

Summary of Halmos' Naive Set Theory

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Chapter 1

Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*, \in . If $x \in A$ we say that x *belongs to* A , x is an *element* of A , or x is *contained in* A . If this does not hold we write $x \notin A$.

Axiom 1.1 (Axiom of Extensionality). *Two sets are equal if and only if they have the same elements.*

Axiom 1.2 (Axiom of Comprehension, Aussonderungsaxiom). *To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

Axiom 1.3. *A set exists.*

Axiom 1.4 (Axiom of Pairing). *For any two sets, there exists a set that they both belong to.*

Axiom 1.5 (Union Axiom). *For every set A , there exists a set that contains all the elements that belong to at least one element of A .*

Definition 1.6 (Subset). Let A and B be sets. We say that A is a *subset* of B , or B *includes* A , and write $A \subseteq B$ or $B \supseteq A$, iff every element of A is an element of B .

Axiom 1.7 (Power Set Axiom). *For any set A , there exists a set that contains all the subsets of A .*

Chapter 2

The Subset Relation

Theorem 2.1. *For any set A , we have $A \subseteq A$.*

PROOF: Every element of A is an element of A . \square

Theorem 2.2. *For any sets A , B and C , if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

PROOF: If every element of A is an element of B , and every element of B is an element of C , then every element of A is an element of C . \square

Theorem 2.3. *For any sets A and B , if $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

PROOF: If every element of A is an element of B , and every element of B is an element of A , then A and B have the same elements, and therefore are equal by the Axiom of Extensionality. \square

Definition 2.4 (Proper Subset). Let A and B be sets. We say that A is a *proper* subset of B , or B *properly* includes A , and write $A \subsetneq B$ or $B \supsetneq A$, iff $A \subseteq B$ and $A \neq B$.

Chapter 3

Comprehension Notation

Definition 3.1. Given a set A and a condition $S(x)$, we write $\{x \in A : S(x)\}$ for the set whose elements are exactly those elements x of A for which $S(x)$ holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality. \square

Theorem 3.2. *There is no set that contains every set.*

PROOF:

$\langle 1 \rangle 1.$ LET: A be a set.

PROVE: There exists a set B such that $B \notin A$.

$\langle 1 \rangle 2.$ LET: $B = \{x \in A : x \notin x\}$

$\langle 1 \rangle 3.$ If $B \in A$ then we have $B \in B$ if and only if $B \notin B$.

$\langle 1 \rangle 4.$ $B \notin A$

\square

Chapter 4

Unordered Pairs

Theorem 4.1. *There exists a set with no elements.*

PROOF: Pick a set A by Axiom 1.3. Then the set $\{x \in A : x \neq x\}$ has no elements. \square

Definition 4.2 (Empty Set). The *empty set* \emptyset is the set with no elements.

Theorem 4.3. *For any set A we have $\emptyset \subset A$.*

PROOF: Vacuous. \square

Definition 4.4 ((Unordered) Pair). For any sets a and b , the *(unordered) pair* $\{a, b\}$ is the set whose elements are just a and b .

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality. \square

Definition 4.5 (Singleton). For any set a , the *singleton* $\{a\}$ is defined to be $\{a, a\}$.

Chapter 5

Unions

Definition 5.1 (Union). For any set \mathcal{C} , the *union* of \mathcal{C} , $\bigcup \mathcal{C}$, is the set whose elements are the elements of the elements of \mathcal{C} .

We write $\bigcup_{X \in \mathcal{A}} t[X]$ for $\bigcup \{t[X] \mid X \in \mathcal{A}\}$.

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality. \square

Proposition 5.2.

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of \emptyset . \square

Proposition 5.3. For any set A , we have $\bigcup \{A\} = A$.

PROOF: For any x , we have x is an element of an element of $\{A\}$ if and only if x is an element of A . \square

Definition 5.4. We write $A \cup B$ for $\bigcup \{A, B\}$.

Proposition 5.5. For any set A , we have $A \cup \emptyset = A$.

PROOF: $x \in A \cup \emptyset$ iff $x \in A$ or $x \in \emptyset$, iff $x \in A$. \square

Proposition 5.6 (Commutativity). For any sets A and B , we have $A \cup B = B \cup A$.

PROOF: $x \in A \cup B$ iff $x \in A$ or $x \in B$, iff $x \in B$ or $x \in A$, iff $x \in B \cup A$. \square

Proposition 5.7 (Associativity). For any sets A , B and C , we have $A \cup (B \cup C) = (A \cup B) \cup C$.

