Mathematics

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Contents

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*.

For any set A, let there be *elements* of A. We write $a \in A$ for: a is an element of A.

For any sets A and B, let there be a set B^A, whose elements are called *functions* from A to B. We write $f : A \to B$ for $f \in B^A$.

For any function $f : A \to B$ and element $a \in A$, let there be an element $f(a) \in B$, the *value* of the function f at the *argument* a.

1.2 Axioms

Axiom Schema 1.2.1 (Choice). *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x : \mathrm{El}\,(X)$ and $y : \mathrm{El}\,(Y)$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a : \mathrm{El}\,(A)$, there exists $b : \mathrm{El}\,(B)$ such that $P[A, B, a, b]$. Then there exists a function $f : A \to B$ such that $\forall a : \mathrm{El}\,(A)\,.P[A, B, a, f(a)]$.

Axiom 1.2.2 (Extensionality). *Let $f, g : A \to B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Axiom 1.2.3 (Pairing). *For any sets A and B, there exists a set $A \times B$, the* Cartesian product *of A and B, and functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that, for all $a : \mathrm{El}\,(A)$ and $b : \mathrm{El}\,(B)$, there exists a unique $(a, b) : \mathrm{El}\,(A \times B)$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Definition 1.2.4 (Injective). A function $f : A \to B$ is *injective* or an *injection* iff, for all $x, y : \mathrm{El}\,(A)$, if $f(x) = f(y)$ then $x = y$.

Axiom Schema 1.2.5 (Separation)**.** *For every property $P[X, x]$ where X is a set variable and $x : \mathrm{El}\,(X)$, the following is an axiom:*

For every set A, there exists a set $S = \{x : \mathrm{El}\,(A) \mid P[A, x]\}$ and an injection $i : S \to A$ such that, for all $x : \mathrm{El}\,(A)$, we have

$$(\exists y : S.i(y) = x) \Leftrightarrow P[A, x] \ \ .$$

Axiom 1.2.6 (Infinity)**.** *There exists a set $\mathbb{N}$, an element $0 : \mathrm{El}\,(\mathbb{N})$, and a function $s : \mathbb{N} \to \mathbb{N}$ such that:*

- $\forall n : \mathrm{El}\,(\mathbb{N})\,.s(n) \neq 0$

- $\forall m, n : \mathrm{El}\,(\mathbb{N})\,.s(m) = s(n) \Rightarrow m = n.$

Chapter 2

Sets and Functions

2.1 Injective Functions

2.2 Surjective Functions

Definition 2.2.1 (Surjective). A function $f : A \to B$ is *surjective* iff, for all $y : \mathrm{El}(B)$, there exists $x : \mathrm{El}(A)$ such that $f(x) = y$.

2.3 Bijections

Definition 2.3.1 (Bijective). A function $f : A \to B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$".

2.4 Identity Function

Definition 2.4.1 (Identity). For any set A, the *identity* function $\mathrm{id}_A : A \to A$ is the function defined by $\mathrm{id}_A(a) = a$.

2.5 Composition

Definition 2.5.1 (Composite). Let $f : A \to B$ and $g : B \to C$. The *composite* $g \circ f : A \to C$ is the function such that, for all $a \in A$, we have

$$(g \circ f)(a) = g(f(a)) \ .$$

Proposition 2.5.2. *Given functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \ .$$

PROOF: Each is the function that maps $a \in A$ to $h(g(f(a)))$. $\square$

Proposition 2.5.3. *The composite of injective functions is injective.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : A \rightarrowtail B$ and $g : B \rightarrowtail C$ be injective.
$\langle 1 \rangle 2$. LET: $x, y \in A$ satisfy $(g \circ f)(x) = (g \circ f)(y)$
$\langle 1 \rangle 3$. $g(f(x)) = g(f(y))$
$\langle 1 \rangle 4$. $f(x) = f(y)$
$\langle 1 \rangle 5$. $x = y$
$\square$

Proposition 2.5.4. *For functions $f : A \to B$ and $g : B \to C$, if $g \circ f$ is injective then f is injective.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $g \circ f$ is injective.
$\langle 1 \rangle 2$. LET: $x, y \in A$
$\langle 1 \rangle 3$. ASSUME: $f(x) = f(y)$
$\langle 1 \rangle 4$. $g(f(x)) = g(f(y))$
$\langle 1 \rangle 5$. $x = y$
$\square$

Proposition 2.5.5. *The composite of surjective functions is surjective.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$ be injective.
$\langle 1 \rangle 2$. LET: $c \in C$
$\langle 1 \rangle 3$. PICK $b \in B$ such that $g(b) = c$.
$\langle 1 \rangle 4$. PICK $a \in A$ such that $f(a) = b$.
$\langle 1 \rangle 5$. $(g \circ f)(a) = c$
$\square$

Proposition 2.5.6. *Let $f : A \to B$. Then the following are equivalent.*

  1. *f is surjective.*

  2. *For any set X and functions $g, h : B \to X$, if $g \circ f = h \circ f$ then $g = h$.*

  3. *There exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
    $\langle 2 \rangle 1$. ASSUME: f is surjective.
    $\langle 2 \rangle 2$. LET: X be a set.
    $\langle 2 \rangle 3$. LET: $g, h : B \to X$

$\langle 2 \rangle 4$. ASSUME: $g \circ f = h \circ f$

$\langle 2 \rangle 5$. LET: $b \in B$

     PROVE: $g(b) = h(b)$

$\langle 2 \rangle 6$. PICK $a \in A$ such that $f(a) = b$

$\langle 2 \rangle 7$. $g(b) = h(b)$

     PROOF: $g(b) = g(f(a)) = h(f(a)) = h(b)$

$\langle 1 \rangle 2$. $1 \Rightarrow 3$

$\langle 2 \rangle 1$. ASSUME: f is surjective.

$\langle 2 \rangle 2$. PICK $g : B \to A$ such that, for all $b \in B$, we have $f(g(b)) = b$.

     PROOF: Axiom of Choice.

$\langle 2 \rangle 3$. $f \circ g = \mathrm{id}_B$.

$\langle 1 \rangle 3$. $3 \Rightarrow 2$

$\langle 2 \rangle 1$. LET: $g : B \to A$ such that $f \circ g = \mathrm{id}_B$

$\langle 2 \rangle 2$. LET: X be a set.

$\langle 2 \rangle 3$. LET: $h, k : B \to X$

$\langle 2 \rangle 4$. ASSUME: $h \circ f = k \circ f$

$\langle 2 \rangle 5$. $h = k$

     PROOF: $h = h \circ f \circ g = k \circ f \circ g = k$

$\langle 1 \rangle 4$. $2 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: 2

$\langle 2 \rangle 2$. LET: $b \in B$

$\langle 2 \rangle 3$. LET: $h : B \to 2$ be the function that maps everything to 1.

$\langle 2 \rangle 4$. LET: $k : B \to 2$ be the function that maps b to 0 and everything else
       to 1.

$\langle 2 \rangle 5$. $h \neq k$

$\langle 2 \rangle 6$. $h \circ f \neq k \circ f$

$\langle 2 \rangle 7$. PICK $a \in A$ such that $h(f(a)) \neq k(f(a))$

$\langle 2 \rangle 8$. $f(a) = b$

$\square$

Proposition 2.5.7. *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is surjective then g is surjective.*

PROOF:

$\langle 1 \rangle 1$. LET: $c \in C$

$\langle 1 \rangle 2$. There exists $a \in A$ such that $g(f(a)) = c$.

$\langle 1 \rangle 3$. There exists $b \in B$ such that $g(b) = c$.

$\square$

Proposition 2.5.8. *The composite of bijections is a bijection.*

PROOF: Propositions 2.5.3 and 2.5.5. $\square$

Proposition 2.5.9. *Let $f : A \to B$. Then f is bijective if and only if there exists a function $f^{-1} : B \to A$, the inverse of f, such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$, in which case the inverse is unique.*

PROOF:

$\langle 1 \rangle 1$. If f is bijective then there exists $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$
      and $f^{-1} \circ f = \mathrm{id}_A$.
   $\langle 2 \rangle 1$. ASSUME: f is bijective.
   $\langle 2 \rangle 2$. PICK $g : B \to A$ such that $f \circ g = \mathrm{id}_B$
      PROOF: Proposition 2.5.6.
   $\langle 2 \rangle 3$. $f \circ g \circ f = f$
   $\langle 2 \rangle 4$. $g \circ f = \mathrm{id}_A$
      PROOF: Proposition **??**.
$\langle 1 \rangle 2$. If there exists $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$,
      then f is bijective.
   $\langle 2 \rangle 1$. LET: $f^{-1} : B \to A$ satisfy $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$
   $\langle 2 \rangle 2$. f is injective.
      PROOF: If $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$.
   $\langle 2 \rangle 3$. f is surjective.
      PROOF: Proposition 2.5.6.
$\langle 1 \rangle 3$. If $g, h : B \to A$ satisfy $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ and
      $h \circ f = \mathrm{id}_A$ then $g = h$.
   PROOF: We have $g = g \circ f \circ h = h$.
□

Proposition 2.5.10. *Let $f : A \to B$. Then $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.*

PROOF: Each is the function that maps a to $f(a)$. □

Proposition 2.5.11.
$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps f to $(\pi_1 \circ f, \pi_2 \circ f)$ is a bijection. □

Proposition 2.5.12.
$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function Φ such that $\Phi(f)(c)(b) = f(b, c)$ is a bijection. □

2.5.1   The Empty Set

Theorem 2.5.13. *There exists a set which has no elements.*

PROOF:
$\langle 1 \rangle 1$. PICK a set A
   PROOF: By the Axiom of Infinity, a set exists.
$\langle 1 \rangle 2$. LET: $S = \{x : \mathrm{El}\,(A) \mid \bot\}$ with injection $i : S \to A$
   PROOF: Axiom of Separation.
$\langle 1 \rangle 3$. S has no elements.
□

Theorem 2.5.14. *If E and E' have no elements then $E \approx E'$.*

PROOF:

$\langle 1 \rangle 1$. LET: E and E' have no elements.
$\langle 1 \rangle 2$. PICK a function $F : E \to E'$.
  PROOF: Axiom of Choice since vacuously $\forall x : \mathrm{El}\,(E)\,.\exists y : \mathrm{El}\,(E')\,.\top$.
$\langle 1 \rangle 3$. F is injective.
  PROOF: Vacuously, for all $x, y : \mathrm{El}\,(E)$, if $F(x) = F(y)$ then $x = y$.
$\langle 1 \rangle 4$. F is surjective.
  PROOF: Vacuously, for all $y : \mathrm{El}\,(E)$, there exists $x : \mathrm{El}\,(E)$ such that $F(x) = y$.
$\square$

Definition 2.5.15 (Empty Set)**.** The *empty set* $\varnothing$ is the set with no elements.

2.5.2   The Singleton

Theorem 2.5.16. *There exists a set that has exactly one element.*

PROOF:
$\langle 1 \rangle 1$. PICK a set A that has an element.
  PROOF: By the Axiom of Infinity, there exists a set that has an element.
$\langle 1 \rangle 2$. PICK $a : \mathrm{El}\,(A)$
$\langle 1 \rangle 3$. PICK a set S and injection $i : S \rightarrowtail A$ such that, for all $x : \mathrm{El}\,(A)$, there exists $s : \mathrm{El}\,(S)$ such that $s = x$ if and only if $x = a$
$\langle 1 \rangle 4$. S has exactly one element.
$\square$

Theorem 2.5.17. *If A and B both have exactly one element then $A \approx B$.*

PROOF:
$\langle 1 \rangle 1$. LET: A and B both have exactly one element a and b respectively.
$\langle 1 \rangle 2$. LET: $F : A \to B$ be the function such that, for all $x : \mathrm{El}\,(A)$, we have $(x = a \wedge F(x) = b)$
$\langle 1 \rangle 3$. F is a bijection.
$\square$

Definition 2.5.18 (Singleton)**.** Let 1 be the set that has exactly one element. Let $*$ be its element.

