

# Encyclopaedia of Mathematics and Physics

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# Chapter 1

## Set Theory

**Proposition 1.1.** *Every infinite subset of a countably infinite set is countable.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $i : A \hookrightarrow \mathbb{N}$  be an infinite subset of  $\mathbb{N}$ .
- $\langle 1 \rangle 2$ . Define  $j : \mathbb{N} \rightarrow A$  by:  $j(k)$  is the element such that  $i(j(k))$  is least such that  $i(j(k)) \notin \{i(j(0)), \dots, i(j(k-1))\}$ .
- $\langle 1 \rangle 3$ .  $j$  is a bijection.

□

**Proposition 1.2.** *A countable union of countable sets is countable.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $(A_n)$  be a sequence of countable sets.
- $\langle 1 \rangle 2$ . For  $n \in \mathbb{N}$ , PICK an enumeration  $(e_{nm})_m$  of  $A_n$ .
- $\langle 1 \rangle 3$ . LET:  $(p_k)$  be the following enumeration of  $\mathbb{N} \times \mathbb{N}$ :  
 $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \dots$
- $\langle 1 \rangle 4$ .  $(e_{\pi_1(p_k)\pi_2(p_k)})_k$  is an enumeration of  $\bigcup_n A_n$ .

□

**Theorem 1.3.**  $2^{\mathbb{N}}$  is uncountable.

PROOF:

- $\langle 1 \rangle 1$ . ASSUME: for a contradiction  $f : \mathbb{N} \approx 2^{\mathbb{N}}$
- $\langle 1 \rangle 2$ . LET:  $S = \{n \in \mathbb{N} : n \notin f(n)\}$
- $\langle 1 \rangle 3$ . For all  $n$ , we have  $n \in S \Leftrightarrow n \notin f(n)$
- $\langle 1 \rangle 4$ . For all  $n$  we have  $S \neq f(n)$ .
- $\langle 1 \rangle 5$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 1$ .

□



## Chapter 2

# Relations

**Definition 2.1** (Antisymmetric). A relation  $R$  on a set  $A$  is *antisymmetric* iff, whenever  $xRy$  and  $yRx$ , then  $x = y$ .

**Definition 2.2** (Transitive). A relation  $R$  on a type  $A$  is *transitive* iff, whenever  $xRy$  and  $yRz$ , then  $xRz$ .





## Chapter 3

# Order Theory

**Definition 3.1** (Linear Order). A *linear order* on a set  $A$  is a binary relation  $\leq$  on  $A$  that is transitive, antisymmetric and:

$$\forall x, y \in A. x \leq y \vee y \leq x .$$

A *linearly ordered set* is a pair  $(A, \leq)$  where  $A$  is a set and  $\leq$  is a binary relation on  $A$ .

We write  $x < y$  for  $x \leq y$  and  $x \neq y$ .

**Definition 3.2** (Upper Bound). Let  $S$  be a linearly ordered set,  $u \in S$  and  $E \subseteq S$ . Then  $u$  is an *upper bound* in  $E$  iff  $\forall x \in E. x \leq u$ . We say  $E$  is *bounded above* iff it has an upper bound.

The *up-set* of  $E$ , denoted  $E \uparrow$ , is the set of upper bounds of  $E$ .

**Definition 3.3** (Lower Bound). Let  $S$  be a linearly ordered set,  $l \in S$  and  $E \subseteq S$ . Then  $l$  is a *lower bound* in  $E$  iff  $\forall x \in E. l \leq x$ . We say  $E$  is *bounded below* iff it has a lower bound.

The *down-set* of  $E$ , denoted  $E \downarrow$ , is the set of lower bounds of  $E$ .

**Definition 3.4** (Supremum). Let  $S$  be a linearly ordered set,  $u \in S$  and  $E \subseteq S$ . Then  $u$  is the *least upper bound* or *supremum* of  $E$  iff  $u$  is an upper bound for  $E$  and, for any upper bound  $u'$  for  $E$ , we have  $u \leq u'$ .

**Definition 3.5** (Infimum). Let  $S$  be a linearly ordered set,  $l \in S$  and  $E \subseteq S$ . Then  $l$  is the *greatest lower bound* or *infimum* of  $E$  iff  $l$  is a lower bound for  $E$  and, for any lower bound  $l'$  for  $E$ , we have  $l' \leq l$ .

**Definition 3.6** (Least Upper Bound Property). A linearly ordered set  $S$  has the *least upper bound property* iff every nonempty subset of  $S$  that is bounded above has a least upper bound.

**Proposition 3.7.** Let  $S$  be a linearly ordered set and  $E \subseteq S$ .

1. If  $E \downarrow$  has a supremum  $l$ , then  $l$  is the infimum of  $E$ .

2. If  $E \uparrow$  has an infimum  $u$ , then  $U$  is the supremum of  $E$ .

PROOF:

- (1)1. If  $E \downarrow$  has a supremum  $l$ , then  $l$  is the infimum of  $E$ .  
 (2)1.  $l$  is a lower bound for  $E$ .  
 (3)1. LET:  $x \in E$   
 (3)2.  $x$  is an upper bound for  $E \downarrow$ .  
 PROOF: For all  $y \in E \downarrow$  we have  $y \leq x$ .  
 (3)3.  $l \leq x$   
 (2)2. For any lower bound  $l'$  for  $E$ , we have  $l' \leq l$ .  
 PROOF: Since  $l$  is an upper bound for  $E \downarrow$ .  
 (1)2. If  $E \uparrow$  has an infimum  $u$ , then  $u$  is the supremum of  $E$ .  
 PROOF: Dual.

□

**Corollary 3.7.1.** *A linearly ordered set has the least upper bound property if and only if every nonempty set bounded below has an infimum.*

**Definition 3.8** (Closed Downwards). Let  $S$  be a linearly ordered set and  $E \subseteq S$ . Then  $E$  is *closed downwards* iff, whenever  $x \in E$  and  $y < x$ , then  $y \in E$ .

**Definition 3.9** (Closed Upwards). Let  $S$  be a linearly ordered set and  $E \subseteq S$ . Then  $E$  is *closed upwards* iff, whenever  $x \in E$  and  $x < y$ , then  $y \in E$ .

**Definition 3.10** (Greatest). Let  $S$  be a linearly ordered set and  $u \in S$ . Then  $u$  is *greatest* in  $S$  iff  $\forall x \in S. x \leq u$ .

**Definition 3.11** (Least). Let  $S$  be a linearly ordered set and  $l \in S$ . Then  $l$  is *least* in  $S$  iff  $\forall x \in S. l \leq x$ .

**Proposition 3.12.** *Let  $\leq$  be a linear order on a set  $S$  and  $E \subseteq S$ . Then  $\leq \cap E^2$  is a linear order on  $E$ .*

PROOF: Easy. □

Given a linearly ordered set  $(S, \leq)$  and  $E \subseteq S$ , we write just  $E$  for the linearly ordered set  $(E, \leq \cap E^2)$ .

**Definition 3.13** (Lexicographic Order). Let  $A$  and  $B$  be linearly ordered sets. The *lexicographic order* or *dictionary order* on  $A \times B$  is the order defined by

$$(a, b) \leq (a', b') \Leftrightarrow a = a' \vee (a < a' \wedge b \leq b') .$$

**Proposition 3.14.** *The lexicographic order is a linear order.*

## Chapter 4

# Field Theory

**Definition 4.1** (Field). A *field*  $F$  consists of a set  $F$ , two operations  $+, \cdot : F^2 \rightarrow F$  and an element  $0 \in F$  such that:

- $+$  is commutative.
- $+$  is associative.
- $\forall x \in F. x + 0 = x$
- $\forall x \in F. \exists y \in F. x + y = 0$
- $\cdot$  is commutative.
- $\cdot$  is associative.
- There exists  $1 \in F$  such that  $1 \neq 0$  and  $\forall x \in F. x1 = x$  and  $\forall x \in F. x \neq 0 \Rightarrow \exists y \in F. xy = 1$
- *Distributive Law*  $\forall x, y, z \in F. x(y + z) = xy + xz$

**Proposition 4.2.** *In any field  $F$ , the element  $0$  is the unique element such that  $\forall x \in F. x + 0 = x$ .*

PROOF: If  $0$  and  $0'$  both have this property then  $0 = 0 + 0' = 0'$ .  $\square$

**Proposition 4.3.** *In any field  $F$ , given  $x \in F$ , there is a unique  $y \in F$  such that  $x + y = 0$ .*

PROOF: If  $x + y = x + y' = 0$  then

$$\begin{aligned} y &= y + 0 \\ &= y + x + y' \\ &= 0 + y' \\ &= y' \end{aligned}$$

$\square$

**Definition 4.4.** Let  $F$  be a field. Let  $x \in F$ . We denote by  $-x$  the unique element of  $F$  such that  $x + (-x) = 0$ .

Given  $x, y \in F$ , we write  $x - y$  for  $x + (-y)$ .

**Proposition 4.5.** In any field  $F$ , if  $x + y = x + z$  then  $y = z$ .

PROOF: If  $x + y = x + z$  we have

$$-x + x + y = -x + x + z$$

$$\therefore 0 + y = 0 + z$$

$$\therefore y = z$$

□

**Proposition 4.6.** In any field  $F$ , we have  $-(-x) = x$ .

PROOF: Since  $x + (-x) = 0$ . □

**Proposition 4.7.** In any field  $F$ , the element  $1$  such that  $\forall x \in F. x1 = x$  is unique.

PROOF: If  $1$  and  $1'$  both have this property then  $1 = 1 \cdot 1' = 1'$ . □

**Proposition 4.8.** In any field  $F$ , given  $x \in F$  with  $x \neq 0$ , the element  $y$  such that  $xy = 1$  is unique.

PROOF: If  $y$  and  $y'$  both have this property then we have

$$y = y1$$

$$= yxy'$$

$$= 1y'$$

$$= y'$$

□

**Definition 4.9.** In any field  $F$ , if  $x \neq 0$ , we write  $x^{-1}$  for the unique element such that  $xx^{-1} = 1$ .

We write  $x/y$  for  $xy^{-1}$ .

**Proposition 4.10.** In any field  $F$ , if  $xy = xz$  and  $x \neq 0$  then  $y = z$ .

PROOF:

$$y = 1y$$

$$= x^{-1}xy$$

$$= x^{-1}xz$$

$$= 1z$$

$$= z$$

□

**Proposition 4.11.** In any field  $F$ , if  $x \neq 0$  then  $x^{-1} \neq 0$  and  $(x^{-1})^{-1} = x$ .

PROOF: Since  $xx^{-1} = 1$ . □

**Proposition 4.12.** In any field  $F$ , we have  $x0 = 0$ .

PROOF:

$$\begin{aligned}
 x0 + 0 &= x0 \\
 &= x(0 + 0) \\
 &= x0 + x0 \\
 \therefore 0 &= x0 \quad \square
 \end{aligned}$$

**Proposition 4.13.** *In any field  $F$ , if  $xy = 0$  then  $x = 0$  or  $y = 0$ .*

PROOF: If  $xy = 0$  and  $x \neq 0$  then we have  $y = x^{-1}xy = x^{-1}0 = 0$ .  $\square$

**Proposition 4.14.** *In any field  $F$ , we have  $(-x)y = -(xy)$ .*

PROOF:

$$\begin{aligned}
 xy + (-x)y &= (x + (-x))y \\
 &= 0y \\
 &= 0 \quad (\text{Proposition 4.12}) \square
 \end{aligned}$$

**Corollary 4.14.1.** *In any field  $F$ , we have  $(-x)(-y) = xy$ .*

PROOF:

$$\begin{aligned}
 (-x)(-y) &= -(x(-y)) \\
 &= -(-(xy)) \\
 &= xy \quad (\text{Proposition 4.6}) \square
 \end{aligned}$$

**Proposition 4.15.** *Let  $K$  be a field. Let  $a, b \in K$ . If  $a^2 = b^2$  then  $a = b$  or  $a = -b$ .*

PROOF:

$$\begin{aligned}
 a^2 - b^2 &= 0 \\
 \therefore (a - b)(a + b) &= 0
 \end{aligned}$$

Hence either  $a - b = 0$  or  $a + b = 0$ , and the conclusion follows.  $\square$

## 4.1 Ordered Fields

**Definition 4.16** (Ordered Field). An *ordered field*  $F$  consists of a field  $F$  and a linear order  $\leq$  on  $F$  such that:

- For all  $x, y, z \in F$ , if  $y < z$  then  $x + y < x + z$
- For all  $x, y \in F$ , if  $x > 0$  and  $y > 0$  then  $xy > 0$ .

We call  $x$  *positive* iff  $x > 0$  and *negative* iff  $x < 0$ .

**Example 4.17.**  $\mathbb{Q}$  is an ordered field.

**Proposition 4.18.** *In any ordered field, if  $x$  is positive then  $-x$  is negative.*

PROOF: If  $x > 0$  then  $0 = x + (-x) > 0 = (-x) = -x$ .  $\square$

**Proposition 4.19.** *In any ordered field, if  $y < z$  and  $x$  is positive then  $xy < xz$ .*

PROOF: If  $y < z$  then we have

$$\begin{aligned} 0 &< z - y \\ \therefore 0 &< x(z - y) \\ &= xz - xy \\ \therefore xy &< xz \end{aligned}$$

□

**Proposition 4.20.** *In any ordered field, if  $y < z$  and  $x$  is negative then  $xy > xz$ .*

PROOF:

- <1>1.  $-x$  is positive.
- <1>2.  $(-x)y < (-x)z$
- <1>3.  $-(xy) < -(xz)$
- <1>4.  $xz < xy$

□

**Proposition 4.21.** *In any ordered field, if  $x \neq 0$  then  $x^2 > 0$ .*

PROOF:

- <1>1. If  $x > 0$  then  $x^2 > 0$ .

PROOF: Proposition 4.19.

- <1>2. If  $x < 0$  then  $x^2 > 0$ .

PROOF: Proposition 4.20.

□

**Corollary 4.21.1.** *In any ordered field, we have  $1 > 0$ .*

**Proposition 4.22.** *In any ordered field, if  $x$  is positive then  $x^{-1}$  is positive.*

PROOF: If  $x^{-1} < 0$  then we would have  $1 = xx^{-1} < x0 = 0$  contradicting Corollary 4.21.1. □

**Proposition 4.23.** *In any ordered field, if  $0 < x < y$  then  $y^{-1} < x^{-1}$ .*

PROOF:

- <1>1. ASSUME:  $0 < x < y$
- <1>2.  $x^{-1}$  and  $y^{-1}$  are positive.

PROOF: Proposition 4.22.

- <1>3.  $xy^{-1} < yy^{-1} = 1$
- <1>4.  $y^{-1} = x^{-1}xy^{-1} < x^{-1}1 = x^{-1}$

□

**Lemma 4.24.** *Let  $K$  be an ordered field. Let  $b \in K$  with  $b > 1$ . Let  $n$  be a positive integer. Then*

$$b^n - 1 \geq n(b - 1)$$

PROOF:

$$\begin{aligned} b^n - 1 &= (b - 1)(b^{n-1} + b^{n-2} + \cdots + 1) \\ &\geq (b - 1)(1 + 1 + \cdots + 1) \\ &= n(b - 1) \end{aligned}$$

□

## Chapter 5

# Real Analysis

### 5.1 Construction of the Real Numbers

**Definition 5.1** (Cut). A *cut* is a subset  $\alpha$  of  $\mathbb{Q}$  such that:

- $\emptyset \neq \alpha \neq \mathbb{Q}$
- $\alpha$  is closed downwards.
- $\alpha$  has no greatest element.

In this section, we write  $R$  for the set of all cuts.

**Proposition 5.2.**  *$R$  is linearly ordered by  $\subseteq$ .*

PROOF: The only difficult part is to prove that, for any cuts  $\alpha$  and  $\beta$ , either  $\alpha \subseteq \beta$  or  $\beta \subseteq \alpha$ .

$\langle 1 \rangle 1$ . ASSUME:  $\alpha \not\subseteq \beta$

PROVE:  $\beta \subseteq \alpha$

$\langle 1 \rangle 2$ . PICK  $q \in \alpha$  such that  $q \notin \beta$

$\langle 1 \rangle 3$ . LET:  $r \in \beta$

$\langle 1 \rangle 4$ .  $q \not\leq r$

$\langle 1 \rangle 5$ .  $r < q$

$\langle 1 \rangle 6$ .  $r \in \alpha$

□

**Proposition 5.3.**  *$R$  has the least upper bound property.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $E \subseteq R$  be nonempty and bounded above.

$\langle 1 \rangle 2$ . LET:  $s = \bigcup E$

PROVE:  $s$  is a cut.

$\langle 1 \rangle 3$ .  $\emptyset \neq s$

PROOF: Since  $E$  is nonempty and every element of  $E$  is nonempty.

$\langle 1 \rangle 4$ .  $s \neq \mathbb{Q}$

- ⟨2⟩1. PICK an upper bound  $u$  for  $E$ .
- ⟨2⟩2. PICK  $q \notin u$   
     PROVE:  $q \notin s$
- ⟨2⟩3.  $\forall \alpha \in E. \alpha \subseteq u$
- ⟨2⟩4.  $s \subseteq u$
- ⟨2⟩5.  $q \notin s$
- ⟨1⟩5.  $s$  is closed downwards.
- ⟨2⟩1. LET:  $q \in s$  and  $r < q$ .
- ⟨2⟩2. PICK  $\alpha \in E$  such that  $q \in \alpha$ .
- ⟨2⟩3.  $r \in \alpha$
- ⟨2⟩4.  $r \in s$
- ⟨1⟩6.  $s$  has no greatest element.
- ⟨2⟩1. LET:  $q \in s$
- ⟨2⟩2. PICK  $\alpha \in E$  such that  $q \in \alpha$ .
- ⟨2⟩3. PICK  $r \in \alpha$  such that  $q < r$ .
- ⟨2⟩4.  $r \in s$

□

**Definition 5.4** (Addition). Given cuts  $\alpha$  and  $\beta$ , we define

$$\alpha + \beta = \{q + r : q \in \alpha, r \in \beta\} .$$

**Proposition 5.5.** *Given cuts  $\alpha$  and  $\beta$ , we have  $\alpha + \beta$  is a cut.*

PROOF:

- ⟨1⟩1.  $\alpha + \beta$  is nonempty.  
     PROOF: Since  $\alpha$  and  $\beta$  are nonempty.
- ⟨1⟩2.  $\alpha + \beta \neq \mathbb{Q}$ 
  - ⟨2⟩1. PICK  $q \in \mathbb{Q} - \alpha$  and  $r \in \mathbb{Q} - \beta$ .  
     PROVE:  $q + r \notin \alpha + \beta$
  - ⟨2⟩2. ASSUME: for a contradiction  $q + r \in \alpha + \beta$ .
  - ⟨2⟩3. PICK  $x \in \alpha$  and  $y \in \beta$  such that  $q + r = x + y$
  - ⟨2⟩4.  $x < q$
  - ⟨2⟩5.  $y < r$
  - ⟨2⟩6.  $x + y < q + r$
  - ⟨2⟩7. Q.E.D.
- PROOF: This is a contradiction.
- ⟨1⟩3.  $\alpha + \beta$  is closed downwards.
  - ⟨2⟩1. LET:  $q \in \alpha, r \in \beta$  and  $x < q + r$
  - ⟨2⟩2.  $x - q < r$
  - ⟨2⟩3.  $x - q \in \beta$
  - ⟨2⟩4.  $x \in \alpha + \beta$
- ⟨1⟩4.  $\alpha + \beta$  has no greatest element.
  - ⟨2⟩1. LET:  $q \in \alpha$  and  $r \in \beta$ .  
     PROVE:  $q + r$  is not greatest in  $\alpha + \beta$ .
  - ⟨2⟩2. PICK  $q' \in \alpha$  with  $q < q'$  and  $r' \in \beta$  with  $r < r'$ .
  - ⟨2⟩3.  $q + r < q' + r' \in \alpha + \beta$



□

**Proposition 5.6.** *Addition is commutative and associative on  $R$ .*

PROOF: Immediate from definitions and the fact that addition is commutative and associative on  $\mathbb{Q}$ . □

**Definition 5.7.** For any  $q \in \mathbb{Q}$ , let  $q^* = \{r \in \mathbb{Q} : r < q\}$ .

**Proposition 5.8.** *For any  $q \in \mathbb{Q}$ , we have  $q^*$  is a cut.*

PROOF:

⟨1⟩1.  $q^* \neq \emptyset$

PROOF: Since  $q - 1 \in q^*$ .

⟨1⟩2.  $q^* \neq \mathbb{Q}$

PROOF: Since  $q \notin q^*$ .

⟨1⟩3.  $q^*$  is closed downwards.

PROOF: Immediate from definition.

⟨1⟩4.  $q^*$  has no greatest element.

PROOF: For all  $r \in q^*$  we have  $r < (q + r)/2 \in q^*$ .

□

**Proposition 5.9.** *For any cut  $\alpha$  we have  $\alpha + 0^* = \alpha$ .*

PROOF:

⟨1⟩1.  $\alpha + 0^* \subseteq \alpha$

⟨2⟩1. LET:  $q \in \alpha$  and  $r \in 0^*$

PROVE:  $q + r \in \alpha$

⟨2⟩2.  $r < 0$

⟨2⟩3.  $q + r < q$

⟨2⟩4.  $q + r \in \alpha$

⟨1⟩2.  $\alpha \subseteq \alpha + 0^*$

⟨2⟩1. LET:  $q \in \alpha$

⟨2⟩2. PICK  $r \in \alpha$  such that  $q < r$

⟨2⟩3.  $q = r + (q - r) \in \alpha + 0^*$

□

**Proposition 5.10.** *For any cut  $\alpha$ , there exists a cut  $\beta$  such that  $\alpha + \beta = 0$ .*

PROOF:

⟨1⟩1. LET:  $\beta = \{p \in \mathbb{Q} : \exists r > 0. -p - r \notin \alpha\}$

⟨1⟩2.  $\beta$  is a cut.

⟨2⟩1.  $\beta \neq \emptyset$

⟨3⟩1. PICK  $q \notin \alpha$

⟨3⟩2.  $-q - 1 \in \beta$

⟨2⟩2.  $\beta \neq \mathbb{Q}$

⟨3⟩1. PICK  $q \in \alpha$

PROVE:  $-q \notin \beta$

⟨3⟩2. ASSUME: for a contradiction  $-q \in \beta$

- $\langle 3 \rangle 3$ . PICK  $r > 0$  such that  $q - r \notin \alpha$
- $\langle 3 \rangle 4$ .  $q - r < q$
- $\langle 3 \rangle 5$ . Q.E.D.

PROOF: This contradicts the fact that  $\alpha$  is closed downwards.

- $\langle 2 \rangle 3$ .  $\beta$  is closed downwards.
  - $\langle 3 \rangle 1$ . LET:  $p \in \beta$  and  $q < p$ .
  - $\langle 3 \rangle 2$ . PICK  $r > 0$  such that  $-p - r \notin \alpha$
  - $\langle 3 \rangle 3$ .  $-p - r < -q - r$
  - $\langle 3 \rangle 4$ .  $-q - r \notin \alpha$
  - $\langle 3 \rangle 5$ .  $q \in \beta$
- $\langle 2 \rangle 4$ .  $\beta$  has no greatest element.
  - $\langle 3 \rangle 1$ . LET:  $p \in \beta$
  - $\langle 3 \rangle 2$ . PICK  $r > 0$  such that  $-p - r \notin \alpha$
  - $\langle 3 \rangle 3$ .  $-(p + r/2) - r/2 \notin \alpha$
  - $\langle 3 \rangle 4$ .  $p + r/2 \in \beta$
- $\langle 1 \rangle 3$ .  $\alpha + \beta \subseteq 0^*$ 
  - $\langle 2 \rangle 1$ . LET:  $p \in \alpha$  and  $q \in \beta$ .
  - $\langle 2 \rangle 2$ . PICK  $r > 0$  such that  $-q - r \notin \alpha$ .
  - $\langle 2 \rangle 3$ .  $p < -q - r$
  - $\langle 2 \rangle 4$ .  $p + q < -r$
  - $\langle 2 \rangle 5$ .  $p + q < 0$
  - $\langle 2 \rangle 6$ .  $p + q \in 0^*$
- $\langle 1 \rangle 4$ .  $0^* \subseteq \alpha + \beta$ 
  - $\langle 2 \rangle 1$ . LET:  $v \in 0^*$
  - $\langle 2 \rangle 2$ . LET:  $w = -v/2$
  - $\langle 2 \rangle 3$ .  $w > 0$
  - $\langle 2 \rangle 4$ . PICK an integer  $n$  such that  $nw \in \alpha$  and  $(n + 1)w \notin \alpha$ .
  - $\langle 2 \rangle 5$ . LET:  $p = -(n + 2)w$
  - $\langle 2 \rangle 6$ .  $p \in \beta$
  - $\langle 2 \rangle 7$ .  $v = nw + p$
  - $\langle 2 \rangle 8$ .  $v \in \alpha + \beta$

□

**Proposition 5.11.** *Given  $\alpha, \beta, \gamma \in R$ , if  $\beta < \gamma$ , then  $\alpha + \beta < \alpha + \gamma$ .*

PROOF:

- $\langle 1 \rangle 1$ .  $\alpha + \beta \subseteq \alpha + \gamma$   
 PROOF: Immediate from definitions.
- $\langle 1 \rangle 2$ .  $\alpha + \beta \neq \alpha + \gamma$   
 PROOF: If  $\alpha + \beta = \alpha + \gamma$  then  $\beta = \gamma$  by cancellation.

