

# Mathematics

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# Chapter 1

## Primitive Terms and Axioms

### 1.1 Primitive Terms

Let there be *sets*.

For any set  $A$ , let there be *elements* of  $A$ . We write  $a \in A$  for:  $a$  is an element of  $A$ .

For any sets  $A$  and  $B$ , let there be a set  $B^A$ , whose elements are called *functions* from  $A$  to  $B$ . We write  $f : A \rightarrow B$  for  $f \in B^A$ .

For any function  $f : A \rightarrow B$  and element  $a \in A$ , let there be an element  $f(a) \in B$ , the *value* of the function  $f$  at the *argument*  $a$ .

### 1.2 Injections, Surjections and Bijections

**Definition 1.2.1** (Injective). A function  $f : A \rightarrow B$  is *injective* or an *injection* iff, for all  $x, y \in A$ , if  $f(x) = f(y)$  then  $x = y$ .

**Definition 1.2.2** (Surjective). A function  $f : A \rightarrow B$  is *surjective* or a *surjection* iff, for all  $y \in B$ , there exists  $x \in A$  such that  $f(x) = y$ .

**Definition 1.2.3** (Bijective). A function  $f : A \rightarrow B$  is *bijective* or a *bijection* iff it is injective and surjective.

Sets  $A$  and  $B$  are *equinumerous*,  $A \approx B$ , iff there exists a bijection between them.

If we prove there exists a set  $X$  such that  $P(X)$ , and that any two sets that satisfy  $P$  are bijective, then we may introduce a constant  $C$  and define "Let  $C$  be the set such that  $P(C)$ ".

### 1.3 Axioms

**Axiom Schema 1.3.1** (Choice). *Let  $P[X, Y, x, y]$  be a formula where  $X$  and  $Y$  are set variables,  $x \in X$  and  $y \in Y$ . Then the following is an axiom.*

*Let  $A$  and  $B$  be sets. Assume that, for all  $a \in A$ , there exists  $b \in B$  such that  $P[A, B, a, b]$ . Then there exists a function  $f : A \rightarrow B$  such that  $\forall a \in A. P[A, B, a, f(a)]$ .*

**Axiom 1.3.2** (Extensionality). *Let  $f, g : A \rightarrow B$ . If, for all  $x \in A$ , we have  $f(x) = g(x)$ , then  $f = g$ .*

**Definition 1.3.3** (Composition). *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . The composite  $g \circ f : A \rightarrow C$  is the function such that, for all  $a \in A$ , we have*

$$(g \circ f)(a) = g(f(a)) .$$

**Axiom 1.3.4** (Pairing). *For any sets  $A$  and  $B$ , there exists a set  $A \times B$ , the Cartesian product of  $A$  and  $B$ , and functions  $\pi_1 : A \times B \rightarrow A$  and  $\pi_2 : A \times B \rightarrow B$  such that, for all  $a \in A$  and  $b \in B$ , there exists a unique  $(a, b) \in A \times B$  such that  $\pi_1(a, b) = a$  and  $\pi_2(a, b) = b$ .*

**Axiom Schema 1.3.5** (Separation). *For every property  $P[X, x]$  where  $X$  is a set variable and  $x \in X$ , the following is an axiom:*

*For every set  $A$ , there exists a set  $S = \{x \in A : P[A, x]\}$  and an injection  $i : S \rightarrow A$  such that, for all  $x \in A$ , we have*

$$(\exists y \in S. i(y) = x) \Leftrightarrow P[A, x] .$$

**Axiom 1.3.6** (Infinity). *There exists a set  $\mathbb{N}$ , an element  $0 \in \mathbb{N}$ , and a function  $s : \mathbb{N} \rightarrow \mathbb{N}$  such that:*

- $\forall n \in \mathbb{N}. s(n) \neq 0$
- $\forall m, n \in \mathbb{N}. s(m) = s(n) \Rightarrow m = n$ .

**Axiom Schema 1.3.7** (Collection). *Let  $P[X, Y, x]$  be a formula with set variables  $X$  and  $Y$  and an element variable  $x \in X$ . Then the following is an axiom.*

*For any set  $A$ , there exist sets  $B$  and  $Y$  and functions  $p : B \rightarrow A$ , and  $m : B \times Y \Rightarrow \mathbb{N}$  such that:*

- $m$  is injective.
- $\forall b \in B. P[A, \{y \in Y : m(b, y) = 0\}, p(b)]$
- For all  $a \in A$ , if  $\exists Y. P[A, Y, a]$ , then there exists  $b \in B$  such that  $a = p(b)$ .

**Axiom 1.3.8** (Universe). *There exists a set  $E$ , a set  $U$  and a function  $el : E \rightarrow U$  such that the following holds.*

*Let us say that a set  $A$  is small iff there exists  $u \in U$  such that  $A \approx \{e \in E : el(e) = u\}$ .*



- $\mathbb{N}$  is small.
- For any  $U$ -small sets  $A$  and  $B$ , the set  $B^A$  is small.
- For any  $U$ -small sets  $A$  and  $B$ , the set  $A \times B$  is small.
- Let  $f : A \rightarrow B$  be a function. If  $B$  is small and  $\{a \in A : f(a) = b\}$  is small for all  $b \in B$ , then  $A$  is small.
- If  $p : B \rightarrow A$  is a surjective function such that  $A$  is small, then there exists a  $U$ -small set  $C$ , a surjection  $q : C \rightarrow A$ , and a function  $f : C \rightarrow B$  such that  $q = p \circ f$ .



## Chapter 2

# Sets and Functions

### 2.1 Injections and Surjections

**Proposition 2.1.1.** *Let  $A$  and  $B$  be sets. If there exists an injective function  $f : A \rightarrow B$ , and  $A$  is nonempty, then there exists a surjective function  $B \rightarrow A$ .*

PROOF: Pick  $a_0 \in A$ . Define  $g : B \rightarrow A$  by:  $g(b)$  is the unique element in  $A$  such that  $f(a) = b$  if there is such an  $a$ , otherwise  $g(b) = a_0$ .  $\square$

### 2.2 Composition

**Proposition 2.2.1.** *Given functions  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h : C \rightarrow D$ , we have*

$$h \circ (g \circ f) = (h \circ g) \circ f .$$

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in A$  we have  $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$ .

$\langle 2 \rangle 1$ . LET:  $x \in A$

$\langle 2 \rangle 2$ .  $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$

PROOF:

$$\begin{aligned} (h \circ (g \circ f))(x) &= h((g \circ f)(x)) && \text{(Definition of composition)} \\ &= h(g(f(x))) && \text{(Definition of composition)} \\ &= (h \circ g)(f(x)) && \text{(Definition of composition)} \\ &= ((h \circ g) \circ f)(x) && \text{(Definition of composition)} \end{aligned}$$

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: By the Axiom of Extensionality.

$\square$

#### 2.2.1 Injections

**Proposition 2.2.2.** *The composite of injective functions is injective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A, B$  and  $C$  be sets.

$\langle 1 \rangle 2$ . LET:  $f : A \rightarrow B$

$\langle 1 \rangle 3$ . LET:  $g : B \rightarrow C$

$\langle 1 \rangle 4$ . ASSUME:  $g$  is injective.

$\langle 1 \rangle 5$ . ASSUME:  $f$  is injective.

$\langle 1 \rangle 6$ . LET:  $x, y \in A$

$\langle 1 \rangle 7$ . ASSUME:  $(g \circ f)(x) = (g \circ f)(y)$

PROVE:  $x = y$

$\langle 1 \rangle 8$ .  $g(f(x)) = g(f(y))$

PROOF:

$$g(f(x)) = (g \circ f)(x) \quad (\text{definition of composition})$$

$$= (g \circ f)(y) \quad (\langle 1 \rangle 7)$$

$$= g(f(y)) \quad (\text{definition of composition})$$

$\langle 1 \rangle 9$ .  $f(x) = f(y)$

PROOF:  $\langle 1 \rangle 4, \langle 1 \rangle 8$

$\langle 1 \rangle 10$ .  $x = y$

PROOF:  $\langle 1 \rangle 5, \langle 1 \rangle 9$

□

**Proposition 2.2.3.** *For functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , if  $g \circ f$  is injective then  $f$  is injective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A, B$  and  $C$  be sets.

$\langle 1 \rangle 2$ . LET:  $f : A \rightarrow B$

$\langle 1 \rangle 3$ . LET:  $g : B \rightarrow C$

$\langle 1 \rangle 4$ . ASSUME:  $g \circ f$  is injective.

$\langle 1 \rangle 5$ . LET:  $x, y \in A$

$\langle 1 \rangle 6$ . ASSUME:  $f(x) = f(y)$

$\langle 1 \rangle 7$ .  $(g \circ f)(x) = (g \circ f)(y)$

PROOF:

$$(g \circ f)(x) = g(f(x)) \quad (\text{definition of composition})$$

$$= g(f(y)) \quad (\langle 1 \rangle 6)$$

$$= (g \circ f)(y) \quad (\text{definition of composition})$$

$\langle 1 \rangle 8$ .  $x = y$

PROOF:  $\langle 1 \rangle 4, \langle 1 \rangle 7$

□

**Proposition 2.2.4.** *Let  $f : A \rightarrow B$  be injective. For every set  $X$  and functions  $x, y : X \rightarrow A$ , if  $f \circ x = f \circ y$  then  $x = y$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $f$  is injective.

$\langle 1 \rangle 2$ . LET:  $X$  be a set.

$\langle 1 \rangle 3$ . LET:  $x, y : X \rightarrow A$

$\langle 1 \rangle 4$ . ASSUME:  $f \circ x = f \circ y$

$\langle 1 \rangle 5$ .  $\forall t \in X. x(t) = y(t)$

$\langle 2 \rangle 1$ . LET:  $t \in X$

$\langle 2 \rangle 2$ .  $f(x(t)) = f(y(t))$

PROOF:

$$f(x(t)) = (f \circ x)(t) \quad (\text{definition of composition})$$

$$= (f \circ y)(t) \quad (\langle 1 \rangle 4)$$

$$= f(y(t)) \quad (\text{definition of composition})$$

$\langle 2 \rangle 3$ .  $x(t) = y(t)$

PROOF:  $\langle 1 \rangle 1, \langle 2 \rangle 2$

$\langle 1 \rangle 6$ .  $x = y$

PROOF: Axiom of Extensionality,  $\langle 1 \rangle 5$

□

## 2.2.2 Surjections

**Proposition 2.2.5.** *The composite of surjective functions is surjective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be surjective.

$\langle 1 \rangle 2$ . LET:  $c \in C$

$\langle 1 \rangle 3$ . PICK  $b \in B$  such that  $g(b) = c$ .

$\langle 1 \rangle 4$ . PICK  $a \in A$  such that  $f(a) = b$ .

$\langle 1 \rangle 5$ .  $(g \circ f)(a) = c$

□

**Proposition 2.2.6.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $g \circ f$  is surjective then  $g$  is surjective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $c \in C$

$\langle 1 \rangle 2$ . There exists  $a \in A$  such that  $g(f(a)) = c$ .

$\langle 1 \rangle 3$ . There exists  $b \in B$  such that  $g(b) = c$ .

□

## 2.2.3 Bijections

**Proposition 2.2.7.** *The composite of bijections is a bijection.*

PROOF: Propositions 2.2.2 and 2.2.5. □

**Proposition 2.2.8.** *Let  $f : A \rightarrow B$ . Then  $f$  is bijective if and only if there exists a function  $f^{-1} : B \rightarrow A$ , the inverse of  $f$ , such that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ , in which case the inverse is unique.*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is bijective then there exists  $f^{-1} : B \rightarrow A$  such that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ .

- ⟨2⟩1. ASSUME:  $f$  is bijective.  
 ⟨2⟩2. PICK  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$   
 PROOF: Proposition 2.3.11.  
 ⟨2⟩3.  $f \circ g \circ f = f$   
 ⟨2⟩4.  $g \circ f = \text{id}_A$   
 PROOF: Proposition 2.2.4.  
 ⟨1⟩2. If there exists  $f^{-1} : B \rightarrow A$  such that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ ,  
 then  $f$  is bijective.  
 ⟨2⟩1. LET:  $f^{-1} : B \rightarrow A$  satisfy  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$   
 ⟨2⟩2.  $f$  is injective.  
 PROOF: If  $f(x) = f(y)$  then  $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$ .  
 ⟨2⟩3.  $f$  is surjective.  
 PROOF: Proposition 2.3.11.  
 ⟨1⟩3. If  $g, h : B \rightarrow A$  satisfy  $f \circ g = \text{id}_B$  and  $g \circ f = \text{id}_A$  and  $f \circ h = \text{id}_B$  and  
 $h \circ f = \text{id}_A$  then  $g = h$ .  
 PROOF: We have  $g = g \circ f \circ h = h$ .  
 □

**Theorem 2.2.9** (Schroeder-Bernstein). *Let  $A$  and  $B$  be sets. If there exist injections  $A \rightarrow B$  and  $B \rightarrow A$ , then  $A \approx B$ .*

PROOF:

- ⟨1⟩1. LET:  $f : A \rightarrow B$  and  $g : B \rightarrow A$  be injections.  
 ⟨1⟩2. Define the subsets  $A_n$  of  $A$  by

$$\begin{aligned}
 A_0 &:= A - g(B) \\
 A_{n+1} &:= g(f(A_n))
 \end{aligned}$$

- ⟨1⟩3. Define  $h : A \rightarrow B$  by

$$h(x) = \begin{cases} f(x) & \text{if } \exists n. x \in A_n \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

- ⟨1⟩4.  $h$  is injective.

- ⟨2⟩1. LET:  $x, y \in A$   
 ⟨2⟩2. ASSUME:  $h(x) = h(y)$   
 ⟨2⟩3. CASE:  $x \in A_m$  and  $y \in A_n$ .

PROOF: Then  $f(x) = f(y)$  so  $x = y$  since  $f$  is injective.

- ⟨2⟩4. CASE:  $x \in A_m$  and there is no  $y$  such that  $y \in A_n$ .

- ⟨3⟩1.  $f(x) = g^{-1}(y)$   
 ⟨3⟩2.  $y = g(f(x))$   
 ⟨3⟩3.  $y \in A_{m+1}$   
 ⟨3⟩4. Q.E.D.

PROOF: This is a contradiction.

- ⟨2⟩5. CASE:  $y \in A_n$  and there is no  $m$  such that  $x \in A_m$ .

PROOF: Similar.

- ⟨2⟩6. CASE: There is no  $m$  such that  $x \in A_m$  and there is no  $n$  such that  $y \in A_n$ .

PROOF: Then  $g^{-1}(x) = g^{-1}(y)$  and so  $x = y$ .

- ⟨1⟩5.  $h$  is surjective.

- ⟨2⟩1. LET:  $y \in B$
- ⟨2⟩2. CASE:  $g(y) \in A_n$ 
  - ⟨3⟩1.  $n \neq 0$
  - ⟨3⟩2. PICK  $x \in A_{n-1}$  such that  $g(y) = g(f(x))$
  - ⟨3⟩3.  $y = f(x)$
  - ⟨3⟩4.  $y = h(x)$
- ⟨2⟩3. CASE: There is no  $n$  such that  $g(y) \in A_n$ .  
PROOF: Then  $h(g(y)) = y$ .

□

**Proposition 2.2.10.**

$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps  $f$  to  $(\pi_1 \circ f, \pi_2 \circ f)$  is a bijection. □**Proposition 2.2.11.**

$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function  $\Phi$  such that  $\Phi(f)(c)(b) = f(b, c)$  is a bijection. □

## 2.3 Identity Function

**Definition 2.3.1** (Identity). For any set  $A$ , the *identity* function  $\text{id}_A : A \rightarrow A$  is the function defined by  $\text{id}_A(a) = a$ .

**Proposition 2.3.2.** Let  $f : A \rightarrow B$ . Then  $\text{id}_B \circ f = f = f \circ \text{id}_A$ .

PROOF: Each is the function that maps  $a$  to  $f(a)$ . □

**Proposition 2.3.3.** Let  $f : A \rightarrow B$ . Then  $f$  is surjective if and only if there exists  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$ .

PROOF:

- ⟨1⟩1.  $1 \Rightarrow 3$ 
  - ⟨2⟩1. ASSUME:  $f$  is surjective.
  - ⟨2⟩2. PICK  $g : B \rightarrow A$  such that, for all  $b \in B$ , we have  $f(g(b)) = b$ .  
PROOF: Axiom of Choice.
  - ⟨2⟩3.  $f \circ g = \text{id}_B$ .
- ⟨1⟩2.  $3 \Rightarrow 2$ 
  - ⟨2⟩1. LET:  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$
  - ⟨2⟩2. LET:  $X$  be a set.
  - ⟨2⟩3. LET:  $h, k : B \rightarrow X$
  - ⟨2⟩4. ASSUME:  $h \circ f = k \circ f$
  - ⟨2⟩5.  $h = k$

PROOF:  $h = h \circ f \circ g = k \circ f \circ g = k$ 

□

### 2.3.1 The Empty Set

**Theorem 2.3.4.** *There exists a set which has no elements.*

PROOF:

$\langle 1 \rangle 1$ . PICK a set  $A$

PROOF: By the Axiom of Infinity, a set exists.

$\langle 1 \rangle 2$ . LET:  $S = \{x \in A : \perp\}$  with injection  $i : S \rightarrow A$

PROOF: Axiom of Separation.

$\langle 1 \rangle 3$ .  $S$  has no elements.

□

**Theorem 2.3.5.** *If  $E$  and  $E'$  have no elements then  $E \approx E'$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $E$  and  $E'$  have no elements.

$\langle 1 \rangle 2$ . PICK a function  $F : E \rightarrow E'$ .

PROOF: Axiom of Choice since vacuously  $\forall x \in E. \exists y \in E'. \top$ .

$\langle 1 \rangle 3$ .  $F$  is injective.

PROOF: Vacuously, for all  $x, y \in E$ , if  $F(x) = F(y)$  then  $x = y$ .

$\langle 1 \rangle 4$ .  $F$  is surjective.

PROOF: Vacuously, for all  $y \in E$ , there exists  $x \in E$  such that  $F(x) = y$ .

□

**Definition 2.3.6** (Empty Set). The *empty set*  $\emptyset$  is the set with no elements.

### 2.3.2 The Singleton

**Theorem 2.3.7.** *There exists a set that has exactly one element.*

PROOF:

$\langle 1 \rangle 1$ . PICK a set  $A$  that has an element.

PROOF: By the Axiom of Infinity, there exists a set that has an element.

$\langle 1 \rangle 2$ . PICK  $a \in A$

$\langle 1 \rangle 3$ . PICK a set  $S$  and injection  $i : S \rightarrow A$  such that, for all  $x \in A$ , there exists  $s \in S$  such that  $s = x$  if and only if  $x = a$

$\langle 1 \rangle 4$ .  $S$  has exactly one element.

□

**Theorem 2.3.8.** *If  $A$  and  $B$  both have exactly one element then  $A \approx B$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $B$  both have exactly one element  $a$  and  $b$  respectively.

$\langle 1 \rangle 2$ . LET:  $F : A \rightarrow B$  be the function such that, for all  $x \in A$ , we have  $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$ .  $F$  is a bijection.

□

**Definition 2.3.9** (Singleton). Let  $1$  be the set that has exactly one element. Let  $*$  be its element.



**Proposition 2.3.10.** *Let  $f : A \rightarrow B$ . Assume that, for every set  $X$  and functions  $x, y : X \rightarrow A$ , if  $f \circ x = f \circ y$  then  $x = y$ . Then  $f$  is injective.*

PROOF: Take  $X = 1$ .  $\square$

### 2.3.3 The Set Two

**Proposition 2.3.11.** *Let  $f : A \rightarrow B$ . Then  $f$  is surjective if and only if, for any set  $X$  and functions  $g, h : B \rightarrow X$ , if  $g \circ f = h \circ f$  then  $g = h$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is surjective then, for any set  $X$  and functions  $g, h : B \rightarrow X$ , if  $g \circ f = h \circ f$  then  $g = h$ .

$\langle 2 \rangle 1$ . ASSUME:  $f$  is surjective.

$\langle 2 \rangle 2$ . LET:  $X$  be a set.

$\langle 2 \rangle 3$ . LET:  $g, h : B \rightarrow X$

$\langle 2 \rangle 4$ . ASSUME:  $g \circ f = h \circ f$

$\langle 2 \rangle 5$ . LET:  $b \in B$

PROVE:  $g(b) = h(b)$

$\langle 2 \rangle 6$ . PICK  $a \in A$  such that  $f(a) = b$

$\langle 2 \rangle 7$ .  $g(b) = h(b)$

PROOF:  $g(b) = g(f(a)) = h(f(a)) = h(b)$

$\langle 1 \rangle 2$ . If, for any set  $X$  and functions  $g, h : B \rightarrow x$ , if  $g \circ f = h \circ f$  then  $g = h$ , then  $f$  is surjective.

$\langle 2 \rangle 1$ . ASSUME: For any set  $X$  and functions  $g, h : B \rightarrow X$ , if  $g \circ f = h \circ f$  then  $g = h$ .

$\langle 2 \rangle 2$ . LET:  $b \in B$

$\langle 2 \rangle 3$ . LET:  $h : B \rightarrow 2$  be the function that maps everything to 1.

$\langle 2 \rangle 4$ . LET:  $k : B \rightarrow 2$  be the function that maps  $b$  to 0 and everything else to 1.

$\langle 2 \rangle 5$ .  $h \neq k$

$\langle 2 \rangle 6$ .  $h \circ f \neq k \circ f$

$\langle 2 \rangle 7$ . PICK  $a \in A$  such that  $h(f(a)) \neq k(f(a))$

$\langle 2 \rangle 8$ .  $f(a) = b$

$\square$

### 2.3.4 Subsets

**Definition 2.3.12** (Subset). A *subset* of a set  $A$  consists of a set  $S$  and an injection  $i : S \rightarrow A$ . We write  $(S, i) \subseteq A$ .

We say two subsets  $(S, i)$  and  $(T, j)$  are *equal*,  $(S, i) = (T, j)$ , iff there exists a bijection  $\phi : S \approx T$  such that  $j \circ \phi = i$ .

**Proposition 2.3.13.** *For any subset  $(S, i)$  of  $A$  we have  $(S, i) = (S, i)$ .*

PROOF: We have  $\text{id}_S : S \approx S$  and  $i \circ \text{id}_S = i$ .

**Proposition 2.3.14.** *If  $(S, i) = (T, j)$  then  $(T, j) = (S, i)$ .*

PROOF: If  $\phi : S \approx T$  and  $j \circ \phi = i$  then  $\phi^{-1} : T \approx S$  and  $i \circ \phi^{-1} = j$ .  $\square$

**Proposition 2.3.15.** *If  $(R, i) = (S, j)$  and  $(S, j) = (T, k)$  then  $(R, i) = (T, k)$ .*

PROOF: If  $\phi : R \approx S$  and  $j \circ \phi = i$ , and  $\psi : S \approx T$  and  $k \circ \psi = j$ , then  $\psi \circ \phi : R \approx T$  and  $k \circ \psi \circ \phi = i$ .  $\square$

**Definition 2.3.16** (Membership). Given  $(S, i) \subseteq A$  and  $a \in A$ , we write  $a \in (S, i)$  for  $\exists s \in S. i(s) = a$ .

**Proposition 2.3.17.** *If  $a \in (S, i)$  and  $(S, i) = (T, j)$  then  $a \in (T, j)$ .*

PROOF: If  $i(s) = a$  then  $j(\phi(s)) = a$ .  $\square$

**Definition 2.3.18** (Union). Given subsets  $S$  and  $T$  of  $A$ , the *union* is the subset  $\{x \in A : x \in S \vee x \in T\}$ .

**Definition 2.3.19** (Intersection). Given subsets  $S$  and  $T$  of  $A$ , the *intersection* is the subset  $\{x \in A : x \in S \wedge x \in T\}$ .

**Proposition 2.3.20** (Distributive Law).

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

**Proposition 2.3.21** (Distributive Law).

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

**Definition 2.3.22.** Given a set  $A$ , we write  $\emptyset$  for the subset  $(\emptyset, !)$  where  $!$  is the unique function  $\emptyset \rightarrow A$ .

**Proposition 2.3.23.**

$$S \cup \emptyset = S$$

**Proposition 2.3.24.**

$$S \cap \emptyset = \emptyset$$

**Definition 2.3.25** (Inclusion). Given subsets  $(S, i)$  and  $(T, j)$  of a set  $A$ , we write  $(S, i) \subseteq (T, j)$  iff there exists  $f : S \rightarrow T$  such that  $j \circ f = i$ .

**Proposition 2.3.26.**

$$\emptyset \subseteq S$$

**Definition 2.3.27** (Disjoint). Subsets  $S$  and  $T$  of  $A$  are *disjoint* iff  $S \cap T = \emptyset$ .

**Definition 2.3.28** (Difference). Given subsets  $S$  and  $T$  of  $A$ , the *difference* of  $S$  and  $T$  is  $S - T = \{x \in A : x \in S \wedge x \notin T\}$ .

**Proposition 2.3.29** (De Morgan's Law).

$$R - (S \cup T) = (R - S) \cap (R - T)$$

**Proposition 2.3.30** (De Morgan's Law).

$$R - (S \cap T) = (R - S) \cup (R - T)$$

### 2.3.5 Union

**Definition 2.3.31** (Union). Given  $\mathcal{A} \in \mathcal{PPX}$ , its *union* is

$$\bigcup \mathcal{A} := \{x \in X : \exists S \in \mathcal{A}. x \in S\} \in \mathcal{PX} .$$

### 2.3.6 Intersection

**Definition 2.3.32** (Intersection). Given  $\mathcal{A} \in \mathcal{PPX}$ , its *intersection* is

$$\bigcap \mathcal{A} := \{x \in X : \forall S \in \mathcal{A}. x \in S\} \in \mathcal{PX} .$$

### 2.3.7 Direct Image

**Definition 2.3.33** (Direct Image). Let  $f : A \rightarrow B$ . Let  $S$  be a subset of  $A$ . The *(direct) image* of  $S$  under  $f$  is the subset of  $B$  given by

$$f(S) := \{f(a) : a \in S\} .$$

**Proposition 2.3.34.**

1. If  $S \subseteq T$  then  $f(S) \subseteq f(T)$
2.  $f(\bigcup S) = \bigcup_{S \in \mathcal{S}} f(S)$

**Example 2.3.35.** It is not true in general that  $f(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f(S)$ . Take  $f$  to be the only function  $\{0, 1\} \rightarrow \{0\}$ , and  $\mathcal{S} = \{\{0\}, \{1\}\}$ . Then  $f(\bigcap \mathcal{S}) = \emptyset$  but  $\bigcap_{S \in \mathcal{S}} f(S) = \{0\}$ .

**Example 2.3.36.** It is not true in general that  $f(S - T) = f(S) - f(T)$ . Take  $f$  to be the only function  $\{0, 1\} \rightarrow \{0\}$ ,  $S = \{0\}$  and  $T = \{1\}$ . Then  $f(S - T) = \{0\}$  but  $f(S) - f(T) = \emptyset$ .

### 2.3.8 Inverse Image

**Definition 2.3.37** (Inverse Image). Let  $f : A \rightarrow B$ . Let  $S$  be a subset of  $B$ . The *inverse image* or *preimage* of  $S$  under  $f$  is the subset of  $A$  given by

$$f^{-1}(S) := \{x \in A : f(x) \in S\} .$$

**Proposition 2.3.38.** 1. If  $S \subseteq T$  then  $f^{-1}(S) \subseteq f^{-1}(T)$

2.  $f^{-1}(\bigcup S) = \bigcup_{S \in \mathcal{S}} f^{-1}(S)$
3.  $f^{-1}(\bigcap S) = \bigcap_{S \in \mathcal{S}} f^{-1}(S)$
4.  $f^{-1}(S - T) = f^{-1}(S) - f^{-1}(T)$
5.  $S \subseteq f^{-1}(f(S))$ . Equality holds if  $f$  is injective.
6.  $f(f^{-1}(T)) \subseteq T$ . Equality holds if  $f$  is surjective.
7.  $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$

## 2.4 Relations

**Definition 2.4.1** (Relation). Let  $A$  and  $B$  be sets. A *relation*  $R$  between  $A$  and  $B$ ,  $R : A \rightarrow B$ , is a subset of  $A \times B$ .

Given  $a \in A$  and  $b \in B$ , we write  $aRb$  for  $(a, b) \in R$ .

A relation *on* a set  $A$  is a relation between  $A$  and  $A$ .

**Definition 2.4.2** (Reflexive). A relation  $R$  on a set  $A$  is *reflexive* iff  $\forall a \in A. aRa$ .

**Definition 2.4.3** (Symmetric). A relation  $R$  on a set  $A$  is *symmetric* iff, whenever  $xRy$ , then  $yRx$ .

**Definition 2.4.4** (Transitive). A relation  $R$  on a set  $A$  is *transitive* iff, whenever  $xRy$  and  $yRz$ , then  $xRz$ .

### 2.4.1 Equivalence Relations

**Definition 2.4.5** (Equivalence Relation). A relation  $R$  on a set  $A$  is an *equivalence relation* iff it is reflexive, symmetric and transitive.

**Definition 2.4.6** (Equivalence Class). Let  $R$  be an equivalence relation on a set  $A$  and  $a \in A$ . The *equivalence class* of  $a$  with respect to  $R$  is

$$\{x \in A : xRa\} .$$

**Proposition 2.4.7.** *Two equivalence classes are either disjoint or equal.*

## 2.5 Power Set

**Definition 2.5.1** (Power Set). The *power set* of a set  $A$  is  $\mathcal{P}A := 2^A$ .

Given  $S \in \mathcal{P}A$  and  $a \in A$ , we write  $a \in S$  for  $S(a) = 1$ .

**Definition 2.5.2** (Pairwise Disjoint). Let  $P \subseteq \mathcal{P}A$ . We say the members of  $P$  are *pairwise disjoint* iff, for all  $S, T \in P$ , if  $S \neq T$  then  $S \cap T = \emptyset$ .

### 2.5.1 Partitions

**Definition 2.5.3** (Partition). Let  $A$  be a set. A *partition* of  $A$  is a set  $P \in \mathcal{P}\mathcal{P}A$  such that:

- $\bigcup P = A$
- Every member of  $P$  is nonempty.
- The members of  $P$  are pairwise disjoint.

## 2.6 Cartesian Product

**Definition 2.6.1** (Cartesian Product). Let  $A$  and  $B$  be sets. The *Cartesian product* of  $A$  and  $B$ ,  $A \times B$ , is the tabulation of the relation  $A \bowtie B$  that holds for all  $a \in A$  and  $b \in B$ . The associated functions  $\pi_1 : A \times B \rightarrow A$  and  $\pi_2 : A \times B \rightarrow B$  are called the *projections*.

Given  $a \in A$  and  $b \in B$ , we write  $(a, b)$  for the unique element of  $A \times B$  such that  $\pi_1(a, b) = a$  and  $\pi_2(a, b) = b$ .

## 2.7 Quotient Sets

**Proposition 2.7.1.** Let  $\sim$  be an equivalence relation on  $X$ . Then there exists a set  $X/\sim$ , the quotient set of  $X$  with respect to  $\sim$ , and a surjective function  $\pi : X \twoheadrightarrow X/\sim$ , the canonical projection, such that, for all  $x, y \in X$ , we have  $x \sim y$  if and only if  $\pi(x) = \pi(y)$ .

Further, if  $p : X \twoheadrightarrow Q$  is another quotient with respect to  $\sim$ , then there exists a unique bijection  $\phi : X/\sim \approx Q$  such that  $\phi \circ \pi = p$ .

## 2.8 Partitions

**Definition 2.8.1** (Partition). A *partition* of a set  $X$  is a set of pairwise disjoint subsets of  $X$  whose union is  $X$ .

## 2.9 Disjoint Union

**Theorem 2.9.1.** For any sets  $A$  and  $B$ , there exists a set  $A + B$ , the disjoint union of  $A$  and  $B$ , and functions  $\kappa_1 : A \rightarrow A + B$  and  $\kappa_2 : B \rightarrow A + B$ , the injections, such that, for every set  $X$  and functions  $f : A \rightarrow X$  and  $g : B \rightarrow X$ , there exists a unique function  $[f, g] : A + B \rightarrow X$  such that  $[f, g] \circ \kappa_1 = f$  and  $[f, g] \circ \kappa_2 = g$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $A + B := \{p \in \mathcal{P}A \times \mathcal{P}B : \exists a \in A. p = (\{a\}, \emptyset) \vee \exists b \in B. p = (\emptyset, \{b\})\}$

**Definition 2.9.2** (Restriction). Let  $f : A \rightarrow B$  and let  $(S, i)$  be a subset of  $A$ . The *restriction* of  $f$  to  $S$  is the function  $f \upharpoonright S : S \rightarrow B$  defined by  $f \upharpoonright S = f \circ i$ .