2.5.3   Subsets

Definition 2.5.19 (Subset)**.** A *subset* of a set A consists of a set S and an injection $i : S \rightarrowtail A$. We write $(S, i) : \mathrm{Sub}(A)$.
  We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Proposition 2.5.20. *For any subset (S, i) of A we have $(S, i) = (S, i)$.*

PROOF: We have $\mathrm{id}_S : S \approx S$ and $i \circ \mathrm{id}_S = i$.

Proposition 2.5.21. *If $(S, i) = (T, j)$ then $(T, j) = (S, i)$.*

PROOF: If $\phi : S \approx T$ and $j \circ \phi = i$ then $\phi^{-1} : T \approx S$ and $i \circ \phi^{-1} = j$. $\square$

Proposition 2.5.22. *If $(R, i) = (S, j)$ and $(S, j) = (T, k)$ then $(R, i) = (T, k)$.*

PROOF: If $\phi : R \approx S$ and $j \circ \phi = i$, and $\psi : S \approx T$ and $k \circ \psi = j$, then $\psi \circ \phi : R \approx T$ and $k \circ \psi \circ \phi = i$. $\square$

Definition 2.5.23 (Membership). Given $(S, i) : \mathrm{Sub}(A)$ and $a \in A$, we write $a \in (S, i)$ for $\exists s : \mathrm{El}\,(S)\,.i(s) = a$.

Proposition 2.5.24. *If $a \in (S, i)$ and $(S, i) = (T, j)$ then $a \in (T, j)$.*

PROOF: If $i(s) = a$ then $j(\phi(s)) = a$. $\square$

2.6   Axioms Part Two

Axiom 2.6.1 (Power Set). *For any set A, there exists a set $\mathcal{P}A$, the* power set *of A, and a relation $\in : A \nrightarrow \mathcal{P}A$, called* membership, *such that, for any subset S of A, there exists a unique $\overline{S} \in \mathcal{P}A$ such that, for all $x \in A$, we have $x \in \overline{S}$ if and only if $x \in S$.*
    We usually write just S for $\overline{S}$.

Axiom Schema 2.6.2 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*
    For any set A, there exists a set B, a function $p : B \to A$, a set Y and a relation $M : B \nrightarrow Y$ such that:

- $\forall b \in B.P[A, \{y \in Y : bMy\}, p(b)]$

- *For all $a \in A$, if $\exists Y.P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 2.6.3 (Universe). Let $E : U \nrightarrow X$ be a relation. Let us say that a set A is *small* iff there exists $u \in U$ such that $A \approx \{x \in X : uEx\}$.
    Then (U, X, E) form a *universe* if and only if:

- $\mathbb{N}$ is U-small.

- For any U-small sets A and B and relation $R : A \nrightarrow B$, the tabulation of R is U-small.

- If A is U-small then so is $\mathcal{P}A$

- Let $f : A \to B$ be a function. If B is U-small and $f^{-1}(b)$ is U-small for all $b \in B$, then A is U-small.

- If $p : B \twoheadrightarrow A$ is a surjective function such that A is U-small, then there exists a U-small set C, a surjection $q : C \twoheadrightarrow A$, and a function $f : C \to B$ such that $q = pf$.

Axiom 2.6.4 (Universe). *There exists a universe.*

    Let $E : U \nrightarrow X$ be a universe. We shall say a set is *small* iff it is U-small, and *large* otherwise.

2.7   Cartesian Product

Definition 2.7.1 (Cartesian Product)**.** Let A and B be sets. The *Cartesian product* of A and B, $A \times B$, is the tabulation of the relation $A \nrightarrow B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

2.8   Quotient Sets

Proposition 2.8.1. *Let $\sim$ be an equivalence relation on X. Then there exists a set $X/\sim$, the* quotient set *of X with respect to $\sim$, and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the* canonical projection*, such that, for all $x, y : \mathrm{El}\,(X)$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to $\sim$, then there exists a unique bijection $\phi : X/\sim\,\approx Q$ such that $\phi \circ \pi = p$.

2.9   Partitions

Definition 2.9.1 (Partition)**.** A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X.

Chapter 3

Order Theory

3.1 Relations

Definition 3.1.1 (Reflexive). A relation $R \subseteq A \times A$ is *reflexive* iff, for all $a \in A$, we have $(a, a) \in R$.

Definition 3.1.2 (Antisymmetric). A relation $R \subseteq A \times A$ is *antisymmetric* iff, for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Definition 3.1.3 (Transitive). A relation $R \subseteq A \times A$ is *transitive* iff, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Definition 3.1.4 (Partial Order). A *partial order* on a set A is a relation on A that is reflexive, antisymmetric and transitive.

We say $(A, \leqslant)$ is a *partially ordered set* or *poset* iff $\leqslant$ is a partial order on A.

Chapter 4

Category Theory

4.1 Categories

Definition 4.1.1. A *category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}\,(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \mathrm{Ob}\,(\mathcal{C})$.

- for any objects X and Y, a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y. We write $f : X \to Y$ for $f \in \mathcal{C}[X, Y]$.

- for any objects X, Y and Z, a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \to \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$

- For any object X, there exists a morphism $\mathrm{id}_X : X \to X$, the *identity morphism* on X, such that:

  - for any object Y and morphism $f : Y \to X$ we have $\mathrm{id}_X \circ f = f$
  - for any object Y and morphism $f : X \to Y$ we have $f \circ \mathrm{id}_X = f$

We write the composite of morphism f_1, ..., f_n as $f_n \circ \cdots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 4.1.2. Let **Set** be the category of small sets and functions.

Definition 4.1.3. We identify a poset $(A, \leqslant)$ with the category with:

- set of objects A

- for $a, b \in A$, the set of homomorphisms is $\{x \in 1 : a \leqslant b\}$

Proposition 4.1.4. *A category is a poset iff, for any two objects, there exists at most one morphism between them.*

Proposition 4.1.5. *The identity morphism on an object is unique.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{C}$ be a category.
$\langle 1 \rangle 2$. LET: $A \in \mathcal{C}$
$\langle 1 \rangle 3$. LET: $i, j : A \to A$ be identity morphisms on A.
$\langle 1 \rangle 4$. $i = j$
    PROOF:

$$i = i \circ j \qquad\qquad (j \text{ is an identity on } A)$$
$$= j \qquad\qquad (i \text{ is an identity on } A)$$

□

Definition 4.1.6. Given $f : A \to B$ and an object C, define the function $f^* : \mathcal{C}[B, C] \to \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 4.1.7. Given $f : A \to B$ and an object C, define the function $f_* : \mathcal{C}[C, A] \to \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

4.1.1   Monomorphisms

Definition 4.1.8 (Monomorphism)**.** Let $f : A \to B$. Then f is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for any object X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

4.1.2   Epimorphisms

Definition 4.1.9 (Epimorphism)**.** Let $f : A \to B$. Then f is *epic* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for any object X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

4.1.3   Sections and Retractions

Definition 4.1.10 (Section, Retraction)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $rs = \mathrm{id}_B$.

Proposition 4.1.11. *Let $f : A \to B$ and $r, s : B \to A$. If r is a retraction of f and s is a section of f then $r = s$.*

PROOF:
$$r = r\mathrm{id}_B \qquad\qquad\qquad (\text{Unit Law})$$
$$= rfs \qquad\qquad\qquad (s \text{ is a section of } f)$$
$$= \mathrm{id}_A s \qquad\qquad\qquad (r \text{ is a retraction of } f)$$
$$= s \qquad\qquad\qquad (\text{Unit Law})□$$

Proposition 4.1.12. *Every section is monic.*

PROOF:

$\langle 1 \rangle$1. LET: $s : B \to A$ be a section of $r : A \to B$.
$\langle 1 \rangle$2. LET: X be an object and $x, y : X \to B$
$\langle 1 \rangle$3. ASSUME: $s \circ x = s \circ y$
$\langle 1 \rangle$4. $x = y$
   PROOF: $x = r \circ s \circ x = r \circ s \circ y = y$.
□

Proposition 4.1.13. *Every retraction is epic.*

PROOF: Dual. □

4.1.4   Isomorphisms

Definition 4.1.14 (Isomorphism). A morphism $f : A \to B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \to A$ that is both a retraction and section of f.

   Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 4.1.15. *The inverse of an isomorphism is unique.*

PROOF: From Proposition 4.1.11. □

Proposition 4.1.16. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $\left(f^{-1}\right)^{-1} = f$.*

PROOF: Since $f f^{-1} = \mathrm{id}_B$ and $f^{-1} f = \mathrm{id}_A$. □

   Isomorphism.
   Define the opposite category.
   Slice categories

Definition 4.1.17. Let $\mathcal{C}$ be a category and $B \in \mathcal{C}$. The category $\mathcal{C}_B^B$ of objects *over and under B* is the category with:

- objects all triples (X, u, p) such that $u : B \to X$ and $p : X \to B$

- morphisms $f : (X, u, p) \to (Y, u', p')$ all morphisms $f : X \to Y$ such that $fu = u'$ and $p'f = p$.

Proposition 4.1.18.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B)\backslash\mathrm{id}_B \cong (\mathcal{C}\backslash B)/\mathrm{id}_B$$

$(B, \mathrm{id}_B, \mathrm{id}_B)$ is the zero object in $\mathcal{C}_B^B$.

4.1.5   Initial Objects

Definition 4.1.19 (Initial Object). An object I is *initial* iff, for any object X, there exists exactly one morphism $I \to X$.

Proposition 4.1.20. *The empty set is initial in* **Set***.*

PROOF: For any set A, the nowhere-defined function is the unique function $\varnothing \to A$. $\square$

Proposition 4.1.21. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:
$\langle 1 \rangle 1$. LET: $i : I \to I'$ be the unique morphism $I \to I'$.
$\langle 1 \rangle 2$. LET: $i^{-1} : I' \to I$ be the unique morphism $I' \to I$.
$\langle 1 \rangle 3$. $ii^{-1} = \mathrm{id}_{I'}$
  PROOF: There is only one morphism $I' \to I'$.
$\langle 1 \rangle 4$. $i^{-1}i = \mathrm{id}_I$
  PROOF: There is only one morphism $I \to I$.
$\square$

4.1.6   Terminal Objects

Definition 4.1.22 (Terminal Object)**.** An object T is *terminal* iff, for any object X, there exists exactly one morphism $X \to T$.

Proposition 4.1.23. *1 is terminal in* **Set***.*

PROOF: For any set A, the constant function to $*$ is the only function $A \to 1$. $\square$

Proposition 4.1.24. *If T and T' are terminal objects, then there exists a unique isomorphism $T \cong T'$.*

PROOF: Dual to Proposition 4.1.21. $\square$

4.1.7   Zero Objects

Definition 4.1.25 (Zero Object)**.** An object Z is a *zero object* iff it is an initial object and a terminal object.

Definition 4.1.26 (Zero Morphism)**.** Let $\mathcal{C}$ be a category with a zero object Z. Let $A, B \in \mathcal{C}$. The *zero morphism* $A \to B$ is the unique morphism $A \to Z \to B$.

Proposition 4.1.27. *There is no zero object in* **Set***.*

PROOF: Since $\varnothing \not\approx 1$. $\square$

4.1.8   Triads

Definition 4.1.28 (Triad)**.** Let $\mathcal{C}$ be a category. A *triad* consists of objects X, Y, M and morphisms $\alpha : X \to M$, $\beta : Y \to M$. We call M the *codomain* of the triad.

4.1.9   Cotriads

Definition 4.1.29 (Cotriad)**.** Let $\mathcal{C}$ be a category. A *cotriad* consists of objects X, Y, W and morphisms $\xi : W \to X$, $\eta : W \to Y$. We call W the *domain* of the triad.