PROOF: Each is the set of all x such that $x \in A$ or $x \in B$ or $x \in C$. \square

Proposition 5.8 (Idempotence). For any set A , we have $A \cup A = A$.

PROOF: $x \in A$ or $x \in A$ is equivalent to $x \in A$. \square

Proposition 5.9. *For any sets A and B , we have $A \subseteq B$ if and only if $A \cup B = B$.*

PROOF: For any x , the statement "if $x \in A$ then $x \in B$ " is equivalent to " $x \in A$ or $x \in B$ if and only if $x \in B$ ". \square

Proposition 5.10. *For any sets a and b , we have $\{a\} \cup \{b\} = \{a, b\}$.*

PROOF: Immediate from definitions. \square

Chapter 6

Intersections

Definition 6.1 (Intersection). For any sets A and B , the *intersection* $A \cap B$ is defined to be $\{x \in A : x \in B\}$.

Proposition 6.2. For any set A , we have $A \cap \emptyset = \emptyset$.

PROOF: There is no x such that $x \in A$ and $x \in \emptyset$. \square

Proposition 6.3. For any sets A and B , we have

$$A \cap B = B \cap A .$$

PROOF: $x \in A$ and $x \in B$ if and only if $x \in B$ and $x \in A$. \square

Proposition 6.4. For any sets A , B and C , we have

$$A \cap (B \cap C) = (A \cap B) \cap C .$$

PROOF: Each is the set of all x such that $x \in A$ and $x \in B$ and $x \in C$. \square

Proposition 6.5. For any set A , we have

$$A \cap A = A .$$

PROOF: We have $x \in A$ and $x \in A$ if and only if $x \in A$. \square

Proposition 6.6. For any sets A and B , we have $A \subseteq B$ if and only if $A \cap B = A$.

PROOF: For any x , the statement "if $x \in A$ then $x \in B$ " is equivalent to " $x \in A$ and $x \in B$ if and only if $x \in A$ ". \square

Proposition 6.7 (Distributive Law). For any sets A , B and C , we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

PROOF:

$$x \in A \cap (B \cup C) \Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) . \square$$

Proposition 6.8 (Distributive Law). *For any sets A , B and C , we have*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

PROOF:

$$x \in A \vee (x \in B \wedge x \in C) \Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \quad \square$$

Proposition 6.9. *For any sets A , B and C , we have $C \subseteq A$ if and only if $(A \cap B) \cup C = A \cap (B \cup C)$.*

PROOF: The statement "if $x \in C$ then $x \in A$ " is equivalent to the statement " $((x \in A \wedge x \in B) \vee x \in C) \Leftrightarrow (x \in A \wedge (x \in B \vee x \in C))$ ". \square

Definition 6.10 (Disjoint). Two sets A and B are *disjoint* if and only if $A \cap B = \emptyset$.

Definition 6.11 (Pairwise Disjoint). Let A be a set. We say the elements of A are *pairwise disjoint* if and only if, for all $x, y \in A$, if $x \cap y \neq \emptyset$ then $x = y$.

Definition 6.12 (Intersection). For any nonempty set \mathcal{C} , the *intersection* of \mathcal{C} , $\bigcap \mathcal{C}$, is the set that contains exactly those sets that belong to every element of \mathcal{C} .

We write $\bigcap_{X \in \mathcal{A}} t[X]$ for $\bigcap \{t[X] \mid X \in \mathcal{A}\}$.

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{C} be a nonempty set.

$\langle 1 \rangle 2$. There exists a set I whose elements are exactly the sets that belong to every element of \mathcal{C} .

PROOF: Pick $A \in \mathcal{C}$, and take $I = \{x \in A : \forall X \in \mathcal{C}. x \in X\}$.

$\langle 1 \rangle 3$. For any sets I , J , if the elements of I and J are exactly the sets that belong to every element of \mathcal{C} then $I = J$.

PROOF: Axiom of Extensionality.

\square

Chapter 7

Unordered Triples

Definition 7.1 ((Unordered) Triple). Given sets a_1, \dots, a_n , define the (*unordered*) n -tuple $\{a_1, \dots, a_n\}$ to be

$$\{a_1, \dots, a_n\} := \{a_1\} \cup \dots \cup \{a_n\} \ .$$

Chapter 8

Relative Complements

Definition 8.1 (Relative Complement). For any sets A and B , the *difference* or *relative complement* $A - B$ is defined to be

$$A - B := \{x \in A : x \notin B\} .$$

Proposition 8.2. For any sets A and E , we have $A \subseteq E$ if and only if

$$E - (E - A) = A$$

PROOF:

$\langle 1 \rangle 1$. LET: A and E be sets.