□

**Definition 5.12.** Given cuts  $\alpha$  and  $\beta$ , define  $\alpha\beta$  by:

$$\alpha\beta = \begin{cases} \{p \in \mathbb{Q} : \exists r \in \alpha. \exists s \in \beta (p \leq rs \wedge r > 0 \wedge s > 0)\} & \text{if } \alpha > 0^* \text{ and } \beta > 0^* \\ (-\alpha)(-\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^* \\ -((-\alpha)\beta) & \text{if } \alpha < 0^* \text{ and } \beta > 0^* \\ -(\alpha(-\beta)) & \text{if } \alpha > 0^* \text{ and } \beta < 0^* \\ 0^* & \text{if } \alpha = 0^* \text{ or } \beta = 0^* \end{cases}$$

**Proposition 5.13.** For any cuts  $\alpha$  and  $\beta$ , we have  $\alpha\beta$  is a cut.

PROOF:

(1)1. If  $\alpha > 0^*$  and  $\beta > 0^*$  then  $\alpha\beta$  is a cut.

(2)1.  $\alpha\beta \neq \emptyset$

(3)1. PICK  $q \in \alpha$  and  $r \in \beta$  such that  $q, r \notin 0^*$

(3)2. ASSUME: w.l.o.g.  $0 < q$  and  $0 < r$ .

PROOF: Since  $\alpha$  and  $\beta$  have no greatest element.

(3)3.  $qr \in \alpha\beta$

(2)2.  $\alpha\beta \neq \mathbb{Q}$

(3)1. PICK  $r \notin \alpha$  and  $s \notin \beta$

PROVE:  $rs \notin \alpha\beta$

(3)2. ASSUME: for a contradiction  $rs \in \alpha\beta$ .

(3)3. PICK  $r' \in \alpha$  and  $s' \in \beta$  such that  $rs \leq r's'$  and  $r' > 0$  and  $s' > 0$ .

(3)4.  $r' < r$  and  $s' < s$

(3)5.  $r's' < rs$

(3)6. Q.E.D.

PROOF: This is a contradiction.

(2)3.  $\alpha\beta$  is closed downwards.

(3)1. LET:  $p \in \alpha\beta$  and  $p' < p$

(3)2. PICK  $r \in \alpha$  and  $s \in \beta$  such that  $p \leq rs$ ,  $r > 0$  and  $s > 0$

(3)3.  $p' \leq rs$

(3)4.  $p' \in \alpha\beta$

(2)4.  $\alpha\beta$  has no greatest element.

(3)1. LET:  $p \in \alpha\beta$

(3)2. PICK  $r \in \alpha$  and  $s \in \beta$  such that  $p \leq rs$ ,  $r > 0$  and  $s > 0$ .

(3)3. PICK  $r' \in \alpha$  and  $s' \in \beta$  with  $r < r'$  and  $s < s'$ .

(3)4.  $p < r's' \in \alpha\beta$

(1)2. For any cuts  $\alpha$  and  $\beta$ , we have  $\alpha\beta$  is a cut.

PROOF: Since if  $\alpha$  is a cut then  $-\alpha$  is a cut.

□

**Proposition 5.14.** For any cuts  $\alpha$  and  $\beta$  we have  $\alpha\beta = \beta\alpha$ .

PROOF: Easy from the definitions. □

**Proposition 5.15.** For any cuts  $\alpha$ ,  $\beta$  and  $\gamma$  we have

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma .$$

PROOF:

$\langle 1 \rangle 1$ . CASE:  $\alpha$ ,  $\beta$  and  $\gamma$  are all positive.

PROOF: In this case  $\alpha(\beta\gamma) = (\alpha\beta)\gamma = \{p \in \mathbb{Q} : \exists r \in \alpha. \exists s \in \beta. \exists t \in \gamma. (p \leq rst \wedge r > 0 \wedge s > 0 \wedge t > 0)\}$ .

$\langle 1 \rangle 2$ . CASE: One of  $\alpha$ ,  $\beta$  or  $\gamma$  is  $0^*$ .

PROOF: Then  $\alpha(\beta\gamma) = (\alpha\beta)\gamma = 0^*$ .

$\langle 1 \rangle 3$ . CASE:  $\alpha$  and  $\beta$  are positive,  $\gamma$  is negative.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\ &= -(\alpha(\beta(-\gamma))) \\ &= -((\alpha\beta)(-\gamma)) && (\langle 1 \rangle 1) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 4$ . CASE:  $\alpha$  is positive,  $\beta$  is negative,  $\gamma$  is positive.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-((- \beta)\gamma)) \\ &= -(\alpha((- \beta)\gamma)) \\ &= -((\alpha(- \beta))\gamma) && (\langle 1 \rangle 1) \\ &= -(\alpha(-\beta))\gamma \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 5$ . CASE:  $\alpha$  is positive,  $\beta$  and  $\gamma$  are negative.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha((- \beta)(- \gamma)) \\ &= (\alpha(- \beta))(- \gamma) && (\langle 1 \rangle 1) \\ &= -(\alpha\beta)(- \gamma) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 6$ . CASE:  $\alpha$  is negative,  $\beta$  and  $\gamma$  are positive.

PROOF: Similar to  $\langle 1 \rangle 3$ .

$\langle 1 \rangle 7$ . CASE:  $\alpha$  is negative,  $\beta$  is positive,  $\gamma$  is negative.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\ &= (-\alpha)(\beta(-\gamma)) \\ &= ((-\alpha)\beta)(-\gamma) && (\langle 1 \rangle 1) \\ &= -(\alpha\beta)(-\gamma) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 8$ . CASE:  $\alpha$  and  $\beta$  are negative,  $\gamma$  is positive.

PROOF: Similar to  $\langle 1 \rangle 5$ .

$\langle 1 \rangle 9$ . CASE:  $\alpha$ ,  $\beta$  and  $\gamma$  are all negative.

PROOF:

$$\begin{aligned}
 \alpha(\beta\gamma) &= \alpha(-(-\beta)(-\gamma)) \\
 &= -((- \alpha)((-\beta)(-\gamma))) \\
 &= -((( - \alpha)(-\beta))(-\gamma)) & ((1)1) \\
 &= -((\alpha\beta)(-\gamma)) \\
 &= (\alpha\beta)\gamma
 \end{aligned}$$

□

**Proposition 5.16.** *For any cut  $\alpha$  we have  $\alpha 1^* = \alpha$ .*

PROOF:

$\langle 1 \rangle 1$ . CASE:  $\alpha$  is positive.

$\langle 2 \rangle 1$ .  $\alpha 1^* \subseteq \alpha$

$\langle 2 \rangle 2$ .  $\alpha \subseteq \alpha 1^*$

$\langle 1 \rangle 2$ . CASE:  $\alpha = 0^*$

$\langle 1 \rangle 3$ . CASE:  $\alpha$  is negative.

□

**Theorem 5.17.** *There exists an ordered field with the least upper bound property.*

**Proposition 5.18.** *There is no rational  $p$  such that  $p^2 = 2$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $p^2 = 2$ .

$\langle 1 \rangle 2$ . PICK integers  $m, n$  not both even such that  $p = m/n$ .

$\langle 1 \rangle 3$ .  $m^2 = 2n^2$

$\langle 1 \rangle 4$ .  $m$  is even.

$\langle 1 \rangle 5$ . PICK an integer  $k$  such that  $m = 2k$ .

$\langle 1 \rangle 6$ .  $4k^2 = 2n^2$

$\langle 1 \rangle 7$ .  $2k^2 = n^2$

$\langle 1 \rangle 8$ .  $n$  is even.

$\langle 1 \rangle 9$ . Q.E.D.

PROOF:  $\langle 1 \rangle 2$ ,  $\langle 1 \rangle 4$  and  $\langle 1 \rangle 8$  form a contradiction.

□

**Theorem 5.19.** *Any two complete ordered fields are isomorphic.*

**Definition 5.20.** Let  $\mathbb{R}$  be the complete ordered field. We call its elements *real numbers*.

## 5.2 Properties of the Real Numbers

**Theorem 5.21.**  $\mathbb{Q}$  is a subfield of  $\mathbb{R}$ .

**Theorem 5.22** (Archimedean Property). *Let  $x, y \in \mathbb{R}$  with  $x > 0$ . There exists a positive integer  $n$  such that  $nx > y$ .*

PROOF:

- (1)1. LET:  $A = \{nx : n \in \mathbb{Z}^+\}$
- (1)2. ASSUME: for a contradiction there is no positive integer  $n$  such that  $nx > y$ .
- (1)3.  $y$  is an upper bound for  $A$ .
- (1)4. LET:  $\alpha = \sup A$
- (1)5.  $\alpha - x$  is not an upper bound for  $A$ .
- (1)6. PICK a positive integer  $m$  such that  $\alpha - x < mx$
- (1)7.  $\alpha < (m+1)x \in A$
- (1)8. Q.E.D.

PROOF: This contradicts (1)4.

□

**Theorem 5.23.**  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

PROOF:

- (1)1. LET:  $x, y \in \mathbb{R}$  with  $x < y$
- (1)2. PICK a positive integer  $n$  such that  $n(y-x) > 1$ .
- PROOF: Archimedean property.
- (1)3. PICK a positive integer  $m_1$  such that  $m_1 > nx$
- PROOF: Archimedean property.
- (1)4. PICK a positive integer  $m_2$  such that  $m_2 > -nx$
- PROOF: Archimedean property.
- (1)5.  $-m_2 < nx < m_1$
- (1)6. LET:  $m$  be the integer such that  $m-1 \leq nx < m$ .
- (1)7.  $nx < m \leq 1 + nx < ny$
- (1)8.  $x < m/n < y$

□

**Theorem 5.24.** For every real number  $x > 0$  and positive integer  $n$ , there exists a unique positive real number  $y$  such that  $y^n = x$ .

PROOF:

- (1)1. There exists a real  $y > 0$  such that  $y^n = x$ .
- (2)1. LET:  $E = \{t \in \mathbb{R}^+ : t^n < x\}$
- (2)2. LET:  $y = \sup E$
- (3)1.  $E \neq \emptyset$
- (4)1. LET:  $t = x/(x+1)$
- (4)2.  $0 < t < 1$
- (4)3.  $t^n < t < x$
- (4)4.  $t \in E$
- (3)2.  $x+1$  is an upper bound for  $E$ .
- (4)1. LET:  $t > x+1$
- (4)2.  $t^n > t > x$
- (4)3.  $t \notin E$

⟨2⟩3.  $y^n = x$

⟨3⟩1.  $y^n \not\leq x$

⟨4⟩1. ASSUME: for a contradiction  $y^n < x$ .

⟨4⟩2. PICK  $h$  such that  $0 < h < 1$  and

$$h < \frac{x - y^n}{n(y + 1)^{n-1}} .$$

⟨4⟩3.  $(y + h)^n - y^n < x - y^n$

PROOF:

$$\begin{aligned} (y + h)^n - y^n &= ((y + h) - y) \sum_{i=0}^{n-1} (y + h)^{n-1-i} y^i \\ &= h \sum_{i=0}^{n-1} (y + h)^{n-1-i} y^i \\ &\leq hn(y + h)^{n-1} \\ &\leq hn(y + 1)^{n-1} \\ &< x - y^n \end{aligned}$$

⟨4⟩4.  $(y + h)^n < x$

⟨4⟩5.  $y + h \in E$

⟨4⟩6. Q.E.D.

PROOF: This contradicts the fact that  $y$  is an upper bound for  $E$ .

⟨3⟩2.  $y^n \not\geq x$

⟨4⟩1. ASSUME: for a contradiction  $y^n > x$

⟨4⟩2. LET:

$$k = \frac{y^n - x}{ny^{n-1}}$$

⟨4⟩3.  $0 < k < y$

⟨4⟩4.  $y - k$  is an upper bound for  $E$ .

⟨5⟩1. LET:  $t \geq y - k$

⟨5⟩2.  $y^n - t^n \leq y^n - x$

PROOF:

$$\begin{aligned} y^n - t^n &\leq y^n - (y - k)^n \\ &= (y - (y - k)) \sum_{i=0}^{n-1} y^{n-i} (y - k)^i \\ &= k \sum_{i=0}^{n-1} y^{n-i} (y - k)^i \\ &\leq kny^{n-1} \\ &= y^n - x \end{aligned}$$

⟨5⟩3.  $t^n \geq x$

⟨5⟩4.  $t \notin E$

⟨4⟩5. Q.E.D.

PROOF: This contradicts the fact that  $y$  is the least upper bound of  $E$ .

⟨1⟩2. If  $y$  and  $y'$  are positive reals with  $y^n = y'^n$  then  $y = y'$ .

PROOF: Since the function that sends  $y$  to  $y^n$  is strictly monotone.  
 $\square$

**Definition 5.25** (*n*th Root). Given any real number  $x > 0$  and positive integer  $n$ , the *n*th root of  $x$ , denoted  $x^{1/n}$ , is the unique positive real such that

$$(x^{1/n})^n = x .$$

We write  $\sqrt{x}$  for  $x^{1/2}$ .

**Proposition 5.26.** Let  $a$  and  $b$  be positive real numbers and  $n$  a positive integer. Then

$$(ab)^{1/n} = a^{1/n}b^{1/n} .$$

PROOF: Since  $(a^{1/n}b^{1/n})^n = ab$ .  $\square$

**Lemma 5.27.** Let  $b$  be a real number with  $b > 1$ . Let  $n$  be a positive integer. Then

$$b - 1 \geq n(b^{1/n} - 1) .$$

PROOF: From Lemma 4.24.  $\square$

**Lemma 5.28.** Let  $b$  and  $t$  be real numbers with  $b > 1$  and  $t > 1$ . For any positive integer  $n$ , if  $n > \frac{b-1}{t-1}$  then  $b^{1/n} < t$ .

PROOF:

$$\begin{aligned} b - 1 &\geq n(b^{1/n} - 1) \\ \therefore \frac{b - 1}{n} &\geq b^{1/n} - 1 \\ \therefore t - 1 &> b^{1/n} - 1 \\ \therefore t &> b^{1/n} \end{aligned} \quad \square$$

**Lemma 5.29.** Let  $b$  be a real number with  $b > 0$ . Let  $m, n, p, q$  be integers with  $n > 0$  and  $q > 0$ . Assume  $m/n = p/q$ . Then

$$(b^m)^{1/n} = (b^p)^{1/q} .$$

PROOF:

$$\langle 1 \rangle 1. (b^m)^{1/n} = (b^{1/n})^m$$

PROOF:

$$\begin{aligned} ((b^{1/n})^m)^n &= ((b^{1/n})^n)^m \\ &= b^m \end{aligned}$$

$$\langle 1 \rangle 2. ((b^m)^{1/n})^q = b^p$$

PROOF:

$$\begin{aligned} ((b^m)^{1/n})^q &= (b^{1/n})^{mq} \\ &= (b^{1/n})^{np} \\ &= b^p \end{aligned}$$

$\square$



**Definition 5.30.** For  $a$  a positive real and  $q$  a rational number, we may therefore define  $a^q$  by

$$a^{m/n} = (a^m)^{1/n}$$

for  $m$  and  $n$  integers with  $n > 0$ .

**Proposition 5.31.** Let  $a$  be a positive real and  $r, s$  rational numbers. Then

$$a^{r+s} = a^r a^s .$$

PROOF:

$$\begin{aligned} a^{m/n+p/q} &= a^{(mq+np)/nq} \\ &= (a^{mq+np})^{1/nq} \\ &= (a^{mq})^{1/nq} (a^{np})^{1/nq} \\ &= a^{m/n} a^{p/q} \end{aligned} \quad \square$$

**Proposition 5.32.** Let  $b > 1$  be a real number and  $q$  a rational number. Then

$$b^q = \sup\{b^t : t \in \mathbb{Q}, t \leq q\}$$

PROOF: It is the greatest element of this set.  $\square$

**Definition 5.33.** Let  $b > 1$  be a real number and  $x$  a real number. Then

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t \leq x\} .$$

**Lemma 5.34.** Let  $b, w$  and  $y$  be real numbers with  $b > 1$ . Assume  $b^w < y$ . Then there exists a positive integer  $n$  such that  $b^{w+1/n} < y$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $t = yb^{-w}$
- $\langle 1 \rangle 2$ . PICK a positive integer  $n$  such that  $n > \frac{b-1}{t-1}$ .
- $\langle 1 \rangle 3$ .  $b^{1/n} < t$

PROOF: Lemma 5.28.

- $\langle 1 \rangle 4$ .  $b^{w+1/n} < y$

$\square$

**Lemma 5.35.** Let  $b, w$  and  $y$  be real numbers with  $b > 1$ . Assume  $b^w > y$ . Then there exists a positive integer  $n$  such that  $b^{w-1/n} < y$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $t = b^w/y$
- $\langle 1 \rangle 2$ . PICK a positive integer  $n$  such that  $n > \frac{b-1}{t-1}$
- $\langle 1 \rangle 3$ .  $b^{1/n} < t$

PROOF: Lemma 5.28.

- $\langle 1 \rangle 4$ .  $y < b^{w-1/n}$

$\square$

**Proposition 5.36.** *For  $b$  and  $x$  real numbers with  $b > 1$  we have*

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t < x\} .$$

PROOF:

- $\langle 1 \rangle 1.$   $b^x$  is an upper bound for  $\{b^t : t \in \mathbb{Q}, t < x\}$ .
- $\langle 1 \rangle 2.$  LET:  $u$  be any upper bound for  $\{b^t : t \in \mathbb{Q}, t < x\}$ .  
PROVE:  $b^x \leq u$
- $\langle 1 \rangle 3.$  LET:  $q$  be a rational number with  $q \leq x$ .  
PROVE:  $b^q \leq u$
- $\langle 1 \rangle 4.$  ASSUME: for a contradiction  $b^q > u$ .
- $\langle 1 \rangle 5.$  PICK a positive integer  $n$  such that  $b^{q-1/n} > u$ .  
PROOF: Lemma 5.35.
- $\langle 1 \rangle 6.$   $b^{q-1/n} \leq u$   
PROOF:  $\langle 1 \rangle 2$
- $\langle 1 \rangle 7.$  Q.E.D.  
PROOF: This contradicts  $\langle 1 \rangle 4$ .

□

**Lemma 5.37.** *Let  $A$  be a set of positive real numbers with supremum  $a > 0$  and  $B$  a set of positive real numbers with supremum  $b > 0$ . Then  $ab$  is the supremum of  $\{xy : x \in A, y \in B\}$ .*

PROOF:

- $\langle 1 \rangle 1.$  For all  $x \in A$  and  $y \in B$  we have  $xy \leq ab$ .
- $\langle 1 \rangle 2.$  If  $u$  is any upper bound for  $\{xy : x \in A, y \in B\}$  then  $ab \leq u$ .  
  - $\langle 2 \rangle 1.$  LET:  $u$  be an upper bound for  $\{xy : x \in A, y \in B\}$ .
  - $\langle 2 \rangle 2.$  For all  $x \in A$  we have  $u/x$  is an upper bound for  $B$ .
  - $\langle 2 \rangle 3.$  For all  $x \in A$  we have  $b \leq u/x$
  - $\langle 2 \rangle 4.$  For all  $x \in A$  we have  $x \leq u/b$
  - $\langle 2 \rangle 5.$   $a \leq u/b$
  - $\langle 2 \rangle 6.$   $ab \leq u$

□

**Proposition 5.38.** *Let  $b, x, y \in \mathbb{R}$  with  $b > 1$ . Then*

$$b^{x+y} = b^x b^y .$$

PROOF:

- $\langle 1 \rangle 1.$  For any rational number  $q < x + y$ , there exist rational numbers  $r < x$  and  $s < y$  such that  $q = r + s$ .  
  - $\langle 2 \rangle 1.$   $q - x < y$
  - $\langle 2 \rangle 2.$  PICK a rational  $t$  such that  $q - x < t < y$
  - $\langle 2 \rangle 3.$   $q = t + (q - t)$  and  $t < y, q - t < x$
- $\langle 1 \rangle 2.$   $b^x b^y = b^{x+y}$

PROOF:

$$\begin{aligned}
 b^x b^y &= \sup\{b^q b^r : q, r \in \mathbb{Q}, q < x, r < y\} \\
 &= \sup\{b^{q+r} : q, r \in \mathbb{Q}, q < x, r < y\} \\
 &= \sup\{b^q : q \in \mathbb{Q}, q < x + y\} \\
 &= b^{x+y}
 \end{aligned}$$

□

### 5.2.1 Logarithms

**Proposition 5.39.** *Let  $b$  and  $y$  be real numbers with  $b > 1$  and  $y > 0$ . There exists a unique real  $x$  such that  $b^x = y$ .*

PROOF:

⟨1⟩1. LET:  $x = \sup\{w : b^w < y\}$

PROVE:  $b^x = y$

⟨2⟩1.  $\{w : b^w < y\} \neq \emptyset$

PROOF: It contains 0.

⟨2⟩2.  $\{w : b^w < y\}$  is bounded above.

⟨3⟩1. LET:  $n$  be the least integer such that

$$n \geq \frac{y-1}{b-1}$$

PROOF: Archimedean property.

⟨3⟩2. LET:  $w$  be a real number with  $b^w < y$

PROVE:  $w < n$

⟨3⟩3.  $b^w < n(b-1) + 1$

⟨3⟩4.  $b^w < b^n$

⟨3⟩5.  $w < n$

⟨1⟩2.  $b^x \leq y$

⟨2⟩1. ASSUME: for a contradiction  $b^x > y$

⟨2⟩2. PICK a positive integer  $n$  such that  $b^{x-1/n} > y$

PROOF: Lemma 5.35.

⟨2⟩3. PICK  $w$  such that  $x - 1/n < w$  and  $b^w < y$

PROOF: Since  $x - 1/n$  is not an upper bound for  $\{w : b^w < y\}$ .

⟨2⟩4.  $b^{x-1/n} < y$

⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩2.

⟨1⟩3.  $b^x \geq y$

⟨2⟩1. ASSUME: for a contradiction  $b^x < y$ .

⟨2⟩2. PICK a positive integer  $n$  such that  $b^{x+1/n} < y$ .

⟨2⟩3.  $x + 1/n \leq x$

⟨2⟩4. Q.E.D.

PROOF: This is a contradiction.

□

**Definition 5.40** (Logarithm). Let  $b$  and  $y$  be real numbers with  $b > 1$  and  $y > 0$ . The *logarithm* of  $y$  to base  $b$ , denoted  $\log_b y$ , is the unique real number

such that

$$b^{\log_b y} = y \ .$$

### 5.2.2 Intervals

**Definition 5.41** (Intervals). Let  $a, b \in \mathbb{R}$ .

The *open interval*  $(a, b)$  is  $\{x \in \mathbb{R} : a < x < b\}$ .

The *closed interval*  $[a, b]$  is  $\{x \in \mathbb{R} : a \leq x \leq b\}$ .

The *half-open intervals*  $[a, b)$  and  $(a, b]$  are defined by

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

**Definition 5.42** ( $k$ -cell). Let  $k$  be a positive integer. A  $k$ -cell is a subset of  $\mathbb{R}^k$  of the form

$$\{\vec{x} \in \mathbb{R}^k : \forall i = 1, \dots, k. a_i \leq x_i \leq b_i\}$$

for some real numbers  $a_1, \dots, a_k, b_1, \dots, b_k$  with  $a_i \leq b_i$  for each  $i$ .