4.1.10 Pullbacks

Definition 4.1.30 (Pullback)**.** A diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\ \xi\ } & X \\
\eta \downarrow & & \downarrow \alpha \\
Y & \xrightarrow{\ \beta\ } & M
\end{array}
$$

is a *pullback* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : Z \to X$ and $g : Z \to Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \to W$ such that $\xi h = f$ and $\eta h = g$.

In this case we also say that η is the *pullback* of β along α.

Proposition 4.1.31. *If $\xi : W \to X$ and $\eta : W \to Y$ form a pullback of $\alpha : X \to M$ and $\beta : Y \to M$, and $\xi' : W' \to X$ and $\eta' : W' \to Y$ also form the pullback of α and β, then there exists a unique isomorphism $\phi : W \cong W'$ such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.*

Proof:
$\langle 1\rangle 1$. Let: $\phi : W \to W'$ be the unique morphism such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.
$\langle 1\rangle 2$. Let: $\phi^{-1} : W' \to W$ be the unique morphism such that $\eta\phi^{-1} = \eta'$ and
$\qquad \xi\phi^{-1} = \xi'$.
$\langle 1\rangle 3$. $\phi\phi^{-1} = \mathrm{id}_{W'}$
$\quad$ Proof: Each is the unique $x : W' \to W'$ such that $\eta'x = \eta'$ and $\xi'x = \xi'$.
$\langle 1\rangle 4$. $\phi^{-1}\phi = \mathrm{id}_W$
$\quad$ Proof: Each is the unique $x : W \to W$ such that $\eta x = \eta$ and $\xi x = \xi$.
□

Proposition 4.1.32. *For any morphism $h : A \to B$, the following diagram is a pullback diagram.*

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & B \\
\| & & \| \\
A & \xrightarrow{\ h\ } & B
\end{array}
$$

Proof:
$\langle 1\rangle 1$. Let: Z be an object.
$\langle 1\rangle 2$. Let: $f : Z \to B$ and $g : Z \to A$ satisfy $\mathrm{id}_B f = hg$
$\langle 1\rangle 3$. $g : Z \to B$ is the unique morphism such that $\mathrm{id}_A g = g$ and $hg = f$.
□

Proposition 4.1.33. *The pullback of an isomorphism is an isomorphism.*

Proof:

$\langle 1 \rangle$1. LET:

$$
\begin{array}{ccc}
W & \xrightarrow{\ \xi\ } & X \\
\eta \downarrow & & \downarrow \alpha \\
Y & \xrightarrow{\ \beta\ } & M
\end{array}
$$

be a pullback diagram.

$\langle 1 \rangle$2. ASSUME: β is an isomorphism.

$\langle 1 \rangle$3. LET: ξ^{-1} be the unique morphism $X \to W$ such that $\xi \xi^{-1} = \mathrm{id}_X$ and $\eta \xi^{-1} = \beta^{-1}\alpha$.

   PROOF: This exists since $\alpha \mathrm{id}_X = \beta \beta^{-1} \alpha = \alpha$.

$\langle 1 \rangle$4. $\xi^{-1}\xi = \mathrm{id}_W$

   PROOF: Each is the unique $x : W \to W$ such that $\xi x = \xi$ and $\eta x = \eta$.

□

Proposition 4.1.34. *Let* $\beta : (Y, y) \to (M, m)$ *and* $\alpha : (X, x) \to (M, m)$ *in* $\mathcal{C} \backslash A$. *Let*

$$
\begin{array}{ccc}
W & \xrightarrow{\ \xi\ } & X \\
\eta \downarrow & & \downarrow \alpha \\
Y & \xrightarrow{\ \beta\ } & M
\end{array}
$$

be a pullback in $\mathcal{C}$. *Let* $w : A \to W$ *be the unique morphism such that* $\xi w = x$ *and* $\eta w = y$. *Then* $\xi : (W, w) \to (X, x)$ *and* $\eta : (W, w) \to (Y, y)$ *is the pullback of* β *and* α *in* $\mathcal{C} \backslash A$.

PROOF:

$\langle 1 \rangle$1. LET: $(Z, z) \in \mathcal{C} \backslash A$

$\langle 1 \rangle$2. LET: $f : (Z, z) \to (X, x)$ and $g : (Z, z) \to (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle$3. LET: $h : Z \to W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle$4. $hz = w$

   $\langle 2 \rangle$1. $\xi h z = \xi w$

     PROOF:

$$
\begin{aligned}
\xi h z &= f z & (\langle 1 \rangle 3) \\
&= x & (\langle 1 \rangle 2) \\
&= \xi w
\end{aligned}
$$

   $\langle 2 \rangle$2. $\eta h z = \eta w$

     PROOF: Similar.

$\langle 1 \rangle$5. $h : (Z, z) \to (W, w)$

□

Proposition 4.1.35. *Let* $\beta : (Y, y) \to (M, m)$ *and* $\alpha : (X, x) \to (M, m)$ *in*

$\mathcal{C}/A$. Let

$$
\begin{array}{ccc}
W & \xrightarrow{\ \xi\ } & X \\
\eta \downarrow & & \downarrow \alpha \\
Y & \xrightarrow[\ \beta\ ]{} & M
\end{array}
$$

be a pullback in $\mathcal{C}$. Let $w = x\xi : W \to A$. Then $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ form a pullback of α and β in $\mathcal{C}/A$.

PROOF:
$\langle 1 \rangle 1.\ \eta : (W, w) \to (Y, y)$
  PROOF:

$$
\begin{aligned}
y\eta &= m\beta\eta \\
&= m\alpha\xi \\
&= x\xi \\
&= w
\end{aligned}
$$

$\langle 1 \rangle 2.$ LET: $(Z, z) \in \mathcal{C}/A$
$\langle 1 \rangle 3.$ LET: $f : (Z, z) \to (X, x)$ and $g : (Z, z) \to (Y, y)$ satisfy $\alpha f = \beta g$.
$\langle 1 \rangle 4.$ LET: $h : Z \to W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.
$\langle 1 \rangle 5.\ h : (Z, z) \to (W, w)$
  PROOF:

$$
\begin{aligned}
wh &= x\xi h \\
&= xf & (\langle 1 \rangle 4) \\
&= z & (\langle 1 \rangle 3)
\end{aligned}
$$

▯

Proposition 4.1.36. *In* **Set***, let $\alpha : X \to M$ and $\beta : Y \to M$. Let $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$ with inclusion $i : W \to X \times Y$. Let $\xi = \pi_1 i : W \to X$ and $\eta : \pi_2 i : W \to Y$. Then ξ and η form the pullback of α and β.*

PROOF:
$\langle 1 \rangle 1.\ \alpha\xi = \beta\eta$
  PROOF: For $w \in W$, if $i(w) = (x, y)$ then then $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$.
$\langle 1 \rangle 2.$ For every set Z and functions $f : Z \to X$, $g : Z \to Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \to W$ such that $\xi h = f$ and $\eta h = g$
  PROOF: For $z \in Z$, let $h(z)$ be the unique element of W such that $i(h(z)) = (f(z), g(z))$.
▯

  Pullback lemma

4.1.11   Pushouts

Definition 4.1.37 (Pushout)**.** A diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\ \xi\ } & X \\
\eta \downarrow & & \downarrow \alpha \\
Y & \xrightarrow{\ \beta\ } & M
\end{array}
\tag{4.1}
$$

is a *pushout* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : X \to Z$ and $g : Y \to Z$ such that $f\xi = g\eta$, there exists a unique $h : M \to Z$ such that $h\alpha = f$ and $h\beta = g$.

We also say that β is the *pushout* of ξ along η.

Proposition 4.1.38. *If $\alpha : X \to M$ and $\beta : Y \to M$ form a pushout of $\xi : W \to X$ and $\eta : W \to Y$, and $\alpha' : X \to M'$ and $\beta' : Y \to M'$ also form a pushout of ξ and η, then there exists a unique isomorphism $\phi : M \cong M'$ such that $\phi\alpha = \alpha'$ and $\phi\beta = \beta'$.*

PROOF: Dual to Proposition 4.1.31. □

Proposition 4.1.39. *For any morphism $h : A \to B$, the following diagram is a pushout diagram.*

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & B \\
\| & & \| \\
A & \xrightarrow{\ h\ } & B
\end{array}
$$

PROOF: Dual to Proposition 4.1.32.

Proposition 4.1.40. *The diagram (4.1) is a pushout in $\mathcal{C}$ iff it is a pullback in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. □

Proposition 4.1.41. *The pushout of an isomorphism is an isomorphism.*

PROOF: Dual to Proposition 4.1.33. □

Proposition 4.1.42. *Let $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ in $\mathcal{C}\backslash A$. Let*

$$
\begin{array}{ccc}
W & \xrightarrow{\ \xi\ } & X \\
\eta \downarrow & & \downarrow \alpha \\
Y & \xrightarrow{\ \beta\ } & M
\end{array}
$$

be a pushout in $\mathcal{C}$. Let $m := \alpha x : A \to M$. Then $\alpha : (X, x) \to (M, m)$ and $\beta : (Y, y) \to (M, m)$ is the pushout of ξ and η in $\mathcal{C}\backslash A$.

PROOF: Dual to Proposition 4.1.35. $\square$

Proposition 4.1.43. *Let $\xi : (W, w) \to (X, x)$ and $\eta : (W, w) \to (Y, y)$ in $\mathcal{C}/A$.*
Let

$$
\begin{array}{ccc}
W & \xrightarrow{\ \xi\ } & X \\
{\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \alpha} \\
Y & \xrightarrow[\ \beta\ ]{} & M
\end{array}
$$

be a pushout in $\mathcal{C}$. Let $m : M \to A$ be the unique morphism such that $m\alpha = x$ and $m\beta = y$. Then $\alpha : (X, x) \to (M, m)$ and $\beta : (Y, y) \to (M, m)$ is the pushout of ξ and η in $\mathcal{C} \backslash A$.

PROOF: Dual to Proposition 4.1.34. $\square$

Proposition 4.1.44. Set *has pushouts.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\xi : W \to X$ and $\eta : W \to Y$.
$\langle 1 \rangle 2.$ LET: $\sim$ be the equivalence relation on $X + Y$ generated by $\xi(w) \sim \eta(w)$
      for all $w \in W$
$\langle 1 \rangle 3.$ LET: $M = (X + Y)/\sim$ with canonical projection $\pi : X + Y \twoheadrightarrow M$.
$\langle 1 \rangle 4.$ LET: $\alpha = \pi \circ \kappa_1 : X \to M$
$\langle 1 \rangle 5.$ LET: $\beta = \pi \circ \kappa_2 : Y \to M$
$\langle 1 \rangle 6.$ LET: Z be any set, $f : X \to Z$ and $g : Y \to Z$.
$\langle 1 \rangle 7.$ ASSUME: $f\xi = g\eta$
$\langle 1 \rangle 8.$ LET: $h : X + Y \to Z$ be the function defined by $h(x) = f(x)$ and $h(y) = g(y)$ for $x \in X$ and $y \in Y$
$\langle 1 \rangle 9.$ h respects $\sim$
  PROOF: For $w \in W$ we have

$$
\begin{aligned}
h(\xi(w)) &= f(\xi(w)) & (\langle 1 \rangle 8) \\
&= g(\eta(w)) & (\langle 1 \rangle 7) \\
&= h(\eta(w)) & (\langle 1 \rangle 8)
\end{aligned}
$$

$\langle 1 \rangle 10.$ LET: $\overline{h} : M \to Z$ be the induced function.
$\langle 1 \rangle 11.$ $\overline{h}\alpha = f$
  PROOF:

$$
\begin{aligned}
\overline{h}(\alpha(x)) &= \overline{h}(\pi(\kappa_1(x))) \\
&= h(\kappa_1(x)) \\
&= f(x)
\end{aligned}
$$

$\langle 1 \rangle 12.$ $\overline{h}\beta = g$
  PROOF: Similar.
$\langle 1 \rangle 13.$ For all $k : M \to Z$, if $k\alpha = f$ and $k\beta = g$ then $k = \overline{h}$.

