 3$. \mathbf{E} satisfies trichotomy.

PROOF: Proposition 7.3.8.

$\langle 1 \rangle 4$. \mathbf{E} linearly orders **On**.

PROOF: Proposition 6.2.3.

$\langle 1 \rangle 5$. \mathbf{E} is well founded.

PROOF: Proposition 9.6.2.

□

Corollary 7.3.10.1 (Burali-Forti Paradox (ZFC)). *The class \mathbf{On} is a proper class.*

PROOF: If it were a set, it would be a transitive set well-ordered by \in , and hence a member of itself, contradicting Proposition 1.5.3.

Proposition 7.3.11 (ZFC). *Any transitive set of ordinal numbers is an ordinal number.*

PROOF: It is well-ordered by \in by Proposition 6.3.3 and Theorem 7.3.10. □

Proposition 7.3.12 (Z). *\emptyset is an ordinal number.*

PROOF: Vacuously, it is a transitive set well-ordered by \in .

Definition 7.3.13. We define $0 = \emptyset$.

Proposition 7.3.14 (ZFC). *If A is a set of ordinal numbers then $\bigcup A$ is an ordinal number.*

PROOF:

$\langle 1 \rangle 1.$ $\bigcup A$ is a transitive set.

PROOF: Proposition 1.6.3.

$\langle 1 \rangle 2.$ $\bigcup A$ is a set of ordinals.

PROOF: Proposition 7.3.7.

$\langle 1 \rangle 3.$ Q.E.D.

PROOF: Proposition 7.3.11.

□

Corollary 7.3.14.1 (ZFC). *The poset \mathbf{On} is complete.*

PROOF: For any nonempty set A of ordinals, $\bigcup A$ is its supremum. □

Proposition 7.3.15 (ZFC). *Let α be an ordinal and $S \subseteq \alpha$. Then S is well-ordered by \in and the ordinal of (S, \in) is $\leq \alpha$.*

PROOF:

$\langle 1 \rangle 1.$ S is well ordered by \in .

$\langle 1 \rangle 2.$ LET: β be the ordinal of (S, \in)

$\langle 1 \rangle 3.$ LET: $E : S \approx \beta$ be the unique isomorphism.

$\langle 1 \rangle 4.$ $\forall \gamma \in S. E(\gamma) \leq \gamma$

$\langle 2 \rangle 1.$ LET: $\gamma \in S$

$\langle 2 \rangle 2.$ ASSUME: as transfinite induction hypothesis $\forall \delta < \gamma. E(\delta) \leq \delta$

$\langle 2 \rangle 3.$ $E(\gamma)$ is the least element of β that is greater than $E(\delta)$ for all $\delta < \gamma$

$\langle 2 \rangle 4.$ γ is greater than $E(\delta)$ for all $\delta < \gamma$

$\langle 2 \rangle 5.$ $E(\gamma) \leq \gamma$

$\langle 1 \rangle 5.$ $\beta \leq \alpha$

- ⟨2⟩1. $\forall \gamma < \beta. \gamma < \alpha$
- ⟨3⟩1. LET: $\gamma < \beta$
- ⟨3⟩2. PICK $\delta \in S$ such that $E(\delta) = \gamma$
- ⟨3⟩3. $\gamma = E(\delta) \leq \delta < \alpha$

□

Proposition 7.3.16 (ZFC). *Let α be a set. Then the following are equivalent.*

- 1. α is an ordinal.
- 2. α is a transitive set and, for all $x, y \in \alpha$, either $x = y$ or $x \in y$ or $y \in x$.
- 3. α is a transitive set of transitive sets.

PROOF:

- ⟨1⟩1. $1 \Rightarrow 2$

PROOF: Immediate from definitions.

- ⟨1⟩2. $2 \Rightarrow 3$

- ⟨2⟩1. ASSUME: α is a transitive set and, for all $x, y \in \alpha$, either $x = y$ or $x \in y$ or $y \in x$

- ⟨2⟩2. LET: $z \in \alpha$

PROVE: z is transitive.

- ⟨2⟩3. LET: $x \in y \in z$

- ⟨2⟩4. $y \in \alpha$

- ⟨2⟩5. $x \in \alpha$

- ⟨2⟩6. Either $x = z$ or $x \in z$ or $z \in x$

- ⟨2⟩7. $x \neq z$

PROOF: We cannot have $x \in y \in x$ by the Axiom of Regularity.

- ⟨2⟩8. $z \notin x$

PROOF: We cannot have $x \in y \in z \in x$ by the Axiom of Regularity.

- ⟨1⟩3. $3 \Rightarrow 1$

- ⟨2⟩1. LET: x be a transitive set of transitive sets.

- ⟨2⟩2. ASSUME: as \in -induction hypothesis that, for all $y \in x$, if y is a transitive set of transitive sets then y is a transitive set of ordinals.

- ⟨2⟩3. Every element of x is an ordinal.

- ⟨3⟩1. LET: $y \in x$

- ⟨3⟩2. y is transitive.

- ⟨3⟩3. Every element of y is transitive.

PROOF: Since every element of y is an element of x , because x is transitive.

- ⟨3⟩4. y is an ordinal.

PROOF: ⟨2⟩2

- ⟨2⟩4. Q.E.D.

PROOF: Proposition 7.3.11.

□

Lemma 7.3.17 (Z). *Let A and B be well-ordered sets. If B is an end extension of A then the ordinal of A is \leq the ordinal of B .*

PROOF:

- $\langle 1 \rangle 1.$ LET: α be the ordinal of A and β the ordinal of B .
- $\langle 1 \rangle 2.$ LET: $E_A : A \cong \alpha$ and $E_B : B \cong \beta$ be the canonical isomorphisms.
- $\langle 1 \rangle 3.$ $\forall a \in A. E_A(a) = E_B(a)$
 - $\langle 2 \rangle 1.$ LET: $a \in A$
 - $\langle 2 \rangle 2.$ ASSUME: as transfinite induction hypothesis $\forall x < a. E_A(x) = E_B(x)$
 - $\langle 2 \rangle 3.$ $E_A(a)$ is the least ordinal that is greater than $E_A(x)$ for all $x < a$
 - $\langle 2 \rangle 4.$ $E_B(a)$ is the least ordinal that is greater than $E_B(x)$ for all $x < b$
 - $\langle 2 \rangle 5.$ $\{x \in A \mid x <_A a\} = \{x \in B \mid x <_B a\}$
 - $\langle 2 \rangle 6.$ $E_A(a) = E_B(a)$
- $\langle 1 \rangle 4.$ $\alpha \subseteq \beta$
- $\langle 1 \rangle 5.$ $\alpha \leq \beta$

□

Lemma 7.3.18. *Let \mathcal{C} be a set of well ordered sets such that, for any $A, B \in \mathcal{C}$, we have that one of A and B is an end extension of the other. Let $W = \bigcup \mathcal{C}$ under $x \leq y$ iff there exists $A \in W$ such that $x, y \in A$ and $x \leq y$. Then W is a well ordered set whose ordinal is the supremum of the ordinals of the members of \mathcal{C} .*

PROOF:

- $\langle 1 \rangle 1.$ \leq is reflexive on W .
 - $\langle 2 \rangle 1.$ LET: $x \in W$
 - $\langle 2 \rangle 2.$ PICK $A \in W$ such that $x \in A$.
 - $\langle 2 \rangle 3.$ $x \leq x$
 - $\langle 1 \rangle 2.$ \leq is antisymmetric on W .
 - $\langle 2 \rangle 1.$ LET: $x, y \in W$
 - $\langle 2 \rangle 2.$ ASSUME: $x \leq y$ and $y \leq x$
 - $\langle 2 \rangle 3.$ PICK $A \in W$ such that $x, y \in A$ and $x \leq_A y$, and $B \in W$ such that $x, y \in B$ and $y \leq_B x$
 - $\langle 2 \rangle 4.$ ASSUME: w.l.o.g. B is an end extension of A
 - $\langle 2 \rangle 5.$ $x \leq_B y$ and $y \leq_B x$
 - $\langle 2 \rangle 6.$ $x = y$
 - $\langle 1 \rangle 3.$ \leq is transitive on W .
 - $\langle 2 \rangle 1.$ ASSUME: $x \leq y \leq z$
 - $\langle 2 \rangle 2.$ PICK $A, B \in W$ such that $x \leq_A y$ and $y \leq_B z$
 - $\langle 2 \rangle 3.$ CASE: A is an end extension of B .
 - $\langle 3 \rangle 1.$ $x \leq_A y$ and $y \leq_A z$
 - $\langle 3 \rangle 2.$ $x \leq_A z$
 - $\langle 3 \rangle 3.$ $x \leq z$
 - $\langle 2 \rangle 4.$ CASE: B is an end extension of A .
- PROOF: Similar.
- $\langle 1 \rangle 4.$ \leq is total on W .
 - $\langle 2 \rangle 1.$ LET: $x, y \in W$
 - $\langle 2 \rangle 2.$ PICK $A, B \in \mathcal{C}$ such that $x \in A$ and $y \in B$
 - $\langle 2 \rangle 3.$ ASSUME: w.l.o.g. B is an end extension of A
 - $\langle 2 \rangle 4.$ $x \leq_B y$ or $y \leq_B x$

- (2)5. $x \leq_W y$ or $y \leq_W x$
- (1)5. Every nonempty subset of W has a least element.
 - (2)1. LET: S be a nonempty subset of W
 - (2)2. PICK $s \in S$
 - (2)3. PICK $A \in \mathcal{C}$ such that $s \in A$
 - (2)4. LET: a be the \leq_A -least element of $S \cap A$
PROVE: a is least in S
 - (2)5. LET: $x \in S$
PROVE: $a \leq x$
 - (2)6. PICK $B \in \mathcal{C}$ such that $x \in B$
 - (2)7. CASE: A is an end extension of B
 - (3)1. $a \leq_A x$
 - (3)2. $a \leq x$
 - (2)8. CASE: B is an end extension of A
 - (3)1. CASE: $x \in A$
 - (4)1. $a \leq_A x$
 - (4)2. $a \leq x$
 - (3)2. CASE: $x \in B - A$
 - (4)1. $a \leq_B x$
 - (4)2. $a \leq x$
- (1)6. For all $A \in \mathcal{C}$, W is an end extension of A .
 - (2)1. For all $x, y \in A$, we have $x \leq_A y$ if and only if $x \leq_W y$
 - (3)1. LET: $x, y \in A$
 - (3)2. If $x \leq_A y$ then $x \leq_W y$
PROOF: Immediate from definitions.
 - (3)3. If $x \leq_W y$ then $x \leq_A y$
 - (4)1. ASSUME: $x \not\leq_A y$
 - (4)2. PICK $B \in \mathcal{C}$ such that $x \leq_B y$
 - (4)3. CASE: A is an end extension of B
PROOF: Then $x \leq_A y$.
 - (4)4. CASE: B is an end extension of A
PROOF: Then $x \leq_A y$.
 - (2)2. For all $x \in A$ and $y \in W - A$ we have $x < y$
 - (3)1. LET: $x \in A$ and $y \in W - A$
 - (3)2. PICK $B \in \mathcal{C}$ such that $y \in B$
 - (3)3. B is an end extension of A
 - (3)4. $x <_B y$
 - (3)5. $x <_W y$
- (1)7. For all $A \in \mathcal{C}$, the ordinal of A is \leq the ordinal of W .
PROOF: Lemma 7.3.17.
- (1)8. For any ordinal α , if for all $A \in \mathcal{C}$ the ordinal of A is $\leq \alpha$, then the ordinal of W is $\leq \alpha$.
 - (2)1. LET: α be an ordinal.
 - (2)2. ASSUME: for all $A \in \mathcal{C}$, the ordinal of A is $\leq \alpha$
 - (2)3. LET: β be the ordinal of W
 - (2)4. LET: $E : W \approx \beta$ be the canonical isomorphism.

- ⟨2⟩5. ASSUME: for a contradiction $\alpha < \beta$
- ⟨2⟩6. LET: $a \in W$ be the element with $E(a) = \alpha$
- ⟨2⟩7. PICK $A \in \mathcal{C}$ such that $a \in A$
- ⟨2⟩8. LET: γ be the ordinal of A and $E_A : A \cong \gamma$ be the canonical isomorphism.
- ⟨2⟩9. For all $x \in A$ we have $E_A(x) = E(x)$
PROOF: Transfinite induction on x .
- ⟨2⟩10. $E_A(a) = \alpha$
- ⟨2⟩11. $\alpha < \gamma$
- ⟨2⟩12. Q.E.D.
PROOF: This contradicts ⟨2⟩2.

□

7.4 Successors

Definition 7.4.1 (Successor). The *successor* of a set a is the set $a^+ := a \cup \{a\}$.

Proposition 7.4.2 (Z). *A set a is a transitive set if and only if*

$$\bigcup(a^+) = a \text{ .}$$

PROOF:

- ⟨1⟩1. If a is a transitive set then $\bigcup(a^+) = a$.
 - ⟨2⟩1. ASSUME: a is a transitive set.
 - ⟨2⟩2. $\bigcup(a^+) \subseteq a$
 - ⟨3⟩1. LET: $x \in \bigcup(a^+)$
PROVE: $x \in a$
 - ⟨3⟩2. PICK $y \in a^+$ such that $x \in y$.
 - ⟨3⟩3. $y \in a$ or $y = a$.
 - ⟨3⟩4. CASE: $y \in a$
PROOF: Then $x \in a$ because a is a transitive set.
 - ⟨3⟩5. CASE: $y = a$
PROOF: Then $x \in a$ immediately.
 - ⟨2⟩3. $a \subseteq \bigcup(a^+)$
PROOF: Since $a \in a^+$.
- ⟨1⟩2. If $\bigcup(a^+) = a$ then a is a transitive set.
 - ⟨2⟩1. ASSUME: $\bigcup(a^+) = a$
 - ⟨2⟩2. $\bigcup a \subseteq a$
PROOF:

$$\begin{aligned} \bigcup a &\subseteq \bigcup(a^+) && \text{(Proposition 1.5.9)} \\ &= a && (\langle 2 \rangle 1) \end{aligned}$$

- ⟨2⟩3. a is a transitive set.
PROOF: Proposition 1.6.2.

□

Proposition 7.4.3. *For any set a , we have a is a transitive set if and only if a^+ is a transitive set.*

PROOF:

$\langle 1 \rangle 1$. If a is a transitive set then a^+ is a transitive set.

PROOF: If a is a transitive set then $\bigcup(a^+) = a \subseteq a^+$ by Proposition 7.4.2 and so a^+ is a transitive set.

$\langle 1 \rangle 2$. If a^+ is a transitive set then a is a transitive set.

$\langle 2 \rangle 1$. ASSUME: a^+ is a transitive set.

$\langle 2 \rangle 2$. LET: $x \in y \in a$

$\langle 2 \rangle 3$. $x \in y \in a^+$

$\langle 2 \rangle 4$. $x \in a^+$

PROOF: $\langle 2 \rangle 1$

$\langle 2 \rangle 5$. $x \neq a$

PROOF: From $\langle 2 \rangle 2$ and the Axiom of Regularity.

$\langle 2 \rangle 6$. $x \in a$

□

Definition 7.4.4. We write 0 for \emptyset , 1 for \emptyset^+ , 2 for \emptyset^{++} , etc.

Proposition 7.4.5. *For any set A we have $\mathcal{P}A \approx 2^A$.*

PROOF: The function $H : \mathcal{P}A \rightarrow 2^A$ defined by $H(S)(a) = \{\emptyset\}$ if $a \in S$ and \emptyset if $a \notin S$ is a bijection. □

Proposition 7.4.6. *For any ordinal number α we have α^+ is an ordinal number.*

PROOF:

$\langle 1 \rangle 1$. α^+ is a transitive set.

PROOF: Proposition 7.4.3.

$\langle 1 \rangle 2$. α^+ is well-ordered by \in .

$\langle 2 \rangle 1$. For all $x, y, z \in \alpha^+$, if $x \in y \in z$ then $x \in z$

$\langle 3 \rangle 1$. CASE: $z = \alpha$

PROOF: Then $x \in \alpha$ since α is a transitive set.

$\langle 3 \rangle 2$. CASE: $z \in \alpha$

PROOF: Then $x \in z$ since α is well-ordered by \in .

$\langle 2 \rangle 2$. For all $x, y \in \alpha^+$ we have $x \in y$ or $x = y$ or $y \in x$

$\langle 3 \rangle 1$. CASE: $x, y \in \alpha$

PROOF: The result follows because α is well-ordered by \in .

$\langle 3 \rangle 2$. CASE: $x \in \alpha, y = \alpha$

PROOF: Then $x \in y$.

$\langle 3 \rangle 3$. CASE: $x = \alpha, y \in \alpha$

PROOF: Then $y \in x$.

$\langle 3 \rangle 4$. CASE: $x = \alpha, y = \alpha$

PROOF: Then $x = y$.

$\langle 2 \rangle 3$. Every nonempty subset of α^+ has an \in -least element.

$\langle 3 \rangle 1$. LET: $S \subseteq \alpha^+$ be nonempty

- ⟨3⟩2. CASE: $S = \{\alpha\}$
 PROOF: α is least in S .
- ⟨3⟩3. CASE: $S \neq \{\alpha\}$
 - ⟨4⟩1. $S - \{\alpha\}$ is a nonempty subset of α
 - ⟨4⟩2. LET: β be least in $S - \{\alpha\}$
 - ⟨4⟩3. β is least in S .

□

Proposition 7.4.7. *For ordinals α and β , if $\alpha^+ = \beta^+$ then $\alpha = \beta$.*

PROOF: If $\alpha^+ = \beta^+$ then

$$\begin{aligned}\alpha &= \bigcup(\alpha^+) && \text{(Proposition 7.4.2)} \\ &= \bigcup(\beta^+) \\ &= \beta && \text{(Proposition 7.4.2)}\end{aligned}$$

Proposition 7.4.8. *For ordinals α and β , we have $\alpha < \beta$ if and only if $\alpha^+ < \beta^+$.*

PROOF:

$$\begin{aligned}\alpha < \beta &\Leftrightarrow \alpha^+ \leq \beta \\ &\Leftrightarrow \alpha^+ < \beta^+ && \square\end{aligned}$$

Definition 7.4.9 (Successor Ordinal). An ordinal α is a *successor ordinal* iff $\alpha = \beta^+$ for some β .

Definition 7.4.10 (Limit Ordinal). A *limit ordinal* is an ordinal that is neither 0 nor a successor ordinal.

Proposition 7.4.11. *If λ is a limit ordinal and $\beta < \lambda$ then $\beta^+ < \lambda$.*

PROOF: Since $\beta^+ \leq \lambda$ and $\beta^+ \neq \lambda$. □

7.5 The Well-Ordering Theorem and Zorn's Lemma

Theorem 7.5.1 (Hartogs). *For any set A , there exists an ordinal not dominated by A .*

PROOF:

- ⟨1⟩1. LET: α be the class of all ordinals β such that $\beta \preccurlyeq A$
 PROVE: α is a set.
- ⟨1⟩2. LET: $W = \{(B, R) \mid B \subseteq A, R \text{ is a well ordering on } B\}$
- ⟨1⟩3. α is the class of the ordinals of the elements of W .
 - ⟨2⟩1. For all $(B, R) \in W$, the ordinal of (B, R) is in α .
 - ⟨3⟩1. LET: $(B, R) \in W$
 - ⟨3⟩2. LET: β be the ordinal of (B, R)
 - ⟨3⟩3. LET: $E : B \cong \beta$ be the canonical isomorphism.
 - ⟨3⟩4. LET: $i : B \hookrightarrow A$ be the inclusion

- ⟨3⟩5. $i \circ E^{-1}$ is an injection $\beta \rightarrow A$
- ⟨3⟩6. $\beta \in \alpha$
- ⟨2⟩2. For all $\beta \in \alpha$, there exists $(B, R) \in W$ such that β is the ordinal number of (B, R) .
- ⟨3⟩1. LET: $\beta \in \alpha$
- ⟨3⟩2. PICK an injection $f : \beta \rightarrow A$
- ⟨3⟩3. Define \leq on $\text{ran } f$ by $f(x) \leq f(y)$ iff $x \leq y$
- ⟨3⟩4. $(\text{ran } f, \leq) \in W$
- ⟨3⟩5. β is the ordinal number of $(\text{ran } f, \leq)$
- ⟨1⟩4. α is a set.
PROOF: By an Axiom of Replacement.
- ⟨1⟩5. α is an ordinal.
PROOF: It is a transitive set of ordinals.
- ⟨1⟩6. $\alpha \not\approx A$
PROOF: Since $\alpha \notin \alpha$.

□

Theorem 7.5.2 (Numeration Theorem). *Every set is equinumerous with some ordinal.*

PROOF:

- ⟨1⟩1. LET: A be any set.
- ⟨1⟩2. PICK an ordinal α not dominated by A .
- ⟨1⟩3. PICK a choice function G for A .
- ⟨1⟩4. PICK $e \notin A$
- ⟨1⟩5. LET: $F : \alpha \rightarrow A \cup \{e\}$ by transfinite recursion:

$$F(\gamma) = \begin{cases} G(A - F(\{\delta \mid \delta < \gamma\})) & \text{if } A - F(\{\delta \mid \delta < \gamma\}) \neq \emptyset \\ e & \text{if } A - F(\{\delta \mid \delta < \gamma\}) = \emptyset \end{cases}$$
- ⟨1⟩6. $e \in \text{ran } F$
 - ⟨2⟩1. ASSUME: for a contradiction $e \notin \text{ran } F$
 - ⟨2⟩2. F is an injection $\alpha \rightarrow A$.
 - ⟨3⟩1. LET: $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$
PROVE: $F(\beta) \neq F(\gamma)$
 - ⟨3⟩2. ASSUME: w.l.o.g. $\beta < \gamma$
 - ⟨3⟩3. $F(\gamma) \in A - F(\{\delta \mid \delta < \gamma\})$
 - ⟨3⟩4. $F(\gamma) \notin F(\{\delta \mid \delta < \gamma\})$
 - ⟨3⟩5. $F(\gamma) \neq F(\beta)$
 - ⟨2⟩3. Q.E.D.
PROOF: This contradicts ⟨1⟩2.
- ⟨1⟩7. LET: δ be least such that $F(\delta) = e$
- ⟨1⟩8. $F \upharpoonright \delta : \delta \approx A$

Theorem 7.5.3 (Well-Ordering Theorem). *Any set can be well ordered.*

PROOF:

- ⟨1⟩1. PICK an ordinal δ and a bijection $F : A \approx \delta$
- ⟨1⟩2. Define \leq on A by $F(x) \leq F(y)$ iff $x \leq y$ for $x, y \in \delta$

$\langle 1 \rangle 3$. \leq is a well ordering on A .

□

Theorem 7.5.4 (Zorn's Lemma). *Let \mathcal{A} be a set such that, for every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$. Then \mathcal{A} has a maximal element.*

PROOF:

$\langle 1 \rangle 1$. PICK a well ordering $<$ on \mathcal{A} .

$\langle 1 \rangle 2$. LET: $F : \mathcal{A} \rightarrow 2$ be the function defined by transfinite recursion by:

$$F(A) = \begin{cases} 1 & \text{if } A \text{ includes every set } B < A \text{ for which } F(B) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\langle 1 \rangle 3$. LET: $\mathcal{C} = \{A \in \mathcal{A} \mid F(A) = 1\}$

PROVE: $\bigcup \mathcal{C}$ is a maximal element of \mathcal{A}

$\langle 1 \rangle 4$. For all $A \in \mathcal{A}$, we have $A \in \mathcal{C}$ iff $\forall B < A. B \in \mathcal{C} \Rightarrow B \subseteq A$

$\langle 1 \rangle 5$. \mathcal{C} is a chain.

$\langle 2 \rangle 1$. LET: $A, A' \in \mathcal{C}$

$\langle 2 \rangle 2$. ASSUME: w.l.o.g. $A \leq A'$

$\langle 2 \rangle 3$. $A \subseteq A'$

PROOF: By $\langle 1 \rangle 4$

$\langle 1 \rangle 6$. $\bigcup \mathcal{C} \in \mathcal{A}$

$\langle 1 \rangle 7$. $\bigcup \mathcal{C}$ is maximal in \mathcal{A} .

$\langle 2 \rangle 1$. LET: $A \in \mathcal{A}$ and $\bigcup \mathcal{C} \subseteq A$

$\langle 2 \rangle 2$. $A \in \mathcal{C}$

PROOF: By $\langle 1 \rangle 4$ since $\forall B \in \mathcal{C}. B \subseteq A$.

$\langle 2 \rangle 3$. $A \subseteq \bigcup \mathcal{C}$

$\langle 2 \rangle 4$. $A = \bigcup \mathcal{C}$

□

Proposition 7.5.5 (Teichmüller-Tukey Lemma). *Let \mathcal{A} be a nonempty set such that, for every B , we have $B \in \mathcal{A}$ if and only if every finite subset of B is a member of \mathcal{A} . Then \mathcal{A} has a maximal element.*

PROOF:

$\langle 1 \rangle 1$. For every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 2 \rangle 1$. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain.

$\langle 2 \rangle 2$. Every finite subset of $\bigcup \mathcal{B}$ is a member of \mathcal{A} .

$\langle 3 \rangle 1$. LET: C be a finite subset of $\bigcup \mathcal{B}$.

$\langle 3 \rangle 2$. PICK $B \in \mathcal{B}$ such that $C \subseteq B$.

$\langle 3 \rangle 3$. $B \in \mathcal{A}$

$\langle 3 \rangle 4$. Every finite subset of B is in \mathcal{A} .

$\langle 3 \rangle 5$. $C \in \mathcal{A}$

$\langle 2 \rangle 3$. $\bigcup \mathcal{B} \in \mathcal{A}$.

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Zorn's lemma.

□

Theorem Schema 7.5.6. *For any class \mathbf{A} , there exists a class \mathbf{F} such that the following is a theorem:*

If \mathbf{A} is a proper class of ordinals, then $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{A}$ is an order isomorphism.

PROOF:

$\langle 1 \rangle 1$. Define $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{A}$ by transfinite recursion as follows: $\mathbf{F}(\alpha)$ is the least element of \mathbf{A} that is different from $\mathbf{F}(\beta)$ for all $\beta < \alpha$.

$\langle 1 \rangle 2$. For all $\alpha, \beta \in \mathbf{On}$, if $\alpha < \beta$ then $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$

PROOF: We have $\mathbf{F}(\alpha) \neq \mathbf{F}(\beta)$ by the definition of $\mathbf{F}(\beta)$, and $\mathbf{F}(\beta) \not< \mathbf{F}(\alpha)$ by the leastness of $\mathbf{F}(\alpha)$.

$\langle 1 \rangle 3$. \mathbf{F} is surjective.

$\langle 2 \rangle 1$. LET: $\alpha \in \mathbf{A}$

$\langle 2 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall \beta \in \mathbf{A}$, if $\beta < \alpha$ then there exists γ such that $\beta = \mathbf{F}(\gamma)$.

$\langle 2 \rangle 3$. LET: $\gamma = \{\delta \in \mathbf{On} \mid \mathbf{F}(\delta) < \alpha\}$

$\langle 2 \rangle 4$. γ is a set.

PROOF: Axiom of Replacement applied to α .

$\langle 2 \rangle 5$. γ is a transitive set.

PROOF: If $\mathbf{F}(\delta) < \alpha$ and $\epsilon < \delta$ then $\mathbf{F}(\epsilon) < \alpha$ by $\langle 1 \rangle 2$.

$\langle 2 \rangle 6$. γ is an ordinal.

PROOF: Proposition 7.3.11.

$\langle 2 \rangle 7$. $\mathbf{F}(\gamma) = \alpha$

$\langle 3 \rangle 1$. $\mathbf{F}(\gamma)$ is the least element of \mathbf{A} different from $\mathbf{F}(\delta)$ for all $\delta < \gamma$

$\langle 3 \rangle 2$. $\mathbf{F}(\gamma)$ is the least element of \mathbf{A} different from x for all $x \in \mathbf{A}$ with $x < \alpha$

$\langle 3 \rangle 3$. $\mathbf{F}(\gamma) = \alpha$

□

7.6 Ordinal Operations

Definition 7.6.1 (Ordinal Operation). An *ordinal operation* is a function $\mathbf{On} \rightarrow \mathbf{On}$.

Definition 7.6.2 (Continuous). An ordinal operation $\mathbf{T} : \mathbf{On} \rightarrow \mathbf{On}$ is *continuous* iff, for every limit ordinal λ , we have $\mathbf{T}(\lambda) = \bigcup_{\alpha < \lambda} \mathbf{T}(\alpha)$.

Definition 7.6.3 (Normal). An ordinal operation is *normal* iff it is continuous and strictly monotone.

Proposition Schema 7.6.4. For any class \mathbf{T} , the following is a theorem.

If \mathbf{T} is a continuous ordinal operation and $\forall \gamma. \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$, then \mathbf{T} is normal.

PROOF:

$\langle 1 \rangle 1$. LET: $P[\beta]$ be the property $\forall \gamma < \beta. \mathbf{T}(\gamma) < \mathbf{T}(\beta)$

$\langle 1 \rangle 2$. $P[0]$

PROOF: Vacuous.

$\langle 1 \rangle 3$. For any ordinal γ , if $P[\gamma]$ then $P[\gamma^+]$

$\langle 2 \rangle 1$. ASSUME: $P[\gamma]$

- ⟨2⟩2. LET: $\delta < \gamma^+$
- ⟨2⟩3. CASE: $\delta < \gamma$
PROOF: Then $\mathbf{T}(\delta) < \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$.
- ⟨2⟩4. CASE: $\delta = \gamma$
PROOF: Then $\mathbf{T}(\delta) = \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$.
- ⟨1⟩4. For any limit ordinal λ , if $\forall \gamma < \lambda. P[\gamma]$ then $P[\lambda]$.
 - ⟨2⟩1. ASSUME: $\forall \gamma < \lambda. P[\gamma]$
 - ⟨2⟩2. LET: $\delta < \lambda$
 - ⟨2⟩3. $\mathbf{T}(\delta) < \mathbf{T}(\lambda)$
 - PROOF:

$$\begin{aligned}
 \mathbf{T}(\delta) &< \mathbf{T}(\delta^+) \\
 &\leq \bigcup_{\epsilon < \lambda} \mathbf{T}(\epsilon) \\
 &= \mathbf{T}(\lambda)
 \end{aligned}$$

□

Proposition Schema 7.6.5. *For any class \mathbf{T} , the following is a theorem:*

Assume \mathbf{T} is a normal ordinal operation. For every ordinal α , we have $\alpha \leq \mathbf{T}(\alpha)$.

PROOF:

- ⟨1⟩1. LET: γ be an ordinal.
- ⟨1⟩2. ASSUME: as induction hypothesis $\forall \delta < \gamma. \mathbf{T}(\delta) \geq \delta$
- ⟨1⟩3. For all $\delta < \gamma$ we have $\delta < \mathbf{T}(\gamma)$
PROOF: \mathbf{T} is strictly monotone.
- ⟨1⟩4. $\gamma \leq \mathbf{T}(\gamma)$

□

Proposition Schema 7.6.6. *For any class \mathbf{T} , the following is a theorem:*

Assume \mathbf{T} is a normal ordinal operation. For any ordinal $\beta \geq \mathbf{T}(0)$, there exists a greatest ordinal γ such that $\mathbf{T}(\gamma) \leq \beta$.

PROOF:

- ⟨1⟩1. There exists γ such that $\mathbf{T}(\gamma) > \beta$
 - ⟨2⟩1. For all γ we have $\mathbf{T}(\gamma) \geq \gamma$
PROOF: Proposition 7.6.5.
 - ⟨2⟩2. $\mathbf{T}(\beta^+) > \beta$
- ⟨1⟩2. LET: δ be least such that $\mathbf{T}(\delta) > \beta$
- ⟨1⟩3. δ is a successor ordinal.
 - ⟨2⟩1. $\delta \neq 0$
PROOF: Since $\mathbf{T}(0) \leq \beta$.
 - ⟨2⟩2. δ is not a limit ordinal.
 - ⟨3⟩1. ASSUME: for a contradiction δ is a limit ordinal.
 - ⟨3⟩2. $\beta < \bigcup_{\epsilon < \delta} \mathbf{T}(\epsilon)$
PROOF: \mathbf{T} is continuous.
 - ⟨3⟩3. There exists $\epsilon < \delta$ such that $\beta < \mathbf{T}(\epsilon)$

⟨3⟩4. Q.E.D.

PROOF: This contradicts the minimality of δ .

⟨1⟩4. LET: $\delta = \gamma^+$

⟨1⟩5. γ is greatest such that $\mathbf{T}(\gamma) \leq \beta$

□

Theorem Schema 7.6.7. *For any class \mathbf{T} , the following is a theorem:*

Assume that \mathbf{T} is a normal ordinal operation. For any nonempty set of ordinals S , we have

$$\mathbf{T}(\sup S) = \sup_{\alpha \in S} \mathbf{T}(\alpha) .$$

PROOF:

⟨1⟩1. $\forall \alpha \in S. \mathbf{T}(\alpha) \leq \mathbf{T}(\sup S)$

PROOF: Since \mathbf{T} is monotone.

⟨1⟩2. For any ordinal β , if $\forall \alpha \in S. \mathbf{T}(\alpha) \leq \beta$, then $\mathbf{T}(\sup S) \leq \beta$

⟨2⟩1. LET: β be an ordinal.

⟨2⟩2. LET: $\gamma = \sup S$

⟨2⟩3. ASSUME: $\forall \alpha \in S. \mathbf{T}(\alpha) \leq \beta$

⟨2⟩4. CASE: γ is 0 or a successor ordinal

PROOF: Then we must have $\gamma \in S$ so $\mathbf{T}(\gamma) \leq \beta$ from ⟨2⟩3.

⟨2⟩5. CASE: γ is a limit ordinal

⟨3⟩1. $\mathbf{T}(\gamma) = \sup_{\alpha < \gamma} \mathbf{T}(\alpha)$

PROOF: \mathbf{T} is continuous.

⟨3⟩2. ASSUME: for a contradiction $\beta < \mathbf{T}(\gamma)$

⟨3⟩3. PICK $\alpha < \gamma$ such that $\beta < \mathbf{T}(\alpha)$

PROOF: ⟨3⟩1, ⟨3⟩2

⟨3⟩4. PICK $\alpha' \in S$ such that $\alpha < \alpha'$

PROOF: ⟨2⟩2, ⟨3⟩3

⟨3⟩5. $\beta < \mathbf{T}(\alpha') \leq \beta$

PROOF: \mathbf{T} is strictly monotone, ⟨3⟩3, ⟨3⟩4, ⟨2⟩3.

⟨3⟩6. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 7.6.8. *For any classes \mathbf{A} and \mathbf{T} , the following is a theorem:*

Assume \mathbf{A} is a proper class of ordinals such that, for every set $S \subseteq \mathbf{A}$, we have $\bigcup S \in \mathbf{A}$. Assume \mathbf{T} is the unique order isomorphism $\mathbf{On} \cong \mathbf{A}$. Then \mathbf{T} is normal.

PROOF:

⟨1⟩1. \mathbf{T} is strictly monotone.

PROOF: Since it is an order isomorphism.

⟨1⟩2. \mathbf{T} is continuous.

⟨2⟩1. LET: λ be a limit ordinal.

⟨2⟩2. $\mathbf{T}'(\lambda)$ is the least member of \mathbf{A} that is greater than $\mathbf{T}'(\alpha)$ for all $\alpha < \lambda$

⟨2⟩3. $\mathbf{T}'(\lambda) = \sup_{\alpha < \lambda} \mathbf{T}'(\alpha)$

□

Proposition Schema 7.6.9. *For any class \mathbf{T} , the following is a theorem:*

If \mathbf{T} is a normal ordinal operation, then for any limit ordinal λ , we have $\mathbf{T}(\lambda)$ is a limit ordinal.

PROOF:

$\langle 1 \rangle 1.$ $\mathbf{T}(\lambda) \neq 0$

PROOF: Since $0 \leq \mathbf{T}(0) < \mathbf{T}(\lambda)$.

$\langle 1 \rangle 2.$ $\mathbf{T}(\lambda)$ is not a successor ordinal.

$\langle 2 \rangle 1.$ ASSUME: for a contradiction $\mathbf{T}(\lambda) = \alpha^+$

$\langle 2 \rangle 2.$ $\alpha < \mathbf{T}(\lambda) = \sup_{\beta < \lambda} \mathbf{T}(\beta)$

$\langle 2 \rangle 3.$ PICK $\beta < \lambda$ such that $\alpha < \mathbf{T}(\beta)$

$\langle 2 \rangle 4.$ $\alpha^+ \leq \mathbf{T}(\beta) < \mathbf{T}(\lambda)$

$\langle 2 \rangle 5.$ Q.E.D.

PROOF: This is a contradiction.

□

7.7 Ordinal Arithmetic

7.7.1 Addition

Definition 7.7.1. Let A and B be disjoint well-ordered sets. The *concatenation* of A and B is the set $A \cup B$ under the relation:

- if $a, a' \in A$ then $a \leq a'$ iff $a \leq a'$ in A
- if $b, b' \in B$ then $b \leq b'$ iff $b \leq b'$ in B
- if $a \in A$ and $b \in B$ then $a \leq b$ and $b \not\leq a$.

Proposition 7.7.2. *If A and B are disjoint well-ordered sets, then their concatenation is well-ordered.*

PROOF:

$\langle 1 \rangle 1.$ \leq is reflexive.

PROOF: For all $a \in A$ we have $a \leq a$, and for all $b \in B$ we have $b \leq b$.

$\langle 1 \rangle 2.$ \leq is antisymmetric.

$\langle 2 \rangle 1.$ ASSUME: $x \leq y \leq x$

$\langle 2 \rangle 2.$ CASE: $x, y \in A$

PROOF: Then $x = y$ since the order on A is antisymmetric.

$\langle 2 \rangle 3.$ CASE: $x \in A$ and $y \in B$

PROOF: This is impossible as it would imply $y \not\leq x$.

$\langle 2 \rangle 4.$ CASE: $x \in B$ and $y \in A$

PROOF: This is impossible as it would imply $x \not\leq y$.

$\langle 2 \rangle 5.$ CASE: $x, y \in B$

PROOF: Then $x = y$ since the order on B is antisymmetric.

$\langle 1 \rangle 3.$ \leq is transitive.

$\langle 2 \rangle 1.$ ASSUME: $x \leq y \leq z$

$\langle 2 \rangle 2.$ CASE: $x, z \in A$

PROOF: In this case $y \in A$ since $y \leq z$, and so $x \leq z$ since the order on A is transitive.

$\langle 2 \rangle 3$. CASE: $x \in A$ and $z \in B$

PROOF: Then $x \leq z$ immediately.

$\langle 2 \rangle 4$. CASE: $x \in B$ and $z \in A$

PROOF: This is impossible because we have $y \notin A$ since $x \leq y$ and $y \notin B$ since $y \leq z$.

$\langle 2 \rangle 5$. CASE: $x, z \in B$

PROOF: In this case $y \in B$ since $x \leq y$, and so $x \leq z$ since the order on B is transitive.

$\langle 1 \rangle 4$. \leq is total.

$\langle 2 \rangle 1$. LET: $x, y \in A \cup B$

$\langle 2 \rangle 2$. CASE: $x, y \in A$

PROOF: Then $x \leq y$ or $y \leq x$ because the order on A is total.

$\langle 2 \rangle 3$. CASE: $x \in A$ and $y \in B$

PROOF: Then $x \leq y$.

$\langle 2 \rangle 4$. CASE: $x \in B$ and $y \in A$

PROOF: Then $y \leq x$.

$\langle 2 \rangle 5$. CASE: $x, y \in B$

PROOF: Then $x \leq y$ or $y \leq x$ because the order on B is total.

$\langle 1 \rangle 5$. Every nonempty subset of $A \cup B$ has a least element.

$\langle 2 \rangle 1$. LET: S be a nonempty subset of $A \cup B$

$\langle 2 \rangle 2$. CASE: $S \cap A = \emptyset$

PROOF: Then $S \subseteq B$ and so S has a least element.

$\langle 2 \rangle 3$. CASE: $S \cap A \neq \emptyset$

PROOF: The least element of $S \cap A$ is the least element of S .

□

Definition 7.7.3 (Ordinal Addition). Let α and β be ordinal numbers. Then $\alpha + \beta$ is the ordinal number of the concatenation of A and B , where A is any well ordered set with ordinal α and B is any well ordered set with ordinal β .

Theorem 7.7.4 (Associative Law for Addition). For any ordinals ρ , σ and τ , we have

$$\rho + (\sigma + \tau) = (\rho + \sigma) + \tau .$$

PROOF: Given disjoint well ordered sets A , B and C , the concatenation of A with (the concatenation of B and C) is the same as the concatenation of (the concatenation of A and B) and C . □

Theorem 7.7.5. For any ordinal ρ we have

$$\rho + 0 = 0 + \rho = \rho .$$

PROOF: For any well ordered set A , the concatenation of A with \emptyset is A , and the concatenation of \emptyset with A is A . □

Theorem 7.7.6. For any ordinal α we have $\alpha + 1 = \alpha^+$.