### 5.2.3 The Cantor Set

**Definition 5.43** (Cantor Set). Define a sequence  $E_n$  of unions of intervals as follows:

- $E_0 = [0, 1]$
- $E_{n+1}$  is formed from  $E_n$  by replacing every interval  $[a, b]$  with  $[a, (2a+b)/3]$  and  $[(a+2b)/3, b]$ .

The *Cantor set* is  $\bigcap_{n=0}^{\infty} E_n$ .

## 5.3 The Extended Real Number System

**Definition 5.44** (Extended Real Number System). The *extended real number system* is the set  $\mathbb{R} \cup \{+\infty, -\infty\}$ .

We extend the ordering  $\leq$  to the extended reals by defining

$$-\infty < x < +\infty$$

for every  $x \in \mathbb{R}$ .

We extend  $+$ ,  $\cdot$  and  $/$  to partial operations on the extended real by defining:

$$\begin{array}{ll}
x + (+\infty) = +\infty & (x \in \mathbb{R}) \\
x + (-\infty) = -\infty & (x \in \mathbb{R}) \\
(+\infty) + x = +\infty & (x \in \mathbb{R}) \\
(+\infty) + (+\infty) \text{ is undefined} & \\
(+\infty) + (-\infty) \text{ is undefined} & \\
(-\infty) + x = -\infty & (x \in \mathbb{R}) \\
(-\infty) + (+\infty) \text{ is undefined} & \\
(-\infty) + (-\infty) \text{ is undefined} & \\
x \cdot (+\infty) = +\infty & (x \in \mathbb{R}) \\
x \cdot (-\infty) = -\infty & (x \in \mathbb{R}) \\
(+\infty) \cdot x = +\infty & (x \in \mathbb{R}) \\
(+\infty) \cdot (+\infty) \text{ is undefined} & \\
(+\infty) \cdot (-\infty) \text{ is undefined} & \\
(-\infty) \cdot x = -\infty & (x \in \mathbb{R}) \\
(-\infty) \cdot (+\infty) \text{ is undefined} & \\
(-\infty) \cdot (-\infty) \text{ is undefined} & \\
x / (+\infty) = 0 & (x \in \mathbb{R}) \\
x / (-\infty) = 0 & (x \in \mathbb{R}) \\
(+\infty) / x \text{ is undefined} & (x \in \mathbb{R}) \\
(+\infty) / (+\infty) \text{ is undefined} & \\
(+\infty) / (-\infty) \text{ is undefined} & \\
(-\infty) / x \text{ is undefined} & (x \in \mathbb{R}) \\
(-\infty) / (+\infty) \text{ is undefined} & \\
(-\infty) / (-\infty) \text{ is undefined} &
\end{array}$$



## Chapter 6

# Complex Analysis

**Definition 6.1** (Complex Numbers). A *complex number* is a pair of real numbers. We write  $\mathbb{C}$  for the set of complex numbers.

Define  $+$  and  $\cdot$  on  $\mathbb{C}$  by:

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b)(c, d) &= (ac - bd, ad + bc)\end{aligned}$$

**Theorem 6.2.** *The complex numbers form a field.*

**Theorem 6.3.** *The function that maps  $a$  to  $(a, 0)$  is an embedding of  $\mathbb{R}$  in  $\mathbb{C}$ .*

**Definition 6.4.**

$$i = (0, 1)$$

**Lemma 6.5.**

$$(a, b) = a + ib$$

PROOF: Since  $(a, 0) + (0, 1)(b, 0) = (a, b)$ .  $\square$

**Lemma 6.6.**

$$i^2 = -1$$

PROOF: Immediate from definitions.  $\square$

**Corollary 6.6.1.** *There is no linear order on  $\mathbb{C}$  that makes  $\mathbb{C}$  into an ordered field.*

**Definition 6.7** (Complex Conjugate). For any complex number  $z$ , the *complex conjugate*  $\bar{z}$  is defined by

$$\overline{a + ib} = a - ib \quad (a, b \in \mathbb{R}) .$$

**Definition 6.8** (Real Part). For any complex number  $z$ , the *real part* of  $z$ , denoted  $\operatorname{Re}(z)$ , is defined by

$$\operatorname{Re}(a + ib) = a \quad (a, b \in \mathbb{R}) .$$

**Definition 6.9** (Imaginary Part). For any complex number  $z$ , the *imaginary part* of  $z$ , denoted  $\text{Im}(z)$ , is defined by

$$\text{Im}(a + ib) = b \quad (a, b \in \mathbb{R}) .$$

**Theorem 6.10.** For all  $z, w \in \mathbb{C}$  we have

$$\overline{z + w} = \bar{z} + \bar{w} .$$

PROOF:

$$\begin{aligned} \overline{(a + ib) + (c + id)} &= \overline{(a + c) + i(b + d)} \\ &= (a + c) - i(b + d) \\ &= (a - ib) + (c - id) \\ &= \overline{a + ib} + \overline{c + id} \end{aligned} \quad \square$$

**Theorem 6.11.** For all  $z, w \in \mathbb{C}$  we have

$$\overline{zw} = \bar{z} \cdot \bar{w} .$$

PROOF:

$$\begin{aligned} \overline{(a + ib)(c + id)} &= \overline{(ac - bd) + i(ad + bc)} \\ &= (ac - bd) - i(ad + bc) \\ &= (a - ib)(c - id) \\ &= \overline{a + ib} \cdot \overline{c + id} \end{aligned} \quad \square$$

**Theorem 6.12.** For all  $z \in \mathbb{C}$  we have

$$\text{Re}(z) = \frac{1}{2}(z + \bar{z}) .$$

PROOF:

$$\begin{aligned} (a + ib) + \overline{a + ib} &= (a + ib) + (a - ib) \\ &= 2a \\ &= 2 \text{Re}(a + ib) \end{aligned} \quad \square$$

**Theorem 6.13.** For all  $z \in \mathbb{C}$  we have

$$\text{Im}(z) = \frac{1}{2i}(z - \bar{z}) .$$

PROOF:

$$\begin{aligned} (a + ib) - \overline{a + ib} &= (a + ib) - (a - ib) \\ &= 2ib \\ &= 2i \text{Im}(a + ib) \end{aligned} \quad \square$$

**Theorem 6.14.** For all  $z \in \mathbb{C}$  we have  $z\bar{z}$  is a non-negative real.



PROOF:

$$\begin{aligned}(a + ib)(\overline{a + ib}) &= (a + ib)(a - ib) \\ &= a^2 + b^2\end{aligned}\quad \square$$

**Theorem 6.15.** *For any  $z \in \mathbb{C}$ , if  $z\bar{z} = 0$  then  $z = 0$ .*

PROOF: Let  $z = a + ib$ . Then  $z\bar{z} = a^2 + b^2 = 0$  iff  $a = b = 0$ .  $\square$

**Definition 6.16** (Absolute Value). For  $z \in \mathbb{C}$ , the *absolute value* of  $z$  is

$$|z| = (z\bar{z})^{1/2}.$$

**Proposition 6.17.** *For  $x$  a non-negative real we have  $|x| = x$ .*

PROOF: Since  $|x| = \sqrt{x^2} = x$ .  $\square$

**Proposition 6.18.** *For  $x$  a negative real we have  $|x| = -x$ .*

PROOF: Since  $|x| = \sqrt{x^2} = -x$ .  $\square$

**Theorem 6.19.** *For any complex number  $z$  we have  $|z| \geq 0$ .*

PROOF: Immediate from definition.  $\square$

**Theorem 6.20.** *For any complex number  $z$ , if  $|z| = 0$  then  $z = 0$ .*

PROOF: From Theorem 6.15.  $\square$

**Theorem 6.21.** *For any complex number  $z$  we have*

$$|\bar{z}| = |z|.$$

PROOF: Immediate from definitions.  $\square$

**Theorem 6.22.** *For any complex numbers  $z$  and  $w$  we have*

$$|zw| = |z||w|.$$

PROOF:

$$\begin{aligned}|zw| &= \sqrt{zw\bar{z}\bar{w}} \\ &= \sqrt{z\bar{z}}\sqrt{w\bar{w}} && \text{(Proposition 5.26)} \\ &= |z||w|\end{aligned}\quad \square$$

**Theorem 6.23.** *For any complex number  $z$  we have*

$$|\operatorname{Re} z| \leq |z|$$

PROOF: Let  $z = a + ib$ . Then

$$|\operatorname{Re} z| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}. \square$$

**Theorem 6.24.** *For any complex numbers  $z$  and  $w$  we have*

$$|z + w| \leq |z| + |w|.$$

PROOF:

$$\begin{aligned}
 |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\
 &= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\
 &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 && \text{(Theorem 6.12)} \\
 &\leq |z|^2 + 2|z\bar{w}| + |w|^2 && \text{(Theorem 6.23)} \\
 &= |z|^2 + 2|z||w| + |w|^2 && \text{(Theorem 6.22)} \\
 &= (|z| + |w|)^2 && \square
 \end{aligned}$$

**Theorem 6.25** (Schwarz Inequality). *Let  $a_1, \dots, a_n, b_1, \dots, b_n$  be complex numbers. Then*

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2 .$$

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } A = \sum_{j=1}^n |a_j|^2$$

$$\langle 1 \rangle 2. \text{ LET: } B = \sum_{j=1}^n |b_j|^2$$

$$\langle 1 \rangle 3. \text{ LET: } C = \sum_{j=1}^n a_j \bar{b}_j$$

$$\langle 1 \rangle 4. \text{ ASSUME: w.l.o.g. } B > 0$$

PROOF: If  $B = 0$  then  $b_1 = \dots = b_n = 0$  and both sides of the inequality are 0.

$$\langle 1 \rangle 5. \sum_{j=1}^n |Ba_j - Cb_j|^2 = B(AB - |C|^2)$$

PROOF:

$$\begin{aligned}
 \sum_{j=1}^n |Ba_j - Cb_j|^2 &= \sum_{j=1}^n (Ba_j - Cb_j)(B\bar{a}_j - \bar{C}\bar{b}_j) \\
 &= B^2 \sum_{j=1}^n |a_j|^2 - B\bar{C} \sum_{j=1}^n a_j \bar{b}_j - BC \sum_{j=1}^n \bar{a}_j b_j + |C|^2 \sum_{j=1}^n |b_j|^2 \\
 &= B^2 A - 2B|C|^2 + B|C|^2 \\
 &= B(AB - |C|^2)
 \end{aligned}$$

$$\langle 1 \rangle 6. B(AB - |C|^2) \geq 0$$

$$\langle 1 \rangle 7. AB \geq |C|^2$$

$\square$

**Proposition 6.26.** *For any non-zero complex number  $w$ , there are exactly two complex numbers  $z$  such that  $z^2 = w$ .*

PROOF:

$$\langle 1 \rangle 1. \text{ There are at most two complex numbers } z \text{ such that } z^2 = w.$$

PROOF: Proposition 4.15.

$$\langle 1 \rangle 2. \text{ There are at least two complex numbers } z \text{ such that } z^2 = w.$$

$$\langle 2 \rangle 1. \text{ LET: } w = u + iv$$

$$\langle 2 \rangle 2. \text{ LET: } a = \sqrt{\frac{|w|+u}{2}}$$

$$\langle 2 \rangle 3. \text{ LET: } b = \sqrt{\frac{|w|-u}{2}}$$

⟨2⟩4. CASE:  $v \geq 0$

⟨3⟩1. LET:  $z = a + ib$

⟨3⟩2.  $z^2 = w$

PROOF:

$$\begin{aligned} z^2 &= (a + ib)^2 \\ &= a^2 - b^2 + 2iab \\ &= u + i\sqrt{|w|^2 - u^2} \\ &= u + iv \\ &= w \end{aligned}$$

⟨3⟩3.  $(-z)^2 = w$

⟨2⟩5. CASE:  $v \leq 0$

⟨3⟩1. LET:  $z = a - ib$

⟨3⟩2.  $z^2 = w$

PROOF:

$$\begin{aligned} z^2 &= (a - ib)^2 \\ &= a^2 - b^2 - 2iab \\ &= u - i\sqrt{|w|^2 - u^2} \\ &= u - i|v| \\ &= w \end{aligned}$$

⟨3⟩3.  $(-z)^2 = w$

□

## 6.1 Algebraic Numbers

**Definition 6.27** (Algebraic). A complex number  $z$  is *algebraic* iff there exist integers  $a_0, a_1, \dots, a_n$  not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0 ;$$

otherwise, it is *transcendental*.

**Proposition 6.28.** *The set of algebraic numbers is countable.*

PROOF: There are countably many finite sequences of integers  $(a_0, a_1, \dots, a_n)$ , and for each one, there are only finitely many complex numbers  $z$  such that  $a_0 z^n + a_1 z^{n-1} + \dots + a_n = 0$ . □



**Part I**

**Linear Algebra**



## Chapter 7

# Vector Spaces

### 7.1 Convex Sets

**Definition 7.1** (Convex). Let  $E \subseteq \mathbb{R}^k$ . Then  $E$  is *convex* iff, for all  $\vec{x}, \vec{y} \in E$  and  $\lambda \in (0, 1)$ ,

$$\lambda \vec{x} + (1 - \lambda) \vec{y} \in E \text{ .}$$

**Proposition 7.2.** *Every  $k$ -cell is convex.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $C = \{\vec{x} \in \mathbb{R}^k : \forall i. a_i \leq x_i \leq b_i\}$  be a  $k$ -cell.

$\langle 1 \rangle 2$ . LET:  $\vec{x}, \vec{y} \in C$  and  $\lambda \in (0, 1)$ .

PROVE:  $\lambda \vec{x} + (1 - \lambda) \vec{y} \in C$

$\langle 1 \rangle 3$ . For each  $i$  we have  $a_i \leq \lambda x_i + (1 - \lambda) y_i \leq b_i$

PROOF: Since  $\lambda a_1 + (1 - \lambda) a_i \leq \lambda x_i + (1 - \lambda) y_i \leq \lambda b_i + (1 - \lambda) b_i$ .

□





## Chapter 8

# Real Inner Product Spaces

**Definition 8.1** (Inner Product). Given  $\vec{x}, \vec{y} \in \mathbb{R}^k$ , define the *inner product*  $\vec{x} \cdot \vec{y}$  by

$$(x_1, \dots, x_k) \cdot (y_1, \dots, y_k) = x_1 y_1 + \dots + x_k y_k .$$

**Definition 8.2** (Norm). Define the *norm* of a vector  $\vec{x} \in \mathbb{R}^k$  by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} .$$

**Proposition 8.3.**

$$\|\vec{x}\| \geq 0$$

PROOF: Immediate from the definition.  $\square$

**Proposition 8.4.** If  $\|\vec{x}\| = 0$  then  $\vec{x} = \vec{0}$ .

PROOF: If  $\|\vec{x}\| = 0$  then  $x_1^2 + \dots + x_n^2 = 0$  so  $x_1 = \dots = x_n = 0$ .  $\square$

**Proposition 8.5.** For  $\alpha \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^k$ ,

$$\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| .$$

PROOF: Easy.  $\square$

**Proposition 8.6.** For  $\vec{x}, \vec{y} \in \mathbb{R}^k$ , we have

$$\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\| \|\vec{y}\| .$$

PROOF: By the Schwarz inequality.  $\square$

**Proposition 8.7.** For  $\vec{x}, \vec{y} \in \mathbb{R}^k$  we have

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| .$$

PROOF:

$$\begin{aligned}
\|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
&= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Proposition 8.6)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square
\end{aligned}$$

**Corollary 8.7.1.** For  $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$  we have

$$\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| .$$

**Definition 8.8** (Bounded Function). Let  $E$  be a set. Let  $f : E \rightarrow \mathbb{R}^k$ . Then  $f$  is *bounded* iff  $f(E)$  is bounded.

## 8.1 Balls

**Definition 8.9** (Closed Ball). Let  $\vec{x} \in \mathbb{R}^k$  and  $r > 0$ . The *closed ball* with centre  $\vec{x}$  and radius  $r$  is

$$\{y \in \mathbb{R}^k : \|y - x\| \leq r\} .$$

**Proposition 8.10.** Every closed ball is convex.

PROOF:

- <1>1. LET:  $B$  be the closed ball with center  $\vec{a}$  and radius  $r$ .
- <1>2. LET:  $\vec{x}, \vec{y} \in B$
- <1>3. LET:  $\lambda \in (0, 1)$
- <1>4.  $\lambda\vec{x} + (1 - \lambda)\vec{y} \in B$

PROOF:

$$\begin{aligned}
\|\lambda\vec{x} + (1 - \lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1 - \lambda)(\vec{y} - \vec{a})\| \\
&= \lambda\|\vec{x} - \vec{a}\| + (1 - \lambda)\|\vec{y} - \vec{a}\| \\
&\leq \lambda r + (1 - \lambda)r \\
&= r && \square
\end{aligned}$$

$\square$

## Chapter 9

# Complex Inner Product Spaces

**Definition 9.1** (Inner Product). Let  $V$  be a complex vector space. An *inner product* on  $V$  is a function  $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{C}$  such that, for all  $x, y, z \in V$  and  $\alpha \in \mathbb{C}$ :

- $\langle y, x \rangle = \overline{\langle x, y \rangle}$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- $\langle x, x \rangle \geq 0$
- If  $\langle x, x \rangle = 0$  then  $x = 0$ .

An *inner product space* consists of a complex vector space  $V$  and an inner product on  $V$ .

**Definition 9.2** (Norm). Let  $V$  be an inner product space and  $x \in V$ . The *norm* of  $x$  is

$$\|x\| = \sqrt{\langle x, x \rangle} .$$

**Proposition 9.3.** *An inner product space is a metric space under*

$$d(x, y) = \|x - y\| .$$

**Definition 9.4** (Bounded). Let  $V_1$  and  $V_2$  be inner product spaces and  $T : V_1 \rightarrow V_2$  a linear transformation. Then  $T$  is *bounded* iff  $\{\|T(x)\| : \|x\| = 1\}$  is bounded above.

**Proposition 9.5.** *Every linear transformation between finite dimensional inner product spaces is bounded.*

**Definition 9.6** (Outer Product). Let  $V$  be an inner product space and  $|\psi\rangle, |\phi\rangle \in V$ . The *outer product* of  $|\psi\rangle$  and  $|\phi\rangle$  is

$$|\psi\rangle \langle \phi| : V \rightarrow V .$$

## 9.1 Hilbert Spaces

**Definition 9.7** (Hilbert Space). A *Hilbert space* is a complete inner product space.

**Theorem 9.8** (Completeness Relation). Let  $\mathcal{H}$  be a Hilbert space. Let  $\{|e_n\rangle\}_{n \in \mathbb{N}}$  be a countable orthonormal basis for  $\mathcal{H}$ . Then

$$\sum_{n=0}^{\infty} |e_n\rangle \langle e_n| = I \quad .$$

PROOF:

(1)1. LET:  $|\psi\rangle \in \mathcal{H}$

(1)2. LET:  $|\psi\rangle = \sum_{n=0}^{\infty} \alpha_n |e_n\rangle$

(1)3.  $\sum_{n=0}^{\infty} \langle e_n | \phi \rangle |e_n\rangle = |\psi\rangle$

PROOF:

$$\begin{aligned} \sum_{n=0}^{\infty} \langle e_n | \phi \rangle |e_n\rangle &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_m \langle e_n | e_m \rangle |e_n\rangle \\ &= \sum_{n=0}^{\infty} \alpha_n |e_n\rangle \\ &= |\psi\rangle \end{aligned}$$

□

□

**Definition 9.9** (Separable). A Hilbert space is *separable* iff it has a countable dense orthonormal basis.

# Chapter 10

## Lie Algebras

**Definition 10.1** (Lie Algebra). Let  $K$  be a field. A *Lie algebra*  $\mathcal{L}$  over  $K$  consists of a vector space  $\mathcal{L}$  over  $K$  and an operation

$$[\cdot, \cdot] : \mathcal{L}^2 \rightarrow \mathcal{L} ,$$

the *Lie bracket* or *commutator*, such that, for all  $x, y, z \in \mathcal{L}$  and  $\alpha \in K$ :

$$\begin{aligned} [x + y, z] &= [x, z] + [y, z] \\ [x, y + z] &= [x, y] + [x, z] \\ [\alpha x, y] &= \alpha[x, y] \\ [x, x] &= 0 \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \end{aligned} \quad \text{(Jacobi identity)}$$

**Lemma 10.2.** *If  $K$  has characteristic 0 then the condition  $[x, x] = 0$  can be replaced with  $[x, y] = -[y, x]$ .*

**Proposition 10.3.** *The commutator is determined by its values on any basis for  $\mathcal{L}$ .*

**Example 10.4.**  $\mathbb{R}^3$  with the cross product is a real Lie algebra.

**Example 10.5.** For any  $n \geq 0$ , we have  $GL(n, K)$  is a Lie algebra over  $K$  under

$$[A, B] = AB - BA .$$

**Definition 10.6** (Linear Lie Algebra). A *linear Lie algebra* over  $K$  is a Lie algebra over  $K$  that is a subalgebra of  $GL(n, K)$  for some  $n$ .

**Example 10.7** (Special Linear Algebra). The *special Linear algebra*  $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \text{tr} = 0\}$  is a real linear Lie algebra.

**Example 10.8** (Orthogonal Lie Algebra). The *orthogonal Lie algebra*  $SO(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A \text{ is skew-symmetric}\}$  is a real linear Lie algebra.

**Example 10.9.** Let  $u(n)$  be the set of all skew-Hermitian  $n \times n$ -matrices as a real Lie algebra.

Let  $su(n) = u(n) \cap SL(n, \mathbb{R})$ .

**Proposition 10.10.**  $SU(2)$  is spanned by the Pauli matrices

$$\sigma_x = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_y = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which satisfy

$$\begin{aligned} [\sigma_x, \sigma_y] &= \sigma_z \\ [\sigma_y, \sigma_z] &= \sigma_x \\ [\sigma_z, \sigma_x] &= \sigma_y \end{aligned}$$

## 10.1 Lie Algebar Homomorphisms

**Definition 10.11** (Homomorphism). Let  $L_1$  and  $L_2$  be Lie algebras over the same field. A *Lie algebra homomorphism*  $\phi : L_1 \rightarrow L_2$  is a linear transformation such that

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all  $x, y \in L_1$ .

**Lemma 10.12.** *Every bijective Lie algebra homomorphism is an isomorphism.*

**Definition 10.13** (Representation). Let  $L$  be a real (complex) Lie algebra. A *representation* of  $L$  is a Lie algebra homomorphism  $L \rightarrow GL(n, \mathbb{R})$  ( $GL(n, \mathbb{C})$ ) for some  $n$ .

**Example 10.14.** The linear transformation  $\mathbb{R}^3 \rightarrow su(2)$  defined by

$$i \mapsto \sigma_x, j \mapsto \sigma_y, k \mapsto \sigma_z$$

is a representation of  $\mathbb{R}^3$ .

# Part II

# Topology





# Chapter 11

## Metric Spaces

**Definition 11.1** (Metric). A *metric* on a set  $X$  is a function  $d : X^2 \rightarrow \mathbb{R}$  such that, for all  $x, y, z \in X$ :

- $d(x, y) \geq 0$
- $d(x, y) = 0$  iff  $x = y$
- $d(x, y) = d(y, x)$
- **Triangle Inequality**  $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space*  $X$  consists of a set  $X$  and a metric on  $X$ .

**Example 11.2.**  $\mathbb{R}^k$  is a metric space under  $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$ . The triangle inequality is Corollary 8.7.1.

**Example 11.3.** For any set  $X$ , the *discrete* metric on  $X$  is defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

**Proposition 11.4.** Let  $(X, d)$  be a metric space and  $Y$  a subset of  $X$ . Then  $d \upharpoonright Y^2$  is a metric on  $Y$ .

PROOF: Easy.  $\square$

### 11.1 Balls

**Definition 11.5** (Open Ball). Let  $\vec{x} \in \mathbb{R}^k$  and  $r > 0$ . The *open ball* with centre  $\vec{x}$  and radius  $r$  is

$$\{y \in \mathbb{R}^k : \|y - x\| < r\} .$$

**Proposition 11.6.** Every open ball in  $\mathbb{R}^k$  is convex.

PROOF:

(1)1. LET:  $B$  be the open ball with center  $\vec{a}$  and radius  $r$ .