PROOF:

$$k(\pi(\kappa_1(x))) = k(\alpha(x))$$
$$= f(x)$$
$$k(\pi(\kappa_2(y))) = k(\beta(y))$$
$$= g(y)$$
$$\therefore k \circ \pi = h$$
$$\therefore k = \overline{h}$$

Definition 4.1.45. Let $u : A \rightarrowtail X$ be an injection. The *pointed set obtained from X by collapsing* (A, u), denoted $X/(A, u)$, is the pushout

$$
\begin{array}{ccc}
A & \longrightarrow & 1 \\
\downarrow{\scriptstyle u} & {\scriptstyle *} & \downarrow \\
X & \longrightarrow & X/(A, u)
\end{array}
$$

Proposition 4.1.46. *In* $\mathbf{Set}_*$, *any two morphisms* $1 \to X$ *and* $1 \to Y$ *have a pushout.*
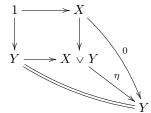
PROOF: The pushout of $a : (1, *) \to (X, x)$ and $b : (1, *) \to (Y, y)$ is $(X + Y/\sim, x)$ where $\sim$ is the equivalence relation generated by $x \sim y$. □

Definition 4.1.47 (Wedge). The *wedge* of pointed sets X and Y, $X \vee Y$, is the pushout of the unique morphism $1 \to X$ and $1 \to Y$.

Definition 4.1.48 (Smash). Let X and Y be pointed sets. Let $\xi : X \vee Y \to X$ be the unique morphism such that the following diagram commutes.

$$
\begin{array}{ccc}
1 & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X \vee Y \\
& {\scriptstyle 0} & \searrow {\scriptstyle \xi} \\
& & X
\end{array}
$$

Let $\eta : X \vee Y \to Y$ be the unique morphism such that the following diagram commutes.

$$
\begin{array}{ccc}
1 & \longrightarrow & X \\
\downarrow & & \searrow {\scriptstyle 0} \\
Y & \longrightarrow & X \vee Y \\
& & \searrow {\scriptstyle \eta} \\
& & Y
\end{array}
$$

Let $\zeta = \langle \xi, \eta \rangle : X \vee Y \to X \times Y$. The *smash* of X and Y, $X \wedge Y$, is the result of collapsing $X \times Y$ with respect to ζ.

Pushout lemma

4.1.12 Subcategories

Definition 4.1.49 (Subcategory)**.** A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ consists of:

- a subset $\mathrm{Ob}\,(\mathcal{C}')$ of $\mathcal{C}$

- for all $A, B \in \mathrm{Ob}\,(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \mathrm{Ob}\,(\mathcal{C}')$, we have $\mathrm{id}_A \in \mathcal{C}'[A, A]$

- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \mathrm{Ob}\,(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

4.1.13 Opposite Category

Definition 4.1.50 (Opposite Category)**.** For any category $\mathcal{C}$, the *opposite* category $\mathcal{C}^{\mathrm{op}}$ is the category with

- $\mathrm{Ob}\,(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}\,(\mathcal{C})$

- $\mathcal{C}^{\mathrm{op}}[A, B] = \mathcal{C}[B, A]$

- Given $f \in \mathcal{C}^{\mathrm{op}}[A, B]$ and $g \in \mathcal{C}^{\mathrm{op}}[B, C]$, their composite in $\mathcal{C}^{\mathrm{op}}$ is $f \circ g$, where $\circ$ is composition in $\mathcal{C}$.

Proposition 4.1.51. *An object is initial in $\mathcal{C}$ iff it is terminal in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. □

Proposition 4.1.52. *An object is terminal in $\mathcal{C}$ iff it is initial in $\mathcal{C}^{\mathrm{op}}$.*

PROOF: Immediate from definitions. □

Corollary 4.1.52.1. *If T and T' are terminal objects in $\mathcal{C}$ then there exists a unique isomorphism $T \cong T'$.*

4.1.14 Groupoids

Definition 4.1.53 (Groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

4.1.15   Concrete Categories

Definition 4.1.54 (Concrete Category)**.** A *concrete category* $\mathcal{C}$ consists of:

- a set $\mathrm{Ob}\,(\mathcal{C})$ of *objects*

- for any object $A \in \mathrm{Ob}\,(\mathcal{C})$, a set $|A|$

- for any objects $A, B \in \mathrm{Ob}\,(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$

- for any object A we have $\mathrm{id}_{|A|} \in \mathcal{C}[A, A]$.

4.1.16   Power of Categories

Definition 4.1.55. Let $\mathcal{C}$ be a category and J a set. The category $\mathcal{C}^J$ is the category with:

- objects all J-indexed families of objects of $\mathcal{C}$

- morphisms $\{X_j\}_{j \in J} \to \{Y_j\}_{j \in J}$ all families $\{f_j\}_{j \in J}$ where $f_j : X_j \to Y_j$

4.1.17   Arrow Category

Definition 4.1.56 (Arrow Category)**.** Let $\mathcal{C}$ be a category. The *arrow category* $\mathcal{C}^{\to}$ is the category with:

- objects all triples (A, B, f) where $f : A \to B$ in $\mathcal{C}$

- morphisms $(A, B, f) \to (C, D, g)$ all pairs $(u : A \to C, v : B \to D)$ such that $vf = gu$.

4.1.18   Slice Category

Definition 4.1.57 (Slice Category)**.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category under A, $\mathcal{C} \backslash A$,* is the category with:

- objects all pairs (B, f) where $B \in \mathcal{C}$ and $f : A \to B$

- morphisms $(B, f) \to (C, g)$ are morphisms $u : B \to C$ such that $uf = g$.

We identify this with the subcategory of $\mathcal{C}^{\to}$ formed by mapping (B, f) to (A, B, f) and u to (id_A, u).

Proposition 4.1.58. *If $s : (B, f) \to (C, g)$ in $\mathcal{C} \backslash A$, then any retraction of s in $\mathcal{C}$ is a retraction of s in $\mathcal{C} \backslash A$.*

PROOF:
$\langle 1 \rangle 1$. LET: $r : C \to B$ be a retraction of s in $\mathcal{C}$.

⟨1⟩2. $rg = f$
  PROOF: $rg = rsf = f$.
⟨1⟩3. $r : (C, g) \to (B, f)$ in $\mathcal{C} \backslash A$
⟨1⟩4. $rs = \mathrm{id}_{(B,f)}$
  PROOF: Because composition is inherited from $\mathcal{C}$.
□

Proposition 4.1.59. id_A *is the initial object in* $\mathcal{C} \backslash A$.

PROOF: For any $(B, f) \in \mathcal{C} \backslash A$, we have f is the only morphism $A \to B$ such that $f\mathrm{id}_A = f$. □

Proposition 4.1.60. *If* A *is terminal in* $\mathcal{C}$ *then* id_A *is the zero object in* $\mathcal{C} \backslash A$.

PROOF: For any $(B, f) \in \mathcal{C} \backslash A$, the unique morphism $! : B \to A$ is the unique morphism such that $!f = \mathrm{id}_A$. □

Definition 4.1.61 (Pointed Sets)**.** The *category of pointed sets* is **Set**$\backslash 1$.

Definition 4.1.62. Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category over* A, $\mathcal{C}/A$, is the category with:

- objects all pairs (B, f) with $f : B \to A$

- morphisms $u : (B, f) \to (C, g)$ all morphisms $u : B \to C$ such that $gu = f$.

Proposition 4.1.63. *Let* $u : (B, f) \to (C, g) : \mathcal{C}/A$. *Any section of* u *in* $\mathcal{C}$ *is a section of* u *in* $\mathcal{C}/A$.

PROOF: Dual to Proposition 4.1.58. □

Proposition 4.1.64. id_A *is terminal in* $\mathcal{C}/A$.

PROOF: Dual to Proposition 4.1.59. □

Proposition 4.1.65. *If* A *is initial in* $\mathcal{C}$ *then* id_A *is the zero object in* $\mathcal{C}/A$.

PROOF: Dual to Proposition 4.1.60. □

Definition 4.1.66. Let $A \in \mathcal{C}$. The category of objects *over and under* A, written $\mathcal{C}_A^A$, is the category with:

- objects all triples (X, u, p) where $u : A \to X$, $p : X \to A$ and $pu = \mathrm{id}_A$

- morphism $f : (X, u, p) \to (Y, v, q)$ all morphisms $f : X \to Y$ such that $fu = v$ and $qf = p$

Proposition 4.1.67. $(A, \mathrm{id}_A, \mathrm{id}_A)$ *is the zero object in* $\mathcal{C}_A^A$.

PROOF: For any object (X, u, p), we have p is the unique morphism $(X, u, p) \to (A, \mathrm{id}_A, \mathrm{id}_A)$, and u is the unique morphism $(A, \mathrm{id}_A, \mathrm{id}_A) \to (X, u, p)$. □

Definition 4.1.68 (Fibre Collapsing)**.** Let B be a set. Let $u : (A, a) \to (X, x)$ in $\mathbf{Set}/B$. Form the pushout

$$
\begin{array}{ccc}
A & \xrightarrow{\ a\ } & B \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle j} \\
X & \xrightarrow{\ i\ } & C
\end{array}
$$

Let $c : C \to B$ be the unique morphism such that $cj = \mathrm{id}_B$ and $ci = x$. Then $(C, j, c) \in \mathbf{Set}_B^B$ is called the set over and under B obtained from X by *fibre collapsing* with respect to u. If (A, u) is a subset of X, we denote this set over and under B by $X/_B(A, u)$.

Definition 4.1.69 (Fibre Wedge)**.** Let B be a small set. Let $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$. The *fibre wedge* of X and Y is the pushout of u_X and u_Y:

$$
\begin{array}{ccc}
B & \xrightarrow{\ u_X\ } & X \\
\downarrow{\scriptstyle u_Y} & & \downarrow \\
Y & \longrightarrow & X \vee_B Y
\end{array}
$$

Definition 4.1.70 (Fibre Smash)**.** Let $X, Y \in \mathbf{Set}_B^B$. Let $\xi : X \vee_B Y \to X$ be the unique morphism such that the following diagram commutes.

Let $\eta : X \vee_B Y \to Y$ be the unique morphism such that the following diagram commutes.

Let $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \to X \times Y$. The *fibre smash* of X and Y, $X \wedge_B Y$, is the result of collapsing $X \times Y$ with respect to ζ.