## 2.10 Natural Numbers

**Theorem 2.10.1** (Principle of Recursive Definition). Let  $A$  be a set. Let  $F$  be the set of all functions  $\{m \in \mathbb{N} : m < n\} \rightarrow A$  for some  $n$ . Let  $\rho : F \rightarrow A$ . Then there exists a unique  $g : \mathbb{N} \rightarrow A$  such that, for all  $n \in \mathbb{N}$ , we have

$$g(n) = \rho(g \upharpoonright \{m \in \mathbb{N} : m < n\}) \ .$$

PROOF:

⟨1⟩1. Given a subset  $B \subseteq \mathbb{N}$ , let us say that a function  $g : B \rightarrow A$  is *acceptable* iff, for all  $n \in B$ , we have

$$\forall m < n. m \in B$$

and

$$g(n) = \rho(g \upharpoonright \{m \in \mathbb{N} : m < n\}) .$$

⟨1⟩2. For all  $n \in \mathbb{N}$ , there exists an acceptable function  $\{m \in \mathbb{N} : m < n\} \rightarrow A$ .

⟨2⟩1. LET:  $P[n]$  be the property: There exists an acceptable function  $\{m \in \mathbb{N} : m < n\} \rightarrow A$ .

⟨2⟩2.  $P[0]$

PROOF: The unique function  $\emptyset \rightarrow A$  is acceptable.

⟨2⟩3. For any natural number  $n$ , if  $P[n]$  then  $P[n+1]$ .

⟨3⟩1. ASSUME:  $P[n]$

⟨3⟩2. PICK an acceptable  $f : \{m \in \mathbb{N} : m < n\} \rightarrow A$ .

⟨3⟩3. LET:  $g : \{m \in \mathbb{N} : m < n+1\} \rightarrow A$  be the function

$$g(m) = \begin{cases} f(m) & \text{if } m < n \\ \rho(f) & \text{if } m = n \end{cases}$$

⟨3⟩4.  $g$  is acceptable.

⟨1⟩3. If  $g : B \rightarrow A$  and  $h : C \rightarrow A$  are acceptable, then  $g$  and  $h$  agree on  $B \cap C$ .

⟨1⟩4. Define  $g : \mathbb{N} \rightarrow A$  by:  $g(n) = a$  iff there exists an acceptable  $h : \{m \in \mathbb{N} : m < n+1\}$  such that  $h(n) = a$ .

⟨1⟩5.  $g$  is acceptable.

⟨1⟩6. If  $g' : \mathbb{N} \rightarrow A$  is acceptable then  $g' = g$ .

□

## 2.11 Finite and Infinite Sets

**Definition 2.11.1** (Finite). A set  $A$  is *finite* iff there exists  $n \in \mathbb{N}$  such that  $A \approx \{m \in \mathbb{N} : m < n\}$ . In this case, we say  $A$  has *cardinality*  $n$ .

**Proposition 2.11.2.** Let  $n \in \mathbb{N}$ . Let  $A$  be a set. Let  $a_0 \in A$ . Then  $A \approx \{m \in \mathbb{N} : m < n+1\}$  if and only if  $A - \{a_0\} \approx \{m \in \mathbb{N} : m < n\}$ .

**Theorem 2.11.3.** Let  $A$  be a set. Suppose that  $A \approx \{m \in \mathbb{N} : m < n\}$ . Let  $B$  be a proper subset of  $A$ . Then  $B \not\approx \{m \in \mathbb{N} : m < n\}$  but there exists  $m < n$  such that  $B \approx \{k \in \mathbb{N} : k < m\}$ .

PROOF:

⟨1⟩1. LET:  $P[n]$  be the property: for every set  $A$ , if  $A \approx \{m \in \mathbb{N} : m < n\}$ , then for every proper subset  $B$  of  $A$ , we have  $B \not\approx \{m \in \mathbb{N} : m < n\}$  but there exists  $m < n$  such that  $B \approx \{k \in \mathbb{N} : k < m\}$ .

⟨1⟩2.  $P[0]$

PROOF: If  $A \approx \{m \in \mathbb{N} : m < 0\}$  then  $A$  is empty and so has no proper subset.

⟨1⟩3. For every natural number  $n$ , if  $P[n]$  then  $P[n+1]$ .

⟨2⟩1. LET:  $n$  be a natural number.

⟨2⟩2. ASSUME:  $P[n]$

- $\langle 2 \rangle 3$ . LET:  $A$  be a set.  
 $\langle 2 \rangle 4$ . ASSUME:  $A \approx \{m \in \mathbb{N} : m < n + 1\}$   
 $\langle 2 \rangle 5$ . LET:  $B$  be a proper subset of  $A$ .  
 $\langle 2 \rangle 6$ . CASE:  $B = \emptyset$   
 PROOF: Then  $B \not\approx \{m \in \mathbb{N} : m < n + 1\}$  but  $B \approx \{k \in \mathbb{N} : k < 0\}$ .  
 $\langle 2 \rangle 7$ . CASE:  $B \neq \emptyset$   
 $\langle 3 \rangle 1$ . PICK  $b_0 \in B$   
 $\langle 3 \rangle 2$ .  $A - \{b_0\} \approx \{m \in \mathbb{N} : m < n\}$   
 $\langle 3 \rangle 3$ .  $B - \{b_0\}$  is a proper subset of  $A - \{b_0\}$   
 $\langle 3 \rangle 4$ .  $B - \{b_0\} \not\approx \{m \in \mathbb{N} : m < n\}$   
 $\langle 3 \rangle 5$ .  $B \approx \{m \in \mathbb{N} : m < n + 1\}$   
 $\langle 3 \rangle 6$ . PICK  $m < n$  such that  $B - \{b_0\} \approx \{k \in \mathbb{N} : k < m\}$   
 $\langle 3 \rangle 7$ .  $m + 1 < n + 1$   
 $\langle 3 \rangle 8$ .  $B \approx \{k \in \mathbb{N} : k < m + 1\}$

□

**Corollary 2.11.3.1.** *If  $A$  is finite then there is no bijection between  $A$  and a proper subset of  $A$ .*

**Corollary 2.11.3.2.**  *$\mathbb{N}$  is infinite.*

**Corollary 2.11.3.3.** *The cardinality of a finite set is unique.*

**Corollary 2.11.3.4.** *A subset of a finite set is finite.*

**Corollary 2.11.3.5.** *If  $A$  is finite and  $B$  is a proper subset of  $A$  then  $|B| < |A|$ .*

**Corollary 2.11.3.6.** *Let  $A$  be a set. Then the following are equivalent:*

1.  $A$  is finite.
2. There exists a surjection from an initial segment of  $\mathbb{N}$  onto  $A$ .
3. There exists an injection from  $A$  to an initial segment of  $\mathbb{N}$ .

**Corollary 2.11.3.7.** *A finite union of finite sets is finite.*

**Corollary 2.11.3.8.** *A finite Cartesian product of finite sets is finite.*

**Theorem 2.11.4.** *Let  $A$  be a set. The following are equivalent:*

1. There exists an injective function  $\mathbb{N} \rightarrow A$ .
2. There exists a bijection between  $A$  and a proper subset of  $A$ .
3.  $A$  is infinite.

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

$\langle 2 \rangle 1$ . LET:  $f : \mathbb{N} \rightarrow A$  be injective.

$\langle 2 \rangle 2$ . LET:  $s : \mathbb{N} \approx \mathbb{N} - \{0\}$  be the function  $s(n) = n + 1$ .

$\langle 2 \rangle 3$ .  $f \circ s \circ f^{-1} : A \approx A - \{f(0)\}$

⟨1⟩2.  $2 \Rightarrow 3$

PROOF: Corollary 2.11.3.1.

⟨1⟩3.  $3 \Rightarrow 1$

PROOF: Choose a function  $f : \mathbb{N} \rightarrow A$  such that  $f(n) \in A - \{f(m) : m < n\}$  for all  $n$ .

□

## 2.12 Countable Sets

**Definition 2.12.1** (Countable). A set  $A$  is *countably infinite* iff  $A \approx \mathbb{N}$ .

**Proposition 2.12.2.**  $\mathbb{N} \times \mathbb{N}$  is *countably infinite*.

PROOF: Define  $f : \mathbb{N} \times \mathbb{N} \approx \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x\}$  by

$$f(x, y) = (x + y, y)$$

Define  $g : \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x\} \approx \mathbb{N}$  by

$$g(x, y) = x(x - 1)/2 + y. \quad \square$$

**Proposition 2.12.3.** Every infinite subset of  $\mathbb{N}$  is *countably infinite*.

PROOF:

⟨1⟩1. LET:  $C$  be an infinite subset of  $\mathbb{N}$

⟨1⟩2. Define  $h : \mathbb{Z} \rightarrow C$  by recursion thus:  $h(n)$  is the smallest element of  $C - \{h(m) : m < n\}$ .

⟨1⟩3.  $h$  is injective.

PROOF: If  $m < n$  then  $h(m) \neq h(n)$  because  $h(n) \in C - \{h(m) : m < n\}$ .

⟨1⟩4.  $h$  is surjective.

⟨2⟩1. For all  $n \in \mathbb{N}$  we have  $n \leq h(n)$ .

⟨2⟩2. LET:  $c \in C$

⟨2⟩3.  $c \leq h(c)$

⟨2⟩4. LET:  $n$  be least such that  $c \leq h(n)$

⟨2⟩5.  $c \in C - \{h(m) : m < n\}$

⟨2⟩6.  $h(n) \leq c$

⟨2⟩7.  $h(n) = c$

□

**Definition 2.12.4** (Countable). A set is *countable* iff it is either finite or countably infinite; otherwise it is *uncountable*.

**Proposition 2.12.5.** Let  $B$  be a nonempty set. Then the following are equivalent.

1.  $B$  is countable.
2. There exists a surjection  $\mathbb{N} \twoheadrightarrow B$ .
3. There exists an injection  $B \hookrightarrow \mathbb{N}$ .

PROOF:



- ⟨1⟩1.  $1 \Rightarrow 2$ 
  - ⟨2⟩1. ASSUME:  $B$  is countable.
  - ⟨2⟩2. CASE:  $B$  is finite.
    - ⟨3⟩1. PICK a natural number  $n$  and bijection  $f : \{m \in \mathbb{N} : m < n\} \approx B$
    - ⟨3⟩2. PICK  $b \in B$
    - ⟨3⟩3. Extend  $f$  to a surjection  $g : \mathbb{N} \rightarrow B$  by setting  $g(m) = b$  for  $m \geq n$ .
  - ⟨2⟩3. CASE:  $B$  is countably infinite.
    - PROOF: Then there exists a bijection  $\mathbb{N} \approx B$ .
- ⟨1⟩2.  $2 \Rightarrow 3$ 
  - PROOF: Given a surjection  $f : \mathbb{N} \rightarrow B$ , define  $g : B \rightarrow \mathbb{N}$  by  $g(b)$  is the smallest number such that  $f(g(b)) = b$ .
- ⟨1⟩3.  $3 \Rightarrow 1$ 
  - ⟨2⟩1. LET:  $f : B \rightarrow \mathbb{N}$  be injective.
  - ⟨2⟩2.  $f(B)$  is countable.
  - ⟨2⟩3.  $B \approx f(B)$
  - ⟨2⟩4.  $B$  is countable.

□

**Corollary 2.12.5.1.** *A subset of a countable set is countable.*

**Corollary 2.12.5.2.**  $\mathbb{N} \times \mathbb{N}$  is countably infinite.

PROOF: The function that maps  $(m, n)$  to  $2^m 3^n$  is injective. □

**Corollary 2.12.5.3.** *The Cartesian product of two countable sets is countable.*

**Theorem 2.12.6.** *A countable union of countable sets is countable.*

PROOF:

- ⟨1⟩1. LET:  $A$  be a set.
- ⟨1⟩2. LET:  $\mathcal{B} \subseteq \mathcal{P}A$  be a countable set of countable sets such that  $\bigcup \mathcal{B} = A$
- ⟨1⟩3. PICK a surjection  $B : \mathbb{N} \rightarrow \mathcal{B}$
- ⟨1⟩4. ASSUME: w.l.o.g. each  $B(n)$  is nonempty.
- ⟨1⟩5. For  $n \in \mathbb{N}$ , PICK a surjective function  $g_n : \mathbb{N} \rightarrow B(n)$
- ⟨1⟩6. LET:  $h : \mathbb{N} \times \mathbb{N} \rightarrow A$  be the function  $h(m, n) = g_m(n)$
- ⟨1⟩7.  $h$  is surjective.

□

**Theorem 2.12.7.**  $2^{\mathbb{N}}$  is uncountable.

PROOF:

- ⟨1⟩1. LET:  $f : \mathbb{N} \rightarrow 2^{\mathbb{N}}$ 
  - PROVE:  $f$  is not surjective.
- ⟨1⟩2. Define  $g : \mathbb{N} \rightarrow 2$  by  $g(n) = 1 - f(n)(n)$ .
- ⟨1⟩3. For all  $n \in \mathbb{N}$  we have  $g(n) \neq f(n)(n)$ .
- ⟨1⟩4. For all  $n \in \mathbb{N}$  we have  $g \neq f(n)$ .

□

**Theorem 2.12.8.** *For any set  $A$ , there is no surjective function  $A \rightarrow \mathcal{P}A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow \mathcal{P}A$

$\langle 1 \rangle 2$ . LET:  $S = \{x \in A : x \notin f(x)\}$

$\langle 1 \rangle 3$ . For all  $a \in A$  we have  $S \neq f(a)$

PROOF: We have  $a \in S$  if and only if  $a \notin f(a)$ .

□

**Corollary 2.12.8.1.** *For any set  $A$ , there is no injective function  $\mathcal{P}A \rightarrow A$ .*

## Chapter 3

# Order Theory

### 3.1 Relations

**Definition 3.1.1** (Reflexive). A relation  $R \subseteq A \times A$  is *reflexive* iff, for all  $a \in A$ , we have  $(a, a) \in R$ .

**Definition 3.1.2** (Antisymmetric). A relation  $R \subseteq A \times A$  is *antisymmetric* iff, for all  $a, b \in A$ , if  $(a, b) \in R$  and  $(b, a) \in R$  then  $a = b$ .

**Definition 3.1.3** (Transitive). A relation  $R \subseteq A \times A$  is *transitive* iff, for all  $a, b, c \in A$ , if  $(a, b) \in R$  and  $(b, c) \in R$  then  $(a, c) \in R$ .

**Definition 3.1.4** (Partial Order). A *partial order* on a set  $A$  is a relation on  $A$  that is reflexive, antisymmetric and transitive.

We say  $(A, \leq)$  is a *partially ordered set* or *poset* iff  $\leq$  is a partial order on  $A$ .

**Definition 3.1.5** (Greatest). Let  $A$  be a poset and  $a \in A$ . Then  $a$  is the *greatest* element iff  $\forall x \in A. x \leq a$ .

**Definition 3.1.6** (Least). Let  $A$  be a poset and  $a \in A$ . Then  $a$  is the *least* element iff  $\forall x \in A. a \leq x$ .

**Definition 3.1.7** (Upper Bound). Let  $A$  be a poset,  $S \subseteq A$ , and  $u \in A$ . Then  $u$  is an *upper bound* for  $S$  iff  $\forall x \in S. x \leq u$ . We say  $S$  is *bounded above* iff it has an upper bound.

**Definition 3.1.8** (Lower Bound). Let  $A$  be a poset,  $S \subseteq A$ , and  $l \in A$ . Then  $l$  is a *lower bound* for  $S$  iff  $\forall x \in S. l \leq x$ . We say  $S$  is *bounded below* iff it has a lower bound.

**Definition 3.1.9** (Supremum). Let  $A$  be a poset,  $S \subseteq A$  and  $s \in A$ . Then  $s$  is the *supremum* or *least upper bound* for  $S$  iff  $s$  is the least element in the sub-poset of upper bounds for  $A$ .

**Definition 3.1.10** (Supremum). Let  $A$  be a poset,  $S \subseteq A$  and  $i \in A$ . Then  $i$  is the *infimum* or *greatest lower bound* for  $S$  iff  $i$  is the greatest element in the sub-poset of lower bounds for  $A$ .

**Definition 3.1.11** (Least Upper Bound Property). A poset  $A$  has the *least upper bound property* iff every nonempty subset of  $A$  that is bounded above has a least upper bound.

**Proposition 3.1.12.** *Let  $A$  be a poset. Then  $A$  has the least upper bound property if and only if every nonempty subset of  $A$  that is bounded below has a greatest lower bound.*

PROOF:

$\langle 1 \rangle 1$ . If  $A$  has the least upper bound property then every subset of  $A$  that is bounded below has a greatest lower bound.

$\langle 2 \rangle 1$ . ASSUME:  $A$  has the least upper bound property.

$\langle 2 \rangle 2$ . LET:  $S \subseteq A$  be nonempty and bounded below.

$\langle 2 \rangle 3$ . LET:  $L$  be the set of lower bounds of  $S$ .

$\langle 2 \rangle 4$ .  $L$  is nonempty.

PROOF: Because  $S$  is bounded below.

$\langle 2 \rangle 5$ .  $L$  is bounded above.

PROOF: Pick an element  $s \in S$ . Then  $s$  is an upper bound for  $L$ .

$\langle 2 \rangle 6$ . LET:  $s$  be the supremum of  $L$ .

$\langle 2 \rangle 7$ .  $s$  is the greatest lower bound of  $S$ .

$\langle 3 \rangle 1$ .  $s$  is a lower bound of  $S$ .

$\langle 4 \rangle 1$ . LET:  $x \in S$

$\langle 4 \rangle 2$ .  $x$  is an upper bound for  $L$ .

$\langle 4 \rangle 3$ .  $s \leq x$

$\langle 3 \rangle 2$ . For any lower bound  $l$  of  $S$  we have  $l \leq s$ .

PROOF: Immediate from  $\langle 2 \rangle 6$ .

$\langle 1 \rangle 2$ . If every subset of  $A$  that is bounded below has a greatest lower bound, then  $A$  has the least upper bound property.

PROOF: Dual.

□

### 3.1.1 Strict Partial Orders

**Definition 3.1.13** (Strict Partial Order). A *strict partial order* on a set  $A$  is a relation on  $A$  that is irreflexive and transitive.

**Proposition 3.1.14.** 1. *If  $\leq$  is a partial order on  $A$  then  $<$  is a strict partial order on  $A$ , where  $x < y$  iff  $x \leq y \wedge x \neq y$ .*

2. *If  $<$  is a strict partial order on  $A$  then  $\leq$  is a partial order on  $A$ , where  $x \leq y$  iff  $x < y \vee x = y$ .*

3. *These two relations are inverses of one another.*

### 3.1.2 Linear Orders

**Definition 3.1.15** (Linear Order). A *linear order* on a set  $A$  is a partial order  $\leq$  on  $A$  such that, for all  $x, y \in A$ , we have  $x \leq y$  or  $y \leq x$ .

A *linearly ordered set* is a pair  $(X, \leq)$  such that  $X$  is a set and  $\leq$  is a linear order on  $X$ .

**Definition 3.1.16** (Open Interval). Let  $X$  be a linearly ordered set and  $a, b \in X$ . The *open interval*  $(a, b)$  is the set

$$\{x \in X : a < x < b\} .$$

**Definition 3.1.17** (Immediate Predecessor, Immediate Successor). Let  $X$  be a linearly ordered set and  $a, b \in X$ . Then  $b$  is the (*immediate*) *successor* of  $a$ , and  $a$  is the (*immediate*) *predecessor* of  $b$ , iff  $a < b$  and there is no  $x$  such that  $a < x < b$ .

**Definition 3.1.18** (Dictionary Order). Let  $A$  and  $B$  be linearly ordered sets. The *dictionary order* on  $A \times B$  is the order defined by

$$(a, b) < (a', b') \Leftrightarrow a < a' \vee (a = a' \wedge b < b') .$$

**Theorem 3.1.19** (Maximum Principle). *Every poset has a maximal linearly ordered subset.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $(A, \leq)$  be a poset.

$\langle 1 \rangle 2$ . PICK a well ordering  $\leq$  of  $A$ .

PROOF: Well Ordering Theorem.

$\langle 1 \rangle 3$ . LET:  $h : A \rightarrow 2$  be the function defined by  $\leq$ -recursion thus:

$$h(a) = \begin{cases} 1 & \text{if } a \text{ is } \leq\text{-comparable with every } b < a \text{ such that } h(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\langle 1 \rangle 4$ . LET:  $B = \{x \in A : h(x) = 1\}$

PROVE:  $B$  is a maximal subset linearly ordered by  $\leq$ .

$\langle 1 \rangle 5$ .  $B$  is linearly ordered by  $\leq$ .

$\langle 2 \rangle 1$ . LET:  $x, y \in B$

$\langle 2 \rangle 2$ . ASSUME: w.l.o.g.  $x \leq y$

$\langle 2 \rangle 3$ .  $y$  is  $\leq$ -comparable with  $x$

$\langle 1 \rangle 6$ . For any subset  $C \subseteq A$  linearly ordered by  $\leq$ , if  $B \subseteq C$  then  $B = C$ .

$\langle 2 \rangle 1$ . LET:  $x \in C$

$\langle 2 \rangle 2$ .  $x$  is comparable with every  $y \leq x$  such that  $h(y) = 1$

$\langle 2 \rangle 3$ .  $x \in B$

□

**Theorem 3.1.20** (Zorn's Lemma). *Let  $A$  be a poset. If every linearly ordered subset of  $A$  is bounded above, then  $A$  has a maximal element.*

PROOF:

⟨1⟩1. PICK a maximal linearly ordered subset  $B$  of  $A$ .

PROOF: Maximal Principle

⟨1⟩2. PICK an upper bound  $c$  for  $B$ .

PROVE:  $c$  is maximal.

⟨1⟩3. LET:  $x \in A$

⟨1⟩4. ASSUME:  $c \leq x$

PROVE:  $x = c$

⟨1⟩5.  $x$  is an upper bound for  $B$ .

⟨1⟩6.  $x \in B$

PROOF: By the maximality of  $B$ , since  $B \cup \{x\}$  is linearly ordered.

⟨1⟩7.  $x \leq c$

PROOF: ⟨1⟩2

⟨1⟩8.  $x = c$

□

**Corollary 3.1.20.1** (Kuratowski's Lemma). *Let  $\mathcal{A} \subseteq \mathcal{P}X$ . Suppose that, for every subset  $\mathcal{B} \subseteq \mathcal{A}$  that is linearly ordered by inclusion, we have  $\bigcup \mathcal{B} \in \mathcal{A}$ . Then  $\mathcal{A}$  has a maximal element.*

**Definition 3.1.21** (Closed Interval). Let  $X$  be a linearly ordered set. Let  $a, b \in X$  with  $a < b$ . The *closed interval*  $[a, b]$  is

$$[a, b] := \{x \in X : a \leq x \leq b\} .$$

**Definition 3.1.22** (Half-Open Interval). Let  $X$  be a linearly ordered set. Let  $a, b \in X$  with  $a < b$ . The *half-open intervals*  $(a, b]$  and  $[a, b)$  are defined by

$$(a, b] := \{x \in X : a < x \leq b\}$$

$$[a, b) := \{x \in X : a \leq x < b\}$$

**Definition 3.1.23** (Open Ray). Let  $X$  be a linearly ordered set and  $a \in X$ . The *open rays*  $(a, +\infty)$  and  $(-\infty, a)$  are defined by:

$$(a, +\infty) := \{x \in X : a < x\}$$

$$(-\infty, a) := \{x \in X : x < a\}$$

**Definition 3.1.24** (Closed Ray). Let  $X$  be a linearly ordered set and  $a \in X$ . The *closed rays*  $[a, +\infty)$  and  $(-\infty, a]$  are defined by:

$$[a, +\infty) := \{x \in X : a \leq x\}$$

$$(-\infty, a] := \{x \in X : x \leq a\}$$

### 3.1.3 Sets of Finite Type

**Definition 3.1.25** (Finite Type). Let  $X$  be a set. Let  $\mathcal{A} \subseteq \mathcal{P}X$ . Then  $\mathcal{A}$  is of *finite type* if and only if, for any  $B \subseteq X$ , we have  $B \in \mathcal{A}$  if and only if every finite subset of  $B$  is in  $\mathcal{A}$ .

**Proposition 3.1.26** (Tukey's Lemma). *Let  $X$  be a set. Let  $\mathcal{A} \subseteq \mathcal{P}X$ . If  $\mathcal{A}$  is of finite type, then  $\mathcal{A}$  has a maximal element.*

PROOF:

$\langle 1 \rangle 1$ . For every subset  $\mathcal{B} \subseteq \mathcal{A}$  that is linearly ordered by inclusion, we have  $\bigcup \mathcal{B} \in \mathcal{A}$ .

$\langle 2 \rangle 1$ . LET:  $\mathcal{B} \subseteq \mathcal{A}$

$\langle 2 \rangle 2$ . ASSUME:  $\mathcal{B}$  is linearly ordered by inclusion.

$\langle 2 \rangle 3$ . Every finite subset of  $\bigcup \mathcal{B}$  is in  $\mathcal{A}$

$\langle 2 \rangle 4$ .  $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: Kuratowski's Lemma.

□

## 3.2 Well Orders

**Definition 3.2.1** (Well Ordered Set). A *well ordered set* is a linearly ordered set such that every nonempty subset has a least element.

**Proposition 3.2.2.** *Any subset of a well ordered set is well ordered.*

**Proposition 3.2.3.** *The product of two well ordered sets is well ordered under the dictionary order.*

**Theorem 3.2.4** (Well Ordering Theorem). *Every set has a well ordering.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a set.

$\langle 1 \rangle 2$ . PICK a choice function  $c : \mathcal{P}X - \{\emptyset\} \rightarrow X$

$\langle 1 \rangle 3$ . Define a *tower* to be a pair  $(T, <)$  where  $T \subseteq X$ ,  $<$  is a well ordering of  $T$ , and

$$\forall x \in T. x = c(X - \{y \in T : y < x\}) .$$

$\langle 1 \rangle 4$ . Given two towers, either they are equal or one is a section of the other.

$\langle 2 \rangle 1$ . LET:  $(T_1, <_1)$  and  $(T_2, <_2)$  be towers.

$\langle 2 \rangle 2$ . ASSUME: w.l.o.g. there exists a strictly monotone function  $h : T_1 \rightarrow T_2$

$\langle 2 \rangle 3$ .  $h(T_1)$  is either  $T_2$  or a section of  $T_2$

PROOF: Proposition 3.2.11.

$\langle 2 \rangle 4$ .  $\forall x \in T_1. h(x) = x$

$\langle 3 \rangle 1$ . LET:  $x \in T_1$

$\langle 3 \rangle 2$ . ASSUME: as transfinite induction hypothesis  $\forall y < x. h(y) = y$

$\langle 3 \rangle 3$ .  $h(x)$  is the least element of  $T_2 - \{h(y) \in T_1 : y < x\}$

$\langle 3 \rangle 4$ .  $h(x)$  is the least element of  $T_2 - \{y \in T_1 : y < x\}$

PROOF:  $\langle 3 \rangle 2$

$\langle 3 \rangle 5$ .  $h(x) = x$

PROOF:

$$\begin{aligned}
 h(x) &= c(X - \{y \in T_2 : y < h(x)\}) && \langle 1 \rangle 3 \\
 &= c(X - \{y \in T_2 : y < x\}) && \langle 3 \rangle 4 \\
 &= c(X - \{y \in T_1 : y < x\}) && \langle 3 \rangle 2 \\
 &= x && \langle 1 \rangle 3
 \end{aligned}$$

$\langle 1 \rangle 5$ . If  $(T, <)$  is a tower and  $T \neq X$ , then there exists a tower of which  $(T, <)$  is a section.

PROOF: Let  $T_1 = T \cup \{c(T)\}$  and  $<_1$  be the extension of  $<$  such that  $x < c(T)$  for all  $x \in T$ .

$\langle 1 \rangle 6$ . LET:  $\mathbf{T} = \bigcup \{T : \exists R. (T, R) \text{ is a tower}\}$  and  $\mathbf{R} = \bigcup \{R : \exists T. (T, R) \text{ is a tower}\}$

$\langle 1 \rangle 7$ .  $(\mathbf{T}, \mathbf{R})$  is a tower.

$\langle 2 \rangle 1$ .  $\mathbf{R}$  is irreflexive.

PROOF: Since for every tower  $(T, <)$  we have  $<$  is irreflexive.

$\langle 2 \rangle 2$ .  $\mathbf{R}$  is transitive.

$\langle 3 \rangle 1$ . ASSUME:  $x\mathbf{R}y$  and  $y\mathbf{R}z$

$\langle 3 \rangle 2$ . PICK towers  $(T_1, <_1)$  and  $(T_2, <_2)$  such that  $x <_1 y$  and  $y <_2 z$

$\langle 3 \rangle 3$ . ASSUME: w.l.o.g.  $(T_1, <_1)$  is either  $(T_2, <_2)$  or a section of  $(T_2, <_2)$

$\langle 3 \rangle 4$ .  $x <_2 y <_2 z$

$\langle 3 \rangle 5$ .  $x <_2 z$

$\langle 3 \rangle 6$ .  $x\mathbf{R}z$

$\langle 2 \rangle 3$ . For all  $x, y \in \mathbf{T}$ , either  $x\mathbf{R}y$  or  $x = y$  or  $y\mathbf{R}x$

PROOF: There exists a tower that has both  $x$  and  $y$ .

$\langle 2 \rangle 4$ . Every nonempty subset of  $\mathbf{T}$  has an  $\mathbf{R}$ -least element.

$\langle 3 \rangle 1$ . LET:  $A \subseteq \mathbf{T}$  be nonempty.

$\langle 3 \rangle 2$ . PICK  $a \in A$

$\langle 3 \rangle 3$ . PICK a tower  $(T, <)$  such that  $a \in T$ .

$\langle 3 \rangle 4$ . LET:  $b$  be the  $<$ -least element of  $A \cap T$

PROVE:  $b$  is  $\mathbf{R}$ -least in  $A$ .

$\langle 3 \rangle 5$ . LET:  $x \in A$

$\langle 3 \rangle 6$ . Etc.

$\langle 2 \rangle 5$ .  $\forall x \in \mathbf{T}. x = c(X - \{y \in \mathbf{T} : y\mathbf{R}x\})$

$\langle 1 \rangle 8$ .  $\mathbf{T} = X$

$\langle 1 \rangle 9$ .  $\mathbf{R}$  is a well ordering of  $X$ .

□

**Proposition 3.2.5.** *There exists a well-ordered set with a largest element  $\Omega$  such that  $(-\infty, \Omega)$  is uncountable but, for all  $\alpha < \Omega$ , we have  $(-\infty, \alpha)$  is countable.*

PROOF:

$\langle 1 \rangle 1$ . PICK an uncountable well ordered set  $B$ .

$\langle 1 \rangle 2$ . LET:  $C = 2 \times B$  under the dictionary order.

$\langle 1 \rangle 3$ . LET:  $\Omega$  be the least element of  $C$  such that  $(-\infty, \Omega)$  is uncountable.

$\langle 1 \rangle 4$ . LET:  $A = (-\infty, \Omega]$

$\langle 1 \rangle 5$ .  $A$  is a well ordered set with largest element  $\Omega$  such that  $(-\infty, \Omega)$  is uncountable but, for all  $\alpha < \Omega$ , we have  $(-\infty, \alpha)$  is countable.