(1)2. LET:  $\vec{x}, \vec{y} \in B$

(1)3. LET:  $\lambda \in (0, 1)$

(1)4.  $\lambda\vec{x} + (1 - \lambda)\vec{y} \in B$

PROOF:

$$\begin{aligned} \|\lambda\vec{x} + (1 - \lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1 - \lambda)(\vec{y} - \vec{a})\| \\ &= \lambda\|\vec{x} - \vec{a}\| + (1 - \lambda)\|\vec{y} - \vec{a}\| \\ &< \lambda r + (1 - \lambda)r \\ &= r \end{aligned}$$

□

□

## 11.2 Limit Points

**Definition 11.7** (Limit Point). Let  $X$  be a metric space. Let  $E \subseteq X$  and  $p \in X$ . Then  $p$  is a *limit point* of  $E$  iff every open ball with centre  $p$  contains a point of  $E$  other than  $p$ .

**Proposition 11.8.** Let  $X$  be a metric space. Let  $E \subseteq X$ . Let  $p$  be a limit point of  $E$ . Then every neighbourhood of  $p$  contains infinitely many points of  $E$ .

PROOF:

(1)1. ASSUME: for a contradiction  $N$  is a neighbourhood of  $p$  that contains only finitely many points  $q_1, \dots, q_n$  of  $E - \{p\}$ .

(1)2. LET:  $r = \min(q_1, \dots, q_n)$

(1)3. LET:  $B$  be the open ball with centre  $p$  and radius  $r$ .

(1)4.  $B$  is a neighbourhood of  $p$  that contains no points of  $E$  other than  $p$ .

□

**Corollary 11.8.1.** A finite set has no limit points.

**Definition 11.9** (Isolated Point). Let  $X$  be a metric space. Let  $E \subseteq X$  and  $p \in X$ . Then  $p$  is an *isolated point* of  $E$  iff  $p \in E$  and  $p$  is not a limit point of  $E$ .

## 11.3 Closed Sets

**Definition 11.10** (Closed Set). Let  $X$  be a metric space. Let  $E \subseteq X$ . Then  $E$  is *closed* iff every limit point of  $E$  is a member of  $E$ .

## 11.4 Interior Points

**Definition 11.11** (Interior Point). Let  $X$  be a metric space. Let  $E \subseteq X$  and  $p \in X$ . Then  $p$  is an *interior point* of  $E$  iff there exists an open ball  $B$  with centre  $p$  such that  $B \subseteq E$ .

**Definition 11.12** (Interior). The *interior* of a set  $E$ , denoted  $E^\circ$ , is the set of all its interior points.

**Proposition 11.13.** *The interior of  $E$  is the largest open set that is included in  $E$ .*

PROOF:

- (1)1. LET:  $I$  be the interior of  $E$ .
- (1)2.  $I$  is open.
  - (2)1. LET:  $p \in I$
  - (2)2. PICK an open ball  $B$  with centre  $p$  such that  $B \subseteq E$ .
  - (2)3.  $B \subseteq I$ 
    - (3)1. LET:  $q \in B$
    - (3)2. There exists an open ball  $B'$  with centre  $q$  such that  $B' \subseteq B$ .
    - (3)3. There exists an open ball  $B'$  with centre  $q$  such that  $B' \subseteq E$ .
    - (3)4.  $q \in I$
- (1)3. If  $J$  is any open set and  $J \subseteq E$  then  $J \subseteq I$ .
  - (2)1. LET:  $J$  be an open set.
  - (2)2. ASSUME:  $J \subseteq E$
  - (2)3. For all  $p \in J$ , there exists an open ball  $B$  with centre  $p$  such that  $B \subseteq J$ .
  - (2)4. For all  $p \in J$ , there exists an open ball  $B$  with centre  $p$  such that  $B \subseteq E$ .
  - (2)5.  $p \in I$

□

## 11.5 Open Sets

**Definition 11.14** (Open Sets). Let  $X$  be a metric space. Let  $E \subseteq X$ . Then  $E$  is *open* iff every point in  $E$  is an interior point of  $E$ .

**Proposition 11.15.** *Every open ball is open.*

PROOF:

- (1)1. LET:  $B$  be an open ball with centre  $c$  and radius  $r$ .
- (1)2. LET:  $x \in B$
- (1)3. LET:  $\epsilon = r - d(x, c)$
- (1)4. LET:  $B'$  be the open ball with centre  $x$  and radius  $\epsilon$ .
  - PROVE:  $B' \subseteq B$
- (1)5. LET:  $y \in B'$
- (1)6.  $d(y, c) < r$

PROOF:

$$\begin{aligned}
 d(y, c) &\leq d(y, x) + d(x, c) && \text{(Triangle Inequality)} \\
 &< \epsilon + d(x, c) && ((1)5) \\
 &= r && ((1)3)
 \end{aligned}$$

□

**Proposition 11.16.** *A set is open if and only if its complement is closed.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $E \subseteq X$

$\langle 1 \rangle 2$ . If  $E$  is open then  $X - E$  is closed.

$\langle 2 \rangle 1$ . ASSUME:  $E$  is open.

$\langle 2 \rangle 2$ . LET:  $p$  be a limit point of  $X - E$ .

PROVE:  $p \in X - E$

$\langle 2 \rangle 3$ . ASSUME: for a contradiction  $p \in E$ .

$\langle 2 \rangle 4$ . PICK an open ball  $B$  with centre  $p$  such that  $B \subseteq E$ .

$\langle 2 \rangle 5$ .  $B$  contains a point of  $X - E$ .

PROOF:  $\langle 2 \rangle 2$

$\langle 2 \rangle 6$ . Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 4$ .

$\langle 1 \rangle 3$ . If  $X - E$  is closed then  $E$  is open.

$\langle 2 \rangle 1$ . ASSUME:  $X - E$  is closed.

$\langle 2 \rangle 2$ . LET:  $p \in E$

$\langle 2 \rangle 3$ . ASSUME: for a contradiction no open ball with centre  $p$  is a subset of  $E$ .

$\langle 2 \rangle 4$ . Every open ball with centre  $p$  intersects  $X - E$ .

$\langle 2 \rangle 5$ .  $p$  is a limit point of  $X - E$ .

$\langle 2 \rangle 6$ .  $p \in X - E$

PROOF:  $\langle 2 \rangle 1$

$\langle 2 \rangle 7$ . Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 2$ .

□

**Corollary 11.16.1.** *A set is closed if and only if its complement is open.*

**Proposition 11.17.** *The union of a set of open sets is open.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{U}$  be a set of open sets.

$\langle 1 \rangle 2$ . LET:  $p \in \bigcup \mathcal{U}$

$\langle 1 \rangle 3$ . PICK  $U \in \mathcal{U}$  such that  $p \in U$ .

$\langle 1 \rangle 4$ . PICK an open ball  $B$  with centre  $p$  such that  $B \subseteq U$ .

$\langle 1 \rangle 5$ .  $B \subseteq \bigcup \mathcal{U}$

□

**Corollary 11.17.1.** *The intersection of a set of closed sets is closed.*

**Proposition 11.18.** *The intersection of two open sets is open.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $U$  and  $V$  be open.

$\langle 1 \rangle 2$ . LET:  $p \in U \cap V$

$\langle 1 \rangle 3$ . PICK open balls  $B_1$  and  $B_2$  with centre  $p$  such that  $B_1 \subseteq U$  and  $B_2 \subseteq V$ .

$\langle 1 \rangle 4$ . ASSUME: w.l.o.g. the radius of  $B_1$  is  $\leq$  the radius of  $B_2$ .

$\langle 1 \rangle 5$ .  $B_1 \subseteq U \cap V$

□

**Corollary 11.18.1.** *The union of two closed sets is closed.*

**Example 11.19.** The intersection of a set of open sets is not necessarily open.

For every positive integer  $n$ , we have  $(-1/n, 1/n)$  is open in  $\mathbb{R}$ , but  $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$  is not open.

**Theorem 11.20.** *Let  $X$  be a metric space. Let  $Y \subseteq X$  and  $E \subseteq Y$ . Then  $E$  is open in  $Y$  if and only if there exists an open subset  $G$  of  $X$  such that  $E = G \cap Y$ .*

PROOF:

(1)1. If  $E$  is open in  $Y$  then there exists an open subset  $G$  of  $X$  such that  $E = G \cap Y$ .

(2)1. ASSUME:  $E$  is open in  $Y$ .

(2)2. For  $p \in E$ , PICK  $r_p > 0$  such that the open ball in  $Y$  with centre  $p$  and radius  $r_p$  is included in  $E$ .

(2)3. For  $p \in E$ ,

LET:  $V_p$  be the open ball in  $X$  with centre  $p$  and radius  $r_p$ .

(2)4. LET:  $G = \bigcup_{p \in E} V_p$

(2)5.  $G$  is open in  $X$ .

PROOF: Proposition 11.17.

(2)6.  $E = G \cap Y$

(3)1.  $E \subseteq G \cap Y$

(4)1. LET:  $p \in E$

(4)2.  $p \in V_p$

(4)3.  $p \in G$

(3)2.  $G \cap Y \subseteq E$

(4)1. LET:  $x \in G \cap Y$

(4)2. PICK  $p \in E$  such that  $x \in V_p$

(4)3.  $d(x, p) < r_p$

(4)4.  $x \in E$

(1)2. For any open subset  $G$  of  $X$ , we have  $G \cap Y$  is open in  $Y$ .

(2)1. LET:  $G$  be an open subset of  $X$ .

(2)2. LET:  $p \in G \cap Y$

(2)3. PICK  $r > 0$  such that the open ball in  $X$  with centre  $p$  and radius  $r$  is included in  $G$ .

(2)4. The open ball in  $Y$  with centre  $p$  and radius  $r$  is included in  $G \cap Y$ .

□

## 11.6 Perfect Sets

**Definition 11.21** (Perfect Set). Let  $X$  be a metric space. Let  $E \subseteq X$ . Then  $E$  is *perfect* iff  $E$  is closed and every point in  $E$  is a limit point of  $E$ .

## 11.7 Bounded Sets

**Definition 11.22** (Bounded Set). Let  $X$  be a metric space. Let  $E \subseteq X$ . Then  $E$  is *bounded* iff there exists a real number  $M$  and  $q \in X$  such that, for all  $p \in E$ , we have  $d(p, q) < M$ .

**Definition 11.23** (Diameter). Let  $X$  be a metric space and  $E \subseteq X$  be bounded. Then the *diameter* of  $E$  is  $\sup\{d(x, y) : x, y \in E\}$ .

**Proposition 11.24.** Let  $X$  be a metric space. Let  $E \subseteq X$  be bounded. Then  $\overline{E}$  is bounded and

$$\text{diam } \overline{E} = \text{diam } E .$$

PROOF:

(1)1.  $\text{diam } E$  is an upper bound for  $\{d(x, y) : x, y \in \overline{E}\}$ .

⟨2⟩1. LET:  $x, y \in \overline{E}$

⟨2⟩2. For all  $\epsilon > 0$  we have  $d(x, y) < \text{diam } E + \epsilon$ .

⟨3⟩1. LET:  $\epsilon > 0$

⟨3⟩2. PICK  $x', y' \in E$  such that  $d(x', x) < \epsilon/2$  and  $d(y', y) < \epsilon/2$

⟨3⟩3.  $d(x', y') < \text{diam } E$

⟨3⟩4.  $d(x, y) < \text{diam } E + \epsilon$

⟨2⟩3.  $d(x, y) \leq \text{diam } E$

(1)2.  $\text{diam } \overline{E}$  is an upper bound for  $\{d(x, y) : x, y \in E\}$ .

PROOF: This follows since  $E \subseteq \overline{E}$ .

□

## 11.8 Dense Sets

**Definition 11.25** (Dense Set). Let  $X$  be a metric space. Let  $E \subseteq X$ . Then  $E$  is *dense* iff every point of  $X$  is either a limit point of  $E$  or a point of  $E$ , or both.

## 11.9 Closure

**Definition 11.26** (Closure). Let  $X$  be a metric space. Let  $E \subseteq X$ . Then the *closure* of  $E$ , denoted  $\overline{E}$ , is the union of  $E$  and the set of limit points of  $E$ .

**Proposition 11.27.**  $\overline{E}$  is the smallest closed set that includes  $E$ .

PROOF:

(1)1.  $\overline{E}$  is closed.

⟨2⟩1. LET:  $p$  be a limit point of  $\overline{E}$ .

⟨2⟩2. ASSUME:  $p \notin E$

PROVE:  $p$  is a limit point of  $E$ .

⟨2⟩3. LET:  $B$  be the open ball with centre  $p$  and radius  $r$ .

PROVE:  $B$  intersects  $E$ .

⟨2⟩4. PICK a point  $q \in B \cap \overline{E}$ .

⟨2⟩5. PICK an open ball  $B'$  with centre  $q$  such that  $B' \subseteq B$ .

- ⟨2⟩6. PICK a point  $r \in E \cap B'$
- ⟨2⟩7.  $r \in E \cap B$
- ⟨1⟩2. If  $C$  is closed and  $E \subseteq C$  then  $\overline{E} \subseteq C$ .
  - ⟨2⟩1. ASSUME:  $C$  is closed.
  - ⟨2⟩2. ASSUME:  $E \subseteq C$
  - ⟨2⟩3. LET:  $p \in \overline{E}$
  - ⟨2⟩4. ASSUME: for a contradiction  $p \notin C$
  - ⟨2⟩5.  $p$  is a limit point of  $C$ .
    - ⟨3⟩1. LET:  $B$  be an open ball with centre  $p$ .
    - ⟨3⟩2.  $B$  intersects  $E$ .
    - ⟨3⟩3.  $B$  intersects  $C$ .
    - ⟨3⟩4.  $B$  intersects  $C$  in a point other than  $p$ .
- PROOF: ⟨2⟩3
- ⟨2⟩6. Q.E.D.
- PROOF: This contradicts ⟨2⟩1.

□

**Corollary 11.27.1.**  $E$  is closed if and only if  $E = \overline{E}$ .

**Theorem 11.28.** Let  $E$  be a nonempty set of real numbers bounded above. Then  $\sup E \in \overline{E}$ .

PROOF:

- ⟨1⟩1. ASSUME:  $\sup E \notin E$   
PROVE:  $\sup E$  is a limit point of  $E$ .
- ⟨1⟩2. LET:  $B$  be an open ball with centre  $\sup E$  and radius  $r$ .
- ⟨1⟩3. There exists  $x \in E$  such that  $x > \sup E - r$ .
- ⟨1⟩4.  $E$  intersects  $B$  in a point other than  $p$ .

□

**Proposition 11.29.**

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:

- ⟨1⟩1.  $\overline{A \cup B}$  is a closed set that includes  $A \cup B$ .
- ⟨1⟩2. If  $C$  is a closed set that includes  $A \cup B$  then  $\overline{A \cup B} \subseteq C$ .

□

**Example 11.30.** It is not true in general. that  $\overline{\bigcup \mathcal{A}} = \bigcup_{A \in \mathcal{A}} \overline{A}$ .

In  $\mathbb{R}$ , let  $\mathcal{A} = \{\{1/n\} : n \in \mathbb{Z}^+\}$ . Then

$$\begin{aligned} \overline{\bigcup \mathcal{A}} &= \{1/n : n \in \mathbb{Z}^+\} \cup \{0\} \\ \bigcup_{A \in \mathcal{A}} \overline{A} &= \{1/n : n \in \mathbb{Z}^+\} \end{aligned}$$

**Proposition 11.31.**

$$X - E^\circ = \overline{X - E}$$

PROOF:

$$\begin{aligned}
 p \in X - E^\circ &\Leftrightarrow p \notin E^\circ \\
 &\Leftrightarrow \forall B \text{ an open ball with centre } p, B \not\subseteq E \\
 &\Leftrightarrow \forall B \text{ an open ball with centre } p, B \text{ intersects } X - E \\
 &\Leftrightarrow p \in \overline{X - E} \quad \square
 \end{aligned}$$

## 11.10 Compact Sets

**Definition 11.32** (Open Cover). Let  $X$  be a metric space. Let  $E \subseteq X$ . An *open cover* of  $E$  is a set  $\mathcal{U}$  of open sets such that  $E \subseteq \bigcup \mathcal{U}$ .

**Definition 11.33** (Compact Set). Let  $X$  be a metric space. Let  $K \subseteq X$ . Then  $K$  is *compact* iff every open cover of  $K$  includes a finite subcover.

**Proposition 11.34.** *Every finite set is compact.*

PROOF: Easy.  $\square$

**Theorem 11.35.** *Let  $X$  be a metric space. Let  $Y \subseteq X$  and  $K \subseteq Y$ . Then  $K$  is compact in  $Y$  if and only if  $K$  is compact in  $X$ .*

PROOF:

- (1)1. If  $K$  is compact in  $Y$  then  $K$  is compact in  $X$ .
  - (2)1. ASSUME:  $K$  is compact in  $Y$ .
  - (2)2. LET:  $\mathcal{U}$  be an open cover of  $K$  in  $X$ .
  - (2)3.  $\{U \cap Y : U \in \mathcal{U}\}$  is an open cover of  $K$  in  $Y$ .
  - (2)4. PICK a finite subcover  $\{U_1 \cap Y, \dots, U_n \cap Y\}$
  - (2)5.  $\{U_1, \dots, U_n\}$  is a finite subset of  $\mathcal{U}$  that is an open cover of  $K$  in  $X$ .
- (1)2. If  $K$  is compact in  $X$  then  $K$  is compact in  $Y$ .
  - (2)1. ASSUME:  $K$  is compact in  $X$ .
  - (2)2. LET:  $\mathcal{U}$  be an open cover of  $K$  in  $Y$ .
  - (2)3.  $\{U \text{ open in } X : U \cap Y \in \mathcal{U}\}$  is an open cover of  $K$  in  $X$ .
  - (2)4. PICK a finite subcover  $\{U_1, \dots, U_n\}$ .
  - (2)5.  $\{U_1 \cap Y, \dots, U_n \cap Y\}$  is a subset of  $\mathcal{U}$  that is an open cover of  $E$  in  $Y$ .

$\square$

**Proposition 11.36.** *Every compact set is closed.*

PROOF:

- (1)1. LET:  $E$  be compact.
- (1)2. LET:  $p \in X - E$ 
  - PROVE: There exists an open ball with centre  $p$  that is a subset of  $X - E$ .
- (1)3. For all  $q \in E$ , there exist disjoint open balls  $B$  with centre  $q$  and  $B'$  with centre  $p$ .
- (1)4. The set of open balls  $B$  such that there exists a disjoint open ball  $B'$  with centre  $p$  is an open cover of  $E$ .
- (1)5. PICK a finite subcover  $\{B_1, \dots, B_n\}$ .



- (1)6. For  $i = 1, \dots, n$ , PICK an open ball  $B'_i$  with centre  $p$  such that  $B_i \cap B'_i = \emptyset$ .  
 (1)7.  $B'_1 \cap \dots \cap B'_n$  is an open ball with centre  $p$  that is a subset of  $X - E$ .  
 $\square$

**Proposition 11.37.** *Every closed subset of a compact set is compact.*

PROOF:

- (1)1. LET:  $E$  be compact and  $C \subseteq E$  be closed.  
 (1)2. LET:  $\mathcal{U}$  be an open cover of  $C$ .  
 (1)3.  $\mathcal{U} \cup \{X - C\}$  is an open cover of  $E$ .  
 (1)4. PICK a finite subcover  $\{U_1, \dots, U_n\}$  or  $\{U_1, \dots, U_n, X - C\}$ .  
 (1)5.  $\{U_1, \dots, U_n\}$  covers  $C$ .  
 $\square$

**Corollary 11.37.1.** *The intersection of a compact set and a closed set is compact.*

**Proposition 11.38.** *Let  $\mathcal{K}$  be a nonempty set of compact sets. If every nonempty finite subset of  $\mathcal{K}$  has nonempty intersection, then  $\bigcap \mathcal{K}$  is nonempty.*

PROOF:

- (1)1. PICK  $K \in \mathcal{K}$   
 (1)2. ASSUME:  $\bigcap \mathcal{K} = \emptyset$   
 (1)3.  $\{X - K' : K' \in \mathcal{K}\}$  is an open cover of  $K$ .  
 (1)4. PICK a finite subcover  $\{X - K_1, \dots, X - K_n\}$ .  
 (1)5. There exists  $p \in K \cap K_1 \cap \dots \cap K_n$   
 (1)6. Q.E.D.

PROOF: (1)4 and (1)5 form a contradiction.

$\square$

**Corollary 11.38.1.** *Let  $(K_n)$  be a sequence of nonempty compact sets such that  $K_0 \supseteq K_1 \supseteq \dots$ . Then  $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$ .*

**Theorem 11.39.** *Let  $X$  be a metric space and  $E \subseteq X$ . Then  $E$  is compact if and only if every infinite subset of  $E$  has a limit point in  $E$ .*

PROOF:

- (1)1. If  $E$  is compact then every infinite subset of  $E$  has a limit point in  $E$ .  
 (2)1. ASSUME:  $E$  is compact.  
 (2)2. LET:  $A \subseteq E$  be infinite.  
 (2)3. ASSUME: for a contradiction  $E$  has no limit point in  $K$ .  
 (2)4. For all  $p \in K$ , there exists an open ball  $B$  with centre  $p$  such that  $B$  does not intersect  $E$  outside  $p$ .  
 (2)5. The set of open balls that intersect  $E$  in at most one point is an open cover for  $K$ .  
 (2)6. PICK a finite subcover  $B_1, \dots, B_n$ .  
 (2)7.  $E$  has at most  $n$  points.  
 (2)8. Q.E.D.

PROOF: This contradicts the fact that  $E$  is finite.

- (1)2. If every infinite subset of  $K$  has a limit point in  $K$  then  $K$  is compact.
- (2)1. ASSUME: Every infinite subset of  $K$  has a limit point in  $K$ .
- (2)2. LET:  $\mathcal{U}$  be an open cover of  $K$ .
- (2)3. ASSUME: w.l.o.g.  $\mathcal{U}$  is countable.
- PROOF: We may replace  $\mathcal{U}$  with the set of all open balls  $B$  with centres in  $\mathbb{Q}^2$  and rational radius such that there exists  $U \in \mathcal{U}$  such that  $B \subseteq U$ .
- (2)4. PICK an enumeration  $\mathcal{U} = \{G_n : n \in \mathbb{N}\}$ .
- (2)5. For  $n \in \mathbb{N}$ ,  
 LET:  $F_n = \bigcup_{i=0}^n G_i$ .
- (2)6. For all  $n \in \mathbb{N}$ , we have  $K - F_n \neq \emptyset$ .
- PROOF: Since  $\{G_0, \dots, G_n\}$  does not cover  $K$ .
- (2)7.  $\bigcap_{n=0}^{\infty} F_n = \emptyset$
- PROOF: Since  $\{G_n : n \in \mathbb{N}\}$  covers  $K$ .
- (2)8. For  $n \in \mathbb{N}$ , PICK  $a_n \in K - F_n$
- (2)9. LET:  $E = \{a_n : n \in \mathbb{N}\}$
- (2)10.  $E$  is infinite.
- (3)1. LET:  $n \in \mathbb{N}$   
 PROVE: there exists  $m$  such that  $a_m \notin \{a_0, a_1, \dots, a_n\}$ .
- (3)2. For  $i = 0, \dots, n$ , PICK  $k_i$  such that  $a_i \in G_{k_i}$ .
- (3)3. LET:  $m = \max(k_0, \dots, k_n)$
- (3)4. ASSUME: for a contradiction  $a_m = a_i$  for some  $i = 0, \dots, n$
- (3)5.  $a_i \in G_{k_i}$
- (3)6.  $a_i \notin F_m$
- (3)7. Q.E.D.
- PROOF: This is a contradiction since  $k_i \leq m$ .
- (2)11. PICK a limit point  $l$  for  $E$  in  $K$ .
- PROOF: From (2)1.
- (2)12. PICK  $n$  such that  $l \in G_n$ .
- (2)13. PICK an open ball  $B$  with centre  $l$  such that  $B \subseteq G_n$
- (2)14.  $B \cap E$  is infinite.
- PROOF: Proposition 11.8.
- (2)15. PICK  $m \geq n$  such that  $a_m \in B$ .
- (2)16.  $a_m \in G_n$
- (2)17. Q.E.D.
- PROOF: This is a contradiction since  $a_m \notin F_m$ .

□

**Theorem 11.40** (Heine-Borel). *Let  $E \subseteq \mathbb{R}^k$ . Then  $E$  is compact if and only if it is closed and bounded.*

PROOF:

- (1)1. If  $E$  is compact then  $E$  is closed.  
 PROOF: Proposition 11.36.
- (1)2. If  $E$  is compact then  $E$  is bounded.  
 PROOF: Otherwise  $\{(-N, N)^k : N \in \mathbb{Z}^+\}$  would be an open cover of  $E$  with no finite subcover.
- (1)3. If  $E$  is closed and bounded then  $E$  is compact.