$\langle 1 \rangle 2$. If $A \subseteq E$ then $E - (E - A) = A$

$\langle 2 \rangle 1$. ASSUME: $A \subseteq E$

$\langle 2 \rangle 2$. $E - (E - A) \subseteq A$

PROOF: If $x \in E$ and $x \notin E - A$ then $x \in A$.

$\langle 2 \rangle 3$. $A \subseteq E - (E - A)$

PROOF: If $x \in A$ then $x \in E$ and $x \notin E - A$.

$\langle 1 \rangle 3$. If $E - (E - A) = A$ then $A \subseteq E$.

PROOF: Since $E - (E - A) \subseteq E$.

□

Proposition 8.3. For any set E we have

$$E - \emptyset = E$$

PROOF: $x \in E$ if and only if $x \in E$ and $x \notin \emptyset$. □

Proposition 8.4. For any set E we have

$$E - E = \emptyset .$$

PROOF: There is no x such that $x \in E$ and $x \notin E$. □

Proposition 8.5. For any sets A and E , we have

$$A \cap (E - A) = \emptyset .$$

PROOF: There is no x such that $x \in A$ and $x \in E - A$. \square

Proposition 8.6. *Let A and E be sets. Then $A \subseteq E$ if and only if*

$$A \cup (E - A) = E .$$

PROOF:

$\langle 1 \rangle 1$. LET: A and E be sets.

$\langle 1 \rangle 2$. If $A \subseteq E$ then $A \cup (E - A) = E$.

$\langle 2 \rangle 1$. ASSUME: $A \subseteq E$

$\langle 2 \rangle 2$. $A \cup (E - A) \subseteq E$

PROOF: If $x \in A$ or $x \in E - A$ then $x \in E$.

$\langle 2 \rangle 3$. $E \subseteq A \cup (E - A)$

PROOF: If $x \in E$ then either $x \in A$ or $x \notin A$. In the latter case, $x \in E - A$.

$\langle 1 \rangle 3$. If $A \cup (E - A) = E$ then $A \subseteq E$

PROOF: Since $A \subseteq A \cup (E - A)$.

\square

Proposition 8.7. *Let A , B and E be sets. Then:*

1. *If $A \subseteq B$ then $E - B \subseteq E - A$.*

2. *If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A , B and E be sets.

$\langle 1 \rangle 2$. If $A \subseteq B$ then $E - B \subseteq E - A$.

PROOF: If $A \subseteq B$, $x \in E$ and $x \notin B$, then we have $x \in E$ and $x \notin A$.

$\langle 1 \rangle 3$. If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.

$\langle 2 \rangle 1$. ASSUME: $A \subseteq E$

$\langle 2 \rangle 2$. ASSUME: $E - B \subseteq E - A$

$\langle 2 \rangle 3$. LET: $x \in A$

$\langle 2 \rangle 4$. $x \in E$

$\langle 2 \rangle 5$. $x \notin E - A$

$\langle 2 \rangle 6$. $x \notin E - B$

$\langle 2 \rangle 7$. $x \in B$

\square

Example 8.8. We cannot remove the hypothesis $A \subseteq E$ in item 2 above. Let $E = \emptyset$, $A = \{\emptyset\}$ and $B = \emptyset$. Then $E - B = E - A = \emptyset$ but $A \not\subseteq B$.

Proposition 8.9 (De Morgan's Law). *For any sets A , B and E , we have $E - (A \cup B) = (E - A) \cap (E - B)$.*

PROOF: $(x \in E \wedge \neg(x \in A \vee x \in B)) \Leftrightarrow (x \in E \wedge x \notin A \wedge x \in E \wedge x \notin B)$. \square

Proposition 8.10 (De Morgan's Law). *For any sets A , B and E , we have $E - (A \cap B) = (E - A) \cup (E - B)$.*

PROOF: $(x \in E \vee \neg(x \in A \wedge x \in B)) \Leftrightarrow (x \in E \wedge x \notin A) \vee (x \in E \wedge x \notin B)$. \square

Proposition 8.11. *For any sets A , B and E , if $A \subseteq E$ then*

$$A - B = A \cap (E - B) .$$

PROOF: If $A \subseteq E$ then we have $(x \in A \wedge x \notin B) \Leftrightarrow (x \in A \wedge x \in E \wedge x \notin B)$. \square

Proposition 8.12. *For any sets A and B , we have $A \subseteq B$ if and only if $A - B = \emptyset$.*

PROOF: Both are equivalent to the statement that there is no x such that $x \in A$ and $x \notin B$. \square