□

**Proposition 3.2.6.** *Every well ordered set has the least upper bound property.*

PROOF: For any subset that is bounded above, the set of upper bounds is nonempty, hence has a least element. □

**Proposition 3.2.7.** *In a well ordered set, every element that is not greatest has a successor.*

PROOF: If  $a$  is not greatest, then  $\{x : x > a\}$  is nonempty, hence has a least element. □

**Theorem 3.2.8** (Transfinite Induction). *Let  $J$  be a well ordered set. Let  $S \subseteq J$ . Assume that, for every  $\alpha \in J$ , if  $\forall x < \alpha. x \in S$  then  $\alpha \in S$ . Then  $S = J$ .*

PROOF: Otherwise  $J - S$  would be a nonempty subset of  $J$  with no least element. □

**Proposition 3.2.9.** *Let  $I$  be a well ordered set. Let  $\{A_i\}_{i \in I}$  be a family of well ordered sets. Define  $<$  on  $\coprod_{i \in I} A_i$  by:  $\kappa_i(a) < \kappa_j(b)$  iff either  $i < j$ , or  $i = j$  and  $a < b$  in  $A_i$ . Then  $<$  well orders  $\coprod_{i \in I} A_i$ .*

PROOF: Easy. □

**Theorem 3.2.10** (Principle of Transfinite Recursion). *Let  $J$  be a well ordered set. Let  $C$  be a set. Let  $\mathcal{F}$  be the set of all functions from a section of  $J$  into  $C$ . Let  $\rho : \mathcal{F} \rightarrow C$ . Then there exists a unique function  $h : J \rightarrow C$  such that, for all  $\alpha \in J$ , we have*

$$h(\alpha) = \rho(h \upharpoonright (-\infty, \alpha)) .$$

PROOF:

⟨1⟩1. For a function  $h$  mapping either a section of  $J$  or all of  $J$  into  $C$ , let us say  $h$  is *acceptable* iff, for all  $x \in \text{dom } h$ , we have  $(-\infty, x) \subseteq \text{dom } h$  and  $h(x) = \rho(h \upharpoonright (-\infty, x))$ .

⟨1⟩2. If  $h$  and  $k$  are acceptable functions then  $h(x) = k(x)$  for all  $x$  in both domains.

⟨2⟩1. LET:  $x \in J$

⟨2⟩2. ASSUME: as transfinite induction hypothesis that, for all  $y < x$  and any acceptable functions  $h$  and  $k$  with  $y \in \text{dom } h \cap \text{dom } k$ , we have  $h(y) = k(y)$

⟨2⟩3. LET:  $h$  and  $k$  be acceptable functions with  $x \in \text{dom } h \cap \text{dom } k$

⟨2⟩4.  $h \upharpoonright (-\infty, x) = k \upharpoonright (-\infty, x)$

PROOF: By ⟨2⟩2.

⟨2⟩5.  $h(x) = k(x)$

PROOF: By ⟨2⟩3, each is the least element of the set in ⟨2⟩4.

⟨1⟩3. For  $\alpha \in J$ , if there exists an acceptable function  $(-\infty, \alpha) \rightarrow C$ , then there exists an acceptable function  $(-\infty, \alpha] \rightarrow C$ .

⟨2⟩1. LET:  $\alpha \in J$

- $\langle 2 \rangle 2$ . LET:  $f : (-\infty, \alpha) \rightarrow C$  be acceptable.  
 $\langle 2 \rangle 3$ . LET:  $g : (-\infty, \alpha] \rightarrow C$  be the function given by
 
$$g(x) = \begin{cases} f(x) & \text{if } x < \alpha \\ \rho(f) & \text{if } x = \alpha \end{cases}$$
 $\langle 2 \rangle 4$ .  $g$  is acceptable.  
 $\langle 1 \rangle 4$ . Let  $K \subseteq J$ . Assume that, for all  $\alpha \in K$ , there exists an acceptable function  $(-\infty, \alpha) \rightarrow C$ . Then there exists an acceptable function  $\bigcup_{\alpha \in K} (-\infty, \alpha) \rightarrow C$ .  
 $\langle 2 \rangle 1$ . Define  $f : \bigcup_{\alpha \in K} (-\infty, \alpha) \rightarrow C$  by:  $f(x) = y$  iff there exists  $\alpha \in K$  and  $g : (-\infty, \alpha) \rightarrow C$  acceptable such that  $g(x) = y$ .  
 $\langle 1 \rangle 5$ . For every  $\beta \in J$ , there exists an acceptable function  $(-\infty, \beta) \rightarrow C$   
 $\langle 2 \rangle 1$ . LET:  $\beta \in J$   
 $\langle 2 \rangle 2$ . ASSUME: as transfinite induction hypothesis that, for all  $\alpha < \beta$ , there exists an acceptable function  $(-\infty, \alpha) \rightarrow C$   
 $\langle 2 \rangle 3$ . CASE:  $\beta$  has a predecessor  
 $\langle 3 \rangle 1$ . LET:  $\alpha$  be the predecessor of  $\beta$ .  
 $\langle 3 \rangle 2$ . There exists an acceptable function  $(-\infty, \alpha) \rightarrow C$ .  
 $\langle 3 \rangle 3$ . There exists an acceptable function  $(-\infty, \beta) \rightarrow C$ .  
 PROOF: By  $\langle 1 \rangle 3$  since  $(-\infty, \beta) = (-\infty, \alpha]$ .  
 $\langle 2 \rangle 4$ . CASE:  $\beta$  has no predecessor.  
 PROOF: The result follows by  $\langle 1 \rangle 4$  since  $(-\infty, \beta) = \bigcup_{\alpha < \beta} (-\infty, \alpha)$ .  
 $\langle 1 \rangle 6$ . There exists an acceptable function  $J \rightarrow C$ .  
 $\langle 2 \rangle 1$ . CASE:  $J$  has a greatest element.  
 $\langle 3 \rangle 1$ . LET:  $g$  be greatest.  
 $\langle 3 \rangle 2$ . There exists an acceptable function  $(-\infty, g) \rightarrow C$ .  
 PROOF:  $\langle 1 \rangle 5$   
 $\langle 3 \rangle 3$ . There exists an acceptable function  $J \rightarrow C$ .  
 PROOF: By  $\langle 1 \rangle 3$  since  $J = (-\infty, g]$ .  
 $\langle 2 \rangle 2$ . CASE:  $J$  has no greatest element.  
 PROOF: By  $\langle 1 \rangle 4$  since  $J = \bigcup_{\alpha \in J} (-\infty, \alpha)$ .  
 $\square$

**Corollary 3.2.10.1** (Cardinal Comparability). *Let  $A$  and  $B$  be sets. Then either  $A \leq B$  or  $B \leq A$ .*

PROOF: Choose well orderings of  $A$  and  $B$ . Then either there exists a surjection  $A \twoheadrightarrow B$ , or there exists an injective function  $h : A \rightarrow B$  defined by transfinite recursion by  $h(x)$  is the least element of  $B - h((-\infty, x))$ .  $\square$

**Proposition 3.2.11.** *Let  $J$  and  $E$  be well ordered sets. Let  $h : J \rightarrow E$ . Then the following are equivalent.*

1.  $h$  is strictly monotone and  $h(J)$  is either  $E$  or a section of  $E$ .
2. For all  $\alpha \in J$ , we have  $h(\alpha)$  is the least element of  $E - h((-\infty, \alpha))$ .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1. \text{ ASSUME: } 1$

$\langle 2 \rangle 2. h(J)$  is closed downwards.

$\langle 2 \rangle 3. \text{ LET: } \alpha \in J$

$\langle 2 \rangle 4. h(\alpha) \in E - h((-\infty, \alpha))$

PROOF: If  $\beta < \alpha$  then  $h(\beta) < h(\alpha)$ .

$\langle 2 \rangle 5. \text{ For all } y \in E - h((-\infty, \alpha)) \text{ we have } h(\alpha) \leq y$

$\langle 3 \rangle 1. \text{ ASSUME: for a contradiction } y < h(\alpha)$

$\langle 3 \rangle 2. y \in h(J)$

$\langle 3 \rangle 3. \text{ PICK } \beta \in J \text{ such that } h(\beta) = y$

$\langle 3 \rangle 4. h(\beta) < h(\alpha)$

$\langle 3 \rangle 5. \beta < \alpha$

$\langle 3 \rangle 6. \text{ Q.E.D.}$

PROOF: This contradicts the fact that  $y \notin h((-\infty, \alpha))$ .

$\langle 1 \rangle 2. 2 \Rightarrow 1$

$\langle 2 \rangle 1. \text{ ASSUME: } 2$

$\langle 2 \rangle 2. h$  is strictly monotone.

$\langle 3 \rangle 1. \text{ LET: } \alpha, \beta \in J \text{ with } \alpha < \beta$

$\langle 3 \rangle 2. h(\alpha) \neq h(\beta)$

PROOF: Because  $h(\beta) \in E - h((-\infty, \beta))$ .

$\langle 3 \rangle 3. h(\alpha) \leq h(\beta)$

PROOF: Because  $h(\alpha)$  is least in  $E - h((-\infty, \alpha))$ .

$\langle 3 \rangle 4. h(\alpha) < h(\beta)$

$\langle 2 \rangle 3. h(J)$  is either  $E$  or a section of  $E$ .

$\langle 3 \rangle 1. \text{ ASSUME: } h(J) \neq E$

$\langle 3 \rangle 2. \text{ LET: } e \text{ be least in } E - h(J)$

PROVE:  $h(J) = (-\infty, e)$

$\langle 3 \rangle 3. h(J) \subseteq (-\infty, e)$

$\langle 4 \rangle 1. \text{ LET: } \alpha \in J$

$\langle 4 \rangle 2. h(\alpha) \neq e$

PROOF:  $e \notin h(J)$

$\langle 4 \rangle 3. h(\alpha) \leq e$

PROOF: Since  $h(\alpha)$  is least in  $E - h((-\infty, \alpha))$ .

$\langle 4 \rangle 4. h(\alpha) < e$

$\langle 3 \rangle 4. (-\infty, e) \subseteq h(J)$

PROOF: If  $e' < e$  then  $e' \in h(J)$  by leastness of  $e$ .

□



## Chapter 4

# Category Theory

### 4.1 Categories

**Definition 4.1.1.** A *category*  $\mathcal{C}$  consists of:

- a set  $\text{Ob}(\mathcal{C})$  of *objects*. We write  $A \in \mathcal{C}$  for  $A \in \text{Ob}(\mathcal{C})$ .
- for any objects  $X$  and  $Y$ , a set  $\mathcal{C}[X, Y]$  of *morphisms* from  $X$  to  $Y$ . We write  $f : X \rightarrow Y$  for  $f \in \mathcal{C}[X, Y]$ .
- for any objects  $X, Y$  and  $Z$ , a function  $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \rightarrow \mathcal{C}[X, Z]$ , called *composition*.

such that:

- Given  $f : X \rightarrow Y, g : Y \rightarrow Z$  and  $h : Z \rightarrow W$ , we have  $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object  $X$ , there exists a morphism  $\text{id}_X : X \rightarrow X$ , the *identity morphism* on  $X$ , such that:
  - for any object  $Y$  and morphism  $f : Y \rightarrow X$  we have  $\text{id}_X \circ f = f$
  - for any object  $Y$  and morphism  $f : X \rightarrow Y$  we have  $f \circ \text{id}_X = f$

We write the composite of morphism  $f_1, \dots, f_n$  as  $f_n \circ \dots \circ f_1$ . This is unambiguous thanks to Associativity.

**Definition 4.1.2.** Let **Set** be the category of small sets and functions.

**Definition 4.1.3.** Let **LPos** be the category of linearly ordered sets and monotone functions.

**Proposition 4.1.4.** Any finite linearly ordered set is isomorphic to  $\{m \in \mathbb{N} : m < n\}$  for some  $n$ .

PROOF:

$\langle 1 \rangle$ 1. Every finite nonempty linearly ordered set has a greatest element.

- $\langle 2 \rangle 1$ . LET:  $P[n]$  be the property: for any linearly ordered set  $A$ , if there exists a bijection  $A \approx \{m \in \mathbb{N} : m < n\}$  and  $A$  is nonempty then  $A$  has a greatest element.
- $\langle 2 \rangle 2$ .  $P[0]$   
 PROOF: Vacuous.
- $\langle 2 \rangle 3$ .  $\forall n \in \mathbb{N}. P[n] \Rightarrow P[n+1]$
- $\langle 3 \rangle 1$ . LET:  $n \in \mathbb{N}$
- $\langle 3 \rangle 2$ . ASSUME:  $P[n]$
- $\langle 3 \rangle 3$ . LET:  $A$  be a nonempty linearly ordered set.
- $\langle 3 \rangle 4$ . LET:  $f : A \approx \{m \in \mathbb{N} : m < n+1\}$
- $\langle 3 \rangle 5$ . LET:  $a = f^{-1}(n)$
- $\langle 3 \rangle 6$ .  $f \upharpoonright (A - \{a\}) : A - \{a\} \approx \{m \in \mathbb{N} : m < n\}$
- $\langle 3 \rangle 7$ . ASSUME: w.l.o.g.  $a$  is not greatest in  $A$ .
- $\langle 3 \rangle 8$ . LET:  $b$  be greatest in  $A - \{a\}$   
 PROOF:  $\langle 3 \rangle 2$
- $\langle 3 \rangle 9$ .  $b$  is greatest in  $A$ .
- $\langle 1 \rangle 2$ . LET:  $P[n]$  be the property: for any linearly ordered set  $A$ , if there exists a bijection  $A \approx \{m \in \mathbb{N} : m < n\}$  then there exists an isomorphism in **LPos**  $A \cong \{m \in \mathbb{N} : m < n\}$ .
- $\langle 1 \rangle 3$ .  $P[0]$   
 PROOF: If there exists a bijection  $A \approx \emptyset$  then  $A$  is empty and so the unique function  $A \rightarrow \emptyset$  is an order isomorphism.
- $\langle 1 \rangle 4$ . For every natural number  $n$ , if  $P[n]$  then  $P[n+1]$ .
- $\langle 2 \rangle 1$ . LET:  $n$  be a natural number.
- $\langle 2 \rangle 2$ . ASSUME:  $P[n]$
- $\langle 2 \rangle 3$ . LET:  $A$  be a linearly ordered set.
- $\langle 2 \rangle 4$ . ASSUME:  $A$  has  $n+1$  elements.
- $\langle 2 \rangle 5$ . LET:  $a$  be the greatest element in  $A$ .
- $\langle 2 \rangle 6$ . LET:  $f : A - \{a\} \cong \{m \in \mathbb{N} : m < n\}$  be an order isomorphism.  
 PROOF:  $\langle 2 \rangle 2$
- $\langle 2 \rangle 7$ . Define  $g : A \rightarrow \{m \in \mathbb{N} : m < n+1\}$  by
 
$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ n & \text{if } x = a \end{cases}$$
- $\langle 2 \rangle 8$ .  $g$  is an order isomorphism.
- $\langle 1 \rangle 5$ .  $\forall n \in \mathbb{N}. P[n]$   
 $\square$

**Corollary 4.1.4.1.** *Any finite linearly ordered set is well ordered.*

**Proposition 4.1.5.** *Let  $J$  and  $E$  be well ordered sets. Suppose there is a strictly monotone map  $J \rightarrow E$ . Then  $J$  is isomorphic either to  $E$  or a section of  $E$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $k : J \rightarrow E$  be strictly monotone.
- $\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $E$  is nonempty.
- $\langle 1 \rangle 3$ . PICK  $e_0 \in E$

⟨1⟩4. LET:  $h : J \rightarrow E$  be the function defined by transfinite recursion thus:

$$h(\alpha) = \begin{cases} \text{the least element in } E - h((-\infty, \alpha)) & \text{if } h((-\infty, \alpha)) \neq E \\ e_0 & \text{if } h((-\infty, \alpha)) = E \end{cases}$$

⟨1⟩5.  $\forall \alpha \in J, h(\alpha) \leq k(\alpha)$

⟨2⟩1. LET:  $\alpha \in J$

⟨2⟩2. ASSUME: as transfinite induction hypothesis  $\forall \beta < \alpha, h(\beta) \leq k(\beta)$ .

⟨2⟩3.  $\forall \beta < \alpha, h(\beta) < k(\alpha)$

⟨2⟩4.  $h((-\infty, \alpha)) \neq E$

⟨2⟩5.  $h(\alpha)$  is the least element in  $E - h((-\infty, \alpha))$ .

⟨2⟩6.  $k(\alpha) \in E - h((-\infty, \alpha))$

⟨2⟩7.  $h(\alpha) \leq k(\alpha)$

⟨1⟩6.  $\forall \alpha \in J, h((-\infty, \alpha)) \neq E$

PROOF: For  $\beta < \alpha$  we have  $h(\beta) \leq k(\beta) < k(\alpha)$  so  $k(\alpha) \notin h((-\infty, \alpha))$ .

⟨1⟩7. For all  $\alpha \in J$ , we have  $h(\alpha)$  is the least element of  $E - h((-\infty, \alpha))$ .

⟨1⟩8.  $h$  is strictly monotone and  $h(J)$  is either  $E$  or a section of  $E$ .

PROOF: Proposition 3.2.11.

□

**Proposition 4.1.6.** *If  $A$  and  $B$  are well ordered sets, then exactly one of the following conditions hold:  $A \cong B$ , or  $A$  is isomorphic to a section of  $B$ , or  $B$  is isomorphic to a section of  $A$ .*

PROOF:

⟨1⟩1. At least one of the conditions holds.

⟨2⟩1.  $B$  is isomorphic to either  $A + B$  or a section of  $A + B$ .

⟨2⟩2. CASE:  $B \cong A + B$

⟨3⟩1. LET:  $\phi$  be the isomorphism  $B \cong A + B$

⟨3⟩2. LET:  $b_0$  be the least element in  $B$ .

⟨3⟩3.  $A$  is isomorphic to the section  $(-\infty, \phi^{-1}(\kappa_2(b_0)))$  of  $B$ .

⟨2⟩3. CASE:  $a \in A$  and  $B \cong (-\infty, \kappa_1(a))$

PROOF: Then  $B$  is isomorphic to the section  $(-\infty, a)$  of  $A$ .

⟨2⟩4. CASE:  $b \in B$  and  $\phi : B \cong (-\infty, \kappa_2(b))$

⟨3⟩1. CASE:  $b$  is least in  $B$ .

PROOF: Then  $A \cong B$ .

⟨3⟩2. CASE:  $b$  is not least in  $B$ .

⟨4⟩1. LET:  $b_0$  be least in  $B$ .

⟨4⟩2.  $A$  is isomorphic to the section  $(-\infty, \phi^{-1}(\kappa_2(b_0)))$  of  $B$ .

⟨1⟩2. At most one of the conditions holds.

PROOF: Since a well ordered set cannot be isomorphic to a section of itself.

□

**Theorem 4.1.7.** *There exists a well ordered set, unique up to order isomorphism, that is uncountable but such that every section is countable.*

PROOF:

⟨1⟩1. There exists a well ordered set that is uncountable but such that every section is countable.

- $\langle 2 \rangle 1$ . PICK a well ordered set  $A$  with an element  $\Omega \in A$  such that  $(-\infty, \Omega)$  is uncountable but  $\forall \alpha < \Omega. (-\infty, \alpha)$  is countable.  
 $\langle 2 \rangle 2$ . LET:  $(-\infty, \Omega)$  is uncountable but every section is countable.  
 $\langle 1 \rangle 2$ . If  $A$  and  $B$  are uncountable well ordered sets such that every section is countable, then  $A \cong B$ .

PROOF: Since it cannot be that one of  $A$  and  $B$  is isomorphic to a section of the other.

□

**Definition 4.1.8** (Minimal Uncountable Well Ordered Set). The *minimal uncountable well ordered set*  $\Omega$  is the well ordered set that is uncountable but such that every section is countable.

We write  $\bar{\Omega}$  for the well ordered set  $\Omega \cup \{\Omega\}$  where  $\Omega$  is greatest.

**Proposition 4.1.9.** *Every countable subset of  $\Omega$  is bounded above.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $A$  be a countable subset of  $\Omega$ .  
 $\langle 1 \rangle 2$ . For all  $a \in A$  we have  $(-\infty, a)$  is countable.  
 $\langle 1 \rangle 3$ .  $\bigcup_{a \in A} (-\infty, a)$  is countable.  
 $\langle 1 \rangle 4$ .  $\bigcup_{a \in A} (-\infty, a) \neq \Omega$   
 $\langle 1 \rangle 5$ . PICK  $x \in \Omega - \bigcup_{a \in A} (-\infty, a)$   
 $\langle 1 \rangle 6$ .  $x$  is an upper bound for  $A$ .

□

**Proposition 4.1.10.**  *$\Omega$  has no greatest element.*

PROOF: For any  $\alpha \in \Omega$  we have  $(-\infty, \alpha]$  is countable and hence not the whole of  $\Omega$ . □

**Proposition 4.1.11.** *There are uncountably many elements of  $\Omega$  that have no predecessor.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $A$  be the set of all elements of  $\Omega$  that have no predecessor.  
 $\langle 1 \rangle 2$ . LET:  $f : A \times \mathbb{N} \rightarrow \Omega$  be the function that maps  $(a, n)$  to the  $n$ th successor of  $a$ .  
 $\langle 1 \rangle 3$ .  $f$  is surjective.  
 $\langle 2 \rangle 1$ . ASSUME: for a contradiction  $x \in \Omega$  and there is no element  $a \in A$  and  $n \in \mathbb{N}$  such that  $x$  is the  $n$ th successor of  $a$ .  
 $\langle 2 \rangle 2$ . LET:  $x_n$  be the  $n$ th predecessor of  $x$  for  $n \in \mathbb{N}$ .  
 $\langle 2 \rangle 3$ .  $\{x_n : n \in \mathbb{N}\}$  is a nonempty subset of  $\Omega$  with no least element.  
 $\langle 1 \rangle 4$ .  $A \times \mathbb{N}$  is uncountable.  
 $\langle 1 \rangle 5$ .  $A$  is uncountable.

□

**Definition 4.1.12.** We identify a poset  $(A, \leq)$  with the category with:

- set of objects  $A$



- for  $a, b \in A$ , the set of homomorphisms is  $\{x \in 1 : a \leq b\}$

**Proposition 4.1.13.** *A category is a poset iff, for any two objects, there exists at most one morphism between them.*

**Proposition 4.1.14.** *The identity morphism on an object is unique.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C}$  be a category.

$\langle 1 \rangle 2$ . LET:  $A \in \mathcal{C}$

$\langle 1 \rangle 3$ . LET:  $i, j : A \rightarrow A$  be identity morphisms on  $A$ .

$\langle 1 \rangle 4$ .  $i = j$

PROOF:

$$\begin{aligned} i &= i \circ j & (j \text{ is an identity on } A) \\ &= j & (i \text{ is an identity on } A) \end{aligned}$$

□

**Proposition 4.1.15.** *Let  $A$  be a linearly ordered set. Then  $A$  is well ordered if and only if it does not contain a subset of order type  $\mathbb{N}^{\text{op}}$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $A$  is well ordered then it does not contain a subset of order type  $\mathbb{N}^{\text{op}}$ .

PROOF: A subset of order type  $\mathbb{N}^{\text{op}}$  would be a subset with no least element.

$\langle 1 \rangle 2$ . If  $A$  is not well ordered then it contains a subset of order type  $\mathbb{N}^{\text{op}}$ .

$\langle 2 \rangle 1$ . ASSUME:  $A$  is not well ordered.

$\langle 2 \rangle 2$ . PICK a nonempty subset  $S$  with no least element.

$\langle 2 \rangle 3$ . PICK  $a_0 \in S$

$\langle 2 \rangle 4$ . Extend to a sequence  $(a_n)$  in  $S$  such that  $a_{n+1} < a_n$  for all  $n$ .

$\langle 2 \rangle 5$ .  $\{a_n : n \in \mathbb{N}\}$  has order type  $\mathbb{N}^{\text{op}}$ .

□

**Corollary 4.1.15.1.** *Let  $A$  be a linearly ordered set. If every countable subset of  $A$  is well ordered, then  $A$  is well ordered.*

**Definition 4.1.16.** Given  $f : A \rightarrow B$  and an object  $C$ , define the function  $f^* : \mathcal{C}[B, C] \rightarrow \mathcal{C}[A, C]$  by  $f^*(g) = g \circ f$ .

**Definition 4.1.17.** Given  $f : A \rightarrow B$  and an object  $C$ , define the function  $f_* : \mathcal{C}[C, A] \rightarrow \mathcal{C}[C, B]$  by  $f_*(g) = f \circ g$ .

### 4.1.1 Monomorphisms

**Definition 4.1.18** (Monomorphism). Let  $f : A \rightarrow B$ . Then  $f$  is *monic* or a *monomorphism*,  $f : A \rightarrowtail B$ , iff, for any object  $X$  and functions  $x, y : X \rightarrow A$ , if  $f \circ x = f \circ y$  then  $x = y$ .

### 4.1.2 Epimorphisms

**Definition 4.1.19** (Epimorphism). Let  $f : A \rightarrow B$ . Then  $f$  is *epic* or an *epimorphism*,  $f : A \twoheadrightarrow B$ , iff, for any object  $X$  and functions  $x, y : B \rightarrow X$ , if  $x \circ f = y \circ f$  then  $x = y$ .

### 4.1.3 Sections and Retractions

**Definition 4.1.20** (Section, Retraction). Let  $r : A \rightarrow B$  and  $s : B \rightarrow A$ . Then  $r$  is a *retraction* of  $s$ , and  $s$  is a *section* of  $r$ , iff  $rs = \text{id}_B$ .

**Proposition 4.1.21.** *Let  $f : A \rightarrow B$  and  $r, s : B \rightarrow A$ . If  $r$  is a retraction of  $f$  and  $s$  is a section of  $f$  then  $r = s$ .*

PROOF:

$$\begin{aligned}
 r &= r \text{id}_B && \text{(Unit Law)} \\
 &= rfs && (s \text{ is a section of } f) \\
 &= \text{id}_A s && (r \text{ is a retraction of } f) \\
 &= s && \text{(Unit Law)} \square
 \end{aligned}$$

**Proposition 4.1.22.** *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $s : B \rightarrow A$  be a section of  $r : A \rightarrow B$ .

$\langle 1 \rangle 2$ . LET:  $X$  be an object and  $x, y : X \rightarrow B$

$\langle 1 \rangle 3$ . ASSUME:  $s \circ x = s \circ y$

$\langle 1 \rangle 4$ .  $x = y$

PROOF:  $x = r \circ s \circ x = r \circ s \circ y = y$ .

$\square$

**Proposition 4.1.23.** *Every retraction is epic.*

PROOF: Dual.  $\square$

### 4.1.4 Isomorphisms

**Definition 4.1.24** (Isomorphism). A morphism  $f : A \rightarrow B$  is an *isomorphism*,  $f : A \cong B$ , iff there exists a morphism  $f^{-1} : B \rightarrow A$  that is both a retraction and section of  $f$ .

Objects  $A$  and  $B$  are *isomorphic*,  $A \cong B$ , iff there exists an isomorphism between them.

**Proposition 4.1.25.** *The inverse of an isomorphism is unique.*

PROOF: From Proposition 4.1.21.  $\square$

**Proposition 4.1.26.** *If  $f : A \cong B$  then  $f^{-1} : B \cong A$  and  $(f^{-1})^{-1} = f$ .*

PROOF: Since  $ff^{-1} = \text{id}_B$  and  $f^{-1}f = \text{id}_A$ .  $\square$

Isomorphism.

Define the opposite category.

Slice categories

**Definition 4.1.27.** Let  $\mathcal{C}$  be a category and  $B \in \mathcal{C}$ . The category  $\mathcal{C}_B^B$  of objects *over and under*  $B$  is the category with:

- objects all triples  $(X, u, p)$  such that  $u : B \rightarrow X$  and  $p : X \rightarrow B$
- morphisms  $f : (X, u, p) \rightarrow (Y, u', p')$  all morphisms  $f : X \rightarrow Y$  such that  $fu = u'$  and  $p'f = p$ .

**Proposition 4.1.28.**

$$\mathcal{C}_B^B \cong (\mathcal{C}/B) \backslash \text{id}_B \cong (\mathcal{C} \backslash B) / \text{id}_B$$

$(B, \text{id}_B, \text{id}_B)$  is the zero object in  $\mathcal{C}_B^B$ .

### 4.1.5 Initial Objects

**Definition 4.1.29** (Initial Object). An object  $I$  is *initial* iff, for any object  $X$ , there exists exactly one morphism  $I \rightarrow X$ .

**Proposition 4.1.30.** *The empty set is initial in Set.*

PROOF: For any set  $A$ , the nowhere-defined function is the unique function  $\emptyset \rightarrow A$ .  $\square$

**Proposition 4.1.31.** *If  $I$  and  $I'$  are initial objects, then there exists a unique isomorphism  $I \cong I'$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $i : I \rightarrow I'$  be the unique morphism  $I \rightarrow I'$ .

$\langle 1 \rangle 2$ . LET:  $i^{-1} : I' \rightarrow I$  be the unique morphism  $I' \rightarrow I$ .

$\langle 1 \rangle 3$ .  $ii^{-1} = \text{id}_{I'}$

PROOF: There is only one morphism  $I' \rightarrow I'$ .

$\langle 1 \rangle 4$ .  $i^{-1}i = \text{id}_I$

PROOF: There is only one morphism  $I \rightarrow I$ .

$\square$

### 4.1.6 Terminal Objects

**Definition 4.1.32** (Terminal Object). An object  $T$  is *terminal* iff, for any object  $X$ , there exists exactly one morphism  $X \rightarrow T$ .

**Proposition 4.1.33.** *1 is terminal in Set.*

PROOF: For any set  $A$ , the constant function to  $*$  is the only function  $A \rightarrow 1$ .  $\square$

**Proposition 4.1.34.** *If  $T$  and  $T'$  are terminal objects, then there exists a unique isomorphism  $T \cong T'$ .*

PROOF: Dual to Proposition 4.1.31.  $\square$

### 4.1.7 Zero Objects

**Definition 4.1.35** (Zero Object). An object  $Z$  is a *zero object* iff it is an initial object and a terminal object.

**Definition 4.1.36** (Zero Morphism). Let  $\mathcal{C}$  be a category with a zero object  $Z$ . Let  $A, B \in \mathcal{C}$ . The *zero morphism*  $A \rightarrow B$  is the unique morphism  $A \rightarrow Z \rightarrow B$ .

**Proposition 4.1.37.** *There is no zero object in **Set**.*

PROOF: Since  $\emptyset \not\approx 1$ .  $\square$

### 4.1.8 Triads

**Definition 4.1.38** (Triad). Let  $\mathcal{C}$  be a category. A *triad* consists of objects  $X, Y, M$  and morphisms  $\alpha : X \rightarrow M, \beta : Y \rightarrow M$ . We call  $M$  the *codomain* of the triad.

### 4.1.9 Cotriads

**Definition 4.1.39** (Cotriad). Let  $\mathcal{C}$  be a category. A *cotriad* consists of objects  $X, Y, W$  and morphisms  $\xi : W \rightarrow X, \eta : W \rightarrow Y$ . We call  $W$  the *domain* of the triad.

### 4.1.10 Pullbacks

**Definition 4.1.40** (Pullback). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

is a *pullback* iff  $\alpha\xi = \beta\eta$  and, for every object  $Z$  and morphism  $f : Z \rightarrow X$  and  $g : Z \rightarrow Y$  such that  $\alpha f = \beta g$ , there exists a unique  $h : Z \rightarrow W$  such that  $\xi h = f$  and  $\eta h = g$ .

In this case we also say that  $\eta$  is the *pullback* of  $\beta$  along  $\alpha$ .

**Proposition 4.1.41.** *If  $\xi : W \rightarrow X$  and  $\eta : W \rightarrow Y$  form a pullback of  $\alpha : X \rightarrow M$  and  $\beta : Y \rightarrow M$ , and  $\xi' : W' \rightarrow X$  and  $\eta' : W' \rightarrow Y$  also form the pullback of  $\alpha$  and  $\beta$ , then there exists a unique isomorphism  $\phi : W \cong W'$  such that  $\eta'\phi = \eta$  and  $\xi'\phi = \xi$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : W \rightarrow W'$  be the unique morphism such that  $\eta'\phi = \eta$  and  $\xi'\phi = \xi$ .

$\langle 1 \rangle 2$ . LET:  $\phi^{-1} : W' \rightarrow W$  be the unique morphism such that  $\eta\phi^{-1} = \eta'$  and  $\xi\phi^{-1} = \xi'$ .

$\langle 1 \rangle 3$ .  $\phi\phi^{-1} = \text{id}_{W'}$

PROOF: Each is the unique  $x : W' \rightarrow W'$  such that  $\eta'x = \eta'$  and  $\xi'x = \xi'$ .

$\langle 1 \rangle 4$ .  $\phi^{-1}\phi = \text{id}_W$

PROOF: Each is the unique  $x : W \rightarrow W$  such that  $\eta x = \eta$  and  $\xi x = \xi$ .

□

**Proposition 4.1.42.** *For any morphism  $h : A \rightarrow B$ , the following diagram is a pullback diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $Z$  be an object.