PROOF: Since α^+ is the concatenation of α and $\{\alpha\}$. \square

Theorem 7.7.7. *For any ordinal α , the operation that maps β to $\alpha + \beta$ is normal.*

PROOF:

$\langle 1 \rangle 1$. For any limit ordinal λ , we have $\alpha + \lambda = \sup_{\beta < \lambda} (\alpha + \beta)$.

$\langle 2 \rangle 1$. LET: λ be a limit ordinal.

$\langle 2 \rangle 2$. $(\{0\} \times \alpha) \cup (\{1\} \times \lambda) = \bigcup_{\beta \in \lambda} ((\{0\} \times \alpha) \cup (\{1\} \times \beta))$, where the order on the right hand side is as in Lemma 7.3.18.

PROOF:

$$\begin{aligned} (\{0\} \times \alpha) \cup (\{1\} \times \lambda) &= (\{0\} \times \alpha) \cup (\{1\} \times \bigcup_{\beta < \lambda} \beta) \\ &= (\{0\} \times \alpha) \cup \bigcup_{\beta < \lambda} (\{1\} \times \beta) \\ &= \bigcup_{\beta < \lambda} ((\{0\} \times \alpha) \cup (\{1\} \times \beta)) \end{aligned}$$

$\langle 1 \rangle 2$. For any ordinal β we have $\alpha + \beta < \alpha + \beta^+$

PROOF: Since $\alpha + \beta^+ = \alpha + \beta + 1 = (\alpha + \beta)^+$.

\square

Corollary 7.7.7.1. *For any ordinals α, β, γ , we have $\beta < \gamma$ if and only if $\alpha + \beta < \alpha + \gamma$.*

Corollary 7.7.7.2 (Left Cancellation for Addition). *For any ordinals α, β and γ , if $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$.*

Theorem 7.7.8. *For any ordinals α, β, γ , if $\beta \leq \gamma$ then $\beta + \alpha \leq \gamma + \alpha$.*

PROOF: Transfinite induction on α . \square

Theorem 7.7.9 (Subtraction Theorem). *Let α and β be ordinals with $\alpha \leq \beta$. Then there exists a unique ordinal δ such that $\alpha + \delta = \beta$.*

PROOF:

$\langle 1 \rangle 1$. For all ordinals α and β with $\alpha \leq \beta$, there exists δ such that $\alpha + \delta = \beta$

$\langle 2 \rangle 1$. LET: α and β be ordinals with $\alpha \leq \beta$

$\langle 2 \rangle 2$. LET: δ be the greatest ordinal such that $\alpha + \delta \leq \beta$

PROOF: Proposition 7.6.6.

$\langle 2 \rangle 3$. $\alpha + \delta = \beta$

PROOF: If $\alpha + \delta < \beta$ then $\alpha + \delta + 1 \leq \beta$ contradicting the greatestness of δ .

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Uniqueness follows from the Left Cancellation Law.

\square

7.7.2 Multiplication

Definition 7.7.10 (Ordinal Multiplication). Let α and β be ordinal numbers. Then $\alpha\beta$ is the ordinal number of $A \times B$ under the lexicographic order, where A

is any well ordered set with ordinal α and B is any well ordered set with ordinal β .

This is well defined by Proposition 6.3.5.

Theorem 7.7.11 (Associative Law). *For any ordinals ρ , σ and τ , we have*

$$\rho(\sigma\tau) = (\rho\sigma)\tau .$$

PROOF: Let A , B and C be well ordered sets with ordinals ρ , σ and τ . Then both $\rho(\sigma\tau)$ and $(\rho\sigma)\tau$ are the ordinal of $A \times B \times C$ under $(a, b, c) \leq (a', b', c') \Leftrightarrow a \leq a' \vee (a = a' \wedge b \leq b') \vee (a = a' \wedge b = b' \wedge c \leq c')$. \square

Theorem 7.7.12 (Left Distributive Law). *For any ordinals ρ , σ and τ , we have*

$$\rho(\sigma + \tau) = \rho\sigma + \rho\tau$$

PROOF: Let A , B and C be well ordered sets with ordinals ρ , σ and τ and with $B \cap C = \emptyset$. Then both $\rho(\sigma + \tau)$ and $\rho\sigma + \rho\tau$ are the ordinal of $A \times (B \cup C)$ under the lexicographic ordering. \square

Theorem 7.7.13. *For any ordinal ρ we have $\rho 0 = 0\rho = 0$.*

PROOF: For any well ordered set A we have $A \times \emptyset = \emptyset \times A = \emptyset$. \square

Theorem 7.7.14. *For any ordinal ρ we have $\rho 1 = 1\rho = \rho$.*

PROOF: Easy. \square

Theorem 7.7.15. *For any ordinals ρ and σ , if $\rho\sigma = 0$ then $\rho = 0$ or $\sigma = 0$.*

PROOF: If $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$. \square

Theorem 7.7.16. *For any non-zero ordinal α , the operation that maps β to $\alpha\beta$ is normal.*

PROOF:

$\langle 1 \rangle 1$. For any limit ordinal λ , we have $\alpha\lambda = \bigcup_{\beta < \lambda} \alpha\beta$

$\langle 2 \rangle 1$. LET: λ be a limit ordinal

$\langle 2 \rangle 2$. $\alpha \times \lambda = \bigcup_{\beta < \lambda} (\alpha \times \beta)$ as well-ordered sets

$\langle 1 \rangle 2$. For any ordinal β we have $\alpha\beta < \alpha\beta^+$

PROOF: $\alpha\beta^+ = \alpha\beta + \alpha > \alpha\beta$

\square

Corollary 7.7.16.1. *For any ordinals α , β , γ , if $\alpha \neq 0$ then $\beta < \gamma$ if and only if $\alpha\beta < \alpha\gamma$.*

Corollary 7.7.16.2 (Left Cancellation for Multiplication). *For any ordinals α , β , γ , if $\alpha \neq 0$ and $\alpha\beta = \alpha\gamma$ then $\beta = \gamma$.*

Theorem 7.7.17. *For any ordinals α , β and γ , if $\beta \leq \gamma$ then $\beta\alpha \leq \gamma\alpha$.*

PROOF: Transfinite induction on α . \square

Theorem 7.7.18 (Division Theorem). *Let α and δ be ordinal numbers with $\delta \neq 0$. Then there exist unique ordinals β and γ with $\gamma < \delta$ and*

$$\alpha = \delta\beta + \gamma .$$

PROOF:

- $\langle 1 \rangle 1$. For any ordinal numbers α and δ with $\delta \neq 0$, there exist ordinals β and γ such that $\gamma < \delta$ and $\alpha = \delta\beta + \gamma$
 $\langle 2 \rangle 1$. LET: α and δ be ordinals with $\delta \neq 0$
 $\langle 2 \rangle 2$. LET: β be the greatest ordinal such that $\delta\beta \leq \alpha$
 PROOF: Proposition 7.6.6.
 $\langle 2 \rangle 3$. There exists an ordinal γ such that $\alpha = \delta\beta + \gamma$
 PROOF: Subtraction Theorem
 $\langle 1 \rangle 2$. For any ordinals $\delta, \beta, \beta', \gamma, \gamma'$, if $\delta\beta + \gamma = \delta\beta' + \gamma'$ and $\delta \neq 0$ and $\gamma, \gamma' < \delta$ then $\beta = \beta'$ and $\gamma = \gamma'$

- $\langle 2 \rangle 1$. LET: $\delta, \beta, \beta', \gamma, \gamma'$ be ordinals.
 $\langle 2 \rangle 2$. ASSUME: $\delta \neq 0$ and $\delta\beta + \gamma = \delta\beta' + \gamma'$
 $\langle 2 \rangle 3$. $\beta = \beta'$
 $\langle 3 \rangle 1$. $\beta \neq \beta'$

PROOF: If $\beta < \beta'$ then

$$\begin{aligned} \delta\beta' + \gamma' &\geq \delta\beta' \\ &\geq \delta(\beta + 1) \\ &= \delta\beta + \delta \\ &> \delta\beta + \gamma \end{aligned}$$

- $\langle 3 \rangle 2$. $\beta' \neq \beta$

PROOF: Similar.

- $\langle 2 \rangle 4$. $\gamma = \gamma'$

PROOF: By Cancellation.

\square

7.7.3 Exponentiation

Definition 7.7.19. Given ordinals α and β , define the ordinal α^β as follows:

$$\begin{aligned} 0^\alpha &:= 0 & (\alpha > 0) \\ \alpha^0 &:= 1 \\ \alpha^{\beta^+} &:= \alpha^\beta \alpha & (\alpha > 0) \\ \alpha^\lambda &:= \sup_{\beta < \lambda} \alpha^\beta & (\alpha > 0, \lambda \text{ a limit ordinal}) \end{aligned}$$

Theorem 7.7.20. *Let α be an ordinal ≥ 2 . The operation that maps β to α^β is normal.*

PROOF:

(1)1. For λ a limit ordinal we have $\alpha^\lambda = \sup_{\beta < \lambda} \alpha^\beta$

PROOF: By definition.

(1)2. For any ordinal β we have $\alpha^\beta < \alpha^{\beta+}$

PROOF: We have $\alpha^{\beta+} = \alpha^\beta \alpha > \alpha^\beta$ by Theorem 7.7.16 since $\alpha > 1$ and $\alpha^\beta \neq 0$.

□

Corollary 7.7.20.1. *For any ordinals α, β, γ , if $\alpha \geq 2$ then $\beta < \gamma$ if and only if $\alpha^\beta < \alpha^\gamma$.*

Corollary 7.7.20.2 (Cancellation for Exponentiation). *For any ordinals α, β, γ , if $\alpha \geq 2$ and $\alpha^\beta = \alpha^\gamma$ then $\beta = \gamma$.*

Theorem 7.7.21. *For any ordinals α, β and γ , if $\beta \leq \gamma$ then $\beta^\alpha \leq \gamma^\alpha$.*

PROOF: Transfinite induction on α .

Theorem 7.7.22 (Logarithm Theorem). *Let α and β be ordinal numbers with $\alpha \neq 0$ and $\beta > 1$. Then there exist unique ordinals γ, δ and ρ such that*

$$\alpha = \beta^\gamma \delta + \rho, \quad 0 \neq \delta < \beta, \quad \rho < \beta^\gamma.$$

PROOF:

(1)1. For any ordinals α and β with $\alpha \neq 0$ and $\beta > 1$, there exist ordinals γ, δ, ρ such that

$$\alpha = \beta^\gamma \delta + \rho, \quad 0 \neq \delta < \beta, \quad \rho < \beta^\gamma.$$

(2)1. LET: α and β be ordinals with $\alpha \neq 0$ and $\beta > 1$.

(2)2. LET: γ be the greatest ordinal such that $\beta^\gamma \leq \alpha$.

PROOF: Proposition 7.6.6.

(2)3. LET: δ and ρ be the unique ordinals with $\rho < \beta^\gamma$ such that $\alpha = \beta^\gamma \delta + \rho$.

PROOF: By the Division Theorem.

(2)4. $\delta \neq 0$

PROOF: If $\delta = 0$ then $\alpha = \beta^\gamma 0 + \rho = \rho < \beta^\gamma \leq \alpha$ which is a contradiction.

(2)5. $\delta < \beta$

PROOF: If $\beta \leq \delta$ then $\alpha \geq \beta^\gamma \delta \geq \beta^\gamma \beta = \beta^{\gamma+1}$, contradicting the greatestness of γ .

(1)2. If $\beta^\gamma \delta + \rho = \beta^{\gamma'} \delta' + \rho'$ with $\beta > 1, 0 \neq \delta < \beta, 0 \neq \delta' < \beta, \rho < \beta^\gamma$ and $\rho' < \beta^{\gamma'}$, then $\gamma = \gamma', \delta = \delta'$ and $\rho = \rho'$.

(2)1. LET: $\alpha = \beta^\gamma \delta + \rho = \beta^{\gamma'} \delta' + \rho'$

(2)2. $\beta^\gamma \leq \alpha < \beta^{\gamma+1}$

(2)3. $\beta^{\gamma'} \leq \alpha < \beta^{\gamma'+1}$

(2)4. $\beta^\gamma < \beta^{\gamma'+1}$ and $\beta^{\gamma'} < \beta^{\gamma+1}$

(2)5. $\gamma < \gamma' + 1$ and $\gamma' < \gamma + 1$

(2)6. $\gamma = \gamma'$

(2)7. $\delta = \delta'$ and $\rho = \rho'$

PROOF: By the Division Theorem.

□

Theorem 7.7.23. *For any ordinal numbers α, β, γ , we have*

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma .$$

PROOF:

$\langle 1 \rangle 1$. LET: $P[\gamma]$ be the property: for any ordinals α and β we have $\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$

$\langle 1 \rangle 2$. $P[0]$

PROOF:

$$\begin{aligned} \alpha^{\beta+0} &= \alpha^\beta \\ &= \alpha^\beta 1 \\ &= \alpha^\beta \alpha^0 \end{aligned}$$

$\langle 1 \rangle 3$. For all γ , if $P[\gamma]$ then $P[\gamma+1]$

PROOF:

$$\begin{aligned} \alpha^{\beta+\gamma+1} &= \alpha^{\beta+\gamma} \alpha \\ &= \alpha^\beta \alpha^\gamma \alpha && \text{(induction hypothesis)} \\ &= \alpha^\beta \alpha^{\gamma+1} \end{aligned}$$

$\langle 1 \rangle 4$. For any limit ordinal λ , if $\forall \gamma < \lambda. P[\gamma]$ then $P[\lambda]$.

$\langle 2 \rangle 1$. LET: λ be a limit ordinal

$\langle 2 \rangle 2$. ASSUME: $\forall \gamma < \lambda. P[\gamma]$

$\langle 2 \rangle 3$. LET: α and β be any ordinals.

$\langle 2 \rangle 4$. CASE: $\alpha = 0$

PROOF: We have $\alpha^{\beta+\lambda} = \alpha^\beta \alpha^\lambda = 0$.

$\langle 2 \rangle 5$. CASE: $\alpha = 1$

PROOF: We have $\alpha^{\beta+\lambda} = \alpha^\beta \alpha^\lambda = 1$.

$\langle 2 \rangle 6$. CASE: $\alpha > 1$

PROOF:

$$\begin{aligned} \alpha^{\beta+\lambda} &= \alpha^{\sup_{\gamma < \lambda} (\beta+\gamma)} \\ &= \sup_{\gamma < \lambda} \alpha^{\beta+\gamma} && \text{(Theorem 7.6.7)} \\ &= \sup_{\gamma < \lambda} \alpha^\beta \alpha^\gamma && (\langle 2 \rangle 2) \\ &= \alpha^\beta \sup_{\gamma < \lambda} \alpha^\gamma && \text{(Theorem 7.6.7)} \\ &= \alpha^\beta \alpha^\lambda \end{aligned}$$

□

Theorem 7.7.24. *For any ordinal numbers α, β and γ , we have*

$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma} .$$

PROOF:

$\langle 1 \rangle 1$. LET: $P[\gamma]$ be the property: For any ordinals α and β , we have $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$

$\langle 1 \rangle 2$. $P[0]$

PROOF:

$$\begin{aligned} (\alpha^\beta)^0 &= 1 \\ &= \alpha^{\beta 0} \end{aligned}$$

$\langle 1 \rangle 3. \forall \gamma \in \mathbf{On}. P[\gamma] \Rightarrow P[\gamma + 1]$

PROOF:

$$\begin{aligned} (\alpha^\beta)^{\gamma+1} &= (\alpha^\beta)^\gamma \alpha^\beta \\ &= \alpha^{\beta\gamma} \alpha^\beta \\ &= \alpha^{\beta\gamma+\beta} \\ &= \alpha^{\beta(\gamma+1)} \end{aligned}$$

$\langle 1 \rangle 4.$ For any limit ordinal λ , if $\forall \gamma < \lambda. P[\gamma]$ then $P[\lambda]$.

$\langle 2 \rangle 1.$ LET: λ be a limit ordinal.

$\langle 2 \rangle 2.$ ASSUME: $\forall \gamma < \lambda. P[\gamma]$

$\langle 2 \rangle 3.$ LET: α and β be any ordinals.

$\langle 2 \rangle 4.$ CASE: $\alpha = 0$ and $\beta = 0$

PROOF:

$$\begin{aligned} (0^\beta)^\lambda &= 1^\lambda \\ &= 1 \\ &= 0^0 \\ &= 0^{0\lambda} \end{aligned}$$

$\langle 2 \rangle 5.$ CASE: $\alpha = 0$ and $\beta \neq 0$

PROOF: $(0^\beta)^\lambda = 0^{\beta\lambda} = 0.$

$\langle 2 \rangle 6.$ CASE: $\alpha = 1$

PROOF: $(1^\beta)^\lambda = 1^{\beta\lambda} = 1$

$\langle 2 \rangle 7.$ CASE: $\alpha > 1$

PROOF:

$$\begin{aligned} (\alpha^\beta)^\lambda &= \sup_{\gamma < \lambda} (\alpha^\beta)^\gamma \\ &= \sup_{\gamma < \lambda} \alpha^{\beta\gamma} \\ &= \alpha^{\sup_{\gamma < \lambda} \beta\gamma} \\ &= \alpha^{\beta\lambda} \end{aligned}$$

□

7.8 Sequences

i

Definition 7.8.1 (Sequence). Given an ordinal α and class \mathbf{A} , an α -sequence in \mathbf{A} is a function $a : \alpha \rightarrow \mathbf{A}$. We write a_β for $a(\beta)$, and $(a_\beta)_{\beta < \alpha}$ for a .

Definition 7.8.2 (Strictly Increasing). A sequence (a_β) of ordinals is *strictly increasing* iff, whenever $\beta < \gamma$, then $a_\beta < a_\gamma$.

Definition 7.8.3 (Subsequence). Let $(a_\beta)_{\beta < \gamma}$ be a sequence in \mathbf{A} . A *subsequence* of (a_β) is a sequence of the form $(a_{\beta_\xi})_{\xi < \delta}$ where $(\beta_\xi)_{\xi < \delta}$ is a strictly increasing sequence in γ .

Definition 7.8.4 (Convergence). Let $(a_\beta)_{\beta < \gamma}$ be a sequence of ordinals and λ an ordinal. Then (a_β) *converges* to the *limit* λ iff $\lambda = \sup_{\beta < \gamma} a_\beta$.

Lemma 7.8.5. *Let $(a_\beta)_{\beta < \gamma}$ be a sequence of ordinals. Then there is a strictly increasing subsequence $(a_{\beta_\xi})_{\xi < \delta}$ such that $\sup_{\xi < \delta} a_{\beta_\xi} = \sup_{\beta < \gamma} a_\beta$.*

PROOF: Define β_ξ by transfinite recursion as follows. β_ξ is the least β such that $a_\beta > a_{\beta_\zeta}$ for all $\zeta < \xi$ if there is such an a_β ; if not, the sequence ends. \square

7.9 Strict Supremum

Definition 7.9.1 (Strict Supremum). For any set S of ordinals, define the *strict supremum* of S , $\text{ssup } S$, to be the least ordinal greater than every member of S .

Chapter 8

Cardinal Numbers

8.1 Cardinal Numbers

Definition 8.1.1 (Cardinality). For any set A , the *cardinality* or *cardinal number* $|A|$ of A is the least ordinal equinumerous with A .

Let **Card** be the class of all cardinal numbers.

Proposition 8.1.2. For any sets A and B , we have $A \approx B$ iff $|A| = |B|$.

PROOF: Easy. \square

Definition 8.1.3 (Addition). Given cardinal numbers κ and λ , we define $\kappa + \lambda$ to be $|A \cup B|$ where A and B are disjoint sets of cardinality κ and λ respectively.

We prove this is well-defined.

PROOF:

$\langle 1 \rangle 1$. ASSUME: $A \approx A'$, $B \approx B'$, and $A \cap B = A' \cap B' = \emptyset$

$\langle 1 \rangle 2$. PICK bijections $f : A \approx A'$ and $g : B \approx B'$

$\langle 1 \rangle 3$. The function $A \cup B \rightarrow A' \cup B'$ that maps $a \in A$ to $f(a)$ and $b \in B$ to $g(b)$ is a bijection.

\square

Proposition 8.1.4. For any cardinal number κ , we have $\kappa + 0 = \kappa$.

PROOF: Let A and B be disjoint sets of cardinality κ and 0 . Then $B = \emptyset$ so $A \cup B = A$ and so $|A \cup B| = \kappa$. \square

Theorem 8.1.5 (Associative Law for Addition). For any cardinal numbers κ , λ , μ we have $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$.

PROOF: Since $A \cup (B \cup C) = (A \cup B) \cup C$. \square

Proposition 8.1.6. For any cardinal numbers κ and λ we have $\kappa + \lambda = \lambda + \kappa$.

PROOF: Since $A \cup B = B \cup A$. \square

Definition 8.1.7 (Multiplication). For κ and λ cardinal numbers, we define $\kappa\lambda$ to be the cardinal number of $A \times B$, where $|A| = \kappa$ and $|B| = \lambda$.

We prove this is well-defined.

PROOF: If $f : A \approx A'$ and $g : B \approx B'$ then the function that maps (a, b) to $(f(a), g(b))$ is a bijection $A \times B \approx A' \times B'$. \square

Proposition 8.1.8. For any cardinal number κ we have $\kappa \cdot 0 = 0$.

PROOF: Since $A \times \emptyset = \emptyset$. \square

Proposition 8.1.9. For any cardinal number κ we have $\kappa \cdot 1 = \kappa$.

PROOF: The function that maps (a, e) to a is a bijection $A \times \{e\} \approx A$. \square

Theorem 8.1.10 (Distributive Law). For any cardinal numbers κ , λ and μ , we have $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$.

PROOF: Since $A \times (B \cup C) = (A \times B) \cup (A \times C)$. \square

Theorem 8.1.11 (Associative Law for Multiplication). For any cardinal numbers κ , λ and μ , we have $\kappa(\lambda\mu) = (\kappa\lambda)\mu$.

PROOF: Since $A \times (B \times C) \approx (A \times B) \times C$. \square

Theorem 8.1.12 (Commutative Law for Multiplication). For any cardinal numbers κ and λ , we have $\kappa\lambda = \lambda\kappa$.

PROOF: Since $A \times B \approx B \times A$. \square

Theorem 8.1.13. For any cardinal numbers κ and λ , if $\kappa\lambda = 0$ then $\kappa = 0$ or $\lambda = 0$.

PROOF: if $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$. \square

Definition 8.1.14 (Exponentiation). Given cardinal numbers κ and λ , we define κ^λ to be $|A^B|$, where $|A| = \kappa$ and $|B| = \lambda$.

We prove this is well-defined.

PROOF: If $f : A \approx A'$ and $g : B \approx B'$, then the function that maps $h : B \rightarrow A$ to $f \circ h \circ g^{-1}$ is a bijection $A^B \approx A'^{B'}$. \square

Proposition 8.1.15. For any cardinal numbers κ , λ and μ ,

$$\kappa^{\lambda+\mu} = (\kappa^\lambda)^\mu$$

PROOF: The function that maps $f : A \times B \rightarrow C$ to $\lambda a \in A. \lambda b \in B. f(a, b)$ is a bijection $A^{B \times C} \approx (A^B)^C$. \square

Proposition 8.1.16. For any cardinal numbers κ , λ and μ ,

$$(\kappa\lambda)^\mu = \kappa^\mu \lambda^\mu.$$

PROOF: The function $f : A^C \times B^C \rightarrow (A \times B)^C$ with $f(g, h)(c) = (g(c), h(c))$ is a bijection. \square

Proposition 8.1.17. *For any cardinal numbers κ , λ and μ , we have*

$$\kappa^{\lambda+\mu} = \kappa^\lambda \kappa^\mu .$$

PROOF: If $B \cap C = \emptyset$, then $f : A^B \times A^C \rightarrow A^{B \cup C}$ given by $f(g, h)(b) = g(b)$ and $f(g, h)(c) = h(c)$ is a bijection. \square

Proposition 8.1.18. *For any cardinal number κ , we have $\kappa^0 = 1$.*

PROOF: For any set A , we have $A^\emptyset = \{\emptyset\}$. \square

Proposition 8.1.19. *For any cardinal number κ , we have $\kappa^1 = \kappa$.*

PROOF: For any sets A and B , if $B = \{b\}$ then the function $f : A \rightarrow A^B$ with $f(a)(b) = a$ is a bijection. \square

Proposition 8.1.20. *For any non-zero cardinal number κ we have $0^\kappa = 0$.*

PROOF: If A is nonempty then there is no function $A \rightarrow \emptyset$. \square

Proposition 8.1.21. *For any set A we have $|\mathcal{P}A| = 2^{|A|}$.*

PROOF: The function $f : \mathcal{P}A \rightarrow 2^A$ where $f(X)(a) = 0$ if $a \notin X$ and $f(X)(a) = 1$ if $a \in X$. \square

Theorem 8.1.22 (König). *Let I be a set. Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets. Assume that $\forall i \in I. |A_i| < |B_i|$. Then $|\bigcup_{i \in I} A_i| < |\prod_{i \in I} B_i|$.*

PROOF:

$\langle 1 \rangle 1$. For all $i \in I$, choose an injection $f_i : A_i \rightarrow B_i$

$\langle 1 \rangle 2$. For all $i \in I$, choose $b_i \in B_i - f_i(A_i)$

$\langle 1 \rangle 3$. $|\bigcup_{i \in I} A_i| \leq |\prod_{i \in I} B_i|$

$\langle 2 \rangle 1$. Define $g : \bigcup_{i \in I} A_i \rightarrow \prod_{i \in I} B_i$ by

$$g(i, a)(j) = \begin{cases} f_i(a) & \text{if } i = j \\ b_j & \text{otherwise} \end{cases}$$

$\langle 2 \rangle 2$. g is injective.

$\langle 1 \rangle 4$. $|\bigcup_{i \in I} A_i| \neq |\prod_{i \in I} B_i|$

$\langle 2 \rangle 1$. LET: $h : \bigcup_{i \in I} A_i \rightarrow \prod_{i \in I} B_i$

PROVE: h is not surjective.

$\langle 2 \rangle 2$. For $i \in I$, PICK $c_i \in B_i - \{h(i, a)(i) \mid i \in I\}$

$\langle 2 \rangle 3$. $c \in \prod_{i \in I} B_i$

$\langle 2 \rangle 4$. $c \notin \text{ran } h$

\square

Corollary 8.1.22.1. *For any cardinal number κ we have $\kappa < 2^\kappa$.*

8.2 Ordering on Cardinal Numbers

Definition 8.2.1. Given cardinal numbers κ and λ , we have $\kappa \leq \lambda$ iff $A \preccurlyeq B$, where $|A| = \kappa$ and $|B| = \lambda$.

PROOF:

$\langle 1 \rangle 1$. LET: $|A| = \kappa$ and $|B| = \lambda$

$\langle 1 \rangle 2$. PICK bijections $f : A \approx \kappa$ and $g : B \approx \lambda$

$\langle 1 \rangle 3$. If $\kappa \leq \lambda$ then $A \preccurlyeq B$

PROOF: Let $i : \kappa \hookrightarrow \lambda$ be the inclusion. Then $g^{-1} \circ i \circ f$ is an injection $A \rightarrow B$.

$\langle 1 \rangle 4$. If $A \preccurlyeq B$ then $\kappa \leq \lambda$

$\langle 2 \rangle 1$. ASSUME: $A \preccurlyeq B$

$\langle 2 \rangle 2$. PICK an injection $h : A \hookrightarrow B$

$\langle 2 \rangle 3$. $g(h(A)) \subseteq B$ is well-ordered by \in

$\langle 2 \rangle 4$. LET: γ be the ordinal number of $(g(h(A)), \in)$

$\langle 2 \rangle 5$. $\gamma \leq \lambda$

PROOF: Proposition 7.3.15.

$\langle 2 \rangle 6$. $\kappa \leq \gamma$

PROOF: By the leastness of κ , since A is equinumerous with γ .

$\langle 2 \rangle 7$. $\kappa \leq \lambda$

□

Corollary 8.2.1.1. *There is no largest cardinal number.*

Proposition 8.2.2. *For any cardinal numbers κ, λ, μ , if $\kappa \leq \lambda$ then $\kappa + \mu \leq \lambda + \mu$.*

PROOF: If $f : A \rightarrow B$ is injective, and $A \cap C = B \cap C = \emptyset$, then the function $A \cup C \rightarrow B \cup C$ that maps a to $f(a)$ and maps c to c is an injection. □

Proposition 8.2.3. *For any cardinal numbers κ, λ, μ , if $\kappa \leq \lambda$ then $\kappa\mu \leq \lambda\mu$.*

PROOF: If $f : A \rightarrow B$ is injective, then the function $A \times C \rightarrow B \times C$ that maps (a, c) to $(f(a), c)$ is injective. □

Proposition 8.2.4. *For any cardinal numbers κ, λ, μ , if $\kappa \leq \lambda$ then $\kappa^\mu \leq \lambda^\mu$.*

PROOF: Given an injection $f : A \rightarrow B$, the function that maps $A^C \rightarrow B^C$ that maps g to $f \circ g$ is an injection. □

Proposition 8.2.5. *For any cardinal numbers κ, λ, μ , if $\kappa \leq \lambda$ and μ and κ are not both 0, then $\mu^\kappa \leq \mu^\lambda$.*

PROOF:

$\langle 1 \rangle 1$. LET: A, B and C be sets with A and C not both empty.

$\langle 1 \rangle 2$. LET: $f : A \rightarrow B$ be an injection.

PROVE: $C^A \preccurlyeq C^B$

$\langle 1 \rangle 3$. CASE: $C = \emptyset$

PROOF: Then $A \neq \emptyset$ so $C^A = \emptyset \preccurlyeq C^B$.

$\langle 1 \rangle 4$. CASE: $C \neq \emptyset$

- ⟨2⟩1. PICK $c \in C$
 ⟨2⟩2. LET: $g : C^A \rightarrow C^B$ be the function $g(h)(f(a)) = h(a)$, $g(h)(b) = c$ if $b \notin f(A)$
 ⟨2⟩3. g is an injection.

□

Proposition 8.2.6. *Let \mathcal{A} be a set such that $\forall X \in \mathcal{A}. |X| \leq \kappa$. Then*

$$\left| \bigcup \mathcal{A} \right| \leq |\mathcal{A}| \kappa .$$

PROOF:

- ⟨1⟩1. For $X \in \mathcal{A}$, choose a surjection $f_X : \kappa \rightarrow X$.
 ⟨1⟩2. Define $g : \mathcal{A} \times \kappa \rightarrow \bigcup \mathcal{A}$ by $g(X, \alpha) = f_X(\alpha)$
 ⟨1⟩3. g is surjective.

□

Lemma 8.2.7. *The union of a set of cardinal numbers is a cardinal number.*

PROOF:

- ⟨1⟩1. LET: A be a set of cardinal numbers.
 PROVE: $\bigcup A$ is the smallest ordinal equinumerous with $\bigcup A$
 ⟨1⟩2. LET: $\alpha < \bigcup A$
 PROVE: $\alpha \not\approx \bigcup A$
 ⟨1⟩3. PICK $\kappa \in A$ such that $\alpha < \kappa$
 ⟨1⟩4. $\alpha \prec \kappa$
 ⟨1⟩5. $\alpha \prec \bigcup A$

□

Chapter 9

Natural Numbers

9.1 Inductive Sets

Definition 9.1.1 (Inductive). A set I is *inductive* iff $0 \in I$ and $\forall x \in I. x^+ \in I$.

Definition 9.1.2 (Natural Number). A *natural number* is a set that belongs to every inductive set.

Theorem 9.1.3. *The class \mathbb{N} of natural numbers is a set.*

PROOF:

$\langle 1 \rangle 1$. PICK an inductive set I .

PROOF: Axiom of Infinity.

$\langle 1 \rangle 2$. $\mathbb{N} \subseteq I$

□

Theorem 9.1.4. *\mathbb{N} is inductive, and is a subset of every other inductive set.*

PROOF:

$\langle 1 \rangle 1$. \mathbb{N} is inductive.

$\langle 2 \rangle 1$. $0 \in \mathbb{N}$

PROOF: Since 0 is a member of every inductive set.

$\langle 2 \rangle 2$. $\forall n \in \mathbb{N}. n^+ \in \mathbb{N}$

$\langle 3 \rangle 1$. LET: $n \in \mathbb{N}$

$\langle 3 \rangle 2$. LET: I be any inductive set.

PROVE: $n^+ \in I$

$\langle 3 \rangle 3$. $n \in I$

PROOF: $\langle 3 \rangle 1$, $\langle 3 \rangle 2$

$\langle 3 \rangle 4$. $n^+ \in I$

PROOF: $\langle 3 \rangle 2$, $\langle 3 \rangle 3$

$\langle 1 \rangle 2$. \mathbb{N} is a subset of every inductive set.

PROOF: Immediate from definitions.

□

Corollary 9.1.4.1 (Induction Principle for \mathbb{N}). *Any inductive subset of \mathbb{N} coincides with \mathbb{N} .*

Theorem 9.1.5. *Every natural number except 0 is the successor of some natural number.*

PROOF: Trivially by induction. \square

Proposition 9.1.6. *Every natural number is an ordinal.*

PROOF: By induction. \square

Proposition 9.1.7. *\mathbb{N} is a transitive set.*

PROOF:

$\langle 1 \rangle 1. 0 \subseteq \mathbb{N}$

$\langle 1 \rangle 2. \forall n \in \mathbb{N}. n \subseteq \mathbb{N} \Rightarrow n^+ \subseteq \mathbb{N}$

$\langle 1 \rangle 3. \forall n \in \mathbb{N}. n \subseteq \mathbb{N}$

PROOF: From $\langle 1 \rangle 1$ and $\langle 1 \rangle 2$ by induction.

\square

Corollary 9.1.7.1. *\mathbb{N} is an ordinal.*

Definition 9.1.8. We define $\omega = \mathbb{N}$.

Proposition 9.1.9 (Dependent Choice). *Let A be a nonempty set and R a relation on A such that $\forall x \in A. \exists y \in A. (y, x) \in R$. Then there exists a function $f : \mathbb{N} \rightarrow A$ such that $\forall n \in \mathbb{N}. (f(n+1), f(n)) \in R$.*

PROOF:

$\langle 1 \rangle 1.$ PICK a choice function F for A .

$\langle 1 \rangle 2.$ PICK $a \in A$

$\langle 1 \rangle 3.$ Define $f : \mathbb{N} \rightarrow A$ by $f(0) = a$ and $f(n+1) = F(\{y \in A \mid (y, f(n)) \in R\})$.

\square

Theorem Schema 9.1.10. *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem:*

Assume \mathbf{R} is a relation on \mathbf{A} and, for all $a \in \mathbf{A}$, the class $\{x \in \mathbf{A} \mid x\mathbf{R}a\}$ is a set. Then \mathbf{R} is well founded if and only if there does not exist a function $f : \mathbb{N} \rightarrow \mathbf{A}$ such that $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$.

PROOF:

$\langle 1 \rangle 1.$ If there exists a function $f : \mathbb{N} \rightarrow \mathbf{A}$ such that $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$ then \mathbf{R} is not well founded.

PROOF: $f(\mathbb{N})$ is a nonempty subset of \mathbf{A} with no \mathbf{R} -minimal element.

$\langle 1 \rangle 2.$ If \mathbf{R} is not well founded then there exists a function $f : \mathbb{N} \rightarrow \mathbf{A}$ such that $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$.

$\langle 2 \rangle 1.$ ASSUME: \mathbf{R} is not well founded.

$\langle 2 \rangle 2.$ PICK a nonempty subset $B \subseteq \mathbf{A}$ that has no \mathbf{R} -minimal element.

$\langle 2 \rangle 3.$ $\forall x \in B. \exists y \in B. y\mathbf{R}x$

- ⟨2⟩4. Choose a function $g : B \rightarrow B$ such that $\forall x \in B. g(x) \mathbf{R} x$
- ⟨2⟩5. PICK $b \in B$
- ⟨2⟩6. Define $f : \mathbb{N} \rightarrow \mathbf{A}$ recursively by $f(0) = b$ and $\forall n \in \mathbb{N}. f(n+1) = g(f(n))$
- ⟨2⟩7. $\forall n \in \mathbb{N}. f(n+1) \mathbf{R} f(n)$

□

9.2 Cardinality

Definition 9.2.1 (Finite). A set is *finite* iff it is equinumerous to some natural number; otherwise it is *infinite*.

Theorem 9.2.2 (Pigeonhole Principle). *No natural number is equinumerous to a proper subset of itself.*

PROOF:

⟨1⟩1. LET: $P(n)$ be the property: any one-to-one function $n \rightarrow n$ is surjective.

⟨1⟩2. $P(0)$

PROOF: The only function $0 \rightarrow 0$ is injective.

⟨1⟩3. For every natural number n , if $P(n)$ then $P(n+1)$.

⟨2⟩1. ASSUME: $P(n)$

⟨2⟩2. LET: f be a one-to-one function $n+1 \rightarrow n+1$

⟨2⟩3. $f \upharpoonright n$ is a one-to-one function $n \rightarrow n+1$

⟨2⟩4. CASE: $n \notin \text{ran } f$

⟨3⟩1. $f \upharpoonright n : n \rightarrow n$

⟨3⟩2. $\text{ran}(f \upharpoonright n) = n$

⟨3⟩3. $f(n) = n$

PROOF: ⟨2⟩1.

⟨3⟩4. $\text{ran } f = n+1$

⟨2⟩5. CASE: $n \in \text{ran } f$

⟨3⟩1. PICK $p \in n$ such that $f(p) = n$

⟨3⟩2. LET: $\hat{f} : n \rightarrow n$ be the function

$$\hat{f}(p) = f(n)$$

$$\hat{f}(x) = f(x) \quad (x \neq p)$$

⟨3⟩3. \hat{f} is one-to-one

⟨3⟩4. $\text{ran } \hat{f} = n$

PROOF: ⟨2⟩1

⟨3⟩5. $\text{ran } f = n+1$

⟨1⟩4. For every natural number n , $P(n)$.

□

Corollary 9.2.2.1. *No finite set is equinumerous to a proper subset of itself.*

Corollary 9.2.2.2. *Every natural number is a cardinal number.*

PROOF: For any natural number n , we have that n is the least ordinal such that $n \approx n$. □

Corollary 9.2.2.3. \mathbb{N} is a cardinal number.

Corollary 9.2.2.4. \mathbb{N} is infinite.

PROOF: The function that maps n to $n+1$ is a bijection between \mathbb{N} and $\mathbb{N}-\{0\}$.
 \square

Corollary 9.2.2.5. If C is a proper subset of a natural number n , then there exists $m < n$ such that $C \approx m$.

PROOF: By Proposition 7.3.15. \square

Corollary 9.2.2.6. Any subset of a finite set is finite.

Proposition 9.2.3. For any natural numbers m and n we have $m+n$ (cardinal addition) is a natural number.

PROOF: Induction on n . \square

Corollary 9.2.3.1. The union of two finite sets is finite.

Corollary 9.2.3.2. The union of a finite set of finite sets is finite.

PROOF: By induction on the number of elements. \square

Proposition 9.2.4. For natural numbers m and n , the cardinal sum $m+n$ is equal to the ordinal sum $m+n$.

PROOF: Induction on n . \square

Proposition 9.2.5. For any natural numbers m and n , we have mn (cardinal multiplication) is a natural number.

Corollary 9.2.5.1. If A and B are finite sets then $A \times B$ is finite.

Proposition 9.2.6. For natural numbers m and n , the cardinal product mn is equal to the ordinal product mn .

PROOF: Induction on n . \square

Proposition 9.2.7. For any natural numbers m and n we have m^n (cardinal exponentiation) is a natural number.

PROOF: Induction on n .

Corollary 9.2.7.1. If A and B are finite sets then A^B are finite.

Proposition 9.2.8. For natural numbers m and n , the cardinal exponentiation m^n and the ordinal exponentiation m^n agree.

PROOF: Induction on n . \square

Proposition 9.2.9. $\mathbb{N}^2 \approx \mathbb{N}$

PROOF: The function $J : \mathbb{N}^2 \rightarrow \mathbb{N}$ defined by $J(m, n) = ((m + n)^2 + 3m + n)/2$ is a bijection. \square

Proposition 9.2.10. *For any infinite cardinal κ we have $\aleph_0 \leq \kappa$.*

PROOF:

$\langle 1 \rangle 1$. LET: A be an infinite set.

PROVE: $\aleph_0 \leq A$

$\langle 1 \rangle 2$. PICK a choice function F for A .

$\langle 1 \rangle 3$. Define $h : \mathbb{N} \rightarrow \{X \in \mathcal{P}A \mid X \text{ is finite}\}$ by

$$h(0) = \emptyset$$

$$h(n+1) = h(n) \cup \{F(A - \{h(m) \mid m < n\})\}$$

$\langle 1 \rangle 4$. Define $g : \mathbb{N} \rightarrow A$ by $g(n) = F(A - \{h(m) \mid m < n\})$

$\langle 1 \rangle 5$. g is injective.

PROOF: If $m < n$ then $g(m) \neq g(n)$.