Proposition 4.1.71. Set *has products and coproducts.*

Proposition 4.1.72. *Let $\mathcal{C}$ be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in $\mathcal{C}$ and $Z \in \mathcal{C}$. Let $\coprod_{\alpha \in I} X_\alpha$ be the coproduct of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[\coprod_{\alpha \in I} X_\alpha, Z] \approx \prod_{\alpha \in I} \mathcal{C}[X_\alpha, Z] \ .$$

Proposition 4.1.73. *Let $\mathcal{C}$ be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in $\mathcal{C}$ and $Z \in \mathcal{C}$. Let $\prod_{\alpha \in I} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[Z, \prod_{\alpha \in I} X_\alpha] \approx \prod_{\alpha \in I} \mathcal{C}[Z, X_\alpha] \ .$$

Proposition 4.1.74. *A product in $\mathcal{C}$ constitutes a product in $\mathcal{C} \backslash A$.*

Proposition 4.1.75. *A coproduct in $\mathcal{C}$ constitutes a product in $\mathcal{C}/A$.*

4.2 Functors

Definition 4.2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor $F : \mathcal{C} \to \mathcal{D}$* consists of:

- a function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$

- for every morphism $f : A \to B$ in $\mathcal{C}$, a morphism $Ff : FA \to FB$ in $\mathcal{D}$

such that:

- for all $A : \mathrm{El}\left(\mathrm{Ob}(C)\right)$ we have $F\mathrm{id}_A = \mathrm{id}_{FA}$

- for any morphism $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, we have $F(g \circ f) = Fg \circ Ff$

Proposition 4.2.2. *Functors preserve isomorphisms.*

Proof:
$\langle 1 \rangle 1$. LET: $F : \mathcal{C} \to \mathcal{D}$ be a functor.
$\langle 1 \rangle 2$. LET: $f : A \cong B$ in $\mathcal{C}$
$\langle 1 \rangle 3$. $Ff^{-1} \circ Ff = \mathrm{id}_{FA}$
   Proof:
$$Ff^{-1} \circ Ff = F(f^{-1} \circ f)$$
$$= F\mathrm{id}_A$$
$$= \mathrm{id}_{FA}$$
$\langle 1 \rangle 4$. $Ff \circ Ff^{-1} = \mathrm{id}_{FB}$
   Proof:
$$Ff \circ Ff^{-1} = F(f \circ f^{-1})$$
$$= F\mathrm{id}_B$$
$$= \mathrm{id}_{FB}$$

$\square$

Definition 4.2.3 (Identity Functor)**.** For any category $\mathcal{C}$, the *identity* functor on $\mathcal{C}$ is the functor $I_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ defined by

$$I_\mathcal{C} A := A \qquad\qquad (A \in \mathcal{C})$$
$$I_\mathcal{C} f := f \qquad\qquad (f : A \to B \text{ in } \mathcal{C})$$

Proposition 4.2.4. *Let* $F : \mathcal{C} \to \mathcal{D}$*. If* $r : A \to B$ *is a retraction of* $s : B \to A$ *in* $\mathcal{C}$ *then* Fr *is a retraction of* Fs*.*

PROOF:
$$Fr \circ Fs = F(r \circ s)$$
$$= F\mathrm{id}_B$$
$$= \mathrm{id}_{FB} \qquad\qquad \square$$

Corollary 4.2.4.1. *Let* $F : \mathcal{C} \to \mathcal{D}$*. If* $\phi : A \cong B$ *is an isomorphism in* $\mathcal{C}$ *then* $F\phi : FA \cong FB$ *is an isomorphism in* $\mathcal{D}$ *with* $(F\phi)^{-1} = F\phi^{-1}$*.*

Definition 4.2.5 (Composition of Functors)**.** Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, the *composite* functor $GF : \mathcal{C} \to \mathcal{E}$ is defined by

$$(GF)A = G(FA) \qquad\qquad (A \in \mathcal{C})$$
$$(GF)f = G(Ff) \qquad\qquad (f : A \to B : \mathcal{C})$$

Definition 4.2.6 (Category of Categories)**.** Let **Cat** be the category of small categories and functors.

Definition 4.2.7 (Isomorphism of Categories)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an *isomorphism of categories* iff there exists a functor $F^{-1} : \mathcal{D} \to \mathcal{C}$, the *inverse* of F, such that $FF^{-1} = I_\mathcal{D}$ and $F^{-1}F = I_\mathcal{C}$.

Categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, iff there exists an isomorphism between them.

Proposition 4.2.8. *If* A *is initial in* $\mathcal{C}$ *then* $\mathcal{C}\backslash A \cong \mathcal{C}$*.*

PROOF:
$\langle 1 \rangle 1$. Define $F : \mathcal{C}\backslash A \to \mathcal{C}$ by
$$F(B, f) = B$$
$$F(u : (B, f) \to (C, g)) = u$$
$\langle 1 \rangle 2$. Define $G : \mathcal{C} \to \mathcal{C}\backslash A$ by
$$GB = (B, !_B) \qquad\qquad \text{where } !_B \text{ is the unique morphism } A \to B$$
$$G(u : B \to C) = u : (B, !_B) \to (C, !_C)$$
$\langle 1 \rangle 3$. $FG = \mathrm{id}_\mathcal{C}$
$\langle 1 \rangle 4$. $GF = \mathrm{id}_{\mathcal{C}\backslash A}$
　　PROOF: Since $GF(B, f) = (B, !_B) = (B, f)$ because the morphism $A \to B$ is unique.
$\square$

Proposition 4.2.9. *If* A *is terminal in* $\mathcal{C}$ *then* $\mathcal{C}/A \cong \mathcal{C}$*.*

PROOF: Dual. $\square$

Proposition 4.2.10.

$$\mathcal{C}_A^A \cong (\mathcal{C}/A)\backslash(A, \mathrm{id}_A) \cong (\mathcal{C}\backslash A)/(A, \mathrm{id}_A)$$

PROOF:
$\langle 1 \rangle 1.$ Define a functor $F : \mathcal{C}_A^A \to (\mathcal{C}/A)\backslash(A, \mathrm{id}_A)$.
  $\langle 2 \rangle 1.$ Given $A \xrightarrow{u} X \xrightarrow{p} A$ in $\mathcal{C}_A^A$, let $F(X, u, p) = ((X, p), u)$
  $\langle 2 \rangle 2.$ Given $f : (A \xrightarrow{u} X \xrightarrow{p} A) \to (A \xrightarrow{v} Y \xrightarrow{q} A)$, let $Ff = f$.
$\langle 1 \rangle 2.$ Define a functor $G : (\mathcal{C}/A)\backslash(A, \mathrm{id}_A) \to \mathcal{C}_A^A$.
$\langle 1 \rangle 3.$ Define a functor $H : \mathcal{C}_A^A \to (\mathcal{C}\backslash A)/(A, \mathrm{id}_A)$.
$\langle 1 \rangle 4.$ Define a functor $K : (\mathcal{C}\backslash A)/(A, \mathrm{id}_A) \to \mathcal{C}_A^A$.
$\square$

Definition 4.2.11 (Forgetful Functor)**.** For any concrete category $\mathcal{C}$, define the *forgetful* functor $U : \mathcal{C} \to \mathbf{Set}$ by:
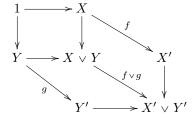
$$UA = |A|$$
$$Uf = f$$

Definition 4.2.12 (Switching Functor)**.** For any category $\mathcal{C}$, define the *switching functor* $T : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ by

$$T(A, B) = (B, A)$$
$$T(f, g) = (g, f)$$

Definition 4.2.13 (Reduction)**.** Let $\Phi : \mathbf{Set} \to \mathbf{Set}$ be a functor. The *reduction* of Φ is the functor $\Phi^* : \mathbf{Set}_* \to \mathbf{Set}_*$ defined by: $\Phi^*(X, a)$ is the collapse of $\Phi(X)$ with respect to $\Phi(a) : \Phi(1) \rightarrowtail \Phi(X)$.

Definition 4.2.14. Extend the wedge $\vee$ to a functor $\mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ by defining, given $f : X \to X'$ and $g : Y \to Y'$, thene $f \vee g$ is the unique morphism that makes the following diagram commute.

Definition 4.2.15. Extend smash to a functor $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ as follows. Given $f : X \to X'$ and $g : Y \to Y'$, let $f \wedge g : X \wedge Y \to X' \wedge Y'$ be the

unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} X \vee Y & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ X \times Y & \longrightarrow & X \wedge Y \\ & \searrow & \searrow \\ & & X' \vee Y' \longrightarrow 1 \\ & & \downarrow \qquad \downarrow \\ & & X' \times Y' \longrightarrow X' \wedge Y' \end{array}$$

Definition 4.2.16 (Reduction). Let B be a small set. Let $\Phi_B : \mathbf{Set}/B \to \mathbf{Set}/B$ be a functor. The *reduction* of Φ_B is the functor $\Phi_B^B : \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ defined as follows.

For $(X, u : B \to X, p : X \to B) \in \mathbf{Set}_B^B$, let $\Phi_B^B(X)$ be the set over and under B obtained from $\Phi_B(X)$ by collapsing with respect to $\Phi_B(u) : \Phi_B(B) \to \Phi_B(X)$.

Definition 4.2.17. Extend $\vee_B$ to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$.

Definition 4.2.18. Extend $\wedge_B$ to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$.

Definition 4.2.19 (Faithful). A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* iff, for any objects $A, B \in \mathcal{C}$ and morphisms $f, g : A \to B : \mathcal{C}$, if $Ff = Fg$ then $f = g$.

Definition 4.2.20 (Full). A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* iff, for any objects $A, B \in \mathcal{C}$ and morphism $g : FA \to FB : \mathcal{D}$, there exists $f : A \to B : \mathcal{C}$ such that $Ff = g$.

Definition 4.2.21 (Fully Faithful). A functor $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* iff it is full and faithful.

Definition 4.2.22 (Full Embedding). A functor $F : \mathcal{C} \to \mathcal{D}$ is a *full embedding* iff it is fully faithful and injective on objects.

4.3   Natural Transformations

Definition 4.3.1 (Natural Transformation). Let $F, G : \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\tau : F \Rightarrow G$ is a family of morphisms $\{\tau_X : FX \to GX\}_{X \in \mathcal{C}}$ such that, for every morphism $f : X \to Y : \mathcal{C}$, we have $Gf \circ \tau_X = \tau_Y \circ Ff$.

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \tau_X \downarrow & & \downarrow \tau_Y \\ GX & \xrightarrow{Gf} & GY \end{array}$$

Definition 4.3.2 (Natural Isomorphism)**.** A natural transformation $\tau : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ is a *natural isomorphism*, $\tau : F \cong G$, iff for all $X \in \mathcal{C}$, τ_X is an isomorphism $FX \cong GX$.

Functors F and G are *naturally isomorphic*, $F \cong G$, iff there exists a natural isomorphism between them.

Definition 4.3.3 (Inverse)**.** Let $\tau : F \cong G$. The *inverse* natural isomorphism $\tau^{-1} : G \cong F$ is defined by $(\tau^{-1})_X = \tau_X{}^{-1}$.

4.4 Bifunctors

Definition 4.4.1 (Commutative)**.** A bifunctor $\square : \mathcal{C}^2 \to \mathcal{C}$ is *commutative* iff $\square \cong \square \circ T$, where $T : \mathcal{C}^2 \to \mathcal{C}^2$ is the swap functor.

Proposition 4.4.2. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is commutative.*

PROOF: Since the pushout of f and g is the pushout of g and f. $\square$

Proposition 4.4.3. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is commutative.*

PROOF: In the diagram defining $X \wedge Y$, construct the isomorphism between the version with X and Y and the version with Y with X for every object. $\square$

Proposition 4.4.4. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is commutative.*

Proposition 4.4.5. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is commutative.*

Definition 4.4.6 (Associative)**.** A bifunctor $\square$ is *associative* iff $\square \circ (\square \times \mathrm{id}) \cong \square \circ (\mathrm{id} \times \square)$.

Proposition 4.4.7. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is associative.*

PROOF: Since $X \vee (Y \vee Z)$ and $(X \vee Y) \vee Z$ are both the pushout of the unique morphisms $1 \to X$, $1 \to Y$ and $1 \to Z$. $\square$

Proposition 4.4.8. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \to \mathbf{Set}_*$ *is associative.*

PROOF: Draw isomorphisms between the diagrams for $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$. $\square$

Product and coproduct are commutative and associative.

Proposition 4.4.9. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is associative.*

Proposition 4.4.10. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \to \mathbf{Set}_B^B$ *is associative.*

Proposition 4.4.11. *Let $\mathcal{C}$ be a category with binary coproducts. Let $\square : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a bifunctor. Then $\square$ distributes over $+$ iff the canonical morphism*

$$(X \square Z) + (Y \square Z) \to (X + Y) \square Z$$

is an isomorphism for all X, Y, Z.

Proposition 4.4.12. *In a category with binary products and binary coproducts, then $\times$ distributes over $+$.*

Proposition 4.4.13. *In $\mathbf{Set}/*$, we have $\times$ does not distribute over $\vee$.*

Proposition 4.4.14. *In $\mathbf{Set}/*$, we have $\wedge$ distributes over $\vee$.*

Proposition 4.4.15. *In $\mathbf{Set}/B$, we have $\times_B$ distributes over $+_B$.*

Proposition 4.4.16. *In $\mathbf{Set}/B^B$, we have $\wedge_B$ distributes over $\vee_B$.*

4.5   Functor Categories

Definition 4.5.1 (Functor Category)**.** Given categories $\mathcal{C}$ and $\mathcal{D}$, define the *functor category* $\mathcal{C}^{\mathcal{D}}$ to be the category with objects the functors from $\mathcal{D}$ to $\mathcal{C}$ and morphisms the natural transformations.