$\langle 1 \rangle 2$ . LET:  $f : Z \rightarrow B$  and  $g : Z \rightarrow A$  satisfy  $\text{id}_B f = hg$

$\langle 1 \rangle 3$ .  $g : Z \rightarrow A$  is the unique morphism such that  $\text{id}_A g = g$  and  $hg = f$ .

□

**Proposition 4.1.43.** *The pullback of an isomorphism is an isomorphism.*

PROOF:

$\langle 1 \rangle 1$ . LET:

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback diagram.

$\langle 1 \rangle 2$ . ASSUME:  $\beta$  is an isomorphism.

$\langle 1 \rangle 3$ . LET:  $\xi^{-1}$  be the unique morphism  $X \rightarrow W$  such that  $\xi\xi^{-1} = \text{id}_X$  and  $\eta\xi^{-1} = \beta^{-1}\alpha$ .

PROOF: This exists since  $\alpha\text{id}_X = \beta\beta^{-1}\alpha = \alpha$ .

$\langle 1 \rangle 4$ .  $\xi^{-1}\xi = \text{id}_W$

PROOF: Each is the unique  $x : W \rightarrow W$  such that  $\xi x = \xi$  and  $\eta x = \eta$ .

□

**Proposition 4.1.44.** *Let  $\beta : (Y, y) \rightarrow (M, m)$  and  $\alpha : (X, x) \rightarrow (M, m)$  in  $\mathcal{C} \setminus A$ . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

*be a pullback in  $\mathcal{C}$ . Let  $w : A \rightarrow W$  be the unique morphism such that  $\xi w = x$  and  $\eta w = y$ . Then  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  is the pullback of  $\beta$  and  $\alpha$  in  $\mathcal{C} \setminus A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $(Z, z) \in \mathcal{C} \backslash A$

$\langle 1 \rangle 2$ . LET:  $f : (Z, z) \rightarrow (X, x)$  and  $g : (Z, z) \rightarrow (Y, y)$  satisfy  $\alpha f = \beta g$ .

$\langle 1 \rangle 3$ . LET:  $h : Z \rightarrow W$  be the unique morphism such that  $\xi h = f$  and  $\eta h = g$ .

$\langle 1 \rangle 4$ .  $hz = w$

$\langle 2 \rangle 1$ .  $\xi hz = \xi w$

PROOF:

$$\xi hz = fz \quad (\langle 1 \rangle 3)$$

$$= x \quad (\langle 1 \rangle 2)$$

$$= \xi w$$

$\langle 2 \rangle 2$ .  $\eta hz = \eta w$

PROOF: Similar.

$\langle 1 \rangle 5$ .  $h : (Z, z) \rightarrow (W, w)$

□

**Proposition 4.1.45.** Let  $\beta : (Y, y) \rightarrow (M, m)$  and  $\alpha : (X, x) \rightarrow (M, m)$  in  $\mathcal{C}/A$ . Let

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in  $\mathcal{C}$ . Let  $w = x\xi : W \rightarrow A$ . Then  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  form a pullback of  $\alpha$  and  $\beta$  in  $\mathcal{C}/A$ .

PROOF:

$\langle 1 \rangle 1$ .  $\eta : (W, w) \rightarrow (Y, y)$

PROOF:

$$y\eta = m\beta\eta$$

$$= m\alpha\xi$$

$$= x\xi$$

$$= w$$

$\langle 1 \rangle 2$ . LET:  $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 3$ . LET:  $f : (Z, z) \rightarrow (X, x)$  and  $g : (Z, z) \rightarrow (Y, y)$  satisfy  $\alpha f = \beta g$ .

$\langle 1 \rangle 4$ . LET:  $h : Z \rightarrow W$  be the unique morphism such that  $\xi h = f$  and  $\eta h = g$ .

$\langle 1 \rangle 5$ .  $h : (Z, z) \rightarrow (W, w)$

PROOF:

$$wh = x\xi h$$

$$= xf \quad (\langle 1 \rangle 4)$$

$$= z \quad (\langle 1 \rangle 3)$$

□

**Proposition 4.1.46.** In **Set**, let  $\alpha : X \rightarrow M$  and  $\beta : Y \rightarrow M$ . Let  $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$  with inclusion  $i : W \rightarrow X \times Y$ . Let  $\xi = \pi_1 i : W \rightarrow X$  and  $\eta = \pi_2 i : W \rightarrow Y$ . Then  $\xi$  and  $\eta$  form the pullback of  $\alpha$  and  $\beta$ .

PROOF:

$\langle 1 \rangle 1. \alpha\xi = \beta\eta$

PROOF: For  $w \in W$ , if  $i(w) = (x, y)$  then  $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$ .

$\langle 1 \rangle 2.$  For every set  $Z$  and functions  $f : Z \rightarrow X, g : Z \rightarrow Y$  such that  $\alpha f = \beta g$ , there exists a unique  $h : Z \rightarrow W$  such that  $\xi h = f$  and  $\eta h = g$

PROOF: For  $z \in Z$ , let  $h(z)$  be the unique element of  $W$  such that  $i(h(z)) = (f(z), g(z))$ .

□

Pullback lemma

#### 4.1.11 Pushouts

**Definition 4.1.47** (Pushout). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array} \quad (4.1)$$

is a *pushout* iff  $\alpha\xi = \beta\eta$  and, for every object  $Z$  and morphism  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  such that  $f\xi = g\eta$ , there exists a unique  $h : M \rightarrow Z$  such that  $h\alpha = f$  and  $h\beta = g$ .

We also say that  $\beta$  is the *pushout* of  $\xi$  along  $\eta$ .

**Proposition 4.1.48.** If  $\alpha : X \rightarrow M$  and  $\beta : Y \rightarrow M$  form a pushout of  $\xi : W \rightarrow X$  and  $\eta : W \rightarrow Y$ , and  $\alpha' : X \rightarrow M'$  and  $\beta' : Y \rightarrow M'$  also form a pushout of  $\xi$  and  $\eta$ , then there exists a unique isomorphism  $\phi : M \cong M'$  such that  $\phi\alpha = \alpha'$  and  $\phi\beta = \beta'$ .

PROOF: Dual to Proposition 4.1.41. □

**Proposition 4.1.49.** For any morphism  $h : A \rightarrow B$ , the following diagram is a pushout diagram.

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF: Dual to Proposition 4.1.42.

**Proposition 4.1.50.** The diagram (4.1) is a pushout in  $\mathcal{C}$  iff it is a pullback in  $\mathcal{C}^{\text{op}}$ .

PROOF: Immediate from definitions. □

**Proposition 4.1.51.** The pushout of an isomorphism is an isomorphism.

PROOF: Dual to Proposition 4.1.43.  $\square$

**Proposition 4.1.52.** *Let  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  in  $\mathcal{C} \setminus A$ . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow[\beta]{} & M \end{array}$$

*be a pushout in  $\mathcal{C}$ . Let  $m := \alpha x : A \rightarrow M$ . Then  $\alpha : (X, x) \rightarrow (M, m)$  and  $\beta : (Y, y) \rightarrow (M, m)$  is the pushout of  $\xi$  and  $\eta$  in  $\mathcal{C} \setminus A$ .*

PROOF: Dual to Proposition 4.1.45.  $\square$

**Proposition 4.1.53.** *Let  $\xi : (W, w) \rightarrow (X, x)$  and  $\eta : (W, w) \rightarrow (Y, y)$  in  $\mathcal{C}/A$ . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow[\beta]{} & M \end{array}$$

*be a pushout in  $\mathcal{C}$ . Let  $m : M \rightarrow A$  be the unique morphism such that  $m\alpha = x$  and  $m\beta = y$ . Then  $\alpha : (X, x) \rightarrow (M, m)$  and  $\beta : (Y, y) \rightarrow (M, m)$  is the pushout of  $\xi$  and  $\eta$  in  $\mathcal{C}/A$ .*

PROOF: Dual to Proposition 4.1.44.  $\square$

**Proposition 4.1.54.** *Set has pushouts.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\xi : W \rightarrow X$  and  $\eta : W \rightarrow Y$ .

$\langle 1 \rangle 2$ . LET:  $\sim$  be the equivalence relation on  $X + Y$  generated by  $\xi(w) \sim \eta(w)$  for all  $w \in W$

$\langle 1 \rangle 3$ . LET:  $M = (X + Y)/\sim$  with canonical projection  $\pi : X + Y \twoheadrightarrow M$ .

$\langle 1 \rangle 4$ . LET:  $\alpha = \pi \circ \kappa_1 : X \rightarrow M$

$\langle 1 \rangle 5$ . LET:  $\beta = \pi \circ \kappa_2 : Y \rightarrow M$

$\langle 1 \rangle 6$ . LET:  $Z$  be any set,  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$ .

$\langle 1 \rangle 7$ . ASSUME:  $f\xi = g\eta$

$\langle 1 \rangle 8$ . LET:  $h : X + Y \rightarrow Z$  be the function defined by  $h(x) = f(x)$  and  $h(y) = g(y)$  for  $x \in X$  and  $y \in Y$

$\langle 1 \rangle 9$ .  $h$  respects  $\sim$

PROOF: For  $w \in W$  we have

$$h(\xi(w)) = f(\xi(w)) \quad (\langle 1 \rangle 8)$$

$$= g(\eta(w)) \quad (\langle 1 \rangle 7)$$

$$= h(\eta(w)) \quad (\langle 1 \rangle 8)$$

$\langle 1 \rangle 10$ . LET:  $\bar{h} : M \rightarrow Z$  be the induced function.

$\langle 1 \rangle 11$ .  $\bar{h}\alpha = f$



PROOF:

$$\begin{aligned}\bar{h}(\alpha(x)) &= \bar{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x)\end{aligned}$$

$\langle 1 \rangle 12.$   $\bar{h}\beta = g$

PROOF: Similar.

$\langle 1 \rangle 13.$  For all  $k : M \rightarrow Z$ , if  $k\alpha = f$  and  $k\beta = g$  then  $k = \bar{h}$ .

PROOF:

$$\begin{aligned}k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\ &= f(x) \\ k(\pi(\kappa_2(y))) &= k(\beta(y)) \\ &= g(y) \\ \therefore k \circ \pi &= h \\ \therefore k &= \bar{h}\end{aligned}$$

□

**Definition 4.1.55.** Let  $u : A \rightarrow X$  be an injection. The *pointed set obtained from  $X$  by collapsing  $(A, u)$* , denoted  $X/(A, u)$ , is the pushout

$$\begin{array}{ccc} A & \xrightarrow{\quad} & 1 \\ \downarrow u & & \downarrow * \\ X & \xrightarrow{\quad} & X/(A, u) \end{array}$$

**Proposition 4.1.56.** In  $\mathbf{Set}_*$ , any two morphisms  $1 \rightarrow X$  and  $1 \rightarrow Y$  have a pushout.

PROOF: The pushout of  $a : (1, *) \rightarrow (X, x)$  and  $b : (1, *) \rightarrow (Y, y)$  is  $(X+Y/\sim, x)$  where  $\sim$  is the equivalence relation generated by  $x \sim y$ . □

**Definition 4.1.57** (Wedge). The *wedge* of pointed sets  $X$  and  $Y$ ,  $X \vee Y$ , is the pushout of the unique morphism  $1 \rightarrow X$  and  $1 \rightarrow Y$ .

**Definition 4.1.58** (Smash). Let  $X$  and  $Y$  be pointed sets. Let  $\xi : X \vee Y \rightarrow X$  be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc} 1 & \xrightarrow{\quad} & X & & \\ \downarrow & & \downarrow & \searrow & \\ Y & \xrightarrow{\quad} & X \vee Y & \xrightarrow{\xi} & X \\ & \searrow 0 & & & \end{array}$$

Let  $\eta : X \vee Y \rightarrow Y$  be the unique morphism such that the following diagram

commutes.



Let  $\zeta = \langle \xi, \eta \rangle : X \vee Y \rightarrow X \times Y$ . The *smash* of  $X$  and  $Y$ ,  $X \wedge Y$ , is the result of collapsing  $X \times Y$  with respect to  $\zeta$ .

Pushout lemma

#### 4.1.12 Subcategories

**Definition 4.1.59** (Subcategory). A *subcategory*  $\mathcal{C}'$  of a category  $\mathcal{C}$  consists of:

- a subset  $\text{Ob}(\mathcal{C}')$  of  $\mathcal{C}$
- for all  $A, B \in \text{Ob}(\mathcal{C}')$ , a subset  $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all  $A \in \text{Ob}(\mathcal{C}')$ , we have  $\text{id}_A \in \mathcal{C}'[A, A]$
- for all  $f \in \mathcal{C}'[A, B]$  and  $g \in \mathcal{C}'[B, C]$ , we have  $g \circ f \in \mathcal{C}'[A, C]$ .

It is a *full* subcategory iff, for all  $A, B \in \text{Ob}(\mathcal{C}')$ , we have  $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$ .

#### 4.1.13 Opposite Category

**Definition 4.1.60** (Opposite Category). For any category  $\mathcal{C}$ , the *opposite* category  $\mathcal{C}^{\text{op}}$  is the category with

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- $\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$
- Given  $f \in \mathcal{C}^{\text{op}}[A, B]$  and  $g \in \mathcal{C}^{\text{op}}[B, C]$ , their composite in  $\mathcal{C}^{\text{op}}$  is  $f \circ g$ , where  $\circ$  is composition in  $\mathcal{C}$ .

**Proposition 4.1.61.** An object is initial in  $\mathcal{C}$  iff it is terminal in  $\mathcal{C}^{\text{op}}$ .

PROOF: Immediate from definitions.  $\square$

**Proposition 4.1.62.** An object is terminal in  $\mathcal{C}$  iff it is initial in  $\mathcal{C}^{\text{op}}$ .

PROOF: Immediate from definitions.  $\square$

**Corollary 4.1.62.1.** If  $T$  and  $T'$  are terminal objects in  $\mathcal{C}$  then there exists a unique isomorphism  $T \cong T'$ .

#### 4.1.14 Groupoids

**Definition 4.1.63** (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

#### 4.1.15 Concrete Categories

**Definition 4.1.64** (Concrete Category). A *concrete category*  $\mathcal{C}$  consists of:

- a set  $\text{Ob}(\mathcal{C})$  of *objects*
- for any object  $A \in \text{Ob}(\mathcal{C})$ , a set  $|A|$
- for any objects  $A, B \in \text{Ob}(\mathcal{C})$ , a set of functions  $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any  $f \in \mathcal{C}[A, B]$  and  $g \in \mathcal{C}[B, C]$ , we have  $g \circ f \in \mathcal{C}[A, C]$
- for any object  $A$  we have  $\text{id}_{|A|} \in \mathcal{C}[A, A]$ .

#### 4.1.16 Power of Categories

**Definition 4.1.65.** Let  $\mathcal{C}$  be a category and  $J$  a set. The category  $\mathcal{C}^J$  is the category with:

- objects all  $J$ -indexed families of objects of  $\mathcal{C}$
- morphisms  $\{X_j\}_{j \in J} \rightarrow \{Y_j\}_{j \in J}$  all families  $\{f_j\}_{j \in J}$  where  $f_j : X_j \rightarrow Y_j$

#### 4.1.17 Arrow Category

**Definition 4.1.66** (Arrow Category). Let  $\mathcal{C}$  be a category. The *arrow category*  $\mathcal{C}^\rightarrow$  is the category with:

- objects all triples  $(A, B, f)$  where  $f : A \rightarrow B$  in  $\mathcal{C}$
- morphisms  $(A, B, f) \rightarrow (C, D, g)$  all pairs  $(u : A \rightarrow C, v : B \rightarrow D)$  such that  $vf = gu$ .

#### 4.1.18 Slice Category

**Definition 4.1.67** (Slice Category). Let  $\mathcal{C}$  be a category and  $A \in \mathcal{C}$ . The *slice category under A*,  $\mathcal{C}_{\backslash A}$ , is the category with:

- objects all pairs  $(B, f)$  where  $B \in \mathcal{C}$  and  $f : A \rightarrow B$
- morphisms  $(B, f) \rightarrow (C, g)$  are morphisms  $u : B \rightarrow C$  such that  $uf = g$ .

We identify this with the subcategory of  $\mathcal{C}^\rightarrow$  formed by mapping  $(B, f)$  to  $(A, B, f)$  and  $u$  to  $(\text{id}_A, u)$ .

**Proposition 4.1.68.** *If  $s : (B, f) \rightarrow (C, g)$  in  $\mathcal{C} \setminus A$ , then any retraction of  $s$  in  $\mathcal{C}$  is a retraction of  $s$  in  $\mathcal{C} \setminus A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $r : C \rightarrow B$  be a retraction of  $s$  in  $\mathcal{C}$ .

$\langle 1 \rangle 2$ .  $rg = f$

PROOF:  $rg = rsf = f$ .

$\langle 1 \rangle 3$ .  $r : (C, g) \rightarrow (B, f)$  in  $\mathcal{C} \setminus A$

$\langle 1 \rangle 4$ .  $rs = \text{id}_{(B, f)}$

PROOF: Because composition is inherited from  $\mathcal{C}$ .

□

**Proposition 4.1.69.**  *$\text{id}_A$  is the initial object in  $\mathcal{C} \setminus A$ .*

PROOF: For any  $(B, f) \in \mathcal{C} \setminus A$ , we have  $f$  is the only morphism  $A \rightarrow B$  such that  $f\text{id}_A = f$ . □

**Proposition 4.1.70.** *If  $A$  is terminal in  $\mathcal{C}$  then  $\text{id}_A$  is the zero object in  $\mathcal{C} \setminus A$ .*

PROOF: For any  $(B, f) \in \mathcal{C} \setminus A$ , the unique morphism  $! : B \rightarrow A$  is the unique morphism such that  $!\text{id}_B = f$ . □

**Definition 4.1.71** (Pointed Sets). The *category of pointed sets* is **Set** \ 1.

**Definition 4.1.72.** Let  $\mathcal{C}$  be a category and  $A \in \mathcal{C}$ . The *slice category over  $A$* ,  $\mathcal{C}/A$ , is the category with:

- objects all pairs  $(B, f)$  with  $f : B \rightarrow A$
- morphisms  $u : (B, f) \rightarrow (C, g)$  all morphisms  $u : B \rightarrow C$  such that  $gu = f$ .

**Proposition 4.1.73.** *Let  $u : (B, f) \rightarrow (C, g) : \mathcal{C}/A$ . Any section of  $u$  in  $\mathcal{C}$  is a section of  $u$  in  $\mathcal{C}/A$ .*

PROOF: Dual to Proposition 4.1.68. □

**Proposition 4.1.74.**  *$\text{id}_A$  is terminal in  $\mathcal{C}/A$ .*

PROOF: Dual to Proposition 4.1.69. □

**Proposition 4.1.75.** *If  $A$  is initial in  $\mathcal{C}$  then  $\text{id}_A$  is the zero object in  $\mathcal{C}/A$ .*

PROOF: Dual to Proposition 4.1.70. □

**Definition 4.1.76.** Let  $A \in \mathcal{C}$ . The category of objects *over and under*  $A$ , written  $\mathcal{C}_A^A$ , is the category with:

- objects all triples  $(X, u, p)$  where  $u : A \rightarrow X$ ,  $p : X \rightarrow A$  and  $pu = \text{id}_A$
- morphism  $f : (X, u, p) \rightarrow (Y, v, q)$  all morphisms  $f : X \rightarrow Y$  such that  $fu = v$  and  $qf = p$

**Proposition 4.1.77.**  *$(A, \text{id}_A, \text{id}_A)$  is the zero object in  $\mathcal{C}_A^A$ .*

PROOF: For any object  $(X, u, p)$ , we have  $p$  is the unique morphism  $(X, u, p) \rightarrow (A, \text{id}_A, \text{id}_A)$ , and  $u$  is the unique morphism  $(A, \text{id}_A, \text{id}_A) \rightarrow (X, u, p)$ .  $\square$

**Definition 4.1.78** (Fibre Collapsing). Let  $B$  be a set. Let  $u : (A, a) \rightarrow (X, x)$  in  $\mathbf{Set}/B$ . Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow u & & \downarrow j \\ X & \xrightarrow{i} & C \end{array}$$

Let  $c : C \rightarrow B$  be the unique morphism such that  $cj = \text{id}_B$  and  $ci = x$ . Then  $(C, j, c) \in \mathbf{Set}_B^B$  is called the set over and under  $B$  obtained from  $X$  by *fibre collapsing* with respect to  $u$ . If  $(A, u)$  is a subset of  $X$ , we denote this set over and under  $B$  by  $X/_B(A, u)$ .

**Definition 4.1.79** (Fibre Wedge). Let  $B$  be a small set. Let  $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$ . The *fibre wedge* of  $X$  and  $Y$  is the pushout of  $u_X$  and  $u_Y$ :

$$\begin{array}{ccc} B & \xrightarrow{u_X} & X \\ \downarrow u_Y & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \end{array}$$

**Definition 4.1.80** (Fibre Smash). Let  $X, Y \in \mathbf{Set}_B^B$ . Let  $\xi : X \vee_B Y \rightarrow X$  be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \xi & \\ & & X \end{array}$$

$0$

Let  $\eta : X \vee_B Y \rightarrow Y$  be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \eta & \\ & & Y \end{array}$$

$0$

Let  $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \rightarrow X \times Y$ . The *fibre smash* of  $X$  and  $Y$ ,  $X \wedge_B Y$ , is the result of collapsing  $X \times Y$  with respect to  $\zeta$ .

**Proposition 4.1.81.** *Set has products and coproducts.*

**Proposition 4.1.82.** *Let  $\mathcal{C}$  be a category. Let  $\{X_\alpha\}_{\alpha \in I}$  be a family of objects in  $\mathcal{C}$  and  $Z \in \mathcal{C}$ . Let  $\coprod_{\alpha \in I} X_\alpha$  be the coproduct of  $\{X_\alpha\}_{\alpha \in I}$ . Then*

$$\mathcal{C}[\coprod_{\alpha \in I} X_\alpha, Z] \approx \prod_{\alpha \in I} \mathcal{C}[X_\alpha, Z] .$$

**Proposition 4.1.83.** *Let  $\mathcal{C}$  be a category. Let  $\{X_\alpha\}_{\alpha \in I}$  be a family of objects in  $\mathcal{C}$  and  $Z \in \mathcal{C}$ . Let  $\prod_{\alpha \in I} X_\alpha$  be the product of  $\{X_\alpha\}_{\alpha \in I}$ . Then*

$$\mathcal{C}[Z, \prod_{\alpha \in I} X_\alpha] \approx \prod_{\alpha \in I} \mathcal{C}[Z, X_\alpha] .$$

**Proposition 4.1.84.** *A product in  $\mathcal{C}$  constitutes a product in  $\mathcal{C}/A$ .*

**Proposition 4.1.85.** *A coproduct in  $\mathcal{C}$  constitutes a product in  $\mathcal{C}/A$ .*

## 4.2 Functors

**Definition 4.2.1** (Functor). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of:

- a function  $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism  $f : A \rightarrow B$  in  $\mathcal{C}$ , a morphism  $Ff : FA \rightarrow FB$  in  $\mathcal{D}$

such that:

- for all  $A \in \text{Ob}(\mathcal{C})$  we have  $F\text{id}_A = \text{id}_{FA}$
- for any morphism  $f : A \rightarrow B$  and  $g : B \rightarrow C$  in  $\mathcal{C}$ , we have  $F(g \circ f) = Fg \circ Ff$

**Proposition 4.2.2.** *Functors preserve isomorphisms.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

$\langle 1 \rangle 2$ . LET:  $f : A \cong B$  in  $\mathcal{C}$

$\langle 1 \rangle 3$ .  $Ff^{-1} \circ Ff = \text{id}_{FA}$

PROOF:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\text{id}_A \\ &= \text{id}_{FA} \end{aligned}$$

$\langle 1 \rangle 4$ .  $Ff \circ Ff^{-1} = \text{id}_{FB}$

PROOF:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

**Definition 4.2.3** (Identity Functor). For any category  $\mathcal{C}$ , the *identity* functor on  $\mathcal{C}$  is the functor  $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  defined by

$$\begin{aligned} I_{\mathcal{C}}A &:= A & (A \in \mathcal{C}) \\ I_{\mathcal{C}}f &:= f & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

**Proposition 4.2.4.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$ . If  $r : A \rightarrow B$  is a retraction of  $s : B \rightarrow A$  in  $\mathcal{C}$  then  $Fr$  is a retraction of  $Fs$ .

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned} \quad \square$$

**Corollary 4.2.4.1.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$ . If  $\phi : A \cong B$  is an isomorphism in  $\mathcal{C}$  then  $F\phi : FA \cong FB$  is an isomorphism in  $\mathcal{D}$  with  $(F\phi)^{-1} = F\phi^{-1}$ .

**Definition 4.2.5** (Composition of Functors). Given functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$ , the *composite* functor  $GF : \mathcal{C} \rightarrow \mathcal{E}$  is defined by

$$\begin{aligned} (GF)A &= G(FA) & (A \in \mathcal{C}) \\ (GF)f &= G(Ff) & (f : A \rightarrow B : \mathcal{C}) \end{aligned}$$

**Definition 4.2.6** (Category of Categories). Let **Cat** be the category of small categories and functors.

**Definition 4.2.7** (Isomorphism of Categories). Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Then  $F$  is an *isomorphism of categories* iff there exists a functor  $F^{-1} : \mathcal{D} \rightarrow \mathcal{C}$ , the *inverse* of  $F$ , such that  $FF^{-1} = I_{\mathcal{D}}$  and  $F^{-1}F = I_{\mathcal{C}}$ .

Categories  $\mathcal{C}$  and  $\mathcal{D}$  are *isomorphic*,  $\mathcal{C} \cong \mathcal{D}$ , iff there exists an isomorphism between them.

**Proposition 4.2.8.** If  $A$  is initial in  $\mathcal{C}$  then  $\mathcal{C} \setminus A \cong \mathcal{C}$ .

PROOF:

⟨1⟩1. Define  $F : \mathcal{C} \setminus A \rightarrow \mathcal{C}$  by

$$F(B, f) = B$$

$$F(u : (B, f) \rightarrow (C, g)) = u$$

⟨1⟩2. Define  $G : \mathcal{C} \rightarrow \mathcal{C} \setminus A$  by

$$GB = (B, !_B)$$

where  $!_B$  is the unique morphism  $A \rightarrow B$

$$G(u : B \rightarrow C) = u : (B, !_B) \rightarrow (C, !_C)$$

⟨1⟩3.  $FG = \text{id}_{\mathcal{C}}$

⟨1⟩4.  $GF = \text{id}_{\mathcal{C} \setminus A}$

PROOF: Since  $GF(B, f) = (B, !_B) = (B, f)$  because the morphism  $A \rightarrow B$  is unique.

□

**Proposition 4.2.9.** *If  $A$  is terminal in  $\mathcal{C}$  then  $\mathcal{C}/A \cong \mathcal{C}$ .*

PROOF: Dual.  $\square$

**Proposition 4.2.10.**

$$\mathcal{C}_A^A \cong (\mathcal{C}/A) \backslash (A, \text{id}_A) \cong (\mathcal{C} \backslash A) / (A, \text{id}_A)$$

PROOF:

- $\langle 1 \rangle 1$ . Define a functor  $F : \mathcal{C}_A^A \rightarrow (\mathcal{C}/A) \backslash (A, \text{id}_A)$ .  
 $\langle 2 \rangle 1$ . Given  $A \xrightarrow{u} X \xrightarrow{p} A$  in  $\mathcal{C}_A^A$ , let  $F(X, u, p) = ((X, p), u)$   
 $\langle 2 \rangle 2$ . Given  $f : (A \xrightarrow{u} X \xrightarrow{p} A) \rightarrow (A \xrightarrow{v} Y \xrightarrow{q} A)$ , let  $Ff = f$ .  
 $\langle 1 \rangle 2$ . Define a functor  $G : (\mathcal{C}/A) \backslash (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$ .  
 $\langle 1 \rangle 3$ . Define a functor  $H : \mathcal{C}_A^A \rightarrow (\mathcal{C} \backslash A) / (A, \text{id}_A)$ .  
 $\langle 1 \rangle 4$ . Define a functor  $K : (\mathcal{C} \backslash A) / (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$ .  
 $\square$

**Definition 4.2.11** (Forgetful Functor). For any concrete category  $\mathcal{C}$ , define the *forgetful* functor  $U : \mathcal{C} \rightarrow \mathbf{Set}$  by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

**Definition 4.2.12** (Switching Functor). For any category  $\mathcal{C}$ , define the *switching* functor  $T : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$  by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

**Definition 4.2.13** (Reduction). Let  $\Phi : \mathbf{Set} \rightarrow \mathbf{Set}$  be a functor. The *reduction* of  $\Phi$  is the functor  $\Phi^* : \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  defined by:  $\Phi^*(X, a)$  is the collapse of  $\Phi(X)$  with respect to  $\Phi(a) : \Phi(1) \rightarrow \Phi(X)$ .

**Definition 4.2.14.** Extend the wedge  $\vee$  to a functor  $\mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  by defining, given  $f : X \rightarrow X'$  and  $g : Y \rightarrow Y'$ , then  $f \vee g$  is the unique morphism that makes the following diagram commute.

$$\begin{array}{ccccc} 1 & \longrightarrow & X & & \\ \downarrow & & \downarrow & \searrow f & \\ Y & \longrightarrow & X \vee Y & & X' \\ & \searrow g & \searrow f \vee g & & \downarrow \\ & & Y' & \longrightarrow & X' \vee Y' \end{array}$$

**Definition 4.2.15.** Extend smash to a functor  $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  as follows. Given  $f : X \rightarrow X'$  and  $g : Y \rightarrow Y'$ , let  $f \wedge g : X \wedge Y \rightarrow X' \wedge Y'$  be the



unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc}
 X \vee Y & \longrightarrow & 1 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 X \times Y & \longrightarrow & X \wedge Y & & \\
 & \searrow & \downarrow & \searrow & \\
 & & X' \vee Y' & \longrightarrow & 1 \\
 & \searrow & \downarrow & \searrow & \\
 & & X' \times Y' & \longrightarrow & X' \wedge Y'
 \end{array}$$

$f \times g$

**Definition 4.2.16** (Reduction). Let  $B$  be a small set. Let  $\Phi_B : \mathbf{Set}/B \rightarrow \mathbf{Set}/B$  be a functor. The *reduction* of  $\Phi_B$  is the functor  $\Phi_B^B : \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  defined as follows.

For  $(X, u : B \rightarrow X, p : X \rightarrow B) \in \mathbf{Set}_B^B$ , let  $\Phi_B^B(X)$  be the set over and under  $B$  obtained from  $\Phi_B(X)$  by collapsing with respect to  $\Phi_B(u) : \Phi_B(B) \rightarrow \Phi_B(X)$ .

**Definition 4.2.17.** Extend  $\vee_B$  to a functor  $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ .

**Definition 4.2.18.** Extend  $\wedge_B$  to a functor  $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ .

**Definition 4.2.19** (Faithful). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *faithful* iff, for any objects  $A, B \in \mathcal{C}$  and morphisms  $f, g : A \rightarrow B : \mathcal{C}$ , if  $Ff = Fg$  then  $f = g$ .

**Definition 4.2.20** (Full). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *full* iff, for any objects  $A, B \in \mathcal{C}$  and morphism  $g : FA \rightarrow FB : \mathcal{D}$ , there exists  $f : A \rightarrow B : \mathcal{C}$  such that  $Ff = g$ .

**Definition 4.2.21** (Fully Faithful). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *fully faithful* iff it is full and faithful.

**Definition 4.2.22** (Full Embedding). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a *full embedding* iff it is fully faithful and injective on objects.

## 4.3 Natural Transformations

**Definition 4.3.1** (Natural Transformation). Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$ . A *natural transformation*  $\tau : F \Rightarrow G$  is a family of morphisms  $\{\tau_X : FX \rightarrow GX\}_{X \in \mathcal{C}}$  such that, for every morphism  $f : X \rightarrow Y : \mathcal{C}$ , we have  $Gf \circ \tau_X = \tau_Y \circ Ff$ .

$$\begin{array}{ccc}
 FX & \xrightarrow{Ff} & FY \\
 \tau_X \downarrow & & \downarrow \tau_Y \\
 GX & \xrightarrow{Gf} & GY
 \end{array}$$

**Definition 4.3.2** (Natural Isomorphism). A natural transformation  $\tau : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$  is a *natural isomorphism*,  $\tau : F \cong G$ , iff for all  $X \in \mathcal{C}$ ,  $\tau_X$  is an isomorphism  $FX \cong GX$ .