\square

Theorem Schema 9.2.11 (König's Lemma). *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem:*

Assume \mathbf{R} is a well founded relation on \mathbf{A} such that, for all $y \in \mathbf{A}$, the class $\{x \in \mathbf{A} \mid x\mathbf{R}y\}$ is a finite set. Let \mathbf{R}^t be the transitive closure of \mathbf{R} . Then, for all $y \in \mathbf{A}$, the class $\{x \in \mathbf{A} \mid x\mathbf{R}^ty\}$ is a finite set.

PROOF:

$\langle 1 \rangle 1$. LET: $y \in \mathbf{A}$

$\langle 1 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall x\mathbf{R}y. \{z \in \mathbf{A} \mid z\mathbf{R}^tx\}$ is a finite set.

$\langle 1 \rangle 3$. $\{x \mid x\mathbf{R}^ty\} = \bigcup_{x\mathbf{R}y} (\{x\} \cup \{z \mid z\mathbf{R}^tx\})$

$\langle 1 \rangle 4$. $\{x \mid x\mathbf{R}^ty\}$ is finite.

PROOF: Corollary 9.2.3.2.

\square

9.3 Countable Sets

Definition 9.3.1 (Countable). A set A is *countable* iff $|A| \leq \aleph_0$.

Theorem 9.3.2. *The union of a countable set of countable sets is countable.*

PROOF: Proposition 8.2.6. \square

9.4 Arithmetic

Definition 9.4.1 (Even). A natural number n is *even* iff there exists $m \in \mathbb{N}$ such that $n = 2m$.

Definition 9.4.2 (Odd). A natural number n is *odd* iff there exists $p \in \mathbb{N}$ such that $n = 2p + 1$.

Proposition 9.4.3. *Every natural number is either even or odd.*

PROOF:

$\langle 1 \rangle 1$. 0 is even.

PROOF: $0 = 2 \times 0$.

$\langle 1 \rangle 2$. For any natural number n , if n is either even or odd then n^+ is either even or odd.

PROOF:

$\langle 2 \rangle 1$. LET: $n \in \mathbb{N}$

$\langle 2 \rangle 2$. If n is even then n^+ is odd.

PROOF: If $n = 2p$ then $n^+ = 2p + 1$.

$\langle 2 \rangle 3$. If n is odd then n^+ is even.

PROOF: If $n = 2p + 1$ then $n^+ = 2(p + 1)$.

□

Proposition 9.4.4. *No natural number is both even and odd.*

PROOF:

$\langle 1 \rangle 1$. 0 is not odd.

PROOF: For any p we have $2p + 1 = (2p)^+ \neq 0$.

$\langle 1 \rangle 2$. For any natural number n , if n is not both even and odd, then n^+ is not both even and odd.

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. If n^+ is even then n is odd.

$\langle 3 \rangle 1$. ASSUME: n^+ is even.

$\langle 3 \rangle 2$. PICK p such that $n^+ = 2p$

$\langle 3 \rangle 3$. $p \neq 0$

PROOF: Since $n^+ \neq 0$.

$\langle 3 \rangle 4$. PICK q such that $p = q^+$

PROOF: Theorem 9.1.5.

$\langle 3 \rangle 5$. $n^+ = 2q + 2$

PROOF: $\langle 3 \rangle 2$, $\langle 3 \rangle 4$.

$\langle 3 \rangle 6$. $n = 2q + 1$

PROOF: Proposition 7.4.7, $\langle 3 \rangle 5$

$\langle 3 \rangle 7$. n is odd.

$\langle 2 \rangle 3$. If n^+ is odd then n is even.

$\langle 3 \rangle 1$. ASSUME: n^+ is odd.

$\langle 3 \rangle 2$. PICK p such that $n^+ = 2p + 1$

$\langle 3 \rangle 3$. $n = 2p$

PROOF: Proposition 7.4.7, $\langle 3 \rangle 2$

$\langle 3 \rangle 4$. n is even.

□

Proposition 9.4.5. *Let m, n, p, q be natural numbers. Assume $m + n = p + q$. Then $m < p$ if and only if $q < n$.*

PROOF:

$\langle 1 \rangle 1$. If $m < p$ then $q < n$.

PROOF: If $m < p$ and $n \leq q$ then $m + n < p + q$.

(1)2. If $q < n$ then $m < p$.

PROOF: Similar.

□

Proposition 9.4.6. *Let m, n, p and q be natural numbers. Assume $n < m$ and $q < p$. Then*

$$mq + np < mp + nq .$$

PROOF:

(1)1. PICK positive natural numbers a and b such that $m = n + a$ and $p = q + b$.

(1)2. $mp + nq > mq + np$

PROOF:

$$\begin{aligned} mp + nq &= (n + a)(q + b) + nq \\ &= 2nq + nb + aq + ab \\ mq + np &= (n + a)q + n(q + b) \\ &= 2nq + aq + nb \\ \therefore mp + nq &= mq + np + ab \\ &> mq + np \end{aligned}$$

□

9.5 Sequences

Definition 9.5.1 (Sequence). Let A be a set. A *finite sequence* in A is a function $a : n \rightarrow A$ for some natural number n ; we write it as $(a(0), a(1), \dots, a(n - 1))$. An *(infinite) sequence* in A is a function $\mathbb{N} \rightarrow A$.

We write A^* for the set of all finite sequences in A .

Proposition 9.5.2. *If A is countable then A^* is countable.*

PROOF: For any n , the set A^n is countable, and A^* is equinumerous with $\bigcup_n A^n$.

□

9.6 Transitive Closure of a Set

Proposition 9.6.1. *For any set A , there exists a unique transitive set C such that:*

- $A \subseteq C$
- For any transitive set X , if $A \subseteq X$ then $C \subseteq X$

PROOF:

(1)1. Define a function $F : \mathbb{N} \rightarrow \mathbf{V}$ by

$$F(0) = A$$

$$F(n + 1) = A \cup \bigcup (F(0) \cup \dots \cup F(n))$$

- ⟨1⟩2. For all $n \in \mathbb{N}$ and $a \in F(n)$ we have $a \subseteq F(n+1)$
 PROOF: $a \in F(0) \cup \dots \cup F(n)$ so $a \subseteq \bigcup(F(0) \cup \dots \cup F(n)) \subseteq F(n+1)$.
 ⟨1⟩3. LET: $C = \bigcup_{n \in \mathbb{N}} F(n)$
 ⟨1⟩4. C is transitive.
 ⟨2⟩1. LET: $x \in y \in C$
 ⟨2⟩2. PICK $n \in \mathbb{N}$ such that $y \in F(n)$
 ⟨2⟩3. $y \subseteq F(n+1)$
 PROOF: ⟨1⟩2
 ⟨2⟩4. $x \in F(n+1)$
 ⟨2⟩5. $x \in C$
 ⟨1⟩5. $A \subseteq C$
 PROOF: Since $F(0) = A$.
 ⟨1⟩6. For any transitive set X , if $A \subseteq X$ then $C \subseteq X$
 ⟨2⟩1. LET: X be a transitive set
 ⟨2⟩2. ASSUME: $A \subseteq X$
 ⟨2⟩3. For all $n \in \mathbb{N}$ we have $F(n) \subseteq X$.
 ⟨3⟩1. $F(0) \subseteq X$
 PROOF: ⟨2⟩2
 ⟨3⟩2. For all $n \in \mathbb{N}$, if $F(n) \subseteq X$, then $F(n+1) \subseteq X$.
 ⟨4⟩1. LET: $n \in \mathbb{N}$
 ⟨4⟩2. ASSUME: $\forall m < n. F(m) \subseteq X$
 ⟨4⟩3. $F(0) \cup \dots \cup F(n) \subseteq X$
 ⟨4⟩4. $\bigcup(F(0) \cup \dots \cup F(n)) \subseteq X$
 PROOF: Since X is transitive.
 ⟨4⟩5. $F(n+1) \subseteq X$
 ⟨2⟩4. $C \subseteq X$
 ⟨1⟩7. Let D be a transitive set such that $A \subseteq D$ and, for any transitive set X ,
 if $A \subseteq X$ then $D \subseteq X$. Then $D = C$.
 PROOF: We have $C \subseteq D$ and $D \subseteq C$.
 □

Proposition 9.6.2 (ZF). *For any class \mathbf{A} , the relation $\mathbf{E} = \{(x, y) \in \mathbf{A}^2 \mid x \in y\}$ is well founded.*

PROOF:

- ⟨1⟩1. Every nonempty set has an \mathbf{E} -minimal element.
 PROOF: Axiom of Regularity.
 ⟨1⟩2. For every set x , there exists a set u such that $x \subseteq u$ and, for all w, y , if
 $y \in u$ and $w \mathbf{E} y$ then $w \in u$.
 PROOF: Take u to be the transitive closure of x .
 □

9.7 The Veblen Fixed Point Theorem

Theorem Schema 9.7.1 (Veblen Fixed Point Theorem). *For any class \mathbf{T} , the following is a theorem:*

Assume \mathbf{T} is a normal ordinal operation. For every ordinal β , there exists $\gamma \geq \beta$ such that $\mathbf{T}(\gamma) = \gamma$.

PROOF:

$\langle 1 \rangle 1$. LET: β be an ordinal.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $\beta < \mathbf{T}(\beta)$

PROOF: We have $\beta \leq \mathbf{T}(\beta)$ by Proposition 7.6.5, and if $\beta = \mathbf{T}(\beta)$ we take $\gamma := \beta$.

$\langle 1 \rangle 3$. Define $f : \mathbb{N} \rightarrow \mathbf{On}$ by recursion thus:

$$f(0) = \beta$$

$$f(n^+) = \mathbf{T}(f(n))$$

$\langle 1 \rangle 4$. LET: $\gamma = \sup_{n \in \mathbb{N}} f(n)$

$\langle 1 \rangle 5$. $\beta \leq \gamma$

PROOF: Since $\beta = f(0)$.

$\langle 1 \rangle 6$. $\mathbf{T}(\gamma) = \gamma$

$\langle 2 \rangle 1$. $\mathbf{T}(\gamma) \leq \gamma$

PROOF:

$$\mathbf{T}(\gamma) = \sup_{n \in \mathbb{N}} \mathbf{T}(f(n)) \quad (\text{Theorem 7.6.7})$$

$$= \sup_{n \in \mathbb{N}} f(n^+) \quad (\langle 1 \rangle 3)$$

$$\leq \sup_{n \in \mathbb{N}} f(n)$$

$$= \gamma$$

$\langle 2 \rangle 2$. $\gamma \leq \mathbf{T}(\gamma)$

PROOF: Proposition 7.6.5.

□

Definition 9.7.2 (Derived Operation). Let \mathbf{T} be a normal ordinal operation. The *derived* operation $\mathbf{T}' : \mathbf{On} \rightarrow \mathbf{V}$ is the unique order isomorphism between \mathbf{On} and the fixed points of \mathbf{T} .

Proposition Schema 9.7.3. For any class \mathbf{T} , the following is a theorem:

If \mathbf{T} is a normal ordinal operation, then the derived operation is normal.

PROOF:

$\langle 1 \rangle 1$. For any set S of fixed points of \mathbf{T} , we have $\bigcup S$ is a fixed point of \mathbf{T}

$\langle 2 \rangle 1$. LET: S be a set of fixed points of \mathbf{T} .

$\langle 2 \rangle 2$. $\mathbf{T}(\sup S) = \sup S$

PROOF:

$$\mathbf{T}(\sup S) = \sup_{\alpha \in S} \mathbf{T}(\alpha) \quad (\text{Theorem 7.6.7})$$

$$= \sup_{\alpha \in S} \alpha \quad (\langle 2 \rangle 1)$$

$$= \sup S$$

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Proposition 7.6.8.

□

9.8 Cantor Normal Form

Theorem 9.8.1. *For any ordinal α , there exist a unique sequence of nonzero natural numbers (n_1, \dots, n_k) and sequence of ordinals $(\gamma_1, \dots, \gamma_k)$ such that*

$$\gamma_k < \gamma_{k-1} < \dots < \gamma_1$$

and

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \dots + \omega^{\gamma_k} n_k .$$

PROOF:

$\langle 1 \rangle 1$. For any ordinal α , there exist a sequence of nonzero natural numbers (n_1, \dots, n_k) and sequence of ordinals $(\gamma_1, \dots, \gamma_k)$ such that

$$\gamma_k < \gamma_{k-1} < \dots < \gamma_1$$

and

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \dots + \omega^{\gamma_k} n_k .$$

$\langle 2 \rangle 1$. LET: α be an ordinal

$\langle 2 \rangle 2$. ASSUME: as an induction hypothesis that, for all $\beta < \alpha$, the theorem holds.

$\langle 2 \rangle 3$. ASSUME: w.l.o.g. $\alpha \neq 0$

$\langle 2 \rangle 4$. LET: γ_1, n_1, ρ_1 be the unique ordinals such that $0 \neq n_1 < \omega$, $\rho_1 < \omega^{\gamma_1}$, and $\alpha = \omega^{\gamma_1} n_1 + \rho_1$

$\langle 2 \rangle 5$. LET: $(\gamma_2, \dots, \gamma_k)$ and (n_2, \dots, n_k) be sequences such that $\gamma_k < \gamma_{k-1} < \dots < \gamma_2$ and $\rho_1 = \omega^{\gamma_2} n_2 + \dots + \omega^{\gamma_k} n_k$

$\langle 2 \rangle 6$. $\gamma_2 < \gamma_1$

PROOF: Since $\omega^{\gamma_2} \leq \rho_1 < \omega^{\gamma_1}$

$\langle 1 \rangle 2$. If

$$\begin{aligned} \gamma_k &< \gamma_{k-1} < \dots < \gamma_1 \\ \gamma'_k &< \gamma'_{k-1} < \dots < \gamma'_1 \end{aligned}$$

and

$$\omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \dots + \omega^{\gamma_k} n_k = \omega^{\gamma'_1} n'_1 + \omega^{\gamma'_2} n'_2 + \dots + \omega^{\gamma'_k} n'_k$$

then $\gamma_i = \gamma'_i$ for all i and $n_i = n'_i$ for all i

PROOF: Prove by induction on i using the Logarithm Theorem.

□

Definition 9.8.2 (Cantor Normal Form). For any ordinal α , the *Cantor normal form* of α is the expression $\alpha = \omega^{\gamma_1} n_1 + \dots + \omega^{\gamma_k} n_k$ such that n_1, \dots, n_k are nonzero natural numbers and $\gamma_k < \gamma_{k-1} < \dots < \gamma_1$.

Chapter 10

The Cumulative Hierarchy

Definition 10.0.1 (Cumulative Hierarchy (ZF)). Define the function $V : \mathbf{On} \rightarrow \mathbf{V}$ by transfinite recursion thus:

$$V_\alpha = \bigcup_{\beta < \alpha} \mathcal{P}V_\beta$$

Proposition 10.0.2 (ZF). *For all $\alpha \in \mathbf{On}$, V_α is a transitive set.*

PROOF:

$\langle 1 \rangle 1$. LET: $\alpha \in \mathbf{On}$

$\langle 1 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha. V_\beta$ is a transitive set.

$\langle 1 \rangle 3$. For all $\beta < \alpha$, $\mathcal{P}V_\beta$ is a transitive set.

PROOF: Proposition 1.6.4.

$\langle 1 \rangle 4$. V_α is a transitive set.

PROOF: Proposition 1.6.3.

□

Proposition 10.0.3 (ZF). *For any ordinals α and β , if $\beta < \alpha$ then $V_\beta \subseteq V_\alpha$.*

PROOF: Since $V_\beta \in \mathcal{P}V_\beta \subseteq V_\alpha$ and V_α is a transitive set. □

Theorem 10.0.4 (ZF).

1. $V_0 = \emptyset$

2. $\forall \alpha \in \mathbf{On}. V_{\alpha+} = \mathcal{P}V_\alpha$

3. For any limit ordinal λ , $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$.

PROOF:

$\langle 1 \rangle 1$. $V_0 = \emptyset$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. $\forall \alpha \in \mathbf{On}. V_{\alpha+} = \mathcal{P}V_\alpha$

PROOF:

- ⟨2⟩1. LET: $\alpha \in \mathbf{On}$
 ⟨2⟩2. For all $\beta < \alpha$ we have $\mathcal{P}V_\beta \subseteq \mathcal{P}V_\alpha$
 PROOF: Propositions 1.5.8 and 10.0.3.
 ⟨2⟩3. $V_{\alpha^+} = \mathcal{P}V_\alpha$

$$\begin{aligned}
 V_{\alpha^+} &= \bigcup_{\beta < \alpha^+} \mathcal{P}V_\beta \\
 &= \bigcup_{\beta < \alpha} \mathcal{P}V_\beta \cup \mathcal{P}V_\alpha \\
 &\quad \mathcal{P}V_\alpha
 \end{aligned}$$

□

- ⟨1⟩3. For any limit ordinal λ , $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$

PROOF:

- ⟨2⟩1. $V_\lambda \subseteq \bigcup_{\alpha < \lambda} V_\alpha$

PROOF:

$$\begin{aligned}
 V_\lambda &= \bigcup_{\alpha < \lambda} \mathcal{P}V_\alpha \\
 &= \bigcup_{\alpha < \lambda} V_{\alpha^+} & (\langle 1 \rangle 2) \\
 &\subseteq \bigcup_{\alpha < \lambda} V_\alpha
 \end{aligned}$$

- ⟨2⟩2. $\bigcup_{\alpha < \lambda} V_\alpha \subseteq V_\lambda$

PROOF: Proposition 10.0.3.

□

Proposition 10.0.5 (ZF). *For every set A , there exists an ordinal α such that $A \in V_\alpha$.*

PROOF:

- ⟨1⟩1. Let us say a set A is *grounded* iff there exists an ordinal α such that $A \in V_\alpha$.
 ⟨1⟩2. For any set A , if every element of A is grounded, then A is grounded.
 ⟨2⟩1. LET: A be a set.
 ⟨2⟩2. $S = \{\alpha \mid \exists a \in A. \alpha \text{ is the least ordinal such that } a \in V_\alpha\}$
 PROOF: S is a set by an Axiom of Replacement.
 ⟨2⟩3. LET: $\beta = \sup S$
 ⟨2⟩4. $A \subseteq V_\beta$
 ⟨3⟩1. LET: $a \in A$
 ⟨3⟩2. LET: α be the least ordinal such that $a \in V_\alpha$
 ⟨3⟩3. $\alpha \in S$
 ⟨3⟩4. $\alpha \leq \beta$
 ⟨3⟩5. $a \in V_\beta$
 ⟨2⟩5. $A \in V_{\beta^+}$
 ⟨1⟩3. ASSUME: for a contradiction there exists an ungrounded set.
 ⟨1⟩4. PICK a transitive set B that has an ungrounded member.
 PROOF: Pick a transitive set c , and take B to be the transitive closure of $\{c\}$.

- ⟨1⟩5. LET: $A = \{x \in B \mid x \text{ is ungrounded}\}$
 ⟨1⟩6. PICK $m \in A$ such that $m \cap A = \emptyset$
 PROOF: Axiom of Regularity.
 ⟨1⟩7. Every member of m is grounded.
 ⟨2⟩1. ASSUME: for a contradiction $x \in m$ is ungrounded.
 ⟨2⟩2. $x \in B$
 PROOF: Since B is transitive (⟨1⟩4).
 ⟨2⟩3. $x \in A$
 PROOF: ⟨1⟩5
 ⟨2⟩4. Q.E.D.
 PROOF: This contradicts ⟨1⟩6.
 ⟨1⟩8. m is grounded.
 PROOF: ⟨1⟩2
 ⟨1⟩9. Q.E.D.
 PROOF: This contradicts ⟨1⟩6.
 □

Definition 10.0.6 (Rank (ZF)). The *rank* of a set A is the least ordinal α such that $A \in V_{\alpha^+}$.

Proposition 10.0.7 (ZF). For any set A we have

$$\text{rank } A = \bigcup_{a \in A} (\text{rank } a)^+$$

PROOF:

- ⟨1⟩1. LET: $\alpha = \bigcup_{a \in A} (\text{rank } a)^+$
 ⟨1⟩2. $A \subseteq V_\alpha$
 ⟨2⟩1. LET: $a \in A$
 ⟨2⟩2. $a \in V_{(\text{rank } a)^+}$
 ⟨2⟩3. $a \in V_\alpha$
 ⟨1⟩3. $A \in V_{\alpha^+}$
 ⟨1⟩4. If $A \subseteq V_\beta$ then $\alpha \leq \beta$
 ⟨2⟩1. ASSUME: $A \subseteq V_\beta$
 ⟨2⟩2. For all $a \in A$ we have $(\text{rank } a)^+ \leq \beta$
 PROOF: Since $a \in V_\beta$.
 ⟨2⟩3. $\alpha \leq \beta$
 □

Corollary 10.0.7.1 (ZF). For any sets a and b , if $a \in b$ then $\text{rank } a < \text{rank } b$.

Proposition 10.0.8 (ZF). For any ordinal number α we have $\text{rank } \alpha = \alpha$.

PROOF:

- ⟨1⟩1. LET: α be an ordinal.
 ⟨1⟩2. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha. \text{rank } \beta = \beta$
 ⟨1⟩3. $\text{rank } \alpha = \bigcup_{\beta < \alpha} \beta^+$

PROOF:

$$\begin{aligned}\text{rank } \alpha &= \bigcup_{\beta < \alpha} (\text{rank } \beta)^+ \\ &= \bigcup_{\beta < \alpha} \beta^+\end{aligned}$$

$$\langle 1 \rangle 4. \bigcup_{\beta < \alpha} \beta^+ \leq \alpha$$

PROOF: Since for all $\beta < \alpha$ we have $\beta^+ \leq \alpha$.

$$\langle 1 \rangle 5. \alpha \leq \bigcup_{\beta < \alpha} \beta^+$$

$$\langle 2 \rangle 1. \text{ LET: } \gamma = \bigcup_{\beta < \alpha} \beta^+$$

$$\langle 2 \rangle 2. \text{ ASSUME: for a contradiction } \gamma < \alpha$$

$$\langle 2 \rangle 3. \gamma^+ \leq \bigcup_{\beta < \alpha} \beta^+ = \gamma$$

$$\langle 2 \rangle 4. \text{ Q.E.D.}$$

PROOF: This is a contradiction.

□

Definition 10.0.9 (Hereditarily Finite (ZF)). A set is *hereditarily finite* iff it is in V_ω .

Proposition 10.0.10 (ZF). V_ω is the smallest set X such that $\emptyset \in X$ and every finite subset of X is an element of X .

Chapter 11

Models of Set Theory

Definition 11.0.1 (Relativization). Let σ be a sentence in the language of set theory and \mathbf{M} a class. The *relativization* of σ to \mathbf{M} is the sentence $\sigma^{\mathbf{M}}$ formed by replacing every quantifier $\forall x$ with $\forall x \in \mathbf{M}$, and $\exists x$ with $\exists x \in \mathbf{M}$.

We write ' \mathbf{M} is a model of σ ' for the sentence $\sigma^{\mathbf{M}}$.

Theorem Schema 11.0.2. *For any class \mathbf{M} , the following is a theorem:*

If \mathbf{M} is a transitive class, then \mathbf{M} is a model of the Axiom of Extensionality.

PROOF:

$\langle 1 \rangle 1$. ASSUME: \mathbf{M} is a transitive class.

PROVE: $\forall x, y \in \mathbf{M} (\forall z \in \mathbf{M} (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$

$\langle 1 \rangle 2$. LET: $x, y \in \mathbf{M}$

$\langle 1 \rangle 3$. ASSUME: $\forall z \in \mathbf{M} (z \in x \Leftrightarrow z \in y)$

$\langle 1 \rangle 4$. $\forall z (z \in x \Leftrightarrow z \in y)$

PROOF: Since $z \in x \Rightarrow z \in \mathbf{M}$ and $z \in y \Rightarrow z \in \mathbf{M}$ by $\langle 1 \rangle 1$.

$\langle 1 \rangle 5$. $x = y$

□

Theorem 11.0.3. *If α is a non-zero ordinal then V_α is a model of the statement: The empty class is a set.*

PROOF:

$\langle 1 \rangle 1$. LET: $\alpha \neq 0$

PROVE: $\exists x \in V_\alpha. \forall y \in V_\alpha. y \notin x$

$\langle 1 \rangle 2$. $\emptyset \in V_\alpha$

$\langle 1 \rangle 3$. $\forall y \in V_\alpha. y \notin \emptyset$

□

Theorem 11.0.4. *For any limit ordinal λ , we have V_λ is a model of the statement: for any sets a and b , the class $\{a, b\}$ is a set.*

PROOF:

$\langle 1 \rangle 1$. LET: λ be a limit ordinal.

PROVE: $\forall a, b \in V_\lambda. \exists c \in V_\lambda. \forall x \in V_\lambda (x \in c \Leftrightarrow x = a \vee x = b)$
 (1)2. LET: $a, b \in V_\lambda$
 (1)3. PICK $\alpha, \beta < \lambda$ such that $a \in V_\alpha$ and $b \in V_\beta$
 (1)4. ASSUME: w.l.o.g. $\alpha \leq \beta$
 (1)5. $a, b \in V_\beta$
 (1)6. $\{a, b\} \in V_{\beta+1}$
 (1)7. $\{a, b\} \in V_\lambda$
 (1)8. $\forall x \in V_\lambda (x \in \{a, b\} \Leftrightarrow x = a \vee x = b)$
 \square

Theorem 11.0.5. *For any ordinal α , we have V_α is a model of the Union Axiom.*

PROOF:

(1)1. LET: α be an ordinal.
 PROVE: $\forall a \in V_\alpha. \exists b \in V_\alpha. \forall x \in V_\alpha (x \in b \Leftrightarrow \exists y \in V_\alpha (x \in y \wedge y \in a))$
 (1)2. LET: $a \in V_\alpha$
 (1)3. PICK $\beta < \alpha$ such that $a \subseteq V_\beta$
 (1)4. $\bigcup a \subseteq V_\beta$
 PROOF: V_β is a transitive set.
 (1)5. $\bigcup a \in V_\alpha$
 (1)6. $\forall x \in V_\alpha (x \in \bigcup a \Leftrightarrow \exists y \in V_\alpha (x \in y \wedge y \in a))$
 PROOF: V_α is a transitive set.
 \square

Theorem 11.0.6. *For any limit ordinal λ , we have V_λ is a model of the Power Set Axiom.*

PROOF:

(1)1. LET: λ be a limit ordinal.
 PROVE: $\forall a \in V_\lambda. \exists b \in V_\lambda. \forall x \in V_\lambda (x \in b \Leftrightarrow \forall y \in V_\lambda (y \in x \Rightarrow y \in a))$
 (1)2. LET: $a \in V_\lambda$
 (1)3. PICK $\alpha < \lambda$ such that $a \in V_\alpha$
 (1)4. $\mathcal{P}a \in V_{\alpha+1}$
 (1)5. $\mathcal{P}a \in V_\lambda$
 (1)6. $\forall x \in V_\lambda (x \in \mathcal{P}a \Leftrightarrow \forall y \in V_\lambda (y \in x \Rightarrow y \in a))$
 \square

Theorem Schema 11.0.7. *For any property $P[x, y_1, \dots, y_n]$, the following is a theorem:*

For any ordinal α , the set V_α is a model of the statement: for any sets a_1, \dots, a_n, B , the class $\{x \in B \mid P[x, a_1, \dots, a_n]\}$ is a set.

PROOF:

(1)1. LET: α be an ordinal.
 (1)2. LET: $a_1, \dots, a_n, B \in V_\alpha$
 (1)3. LET: $C = \{x \in B \mid P[x, a_1, \dots, a_n]^{V_\alpha}\}$
 (1)4. $C \in V_\alpha$

$\langle 1 \rangle 5. \forall x \in V_\alpha (x \in C \Leftrightarrow x \in B \wedge P[x, a_1, \dots, a_n]^{V_\alpha})$

□

Theorem 11.0.8. *For any ordinal $\alpha > \omega$, we have: V_α is a model of the Axiom of Infinity.*

PROOF:

$\langle 1 \rangle 1.$ LET: $\alpha > \omega$

$\langle 1 \rangle 2.$ $\mathbb{N} \in V_\alpha$

$\langle 1 \rangle 3.$ $\exists e \in V_\alpha (e \in \mathbb{N} \wedge \forall x \in V_\alpha. x \notin e)$

$\langle 1 \rangle 4.$ $\forall x \in V_\alpha (x \in \mathbb{N} \Rightarrow \exists y \in V_\alpha \forall z \in V_\alpha (z \in y \Leftrightarrow z \in x \vee z = x))$

□

Theorem 11.0.9. *For any ordinal α , we have V_α is a model of the Axiom of Choice.*

PROOF:

$\langle 1 \rangle 1.$ LET: α be an ordinal.

$\langle 1 \rangle 2.$ LET: $A \in V_\alpha$

$\langle 1 \rangle 3.$ ASSUME: $\forall x \in V_\alpha (x \in A \Rightarrow \exists y \in V_\alpha. y \in A)$

$\langle 1 \rangle 4.$ ASSUME: $\forall x, y, z \in V_\alpha (x \in A \wedge y \in A \wedge z \in x \wedge z \in y \Rightarrow x = y)$

$\langle 1 \rangle 5.$ A is a set of pairwise disjoint nonempty sets.

$\langle 1 \rangle 6.$ PICK c such that, for all $x \in A$, $x \cap c = \emptyset$

$\langle 1 \rangle 7.$ $c \cap \bigcup A \in V_\alpha$

$\langle 1 \rangle 8.$ $\forall x \in V_\alpha (x \in A \Rightarrow \exists y \in V_\alpha \forall z \in V_\alpha (z \in x \wedge z \in c \cap \bigcup A \Leftrightarrow z = y))$

□

Theorem 11.0.10. *For any ordinal α , we have V_α is a model of the Axiom of Regularity.*

PROOF:

$\langle 1 \rangle 1.$ LET: α be an ordinal.

$\langle 1 \rangle 2.$ LET: $A \in V_\alpha$

$\langle 1 \rangle 3.$ ASSUME: $\exists x \in V_\alpha. x \in A$

$\langle 1 \rangle 4.$ PICK $m \in A$ of least rank.

$\langle 1 \rangle 5.$ $m \in V_\alpha$

$\langle 1 \rangle 6.$ $\neg \exists x \in V_\alpha (x \in m \wedge x \in A)$

□

Theorem Schema 11.0.11. *For any axiom α of Zermelo set theory, the following is a theorem:*

For any limit ordinal $\lambda > \omega$, we have V_λ is a model of α .

PROOF: Theorems 11.0.2, 11.0.3, 11.0.4, 11.0.5, 11.0.6, 11.0.7, 11.0.8, 11.0.9, 11.0.10. □

Corollary Schema 11.0.11.1. *for any axiom α of Zermelo set theory, the following is a theorem:*

V_{ω_2} is a model of α .

Lemma 11.0.12. *There exists a well-ordered structure in V_{ω^2} whose ordinal is not in V_{ω^2} .*

PROOF: Take the well-ordered set $\mathbb{N} \times \{0, 1\}$ whose ordinal is ω^2 . \square

Corollary Schema 11.0.12.1. *There exists an instance α of the Axiom Schema of Replacement such that the following is a theorem:
 V_{ω^2} is not a model of α .*

Chapter 12

Infinite Cardinals

12.1 Arithmetic of Infinite Cardinals

Proposition 12.1.1. *For any infinite cardinal κ we have $\kappa\kappa = \kappa$.*

PROOF:

- $\langle 1 \rangle 1$. PICK a set B with $|B| = \kappa$
- $\langle 1 \rangle 2$. LET: $\mathcal{H} = \{f \mid f = \emptyset \vee \exists A \subseteq B. (A \text{ is infinite} \wedge f : A \times A \approx A)\}$
- $\langle 1 \rangle 3$. For any chain $\mathcal{C} \subseteq \mathcal{H}$ we have $\bigcup \mathcal{C} \in \mathcal{H}$
 - $\langle 2 \rangle 1$. LET: $\mathcal{C} \subseteq \mathcal{H}$ be a chain.
 - $\langle 2 \rangle 2$. ASSUME: w.l.o.g. \mathcal{C} has a nonempty element.
 - $\langle 2 \rangle 3$. $\bigcup \mathcal{C}$ is a function.
 - $\langle 3 \rangle 1$. ASSUME: $(x, y), (x, z) \in \bigcup \mathcal{C}$
 - $\langle 3 \rangle 2$. PICK $f, g \in \mathcal{C}$ such that $f(x) = y$ and $g(x) = z$
 - $\langle 3 \rangle 3$. ASSUME: w.l.o.g. $f \subseteq g$
 - $\langle 3 \rangle 4$. $y = z$
 - $\langle 2 \rangle 4$. $\bigcup \mathcal{C}$ is injective.
- PROOF: Similar.
- $\langle 2 \rangle 5$. LET: $A = \text{ran } \bigcup \mathcal{C}$
- $\langle 2 \rangle 6$. A is infinite.
 - $\langle 3 \rangle 1$. PICK a nonzero $f \in \mathcal{C}$
 - $\langle 3 \rangle 2$. LET: A' be the infinite subset of B such that $f : A'^2 \approx A'$
 - $\langle 3 \rangle 3$. $A' \subseteq A$
- $\langle 2 \rangle 7$. $\text{dom } \bigcup \mathcal{C} = A^2$
 - $\langle 3 \rangle 1$. LET: $x, y \in A$
 - $\langle 3 \rangle 2$. PICK $f, g \in \mathcal{C}$ such that $x \in \text{ran } f$ and $y \in \text{ran } g$
 - $\langle 3 \rangle 3$. ASSUME: w.l.o.g. $f \subseteq g$
 - $\langle 3 \rangle 4$. LET: A' be the infinite subset of B such that $g : A'^2 \approx A'$
 - $\langle 3 \rangle 5$. $x, y \in A'$
 - $\langle 3 \rangle 6$. $(x, y) \in \text{dom } g$
 - $\langle 3 \rangle 7$. $(x, y) \in \text{dom } \bigcup \mathcal{C}$
- $\langle 2 \rangle 8$. $\bigcup \mathcal{C} \in \mathcal{H}$

- ⟨1⟩4. PICK a maximal $f_0 \in \mathcal{H}$
- ⟨1⟩5. $f_0 \neq \emptyset$
 - ⟨2⟩1. PICK a countably infinite subset A of B .
PROOF: Proposition 9.2.10.
 - ⟨2⟩2. PICK a bijection $f : A^2 \approx A$
PROOF: Proposition 9.2.9.
 - ⟨2⟩3. $\emptyset \subseteq f \in \mathcal{H}$
 - ⟨2⟩4. \emptyset is not maximal in \mathcal{H}
- ⟨1⟩6. LET: A_0 be the infinite subset of B such that $f_0 : A_0^2 \approx A_0$
- ⟨1⟩7. LET: $\lambda = |A_0|$
- ⟨1⟩8. λ is infinite.
- ⟨1⟩9. $\lambda^2 = \lambda$
- ⟨1⟩10. $\lambda = \kappa$
 - ⟨2⟩1. ASSUME: for a contradiction $\lambda < \kappa$
 - ⟨2⟩2. $\lambda \leq |B - A_0|$
 - ⟨2⟩3. PICK a subset $D \subseteq B - A_0$ with $|D| = \lambda$
 - ⟨2⟩4. $(A_0 \cup D)^2 = A_0^2 \cup (A_0 \times D) \cup (D \times A_0) \cup D^2$
 - ⟨2⟩5. LET: $C = (A_0 \times D) \cup (D \times A_0) \cup D^2$
 - ⟨2⟩6. $|C| = \lambda$
PROOF:

$$|(A_0 \times D) \cup (D \times A_0) \cup D^2| = \lambda^2 + \lambda^2 + \lambda^2$$

$$= \lambda + \lambda + \lambda \quad (\langle 1 \rangle 9)$$

$$= 3\lambda$$

$$\leq \lambda \cdot \lambda$$

$$= \lambda \quad (\langle 1 \rangle 9)$$
 - ⟨2⟩7. PICK a bijection $g : C \approx D$
 - ⟨2⟩8. $f_0 \cup g : (A_0 \cup D)^2 \approx A_0 \cup D$
 - ⟨2⟩9. Q.E.D.

PROOF: This contradicts the maximality of f_0 .

□

Theorem 12.1.2 (Absorption Law of Cardinal Arithmetic). *Let κ and λ be nonzero cardinal numbers such that at least one is infinite. Then*

$$\kappa + \lambda = \kappa\lambda = \max(\kappa, \lambda)$$

PROOF:

- ⟨1⟩1. ASSUME: w.l.o.g. $\lambda \leq \kappa$
- ⟨1⟩2. $\kappa + \lambda = \kappa\lambda = \kappa$

PROOF:

$$\begin{aligned}
 \kappa &\leq \kappa + \lambda \\
 &\leq \kappa + \kappa \\
 &= 2\kappa \\
 &\leq \kappa\lambda \\
 &\leq \kappa\kappa \\
 &= \kappa
 \end{aligned}$$

(Proposition 12.1.1)

□

12.2 Alephs

Definition 12.2.1 (Aleph). Let \aleph be the unique order isomorphism between \mathbf{On} and the class of infinite cardinals.

Proposition 12.2.2. *The operation \aleph is normal.*

PROOF: Proposition 7.6.8 and Lemma 8.2.7. □

Definition 12.2.3 (Continuum Hypothesis). The *continuum hypothesis* is the statement that $\aleph_1 = 2^{\aleph_0}$.

Definition 12.2.4 (Generalised Continuum Hypothesis). The *generalised continuum hypothesis* is the statement that, for all α , $\aleph_{\alpha+} = 2^{\aleph_\alpha}$.

12.3 Beths

Definition 12.3.1 (Beth). Define the operation $\beth : \mathbf{On} \rightarrow \mathbf{Card}$ by transfinite recursion as follows:

$$\begin{aligned}
 \beth_0 &:= \aleph_0 \\
 \beth_{\alpha+} &:= 2^{\beth_\alpha} \\
 \beth_\lambda &:= \bigcup_{\alpha < \lambda} \beth_\alpha \quad (\lambda \text{ a limit ordinal})
 \end{aligned}$$

Proposition 12.3.2. *\beth is a normal operation.*

PROOF: It is continuous by definition, and $\beth_\alpha < \beth_{\alpha+}$ by Cantor's Theorem. □

Proposition 12.3.3. *The continuum hypothesis is equivalent to the statement $\beth_1 = \aleph_1$.*

The generalised continuum hypothesis is equivalent to the statement $\beth = \aleph$.

PROOF: Immediate from definitions. □

Lemma 12.3.4. *For any ordinal number α , we have $|V_{\omega+\alpha}| = \beth_\alpha$.*

PROOF:

(1)1. $|V_\omega| = \beth_0$

PROOF: Since V_ω is the union of \aleph_0 finite sets of increasing size.

(1)2. For any ordinal α , if $|V_{\omega+\alpha}| = \beth_\alpha$ then $|V_{\omega+\alpha+1}| = \beth_{\alpha+1}$

PROOF: Since $V_{\omega+\alpha+1} = \mathcal{P}V_{\omega+\alpha}$.

(1)3. For any limit ordinal λ , if $\forall \alpha < \lambda. |V_{\omega+\alpha}| = \beth_\alpha$ then $|V_{\omega+\lambda}| = \beth_\lambda$.

PROOF:

$$\begin{aligned} |V_{\omega+\lambda}| &= \left| \bigcup_{\alpha < \lambda} V_{\omega+\alpha} \right| \\ &= \sup_{\alpha < \lambda} |V_{\omega+\alpha}| \\ &= \sup_{\alpha < \lambda} \beth_\alpha \\ &= \beth_\lambda \end{aligned}$$

□

12.4 Cofinality

Definition 12.4.1 (Cofinal). Let λ be a limit ordinal and S a set of ordinals smaller than λ . Then S is *cofinal* in λ if and only if $\lambda = \sup S$.

Definition 12.4.2 (Cofinality). For any ordinal α , define the *cofinality* of α , $\text{cf } \alpha$, as follows:

- $\text{cf } 0 = 0$
- For any ordinal α , $\text{cf } \alpha^+ = 1$
- For any limit ordinal λ , $\text{cf } \lambda$ is the smallest cardinal such that there exists a set S of ordinals cofinal in λ with $|S| = \text{cf } \lambda$.

Definition 12.4.3 (Regular). A cardinal κ is *regular* iff $\text{cf } \kappa = \kappa$; otherwise it is *singular*.

Proposition 12.4.4. \aleph_0 is regular.

PROOF: \aleph_0 is not the supremum of $< \aleph_0$ smaller ordinals, because a finite union of finite ordinals is finite. □

Proposition 12.4.5. For every ordinal α , $\aleph_{\alpha+1}$ is regular.

PROOF: If S is a set of ordinals with $|S| < \aleph_{\alpha+1}$ and $\forall \beta \in S. \beta < \aleph_{\alpha+1}$, then we have $|S| \leq \aleph_\alpha$ and $\forall \beta \in S. \beta \leq \aleph_\alpha$, hence

$$\begin{aligned} \left| \bigcup S \right| &\leq \aleph_\alpha^2 && \text{(Proposition 8.2.6)} \\ &= \aleph_\alpha && \text{(Proposition 12.1.1)} \end{aligned}$$

Proposition Schema 12.4.6. For any class \mathbf{T} , the following is a theorem.

Assume $\mathbf{T} : \mathbf{On} \rightarrow \mathbf{On}$ is a normal operation. For any limit ordinal λ we have $\text{cf } \mathbf{T}(\lambda) = \text{cf } \lambda$.

PROOF:

- $\langle 1 \rangle 1.$ $\text{cf } \mathbf{T}(\lambda) \leq \text{cf } \lambda$
 - $\langle 2 \rangle 1.$ PICK a set S of ordinals $< \lambda$ with $|S| = \text{cf } \lambda$ and $\sup S = \lambda$
 - $\langle 2 \rangle 2.$ $\mathbf{T}(\lambda) = \sup_{\alpha \in S} \mathbf{T}(\alpha)$
 PROOF: Theorem 7.6.7.
- $\langle 1 \rangle 2.$ $\text{cf } \lambda \leq \text{cf } \mathbf{T}(\lambda)$
 - $\langle 2 \rangle 1.$ PICK a set A of ordinals $< \mathbf{T}(\lambda)$ such that $|A| = \text{cf } \mathbf{T}(\lambda)$ and $\sup A = \mathbf{T}(\lambda)$
 - $\langle 2 \rangle 2.$ LET: $B = \{\gamma < \lambda \mid \exists \alpha \in A. |\alpha| = \mathbf{T}(\gamma)\}$
 - $\langle 2 \rangle 3.$ $|B| \leq |A| = \text{cf } \mathbf{T}(\lambda)$
 PROVE: $\sup B = \lambda$
 - $\langle 2 \rangle 4.$ $\forall \alpha \in A. |\alpha| \leq \mathbf{T}(\sup B)$
 - $\langle 2 \rangle 5.$ $\forall \alpha \in A. \alpha < \mathbf{T}(\sup B + 1)$
 - $\langle 2 \rangle 6.$ $\aleph_\lambda = \sup A \leq \mathbf{T}(\sup B + 1)$
 - $\langle 2 \rangle 7.$ $\lambda \leq \sup B + 1$
 - $\langle 2 \rangle 8.$ $\lambda \leq \sup B$
 PROOF: λ is a limit ordinal.
 - $\langle 2 \rangle 9.$ $\sup B = \lambda$

□

Corollary 12.4.6.1. \aleph_ω is singular.

PROOF: $\text{cf } \aleph_\omega = \text{cf } \aleph_0 = \aleph_0$. □

Corollary 12.4.6.2. The operation cf is not strictly monotone or continuous.

PROOF: $\text{cf } \aleph_\omega < \text{cf } \aleph_1$ □

Definition 12.4.7 (Weakly Inaccessible). A cardinal is *weakly inaccessible* iff it is \aleph_λ for some limit ordinal λ and regular.

Lemma 12.4.8. Let λ be a limit ordinal. Then there exists a strictly increasing $\text{cf } \lambda$ -sequence that converges to λ .

PROOF:

- $\langle 1 \rangle 1.$ PICK a set S of ordinals $< \lambda$ with $|S| = \text{cf } \lambda$ and $\sup S = \lambda$
- $\langle 1 \rangle 2.$ PICK a bijection $a : \text{cf } \lambda \approx S$
- $\langle 1 \rangle 3.$ PICK a strictly increasing subsequence $(b_\delta)_{\delta < \beta}$ of a that converges to λ .
 PROOF: Lemma 7.8.5.
- $\langle 1 \rangle 4.$ $\beta = \text{cf } \lambda$
 PROOF: By minimality of $\text{cf } \lambda$.

□

Corollary 12.4.8.1. Let λ be a limit ordinal. Then $\text{cf } \lambda$ is the least ordinal such that there exists a strictly increasing $\text{cf } \lambda$ -sequence that converges to λ .

Proposition 12.4.9. For any ordinal λ , $\text{cf } \lambda$ is a regular cardinal.

PROOF:

- (1)1. LET: λ be an ordinal.
 (1)2. ASSUME: w.l.o.g. λ is a limit ordinal.
 (1)3. PICK a strictly increasing sequence $(a_\alpha)_{\alpha < \text{cf } \lambda}$ that converges to λ .
 (1)4. LET: S be a set of ordinals $< \text{cf } \lambda$ such that $|S| = \text{cf } \lambda$ and $\sup S = \text{cf } \lambda$.
 (1)5. LET: $a(S) = \{a_\alpha \mid \alpha \in S\}$
 (1)6. $a(S)$ is cofinal in λ .
 (2)1. LET: $\beta < \lambda$
 (2)2. PICK $\gamma < \text{cf } \lambda$ such that $\beta < a_\gamma$
 (2)3. PICK $\delta \in S$ such that $\gamma < \delta$
 (2)4. $a_\delta \in a(S)$ and $\beta < a_\gamma < a_\delta$
 (1)7. $\text{cf } \lambda \leq \text{cf } \text{cf } \lambda$
 PROOF: Since $a(S)$ is a set of ordinals $< \lambda$ with $|a(S)| = \text{cf } \text{cf } \lambda$ and $\sup a(S) = \lambda$.
 (1)8. $\text{cf } \text{cf } \lambda = \text{cf } \lambda$
 □

Theorem 12.4.10. *Let λ be an infinite cardinal. Then $\text{cf } \lambda$ is the least cardinal such that λ can be partitioned into $\text{cf } \lambda$ sets, each of cardinality $< \lambda$.*

PROOF:

- (1)1. λ can be partitioned into $\text{cf } \lambda$ sets, each of cardinality $< \lambda$
 (2)1. PICK a strictly increasing sequence of ordinals $(a_\alpha)_{\alpha < \text{cf } \lambda}$ that converges to λ
 (2)2. $\{\{\beta \mid a_\alpha \leq \beta < a_{\alpha+1}\} \mid \alpha < \text{cf } \lambda\}$ is a partition of λ into $\text{cf } \lambda$ sets, each of cardinality $< \lambda$
 (1)2. If λ can be partitioned into κ sets, each of cardinality $< \lambda$, then $\text{cf } \lambda \leq \kappa$.
 (2)1. LET: \mathcal{A} be a partition of λ into sets of cardinality $< \lambda$
 (2)2. LET: $\kappa = |P|$
 (2)3. PICK a bijection $A : \kappa \approx P$
 (2)4. $\lambda = \bigcup_{\xi < \kappa} A(\xi)$
 (2)5. For all $\xi < \kappa$ we have $|A(\xi)| < \lambda$
 (2)6. LET: $\mu = \sup_{\xi < \kappa} |A(\xi)|$
 (2)7. $\mu \leq \lambda$
 (2)8. For all $\xi < \kappa$ we have $|A(\xi)| \leq \mu$
 (2)9. $\lambda \leq \mu\kappa$
 PROOF: Proposition 8.2.6.
 (2)10. ASSUME: w.l.o.g. $\kappa < \lambda$
 PROOF: If $\lambda \leq \kappa$ then $\text{cf } \lambda \leq \kappa$ since $\text{cf } \lambda \leq \lambda$.
 (2)11. $\lambda = \mu$
 PROOF:

$$\lambda \leq \mu\kappa \quad (\langle 2 \rangle 9)$$

$$\leq \lambda\lambda \quad (\langle 2 \rangle 7, \langle 2 \rangle 10)$$

$$= \lambda \quad (\text{Proposition 12.1.1})$$

- (2)12. $\{|A(\xi)| \mid \xi < \kappa\}$ is a set of $\leq \kappa$ ordinals all $< \lambda$ whose supremum is λ
 (2)13. $\text{cf } \lambda \leq \kappa$

□

Theorem 12.4.11 (König). *For any infinite cardinal κ we have $\kappa < \text{cf } 2^\kappa$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $\text{cf } 2^\kappa \leq \kappa$

$\langle 1 \rangle 2$. LET: $S = 2^\kappa$

$\langle 1 \rangle 3$. PICK a partition $\{A_\xi \mid \xi < \kappa\}$ of S^κ with $\forall \xi < \kappa. |A_\xi| < 2^\kappa$.

PROOF: Theorem 12.4.10.

$\langle 1 \rangle 4$. $\forall \xi < \kappa. \{g(\xi) \mid g \in A_\xi\} \subsetneq S$

PROOF: We do not have equality because $|\{g(\xi) \mid g \in A_\xi\}| \leq |A_\xi| < 2^\kappa$.

$\langle 1 \rangle 5$. For all $\xi < \kappa$, choose $s_\xi \in S - \{g(\xi) \mid g \in A_\xi\}$

$\langle 1 \rangle 6$. $s \in S^\kappa$

$\langle 1 \rangle 7$. For all $\xi < \kappa$ we have $s \notin A_\xi$

PROOF: Since for all $\xi < \kappa$ and $g \in A_\xi$ we have $s_\xi(\xi) \neq g(\xi)$.

$\langle 1 \rangle 8$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 3$.

□

Corollary 12.4.11.1.

$$2^{\aleph_0} \neq \aleph_\omega$$

Proposition 12.4.12. *For any ordinal α , we have $\text{cf } \alpha$ is the least cardinal such that α is the strict supremum of $\text{cf } \alpha$ smaller ordinals.*

PROOF:

$\langle 1 \rangle 1$. CASE: $\alpha = 0$

PROOF: Since $0 = \text{ssup } \emptyset$.

$\langle 1 \rangle 2$. CASE: $\alpha = \beta^+$

PROOF: Since $\beta^+ = \text{ssup}\{\beta\}$.

$\langle 1 \rangle 3$. CASE: α is a limit ordinal.

$\langle 2 \rangle 1$. There exists a set S of ordinals $< \alpha$ such that $|S| = \text{cf } \alpha$ and $\alpha = \text{ssup } S$.

$\langle 3 \rangle 1$. PICK a set S of ordinals $< \alpha$ such that $|S| = \text{cf } \alpha$ and $\sup S = \alpha$

PROVE: $\alpha = \text{ssup } S$

$\langle 3 \rangle 2$. $\forall \beta \in S. \beta < \alpha$

$\langle 3 \rangle 3$. For any ordinal γ , if $\forall \beta \in S. \beta < \gamma$ then $\alpha \leq \gamma$

$\langle 2 \rangle 2$. If T is a set of ordinals $< \alpha$ such that $\alpha = \text{ssup } T$, then $\text{cf } \alpha \leq |T|$.

$\langle 3 \rangle 1$. LET: T be a set of ordinals $< \alpha$ such that $\alpha = \text{ssup } T$

$\langle 3 \rangle 2$. $\alpha = \sup T$

$\langle 4 \rangle 1$. For all $\beta \in T$ we have $\beta \leq \alpha$

$\langle 4 \rangle 2$. LET: μ be any upper bound for T

PROVE: $\alpha \leq \mu$

$\langle 4 \rangle 3$. $\alpha \leq \mu + 1$

PROOF: Since $\forall \beta \in T. \beta < \mu + 1$.

$\langle 4 \rangle 4$. $\alpha \neq \mu + 1$

PROOF: Since α is a limit ordinal.

$\langle 4 \rangle 5$. $\alpha < \mu + 1$

$\langle 4 \rangle 6$. $\alpha \leq \mu$

$\langle 3 \rangle 3$. $\text{cf } \alpha \leq |T|$

□

12.5 Inaccessible Cardinals

Definition 12.5.1 (Inaccessible Cardinal). A cardinal number κ is *inaccessible* iff:

- $\kappa > \aleph_0$
- $\forall \lambda < \kappa. 2^\lambda < \kappa$ (cardinal exponentiation)
- κ is regular.

Any inaccessible cardinal is weakly inaccessible.

PROOF:

$\langle 1 \rangle 1$. LET: $\kappa = \aleph_\lambda$ be weakly inaccessible.

PROVE: λ is a limit ordinal.

$\langle 1 \rangle 2$. $\lambda \neq 0$

$\langle 1 \rangle 3$. ASSUME: for a contradiction $\lambda = \beta + 1$

$\langle 1 \rangle 4$. $\aleph_\beta < \kappa$

$\langle 1 \rangle 5$. $2^{\aleph_\beta} < \kappa$

$\langle 1 \rangle 6$. $\aleph_{\beta+1} < \kappa$

PROOF: Since $\aleph_{\beta+1} \leq 2^{\aleph_\beta}$.

$\langle 1 \rangle 7$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 3$.

□

Proposition 12.5.2. *If the Generalized Continuum Hypothesis is true, then every weakly inaccessible cardinal is inaccessible.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: The Generalized Continuum Hypothesis.

$\langle 1 \rangle 2$. LET: $\kappa = \aleph_\lambda$ be weakly inaccessible.

$\langle 1 \rangle 3$. $\kappa > \aleph_0$

PROOF: $\lambda > 0$ because λ is a limit ordinal.

$\langle 1 \rangle 4$. For all $\mu < \kappa$ we have $2^\mu < \kappa$

$\langle 2 \rangle 1$. LET: $\mu < \kappa$

$\langle 2 \rangle 2$. LET: $\mu = \aleph_\alpha$

$\langle 2 \rangle 3$. $\alpha < \lambda$

$\langle 2 \rangle 4$. $\alpha + 1 < \lambda$

PROOF: λ is a limit ordinal.

$\langle 2 \rangle 5$. $2^\mu < \kappa$

PROOF:

$$2^\mu = 2^{\aleph_\alpha} \quad (\langle 2 \rangle 2)$$

$$= 2^{\beth_\alpha} \quad (\langle 1 \rangle 1)$$

$$= \beth_{\alpha+1}$$

$$= \aleph_{\alpha+1} \quad (\langle 1 \rangle 1)$$

$$< \aleph_\lambda \quad (\langle 2 \rangle 4)$$

$$= \kappa \quad (\langle 1 \rangle 2)$$

$\langle 1 \rangle 5.$ κ is regular.

PROOF: $\langle 1 \rangle 2$

□

Lemma 12.5.3. *Let κ be an inaccessible cardinal. For every ordinal $\alpha < \kappa$ we have $\beth_\alpha < \kappa$.*

PROOF:

$\langle 1 \rangle 1.$ $\beth_0 < \kappa$

PROOF: Since $\kappa > \aleph_0$.

$\langle 1 \rangle 2.$ For any ordinal α , if $\beth_\alpha < \kappa$ then $\beth_{\alpha+1} < \kappa$.

PROOF: Since $\beth_{\alpha+1} = 2^{\beth_\alpha} < \kappa$.

$\langle 1 \rangle 3.$ For any limit ordinal λ , if $\forall \alpha < \lambda. \beth_\alpha < \kappa$ and $\lambda < \kappa$ then $\beth_\lambda < \kappa$.

PROOF: By regularity of κ , since \beth_λ is the union of $|\lambda|$ cardinals all $< \kappa$.

□

Lemma 12.5.4. *Let κ be an inaccessible cardinal. For all $A \in V_\kappa$ we have $|A| < \kappa$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $A \in V_\kappa$

$\langle 1 \rangle 2.$ PICK $\alpha < \kappa$ such that $A \in V_\alpha$

$\langle 1 \rangle 3.$ $A \subseteq V_\alpha$

$\langle 1 \rangle 4.$ $|A| \leq |V_\alpha| \leq \beth_\alpha < \kappa$

□

Theorem Schema 12.5.5. *For every axiom α of ZFC, the following is a theorem:*

For any inaccessible cardinal κ , we have V_κ is a model of α .

PROOF: For every axiom except the Replacement Axioms, we have Corollary 11.0.11.1.

For an Axiom of Replacement using the property $P[x, y, z_1, \dots, z_n]$, we reason as follows:

$\langle 1 \rangle 1.$ LET: κ be an inaccessible cardinal

PROVE:

$$\begin{aligned} & \forall a_1, \dots, a_n, B \in V_\kappa (\forall x \in B. \forall y, y' \in V_\kappa \\ & (P[x, y, a_1, \dots, a_n]^{V_\kappa} \wedge P[x, y', a_1, \dots, a_n]^{V_\kappa} \Rightarrow y = y') \Rightarrow \\ & \exists C \in V_\kappa \forall y \in V_\kappa (y \in C \Leftrightarrow \exists x \in B. P[x, y, a_1, \dots, a_n]^{V_\kappa})) \end{aligned}$$

$\langle 1 \rangle 2.$ LET: $a_1, \dots, a_n, B \in V_\kappa$

$\langle 1 \rangle 3.$ ASSUME: for all $x \in B$, there exists at most one $y \in V_\kappa$ such that

$$P[x, y, a_1, \dots, a_n]^{V_\kappa}.$$

$\langle 1 \rangle 4.$ LET: $F = \{(x, y) \in B \times V_\kappa \mid P[x, y, a_1, \dots, a_n]^{V_\kappa}\}$

$\langle 1 \rangle 5.$ LET: $C = \text{ran } F$

PROVE: $C \in V_\kappa$

$\langle 1 \rangle 6.$ LET: $S = \{\text{rank } F(x) \mid x \in \text{dom } F\}$

$\langle 1 \rangle 7.$ $|S| < \kappa$

PROOF: Since $|S| \leq |\text{dom } F| \leq |B| < \kappa$.

$\langle 1 \rangle 8. \forall \alpha \in S. \alpha < \kappa$

PROOF: Since $F(x) \in V_\kappa$ for all $x \in \text{dom } F$.

$\langle 1 \rangle 9. \sup S < \kappa$

PROOF: Since κ is regular.

$\langle 1 \rangle 10. \text{rank } C \leq \sup S + 1$

$\langle 1 \rangle 11. \text{rank } C < \kappa$

$\langle 1 \rangle 12. C \in V_\kappa$

□

Chapter 13

Group Theory

13.1 Groups

Definition 13.1.1 (Group). A *group* G consists of a set G and a function $\cdot : G^2 \rightarrow G$ such that:

1. \cdot is associative
2. There exists $e \in G$ such that $\forall x \in G. xe = x$ and $\forall x \in G. \exists y \in G. xy = e$.

Proposition 13.1.2. *The inverse of an element in a group is unique.*

PROOF:

$\langle 1 \rangle$ 1. ASSUME: b and b' are inverses of a .

$\langle 1 \rangle$ 2. $b = b'$

PROOF:

$$\begin{aligned} b &= be \\ &= bab' \\ &= eb' \\ &= b' \end{aligned}$$

□

Definition 13.1.3. We write x^{-1} for the inverse of x .

Proposition 13.1.4. *In any group, if $ab = ac$ then $b = c$.*

PROOF:

$$\begin{aligned} b &= eb \\ &= a^{-1}ab \\ &= a^{-1}ac \\ &= ec \\ &= c \end{aligned}$$

□

Definition 13.1.5. Let \mathbf{Grp} be the category of groups and group homomorphisms.

Definition 13.1.6. We identify any group G with the category G with one object whose morphisms are the elements of G , with composition given by the multiplication in G .

13.2 Abelian Groups

Definition 13.2.1 (Abelian group). An *Abelian group* is a group whose multiplication is commutative.

We may say we are writing an Abelian group *additively*, meaning we write $a + b$ for ab , 0 for e and $-a$ for a^{-1} . In this case we write $a - b$ for ab^{-1} .

Chapter 14

Ring Theory

14.1 Rings

Definition 14.1.1 (Commutative Ring). A *commutative ring* consists of a set R and two binary operations $+$, \cdot on R such that:

- D is an Abelian group under $+$. Let us write 0 for its identity element.
- \cdot is commutative and associative, and distributes over $+$.
- \cdot has an identity element 1 that is different from 0 .

Proposition 14.1.2. *In any commutative ring, $0x = 0$.*

PROOF:

$$\begin{aligned}(0 + 0)x &= 0x \\ \therefore 0x + 0x &= 0x + 0 \\ \therefore 0x &= 0 && \text{(Proposition 13.1.4)} \square\end{aligned}$$

Proposition 14.1.3. *In any commutative ring, $(-a)b = -(ab)$.*

PROOF:

$$\begin{aligned}ab + (-a)b &= (a + (-a))b \\ &= 0b \\ &= 0 && \text{(Proposition 14.1.2)} \square\end{aligned}$$

14.2 Ordered Rings

Definition 14.2.1 (Ordered Commutative Ring). An *ordered commutative ring* consists of a commutative ring R with a linear order $<$ on R such that:

- for all $x, y, z \in R$, we have $x < y$ if and only if $x + z < y + z$.

- for all $x, y, z \in R$, if $0 < z$ then we have $x < y$ if and only if $xz < yz$.

Proposition 14.2.2. *In any ordered commutative ring, $0 < 1$.*

PROOF: If $1 < 0$ then we have $0 < -1$ and so $0 < (-1)(-1) = 1$, which is a contradiction. \square

Proposition 14.2.3. *The ordering on an ordered commutative ring is dense; that is, if $x < y$ then there exists z such that $x < z < y$.*

PROOF: Take $z = (x + y)/2$. \square

14.3 Integral Domains

Definition 14.3.1 (Integral Domain). An *integral domain* is a commutative ring such that, for all $a, b \in D$, if $ab = 0$ then $a = 0$ or $b = 0$.

Proposition 14.3.2. *In any integral domain, if $ab = ac$ and $a \neq 0$ then $b = c$.*

PROOF: We have $a(b - c) = 0$ and $a \neq 0$ so $b - c = 0$ hence $b = c$. \square

Definition 14.3.3 (Ordered Integral Domain). An *ordered integral domain* is an ordered commutative ring that is an integral domain.

Chapter 15

Field Theory

15.1 Fields

Definition 15.1.1 (Field). A *field* F is a commutative ring such that $0 \neq 1$ and, for all $x \in F$, if $x \neq 0$ then there exists $y \in F$ such that $xy = 1$.

Proposition 15.1.2. *Every field is an integral domain.*

PROOF: If $ab = 0$ and $a \neq 0$ then $b = a^{-1}ab = 0$. \square

Proposition 15.1.3. *In any field F , we have $F - \{0\}$ is an Abelian group under multiplication.*

PROOF: Immediate from the definition. \square

Definition 15.1.4 (Field of Fractions). Let D be an integral domain. The *field of fractions* of D is the quotient set $F = (D \times (D - \{0\})) / \sim$ where

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

under

$$\begin{aligned} [(a, b)] + [(c, d)] &= [(ad + bc, bd)] \\ [(a, b)][(c, d)] &= [(ac, bd)] \end{aligned}$$

We prove this is a field.

PROOF:

$\langle 1 \rangle 1.$ \sim is an equivalence relation on $D \times (D - \{0\})$.

PROOF:

$\langle 2 \rangle 1.$ \sim is reflexive.

PROOF: We always have $ab = ba$.

$\langle 2 \rangle 2.$ \sim is symmetric.

PROOF: If $ad = bc$ then $cb = da$.

$\langle 2 \rangle 3$. \sim is transitive.

$\langle 3 \rangle 1$. ASSUME: $(a, b) \sim (c, d) \sim (e, f)$

$\langle 3 \rangle 2$. $ad = bc$ and $cf = de$

$\langle 3 \rangle 3$. $adf = bde$

PROOF: $adf = bcf = bde$

$\langle 3 \rangle 4$. $af = be$

PROOF: Proposition 14.3.2.

□

$\langle 1 \rangle 2$. Addition is well-defined.

PROOF:

$\langle 2 \rangle 1$. If $b \neq 0$ and $d \neq 0$ then $bd \neq 0$.

PROOF: Since D is an integral domain.

$\langle 2 \rangle 2$. If $ab' = a'b$ and $cd' = c'd$ then $(ad + bc)b'd' = (a'd' + b'c')bd$.

PROOF:

$$\begin{aligned} (ad + bc)b'd' &= ab'dd' + bb'cd' \\ &= a'bdd' + bb'c'd \\ &= (a'd' + b'c')bd \end{aligned}$$

□

$\langle 1 \rangle 3$. Multiplication is well-defined.

PROOF:

$\langle 2 \rangle 1$. If $b \neq 0$ and $d \neq 0$ then $bd \neq 0$.

PROOF: Since D is an integral domain.

$\langle 2 \rangle 2$. If $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$ then $[(ac, bd)] = [(a'c', b'd')]$.

PROOF: If $ab' = a'b$ and $cd' = c'd$ then $acb'd' = a'c'bd$.

□

$\langle 1 \rangle 4$. Addition is commutative.

PROOF: $[(a, b)] + [(c, d)] = [(ad + bc, bd)] = [(cb + da, db)] = [(c, d)] + [(a, b)]$ □

$\langle 1 \rangle 5$. Addition is associative.

PROOF:

$$\begin{aligned} [(a, b)] + ([[(c, d)] + [(e, f)]] &= [(a, b)] + [(cf + de, df)] \\ &= [(adf + bcf + bde, bdf)] \\ &= [(ad + bc, bd)] + [(e, f)] \\ &= ([[(a, b)] + [(c, d)]] + [(e, f)]) \quad \square \end{aligned}$$

$\langle 1 \rangle 6$. For any $x \in F$ we have $x + [(0, 1)] = x$

PROOF: $[(a, b)] + [(0, 1)] = [(a \cdot 1 + b \cdot 0, b \cdot 1)] = [(a, b)]$ □

$\langle 1 \rangle 7$. For any $x \in F$, there exists $y \in F$ such that $x + y = [(0, 1)]$.

PROOF: $[(a, b)] + [(-a, b)] = [(ab - ab, b^2)] = [(0, b^2)] = [(0, 1)]$ □

$\langle 1 \rangle 8$. Multiplication is commutative.

PROOF: $[(a, b)][(c, d)] = [(c, d)][(a, b)] = [(ac, bd)]$. □

$\langle 1 \rangle 9$. Multiplication is associative.

PROOF: $[(a, b)]([[(c, d)][(e, f)]]) = ([[(a, b)][(c, d)]])([e, f]) = [(ace, bdf)]$. □

$\langle 1 \rangle 10$. For any $x \in F$ we have $x[(1, 1)] = x$

PROOF: $[(a, b)][(1, 1)] = [(a, b)]$ □

$\langle 1 \rangle 11$. For any non-zero $x \in F$, there exists $y \in F$ such that $xy = [(1, 1)]$.

PROOF:

- $\langle 2 \rangle 1$. LET: $[(a, b)] \in \mathbb{Q}$
- $\langle 2 \rangle 2$. ASSUME: $[(a, b)] \neq [(0, 1)]$
- $\langle 2 \rangle 3$. $a \neq 0$
- $\langle 2 \rangle 4$. $[(a, b)][(b, a)] = [(1, 1)]$

□

□

Definition 15.1.5. For any field F , let $N(F)$ be the intersection of all the subsets $S \subseteq F$ such that $1 \in S$ and $\forall x \in S. x + 1 \in S$.

Definition 15.1.6 (Characteristic Zero). A field F has *characteristic 0* iff $0 \notin N(F)$.

Proposition 15.1.7. In a field F with characteristic 0, the function $n : \mathbb{N} \rightarrow N(F)$ defined by

$$\begin{aligned} n(0) &= 1 \\ n(x + 1) &= n(x) + 1 \end{aligned}$$

is a bijection.

PROOF:

- $\langle 1 \rangle 1$. n is injective.
- $\langle 2 \rangle 1$. ASSUME: for a contradiction $n(i) = n(j)$ with $i \neq j$
- $\langle 2 \rangle 2$. ASSUME: w.l.o.g. $i < j$
- $\langle 2 \rangle 3$. $n(j - i) = 0$
- $\langle 2 \rangle 4$. Q.E.D.

PROOF: This contradicts the fact that F has characteristic 0.

- $\langle 1 \rangle 2$. n is surjective.

PROOF: Since $\text{ran } n$ is a subset of F that includes 1 and is closed under $+1$.

□

Definition 15.1.8. In any field F , let

$$I(F) = N(F) \cup \{0\} \cup \{-x \mid x \in N(F)\}$$

Definition 15.1.9. In any field F , let

$$Q(F) = \{x/y \mid x, y \in I(F), y \neq 0\}$$

Proposition 15.1.10. $Q(F)$ is the smallest subfield of F .

PROOF: $Q(F)$ is closed under $+$ and \cdot , and any subset of F closed under $+$ and \cdot that contains 0 and 1 must include $Q(F)$. □

Theorem 15.1.11. Let F and G be fields of characteristic 0. Then there exists a unique field isomorphism between $Q(F)$ and $Q(G)$.

PROOF:

- (1)1. LET: $\phi : N(F) \rightarrow N(G)$ be the unique function such that $\phi(1) = 1$ and $\forall x \in N(F). \phi(x+1) = \phi(x) + 1$.
- (1)2. ϕ is a bijection.
 PROOF: Similar to Proposition 15.1.7.
- (1)3. $\forall x, y \in N(F). \phi(x+y) = \phi(x) + \phi(y)$
 PROOF: Induction on y .
- (1)4. $\forall x, y \in N(F). \phi(xy) = \phi(x)\phi(y)$
 PROOF: Induction on y .
- (1)5. Extend ϕ to a bijection $I(F) \cong I(G)$ such that $\forall x, y \in I(F). \phi(x+y) = \phi(x) + \phi(y)$ and $\forall x, y \in I(F). \phi(xy) = \phi(x)\phi(y)$
- (2)1. Define $\phi(0) = 0$ and $\phi(-x) = -\phi(x)$ for $x \in N(F)$
- (3)1. $0 \notin N(F)$
- (3)2. For all $x \in N(F)$ we have $-x \notin N(F)$
 PROOF: Then we would have $x + -x = 0 \in N(F)$.
- (3)3. For all $x \in N(F)$ we have $-x \neq 0$
- (2)2. For all $x, y \in I(F)$ we have $\phi(x+y) = \phi(x) + \phi(y)$
 PROOF: Case analysis on x and y .
- (2)3. For all $x, y \in I(F)$ we have $\phi(xy) = \phi(x)\phi(y)$
 PROOF: Case analysis on x and y .
- (1)6. Extend ϕ to a bijection $Q(F) \cong Q(G)$ such that $\forall x, y \in Q(F). \phi(x+y) = \phi(x) + \phi(y)$ and $\forall x, y \in Q(F). \phi(xy) = \phi(x)\phi(y)$
- (2)1. Define $\phi(x/y) = \phi(x)/\phi(y)$
- (1)7. ϕ is unique.
- (2)1. LET: θ satisfy the theorem.
- (2)2. For all $x \in N(F)$ we have $\theta(x) = \phi(x)$
- (2)3. For all $x \in I(F)$ we have $\theta(x) = \phi(x)$
- (2)4. For all $x \in Q(F)$ we have $\theta(x) = \phi(x)$

□

15.2 Ordered Fields

Definition 15.2.1 (Ordered Field). An *ordered field* is an ordered commutative ring that is a field.

Proposition 15.2.2. Every ordered field F has characteristic 0.

PROOF: We have $0 < n$ for all $n \in N(F)$. □

Proposition 15.2.3. Let F be a field of characteristic 0. Then there exists a unique relation $<$ on $Q(F)$ that makes $Q(F)$ into an ordered field.

PROOF: Easy. □

Corollary 15.2.3.1. Let F and G be ordered fields. Let ϕ be the unique field isomorphism between $Q(F)$ and $Q(G)$. Then ϕ is an ordered field isomorphism.

Definition 15.2.4 (Archimedean). An ordered field F is *Archimedean* iff

$$\forall x \in F. \exists n \in N(F). n > x .$$

Proposition 15.2.5. *Let F be an Archimedean ordered field. Let $x, y \in F$ with $x > 0$. Then there exists $n \in N(F)$ such that $nx > y$.*

PROOF: Pick $n > y/x$. \square

Proposition 15.2.6. *Let F be an Archimedean ordered field. For all $x, y \in F$, if $x < y$, then there exists $r \in Q(F)$ such that $x < r < y$.*

PROOF:

$\langle 1 \rangle 1$. CASE: $x > 0$

$\langle 2 \rangle 1$. PICK $n \in N(F)$ such that $n(y - x) > 1$

PROOF: Proposition 15.2.5.

$\langle 2 \rangle 2$. $ny > 1 + nx$

$\langle 2 \rangle 3$. LET: m be the least element of $N(F)$ such that $m > nx$.

$\langle 2 \rangle 4$. $m - 1 \leq nx$

$\langle 2 \rangle 5$. $nx < m < ny$

$\langle 2 \rangle 6$. $x < m/n < y$

$\langle 1 \rangle 2$. CASE: $x \leq 0$

$\langle 2 \rangle 1$. PICK $k \in N(F)$ such that $k > -x$

$\langle 2 \rangle 2$. $0 < x + k < y + k$

$\langle 2 \rangle 3$. PICK $r \in Q(F)$ such that $x + k < r < y + k$

PROOF: $\langle 1 \rangle 1$

$\langle 2 \rangle 4$. $x < r - k < y$

Definition 15.2.7 (Complete). An ordered field F is *complete* iff every nonempty subset of F bounded above has a least upper bound.

Proposition 15.2.8. *Every complete ordered field is Archimedean.*

PROOF:

$\langle 1 \rangle 1$. LET: F be a complete ordered field.

$\langle 1 \rangle 2$. LET: $x \in F$

$\langle 1 \rangle 3$. ASSUME: for a contradiction there is no member of $N(F)$ greater than x .

$\langle 1 \rangle 4$. x is an upper bound for $N(F)$.

$\langle 1 \rangle 5$. LET: $y = \sup N(F)$

$\langle 1 \rangle 6$. PICK $n \in N(F)$ such that $y - 1 < n$

$\langle 1 \rangle 7$. $y < n + 1$

$\langle 1 \rangle 8$. Q.E.D.

PROOF: This is a contradiction.

\square

Proposition 15.2.9. *Let F be a complete ordered field and $a \in F$ be nonnegative. Then there exists $b \in F$ such that $b^2 = a$.*

PROOF:

$\langle 1 \rangle 1$. LET: $B = \{x \in F \mid 0 \leq x \leq 1 + a\}$

$\langle 1 \rangle 2$. LET: $\phi : B \rightarrow B$ be the function

$$\phi(x) = x + \frac{1}{2(1+a)}(a - x^2) .$$

- ⟨1⟩3. ϕ is strictly monotone.
 ⟨2⟩1. LET: $0 \leq x < y \leq 1 + a$
 ⟨2⟩2. $1 - \frac{x+y}{2(1+a)} > 0$
 ⟨2⟩3. $\phi(y) - \phi(x) = (y - x)(1 - \frac{x+y}{2(1+a)}) > 0$
 ⟨2⟩4. $\phi(x) < \phi(y)$
 ⟨1⟩4. PICK $b \in B$ such that $\phi(b) = b$.
 PROOF: Knaster Fixed-Point Theorem.
 ⟨1⟩5. $b^2 = a$
 \square

Theorem 15.2.10 (Uniqueness of the Complete Ordered Field). *If F and G are complete ordered fields, then there exists a unique bijection $\phi : F \cong G$ such that, for all $x, y \in F$,*

$$\begin{aligned}\phi(x + y) &= \phi(x) + \phi(y) \\ \phi(xy) &= \phi(x)\phi(y)\end{aligned}$$

This bijection also satisfies: for all $x, y \in F$,

$$x < y \Leftrightarrow \phi(x) < \phi(y) .$$

PROOF:

- ⟨1⟩1. PICK a bijection $\phi : Q(F) \cong Q(G)$ such that, for all $x, y \in Q(F)$,
 $\phi(x + y) = \phi(x) + \phi(y)$
 $\phi(xy) = \phi(x)\phi(y)$
 $x < y \Leftrightarrow \phi(x) < \phi(y)$

PROOF: Corollary 15.2.3.1.

- ⟨1⟩2. $Q(F)$ intersects every interval in F .

PROOF: Proposition 15.2.6.

- ⟨1⟩3. $Q(G)$ intersects every interval in G .

PROOF: Proposition 15.2.6.

- ⟨1⟩4. PICK an order isomorphism $\psi : F \cong G$ that extends ϕ .

PROOF: Theorem 6.1.21.

- ⟨1⟩5. $\forall x, y \in F. \psi(x + y) = \psi(x) + \psi(y)$
 ⟨2⟩1. LET: $x, y \in F$
 ⟨2⟩2. $\psi(x) + \psi(y) \not\leq \psi(x + y)$
 ⟨3⟩1. ASSUME: for a contradiction $\psi(x) + \psi(y) < \psi(x + y)$
 ⟨3⟩2. PICK $r' \in Q(G)$ such that $\psi(x) < r' < \psi(x + y) - \psi(y)$
 ⟨3⟩3. PICK $s' \in Q(G)$ such that $\psi(y) < s' < \psi(x + y) - r'$
 ⟨3⟩4. $r' + s' < \psi(x + y)$
 ⟨3⟩5. PICK $r, s \in Q(F)$ such that $\phi(r) = r'$ and $\phi(s) = s'$
 ⟨3⟩6. $\phi(r + s) = r' + s'$
 ⟨3⟩7. $\psi(x) < \psi(r)$
 ⟨3⟩8. $\psi(y) < \psi(s)$
 ⟨3⟩9. $\psi(x + y) > \psi(r + s)$
 ⟨3⟩10. $x < r$

- $\langle 3 \rangle 11. y < s$
- $\langle 3 \rangle 12. x + y > r + s$
- $\langle 3 \rangle 13. \text{Q.E.D.}$

PROOF: This is a contradiction.

- $\langle 2 \rangle 3. \psi(x + y) \not\leq \psi(x) + \psi(y)$

PROOF: Similar.

- $\langle 1 \rangle 6. \forall x, y \in F. \psi(xy) = \psi(x)\psi(y)$

- $\langle 2 \rangle 1. \text{LET: } x, y \in F$

- $\langle 2 \rangle 2. \text{CASE: } x \text{ and } y \text{ are positive.}$

- $\langle 3 \rangle 1. \psi(x)\psi(y) \not\leq \psi(xy)$

- $\langle 4 \rangle 1. \text{ASSUME: for a contradiction } \psi(x)\psi(y) < \psi(xy)$

- $\langle 4 \rangle 2. \text{PICK } r' \in Q(G) \text{ such that } \psi(x) < r' < \psi(xy)/\psi(y)$

- $\langle 4 \rangle 3. \text{PICK } s' \in Q(G) \text{ such that } \psi(y) < s' < \psi(xy)/r'$

- $\langle 4 \rangle 4. r's' < \psi(xy)$

- $\langle 4 \rangle 5. \text{PICK } r, s \in Q(F) \text{ such that } \phi(r) = r' \text{ and } \phi(s) = s'$

- $\langle 4 \rangle 6. \phi(rs) = r's'$

- $\langle 4 \rangle 7. x < r, y < s \text{ and } rs < xy$

- $\langle 4 \rangle 8. \text{Q.E.D.}$

PROOF: This is a contradiction.

- $\langle 3 \rangle 2. \psi(xy) \not\leq \psi(x)\psi(y)$

PROOF: Similar.

- $\langle 2 \rangle 3. \text{CASE: } x \text{ and } y \text{ are not both positive.}$

PROOF: Follows from $\langle 2 \rangle 2$ since $\psi(-x) = -\psi(x)$ by $\langle 1 \rangle 5$.

- $\langle 1 \rangle 7. \text{For any field isomorphism } \theta : F \cong G, \text{ we have } \theta = \psi.$

- $\langle 2 \rangle 1. \theta \upharpoonright Q(F) = \phi$

PROOF: Theorem 15.1.11.

- $\langle 2 \rangle 2. \theta \text{ is strictly monotone.}$

- $\langle 3 \rangle 1. \text{LET: } x, y \in F \text{ with } x < y$

- $\langle 3 \rangle 2. y - x > 0$

- $\langle 3 \rangle 3. \text{PICK } z \in F \text{ such that } z^2 = y - x$

- $\langle 3 \rangle 4. \theta(z)^2 = \theta(y) - \theta(x)$

- $\langle 3 \rangle 5. \theta(y) - \theta(x) > 0$

- $\langle 3 \rangle 6. \theta(x) < \theta(y)$

- $\langle 2 \rangle 3. \theta = \psi$

PROOF: By the uniqueness of ψ .

□

Chapter 16

Number Systems

16.1 The Integers

Definition 16.1.1. The set of *integers* \mathbb{Z} is the quotient set \mathbb{N}^2 / \sim , where $(m, n) \sim (p, q)$ iff $m + q = n + p$.

We prove \sim is an equivalence relation on \mathbb{N}^2 .

PROOF:

$\langle 1 \rangle 1.$ \sim is reflexive.

PROOF: For all $m, n \in \mathbb{N}$ we have $m + n = n + m$.

$\langle 1 \rangle 2.$ \sim is symmetric.

PROOF: If $m + q = n + p$ then $p + n = q + m$.

$\langle 1 \rangle 3.$ \sim is transitive.

$\langle 2 \rangle 1.$ ASSUME: $(m, n) \sim (p, q) \sim (r, s)$

$\langle 2 \rangle 2.$ $m + q = n + p$ and $p + s = q + r$

$\langle 2 \rangle 3.$ $m + q + s = n + q + r$

$\langle 2 \rangle 4.$ $m + s = n + r$

PROOF: By cancellation.

□

Definition 16.1.2 (Addition). Define *addition* $+$ on \mathbb{Z} by $[(m, n)] + [(p, q)] = [(m + p, n + q)]$.

We prove this is well-defined.

PROOF: If $m + n' = n + m'$ and $p + q' = q + p'$ then $m + p + n' + q' = n + q + m' + p'$.

□

Proposition 16.1.3. *Addition on \mathbb{Z} is commutative.*

PROOF: $[(m, n)] + [(p, q)] = [(m + p, n + q)] = [(p + m, q + n)] = [(p, q)] + [(m, n)]$.

□

Proposition 16.1.4. *Addition on \mathbb{Z} is associative.*

PROOF: $[(m, n)] + [(p, q)] + [(r, s)] = [(m + p + r, n + q + s)] = [(m, n)] + [(p, q)] + [(r, s)]$. \square

Proposition 16.1.5. *Given natural numbers m and n , we have $[(m, 0)] = [(n, 0)]$ iff $m = n$.*

PROOF: Immediate from definitions. \square

Definition 16.1.6. We identify any natural number n with the integer $[(n, 0)]$.

Proposition 16.1.7. *Addition on integers agrees with addition on natural numbers.*

PROOF: Since $[(m, 0)] + [(n, 0)] = [(m + n, 0)]$. \square

Proposition 16.1.8. *For all $a \in \mathbb{Z}$ we have $a + 0 = a$.*

PROOF: $[(m, n)] + [(0, 0)] = [(m + 0, n + 0)] = [(m, n)]$. \square

Proposition 16.1.9. *For all $a \in \mathbb{Z}$, there exists $b \in \mathbb{Z}$ such that $a + b = 0$.*

PROOF: $[(m, n)] + [(n, m)] = [(m + n, m + n)] = [(0, 0)]$. \square

Proposition 16.1.10. *The integers form an Abelian group under addition.*

PROOF: Proposition 16.1.3, 16.1.4, 16.1.8, 16.1.9. \square

Definition 16.1.11. Define multiplication \cdot on \mathbb{Z} by: $[(m, n)][(p, q)] = [(mp + nq, mq + np)]$.

We prove this is well defined.

PROOF:

$\langle 1 \rangle 1$. ASSUME: $m + n' = n + m'$ and $p + q' = q + p'$

PROVE: $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$

$\langle 1 \rangle 2$. $mp + n'p = np + m'p$

$\langle 1 \rangle 3$. $nq + m'q = mq + n'q$

$\langle 1 \rangle 4$. $m'p + m'q' = m'q + m'p'$

$\langle 1 \rangle 5$. $n'q + n'p' = n'p + n'q'$

$\langle 1 \rangle 6$. $mp + n'p + nq + m'q + m'p + m'q' + n'q + n'p' = np + m'p + mq + n'q + m'q + m'p' + n'p + n'q'$

$\langle 1 \rangle 7$. $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$

PROOF: By cancellation.

\square

Proposition 16.1.12. *Multiplication on integers agrees with multiplication on natural numbers.*

PROOF: Since $[(m, 0)][(n, 0)] = [(mn + 0, m0 + n0)] = [(mn, 0)]$. \square

Proposition 16.1.13. *Multiplication on \mathbb{Z} is commutative.*

PROOF: $[(m, n)][(p, q)] = [(mp + nq, mq + np)] = [(pm + qn, pn + qm)] = [(p, q)][(m, n)]$. \square

Proposition 16.1.14. *Multiplication on \mathbb{Z} is associative.*

PROOF:

$$\begin{aligned}
 [(m, n)][(p, q)][(r, s)] &= [(m, n)][(pr + qs, ps + qr)] \\
 &= [(mpr + mqs + nps + nqr, mps + mqr + npr + nqs)] \\
 &= [(mp + nq, mq + np)][(r, s)] \\
 &= [(m, n)][(p, q)][(r, s)] \quad \square
 \end{aligned}$$

Proposition 16.1.15. *Multiplication distributes over addition.*

PROOF:

$$\begin{aligned}
 [(m, n)][(p, q)] + [(m, n)][(r, s)] &= [(m, n)][(p + r, q + s)] \\
 &= [(mp + mr + nq + ns, mp + nr + mq + ms)] \\
 [(m, n)][(p, q)] + [(m, n)][(r, s)] &= [(mp + nq, mq + np)] + [(mr + ns, ms + nr)] \\
 &= [(mp + nq + mr + ns, mq + np + ms + nr)] \quad \square
 \end{aligned}$$

Proposition 16.1.16. *For any integer a we have $a1 = a$.*

PROOF: Since $[(m, n)][(1, 0)] = [(m1 + n0, m0 + n1)] = [(m, n)]$. \square

Proposition 16.1.17. *For any integers a and b , if $ab = 0$ then $a = 0$ or $b = 0$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: $[(m, n)][(p, q)] = [(0, 0)]$

$\langle 1 \rangle 2$. $mp + nq = mq + np$

$\langle 1 \rangle 3$. ASSUME: $[(m, n)] \neq [(0, 0)]$

$\langle 1 \rangle 4$. $m \neq n$

PROVE: $p = q$

$\langle 1 \rangle 5$. CASE: $m < n$

$\langle 2 \rangle 1$. $p \not< q$

PROOF: If $p < q$ then $mq + np < mp + nq$ by Proposition 9.4.6.

$\langle 2 \rangle 2$. $q \not< p$

PROOF: If $q < p$ then $mp + nq < mq + np$ by Proposition 9.4.6.

$\langle 2 \rangle 3$. $p = q$

PROOF: By trichotomy.

$\langle 1 \rangle 6$. CASE: $n < m$

PROOF: Similar.

\square

Proposition 16.1.18. *The integers \mathbb{Z} form an integral domain.*

PROOF: Propositions 16.1.13, 16.1.14, 16.1.15, 16.1.16, 16.1.17, 16.1.10. \square

Definition 16.1.19. Define $<$ on \mathbb{Z} by $[(m, n)] < [(p, q)]$ if and only if $m + q < n + p$.

We prove this is well-defined.

PROOF:

$\langle 1 \rangle 1$. ASSUME: $m + n' = n + m'$ and $p + q' = q + p'$.

PROVE: $m + q < n + p$ if and only if $m' + q' < n' + p'$

$\langle 1 \rangle 2$. $m + q < n + p$ if and only if $m' + q' < n' + p'$

PROOF:

$$m + q < n + p \Leftrightarrow m + n' + q < n + n' + p \quad (\text{Corollary 7.7.7.1})$$

$$\Leftrightarrow m' + n + q < n + n' + p$$

$$\Leftrightarrow m' + q < n' + p \quad (\text{Corollary 7.7.7.1})$$

$$\Leftrightarrow m' + q + p' < n' + p + p' \quad (\text{Corollary 7.7.7.1})$$

$$\Leftrightarrow m' + q' + p < n' + p + p'$$

$$\Leftrightarrow m' + q' < n' + p' \quad (\text{Corollary 7.7.7.1}) \square$$

Proposition 16.1.20. *The ordering on the integers agrees with the ordering on the natural numbers.*

PROOF: We have $[(m, 0)] < [(n, 0)]$ iff $m < n$. \square

Proposition 16.1.21. *$<$ is a linear order on \mathbb{Z} .*

PROOF:

$\langle 1 \rangle 1$. $<$ is irreflexive.

PROOF: We never have $m + n < m + n$.

$\langle 1 \rangle 2$. $<$ is transitive.

$\langle 2 \rangle 1$. ASSUME: $[(m, n)] < [(p, q)] < [(r, s)]$

$\langle 2 \rangle 2$. $m + q < n + p$ and $p + s < q + r$

$\langle 2 \rangle 3$. $m + q + s < n + q + r$

PROOF: $m + q + s < n + p + s < n + q + r$

$\langle 2 \rangle 4$. $m + s < n + r$

PROOF: Corollary 7.7.7.1.

$\langle 1 \rangle 3$. $<$ is total.

PROOF: Given natural numbers m, n, p and q , either $m + q < n + p$, or $m + q = n + p$, or $n + p < m + q$.

\square

Definition 16.1.22 (Positive). An integer a is *positive* iff $a > 0$.

Theorem 16.1.23. *For any integers a, b and c , we have $a < b$ if and only if $a + c < b + c$.*

PROOF:

$\langle 1 \rangle 1$. If $a < b$ then $a + c < b + c$.

$\langle 2 \rangle 1$. LET: $a = [(m, n)]$, $b = [(p, q)]$ and $c = [(r, s)]$.

$\langle 2 \rangle 2$. ASSUME: $a < b$

$\langle 2 \rangle 3$. $m + q < n + p$

$\langle 2 \rangle 4$. $m + r + q + s < n + r + p + s$

$\langle 2 \rangle 5$. $[(m + r, n + s)] < [(p + r, q + s)]$

$\langle 2 \rangle 6$. $a + c < b + c$

$\langle 1 \rangle 2$. If $a + c < b + c$ then $a < b$.

PROOF: From $\langle 1 \rangle 1$ and Proposition 6.2.6.

□

Proposition 16.1.24. *Let a , b and c be integers. If $0 < c$, then $a < b$ if and only if $ac < bc$.*

PROOF:

$\langle 1 \rangle 1$. LET: $c = [(r, s)]$

$\langle 1 \rangle 2$. ASSUME: $0 < c$

$\langle 1 \rangle 3$. $s < r$

$\langle 1 \rangle 4$. For all integers a and b , if $a < b$ then $ac < bc$

$\langle 2 \rangle 1$. LET: $a = [(m, n)]$, $b = [(p, q)]$.

$\langle 2 \rangle 2$. ASSUME: $a < b$

$\langle 2 \rangle 3$. $m + q < n + p$

$\langle 2 \rangle 4$. $(m + q)r + (p + n)s < (m + q)s + (p + n)r$

PROOF: Proposition 9.4.6, $\langle 1 \rangle 3$, $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. $mr + ns + ps + qr < ms + nr + pr + qs$

$\langle 2 \rangle 6$. $[(mr + ns, ms + nr)] < [(pr + qs, ps + qr)]$

$\langle 2 \rangle 7$. $ac < bc$

$\langle 1 \rangle 5$. For all integers a and b , if $ac < bc$ then $a < b$

PROOF: From $\langle 1 \rangle 4$ and Proposition 6.2.6.

□

Proposition 16.1.25. *Let a be a positive integer. For any integer b , there exists $k \in \mathbb{N}$ such that $b < ak$.*

PROOF:

$\langle 1 \rangle 1$. CASE: $b \leq 0$

PROOF: Take $k = 1$.

$\langle 1 \rangle 2$. CASE: $b > 0$

PROOF: Take $k = b + 1$.

□

16.2 The Rationals

Definition 16.2.1 (Rational Numbers). The set \mathbb{Q} of *rational numbers* is the field of fractions over the integers.

Proposition 16.2.2. *For any integers a and b , we have $[(a, 1)] = [(b, 1)]$ iff $a = b$.*

PROOF: Immediate from definitions. □

Henceforth we identify any integer a with the rational number $[(a, 1)]$.

Proposition 16.2.3. *Addition on the rationals agrees with addition on the integers.*

PROOF: $[(a, 1)] + [(b, 1)] = [(a \cdot 1 + b \cdot 1, 1 \cdot 1)] = [(a + b, 1)]$. \square

Proposition 16.2.4. *Multiplication on the rationals agrees with multiplication on the integers.*

PROOF: $[(a, 1)][(b, 1)] = [(ab, 1)]$ \square

Definition 16.2.5. Define the ordering $<$ on the rationals by: if b and d are positive, then $[(a, b)] < [(c, d)]$ iff $ad < bc$.

We prove this is well-defined.

PROOF:

$\langle 1 \rangle 1$. For any rational q , there exist integers a, b with b positive such that $q = [(a, b)]$.

PROOF: Since $[(a, b)] = [(-a, -b)]$, and if $b \neq 0$ then one of b and $-b$ is positive.

$\langle 1 \rangle 2$. If b, b', d and d' are positive, $[(a, b)] = [(a', b')]$, and $[(c, d)] = [(c', d')]$, then $ad < bc$ iff $a'd' < b'c'$.

PROOF:

$\langle 2 \rangle 1$. If $ad < bc$ then $a'd' < b'c'$.

$\langle 3 \rangle 1$. ASSUME: $ad < bc$

$\langle 3 \rangle 2$. $ab'd < bb'c$

$\langle 3 \rangle 3$. $a'bd < bb'c$

$\langle 3 \rangle 4$. $a'd < b'c$

$\langle 3 \rangle 5$. $a'dd' < b'cd'$

$\langle 3 \rangle 6$. $a'dd' < b'c'd$

$\langle 3 \rangle 7$. $a'd' < b'c'$

$\langle 2 \rangle 2$. If $a'd' < b'c'$ then $ad < bc$.

PROOF: Similar.

\square

Proposition 16.2.6. *The ordering on the rationals agrees with the ordering on the integers.*

PROOF: We have $[(a, 1)] < [(b, 1)]$ if and only if $a < b$. \square

Proposition 16.2.7. *The relation $<$ is a linear ordering on \mathbb{Q} .*

PROOF:

$\langle 1 \rangle 1$. $<$ is irreflexive.

PROOF: We never have $ab < ab$.

$\langle 1 \rangle 2$. $<$ is transitive.

$\langle 2 \rangle 1$. ASSUME: $[(a, b)] < [(c, d)] < [(e, f)]$ where b, d and f are positive.

$\langle 2 \rangle 2$. $ad < bc$ and $cf < de$

$\langle 2 \rangle 3$. $adf < bde$

PROOF: $adf < bcf < bde$

$\langle 2 \rangle 4$. $af < be$

$\langle 1 \rangle 3$. $<$ is total.

PROOF: For any integers a, b, c, d , we have $ad < bc$ or $ad = bc$ or $bc < ad$.

□

Proposition 16.2.8. *For any rationals r , s and t , we have $r < s$ if and only if $r + t < s + t$.*

PROOF:

⟨1⟩1. LET: a, b, c, d, e, f be integers with b, d and f positive.

⟨1⟩2. $[(a, b)] + [(e, f)] < [(c, d)] + [(e, f)]$ if and only if $[(a, b)] < [(c, d)]$.

PROOF:

$$\begin{aligned}
 [(a, b)] + [(e, f)] < [(c, d)] + [(e, f)] &\Leftrightarrow [(af + be, bf)] < [(cf + de, df)] \\
 &\Leftrightarrow (af + be)df < (cf + de)bf \\
 &\Leftrightarrow afd f + bedf < cfbf + debf \\
 &\Leftrightarrow afd f < cfbf \\
 &\Leftrightarrow ad < bc \\
 &\Leftrightarrow [(a, b)] < [(c, d)]
 \end{aligned}$$

□

Corollary 16.2.8.1. *For any rational r , we have $r < 0$ if and only if $0 < -r$.*

Definition 16.2.9 (Absolute Value). For any rational r , the *absolute value* of r is defined by

$$|r| := \begin{cases} -r & \text{if } 0 < -r \\ r & \text{otherwise} \end{cases}$$

Proposition 16.2.10. *For any rationals r , s and t , if t is positive then $r < s$ iff $rt < st$.*

PROOF:

⟨1⟩1. LET: $r = [(a, b)]$, $s = [(c, d)]$ and $t = [(e, f)]$ where b, d and f are positive.

⟨1⟩2. ASSUME: $0 < t$

⟨1⟩3. $e > 0$

⟨1⟩4. $rt < st$ iff $r < s$

PROOF:

$$\begin{aligned}
 rt < st &\Leftrightarrow [(ae, bf)] < [(ce, df)] \\
 &\Leftrightarrow aedf < cebf \\
 &\Leftrightarrow ad < bc \\
 &\Leftrightarrow r < s
 \end{aligned}$$

□

Corollary 16.2.10.1. *The rationals form an ordered field.*

Proposition 16.2.11. *Let p be a positive rational. For any rational number r , there exists $k \in \mathbb{N}$ such that $r < pk$.*

PROOF:

⟨1⟩1. LET: $p = a/b$ and $r = c/d$ where a, b and d are positive.

⟨1⟩2. PICK $k \in \mathbb{N}$ such that $bc < adk$

PROOF: Proposition 16.1.25.

⟨1⟩3. $r < pk$

□

Proposition 16.2.12. $\mathbb{Q} \approx \mathbb{N}$

PROOF: Arrange the rationals in order $0/1, 1/1, 1/2, 0/2, -1/2, -1/1, -2/1, -2/2, -2/3, -1/3, 0/3, 1/3, 2/3$, etc. then remove all duplicates. □

16.3 The Real Numbers

Definition 16.3.1 (Cauchy Sequence). A *Cauchy sequence* is a sequence (q_n) of rationals such that, for every positive rational ϵ , there exists $k \in \mathbb{N}$ such that $\forall m, n > k. |q_m - q_n| < \epsilon$.

Definition 16.3.2 (Dedekind Cut). A *Dedekind cut* is a set $x \subseteq \mathbb{Q}$ such that:

1. $\emptyset \neq x \neq \mathbb{Q}$
2. x is closed downwards.
3. x has no greatest member.

The set \mathbb{R} of *real numbers* is the set of Dedekind cuts.

Proposition 16.3.3. For any rational q , we have $\{r \in \mathbb{Q} \mid r < q\} \in \mathbb{R}$.

PROOF:

⟨1⟩1. LET: $q \in \mathbb{Q}$

⟨1⟩2. LET: $q \downarrow = \{r \mid r < q\}$

⟨1⟩3. $q \notin q \downarrow$

PROOF: We have $q - 1 \in q \downarrow$.

⟨1⟩4. $q \notin q \downarrow$

PROOF: Since $q \notin q \downarrow$.

⟨1⟩5. $q \downarrow$ is closed downwards.

PROOF: Trivial.

⟨1⟩6. $q \downarrow$ has no greatest element.

PROOF: For all $r \in q \downarrow$ we have $r < (q + r)/2 \in q \downarrow$.

□

Proposition 16.3.4. For rationals q and r , we have $q = r$ if and only if $\{s \in \mathbb{Q} \mid s < q\} = \{s \in \mathbb{Q} \mid s < r\}$.

PROOF:

⟨1⟩1. LET: $q \downarrow = \{s \in \mathbb{Q} \mid s < q\}$

⟨1⟩2. LET: $r \downarrow = \{s \in \mathbb{Q} \mid s < r\}$

⟨1⟩3. If $q = r$ then $q \downarrow = r \downarrow$

PROOF: Trivial.

(1)4. If $q < r$ then $q \downarrow \neq r \downarrow$

PROOF: We have $q \in r \downarrow$ and $q \notin q \downarrow$.

(1)5. If $r < q$ then $q \downarrow \neq r \downarrow$

PROOF: We have $r \in q \downarrow$ and $q \notin q \downarrow$.

□

Henceforth we identify a rational q with the real number $\{r \in \mathbb{Q} \mid r < q\}$.

Definition 16.3.5. Define the ordering $<$ on \mathbb{R} by: $x < y$ iff $x \subsetneq y$.

Proposition 16.3.6. *The ordering on the reals agrees with the ordering on the rationals.*

PROOF:

(1)1. LET: $q, r \in \mathbb{Q}$

(1)2. LET: $q \downarrow = \{s \in \mathbb{Q} \mid s < q\}$.

(1)3. LET: $r \downarrow = \{s \in \mathbb{Q} \mid s < r\}$.

PROVE: $q < r$ iff $q \downarrow \subsetneq r \downarrow$

(1)4. If $q < r$ then $q \downarrow \subsetneq r \downarrow$

(2)1. ASSUME: $q < r$

(2)2. $q \downarrow \subseteq r \downarrow$

PROOF: If $s < q$ then $s < r$.

(2)3. $q \downarrow \neq r \downarrow$

PROOF: Proposition 16.3.4.

(1)5. If $q \downarrow \subsetneq r \downarrow$ then $q < r$

(2)1. ASSUME: $q \downarrow \subsetneq r \downarrow$

(2)2. PICK $s \in r \downarrow$ such that $s \notin q \downarrow$

(2)3. $q \leq s < r$

□

Proposition 16.3.7. *The ordering $<$ is a linear ordering on \mathbb{R} .*

PROOF:

(1)1. $<$ is irreflexive.

PROOF: No set is a proper subset of itself.

(1)2. $<$ is transitive.

PROOF: Since the relationship \subsetneq is transitive on the class of all sets.

(1)3. $<$ is total.

(2)1. LET: x, y be Dedekind cuts.

(2)2. ASSUME: $x \not\subseteq y$

PROVE: $y \subsetneq x$

(2)3. PICK $q \in x$ such that $q \notin y$

(2)4. LET: $r \in y$

PROVE: $r \in x$

(2)5. $q \not\leq r$

PROOF: Since y is closed downwards.

(2)6. $r < q$

(2)7. $r \in x$

PROOF: Since x is closed downwards.

□

Proposition 16.3.8. *Any bounded nonempty subset of \mathbb{R} has a least upper bound.*

PROOF:

⟨1⟩1. LET: A be a bounded nonempty subset of \mathbb{R} .

⟨1⟩2. $\bigcup A$ is a Dedekind cut.

⟨2⟩1. $\bigcup A \neq \emptyset$

⟨3⟩1. PICK $x \in A$

⟨3⟩2. PICK $q \in x$

⟨3⟩3. $q \in \bigcup A$

⟨2⟩2. $\bigcup A \neq \mathbb{Q}$

⟨3⟩1. PICK an upper bound u for A

⟨3⟩2. PICK $q \notin u$

PROVE: $q \notin \bigcup A$

⟨3⟩3. ASSUME: for a contradiction $q \in \bigcup A$

⟨3⟩4. PICK $x \in A$ such that $q \in x$

⟨3⟩5. $x \leq u$

⟨3⟩6. $q \in u$

⟨3⟩7. Q.E.D.

PROOF: This is a contradiction.

⟨2⟩3. $\bigcup A$ is closed downwards.

⟨3⟩1. LET: $q \in \bigcup A$ and $r < q$

⟨3⟩2. PICK $x \in A$ such that $q \in x$

⟨3⟩3. $r \in x$

⟨3⟩4. $r \in \bigcup A$

⟨2⟩4. $\bigcup A$ has no greatest element.

⟨3⟩1. LET: $q \in \bigcup A$

⟨3⟩2. PICK $x \in A$ such that $q \in x$

⟨3⟩3. PICK $r \in x$ such that $q < r$

⟨3⟩4. $r \in \bigcup A$

⟨1⟩3. $\bigcup A$ is an upper bound for A .

PROOF: For all $x \in A$ we have $x \subseteq \bigcup A$.

⟨1⟩4. For any upper bound u for $\bigcup A$ we have $\bigcup A \leq u$.

PROOF: If $\forall x \in A. x \subseteq u$ we have $\bigcup A \subseteq u$.

□

Definition 16.3.9 (Addition). Define *addition* $+$ on the reals by

$$x + y := \{q + r \mid q \in x, r \in y\} .$$

We prove this is well-defined.

PROOF:

⟨1⟩1. LET: $x, y \in \mathbb{R}$

PROVE: $X + y$ is a Dedekind cut.

$\langle 1 \rangle 2. x + y \neq \emptyset$

PROOF: Pick $q \in x$ and $r \in y$; then $q + r \in x + y$.

$\langle 1 \rangle 3. x + y \neq \mathbb{Q}$

$\langle 2 \rangle 1. \text{ PICK } q \notin x \text{ and } r \notin y$

PROVE: $q + r \notin x + y$

$\langle 2 \rangle 2. \text{ ASSUME: for a contradiction } q + r \in x + y$

$\langle 2 \rangle 3. \text{ PICK } q' \in x \text{ and } r' \in y \text{ such that } q + r = q' + r'$

$\langle 2 \rangle 4. q' < q \text{ and } r' < r$

$\langle 2 \rangle 5. q' + r' < q + r$

$\langle 2 \rangle 6. \text{ Q.E.D.}$

PROOF: This is a contradiction.

$\langle 1 \rangle 4. x + y$ is closed downwards.

$\langle 2 \rangle 1. \text{ LET: } q \in x \text{ and } r \in y$

$\langle 2 \rangle 2. \text{ LET: } s < q + r$

PROVE: $s \in x + y$

$\langle 2 \rangle 3. s - r < q$

$\langle 2 \rangle 4. s - r \in x$

$\langle 2 \rangle 5. s = (s - r) + r \in x + y$

$\langle 1 \rangle 5. x + y$ has no greatest element.

$\langle 2 \rangle 1. \text{ LET: } q \in x \text{ and } r \in y$

PROVE: There exists $s \in x + y$ such that $q + r < s$

$\langle 2 \rangle 2. \text{ PICK } q' \in x \text{ and } r' \in y \text{ such that } q < q' \text{ and } r < r'$

$\langle 2 \rangle 3. q + r < q' + r' \in x + y$

□

Proposition 16.3.10. *Addition on the reals agrees with addition on the rationals.*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } q, r \in \mathbb{Q}$

$\langle 1 \rangle 2. q \downarrow + r \downarrow \subseteq (q + r) \downarrow$

PROOF: If $s_1 < q$ and $s_2 < r$ then $s_1 + s_2 < q + r$.

$\langle 1 \rangle 3. (q + r) \downarrow \subseteq q \downarrow + r \downarrow$

$\langle 2 \rangle 1. \text{ LET: } s < q + r$

$\langle 2 \rangle 2. s - r < q$

$\langle 2 \rangle 3. \text{ PICK } t \text{ such that } s - r < t < q$

$\langle 2 \rangle 4. s - t < r$

$\langle 2 \rangle 5. s = t + (s - t) \in q \downarrow + r \downarrow$

□

Proposition 16.3.11. *Addition is associative.*

PROOF:

$$\begin{aligned} x + (y + z) &= \{q + r \mid q \in x, r \in y + z\} \\ &= \{q + s_1 + s_2 \mid q \in x, s_1 \in y, s_2 \in z\} \\ &= \{r + s_2 \mid r \in x + y, s_2 \in z\} \\ &= (x + y) + z \end{aligned}$$

□

Proposition 16.3.12. *Addition is commutative.*

PROOF:

$$\begin{aligned} x + y &= \{q + r \mid q \in x, r \in y\} \\ &= \{r + q \mid r \in y, q \in x\} \\ &= y + x \end{aligned}$$

□

Proposition 16.3.13. *For any $x \in \mathbb{R}$ we have $x + 0 = x$.*

PROOF:

⟨1⟩1. $x + 0 \subseteq x$

PROOF: If $q \in x$ and $r < 0$ then $q + r < q$ so $q + r \in x$.

⟨1⟩2. $x \subseteq x + 0$

⟨2⟩1. LET: $q \in x$

⟨2⟩2. PICK $r \in x$ such that $q < r$.

PROOF: x has no greatest element.

⟨2⟩3. $q - r < 0$

⟨2⟩4. $q = r + (q - r) \in x + 0$

□

Definition 16.3.14. For $x \in \mathbb{R}$, define $-x := \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$.

Proposition 16.3.15. *For all $x \in \mathbb{R}$ we have $-x \in \mathbb{R}$.*

PROOF:

⟨1⟩1. LET: $x \in \mathbb{R}$

⟨1⟩2. $-x \neq \emptyset$

⟨2⟩1. PICK $s \notin x$

⟨2⟩2. $-s - 1 \in -x$

⟨1⟩3. $-x \neq \mathbb{Q}$

⟨2⟩1. PICK $s \in x$

PROVE: $-s \notin -x$

⟨2⟩2. ASSUME: for a contradiction $-s \in -x$

⟨2⟩3. PICK $r > -s$ such that $-r \notin x$

⟨2⟩4. $-r < s$

⟨2⟩5. Q.E.D.

PROOF: This contradicts the fact that x is closed downwards.

⟨1⟩4. $-x$ is closed downwards.

PROOF: Immediate from definition.

⟨1⟩5. $-x$ has no greatest element.

⟨2⟩1. LET: $q \in -x$

⟨2⟩2. PICK $r > q$ such that $-r \notin x$

⟨2⟩3. PICK s such that $q < s < r$

⟨2⟩4. $s \in -x$

□

Lemma 16.3.16. *Let p be a positive rational number. For any real number x , there exists a rational $q \in x$ such that $p + q \notin x$.*

PROOF:

- $\langle 1 \rangle 1$. PICK $q_0 \in x$
- $\langle 1 \rangle 2$. There exists $k \in \mathbb{N}$ such that $q_0 + kp \notin x$
 - $\langle 2 \rangle 1$. PICK $q_1 \notin x$
 - $\langle 2 \rangle 2$. PICK $k \in \mathbb{N}$ such that $q_1 - q_0 < pk$
 - PROOF: Proposition 16.2.11.
 - $\langle 2 \rangle 3$. $q_1 < q_0 + kp$
 - $\langle 2 \rangle 4$. $q_0 + kp \notin x$
- $\langle 1 \rangle 3$. LET: k be the least natural number such that $q_0 + kp \notin x$
- $\langle 1 \rangle 4$. $k \neq 0$
- PROOF: $\langle 1 \rangle 1$
- $\langle 1 \rangle 5$. LET: $q = q_0 + (k-1)p$
- $\langle 1 \rangle 6$. $q \in x$ and $q + p \notin x$.

□

Proposition 16.3.17. *For every real x we have $x + (-x) = 0$.*

PROOF:

- $\langle 1 \rangle 1$. LET: x be a real number.
- $\langle 1 \rangle 2$. $x + (-x) \subseteq 0$
 - $\langle 2 \rangle 1$. LET: $q_1 \in x$ and $q_2 \in -x$
 - $\langle 2 \rangle 2$. PICK $r > q_2$ such that $-r \notin x$
 - $\langle 2 \rangle 3$. $q_1 < -r$
 - $\langle 2 \rangle 4$. $r < -q_1$
 - $\langle 2 \rangle 5$. $q_2 < -q_1$
 - $\langle 2 \rangle 6$. $q_1 + q_2 < 0$
- $\langle 1 \rangle 3$. $0 \subseteq x + (-x)$
 - $\langle 2 \rangle 1$. LET: $p < 0$
 - $\langle 2 \rangle 2$. $0 < -p$
 - $\langle 2 \rangle 3$. PICK $q \in x$ such that $q - p/2 \notin x$
 - PROOF: Lemma 16.3.16.
 - $\langle 2 \rangle 4$. LET: $s = p/2 - q$
 - $\langle 2 \rangle 5$. $-s \notin x$
 - $\langle 2 \rangle 6$. $p - q < s$
 - $\langle 2 \rangle 7$. $p - q \in -x$
 - $\langle 2 \rangle 8$. $p \in x + (-x)$

□

Corollary 16.3.17.1. *The reals form an Abelian group under addition.*

Proposition 16.3.18. *For any reals x, y and z , we have $x < y$ if and only if $x + z < y + z$.*

PROOF:

- $\langle 1 \rangle 1$. $\forall x, y, z \in \mathbb{R}. x \leq y \Rightarrow x + z \leq y + z$
 - $\langle 2 \rangle 1$. LET: $x, y, z \in \mathbb{R}$
 - $\langle 2 \rangle 2$. ASSUME: $x \leq y$
 - $\langle 2 \rangle 3$. For all $q \in x$ and $r \in z$ we have $q + r \in y + z$

⟨1⟩2. $\forall x, y, z \in \mathbb{R}. x + z = y + z \Leftrightarrow x = y$

PROOF: Proposition 13.1.4.

⟨1⟩3. $\forall x, y, z \in \mathbb{R}. x < y \Rightarrow x + z < y + z$

⟨1⟩4. Q.E.D.

PROOF: Proposition 6.2.6.

□

Definition 16.3.19 (Absolute Value). The *absolute value* of a real number x is defined to be

$$|x| = \begin{cases} x & \text{if } 0 \leq x \\ -x & \text{if } x < 0 \end{cases}$$

Definition 16.3.20 (Multiplication). Define *multiplication* \cdot on \mathbb{R} as follows:

- If x and y are non-negative then

$$xy = 0 \cup \{rs \mid 0 \leq r \in x \wedge 0 \leq s \in y\} .$$

- If x and y are both negative then

$$xy = (-x)(-y) .$$

- If one of x and y is negative and one is non-negative then

$$xy = -(|x||y|) .$$

We prove this is well-defined.

PROOF:

⟨1⟩1. LET: x and y be non-negative reals.

PROVE: xy is real.

⟨1⟩2. $xy \neq \emptyset$

PROOF: Since $-1 \in xy$.

⟨1⟩3. $xy \neq \mathbb{Q}$

⟨2⟩1. PICK $r \notin x$ and $s \notin y$

PROVE: $rs \notin xy$

⟨2⟩2. $0 \leq r$ and $0 \leq s$

PROOF: Since $0 \subseteq x$ and $0 \subseteq y$.

⟨2⟩3. ASSUME: for a contradiction $rs \in xy$

⟨2⟩4. PICK r' and s' such that $0 \leq r' \in x$, $0 \leq s' \in y$ and $rs = r's'$

⟨2⟩5. $r' < r$

⟨2⟩6. $s' < s$

⟨2⟩7. $r's' < rs$

⟨2⟩8. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩4. xy is closed downwards.

⟨2⟩1. LET: $q \in xy$ and $r < q$

- ⟨2⟩2. CASE: $q \in 0$
 PROOF: Then $r < q < 0$ so $r \in xy$
- ⟨2⟩3. CASE: $q = s_1 s_2$ where $0 \leq s_1 \in x$ and $0 \leq s_2 \in y$
 - ⟨3⟩1. ASSUME: w.l.o.g. $0 \leq r$
 - ⟨3⟩2. $0 < s_1$ and $0 < s_2$
 - ⟨3⟩3. $r/s_2 < s_1$
 - ⟨3⟩4. $r/s_2 \in x$
 - ⟨3⟩5. $r = (r/s_2)s_2 \in xy$
- ⟨1⟩5. xy has no greatest element.
 - ⟨2⟩1. LET: $q \in xy$
 - ⟨2⟩2. CASE: $q \in 0$
 PROOF: $q < q/2 \in 0$
 - ⟨2⟩3. CASE: $q = rs$ where $0 \leq r \in x$ and $0 \leq s \in y$
 - ⟨3⟩1. PICK r' and s' with $r < r' \in x$ and $s < s' \in y$
 - ⟨3⟩2. $q < r's' \in xy$

□

Proposition 16.3.21. *Multiplication is commutative.*

PROOF: Immediate from definition. □

Proposition 16.3.22. *Multiplication is associative.*

PROOF:

- ⟨1⟩1. For non-negative reals x, y and z , we have $x(yz) = (xy)z$
 PROOF: It computes to $0 \cup \{qrs \mid 0 \leq q \in x, 0 \leq r \in y, 0 \leq s \in z\}$.
- ⟨1⟩2. For all reals x, y and z , we have $x(yz) = (xy)z$
 PROOF: It is equal to $|x||y||z|$ if an even number of them are negative, and $-(|x||y||z|)$ otherwise.

□

Proposition 16.3.23. *Multiplication distributes over addition.*

PROOF:

- ⟨1⟩1. For all non-negative reals x, y and z , we have $x(y + z) = xy + xz$
 - ⟨2⟩1. LET: x, y and z be non-negative reals.
 - ⟨2⟩2. $x(y + z) \subseteq xy + xz$
 - ⟨3⟩1. LET: $q \in x(y + z)$
 - ⟨3⟩2. CASE: $q < 0$
 PROOF: Then we have $q/2 \in xy$ and $q/2 \in xz$ so $q \in xy + xz$.
 - ⟨3⟩3. CASE: $q = rs$ where $0 \leq r \in x$ and $0 \leq s \in y + z$
 - ⟨4⟩1. PICK $s_1 \in y$ and $s_2 \in z$ such that $s = s_1 + s_2$
 - ⟨4⟩2. $rs_1 \in xy$
 PROOF: If $s_1 < 0$ then $rs_1 < 0$ so $rs_1 \in xy$. If $0 \leq s_1$ then we also have $rs_1 \in xy$.
 - ⟨4⟩3. $rs_2 \in xz$
 PROOF: Similar.
 - ⟨4⟩4. $q \in xy + xz$

PROOF: Since $q = rs_1 + rs_2$.

$\langle 2 \rangle 3$. $xy + xz \subseteq x(y + z)$

$\langle 3 \rangle 1$. LET: $q \in xy$ and $r \in xz$.

PROVE: $q + r \in x(y + z)$

$\langle 3 \rangle 2$. CASE: $q < 0$ and $r < 0$

PROOF: Then $q + r < 0$ so $q + r \in x(y + z)$.

$\langle 3 \rangle 3$. CASE: $q < 0$ and $r = r_1r_2$ where $0 \leq r_1 \in x$ and $0 \leq r_2 \in z$

$\langle 4 \rangle 1$. $q + r < r$

$\langle 4 \rangle 2$. $q + r \in xz$

$\langle 4 \rangle 3$. ASSUME: w.l.o.g. $0 \leq q + r$

PROOF: Otherwise $q + r \in x(y + z)$ immediately.

$\langle 4 \rangle 4$. PICK s_1, s_2 with $0 \leq s_1 \in x, 0 \leq s_2 \in y$ and $q + r = s_1s_2$

$\langle 4 \rangle 5$. $s_2 \in y + z$

PROOF: Since $0 \in z$ so $s_2 = s_2 + 0 \in y + z$.

$\langle 4 \rangle 6$. $q + r \in x(y + z)$

$\langle 3 \rangle 4$. CASE: $q = q_1q_2$ where $0 \leq q_1 \in x$ and $0 \leq q_2 \in y$ and $r < 0$

PROOF: Similar.

$\langle 3 \rangle 5$. CASE: $q = q_1q_2$ where $0 \leq q_1 \in x$ and $0 \leq q_2 \in y$ and $r = r_1r_2$ where $0 \leq r_1 \in x$ and $0 \leq r_2 \in z$

$\langle 4 \rangle 1$. ASSUME: w.l.o.g. $q_1 \leq r_1$

$\langle 4 \rangle 2$. $q + r \leq r_1(q_2 + r_2) \in x(y + z)$

$\langle 1 \rangle 2$. For any negative real x and non-negative reals y and z , we have $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned} x(y + z) &= -(-x)(y + z) = -((-x)y + (-x)z) & (\langle 1 \rangle 1) \\ &= -((-x)y) - ((-x)z) \\ &= xy + xz \end{aligned}$$

$\langle 1 \rangle 3$. For any non-negative real x and reals y and z with one negative and one non-negative, we have $x(y + z) = xy + xz$

$\langle 2 \rangle 1$. ASSUME: w.l.o.g. y is negative and z is non-negative.

$\langle 2 \rangle 2$. CASE: $0 \leq y + z$

PROOF:

$$\begin{aligned} xy + xz &= xy + x(-y + y + z) \\ &= -(x(-y)) + x(-y + y + z) \\ &= -(x(-y)) + x(-y) + x(y + z) & (\langle 1 \rangle 1) \\ &= x(y + z) \end{aligned}$$

$\langle 2 \rangle 3$. CASE: $y + z < 0$

$\langle 3 \rangle 1$. $-y - z > 0$

$\langle 3 \rangle 2$. $-y = z - y - z$

$\langle 3 \rangle 3$. $xy + xz = x(y + z)$

PROOF:

$$\begin{aligned}
 xy + xz &= -(x(-y)) + xz \\
 &= -(x(z - y - z)) + xz \\
 &= -(xz + x(-y - z)) + xz & ((1)1) \\
 &= -xy - x(-y - z) + xz \\
 &= -x(-y - z) \\
 &= x(y + z)
 \end{aligned}$$

(1)4. For any non-negative real x and negative reals y and z , we have $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned}
 x(y + z) &= -x(-y - z) \\
 &= -(x(-y) + x(-z)) & ((1)1) \\
 &= -x(-y) - x(-z) \\
 &= xy + xz
 \end{aligned}$$

(1)5. For any negative real x and reals y and z with one negative and one non-negative, we have $x(y + z) = xy + xz$

(2)1. ASSUME: w.l.o.g. y is negative and z is non-negative.

(2)2. CASE: $0 \leq y + z$

PROOF:

$$\begin{aligned}
 x(y + z) &= -((-x)(y + z)) \\
 &= -((-x)y + (-x)z) & ((1)3) \\
 &= -((-x)y) - ((-x)z) \\
 &= (-x)(-y) - ((-x)z) \\
 &= xy + xz
 \end{aligned}$$

(2)3. CASE: $y + z < 0$

PROOF:

$$\begin{aligned}
 x(y + z) &= (-x)(-y - z) \\
 &= (-x)(-y) + (-x)(-z) & ((1)3) \\
 &= xy + xz
 \end{aligned}$$

(1)6. For any negative reals x , y and z , we have $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned}
 x(y + z) &= (-x)(-y - z) \\
 &= (-x)(-y) + (-x)(-z) & ((1)1) \\
 &= xy + xz
 \end{aligned}$$

□

Proposition 16.3.24. *For any real x we have $x1 = x$.*

PROOF:

(1)1. CASE: $0 \leq x$

(2)1. $x1 \subseteq x$

(3)1. LET: $q \in x1$

- ⟨3⟩2. CASE: $q < 0$
 PROOF: Then $q \in x$ because $0 \leq x$.
 ⟨3⟩3. $q = rs$ where $0 \leq r \in x$ and $0 \leq s < 1$
 PROOF: Then $q < r$ so $q \in x$.
 ⟨2⟩2. $x \subseteq x1$
 ⟨3⟩1. LET: $q \in x$
 ⟨3⟩2. ASSUME: w.l.o.g. $0 \leq q$
 ⟨3⟩3. PICK r such that $q < r \in x$
 ⟨3⟩4. $0 \leq q/r < 1$
 ⟨3⟩5. $q = r(q/r) \in x1$
 ⟨1⟩2. CASE: $x < 0$
 PROOF:

$$\begin{aligned}
 x1 &= -((-x)1) \\
 &= -(-x) && (\langle 1 \rangle 1) \\
 &= x
 \end{aligned}$$

□

Lemma 16.3.25. *Let $x \in \mathbb{R}$ and c be a positive rational. Then there exists $a \in x$ and a non-least rational upper bound b for x such that $b - a = c$.*

PROOF:

- ⟨1⟩1. PICK $a_1 \in x$ such that if x has a rational supremum s then $a_1 > s - c$
 ⟨1⟩2. There exists a natural number n such that $a_1 + nc$ is an upper bound for x .
 ⟨2⟩1. PICK a non-least upper bound b_1 for x .
 ⟨2⟩2. PICK a natural number n such that $nc > b_1 - a_1$
 PROOF: Proposition 16.2.11.
 ⟨2⟩3. $a_1 + nc > b_1$
 ⟨2⟩4. $a_1 + nc$ is an upper bound for x .
 ⟨1⟩3. LET: k be the least natural number such that $a_1 + kc$ is an upper bound for x .
 ⟨1⟩4. $a_1 + (k - 1)c \in x$
 ⟨1⟩5. $a_1 + kc$ is not the supremum of x .
 ⟨2⟩1. ASSUME: for a contradiction $a_1 + kc$ is the supremum of x .
 ⟨2⟩2. $a_1 > a_1 + (k - 1)c$
 PROOF: ⟨1⟩1
 ⟨2⟩3. Q.E.D.
 PROOF: This is a contradiction.
 ⟨1⟩6. LET: $a = a_1 + (k - 1)c$
 ⟨1⟩7. LET: $b = a_1 + kc$
 ⟨1⟩8. $b - a = c$

□

Proposition 16.3.26. *For any non-zero real x , there exists a real y such that $xy = 1$.*

PROOF:

- ⟨1⟩1. CASE: $0 < x$
- ⟨2⟩1. LET: $y = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{u^{-1} \mid u \text{ is an upper bound for } x \text{ but not the supremum of } x\}$
- ⟨2⟩2. y is a real number.
 - ⟨3⟩1. $y \neq \emptyset$
PROOF: Since $0 \in y$.
 - ⟨3⟩2. $y \neq \mathbb{Q}$
 - ⟨4⟩1. PICK $q \in x$ such that $0 < q$
 - ⟨4⟩2. $q^{-1} \notin y$
 - ⟨3⟩3. y is closed downwards.
 - ⟨4⟩1. LET: $q \in y$ and $r < q$
PROVE: $r \in y$
 - ⟨4⟩2. ASSUME: w.l.o.g. $0 < r$
 - ⟨4⟩3. q^{-1} is a non-least upper bound for x .
 - ⟨4⟩4. $q^{-1} < r^{-1}$
 - ⟨4⟩5. r^{-1} is a non-least upper bound for x .
 - ⟨4⟩6. $r \in y$
 - ⟨3⟩4. y has no greatest element.
 - ⟨4⟩1. LET: $q \in y$
PROVE: There exists $r \in y$ such that $q < r$
 - ⟨4⟩2. CASE: $q \leq 0$
 - ⟨5⟩1. PICK a non-least upper bound u for x .
 - ⟨5⟩2. $q < u^{-1} \in x$
 - ⟨4⟩3. CASE: $q = u^{-1}$ where u is a non-least upper bound for x .
 - ⟨5⟩1. PICK a non-least upper bound v with $v < u$
 - ⟨5⟩2. $u^{-1} < v^{-1} \in y$
- ⟨2⟩3. $0 < y$
- ⟨2⟩4. $xy \subseteq 1$
 - ⟨3⟩1. LET: $q \in xy$
 - ⟨3⟩2. ASSUME: w.l.o.g. $0 < q$
 - ⟨3⟩3. PICK $0 < r \in x$ and $0 < s \in y$ such that $q = rs$
 - ⟨3⟩4. s^{-1} is a non-least upper bound for x
 - ⟨3⟩5. $r < s^{-1}$
 - ⟨3⟩6. $rs < 1$
- ⟨2⟩5. $1 \subseteq xy$
 - ⟨3⟩1. LET: $q < 1$
PROVE: $q \in xy$
 - ⟨3⟩2. ASSUME: w.l.o.g. $0 < q$
 - ⟨3⟩3. PICK a_1 with $0 < a_1 \in x$
 - ⟨3⟩4. $(1 - q)a_1 > 0$
 - ⟨3⟩5. PICK $a \in x$ and a non-least upper bound w of x such that $w - a = (1 - q)a_1$
PROOF: Lemma 16.3.25.
 - ⟨3⟩6. $w - a < (1 - q)w$
 - ⟨3⟩7. $qw < a$
 - ⟨3⟩8. $w < a/q$
 - ⟨3⟩9. a/q is a non-least upper bound for x

1

PROOF:

PROOF:

1

Proposition 16.3.28.

PROOF: The function $f(x) = (2x - 1)/(x - x^2)$ is a bijection between $(0, 1)$ and \mathbb{R} . \square

PROOF:

1

Proposition 16.3.30. *The set of algebraic numbers is countable.*

PROOF: There are countably many integer polynomials, each with finitely many roots. \square

Corollary 16.3.30.1. *There are uncountably many transcendental numbers.*

Proposition 16.3.31. *Let A be a set of disks in the plane, no two of which intersect. Then A is countable.*

PROOF: Every circle includes a point with rational coordinates. Define $f : \{q \in \mathbb{Q}^2 \mid \exists C \in A. q \in C\} \rightarrow A$ by $f(q) = C$ iff $q \in C$. Then f is surjective. \square

Proposition 16.3.32. *There exists an uncountable set of circles in the plane that do not intersect.*

PROOF: The set of all circles with origin O is uncountable. \square

Chapter 17

Real Analysis

Theorem 17.0.1 (Weierstrass). *Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. For every $\epsilon > 0$, there exists a polynomial p such that $\forall x \in [a, b]. |f(x) - p(x)| < \epsilon$.*

Theorem 17.0.2 (Bolzano-Weierstrass). *Every bounded sequence in \mathbb{R}^n has a convergent subsequence.*

17.1 Step Functions

Definition 17.1.1 (Step Function). A *step function* on \mathbb{R} is a function $f : \mathbb{R} \rightarrow \mathbb{R}$, where $[a_1, b_1), \dots, [a_n, b_n)$ are disjoint half-open intervals, such that f is constant on each $[a_i, b_i)$, and zero outside them.

Definition 17.1.2 (Basic Representation). Let f be a step function. The *basic representation* of f is defined as follows.

Let a_0, a_1, \dots, a_n be the points of discontinuity of f . For $k = 1, \dots, n$, let $\alpha_k = f(a_{k-1})$ and g_k be the characteristic function of $[a_{k-1}, a_k)$. Then the basic representation of f is

$$f = \alpha_1 g_1 + \dots + \alpha_n g_n .$$

Proposition 17.1.3. *If f and g are step functions then $\lambda x.f(x) + g(x)$ is a step function.*

Proposition 17.1.4. *If $c \in \mathbb{R}$ and f is a step function then $\lambda x.cf(x)$ is a step function.*

Proposition 17.1.5. *If f is a step function then $\lambda x.|f(x)|$ is a step function.*

Proposition 17.1.6. *If f and g are step functions then $\lambda x.\min(f(x), g(x))$ and $\lambda x.\max(f(x), g(x))$ are step functions.*

Proposition 17.1.7. *If f is a step function and $c \in \mathbb{R}$ then $\lambda x.f(x - c)$ is a step function.*

Definition 17.1.8 (Support). Given $f : \mathbb{R} \rightarrow \mathbb{R}$, the *support* of f is

$$\text{supp } f := \{x \in \mathbb{R} \mid f(x) \neq 0\} .$$

Definition 17.1.9 (Integral of a Step Function). Given a step function f , define $\int f = \int f(x)dx \in \mathbb{R}$ as follows.

Let $f(x) = \lambda_1 f_1(x) + \cdots + \lambda_n f_n(x)$, where f_i is the characteristic function of $[a_i, b_i)$. Then

$$\int f = \lambda_1(b_1 - a_1) + \cdots + \lambda_n(b_n - a_n) .$$

We prove this is well defined.

PROOF:

- (1)1. LET: $f = \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x) = \mu_1 g_1(x) + \cdots + \mu_n g_n(x)$, where f_i is the characteristic function of $[a_i, b_i)$ and g_i is the characteristic function of $[c_i, d_i)$, with $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$ and $c_1 < d_1 \leq c_2 < d_2 \leq \cdots \leq c_n < d_n$.
- (1)2. ASSUME: w.l.o.g. none of the λ_i or μ_i is zero.
- (1)3. ASSUME: w.l.o.g. we never have $\lambda_i = \lambda_{i+1}$ and $b_i = a_{i+1}$, and we never have $\mu_i = \mu_{i+1}$ and $d_i = c_{i+1}$.
- (1)4. We have $m = n$ and for all i , $a_i = b_i$ and $c_i = d_i$ and $\lambda_i = \mu_i$.
 - (2)1. ASSUME: w.l.o.g. $m \leq n$
 - (2)2. ASSUME: as induction hypothesis $a_i = b_i$, $c_i = d_i$, $\lambda_i = \mu_i$ for $i = 1, \dots, k$ with $k < m$.
 - (2)3. $a_{k+1} = b_{k+1}$
PROOF: It is $\inf\{x \in (a_k, +\infty) : f(x) \neq 0\}$ (or $\inf\{x \in \mathbb{R} : f(x) \neq 0\}$ if $k = 0$).
 - (2)4. $\lambda_{k+1} = \mu_{k+1}$
PROOF: It is $f(a_{k+1})$.
 - (2)5. $c_{k+1} = d_{k+1}$
PROOF: It is $\sup\{x \in (a_{k+1}, +\infty) : f(x) = \lambda_{k+1}\}$.
 - (2)6. $m = n$
PROOF: For all $x > b_m$ we have $f(x) = 0$.

□

Proposition 17.1.10. If f and g are step functions then $\int(f(x) + g(x))dx = \int f + \int g$.

Proposition 17.1.11. If f is a step function then $\int cf(x)dx = c \int f$.

Proposition 17.1.12. If f and g are step functions and $\forall x. f(x) \leq g(x)$ then $\int f \leq \int g$.

PROOF: We have $g(x) - f(x) \geq 0$ for all x and so $\int(g(x) - f(x))dx \geq 0$. □

Proposition 17.1.13. If f is a step function then $|\int f| \leq \int |f(x)|dx$.

Proposition 17.1.14. If f is a step function and $c \in \mathbb{R}$ then $\int f(x-c)dx = \int f$.

Lemma 17.1.15. *Let f be a step function with $\text{supp } f \subseteq [a_1, b_1) \cup \cdots \cup [a_n, b_n)$. Let M be a constant. If $\forall x. |f(x)| < M$ then*

$$\int |f(x)| dx \leq M \sum_{k=1}^n (b_k - a_k) .$$

Lemma 17.1.16. *Let $\{[a_i, b_i) \mid i \in \mathbb{N}\}$ be a partition of $[a, b)$. Then*

$$\sum_{i=0}^{\infty} (b_i - a_i) = b - a .$$

PROOF:

$\langle 1 \rangle 1$. For all $c \in (a, b]$ we have $\{[a_i, b_i) \cap [a, c) \mid i \in \mathbb{N}\}$ is a partition of $[a, c)$.

$\langle 1 \rangle 2$. For $c \in (a, b]$ and $n \in \mathbb{N}$,

LET: $b_{cn} := \min(b_n, c)$

$\langle 1 \rangle 3$. LET: $S = \{c \in (a, b] \mid \sum_{a_n < b_{cn}} (b_{cn} - a_n) = c - a\}$

$\langle 1 \rangle 4$. $S \neq \emptyset$

$\langle 2 \rangle 1$. PICK n such that $a_n = a$

$\langle 2 \rangle 2$. $b_n \in S$

$\langle 1 \rangle 5$. LET: $s := \sup S$

$\langle 1 \rangle 6$. $s \in S$

$\langle 2 \rangle 1$. PICK an increasing sequence (s_n) in S that converges to S .

$\langle 2 \rangle 2$. For all n we have $s_n - a \leq \sum_{a_m < b_{sm}} (b_{sm} - a_m) \leq s - a$.

PROOF:

$$s_n - a = \sum_{a_m < b_{s_n m}} (b_{s_n m} - a_m) \quad (s_n \in S)$$

$$\leq \sum_{a_m < b_{sm}} (b_{sm} - a_m) \quad (s_n \leq s)$$

$$\leq s - a \quad (\langle 1 \rangle 1)$$

$\langle 2 \rangle 3$. $\sum_{a_m < b_{sm}} (b_{sm} - a_m) = s - a$

PROOF: Sandwich Theorem

$\langle 1 \rangle 7$. $s = b$

$\langle 2 \rangle 1$. ASSUME: for a contradiction $s < b$

$\langle 2 \rangle 2$. PICK k such that $s \in [a_k, b_k)$

$\langle 2 \rangle 3$. $b_k \in S$

PROOF: Since $\sum_{a_m < b_{sm}} (b_{sm} - a_m) = \sum_{a_m < b_{b_k m}} (b_{b_k m} - a_m)$.

$\langle 2 \rangle 4$. $b_k \leq s$

$\langle 2 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 8$. $b \in S$

$\langle 1 \rangle 9$. $\sum_{a_n < b_n} (b_n - a_n) = b - a$

□

Theorem 17.1.17. *Let (f_n) be a decreasing sequence of non-negative step functions such that, for all $x \in \mathbb{R}$, we have $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$. Then $\int f_n \rightarrow 0$ as $n \rightarrow \infty$.*

PROOF:

$\langle 1 \rangle 1.$ $(\int f_n)$ is decreasing and bounded below by 0.

$\langle 1 \rangle 2.$ LET: $\epsilon := \lim_{n \rightarrow \infty} \int f_n$

$\langle 1 \rangle 3.$ ASSUME: for a contradiction $\epsilon > 0$.

$\langle 1 \rangle 4.$ PICK $a, b \in \mathbb{R}$ such that $\text{supp } f_0 \subseteq [a, b]$

$\langle 1 \rangle 5.$ LET: $\alpha := \epsilon/2(b - a)$

$\langle 1 \rangle 6.$ For $n \in \mathbb{N}$,

LET:

$$A_n := \{x \in [a, b) : f_n(x) < \alpha\} .$$

$\langle 1 \rangle 7.$ LET: $B_0 := A_0$

$\langle 1 \rangle 8.$ For n a positive integer,

LET: $B_n = A_n - A_{n-1}$.

$\langle 1 \rangle 9.$ For all n we have $A_n \subseteq A_{n+1}$.

$\langle 1 \rangle 10.$ For $m \neq n$ we have $B_m \cap B_n = \emptyset$.

$\langle 1 \rangle 11.$ $\bigcup_{n=0}^{\infty} A_n = [a, b)$

PROOF: For all $x \in [a, b)$, there exists N such that $f_N(x) < \alpha$ because $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$.

$\langle 1 \rangle 12.$ $\bigcup_{n=0}^{\infty} B_n = [a, b)$

$\langle 1 \rangle 13.$ For $n \in \mathbb{N}$,

LET: $B_n = [a_{n1}, b_{n1}) \cup \dots \cup [a_{nk_n}, b_{nk_n})$.

$\langle 1 \rangle 14.$ $\sum_{n=0}^{\infty} \sum_{k=1}^{k_n} (b_{nk} - a_{nk}) = b - a$

PROOF: Lemma 17.1.16.

$\langle 1 \rangle 15.$ LET:

$$\delta = \frac{\epsilon}{2 \max_x |f_0(x)|} .$$

$\langle 1 \rangle 16.$ PICK n_0 such that

$$\sum_{n=n_0}^{\infty} \sum_{k=1}^{k_n} (b_{nk} - a_{nk}) < \delta .$$

$\langle 1 \rangle 17.$ $A_{n_0} = B_0 \cup \dots \cup B_{n_0}$

$\langle 1 \rangle 18.$ LET: $g : \mathbb{R} \rightarrow \mathbb{R}$,

$$g(x) = \begin{cases} f_{n_0}(x) & \text{if } x \in A_{n_0} \\ 0 & \text{if } x \notin A_{n_0} \end{cases}$$

$\langle 1 \rangle 19.$ LET: $h : \mathbb{R} \rightarrow \mathbb{R}$,

$$h(x) = \begin{cases} 0 & \text{if } x \in A_{n_0} \\ f_{n_0}(x) & \text{if } x \notin A_{n_0} \end{cases}$$

$\langle 1 \rangle 20.$ $\forall x \in B. f_{n_0}(x) < \alpha$

$\langle 1 \rangle 21.$ $\forall x \in \mathbb{R}. g(x) < \alpha$

$\langle 1 \rangle 22.$

$$\int g < \frac{\epsilon}{2}$$

$\langle 1 \rangle 23.$

$$\int h < \frac{\epsilon}{2}$$

PROOF:

$$\begin{aligned} \int h &< \delta \max_x |f_{n_0}(x)| \\ &\leq \delta \max_x |f_0(x)| \\ &= \epsilon/2 \end{aligned}$$

(1)24.

$$\int f_{n_0} < \epsilon$$

PROOF: Since $\forall x. f_{n_0}(x) = g(x) + h(x)$.

(1)25.

$$\lim_{n \rightarrow \infty} \int f_n < \epsilon$$

(1)26. Q.E.D.

PROOF: This contradicts (1)2.

□

Corollary 17.1.17.1. *Let (f_n) be an increasing sequence of step functions. If $\lim_{n \rightarrow \infty} f_n(x) \geq 0$ for all x , then $\lim_{n \rightarrow \infty} \int f_n \geq 0$.*

PROOF:

(1)1. For $n \in \mathbb{N}$,

LET: $g_n : \mathbb{R} \rightarrow \mathbb{R}$ be the function $g_n(x) = \max(0, -f_n(x))$.

(1)2. (g_n) is a decreasing sequence of step functions and $g_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all x .

(1)3. $\int g_n \rightarrow 0$ as $n \rightarrow \infty$

(1)4. For all $n \in \mathbb{N}$ we have $f_n(x) = \max(0, f_n(x)) - \max(0, -f_n(x))$.

(1)5.

$$\int f_n = \int \max(0, f_n(x)) dx - \int \max(0, -f_n(x)) dx$$

(1)6.

$$\lim_{n \rightarrow \infty} \int f_n = \lim_{n \rightarrow \infty} \int \max(0, f_n(x)) dx$$

(1)7. $\lim_{n \rightarrow \infty} \int f_n \geq 0$

□

17.2 Lebesgue Integration

Definition 17.2.1 (Lebesgue Integration). Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Then f is *Lebesgue integrable* iff there exists a sequence (f_n) of step functions such that:

•

$$\sum_{n=0}^{\infty} \int |f_n(x)| dx < \infty$$

• For all $x \in \mathbb{R}$, if $\sum_{n=0}^{\infty} |f_n(x)| < \infty$, then $f(x) = \sum_{n=0}^{\infty} f_n(x)$.

We write $f \simeq \sum_{n=0}^{\infty} f_n$ for these two conditions.

The *integral* of f is then

$$\int f := \sum_{n=0}^{\infty} \int f_n .$$

Chapter 18

Complex Analysis

Theorem 18.0.1 (Hölder's Inequality). *Let p and q be real numbers with $p > 1$, $q > 1$ and $1/p + 1/q = 1$. If $(x_n) \in l^p$ and $(y_n) \in l^q$ then*

$$\sum_{n=0}^{\infty} |x_n y_n| \leq \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=0}^{\infty} |y_n|^q \right)^{1/q}$$

PROOF:

- $\langle 1 \rangle 1$. LET: p and q be real numbers with $p > 1$ and $q > 1$
- $\langle 1 \rangle 2$. ASSUME: $1/p + 1/q = 1$
- $\langle 1 \rangle 3$. LET: $(x_n) \in l^p$
- $\langle 1 \rangle 4$. LET: $(y_n) \in l^q$
- $\langle 1 \rangle 5$. ASSUME: w.l.o.g. $x_0 \neq 0$ and $y_0 \neq 0$
- $\langle 1 \rangle 6$. For all $x \in [0, 1]$, we have

$$x^{1/p} \leq \frac{1}{p}x + \frac{1}{q}.$$

PROOF:

- $\langle 2 \rangle 1$. LET: $f : [0, 1] \rightarrow \mathbb{R}$ be the function

$$f(x) = \frac{1}{p}x + \frac{1}{q} - x^{1/p}.$$

- $\langle 2 \rangle 2$. $f'(x) = \frac{1}{p} - \frac{1}{p}x^{-1/q}$ for $x \in (0, 1]$
- $\langle 2 \rangle 3$. $f'(x) < 0$ for $x \in (0, 1]$
- $\langle 2 \rangle 4$. $f(1) = 1/p + 1/q - 1 = 0$
- $\langle 2 \rangle 5$. $f(x) \geq 0$ for all $x \in [0, 1]$
- $\langle 1 \rangle 7$. For all non-negative reals a and b , we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

- $\langle 2 \rangle 1$. LET: a and b be non-negative reals.
- $\langle 2 \rangle 2$. CASE: $a^p \leq b^q$
 - $\langle 3 \rangle 1$. $0 \leq a^p/b^q \leq 1$
 - $\langle 3 \rangle 2$.

$$ab^{-q/p} \leq \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q}$$

PROOF: Taking $x = a^p/b^q$ in $\langle 1 \rangle 6$.

$\langle 3 \rangle 3$.

$$ab^{1-q} \leq \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q}$$

PROOF: $-q/p = 1 - q$ from $\langle 1 \rangle 2$.

$\langle 3 \rangle 4$.

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$\langle 2 \rangle 3$. CASE: $b^q \leq a^p$

PROOF: Similar.

$\langle 1 \rangle 8$. For $j = 1, \dots, n$, we have

$$\frac{|x_j|}{(\sum_{k=0}^n |x_k|^p)^{1/p}} \frac{|y_j|}{(\sum_{k=0}^n |y_k|^q)^{1/q}} \leq \frac{1}{p} \frac{|x_j|^p}{\sum_{k=0}^n |x_k|^p} + \frac{1}{q} \frac{|y_j|^q}{\sum_{k=0}^n |y_k|^q}$$

PROOF: From $\langle 1 \rangle 7$ with

$$a = \frac{|x_j|}{(\sum_{k=0}^n |x_k|^p)^{1/p}} \text{ and } b = \frac{|y_j|}{(\sum_{k=0}^n |y_k|^q)^{1/q}}.$$

$\langle 1 \rangle 9$.

$$\frac{\sum_{j=0}^n |x_j| |y_j|}{(\sum_{k=0}^n |x_k|^p)^{1/p} (\sum_{k=0}^n |y_k|^q)^{1/q}} \leq 1$$

PROOF:

$$\begin{aligned} \frac{\sum_{j=0}^n |x_j| |y_j|}{(\sum_{k=0}^n |x_k|^p)^{1/p} (\sum_{k=0}^n |y_k|^q)^{1/q}} &\leq \frac{1}{p} + \frac{1}{q} \quad (\text{Taking the sum } j = 0 \text{ to } n \text{ in } \langle 1 \rangle 8) \\ &= 1 \end{aligned} \quad (\langle 1 \rangle 2)$$

$\langle 1 \rangle 10$. Q.E.D.

PROOF: Taking the limit $n \rightarrow \infty$ in $\langle 1 \rangle 9$.

□

Theorem 18.0.2 (Minkowski's Inequality). *Let p be a real number, $p \geq 1$. Let $(x_n), (y_n) \in l^p$. Then*

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p}$$

PROOF:

$\langle 1 \rangle 1$. LET: p be a real number with $p \geq 1$

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $p > 1$

PROOF: The case $p = 1$ is just the Triangle Inequality.

$\langle 1 \rangle 3$. LET: q be the real such that $1/p + 1/q = 1$

$\langle 1 \rangle 4$.

$$\begin{aligned} \sum_{n=0}^{\infty} |x_n + y_n|^p &\leq \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \\ &\quad + \left(\sum_{n=0}^{\infty} |y_n|^p \right)^{1/p} \left(\sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \end{aligned}$$

PROOF:

$$\begin{aligned}
\sum_{n=0}^{\infty} |x_n + y_n|^p &= \sum_{n=0}^{\infty} |x_n + y_n| |x_n + y_n|^{p-1} \\
&\leq \sum_{n=0}^{\infty} |x_n| |x_n + y_n|^{p-1} + \sum_{n=0}^{\infty} |y_n| |x_n + y_n|^{p-1} \quad (\text{Triangle Inequality}) \\
&\leq \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \\
&\quad + \left(\sum_{n=0}^{\infty} |y_n|^p \right)^{1/p} \left(\sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \quad (\text{Hölder's Inequality})
\end{aligned}$$

⟨1⟩5.

$$\sum_{n=0}^{\infty} |x_n + y_n|^p \leq \left\{ \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} + \left(\sum_{n=0}^{\infty} |y_n|^p \right)^{1/p} \right\} \left(\sum_{n=0}^{\infty} |x_n + y_n|^p \right)^{1/q}$$

⟨1⟩6. Q.E.D.

□

Chapter 19

Topology

19.1 Topological Spaces

Definition 19.1.1 (Topology). Let X be a set. A *topology* on X is a set $\mathcal{T} \subseteq \mathcal{P}X$, whose elements are called *open sets*, such that:

- $X \in \mathcal{T}$
- $\forall \mathcal{U} \subseteq \mathcal{T}. \bigcup \mathcal{U} \in \mathcal{T}$
- $\forall U, V \in \mathcal{T}. U \cap V \in \mathcal{T}$

A *topological space* is a pair (X, \mathcal{T}) such that X is a set and \mathcal{T} is a topology on X . We refer to the elements of X as *points*.

An *open neighbourhood* of a point x is an open set U such that $x \in U$. We write \mathcal{T}_x for the set of all open neighbourhoods of x .

Definition 19.1.2 (Closed Set). In a topological space X , a set C is *closed* iff $X - C$ is open.

Definition 19.1.3 (Discrete Topology). The *discrete topology* on a set X is $\mathcal{P}X$.

Definition 19.1.4 (Indiscrete Topology). The *indiscrete topology* or *trivial topology* on a set X is $\{\emptyset, X\}$.

Definition 19.1.5 (Finer, Coarser). Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . Then \mathcal{T} is *finer*, *larger* or *stronger* than \mathcal{T}' , and \mathcal{T}' is *coarser*, *smaller* or *weaker* than \mathcal{T} , iff $\mathcal{T}' \subseteq \mathcal{T}$.

Definition 19.1.6 (Basis). Let X be a set. A *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$, whose elements we call *basic open neighbourhoods*, such that:

- $\bigcup \mathcal{B} = X$
- $\forall A, B \in \mathcal{B}. \forall x \in A \cap B. \exists C \in \mathcal{B}. x \in C \subseteq A \cap B.$

The topology *generated* by \mathcal{B} is the coarsest topology that includes \mathcal{B} .

Proposition 19.1.7. *The topology generated by \mathcal{B} is $\{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$*

19.2 Continuous Functions

Definition 19.2.1 (Continuous). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is *continuous* iff, for every open set U in Y , the set $f^{-1}(U)$ is open in X .

Proposition 19.2.2. *For any topological space X , the identity function $\text{id}_X : X \rightarrow X$ is continuous.*

Proposition 19.2.3. *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous then $g \circ f : X \rightarrow Z$ is continuous.*

Definition 19.2.4. Let Top be the category of topological spaces and continuous functions.

Proposition 19.2.5. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is continuous if and only if, for every closed set C in Y , we have $f^{-1}(C)$ is closed in X .*

Definition 19.2.6 (Continuous at a Point). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let $x \in X$. Then f is *continuous at x* iff, for every open neighbourhood V of $f(x)$, we have $f^{-1}(V)$ is open.

Definition 19.2.7. The category of *pointed* topological spaces, Top_* , is the category with:

- objects all pairs (A, a) where A is a topological space and $a \in A$;
- morphisms $f : (A, a) \rightarrow (B, b)$ all continuous functions $f : A \rightarrow B$ such that $f(a) = b$.

Definition 19.2.8 (Homeomorphism). A *homeomorphism* is an isomorphism in Top . Two isomorphic topological spaces are called *homeomorphic*.

Definition 19.2.9 (Topological Property). A property of topological spaces is a *topological* property iff it is preserved by homeomorphism.

Proposition 19.2.10. *Cardinality of a topological space is a topological property.*

19.3 Convergence

Definition 19.3.1 (Convergence). Let X be a topological space. Let (x_n) be a sequence in X and $l \in X$. Then (x_n) *converges* to l , $x_n \rightarrow l$ as $n \rightarrow \infty$, if and only if, for every open neighbourhood U of l , there exists N such that $\forall n \geq N. x_n \in U$.

Theorem 19.3.2. *Let X and Y be topological spaces. Let Z be a closed subspace of X and $f : Z \rightarrow Y$ a continuous function. Then the graph of f , $G = \{(x, f(x)) \mid x \in Z\}$, is closed in $X \times Y$.*

PROOF:

$\langle 1 \rangle 1$. LET: $((x_n, f(x_n)))$ be a sequence in G .

$\langle 1 \rangle 2$. LET: $(x_n, f(x_n)) \rightarrow (x, y)$ as $n \rightarrow \infty$

$\langle 1 \rangle 3$. $x \in Z$

PROOF: Since $x_n \rightarrow x$ and Z is closed.

$\langle 1 \rangle 4$. $y = f(x)$

PROOF:

$$\begin{aligned} y &= \lim_{n \rightarrow \infty} f(x_n) \\ &= f\left(\lim_{n \rightarrow \infty} x_n\right) \\ &= f(x) \end{aligned}$$

□

19.4 Homotopy

Definition 19.4.1. Let \mathbf{hTop} be the category whose objects are topological spaces, and whose morphisms are homotopy classes of continuous functions.

Definition 19.4.2. A *homotopy equivalence* is an isomorphism in \mathbf{hTop} . Isomorphic topological spaces are called *homotopic*.

19.5 Metric Spaces

Definition 19.5.1 (Metric). Let X be a set. A *metric* on a set X is a function $d : X^2 \rightarrow \mathbb{R}$ such that:

- $\forall x, y \in X. d(x, y) \geq 0$
- $\forall x, y \in X. d(x, y) = d(y, x)$
- *Triangle Inequality* $\forall x, y, z \in X. d(x, y) + d(y, z) \geq d(x, z)$
- $\forall x, y \in X. d(x, y) = 0$ iff $x = y$.

A *metric space* is a pair (X, d) such that d is a metric on X .

Definition 19.5.2 (Open Ball). In a metric space X , let $c \in X$ and $r > 0$. The *open ball* with *centre* c and *radius* r is

$$B(c, r) := \{x \in X \mid d(x, c) < r\} .$$

Proposition 19.5.3. *In a metric space, the set of open balls forms a basis for a topology.*

Definition 19.5.4 (Metric Topology). Given a metric space X , the *metric topology* on X is the topology generated by the basis of open balls.

A topological space (X, \mathcal{T}) is *metrizable* iff there exists a metric d on X such that \mathcal{T} is the metric topology induced by d .

We identify a metric space with this topological space.

Proposition 19.5.5. *If d is a metric on X and $Y \subseteq X$ then $d \upharpoonright Y^2$ is a metric on Y .*

We write just Y for the metric space $(Y, d \upharpoonright Y^2)$.

Proposition 19.5.6. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$. Then f is continuous if and only if, for every sequence (x_n) in X and $l \in X$, if $x_n \rightarrow l$ as $n \rightarrow \infty$ then $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$.*

PROOF:

- ⟨1⟩1. If f is continuous then, for every sequence (x_n) in X and $l \in X$, if $x_n \rightarrow l$ as $n \rightarrow \infty$, then $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$.
- ⟨2⟩1. ASSUME: f is continuous.
- ⟨2⟩2. LET: (x_n) be a sequence in X .
- ⟨2⟩3. LET: $l \in X$
- ⟨2⟩4. ASSUME: $x_n \rightarrow l$ as $n \rightarrow \infty$.
- ⟨2⟩5. LET: V be an open neighbourhood of $f(l)$
- ⟨2⟩6. $f^{-1}(V)$ is an open neighbourhood of l .
- ⟨2⟩7. PICK N such that $\forall n \geq N. x_n \in f^{-1}(V)$
- ⟨2⟩8. $\forall n \geq N. f(x_n) \in V$
- ⟨1⟩2. If, for every sequence (x_n) in X and $l \in X$, if $x_n \rightarrow l$ as $n \rightarrow \infty$, then $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$, then f is continuous.
- ⟨2⟩1. ASSUME: f is not continuous.
- ⟨2⟩2. PICK an open set V in Y such that $f^{-1}(V)$ is not open in X
- ⟨2⟩3. PICK $l \in f^{-1}(V)$ such that, for all $\epsilon > 0$, $B(l, \epsilon) \not\subseteq f^{-1}(V)$.
- ⟨2⟩4. For $n \in \mathbb{N}$, PICK $x_n \in B(l, 1/(n+1))$ such that $x_n \notin f^{-1}(V)$.
- ⟨2⟩5. $x_n \rightarrow l$ as $n \rightarrow \infty$.
- ⟨2⟩6. $f(x_n) \not\rightarrow f(l)$ as $n \rightarrow \infty$

□

Proposition 19.5.7. *Completeness is not a topological property.*

PROOF: We have $(-1, 1) \cong \mathbb{R}$, but \mathbb{R} is complete and $(-1, 1)$ is not. □

Chapter 20

Ring Theory

Definition 20.0.1. Given a ring R , let $R\text{-Mod}$ be the category of modules over R and R -linear maps.

Chapter 21

Linear Algebra

21.1 Vector Spaces

Definition 21.1.1 (Vector Space). Let K be a field. A *vector space* over K consists of:

- a set V , whose elements are called *vectors*;
- an operation $+$: $V^2 \rightarrow V$, *addition*;
- an operation \cdot : $K \times V \rightarrow V$, *scalar multiplication*

such that:

- V is an Abelian group under $+$
- $\forall \alpha, \beta \in K. \forall x \in V. \alpha(\beta x) = (\alpha\beta)x$
- $\forall \alpha, \beta \in K. \forall x \in V. (\alpha + \beta)x = \alpha x + \beta x$
- $\forall \alpha \in K. \forall x, y \in V. \alpha(x + y) = \alpha x + \alpha y$
- $\forall x \in V. 1x = x$

We call the elements of K *scalars*. A *real vector space* is a vector space over \mathbb{R} , and a *complex vector space* is a vector space over \mathbb{C} .

Proposition 21.1.2. *Let K be a field. Let V be a vector space over K . For any $\lambda \in K$ we have $\lambda 0 = 0$.*

PROOF:

$$\begin{aligned}\lambda 0 &= \lambda(0 + 0) \\ &= \lambda 0 + \lambda 0 \\ \therefore 0 &= \lambda 0\end{aligned}$$

□

Proposition 21.1.3. *Let K be a field. Let V be a vector space over K . Let $\lambda \in K$ and $x \in V$. If $\lambda x = 0$ then either $\lambda = 0$ or $x = 0$.*

PROOF: If $\lambda \neq 0$ then $x = 1x = \lambda^{-1}\lambda x = \lambda^{-1}0 = 0$. \square

Proposition 21.1.4. *Let K be a field. Let V be a vector space over K . For any $x \in V$ we have $0x = 0$.*

PROOF:

$$\begin{aligned} 0x &= (0 + 0)x \\ &= 0x + 0x \\ \therefore 0 &= 0x \end{aligned} \quad \square$$

Proposition 21.1.5. *Let K be a field. Let V be a vector space over K . For any $x \in V$, we have $(-1)x = -x$.*

PROOF:

$$\begin{aligned} x + (-1)x &= 1x + (-1)x \\ &= (1 + (-1))x \\ &= 0x \\ &= 0 \\ \therefore (-1)x &= -x \end{aligned} \quad \square$$

Proposition 21.1.6. *Let K be a field. Then K is a vector space over K under addition and multiplication in K .*

PROOF: Easy. \square

Proposition 21.1.7. \mathbb{C} is a vector space over \mathbb{R} .

PROOF: Easy. \square

Proposition 21.1.8. *Let K be a field. Let $\{V_i\}_{i \in I}$ be a family of vector spaces over K . Then $\prod_{i \in I} V_i$ is a vector space under*

$$\begin{aligned} (f + g)(i) &= f(i) + g(i) & (f, g \in \prod_{i \in I} V_i, x \in X) \\ (\lambda f)(x) &= \lambda f(x) & (\lambda \in K, f \in \prod_{i \in I} V_i, x \in X) \end{aligned}$$

PROOF: Easy. \square

21.2 Subspaces

Definition 21.2.1 (Vector Subspace). Let K be a field. Let V be a vector space over K . A *vector subspace* of V is a subset $U \subseteq V$ such that, for all $\alpha, \beta \in K$ and $x, y \in U$, we have $\alpha x + \beta y \in U$.

It is a *proper* subspace iff $U \neq V$.

Proposition 21.2.2. *If U is a subspace of V then U is a vector space under the restrictions of $+$ and \cdot to U .*

PROOF: Easy. \square

Proposition 21.2.3. *V is a subspace of V .*

PROOF: Easy. \square

Proposition 21.2.4. *If U is a subspace of V and V is a subspace of W then U is a subspace of W .*

PROOF: Easy. \square

Definition 21.2.5. Let Ω be a topological space. Then $\mathcal{C}(\Omega)$ is the complex vector space of all continuous functions from Ω to \mathbb{C} . This is a subspace of \mathbb{C}^Ω .

Definition 21.2.6. Let $n, k \in \mathbb{N}$. Let Ω be an open subset of \mathbb{R}^n . Then $\mathcal{C}^k(\Omega)$ is the complex vector space of all functions $\Omega \rightarrow \mathbb{C}$ that have all continuous partial derivatives of order k . This is a subspace of $\mathcal{C}(\Omega)$. If $l > k$ then $\mathcal{C}^l(\Omega)$ is a subspace of $\mathcal{C}^k(\Omega)$.

Definition 21.2.7. Let $n \in \mathbb{N}$. Let Ω be an open subset of \mathbb{R}^n . Then $\mathcal{C}^\infty(\Omega)$ is the complex vector space of all infinitely differentiable functions $\Omega \rightarrow \mathbb{C}$. This is a subspace of $\mathcal{C}^k(\Omega)$ for all k .

Definition 21.2.8. Let $n \in \mathbb{N}$. Let Ω be an open subset of \mathbb{R}^n . Then $\mathcal{P}(\Omega)$ is the complex vector space of all complex polynomials of n variables, considered as functions $\Omega \rightarrow \mathbb{C}$. This is a subspace of $\mathcal{C}^\infty(\Omega)$.

Proposition 21.2.9. *The space of all convergent sequences in \mathbb{C} is a subspace of the space of all bounded sequences in \mathbb{C} , which is a subspace of $\mathbb{C}^\mathbb{N}$.*

PROOF: Easy. \square

Definition 21.2.10. Let p be a real number, $p \geq 1$. Let l^p be the set of all complex sequences (z_n) such that $\sum_{n=1}^\infty |z_n|^p < \infty$.

Proposition 21.2.11. *For p a real number ≥ 1 , we have that l^p is a subspace of $\mathbb{C}^\mathbb{N}$.*

PROOF:

(1)1. For all $(x_n), (y_n) \in l^p$, we have $(x_n + y_n) \in l^p$.

PROOF: From Minkowski's Inequality.

(1)2. For all $\lambda \in \mathbb{C}$ and $(x_n) \in l^p$ we have $(\lambda x_n) \in l^p$

PROOF:

$$\sum_{n=1}^\infty |\lambda x_n|^p = |\lambda|^p \sum_{n=1}^\infty |x_n|^p < \infty$$

\square

Definition 21.2.12 (Linear Combination). Let K be a field. Let V be a vector space over K . Let $x, x_1, \dots, x_n \in V$. Then x is a *linear combination* of x_1, \dots, x_n iff there exist $\alpha_1, \dots, \alpha_n \in K$ such that

$$x = \alpha_1 x_1 + \dots + \alpha_n x_n .$$

Definition 21.2.13 (Linearly Independent). A finite set of vectors $\{x_1, \dots, x_n\}$ is *linearly independent* iff, whenever $\alpha_1 x_1 + \dots + \alpha_n x_n = 0$, then $\alpha_1 = \dots = \alpha_n = 0$.

A set of vectors is *linearly independent* iff every finite subset is linearly independent; otherwise, it is *linearly dependent*.

Definition 21.2.14 (Span). Let \mathcal{A} be a set of vectors. The *span* of \mathcal{A} , $\text{span } \mathcal{A}$, is the set of all linear combinations of elements of \mathcal{A} .

Proposition 21.2.15. $\text{span } \mathcal{A}$ is the smallest subspace of V that includes \mathcal{A} .

PROOF: Easy. \square

Definition 21.2.16 (Basis). A *basis* for V is a linearly independent set of vectors \mathcal{B} such that $\text{span } \mathcal{B} = V$.

Definition 21.2.17 (Finite Dimensional). A vector space is *finite dimensional* iff it has a finite basis; otherwise it is *infinite dimensional*.

Proposition 21.2.18. In a finite dimensional vector space, any two bases have the same number of elements.