Definition 4.5.2 (Yoneda Embedding)**.** Let $\mathcal{C}$ be a category. The *Yoneda embedding* $Y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the functor that maps an object A to $\mathcal{C}[-, A]$ and morphisms similarly.

Theorem 4.5.3 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category. There exists a natural isomorphism*
$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}[\mathcal{C}[-, X], F] \cong FX$$
that maps $\tau : \mathcal{C}[-, X] \Rightarrow F$ to $\tau_X(\mathrm{id}_X)$.

Proof:
$\langle 1 \rangle 1$. ϕ is natural in X.
　　Proof:
　　$\langle 2 \rangle 1$. Let: $f : X \to Y : \mathcal{C}$
　　$\langle 2 \rangle 2$. Let: $\tau : \mathcal{C}[-, X] \Rightarrow F$
　　$\langle 2 \rangle 3$. $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$
　　　　Proof:
$$\begin{aligned}
\phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\mathrm{id}_Y \circ f) \\
&= \tau_Y(f) \\
&= \tau_Y(f \circ \mathrm{id}_X) \\
&= Ff(\tau_X(\mathrm{id}_X)) \qquad\qquad (\tau \text{ natural}) \\
&= Ff(\phi(\tau))
\end{aligned}$$
$\langle 1 \rangle 2$. ϕ is natural in F.
　　$\langle 2 \rangle 1$. Let: $\alpha : F \Rightarrow G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$
　　$\langle 2 \rangle 2$. Let: $\tau : \mathcal{C}[-, X] \Rightarrow F$
　　$\langle 2 \rangle 3$. $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$
　　　　Proof: $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\mathrm{id}_X)) = \alpha_X(\phi(\tau))$
$\langle 1 \rangle 3$. Each ϕ_{XF} is injective.
　　$\langle 2 \rangle 1$. Let: $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$
　　$\langle 2 \rangle 2$. Assume: $\phi(\sigma) = \phi(\tau)$

$\langle 2 \rangle 3$. LET: $f : Y \to X$

$\langle 2 \rangle 4$. $\sigma_Y(f) = \tau_Y(f)$

PROOF:

$$
\begin{aligned}
\sigma_Y(f) &= \sigma_Y(\mathrm{id}_X \circ f) \\
&= Ff(\sigma_X(\mathrm{id}_X)) && (\sigma \text{ is natural}) \\
&= Ff(\tau_X(\mathrm{id}_X)) && (\langle 2 \rangle 2) \\
&= \tau_Y(\mathrm{id}_X \circ f) && (\tau \text{ is natural}) \\
&= \tau_Y(f)
\end{aligned}
$$

$\langle 1 \rangle 4$. Each ϕ_{XF} is surjective.

$\quad \langle 2 \rangle 1$. LET: $X \in \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$

$\quad \langle 2 \rangle 2$. LET: $a \in FX$

$\quad \langle 2 \rangle 3$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$ be given by $\tau_Y(g) = Fg(a)$ for $g : Y \to X$

$\quad \langle 2 \rangle 4$. τ is natural.

$\quad\quad \langle 3 \rangle 1$. LET: $h : Y \to Z : \mathcal{C}$

$\quad\quad\quad$ PROVE: $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \mathrm{id}_X]$

$\quad\quad \langle 3 \rangle 2$. LET: $g : Z \to X$

$\quad\quad \langle 3 \rangle 3$. $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$

$\quad\quad\quad$ PROOF:

$$
\begin{aligned}
\tau_Y(g \circ h) &= F(g \circ h)(a) \\
&= Fh(Fg(a)) \\
&= Fh(\tau_Z(g))
\end{aligned}
$$

$\quad \langle 2 \rangle 5$. $\phi(\tau) = a$

$\quad\quad$ PROOF:

$$
\begin{aligned}
\phi_X(\tau) &= \tau_X(\mathrm{id}_X) \\
&= F\mathrm{id}_X(a) \\
&= a
\end{aligned}
$$

□

Corollary 4.5.3.1. *The Yoneda embedding is fully faithful.*

Corollary 4.5.3.2. *Given objects A and B in $\mathcal{C}$, we have $A \cong B$ if and only if $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$.*

Chapter 5

Monoid Theory

Definition 5.0.1 (Monoid)**.** A *monoid* is a category with one object.

Definition 5.0.2. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. The monoid $\mathrm{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \to X$ under composition.

Proposition 5.0.3. *For any functor* $F : \mathcal{C} \to \mathcal{D}$ *and* $X \in \mathcal{C}$, *we have that* $F : \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{D}}(FX)$ *is a monoid homomorphism.*

PROOF: Since $F\mathrm{id}_X = \mathrm{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$. $\square$

Chapter 6

Group Theory

Definition 6.0.1. Let **Grp** be the category of small groups and group homomorphisms.

Definition 6.0.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G.

Proposition 6.0.3. *The trivial group is a zero object in* **Grp**.

PROOF: Easy. $\square$

The zero morphism $G \to H$ maps every element in G to e.

Definition 6.0.4. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. We write $\mathrm{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Proposition 6.0.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $X \in \mathcal{C}$. Then $F : \mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{D}}(FX)$ is a group homomorphism.*

PROOF: Since $F\mathrm{id}_X = \mathrm{id}_{FX}$, $F(g \circ f) = Fg \circ Ff$, and $Ff^{-1} = (Ff)^{-1}$. $\square$

Proposition 6.0.6. **Grp** *has products.*

Definition 6.0.7 (Free Product). The product of a family of groups in **Grp** is called the *free product*.

Proposition 6.0.8. **Ab** *has products given by direct sums.*

Chapter 7

Ring Theory

Definition 7.0.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Definition 7.0.2. For any ring R, let $R - \mathbf{Mod}$ be the category of small R-modules and R-module homomorphisms.

Proposition 7.0.3. $R - \mathbf{Mod}$ *has products and coproducts.*

Chapter 8

Field Theory

Proposition 8.0.1. Field *does not have binary products.*

PROOF: There cannot be a field K with field homomorphisms $K \to \mathbb{Z}_2$ and $K \to \mathbb{Z}_3$, because its characteristic would be both 2 and 3. □

Chapter 9

Linear Algebra

Definition 9.0.1. For any field K, we write $\mathbf{Vect}_K$ for $K - \mathbf{Mod}$.

Dual space functor $\mathbf{Vect}_K^{\mathrm{op}} \to \mathbf{Vect}_K$.

Chapter 10

Topology

10.1 Topological Spaces

Definition 10.1.1 (Topological Space). Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say $(X, \mathcal{O})$ is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.

- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.

- $X \in \mathcal{O}$

We call $\mathcal{O}$ the *topology* of the toplogical space, and call its elements *open* sets. We shall often write X for the topological space $(X, \mathcal{O})$.

Definition 10.1.2 (Discrete Topology). For any set X, the power set $\mathcal{P}X$ is called the *discrete* topology on X.

Proposition 10.1.3. *For any set X, the discrete topology on X is a topology on X.*

Definition 10.1.4 (Indiscrete Topology). For any set X, the *indiscrete* or *trivial* topology on X is $\{\varnothing, X\}$.

Proposition 10.1.5. *For any set X, the indiscrete topology on X is a topology on X.*

Definition 10.1.6 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 10.1.7. *A set B is open if and only if $X - B$ is closed.*

Proposition 10.1.8. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that $\mathcal{C}$ is the set of closed sets if and only if:*

- *For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$*

- *For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*

- $\varnothing \in \mathcal{C}$

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Definition 10.1.9 (Neighbourhood). Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x, and x is an *interior* point of U, iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 10.1.10. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 10.1.11. *Let X be a set and $\mathcal{N} : X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ on X such that, for all $x \in X$, we have $\mathcal{N}_x$ is the set of neighbourhoods of x, if and only if:*

- *For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$*

- *For all $x \in X$ we have $X \in \mathcal{N}_x$*

- *For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$*

- *For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$*

- *For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M.M \in \mathcal{N}_y$.*

In this case, $\mathcal{O}$ is unique and is given by $\mathcal{O} = \{U : \forall x \in U.U \in \mathcal{N}_x\}$.

Definition 10.1.12 (Exterior Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x.

Definition 10.1.13 (Boundary Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B.

Definition 10.1.14 (Interior). Let X be a topological space and $B \subseteq X$. The *interior* of B, B°, is the set of all interior points of B.

Proposition 10.1.15. *The interior of B is the union of all the open sets included in B.*

Definition 10.1.16 (Closure). Let X be a topological space and $B \subseteq X$. The *closure* of B, $\overline{B}$, is the set of all points that are not exterior points of B.

Proposition 10.1.17. *The closure of B is the intersection of all the closed sets that include B.*

Proposition 10.1.18. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 10.1.19 (Kuratowski Closure Axioms). *Let X be a set and $- : \mathcal{P}X \to \mathcal{P}X$. Then there exists a topology $\mathcal{O}$ such that, for all $B \subseteq X$, $\overline{B}$ is the closure of B, if and only if:*

- $\overline{\varnothing} = \varnothing$

- *For all $A \subseteq X$ we have $A \subseteq \overline{A}$*

- *For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$*

- *For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

In this case, $\mathcal{O}$ is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

Definition 10.1.20 (Finer, Coarser)**.** Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the set X. Then $\mathcal{T}$ is *coarser, smaller* or *weaker* than $\mathcal{T}'$, or $\mathcal{T}'$ is *finer, larger* or *weaker* than $\mathcal{T}$, iff $\mathcal{T} \subseteq \mathcal{T}'$.

10.1.1 Subspaces

Definition 10.1.21 (Subspace)**.** Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

Example 10.1.22. The *unit sphere* S^2 is $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a subspace of $\mathbb{R}^3$.

10.1.2 Topological Disjoint Union

Definition 10.1.23. Let X and Y be topological spaces. The *disjoint union* is $X + Y$ where $U \subseteq X + Y$ is open if and only if $\kappa_1{}^{-1}(U)$ is open in X and $\kappa_2{}^{-1}(U)$ is open in Y.

10.1.3 Product Topology

Definition 10.1.24 (Product Topology)**.** Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

10.1.4 Bases

Definition 10.1.25 (Basis)**.** Let X be a topological space. A *basis* for the topology on X is a set of open sets $\mathcal{B}$ such that every open set is the union of a subset of $\mathcal{B}$. The elements of $\mathcal{B}$ are called *basic open neighbourhoods* of their elements.

Proposition 10.1.26. *Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a basis for a topology on X if and only if:*

1. *$\bigcup \mathcal{B} = X$*

2. *For all $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.*

In this case, the topology is unique and is the set of all unions of subsets of $\mathcal{B}$. We call it the topology generated by $\mathcal{B}$.

10.1.5   Subbases

Definition 10.1.27 (Subbasis)**.** Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of $\mathcal{S}$.

Definition 10.1.28 (Space with Basepoint)**.** A *space with basepoint* is a pair (X, x) where X is a topological space and $x : \mathrm{El}\,(X)$.

10.1.6   Countability Axioms

Definition 10.1.29 (Neighbourhood Basis)**.** Let X be a topological space and $x_0 : \mathrm{El}\,(X)$. A *neighbourhood basis* of x_0 is a set $\mathcal{U}$ of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of $\mathcal{U}$.

Definition 10.1.30 (First Countable)**.** A topological space is *first countable* iff every point has a countable neighbourhood basis.

Definition 10.1.31 (Second Countable)**.** A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.
A subspace of a first countable space is first countable.
A subspace of a second countable space is second countable.
$\mathbb{R}^n$ is second countable.
An uncountable discrete space is first countable but not second countable.