- $\langle 2 \rangle 1$ . ASSUME:  $E$  is closed and bounded.
- $\langle 2 \rangle 2$ . PICK  $\vec{c}$  and  $M$  such that  $\forall \vec{x} \in E, \|\vec{x} - \vec{c}\| < M$ .
- $\langle 2 \rangle 3$ .  $E \subseteq \prod_{i=1}^k [c_i - M, c_i + M]$
- $\langle 2 \rangle 4$ .  $E$  is compact.

PROOF: Proposition 11.37.

□

**Corollary 11.40.1** (Weierstrass's Theorem). *Every bounded infinite subset of  $\mathbb{R}^k$  has a limit point.*

PROOF: It is a bounded infinite subset of some  $k$ -cell and therefore has a limit point in that  $k$ -cell. □

**Example 11.41.** It is not true that, in any metric space, a set is compact if and only if it is closed and bounded.

In  $\mathbb{Q}$ , the set  $\{p \in \mathbb{Q} : 2 < p^2 < 3\}$  is closed and bounded but not compact.

**Theorem 11.42.** *Every nonempty perfect set in  $\mathbb{R}^k$  is uncountable.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $P$  be a nonempty perfect set in  $\mathbb{R}^k$ .
- $\langle 1 \rangle 2$ .  $P$  is infinite.

PROOF: Corollary 11.8.1.

- $\langle 1 \rangle 3$ . ASSUME: for a contradiction  $P$  is countable.
- $\langle 1 \rangle 4$ . PICK an enumeration  $P = \{x_n : n \in \mathbb{N}\}$ .
- $\langle 1 \rangle 5$ . PICK a sequence  $(V_n)$  of open balls such that, for all  $n$ , we have  $\overline{V_{n+1}} \subseteq V_n$  and  $x_n \notin \overline{V_{n+1}}$  and  $V_n \cap P \neq \emptyset$ 
  - $\langle 2 \rangle 1$ . ASSUME: as induction hypothesis we have picked  $V_0, \dots, V_{n-1}$  that satisfy these conditions.
  - $\langle 2 \rangle 2$ . PICK  $p \in P \cap V_n$  such that  $p \neq x_n$ 

PROOF: We cannot have  $P \cap V_n = \{x_n\}$  because then  $V_n$  would be a neighbourhood of  $x_n$  that only intersects  $P$  at  $x_n$ .
  - $\langle 2 \rangle 3$ . PICK an open ball  $B$  with centre  $p$  such that  $B \subseteq V_n \cap P - \{x_n\}$
  - $\langle 2 \rangle 4$ . LET:  $V_{n+1}$  be the open ball with centre  $p$  and half the radius of  $B$ .
  - $\langle 2 \rangle 5$ .  $\overline{V_{n+1}} \subseteq V_n$ 

PROOF: Since  $\overline{V_{n+1}} \subseteq B \subseteq V_n$ .
  - $\langle 2 \rangle 6$ .  $x_n \notin \overline{V_{n+1}}$ 

PROOF: Since  $\overline{V_{n+1}} \subseteq B \subseteq P - \{x_n\}$ .
  - $\langle 2 \rangle 7$ .  $V_{n+1} \cap P \neq \emptyset$ 

PROOF: Since  $p \in V_{n+1} \cap P$ .
- $\langle 1 \rangle 6$ . For  $n \in \mathbb{N}$ ,
 

LET:  $K_n = \overline{V_n} \cap P$ .
- $\langle 1 \rangle 7$ . For all  $n \in \mathbb{N}$ ,  $K_n$  is compact.
 

PROOF: By the Heine-Borel Theorem.
- $\langle 1 \rangle 8$ .  $\bigcap_{n=0}^{\infty} K_n \cap P = \emptyset$ 

PROOF: Since for each  $n$  we have  $x_n \notin K_{n+1}$ .
- $\langle 1 \rangle 9$ .  $\bigcap_{n=0}^{\infty} K_n = \emptyset$ 

PROOF: Since  $\bigcap_{n=0}^{\infty} K_n \subseteq P$ .

(1)10. Q.E.D.

PROOF: This contradicts Proposition 11.38.

□

**Corollary 11.42.1.** *For any  $a, b \in \mathbb{R}$  with  $a < b$ , the closed interval  $[a, b]$  is uncountable.*

**Corollary 11.42.2.**  *$\mathbb{R}$  is uncountable.*

**Corollary 11.42.3.** *The set of transcendental numbers is uncountable.*

PROOF: Since the set of algebraic numbers is countable. □

**Example 11.43.** The Cantor set is a perfect set in  $\mathbb{R}$  that does not include any open interval.

PROOF:

(1)1. LET:  $(E_n)$  be the sequence of unions of closed intervals from the definition of the Cantor set, and  $C$  be the Cantor set.

(1)2.  $C \neq \emptyset$

PROOF: Since  $0 \in C$ .

(1)3.  $C$  is closed.

PROOF: Each  $E_n$  is closed and  $C$  is their intersection.

(1)4. Every point of  $C$  is a limit point of  $C$ .

(2)1. LET:  $p \in C$

(2)2. LET:  $B$  be an open ball with centre  $p$  and radius  $r$ .

(2)3. PICK  $n$  such that each of the intervals that make up  $E_n$  has length  $< r/2$ .

(2)4. LET:  $I$  be the interval in  $E_n$  that contains  $p$ .

(2)5.  $I \subseteq B$

(2)6. The endpoint of  $I$  that is not  $p$  is in  $P \cap B$ .

(1)5.  $C$  does not include any open interval.

(2)1. LET:  $(\alpha, \beta)$  be any open interval.

(2)2. PICK  $m$  such that  $3^{-m} < (\beta - \alpha)/6$

(2)3. PICK  $k$  such that  $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}) \subseteq (\alpha, \beta)$

(2)4.  $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}) \subseteq P$

(2)5.  $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}) \cap E_m = \emptyset$

(2)6. Q.E.D.

PROOF: This is a contradiction.

□

**Corollary 11.43.1.** *The Cantor set is uncountable.*

**Proposition 11.44.** *Let  $X$  be a metric space. Let  $(K_n)$  be a sequence of compact sets in  $X$  such that  $K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$ . Assume  $\text{diam } K_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\bigcap_{n=0}^{\infty} K_n$  is a singleton.*

PROOF:

(1)1.  $\bigcap_n K_n \neq \emptyset$

PROOF: Corollary 11.38.1.

- (1)2.  $\bigcap_n K_n$  has no more than one point.
- (2)1. ASSUME: for a contradiction  $a, b \in \bigcap_n K_n$  with  $a \neq b$ .
- (2)2. LET:  $\epsilon = d(a, b)$
- (2)3. PICK  $n$  such that  $\text{diam } K_n < \epsilon$
- (2)4.  $a, b \in K_n$
- (2)5. Q.E.D.

PROOF: This is a contradiction.

□

## 11.11 Connected Sets

**Definition 11.45** (Separated). Let  $X$  be a metric space. Let  $A, B \subseteq X$ . Then  $A$  and  $B$  are *separated* iff  $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ .

**Proposition 11.46.** *Any two disjoint open sets are separated.*

PROOF:

- (1)1. LET:  $A$  and  $B$  be disjoint open sets.
- (1)2. ASSUME: for a contradiction  $p \in \overline{A} \cap B$ .
- (1)3.  $B$  is a neighbourhood of  $p$ .
- (1)4.  $B$  intersects  $A$ .

□

**Definition 11.47** (Connected). Let  $X$  be a metric space. Let  $E \subseteq X$ . Then  $E$  is *connected* iff  $E$  is not the union of two nonempty separated sets.

**Theorem 11.48.** *A subset  $E$  of the real line is connected if and only if it is convex.*

PROOF:

- (1)1. If  $E$  is connected then  $E$  is convex.
  - (2)1. ASSUME:  $E$  is connected.
  - (2)2. LET:  $x, y \in E$
  - (2)3. LET:  $z \in (x, y)$
  - (2)4.  $z \in E$ 

PROOF: Otherwise  $E \cap (-\infty, z)$  and  $E \cap (z, +\infty)$  would be a separation of  $E$ .
- (1)2. If  $E$  is convex then  $E$  is connected.
  - (2)1. ASSUME:  $E$  is convex.
  - (2)2. ASSUME: for a contradiction  $E = A \cup B$  where  $A$  and  $B$  are nonempty and separated.
  - (2)3. PICK  $a \in A$  and  $b \in B$ .
  - (2)4. ASSUME: w.l.o.g.  $a < b$
  - (2)5. LET:  $z = \sup(A \cap [a, b])$
  - (2)6.  $z \in \overline{A}$
  - (2)7.  $z \notin B$

$\langle 2 \rangle 8. z < b$

$\langle 2 \rangle 9.$  CASE:  $z \in A$

$\langle 3 \rangle 1. z \notin \overline{B}$

$\langle 3 \rangle 2.$  PICK  $z_1 \in (z, b)$  such that  $z_1 \notin B$

$\langle 3 \rangle 3. a < z_1 < b$

$\langle 3 \rangle 4. z_1 \notin E$

PROOF: We have  $z_1 \notin A$  from  $\langle 2 \rangle 5$  since  $z_1 \in [a, b]$  and  $z_1 > z$ , and  $z_1 \notin B$  from  $\langle 3 \rangle 2$ .

$\langle 3 \rangle 5.$  Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 10.$  CASE:  $z \notin A$

PROOF: Then  $a < z < b$  and  $z \notin E$  contradicting  $\langle 2 \rangle 1$ .

□

**Proposition 11.49.** *Every connected metric space with more than one point is uncountable.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a connected metric space with more than one points.

$\langle 1 \rangle 2.$  PICK distinct points  $p, q \in X$ .

$\langle 1 \rangle 3.$  LET:  $\epsilon = d(p, q)$

$\langle 1 \rangle 4.$  For every  $r \in (0, \epsilon)$ , there exists a point  $x \in X$  such that  $d(p, x) = r$ .

PROOF: Otherwise  $\{x \in X : d(p, x) < r\}$  and  $\{x \in X : d(p, x) > r\}$  would form a separation of  $X$ .

□

**Proposition 11.50.** *The closure of a connected set is connected.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a metric space.

$\langle 1 \rangle 2.$  LET:  $E$  be a connected subspace of  $X$ .

$\langle 1 \rangle 3.$  ASSUME: for a contradiction  $A$  and  $B$  form a separation of  $\overline{E}$

PROVE:  $A \cap E$  and  $B \cap E$  form a separation of  $E$ .

$\langle 1 \rangle 4. A \cap E \neq \emptyset$

$\langle 2 \rangle 1.$  ASSUME: for a contradiction  $A \cap E = \emptyset$

$\langle 2 \rangle 2. E \subseteq B$

$\langle 2 \rangle 3. \overline{E} \subseteq \overline{B}$

$\langle 2 \rangle 4. A \subseteq \overline{B}$

$\langle 2 \rangle 5. A \cap \overline{B} = A \neq \emptyset$

$\langle 2 \rangle 6.$  Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 3$ .

$\langle 1 \rangle 5. B \cap E \neq \emptyset$

PROOF: Similar.

$\langle 1 \rangle 6. \overline{A \cap E} \cap B \cap E = \emptyset$

PROOF: Since  $\overline{A \cap E} \cap B \cap E \subseteq \overline{A} \cap B$ .

$\langle 1 \rangle 7. A \cap E \cap \overline{B \cap E} = \emptyset$

PROOF: Similar.

□

**Example 11.51.** The interior of a connected set is not necessarily connected.

Two touching discs in  $\mathbb{R}^2$  form a connected set but the interior is disconnected.

**Proposition 11.52.** *Every convex set in  $\mathbb{R}^k$  is connected.*

PROOF:

⟨1⟩1. LET:  $E$  be a convex set in  $\mathbb{R}^k$ .

⟨1⟩2. ASSUME: for a contradiction  $A$  and  $B$  form a separation of  $E$ .

⟨1⟩3. PICK  $\vec{a} \in A$  and  $\vec{b} \in B$ .

⟨1⟩4. Define  $p : [0, 1] \rightarrow \mathbb{R}^k$  by  $p(t) = (1 - t)\vec{a} + t\vec{b}$ .

⟨1⟩5.  $p^{-1}(A)$  and  $p^{-1}(B)$  are separated sets in  $\mathbb{R}$ .

⟨1⟩6. PICK  $x \in [0, 1]$  such that  $x \notin p^{-1}(A)$  and  $x \notin p^{-1}(B)$ .

PROOF: There exists such an  $x$  since  $[0, 1]$  is connected.

⟨1⟩7.  $p(x) \in E$

PROOF: Since  $E$  is convex.

⟨1⟩8.  $p(x) \notin A \cup B$

⟨1⟩9. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

## 11.12 Separable Spaces

**Definition 11.53** (Separable). A metric space is *separable* iff it has a countable dense subset.

**Example 11.54.**  $\mathbb{R}^k$  is separable since  $\mathbb{Q}^k$  is dense.

**Proposition 11.55.** *Every compact metric space is separable.*

PROOF:

⟨1⟩1. LET:  $X$  be a compact metric space.

⟨1⟩2. For  $n \in \mathbb{Z}^+$ , pick finitely many points  $a_{n1}, \dots, a_{nr_n}$  such that  $\{B(a_{ni}, 1/n) : 1 \leq i \leq r_n\}$  covers  $X$ .

PROOF: Since  $\{B(x, 1/n) : x \in X\}$  covers  $X$ .

⟨1⟩3.  $\{a_{ni} : n \in \mathbb{Z}^+, 1 \leq i \leq r_n\}$  is dense.

⟨2⟩1. LET:  $U$  be an open set and  $p \in U$ .

⟨2⟩2. PICK  $\epsilon > 0$  such that  $B(p, \epsilon) \subseteq U$ .

⟨2⟩3. PICK  $n$  such that  $1/n < \epsilon$ .

⟨2⟩4. PICK  $i$  such that  $p \in B(a_{ni}, 1/n)$

⟨2⟩5.  $a_{ni} \in U$

□

### 11.13 Bases

**Definition 11.56** (Basis). A *basis* for a metric space  $X$  is a set  $\mathcal{B}$  of open sets such that, for every open set  $U$  and point  $p \in U$ , there exists  $B \in \mathcal{B}$  such that  $p \in B \subseteq U$ .

**Proposition 11.57.** *Every separable metric space has a countable basis.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X$  be a separable metric space.
- $\langle 1 \rangle 2$ . PICK a countable dense set  $D$  in  $X$ .
- $\langle 1 \rangle 3$ . LET:  $\mathcal{B} = \{B(p, \epsilon) : p \in D, \epsilon \in \mathbb{Q}^+\}$
- PROVE:  $\mathcal{B}$  is a basis.
- $\langle 1 \rangle 4$ . LET:  $U$  be an open set in  $X$  and  $p \in U$
- $\langle 1 \rangle 5$ . PICK  $\epsilon > 0$  such that  $B(p, \epsilon) \subseteq U$
- $\langle 1 \rangle 6$ . PICK  $q \in B(p, \epsilon) \cap D$
- $\langle 1 \rangle 7$ . PICK a rational  $\delta$  such that  $d(p, q) < \delta < \epsilon$ .
- $\langle 1 \rangle 8$ .  $B(q, \delta) \in \mathcal{B}$  and  $B(q, \delta) \subseteq U$ .

□

### 11.14 Condensation Points

**Definition 11.58** (Condensation Point). Let  $X$  be a metric space,  $p \in X$  and  $E \subseteq X$ . Then  $p$  is a *condensation point* of  $E$  iff every neighbourhood of  $p$  contains uncountably many points in  $E$ .

**Proposition 11.59.** *Let  $X$  be a metric space. Let  $E \subseteq X$ . Let  $P$  be the set of condensation points of  $E$ . Then  $P$  is perfect.*

PROOF:

- $\langle 1 \rangle 1$ .  $P$  is closed.
- $\langle 2 \rangle 1$ . LET:  $p \in X - P$
- $\langle 2 \rangle 2$ . PICK a neighbourhood  $U$  of  $p$  that contains only countably many points of  $E$ .
- $\langle 2 \rangle 3$ . For every  $x \in U$ , we have that  $U$  is a neighbourhood of  $x$  that contains only countably many points of  $E$ .
- $\langle 2 \rangle 4$ .  $p \in U \subseteq X - P$
- $\langle 1 \rangle 2$ . Every point in  $P$  is a limit point of  $P$ .

PROOF: Immediate from definitions.

□

**Proposition 11.60.** *Let  $X$  be a metric space with a countable basis. Let  $E \subseteq X$  be uncountable. Let  $P$  be the set of condensation points of  $E$ . Then  $E - P$  is countable.*

PROOF:

- $\langle 1 \rangle 1$ . PICK a countable basis  $\mathcal{B}$  for  $X$ .
- $\langle 1 \rangle 2$ . LET:  $W = \bigcup \{B \in \mathcal{B} : E \cap B \text{ is countable}\}$



- ⟨1⟩3.  $P = X - W$
- ⟨2⟩1.  $P \subseteq X - W$ 
  - ⟨3⟩1. ASSUME: for a contradiction  $p \in P \cap W$
  - ⟨3⟩2. PICK  $B \in \mathcal{B}$  such that  $p \in B$  and  $E \cap B$  is countable.
  - ⟨3⟩3.  $E \cap B$  is uncountable.
  - ⟨3⟩4. Q.E.D.
- PROOF: This is a contradiction.
- ⟨2⟩2.  $X - W \subseteq P$ 
  - ⟨3⟩1. LET:  $p \in X - W$
  - ⟨3⟩2. LET:  $U$  be a neighbourhood of  $p$ .
  - ⟨3⟩3. PICK  $B \in \mathcal{B}$  such that  $p \in B \subseteq U$ .
  - ⟨3⟩4.  $E \cap B$  is uncountable.
  - PROOF: Since  $p \notin W$ .
  - ⟨3⟩5.  $E \cap W$  is uncountable.
- ⟨1⟩4.  $E - P = E \cap W$
- ⟨1⟩5.  $E - P$  is countable.

□

**Corollary 11.60.1.** *Every closed subset of a metric space with a countable basis is the union of a perfect set and a countable set.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a metric space with a countable basis.
- ⟨1⟩2. LET:  $E$  be a closed subset of  $X$ .
- ⟨1⟩3. LET:  $P$  be the set of condensation points of  $E$ .
- ⟨1⟩4.  $E - P$  is countable.

PROOF: Proposition 11.60.

- ⟨1⟩5.  $P \cap E$  is perfect.

- ⟨2⟩1.  $P \cap E$  is closed.

PROOF: Proposition 11.59.

- ⟨2⟩2. Every point in  $P \cap E$  is a limit point of  $P \cap E$ .

- ⟨3⟩1. LET:  $l \in P \cap E$
- ⟨3⟩2. LET:  $U$  be a neighbourhood of  $l$ .
- ⟨3⟩3. PICK  $x \in P \cap U$
- ⟨3⟩4.  $U$  is a neighbourhood of  $x$ .
- ⟨3⟩5.  $U$  contains uncountably many points of  $E$ .
- ⟨3⟩6.  $U$  intersects  $P \cap E$

PROOF: It cannot be that every point in  $U$  and  $E$  is not in  $P$  since  $E - P$  is countable.

□

**Corollary 11.60.2.** *Let  $X$  be a metric space with a countable basis. Then every countable set in  $X$  has an isolated point.*



## Chapter 12

# Convergence

**Definition 12.1** (Converge). Let  $X$  be a metric space. Let  $(p_n)$  be a sequence in  $X$  and  $l \in X$ . Then we say  $(p_n)$  *converges* to the *limit*  $l$ , and write

$$p_n \rightarrow l \text{ as } n \rightarrow \infty ,$$

iff for every  $\epsilon > 0$ , there exists an integer  $N$  such that, for all  $n \geq N$ , we have  $d(p_n, l) < \epsilon$ .

We say  $(p_n)$  *diverges* iff it does not converge to any limit.

**Proposition 12.2.** *A sequence has at most one limit.*

PROOF:

(1)1. ASSUME:  $p_n \rightarrow l$  and  $p_n \rightarrow m$  as  $n \rightarrow \infty$ .

(1)2. ASSUME: for a contradiction  $l \neq m$ .

(1)3. LET:  $\epsilon = d(l, m)/2$

(1)4. There exists  $N$  such that  $\forall n \geq N. d(p_n, l) < \epsilon$  and  $d(p_n, m) < \epsilon$

(1)5.  $d(l, m) < 2\epsilon$

(1)6. Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 12.3.** *Every convergent sequence is bounded.*

PROOF:

(1)1. LET:  $p_n \rightarrow l$  as  $n \rightarrow \infty$

(1)2. PICK  $N$  such that  $\forall n \geq N. d(p_n, l) < 1$

(1)3. LET:  $M = \max(d(p_0, l), \dots, d(p_{N-1}, l), 1)$

(1)4. For all  $n$ , we have  $d(p_n, l) \leq M$ .

□

**Proposition 12.4.** *If  $l$  is a limit point of  $E$ , then there exists a sequence in  $E$  that converges to  $l$ .*

PROOF:

$\langle 1 \rangle 1$ . For  $n \in \mathbb{Z}^+$ , PICK a point  $a_n \in E$  such that  $d(a_n, l) < 1/n$ .

PROOF: Since  $B(l, 1/n)$  intersects  $E$ .

$\langle 1 \rangle 2$ .  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .

□

**Corollary 12.4.1.** *Every sequence in a compact metric space has a convergent subsequence.*

PROOF: By Theorem 11.39. □

**Proposition 12.5.** *Assume  $s_n \rightarrow s$  and  $t_n \rightarrow t$  in  $\mathbb{R}^k$ . Then  $s_n + t_n \rightarrow s + t$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 2$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $\|s_n - s\| < \epsilon/2$  and  $\|t_n - t\| < \epsilon/2$ .

$\langle 1 \rangle 3$ . For all  $n \geq N$  we have  $\|(s_n + t_n) - (s + t)\| < \epsilon$ .

PROOF: Since  $\|(s_n + t_n) - (s + t)\| \leq \|s_n - s\| + \|t_n - t\|$ .

□

**Lemma 12.6.** *If  $s_n \rightarrow s$  as  $n \rightarrow \infty$  in  $\mathbb{C}$ , and  $c \in \mathbb{C}$ , then  $cs_n \rightarrow cs$  as  $n \rightarrow \infty$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $c \neq 0$

$\langle 1 \rangle 3$ . PICK  $N$  such that  $\forall n \geq N, |s_n - s| < \epsilon/|c|$ .

$\langle 1 \rangle 4$ .  $\forall n \geq N, |cs_n - cs| < \epsilon$

□

**Proposition 12.7.** *If  $s_n \rightarrow s$  and  $t_n \rightarrow t$  in  $\mathbb{C}$  then  $s_n t_n \rightarrow st$ .*

PROOF:

$\langle 1 \rangle 1$ .  $(s_n - s)(t_n - t) \rightarrow 0$  as  $n \rightarrow \infty$

$\langle 2 \rangle 1$ . LET:  $\epsilon > 0$

$\langle 2 \rangle 2$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $|s_n - s| < \sqrt{\epsilon}$  and  $|t_n - t| < \sqrt{\epsilon}$ .

$\langle 2 \rangle 3$ . For all  $n \geq N$  we have  $|(s_n - s)(t_n - t)| < \epsilon$

$\langle 1 \rangle 2$ .  $s_n t_n - st \rightarrow 0$  as  $n \rightarrow \infty$

PROOF:

$$s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$$

$$\rightarrow 0$$

$$\text{as } n \rightarrow \infty$$

□

**Proposition 12.8.** *If  $s_n \rightarrow s$  as  $n \rightarrow \infty$  in  $\mathbb{C}$ , and every  $s_n$  and  $s$  is nonzero, then  $1/s_n \rightarrow 1/s$  as  $n \rightarrow \infty$ .*

PROOF:

$\langle 1 \rangle 1$ . PICK  $m$  such that, for all  $n \geq m$ , we have  $|s_n - s| < \frac{1}{2}|s|$ .