Functors  $F$  and  $G$  are *naturally isomorphic*,  $F \cong G$ , iff there exists a natural isomorphism between them.

**Definition 4.3.3** (Inverse). Let  $\tau : F \cong G$ . The *inverse* natural isomorphism  $\tau^{-1} : G \cong F$  is defined by  $(\tau^{-1})_X = \tau_X^{-1}$ .

## 4.4 Bifunctors

**Definition 4.4.1** (Commutative). A bifunctor  $\square : \mathcal{C}^2 \rightarrow \mathcal{C}$  is *commutative* iff  $\square \cong \square \circ T$ , where  $T : \mathcal{C}^2 \rightarrow \mathcal{C}^2$  is the swap functor.

**Proposition 4.4.2.**  $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is commutative.

PROOF: Since the pushout of  $f$  and  $g$  is the pushout of  $g$  and  $f$ .  $\square$

**Proposition 4.4.3.**  $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is commutative.

PROOF: In the diagram defining  $X \wedge Y$ , construct the isomorphism between the version with  $X$  and  $Y$  and the version with  $Y$  with  $X$  for every object.  $\square$

**Proposition 4.4.4.**  $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is commutative.

**Proposition 4.4.5.**  $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is commutative.

**Definition 4.4.6** (Associative). A bifunctor  $\square$  is *associative* iff  $\square \circ (\square \times \text{id}) \cong \square \circ (\text{id} \times \square)$ .

**Proposition 4.4.7.**  $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is associative.

PROOF: Since  $X \vee (Y \vee Z)$  and  $(X \vee Y) \vee Z$  are both the pushout of the unique morphisms  $1 \rightarrow X$ ,  $1 \rightarrow Y$  and  $1 \rightarrow Z$ .  $\square$

**Proposition 4.4.8.**  $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$  is associative.

PROOF: Draw isomorphisms between the diagrams for  $X \wedge (Y \wedge Z)$  and  $(X \wedge Y) \wedge Z$ .  $\square$

Product and coproduct are commutative and associative.

**Proposition 4.4.9.**  $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is associative.

**Proposition 4.4.10.**  $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$  is associative.

**Proposition 4.4.11.** Let  $\mathcal{C}$  be a category with binary coproducts. Let  $\square : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  be a bifunctor. Then  $\square$  distributes over  $+$  iff the canonical morphism

$$(X \square Z) + (Y \square Z) \rightarrow (X + Y) \square Z$$

is an isomorphism for all  $X, Y, Z$ .

**Proposition 4.4.12.** *In a category with binary products and binary coproducts, then  $\times$  distributes over  $+$ .*

**Proposition 4.4.13.** *In  $\mathbf{Set}/*$ , we have  $\times$  does not distribute over  $\vee$ .*

**Proposition 4.4.14.** *In  $\mathbf{Set}/*$ , we have  $\wedge$  distributes over  $\vee$ .*

**Proposition 4.4.15.** *In  $\mathbf{Set}/B$ , we have  $\times_B$  distributes over  $+_B$ .*

**Proposition 4.4.16.** *In  $\mathbf{Set}/B^B$ , we have  $\wedge_B$  distributes over  $\vee_B$ .*

## 4.5 Functor Categories

**Definition 4.5.1** (Functor Category). Given categories  $\mathcal{C}$  and  $\mathcal{D}$ , define the *functor category*  $\mathcal{C}^{\mathcal{D}}$  to be the category with objects the functors from  $\mathcal{D}$  to  $\mathcal{C}$  and morphisms the natural transformations.

**Definition 4.5.2** (Yoneda Embedding). Let  $\mathcal{C}$  be a category. The *Yoneda embedding*  $Y : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$  is the functor that maps an object  $A$  to  $\mathcal{C}[-, A]$  and morphisms similarly.

**Theorem 4.5.3** (Yoneda Lemma). *Let  $\mathcal{C}$  be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\text{op}}}[\mathcal{C}[-, X], F] \cong FX$$

*that maps  $\tau : \mathcal{C}[-, X] \Rightarrow F$  to  $\tau_X(\text{id}_X)$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\phi$  is natural in  $X$ .

PROOF:

$\langle 2 \rangle 1$ . LET:  $f : X \rightarrow Y : \mathcal{C}$

$\langle 2 \rangle 2$ . LET:  $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$ .  $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$

PROOF:

$$\begin{aligned} \phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\text{id}_Y \circ f) \\ &= \tau_Y(f) \\ &= \tau_Y(f \circ \text{id}_X) \\ &= Ff(\tau_X(\text{id}_X)) && (\tau \text{ natural}) \\ &= Ff(\phi(\tau)) \end{aligned}$$

$\langle 1 \rangle 2$ .  $\phi$  is natural in  $F$ .

$\langle 2 \rangle 1$ . LET:  $\alpha : F \Rightarrow G : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$

$\langle 2 \rangle 2$ . LET:  $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$ .  $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$

PROOF:  $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\text{id}_X)) = \alpha_X(\phi(\tau))$

$\langle 1 \rangle 3$ . Each  $\phi_{XF}$  is injective.

$\langle 2 \rangle 1$ . LET:  $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 2$ . ASSUME:  $\phi(\sigma) = \phi(\tau)$

⟨2⟩3. LET:  $f : Y \rightarrow X$

⟨2⟩4.  $\sigma_Y(f) = \tau_Y(f)$

PROOF:

$$\begin{aligned}
 \sigma_Y(f) &= \sigma_Y(\text{id}_X \circ f) \\
 &= Ff(\sigma_X(\text{id}_X)) && (\sigma \text{ is natural}) \\
 &= Ff(\tau_X(\text{id}_X)) && (\langle 2 \rangle 2) \\
 &= \tau_Y(\text{id}_X \circ f) && (\tau \text{ is natural}) \\
 &= \tau_Y(f)
 \end{aligned}$$

⟨1⟩4. Each  $\phi_{XF}$  is surjective.

⟨2⟩1. LET:  $X \in \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$

⟨2⟩2. LET:  $a \in FX$

⟨2⟩3. LET:  $\tau : \mathcal{C}[-, X] \Rightarrow F$  be given by  $\tau_Y(g) = Fg(a)$  for  $g : Y \rightarrow X$

⟨2⟩4.  $\tau$  is natural.

⟨3⟩1. LET:  $h : Y \rightarrow Z : \mathcal{C}$

PROVE:  $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \text{id}_X]$

⟨3⟩2. LET:  $g : Z \rightarrow X$

⟨3⟩3.  $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$

PROOF:

$$\begin{aligned}
 \tau_Y(g \circ h) &= F(g \circ h)(a) \\
 &= Fh(Fg(a)) \\
 &= Fh(\tau_Z(g))
 \end{aligned}$$

⟨2⟩5.  $\phi(\tau) = a$

PROOF:

$$\begin{aligned}
 \phi_X(\tau) &= \tau_X(\text{id}_X) \\
 &= F\text{id}_X(a) \\
 &= a
 \end{aligned}$$

□

**Corollary 4.5.3.1.** *The Yoneda embedding is fully faithful.*

**Corollary 4.5.3.2.** *Given objects  $A$  and  $B$  in  $\mathcal{C}$ , we have  $A \cong B$  if and only if  $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$ .*

## Chapter 5

# The Real Numbers

**Theorem 5.0.1.** *The following hold in the real numbers:*

1.  $x + (y + z) = (x + y) + z$
2.  $x(yz) = (xy)z$
3.  $x + y = y + x$
4.  $xy = yx$
5.  $x + 0 = x$
6.  $x1 = x$
7.  $x + (-x) = 0$
8. *If  $x \neq 0$  then  $x \cdot (1/x) = 1$*
9.  $x(y + z) = xy + xz$
10. *If  $x > y$  then  $x + z > y + z$ .*
11. *If  $x > y$  and  $z > 0$  then  $xz > yz$ .*
12.  $\mathbb{R}$  has the least upper bound property.
13. *If  $x < y$  then there exists  $z$  such that  $x < z < y$ .*

**Definition 5.0.2** (Subtraction). We write  $x - y$  for  $x + (-y)$ .

**Definition 5.0.3.** Given real numbers  $x$  and  $y$  with  $y \neq 0$ , we write  $x/y$  for  $xy^{-1}$ .

**Theorem 5.0.4.** *For any real numbers  $x$  and  $y$ , if  $x + y = x$  then  $y = 0$ .*

PROOF:

$\langle 1 \rangle$ 1. LET:  $x, y \in \mathbb{R}$

$\langle 1 \rangle 2$ . ASSUME:  $x + y = x$

$\langle 1 \rangle 3$ .  $y = 0$

PROOF:

$$\begin{aligned}
 y &= y + 0 && \text{(Definition of zero)} \\
 &= y + (x + (-x)) && \text{(Definition of } -x) \\
 &= (y + x) + (-x) && \text{(Associativity of Addition)} \\
 &= (x + y) + (-x) && \text{(Commutativity of Addition)} \\
 &= x + (-x) && (\langle 1 \rangle 2) \\
 &= 0 && \text{(Definition of } -x)
 \end{aligned}$$

□

**Theorem 5.0.5.**

$$\forall x \in \mathbb{R}. 0x = 0$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in \mathbb{R}$

$\langle 1 \rangle 2$ .  $xx + 0x = xx$

PROOF:

$$\begin{aligned}
 xx + 0x &= (x + 0)x && \text{(Distributive Law)} \\
 &= xx && \text{(Definition of 0)}
 \end{aligned}$$

$\langle 1 \rangle 3$ .  $0x = 0$

PROOF: Theorem 5.0.4,  $\langle 1 \rangle 2$ .

□

**Theorem 5.0.6.**

$$-0 = 0$$

PROOF: Since  $0 + 0 = 0$ . □

**Theorem 5.0.7.**

$$\forall x \in \mathbb{R}. -(-x) = x$$

PROOF: Since  $-x + x = 0$ . □

**Theorem 5.0.8.**

$$\forall x, y \in \mathbb{R}. x(-y) = -(xy)$$

PROOF:

$$\begin{aligned}
 x(-y) + xy &= x((-y) + y) && \text{(Distributive Law)} \\
 &= x0 && \text{(Definition of } -y) \\
 &= 0 && \text{(Theorem 5.0.5)} \quad \square
 \end{aligned}$$

**Theorem 5.0.9.**

$$\forall x \in \mathbb{R}. (-1)x = -x$$

PROOF:

$$\begin{aligned}
 (-1)x &= -(1 \cdot x) && \text{(Theorem 5.0.8)} \\
 &= -x && \text{(Definition of 1)} \quad \square
 \end{aligned}$$

### 5.0.1 Subtraction

**Theorem 5.0.10.**

$$\forall x, y, z \in \mathbb{R}. x(y - z) = xy - xz$$

PROOF:

$$\begin{aligned} x(y - z) &= x(y + (-z)) && \text{(Definition of subtraction)} \\ &= xy + x(-z) && \text{(Distributive Law)} \\ &= xy + (-(xz)) && \text{(Theorem 5.0.8)} \\ &= xy - xz && \text{(Definition of subtraction)} \quad \square \end{aligned}$$

**Theorem 5.0.11.**

$$\forall x, y \in \mathbb{R}. -(x + y) = -x - y$$

PROOF:

$$\begin{aligned} -(x + y) &= (-1)(x + y) && \text{(Theorem 5.0.9)} \\ &= (-1)x + (-1)y && \text{(Distributive Law)} \\ &= -x + (-y) && \text{(Theorem 5.0.9)} \\ &= -x - y && \text{(Definition of subtraction)} \quad \square \end{aligned}$$

**Theorem 5.0.12.**

$$\forall x, y \in \mathbb{R}. -(x - y) = -x + y$$

PROOF:

$$\begin{aligned} -(x - y) &= -(x + (-y)) && \text{(Definition of subtraction)} \\ &= -x - (-y) && \text{(Theorem 5.0.11)} \\ &= -x + (-(-y)) && \text{(Definition of subtraction)} \\ &= -x + y && \text{(Theorem 5.0.7)} \quad \square \end{aligned}$$

**Definition 5.0.13** (Reciprocal). Given  $x \in \mathbb{R}$  with  $x \neq 0$ , the *reciprocal* of  $x$ ,  $1/x$ , is the unique real number such that  $x \cdot 1/x = 1$ .

**Theorem 5.0.14.** For any real numbers  $x$  and  $y$ , if  $x \neq 0$  and  $xy = x$  then  $y = 1$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $x, y \in \mathbb{R}$

$\langle 1 \rangle 2$ . ASSUME:  $x \neq 0$

$\langle 1 \rangle 3$ . ASSUME:  $xy = x$

$\langle 1 \rangle 4$ .  $y = 1$

PROOF:

$$\begin{aligned} y &= y1 && \text{(Definition of 1)} \\ &= y(x \cdot 1/x) && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\ &= (yx)1/x && \text{(Associativity of Multiplication)} \\ &= (xy)1/x && \text{(Commutativity of Multiplication)} \\ &= x \cdot 1/x && (\langle 1 \rangle 3) \\ &= 1 && \text{(Definition of } 1/x, \langle 1 \rangle 2) \end{aligned}$$

□

**Definition 5.0.15** (Quotient). Given real numbers  $x$  and  $y$  with  $y \neq 0$ , the quotient  $x/y$  is defined by

$$x/y = x \cdot 1/y .$$

**Theorem 5.0.16.** For any real number  $x$ , if  $x \neq 0$  then  $x/x = 1$ .

PROOF: Immediate from definitions. □

**Theorem 5.0.17.**

$$\forall x \in \mathbb{R}. x/1 = x$$

PROOF:

⟨1⟩1. LET:  $x \in \mathbb{R}$

⟨1⟩2.  $1/1 = 1$

PROOF: Since  $1 \cdot 1 = 1$ .

⟨1⟩3.  $x/1 = x$

PROOF: Since  $x/1 = x \cdot 1/1 = x \cdot 1 = x$ .

□

**Theorem 5.0.18.** For any real numbers  $x$  and  $y$ , if  $x \neq 0$  and  $y \neq 0$  then  $xy \neq 0$ .

PROOF:

⟨1⟩1. LET:  $x, y \in \mathbb{R}$

⟨1⟩2. ASSUME:  $xy = 0$  and  $x \neq 0$

PROVE:  $y = 0$

⟨1⟩3.  $y = 0$

PROOF:

$$\begin{aligned} y &= 1y && \text{(Definition of 1)} \\ &= (1/x)xy && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\ &= (1/x)0 && \text{(\langle 1 \rangle 2)} \\ &= 0 && \text{(Theorem 5.0.5)} \end{aligned}$$

□

**Theorem 5.0.19.** For any real numbers  $y$  and  $z$ , if  $y \neq 0$  and  $z \neq 0$  then  $(1/y)(1/z) = 1/(yz)$ .

PROOF: Since  $yz(1/y)(1/z) = 1 \cdot 1 = 1$ . □

**Corollary 5.0.19.1.** For any real numbers  $x, y, z, w$  with  $y \neq 0 \neq w$ , we have  $(x/y)(z/w) = (xz)/(yw)$ .

**Theorem 5.0.20.** For any real numbers  $x, y, z, w$  with  $y \neq 0 \neq w$ , we have

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + yz}{yw}$$



PROOF:

$$\begin{aligned} yw \left( \frac{x}{y} + \frac{z}{w} \right) &= yw \frac{x}{y} + yw \frac{z}{w} \\ &= wx + yz \end{aligned} \quad \square$$

**Theorem 5.0.21.** *For any real number  $x$ , if  $x \neq 0$  then  $1/x \neq 0$ .*

PROOF: Since  $x \cdot 1/x = 1 \neq 0$ .  $\square$

**Theorem 5.0.22.** *For any real numbers  $w, z$ , if  $w \neq 0 \neq z$  then  $1/(w/z) = z/w$ .*

PROOF: Since  $(z/w)(w/z) = (wz)/(wz) = 1$ .  $\square$

**Theorem 5.0.23.** *For any real numbers  $a, x$  and  $y$ , if  $y \neq 0$  then  $(ax)/y = a(x/y)$*

PROOF: Since  $ya(x/y) = ax$ .  $\square$

**Theorem 5.0.24.** *For any real numbers  $x$  and  $y$ , if  $y \neq 0$  then  $(-x)/y = x/(-y) = -(x/y)$ .*

PROOF:

$\langle 1 \rangle 1.$   $(-x)/y = -(x/y)$

PROOF: Take  $a = -1$  in Theorem 5.0.23.

$\langle 1 \rangle 2.$   $x/(-y) = -(x/y)$

PROOF: Since  $(-y)(-(x/y)) = y(x/y) = x$ .

$\square$

**Theorem 5.0.25.** *For any real numbers  $x, y, z$  and  $w$ , if  $x > y$  and  $w > z$  then  $x + w > y + z$ .*

PROOF: We have  $y + z < x + z < x + w$  by Monotonicity of Addition twice.  $\square$

**Corollary 5.0.25.1.** *For any real numbers  $x$  and  $y$ , if  $x > 0$  and  $y > 0$  then  $x + y > 0$ .*

**Theorem 5.0.26.** *For any real numbers  $x$  and  $y$ , if  $x > 0$  and  $y > 0$  then  $xy > 0$ .*

PROOF:

$$\begin{aligned} xy &> 0y && \text{(Monotonicity of Multiplication)} \\ &= 0 && \text{(Theorem 5.0.5)} \end{aligned} \quad \square$$

**Theorem 5.0.27.** *For any real number  $x$ , we have  $x > 0$  iff  $-x < 0$ .*

PROOF:

$\langle 1 \rangle 1.$  If  $0 < x$  then  $-x < 0$

PROOF: By Monotonicity of Addition adding  $-x$  to both sides.

$\langle 1 \rangle 2.$  If  $-x < 0$  then  $0 < x$

PROOF: By Monotonicity of Addition adding  $x$  to both sides.

$\square$

**Theorem 5.0.28.** *For any real numbers  $x$  and  $y$ , we have  $x > y$  iff  $-x < -y$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $y < x$  then  $-x < -y$ .

PROOF: By Monotonicity of Addition adding  $-x - y$  to both sides.

$\langle 1 \rangle 2$ . If  $-x < -y$  then  $y < x$ .

PROOF: By Monotonicity of Addition adding  $x + y$  to both sides.

□

**Theorem 5.0.29.** *For any real numbers  $x$ ,  $y$  and  $z$ , if  $x > y$  and  $z < 0$  then  $xz < yz$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $x$ ,  $y$  and  $z$  be real numbers.

$\langle 1 \rangle 2$ . ASSUME:  $x > y$

$\langle 1 \rangle 3$ . ASSUME:  $z < 0$

$\langle 1 \rangle 4$ .  $-z > 0$

PROOF: Theorem 5.0.27,  $\langle 1 \rangle 3$ .

$\langle 1 \rangle 5$ .  $x(-z) > y(-z)$

PROOF:  $\langle 1 \rangle 2$ ,  $\langle 1 \rangle 4$ , Monotonicity of Multiplication.

$\langle 1 \rangle 6$ .  $-(xz) > -(yz)$

PROOF: Theorem 5.0.8,  $\langle 1 \rangle 5$ .

$\langle 1 \rangle 7$ .  $xz < yz$

PROOF: Theorem 5.0.27,  $\langle 1 \rangle 6$ .

□

**Theorem 5.0.30.** *For any real number  $x$ , if  $x \neq 0$  then  $xx > 0$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $x > 0$  then  $xx > 0$

PROOF: By Monotonicity of Multiplication.

$\langle 1 \rangle 2$ . If  $x < 0$  then  $xx > 0$

PROOF: Theorem 5.0.29.

□

**Theorem 5.0.31.**

$$0 < 1$$

PROOF: By Theorem 5.0.30 since  $1 = 1 \cdot 1$ . □

**Definition 5.0.32** (Positive). A real number  $x$  is *positive* iff  $x > 0$ .

We write  $\mathbb{R}_+$  for the set of positive reals.

**Theorem 5.0.33.** *For any real numbers  $x$  and  $y$ , we have  $xy$  is positive if and only if  $x$  and  $y$  are both positive or both negative.*

PROOF: By the Monotonicity of Multiplication and Theorem 5.0.29. □

**Corollary 5.0.33.1.** *For any real number  $x$ , if  $x > 0$  then  $1/x > 0$ .*

PROOF: Since  $x \cdot 1/x = 1$  is positive. □

**Theorem 5.0.34.** For any real numbers  $x$  and  $y$ , if  $x > y > 0$  then  $1/x < 1/y$ .

PROOF: If  $1/y \leq 1/x$  then  $1 < 1$  by Monotonicity of Multiplication.  $\square$

**Theorem 5.0.35.** For any real numbers  $x$  and  $y$ , if  $x < y$  then  $x < (x+y)/2 < y$ .

PROOF: We have  $2x < x+y$  and  $x+y < 2y$  by Monotonicity of Addition, hence  $x < (x+y)/2 < y$  by Monotonicity of Multiplication since  $1/2 > 0$ .  $\square$

**Corollary 5.0.35.1.**  $\mathbb{R}$  is a linear continuum.

**Definition 5.0.36** (Negative). A real number  $x$  is *negative* iff  $x < 0$ .

We write  $\overline{\mathbb{R}_+}$  for the set of nonnegative reals.

**Theorem 5.0.37.** For every positive real number  $a$ , there exists a unique positive real  $\sqrt{a}$  such that  $\sqrt{a}^2 = a$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $a$  be a positive real.

$\langle 1 \rangle 2$ . For any real numbers  $x$  and  $h$ , if  $0 \leq h < 1$ , then  

$$(x+h)^2 < x^2 + h(2x+1) .$$

$\langle 2 \rangle 1$ . LET:  $x$  and  $h$  be real numbers.

$\langle 2 \rangle 2$ . ASSUME:  $0 \leq h < 1$

$\langle 2 \rangle 3$ .  $(x+h)^2 < x^2 + h(2x+1)$

PROOF:

$$\begin{aligned} (x+h)^2 &= x^2 + 2hx + h^2 \\ &< x^2 + 2hx + h & (\langle 2 \rangle 2) \\ &= x^2 + h(2x+1) \end{aligned}$$

$\langle 1 \rangle 3$ . For any real numbers  $x$  and  $h$ , if  $h > 0$  then  

$$(x-h)^2 > x^2 - 2hx .$$

$\langle 2 \rangle 1$ . LET:  $x$  and  $h$  be real numbers.

$\langle 2 \rangle 2$ . ASSUME:  $h > 0$

$\langle 2 \rangle 3$ .  $(x-h)^2 > x^2 - 2hx$

PROOF:

$$\begin{aligned} (x-h)^2 &= x^2 - 2hx + h^2 \\ &> x^2 - 2hx & (\langle 2 \rangle 2) \end{aligned}$$

$\langle 1 \rangle 4$ . For any positive real  $x$ , if  $x^2 < a$  then there exists  $h > 0$  such that  

$$(x+h)^2 < a.$$

$\langle 2 \rangle 1$ . LET:  $x$  be a positive real.

$\langle 2 \rangle 2$ . ASSUME:  $x^2 < a$

$\langle 2 \rangle 3$ . LET:  $h = \min((a-x^2)/(2x+1), 1/2)$

$\langle 2 \rangle 4$ .  $0 < h < 1$

$\langle 2 \rangle 5$ .  $(x+h)^2 < a$

PROOF:

$$\begin{aligned} (x+h)^2 &< x^2 + h(2x+1) & (\langle 1 \rangle 2) \\ &\leq a \end{aligned}$$

⟨1⟩5. For any positive real  $x$ , if  $x^2 > a$  then there exists  $h > 0$  such that  $(x - h)^2 > a$ .

⟨2⟩1. LET:  $x$  be a positive real.

⟨2⟩2. ASSUME:  $x^2 > a$

⟨2⟩3. LET:  $h = (x^2 - a)/2x$

⟨2⟩4.  $h > 0$

⟨2⟩5.  $(x - h)^2 > a$

PROOF:

$$(x - h)^2 > x^2 - 2hx$$

$$= a$$

(⟨2⟩3)

⟨1⟩6. LET:  $B = \{x \in \mathbb{R} : x^2 < a\}$

⟨1⟩7.  $B$  is bounded above.

PROOF: If  $a \geq 1$  then  $a$  is an upper bound. If  $a < 1$  then 1 is an upper bound.

⟨1⟩8.  $B$  contains at least one positive real.

PROOF: If  $a \geq 1$  then  $1 \in B$ . If  $a < 1$  then  $a \in B$ .

⟨1⟩9. LET:  $b = \sup B$

⟨1⟩10.  $b^2 = a$

⟨2⟩1.  $b^2 \geq a$

⟨3⟩1. ASSUME: for a contradiction  $b^2 < a$

⟨3⟩2. PICK  $h > 0$  such that  $(b + h)^2 < a$

PROOF: ⟨1⟩4

⟨3⟩3.  $b + h \in B$

⟨3⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩9.

⟨2⟩2.  $b^2 \leq a$

⟨3⟩1. ASSUME: for a contradiction  $b^2 > a$

⟨3⟩2. PICK  $h > 0$  such that  $(b - h)^2 > a$

PROOF: ⟨1⟩5

⟨3⟩3. PICK  $x \in B$  such that  $b - h < x$

PROOF: ⟨1⟩9

⟨3⟩4.  $(b - h)^2 < x^2 < a$

⟨3⟩5. Q.E.D.

PROOF: This contradicts ⟨3⟩2

⟨1⟩11. For any positive reals  $b$  and  $c$ , if  $b^2 = c^2$  then  $b = c$ .

⟨2⟩1. LET:  $b$  and  $c$  be positive reals.

⟨2⟩2. ASSUME:  $b^2 = c^2$

⟨2⟩3.  $b^2 - c^2 = 0$

⟨2⟩4.  $(b - c)(b + c) = 0$

⟨2⟩5.  $b - c = 0$  or  $b + c = 0$

⟨2⟩6.  $b + c \neq 0$

PROOF: Since  $b + c > 0$

⟨2⟩7.  $b - c = 0$

⟨2⟩8.  $b = c$

□

**Theorem 5.0.38.** *The set of real numbers is uncountable.*

## Chapter 6

# Integers and Rationals

### 6.1 Positive Integers

**Definition 6.1.1** (Inductive). A set of real numbers  $A$  is *inductive* iff  $1 \in A$  and  $\forall x \in A. x + 1 \in A$ .

**Definition 6.1.2** (Positive Integer). The set  $\mathbb{Z}_+$  of *positive integers* is the intersection of the set of inductive sets.

**Proposition 6.1.3.** *Every positive integer is positive.*

PROOF: The set of positive reals is inductive.  $\square$

**Proposition 6.1.4.** *1 is the least element of  $\mathbb{Z}_+$ .*

PROOF: Since  $\{x \in \mathbb{R} : x \geq 1\}$  is inductive.  $\square$

**Proposition 6.1.5.**  *$\mathbb{Z}_+$  is inductive.*

PROOF: 1 is an element of every inductive set, and for all  $x \in \mathbb{R}$ , if  $x$  is an element of every inductive set then so is  $x + 1$ .  $\square$

**Theorem 6.1.6** (Principle of Induction). *If  $A$  is an inductive set of positive integers then  $A = \mathbb{Z}_+$ .*

PROOF: Immediate from definitions.  $\square$

**Theorem 6.1.7** (Well-Ordering Property).  *$\mathbb{Z}_+$  is well ordered.*

PROOF: Construct the obvious order isomorphism  $\omega \cong \mathbb{Z}_+$ .  $\square$

**Theorem 6.1.8** (Archimedean Ordering Property). *The set  $\mathbb{Z}_+$  is unbounded above.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $\mathbb{Z}_+$  is bounded above.

⟨1⟩2. LET:

$$s = \sup \mathbb{Z}_+$$

⟨1⟩3. PICK  $n \in \mathbb{Z}_+$  such that  $s - 1 < n$

⟨1⟩4.  $s < n + 1$

⟨1⟩5. Q.E.D.

PROOF: ⟨1⟩2 and ⟨1⟩4 form a contradiction.

□

### 6.1.1 Exponentiation

**Definition 6.1.9.** For  $a$  a real number and  $n$  a positive integer, define the real number  $a^n$  recursively as follows:

$$\begin{aligned} a^1 &= a \\ a^{n+1} &= a^n a \end{aligned}$$

**Theorem 6.1.10.** For all  $a \in \mathbb{R}$  and  $m, n \in \mathbb{Z}_+$ , we have

$$a^n a^m = a^{n+m}$$

PROOF:

⟨1⟩1. LET:  $P(m)$  be the property  $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. a^n a^m = a^{n+m}$

⟨1⟩2.  $P(1)$

PROOF:  $a^n a^1 = a^n a = a^{n+1}$ .

⟨1⟩3.  $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$

⟨2⟩1. LET:  $m$  be a positive integer.

⟨2⟩2. ASSUME:  $P(m)$

⟨2⟩3. LET:  $a \in \mathbb{R}$

⟨2⟩4. LET:  $n \in \mathbb{Z}_+$

⟨2⟩5.  $a^n a^{m+1} = a^{n+m+1}$

PROOF:

$$\begin{aligned} a^n a^{m+1} &= a^n a^m a \\ &= a^{n+m} a && (\langle 2 \rangle 2) \\ &= a^{n+m+1} \end{aligned}$$

⟨1⟩4. Q.E.D.

PROOF: By induction.

□

**Theorem 6.1.11.** For all  $a \in \mathbb{R}$  and  $m, n \in \mathbb{Z}_+$ ,

$$(a^n)^m = a^{nm}.$$

PROOF:

⟨1⟩1. LET:  $P(m)$  be the property  $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. (a^n)^m = a^{nm}$ .

⟨1⟩2.  $P(1)$

PROOF:  $(a^n)^1 = a^n = a^{n \cdot 1}$

⟨1⟩3.  $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$

PROOF:

$$\begin{aligned} (a^n)^{m+1} &= (a^n)^m a^n \\ &= a^{nm} a^n \\ &= a^{nm+n} && (\text{Theorem 6.1.10}) \\ &= a^{n(m+1)} \end{aligned}$$

□

**Theorem 6.1.12.** *For any real numbers  $a$  and  $b$  and positive integer  $m$ ,*

$$a^m b^m = (ab)^m .$$

PROOF: Induction on  $m$ . □

## 6.2 Integers

**Definition 6.2.1** (Integer). The set  $\mathbb{Z}$  of *integers* is

$$\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \{-x : x \in \mathbb{Z}_+\} .$$

**Proposition 6.2.2.** *The sum, difference and product of two integers is an integer.*

PROOF: Easy. □

**Example 6.2.3.**  $1/2$  is not an integer.

**Proposition 6.2.4.** *For any integer  $n$ , there is no integer  $a$  such that  $n < a < n+1$ .*

PROOF:

⟨1⟩1. For any positive integer  $n$ , there is no integer  $a$  such that  $n < a < n+1$ .

⟨2⟩1. There is no integer  $a$  such that  $1 < a < 2$ .

⟨3⟩1. There is no positive integer  $a$  such that  $1 < a < 2$ .

⟨4⟩1. We do not have  $1 < 1 < 2$ .

⟨4⟩2. For any positive integer  $n$ , we do not have  $1 < n+1 < 2$ .

PROOF: Since  $n \geq 1$  so  $n+1 \geq 2$ .

⟨3⟩2. We do not have  $1 < 0 < 2$ .

⟨3⟩3. For any positive integer  $a$ , we do not have  $1 < -a < 2$ .

PROOF: Since  $-a < 0 < 1$ .

⟨2⟩2. For any positive integer  $n$ , if there is no integer  $a$  such that  $n < a < n+1$ , then there is no integer  $a$  such that  $n+1 < a < n+2$ .

PROOF: If  $n+1 < a < n+2$  then  $n < a-1 < n+1$ .

⟨1⟩2. There is no integer  $a$  such that  $0 < a < 1$ .

PROOF: If  $0 < a < 1$  then  $1 < a+1 < 2$ .

⟨1⟩3. For any positive integer  $n$ , there is no integer  $a$  such that  $-n < a < -n+1$ .

PROOF: If  $-n < a < -n+1$  then  $n-1 < -a < n$ .

□

**Theorem 6.2.5.** *Every nonempty subset of  $\mathbb{Z}$  bounded above has a largest element.*

PROOF:

⟨1⟩1. LET:  $S$  be a nonempty subset of  $\mathbb{Z}$  bounded above.

⟨1⟩2. LET:  $u$  be an upper bound for  $S$ .

⟨1⟩3. PICK an integer  $n > u$

PROOF: Archimedean property.

⟨1⟩4. LET:  $k$  be the least positive integer such that  $n - k \in S$ .