Proposition 10.1.32. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.*

PROOF:
$\langle 1 \rangle 1.$ For all $\lambda : \mathrm{El}\,(\Lambda)$, PICK U_λ open in X_λ such that $\varnothing \neq U_\lambda \neq X_\lambda$.
$\langle 1 \rangle 2.$ For all $\lambda : \mathrm{El}\,(\lambda)$, PICK $x_\lambda \in U_\lambda$.
$\langle 1 \rangle 3.$ ASSUME: for a contradiction B is a countable neighbourhood basis for
$\qquad \qquad (x_\lambda)_{\lambda \in \Lambda}$.
$\langle 1 \rangle 4.$ PICK $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$
$\langle 1 \rangle 5.$ There is no $U \in \lambda$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$
$\langle 1 \rangle 6.$ Q.E.D.
$\quad$ PROOF: This is a contradiction.
$\square$

10.2   Continuous Functions

Definition 10.2.1 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* iff, for every open set V in Y, the inverse image $f^{-1}(V)$ is open in X.

Proposition 10.2.2.   *1.* id_X *is continuous*

2. *The composite of two continuous functions is continuous.*

3. *If $f : X \to Y$ is continuous and $X_0 \subseteq X$ then $f{\upharpoonright}X_0 : X_0 \to Y$ is continuous.*

4. *If $f : X + Y \to Z$, then f is continuous iff $f \circ \kappa_1 : X \to Z$ and $f \circ \kappa_2 : Y \to Z$ are continuous.*

5. *If $f : Z \to X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.*

Proposition 10.2.3. *Let X and Y be topological spaces. Let $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

PROOF:
$\langle 1 \rangle 1.$ If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
   PROOF: Since every element of $\mathcal{B}$ is open in Y.
$\langle 1 \rangle 2.$ If, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X, then f is continuous.
   $\langle 2 \rangle 1.$ ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
   $\langle 2 \rangle 2.$ LET: U be open in Y.
   $\langle 2 \rangle 3.$ LET: $x \in f^{-1}(U)$
   $\langle 2 \rangle 4.$ PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq U$.
   $\langle 2 \rangle 5.$ $x \in f^{-1}(B) \subseteq f^{-1}(U)$
$\square$

Definition 10.2.4 (Homeomorphism)**.** Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Definition 10.2.5 (Retraction)**.** Let X be a topological space and A a subspace of X. A continuous function $\rho : X \to A$ is a *retraction* iff $\rho{\upharpoonright}A = \mathrm{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 10.2.6. Let **Top** be the category of small topological spaces and continuous functions.

Forgetful functor **Top** $\to$ **Set**.
Basepoint preserving continuous functor.

10.3   Convergence

Definition 10.3.1 (Convergence)**.** Let X be a topological space. Let (x_n) be a sequence in X. A point $a : \mathrm{El}(X)$ is a *limit* of the sequence iff, for every neighbourhood U of a, there exists n_0 such that $\forall n \geqslant n_0.x_n \in U$.

Convergence in a product space is pointwise convergence.
If $f : X \to Y$ is continuous and $x_n \to l$ in X then $f(x_n) \to f(l)$ in Y.

Example 10.3.2. The converse does not hold.

Let X be the set of all continuous functions $[0,1] \to [-1,1]$ under the product topology. Let $i : X \to L^2([0,1])$ be the inclusion.

If $f_n \to f$ then $i(f_n) \to i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K_\epsilon. \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0,1]} U_\lambda$ where $U_\lambda = [-1,1]$ except for $\lambda = \lambda_1, \ldots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \ldots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 10.3.3. *The converse does hold for first countable spaces. If $f :$ $X \to Y$ where X is first countable, and Y is a topological space, and whenever $x_n \to x$ then $f(x_n) \to f(x)$, then f is continuous.*

10.4   Connected Spaces

Definition 10.4.1 (Connected). A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 10.4.2. *The continuous image of a connected space is connected.*

Proposition 10.4.3. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \varnothing$, and A and B are connected, then X is connected.*

Proposition 10.4.4. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 10.4.5 (Path-connected). A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0,1] \to X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 10.4.6. *The continuous image of a path connected space is path connected.*

Proposition 10.4.7. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \varnothing$, and A and B are path connected, then X is path connected.*

Proposition 10.4.8. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

10.5   Hausdorff Spaces

Definition 10.5.1 (Hausdorff). A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 10.5.2. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 10.5.3.   *1. Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

Proposition 10.5.4. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \to B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X, then $f = g$.*

PROOF:
$\langle 1 \rangle$1. ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.
$\langle 1 \rangle$2. PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.
$\langle 1 \rangle$3. PICK $x \in f^{-1}(U) \cap g^{-1}(V)$
$\langle 1 \rangle$4. $f(x) = g(x) \in U \cap V$
$\langle 1 \rangle$5. Q.E.D.
  PROOF: This is a contradiction.
□

Proposition 10.5.5. *Let X and Y be metric spaces. Let $f : X \to Y$ be uniformly continuous. Let $\hat{X}$ and $\hat{Y}$ be the completions of X and Y. Then f extends uniquely to a continuous map $\hat{X} \to \hat{Y}$.*

PROOF: The extension maps $\lim_{n \to \infty} x_n$ to $\lim_{n \to \infty} f(x_n)$. □

10.6 Separable Spaces

Definition 10.6.1 (Separable)**.** A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

10.7 Sequential Compactness

Definition 10.7.1 (Sequentially Compact)**.** A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

10.8 Compactness

Definition 10.8.1 (Compact)**.** A topological space is *compact* iff every open cover has a finite subcover.

Proposition 10.8.2. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P. Then $X \in P$.*

PROOF:
$\langle 1 \rangle$1. P is an open cover of X
$\langle 1 \rangle$2. PICK a finite subcover $U_1, \ldots, U_n \in P$

$\langle 1 \rangle 3.\ \ X = U_1 \cup \cdots \cup U_n \in P$

$\Box$

Corollary 10.8.2.1. *Let f be a compact space and $f : X \to \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. $\Box$

Proposition 10.8.3. *The continuous image of a compact space is compact.*

Proposition 10.8.4. *A closed subspace of a compact space is compact.*

Proposition 10.8.5. *Let X and Y be nonempty spaces. Then the following are equivalent.*

   1. *X and Y are compact.*

   2. *$X + Y$ is compact.*

   3. *$X \times Y$ is compact.*

Proposition 10.8.6. *A compact subspace of a Hausdorff space is closed.*

Proposition 10.8.7. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 10.8.8. *A first countable compact space is sequentially compact.*

10.9   Quotient Spaces

Definition 10.9.1 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. The *quotient topology* on $X/\sim$ is defined by: $U : \mathrm{El}\,(\mathcal{P}X)$ is open in $X/\sim$ if and only if $\pi^{-1}(U)$ is open in X.

Proposition 10.9.2. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $f : X/\sim \to Y$. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proposition 10.9.3. *Let X and Y be topological spaces. Let $\sim$ be an equivalence relation on X. Let $\phi : Y \to X/\sim$.*
   Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \to X$ such that $\pi \circ \Phi = \phi{\restriction}U$. Then ϕ is continuous.

Proposition 10.9.4. *A quotient of a connected space is connected.*

Proposition 10.9.5. *A quotient of a path connected space is path connected.*

Proposition 10.9.6. *Let X be a topological space and $\sim$ an equivalence relation on X. If $X/\sim$ is Hausdorff then every equivalence class of $\sim$ is closed in X.*

Definition 10.9.7. Let X be a topological space and $A_1, \ldots, A_r \subseteq X$. Then $X/A_1, \ldots, A_r$ is the quotient space of X with respect to $\sim$ where $x \sim y$ iff $x = y$ or $\exists i (x \in A_i \wedge y \in A_i)$.

Definition 10.9.8 (Cone)**.** Let X be a topological space. The *cone over X* is the space $(X \times [0,1])/(X \times \{1\})$.

Definition 10.9.9 (Suspension)**.** Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1,1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 10.9.10 (Wedge Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product $X \vee Y$* is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 10.9.11 (Smash Product)**.** Let $x_0 \in X$ and $y_0 \in Y$. The *smash product $X \wedge Y$* is $(X \times Y)/(X \vee Y)$.

Example 10.9.12. $D^n/S^{n-1} \cong S^n$

PROOF:
$\langle 1 \rangle 1$. LET: $\phi : D^n/S^{n-1} \to S^n$ be the function induced by the map $D^n \to S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.
$\langle 1 \rangle 2$. ϕ is a bijection.
$\langle 1 \rangle 3$. ϕ is a homeomorphism.
　　PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.
$\square$

10.10　Gluing

Definition 10.10.1 (Gluing)**.** Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \to Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where $\sim$ is the equivalence relation generated by $x \sim \phi(x)$ for all $x : \mathrm{El}\,(X)$.

Proposition 10.10.2. *Y is a subspace of $Y \cup_\phi X$.*

Definition 10.10.3. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0,1])/\alpha$ is the quotient space of $X \times [0,1]$ by the equivalence relation generated by $(x,0) \sim (\alpha(x),1)$ for all $x : \mathrm{El}\,(X)$.

Definition 10.10.4 (Möbius Strip)**.** The *Möbius strip* is $([-1,1] \times [0,1])/\alpha$ where $\alpha(x) = -x$.

Definition 10.10.5 (Klein Bottle)**.** The *Klein bottle* is $(S^1 \times [0,1])/\alpha$ where $\alpha(z) = \overline{z}$.

Proposition 10.10.6. *Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\mathrm{id}_{\partial M}} M \cong K$.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : ([-1,1] \times [0,1]) + ([-1,1] \times [0,1]) \to S^1 \times [0,1]$ be the function that maps $\kappa_1(\theta,t)$ to $(e^{\pi i \theta/2}, t)$ and $\kappa_2(\theta,t)$ to $(-e^{-\pi i \theta/2}, t)$.
$\langle 1 \rangle 2$. f induces a bijection $M \cup_{\mathrm{id}_{\partial M}} M \approx K$
$\langle 1 \rangle 3$. f is a homeomorphism.
$\square$

10.11   Metric Spaces

Definition 10.11.1 (Metric Space)**.** Let X be a set and $d : X^2 \to \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geqslant 0$

- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$

- For all $x, y \in X$ we have $d(x, y) = d(y, x)$

- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leqslant d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d). We often write X for the metric space (X, d).

Definition 10.11.2 (Ball)**.** Let X be a metric space. Let $x \in X$ and $r > 0$. The *ball* with *centre* x and *radius* r is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} \ .$$

Definition 10.11.3 (Metric Topology)**.** Let (X, d) be a metric space. The *metric topology* on X is the topology generated by the basis consisting of the balls.

Definition 10.11.4 (Metrizable)**.** A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 10.11.5. *Every metrizable space is Hausdorff.*

Every metrizable space is first countable.
A metric space is compact if and only if it is sequentially compact.
A metric space is separable if and only if it is second countable.

10.12   Complete Metric Spaces

Definition 10.12.1 (Complete)**.** A metric space is *complete* iff every Cauchy sequence converges.

Example 10.12.2. $\mathbb{R}$ is complete.

Proposition 10.12.3. *The product of two complete metric spaces is complete.*

Proposition 10.12.4. *Every compact metric space is complete.*

Proposition 10.12.5. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 10.12.6 (Completion)**.** Let X be a metric space. A *completion* of X is a complete metric space $\hat{X}$ and injection $i : X \rightarrowtail \hat{X}$ such that:

- The metric on X is the restriction of the metric on $\hat{X}$

- X is dense in $\hat{X}$.

Proposition 10.12.7. *Let $i_1 : X \to Y_1$ and $i_2 : X \to Y_2$ be completions of X. Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

PROOF: Define $\phi(\lim_{n \to \infty} i_1(x_n)) = \lim_{n \to \infty} i_2(x_n)$. □

Theorem 10.12.8. *Every metric space has a completion.*

PROOF: Let $\hat{X}$ be the set of Cauchy sequences in X quotiented by $\sim$ where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \to 0$. □

10.13 Manifolds

Definition 10.13.1 (Manifold). An *n-dimensional manifold* is a second countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$.