$\langle 1 \rangle 2$ .  $\forall n \geq m, |s_n| > \frac{1}{2}|s|$

$\langle 1 \rangle 3$ . LET:  $\epsilon > 0$

(1)4. PICK  $N > m$  such that, for all  $n \geq N$ , we have

$$|s_n - s| < \frac{1}{2}|s|^2\epsilon .$$

(1)5. For all  $n \geq N$ , we have

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| < \epsilon .$$

PROOF:

$$\begin{aligned} \left| \frac{1}{s_n} - \frac{1}{s} \right| &= \frac{|s_n - s|}{|s_n||s|} \\ &< \frac{|s|^2\epsilon}{2|s_n||s|} \\ &= \frac{|s|\epsilon}{2|s_n|} \\ &< \epsilon \end{aligned}$$

□

**Theorem 12.9.** Let  $(\vec{x}_n)$  be a sequence in  $\mathbb{R}^k$  and  $\vec{l} \in \mathbb{R}^k$ . Then  $\vec{x}_n \rightarrow \vec{l}$  as  $n \rightarrow \infty$  iff, for  $i = 1, \dots, k$ , we have  $\pi_i(\vec{x}_n) \rightarrow \pi_i(\vec{l})$  as  $n \rightarrow \infty$ .

PROOF:

(1)1. If  $\vec{x}_n \rightarrow \vec{l}$  then  $\pi_i(\vec{x}_n) \rightarrow \pi_i(l)$ .

(2)1.  $\|\vec{x}_n - \vec{l}\| \rightarrow 0$  as  $n \rightarrow \infty$ .

(2)2.  $\sqrt{\sum_{i=1}^k (\pi_i(\vec{x}_n) - \pi_i(l))^2} \rightarrow 0$  as  $n \rightarrow \infty$ .

(2)3.  $\sum_{i=1}^k (\pi_i(\vec{x}_n) - \pi_i(l))^2 \rightarrow 0$  as  $n \rightarrow \infty$ .

(2)4.  $(\pi_i(\vec{x}_n) - \pi_i(l))^2 \rightarrow 0$  as  $n \rightarrow \infty$

(2)5.  $\pi_i(\vec{x}_n) - \pi_i(l) \rightarrow 0$  as  $n \rightarrow \infty$ .

(1)2. If  $\pi_i(\vec{x}_n) \rightarrow \pi_i(\vec{l})$  for every  $i$  then  $\vec{x}_n \rightarrow l$ .

(2)1. ASSUME:  $\pi_i(\vec{x}_n) \rightarrow \pi_i(\vec{l})$  for every  $i$ .

(2)2.  $\vec{x}_n \rightarrow \vec{l}$

PROOF:

$$\begin{aligned} \|\vec{x}_n - \vec{l}\|^2 &= \sum_{i=1}^k (\pi_i(\vec{x}_n) - \pi_i(\vec{l}))^2 \\ &\rightarrow 0 \end{aligned}$$

□

**Corollary 12.9.1.** If  $\beta_n \rightarrow \beta$  in  $\mathbb{R}$  and  $\vec{x}_n \rightarrow \vec{l}$  in  $\mathbb{R}^k$ , then  $\beta_n \vec{x}_n \rightarrow \beta \vec{l}$ .

**Proposition 12.10.** If  $\vec{x}_n \rightarrow \vec{x}$  and  $\vec{y}_n \rightarrow \vec{y}$  in  $\mathbb{R}^k$ , then  $\vec{x}_n \cdot \vec{y}_n \rightarrow \vec{x} \cdot \vec{y}$ .

PROOF:

$$\begin{aligned}
 \vec{x}_n \cdot \vec{y}_n &= \sum_{i=1}^k \pi_i(\vec{x}_n) \pi_i(\vec{y}_n) \\
 &\rightarrow \sum_{i=1}^k \pi_i(\vec{x}) \pi_i(\vec{y}) \\
 &= \vec{x} \cdot \vec{y}
 \end{aligned}
 \quad \square$$

**Proposition 12.11.** *Let  $(p_n)$  be a sequence in the metric space  $X$ . The set  $E^*$  of all limits of convergent subsequences is a closed set.*

PROOF:

- $\langle 1 \rangle 1$ . ASSUME: w.l.o.g.  $\{p_n : n \in \mathbb{N}\}$  is infinite.
- $\langle 1 \rangle 2$ . LET:  $q$  be a limit point of  $E^*$ .  
     PROVE:  $q \in E^*$
- $\langle 1 \rangle 3$ . PICK an integer  $n_0$  such that  $q \neq p_{n_0}$ .
- $\langle 1 \rangle 4$ . Extend a strictly increasing sequence of integers  $(n_i)$  such that, for all  $i$ , we have  $d(q, p_{n_i}) \leq 2^i d(q, p_{n_0})$ .
- $\langle 2 \rangle 1$ . ASSUME: as induction hypothesis we have picked  $n_0 < n_1 < \dots < n_i$  such that, for  $0 \leq j \leq i$ , we have  $d(q, p_{n_j}) \leq 2^j d(q, p_{n_0})$ .
- $\langle 2 \rangle 2$ . PICK  $x \in E^*$  such that  $d(x, q) < 2^{-(i+2)} \delta$
- $\langle 2 \rangle 3$ . There exists a subsequence of  $(p_n)$  that converges to  $x$ .
- $\langle 2 \rangle 4$ . There exists  $n_{i+1} > n_i$  such that  $d(p_{n_{i+1}}, x) < 2^{-(i+2)} \delta$ .
- $\langle 2 \rangle 5$ .  $d(p_{n_{i+1}}, q) < 2^{-(i+1)} \delta$
- $\langle 1 \rangle 5$ .  $p_{n_i} \rightarrow q$  as  $i \rightarrow \infty$ .
- $\langle 1 \rangle 6$ .  $q \in E^*$

$\square$

**Theorem 12.12.** *Every monotonically increasing sequence in  $\mathbb{R}$  that is bounded above converges to its supremum.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $(s_n)$  be a monotonically increasing sequence with supremum  $s$ .
- $\langle 1 \rangle 2$ . LET:  $\epsilon > 0$
- $\langle 1 \rangle 3$ . PICK  $N$  such that  $|s_N - s| < \epsilon$
- $\langle 1 \rangle 4$ . For all  $n \geq N$ , we have  $s - \epsilon < s - s_N \leq s - s_n \leq s$ .
- $\langle 1 \rangle 5$ .  $\forall n \geq N, |s_n - s| < \epsilon$

$\square$

**Theorem 12.13.** *Every monotonically decreasing sequence in  $\mathbb{R}$  that is bounded below converges to its infimum.*

PROOF: Similar.  $\square$

**Proposition 12.14** (Sandwich Theorem). *Let  $(a_n)$ ,  $(b_n)$  and  $(c_n)$  be sequences of real numbers and  $l \in \mathbb{R}$ . Assume  $\forall n, a_n \leq b_n \leq c_n$  and  $a_n \rightarrow l$  and  $c_n \rightarrow l$ . Then  $b_n \rightarrow l$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 2$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $|a_n - l| < \epsilon$  and  $|c_n - l| < \epsilon$ .

$\langle 1 \rangle 3$ .  $\forall n \geq N. |b_n - l| < \epsilon$

□

**Theorem 12.15.** *For any real  $p > 0$  we have*

$$\frac{1}{(n+1)^p} \rightarrow 0$$

as  $n \rightarrow \infty$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 2$ . PICK  $N$  such that  $N > (1/\epsilon)^{1/p}$ .

$\langle 1 \rangle 3$ . LET:  $n \geq N$

$\langle 1 \rangle 4$ .  $1/n^p < \epsilon$

□

**Theorem 12.16.** *For any real  $p > 0$  we have*

$$p^{\frac{1}{n+1}} \rightarrow 1$$

as  $n \rightarrow \infty$ .

PROOF:

$\langle 1 \rangle 1$ . CASE:  $p > 1$

$\langle 2 \rangle 1$ . For  $n \in \mathbb{N}$

LET:  $x_n = p^{\frac{1}{n+1}} - 1$ .

$\langle 2 \rangle 2$ .  $\forall n \in \mathbb{N}. x_n > 0$

$\langle 2 \rangle 3$ .  $\forall n \in \mathbb{N}$ .

$$1 + (n+1)x_n \leq p$$

PROOF: Since  $1 + (n+1)x_n \leq (1+x_n)^{n+1}$  by the Binomial Theorem.

$\langle 2 \rangle 4$ .  $\forall n \in \mathbb{N}$ .

$$0 < x_n \leq \frac{p-1}{n+1}$$

$\langle 2 \rangle 5$ .  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ .

PROOF: Sandwich Theorem.

$\langle 1 \rangle 2$ . CASE:  $p = 1$

PROOF: Trivial.

$\langle 1 \rangle 3$ . CASE:  $p < 1$

PROOF: Then  $p^{1/(n+1)} = 1/((1/p)^{1/(n+1)}) \rightarrow 1/1 = 1$  by  $\langle 1 \rangle 1$ .

□

**Theorem 12.17.**

$$(n+1)^{1/(n+1)} \rightarrow 1 \text{ as } n \rightarrow \infty$$

PROOF:

(1)1. For  $n \in \mathbb{N}$ ,

LET:  $x_n = (n+1)^{1/(n+1)} - 1$ .

(1)2.  $\forall n \in \mathbb{N}. x_n \geq 0$

(1)3.  $\forall n \in \mathbb{N}$

$$n+1 \geq \frac{n(n+1)}{2} x_n^2 .$$

PROOF: Since  $(1+x_n)^{n+1} \geq \frac{n(n+1)}{2} x_n^2$  by the Binomial Theorem.

(1)4.  $\forall n \geq 1$

$$0 \leq x_n \leq \sqrt{\frac{2}{n}}$$

(1)5.  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ .

PROOF: Sandwich Theorem.

□

**Theorem 12.18.** Let  $p$  and  $\alpha$  be real numbers with  $p > 0$ . Then

$$\frac{n^\alpha}{(1+p)^n} \rightarrow 0 \text{ as } n \rightarrow \infty .$$

PROOF:

(1)1. PICK a positive integer  $k$  such that  $k > \alpha$ .

PROOF: Archimedean Property.

(1)2.  $\forall n > 2k$

$$(1+p)^n > \frac{n^k p^k}{2^k k!} .$$

$$\begin{aligned} (1+p)^n &> \binom{n}{k} p^k && \text{(Binomial Theorem)} \\ &= \frac{n(n-1) \cdots (n-k+1)}{k!} p^k \\ &> \frac{n^k p^k}{2^k k!} && (n > 2k \text{ so if } n-k < i \leq n \text{ then } i > n/2) \end{aligned}$$

(1)3.  $\forall n > 2k$

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k} .$$

(1)4.  $n^{\alpha-k} \rightarrow 0$  as  $n \rightarrow \infty$

PROOF: Theorem 12.15.

(1)5.  $\frac{n^\alpha}{(1+p)^n} \rightarrow 0$  as  $n \rightarrow \infty$ .

PROOF: Sandwich Theorem.

□

**Corollary 12.18.1.** For any real number  $x$  with  $|x| < 1$  we have  $x^n \rightarrow 0$  as  $n \rightarrow \infty$ .

PROOF: Taking  $\alpha = 0$ . □



## 12.1 Cauchy Sequences

**Definition 12.19** (Cauchy Sequence). Let  $(p_n)$  be a sequence in the metric space  $X$ . Then  $(p_n)$  is a *Cauchy sequence* iff, for every  $\epsilon > 0$ , there exists  $N$  such that, for all  $m, n \geq N$ , we have  $d(p_m, p_n) < \epsilon$ .

**Proposition 12.20.** Let  $(p_n)$  be a sequence in the metric space  $X$  and let  $E_N = \{p_n : n \geq N\}$  for all  $N$ . Then  $(p_n)$  is a Cauchy sequence if and only if  $\text{diam } E_N \rightarrow 0$  as  $N \rightarrow \infty$ .

PROOF: Immediate from definitions.  $\square$

**Theorem 12.21.** Every convergent sequence is Cauchy.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $(p_n)$  be a convergent sequence with limit  $l$ .
- $\langle 1 \rangle 2$ . LET:  $\epsilon > 0$
- $\langle 1 \rangle 3$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $d(p_n, l) < \epsilon/2$
- $\langle 1 \rangle 4$ .  $\forall m, n \geq N. d(p_m, p_n) < \epsilon$

$\square$

## 12.2 Complete Metric Spaces

**Definition 12.22** (Complete Metric Space). A metric space is *complete* iff every Cauchy sequence converges.

**Theorem 12.23.** Every compact metric space is complete.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X$  be a compact metric space.
- $\langle 1 \rangle 2$ . LET:  $(p_n)$  be a Cauchy sequence in  $X$ .
- $\langle 1 \rangle 3$ . For  $N \in \mathbb{N}$ ,  
LET:  $E_N = \{p_n : n \geq N\}$ .
- $\langle 1 \rangle 4$ .  $\text{diam } \overline{E_N} \rightarrow 0$  as  $N \rightarrow \infty$ .
- $\langle 1 \rangle 5$ . For all  $N$ , every  $\overline{E_N}$  is compact.  
PROOF: Proposition 11.37.
- $\langle 1 \rangle 6$ . For all  $N$  we have  $\overline{E_N} \supseteq \overline{E_{N+1}}$ .
- $\langle 1 \rangle 7$ . LET:  $l$  be the unique point in  $\bigcap_{N=0}^{\infty} \overline{E_N}$   
PROVE:  $p_n \rightarrow l$  as  $n \rightarrow \infty$ .  
PROOF: Proposition 11.44.
- $\langle 1 \rangle 8$ . LET:  $\epsilon > 0$
- $\langle 1 \rangle 9$ . PICK  $N_0$  such that  $\forall N \geq N_0. \text{diam } \overline{E_N} < \epsilon$ .
- $\langle 1 \rangle 10$ .  $\forall q \in E_N. d(l, q) < \epsilon$
- $\langle 1 \rangle 11$ .  $\forall n \geq N. d(l, p_n) < \epsilon$

$\square$

**Corollary 12.23.1.** Let  $X$  be a metric space. If every closed bounded set in  $X$  is compact, then  $X$  is complete.

PROOF:

- (1)1. LET:  $S$  be a Cauchy sequence in  $X$ .
- (1)2.  $S$  is bounded.
- (1)3.  $\overline{S}$  is closed and bounded.
- (1)4.  $\overline{S}$  is compact.
- (1)5.  $S$  is a Cauchy sequence in  $\overline{S}$ .
- (1)6.  $S$  converges.

□

**Corollary 12.23.2.** *For every natural number  $k$ , we have  $\mathbb{R}^k$  is complete.*

**Corollary 12.23.3.** *Every closed subspace of a complete metric space is complete.*

**Proposition 12.24.** *Let  $X$  be a complete metric space. Let  $(E_n)$  be a sequence of nonempty closed bounded sets in  $X$  with*

$$E_0 \supseteq E_1 \supseteq \cdots$$

*and  $\text{diam } E_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\bigcap_{n=0}^{\infty} E_n$  consists of exactly one point.*

PROOF:

- (1)1. LET:  $K = \bigcap_{n=0}^{\infty} E_n$
- (1)2.  $K$  has at least one point.
  - (2)1. For each  $n$ , PICK  $a_n \in E_n$
  - (2)2.  $(a_n)$  is Cauchy.
    - (3)1. LET:  $\epsilon > 0$
    - (3)2. PICK  $N$  such that  $\forall n \geq N, \text{diam } E_n < \epsilon$
    - (3)3.  $\forall m, n \geq N, d(a_m, a_n) < \epsilon$
  - (2)3. LET:  $l = \lim_{n \rightarrow \infty} a_n$
  - (2)4.  $l \in K$ 
    - (3)1. LET:  $n \in \mathbb{N}$
    - (3)2. For all  $m \geq n$  we have  $a_m \in E_n$
    - (3)3.  $l \in E_n$
- (1)3.  $K$  has at most one point.
  - (2)1. ASSUME: for a contradiction  $a, b \in K$  such that  $a \neq b$
  - (2)2. PICK  $n$  such that  $\text{diam } E_n < d(a, b)$
  - (2)3.  $a, b \in E_n$
  - (2)4. Q.E.D.

PROOF: This is a contradiction.

□

**Theorem 12.25** (Baire's Theorem). *Let  $X$  be a complete metric space. Let  $(G_n)$  be a sequence of dense open subsets of  $X$ . Then  $\bigcap_{n=0}^{\infty} G_n$  is not empty.*

PROOF:

- (1)1. PICK a sequence  $(E_n)$  of open balls such that  $E_0 \supseteq E_1 \supseteq \cdots$  and  $\text{diam } E_n \leq 1/2^n$  and  $\overline{E_n} \subseteq G_n$ .

- ⟨2⟩1. ASSUME: as induction hypothesis we have chosen  $E_0, \dots, E_n$  with centres  $c_0, \dots, c_n$ .  
 ⟨2⟩2. PICK  $x \in E_n \cap G_{n+1}$   
 ⟨2⟩3. PICK  $0 < \epsilon \leq 1/2^{n+2}$  such that  $B(x, \epsilon) \subseteq E_n \cap G_{n+1}$   
 ⟨2⟩4. LET:  $E_{n+1} = B(x, \epsilon/2)$   
 ⟨2⟩5.  $E_{n+1} \subseteq E_n$   
 ⟨2⟩6.  $\text{diam } E_{n+1} \leq 1/2^{n+1}$   
 ⟨2⟩7.  $\overline{E_{n+1}} \subseteq G_{n+1}$   
 ⟨1⟩2. LET:  $\bigcap_{n=0}^{\infty} \overline{E_n} = \{p\}$   
 PROOF: Proposition 12.24.  
 ⟨1⟩3.  $p \in \bigcap_{n=0}^{\infty} G_n$   
 $\square$

## 12.3 Divergent Sequences

**Definition 12.26.** Let  $(s_n)$  be a sequence in  $\mathbb{R}$ . Then we say  $s_n$  *diverges to*  $+\infty$ , and write

$$s_n \rightarrow +\infty \text{ as } n \rightarrow \infty ,$$

iff for every real number  $M$ , there exists an integer  $N$  such that

$$\forall n \geq N. s_n \geq M .$$

We say  $s_n$  *diverges to*  $-\infty$ , and write

$$s_n \rightarrow -\infty \text{ as } n \rightarrow \infty ,$$

iff for every real number  $M$ , there exists an integer  $N$  such that

$$\forall n \geq N. s_n \leq M .$$

**Definition 12.27** (Limit Supremum, Limit Infimum). Let  $(s_n)$  be a sequence in  $\mathbb{R}$ . Let  $E$  be the set of all  $l \in \mathbb{R} \cup \{+\infty, -\infty\}$  such that there exists a subsequence of  $(s_n)$  that converges to  $l$ .

The *limit supremum* of  $(s_n)$ , denoted

$$\limsup_{n \rightarrow \infty} s_n ,$$

is the supremum of  $E$  in the extended reals.

The *limit infimum* of  $(s_n)$ , denoted

$$\liminf_{n \rightarrow \infty} s_n ,$$

is the infimum of  $E$  in the extended reals.

PROOF: The set  $E$  is always nonempty because: if  $(s_n)$  is unbounded above then  $+\infty \in E$ ; if it is unbounded below then  $-\infty \in E$ ; and if it is bounded above and below then there is a real number in  $E$  by Corollary 12.4.1.  $\square$

**Theorem 12.28.** *Let  $(s_n)$  be a sequence in  $\mathbb{R}$ . Then there exists a subsequence of  $(s_n)$  that converges or diverges to  $\limsup_{n \rightarrow \infty} s_n$*

PROOF:

(1)1. CASE:  $\limsup_n s_n = +\infty$

PROOF:  $(s_n)$  is unbounded above and so has a subsequence that diverges to  $+\infty$ .

(1)2. CASE:  $\limsup_n s_n \in \mathbb{R}$

PROOF: Then  $\limsup_n s_n$  is in the set of limits of subsequences of  $(s_n)$  by Proposition 12.11.

(1)3. CASE:  $\limsup_n s_n = -\infty$

PROOF:  $(s_n)$  is unbounded below and so has a subsequence that diverges to  $-\infty$ .

□

**Theorem 12.29.** *Let  $(s_n)$  be a sequence in  $\mathbb{R}$ . Then there exists a subsequence of  $(s_n)$  that converges or diverges to  $\liminf_{n \rightarrow \infty} s_n$*

PROOF: Similar. □

**Theorem 12.30.** *Let  $(s_n)$  be a sequence in  $\mathbb{R}$ . If  $x > \limsup_n s_n$ , then there exists  $N$  such that  $\forall n \geq N, s_n < x$ .*

PROOF: If not, we could choose a subsequence of  $(s_n)$  that converges to a value  $\geq x$ , contradicting the definition of  $\limsup_n s_n$ . □

**Theorem 12.31.** *Let  $(s_n)$  be a sequence in  $\mathbb{R}$ . If  $x < \liminf_n s_n$ , then there exists  $N$  such that  $\forall n \geq N, s_n > x$ .*

PROOF: Similar. □

**Theorem 12.32.** *Let  $(s_n)$  be a sequence in  $\mathbb{R}$ . Let  $s^*$  be an extended real such that:*

- *There exists a subsequence of  $(s_n)$  that converges or diverges to  $s^*$ .*
- *For any  $x > s^*$ , there exists  $N$  such that  $\forall n \geq N, s_n < x$ .*

*Then  $s^* = \limsup_n s_n$ .*

PROOF:

(1)1. LET:  $E$  be the set of subsequential limits of  $(s_n)$ .

(1)2.  $s^*$  is an upper bound for  $E$ .

(2)1. LET:  $x \in E$

(2)2. ASSUME: for a contradiction  $x > s^*$ .

(2)3.  $s^* \in \mathbb{R}$

(2)4. LET:  $y = x$  if  $x \in \mathbb{R}$ , or  $s^* + 1$  if  $x = +\infty$

(2)5. There exists  $N$  such that  $\forall n \geq N, s_n < y$ .

(2)6. Q.E.D.

PROOF: This contradicts the fact that some subsequence of  $(s_n)$  converges or diverges to  $x$ .

(1)3. If  $u$  is an upper bound for  $E$  then  $s^* \leq u$ .

□

**Theorem 12.33.** Let  $(s_n)$  be a sequence in  $\mathbb{R}$ . Let  $s^*$  be an extended real such that:

- There exists a subsequence of  $(s_n)$  that converges or diverges to  $s^*$ .
- For any  $x < s^*$ , there exists  $N$  such that  $\forall n \geq N, s_n > x$ .

Then  $s^* = \liminf_n s_n$ .

PROOF: Similar. □

**Proposition 12.34.** Let  $(s_n)$  be a sequence of real numbers and  $l \in \mathbb{R}$ . Then  $(s_n)$  converges to  $l$  iff  $\limsup_n s_n = \liminf_n s_n = l$ .

PROOF:

(1)1. If  $(s_n)$  converges to  $l$  then  $\limsup_n s_n = \liminf_n s_n = l$ .

PROOF: If  $(s_n)$  converges to  $l$  then every subsequence of  $(s_n)$  converges to  $l$ .

(1)2. If  $\limsup_n s_n = \liminf_n s_n = l$  then  $(s_n)$  converges to  $l$ .

⟨2⟩1. ASSUME:  $\limsup_n s_n = \liminf_n s_n = l$

⟨2⟩2. For all  $\epsilon > 0$ , there exists  $N$  such that  $\forall n \geq N, l - \epsilon < s_n < l + \epsilon$ .

PROOF: Theorem 12.32 and 12.33.

⟨2⟩3.  $s_n \rightarrow l$  as  $n \rightarrow \infty$ .

□

**Theorem 12.35.** Let  $(s_n)$  and  $(t_n)$  be sequences of real numbers and  $N \in \mathbb{N}$ . Assume  $\forall n \geq N, s_n \leq t_n$ . Then

$$\liminf_{n \rightarrow \infty} s_n \leq \liminf_{n \rightarrow \infty} t_n .$$

PROOF:

(1)1. For any subsequence  $(t_{n_r})$  of  $(t_n)$  that converges or diverges to  $\pm\infty$ , we have  $\liminf_n s_n \leq \lim_r t_{n_r}$ .

⟨2⟩1. LET:  $(t_{n_r})$  be a subsequence of  $(t_n)$  with limit  $l$ .

⟨2⟩2. PICK  $m$  such that a subsequence of  $(s_{n_r})$  has limit  $m$ .

⟨2⟩3.  $\forall r, s_{n_r} \leq t_{n_r}$

⟨2⟩4.  $m \leq l$

⟨2⟩5.  $\liminf_n s_n \leq l$

(1)2.  $\liminf_n s_n \leq \liminf_n t_n$

□

**Theorem 12.36.** Let  $(s_n)$  and  $(t_n)$  be sequences of real numbers and  $N \in \mathbb{N}$ . Assume  $\forall n \geq N, s_n \leq t_n$ . Then

$$\limsup_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} t_n .$$

PROOF: Similar. □

**Theorem 12.37.** *For any sequence  $(c_n)$  of positive real numbers, we have*

$$\limsup_{n \rightarrow \infty} c_n^{1/n} \leq \limsup_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} .$$

PROOF:

(1)1. LET:  $\alpha = \limsup_n c_{n+1}/c_n$

(1)2. ASSUME: w.l.o.g.  $\alpha < +\infty$

(1)3. For all  $\beta > \alpha$  we have  $\limsup_{n \rightarrow \infty} c_n^{1/n} \leq \beta$ .