⟨2⟩1. PICK  $m \in S$

⟨2⟩2.  $n - m$  is a positive integer.

⟨2⟩3. There exists a positive integer  $k$  such that  $n - k \in S$ .

⟨1⟩5.  $n - k$  is the greatest element in  $S$ .

⟨2⟩1. LET:  $m \in S$

⟨2⟩2.  $n - m \geq k$

⟨2⟩3.  $m \leq n - k$

□

**Theorem 6.2.6.** *For any real number  $x$ , if  $x$  is not an integer then there exists a unique integer  $n$  such that  $n < x < n + 1$ .*

PROOF:

⟨1⟩1.  $\{n \in \mathbb{Z} : n < x\}$  is a nonempty set of integers bounded above.

⟨2⟩1. PICK  $m > -x$

PROOF: Archimedean property.

⟨2⟩2.  $-m < x$

⟨2⟩3.  $\{n \in \mathbb{Z} : n < x\}$  is nonempty.

⟨1⟩2. LET:  $n$  be the greatest integer such that  $n < x$

⟨1⟩3.  $x < n + 1$

⟨1⟩4. If  $n'$  is an integer with  $n' < x < n' + 1$  then  $n' = n$ .

PROOF: We have  $n' < n + 1$  so  $n' \leq n$ , and  $n < n' + 1$  so  $n \leq n'$ .

□

**Definition 6.2.7** (Even). An integer  $n$  is *even* iff  $n/2$  is an integer; otherwise,  $n$  is *odd*.

**Theorem 6.2.8.** *If the integer  $m$  is odd then there exists an integer  $n$  such that  $m = 2n + 1$ .*

PROOF:

⟨1⟩1. LET:  $n$  be the integer such that  $n < m/2 < n + 1$

PROOF: Theorem 6.2.6.

⟨1⟩2.  $2n < m < 2n + 2$

⟨1⟩3.  $m = 2n + 1$

□

**Theorem 6.2.9.** *The product of two odd integers is odd.*



PROOF:  $(2m + 1)(2n + 1) = 2(2mn + m + n) + 1$ .  $\square$

**Corollary 6.2.9.1.** *If  $p$  is an odd integer and  $n$  is a positive integer then  $p^n$  is an odd integer.*

**Definition 6.2.10** (Exponentiation). Extend the definition of exponentiation so  $a^n$  is defined for:

- all real numbers  $a$  and non-negative integers  $n$
- all non-zero real numbers  $a$  and integers  $n$

as follows:

$$\begin{aligned} a^0 &= 1 \\ a^{-n} &= 1/a^n \end{aligned} \quad (n \text{ a positive integer})$$

**Theorem 6.2.11** (Laws of Exponents). *For all non-zero reals  $a$  and  $b$  and integers  $m$  and  $n$ ,*

$$\begin{aligned} a^n a^m &= a^{n+m} \\ (a^n)^m &= a^{nm} \\ a^m b^m &= (ab)^m \end{aligned}$$

PROOF: Easy.  $\square$

**Theorem 6.2.12.**  $\mathbb{Z}$  is countable.

PROOF: The function that maps an integer  $n$  to  $2n$  if  $n \geq 0$  and  $-1 - 2n$  if  $n < 0$  is a bijection  $\mathbb{Z} \approx \mathbb{N}$ .  $\square$

## 6.3 Rational Numbers

**Definition 6.3.1** (Rational Number). The set  $\mathbb{Q}$  of *rational numbers* is the set of all real numbers that are the quotient of two integers. A real that is not rational is *irrational*.

**Theorem 6.3.2.**  $\sqrt{2}$  is irrational.

PROOF:

- $\langle 1 \rangle$ 1. For any positive rational  $a$ , there exist positive integers  $m$  and  $n$  not both even such that  $a = m/n$ .
- $\langle 2 \rangle$ 1. LET:  $a$  be a positive rational.
- $\langle 2 \rangle$ 2. LET:  $n$  be the least positive integer such that  $na$  is a positive integer.
- $\langle 2 \rangle$ 3. LET:  $m = na$
- $\langle 2 \rangle$ 4. ASSUME: for a contradiction  $m$  and  $n$  are both even.
- $\langle 2 \rangle$ 5.  $m/2 = (n/2)a$
- $\langle 2 \rangle$ 6. Q.E.D.

PROOF: This contradicts the leastness of  $n$  ( $\langle 2 \rangle 2$ ).

$\langle 1 \rangle 2$ . ASSUME: for a contradiction  $\sqrt{2}$  is rational.

$\langle 1 \rangle 3$ . PICK positive integers  $m$  and  $n$  not both even such that  $\sqrt{2} = m/n$ .

$\langle 1 \rangle 4$ .  $m^2 = 2n^2$

$\langle 1 \rangle 5$ .  $m^2$  is even.

$\langle 1 \rangle 6$ .  $m$  is even.

PROOF: Theorem 6.2.9.

$\langle 1 \rangle 7$ . LET:  $k = m/2$

$\langle 1 \rangle 8$ .  $4k^2 = 2n^2$

$\langle 1 \rangle 9$ .  $n^2 = 2k^2$

$\langle 1 \rangle 10$ .  $n^2$  is even.

$\langle 1 \rangle 11$ .  $n$  is even.

PROOF: Theorem 6.2.9.

$\langle 1 \rangle 12$ . Q.E.D.

PROOF:  $\langle 1 \rangle 3$ ,  $\langle 1 \rangle 6$  and  $\langle 1 \rangle 11$  form a contradiction.

□

**Theorem 6.3.3.**  $\mathbb{Q}$  is countably infinite.

PROOF: The function  $\mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$  that maps  $(m, n)$  to  $m/(n+1)$  is a surjection.

□

## 6.4 Algebraic Numbers

**Definition 6.4.1** (Algebraic Number). A real number  $r$  is *algebraic* iff there exists a natural number  $n$  and rational numbers  $a_0, a_1, \dots, a_{n-1}$  such that

$$r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0$$

Otherwise,  $r$  is *transcendental*.

**Proposition 6.4.2.** The set of algebraic numbers is countably infinite.

PROOF: There are countably many finite sequences of rational numbers, and each corresponding polynomial has only finitely many roots. □

**Corollary 6.4.2.1.** The set of transcendental numbers is uncountable.

## Chapter 7

# Monoid Theory

**Definition 7.0.1** (Monoid). A *monoid* is a category with one object.

**Definition 7.0.2.** Let  $\mathcal{C}$  be a category and  $X \in \mathcal{C}$ . The monoid  $\text{End}_{\mathcal{C}}(X)$  is the set of all morphisms  $X \rightarrow X$  under composition.

**Proposition 7.0.3.** *For any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $X \in \mathcal{C}$ , we have that  $F : \text{End}_{\mathcal{C}}(X) \rightarrow \text{End}_{\mathcal{D}}(FX)$  is a monoid homomorphism.*

PROOF: Since  $F\text{id}_X = \text{id}_{FX}$  and  $F(g \circ f) = Fg \circ Ff$ .  $\square$



## Chapter 8

# Group Theory

**Definition 8.0.1.** Let **Grp** be the category of small groups and group homomorphisms.

**Definition 8.0.2.** We identify any group  $G$  with the category with one object whose morphisms are the elements of  $G$  with composition given by the multiplication in  $G$ .

**Proposition 8.0.3.** *The trivial group is a zero object in **Grp**.*

PROOF: Easy.  $\square$

The zero morphism  $G \rightarrow H$  maps every element in  $G$  to  $e$ .

**Definition 8.0.4.** Let  $\mathcal{C}$  be a category and  $X \in \mathcal{C}$ . We write  $\text{Aut}_{\mathcal{C}}(X)$  for the set of all isomorphisms  $X \cong X$  under composition.

**Proposition 8.0.5.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor and  $X \in \mathcal{C}$ . Then  $F : \text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(FX)$  is a group homomorphism.*

PROOF: Since  $F\text{id}_X = \text{id}_{FX}$ ,  $F(g \circ f) = Fg \circ Ff$ , and  $Ff^{-1} = (Ff)^{-1}$ .  $\square$

**Proposition 8.0.6.** **Grp** has products.

**Definition 8.0.7** (Free Product). The product of a family of groups in **Grp** is called the *free product*.

**Proposition 8.0.8.** **Ab** has products given by direct sums.



## Chapter 9

# Ring Theory

**Definition 9.0.1.** Let **Ring** be the concrete category of rings and ring homomorphisms.

**Definition 9.0.2** (Spectrum). Let  $R$  be a commutative ring. The *spectrum* of  $R$ ,  $\text{spec } R$ , is the set of all prime ideals of  $R$ .

**Definition 9.0.3** (Zariski Topology). Let  $R$  be a commutative ring. The *Zariski topology* on  $\text{spec } R$  is the topology where the closed sets are the sets of the form

$$VE := \{p \in \text{spec } R : E \subseteq p\}$$

for any  $E \in \mathcal{P}R$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$

$\langle 1 \rangle 2$ . For all  $\mathcal{A} \subseteq \mathcal{C}$  we have  $\bigcap \mathcal{A} \in \mathcal{C}$

$\langle 2 \rangle 1$ . LET:  $\mathcal{A} \subseteq \mathcal{C}$

$\langle 2 \rangle 2$ . LET:  $E = \bigcup \{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$

PROVE:  $VE = \bigcap \mathcal{A}$

$\langle 2 \rangle 3$ . For all  $p \in \text{spec } R$ , if  $E \subseteq p$  then  $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 1$ . LET:  $p \in \text{spec } R$

$\langle 3 \rangle 2$ . ASSUME:  $E \subseteq p$

$\langle 3 \rangle 3$ . LET:  $E' \in \mathcal{P}R$  with  $VE' \in \mathcal{A}$

$\langle 3 \rangle 4$ .  $E' \subseteq E$

$\langle 3 \rangle 5$ .  $E' \subseteq p$

$\langle 3 \rangle 6$ .  $p \in VE'$

$\langle 2 \rangle 4$ . For all  $p \in \text{spec } R$ , if  $p \in \bigcap \mathcal{A}$  then  $E \subseteq p$

$\langle 3 \rangle 1$ . LET:  $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 2$ . For all  $E' \in \mathcal{P}R$  with  $VE' \in \mathcal{A}$  we have  $E' \subseteq p$

$\langle 3 \rangle 3$ .  $E \subseteq p$

$\langle 1 \rangle 3$ . For all  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .

PROOF: Since  $VE \cup VE' = V(E \cap E')$

$\langle 1 \rangle 4. \emptyset \in \mathcal{C}$

$\langle 2 \rangle 1. VR = \emptyset$

PROOF: If  $p \in VR$  then  $R \subseteq p$  contradicting the fact that  $p$  is a prime ideal.

□

**Definition 9.0.4.** For any ring  $R$ , let  $R - \mathbf{Mod}$  be the category of small  $R$ -modules and  $R$ -module homomorphisms.

**Proposition 9.0.5.**  $R - \mathbf{Mod}$  has products and coproducts.



## Chapter 10

# Field Theory

**Proposition 10.0.1.** *Field does not have binary products.*

PROOF: There cannot be a field  $K$  with field homomorphisms  $K \rightarrow \mathbb{Z}_2$  and  $K \rightarrow \mathbb{Z}_3$ , because its characteristic would be both 2 and 3.  $\square$



# Chapter 11

## Linear Algebra

**Definition 11.0.1** (Span). Let  $V$  be a vector space and  $A \subseteq V$ . The *span* of  $A$  is the set of all linear combinations of elements of  $A$ .

**Definition 11.0.2** (Independent). Let  $V$  be a vector space and  $A \subseteq V$ . Then  $A$  is *linearly independent* iff, whenever

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

where  $v_1, \dots, v_n \in A$ , then

$$\alpha_1 = \cdots = \alpha_n = 0 .$$

**Proposition 11.0.3.** *Let  $V$  be a vector space,  $A \subseteq V$  and  $v \in V$ . If  $A$  is linearly independent and  $v \notin \text{span } A$ , then  $A \cup \{v\}$  is independent.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\alpha_1 v_1 + \cdots + \alpha_n v_n + \beta v = 0$  where  $v_1, \dots, v_n \in A$

$\langle 1 \rangle 2$ .  $\beta = 0$

PROOF: Otherwise  $v = (\alpha_1/\beta)v_1 + \cdots + (\alpha_n/\beta)v_n \in \text{span } A$ .

$\langle 1 \rangle 3$ .  $\alpha_1 = \cdots = \alpha_n = 0$

PROOF: Since  $A$  is linearly independent.

□

**Theorem 11.0.4.** *Every vector space has a basis.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $V$  be a vector space.

$\langle 1 \rangle 2$ . PICK a maximal linearly independent set  $\mathcal{B}$ .

PROOF: By Tukey's Lemma.

$\langle 1 \rangle 3$ .  $\text{span } \mathcal{B} = V$

PROOF: Proposition 11.0.3.

□

**Definition 11.0.5.** For any field  $K$ , we write  $\mathbf{Vect}_K$  for  $K - \mathbf{Mod}$ .

Dual space functor  $\mathbf{Vect}_K^{\text{op}} \rightarrow \mathbf{Vect}_K$ .



# Chapter 12

## Topology

### 12.1 Topological Spaces

**Definition 12.1.1** (Topological Space). Let  $X$  be a set and  $\mathcal{O} \subseteq \mathcal{P}X$ . Then we say  $(X, \mathcal{O})$  is a *topological space* iff:

- For any  $\mathcal{U} \subseteq \mathcal{O}$  we have  $\bigcup \mathcal{U} \in \mathcal{O}$ .
- For any  $U, V \in \mathcal{O}$  we have  $U \cap V \in \mathcal{O}$ .
- $X \in \mathcal{O}$

We call  $\mathcal{O}$  the *topology* of the topological space, and call its elements *open* sets. We shall often write  $X$  for the topological space  $(X, \mathcal{O})$ .

**Definition 12.1.2** (Discrete Topology). For any set  $X$ , the power set  $\mathcal{P}X$  is called the *discrete* topology on  $X$ .

**Proposition 12.1.3.** *For any set  $X$ , the discrete topology on  $X$  is a topology on  $X$ .*

**Definition 12.1.4** (Indiscrete Topology). For any set  $X$ , the *indiscrete* or *trivial* topology on  $X$  is  $\{\emptyset, X\}$ .

**Proposition 12.1.5.** *For any set  $X$ , the indiscrete topology on  $X$  is a topology on  $X$ .*

**Definition 12.1.6** (Cofinite Topology). For any set  $X$ , the *cofinite* topology is  $\{X - U : U \subseteq X \text{ is finite}\}$ .

**Definition 12.1.7** (Cocountable Topology). For any set  $X$ , the *cocountable* topology is  $\{X - U : U \subseteq X \text{ is countable}\}$ .

**Definition 12.1.8** (Sierpiński Two-Point Space). The *Sierpiński two-point space* is  $\{0, 1\}$  under the topology  $\{\emptyset, \{1\}, \{0, 1\}\}$ .

**Proposition 12.1.9.** *Let  $X$  be a topological space and  $U \subseteq X$ . Then  $U$  is open if and only if, for all  $x \in U$ , there exists an open set  $V$  such that  $x \in V \subseteq U$ .*

**Proposition 12.1.10.** *The intersection of a set of topologies on a set  $X$  is a topology on  $X$ .*

**Definition 12.1.11** (Closed Set). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is *closed* iff  $X - A$  is open.

**Proposition 12.1.12.** *A set  $B$  is open if and only if  $X - B$  is closed.*

**Proposition 12.1.13.** *Let  $X$  be a set and  $\mathcal{C} \subseteq \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  on  $X$  such that  $\mathcal{C}$  is the set of closed sets if and only if:*

- For any  $\mathcal{D} \subseteq \mathcal{C}$  we have  $\bigcap \mathcal{D} \in \mathcal{C}$
- For any  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .
- $\emptyset \in \mathcal{C}$

*In this case,  $\mathcal{O}$  is unique and is given by  $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$ .*

**Theorem 12.1.14.** *There are infinitely many primes.*

Furstenberg's proof:

PROOF:

$\langle 1 \rangle 1$ . For  $a \in \mathbb{Z} - \{0\}$  and  $b \in \mathbb{Z}$ ,

LET:  $S(a, b) := \{an + b : n \in \mathbb{N}\}$

$\langle 1 \rangle 2$ . LET:  $\mathcal{T}$  be the topology generated by the basis  $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$

$\langle 2 \rangle 1$ . For every  $n \in \mathbb{Z}$ , there exist  $a, b$  such that  $n \in S(a, b)$ .

PROOF:  $n \in S(n, 0)$

$\langle 2 \rangle 2$ . If  $n \in S(a_1, b_1) \cap S(a_2, b_2)$  then there exist  $a_3, b_3$  such that  $n \in S(a_3, b_3) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 1$ . LET:  $d = \text{lcm}(a_1, a_2)$

PROVE:  $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 2$ . LET:  $d = a_1k = a_2l$

$\langle 3 \rangle 3$ . LET:  $n = a_1c + b_1 = a_2d + b_2$

$\langle 3 \rangle 4$ . LET:  $z \in \mathbb{Z}$

PROVE:  $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 5$ .  $dz + n \in S(a_1, b_1)$

PROOF:

$$\begin{aligned} dz + n &= a_1kz + a_1c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

$\langle 3 \rangle 6$ .  $dz + n \in S(a_2, b_2)$

PROOF: Similar.

$\langle 1 \rangle 3$ . For all  $a \in \mathbb{Z} - \{0\}$  and  $b \in \mathbb{Z}$  we have  $S(a, b)$  is closed.

$\langle 2 \rangle 1$ . LET:  $a \in \mathbb{Z} - \{0\}$  and  $b \in \mathbb{Z}$

$\langle 2 \rangle 2$ . LET:  $n \in \mathbb{Z} - S(a, b)$

$\langle 2 \rangle 3$ .  $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$

- ⟨3⟩1. LET:  $x \in S(a, n)$
- ⟨3⟩2. ASSUME: for a contradiction  $x \in S(a, b)$
- ⟨3⟩3. PICK  $m$  such that  $x = am + b$
- ⟨3⟩4. PICK  $l$  such that  $x = al + n$
- ⟨3⟩5.  $n = a(m - l) + b$
- ⟨3⟩6.  $n \in S(a, b)$
- ⟨3⟩7. Q.E.D.

PROOF: This contradicts ⟨2⟩2.

⟨1⟩4.

$$\mathbb{Z} - \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$$

PROOF: Since every integer except 1 and  $-1$  is divisible by a prime.

⟨1⟩5. No nonempty finite set is open.

- ⟨2⟩1. LET:  $U$  be a nonempty open set
- ⟨2⟩2. PICK  $n \in U$
- ⟨2⟩3. There exist  $a, b$  such that  $n \in S(a, b) \subseteq U$
- ⟨2⟩4.  $U$  is infinite.

⟨1⟩6.  $\mathbb{Z} - \{1, -1\}$  is not closed.

⟨1⟩7.  $\bigcup_{p \text{ prime}} S(p, 0)$  is not closed.

⟨1⟩8. The union of finitely many closed sets is closed.

⟨1⟩9. There are infinitely many primes.

□

**Definition 12.1.15** (Neighbourhood). Let  $X$  be a topological space,  $Sx \in X$  and  $U \subseteq X$ . Then  $U$  is a *neighbourhood* of  $x$ , and  $x$  is an *interior* point of  $U$ , iff there exists an open set  $V$  such that  $x \in V \subseteq U$ .

**Proposition 12.1.16.** *A set  $B$  is open if and only if it is a neighbourhood of each of its points.*

**Proposition 12.1.17.** *Let  $X$  be a set and  $\mathcal{N} : X \rightarrow \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  on  $X$  such that, for all  $x \in X$ , we have  $\mathcal{N}_x$  is the set of neighbourhoods of  $x$ , if and only if:*

- For all  $x \in X$  and  $N \in \mathcal{N}_x$  we have  $x \in N$
- For all  $x \in X$  we have  $X \in \mathcal{N}_x$
- For all  $x \in X$ ,  $N \in \mathcal{N}_x$  and  $V \subseteq \mathcal{P}X$ , if  $N \subseteq V$  then  $V \in \mathcal{N}_x$
- For all  $x \in X$  and  $M, N \in \mathcal{N}_x$  we have  $M \cap N \in \mathcal{N}_x$
- For all  $x \in X$  and  $N \in \mathcal{N}_x$ , there exists  $M \in \mathcal{N}_x$  such that  $M \subseteq N$  and  $\forall y \in M. M \in \mathcal{N}_y$ .

In this case,  $\mathcal{O}$  is unique and is given by  $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$ .

**Definition 12.1.18** (Exterior Point). Let  $X$  be a topological space,  $x \in X$  and  $B \subseteq X$ . Then  $x$  is an *exterior point* of  $B$  iff  $B - X$  is a neighbourhood of  $x$ .

**Definition 12.1.19** (Boundary Point). Let  $X$  be a topological space,  $x \in X$  and  $B \subseteq X$ . Then  $x$  is a *boundary point* of  $B$  iff it is neither an interior point nor an exterior point of  $B$ .

**Definition 12.1.20** (Interior). Let  $X$  be a topological space and  $B \subseteq X$ . The *interior* of  $B$ ,  $B^\circ$ , is the set of all interior points of  $B$ .

**Proposition 12.1.21.** *The interior of  $B$  is the union of all the open sets included in  $B$ .*

**Definition 12.1.22** (Closure). Let  $X$  be a topological space and  $B \subseteq X$ . The *closure* of  $B$ ,  $\overline{B}$ , is the set of all points that are not exterior points of  $B$ .

**Proposition 12.1.23.** *The closure of  $B$  is the intersection of all the closed sets that include  $B$ .*

**Proposition 12.1.24.** *A set  $B$  is open iff  $X - B = \overline{X - B}$ .*

**Proposition 12.1.25** (Kuratowski Closure Axioms). *Let  $X$  be a set and  $- : \mathcal{P}X \rightarrow \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  such that, for all  $B \subseteq X$ ,  $\overline{B}$  is the closure of  $B$ , if and only if:*

- $\overline{\emptyset} = \emptyset$
- For all  $A \subseteq X$  we have  $A \subseteq \overline{A}$
- For all  $A \subseteq X$  we have  $\overline{\overline{A}} = \overline{A}$
- For all  $A, B \subseteq X$  we have  $\overline{A \cup B} = \overline{A} \cup \overline{B}$

*In this case,  $\mathcal{O}$  is unique and is defined by  $\mathcal{O} = \{U : X - U = \overline{X - U}\}$ .*

**Definition 12.1.26** (Finer, Coarser). Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the set  $X$ . Then  $\mathcal{T}$  is *coarser*, *smaller* or *weaker* than  $\mathcal{T}'$ , or  $\mathcal{T}'$  is *finer*, *larger* or *weaker* than  $\mathcal{T}$ , iff  $\mathcal{T} \subseteq \mathcal{T}'$ .

### 12.1.1 Subspaces

**Definition 12.1.27** (Subspace). Let  $X$  be a topological space and  $X_0 \subseteq X$ . The *subspace topology* on  $X_0$  is  $\{U \cap X_0 : U \text{ is open in } X\}$ .

**Example 12.1.28.** The *unit sphere*  $S^2$  is  $\{x \in \mathbb{R}^3 : \|x\| = 1\}$  as a subspace of  $\mathbb{R}^3$ .

**Theorem 12.1.29.** *Let  $X$  be a topological space and  $(Y, i)$  a subset of  $X$ . Then the subspace topology on  $Y$  is the unique topology such that, for every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f : Z \rightarrow X$  is continuous.*

PROOF:



- ⟨1⟩1. If we give  $Y$  the subspace topology then, for every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f$  is continuous.
- ⟨2⟩1. Given  $Y$  the subspace topology.
- ⟨2⟩2. LET:  $Z$  be a topological space.
- ⟨2⟩3. LET:  $f : Z \rightarrow Y$
- ⟨2⟩4. If  $f$  is continuous then  $i \circ f$  is continuous.
- PROOF: Since  $i$  is continuous.
- ⟨2⟩5. If  $i \circ f$  is continuous then  $f$  is continuous.
- ⟨3⟩1. ASSUME:  $i \circ f$  is continuous.
- ⟨3⟩2. LET:  $U$  be open in  $Y$ .
- ⟨3⟩3.  $f^{-1}(i^{-1}(i(U)))$  is open in  $Z$ .
- ⟨3⟩4.  $f^{-1}(U)$  is open in  $Z$ .
- ⟨1⟩2. If, for every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f$  is continuous.
- ⟨2⟩1. ASSUME: For every topological space  $Z$  and function  $f : Z \rightarrow Y$ , we have  $f$  is continuous if and only if  $i \circ f$  is continuous.
- ⟨2⟩2.  $i$  is continuous.
- ⟨2⟩3. For every open set  $U$  in  $X$ , we have  $i^{-1}(U)$  is open in  $Y$
- ⟨2⟩4. LET:  $Z$  be the set  $Y$  under the subspace topology and  $f : Z \rightarrow Y$  the identity function.
- ⟨2⟩5.  $i \circ f$  is continuous.
- ⟨2⟩6.  $f$  is continuous.
- ⟨2⟩7. Every set open in  $Y$  is open in  $Z$ .

□

### 12.1.2 Topological Disjoint Union

**Definition 12.1.30** (Coproduct Topology). Let  $\{X_\alpha\}_{\alpha \in A}$  be a family of topological spaces. The *coproduct topology* on  $\coprod_{\alpha \in A} X_\alpha$  is

$$\mathcal{T} = \left\{ \coprod_{\alpha \in A} U_\alpha : \{U_\alpha\}_{\alpha \in A} \text{ is a family with } U_\alpha \text{ open in } X_\alpha \text{ for all } \alpha \right\}.$$

We prove this is a topology.

PROOF:

- ⟨1⟩1. For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF:

$$\bigcup_{i \in I} \coprod_{\alpha \in A} U_{i\alpha} = \coprod_{\alpha \in A} \bigcup_{i \in I} U_{i\alpha}$$

- ⟨1⟩2. For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF:

$$\coprod_{\alpha \in A} U_\alpha \cap \coprod_{\alpha \in A} V_\alpha = \coprod_{\alpha \in A} (U_\alpha \cap V_\alpha)$$

- ⟨1⟩3.  $\coprod_{\alpha \in A} X_\alpha \in \mathcal{T}$

PROOF: Since every  $X_\alpha$  is open in  $X_\alpha$ .

□

**Proposition 12.1.31.** *The coproduct topology is the finest topology on  $\coprod_{\alpha \in A} X_\alpha$  such that every injection  $\kappa_\alpha : X_\alpha \rightarrow \coprod_{\alpha \in A} X_\alpha$  is continuous.*

PROOF:

⟨1⟩1. LET:  $P = \coprod_{\alpha \in A} X_\alpha$

⟨1⟩2. LET:  $\mathcal{T}_c$  be the coproduct topology.

⟨1⟩3. LET:  $\mathcal{T}$  be any topology on  $P$

⟨1⟩4. For all  $\alpha \in A$ , the injection  $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T}_c)$  is continuous.

⟨2⟩1. LET:  $\alpha \in A$

⟨2⟩2. LET:  $\{U_\alpha\}_{\alpha \in A}$  be a family with each  $U_\alpha$  open in  $X_\alpha$ .

⟨2⟩3. For all  $\alpha \in A$ , we have  $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha)$  is open in  $X_\alpha$ .

PROOF: Since  $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha) = U_\alpha$ .

⟨1⟩5. If, for all  $\alpha \in A$ , the injection  $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$  is continuous, then  $\mathcal{T} \subseteq \mathcal{T}_c$ .

⟨2⟩1. ASSUME: For all  $\alpha \in A$ , the injection  $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$  is continuous.

⟨2⟩2. LET:  $U \in \mathcal{T}$

⟨2⟩3. For all  $\alpha \in A$ , we have  $\kappa_\alpha^{-1}(U)$  is open in  $X_\alpha$ .

⟨2⟩4.  $U = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(U) \in \mathcal{T}_c$

□

**Theorem 12.1.32.** *Let  $\{X_\alpha\}_{\alpha \in A}$  be a family of topological spaces. The coproduct topology is the unique topology on  $\coprod_{\alpha \in A} X_\alpha$  such that, for every topological space  $Z$  and function  $f : \coprod_{\alpha \in A} X_\alpha \rightarrow Z$ , we have  $f$  is continuous if and only if  $\forall \alpha \in A, f \circ \kappa_\alpha$  is continuous.*

PROOF:

⟨1⟩1. LET:  $X = \coprod_{\alpha \in A} X_\alpha$

⟨1⟩2. LET:  $\mathcal{T}_c$  be the coproduct topology.

⟨1⟩3. For every topological space  $Z$  and function  $f : (X, \mathcal{T}_c) \rightarrow Z$ , we have  $f$  is continuous if and only if  $\forall \alpha \in A, f \circ \kappa_\alpha$  is continuous.

⟨2⟩1. LET:  $Z$  be a topological space.

⟨2⟩2. LET:  $f : X \rightarrow Z$

⟨2⟩3. If  $f$  is continuous then  $\forall \alpha \in A, f \circ \kappa_\alpha$  is continuous.

PROOF: Because the composite of two continuous functions is continuous.

⟨2⟩4. If  $\forall \alpha \in A, f \circ \kappa_\alpha$  is continuous then  $f$  is continuous.

⟨3⟩1. ASSUME:  $\forall \alpha \in A, f \circ \kappa_\alpha$  is continuous.

⟨3⟩2. LET:  $U$  be open in  $Z$

⟨3⟩3. For all  $\alpha \in A$  we have  $\kappa_\alpha^{-1}(f^{-1}(U))$  is open in  $X_\alpha$

⟨3⟩4.  $f^{-1}(U) = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(f^{-1}(U))$

⟨3⟩5.  $f^{-1}(U)$  is open in  $X$

⟨1⟩4. For any topology  $\mathcal{T}$  on  $X$ , if for every topological space  $Z$  and function  $f : (X, \mathcal{T}) \rightarrow Z$ , we have  $f$  is continuous if and only if  $\forall \alpha \in A, f \circ \kappa_\alpha$  is continuous, then  $\mathcal{T} = \mathcal{T}_c$ .

⟨2⟩1. LET:  $\mathcal{T}$  be a topology on  $X$ .

- $\langle 2 \rangle 2$ . ASSUME: For every topological space  $Z$  and function  $f : (X, \mathcal{T}) \rightarrow Z$ , we have  $f$  is continuous if and only if  $\forall \alpha \in A. f \circ \kappa_\alpha$  is continuous.  
 $\langle 2 \rangle 3$ .  $\mathcal{T} \subseteq \mathcal{T}_c$   
 $\langle 3 \rangle 1$ . For all  $\alpha \in A$  we have  $\kappa_\alpha : X_\alpha \rightarrow (X, \mathcal{T})$  is continuous.  
 PROOF: From  $\langle 2 \rangle 1$  since  $\text{id}_X$  is continuous.  
 $\langle 3 \rangle 2$ .  $\mathcal{T} \subseteq \mathcal{T}_c$   
 PROOF: Proposition 12.1.31.  
 $\langle 2 \rangle 4$ .  $\mathcal{T}_c \subseteq \mathcal{T}$   
 $\langle 3 \rangle 1$ . LET:  $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}_c)$  be the identity function.  
 $\langle 3 \rangle 2$ .  $f \circ \kappa_\alpha$  is continuous for all  $\alpha$ .  
 $\langle 3 \rangle 3$ .  $f$  is continuous.  
 PROOF:  $\langle 2 \rangle 1$   
 $\langle 3 \rangle 4$ .  $\mathcal{T}_c \subseteq \mathcal{T}$

□

### 12.1.3 Product Topology

**Definition 12.1.33** (Product Topology). Let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be a family of topological spaces. The *product topology* on  $\prod_{\lambda \in \Lambda} X_\lambda$  is the coarsest topology such that every projection onto  $X_\lambda$  is continuous.

**Proposition 12.1.34.** Let  $\{X_\alpha\}_{\alpha \in A}$  be a family of topological spaces. The product topology on  $\prod_{\alpha \in A} X_\alpha$  is the topology generated by the basis  $\mathcal{B} = \{\prod_{\alpha \in A} U_\alpha : \text{for all } \alpha \in A, U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in A\}$ .