Proposition 8.13. *For any sets A and B , we have*

$$A - (A - B) = A \cap B .$$

PROOF: $(x \in A \wedge \neg(x \in A \wedge x \notin B)) \Leftrightarrow x \in A \wedge x \in B$. \square

Proposition 8.14. *For any sets A , B and C , we have*

$$A \cap (B - C) = (A \cap B) - (A \cap C) .$$

PROOF: $(x \in A \wedge x \in B \wedge x \notin C) \Leftrightarrow (x \in A \wedge x \in B \wedge \neg(x \in A \wedge x \in C))$. \square

Proposition 8.15. *For any sets A , B , C and E , if $(A \cap B) - C \subseteq E$ then we have*

$$A \cap B \subseteq (A \cap C) \cup (B \cap (E - C)) .$$

PROOF:

<1>1. LET: $x \in A \cap B$

PROVE: $x \in (A \cap C) \cup (B \cap (E - C))$

<1>2. CASE: $x \in C$

PROOF: Then $x \in A \cap C$.

<1>3. CASE: $x \notin C$

PROOF: Then $x \in E$ and so $x \in B \cap (E - C)$.

\square

Proposition 8.16. *For any sets A , B , C and E , we have*

$$(A \cup C) \cap (B \cup (E - C)) \subseteq A \cup B .$$

PROOF: The statement $(x \in A \vee x \in C) \wedge (x \in B \vee (x \in E \wedge x \notin C))$ implies $x \in A \vee x \in B$. \square

Proposition 8.17 (De Morgan's Law). *Let E be a set and \mathcal{C} a nonempty set. Then*

$$E - \bigcup \mathcal{C} = \bigcap_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy. \square

Proposition 8.18 (De Morgan's Law). *Let E be a set and \mathcal{C} a nonempty set. Then*

$$E - \bigcap \mathcal{C} = \bigcup_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy. \square

Chapter 9

Symmetric Difference

Definition 9.1 (Symmetric Difference). For any sets A and B , the *symmetric difference* $A + B$ is defined to be

$$A + B := (A - B) \cup (B - A) .$$

Proposition 9.2. For any sets A and B , we have

$$A + B = B + A$$

PROOF: From the commutativity of union. \square

Proposition 9.3. For any sets A , B and C , we have

$$A + (B + C) = (A + B) + C .$$

PROOF: Each is the set of all x that belong to either exactly one or all three of A , B and C . \square

Proposition 9.4. For any set A , we have

$$A + \emptyset = A .$$

PROOF:

$$\begin{aligned} A + \emptyset &= (A - \emptyset) \cup (\emptyset - A) \\ &= A \cup \emptyset \\ &= A \end{aligned}$$

\square

Proposition 9.5. For any set A we have

$$A + A = \emptyset .$$

PROOF:

$$\begin{aligned} A + A &= (A - A) \cup (A - A) \\ &= \emptyset \cup \emptyset \\ &= \emptyset \end{aligned}$$

\square

Chapter 10

Power Sets

Definition 10.1 (Power Set). For any set A , the *power set* of A , $\mathcal{P}A$, is the set whose elements are exactly the subsets of A .

PROOF: This exists by the Power Set Axiom and Axiom of Comprehension, and is unique by the Axiom of Extensionality. \square

Proposition 10.2.

$$\mathcal{P}\emptyset = \{\emptyset\}$$

PROOF: The only subset of \emptyset is \emptyset . \square

Proposition 10.3. For any set a , we have

$$\mathcal{P}\{a\} = \{\emptyset, \{a\}\} .$$

PROOF: The only subsets of $\{a\}$ are \emptyset and $\{a\}$. \square

Proposition 10.4. For any sets a and b , we have

$$\mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} .$$

PROOF: The only subsets of $\{a, b\}$ are \emptyset , $\{a\}$, $\{b\}$ and $\{a, b\}$. \square

Proposition 10.5. For any nonempty set \mathcal{C} we have

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}X = \mathcal{P}\left(\bigcap \mathcal{C}\right) .$$

PROOF:

$$x \in \bigcup_{X \in \mathcal{C}} \mathcal{P}X \Leftrightarrow \forall X \in \mathcal{C}. x \subseteq X$$

$$\Leftrightarrow \forall X \in \mathcal{C}. \forall y \in x. y \in X$$

$$\Leftrightarrow \forall y \in x. \forall X \in \mathcal{C}. y \in X$$

$$\Leftrightarrow x \subseteq \bigcap \mathcal{C}$$

\square

Proposition 10.6. *For any set \mathcal{C} we have*

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}X \subseteq \mathcal{P}\bigcup \mathcal{C} .$$

PROOF: If there exists $X \in \mathcal{C}$ such that $x \subseteq X$ then $x \subseteq \bigcup \mathcal{C}$. \square

Proposition 10.7. *For any set E , we have*

$$\bigcap \mathcal{P}E = \emptyset .$$

PROOF: Since $\emptyset \in \mathcal{P}E$. \square

Proposition 10.8. *For any sets E and F , if $E \subseteq F$ then $\mathcal{P}E \subseteq \mathcal{P}F$.*

PROOF: If $E \subseteq F$ and $X \subseteq E$ then $X \subseteq F$. \square

Chapter 11

Ordered Pairs

Definition 11.1 (Ordered Pair). For any sets a and b , the *ordered pair* (a, b) is defined by

$$(a, b) := \{\{a\}, \{a, b\}\} .$$

Proposition 11.2. For any sets a, b, x and y , if $(a, b) = (x, y)$ then $a = x$ and $b = y$.

PROOF:

$\langle 1 \rangle 1$. LET: a, b, x and y be sets.

$\langle 1 \rangle 2$. ASSUME: $(a, b) = (x, y)$

$\langle 1 \rangle 3$. $a = x$

PROOF: $\{a\} = \bigcap (a, b) = \bigcap (x, y) = \{x\}$.

$\langle 1 \rangle 4$. $\{a, b\} = \{x, y\}$

$\langle 1 \rangle 5$. CASE: $a = b$

$\langle 2 \rangle 1$. $x = y$

PROOF: Since $\{x, y\} = \{a, b\}$ is a singleton.

$\langle 2 \rangle 2$. $b = y$

PROOF: $b = a = x = y$

$\langle 1 \rangle 6$. CASE: $a \neq b$

$\langle 2 \rangle 1$. $x \neq y$

PROOF: Since $\{x, y\} = \{a, b\}$ is not a singleton.

$\langle 2 \rangle 2$. $b = y$

PROOF: $\{b\} = \{a, b\} - \{a\} = \{x, y\} - \{x\} = \{y\}$.

□

Definition 11.3 (Cartesian Product). For any sets A and B , the *Cartesian product* $A \times B$ is

$$A \times B := \{p \in \mathcal{PP}(A \cup B) : \exists a \in A. \exists b \in B. p = (a, b)\} .$$

Proposition 11.4. For any sets A, B and X , we have

$$(A \cup B) \times X = (A \times X) \cup (B \times X) .$$

PROOF: Easy. \square

Proposition 11.5. *For any sets A , B , X and Y , we have*

$$(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y) .$$

PROOF: Easy. \square

Proposition 11.6. *For any sets A , B and X , we have*

$$(A - B) \times X = (A \times X) - (B \times X) .$$

PROOF: Easy. \square

Proposition 11.7. *For any sets A and B , we have $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.*

PROOF: Easy. \square

Proposition 11.8. *For any sets A , B , X and Y , if $A \subseteq X$ and $B \subseteq Y$ then $A \times B \subseteq X \times Y$. The converse holds assuming $A \neq \emptyset$ and $B \neq \emptyset$.*

PROOF: Easy. \square

Chapter 12

Relations

Definition 12.1 (Relation). A *relation* is a set of ordered pairs.

If R is a relation, we write xRy for $(x, y) \in R$.

Given sets X and Y , a relation *between X and Y* is a subset of $X \times Y$.

Given a set X , a relation *on X* is a relation between X and X .

Definition 12.2 (Domain). The *domain* of a relation R is the set

$$\text{dom } R := \{x \in \bigcup \bigcup R : \exists y. (x, y) \in R\} .$$

Definition 12.3 (Range). The *range* of a relation R is the set

$$\text{ran } R := \{y \in \bigcup \bigcup R : \exists x. (x, y) \in R\} .$$

Definition 12.4 (Reflexive). Let R be a relation on X . Then R is *reflexive* iff, for all $x \in X$, we have xRx .

Definition 12.5 (Symmetric). Let R be a relation on X . Then R is *symmetric* iff, whenever xRy , then yRx .

Definition 12.6 (Transitive). Let R be a relation on X . Then R is *transitive* iff, whenever xRy and yRz , then xRz .

Definition 12.7 (Equivalence Relation). Let R be a relation on X . Then R is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Definition 12.8 (Partition). Let X be a set. A *partition* of X is a pairwise disjoint set of nonempty subsets of X whose union is X .

Definition 12.9 (Equivalence Class). Let R be an equivalence relation on X . Let $x \in X$. The *equivalence class* of x with respect to R is

$$x/R := \{y \in X : xRy\} .$$

We write X/R for the set of all equivalence classes with respect to R .