(2)1. LET:  $\beta > \alpha$

(2)2. PICK  $N$  such that, for all  $n \geq N$ , we have  

$$\frac{c_{n+1}}{c_n} \leq \beta .$$

PROOF: Theorem 12.30.

(2)3. For all  $k \geq 0$  we have

$$c_{N+k+1} \leq \beta c_{N+k} .$$

(2)4. For all  $n \geq N$  we have

$$c_n \leq c_N \beta^{-N} \beta^n .$$

PROOF: Induction on  $n$ .

(2)5. For all  $n \geq N$  we have

$$c_n^{1/n} \leq (c_N \beta^{-N})^{1/n} \beta .$$

(2)6.

$$\limsup_{n \rightarrow \infty} c_n^{1/n} \leq \beta$$

PROOF:

$$\limsup_{n \rightarrow \infty} c_n^{1/n} \leq \limsup_{n \rightarrow \infty} (c_N \beta^{-N})^{1/n} \beta \quad (\text{Theorem 12.36})$$

$$= \beta \quad (\text{Theorem 12.16})$$

(1)4.

$$\limsup_{n \rightarrow \infty} c_n^{1/n} \leq \alpha$$

□

**Theorem 12.38.** *For any sequence  $(c_n)$  of positive real numbers, we have*

$$\liminf_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n \rightarrow \infty} c_n^{1/n} .$$

PROOF: Similar. □

**Proposition 12.39.** *Let  $(a_n)$  and  $(b_n)$  be sequences of reals. Assume that it is not the case that one of  $\limsup_n a_n$ ,  $\limsup_n b_n$  is  $+\infty$  and the other is  $-\infty$ . Then*

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n .$$

## 12.4 Infinite Series

**Definition 12.40.** Let  $(a_n)$  be a sequence in  $\mathbb{R}^k$  and  $s \in \mathbb{R}^k$ . We say the infinite series  $\sum_{n=0}^{\infty} a_n$  converges to  $s$ , and write

$$\sum_{n=0}^{\infty} a_n = s$$

iff

$$\sum_{n=0}^N a_n \rightarrow s \text{ as } N \rightarrow \infty .$$

If  $(\sum_{n=0}^N a_n)$  diverges, we say the infinite series  $\sum_{n=0}^{\infty} a_n$  diverges.

**Theorem 12.41.** Let  $(a_n)$  be a sequence in  $\mathbb{R}^k$ . Then  $\sum_{n=0}^{\infty} a_n$  converges if and only if, for all  $\epsilon > 0$ , there exists  $N$  such that, for all  $m, n \geq N$ ,

$$\left\| \sum_{i=m}^n a_i \right\| \leq \epsilon .$$

PROOF: This is what it means for  $(\sum_{i=0}^n a_i)$  to be a Cauchy sequence.  $\square$

**Corollary 12.41.1.** If  $\sum_{n=0}^{\infty} a_n$  converges then  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**Theorem 12.42.** A series of nonnegative reals converges if and only if its partial sums form a bounded sequence.

PROOF: Its partial sums are a monotonically increasing sequence, and so converge if and only if they are bounded above.  $\square$

**Theorem 12.43** (Comparison Test). Let  $(a_n)$  be a sequence in  $\mathbb{R}^k$  and  $(c_n)$  a sequence of real numbers. If there exists  $N$  such that  $\forall n \geq N, \|a_n\| \leq c_n$ , and if  $\sum_n c_n$  converges, then  $\sum_n a_n$  converges.

PROOF:

$\langle 1 \rangle$  1. LET:  $\epsilon > 0$

$\langle 1 \rangle$  2. PICK  $N$  such that  $\forall n \geq N, \|a_n\| \leq c_n$  and  $\forall m, n \geq N, \sum_{k=m}^n c_k < \epsilon$ .

$\langle 1 \rangle$  3.  $\forall m, n \geq N, \|\sum_{k=m}^n a_k\| \leq \epsilon$

$\square$

**Corollary 12.43.1.** Let  $(a_n)$  and  $(d_n)$  be sequences of real numbers. If there exists  $N$  such that  $\forall n \geq N, a_n \geq d_n \geq 0$ , and if  $\sum_n d_n$  diverges, then  $\sum_n a_n$  diverges.

**Theorem 12.44** (Geometric Series). For  $x$  a real number with  $0 \leq x < 1$  we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} .$$

PROOF: Since  $\sum_{n=0}^N x^n = \frac{1-x^{N+1}}{1-x} \rightarrow \frac{1}{1-x}$  as  $n \rightarrow \infty$ .  $\square$

**Theorem 12.45.** For  $x$  a real number with  $x \geq 1$  we have  $\sum_{n=0}^{\infty} x^n$  diverges.

PROOF: If  $x = 1$  then  $\sum_{n=0}^N x^n = N + 1$ . If  $x > 1$  then  $\sum_{n=0}^N x^n = \frac{1-x^{N+1}}{1-x}$ . Both of these sequences diverge.  $\square$

**Theorem 12.46.** Let  $(a_n)$  be a monotonically decreasing sequence of nonnegative real numbers. Then  $\sum_n a_n$  converges if and only if  $\sum_n 2^n a_{2^n}$  converges.

PROOF:

(1)1. For  $N \in \mathbb{N}$ ,

$$\text{LET: } s_N = \sum_{n=0}^N a_n.$$

(1)2. For  $N \in \mathbb{N}$ ,

$$\text{LET: } t_N = \sum_{n=0}^N 2^n a_{2^n}.$$

(1)3. For natural number  $N$  and  $k$  with  $N < 2^k$  we have  $s_N \leq a_0 + t_{k-1}$ .

PROOF:

$$\begin{aligned} s_N &\leq \sum_{n=0}^{2^k-1} a_n \\ &= a_0 + \sum_{i=0}^{k-1} \sum_{n=2^i}^{2^{i+1}-1} a_n \\ &\leq a_0 + \sum_{i=0}^{k-1} 2^i a_{2^i} \\ &= a_0 + t_{k-1} \end{aligned}$$

(1)4. For natural number  $N$  and  $k$  with  $N > 2^k$  we have  $t_k < 2s_N$ .

PROOF:

$$\begin{aligned} s_N &\geq \sum_{n=1}^{2^k} a_n \\ &\geq \sum_{i=0}^k \sum_{n=2^i}^{2^{i+1}-1} a_n \\ &\geq \sum_{i=0}^k 2^i a_{2^{i+1}} \\ &= (1/2)t_k \end{aligned}$$

(1)5.  $(s_N)$  converges if and only if  $(t_k)$  converges.

$\square$

**Theorem 12.47.** If  $p$  is a real number with  $p > 1$  then  $\sum_n 1/n^p$  converges.

PROOF: Since

$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}$$

which converges since  $2^{1-p} < 1$ .  $\square$



**Theorem 12.48.** *If  $p$  is a real number with  $p \leq 1$  then  $\sum_n 1/n^p$  diverges.*

PROOF: If  $p \leq 0$  then  $1/n^p$  does not converge to 0.

If  $0 < p \leq 1$  we have

$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}$$

which diverges since  $2^{1-p} \geq 1$ .  $\square$

**Theorem 12.49.** *Let  $p$  be a real number. The series*

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

*converges if and only if  $p > 1$ .*

PROOF:

$$\begin{aligned} 2^k \frac{1}{2^k (\ln 2^k)^p} &= \frac{1}{(k \ln 2)^p} \\ &= \frac{1}{(\ln 2)^p} \cdot \frac{1}{k^p} \end{aligned}$$

and this series converges iff  $\sum_k \frac{1}{k^p}$  converges iff  $p > 1$ .  $\square$

**Theorem 12.50** (Root Test). *Let  $(a_n)_{n \geq 1}$  be a sequence in  $\mathbb{R}^k$ . Let  $\alpha = \limsup_{n \rightarrow \infty} \|a_n\|^{1/n}$ .*

1. *If  $\alpha < 1$  then  $\sum_{n=1}^{\infty} a_n$  converges.*
2. *If  $\alpha > 1$  then  $\sum_{n=1}^{\infty} a_n$  diverges.*

PROOF:

(1)1. If  $\alpha < 1$  then  $\sum_{n=1}^{\infty} a_n$  converges.

(2)1. ASSUME:  $\alpha < 1$

(2)2. PICK  $\beta$  such that  $\alpha < \beta < 1$

(2)3. PICK  $N$  such that  $\forall n \geq N, \|a_n\|^{1/n} < \beta$

PROOF: Theorem 12.30.

(2)4.  $\forall n \geq N, \|a_n\| < \beta^n$

(2)5.  $\sum_{n=1}^{\infty} \beta^n$  converges.

PROOF: Theorem 12.44.

(2)6.  $\sum_{n=1}^{\infty} a_n$  converges.

PROOF: Comparison Test.

(1)2. If  $\alpha > 1$  then  $\sum_{n=1}^{\infty} a_n$  diverges.

(2)1. ASSUME:  $\alpha > 1$

(2)2. There exists a sequence of positive integers  $(n_k)$  such that  $\|a_{n_k}\|^{1/n_k} \rightarrow \alpha$  as  $k \rightarrow \infty$ .

PROOF: Theorem 12.28.

(2)3. There are infinitely many  $n$  such that  $\|a_n\| > 1$ .

(2)4.  $a_n \not\rightarrow 0$  as  $n \rightarrow \infty$ .

(2)5.  $\sum_{n=1}^{\infty} a_n$  diverges.

PROOF: Corollary 12.41.1.

□

**Example 12.51.** If  $a_n = 1/n$  then  $|a_n|^{1/n} \rightarrow 1$  and  $\sum_n a_n$  diverges.

If  $a_n = 1/n^2$  then  $|a_n|^{1/n} \rightarrow 1$  and  $\sum_n a_n$  converges.

**Theorem 12.52** (Ratio Test). *Let  $(a_n)_{n \geq 0}$  be a sequence in  $\mathbb{R}^k$ .*

1. *If*

$$\limsup_{n \rightarrow \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$$

*then  $\sum_{n=0}^{\infty} a_n$  converges.*

2. *If there exists  $N$  such that  $\forall n \geq N, \frac{\|a_{n+1}\|}{\|a_n\|} > 1$  then  $\sum_{n=0}^{\infty} a_n$  diverges.*

PROOF:

$\langle 1 \rangle 1$ . If  $\limsup_{n \rightarrow \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$  then  $\sum_{n=0}^{\infty} a_n$  converges.

$\langle 2 \rangle 1$ . ASSUME:  $\limsup_{n \rightarrow \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$

$\langle 2 \rangle 2$ .  $\limsup_{n \rightarrow \infty} \|a_n\|^{1/n} < 1$

PROOF: Theorem 12.37.

$\langle 2 \rangle 3$ .  $\sum_{n=0}^{\infty} a_n$  converges.

PROOF: Root Test

$\langle 1 \rangle 2$ . If there exists  $N$  such that  $\forall n \geq N, \frac{\|a_{n+1}\|}{\|a_n\|} > 1$  then  $\sum_{n=0}^{\infty} a_n$  diverges.

PROOF: Since  $a_n \not\rightarrow 0$  as  $n \rightarrow \infty$ .

□

**Example 12.53.** If  $a_n = 1/n$  then  $a_{n+1}/a_n \rightarrow 1$  and  $\sum_n a_n$  diverges.

If  $a_n = 1/n^2$  then  $a_{n+1}/a_n \rightarrow 1$  and  $\sum_n a_n$  converges.

## 12.5 The Number $e$

**Lemma 12.54.** *The series  $\sum_{n=0}^{\infty} \frac{1}{n!}$  converges.*

PROOF:

$$\sum_{n=0}^N \frac{1}{n!} \leq 1 + \sum_{n=1}^N \frac{1}{2^{n-1}} < 3$$

□

**Definition 12.55.** The number  $e$  is

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} .$$

**Theorem 12.56.**

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e \text{ as } n \rightarrow \infty$$

PROOF:

(1)1. For  $n \in \mathbb{N}$ ,

$$\text{LET: } s_n = \sum_{k=0}^n \frac{1}{k!}$$

(1)2. For  $n \in \mathbb{Z}^+$ ,

$$\text{LET: } t_n = \left(1 + \frac{1}{n}\right)^n$$

(1)3. For  $n \in \mathbb{Z}^+$  we have

$$t_n = \sum_{k=0}^n \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) .$$

PROOF:

$$\begin{aligned} t_n &= \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} && \text{(Binomial Theorem)} \\ &= \sum_{k=0}^n \frac{\prod_{i=0}^{k-1} (n-i)}{k!} \frac{1}{n^k} \\ &= \sum_{k=0}^n \frac{1}{k!} \prod_{i=0}^{k-1} \frac{n-i}{n} \\ &= \sum_{k=0}^n \frac{1}{k!} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) \\ &= \sum_{k=0}^n \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \end{aligned}$$

(1)4. For  $n \in \mathbb{Z}^+$  we have  $t_n \leq s_n$ .

(1)5.  $\limsup_{n \rightarrow \infty} t_n \leq e$

(1)6. For  $m, n \in \mathbb{Z}^+$  with  $n \geq m$  we have

$$t_n \geq \sum_{k=0}^m \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) .$$

(1)7. For  $m \in \mathbb{Z}^+$  we have

$$\liminf_{n \rightarrow \infty} t_n \geq \sum_{k=0}^m \frac{1}{k!} .$$

(1)8. For  $m \in \mathbb{Z}^+$  we have

$$s_m \leq \liminf_{n \rightarrow \infty} t_n .$$

(1)9.

$$e \leq \liminf_{n \rightarrow \infty} t_n$$

(1)10.  $t_n \rightarrow e$  as  $n \rightarrow \infty$ .

PROOF: From (1)5 and (1)9.

□

**Theorem 12.57.**  $e$  is irrational.

PROOF:

(1)1. ASSUME: for a contradiction  $e = p/q$  where  $p$  and  $q$  are positive integers.

(1)2. For  $n \in \mathbb{N}$ ,

LET:  $s_n = \sum_{k=0}^n \frac{1}{k!}$ .  
 (1)3. For  $n \in \mathbb{Z}^+$  we have

$$0 < e - s_n < \frac{1}{n!n}.$$

PROOF:

$$\begin{aligned} e - s_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &< \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \frac{1}{(n+1)^k} \\ &= \frac{1}{n!n} \end{aligned}$$

(1)4.

$$0 < q!(e - s_q) < \frac{1}{q}$$

(1)5.  $q!e$  is an integer.

(1)6.  $q!(e - s_q)$  is an integer.

(1)7. There exists an integer between 0 and 1.

(1)8. Q.E.D.

PROOF: This is a contradiction.

□

**Theorem 12.58.**  $e$  is transcendental.

PROOF: See I. M. Niven. *Irrational Numbers* p. 25. □

## 12.6 Power Series

**Definition 12.59** (Power Series). Let  $(c_n)$  be a sequence of complex numbers. The *power series* with *coefficients*  $(c_n)$  is the function that maps a complex number  $z$  to the series

$$\sum_{n=0}^{\infty} c_n z^n.$$

**Definition 12.60** (Radius of Convergence). Let  $(c_n)$  be a sequence of complex numbers. Let

$$\begin{aligned} \alpha &:= \limsup_{n \rightarrow \infty} |c_n|^{1/n} \\ R &:= \frac{1}{\alpha} \end{aligned}$$

where  $R = +\infty$  if  $\alpha = 0$  and  $R = 0$  if  $\alpha = +\infty$ . Then  $R$  is called the *radius of convergence* of the power series  $\sum_n c_n z^n$ .

**Theorem 12.61.** Let  $R$  be the radius of convergence of  $\sum_n c_n z^n$ .

1. If  $|z| < R$  then  $\sum_{n=0}^{\infty} c_n z^n$  converges.

2. If  $|z| > R$  then  $\sum_{n=0}^{\infty} c_n z^n$  diverges.

PROOF:

(1)1. For  $z \in \mathbb{C}$  and  $n \in \mathbb{N}$ ,

LET:  $a_n(z) = c_n z^n$

(1)2.

$$\limsup_{n \rightarrow \infty} |a_n(z)|^{1/n} = |z|/R$$

(1)3. If  $|z| < R$  then  $\sum_{n=0}^{\infty} a_n(z)$  converges.

PROOF: Root Test.

(1)4. If  $|z| > R$  then  $\sum_{n=0}^{\infty} a_n(z)$  diverges.

PROOF: Root Test.

□

## 12.7 Summation by Parts

**Theorem 12.62.** Let  $(a_n), (b_n)$  be two sequences in  $\mathbb{R}^k$ . Let

$$A_n = \sum_{k=0}^n a_k \quad (n \geq -1) .$$

Let  $p$  and  $q$  be integers with  $0 \leq p \leq q$ . Then

$$\sum_{n=p}^q a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p .$$

PROOF:

$$\begin{aligned} \sum_{n=p}^q a_n b_n &= \sum_{n=p}^q (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^q A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \quad \square \end{aligned}$$

**Theorem 12.63.** Let  $(a_n)$  be a sequence in  $\mathbb{R}^k$  and  $(b_n)$  be a sequence of real numbers. Assume that:

1. The partial sums  $\sum_{n=0}^N a_n$  form a bounded sequence.
2.  $(b_n)$  is monotone decreasing.
3.  $b_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Then  $\sum_{n=0}^{\infty} a_n b_n$  converges.

PROOF:

(1)1. PICK  $M$  such that, for all  $N$ , we have  $\|\sum_{n=0}^N a_n\| \leq M$ .

(1)2. LET:  $\epsilon > 0$

(1)3. PICK  $N$  such that  $b_N \leq \epsilon/2M$ .

(1)4. LET:  $N \leq p \leq q$

(1)5. For any integer  $k$ ,

LET:  $A_k = \sum_{n=0}^k a_n$ .

(1)6.  $\|\sum_{n=p}^q a_n b_n\| \leq \epsilon$

PROOF:

$$\left\| \sum_{n=p}^q a_n b_n \right\| = \left\| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right\| \quad (\text{Summation by Parts})$$

$$\leq M \left\| \sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q - b_p \right\|$$

$$= 2M b_p$$

$$\leq 2M b_N$$

$$\leq \epsilon$$

(1)7. Q.E.D.

PROOF: Cauchy criterion.

□

**Corollary 12.63.1** (Alternating Series). *Let  $(c_n)$  be a sequence of real numbers. Assume that*

1.  $(|c_n|)$  *is monotone decreasing.*

2.  $c_n \geq 0$  *for all odd  $n$ , and  $c_n \leq 0$  for all even  $n$ .*

3.  $c_n \rightarrow 0$  *as  $n \rightarrow \infty$*

*Then  $\sum_{n=0}^{\infty} c_n$  converges.*

PROOF: Take  $a_n = (-1)^{n+1}$  and  $b_n = |c_n|$ . □

**Theorem 12.64.** *Let  $\sum_n c_n z^n$  be a power series with radius of convergence 1. Suppose  $(c_n)$  is monotone decreasing with limit 0. Then  $\sum_n c_n z^n$  converges at every point on the circle  $|z| = 1$  except possibly  $z = 1$ .*

PROOF:

(1)1. LET:  $z$  be a complex number with  $|z| = 1$  and  $z \neq 1$ .

(1)2. For  $n \in \mathbb{N}$ ,

LET:  $a_n = z^n$ .

(1)3. For  $n \in \mathbb{N}$ ,

LET:  $b_n = c_n$ .

(1)4. The partial sums  $\sum_{n=0}^N a_n$  form a bounded sequence.

PROOF:

$$\begin{aligned} \left| \sum_{n=0}^N a_n \right| &= \left| \sum_{n=0}^N z^n \right| \\ &= \left| \frac{1 - z^{N+1}}{1 - z} \right| \\ &\leq \frac{2}{|1 - z|} \end{aligned}$$

(1)5.  $(b_n)$  is monotone decreasing with limit 0.

(1)6. Q.E.D.

PROOF: Theorem 12.63.

□

## 12.8 Absolute Convergence

**Definition 12.65** (Absolute Convergence). Let  $(a_n)$  be a sequence in  $\mathbb{R}^k$ . Then the series  $\sum_{n=0}^{\infty} a_n$  *converges absolutely* iff  $\sum_{n=0}^{\infty} \|a_n\|$  converges.

**Theorem 12.66.** If  $\sum_{n=0}^{\infty} a_n$  converges absolutely then  $\sum_{n=0}^{\infty} a_n$  converges.

PROOF:

(1)1. LET:  $\epsilon > 0$

(1)2. PICK  $N$  such that, for all  $p, q \geq N$ , we have

$$\sum_{n=p}^q \|a_n\| \leq \epsilon .$$

(1)3. For  $p, q \geq N$ , we have

$$\left\| \sum_{n=p}^q a_n \right\| \leq \epsilon .$$

S

(1)4. Q.E.D.

PROOF: Cauchy criterion.

□

## 12.9 Addition and Multiplication of Series

**Theorem 12.67.** If  $\sum_n a_n = A$  and  $\sum_n b_n = B$  then  $\sum_n (a_n + b_n) = A + B$ .

PROOF:

$$\begin{aligned} \sum_{n=0}^N (a_n + b_n) &= \sum_{n=0}^N a_n + \sum_{n=0}^N b_n \\ &\rightarrow A + B \qquad \text{as } N \rightarrow \infty \square \end{aligned}$$

**Theorem 12.68.** If  $\sum_n a_n = A$  then  $\sum_n (ca_n) = cA$ .

PROOF:

$$\begin{aligned} \sum_{n=0}^N ca_n &= c \sum_{n=0}^N a_n \\ &\rightarrow cA \end{aligned} \quad \text{as } N \rightarrow \infty \square$$

**Definition 12.69** (Cauchy Product). The (*Cauchy*) *product* of the series  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  is

$$\sum_{n=0}^{\infty} \sum_{k=0}^n a_k b_{n-k} .$$

**Theorem 12.70.** Let  $(a_n)$  and  $(b_n)$  be sequences of complex numbers. Assume:

1.  $\sum_{n=0}^{\infty} a_n$  converges absolutely.
2.  $\sum_{n=0}^{\infty} b_n$  converges.

For  $n \in \mathbb{N}$ , let  $c_n = \sum_{k=0}^n a_k b_{n-k}$ . Then

$$\sum_{n=0}^{\infty} c_n = \left( \sum_{n=0}^{\infty} a_n \right) \left( \sum_{n=0}^{\infty} b_n \right) .$$

PROOF:

$\langle 1 \rangle 1.$  LET:

$$A = \sum_{n=0}^{\infty} a_n$$

$\langle 1 \rangle 2.$  LET:

$$B = \sum_{n=0}^{\infty} b_n$$

$\langle 1 \rangle 3.$  For  $n \in \mathbb{N}$ ,

LET:

$$A_n = \sum_{k=0}^n a_k .$$

$\langle 1 \rangle 4.$  For  $n \in \mathbb{N}$ ,

LET:

$$B_n = \sum_{k=0}^n b_k .$$

$\langle 1 \rangle 5.$  For  $n \in \mathbb{N}$ ,

LET:

$$C_n = \sum_{k=0}^n a_k b_{n-k} .$$

$\langle 1 \rangle 6.$  For  $n \in \mathbb{N}$ ,

LET:

$$\beta_n = B_n - B$$



(1)7. For  $n \in \mathbb{N}$ ,

$$C_n = A_n B + \sum_{k=0}^n a_k \beta_{n-k} .$$

(1)8. For  $n \in \mathbb{N}$ ,

LET:

$$\gamma_n = \sum_{k=0}^n a_k \beta_{n-k}$$

(1)9.  $A_n B \rightarrow AB$  as  $n \rightarrow \infty$ .

(1)10.  $\gamma_n \rightarrow 0$  as  $n \rightarrow \infty$ .

(2)1. LET:  $\alpha = \sum_{n=0}^{\infty} |a_n|$

(2)2. For all  $\epsilon > 0$  we have  $\limsup_n |\gamma_n| \leq \epsilon \alpha$ .

(3)1. LET:  $\epsilon > 0$

(3)2. PICK  $N$  such that  $\forall n \geq N. |\beta_n| \leq \epsilon$ .