PROOF:

- $\langle 1 \rangle 1$ .  $\mathcal{B}$  is a basis for a topology.  
 $\langle 1 \rangle 2$ . LET:  $\mathcal{T}$  be the topology generated by  $\mathcal{B}$ .  
 $\langle 1 \rangle 3$ . LET:  $\mathcal{T}_p$  be the product topology.  
 $\langle 1 \rangle 4$ .  $\mathcal{T} \subseteq \mathcal{T}_p$   
 $\langle 2 \rangle 1$ . LET:  $B \in \mathcal{B}$   
 $\langle 2 \rangle 2$ . LET:  $B = \prod_{\alpha \in A} U_\alpha$  with each  $U_\alpha$  open in  $X_\alpha$  and  $U_\alpha = X_\alpha$  except for  $\alpha = \alpha_1, \dots, \alpha_n$   
 $\langle 2 \rangle 3$ .  $B = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$   
 $\langle 2 \rangle 4$ .  $B \in \mathcal{T}_p$   
 $\langle 1 \rangle 5$ .  $\mathcal{T}_p \subseteq \mathcal{T}$   
 $\langle 2 \rangle 1$ . For every  $\alpha \in A$  we have  $\pi_\alpha$  is continuous.  
 PROOF: Since  $\pi^{-1}(U)$  is open for every  $U$  open in  $X_\alpha$ .

□

**Theorem 12.1.35.** Let  $\{X_\alpha\}_{\alpha \in A}$  be a family of topological spaces. Then the product topology on  $\prod_{\alpha \in A} X_\alpha$  is the unique topology such that, for every topological space  $Z$  and function  $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$ , we have  $f$  is continuous if and only if, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f : Z \rightarrow X_\alpha$  is continuous.

PROOF:

- ⟨1⟩1. If we give  $\prod_{\alpha \in A} X_\alpha$  the product topology, then for every topological space  $Z$  and function  $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$ , we have  $f$  is continuous if and only if, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f$  is continuous.  
 ⟨2⟩1. Give  $\prod_{\alpha \in A} X_\alpha$  the product topology.  
 ⟨2⟩2. LET:  $Z$  be a topological space.  
 ⟨2⟩3. LET:  $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$   
 ⟨2⟩4. If  $f$  is continuous then, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f$  is continuous.  
 PROOF: Since the composite of two continuous functions is continuous.  
 ⟨2⟩5. If, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f$  is continuous, then  $f$  is continuous.  
 ⟨3⟩1. ASSUME: For all  $\alpha \in A$  we have  $\pi_\alpha \circ f$  is continuous.  
 ⟨3⟩2. LET:  $\{U_\alpha\}_{\alpha \in A}$  be a family with  $U_\alpha$  open in  $X_\alpha$  such that  $U_\alpha = X_\alpha$  for all  $\alpha$  except  $\alpha = \alpha_1, \dots, \alpha_n$ .  
 ⟨3⟩3. For all  $\alpha$  we have  $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$  is open in  $Z$ .  
 ⟨3⟩4.  $f^{-1}(\prod_{\alpha} U_\alpha)$  is open in  $Z$   
 PROOF: Since  $f^{-1}(\prod_{\alpha} U_\alpha) = f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \dots \cap f^{-1}(\pi_{\alpha_n}^{-1}(U_{\alpha_n}))$ .  
 ⟨1⟩2. If  $\mathcal{T}$  is a topology on  $\prod_{\alpha \in A} X_\alpha$  such that, for every topological space  $Z$  and function  $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$ , we have  $f$  is continuous if and only if, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f$  is continuous, then  $\mathcal{T}$  is the product topology.  
 ⟨2⟩1. ASSUME:  $\mathcal{T}$  is a topology on  $\prod_{\alpha \in A} X_\alpha$  such that, for every topological space  $Z$  and function  $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$ , we have  $f$  is continuous if and only if, for all  $\alpha \in A$ , we have  $\pi_\alpha \circ f$  is continuous.  
 ⟨2⟩2. LET:  $\mathcal{T}_p$  be the product topology.  
 ⟨2⟩3.  $\mathcal{T} \subseteq \mathcal{T}_p$   
 ⟨3⟩1. LET:  $Z = (\prod_{\alpha} X_\alpha, \mathcal{T}_p)$   
 ⟨3⟩2. LET:  $f : Z \rightarrow \prod_{\alpha} X_\alpha$  be the identity function  
 ⟨3⟩3. For all  $\alpha$  we have  $\pi_\alpha \circ f$  is continuous.  
 ⟨3⟩4.  $f$  is continuous.  
 PROOF: ⟨2⟩1  
 ⟨3⟩5. Every set open in  $\mathcal{T}$  is open in  $\mathcal{T}_p$   
 ⟨2⟩4.  $\mathcal{T}_p \subseteq \mathcal{T}$   
 ⟨3⟩1.  $\text{id}_{\prod_{\alpha} X_\alpha}$  is continuous.  
 ⟨3⟩2. For all  $\alpha$  we have  $\pi_\alpha$  is continuous.  
 PROOF: ⟨2⟩1  
 ⟨3⟩3.  $\mathcal{T}_p \subseteq \mathcal{T}$   
 PROOF: Since  $\mathcal{T}_p$  is the coarsest topology such that every  $\pi_\alpha$  is continuous.

□

**Example 12.1.36.** It is not true that, for any function  $f : \prod_{\alpha \in A} X_\alpha \rightarrow Y$ , if  $f$  is continuous in every variable separately then  $f$  is continuous.

Define  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

Then  $f$  is continuous in  $x$  and in  $y$ , but is not continuous.

**Proposition 12.1.37.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $Y_i$  be a subspace of  $X_i$  for all  $i \in I$ . Then the product topology on  $\prod_{i \in I} Y_i$  is the same as the subspace topology on  $\prod_{i \in I} Y_i$  as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF:

$\langle 1 \rangle 1$ . Given  $\prod_{i \in I} Y_i$  the subspace topology.

$\langle 1 \rangle 2$ . LET:  $\iota : \prod_{i \in I} Y_i$  be the inclusion.

$\langle 1 \rangle 3$ . LET:  $Z$  be any topological space.

$\langle 1 \rangle 4$ . LET:  $f : Z \rightarrow \prod_{i \in I} Y_i$

$\langle 1 \rangle 5$ .  $f$  is continuous if and only if, for all  $i \in I$ , we have  $\pi_i \circ f$  is continuous.

PROOF:

$f$  is continuous  $\Leftrightarrow \iota \circ f : Z \rightarrow \prod_{i \in I} X_i$  is continuous (Theorem 12.1.29)

$\Leftrightarrow \forall i \in I, \pi_i \circ \iota \circ f : Z \rightarrow X_i$  is continuous (Theorem 12.1.35)

$\Leftrightarrow \forall i \in I, \iota_i \circ \pi_i \circ f : Z \rightarrow X_i$  is continuous

$\Leftrightarrow \forall i \in I, \pi_i \circ f : Z \rightarrow Y_i$  is continuous (Theorem 12.1.29)

where  $\iota_i$  is the inclusion  $Y_i \rightarrow X_i$ .

□

#### 12.1.4 Bases

**Definition 12.1.38** (Basis). Let  $X$  be a topological space. A *basis* for the topology on  $X$  is a set of open sets  $\mathcal{B}$  such that every open set is the union of a subset of  $\mathcal{B}$ . The elements of  $\mathcal{B}$  are called *basic open neighbourhoods* of their elements.

**Proposition 12.1.39.** *Let  $X$  be a set. The set of all one-element subsets of  $X$  is a basis for the discrete topology on  $X$ .*

**Proposition 12.1.40.** *Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Then the topology on  $X$  is the coarsest topology that includes  $\mathcal{B}$ .*

**Proposition 12.1.41.** *Let  $X$  and  $Y$  be topological spaces. Let  $\mathcal{B}$  be a basis for the topology on  $X$  and  $\mathcal{C}$  a basis for the topology on  $Y$ . Then*

$$\{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$$

*is a basis for the product topology on  $X \times Y$ .*

**Definition 12.1.42** (Order Topology). Let  $X$  be a linearly ordered set. The *order topology* on  $X$  is the topology generated by the open interval  $(a, b)$  as well as the open rays  $(a, +\infty)$  and  $(-\infty, b)$  for  $a, b \in X$ .

The *standard topology* on  $\mathbb{R}$  is the order topology.

**Proposition 12.1.43.** *Let  $X$  be a linearly ordered set. Then the order topology is generated by the basis consisting of:*

- all open intervals  $(a, b)$

- all intervals of the form  $[\perp, b)$  where  $\perp$  is the least element of  $X$ , if any
- all intervals of the form  $(a, \top]$  where  $\top$  is the greatest element of  $X$ , if any.

**Proposition 12.1.44.** *Let  $X$  be a linearly ordered set. The open rays in  $X$  form a subbasis for the order topology.*

**Definition 12.1.45** (Lower Limit Topology). The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology on  $\mathbb{R}$  generated by the basis consisting of all half-open intervals  $[a, b)$ .

We write  $\mathbb{R}_l$  for  $\mathbb{R}$  under the lower limit topology.

**Definition 12.1.46** ( $K$ -topology). Let  $K = \{1/n : n \in \mathbb{Z}_+\}$ . The  $K$ -topology on  $\mathbb{R}$  is the topology generated by the basis consisting of all open intervals  $(a, b)$  and all sets of the form  $(a, b) - K$ .

We write  $\mathbb{R}_K$  for  $\mathbb{R}$  under the  $K$ -topology.

**Proposition 12.1.47.** *Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology on  $X$  and  $U \subseteq X$ . Then  $U$  is open if and only if, for all  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .*

**Proposition 12.1.48.** *Let  $X$  be a topological space and  $\mathcal{B} \subseteq \mathcal{P}X$ . Assume that, for every open set  $U$  and element  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ . Then  $\mathcal{B}$  is a basis for the topology on  $X$ .*

**Proposition 12.1.49.** *Let  $X$  be a topological space and  $\mathcal{B} \subseteq \mathcal{P}X$ . Then  $\mathcal{B}$  is a basis for a topology on  $X$  if and only if:*

1.  $\bigcup \mathcal{B} = X$
2. For all  $A, B \in \mathcal{B}$  and  $x \in A \cap B$ , there exists  $C \in \mathcal{B}$  such that  $x \in C \subseteq A \cap B$ .

*In this case, the topology is unique and is the set of all unions of subsets of  $\mathcal{B}$ . We call it the topology generated by  $\mathcal{B}$ .*

**Proposition 12.1.50.** *Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively, on  $X$ . Then  $\mathcal{T}'$  is finer than  $\mathcal{T}$  if and only if, for every  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .*

**Corollary 12.1.50.1.** *The topologies of  $\mathbb{R}_l$  and  $\mathbb{R}_K$  are strictly finer than the standard topology on  $\mathbb{R}$  but are not comparable to one another.*

### 12.1.5 Subbases

**Definition 12.1.51** (Subbasis). Let  $X$  be a topological space. A *subbasis* for the topology on  $X$  is a set  $\mathcal{S}$  of open sets such that every open set is a union of finite intersections of  $\mathcal{S}$ .

**Proposition 12.1.52.** *Let  $X$  be a set and  $\mathcal{S} \subseteq \mathcal{P}X$ . Then  $\mathcal{S}$  is a subbasis for a topology on  $X$  if and only if  $\bigcup \mathcal{S} = X$ , in which case the topology is unique and is the set of all unions of finite intersections of elements of  $\mathcal{S}$ .*

**Proposition 12.1.53.** *Let  $X$  be a topological space. Let  $\mathcal{S}$  be a subbasis for the topology on  $X$ . Then the topology on  $X$  is the coarsest topology that includes  $\mathcal{S}$ .*

**Proposition 12.1.54.** *Let  $X$  and  $Y$  be topological spaces. Then*

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$$

*is a subbasis for the product topology on  $X \times Y$ .*

PROOF:

$\langle 1 \rangle 1$ . Every element of  $\mathcal{S}$  is open.

PROOF: Since  $\pi_1^{-1}(U) = U \times Y$  and  $\pi_2^{-1}(V) = X \times V$ .

$\langle 1 \rangle 2$ . Every open set is a union of finite intersections of elements of  $\mathcal{S}$ .

PROOF: Since, for  $U$  open in  $X$  and  $V$  open in  $Y$ , we have  $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ .

□

**Definition 12.1.55** (Space with Basepoint). A *space with basepoint* is a pair  $(X, x)$  where  $X$  is a topological space and  $x \in X$ .

### 12.1.6 Countability Axioms

**Definition 12.1.56** (Neighbourhood Basis). Let  $X$  be a topological space and  $x_0 \in X$ . A *neighbourhood basis* of  $x_0$  is a set  $\mathcal{U}$  of neighbourhoods of  $x_0$  such that every neighbourhood of  $x_0$  includes an element of  $\mathcal{U}$ .

**Definition 12.1.57** (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

**Definition 12.1.58** (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

$\mathbb{R}^n$  is second countable.

An uncountable discrete space is first countable but not second countable.

**Proposition 12.1.59.** *Let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be a family of topological spaces such that no  $X_\lambda$  is indiscrete. If  $\Lambda$  is uncountable, then  $\prod_{\lambda \in \Lambda} X_\lambda$  is not first countable.*

PROOF:

$\langle 1 \rangle 1$ . For all  $\lambda \in \Lambda$ , PICK  $U_\lambda$  open in  $X_\lambda$  such that  $\emptyset \neq U_\lambda \neq X_\lambda$ .

$\langle 1 \rangle 2$ . For all  $\lambda \in \Lambda$ , PICK  $x_\lambda \in U_\lambda$ .

$\langle 1 \rangle 3$ . ASSUME: for a contradiction  $B$  is a countable neighbourhood basis for  $(x_\lambda)_{\lambda \in \Lambda}$ .

$\langle 1 \rangle 4$ . PICK  $\lambda \in \Lambda$  such that, for all  $U \in B$ , we have  $\pi_\lambda(U) = X_\lambda$

$\langle 1 \rangle 5$ . There is no  $U \in B$  such that  $U \subseteq \pi_\lambda^{-1}(U_\lambda)$

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This is a contradiction.

□

## 12.2 Continuous Functions

**Definition 12.2.1** (Continuous). Let  $X$  and  $Y$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous* iff, for every open set  $V$  in  $Y$ , the inverse image  $f^{-1}(V)$  is open in  $X$ .

**Proposition 12.2.2.** 1.  $\text{id}_X$  is continuous

2. The composite of two continuous functions is continuous.

3. If  $f : X \rightarrow Y$  is continuous and  $X_0 \subseteq X$  then  $f|_{X_0} : X_0 \rightarrow Y$  is continuous.

4. If  $f : X + Y \rightarrow Z$ , then  $f$  is continuous iff  $f \circ \kappa_1 : X \rightarrow Z$  and  $f \circ \kappa_2 : Y \rightarrow Z$  are continuous.

5. If  $f : Z \rightarrow X \times Y$ , then  $f$  is continuous iff  $\pi_1 \circ f$  and  $\pi_2 \circ f$  are continuous.

**Proposition 12.2.3.** Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous then, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{B}$  is open in  $Y$ .

$\langle 1 \rangle 2$ . If, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ , then  $f$  is continuous.

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

$\langle 2 \rangle 2$ . LET:  $U$  be open in  $Y$ .

$\langle 2 \rangle 3$ . LET:  $x \in f^{-1}(U)$

$\langle 2 \rangle 4$ . PICK  $B \in \mathcal{B}$  such that  $f(x) \in B \subseteq U$ .

$\langle 2 \rangle 5$ .  $x \in f^{-1}(B) \subseteq f^{-1}(U)$

□

**Definition 12.2.4** (Homeomorphism). Let  $X$  and  $Y$  be topological spaces. A *homeomorphism* between  $X$  and  $Y$  is a bijection  $f : X \approx Y$  such that  $f$  and  $f^{-1}$  are continuous.

**Definition 12.2.5** (Retraction). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A continuous function  $\rho : X \rightarrow A$  is a *retraction* iff  $\rho|_A = \text{id}_A$ . We say  $A$  is a *retract* of  $X$  iff there exists a retraction.

**Definition 12.2.6.** Let **Top** be the category of small topological spaces and continuous functions.

**Proposition 12.2.7.**  $\emptyset$  is initial in **Top**.

**Proposition 12.2.8.**  $1$  is terminal in **Top**.

Forgetful functor **Top**  $\rightarrow$  **Set**.

Basepoint preserving continuous functor.



**Proposition 12.2.9.** *Let  $(X, \mathcal{T})$  be a topological space. Let  $S$  be the Sierpiński two-point space. Define  $\Phi : \mathcal{T} \rightarrow \mathbf{Top}[X, S]$  by  $\Phi(U)(x) = 1$  iff  $x \in U$ . Then  $\Phi$  is a bijection.*

PROOF:

$\langle 1 \rangle 1$ . For all  $U \in \mathcal{T}$  we have  $\Phi(U)$  is continuous.

$\langle 2 \rangle 1$ . LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 2$ .  $\Phi(U)(\{1\})$  is open.

PROOF: Since  $\Phi(U)(\{1\}) = U$ .

$\langle 1 \rangle 2$ .  $\Phi$  is injective.

PROOF: If  $\Phi(U) = \Phi(V)$  then we have  $\forall x (x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$ .

$\langle 1 \rangle 3$ .  $\Phi$  is surjective.

PROOF: Given  $f : X \rightarrow S$  continuous we have  $\Phi(f^{-1}(1)) = f$ .

□

### 12.2.1 Paths

**Definition 12.2.10** (Path). A *path* in a topological space  $X$  is a continuous function  $[0, 1] \rightarrow X$ .

### 12.2.2 Loops

**Definition 12.2.11** (Loop). A *loop* in a topological space  $X$  is a path  $\alpha : [0, 1] \rightarrow X$  such that  $\alpha(0) = \alpha(1)$ .

## 12.3 Convergence

**Definition 12.3.1** (Convergence). Let  $X$  be a topological space. Let  $(x_n)$  be a sequence in  $X$ . A point  $a \in X$  is a *limit* of the sequence iff, for every neighbourhood  $U$  of  $a$ , there exists  $n_0$  such that  $\forall n \geq n_0. x_n \in U$ .

Convergence in a product space is pointwise convergence.

If  $f : X \rightarrow Y$  is continuous and  $x_n \rightarrow l$  in  $X$  then  $f(x_n) \rightarrow f(l)$  in  $Y$ .

**Example 12.3.2.** The converse does not hold.

Let  $X$  be the set of all continuous functions  $[0, 1] \rightarrow [-1, 1]$  under the product topology. Let  $i : X \rightarrow L^2([0, 1])$  be the inclusion.

If  $f_n \rightarrow f$  then  $i(f_n) \rightarrow i(f)$  — Lebesgue convergence theorem.

We prove that  $i$  is not continuous.

Assume for a contradiction  $i$  is continuous. Choose a neighbourhood  $K$  of 0 in  $X$  such that  $\forall \phi \in K. \int \phi^2 < 1/2$ . Let  $K = \prod_{\lambda \in [0, 1]} U_\lambda$  where  $U_\lambda = [-1, 1]$  except for  $\lambda = \lambda_1, \dots, \lambda_n$ . Let  $\phi$  be the function that is 0 at  $\lambda_1, \dots, \lambda_n$  and 1 everywhere else. Then  $\phi \in K$  but  $\int \phi^2 = 1$ .

**Proposition 12.3.3.** *The converse does hold for first countable spaces. If  $f : X \rightarrow Y$  where  $X$  is first countable, and  $Y$  is a topological space, and whenever  $x_n \rightarrow x$  then  $f(x_n) \rightarrow f(x)$ , then  $f$  is continuous.*

## 12.4 Subspaces

**Definition 12.4.1** (Subspace). Let  $X$  be a topological space,  $Y$  a set, and  $f : Y \rightarrow X$ . The *subspace topology* on  $Y$  induced by  $f$  is  $\mathcal{T} = \{f^{-1}(U) : U \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF: Since  $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$ .

$\langle 1 \rangle 2$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF: Since  $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$ .

$\langle 1 \rangle 3$ .  $Y \in \mathcal{T}$

PROOF: Since  $Y = f^{-1}(X)$ .

□

**Proposition 12.4.2.** Let  $X$  be a topological space,  $Y$  a set and  $f : Y \rightarrow X$  a function. Then the subspace topology on  $Y$  is the coarsest topology such that  $f$  is continuous.

PROOF: Immediate from definition. □

## 12.5 Embedding

**Definition 12.5.1** (Embedding). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *embedding* iff  $f$  is injective and the topology on  $X$  is the subspace induced by  $f$ .

## 12.6 Quotient Spaces

**Definition 12.6.1** (Quotient Topology). Let  $X$  be a topological space,  $S$  a set, and  $\pi : X \twoheadrightarrow S$  be a surjection. The *quotient topology* on  $S$  induced by  $\pi$  is  $\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ .

PROOF: Since  $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}$ .

$\langle 1 \rangle 2$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .

PROOF: Since  $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$ .

$\langle 1 \rangle 3$ .  $X \in \mathcal{T}$

PROOF: Since  $X = \pi^{-1}(Y)$ .

□

**Proposition 12.6.2.** Let  $X$  be a topological space,  $S$  a set and  $\pi : X \twoheadrightarrow S$  a surjection. Then the quotient topology on  $S$  is the finest topology such that  $\pi$  is continuous.

PROOF: Immediate from definitions.  $\square$

**Definition 12.6.3** (Quotient Map). Let  $X$  and  $S$  be topological spaces and  $\pi : X \rightarrow S$ . Then  $\pi$  is a *quotient map* iff  $\pi$  is surjective and the topology on  $S$  is the quotient topology induced by  $\pi$ .

**Theorem 12.6.4.** Let  $X$  be a topological space, let  $S$  be a set, and let  $\pi : X \twoheadrightarrow S$  be surjective. Then the quotient topology on  $S$  is the unique topology such that, for every topological space  $Z$  and function  $f : S \rightarrow Z$ , we have  $f$  is continuous if and only if  $f \circ \pi$  is continuous.

PROOF:

- $\langle 1 \rangle 1$ . If  $S$  is given the quotient topology, then for every topological space  $Z$  and function  $f : S \rightarrow Z$ , we have  $f$  is continuous if and only if  $f \circ \pi$  is continuous.
  - $\langle 2 \rangle 1$ . Give  $S$  the quotient topology.
  - $\langle 2 \rangle 2$ . LET:  $Z$  be a topological space.
  - $\langle 2 \rangle 3$ . LET:  $f : S \rightarrow Z$
  - $\langle 2 \rangle 4$ . If  $f$  is continuous then  $f \circ \pi$  is continuous.
 

PROOF: The composite of two continuous functions is continuous.
  - $\langle 2 \rangle 5$ . If  $f \circ \pi$  is continuous then  $f$  is continuous.
    - $\langle 3 \rangle 1$ . ASSUME:  $f \circ \pi$  is continuous.
    - $\langle 3 \rangle 2$ . LET:  $U$  be open in  $Z$ .
    - $\langle 3 \rangle 3$ .  $\pi^{-1}(f^{-1}(U))$  is open in  $X$ .
    - $\langle 3 \rangle 4$ .  $f^{-1}(U)$  is open in  $S$ .
- $\langle 1 \rangle 2$ . If  $S$  is given a topology such that, for every topological space  $Z$  and function  $f : S \rightarrow Z$ , we have  $f$  is continuous if and only if  $f \circ \pi$  is continuous, then that topology is the quotient topology.
  - $\langle 2 \rangle 1$ . Give  $S$  a topology such that, for every topological space  $Z$  and function  $f : S \rightarrow Z$ , we have  $f$  is continuous if and only if  $f \circ \pi$  is continuous.
  - $\langle 2 \rangle 2$ . LET:  $U \subseteq S$
  - $\langle 2 \rangle 3$ . If  $\pi^{-1}(U)$  is open in  $X$  then  $U$  is open in  $S$ .
    - $\langle 3 \rangle 1$ . LET:  $Z$  be  $S$  under the quotient topology induced by  $\pi$ .
    - $\langle 3 \rangle 2$ . LET:  $f : S \rightarrow Z$  be the identity function.
    - $\langle 3 \rangle 3$ .  $f \circ \pi$  is continuous.
    - $\langle 3 \rangle 4$ .  $f$  is continuous.
 

PROOF:  $\langle 2 \rangle 1$
    - $\langle 3 \rangle 5$ .  $U$  is open in  $Z$ .
    - $\langle 3 \rangle 6$ .  $U$  is open in  $X$ .
  - $\langle 2 \rangle 4$ . If  $U$  is open in  $S$  then  $\pi^{-1}(U)$  is open in  $X$ .
 

PROOF: Since  $\pi$  is continuous (taking  $Z = S$  and  $f = \text{id}_S$  in  $\langle 2 \rangle 1$ ).

$\square$

**Corollary 12.6.4.1.** Let  $\pi : X \twoheadrightarrow S$  be a quotient map. Let  $Z$  be a topological space. Let  $f : X \rightarrow Z$  be continuous. Then there exists a continuous map  $g : S \rightarrow Z$  such that  $f = g \circ \pi$  if and only if, for all  $s \in S$ , we have  $f$  is constant on  $\pi^{-1}(s)$ .

**Proposition 12.6.5.** Let  $Z$  be a topological space. Define  $\pi : [0, 1] \rightarrow S^1$  by  $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$ . Given any continuous function  $f : S^1 \rightarrow Z$ , we have  $f \circ \pi$  is a loop in  $Z$ . This defines a bijection between  $\mathbf{Top}[S^1, Z]$  and the set of loops in  $Z$ .

PROOF: Since  $\pi$  is a quotient map.  $\square$

**Definition 12.6.6** (Projective Space). The *projective space*  $\mathbb{RP}^n$  is the quotient of  $\mathbb{R}^{n+1} - \{0\}$  by  $\sim$  where  $x \sim \lambda x$  for all  $x \in \mathbb{R}^{n+1} - \{0\}$  and  $\lambda \in \mathbb{R}$ .

**Definition 12.6.7** (Torus). The *torus*  $T$  is the quotient of  $[0, 1]^2$  by  $\sim$  where  $(x, 0) \sim (x, 1)$  and  $(0, y) \sim (1, y)$ .

**Definition 12.6.8** (Möbius Band). The *Möbius band* is the quotient of  $[0, 1]^2$  by  $\sim$  where  $(0, y) \sim (1, 1 - y)$ .

**Definition 12.6.9** (Klein Bottle). The *Klein bottle* is the quotient of  $[0, 1]^2$  by  $\sim$  where  $(x, 0) \sim (x, 1)$  and  $(0, y) \sim (1, 1 - y)$ .

**Proposition 12.6.10.**  $\mathbb{RP}^2$  is the quotient of  $[0, 1]^2$  by  $\sim$  where  $(x, 0) \sim (1 - x, 1)$  and  $(0, y) \sim (1, 1 - y)$ .

PROOF:TODO

**Example 12.6.11.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces and  $\{Y_i\}_{i \in I}$  a family of sets. Let  $q_i : X_i \twoheadrightarrow Y_i$  be a surjective function for all  $i \in I$ . Give each  $Y_i$  the quotient topology. It is not true in general that the product topology on  $\prod_{i \in I} Y_i$  is the same as the quotient topology induced by  $\prod_{i \in I} q_i : \prod_{i \in I} X_i \twoheadrightarrow \prod_{i \in I} Y_i$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $X^* = \mathbb{R} - \mathbb{Z}_+ + \{b\}$  be the quotient space obtained from  $\mathbb{R}$  by identifying the subset  $\mathbb{Z}_+$  to the point  $b$ .

$\langle 1 \rangle 2$ . LET:  $p : \mathbb{R} \rightarrow X^*$  be the quotient map.

PROVE:  $p \times \text{id}_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$  is not a quotient map.

$\langle 1 \rangle 3$ . For  $n \in \mathbb{Z}_+$ ,

LET:  $c_n = \sqrt{2}/n$

$\langle 1 \rangle 4$ . For  $n \in \mathbb{Z}_+$ ,

LET:  $U_n = \{(x, y) \in \mathbb{Q} \times \mathbb{R} : n - 1/4 < x < n + 1/4 \text{ and } ((y > x + c_n - n \text{ and } y > -x + c_n + n) \text{ or } (y < x + c_n - n \text{ and } y < -x + c_n + n))\}$

$\langle 1 \rangle 5$ . For all  $n \in \mathbb{Z}_+$ ,  $U_n$  is open in  $\mathbb{R} \times \mathbb{Q}$

$\langle 1 \rangle 6$ . For all  $n \in \mathbb{Z}_+$  we have  $\{n\} \times \mathbb{Q} \subseteq U_n$

$\langle 1 \rangle 7$ . LET:  $U = \bigcup_{n \in \mathbb{Z}_+} U_n$

$\langle 1 \rangle 8$ .  $U$  is open in  $\mathbb{R} \times \mathbb{Q}$ .

$\langle 1 \rangle 9$ .  $U$  is saturated with respect to  $p \times \text{id}_{\mathbb{Q}}$ .

$\langle 1 \rangle 10$ . LET:  $U' = (p \times \text{id}_{\mathbb{Q}})(U)$

$\langle 1 \rangle 11$ . ASSUME: for a contradiction  $U'$  is open in  $X^* \times \mathbb{Q}$ .

## 12.7 Connected Spaces

**Definition 12.7.1** (Connected). A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

**Proposition 12.7.2.** *The continuous image of a connected space is connected.*

**Proposition 12.7.3.** *Let  $X$  be a topological space and  $A, B \subseteq X$ . If  $X = A \cup B$ ,  $A \cap B \neq \emptyset$ , and  $A$  and  $B$  are connected, then  $X$  is connected.*

**Proposition 12.7.4.** *If  $X$  and  $Y$  are nonempty topological spaces, then  $X \times Y$  is connected if and only if  $X$  and  $Y$  are connected.*

**Definition 12.7.5** (Path-connected). A topological space  $X$  is *path-connected* iff, for any points  $a, b \in X$ , there exists a continuous function  $\alpha : [0, 1] \rightarrow X$ , called a *path*, such that  $\alpha(0) = a$  and  $\alpha(1) = b$ .

**Proposition 12.7.6.** *The continuous image of a path connected space is path connected.*

**Proposition 12.7.7.** *Let  $X$  be a topological space and  $A, B \subseteq X$ . If  $X = A \cup B$ ,  $A \cap B \neq \emptyset$ , and  $A$  and  $B$  are path connected, then  $X$  is path connected.*

**Proposition 12.7.8.** *If  $X$  and  $Y$  are nonempty topological spaces, then  $X \times Y$  is path connected if and only if  $X$  and  $Y$  are path connected.*

## 12.8 Hausdorff Spaces

**Definition 12.8.1** (Hausdorff). A topological space is a *Hausdorff* space or a  $T_2$  space iff any two distinct points have disjoint neighbourhoods.

**Proposition 12.8.2.** *In a Hausdorff space, a sequence has at most one limit.*

**Proposition 12.8.3.** 1. *Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

**Proposition 12.8.4.** *Let  $A$  be a topological space and  $B$  a Hausdorff space. Let  $f, g : A \rightarrow B$  be continuous. Let  $X \subseteq A$  be dense. If  $f$  and  $g$  agree on  $X$ , then  $f = g$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $a \in A$  and  $f(a) \neq g(a)$ .

$\langle 1 \rangle 2$ . PICK disjoint neighbourhoods  $U$  and  $V$  of  $f(a)$  and  $g(a)$  respectively.

$\langle 1 \rangle 3$ . PICK  $x \in f^{-1}(U) \cap g^{-1}(V)$

$\langle 1 \rangle 4$ .  $f(x) = g(x) \in U \cap V$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 12.8.5.** *Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$  be uniformly continuous. Let  $\hat{X}$  and  $\hat{Y}$  be the completions of  $X$  and  $Y$ . Then  $f$  extends uniquely to a continuous map  $\hat{X} \rightarrow \hat{Y}$ .*

PROOF: The extension maps  $\lim_{n \rightarrow \infty} x_n$  to  $\lim_{n \rightarrow \infty} f(x_n)$ . □

## 12.9 Separable Spaces

**Definition 12.9.1** (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

## 12.10 Sequential Compactness

**Definition 12.10.1** (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

## 12.11 Compactness

**Definition 12.11.1** (Compact). A topological space is *compact* iff every open cover has a finite subcover.

**Proposition 12.11.2.** *Let  $X$  be a compact topological space. Let  $P$  be a set of open sets such that, for all  $U, V \in P$ , we have  $U \cup V \in P$ . Assume that every point has an open neighbourhood in  $P$ . Then  $X \in P$ .*

PROOF:

- ⟨1⟩1.  $P$  is an open cover of  $X$
- ⟨1⟩2. PICK a finite subcover  $U_1, \dots, U_n \in P$
- ⟨1⟩3.  $X = U_1 \cup \dots \cup U_n \in P$

□

**Corollary 12.11.2.1.** *Let  $f$  be a compact space and  $f : X \rightarrow \mathbb{R}$  be locally bounded. Then  $f$  is bounded.*

PROOF: Take  $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$ . □

**Proposition 12.11.3.** *The continuous image of a compact space is compact.*

**Proposition 12.11.4.** *A closed subspace of a compact space is compact.*

**Proposition 12.11.5.** *Let  $X$  and  $Y$  be nonempty spaces. Then the following are equivalent.*

1.  $X$  and  $Y$  are compact.