Definition 21.2.19 (Dimension). The *dimension* of a finite dimensional vector space V , $\dim V$, is the number of elements in any basis.

Proposition 21.2.20.

$$\dim K^n = n$$

PROOF: The standard basis is the set of vectors with one coordinate 1 and all others 0. \square

Proposition 21.2.21. The dimension of \mathbb{C}^n as a real vector space is $2n$.

Proposition 21.2.22. The set of all step functions is a subspace of $\mathbb{R}^{\mathbb{R}}$.

21.3 Linear Transformations

Definition 21.3.1 (Linear Transformation). Let K be a field. Let U and V be vector spaces over K . Let $T : U \rightarrow V$. Then T is a *linear transformation* iff

$$\forall \alpha, \beta \in K. \forall x, y \in U. T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) .$$

Let Vect_K be the category of vector spaces over K and linear transformations.

Proposition 21.3.2. *If $T : U \rightarrow V$ is a linear transformation then $T(U)$ is a subspace of V .*

Proposition 21.3.3. *If $T : U \rightarrow V$ is a linear transformation then the graph of T , $\{(x, T(x)) \mid x \in U\}$, is a subspace of $U \times V$.*

Definition 21.3.4 (Null Space). Let U and V be vector spaces over K and $T : U \rightarrow V$. The *null space* of T is

$$\mathcal{N}(T) := \{x \in U \mid T(x) = 0\} .$$

Proposition 21.3.5. *If $T : U \rightarrow V$ is a linear transformation then $\mathcal{N}(T)$ is a subspace of U .*

Proposition 21.3.6. *Let U and V be vector spaces over K . The set of all linear transformations $U \rightarrow V$ is a vector space over K under*

$$\begin{aligned} (S + T)(u) &= S(u) + T(u) \\ (\lambda S)(u) &= \lambda S(u) \end{aligned}$$

21.4 Normed Spaces

Definition 21.4.1 (Norm). Let K be either \mathbb{R} or \mathbb{C} . A *norm* on a vector space V over K is a function $\| \cdot \| : V \rightarrow \mathbb{R}$ such that:

1. $\forall x \in V. \|x\| = 0 \Rightarrow x = 0$
2. $\forall \lambda \in K. \forall x \in V. \|\lambda x\| = |\lambda| \|x\|$
3. *Triangle Inequality* $\forall x, y \in V. \|x + y\| \leq \|x\| + \|y\|$

Proposition 21.4.2. *Let K be either \mathbb{R} or \mathbb{C} . Let V be a normed space over K . Define $d : V^2 \rightarrow \mathbb{R}$ by $d(x, y) = \|x - y\|$. Then d is a metric on V .*

PROOF: Easy. \square

We identify any normed space V with this metric space.

Proposition 21.4.3. *Let K be either \mathbb{R} or \mathbb{C} . Let V be a vector space over K . Let $\| \cdot \|$ be a norm on V . Then*

$$\|0\| = 0 .$$

PROOF:

$$\begin{aligned} \|0\| &= \|0 \cdot 0\| && \text{(Proposition 21.1.4)} \\ &= |0| \|0\| && \text{(Axiom 2 for a norm)} \\ &= 0 && \square \end{aligned}$$

Proposition 21.4.4. *Let K be either \mathbb{R} or \mathbb{C} . Let V be a vector space over K . Let $\| \cdot \|$ be a norm on V . Let $x \in V$. Then*

$$\|x\| \geq 0 .$$

PROOF:

$$\begin{aligned}
 0 &= \|0\| && \text{(Proposition 21.4.3)} \\
 &= \|x - x\| \\
 &\leq \|x\| + \|-x\| && \text{(Triangle Inequality)} \\
 &= \|x\| + \|x\| && \text{(Axiom 2 for a norm)} \\
 &= 2\|x\| && \square
 \end{aligned}$$

Proposition 21.4.5. *Let K be either \mathbb{R} or \mathbb{C} . Let V be a vector space over K . Let $\|\cdot\|$ be a norm on V . Let $x, y \in V$. Then*

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

PROOF:

$$\langle 1 \rangle 1. \quad \|x\| - \|y\| \leq \|x - y\|$$

PROOF: $\|x\| \leq \|x - y\| + \|y\|$ by the Triangle Inequality.

$$\langle 1 \rangle 2. \quad \|y\| - \|x\| \leq \|x - y\|$$

PROOF:

$$\begin{aligned}
 \|x\| + \|x - y\| &= \|x\| + \|y - x\| && \text{(Axiom 2 for a norm)} \\
 &\leq \|y\| && \text{(Triangle Inequality)}
 \end{aligned}$$

\square

Corollary 21.4.5.1. *Let V be a normed space. Then $\|\cdot\| : V \rightarrow \mathbb{R}$ is continuous.*

Definition 21.4.6 (Euclidean Norm). The *Euclidean norm* on \mathbb{C}^n is defined by

$$\|(z_1, \dots, z_n)\| = \sqrt{|z_1|^2 + \dots + |z_n|^2}$$

Proposition 21.4.7. *Define $\|\cdot\| : \mathbb{C}^n \rightarrow \mathbb{R}$ by*

$$\|(z_1, \dots, z_n)\| = |z_1| + \dots + |z_n|$$

Then this defines a norm on \mathbb{C}^n .

PROOF: Easy. \square

Proposition 21.4.8. *Define $\|\cdot\| : \mathbb{C}^n \rightarrow \mathbb{R}$ by*

$$\|(z_1, \dots, z_n)\| = \max(|z_1|, \dots, |z_n|)$$

Then this defines a norm on \mathbb{C}^n .

PROOF: Easy. \square

Proposition 21.4.9. *Let Ω be a closed bounded subset of \mathbb{R}^n . Define $\|\cdot\| : \mathcal{C}(\Omega) \rightarrow \mathbb{R}$ by $\|f\| = \max_{x \in \Omega} |f(x)|$. Then $\|\cdot\|$ defines a norm on $\mathcal{C}(\Omega)$.*

PROOF: Easy. \square

Proposition 21.4.10. Let p be a real number, $p \geq 1$. Define $\| \cdot \| : l^p \rightarrow \mathbb{R}$ by

$$\|(z_n)\| = \left(\sum_{n=0}^{\infty} |z_n|^p \right)^{1/p}.$$

Then this defines a norm on l^p .

PROOF: Easy. The triangle inequality is Minkowski's Inequality. \square

Definition 21.4.11 (Normed Space). Let K be either \mathbb{R} or \mathbb{C} . A *normed space* over K consists of a vector space V over K and a norm on V .

We shall write simply:

- K^n for the normed space K^n under the Euclidean norm
- l^p for the normed space l^p under the norm $\|(z_n)\| = (\sum_{n=0}^{\infty} |z_n|^p)^{1/p}$.

Proposition 21.4.12. Let K be either \mathbb{R} or \mathbb{C} . Let V be a normed space over K . If $x_n \rightarrow l$ as $n \rightarrow \infty$ in V and $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ in K , then $\lambda_n x_n \rightarrow \lambda l$ as $n \rightarrow \infty$.

PROOF:

$\langle 1 \rangle 1$. LET: $\epsilon > 0$

$\langle 1 \rangle 2$. LET: $K = |\lambda| + \epsilon/2 \|l\|$

$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $|\lambda_n - \lambda| < \epsilon/2 \|l\|$ and $\|x_n - l\| < \epsilon/(2K)$

$\langle 1 \rangle 4$. For all $n \geq N$ we have $|\lambda_n| < K$

$\langle 1 \rangle 5$. $\|\lambda_n x_n - \lambda l\| < \epsilon$

PROOF:

$$\begin{aligned} \|\lambda_n x_n - \lambda l\| &\leq \|\lambda_n x_n - \lambda_n l\| + \|\lambda_n l - \lambda l\| \\ &= |\lambda_n| \|x_n - l\| + |\lambda_n - \lambda| \|l\| \\ &< K \frac{\epsilon}{2K} + \frac{\epsilon}{2\|l\|} \|l\| \\ &= \epsilon \end{aligned}$$

\square

Proposition 21.4.13. In a normed space, if $x_n \rightarrow l$ and $y_n \rightarrow m$ then $x_n + y_n \rightarrow l + m$

PROOF:

$$\begin{aligned} \|(x_n + y_n) - (l + m)\| &\leq \|x_n - l\| + \|y_n - m\| \\ &\rightarrow 0 \end{aligned}$$

\square

Definition 21.4.14 (Uniform Convergence). Let Ω be a closed bounded set in \mathbb{R}^n . Let (f_n) be a sequence in $\mathcal{C}(\Omega)$ and $f \in \mathcal{C}(\Omega)$. Then (f_n) *converges uniformly* to f if and only if, for every $\epsilon > 0$, there exists N such that $\forall x \in \Omega, \forall n \geq N, |f_n(x) - f(x)| < \epsilon$.

Proposition 21.4.15. (f_n) converges uniformly to f iff (f_n) converges to f under the uniform convergence norm.

PROOF: Easy. \square

Proposition 21.4.16. There is no norm on $\mathcal{C}([0, 1])$ that induces pointwise convergence.

PROOF:

$\langle 1 \rangle 1$. LET: $\| \cdot \|$ be any norm on $\mathcal{C}([0, 1])$

$\langle 1 \rangle 2$. For $n \in \mathbb{N}$, define $g_n \in \mathcal{C}([0, 1])$ by

$$g_n(t) = \begin{cases} 2^n t & \text{if } 0 \leq t \leq 2^{-n} \\ 2 - 2^n t & \text{if } 2^{-n} \leq t \leq 2^{1-n} \\ 0 & \text{otherwise} \end{cases}$$

$\langle 1 \rangle 3$. For all $n \in \mathbb{N}$ we have $\|g_n\| \neq 0$

$\langle 1 \rangle 4$. For $n \in \mathbb{N}$,

LET: $f_n = g_n / \|g_n\|$

$\langle 1 \rangle 5$. For all $n \in \mathbb{N}$, $\|f_n\| = 1$

$\langle 1 \rangle 6$. f_n does not converge to 0

$\langle 1 \rangle 7$. $f_n \rightarrow 0$ as $n \rightarrow \infty$ pointwise.

\square

Definition 21.4.17 (Equivalent Norms). Let K be either \mathbb{R} or \mathbb{C} . Let V be a vector spaces over K . Then two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ are *equivalent* if and only if, for any sequence (x_n) in V and $l \in V$, we have $x_n \rightarrow l$ under $\| \cdot \|_1$ if and only if $x_n \rightarrow l$ under $\| \cdot \|_2$.

Proposition 21.4.18. Let K be either \mathbb{R} or \mathbb{C} . Let V be a vector spaces over K . Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be norms on V . Then $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent if and only if there exist positive reals α and β such that, for all $x \in V$,

$$\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1 \quad (21.1)$$

PROOF:

$\langle 1 \rangle 1$. If $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent then (21.1) holds.

$\langle 2 \rangle 1$. ASSUME: $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent.

$\langle 2 \rangle 2$. There exists $\alpha > 0$ such that, for all $x \in V$, we have $\alpha \|x\|_1 \leq \|x\|_2$

$\langle 3 \rangle 1$. ASSUME: for a contradiction $\forall \alpha > 0. \exists x \in V. \alpha \|x\|_1 > \|x\|_2$

$\langle 3 \rangle 2$. For $n \in \mathbb{Z}^+$, choose $x_n \in V$ such that $1/n \|x_n\|_1 > \|x_n\|_2$

$\langle 3 \rangle 3$. For $n \in \mathbb{Z}^+$,

LET:

$$y_n = \frac{1}{\sqrt{n}} \frac{x_n}{\|x_n\|_2}$$

$\langle 3 \rangle 4$. $\|y_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$

$\langle 3 \rangle 5$. For all $n \in \mathbb{Z}^+$, $\|y_n\|_1 > \sqrt{n}$

$\langle 3 \rangle 6$. $\|y_n\| \not\rightarrow 0$ as $n \rightarrow \infty$

$\langle 2 \rangle 3$. There exists $\beta > 0$ such that, for all $x \in V$, we have $\|x\|_2 \leq \beta \|x\|_1$

PROOF: Similar.

(1)2. If (21.1) holds then $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent.

(2)1. ASSUME: (21.1) holds.

(2)2. LET: (x_n) be a sequence in V and $l \in V$

(2)3. If $x_n \rightarrow l$ under $\| \cdot \|_1$ then $x_n \rightarrow l$ under $\| \cdot \|_2$.

(3)1. ASSUME: $x_n \rightarrow l$ und $\| \cdot \|_1$.

(3)2. LET: $\epsilon > 0$

(3)3. PICK N such that $\forall n \geq N. \|x_n - l\| < \epsilon/\beta$

(3)4. LET: $n \geq N$

(3)5. $\|x_n - l\|_2 < \epsilon$

PROOF:

$$\begin{aligned} \|x_n - l\|_2 &\leq \beta \|x_n - l\|_1 \\ &< \epsilon \end{aligned}$$

(2)4. If $x_n \rightarrow l$ under $\| \cdot \|_2$ then $x_n \rightarrow l$ under $\| \cdot \|_1$.

PROOF: Similar.

□

Proposition 21.4.19. *Let K be either \mathbb{R} or \mathbb{C} . Let V be a normed space over K . If $x_1, \dots, x_n \in V$ are linearly independent, then there exists $c > 0$ such that, for all $\alpha_1, \dots, \alpha_n \in K$,*

$$\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \geq c(|\alpha_1| + \dots + |\alpha_n|) .$$

PROOF:

(1)1. LET: $B = \{(\beta_1, \dots, \beta_n) \in K^n \mid |\beta_1| + \dots + |\beta_n| = 1\}$

(1)2. LET: $f : B \rightarrow \mathbb{R}$ be the function

$$f(\beta_1, \dots, \beta_n) = \|\beta_1 x_1 + \dots + \beta_n x_n\| .$$

(1)3. LET: c be the minimum value in $f(B)$

PROOF: f is continuous and B is compact.

(1)4. $c > 0$

PROOF: We never have $f(\beta_1, \dots, \beta_n) = 0$ by linear independence.

(1)5. LET: $\alpha_1, \dots, \alpha_n \in K$

(1)6. ASSUME: w.l.o.g. $\alpha_1, \dots, \alpha_n$ are not all zero.

(1)7. For $i = 1, \dots, n$,

$$\text{LET: } \beta_i = \alpha_i / (|\alpha_1| + \dots + |\alpha_n|)$$

(1)8. $(\beta_1, \dots, \beta_n) \in B$

(1)9. $f(\beta_1, \dots, \beta_n) \geq c$

(1)10. $\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \geq c(|\alpha_1| + \dots + |\alpha_n|)$

□

Theorem 21.4.20. *Let K be either \mathbb{R} or \mathbb{C} . Let V be a finite dimensional vector space over K . Then any two norms on V are equivalent.*

PROOF:

(1)1. PICK a basis $\{e_1, \dots, e_n\}$ for V

(1)2. LET: $\| \cdot \|_0 : V \rightarrow \mathbb{R}$ be the function

$$\|\alpha_1 e_1 + \dots + \alpha_n e_n\|_0 = |\alpha_1| + \dots + |\alpha_n| .$$

(1)3. $\| \cdot \|_0$ is a norm.

⟨2⟩1. $\forall x \in V. \|x\|_0 = 0 \Rightarrow x = 0$

PROOF: If $|\alpha_1| + \cdots + |\alpha_n| = 0$ then $\alpha_1 = \cdots = \alpha_n = 0$.

⟨2⟩2. $\forall \lambda \in K. \forall x \in V. \|\lambda x\| = |\lambda| \|x\|$

PROOF:

$$\begin{aligned} \|\lambda(\alpha_1 e_1 + \cdots + \alpha_n e_n)\|_0 &= \|\lambda \alpha_1 e_1 + \cdots + \lambda \alpha_n e_n\|_0 \\ &= |\lambda \alpha_1| + \cdots + |\lambda \alpha_n| \\ &= |\lambda|(|\alpha_1| + \cdots + |\alpha_n|) \\ &= |\lambda| \|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \end{aligned}$$

⟨2⟩3. The triangle inequality holds.

PROOF:

$$\begin{aligned} \|(\alpha_1 e_1 + \cdots + \alpha_n e_n) + (\beta_1 e_1 + \cdots + \beta_n e_n)\| &= \|(\alpha_1 + \beta_1) e_1 + \cdots + (\alpha_n + \beta_n) e_n\| \\ &= |\alpha_1 + \beta_1| + \cdots + |\alpha_n + \beta_n| \\ &\leq (|\alpha_1| + \cdots + |\alpha_n|) + (|\beta_1| + \cdots + |\beta_n|) \\ &= \|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 + \|\beta_1 e_1 + \cdots + \beta_n e_n\|_0 \end{aligned}$$

⟨1⟩4. LET: $\| \cdot \|$ be any norm on V .

PROVE: $\| \cdot \|$ is equivalent to $\| \cdot \|_0$.

⟨1⟩5. For all $\alpha_1, \dots, \alpha_n \in K$,

$$\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \leq \max(\|e_1\|, \dots, \|e_n\|)(|\alpha_1| + \cdots + |\alpha_n|)$$

⟨2⟩1. LET: $\alpha_1, \dots, \alpha_n \in K$

⟨2⟩2. $\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \leq \max(\|e_1\|, \dots, \|e_n\|)(|\alpha_1| + \cdots + |\alpha_n|)$

PROOF:

$$\begin{aligned} \|\alpha_1 e_1 + \cdots + \alpha_n e_n\| &\leq |\alpha_1| \|e_1\| + \cdots + |\alpha_n| \|e_n\| \\ &\leq (|\alpha_1| + \cdots + |\alpha_n|) \max(\|e_1\|, \dots, \|e_n\|) \quad \square \end{aligned}$$

⟨1⟩6. LET: $\beta = \max(\|e_1\|, \dots, \|e_n\|)$

⟨1⟩7. For all $x \in V$,

$$\|x\| \leq \beta \|x\|_0 .$$

⟨1⟩8. There exists $\alpha > 0$ such that, for all $x \in V$,

$$\alpha \|x\|_0 \leq \|x\| .$$

PROOF: Proposition 21.4.19.

□

Definition 21.4.21 (Closed Ball). Let K be either \mathbb{R} or \mathbb{C} . Let V be a vector space over K . Let $x \in V$ and let r be a positive real number. The *closed ball* with *centre* x and *radius* r is

$$\overline{B(x, r)} := \{y \in V \mid \|x - y\| \leq r\} .$$

Definition 21.4.22 (Sphere). Let K be either \mathbb{R} or \mathbb{C} . Let V be a vector space over K . Let $x \in V$ and let r be a positive real number. The *sphere* with *centre* x and *radius* r is

$$S(x, r) := \{y \in V \mid \|x - y\| = r\} .$$

Proposition 21.4.23. *Every closed ball is closed.*

Proposition 21.4.24. *Every sphere is closed.*

Proposition 21.4.25. *The union of two closed sets is closed.*

Proposition 21.4.26. *The intersection of a nonempty set of closed sets is closed.*

Proposition 21.4.27. *In a normed space V , both \emptyset and V are closed.*

Proposition 21.4.28. *Let V be a normed space and $C \subseteq V$. Then C is closed iff, for every sequence (x_n) in C and $l \in V$, if $x_n \rightarrow l$ then $l \in C$.*

PROOF:

$\langle 1 \rangle 1$. If C is closed then, for every sequence (x_n) in C and $l \in V$, if $x_n \rightarrow l$ then $l \in C$.

$\langle 2 \rangle 1$. ASSUME: C is closed.

$\langle 2 \rangle 2$. LET: (x_n) be a sequence in C .

$\langle 2 \rangle 3$. LET: $l \in V$

$\langle 2 \rangle 4$. ASSUME: $x_n \rightarrow l$ as $n \rightarrow \infty$

$\langle 2 \rangle 5$. ASSUME: for a contradiction $l \notin C$

$\langle 2 \rangle 6$. PICK $\epsilon > 0$ such that $B(l, \epsilon) \subseteq V - C$

$\langle 2 \rangle 7$. PICK N such that $\forall n \geq N, \|x_n - l\| < \epsilon$

$\langle 2 \rangle 8$. $x_N \in C$

$\langle 2 \rangle 9$. $\|x_N - l\| < \epsilon$

$\langle 2 \rangle 10$. Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 2$. If, for every sequence (x_n) in C and $l \in V$, if $x_n \rightarrow l$ then $l \in C$, then C is closed.

$\langle 2 \rangle 1$. ASSUME: for every sequence (x_n) in C and $l \in V$, if $x_n \rightarrow l$ then $l \in C$.

$\langle 2 \rangle 2$. LET: $x \in V - C$

$\langle 2 \rangle 3$. ASSUME: for a contradiction there is no $\epsilon > 0$ such that $B(x, \epsilon) \subseteq V - C$

$\langle 2 \rangle 4$. For $n \in \mathbb{Z}^+$, PICK $x_n \in B(x, \epsilon) \cap C$

$\langle 2 \rangle 5$. $x_n \rightarrow x$ as $n \rightarrow \infty$

$\langle 2 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

□

Definition 21.4.29 (Closure). Let V be a normed space and $A \subseteq V$. The *closure* of A , $\text{cl } A$, is the intersection of all the closed sets that include A .

Theorem 21.4.30. *Let V be a normed space and $S \subseteq V$. Then the closure of S is the set of all limits of convergent sequences in S .*

PROOF:

$\langle 1 \rangle 1$. For all $l \in \text{cl } S$, there exists a sequence (x_n) that converges to l .

$\langle 2 \rangle 1$. LET: $l \in \text{cl } S$

$\langle 2 \rangle 2$. For $n \in \mathbb{N}$, PICK $x_n \in B(l, 1/(n+1)) \cap S$

$\langle 3 \rangle 1$. ASSUME: for a contradiction $B(l, 1/(n+1))$ does not intersect S .

$\langle 3 \rangle 2$. $V - B(l, 1/(n+1))$ is a closed set that includes S .

$\langle 3 \rangle 3$. $\text{cl } S \subseteq V - B(l, 1/(n+1))$

$\langle 3 \rangle 4.$ $l \notin B(l, 1/(n+1))$

$\langle 3 \rangle 5.$ Q.E.D.

PROOF: This is a contradiction.

$\langle 2 \rangle 3.$ $x_n \rightarrow l$

$\langle 1 \rangle 2.$ For every sequence (x_n) in S , if $x_n \rightarrow l$ then $l \in \text{cl } S$.

PROOF: Proposition 21.4.28.

□

Definition 21.4.31 (Dense). Let V be a normed space and $S \subseteq V$. Then S is *dense* iff $\text{cl } S = V$.

Proposition 21.4.32. In $\mathcal{C}([a, b])$, the set of polynomials is dense.

PROOF: By the Weierstrass Theorem. □

Proposition 21.4.33. For any real $p \geq 1$, the set of all sequences with only finitely many non-zero terms is dense in l^p .

PROOF:

$\langle 1 \rangle 1.$ LET: $p \geq 1$

$\langle 1 \rangle 2.$ LET: $(z_n) \in l^p$

$\langle 1 \rangle 3.$ LET: $\epsilon > 0$

$\langle 1 \rangle 4.$ PICK N such that $\forall n \geq N, |z_n| < \epsilon/2$

$\langle 1 \rangle 5.$ LET: (y_n) be the sequence with $y_n = z_n$ for $n < N$, and $y_n = 0$ for $n \geq N$

$\langle 1 \rangle 6.$ $\|(z_n) - (y_n)\| \leq \epsilon/2$

$\langle 1 \rangle 7.$ $\|(z_n) - (y_n)\| < \epsilon$

□

Theorem 21.4.34. Let V be a normed space. Let $S \subseteq V$. Then the following are equivalent:

1. S is dense.
2. For all $x \in V$, there exists a sequence (x_n) in S such that $x_n \rightarrow x$.
3. Every nonempty open subset of V intersects S .

PROOF:

$\langle 1 \rangle 1.$ $1 \Leftrightarrow 2$

PROOF: Theorem 21.4.30.

$\langle 1 \rangle 2.$ $1 \Leftrightarrow 3$

PROOF:

S is dense \Leftrightarrow the only closed set that includes S is V

\Leftrightarrow the only open set that does not intersect S is empty

□

Definition 21.4.35 (Compact). Let V be a normed space and $S \subseteq V$. Then S is *compact* if and only if every sequence in S has a subsequence that converges to a limit in S .

Proposition 21.4.36. *Every compact set is closed.*

PROOF:

- ⟨1⟩1. LET: V be a normed space.
- ⟨1⟩2. LET: $C \subseteq V$ be compact.
- ⟨1⟩3. LET: (x_n) be a sequence in C that converges to $l \in V$.
- ⟨1⟩4. PICK a subsequence (y_n) of (x_n) that converges to $m \in C$.
- ⟨1⟩5. $l = m$
- ⟨1⟩6. $l \in C$
- ⟨1⟩7. Q.E.D.

PROOF: Proposition 21.4.28.

□

Definition 21.4.37 (Bounded). Let V be a normed space and $S \subseteq V$. Then S is *bounded* iff there exists $r > 0$ such that $S \subseteq B(0, r)$.

Proposition 21.4.38. *In \mathbb{R}^n and \mathbb{C}^n , the compact sets are the closed bounded sets.*

PROOF:

- ⟨1⟩1. LET: $C \subseteq K^n$
- ⟨1⟩2. If C is compact then C is closed.
- PROOF: Proposition 21.4.36.
- ⟨1⟩3. If C is compact then C is bounded.
- ⟨2⟩1. ASSUME: C is compact.
- ⟨2⟩2. ASSUME: for a contradiction C is not bounded.
- ⟨2⟩3. For $n \in \mathbb{N}$, PICK $x_n \in C$ with $\|x_n\| > n + 1$.
- ⟨2⟩4. PICK a convergent subsequence (x_{n_r}) that converges to $l \in C$
- ⟨2⟩5. $\|x_{n_r}\| \rightarrow \|l\|$
- ⟨2⟩6. $\|x_{n_r}\| \rightarrow +\infty$
- ⟨2⟩7. Q.E.D.

PROOF: This is a contradiction.

- ⟨1⟩4. If C is closed and bounded then C is compact.

PROOF: By the Bolzano-Weierstrass Theorem.

□

Proposition 21.4.39. *Let V be a normed space and $S \subseteq V$. Then S is bounded if and only if, for every sequence (x_n) in S and every sequence (λ_n) in K , if $\lambda_n \rightarrow 0$ then $\|\lambda_n x_n\| \rightarrow 0$.*

PROOF:

- ⟨1⟩1. If S is bounded then, for every sequence (x_n) in S and every sequence (λ_n) in K , if $\lambda_n \rightarrow 0$ then $\|\lambda_n x_n\| \rightarrow 0$.
- ⟨2⟩1. ASSUME: S is bounded.
- ⟨2⟩2. PICK $r > 0$ such that $S \subseteq B(0, r)$.
- ⟨2⟩3. LET: (x_n) be a sequence in S .
- ⟨2⟩4. LET: (λ_n) be a sequence in K .
- ⟨2⟩5. ASSUME: $\lambda_n \rightarrow 0$

- ⟨2⟩6. LET: $\epsilon > 0$
- ⟨2⟩7. PICK N such that $\forall n \geq N, |\lambda_n| < \epsilon/r$
- ⟨2⟩8. $\forall n \geq N, \|\lambda_n x_n\| < \epsilon$
- ⟨1⟩2. If S is unbounded then there exists a sequence (x_n) in S and (λ_n) in K such that $\lambda_n \rightarrow 0$ and $\|\lambda_n x_n\| \not\rightarrow 0$.
- ⟨2⟩1. S is unbounded.
- ⟨2⟩2. For $n \in \mathbb{N}$, PICK $x_n \in S$ such that $\|x_n\| > n$.
- ⟨2⟩3. For $n \in \mathbb{N}$,
LET: $\lambda_n = 1/n$ if $n > 0$, 1 if $n = 0$
- ⟨2⟩4. $\lambda_n \rightarrow 0$
- ⟨2⟩5. $\|\lambda_n x_n\| > 1$ for all $n > 1$

□

Proposition 21.4.40. *In $\mathcal{C}([0, 1])$, the unit ball $\overline{B(0, 1)}$ is closed and bounded but not compact.*

PROOF:

- ⟨1⟩1. $\overline{B(0, 1)}$ is closed.

PROOF: Proposition 21.4.23.

- ⟨1⟩2. $\overline{B(0, 1)}$ is bounded.

PROOF: $\overline{B(0, 1)} \subseteq B(0, 2)$.

- ⟨1⟩3. $\overline{B(0, 1)}$ is not compact.

- ⟨2⟩1. For $n \in \mathbb{N}$,

LET: $x_n : [0, 1] \rightarrow \mathbb{R}$ be the function $x_n(t) = t^n$.

- ⟨2⟩2. For $n \in \mathbb{N}$, we have $x_n \in \overline{B(0, 1)}$.

- ⟨2⟩3. No subsequence of (x_n) converges.

□

Theorem 21.4.41 (Riesz's Lemma). *Let X be a closed proper subspace of a normed space V . For every $\epsilon \in (0, 1)$, there exists $x_\epsilon \in V$ such that $\|x_\epsilon\| = 1$ and $\forall x \in X, \|x_\epsilon - x\| \geq \epsilon$.*

PROOF:

- ⟨1⟩1. PICK $z \in E - X$

- ⟨1⟩2. LET: $d = \inf_{x \in X} \|z - x\|$

- ⟨1⟩3. $d > 0$

- ⟨2⟩1. PICK- $\epsilon > 0$ such that $B(z, \epsilon) \subseteq E - X$

- ⟨2⟩2. $d \geq \epsilon$

- ⟨1⟩4. For all $\epsilon \in (0, 1)$, choose $y_\epsilon \in X$ such that

$$d \leq \|z - y_\epsilon\| < d/\epsilon .$$

- ⟨1⟩5. For $\epsilon \in (0, 1)$,

LET:

$$x_\epsilon = \frac{z - y_\epsilon}{\|z - y_\epsilon\|} .$$

- ⟨1⟩6. For all $x \in X$ we have $\|x_\epsilon - x\| > \epsilon$

PROOF:

$$\begin{aligned}
 \|x_\epsilon - x\| &= \left\| \frac{z - y_\epsilon}{\|z - y_\epsilon\|} - x \right\| \\
 &= \frac{1}{\|z - y_\epsilon\|} \|z - y_\epsilon - \|z - y_\epsilon\|x\| \quad (y_\epsilon + \|z - y_\epsilon\|x \in X) \\
 &\geq \frac{1}{\|z - y_\epsilon\|} d \\
 &> \epsilon
 \end{aligned}$$

□

Theorem 21.4.42. *Let V be a normed space. Then V is finite dimensional if and only if $\overline{B(0,1)}$ is compact.*

PROOF:

⟨1⟩1. If V is finite dimensional then $\overline{B(0,1)}$ is compact.

⟨2⟩1. ASSUME: V is finite dimensional.

⟨2⟩2. PICK a basis $\{e_1, \dots, e_n\}$.

⟨2⟩3. ASSUME: w.l.o.g. $\forall \alpha_1, \dots, \alpha_n \in K, \|\alpha_1 e_1 + \dots + \alpha_n e_n\| = |\alpha_1| + \dots + |\alpha_n|$

⟨2⟩4. LET: (x_m) be a sequence in $\overline{B(0,1)}$

⟨2⟩5. For $m \in \mathbb{N}$,

$$\text{LET: } x_m = \alpha_{m1}e_1 + \dots + \alpha_{mn}e_n$$

⟨2⟩6. For $m \in \mathbb{N}$ and $i = 1, \dots, n$, we have $|\alpha_{mi}| \leq 1$

⟨2⟩7. For $i = 1, \dots, n$, PICK a convergent subsequence $(\alpha_{m_r i})$ of (α_{mi}) in \mathbb{C} that converges to l_i

PROOF: Since $\overline{B(0,1)}$ is compact in K .

⟨2⟩8. x_m converges to $l_1 e_1 + \dots + l_n e_n$

PROOF:

$$\begin{aligned}
 \|x_m - l_1 e_1 - \dots - l_n e_n\| &= \|(\alpha_{m1} - l_1)e_1 + \dots + (\alpha_{mn} - l_n)e_n\| \\
 &= |\alpha_{m1} - l_1| + \dots + |\alpha_{mn} - l_n| \\
 &\rightarrow 0 \quad \text{as } m \rightarrow \infty
 \end{aligned}$$

⟨1⟩2. If V is infinite dimensional then $\overline{B(0,1)}$ is not compact.

⟨2⟩1. ASSUME: V is infinite dimensional.

⟨2⟩2. Choose a sequence (x_n) such that $\|x_n\| = 1$ and $\|x_m - x_n\| \geq 1/2$ for all $m \neq n$.

⟨3⟩1. ASSUME: as induction hypothesis there exists a sequence (x_0, x_1, \dots, x_n) such that $\|x_i\| = 1$ and $\|x_i - x_j\| \geq 1/2$ for $i \neq j$

⟨3⟩2. PICK x_{n+1} such that $\|x_{n+1}\| = 1$ and $\|x_{n+1} - x\| \geq 1/2$ for $x \in \{x_1, \dots, x_n\}$.

⟨2⟩3. ASSUME: for a contradiction (x_{n_r}) is a subsequence that converges to l

⟨2⟩4. For all r we have $1/2 \leq \|x_{n_r} - l\| + \|x_{n_{r+1}} - l\|$

PROOF:

$$\begin{aligned}
 1/2 &\leq \|x_{n_r} - x_{n_{r+1}}\| && (\langle 2 \rangle 2) \\
 &\leq \|x_{n_r} - l\| + \|x_{n_{r+1}} - l\| && (\text{Triangle Inequality})
 \end{aligned}$$

⟨2⟩5. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 21.4.43. *Let U and V be normed spaces. Let $f : U \rightarrow V$. If f is continuous at one point, then it is continuous.*

PROOF:

- ⟨1⟩1. ASSUME: f is continuous at $x_0 \in U$.
- ⟨1⟩2. LET: (x_n) be a sequence in U that converges to $l \in U$.
- ⟨1⟩3. $x_n - l + x_0 \rightarrow x_0$ as $n \rightarrow \infty$
- ⟨1⟩4. $f(x_n - l + x_0) \rightarrow f(x_0)$ as $n \rightarrow \infty$
- ⟨1⟩5. $f(x_n) - f(l) + f(x_0) \rightarrow f(x_0)$ as $n \rightarrow \infty$
- ⟨1⟩6. $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$.

□

Definition 21.4.44 (Bounded). Let U and V be normed spaces and $L : U \rightarrow V$ be a linear transformation. Then L is *bounded* iff there exists $\alpha > 0$ such that $\forall x \in U, \|L(x)\| \leq \alpha\|x\|$.

Theorem 21.4.45. *Let U and V be normed spaces. Let $L : U \rightarrow V$ be a linear transformation. Then L is continuous if and only if it is bounded.*

PROOF:

- ⟨1⟩1. If L is continuous then L is bounded.
 - ⟨2⟩1. ASSUME: L is not bounded.
 - ⟨2⟩2. For $n \in \mathbb{N}$, choose $x_n \in U$ such that $\|L(x_n)\| > (n+1)\|x_n\|$
 - ⟨2⟩3. For $n \in \mathbb{N}$,
LET: $y_n = x_n/(n+1)\|x_n\|$
 - ⟨2⟩4. $y_n \rightarrow 0$ as $n \rightarrow \infty$
 - ⟨2⟩5. For $n \in \mathbb{N}$, we have $\|L(y_n)\| > 1$
 - ⟨2⟩6. $L(y_n) \not\rightarrow 0 = L(0)$ as $n \rightarrow \infty$
 - ⟨2⟩7. L is not continuous.
- ⟨1⟩2. If L is bounded then L is continuous.
 - ⟨2⟩1. LET: $\alpha > 0$ be such that $\forall x \in U, \|L(x)\| \leq \alpha\|x\|$
PROVE: L is continuous at 0.
 - ⟨2⟩2. LET: (x_n) be a sequence in U that converges to 0.
 - ⟨2⟩3. $L(x_n) \rightarrow 0$ as $n \rightarrow \infty$
 - ⟨2⟩4. Q.E.D.

PROOF: Proposition 21.4.43.

□

□

Corollary 21.4.45.1. *If U and V are finite dimensional normed spaces, then every linear transformation $U \rightarrow V$ is continuous.*

Definition 21.4.46. For U and V normed spaces, let $\mathcal{B}(U, V)$ be the space of all bounded linear transformations $U \rightarrow V$. This is a subspace of the space of all linear transformations $U \rightarrow V$.

Define the *uniform convergence norm* on $\mathcal{B}(U, V)$ by

$$\|L\| = \sup\{\|L(x)\| \mid x \in U, \|x\| = 1\} .$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1.$ $\forall L \in \mathcal{B}(U, V). \|L\| = 0 \Rightarrow L = 0$

$\langle 2 \rangle 1.$ LET: $L \in \mathcal{B}(U, V)$

$\langle 2 \rangle 2.$ ASSUME: $\|L\| = 0$

$\langle 2 \rangle 3.$ For all $x \in U$, if $\|x\| = 1$ then $\|L(x)\| = 0$

$\langle 2 \rangle 4.$ LET: $x \in U$

PROVE: $L(x) = 0$

$\langle 2 \rangle 5.$ ASSUME: w.l.o.g. $x \neq 0$

$\langle 2 \rangle 6.$ $\|L(x/\|x\|)\| = 0$

PROOF: $\langle 2 \rangle 3$

$\langle 2 \rangle 7.$ $L(x/\|x\|) = 0$

$\langle 2 \rangle 8.$ $L(x)/\|x\| = 0$

$\langle 2 \rangle 9.$ $L(x) = 0$

$\langle 1 \rangle 2.$ $\forall \lambda \in K. \forall L \in \mathcal{B}(U, V). \|\lambda L\| = |\lambda| \|L\|$

$\langle 2 \rangle 1.$ LET: $\lambda \in K$

$\langle 2 \rangle 2.$ LET: $L \in \mathcal{B}(U, V)$

$\langle 2 \rangle 3.$ $\|\lambda L\| = |\lambda| \|L\|$

PROOF:

$$\begin{aligned} \|\lambda L\| &= \sup_{\|x\|=1} \|\lambda L(x)\| \\ &= \sup_{\|x\|=1} (|\lambda| \|L(x)\|) \\ &= |\lambda| \sup_{\|x\|=1} \|L(x)\| \\ &= |\lambda| \|L\| \end{aligned}$$

$\langle 1 \rangle 3.$ The triangle inequality holds.

$\langle 2 \rangle 1.$ LET: $L, M \in \mathcal{B}(U, V)$

$\langle 2 \rangle 2.$ For all $x \in U$, if $\|x\| = 1$ then

$$\|L(x) + M(x)\| \leq \|L(x)\| + \|M(x)\| .$$

$\langle 2 \rangle 3.$ For all $x \in U$, if $\|x\| = 1$ then

$$\|L(x) + M(x)\| \leq \|L\| + \|M\| .$$

$\langle 2 \rangle 4.$ $\|L + M\| \leq \|L\| + \|M\|$

□

Proposition 21.4.47. *Let U and V be normed spaces and $L \in \mathcal{B}(U, V)$. Then $\|L\|$ is the least number such that $\forall x \in U. \|L(x)\| \leq \|L\| \|x\|$.*

Theorem 21.4.48. *Let U and V be normed spaces. Let $T : U \rightarrow V$ be a continuous linear transformation. Then the null space $\mathcal{N}(T)$ is closed in U .*

PROOF: If (x_n) is a sequence in $\mathcal{N}(T)$ and $x_n \rightarrow l$ then $T(l) = \lim_{n \rightarrow \infty} T(x_n) = 0$ so $l \in \mathcal{N}(T)$. □

Theorem 21.4.49 (Diagonal Theorem). *Let V be a normed space. Let $(x_{ij})_{i,j \in \mathbb{N}}$ be an infinite matrix in V . If:*

1. $\forall j \in \mathbb{N}. x_{ij} \rightarrow 0$ as $i \rightarrow \infty$;
2. Every strictly increasing sequence of natural numbers (p_i) has a subsequence (q_i) such that

$$\sum_{j=0}^{\infty} x_{q_i q_j} \rightarrow 0 \text{ as } i \rightarrow \infty$$

then $x_{ii} \rightarrow 0$ as $i \rightarrow \infty$.

PROOF:

- $\langle 1 \rangle 1$. LET: V be a normed space.
- $\langle 1 \rangle 2$. LET: $(x_{ij})_{i,j \in \mathbb{N}}$ be an infinite matrix in V .
- $\langle 1 \rangle 3$. ASSUME: $\forall j \in \mathbb{N}. x_{ij} \rightarrow 0$ as $i \rightarrow \infty$
- $\langle 1 \rangle 4$. ASSUME: Every strictly increasing sequence of natural numbers (p_i) has a subsequence (q_i) such that

$$\sum_{j=0}^{\infty} x_{q_i q_j} \rightarrow 0 \text{ as } i \rightarrow \infty$$
- $\langle 1 \rangle 5$. ASSUME: for a contradiction $x_{ii} \not\rightarrow 0$ as $i \rightarrow \infty$
- $\langle 1 \rangle 6$. PICK a strictly increasing sequence (p_i) and $\epsilon > 0$ such that $\forall i \in \mathbb{N}. \|x_{p_i p_i}\| \geq \epsilon$
 - $\langle 2 \rangle 1$. PICK $\epsilon > 0$ such that, for all N , there exists $n \geq N$ such that $\|x_{ii}\| \geq \epsilon$
 - $\langle 2 \rangle 2$. Choose a sequence (p_i) such that, for all i , we have $p_{i+1} \geq p_i + 1$ and $\|x_{p_i p_i}\| \geq \epsilon$
- $\langle 1 \rangle 7$. PICK a subsequence (q_i) of (p_i) such that $\sum_{j=0}^{\infty} x_{q_i q_j} \rightarrow 0$ as $i \rightarrow \infty$.

PROOF: $\langle 1 \rangle 4$
- $\langle 1 \rangle 8$. For all i we have $x_{q_i q_j} \rightarrow 0$ as $j \rightarrow \infty$.

PROOF: Since $\sum_{j=0}^{\infty} x_{q_i q_j} < \infty$.
- $\langle 1 \rangle 9$. For all j we have $x_{q_i q_j} \rightarrow 0$ as $i \rightarrow \infty$.

PROOF: $\langle 1 \rangle 3$
- $\langle 1 \rangle 10$. PICK a subsequence (r_i) of (q_i) such that, for all i, j with $i \neq j$, we have $\|x_{r_i r_j}\| < \epsilon/2^{j+2}$.
 - $\langle 2 \rangle 1$. LET: $r_1 = q_1$
 - $\langle 2 \rangle 2$. ASSUME: as induction hypothesis we have defined (r_1, \dots, r_n) such that:
 - $r_1 < r_2 < \dots < r_n$
 - Whenever $q_i > r_j$ then $\|x_{q_i r_j}\| < \epsilon/2^{j+2}$
 - Whenever $i < j$ then $\|x_{r_i r_j}\| < \epsilon/2^{j+2}$
 - $\langle 2 \rangle 3$. LET: r_{n+1} be the first element of (q_i) such that $r_{n+1} > r_n$, $\|x_{q_i r_{n+1}}\| < \epsilon/2^{n+2}$ whenever $q_i \geq r_{n+1}$, and $\|x_{r_j r_{n+1}}\| < \epsilon/2^{n+3}$ for $j = 1, \dots, n$.

PROOF: One exists by $\langle 1 \rangle 8$.
- $\langle 2 \rangle 4$. $r_1 < \dots < r_n < r_{n+1}$
- $\langle 2 \rangle 5$. Whenever $q_i > r_j$ then $\|x_{q_i r_j}\| < \epsilon/2^{j+2}$
- $\langle 2 \rangle 6$. Whenever $i < j$ then $\|x_{r_i r_j}\| < \epsilon/2^{j+2}$
- $\langle 1 \rangle 11$. PICK a subsequence (s_i) of (r_i) such that $\sum_{j=0}^{\infty} x_{s_i s_j} \rightarrow 0$ as $i \rightarrow \infty$

PROOF: $\langle 1 \rangle 4$

$$\langle 1 \rangle 12. \forall i \in \mathbb{N}. \left\| \sum_{j=0}^{\infty} x_{s_i s_j} \right\| > \epsilon/2$$

PROOF:

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} x_{s_i s_j} \right\| &= \left\| x_{s_i s_i} + \sum_{j \neq i} x_{s_i s_j} \right\| \\ &\geq \left\| x_{s_i s_i} \right\| - \left\| \sum_{j \neq i} x_{s_i s_j} \right\| && \text{(Proposition 21.4.5)} \\ &= \left\| x_{s_i s_i} \right\| - \left\| \sum_{j \neq i} x_{s_i s_j} \right\| && \left(\left\| x_{s_i s_i} \right\| > \epsilon, \left\| \sum_j x_{s_i s_j} \right\| \leq \epsilon/2 \right) \\ &\geq \left\| x_{s_i s_i} \right\| - \sum_{j \neq i} \left\| x_{s_i s_j} \right\| && \text{(Triangle Inequality,)} \\ &> \epsilon - \sum_{j \neq i} \epsilon/2^{j+2} && (\langle 1 \rangle 6, \langle 1 \rangle 10) \\ &> \epsilon/2 \end{aligned}$$

$\langle 1 \rangle 13$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 11$.

□

21.4.1 Functionals

Definition 21.4.50 (Functional). Let K be either \mathbb{R} or \mathbb{C} . Let V be a normed space over K . A *functional* over V is a bounded linear transformation $V \rightarrow K$. The *dual space* of V is

$$V' := \mathcal{B}(V, K) .$$

21.4.2 Contraction Mappings

Definition 21.4.51 (Contraction). Let V be a normed space and $A \subseteq V$. Let $f : A \rightarrow V$. Then f is a *contraction* iff there exists a real number α with $0 < \alpha < 1$ such that, for all $x, y \in A$, we have

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| .$$

Proposition 21.4.52. *Every contraction mapping is continuous.*

21.5 Banach Spaces

Definition 21.5.1 (Cauchy Sequence). A sequence (x_n) in a normed space is a *Cauchy sequence* iff, for every $\epsilon > 0$, there exists N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$.

Theorem 21.5.2. *Let V be a normed space. Let (x_n) be a sequence in V . Then the following are equivalent.*

1. (x_n) is Cauchy.
2. For every pair of strictly increasing sequences of natural numbers (p_n) and (q_n) , we have $\|x_{p_n} - x_{q_n}\| \rightarrow 0$ as $n \rightarrow \infty$.
3. For every strictly increasing sequence of natural numbers (p_n) , we have $\|x_{p_{n+1}} - x_{p_n}\| \rightarrow 0$ as $n \rightarrow \infty$.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

- $\langle 2 \rangle 1.$ ASSUME: (x_n) is Cauchy.
- $\langle 2 \rangle 2.$ LET: (p_n) and (q_n) be a pair of increasing sequences of natural numbers.
- $\langle 2 \rangle 3.$ LET: $\epsilon > 0$
- $\langle 2 \rangle 4.$ PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$
- $\langle 2 \rangle 5.$ $\forall n \geq N. p_n, q_n \geq N$
- $\langle 2 \rangle 6.$ $\forall n \geq N. \|x_{p_n} - x_{q_n}\| < \epsilon$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: Trivial.

$\langle 1 \rangle 3. 2 \Rightarrow 1$

- $\langle 2 \rangle 1.$ ASSUME: (x_n) is not Cauchy.
- $\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that, for all N , there exist $m, n \geq N$ such that $\|x_m - x_n\| \geq \epsilon$
- $\langle 2 \rangle 3.$ PICK strictly increasing sequences of natural numbers (p_n) and (q_n) such that, for all n , $\|x_{p_n} - x_{q_n}\| \geq \epsilon$
- $\langle 3 \rangle 1.$ ASSUME: as induction hypothesis we have chosen (p_1, \dots, p_n) and (q_1, \dots, q_n) strictly increasing such that $\forall i. \|x_{p_i} - x_{q_i}\| \geq \epsilon$
- $\langle 3 \rangle 2.$ PICK $p_{n+1}, q_{n+1} \geq \max(p_n, q_n)$ such that $\|x_{p_{n+1}} - x_{q_{n+1}}\| \geq \epsilon$
- $\langle 2 \rangle 4.$ 2 is false.

$\langle 1 \rangle 4. 3 \Rightarrow 2$

- $\langle 2 \rangle 1.$ ASSUME: (p_n) and (q_n) are strictly increasing sequences such that $\|x_{p_n} - x_{q_n}\| \not\rightarrow 0$ as $n \rightarrow \infty$.
- $\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that, for all N , there exists $n \geq N$ such that $\|x_{p_n} - x_{q_n}\| \geq \epsilon$
- $\langle 2 \rangle 3.$ Choose a strictly increasing sequence (r_n) such that, for all n , we have $\|x_{r_{2n}} - x_{r_{2n+1}}\| \geq \epsilon$
- $\langle 3 \rangle 1.$ ASSUME: as induction hypothesis we have chosen $(r_0, r_1, \dots, r_{2n+1})$ such that, for $i = 0, 1, \dots, n$, we have $\|x_{r_{2i}} - x_{r_{2i+1}}\| \geq \epsilon$
- $\langle 3 \rangle 2.$ PICK $i, j \geq r_{2n+1}$ such that $\|x_i - x_j\| \geq \epsilon$
- $\langle 3 \rangle 3.$ Set $r_{2n+2} = \min(i, j)$ and $r_{2n+3} = \max(i, j)$
- $\langle 2 \rangle 4.$ 3 is false.

□

Proposition 21.5.3. *Every convergent sequence is a Cauchy sequence.*

PROOF:

- ⟨1⟩1. LET: (x_n) be a convergent sequence in a normed space V with limit l .
 ⟨1⟩2. LET: $\epsilon > 0$
 ⟨1⟩3. PICK N such that $\forall n \geq N. \|x_n - l\| < \epsilon/2$
 ⟨1⟩4. $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$
 \square

Proposition 21.5.4. *Let $\mathcal{P}([0, 1])$ be the space of polynomials on $[0, 1]$ under the norm of uniform convergence. For $n \in \mathbb{N}$, let $P_n = 1 + x + x^2/2! + \cdots + x^n/n!$. Then (P_n) is Cauchy but does not converge, since e^x is not a polynomial.*

PROOF: Easy. \square

Proposition 21.5.5. *If (x_n) is a Cauchy sequence in a normed space V , then $(\|x_n\|)$ converges in \mathbb{R} .*

PROOF:

- ⟨1⟩1. $(\|x_n\|)$ is Cauchy.
 ⟨2⟩1. LET: $\epsilon > 0$
 ⟨2⟩2. PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$
 ⟨2⟩3. $\forall m, n \geq N. \|\|x_m\| - \|x_n\|\| < \epsilon$

PROOF: Proposition 21.4.5.

- ⟨1⟩2. Q.E.D.

PROOF: Since \mathbb{R} is complete.

\square

Corollary 21.5.5.1. *Every Cauchy sequence is bounded.*

Definition 21.5.6 (Banach space). A normed space is *complete* or a *Banach space* iff every Cauchy sequence converges.

Proposition 21.5.7. *For all $p \geq 1$, the space l^p is complete.*

PROOF:

- ⟨1⟩1. LET: (a_n) be a Cauchy sequence in l^p .
 ⟨1⟩2. For $n \in \mathbb{N}$,
 LET: $a_n = (\alpha_{n0}, \alpha_{n1}, \dots)$
 ⟨1⟩3. For all $\epsilon > 0$, there exists N such that $\forall m, n \geq N$

$$\sum_{k=0}^{\infty} |\alpha_{mk} - \alpha_{nk}|^p < \epsilon .$$

PROOF: ⟨1⟩1

- ⟨1⟩4. For all $k \in \mathbb{N}$ and $\epsilon > 0$, there exists N such that $\forall m, n \geq N$
 $|\alpha_{mk} - \alpha_{nk}| < \epsilon .$

PROOF: ⟨1⟩3

- ⟨1⟩5. For all $k \in \mathbb{N}$, the sequence $(\alpha_{nk})_n$ converges in \mathbb{C} .

PROOF: \mathbb{C} is complete.

- ⟨1⟩6. For $k \in \mathbb{N}$,

LET:

$$\alpha_k = \lim_{n \rightarrow \infty} \alpha_{nk} .$$

⟨1⟩7. LET: $a = (\alpha_k)_k$

⟨1⟩8. For all $\epsilon > 0$, there exists N such that $\forall n \geq N$

$$\sum_{k=0}^{\infty} |\alpha_k - \alpha_{nk}|^p < \epsilon .$$

PROOF: Take the limit $m \rightarrow \infty$ in ⟨1⟩4.

⟨1⟩9. $a \in l^p$

⟨2⟩1. PICK N such that $\forall n \geq N, \sum_{k=0}^{\infty} |\alpha_k - \alpha_{nk}|^p < 1$

⟨2⟩2. $a - a_N \in l^p$

⟨2⟩3. Q.E.D.

PROOF: Since l^p is closed under $+$.

⟨1⟩10. $a_n \rightarrow a$ as $n \rightarrow \infty$.

PROOF: Immediate from ⟨1⟩8.

□

Proposition 21.5.8. *For any real number a, b with $a < b$, the space $\mathcal{C}([a, b])$ is complete.*

PROOF:

⟨1⟩1. LET: (f_n) be a Cauchy sequence in $\mathcal{C}([a, b])$

⟨1⟩2. For all $\epsilon > 0$, there exists N such that, for all $m, n \geq N$ and $x \in [a, b]$,

$$|f_m(x) - f_n(x)| < \epsilon .$$

⟨1⟩3. For all $x \in [a, b]$, $(f_n(x))_n$ is Cauchy.

⟨1⟩4. LET: $f : [a, b] \rightarrow \mathbb{C}$ be the function

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) .$$

⟨1⟩5. For all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in [a, b]$,

$$|f_n(x) - f(x)| < \epsilon .$$

PROOF: Take the limit $m \rightarrow \infty$ in ⟨1⟩2.

⟨1⟩6. f is continuous.

⟨2⟩1. LET: $x_0 \in [a, b]$

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. PICK N such that, for all $n \geq N$ and $y \in [a, b]$, we have

$$|f_n(y) - f(y)| < \epsilon/3 .$$

PROOF: ⟨1⟩5

⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in [a, b]$ with $|x_0 - y| < \delta$, we have

$$|f_N(x_0) - f_N(y)| < \epsilon/3$$

⟨2⟩5. For all $y \in [a, b]$, if $|x_0 - y| < \delta$ then $|f(x_0) - f(y)| < \epsilon$

PROOF:

$$|f(x_0) - f(y)|$$

$$\leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(y)| + |f_N(y) - f(y)| \quad (\text{Triangle Inequality})$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 \quad (\langle 2 \rangle 3, \langle 2 \rangle 4)$$

$$= \epsilon$$

⟨1⟩7. $f_n \rightarrow f$ as $n \rightarrow \infty$.

PROOF: Immediate from ⟨1⟩5.

□

Definition 21.5.9 (Convergent Series). Let (x_n) be a sequence in a normed space V . We say the series $\sum_{n=0}^{\infty} x_n$ is *convergent* iff the sequence $(\sum_{n=0}^N x_n)_N$ is convergent, and then we write $\sum_{n=0}^{\infty} x_n = l$ for $\lim_{N \rightarrow \infty} \sum_{n=0}^N x_n = l$.

Definition 21.5.10 (Absolutely Convergent Series). Let (x_n) be a sequence in a normed space V . We say the series $\sum_{n=0}^{\infty} x_n$ is *absolutely convergent* iff the series $\sum_{n=0}^{\infty} \|x_n\|$ converges in \mathbb{R} .

Theorem 21.5.11. *A normed space is complete if and only if every absolutely convergent series is convergent.*

PROOF:

$\langle 1 \rangle 1$. LET: V be a normed space.

$\langle 1 \rangle 2$. If V is complete then every absolutely convergent series is convergent.

$\langle 2 \rangle 1$. ASSUME: V is complete.

$\langle 2 \rangle 2$. LET: (x_n) be a sequence in V .

$\langle 2 \rangle 3$. ASSUME: $\sum_{n=0}^{\infty} \|x_n\|$ converges.

PROVE: $(\sum_{n=0}^N x_n)_N$ is Cauchy.

$\langle 2 \rangle 4$. LET: $\epsilon > 0$

$\langle 2 \rangle 5$. PICK N such that $\sum_{n=N}^{\infty} \|x_n\| < \epsilon$

$\langle 2 \rangle 6$. LET: $m > n \geq N$

$\langle 2 \rangle 7$. $\|\sum_{k=0}^m x_k - \sum_{k=0}^n x_k\| < \epsilon$

PROOF:

$$\begin{aligned} \left\| \sum_{k=0}^m x_k - \sum_{k=0}^n x_k \right\| &= \left\| \sum_{k=n+1}^m x_k \right\| \\ &\leq \sum_{k=n+1}^m \|x_k\| \\ &\leq \sum_{k=n+1}^{\infty} \|x_k\| \\ &< \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. If every absolutely convergent series is convergent then V is complete.

$\langle 2 \rangle 1$. ASSUME: Every absolutely convergent series is convergent.

$\langle 2 \rangle 2$. LET: (x_n) be a Cauchy sequence in V .

$\langle 2 \rangle 3$. Choose an increasing sequence of natural numbers (p_k) such that, for all $m, n \geq p_k$, we have

$$\|x_m - x_n\| < 2^{-k}.$$

$\langle 2 \rangle 4$. $\sum_{k=0}^{\infty} \|x_{p_{k+1}} - x_{p_k}\|$ is absolutely convergent.

$\langle 2 \rangle 5$. $\sum_{k=0}^{\infty} \|x_{p_{k+1}} - x_{p_k}\|$ is convergent.

$\langle 2 \rangle 6$. (x_{p_k}) converges.

$\langle 2 \rangle 7$. LET: $l = \lim_{k \rightarrow \infty} x_{p_k}$

$\langle 2 \rangle 8$. $x_n \rightarrow l$ as $n \rightarrow \infty$.

PROOF:

$$\begin{aligned} \|x_n - l\| &\leq \|x_n - x_{p_n}\| + \|x_{p_n} - l\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ (Theorem 21.5.2)} \end{aligned}$$

□

Proposition 21.5.12. *A closed subspace of a Banach space is a Banach space.*

PROOF:

- ⟨1⟩1. LET: V be a Banach space.
- ⟨1⟩2. LET: U be a closed subspace of V .
- ⟨1⟩3. LET: (x_n) be a Cauchy sequence in U .
- ⟨1⟩4. (x_n) is a Cauchy sequence in V .
- ⟨1⟩5. LET: l be the limit of (x_n) in V .
- ⟨1⟩6. $l \in U$

PROOF: Proposition 21.4.28.

□

Definition 21.5.13 (Completion). Let V be a normed space. A *completion* of V consists of a Banach space W and a function $\phi : V \rightarrow W$ such that:

- ϕ is injective.
- ϕ is a linear transformation.
- ϕ preserves the norm.
- $\phi(V)$ is dense in W .

Definition 21.5.14 (Equivalent Cauchy Sequences). Let V be a normed space. Let (x_n) and (y_n) be Cauchy sequences in V . Then (x_n) and (y_n) are *equivalent*, $(x_n) \sim (y_n)$, iff $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Proposition 21.5.15. *Equivalence is an equivalence relation on the set of Cauchy sequences.*

Proposition 21.5.16. *If $(x_n) \sim (y_n)$ then $\lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|y_n\|$.*

Theorem 21.5.17. *Let V be a normed space. Let W be the quotient set of all Cauchy sequences modulo \sim . Define $+$, \cdot and $\| \cdot \|$ on W by*

$$\begin{aligned} [(x_n)] + [(y_n)] &= [(x_n + y_n)] \\ \lambda[(x_n)] &= [(\lambda x_n)] \\ \|[(x_n)]\| &= \lim_{n \rightarrow \infty} \|x_n\| \end{aligned}$$

Define $\phi : V \rightarrow W$ by $\phi(x)$ is the constant sequence (x) . Then $\phi : V \rightarrow W$ is the completion of V .

PROOF:

- ⟨1⟩1. $+$, \cdot and $\| \cdot \|$ are well defined.
- ⟨1⟩2. W is a normed space.
- ⟨1⟩3. $\phi(V)$ is dense in W .

PROOF: For any $[(x_n)] \in W$ we have $[(x_n)] = \lim_{n \rightarrow \infty} \phi(x_n)$.

- ⟨1⟩4. W is complete.

- (2)1. LET: (X_n) be a Cauchy sequence in W .
 (2)2. For all $n \in \mathbb{N}$, PICK $x_n \in V$ such that $\|\phi(x_n) - X_n\| < 1/n$
 PROOF: (1)3
 (2)3. For all m, n we have $\|x_n - x_m\| \leq \|X_n - X_m\| + 1/n + 1/m$
 PROOF:

$$\begin{aligned} \|x_n - x_m\| &= \|\phi(x_n) - \phi(x_m)\| \\ &\leq \|\phi(x_n) - X_n\| + \|X_n - X_m\| + \|\phi(x_m) - X_m\| \\ &\leq \|X_n - X_m\| + 1/n + 1/m \end{aligned}$$

 (2)4. (x_n) is a Cauchy sequence in V .
 (2)5. LET: $X = [(x_n)]$
 PROVE: $X_n \rightarrow X$ as $n \rightarrow \infty$
 (2)6. $\|X_n - X\| \rightarrow 0$ as $n \rightarrow \infty$
 PROOF:

$$\begin{aligned} \|X_n - X\| &\leq \|X_n - \phi(x_n)\| + \|\phi(x_n) - X\| \\ &< \| \phi(x_n) - X \| + 1/n \\ &\rightarrow 0 \end{aligned}$$

- (1)5. ϕ is injective.
 (1)6. ϕ is a linear transformation.
 (1)7. ϕ preserves the norm.
 \square

Theorem 21.5.18. *Let U be a normed space and V a Banach space. Then $\mathcal{B}(U, V)$ is a Banach space.*

PROOF:

- (1)1. LET: (L_n) be a Cauchy sequence in $\mathcal{B}(U, V)$
 (1)2. For all $x \in U$, we have $(L_n(x))$ is a Cauchy sequence in V .
 (2)1. LET: $x \in U$
 (2)2. ASSUME: w.l.o.g. $x \neq 0$
 (2)3. LET: $\epsilon > 0$
 (2)4. PICK N such that $\forall m, n \geq N. \|L_m - L_n\| < \epsilon/\|x\|$
 (2)5. $\forall m, n \geq N. \|L_m(x) - L_n(x)\| < \epsilon$
 PROOF: $\|L_m(x) - L_n(x)\| \leq \|L_m - L_n\| \|x\| < \epsilon$
 (1)3. Define $L : U \rightarrow V$ by $L(x) = \lim_{n \rightarrow \infty} L_n(x)$
 (1)4. $L \in \mathcal{B}(U, V)$
 (2)1. L is linear.
 (3)1. LET: $\lambda, \mu \in K$ and $x, y \in U$
 (3)2. $L(\lambda x + \mu y) = \lambda L(x) + \mu L(y)$
 PROOF:

$$\begin{aligned} L(\lambda x + \mu y) &= \lim_{n \rightarrow \infty} L_n(\lambda x + \mu y) \\ &= \lim_{n \rightarrow \infty} (\lambda L_n(x) + \mu L_n(y)) \\ &= \lambda L(x) + \mu L(y) \end{aligned}$$

 (2)2. L is bounded.
 (3)1. PICK N such that $\forall m, n \geq N. \|L_m - L_n\| < 1$

PROVE: $\forall x \in U, \|L(x)\| \leq (\|L_N\| + 1)\|x\|$
 (3)2. $\forall n \geq N, \|L_n\| \leq \|L_N\| + 1$
 PROOF: Since $\|L_n - L_N\| \leq \|L_n - L_N\| < 1$.
 (3)3. LET: $x \in U$
 (3)4. $\|L(x)\| \leq (\|L_N\| + 1)\|x\|$
 PROOF:

$$\begin{aligned} \|L(x)\| &= \left\| \lim_{n \rightarrow \infty} L_n(x) \right\| \\ &= \lim_{n \rightarrow \infty} \|L_n(x)\| \\ &\leq \lim_{n \rightarrow \infty} \|L_n\| \|x\| \\ &\leq (\|L_N\| + 1)\|x\| \end{aligned}$$

(1)5. $L_n \rightarrow L$ as $n \rightarrow \infty$
 (2)1. LET: $\epsilon > 0$
 (2)2. PICK N such that $\forall m, n \geq N, \|L_m - L_n\| < \epsilon/4$
 (2)3. LET: $n \geq N$
 (2)4. For all $x \in U$ we have $\|L_n(x) - L(x)\| < (\epsilon/2)\|x\|$
 (3)1. LET: $x \in U$
 (3)2. For all $m \geq N$ we have $\|L_n(x) - L_m(x)\| < (\epsilon/4)\|x\|$
 (3)3. $\|L_n(x) - L(x)\| \leq (\epsilon/4)\|x\|$
 PROOF: Taking the limit as $m \rightarrow \infty$.
 (2)5. $\|L_n - L\| \leq \epsilon/2$
 (2)6. $\|L_n - L\| < \epsilon$

□

Corollary 21.5.18.1. *The dual space of a normed space is a Banach space.*

Theorem 21.5.19. *Let U be a normed space and V a Banach space. Let W be a subspace of U . Let $L : W \rightarrow V$ be a bounded linear transformation. Then L has a unique extension to a bounded linear transformation $\bar{W} \rightarrow V$.*

PROOF:

(1)1. Define $L' : \bar{W} \rightarrow V$ as follows. Given $x \in \bar{W}$, pick a sequence (x_n) in W that converges to x . Then $L'(x) = \lim_{n \rightarrow \infty} L(x_n)$
 (2)1. For all $x \in \bar{W}$, there exists a sequence (x_n) in W that converges to x .
 PROOF: Theorem 21.4.30.
 (2)2. For any sequence (x_n) in W that converges in \bar{W} , we have $(L(x_n))$ converges in V .
 (3)1. LET: (x_n) be a sequence in W that converges in \bar{W}
 (3)2. (x_n) is Cauchy.
 (3)3. $(L(x_n))$ is Cauchy.
 PROOF: For any strictly increasing sequence of natural numbers (p_n) , we have $\|L(x_{p_{n+1}}) - L(x_{p_n})\| \leq \|L\| \|x_{p_{n+1}} - x_{p_n}\| \rightarrow 0$ as $n \rightarrow \infty$.
 (3)4. Q.E.D.
 PROOF: W is a Banach space.
 (2)3. If (x_n) and (y_n) are sequences in W that converge to the same point in \bar{W} , then $\lim_{n \rightarrow \infty} L(x_n) = \lim_{n \rightarrow \infty} L(y_n)$

PROOF: Since $\|L(x_n) - L(y_n)\| \leq \|L\|\|x_n - y_n\| \rightarrow 0$.

(1)2. L' extends L

PROOF: For $x \in W$ we have the constant sequence (x) converges to x , and the constant sequence $(L(x))$ converges to $L(x)$, so $L'(x) = L(x)$.

(1)3. L' is a linear transformation.

(2)1. LET: $\lambda, \mu \in K$ and $x, y \in \overline{W}$

(2)2. PICK sequences $(x_n), (y_n)$ in W that converge to x and y respectively.

(2)3. $\lambda x_n + \mu y_n \rightarrow \lambda x + \mu y$

(2)4. Q.E.D.

PROOF:

$$\begin{aligned} L'(\lambda x + \mu y) &= \lim_{n \rightarrow \infty} L(\lambda x_n + \mu y_n) \\ &= \lim_{n \rightarrow \infty} (\lambda L(x_n) + \mu L(y_n)) \\ &= \lambda L'(x) + \mu L'(y) \end{aligned}$$

(1)4. L' is bounded.

(2)1. LET: $x \in \overline{W}$

(2)2. PICK a sequence (x_n) in W that converges to x .

(2)3. $\|L'(x)\| \leq \|L\|\|x\|$

PROOF:

$$\begin{aligned} \|L'(x)\| &= \left\| \lim_{n \rightarrow \infty} L(x_n) \right\| \\ &= \lim_{n \rightarrow \infty} \|L(x_n)\| \\ &\leq \|L\| \lim_{n \rightarrow \infty} \|x_n\| \\ &= \|L\|\|x\| \end{aligned}$$

(1)5. If $L'' : \overline{W} \rightarrow V$ is a bounded linear transformation that extends L , then $L'' = L'$.

(2)1. LET: $x \in \overline{W}$

(2)2. PICK a sequence (x_n) in W that converges to x .

(2)3. $L''(x) = L'(x)$

PROOF:

$$\begin{aligned} L''(x) &= \lim_{n \rightarrow \infty} L''(x_n) \\ &= \lim_{n \rightarrow \infty} L(x_n) \\ &= L'(x_n) \end{aligned}$$

□

Theorem 21.5.20 (Banach-Steinhaus). *Let X be a Banach space and Y a normed space. Let \mathcal{T} be a set of bounded linear transformations from X into Y . Assume that, for all $x \in X$, there exists $M_x > 0$ such that, for all $T \in \mathcal{T}$, we have $\|T(x)\| \leq M_x$. Then there exists $M > 0$ such that, for all $T \in \mathcal{T}$, we have $\|T\| \leq M$.*

PROOF:

(1)1. LET: X be a Banach space.

(1)2. LET: Y be a normed space.

- ⟨1⟩3. LET: \mathcal{T} be a set of bounded linear transformations.
 ⟨1⟩4. ASSUME: For all $x \in X$, there exists $M_x > 0$ such that, for all $T \in \mathcal{T}$, we have $\|T(x)\| \leq M_x$.
 ⟨1⟩5. ASSUME: for a contradiction there is no $M > 0$ such that, for all $T \in \mathcal{T}$, we have $\|T\| \leq M$
 ⟨1⟩6. For every positive integer n , choose $T_n \in \mathcal{T}$ such that $\|T_n\| > n2^n$.
 ⟨1⟩7. For every positive integer n , choose $x_n \in U$ such that $\|x_n\| = 1$ and $\|T_n(x_n)\| > n2^n$.
 ⟨1⟩8. For every positive integer n we have

$$\left\| \frac{1}{n} T_n \left(\frac{x_n}{2^n} \right) \right\| > 1 .$$

- ⟨1⟩9. For positive integers i and j ,

LET:

$$y_{ij} = \frac{1}{i} T_i \left(\frac{x_j}{2^j} \right)$$

- ⟨1⟩10. LET: $z = \sum_{j=1}^{\infty} \frac{x_j}{2^j}$

- ⟨2⟩1. $\sum_{j=1}^{\infty} \frac{x_j}{2^j}$ is absolutely convergent.

PROOF:

$$\begin{aligned} \sum_{j=1}^{\infty} \left\| \frac{x_j}{2^j} \right\| &= \sum_{j=1}^{\infty} \frac{1}{2^j} \\ &= 1 \end{aligned} \quad (\langle 1 \rangle 7)$$

- ⟨2⟩2. Q.E.D.

PROOF: Theorem 21.5.11.

- ⟨1⟩11. PICK $C > 0$ such that, for all i , we have $\left\| \sum_{j=1}^{\infty} y_{ij} \right\| \leq C/i$.

- ⟨2⟩1. PICK $C > 0$ such that, for all $T \in \mathcal{T}$, we have $\|T(z)\| \leq C$.

PROOF: ⟨1⟩4.

- ⟨2⟩2. For all i we have $\left\| \sum_{j=1}^{\infty} y_{ij} \right\| \leq C/i$

PROOF:

$$\left\| \sum_{j=1}^{\infty} y_{ij} \right\| = \left\| \sum_{j=1}^{\infty} \frac{1}{i} T_i \left(\frac{x_j}{2^j} \right) \right\| \quad (\langle 1 \rangle 9)$$

$$= \frac{1}{i} \left\| T_i \left(\sum_{j=1}^{\infty} \frac{x_j}{2^j} \right) \right\| \quad (T_i \text{ continuous by } \langle 1 \rangle 3)$$

$$= \frac{1}{i} \|T_i(z)\| \quad (\langle 1 \rangle 10)$$

$$\leq \frac{C}{i} \quad (\langle 2 \rangle 1)$$

- ⟨1⟩12. $\sum_{j=1}^{\infty} y_{ij} \rightarrow 0$ as $i \rightarrow \infty$

- ⟨1⟩13. For any increasing sequence of positive integers (q_i) , we have $\sum_{j=0}^{\infty} y_{q_i q_j} \rightarrow 0$ as $i \rightarrow \infty$

PROOF: Similar.

- ⟨1⟩14. For all j we have $y_{ij} \rightarrow 0$ as $i \rightarrow \infty$

PROOF: From ⟨1⟩9 and the fact that T_i is continuous.

$\langle 1 \rangle 15.$ $y_{ii} \rightarrow 0$ as $i \rightarrow \infty$.

PROOF: By the Diagonal Theorem.

$\langle 1 \rangle 16.$ Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 8$.

□

Theorem 21.5.21 (Banach Fixed Point Theorem). *Let V be a Banach space. Let $F \subseteq V$ be closed and nonempty. Let $f : F \rightarrow F$ be a contraction. Then there exists a unique $z \in F$ such that $f(z) = z$.*

PROOF:

$\langle 1 \rangle 1.$ PICK α such that $0 < \alpha < 1$ and $\forall x, y \in V. \|f(x) - f(y)\| \leq \alpha \|x - y\|$.

$\langle 1 \rangle 2.$ PICK $x_0 \in F$

$\langle 1 \rangle 3.$ Extend to the sequence (x_n) in F by defining $x_{n+1} := f(x_n)$.

$\langle 1 \rangle 4.$ (x_n) is Cauchy.

$\langle 2 \rangle 1.$ $\forall n \in \mathbb{N}. \|x_{n+1} - x_n\| \leq \alpha^n \|x_1 - x_0\|$

$\langle 2 \rangle 2.$ LET: $\epsilon > 0$

$\langle 2 \rangle 3.$ PICK N such that $\frac{\|x_1 - x_0\|}{1 - \alpha} \alpha^N \leq \epsilon$

$\langle 2 \rangle 4.$ $\forall m, n \geq N. \|x_n - x_m\| < \epsilon$

$\langle 3 \rangle 1.$ LET: $m, n \geq N$

$\langle 3 \rangle 2.$ ASSUME: w.l.o.g. $m < n$

$\langle 3 \rangle 3.$ $\|x_n - x_m\| < \epsilon$

PROOF:

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\| \\ &\leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m) \|x_1 - x_0\| \\ &< \frac{\|x_1 - x_0\|}{1 - \alpha} \alpha^m \\ &\leq \frac{\|x_1 - x_0\|}{1 - \alpha} \alpha^N \\ &\leq \epsilon \end{aligned}$$

$\langle 1 \rangle 5.$ LET: $z = \lim_{n \rightarrow \infty} x_n$

$\langle 1 \rangle 6.$ z is unique such that $f(z) = z$.

$\langle 2 \rangle 1.$ $f(z) = z$

PROOF:

$$\begin{aligned} f(z) &= f\left(\lim_{n \rightarrow \infty} x_n\right) \\ &= \lim_{n \rightarrow \infty} f(x_n) \\ &= \lim_{n \rightarrow \infty} x_{n+1} \\ &= z \end{aligned}$$

$\langle 2 \rangle 2.$ If $f(w) = w$ then $w = z$.

$\langle 3 \rangle 1.$ $\|w - z\| \leq \alpha \|w - z\|$

PROOF: $\|w - z\| = \|f(w) - f(z)\| \leq \alpha \|w - z\|$

$\langle 3 \rangle 2.$ $\|w - z\| = 0$

PROOF: Otherwise $\|w - z\| < \|w - z\|$.

$\langle 3 \rangle 3.$ $w = z$

□

21.6 Inner Product Spaces

Definition 21.6.1 (Inner Product Space). An *inner product* on a complex vector space V is a function $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{C}$ such that:

- $\forall x, y \in V. \langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\forall \lambda \in \mathbb{C}. \forall x, y \in V. \langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
- $\forall x, y, z \in V. \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\forall x \in V. \langle x, x \rangle = 0 \Rightarrow x = 0$

An *inner product space* or *pre-Hilbert space* is a complex vector space with an inner product.

Proposition 21.6.2. \mathbb{C}^n is an inner product space under

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = x_1 \overline{y_1} + \dots + x_n \overline{y_n} .$$

Proposition 21.6.3. l^2 is an inner product under

$$\langle (x_n), (y_n) \rangle = \sum_{n=0}^{\infty} x_n \overline{y_n} .$$

PROOF:

$\langle 1 \rangle 1$. For all $(x_n), (y_n) \in l^2$ we have $\sum_n x_n \overline{y_n} < \infty$

$\langle 2 \rangle 1$. LET: $(x_n), (y_n) \in l^2$

$\langle 2 \rangle 2$. $\sum_{n=1}^N |x_n \overline{y_n}| \leq (\sum_{n=1}^{\infty} |x_n|^2)^{1/2} (\sum_{n=1}^{\infty} |y_n|^2)^{1/2}$

PROOF:

$$\begin{aligned} \sum_{n=1}^N |x_n \overline{y_n}| &= \sum_{n=1}^N |x_n| |y_n| \\ &\leq \left(\sum_{n=1}^N |x_n|^2 \right)^{1/2} \left(\sum_{n=1}^N |y_n|^2 \right)^{1/2} \quad (\text{Cauchy-Schwarz}) \\ &\leq \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} |y_n|^2 \right)^{1/2} \end{aligned}$$

$\langle 2 \rangle 3$. $\sum_{n=1}^N x_n \overline{y_n}$ is absolutely convergent.

$\langle 1 \rangle 2$. $\forall (x_n), (y_n) \in l^2. \langle (x_n), (y_n) \rangle = \overline{\langle (y_n), (x_n) \rangle}$

$\langle 1 \rangle 3$. $\forall \lambda \in \mathbb{C}. \forall (x_n), (y_n) \in l^2. \langle \lambda (x_n), (y_n) \rangle = \lambda \langle (x_n), (y_n) \rangle$

$\langle 1 \rangle 4$. $\forall (x_n), (y_n), (z_n) \in l^2. \langle (x_n) + (y_n), (z_n) \rangle = \langle (x_n), (z_n) \rangle + \langle (y_n), (z_n) \rangle$

$\langle 1 \rangle 5$. $\forall (x_n) \in l^2. \langle (x_n), (x_n) \rangle = 0 \Rightarrow (x_n) = 0$

□

Proposition 21.6.4. *The space $C[0, 1]$ of all continuous functions $[0, 1] \rightarrow \mathbb{C}$ is an inner product space under*

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt .$$

Proposition 21.6.5. *The space \mathbb{C}^{mn} of all $m \times n$ complex matrices is an inner product space under*

$$\langle A, B \rangle = \text{tr}(B^* A)$$

where B^* is the conjugate transpose of B .

Theorem 21.6.6. *Let V be an inner product space. Let $x, y, z \in V$ and $\lambda \in \mathbb{C}$. Then:*

1. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
2. $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$
3. $\langle x, 0 \rangle = \langle 0, x \rangle = 0$
4. If $\forall w \in V. \langle x, w \rangle = \langle y, w \rangle$ then $x = y$
5. $\langle x, x \rangle$ is a non-negative real.

PROOF: For part 4, take $w = x - y$. \square

Theorem 21.6.7 (Cauchy-Schwarz Inequality). *Let V be an inner product space. Let $x, y \in V$. Then*

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} .$$

Equality holds if and only if x and y are linearly dependent.

PROOF:

$\langle 1 \rangle 1$. If x and y are linearly dependent then $|\langle x, y \rangle| = \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$.

PROOF: If $y = \lambda x$ then both sides are equal to $|\lambda| \langle x, x \rangle$.

$\langle 1 \rangle 2$. If x and y are linearly independent then $|\langle x, y \rangle| < \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$.

$\langle 2 \rangle 1$. For any $\lambda \in \mathbb{C}$ with $x + \lambda y \neq 0$ we have

$$\langle x, x \rangle + 2\Re(\overline{\lambda} \langle x, y \rangle) + |\lambda|^2 \langle y, y \rangle > 0 .$$

PROOF:

$$\begin{aligned} 0 &< \langle x + \lambda y, x + \lambda y \rangle \\ &= \langle x, x \rangle + \overline{\lambda} \langle x, y \rangle + \lambda \langle y, x \rangle + |\lambda|^2 \langle y, y \rangle \\ &= \langle x, x \rangle + 2\Re(\overline{\lambda} \langle x, y \rangle) + |\lambda|^2 \langle y, y \rangle \end{aligned}$$

$\langle 2 \rangle 2$. LET: $u = |\langle x, y \rangle| / \langle x, y \rangle$ or $u = 1$ if $\langle x, y \rangle = 0$

$\langle 2 \rangle 3$. For any $t \in \mathbb{R}$,

$$\langle x, x \rangle + 2|\langle x, y \rangle|t + \langle y, y \rangle t^2 > 0$$

PROOF: Take $\lambda = tu$ in $\langle 2 \rangle 1$

$\langle 2 \rangle 4$.

$$4|\langle x, y \rangle|^2 - 4\langle x, x \rangle \langle y, y \rangle < 0$$

PROOF: The quadratic $\langle 2 \rangle 3$ must have negative discriminant.
 $\langle 2 \rangle 5$.

$$|\langle x, y \rangle| < \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$$

□

Theorem 21.6.8. *Every inner product space is a normed space under $\|x\| = \langle x, x \rangle^{1/2}$.*

PROOF:

$\langle 1 \rangle 1$. If $\|x\| = 0$ then $x = 0$

$\langle 1 \rangle 2$. $\|\lambda x\| = |\lambda| \|x\|$

$\langle 1 \rangle 3$. $\|x + y\| \leq \|x\| + \|y\|$

PROOF:

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 2\Re \langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \\ &< \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 && \text{(Cauchy-Schwarz)} \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

□

Theorem 21.6.9 (Parallelogram Law). *Let V be an inner product space. Let $x, y \in V$. Then*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

PROOF:

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 \\ \|x - y\|^2 &= \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2 \end{aligned}$$

Theorem 21.6.10 (Polarization Identity). *Let V be an inner product space. Let $x, y \in V$. Then*

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 .$$

PROOF: Straightforward calculation. □