(3)3. For all  $n \geq N$  we have  $|\gamma_n| \leq \left| \sum_{k=0}^N \beta_k \alpha_{n-k} \right| + \epsilon \alpha$ .

PROOF:

$$\begin{aligned} |\gamma_n| &\leq \left| \sum_{k=0}^N \beta_k \alpha_{n-k} \right| + \left| \sum_{k=N+1}^n \beta_k \alpha_{n-k} \right| \\ &\leq \left| \sum_{k=0}^N \beta_k \alpha_{n-k} \right| + \epsilon \alpha \end{aligned}$$

(3)4.

$$\limsup_{n \rightarrow \infty} |\gamma_n| \leq \epsilon \alpha$$

(2)3.  $\limsup_n \gamma_n = 0$

(1)11.  $C_n \rightarrow AB$  as  $n \rightarrow \infty$ .

□

**Theorem 12.71** (Abel). *Let  $(a_n)$  and  $(b_n)$  be sequences of complex numbers. Let*

$$c_n = \sum_{k=0}^n a_k b_{n-k}$$

*for all  $n$ . If the series  $\sum_n a_n$ ,  $\sum_n b_n$  and  $\sum_n c_n$  all converge, then*

$$\sum_n c_n = \left( \sum_n a_n \right) \left( \sum_n b_n \right) .$$

**Proposition 12.72.** *The Cauchy product of two absolutely convergent series is absolutely convergent.*

PROOF:

(1)1. LET:  $\sum_n a_n$  and  $\sum_n b_n$  be two absolutely convergent series.

(1)2. LET:  $c_n = \sum_{k=0}^n a_k b_{n-k}$

(1)3.  $\sum_n |c_n|$  converges.

PROOF:

$$\begin{aligned} \sum_{n=0}^{\infty} |c_n| &= \sum_{n=0}^{\infty} \left| \sum_{k=0}^n a_k b_{n-k} \right| \\ &\leq \sum_{n=0}^{\infty} \sum_{k=0}^n |a_k| |b_{n-k}| \end{aligned}$$

which converges by Theorem 12.70.

## 12.10 Rearrangements

**Definition 12.73** (Rearrangement). A *rearrangement* of a sequence  $(a_n)$  is a sequence  $(a_{\phi(n)})$  for some bijection  $\phi : \mathbb{N} \approx \mathbb{N}$ .

**Theorem 12.74** (Riemann). Let  $\sum_{n=1}^{\infty} a_n$  be a series that converges but not absolutely. Let  $\alpha$  and  $\beta$  be extended reals with  $\alpha \leq \beta$ . Then there exists a rearrangement of  $\sum_n a_n$  with partial sums  $s'_n$  such that

$$\limsup_{n \rightarrow \infty} s'_n = \alpha, \quad \liminf_{n \rightarrow \infty} s'_n = \beta.$$

PROOF:

(1)1. For  $n \in \mathbb{Z}^+$ ,

LET:

$$p_n = \frac{|a_n| + a_n}{2}.$$

(1)2. For  $n \in \mathbb{Z}^+$ ,

LET:

$$q_n = \frac{|a_n| - a_n}{2}.$$

(1)3.  $\forall n \in \mathbb{Z}^+. p_n - q_n = a_n$

(1)4.  $\forall n \in \mathbb{Z}^+. p_n + q_n = |a_n|$

(1)5.  $\forall n \in \mathbb{Z}^+. p_n \geq 0$

(1)6.  $\forall n \in \mathbb{Z}^+. q_n \geq 0$

(1)7.  $\sum_n p_n$  and  $\sum_n q_n$  both diverge.

(2)1. It is not the case that  $\sum_n p_n$  and  $\sum_n q_n$  both converge.

PROOF: This would imply that  $\sum_n |a_n|$  converges by (1)4.

(2)2. It is not the case that  $\sum_n p_n$  converges and  $\sum_n q_n$  diverges.

PROOF: This would imply that  $\sum_n a_n$  diverges by (1)3.

(2)3. It is not the case that  $\sum_n p_n$  diverges and  $\sum_n q_n$  converges.

PROOF: This would imply that  $\sum_n a_n$  diverges by (1)3.

(1)8. LET:  $(P_n)$  be the subsequence of  $(a_n)$  consisting of the non-negative terms.

(1)9. LET:  $(Q_n)$  be the subsequence of  $(|a_n|)$  consisting only of the terms such that  $a_n$  is negative.

(1)10.  $\sum_n P_n$  diverges.

PROOF: It is the series  $\sum_n p_n$  with the zero terms removed.

(1)11.  $\sum_n Q_n$  diverges.

PROOF: It is the series  $\sum_n q_n$  with the zero terms removed.

(1)12. PICK sequences of real numbers  $(\alpha_n)$ ,  $(\beta_n)$  such that  $\alpha_n \rightarrow \alpha$ ,  $\beta_n \rightarrow \beta$ ,  $\alpha_n < \beta_n$  for all  $n$ , and  $\beta_1 > 0$ .

(1)13. PICK strictly increasing sequences of natural numbers  $(m_n)_{n \geq 1}$ ,  $(k_n)_{n \geq 1}$  such that, for all  $n$ ,

$$\sum_{i=1}^{n-1} \left( \sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) + \sum_{j=m_{n-1}+1}^{m_n} P_j > \beta_n$$

$$\sum_{i=1}^n \left( \sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) < \alpha_n$$

and  $m_n$  and  $k_n$  are the smallest integers that make these inequalities true.

PROOF: Given the choice of  $m_1, \dots, m_n$  and  $k_1, \dots, k_n$ , there must exist such an  $m_{n+1}$  by (1)10, and then there must exist such a  $k_{n+1}$  by (1)11.

(1)14. For  $n \in \mathbb{Z}^+$ ,

$$\text{LET: } x_n = \sum_{i=1}^{n-1} \left( \sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) + \sum_{j=m_{n-1}+1}^{m_n} P_j$$

(1)15. For  $n \in \mathbb{Z}^+$ ,

$$\text{LET: } y_n = \sum_{i=1}^n \left( \sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right)$$

(1)16. For  $n \in \mathbb{Z}^+$  we have

$$|x_n - \beta_n| \leq P_{m_n}.$$

PROOF: By minimality of  $m_n$ .

(1)17. For  $n \in \mathbb{Z}^+$  we have

$$|y_n - \alpha_n| \leq Q_{k_n}.$$

PROOF: By minimality of  $k_n$ .

(1)18.  $P_n \rightarrow 0$  as  $n \rightarrow \infty$ .

PROOF: Since  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

(1)19.  $Q_n \rightarrow 0$  as  $n \rightarrow \infty$ .

PROOF: Since  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

(1)20.  $x_n \rightarrow \beta$  as  $n \rightarrow \infty$ .

PROOF: (1)16, (1)18

(1)21.  $y_n \rightarrow \alpha$  as  $n \rightarrow \infty$ .

PROOF: (1)17, (1)19

(1)22. No number less than  $\alpha$  or greater than  $\beta$  is a subsequential limit of the partial sums of the series  $P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + P_{m_1+1} + \dots + P_{m_2} - Q_{k_1+1} - \dots - Q_{k_2} + \dots$

PROOF: Since every partial sum after the  $m_n + k_n$  term is between  $\alpha_n - Q_{k_n}$  and  $\beta_n + P_{m_n}$ .

□

**Theorem 12.75.** If  $\sum_n a_n$  is a series in  $\mathbb{R}^k$  that converges absolutely to  $s$ , then every rearrangement of  $\sum_n a_n$  converges to  $s$ .

PROOF:

(1)1. LET:  $\sum_n a'_n = \sum_n a_{k_n}$  be a rearrangement with partial sums  $s'_n$ .

(1)2. LET:  $\epsilon > 0$

(1)3. PICK  $N$  such that, for all  $m \geq n \geq N$ , we have

$$\sum_{i=n}^m \|a_i\| \leq \epsilon/3 .$$

(1)4. PICK  $p$  such that  $\{1, \dots, N\} \subseteq \{k_1, k_2, \dots, k_p\}$ .

(1)5. For all  $n > p$  we have  $\|s_n - s'_n\| \leq \epsilon$ .

PROOF:

$$\begin{aligned} \|s_n - s'_n\| &= \left\| \sum_{i=1}^N a_i + \sum_{i=N+1}^n a_i - \sum_{i=1}^p a_{k_i} - \sum_{i=p+1}^n a_{k_i} \right\| \\ &\leq \left\| \sum_{i=N+1}^n a_i \right\| + \left\| \sum_{\substack{1 \leq i \leq p \\ k_i > N}} a_{k_i} \right\| + \left\| \sum_{i=p+1}^n a_{k_i} \right\| \\ &\leq \epsilon \end{aligned} \tag{1}3$$

(1)6.  $s'_n \rightarrow s$  as  $n \rightarrow \infty$ .

□

## 12.11 Completion of a Metric Space

**Definition 12.76** (Completion). Let  $X$  be a metric space. Let  $X^*$  be the set of all Cauchy sequences in  $X$ , quotiented by:  $(p_n) \sim (q_n)$  iff  $d(p_n, q_n) \rightarrow 0$ . Define the distance function on  $X^*$  by:

$$\Delta((p_n), (q_n)) = \lim_{n \rightarrow \infty} d(p_n, q_n) .$$

Then the metric space  $X^*$  is called the *completion* of  $X$ .

**Theorem 12.77.** *The completion of  $X^*$  is a complete metric space, and  $X$  is a dense subspace under the embedding that maps  $p \in X$  to the constant sequence  $(p)$ .*

**Example 12.78.**  $\mathbb{R}$  is the completion of  $\mathbb{Q}$ .

# Chapter 13

## Continuity

### 13.1 Limit of a Function

**Definition 13.1** (Limit). Let  $X$  and  $Y$  be metric spaces. Let  $E \subseteq X$  and  $f : E \rightarrow Y$ . Let  $p$  be a limit point of  $E$  and  $q \in Y$ . Then we say  $q$  is the *limit* of  $f$  at  $p$ , and write

$$f(x) \rightarrow q \text{ as } x \rightarrow p, \text{ or } \lim_{x \rightarrow p} f(x) = q ,$$

iff for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x \in E$ , if  $0 < d(x, p) < \delta$  then  $d(f(x), q) < \epsilon$ .

**Theorem 13.2.** *Let  $X$  and  $Y$  be metric spaces. Let  $E \subseteq X$  and  $f : E \rightarrow Y$ . Let  $p$  be a limit point of  $E$  and  $q \in Y$ . Then  $f(x) \rightarrow q$  as  $x \rightarrow p$  if and only if, for every sequence  $(p_n)$  in  $E - \{p\}$  with limit  $p$ , we have  $f(p_n) \rightarrow q$  as  $n \rightarrow \infty$ .*

PROOF:

- (1)1. If  $f(x) \rightarrow q$  as  $x \rightarrow p$  then, for every sequence  $(p_n)$  in  $E - \{p\}$  with limit  $p$ , we have  $f(p_n) \rightarrow q$  as  $n \rightarrow \infty$ .
- (2)1. ASSUME:  $f(x) \rightarrow q$  as  $x \rightarrow p$ .
- (2)2. LET:  $(p_n)$  be a sequence in  $E - \{p\}$  with limit  $p$ .
- (2)3. LET:  $\epsilon > 0$
- (2)4. PICK  $\delta > 0$  such that, for all  $x \in E$ , if  $0 < d(x, p) < \delta$  then  $d(f(x), q) < \epsilon$ .
- (2)5. PICK  $N$  such that, for all  $n \geq N$ , we have  $d(p_n, p) < \delta$
- (2)6.  $\forall n \geq N. d(f(p_n), q) < \epsilon$
- (1)2. If, for every sequence  $(p_n)$  in  $E - \{p\}$  with limit  $p$ , we have  $f(p_n) \rightarrow q$  as  $n \rightarrow \infty$ , then  $f(x) \rightarrow q$  as  $x \rightarrow p$ .
- (2)1. ASSUME:  $f(x) \nrightarrow q$  as  $x \rightarrow p$ .
- (2)2. PICK  $\epsilon > 0$  such that, for all  $\delta > 0$ , there exists a  $x \in E$  such that  $0 < d(x, p) < \delta$  and  $d(f(x), q) \geq \epsilon$ .
- (2)3. For all  $n \in \mathbb{Z}^+$ , PICK  $p_n \in E$  such that  $0 < d(p_n, p) < 1/n$  and  $d(f(p_n), q) \geq \epsilon$ .

- (2)4.  $p_n \rightarrow p$  as  $n \rightarrow \infty$ .  
 (2)5.  $f(p_n) \nrightarrow q$  as  $n \rightarrow \infty$ .

□

**Corollary 13.2.1.** *A function has at most one limit at any point.*

**Theorem 13.3.** *Let  $X$  be a metric space,  $E \subseteq X$ , and  $p$  a limit point of  $E$ . Let  $f, g : E \rightarrow \mathbb{R}^k$ . Assume  $f(x) \rightarrow a$  as  $x \rightarrow p$  and  $g(x) \rightarrow b$  as  $x \rightarrow p$ . Then*

$$f(x) + g(x) \rightarrow a + b \text{ as } x \rightarrow p .$$

PROOF:

- (1)1. LET:  $(p_n)$  be a sequence in  $E$  that converges to  $p$ .  
 (1)2.  $f(p_n) \rightarrow a$  as  $n \rightarrow \infty$ .  
 (1)3.  $g(p_n) \rightarrow b$  as  $n \rightarrow \infty$ .  
 (1)4.  $f(p_n) + g(p_n) \rightarrow a + b$  as  $n \rightarrow \infty$ .

PROOF: Proposition 12.5.

- (1)5. Q.E.D.

PROOF: Theorem 13.2.

□

**Theorem 13.4.** *Let  $X$  be a metric space,  $E \subseteq X$ , and  $p$  a limit point of  $E$ . Let  $f, g : E \rightarrow \mathbb{C}$ . Assume  $f(x) \rightarrow a$  as  $x \rightarrow p$  and  $g(x) \rightarrow b$  as  $x \rightarrow p$ . Then*

$$f(x)g(x) \rightarrow ab \text{ as } x \rightarrow p .$$

PROOF:

- (1)1. LET:  $(p_n)$  be a sequence in  $E$  that converges to  $p$ .  
 (1)2.  $f(p_n) \rightarrow a$  as  $n \rightarrow \infty$ .  
 (1)3.  $g(p_n) \rightarrow b$  as  $n \rightarrow \infty$ .  
 (1)4.  $f(p_n)g(p_n) \rightarrow ab$  as  $n \rightarrow \infty$ .

PROOF: Proposition 12.7.

- (1)5. Q.E.D.

PROOF: Theorem 13.2.

□

**Theorem 13.5.** *Let  $X$  be a metric space,  $E \subseteq X$ , and  $p$  a limit point of  $E$ . Let  $f : E \rightarrow \mathbb{C} - \{0\}$ . Assume  $f(x) \rightarrow a \neq 0$  as  $x \rightarrow p$ . Then*

$$f(x)^{-1} \rightarrow a^{-1} \text{ as } x \rightarrow p .$$

PROOF:

- (1)1. LET:  $(p_n)$  be a sequence in  $E$  that converges to  $p$ .  
 (1)2.  $f(p_n) \rightarrow a$  as  $n \rightarrow \infty$ .  
 (1)3.  $f(p_n)^{-1} \rightarrow a^{-1}$  as  $n \rightarrow \infty$ .

PROOF: Proposition 12.8.

- (1)4. Q.E.D.

PROOF: Theorem 13.2.

□

**Theorem 13.6.** *Let  $X$  be a metric space,  $E \subseteq X$ , and  $p$  a limit point of  $E$ . Let  $f, g : E \rightarrow \mathbb{R}^k$ . Assume  $f(x) \rightarrow a$  as  $x \rightarrow p$  and  $g(x) \rightarrow b$  as  $x \rightarrow p$ . Then*

$$f(x) \cdot g(x) \rightarrow a \cdot b \text{ as } x \rightarrow p .$$

PROOF:

<1>1. LET:  $(p_n)$  be a sequence in  $E$  that converges to  $p$ .

<1>2.  $f(p_n) \rightarrow a$  as  $n \rightarrow \infty$ .

<1>3.  $g(p_n) \rightarrow b$  as  $n \rightarrow \infty$ .

<1>4.  $f(p_n) \cdot g(p_n) \rightarrow a \cdot b$  as  $n \rightarrow \infty$ .

PROOF: Proposition 12.10.

<1>5. Q.E.D.

PROOF: Theorem 13.2.

□

## 13.2 Continuous Functions

**Definition 13.7** (Continuous). Let  $X$  be a metric space,  $E \subseteq X$  and  $p \in E$ . Then  $f$  is *continuous* at  $p$  iff, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x \in E$ , if  $d(x, p) < \delta$  then

$$d(f(x), f(p)) < \epsilon .$$

$f$  is *continuous* or *continuous on  $E$*  iff  $f$  is continuous at every point.

**Theorem 13.8.** *Let  $X$  be a metric space,  $E \subseteq X$  and  $p \in E$  be a limit point of  $E$ . Then  $f$  is continuous at  $p$  iff  $f(x) \rightarrow f(p)$  as  $x \rightarrow p$ .*

PROOF: Easy. □

**Theorem 13.9.** *Let  $X, Y$  and  $Z$  be metric spaces. Let  $E \subseteq X$ . Let  $f : E \rightarrow Y$  and  $g : f(E) \rightarrow Z$ . Let  $p \in E$ . If  $f$  is continuous at  $p$  and  $g$  is continuous at  $f(p)$  then  $g \circ f$  is continuous at  $p$ .*

PROOF:

<1>1. LET:  $\epsilon > 0$

<1>2. PICK  $\delta_1 > 0$  such that, for all  $y \in f(E)$ , if  $d(y, f(p)) < \delta_1$  then  $d(g(y), g(f(p))) < \epsilon$ .

<1>3. PICK  $\delta_2 > 0$  such that, for all  $x \in E$ , if  $d(x, p) < \delta_2$  then  $d(f(x), f(p)) < \delta_1$ .

<1>4. For all  $x \in E$ , if  $d(x, p) < \delta_2$  then  $d(g(f(x)), g(f(p))) < \epsilon$ .

□

**Theorem 13.10.** *Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for every open set  $V \subseteq Y$ , we have  $f^{-1}(V)$  is open in  $X$ .*

PROOF:

- (1)1. If  $f$  is continuous then, for every open set  $V$  in  $Y$ , we have  $f^{-1}(V)$  is open in  $X$ .  
 (2)1. ASSUME:  $f$  is continuous.  
 (2)2. LET:  $V$  be an open set in  $Y$ .  
 PROVE:  $f^{-1}(V)$  is open.  
 (2)3. LET:  $x \in f^{-1}(V)$   
 (2)4. PICK  $\epsilon > 0$  such that  $B(f(x), \epsilon) \subseteq V$ .  
 (2)5. PICK  $\delta > 0$  such that, for all  $x' \in X$ , if  $d(x', x) < \delta$  then  $d(f(x'), f(x)) < \epsilon$ .  
 (2)6.  $B(x, \delta) \subseteq f^{-1}(V)$   
 (1)2. If, for every open set  $V$  in  $Y$ , we have  $f^{-1}(V)$  is open in  $X$ , then  $f$  is continuous.  
 (2)1. ASSUME: For every open set  $V$  in  $Y$ , we have  $f^{-1}(V)$  is open in  $X$ .  
 (2)2. LET:  $p \in X$   
 (2)3. LET:  $\epsilon > 0$   
 (2)4.  $f^{-1}(B(f(p), \epsilon))$  is open in  $X$ .  
 (2)5. PICK  $\delta > 0$  such that  $B(p, \delta) \subseteq f^{-1}(B(f(p), \epsilon))$ .  
 (2)6. For all  $x \in X$ , if  $d(x, p) < \delta$  then  $d(f(x), f(p)) < \epsilon$ .

□

**Corollary 13.10.1.** *Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for every closed set  $C$  in  $Y$ , we have  $f^{-1}(C)$  is closed in  $X$ .*

**Theorem 13.11.** *Let  $X$  be a metric space. Let  $f : X \rightarrow \mathbb{R}^k$ . Then  $f$  is continuous if and only if, for  $i = 1, \dots, k$ , we have  $\pi_i \circ f$  is continuous.*

PROOF:

- (1)1. Each  $\pi_i$  is continuous.  
 (2)1. LET:  $\vec{p} \in \mathbb{R}^k$   
 (2)2. LET:  $\epsilon > 0$   
 (2)3. LET:  $\vec{q} \in \mathbb{R}^k$   
 (2)4. ASSUME:  $\|\vec{p} - \vec{q}\| < \epsilon$   
 (2)5.  $|p_i - q_i| < \epsilon$   
 (1)2. If, for all  $i$ , we have  $\pi_i \circ f$  is continuous, then  $f$  is continuous.  
 (2)1. ASSUME: For all  $i$ , we have  $\pi_i \circ f$  is continuous.  
 (2)2. LET:  $p \in X$   
 (2)3. LET:  $\epsilon > 0$   
 (2)4. For  $i = 1, \dots, k$ , PICK  $\delta_i > 0$  such that, for all  $x \in X$ , we have if  $d(x, p) < \delta_i$  then  $|\pi_i(f(p)) - \pi_i(f(x))| < \epsilon/\sqrt{k}$   
 (2)5. LET:  $\delta = \min(\delta_1, \dots, \delta_k)$   
 (2)6. LET:  $q \in X$  with  $d(p, q) < \delta$ .  
 (2)7.  $\|f(p) - f(q)\| < \epsilon$



PROOF:

$$\begin{aligned}\|f(p) - f(q)\| &= \sqrt{\sum_{i=1}^k |\pi_i(f(p)) - \pi_i(f(q))|^2} \\ &< \sqrt{\sum_{i=1}^k \epsilon^2/k} \\ &= \epsilon\end{aligned}$$

□

**Theorem 13.12.** *Let  $X$  be a compact metric space and  $Y$  a metric space. Let  $f : X \rightarrow Y$  be continuous. Then  $f(X)$  is compact.*

PROOF:

- (1)1. LET:  $\mathcal{V}$  be an open cover of  $f(X)$ .
- (1)2.  $\{f^{-1}(V) : V \in \mathcal{V}\}$  is an open cover of  $X$ .
- (1)3. PICK a finite subcover  $\{f^{-1}(V_1), \dots, f^{-1}(V_n)\}$ .
- (1)4.  $\{V_1, \dots, V_n\}$  covers  $f(X)$ .

□

**Corollary 13.12.1.** *Every continuous function from a compact metric space to  $\mathbb{R}^k$  is bounded.*

**Theorem 13.13** (Extreme Values Theorem). *Let  $X$  be a compact metric space. Let  $f : X \rightarrow \mathbb{R}$ . Let  $M = \sup f(X)$  and  $m = \inf f(X)$ . Then there exist  $p, q \in X$  such that  $f(p) = M$  and  $f(q) = m$ .*

PROOF: Since  $f(X)$  is compact and hence closed. □

**Theorem 13.14.** *Let  $X$  be a compact metric space and  $Y$  a metric space. Let  $f : X \rightarrow Y$  be a continuous bijection. Then  $f^{-1}$  is continuous.*

PROOF:

- (1)1. LET:  $V$  be open in  $Y$ .
- (1)2.  $X - f^{-1}(V)$  is compact.
- (1)3.  $f(X - f^{-1}(V))$  is compact.
- (1)4.  $Y - f(f^{-1}(V))$  is compact.
- (1)5.  $Y - f(f^{-1}(V))$  is closed.
- (1)6.  $f^{-1}(V)$  is open.

□



**Part III**

**More Algebra**



## Chapter 14

# Lie Groups

**Definition 14.1** (Lie Group). A *Lie group*  $G$  is a group  $G$  that is also an analytic differentiable manifold such that the group operation and inverse operation are analytic.

A *homomorphism of Lie groups* is a group homomorphism that is an analytic function.

**Lemma 14.2.** *Every bijective Lie group homomorphism is an isomorphism.*

**Definition 14.3** (Unitary Group). The *unitary group*  $U(n)$  is the Lie group of all  $n \times n$  unitary matrices.

**Definition 14.4** (Special Unitary Group). The *special unitary group*  $SU(n)$  is the Lie group of all  $n \times n$  unitary matrices with determinant 1.

**Definition 14.5** (Lie Subgroup). Let  $G$  be a Lie group. A *Lie subgroup* of  $G$  is a subgroup that is also an analytic submanifold of  $G$ .

**Example 14.6.**  $U(n)$  and  $SU(n)$  are Lie subgroups of  $GL(n, \mathbb{C})$ .