2.  $X + Y$  is compact.

3.  $X \times Y$  is compact.

**Proposition 12.11.6.** *A compact subspace of a Hausdorff space is closed.*

**Proposition 12.11.7.** *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

**Proposition 12.11.8.** *A first countable compact space is sequentially compact.*

## 12.12 Quotient Spaces

**Definition 12.12.1** (Quotient Space). Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . The *quotient topology* on  $X/\sim$  is defined by:  $U \in \mathcal{P}X$  is open in  $X/\sim$  if and only if  $\pi^{-1}(U)$  is open in  $X$ .

**Proposition 12.12.2.** *Let  $X$  and  $Y$  be topological spaces. Let  $\sim$  be an equivalence relation on  $X$ . Let  $f : X/\sim \rightarrow Y$ . Then  $f$  is continuous if and only if  $f \circ \pi$  is continuous.*

**Proposition 12.12.3.** *Let  $X$  and  $Y$  be topological spaces. Let  $\sim$  be an equivalence relation on  $X$ . Let  $\phi : Y \rightarrow X/\sim$ .*

*Assume that, for all  $y \in Y$ , there exists a neighbourhood  $U$  of  $y$  and a continuous function  $\Phi : U \rightarrow X$  such that  $\pi \circ \Phi = \phi|_U$ . Then  $\phi$  is continuous.*

**Proposition 12.12.4.** *A quotient of a connected space is connected.*

**Proposition 12.12.5.** *A quotient of a path connected space is path connected.*

**Proposition 12.12.6.** *Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . If  $X/\sim$  is Hausdorff then every equivalence class of  $\sim$  is closed in  $X$ .*

**Definition 12.12.7.** Let  $X$  be a topological space and  $A_1, \dots, A_r \subseteq X$ . Then  $X/A_1, \dots, A_r$  is the quotient space of  $X$  with respect to  $\sim$  where  $x \sim y$  iff  $x = y$  or  $\exists i(x \in A_i \wedge y \in A_i)$ .

**Definition 12.12.8** (Cone). Let  $X$  be a topological space. The *cone over  $X$*  is the space  $(X \times [0, 1])/(X \times \{1\})$ .

**Definition 12.12.9** (Suspension). Let  $X$  be a topological space. The *suspension* of  $X$  is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

**Definition 12.12.10** (Wedge Product). Let  $x_0 \in X$  and  $y_0 \in Y$ . The *wedge product*  $X \vee Y$  is  $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$  as a subspace of  $X \times Y$ .

**Definition 12.12.11** (Smash Product). Let  $x_0 \in X$  and  $y_0 \in Y$ . The *smash product*  $X \wedge Y$  is  $(X \times Y)/(X \vee Y)$ .

**Example 12.12.12.**  $D^n/S^{n-1} \cong S^n$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : D^n/S^{n-1} \rightarrow S^n$  be the function induced by the map  $D^n \rightarrow S^n$  that maps the radii of  $D^n$  onto the meridians of  $S^n$  from the north to the south pole.

$\langle 1 \rangle 2$ .  $\phi$  is a bijection.

$\langle 1 \rangle 3$ .  $\phi$  is a homeomorphism.

PROOF: Since  $D^n/S^{n-1}$  is compact and  $S^n$  is Hausdorff.

□

## 12.13 Gluing

**Definition 12.13.1** (Gluing). Let  $X$  and  $Y$  be topological spaces,  $X_0 \subseteq X$  and  $\phi : X_0 \rightarrow Y$  a continuous map. Then  $Y \cup_\phi X$  is the quotient space  $(X + Y)/\sim$ , where  $\sim$  is the equivalence relation generated by  $x \sim \phi(x)$  for all  $x \in X$ .

**Proposition 12.13.2.**  $Y$  is a subspace of  $Y \cup_\phi X$ .

**Definition 12.13.3.** Let  $X$  be a topological space and  $\alpha : X \cong X$  a homeomorphism. Then  $(X \times [0, 1])/\alpha$  is the quotient space of  $X \times [0, 1]$  by the equivalence relation generated by  $(x, 0) \sim (\alpha(x), 1)$  for all  $x \in X$ .

**Definition 12.13.4** (Möbius Strip). The *Möbius strip* is  $([-1, 1] \times [0, 1])/\alpha$  where  $\alpha(x) = -x$ .

**Definition 12.13.5** (Klein Bottle). The *Klein bottle* is  $(S^1 \times [0, 1])/\alpha$  where  $\alpha(z) = \bar{z}$ .

**Proposition 12.13.6.** Let  $M$  be the Möbius strip and  $K$  the Klein bottle. Then  $M \cup_{\text{id}_M} M \cong K$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$  be the function that maps  $\kappa_1(\theta, t)$  to  $(e^{i\theta/2}, t)$  and  $\kappa_2(\theta, t)$  to  $(-e^{-i\theta/2}, t)$ .

$\langle 1 \rangle 2$ .  $f$  induces a bijection  $M \cup_{\text{id}_M} M \approx K$

$\langle 1 \rangle 3$ .  $f$  is a homeomorphism.

□

## 12.14 Metric Spaces

**Definition 12.14.1** (Metric Space). Let  $X$  be a set and  $d : X^2 \rightarrow \mathbb{R}$ . We say  $(X, d)$  is a *metric space* iff:

- For all  $x, y \in X$  we have  $d(x, y) \geq 0$
- For all  $x, y \in X$  we have  $d(x, y) = 0$  iff  $x = y$
- For all  $x, y \in X$  we have  $d(x, y) = d(y, x)$



- (*Triangle Inequality*) For all  $x, y, z \in X$  we have  $d(x, z) \leq d(x, y) + d(y, z)$

We call  $d$  the *metric* of the metric space  $(X, d)$ . We often write  $X$  for the metric space  $(X, d)$ .

**Definition 12.14.2** (Ball). Let  $X$  be a metric space. Let  $x \in X$  and  $r > 0$ . The *ball* with *centre*  $x$  and *radius*  $r$  is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} .$$

**Definition 12.14.3** (Metric Topology). Let  $(X, d)$  be a metric space. The *metric topology* on  $X$  is the topology generated by the basis consisting of the balls.

**Definition 12.14.4** (Metrisable). A topological space is *metrisable* iff there exists a metric that induces its topology.

**Proposition 12.14.5.** *Every metrisable space is Hausdorff.*

Every metrisable space is first countable.

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

### 12.14.1 Products

**Definition 12.14.6** (Euclidean Metric). Let  $X$  and  $Y$  be metric spaces. The *Euclidean metric* on  $X \times Y$  is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} .$$

We write  $X \times Y$  for the set  $X \times Y$  under this metric.

We prove this is a metric.

PROOF:

<1>1.  $d((x_1, y_1), (x_2, y_2)) \geq 0$

PROOF: Immediate from definition.

<1>2.  $d((x_1, y_1), (x_2, y_2)) = 0$  iff  $(x_1, y_1) = (x_2, y_2)$

PROOF:  $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = 0$  iff  $d(x_1, x_2) = d(y_1, y_2) = 0$  iff  $x_1 = x_2$  and  $y_1 = y_2$ .

<1>3.  $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$

PROOF: Since  $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = \sqrt{d(x_2, x_1)^2 + d(y_2, y_1)^2}$ .

<1>4. The triangle inequality holds.

PROOF:

$$\begin{aligned}
& (d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)))^2 \\
&= d((x_1, y_1), (x_2, y_2))^2 + 2d((x_1, y_1), (x_2, y_2))d((x_2, y_2), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3))^2 \\
&= d(x_1, x_2)^2 + d(y_1, y_2)^2 + 2\sqrt{(d(x_1, x_2)^2 + d(y_1, y_2)^2)(d(x_2, x_3)^2 + d(y_2, y_3)^2)} + d(x_2, x_3)^2 + d(y_2, y_3)^2 \\
&\geq d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(y_1, y_2)^2 + d(y_2, y_3)^2 + 2(d(x_1, x_2)d(x_2, x_3) + d(y_1, y_2)d(y_2, y_3)) \\
&\quad \text{(Cauchy-Schwarz)} \\
&= (d(x_1, x_2) + d(x_2, x_3))^2 + (d(y_1, y_2) + d(y_2, y_3))^2 \\
&\geq d(x_1, x_3)^2 + d(y_1, y_3)^2 \\
&= d((x_1, y_1), (x_3, y_3))^2
\end{aligned}$$

□

**Proposition 12.14.7.** *Let  $X$  and  $Y$  be metric spaces. The Euclidean metric on  $X \times Y$  induces the product topology on  $X \times Y$ .*

PROOF:

⟨1⟩1. Every open ball is open in the product topology.

⟨2⟩1.  $(x, y) \in B((a, b), \epsilon)$

PROVE:  $B(x, \sqrt{\epsilon}) \times B(y, \sqrt{\epsilon}) \subseteq B((a, b), \epsilon)$

⟨2⟩2. LET:  $x' \in B(x, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$  and  $y' \in B(y, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$

PROVE:  $d((x', y'), (a, b)) < \epsilon$

⟨2⟩3.  $d((x', y'), (x, y)) < \epsilon - d((x, y), (a, b))$

PROOF:

$$\begin{aligned}
d((x', y'), (x, y)) &= \sqrt{d(x', x)^2 + d(y', y)^2} \\
&< \sqrt{(\epsilon - d((x, y), (a, b)))^2/2 + (\epsilon - d((x, y), (a, b)))^2/2} \\
&= \epsilon - d((x, y), (a, b))
\end{aligned}$$

⟨2⟩4.  $d((x', y'), (a, b)) < \epsilon$

PROOF:

$$\begin{aligned}
d((x', y'), (a, b)) &\leq d((x', y'), (x, y)) + d((x, y), (a, b)) \quad \text{(Triangle Inequality)} \\
&< \epsilon \quad \quad \quad (\langle 2 \rangle 3)
\end{aligned}$$

⟨1⟩2. If  $U$  is open in  $X$  and  $V$  is open in  $Y$  then  $U \times V$  is open under the Euclidean metric.

⟨2⟩1. LET:  $(x, y) \in U \times V$

⟨2⟩2. PICK  $\delta, \epsilon > 0$  such that  $B(x, \delta) \subseteq U$  and  $B(y, \epsilon) \subseteq V$

PROVE:  $B((x, y), \min(\delta, \epsilon)) \subseteq U \times V$

⟨2⟩3. LET:  $(x', y') \in B((x, y), \min(\delta, \epsilon))$

⟨2⟩4.  $d(x', x) < \delta$

⟨3⟩1.  $d((x', y'), (x, y)) < \min(\delta, \epsilon)$

⟨3⟩2.  $d(x', x)^2 + d(y', y)^2 < \delta^2$

⟨3⟩3.  $d(x', x)^2 < \delta^2$

⟨2⟩5.  $d(y', y) < \epsilon$

PROOF: Similar.

⟨2⟩6.  $(x', y') \in U \times V$

□

## 12.15 Complete Metric Spaces

**Definition 12.15.1** (Complete). A metric space is *complete* iff every Cauchy sequence converges.

**Example 12.15.2.**  $\mathbb{R}$  is complete.

**Proposition 12.15.3.** *The product of two complete metric spaces is complete.*

**Proposition 12.15.4.** *Every compact metric space is complete.*

**Proposition 12.15.5.** *Let  $X$  be a complete metric space and  $A \subseteq X$ . Then  $A$  is complete if and only if  $A$  is closed.*

**Definition 12.15.6** (Completion). Let  $X$  be a metric space. A *completion* of  $X$  is a complete metric space  $\hat{X}$  and injection  $i : X \rightarrow \hat{X}$  such that:

- The metric on  $X$  is the restriction of the metric on  $\hat{X}$
- $X$  is dense in  $\hat{X}$ .

**Proposition 12.15.7.** *Let  $i_1 : X \rightarrow Y_1$  and  $i_2 : X \rightarrow Y_2$  be completions of  $X$ . Then there exists a unique isometry  $\phi : Y_1 \cong Y_2$  such that  $\phi \circ i_1 = i_2$ .*

PROOF: Define  $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$ .  $\square$

**Theorem 12.15.8.** *Every metric space has a completion.*

PROOF: Let  $\hat{X}$  be the set of Cauchy sequences in  $X$  quotiented by  $\sim$  where  $(x_n) \sim (y_n)$  if and only if  $d(x_n, y_n) \rightarrow 0$ .  $\square$

## 12.16 Manifolds

**Definition 12.16.1** (Manifold). An  *$n$ -dimensional manifold* is a second countable Hausdorff space locally homeomorphic to  $\mathbb{R}^n$ .



## Chapter 13

# Homotopy Theory

### 13.1 Homotopies

**Definition 13.1.1** (Homotopy). Let  $X$  and  $Y$  be topological spaces. Let  $f, g : X \rightarrow Y$  be continuous. A *homotopy* between  $f$  and  $g$  is a continuous function  $h : X \times [0, 1] \rightarrow Y$  such that

- $\forall x \in X. h(x, 0) = f(x)$
- $\forall x \in X. h(x, 1) = g(x)$

We say  $f$  and  $g$  are *homotopic*,  $f \simeq g$ , iff there exists a homotopy between them.

Let  $[X, Y]$  be the set of all homotopy classes of functions  $X \rightarrow Y$ .

**Proposition 13.1.2.** Let  $f, f' : X \rightarrow Y$  and  $g, g' : Y \rightarrow Z$  be continuous. If  $f \simeq f'$  and  $g \simeq g'$  then  $g \circ f \simeq g' \circ f'$ .

**Definition 13.1.3.** Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

A *homotopy functor* is a functor  $\mathbf{Top} \rightarrow \mathcal{C}$  that factors through the canonical functor  $\mathbf{Top} \rightarrow \mathbf{HTop}$ .

**Definition 13.1.4.** A functor  $F : \mathbf{Top} \rightarrow \mathcal{C}$  is *homotopy invariant* iff, for any topological spaces  $X, Y$  and continuous functions  $f, g : X \rightarrow Y$ , if  $f \simeq g$  then  $Hf = Hg$ .

Basepoint-preserving homotopy.

### 13.2 Homotopy Equivalence

**Definition 13.2.1** (Homotopy Equivalence). Let  $X$  and  $Y$  be topological spaces. A *homotopy equivalence* between  $X$  and  $Y$ ,  $f : X \simeq Y$ , is a continuous function  $f : X \rightarrow Y$  such that there exists a continuous function  $g : Y \rightarrow X$ , the *homotopy inverse* to  $f$ , such that  $g \circ f \simeq \text{id}_X$  and  $f \circ g \simeq \text{id}_Y$ .

**Definition 13.2.2** (Contractible). A topological space  $X$  is *contractible* iff  $X \simeq 1$ .

**Example 13.2.3.**  $\mathbb{R}^n$  is contractible.

**Example 13.2.4.**  $D^n$  is contractible.

**Definition 13.2.5** (Deformation Retract). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A retraction  $\rho : X \rightarrow A$  is a *deformation retraction* iff  $i \circ \rho \simeq \text{id}_X$ , where  $i$  is the inclusion  $A \hookrightarrow X$ . We say  $A$  is a *deformation retract* of  $X$  iff there exists a deformation retraction.

**Definition 13.2.6** (Strong Deformation Retract). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A *strong deformation retraction*  $\rho : X \rightarrow A$  is a continuous function such that there exists a homotopy  $h : X \times [0, 1] \rightarrow X$  between  $i \circ \rho$  and  $\text{id}_X$  such that, for all  $a \in X$  and  $t \in [0, 1]$ , we have  $h(a, t) = a$ .

We say  $A$  is a *strong deformation retract* of  $X$  iff a strong deformation retraction exists.

**Example 13.2.7.**  $\{0\}$  is a strong deformation retract of  $\mathbb{R}^n$  and of  $D^n$ .

**Example 13.2.8.**  $S^1$  is a strong deformation retract of the torus  $S^1 \times D^2$ .

**Example 13.2.9.**  $S^{n-1}$  is a strong deformation retract of  $D^n - \{0\}$ .

**Example 13.2.10.** For any topological space  $X$ , the singleton consisting of the vertex is a strong deformation retract of the cone over  $X$ .

## Chapter 14

# Simplicial Complexes

**Definition 14.0.1** (Simplex). A  $k$ -dimensional simplex or  $k$ -simplex in  $\mathbb{R}^n$  is the convex hull  $s(x_0, \dots, x_k)$  of  $k + 1$  points in general position.

**Definition 14.0.2** (Face). A *sub-simplex* or *face* of  $s(x_0, \dots, x_k)$  is the convex hull of a subset of  $\{x_0, \dots, x_k\}$ .

**Definition 14.0.3** (Simplicial Complex). A *simplicial complex* in  $\mathbb{R}^n$  is a set  $K$  of simplices such that:

- for every simplex  $s$  in  $K$ , every face of  $s$  is in  $K$ .
- The intersection of two simplices  $s_1, s_2 \in K$  is either empty or is a face of both  $s_1$  and  $s_2$ .
- $K$  is locally finite, i.e. every point of  $\mathbb{R}^n$  has a neighbourhood that only intersects finitely many elements of  $K$ .

The topological space *underlying*  $K$  is  $|K| = \bigcup K$  as a subspace of  $\mathbb{R}^n$ .

### 14.1 Cell Decompositions

**Definition 14.1.1** ( $n$ -cell). An  $n$ -cell is a topological space homeomorphic to  $\mathbb{R}^n$ .

**Definition 14.1.2** (Cell Decomposition). Let  $X$  be a topological space. A *cell decomposition* of  $X$  is a partition of  $X$  into subspaces that are  $n$ -cells.

**Definition 14.1.3** ( $n$ -skeleton). Given a cell decomposition of  $X$ , the  $n$ -skeleton  $X^n$  is the union of all the cells of dimension  $\leq n$ .

### 14.2 CW-complexes

**Definition 14.2.1** (CW-Complex). A *CW-complex* consists of a topological space  $X$  and a cell decomposition  $\mathcal{E}$  of  $X$  such that:

1. *Characteristic Maps* For every  $n$ -cell  $e \in \mathcal{E}$ , there exists a continuous map  $\Phi_e : D^n \rightarrow X$  such that  $\Phi_e((D^n)^\circ) = e$ , the corestriction  $\Phi_e : (D^n)^\circ \approx e$  is a homeomorphism, and  $\Phi_e(S^n)$  is the union of all the cells in  $\mathcal{E}$  of dimension  $< n$ .
2. *Closure Finiteness* For all  $e \in \mathcal{E}$ , we have  $\bar{e}$  intersects only finitely many other cells in  $\mathcal{E}$ .
3. *Weak Topology* Given  $A \subseteq X$ , we have  $A$  is closed iff for all  $e \in \mathcal{E}$ ,  $A \cap \bar{e}$  is closed.

**Proposition 14.2.2.** *If a cell decomposition  $\mathcal{E}$  satisfies the Characteristic Maps axiom, then for every  $n$ -cell  $e \in \mathcal{E}$  we have  $\bar{e} = \Phi_e(D^n)$ . Therefore  $\bar{e}$  is compact and  $\bar{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$ .*

PROOF:

$\langle 1 \rangle 1.$   $e \subseteq \Phi_e(D^n) \subseteq \bar{e}$

PROOF:

$$\begin{aligned}
 e &= \Phi_e((D^n)^\circ) \\
 &\subseteq \Phi_e(D^n) \\
 &= \Phi_e(\overline{(D^n)^\circ}) \\
 &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
 &= \bar{e}
 \end{aligned}$$

$\langle 1 \rangle 2.$   $\Phi_e(D^n)$  is compact.

PROOF: Because  $D^n$  is compact.

$\langle 1 \rangle 3.$   $\Phi_e(D^n)$  is closed.

$\langle 1 \rangle 4.$   $\Phi_e(D^n) = \bar{e}$

□



## Chapter 15

# Topological Groups

**Definition 15.0.1** (Topological Group). A *topological group* is a group  $G$  with a topology such that the function  $G^2 \rightarrow G$  that maps  $(x, y)$  to  $xy^{-1}$  is continuous.

**Example 15.0.2.**  $GL(n, \mathbb{R})$  and  $GL(n, \mathbb{C})$  are topological groups.

**Proposition 15.0.3.** Any subgroup of a topological group is a topological group under the subspace topology.

**Definition 15.0.4** (Homogeneous Space). A *homogeneous space* is a topological space of the form  $G/H$ , where  $G$  is a topological group and  $H$  is a normal subgroup of  $G$ , under the quotient topology.

**Proposition 15.0.5.** Let  $G$  be a topological group and  $H$  a normal subgroup of  $G$ . Then  $G/H$  is Hausdorff if and only if  $H$  is closed.

PROOF: See Bourbaki, N., General Topology. III.12  $\square$

### 15.1 Continuous Actions

**Definition 15.1.1** (Continuous Action). Let  $G$  be a topological group and  $X$  a topological space. A *continuous action* of  $G$  on  $X$  is a continuous function  $\cdot : G \times X \rightarrow X$  such that:

- $\forall x \in X. ex = x$
- $\forall g, h \in G. \forall x \in X. g(hx) = (gh)x$

A  $G$ -space consists of a topological space  $X$  and a continuous action of  $G$  on  $X$ .

**Definition 15.1.2** (Orbit). Let  $X$  be a  $G$ -space and  $x \in X$ . The *orbit* of  $x$  is  $\{gx : g \in G\}$ .

The *orbit space*  $X/G$  is the set of all orbits under the quotient topology.

**Proposition 15.1.3.** *Define an action of  $SO(2)$  on  $S^2$  by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

*Then  $S^2/SO(2) \cong [-1, 1]$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f_3 : S^2/SO(2) \rightarrow [-1, 1]$  be the function induced by  $\pi_3 : S^2 \rightarrow [-1, 1]$

$\langle 1 \rangle 2$ .  $f_3$  is bijective.

$\langle 1 \rangle 3$ .  $S^2/SO(2)$  is compact.

PROOF: It is the continuous image of  $S^2$  which is compact.

$\langle 1 \rangle 4$ .  $[-1, 1]$  is Hausdorff.

$\langle 1 \rangle 5$ .  $f_3$  is a homeomorphism.

□

**Definition 15.1.4** (Stabilizer). Let  $X$  be a  $G$ -space and  $x \in X$ . The *stabilizer* of  $x$  is  $G_x := \{g \in G : gx = x\}$ .

**Proposition 15.1.5.** *The function that maps  $gG_x$  to  $gx$  is a continuous bijection from  $G/G_x$  to  $Gx$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $gG_x = hG_x$  then  $gx = hx$ .

$\langle 2 \rangle 1$ . ASSUME:  $gG_x = hG_x$

$\langle 2 \rangle 2$ .  $g^{-1}h \in G_x$

$\langle 2 \rangle 3$ .  $g^{-1}hx = x$

$\langle 2 \rangle 4$ .  $gx = hx$

$\langle 1 \rangle 2$ . If  $gx = hx$  then  $gG_x = hG_x$ .

PROOF: Similar.

$\langle 1 \rangle 3$ . The function is continuous.

PROOF: Proposition 12.12.2.

□

## Chapter 16

# Topological Vector Spaces

**Definition 16.0.1** (Topological Vector Space). Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . A *topological vector space* over  $K$  consists of a vector space  $E$  over  $K$  and a topology on  $E$  such that:

- Subtraction is a continuous function  $E^2 \rightarrow E$
- Multiplication is a continuous function  $K \times E \rightarrow E$

**Proposition 16.0.2.** *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition.  $\square$

**Theorem 16.0.3.** *The usual topology on a finite dimensional vector space over  $K$  is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18  $\square$

**Proposition 16.0.4.** *Let  $E$  be a topological vector space and  $E_0$  a subspace of  $E$ . Then  $\overline{E_0}$  is a subspace of  $E$ .*

**Definition 16.0.5.** Let  $E$  be a topological vector space. The topological space associated with  $E$  is  $E/\overline{\{0\}}$ .

### 16.1 Cauchy Sequences

**Definition 16.1.1** (Cauchy Sequence). Let  $E$  be a topological vector space. A sequence  $(x_n)$  in  $E$  is a *Cauchy sequence* iff, for every neighbourhood  $U$  of 0, there exists  $n_0$  such that  $\forall m, n \geq n_0, x_n - x_m \in U$ .

**Definition 16.1.2** (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

## 16.2 Seminorms

**Definition 16.2.1** (Seminorm). Let  $E$  be a vector space over  $K$ . A *seminorm* on  $E$  is a function  $\| \cdot \| : E \rightarrow \mathbb{R}$  such that:

1.  $\forall x \in E, \|x\| \geq 0$
2.  $\forall \alpha \in K, \forall x \in E, \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality*  $\forall x, y \in E, \|x + y\| \leq \|x\| + \|y\|$

**Example 16.2.2.** The function that maps  $(x_1, \dots, x_n)$  to  $|x_i|$  is a seminorm on  $\mathbb{R}^n$ .

**Definition 16.2.3.** Let  $E$  be a vector space over  $K$ . Let  $\Lambda$  be a set of seminorms on  $E$ . The topology *generated* by  $\Lambda$  is the topology generated by the subbasis consisting of all sets of the form  $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$  for  $\epsilon > 0$ ,  $\lambda \in \Lambda$  and  $x \in E$ .

**Proposition 16.2.4.**  $E$  is a topological vector space under this topology. It is Hausdorff iff, for all  $x \in E$ , if  $\forall \lambda \in \Lambda, \lambda(x) = 0$  then  $x = 0$ .

## 16.3 Fréchet Spaces

**Definition 16.3.1** (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

**Proposition 16.3.2.** Let  $E$  be a pre-Fréchet space whose topology is generated by the family of seminorms  $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$ . Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

**Definition 16.3.3** (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

## 16.4 Normed Spaces

**Definition 16.4.1** (Normed Space). Let  $E$  be a vector space over  $K$ . A *norm* on  $E$  is a function  $\| \cdot \| : E \rightarrow \mathbb{R}$  is a seminorm such that,  $\forall x \in E, \|x\| = 0 \Leftrightarrow x = 0$ .

A *normed space* consists of a vector space with a norm.

**Proposition 16.4.2.** If  $E$  is a normed space then  $d(x, y) = \|x - y\|$  is a metric on  $E$  that makes  $E$  into a topological vector space. The two definitions of Cauchy sequence agree on  $E$ .

**Definition 16.4.3** ( $p$ -norm). For any  $p \geq 1$ , the  $p$ -norm on  $\mathbb{R}^n$  is defined by

$$\|\vec{x}\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1$ . For all  $\vec{x} \in \mathbb{R}^n$  we have  $\|\vec{x}\|_p \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$ . For all  $\alpha \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^n$  we have  $\|\alpha \vec{x}\|_p = |\alpha| \|\vec{x}\|_p$

PROOF:

$$\begin{aligned} \|\alpha(x_1, \dots, x_n)\| &= \|(\alpha x_1, \dots, \alpha x_n)\| \\ &= \left( \sum_{i=1}^n (\alpha x_i)^p \right)^{\frac{1}{p}} \\ &= \left( |\alpha|^p \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|\vec{x}\|_p \end{aligned}$$

$\langle 1 \rangle 3$ . The triangle inequality holds.

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\ &= \sum_{i=1}^n |x_i + y_i| |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^n (|x_i| + |y_i|) |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} \\ &\leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} \quad (\text{Hölder's Inequality}) \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p) \|\vec{x} + \vec{y}\|_p^{p-1} \end{aligned}$$

Assuming w.l.o.g.  $\|\vec{x} + \vec{y}\|_p^{p-1} \neq 0$  (using ??) we have  $\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$ .

$\langle 1 \rangle 4$ . For any  $\vec{x} \in \mathbb{R}^n$ , we have  $\|\vec{x}\| = 0$  iff  $\vec{x} = \vec{0}$ .

PROOF:  $\sum_{i=1}^n x_i^p = 0$  iff  $x_1 = \dots = x_n = 0$ .

□

**Definition 16.4.4** (Sup-norm). The *sup-norm* on  $\mathbb{R}^n$  is defined by

$$\|(x_1, \dots, x_n)\|_\infty := \max(|x_1|, \dots, |x_n|) .$$

**Proposition 16.4.5.** *The 2-norm on  $\mathbb{R}^n$  induces the standard metric.*

PROOF: Immediate from definitions.  $\square$

**Definition 16.4.6.** For  $p \geq 1$ , the normed space  $l_p$  is the set of all sequences  $(x_n)$  in  $\mathbb{R}$  such that  $\sum_{n=1}^\infty x_n^p$  converges, under

$$\|(x_n)\|_p := \left( \sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} .$$

**Proposition 16.4.7.** *The spaces  $l_p$  for  $p \geq 1$  are all homeomorphic.*

PROOF: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. <http://dx.doi.org/10.1007/BF01075865>.

**Definition 16.4.8.** Let  $l_\infty$  be the set of all bounded sequences in  $\mathbb{R}$  under

$$\|(x_n)\| := \sup_n |x_n|$$

**Proposition 16.4.9.** *For all  $p \geq 1$  we have  $l_p$  is not homeomorphic to  $l_\infty$ .*

**Proposition 16.4.10.** *Let  $\| \cdot \|$  be a seminorm on the vector space  $E$ . Then  $\| \cdot \|$  defines a norm on  $E/\{0\}$ .*

**Proposition 16.4.11.** *Let  $E$  and  $F$  be normed spaces. Any continuous linear map  $E \rightarrow F$  is uniformly continuous.*

**Definition 16.4.12.** For  $p \geq 1$ . let  $\mathcal{L}^p(\mathbb{R}^n)$  be the vector space of all Lebesgue-measurable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $|f|^p$  is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on  $\mathcal{L}^p(\mathbb{R}^n)$ . Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\{0\} .$$

## 16.5 Inner Product Spaces

**Proposition 16.5.1.** *If  $E$  is an inner product space then  $\|x\| = \sqrt{\langle x, x \rangle}$  is a norm on  $E$ .*

## 16.6 Banach Spaces

**Definition 16.6.1** (Banach Space). A *Banach space* is a complete normed space.

**Example 16.6.2.** For any topological space  $X$ , the set  $C(X)$  of bounded continuous functions  $X \rightarrow \mathbb{R}$  is a Banach space under  $\|f\| = \sup_{x \in X} |f(x)|$ .

**Proposition 16.6.3.** *The completion of a normed space is a Banach space.*

**Proposition 16.6.4.** *Let  $E$  and  $F$  be normed spaces. Let  $f : E \rightarrow F$  be a continuous linear map. Then the extension to the completions  $\hat{E} \rightarrow \hat{F}$  is linear.*

**Proposition 16.6.5.**  $L^p(\mathbb{R}^n)$  is a Banach space.

**Proposition 16.6.6.**  $C(\mathbb{R})$  is first countable but not second countable.

PROOF: For every sequence of 0s and 1s  $s = (s_n)$ , let  $f_s$  be a continuous bounded function whose value at  $n$  is  $s_n$ . Then the set of all  $f_s$  is an uncountable discrete set in  $C(\mathbb{R})$ . Hence  $C(\mathbb{R})$  is not second countable.

It is first countable because it is metrizable.  $\square$

## 16.7 Hilbert Spaces

**Definition 16.7.1** (Hilbert Space). A *Hilbert space* is a complete inner product space.

**Example 16.7.2.** The set of *square-integrable functions* is the set of Lebesgue integrable functions  $[-\pi, \pi] \rightarrow \mathbb{R}$  quotiented by:  $f \sim g$  iff  $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$  has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

**Proposition 16.7.3.** *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

## 16.8 Locally Convex Spaces

**Definition 16.8.1** (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

**Proposition 16.8.2.** *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18.  $\square$

**Proposition 16.8.3.** *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18.  $\square$

**Example 16.8.4.** Let  $E$  be an infinite dimensional Hilbert space. Let  $E'$  be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map  $E \rightarrow \mathbb{R}$  is continuous as a map  $E' \rightarrow \mathbb{R}$ . Then  $E$  is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

**Definition 16.8.5** (Thom Space). Let  $E$  be a vector bundle with a Riemannian metric,  $DE = \{x \in E : \|x\| \leq 1\}$  its disc bundle and  $SE := \{v \in E : \|v\| = 1\}$  its sphere bundle. The *Thom space* of  $E$  is the quotient space  $DE/SE$ .