Chapter 11

Homotopy Theory

11.1 Homotopies

Definition 11.1.1 (Homotopy)**.** Let X and Y be topological spaces. Let $f, g : X \to Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0, 1] \to Y$ such that

- $\forall x : \mathrm{El}\,(X)\,.h(x, 0) = f(x)$

- $\forall x : \mathrm{El}\,(X)\,.h(x, 1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.
   Let $[X, Y]$ be the set of all homotopy classes of functions $X \to Y$.

Proposition 11.1.2. *Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.*

Definition 11.1.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

Definition 11.1.4. A functor $F : \mathbf{Top} \to \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \to Y$, if $f \simeq g$ then $Hf = Hg$.

   Basepoint-preserving homotopy.

11.2 Homotopy Equivalence

Definition 11.2.1 (Homotopy Equivalence)**.** Let X and Y be topological spaces. A *homotopy equivalence* between X and Y, $f : X \simeq Y$, is a continuous function $f : X \to Y$ such that there exists a continuous function $g : Y \to X$, the *homotopy inverse* to f, such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Definition 11.2.2 (Contractible). A topological space X is *contractible* iff $X \simeq 1$.

Example 11.2.3. $\mathbb{R}^n$ is contractible.

Example 11.2.4. D^n is contractible.

Definition 11.2.5 (Deformation Retract). Let X be a topological space and A a subspace of X. A retraction $\rho : X \to A$ is a *deformation retraction* iff $i \circ \rho \simeq \mathrm{id}_X$, where i is the inclusion $A \rightarrowtail X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 11.2.6 (Strong Deformation Retract). Let X be a topological space and A a subspace of X. A *strong deformation retraction* $\rho : X \to A$ is a continuous function such that there exists a homotopy $h : X \times [0,1] \to X$ between $i \circ \rho$ and id_X such that, for all $a : \mathrm{El}(X)$ and $t : \mathrm{El}([0,1])$, we have $h(a,t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 11.2.7. $\{0\}$ is a strong deformation retract of $\mathbb{R}^n$ and of D^n.

Example 11.2.8. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 11.2.9. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 11.2.10. For any topological space X, the singleton consisting of the vertex is a strong deformation retract of the cone over X.

Chapter 12

Simplicial Complexes

Definition 12.0.1 (Simplex). A *k-dimensional simplex* or *k-simplex* in $\mathbb{R}^n$ is the convex hull $s(x_0, \ldots, x_k)$ of $k + 1$ points in general position.

Definition 12.0.2 (Face). A *sub-simplex* or *face* of $s(x_0, \ldots, x_k)$ is the convex hull of a subset of $\{x_0, \ldots, x_k\}$.

Definition 12.0.3 (Simplicial Complex). A *simplicial complex* in $\mathbb{R}^n$ is a set K of simplices such that:

- for every simplex s in K, every face of s is in K.

- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2.

- K is locally finite, i.e. every point of $\mathbb{R}^n$ has a neighbourhood that only intersects finitely many elements of K.

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of $\mathbb{R}^n$.

12.1 Cell Decompositions

Definition 12.1.1 (*n*-cell). An *n-cell* is a topological space homeomorphic to $\mathbb{R}^n$.

Definition 12.1.2 (Cell Decomposition). Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are *n*-cells.

Definition 12.1.3 (*n*-skeleton). Given a cell decomposition of X, the *n-skeleton* X^n is the union of all the cells of dimension $\leqslant n$.

12.2 CW-complexes

Definition 12.2.1 (CW-Complex). A *CW-complex* consists of a topological space X and a cell decomposition $\mathcal{E}$ of X such that:

1. *Characteristic Maps* For every n-cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \to X$ such that $\Phi((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in $\mathcal{E}$ of dimension $< n$.

2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have $\overline{e}$ intersects only finitely many other cells in $\mathcal{E}$.

3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \overline{e}$ is closed.

Proposition 12.2.2. *If a cell decomposition $\mathcal{E}$ satisfies the Characteristic Maps axiom, then for every n-cell $e \in \mathcal{E}$ we have $\overline{e} = \Phi_e(D^n)$. Therefore $\overline{e}$ is compact and $\overline{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

Proof:
$\langle 1 \rangle 1.\ e \subseteq \Phi_e(D^n) \subseteq \overline{e}$
    Proof:

$$
\begin{aligned}
e &= \Phi_e((D^n)^\circ) \\
  &\subseteq \Phi_e(D^n) \\
  &= \Phi_e(\overline{(D^n)^\circ}) \\
  &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
  &= \overline{e}
\end{aligned}
$$

$\langle 1 \rangle 2.\ \Phi_e(D^n)$ is compact.
    Proof: Because D^n is compact.
$\langle 1 \rangle 3.\ \Phi_e(D^n)$ is closed.
$\langle 1 \rangle 4.\ \Phi_e(D^n) = \overline{e}$
$\square$

Chapter 13

Topological Groups

Definition 13.0.1 (Topological Group). A *topological group* is a group G with a topology such that the function $G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.

Example 13.0.2. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 13.0.3. *Any subgroup of a topological group is a topological group under the subspace topology.*

Definition 13.0.4 (Homogeneous Space). A *homogeneous space* is a topological space of the form G/H, where G is a topological group and H is a normal subgroup of G, under the quotient topology.

Proposition 13.0.5. *Let G be a topological group and H a normal subgroup of G. Then G/H is Hausdorff if and only if H is closed.*

PROOF: See Bourbaki, N., General Topology. III.12 □

13.1 Continuous Actions

Definition 13.1.1 (Continuous Action). Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \to X$ such that:

- $\forall x : \mathrm{El}\,(X)\,.ex = x$

- $\forall g, h : \mathrm{El}\,(G)\,.\forall x : \mathrm{El}\,(X)\,.g(hx) = (gh)x$

A *G-space* consists of a topological space X and a continuous action of G on X.

Definition 13.1.2 (Orbit). Let X be a G-space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.
The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 13.1.3. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:
$\langle 1 \rangle 1$. LET: $f_3 : S^2/SO(2) \to [-1, 1]$ be the function induced by $\pi_3 : S^2 \to [-1, 1]$
$\langle 1 \rangle 2$. f_3 is bijective.
$\langle 1 \rangle 3$. $S^2/SO(2)$ is compact.
      PROOF: It is the continuous image of S^2 which is compact.
$\langle 1 \rangle 4$. $[-1, 1]$ is Hausdorff.
$\langle 1 \rangle 5$. f_3 is a homeomorphism.
☐

Definition 13.1.4 (Stabilizer)**.** Let X be a G-space and $x \in X$. The *stabilizer* of x is $G_x := \{g : \mathrm{El}\,(G) \mid gx = x\}$.

Proposition 13.1.5. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx.*

PROOF:
$\langle 1 \rangle 1$. If $gG_x = hG_x$ then $gx = hx$.
   $\langle 2 \rangle 1$. ASSUME: $gG_x = hG_x$
   $\langle 2 \rangle 2$. $g^{-1}h \in G_x$
   $\langle 2 \rangle 3$. $g^{-1}hx = x$
   $\langle 2 \rangle 4$. $gx = hx$
$\langle 1 \rangle 2$. If $gx = hx$ then $gG_x = hG_x$.
   PROOF: Similar.
$\langle 1 \rangle 3$. The function is continuous.
   PROOF: Proposition **??**.
☐

Chapter 14

Topological Vector Spaces

Definition 14.0.1 (Topological Vector Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Substraction is a continuous function $E^2 \to E$

- Multiplication is a continuous function $K \times E \to E$

Proposition 14.0.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. □

Theorem 14.0.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 □

Proposition 14.0.4. *Let E be a topological vector space and E_0 a subspace of E. Then $\overline{E_0}$ is a subspace of E.*

Definition 14.0.5. Let E be a topological vector space. The topological space *associated* with E is $E/\overline{\{0\}}$.

14.1 Cauchy Sequences

Definition 14.1.1 (Cauchy Sequence)**.** Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geqslant n_0. x_n - x_m \in U$.

Definition 14.1.2 (Complete Topological Vector Space)**.** A topological vector space is *complete* iff every Cauchy sequence converges.

14.2   Seminorms

Definition 14.2.1 (Seminorm)**.** Let E be a vector space over K. A *seminorm* on E is a function $\| \ \| : E \to \mathbb{R}$ such that:

1. $\forall x : \mathrm{El}\,(E)\,.\|x\| \geqslant 0$

2. $\forall \alpha : \mathrm{El}\,(K)\,.\forall x : \mathrm{El}\,(E)\,.\|\alpha x\| = |\alpha|\|x\|$

3. *Triangle Inequality* $\forall x,y : \mathrm{El}\,(E)\,.\|x + y\| \leqslant \|x\| + \|y\|$

Example 14.2.2. The function that maps $(x_1, \ldots, x_n)$ to $|x_i|$ is a seminorm on $\mathbb{R}^n$.

Definition 14.2.3. Let E be a vector space over K. Let Λ be a set of seminorms on E. The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x : \mathrm{El}\,(E)$.

Proposition 14.2.4. *E is a topological vector space under this topology. It is Hausdorff iff, for all $x : \mathrm{El}\,(E)$, if $\forall \lambda \in \Lambda.\lambda(x) = 0$ then $x = 0$.*

14.3   Fréchet Spaces

Definition 14.3.1 (Pre-Fréchet Space)**.** A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 14.3.2. *Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \ \|_n : n \in \mathbb{Z}^+\}$. Then*

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 14.3.3 (Fréchet Space)**.** A *Fréchet space* is a complete pre-Fréchet space.

14.4   Normed Spaces

Definition 14.4.1 (Normed Space)**.** Let E be a vector space over K. A *norm* on E is a function $\| \ \| : E \to \mathbb{R}$ is a seminorm such that, $\forall x \in E.\|x\| = 0 \Leftrightarrow x = 0$.
   A *normed space* consists of a vector space with a norm.

Proposition 14.4.2. *If E is a normed space then $d(x,y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E.*

Proposition 14.4.3. *Let $\| \ \|$ be a seminorm on the vector space E. Then $\| \ \|$ defines a norm on $E/\overline{\{0\}}$.*

Proposition 14.4.4. *Let E and F be normed spaces. Any continuous linear map $E \to F$ is uniformly continuous.*

Definition 14.4.5. For $p \geqslant 1$. let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt{p} \int_{\mathbb{R}^n} |f(x)|^p dx$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\overline{\{0\}} \ .$$

14.5 Inner Product Spaces

Proposition 14.5.1. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E.*

14.6 Banach Spaces

Definition 14.6.1 (Banach Space)**.** A *Banach space* is a complete normed space.

Example 14.6.2. For any topological space X, the set $C(X)$ of bounded continuous functions $X \to \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 14.6.3. *The completion of a normed space is a Banach space.*

Proposition 14.6.4. *Let E and F be normed spaces. Let $f : E \to F$ be a continuous linear map. Then the extension to the completions $\hat{E} \to \hat{F}$ is linear.*

Proposition 14.6.5. *$L^p(\mathbb{R}^n)$ is a Banach space.*

Proposition 14.6.6. *$C(\mathbb{R})$ is first countable but not second countable.*

PROOF: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n. Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.
It is first countable because it is metrizable. $\square$

14.7 Hilbert Spaces

Definition 14.7.1 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Example 14.7.2. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \to \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi} \pi f(x) g(x) dx \ .$$

Proposition 14.7.3. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

14.8   Locally Convex Spaces

Definition 14.8.1 (Locally Convex Space)**.** A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 14.8.2. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Proposition 14.8.3. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. □

Example 14.8.4. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \to \mathbb{R}$ is continuous as a map $E' \to \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.
    Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 14.8.5 (Thom Space)**.** Let E be a vector bundle with a Riemannian metric, $DE = \{x : \mathrm{El}\,(E) \mid \|x\| \leqslant 1\}$ its disc bundle and $SE := \{v : \mathrm{El}\,(E) \mid \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE.