

Mathematics

Robin Adams

October 5, 2023

Contents

1	Primitive Terms and Axioms	7
1.1	Primitive Terms	7
1.2	Axioms	7
2	Sets and Functions	9
2.1	Surjective Functions	9
2.2	Bijections	9
2.3	Identity Function	9
2.4	Composition	9
2.4.1	The Empty Set	13
2.4.2	The Singleton	13
2.4.3	Subsets	14
2.5	Relations	15
2.6	Power Set	15
2.7	Cartesian Product	15
2.8	Quotient Sets	15
2.9	Partitions	16
2.10	Disjoint Union	16
3	Order Theory	17
3.1	Relations	17
4	Category Theory	19
4.1	Categories	19
4.1.1	Monomorphisms	20
4.1.2	Epimorphisms	20
4.1.3	Sections and Retractions	20
4.1.4	Isomorphisms	21
4.1.5	Initial Objects	21
4.1.6	Terminal Objects	22
4.1.7	Zero Objects	22
4.1.8	Triads	22
4.1.9	Cotriads	22
4.1.10	Pullbacks	23

4.1.11	Pushouts	26
4.1.12	Subcategories	29
4.1.13	Opposite Category	29
4.1.14	Groupoids	29
4.1.15	Concrete Categories	30
4.1.16	Power of Categories	30
4.1.17	Arrow Category	30
4.1.18	Slice Category	30
4.2	Functors	33
4.3	Natural Transformations	36
4.4	Bifunctors	37
4.5	Functor Categories	38
5	Monoid Theory	41
6	Group Theory	43
7	Ring Theory	45
8	Field Theory	47
9	Linear Algebra	49
10	Topology	51
10.1	Topological Spaces	51
10.1.1	Subspaces	54
10.1.2	Topological Disjoint Union	55
10.1.3	Product Topology	57
10.1.4	Bases	59
10.1.5	Subbases	59
10.1.6	Countability Axioms	59
10.2	Continuous Functions	60
10.2.1	Paths	61
10.2.2	Loops	62
10.3	Convergence	62
10.4	Subspaces	62
10.5	Embedding	63
10.6	Quotient Spaces	63
10.7	Connected Spaces	65
10.8	Hausdorff Spaces	66
10.9	Separable Spaces	66
10.10	Sequential Compactness	66
10.11	Compactness	66
10.12	Quotient Spaces	67
10.13	Gluing	68
10.14	Metric Spaces	69

<i>CONTENTS</i>	5
10.14.1 Products	70
10.15 Complete Metric Spaces	71
10.16 Manifolds	72
11 Homotopy Theory	73
11.1 Homotopies	73
11.2 Homotopy Equivalence	73
12 Simplicial Complexes	75
12.1 Cell Decompositions	75
12.2 CW-complexes	75
13 Topological Groups	77
13.1 Continuous Actions	77
14 Topological Vector Spaces	79
14.1 Cauchy Sequences	79
14.2 Seminorms	80
14.3 Fréchet Spaces	80
14.4 Normed Spaces	80
14.5 Inner Product Spaces	82
14.6 Banach Spaces	83
14.7 Hilbert Spaces	83
14.8 Locally Convex Spaces	83

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*.

For any set A , let there be *elements* of A . We write $a \in A$ for: a is an element of A .

For any sets A and B , let there be a set B^A , whose elements are called *functions* from A to B . We write $f : A \rightarrow B$ for $f \in B^A$.

For any function $f : A \rightarrow B$ and element $a \in A$, let there be an element $f(a) \in B$, the *value* of the function f at the *argument* a .

1.2 Injections, Surjections and Bijections

Definition 1.2.1 (Injective). A function $f : A \rightarrow B$ is *injective* or an *injection* iff, for all $x, y : \text{El}(A)$, if $f(x) = f(y)$ then $x = y$.

Definition 1.2.2 (Surjective). A function $f : A \rightarrow B$ is *surjective* or a *surjection* iff, for all $y : \text{El}(B)$, there exists $x : \text{El}(A)$ such that $f(x) = y$.

Definition 1.2.3 (Bijective). A function $f : A \rightarrow B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$ ".

1.3 Axioms

Axiom Schema 1.3.1 (Choice). *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x : \text{El}(X)$ and $y : \text{El}(Y)$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a : \text{El}(A)$, there exists $b : \text{El}(B)$ such that $P[A, B, a, b]$. Then there exists a function $f : A \rightarrow B$ such that $\forall a : \text{El}(A). P[A, B, a, f(a)]$.

Axiom 1.3.2 (Extensionality). *Let $f, g : A \rightarrow B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Axiom 1.3.3 (Pairing). *For any sets A and B , there exists a set $A \times B$, the Cartesian product of A and B , and functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ such that, for all $a : \text{El}(A)$ and $b : \text{El}(B)$, there exists a unique $(a, b) : \text{El}(A \times B)$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Axiom Schema 1.3.4 (Separation). *For every property $P[X, x]$ where X is a set variable and $x : \text{El}(X)$, the following is an axiom:*

For every set A , there exists a set $S = \{x : \text{El}(A) \mid P[A, x]\}$ and an injection $i : S \rightarrow A$ such that, for all $x : \text{El}(A)$, we have

$$(\exists y : S. i(y) = x) \Leftrightarrow P[A, x] .$$

Axiom 1.3.5 (Infinity). *There exists a set \mathbb{N} , an element $0 : \text{El}(\mathbb{N})$, and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that:*

- $\forall n : \text{El}(\mathbb{N}). s(n) \neq 0$
- $\forall m, n : \text{El}(\mathbb{N}). s(m) = s(n) \Rightarrow m = n$.

Axiom Schema 1.3.6 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A , there exists a set B , a function $p : B \rightarrow A$, a set Y and a relation $M : B \rightarrowtail Y$ such that:

- $\forall b \in B. P[A, \{y \in Y : bMy\}, p(b)]$
- *For all $a \in A$, if $\exists Y. P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 1.3.7 (Composite). *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. The composite $g \circ f : A \rightarrow C$ is the function such that, for all $a \in A$, we have*

$$(g \circ f)(a) = g(f(a)) .$$

Axiom 1.3.8 (Universe). *There exists a set E , a set U and a function $el : E \rightarrow U$ such that the following holds.*

Let us say that a set A is small iff there exists $u \in U$ such that $A \approx \{e \in E : el(e) = u\}$.

- \mathbb{N} is small.

- For any U -small sets A and B , the set B^A is small.
- For any U -small sets A and B , the set $A \times B$ is small.
- Let $f : A \rightarrow B$ be a function. If B is small and $f^{-1}(b)$ is U -small for all $b \in B$, then A is small.
- If $p : B \twoheadrightarrow A$ is a surjective function such that A is small, then there exists a U -small set C , a surjection $q : C \twoheadrightarrow A$, and a function $f : C \rightarrow B$ such that $q = p \circ f$.

Chapter 2

Sets and Functions

2.1 Identity Function

Definition 2.1.1 (Identity). For any set A , the *identity* function $\text{id}_A : A \rightarrow A$ is the function defined by $\text{id}_A(a) = a$.

2.2 Composition

Proposition 2.2.1. *Given functions $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f .$$

PROOF: Each is the function that maps $a \in A$ to $h(g(f(a)))$. \square

Proposition 2.2.2. *The composite of injective functions is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$ be injective.

$\langle 1 \rangle 2$. LET: $x, y \in A$ satisfy $(g \circ f)(x) = (g \circ f)(y)$

$\langle 1 \rangle 3$. $g(f(x)) = g(f(y))$

$\langle 1 \rangle 4$. $f(x) = f(y)$

$\langle 1 \rangle 5$. $x = y$

\square

Proposition 2.2.3. *For functions $f : A \rightarrow B$ and $g : B \rightarrow C$, if $g \circ f$ is injective then f is injective.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: $g \circ f$ is injective.

$\langle 1 \rangle 2$. LET: $x, y \in A$

$\langle 1 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 1 \rangle 4$. $g(f(x)) = g(f(y))$

$\langle 1 \rangle 5$. $x = y$

□

Proposition 2.2.4. *Let $f : A \rightarrow B$. Then f is injective if and only if, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.*

PROOF:

⟨1⟩1. If f is injective then, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

⟨2⟩1. ASSUME: f is injective.

⟨2⟩2. LET: X be a set.

⟨2⟩3. LET: $x, y : X \rightarrow A$

⟨2⟩4. ASSUME: $f \circ x = f \circ y$

⟨2⟩5. $\forall t \in X. x(t) = y(t)$

⟨3⟩1. LET: $t \in X$

⟨3⟩2. $f(x(t)) = f(y(t))$

PROOF: ⟨2⟩4

⟨3⟩3. $x(t) = y(t)$

PROOF: ⟨2⟩1

⟨2⟩6. $x = y$

PROOF: Axiom of Extensionality.

⟨1⟩2. If, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

PROOF: Take $X = 1$.

□

Proposition 2.2.5. *The composite of surjective functions is surjective.*

PROOF:

⟨1⟩1. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$ be injective.

⟨1⟩2. LET: $c \in C$

⟨1⟩3. PICK $b \in B$ such that $g(b) = c$.

⟨1⟩4. PICK $a \in A$ such that $f(a) = b$.

⟨1⟩5. $(g \circ f)(a) = c$

□

Proposition 2.2.6. *Let $f : A \rightarrow B$. Then the following are equivalent.*

1. f is surjective.

2. For any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$.

3. There exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

⟨2⟩1. ASSUME: f is surjective.

⟨2⟩2. LET: X be a set.

⟨2⟩3. LET: $g, h : B \rightarrow X$

⟨2⟩4. ASSUME: $g \circ f = h \circ f$

⟨2⟩5. LET: $b \in B$

PROVE: $g(b) = h(b)$
 $\langle 2 \rangle 6$. PICK $a \in A$ such that $f(a) = b$
 $\langle 2 \rangle 7$. $g(b) = h(b)$
 PROOF: $g(b) = g(f(a)) = h(f(a)) = h(b)$
 $\langle 1 \rangle 2$. $1 \Rightarrow 3$
 $\langle 2 \rangle 1$. ASSUME: f is surjective.
 $\langle 2 \rangle 2$. PICK $g : B \rightarrow A$ such that, for all $b \in B$, we have $f(g(b)) = b$.
 PROOF: Axiom of Choice.
 $\langle 2 \rangle 3$. $f \circ g = \text{id}_B$.
 $\langle 1 \rangle 3$. $3 \Rightarrow 2$
 $\langle 2 \rangle 1$. LET: $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$
 $\langle 2 \rangle 2$. LET: X be a set.
 $\langle 2 \rangle 3$. LET: $h, k : B \rightarrow X$
 $\langle 2 \rangle 4$. ASSUME: $h \circ f = k \circ f$
 $\langle 2 \rangle 5$. $h = k$
 PROOF: $h = h \circ f \circ g = k \circ f \circ g = k$
 $\langle 1 \rangle 4$. $2 \Rightarrow 1$
 $\langle 2 \rangle 1$. ASSUME: 2
 $\langle 2 \rangle 2$. LET: $b \in B$
 $\langle 2 \rangle 3$. LET: $h : B \rightarrow 2$ be the function that maps everything to 1.
 $\langle 2 \rangle 4$. LET: $k : B \rightarrow 2$ be the function that maps b to 0 and everything else to 1.
 $\langle 2 \rangle 5$. $h \neq k$
 $\langle 2 \rangle 6$. $h \circ f \neq k \circ f$
 $\langle 2 \rangle 7$. PICK $a \in A$ such that $h(f(a)) \neq k(f(a))$
 $\langle 2 \rangle 8$. $f(a) = b$
 \square

Proposition 2.2.7. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is surjective then g is surjective.*

PROOF:
 $\langle 1 \rangle 1$. LET: $c \in C$
 $\langle 1 \rangle 2$. There exists $a \in A$ such that $g(f(a)) = c$.
 $\langle 1 \rangle 3$. There exists $b \in B$ such that $g(b) = c$.
 \square

Proposition 2.2.8. *The composite of bijections is a bijection.*

PROOF: Propositions 2.4.2 and 2.4.5. \square

Proposition 2.2.9. *Let $f : A \rightarrow B$. Then f is bijective if and only if there exists a function $f^{-1} : B \rightarrow A$, the inverse of f , such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$, in which case the inverse is unique.*

PROOF:
 $\langle 1 \rangle 1$. If f is bijective then there exists $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$.

- ⟨2⟩1. ASSUME: f is bijective.
 ⟨2⟩2. PICK $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$
 PROOF: Proposition 2.4.6.
 ⟨2⟩3. $f \circ g \circ f = f$
 ⟨2⟩4. $g \circ f = \text{id}_A$
 PROOF: Proposition 2.4.4.
 ⟨1⟩2. If there exists $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$,
 then f is bijective.
 ⟨2⟩1. LET: $f^{-1} : B \rightarrow A$ satisfy $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$
 ⟨2⟩2. f is injective.
 PROOF: If $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$.
 ⟨2⟩3. f is surjective.
 PROOF: Proposition 2.4.6.
 ⟨1⟩3. If $g, h : B \rightarrow A$ satisfy $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$ and $f \circ h = \text{id}_B$ and
 $h \circ f = \text{id}_A$ then $g = h$.
 PROOF: We have $g = g \circ f \circ h = h$.
 □

Proposition 2.2.10. *Let $f : A \rightarrow B$. Then $\text{id}_B \circ f = f = f \circ \text{id}_A$.*

PROOF: Each is the function that maps a to $f(a)$. □

Proposition 2.2.11.

$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps f to $(\pi_1 \circ f, \pi_2 \circ f)$ is a bijection. □

Proposition 2.2.12.

$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function Φ such that $\Phi(f)(c)(b) = f(b, c)$ is a bijection. □

2.2.1 The Empty Set

Theorem 2.2.13. *There exists a set which has no elements.*

PROOF:

- ⟨1⟩1. PICK a set A
 PROOF: By the Axiom of Infinity, a set exists.
 ⟨1⟩2. LET: $S = \{x : \text{El}(A) \mid \perp\}$ with injection $i : S \rightarrow A$
 PROOF: Axiom of Separation.
 ⟨1⟩3. S has no elements.
 □

Theorem 2.2.14. *If E and E' have no elements then $E \approx E'$.*

PROOF:

- ⟨1⟩1. LET: E and E' have no elements.
 ⟨1⟩2. PICK a function $F : E \rightarrow E'$.

PROOF: Axiom of Choice since vacuously $\forall x : \text{El}(E) . \exists y : \text{El}(E') . \top$.

$\langle 1 \rangle 3$. F is injective.

PROOF: Vacuously, for all $x, y : \text{El}(E)$, if $F(x) = F(y)$ then $x = y$.

$\langle 1 \rangle 4$. F is surjective.

PROOF: Vacuously, for all $y : \text{El}(E)$, there exists $x : \text{El}(E)$ such that $F(x) = y$.

□

Definition 2.2.15 (Empty Set). The *empty set* \emptyset is the set with no elements.

2.2.2 The Singleton

Theorem 2.2.16. *There exists a set that has exactly one element.*

PROOF:

$\langle 1 \rangle 1$. PICK a set A that has an element.

PROOF: By the Axiom of Infinity, there exists a set that has an element.

$\langle 1 \rangle 2$. PICK $a : \text{El}(A)$

$\langle 1 \rangle 3$. PICK a set S and injection $i : S \rightarrowtail A$ such that, for all $x : \text{El}(A)$, there exists $s : \text{El}(S)$ such that $s = x$ if and only if $x = a$

$\langle 1 \rangle 4$. S has exactly one element.

□

Theorem 2.2.17. *If A and B both have exactly one element then $A \approx B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B both have exactly one element a and b respectively.

$\langle 1 \rangle 2$. LET: $F : A \rightarrow B$ be the function such that, for all $x : \text{El}(A)$, we have
 $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$. F is a bijection.

□

Definition 2.2.18 (Singleton). Let 1 be the set that has exactly one element. Let $*$ be its element.

2.2.3 Subsets

Definition 2.2.19 (Subset). A *subset* of a set A consists of a set S and an injection $i : S \rightarrowtail A$. We write $(S, i) \subseteq A$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Proposition 2.2.20. *For any subset (S, i) of A we have $(S, i) = (S, i)$.*

PROOF: We have $\text{id}_S : S \approx S$ and $i \circ \text{id}_S = i$.

Proposition 2.2.21. *If $(S, i) = (T, j)$ then $(T, j) = (S, i)$.*

PROOF: If $\phi : S \approx T$ and $j \circ \phi = i$ then $\phi^{-1} : T \approx S$ and $i \circ \phi^{-1} = j$. □

Proposition 2.2.22. *If $(R, i) = (S, j)$ and $(S, j) = (T, k)$ then $(R, i) = (T, k)$.*

PROOF: If $\phi : R \approx S$ and $j \circ \phi = i$, and $\psi : S \approx T$ and $k \circ \psi = j$, then $\psi \circ \phi : R \approx T$ and $k \circ \psi \circ \phi = i$. \square

Definition 2.2.23 (Membership). Given $(S, i) \subseteq A$ and $a \in A$, we write $a \in (S, i)$ for $\exists s \in S. i(s) = a$.

Proposition 2.2.24. *If $a \in (S, i)$ and $(S, i) = (T, j)$ then $a \in (T, j)$.*

PROOF: If $i(s) = a$ then $j(\phi(s)) = a$. \square

Definition 2.2.25 (Union). Given subsets S and T of A , the *union* is the subset $\{x \in A : x \in S \vee x \in T\}$.

Definition 2.2.26 (Intersection). Given subsets S and T of A , the *intersection* is the subset $\{x \in A : x \in S \wedge x \in T\}$.

Proposition 2.2.27 (Distributive Law).

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

Proposition 2.2.28 (Distributive Law).

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

Definition 2.2.29. Given a set A , we write \emptyset for the subset $(\emptyset, !)$ where $!$ is the unique function $\emptyset \rightarrow A$.

Proposition 2.2.30.

$$S \cup \emptyset = S$$

Proposition 2.2.31.

$$S \cap \emptyset = \emptyset$$

Definition 2.2.32 (Inclusion). Given subsets (S, i) and (T, j) of a set A , we write $(S, i) \subseteq (T, j)$ iff there exists $f : S \rightarrow T$ such that $j \circ f = i$.

Proposition 2.2.33.

$$\emptyset \subseteq S$$

Definition 2.2.34 (Disjoint). Subsets S and T of A are *disjoint* iff $S \cap T = \emptyset$.

Definition 2.2.35 (Difference). Given subsets S and T of A , the *difference* of S and T is $S - T = \{x \in A : x \in S \wedge x \notin T\}$.

Proposition 2.2.36 (De Morgan's Law).

$$R - (S \cup T) = (R - S) \cap (R - T)$$

Proposition 2.2.37 (De Morgan's Law).

$$R - (S \cap T) = (R - S) \cup (R - T)$$

2.3 Relations

Definition 2.3.1 (Relation). Let A and B be sets. A *relation* R between A and B , $R : A \rightarrowtail B$, is a subset of $A \times B$.

Given $a \in A$ and $b \in B$, we write aRb for $(a, b) \in R$.

2.4 Power Set

Definition 2.4.1 (Power Set). The *power set* of a set A is $\mathcal{P}A := 2^A$.

Given $S \in \mathcal{P}A$ and $a \in A$, we write $a \in A$ for $S(a) = 1$.

2.5 Cartesian Product

Definition 2.5.1 (Cartesian Product). Let A and B be sets. The *Cartesian product* of A and B , $A \times B$, is the tabulation of the relation $A \rightarrowtail B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

2.6 Quotient Sets

Proposition 2.6.1. Let \sim be an equivalence relation on X . Then there exists a set X/\sim , the quotient set of X with respect to \sim , and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the canonical projection, such that, for all $x, y : \text{El}(X)$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to \sim , then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

2.7 Partitions

Definition 2.7.1 (Partition). A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X .

2.8 Disjoint Union

Theorem 2.8.1. For any sets A and B , there exists a set $A + B$, the disjoint union of A and B , and functions $\kappa_1 : A \rightarrow A + B$ and $\kappa_2 : B \rightarrow A + B$, the injections, such that, for every set X and functions $f : A \rightarrow X$ and $g : B \rightarrow X$, there exists a unique function $[f, g] : A + B \rightarrow X$ such that $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$.

PROOF:

$\langle 1 \rangle 1$. LET: $A + B := \{p \in \mathcal{P}A \times \mathcal{P}B : \exists a \in A. p = (\{a\}, \emptyset) \vee \exists b \in B. p = (\emptyset, \{b\})\}$

Definition 2.8.2.

Chapter 3

Order Theory

3.1 Relations

Definition 3.1.1 (Reflexive). A relation $R \subseteq A \times A$ is *reflexive* iff, for all $a \in A$, we have $(a, a) \in R$.

Definition 3.1.2 (Antisymmetric). A relation $R \subseteq A \times A$ is *antisymmetric* iff, for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Definition 3.1.3 (Transitive). A relation $R \subseteq A \times A$ is *transitive* iff, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Definition 3.1.4 (Partial Order). A *partial order* on a set A is a relation on A that is reflexive, antisymmetric and transitive.

We say (A, \leq) is a *partially ordered set* or *poset* iff \leq is a partial order on A .

Chapter 4

Category Theory

4.1 Categories

Definition 4.1.1. A *category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \text{Ob}(\mathcal{C})$.
- for any objects X and Y , a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y . We write $f : X \rightarrow Y$ for $f \in \mathcal{C}[X, Y]$.
- for any objects X, Y and Z , a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \rightarrow \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \rightarrow Y, g : Y \rightarrow Z$ and $h : Z \rightarrow W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X , there exists a morphism $\text{id}_X : X \rightarrow X$, the *identity morphism* on X , such that:
 - for any object Y and morphism $f : Y \rightarrow X$ we have $\text{id}_X \circ f = f$
 - for any object Y and morphism $f : X \rightarrow Y$ we have $f \circ \text{id}_X = f$

We write the composite of morphism f_1, \dots, f_n as $f_n \circ \dots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 4.1.2. Let **Set** be the category of small sets and functions.

Definition 4.1.3. We identify a poset (A, \leq) with the category with:

- set of objects A
- for $a, b \in A$, the set of homomorphisms is $\{x \in 1 : a \leq b\}$

Proposition 4.1.4. A category is a poset iff, for any two objects, there exists at most one morphism between them.

Proposition 4.1.5. *The identity morphism on an object is unique.*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{C} be a category.

$\langle 1 \rangle 2$. LET: $A \in \mathcal{C}$

$\langle 1 \rangle 3$. LET: $i, j : A \rightarrow A$ be identity morphisms on A .

$\langle 1 \rangle 4$. $i = j$

PROOF:

$$\begin{aligned} i &= i \circ j && (j \text{ is an identity on } A) \\ &= j && (i \text{ is an identity on } A) \end{aligned}$$

□

Definition 4.1.6. Given $f : A \rightarrow B$ and an object C , define the function $f^* : \mathcal{C}[B, C] \rightarrow \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 4.1.7. Given $f : A \rightarrow B$ and an object C , define the function $f_* : \mathcal{C}[C, A] \rightarrow \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

4.1.1 Monomorphisms

Definition 4.1.8 (Monomorphism). Let $f : A \rightarrow B$. Then f is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for any object X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

4.1.2 Epimorphisms

Definition 4.1.9 (Epimorphism). Let $f : A \rightarrow B$. Then f is *epic* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for any object X and functions $x, y : B \rightarrow X$, if $x \circ f = y \circ f$ then $x = y$.

4.1.3 Sections and Retractions

Definition 4.1.10 (Section, Retraction). Let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $rs = \text{id}_B$.

Proposition 4.1.11. *Let $f : A \rightarrow B$ and $r, s : B \rightarrow A$. If r is a retraction of f and s is a section of f then $r = s$.*

PROOF:

$$\begin{aligned} r &= r \text{id}_B && (\text{Unit Law}) \\ &= rfs && (s \text{ is a section of } f) \\ &= \text{id}_A s && (r \text{ is a retraction of } f) \\ &= s && (\text{Unit Law}) \square \end{aligned}$$

Proposition 4.1.12. *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $s : B \rightarrow A$ be a section of $r : A \rightarrow B$.

$\langle 1 \rangle 2$. LET: X be an object and $x, y : X \rightarrow B$

$\langle 1 \rangle 3$. ASSUME: $s \circ x = s \circ y$

$\langle 1 \rangle 4$. $x = y$

PROOF: $x = r \circ s \circ x = r \circ s \circ y = y$.

□

Proposition 4.1.13. *Every retraction is epic.*

PROOF: Dual. □

4.1.4 Isomorphisms

Definition 4.1.14 (Isomorphism). A morphism $f : A \rightarrow B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$ that is both a retraction and section of f .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 4.1.15. *The inverse of an isomorphism is unique.*

PROOF: From Proposition 4.1.11. □

Proposition 4.1.16. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $ff^{-1} = \text{id}_B$ and $f^{-1}f = \text{id}_A$. □

Isomorphism.

Define the opposite category.

Slice categories

Definition 4.1.17. Let \mathcal{C} be a category and $B \in \mathcal{C}$. The category \mathcal{C}_B^B of objects *over and under* B is the category with:

- objects all triples (X, u, p) such that $u : B \rightarrow X$ and $p : X \rightarrow B$
- morphisms $f : (X, u, p) \rightarrow (Y, u', p')$ all morphisms $f : X \rightarrow Y$ such that $fu = u'$ and $p'f = p$.

Proposition 4.1.18.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B) \backslash \text{id}_B \cong (\mathcal{C} \backslash B) / \text{id}_B$$

$(B, \text{id}_B, \text{id}_B)$ is the zero object in \mathcal{C}_B^B .

4.1.5 Initial Objects

Definition 4.1.19 (Initial Object). An object I is *initial* iff, for any object X , there exists exactly one morphism $I \rightarrow X$.

Proposition 4.1.20. *The empty set is initial in **Set**.*

PROOF: For any set A , the nowhere-defined function is the unique function $\emptyset \rightarrow A$. \square

Proposition 4.1.21. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : I \rightarrow I'$ be the unique morphism $I \rightarrow I'$.

$\langle 1 \rangle 2$. LET: $i^{-1} : I' \rightarrow I$ be the unique morphism $I' \rightarrow I$.

$\langle 1 \rangle 3$. $ii^{-1} = \text{id}_{I'}$

PROOF: There is only one morphism $I' \rightarrow I'$.

$\langle 1 \rangle 4$. $i^{-1}i = \text{id}_I$

PROOF: There is only one morphism $I \rightarrow I$.

\square

4.1.6 Terminal Objects

Definition 4.1.22 (Terminal Object). An object T is *terminal* iff, for any object X , there exists exactly one morphism $X \rightarrow T$.

Proposition 4.1.23. *1 is terminal in Set.*

PROOF: For any set A , the constant function to $*$ is the only function $A \rightarrow 1$. \square

Proposition 4.1.24. *If T and T' are terminal objects, then there exists a unique isomorphism $T \cong T'$.*

PROOF: Dual to Proposition 4.1.21. \square

4.1.7 Zero Objects

Definition 4.1.25 (Zero Object). An object Z is a *zero object* iff it is an initial object and a terminal object.

Definition 4.1.26 (Zero Morphism). Let \mathcal{C} be a category with a zero object Z . Let $A, B \in \mathcal{C}$. The *zero morphism* $A \rightarrow B$ is the unique morphism $A \rightarrow Z \rightarrow B$.

Proposition 4.1.27. *There is no zero object in Set.*

PROOF: Since $\emptyset \not\approx 1$. \square

4.1.8 Triads

Definition 4.1.28 (Triad). Let \mathcal{C} be a category. A *triad* consists of objects X, Y, M and morphisms $\alpha : X \rightarrow M, \beta : Y \rightarrow M$. We call M the *codomain* of the triad.

4.1.9 Cotriads

Definition 4.1.29 (Cotriad). Let \mathcal{C} be a category. A *cotriad* consists of objects X, Y, W and morphisms $\xi : W \rightarrow X, \eta : W \rightarrow Y$. We call W the *domain* of the triad.

4.1.10 Pullbacks

Definition 4.1.30 (Pullback). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

is a *pullback* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$.

In this case we also say that η is the *pullback* of β along α .

Proposition 4.1.31. *If $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$ form a pullback of $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$, and $\xi' : W' \rightarrow X$ and $\eta' : W' \rightarrow Y$ also form the pullback of α and β , then there exists a unique isomorphism $\phi : W \cong W'$ such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : W \rightarrow W'$ be the unique morphism such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.

$\langle 1 \rangle 2$. LET: $\phi^{-1} : W' \rightarrow W$ be the unique morphism such that $\eta\phi^{-1} = \eta'$ and $\xi\phi^{-1} = \xi'$.

$\langle 1 \rangle 3$. $\phi\phi^{-1} = \text{id}_{W'}$

PROOF: Each is the unique $x : W' \rightarrow W'$ such that $\eta'x = \eta'$ and $\xi'x = \xi'$.

$\langle 1 \rangle 4$. $\phi^{-1}\phi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\eta x = \eta$ and $\xi x = \xi$.

□

Proposition 4.1.32. *For any morphism $h : A \rightarrow B$, the following diagram is a pullback diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: Z be an object.

$\langle 1 \rangle 2$. LET: $f : Z \rightarrow B$ and $g : Z \rightarrow A$ satisfy $\text{id}_B f = hg$

$\langle 1 \rangle 3$. $g : Z \rightarrow A$ is the unique morphism such that $\text{id}_A g = g$ and $hg = f$.

□

Proposition 4.1.33. *The pullback of an isomorphism is an isomorphism.*

PROOF:

⟨1⟩1. LET:

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback diagram.

⟨1⟩2. ASSUME: β is an isomorphism.

⟨1⟩3. LET: ξ^{-1} be the unique morphism $X \rightarrow W$ such that $\xi\xi^{-1} = \text{id}_X$ and $\eta\xi^{-1} = \beta^{-1}\alpha$.

PROOF: This exists since $\alpha\text{id}_X = \beta\beta^{-1}\alpha = \alpha$.

⟨1⟩4. $\xi^{-1}\xi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\xi x = \xi$ and $\eta x = \eta$.

□

Proposition 4.1.34. *Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w : A \rightarrow W$ be the unique morphism such that $\xi w = x$ and $\eta w = y$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ is the pullback of β and α in $\mathcal{C} \setminus A$.

PROOF:

⟨1⟩1. LET: $(Z, z) \in \mathcal{C} \setminus A$

⟨1⟩2. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

⟨1⟩3. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

⟨1⟩4. $hz = w$

⟨2⟩1. $\xi hz = \xi w$

PROOF:

$$\begin{aligned} \xi hz &= fz & (\langle 1 \rangle 3) \\ &= x & (\langle 1 \rangle 2) \\ &= \xi w \end{aligned}$$

⟨2⟩2. $\eta hz = \eta w$

PROOF: Similar.

⟨1⟩5. $h : (Z, z) \rightarrow (W, w)$

□

Proposition 4.1.35. *Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in*

\mathcal{C}/A . Let

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w = x\xi : W \rightarrow A$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ form a pullback of α and β in \mathcal{C}/A .

PROOF:

$\langle 1 \rangle 1$. $\eta : (W, w) \rightarrow (Y, y)$

PROOF:

$$\begin{aligned} y\eta &= m\beta\eta \\ &= m\alpha\xi \\ &= x\xi \\ &= w \end{aligned}$$

$\langle 1 \rangle 2$. LET: $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 3$. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle 4$. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle 5$. $h : (Z, z) \rightarrow (W, w)$

PROOF:

$$\begin{aligned} wh &= x\xi h \\ &= xf && (\langle 1 \rangle 4) \\ &= z && (\langle 1 \rangle 3) \end{aligned}$$

□

Proposition 4.1.36. In **Set**, let $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$. Let $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$ with inclusion $i : W \rightarrow X \times Y$. Let $\xi = \pi_1 i : W \rightarrow X$ and $\eta = \pi_2 i : W \rightarrow Y$. Then ξ and η form the pullback of α and β .

PROOF:

$\langle 1 \rangle 1$. $\alpha\xi = \beta\eta$

PROOF: For $w \in W$, if $i(w) = (x, y)$ then $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$.

$\langle 1 \rangle 2$. For every set Z and functions $f : Z \rightarrow X$, $g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$

PROOF: For $z \in Z$, let $h(z)$ be the unique element of W such that $i(h(z)) = (f(z), g(z))$.

□

Pullback lemma

4.1.11 Pushouts

Definition 4.1.37 (Pushout). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array} \quad (4.1)$$

is a *pushout* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ such that $f\xi = g\eta$, there exists a unique $h : M \rightarrow Z$ such that $h\alpha = f$ and $h\beta = g$.

We also say that β is the *pushout* of ξ along η .

Proposition 4.1.38. *If $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$ form a pushout of $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$, and $\alpha' : X \rightarrow M'$ and $\beta' : Y \rightarrow M'$ also form a pushout of ξ and η , then there exists a unique isomorphism $\phi : M \cong M'$ such that $\phi\alpha = \alpha'$ and $\phi\beta = \beta'$.*

PROOF: Dual to Proposition 4.1.31. \square

Proposition 4.1.39. *For any morphism $h : A \rightarrow B$, the following diagram is a pushout diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF: Dual to Proposition 4.1.32.

Proposition 4.1.40. *The diagram (4.1) is a pushout in \mathcal{C} iff it is a pullback in \mathcal{C}^{op} .*

PROOF: Immediate from definitions. \square

Proposition 4.1.41. *The pushout of an isomorphism is an isomorphism.*

PROOF: Dual to Proposition 4.1.33. \square

Proposition 4.1.42. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m := \alpha x : A \rightarrow M$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in $\mathcal{C} \setminus A$.

PROOF: Dual to Proposition 4.1.35. \square

Proposition 4.1.43. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in \mathcal{C}/A . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m : M \rightarrow A$ be the unique morphism such that $m\alpha = x$ and $m\beta = y$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in \mathcal{C}/A .

PROOF: Dual to Proposition 4.1.34. \square

Proposition 4.1.44. *Set has pushouts.*

PROOF:

$\langle 1 \rangle 1$. LET: $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$.

$\langle 1 \rangle 2$. LET: \sim be the equivalence relation on $X + Y$ generated by $\xi(w) \sim \eta(w)$ for all $w \in W$

$\langle 1 \rangle 3$. LET: $M = (X + Y) / \sim$ with canonical projection $\pi : X + Y \twoheadrightarrow M$.

$\langle 1 \rangle 4$. LET: $\alpha = \pi \circ \kappa_1 : X \rightarrow M$

$\langle 1 \rangle 5$. LET: $\beta = \pi \circ \kappa_2 : Y \rightarrow M$

$\langle 1 \rangle 6$. LET: Z be any set, $f : X \rightarrow Z$ and $g : Y \rightarrow Z$.

$\langle 1 \rangle 7$. ASSUME: $f\xi = g\eta$

$\langle 1 \rangle 8$. LET: $h : X + Y \rightarrow Z$ be the function defined by $h(x) = f(x)$ and $h(y) = g(y)$ for $x \in X$ and $y \in Y$

$\langle 1 \rangle 9$. h respects \sim

PROOF: For $w \in W$ we have

$$\begin{aligned} h(\xi(w)) &= f(\xi(w)) && (\langle 1 \rangle 8) \\ &= g(\eta(w)) && (\langle 1 \rangle 7) \\ &= h(\eta(w)) && (\langle 1 \rangle 8) \end{aligned}$$

$\langle 1 \rangle 10$. LET: $\bar{h} : M \rightarrow Z$ be the induced function.

$\langle 1 \rangle 11$. $\bar{h}\alpha = f$

PROOF:

$$\begin{aligned} \bar{h}(\alpha(x)) &= \bar{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x) \end{aligned}$$

$\langle 1 \rangle 12$. $\bar{h}\beta = g$

PROOF: Similar.

$\langle 1 \rangle 13$. For all $k : M \rightarrow Z$, if $k\alpha = f$ and $k\beta = g$ then $k = \bar{h}$.

PROOF:

$$\begin{aligned}
 k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\
 &= f(x) \\
 k(\pi(\kappa_2(y))) &= k(\beta(y)) \\
 &= g(y) \\
 \therefore k \circ \pi &= h \\
 \therefore k &= \bar{h}
 \end{aligned}$$

□

Definition 4.1.45. Let $u : A \rightarrow X$ be an injection. The *pointed set obtained from X by collapsing (A, u)* , denoted $X/(A, u)$, is the pushout

$$\begin{array}{ccc}
 A & \xrightarrow{\quad} & 1 \\
 \downarrow u & & \downarrow * \\
 X & \longrightarrow & X/(A, u)
 \end{array}$$

Proposition 4.1.46. In \mathbf{Set}_* , any two morphisms $1 \rightarrow X$ and $1 \rightarrow Y$ have a pushout.

PROOF: The pushout of $a : (1, *) \rightarrow (X, x)$ and $b : (1, *) \rightarrow (Y, y)$ is $(X+Y/\sim, x)$ where \sim is the equivalence relation generated by $x \sim y$. □

Definition 4.1.47 (Wedge). The *wedge* of pointed sets X and Y , $X \vee Y$, is the pushout of the unique morphism $1 \rightarrow X$ and $1 \rightarrow Y$.

Definition 4.1.48 (Smash). Let X and Y be pointed sets. Let $\xi : X \vee Y \rightarrow X$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & X \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{\quad} & X \vee Y \\
 & \searrow \xi & \nearrow \\
 & & X
 \end{array}$$

0

Let $\eta : X \vee Y \rightarrow Y$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & X \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{\quad} & X \vee Y \\
 & \searrow \eta & \nearrow \\
 & & Y
 \end{array}$$

0

Let $\zeta = \langle \xi, \eta \rangle : X \vee Y \rightarrow X \times Y$. The *smash* of X and Y , $X \wedge Y$, is the result of collapsing $X \times Y$ with respect to ζ .

Pushout lemma

4.1.12 Subcategories

Definition 4.1.49 (Subcategory). A *subcategory* \mathcal{C}' of a category \mathcal{C} consists of:

- a subset $\text{Ob}(\mathcal{C}')$ of \mathcal{C}
- for all $A, B \in \text{Ob}(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \text{Ob}(\mathcal{C}')$, we have $\text{id}_A \in \mathcal{C}'[A, A]$
- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \text{Ob}(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

4.1.13 Opposite Category

Definition 4.1.50 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- $\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$
- Given $f \in \mathcal{C}^{\text{op}}[A, B]$ and $g \in \mathcal{C}^{\text{op}}[B, C]$, their composite in \mathcal{C}^{op} is $f \circ g$, where \circ is composition in \mathcal{C} .

Proposition 4.1.51. *An object is initial in \mathcal{C} iff it is terminal in \mathcal{C}^{op} .*

PROOF: Immediate from definitions. \square

Proposition 4.1.52. *An object is terminal in \mathcal{C} iff it is initial in \mathcal{C}^{op} .*

PROOF: Immediate from definitions. \square

Corollary 4.1.52.1. *If T and T' are terminal objects in \mathcal{C} then there exists a unique isomorphism $T \cong T'$.*

4.1.14 Groupoids

Definition 4.1.53 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

4.1.15 Concrete Categories

Definition 4.1.54 (Concrete Category). A *concrete category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*
- for any object $A \in \text{Ob}(\mathcal{C})$, a set $|A|$
- for any objects $A, B \in \text{Ob}(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$
- for any object A we have $\text{id}_{|A|} \in \mathcal{C}[A, A]$.

4.1.16 Power of Categories

Definition 4.1.55. Let \mathcal{C} be a category and J a set. The category \mathcal{C}^J is the category with:

- objects all J -indexed families of objects of \mathcal{C}
- morphisms $\{X_j\}_{j \in J} \rightarrow \{Y_j\}_{j \in J}$ all families $\{f_j\}_{j \in J}$ where $f_j : X_j \rightarrow Y_j$

4.1.17 Arrow Category

Definition 4.1.56 (Arrow Category). Let \mathcal{C} be a category. The *arrow category* \mathcal{C}^\rightarrow is the category with:

- objects all triples (A, B, f) where $f : A \rightarrow B$ in \mathcal{C}
- morphisms $(A, B, f) \rightarrow (C, D, g)$ all pairs $(u : A \rightarrow C, v : B \rightarrow D)$ such that $vf = gu$.

4.1.18 Slice Category

Definition 4.1.57 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category under A*, $\mathcal{C} \backslash A$, is the category with:

- objects all pairs (B, f) where $B \in \mathcal{C}$ and $f : A \rightarrow B$
- morphisms $(B, f) \rightarrow (C, g)$ are morphisms $u : B \rightarrow C$ such that $uf = g$.

We identify this with the subcategory of \mathcal{C}^\rightarrow formed by mapping (B, f) to (A, B, f) and u to (id_A, u) .

Proposition 4.1.58. If $s : (B, f) \rightarrow (C, g)$ in $\mathcal{C} \backslash A$, then any retraction of s in \mathcal{C} is a retraction of s in $\mathcal{C} \backslash A$.

PROOF:

$\langle 1 \rangle 1$. LET: $r : C \rightarrow B$ be a retraction of s in \mathcal{C} .

$\langle 1 \rangle 2. rg = f$

PROOF: $rg = rsf = f$.

$\langle 1 \rangle 3. r : (C, g) \rightarrow (B, f)$ in $\mathcal{C} \backslash A$

$\langle 1 \rangle 4. rs = \text{id}_{(B, f)}$

PROOF: Because composition is inherited from \mathcal{C} .

□

Proposition 4.1.59. id_A is the initial object in $\mathcal{C} \backslash A$.

PROOF: For any $(B, f) \in \mathcal{C} \backslash A$, we have f is the only morphism $A \rightarrow B$ such that $f \text{id}_A = f$. □

Proposition 4.1.60. If A is terminal in \mathcal{C} then id_A is the zero object in $\mathcal{C} \backslash A$.

PROOF: For any $(B, f) \in \mathcal{C} \backslash A$, the unique morphism $! : B \rightarrow A$ is the unique morphism such that $!f = \text{id}_A$. □

Definition 4.1.61 (Pointed Sets). The category of pointed sets is **Set** $\backslash 1$.

Definition 4.1.62. Let \mathcal{C} be a category and $A \in \mathcal{C}$. The slice category over A , \mathcal{C}/A , is the category with:

- objects all pairs (B, f) with $f : B \rightarrow A$
- morphisms $u : (B, f) \rightarrow (C, g)$ all morphisms $u : B \rightarrow C$ such that $gu = f$.

Proposition 4.1.63. Let $u : (B, f) \rightarrow (C, g) : \mathcal{C}/A$. Any section of u in \mathcal{C} is a section of u in \mathcal{C}/A .

PROOF: Dual to Proposition 4.1.58. □

Proposition 4.1.64. id_A is terminal in \mathcal{C}/A .

PROOF: Dual to Proposition 4.1.59. □

Proposition 4.1.65. If A is initial in \mathcal{C} then id_A is the zero object in \mathcal{C}/A .

PROOF: Dual to Proposition 4.1.60. □

Definition 4.1.66. Let $A \in \mathcal{C}$. The category of objects over and under A , written \mathcal{C}_A^A , is the category with:

- objects all triples (X, u, p) where $u : A \rightarrow X$, $p : X \rightarrow A$ and $pu = \text{id}_A$
- morphism $f : (X, u, p) \rightarrow (Y, v, q)$ all morphisms $f : X \rightarrow Y$ such that $fu = v$ and $qf = p$

Proposition 4.1.67. $(A, \text{id}_A, \text{id}_A)$ is the zero object in \mathcal{C}_A^A .

PROOF: For any object (X, u, p) , we have p is the unique morphism $(X, u, p) \rightarrow (A, \text{id}_A, \text{id}_A)$, and u is the unique morphism $(A, \text{id}_A, \text{id}_A) \rightarrow (X, u, p)$. □

Definition 4.1.68 (Fibre Collapsing). Let B be a set. Let $u : (A, a) \rightarrow (X, x)$ in \mathbf{Set}/B . Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow u & & \downarrow j \\ X & \xrightarrow{i} & C \end{array}$$

Let $c : C \rightarrow B$ be the unique morphism such that $cj = \text{id}_B$ and $ci = x$. Then $(C, j, c) \in \mathbf{Set}_B^B$ is called the set over and under B obtained from X by *fibre collapsing* with respect to u . If (A, u) is a subset of X , we denote this set over and under B by $X/_B(A, u)$.

Definition 4.1.69 (Fibre Wedge). Let B be a small set. Let $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$. The *fibre wedge* of X and Y is the pushout of u_X and u_Y :

$$\begin{array}{ccc} B & \xrightarrow{u_X} & X \\ \downarrow u_Y & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \end{array}$$

Definition 4.1.70 (Fibre Smash). Let $X, Y \in \mathbf{Set}_B^B$. Let $\xi : X \vee_B Y \rightarrow X$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \xi & \downarrow \\ & & X \end{array}$$

0

Let $\eta : X \vee_B Y \rightarrow Y$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \eta & \downarrow \\ & & Y \end{array}$$

0

Let $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \rightarrow X \times Y$. The *fibre smash* of X and Y , $X \wedge_B Y$, is the result of collapsing $X \times Y$ with respect to ζ .

Proposition 4.1.71. *Set has products and coproducts.*

Proposition 4.1.72. *Let \mathcal{C} be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in \mathcal{C} and $Z \in \mathcal{C}$. Let $\coprod_{\alpha \in I} X_\alpha$ be the coproduct of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[\coprod_{\alpha \in I} X_\alpha, Z] \approx \prod_{\alpha \in I} \mathcal{C}[X_\alpha, Z] .$$

Proposition 4.1.73. *Let \mathcal{C} be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in \mathcal{C} and $Z \in \mathcal{C}$. Let $\prod_{\alpha \in I} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[Z, \prod_{\alpha \in I} X_\alpha] \approx \prod_{\alpha \in I} \mathcal{C}[Z, X_\alpha] .$$

Proposition 4.1.74. *A product in \mathcal{C} constitutes a product in \mathcal{C}/A .*

Proposition 4.1.75. *A coproduct in \mathcal{C} constitutes a product in \mathcal{C}/A .*

4.2 Functors

Definition 4.2.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- a function $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism $f : A \rightarrow B$ in \mathcal{C} , a morphism $Ff : FA \rightarrow FB$ in \mathcal{D}

such that:

- for all $A : \text{El}(\text{Ob}(\mathcal{C}))$ we have $F\text{id}_A = \text{id}_{FA}$
- for any morphism $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathcal{C} , we have $F(g \circ f) = Fg \circ Ff$

Proposition 4.2.2. *Functors preserve isomorphisms.*

PROOF:

$\langle 1 \rangle 1$. LET: $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

$\langle 1 \rangle 2$. LET: $f : A \cong B$ in \mathcal{C}

$\langle 1 \rangle 3$. $Ff^{-1} \circ Ff = \text{id}_{FA}$

PROOF:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\text{id}_A \\ &= \text{id}_{FA} \end{aligned}$$

$\langle 1 \rangle 4$. $Ff \circ Ff^{-1} = \text{id}_{FB}$

PROOF:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

Definition 4.2.3 (Identity Functor). For any category \mathcal{C} , the *identity* functor on \mathcal{C} is the functor $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$\begin{aligned} I_{\mathcal{C}} A &:= A & (A \in \mathcal{C}) \\ I_{\mathcal{C}} f &:= f & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

Proposition 4.2.4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $r : A \rightarrow B$ is a retraction of $s : B \rightarrow A$ in \mathcal{C} then Fr is a retraction of Fs .

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned} \quad \square$$

Corollary 4.2.4.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $\phi : A \cong B$ is an isomorphism in \mathcal{C} then $F\phi : FA \cong FB$ is an isomorphism in \mathcal{D} with $(F\phi)^{-1} = F\phi^{-1}$.

Definition 4.2.5 (Composition of Functors). Given functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$, the *composite* functor $GF : \mathcal{C} \rightarrow \mathcal{E}$ is defined by

$$\begin{aligned} (GF)A &= G(FA) & (A \in \mathcal{C}) \\ (GF)f &= G(Ff) & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

Definition 4.2.6 (Category of Categories). Let **Cat** be the category of small categories and functors.

Definition 4.2.7 (Isomorphism of Categories). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F is an *isomorphism of categories* iff there exists a functor $F^{-1} : \mathcal{D} \rightarrow \mathcal{C}$, the *inverse* of F , such that $FF^{-1} = I_{\mathcal{D}}$ and $F^{-1}F = I_{\mathcal{C}}$.

Categories \mathcal{C} and \mathcal{D} are *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, iff there exists an isomorphism between them.

Proposition 4.2.8. If A is initial in \mathcal{C} then $\mathcal{C} \setminus A \cong \mathcal{C}$.

PROOF:

$\langle 1 \rangle 1$. Define $F : \mathcal{C} \setminus A \rightarrow \mathcal{C}$ by

$$F(B, f) = B$$

$$F(u : (B, f) \rightarrow (C, g)) = u$$

$\langle 1 \rangle 2$. Define $G : \mathcal{C} \rightarrow \mathcal{C} \setminus A$ by

$$GB = (B, !_B)$$

where $!_B$ is the unique morphism $A \rightarrow B$

$$G(u : B \rightarrow C) = u : (B, !_B) \rightarrow (C, !_C)$$

$\langle 1 \rangle 3$. $FG = \text{id}_{\mathcal{C}}$

$\langle 1 \rangle 4$. $GF = \text{id}_{\mathcal{C} \setminus A}$

PROOF: Since $GF(B, f) = (B, !_B) = (B, f)$ because the morphism $A \rightarrow B$ is unique.

\square

Proposition 4.2.9. If A is terminal in \mathcal{C} then $\mathcal{C}/A \cong \mathcal{C}$.

PROOF: Dual. \square

Proposition 4.2.10.

$$\mathcal{C}_A^A \cong (\mathcal{C}/A) \backslash (A, \text{id}_A) \cong (\mathcal{C} \backslash A) / (A, \text{id}_A)$$

PROOF:

$\langle 1 \rangle 1$. Define a functor $F : \mathcal{C}_A^A \rightarrow (\mathcal{C}/A) \backslash (A, \text{id}_A)$.

$\langle 2 \rangle 1$. Given $A \xrightarrow{u} X \xrightarrow{p} A$ in \mathcal{C}_A^A , let $F(X, u, p) = ((X, p), u)$

$\langle 2 \rangle 2$. Given $f : (A \xrightarrow{u} X \xrightarrow{p} A) \rightarrow (A \xrightarrow{v} Y \xrightarrow{q} A)$, let $Ff = f$.

$\langle 1 \rangle 2$. Define a functor $G : (\mathcal{C}/A) \backslash (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.

$\langle 1 \rangle 3$. Define a functor $H : \mathcal{C}_A^A \rightarrow (\mathcal{C} \backslash A) / (A, \text{id}_A)$.

$\langle 1 \rangle 4$. Define a functor $K : (\mathcal{C} \backslash A) / (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.

\square

Definition 4.2.11 (Forgetful Functor). For any concrete category \mathcal{C} , define the *forgetful* functor $U : \mathcal{C} \rightarrow \mathbf{Set}$ by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

Definition 4.2.12 (Switching Functor). For any category \mathcal{C} , define the *switching* functor $T : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

Definition 4.2.13 (Reduction). Let $\Phi : \mathbf{Set} \rightarrow \mathbf{Set}$ be a functor. The *reduction* of Φ is the functor $\Phi^* : \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ defined by: $\Phi^*(X, a)$ is the collapse of $\Phi(X)$ with respect to $\Phi(a) : \Phi(1) \rightarrow \Phi(X)$.

Definition 4.2.14. Extend the wedge \vee to a functor $\mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ by defining, given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, there $f \vee g$ is the unique morphism that makes the following diagram commute.

$$\begin{array}{ccccc} 1 & \longrightarrow & X & & \\ \downarrow & & \downarrow & \searrow f & \\ Y & \longrightarrow & X \vee Y & & X' \\ & \searrow g & \downarrow f \vee g & \downarrow & \\ & & Y' & \longrightarrow & X' \vee Y' \end{array}$$

Definition 4.2.15. Extend smash to a functor $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ as follows. Given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, let $f \wedge g : X \wedge Y \rightarrow X' \wedge Y'$ be the

unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc}
 X \vee Y & \xrightarrow{\quad} & 1 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 X \times Y & \xrightarrow{\quad} & X \wedge Y & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & X' \vee Y' & \xrightarrow{\quad} & 1 \\
 f \times g \swarrow & & \downarrow & & \downarrow \\
 & & X' \times Y' & \xrightarrow{\quad} & X' \wedge Y'
 \end{array}$$

Definition 4.2.16 (Reduction). Let B be a small set. Let $\Phi_B : \mathbf{Set}/B \rightarrow \mathbf{Set}/B$ be a functor. The *reduction* of Φ_B is the functor $\Phi_B^B : \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ defined as follows.

For $(X, u : B \rightarrow X, p : X \rightarrow B) \in \mathbf{Set}_B^B$, let $\Phi_B^B(X)$ be the set over and under B obtained from $\Phi_B(X)$ by collapsing with respect to $\Phi_B(u) : \Phi_B(B) \rightarrow \Phi_B(X)$.

Definition 4.2.17. Extend \vee_B to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$.

Definition 4.2.18. Extend \wedge_B to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$.

Definition 4.2.19 (Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *faithful* iff, for any objects $A, B \in \mathcal{C}$ and morphisms $f, g : A \rightarrow B : \mathcal{C}$, if $Ff = Fg$ then $f = g$.

Definition 4.2.20 (Full). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *full* iff, for any objects $A, B \in \mathcal{C}$ and morphism $g : FA \rightarrow FB : \mathcal{D}$, there exists $f : A \rightarrow B : \mathcal{C}$ such that $Ff = g$.

Definition 4.2.21 (Fully Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *fully faithful* iff it is full and faithful.

Definition 4.2.22 (Full Embedding). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *full embedding* iff it is fully faithful and injective on objects.

4.3 Natural Transformations

Definition 4.3.1 (Natural Transformation). Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$. A *natural transformation* $\tau : F \Rightarrow G$ is a family of morphisms $\{\tau_X : FX \rightarrow GX\}_{X \in \mathcal{C}}$ such that, for every morphism $f : X \rightarrow Y : \mathcal{C}$, we have $Gf \circ \tau_X = \tau_Y \circ Ff$.

$$\begin{array}{ccc}
 FX & \xrightarrow{Ff} & FY \\
 \tau_X \downarrow & & \downarrow \tau_Y \\
 GX & \xrightarrow{Gf} & GY
 \end{array}$$

Definition 4.3.2 (Natural Isomorphism). A natural transformation $\tau : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$ is a *natural isomorphism*, $\tau : F \cong G$, iff for all $X \in \mathcal{C}$, τ_X is an isomorphism $FX \cong GX$.

Functors F and G are *naturally isomorphic*, $F \cong G$, iff there exists a natural isomorphism between them.

Definition 4.3.3 (Inverse). Let $\tau : F \cong G$. The *inverse* natural isomorphism $\tau^{-1} : G \cong F$ is defined by $(\tau^{-1})_X = \tau_X^{-1}$.

4.4 Bifunctors

Definition 4.4.1 (Commutative). A bifunctor $\square : \mathcal{C}^2 \rightarrow \mathcal{C}$ is *commutative* iff $\square \cong \square \circ T$, where $T : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ is the swap functor.

Proposition 4.4.2. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is commutative.

PROOF: Since the pushout of f and g is the pushout of g and f . \square

Proposition 4.4.3. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is commutative.

PROOF: In the diagram defining $X \wedge Y$, construct the isomorphism between the version with X and Y and the version with Y with X for every object. \square

Proposition 4.4.4. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is commutative.

Proposition 4.4.5. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is commutative.

Definition 4.4.6 (Associative). A bifunctor \square is *associative* iff $\square \circ (\square \times \text{id}) \cong \square \circ (\text{id} \times \square)$.

Proposition 4.4.7. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is associative.

PROOF: Since $X \vee (Y \vee Z)$ and $(X \vee Y) \vee Z$ are both the pushout of the unique morphisms $1 \rightarrow X$, $1 \rightarrow Y$ and $1 \rightarrow Z$. \square

Proposition 4.4.8. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is associative.

PROOF: Draw isomorphisms between the diagrams for $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$. \square

Product and coproduct are commutative and associative.

Proposition 4.4.9. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is associative.

Proposition 4.4.10. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is associative.

Proposition 4.4.11. Let \mathcal{C} be a category with binary coproducts. Let $\square : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ be a bifunctor. Then \square distributes over $+$ iff the canonical morphism

$$(X \square Z) + (Y \square Z) \rightarrow (X + Y) \square Z$$

is an isomorphism for all X, Y, Z .

Proposition 4.4.12. *In a category with binary products and binary coproducts, then \times distributes over $+$.*

Proposition 4.4.13. *In $\mathbf{Set}/*$, we have \times does not distribute over \vee .*

Proposition 4.4.14. *In $\mathbf{Set}/*$, we have \wedge distributes over \vee .*

Proposition 4.4.15. *In \mathbf{Set}/B , we have \times_B distributes over $+_B$.*

Proposition 4.4.16. *In \mathbf{Set}/B^B , we have \wedge_B distributes over \vee_B .*

4.5 Functor Categories

Definition 4.5.1 (Functor Category). Given categories \mathcal{C} and \mathcal{D} , define the *functor category* $\mathcal{C}^{\mathcal{D}}$ to be the category with objects the functors from \mathcal{D} to \mathcal{C} and morphisms the natural transformations.

Definition 4.5.2 (Yoneda Embedding). Let \mathcal{C} be a category. The *Yoneda embedding* $Y : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ is the functor that maps an object A to $\mathcal{C}[-, A]$ and morphisms similarly.

Theorem 4.5.3 (Yoneda Lemma). *Let \mathcal{C} be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\text{op}}}[\mathcal{C}[-, X], F] \cong FX$$

that maps $\tau : \mathcal{C}[-, X] \Rightarrow F$ to $\tau_X(\text{id}_X)$.

PROOF:

$\langle 1 \rangle 1$. ϕ is natural in X .

PROOF:

$\langle 2 \rangle 1$. LET: $f : X \rightarrow Y : \mathcal{C}$

$\langle 2 \rangle 2$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$. $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$

PROOF:

$$\begin{aligned} \phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\text{id}_Y \circ f) \\ &= \tau_Y(f) \\ &= \tau_Y(f \circ \text{id}_X) \\ &= Ff(\tau_X(\text{id}_X)) && (\tau \text{ natural}) \\ &= Ff(\phi(\tau)) \end{aligned}$$

$\langle 1 \rangle 2$. ϕ is natural in F .

$\langle 2 \rangle 1$. LET: $\alpha : F \Rightarrow G : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$

$\langle 2 \rangle 2$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$. $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$

PROOF: $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\text{id}_X)) = \alpha_X(\phi(\tau))$

$\langle 1 \rangle 3$. Each ϕ_{XF} is injective.

$\langle 2 \rangle 1$. LET: $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 2$. ASSUME: $\phi(\sigma) = \phi(\tau)$

$\langle 2 \rangle 3$. LET: $f : Y \rightarrow X$

$\langle 2 \rangle 4$. $\sigma_Y(f) = \tau_Y(f)$

PROOF:

$$\begin{aligned}
 \sigma_Y(f) &= \sigma_Y(\text{id}_X \circ f) \\
 &= Ff(\sigma_X(\text{id}_X)) && (\sigma \text{ is natural}) \\
 &= Ff(\tau_X(\text{id}_X)) && (\langle 2 \rangle 2) \\
 &= \tau_Y(\text{id}_X \circ f) && (\tau \text{ is natural}) \\
 &= \tau_Y(f)
 \end{aligned}$$

$\langle 1 \rangle 4$. Each ϕ_{XF} is surjective.

$\langle 2 \rangle 1$. LET: $X \in \mathcal{C}$ and $F : \mathcal{C} \rightarrow \mathcal{D}$

$\langle 2 \rangle 2$. LET: $a \in FX$

$\langle 2 \rangle 3$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$ be given by $\tau_Y(g) = Fg(a)$ for $g : Y \rightarrow X$

$\langle 2 \rangle 4$. τ is natural.

$\langle 3 \rangle 1$. LET: $h : Y \rightarrow Z : \mathcal{C}$

PROVE: $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \text{id}_X]$

$\langle 3 \rangle 2$. LET: $g : Z \rightarrow X$

$\langle 3 \rangle 3$. $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$

PROOF:

$$\begin{aligned}
 \tau_Y(g \circ h) &= F(g \circ h)(a) \\
 &= Fh(Fg(a)) \\
 &= Fh(\tau_Z(g))
 \end{aligned}$$

$\langle 2 \rangle 5$. $\phi(\tau) = a$

PROOF:

$$\begin{aligned}
 \phi_X(\tau) &= \tau_X(\text{id}_X) \\
 &= F\text{id}_X(a) \\
 &= a
 \end{aligned}$$

□

Corollary 4.5.3.1. *The Yoneda embedding is fully faithful.*

Corollary 4.5.3.2. *Given objects A and B in \mathcal{C} , we have $A \cong B$ if and only if $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$.*

Chapter 5

Monoid Theory

Definition 5.0.1 (Monoid). A *monoid* is a category with one object.

Definition 5.0.2. Let \mathcal{C} be a category and $X \in \mathcal{C}$. The monoid $\text{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \rightarrow X$ under composition.

Proposition 5.0.3. *For any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and $X \in \mathcal{C}$, we have that $F : \text{End}_{\mathcal{C}}(X) \rightarrow \text{End}_{\mathcal{D}}(FX)$ is a monoid homomorphism.*

PROOF: Since $F\text{id}_X = \text{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$. \square

Chapter 6

Group Theory

Definition 6.0.1. Let \mathbf{Grp} be the category of small groups and group homomorphisms.

Definition 6.0.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G .

Proposition 6.0.3. *The trivial group is a zero object in \mathbf{Grp} .*

PROOF: Easy. \square

The zero morphism $G \rightarrow H$ maps every element in G to e .

Definition 6.0.4. Let \mathcal{C} be a category and $X \in \mathcal{C}$. We write $\text{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Proposition 6.0.5. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $X \in \mathcal{C}$. Then $F : \text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(FX)$ is a group homomorphism.*

PROOF: Since $F\text{id}_X = \text{id}_{FX}$, $F(g \circ f) = Fg \circ Ff$, and $Ff^{-1} = (Ff)^{-1}$. \square

Proposition 6.0.6. \mathbf{Grp} has products.

Definition 6.0.7 (Free Product). The product of a family of groups in \mathbf{Grp} is called the *free product*.

Proposition 6.0.8. \mathbf{Ab} has products given by direct sums.

Chapter 7

Ring Theory

Definition 7.0.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Definition 7.0.2 (Spectrum). Let R be a commutative ring. The *spectrum* of R , $\text{spec } R$, is the set of all prime ideals of R .

Definition 7.0.3 (Zariski Topology). Let R be a commutative ring. The *Zariski topology* on $\text{spec } R$ is the topology where the closed sets are the sets of the form

$$VE := \{p \in \text{spec } R : E \subseteq p\}$$

for any $E \in \mathcal{P}R$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$

$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{A} \in \mathcal{C}$

$\langle 2 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{C}$

$\langle 2 \rangle 2$. LET: $E = \bigcup \{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$

PROVE: $VE = \bigcap \mathcal{A}$

$\langle 2 \rangle 3$. For all $p \in \text{spec } R$, if $E \subseteq p$ then $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 1$. LET: $p \in \text{spec } R$

$\langle 3 \rangle 2$. ASSUME: $E \subseteq p$

$\langle 3 \rangle 3$. LET: $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$

$\langle 3 \rangle 4$. $E' \subseteq E$

$\langle 3 \rangle 5$. $E' \subseteq p$

$\langle 3 \rangle 6$. $p \in VE'$

$\langle 2 \rangle 4$. For all $p \in \text{spec } R$, if $p \in \bigcap \mathcal{A}$ then $E \subseteq p$

$\langle 3 \rangle 1$. LET: $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 2$. For all $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$ we have $E' \subseteq p$

$\langle 3 \rangle 3$. $E \subseteq p$

$\langle 1 \rangle 3$. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

PROOF: Since $VE \cup VE' = V(E \cap E')$

$\langle 1 \rangle 4. \emptyset \in \mathcal{C}$

$\langle 2 \rangle 1. VR = \emptyset$

PROOF: If $p \in VR$ then $R \subseteq p$ contradicting the fact that p is a prime ideal.

□

Definition 7.0.4. For any ring R , let $R - \mathbf{Mod}$ be the category of small R -modules and R -module homomorphisms.

Proposition 7.0.5. $R - \mathbf{Mod}$ has products and coproducts.

Chapter 8

Field Theory

Proposition 8.0.1. *Field does not have binary products.*

PROOF: There cannot be a field K with field homomorphisms $K \rightarrow \mathbb{Z}_2$ and $K \rightarrow \mathbb{Z}_3$, because its characteristic would be both 2 and 3. \square

Chapter 9

Linear Algebra

Definition 9.0.1. For any field K , we write \mathbf{Vect}_K for $K\text{-Mod}$.

Dual space functor $\mathbf{Vect}_K^{\text{op}} \rightarrow \mathbf{Vect}_K$.

Chapter 10

Topology

10.1 Topological Spaces

Definition 10.1.1 (Topological Space). Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say (X, \mathcal{O}) is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call \mathcal{O} the *topology* of the topological space, and call its elements *open* sets. We shall often write X for the topological space (X, \mathcal{O}) .

Definition 10.1.2 (Discrete Topology). For any set X , the power set $\mathcal{P}X$ is called the *discrete* topology on X .

Proposition 10.1.3. *For any set X , the discrete topology on X is a topology on X .*

Definition 10.1.4 (Indiscrete Topology). For any set X , the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Proposition 10.1.5. *For any set X , the indiscrete topology on X is a topology on X .*

Definition 10.1.6 (Cofinite Topology). For any set X , the *cofinite* topology is $\{X - U : U \subseteq X \text{ is finite}\}$.

Definition 10.1.7 (Cocountable Topology). For any set X , the *cocountable* topology is $\{X - U : U \subseteq X \text{ is countable}\}$.

Definition 10.1.8 (Sierpiński Two-Point Space). The *Sierpiński two-point space* is $\{0, 1\}$ under the topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Definition 10.1.9 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 10.1.10. *A set B is open if and only if $X - B$ is closed.*

Proposition 10.1.11. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that \mathcal{C} is the set of closed sets if and only if:*

- For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$
- For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.
- $\emptyset \in \mathcal{C}$

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Theorem 10.1.12. *There are infinitely many primes.*

Furstenberg's proof:

PROOF:

$\langle 1 \rangle 1$. For $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$,

LET: $S(a, b) := \{an + b : n \in \mathbb{N}\}$

$\langle 1 \rangle 2$. LET: \mathcal{T} be the topology generated by the basis $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$

$\langle 2 \rangle 1$. For every $n \in \mathbb{Z}$, there exist a, b such that $n \in S(a, b)$.

PROOF: $n \in S(n, 0)$

$\langle 2 \rangle 2$. If $n \in S(a_1, b_1) \cap S(a_2, b_2)$ then there exist a_3, b_3 such that $n \in S(a_3, b_3) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 1$. LET: $d = \text{lcm}(a_1, a_2)$

PROVE: $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 2$. LET: $d = a_1k = a_2l$

$\langle 3 \rangle 3$. LET: $n = a_1c + b_1 = a_2d + b_2$

$\langle 3 \rangle 4$. LET: $z \in \mathbb{Z}$

PROVE: $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 5$. $dz + n \in S(a_1, b_1)$

PROOF:

$$\begin{aligned} dz + n &= a_1kz + a_1c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

$\langle 3 \rangle 6$. $dz + n \in S(a_2, b_2)$

PROOF: Similar.

$\langle 1 \rangle 3$. For all $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$ we have $S(a, b)$ is closed.

$\langle 2 \rangle 1$. LET: $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$

$\langle 2 \rangle 2$. LET: $n \in \mathbb{Z} - S(a, b)$

$\langle 2 \rangle 3$. $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$

$\langle 3 \rangle 1$. LET: $x \in S(a, n)$

$\langle 3 \rangle 2$. ASSUME: for a contradiction $x \in S(a, b)$

$\langle 3 \rangle 3$. PICK m such that $x = am + b$

$\langle 3 \rangle 4$. PICK l such that $x = al + n$

$\langle 3 \rangle 5$. $n = a(m - l) + b$

⟨3⟩6. $n \in S(a, b)$

⟨3⟩7. Q.E.D.

PROOF: This contradicts ⟨2⟩2.

⟨1⟩4.

$$\mathbb{Z} - \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$$

PROOF: Since every integer except 1 and -1 is divisible by a prime.

⟨1⟩5. No nonempty finite set is open.

⟨2⟩1. LET: U be a nonempty open set

⟨2⟩2. PICK $n \in U$

⟨2⟩3. There exist a, b such that $n \in S(a, b) \subseteq U$

⟨2⟩4. U is infinite.

⟨1⟩6. $\mathbb{Z} - \{1, -1\}$ is not closed.

⟨1⟩7. $\bigcup_{p \text{ prime}} S(p, 0)$ is not closed.

⟨1⟩8. The union of finitely many closed sets is closed.

⟨1⟩9. There are infinitely many primes.

□

Definition 10.1.13 (Neighbourhood). Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x , and x is an *interior* point of U , iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 10.1.14. A set B is open if and only if it is a neighbourhood of each of its points.

Proposition 10.1.15. Let X be a set and $\mathcal{N} : X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that, for all $x \in X$, we have \mathcal{N}_x is the set of neighbourhoods of x , if and only if:

- For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$
- For all $x \in X$ we have $X \in \mathcal{N}_x$
- For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$
- For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$
- For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M. M \in \mathcal{N}_y$.

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$.

Definition 10.1.16 (Exterior Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x .

Definition 10.1.17 (Boundary Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B .

Definition 10.1.18 (Interior). Let X be a topological space and $B \subseteq X$. The *interior* of B , B° , is the set of all interior points of B .

Proposition 10.1.19. *The interior of B is the union of all the open sets included in B .*

Definition 10.1.20 (Closure). Let X be a topological space and $B \subseteq X$. The closure of B , \overline{B} , is the set of all points that are not exterior points of B .

Proposition 10.1.21. *The closure of B is the intersection of all the closed sets that include B .*

Proposition 10.1.22. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 10.1.23 (Kuratowski Closure Axioms). *Let X be a set and $- : \mathcal{P}X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} such that, for all $B \subseteq X$, \overline{B} is the closure of B , if and only if:*

- $\overline{\emptyset} = \emptyset$
- For all $A \subseteq X$ we have $A \subseteq \overline{A}$
- For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$
- For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$

In this case, \mathcal{O} is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

Definition 10.1.24 (Finer, Coarser). Let \mathcal{T} and \mathcal{T}' be topologies on the set X . Then \mathcal{T} is *coarser*, *smaller* or *weaker* than \mathcal{T}' , or \mathcal{T}' is *finer*, *larger* or *weaker* than \mathcal{T} , iff $\mathcal{T} \subseteq \mathcal{T}'$.

10.1.1 Subspaces

Definition 10.1.25 (Subspace). Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

Example 10.1.26. The *unit sphere* S^2 is $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a subspace of \mathbb{R}^3 .

Theorem 10.1.27. *Let X be a topological space and (Y, i) a subset of X . Then the subspace topology on Y is the unique topology such that, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f : Z \rightarrow X$ is continuous.*

PROOF:

⟨1⟩1. If we give Y the subspace topology then, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.

⟨2⟩1. Given Y the subspace topology.

⟨2⟩2. LET: Z be a topological space.

⟨2⟩3. LET: $f : Z \rightarrow Y$

⟨2⟩4. If f is continuous then $i \circ f$ is continuous.

PROOF: Since i is continuous.

- ⟨2⟩5. If $i \circ f$ is continuous then f is continuous.
- ⟨3⟩1. ASSUME: $i \circ f$ is continuous.
- ⟨3⟩2. LET: U be open in Y .
- ⟨3⟩3. $f^{-1}(i^{-1}(i(U)))$ is open in Z .
- ⟨3⟩4. $f^{-1}(U)$ is open in Z .
- ⟨1⟩2. If, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.
- ⟨2⟩1. ASSUME: For every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.
- ⟨2⟩2. i is continuous.
- ⟨2⟩3. For every open set U in X , we have $i^{-1}(U)$ is open in Y .
- ⟨2⟩4. LET: Z be the set Y under the subspace topology and $f : Z \rightarrow Y$ the identity function.
- ⟨2⟩5. $i \circ f$ is continuous.
- ⟨2⟩6. f is continuous.
- ⟨2⟩7. Every set open in Y is open in Z .

□

10.1.2 Topological Disjoint Union

Definition 10.1.28 (Coproduct Topology). Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The *coproduct topology* on $\coprod_{\alpha \in A} X_\alpha$ is

$$\mathcal{T} = \left\{ \coprod_{\alpha \in A} U_\alpha : \{U_\alpha\}_{\alpha \in A} \text{ is a family with } U_\alpha \text{ open in } X_\alpha \text{ for all } \alpha \right\}.$$

We prove this is a topology.

PROOF:

- ⟨1⟩1. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF:

$$\bigcup_{i \in I} \coprod_{\alpha \in A} U_{i\alpha} = \coprod_{\alpha \in A} \bigcup_{i \in I} U_{i\alpha}$$

- ⟨1⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF:

$$\coprod_{\alpha \in A} U_\alpha \cap \coprod_{\alpha \in A} V_\alpha = \coprod_{\alpha \in A} (U_\alpha \cap V_\alpha)$$

- ⟨1⟩3. $\coprod_{\alpha \in A} X_\alpha \in \mathcal{T}$

PROOF: Since every X_α is open in X_α .

□

Proposition 10.1.29. The coproduct topology is the finest topology on $\coprod_{\alpha \in A} X_\alpha$ such that every injection $\kappa_\alpha : X_\alpha \rightarrow \coprod_{\alpha \in A} X_\alpha$ is continuous.

PROOF:

- ⟨1⟩1. LET: $P = \coprod_{\alpha \in A} X_\alpha$

- ⟨1⟩2. LET: \mathcal{T}_c be the coproduct topology.

- ⟨1⟩3. LET: \mathcal{T} be any topology on P
- ⟨1⟩4. For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T}_c)$ is continuous.
 - ⟨2⟩1. LET: $\alpha \in A$
 - ⟨2⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with each U_α open in X_α .
 - ⟨2⟩3. For all $\alpha \in A$, we have $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha)$ is open in X_α .

PROOF: Since $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha) = U_\alpha$.
- ⟨1⟩5. If, for all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$ is continuous, then $\mathcal{T} \subseteq \mathcal{T}_c$.
 - ⟨2⟩1. ASSUME: For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$ is continuous.
 - ⟨2⟩2. LET: $U \in \mathcal{T}$
 - ⟨2⟩3. For all $\alpha \in a$, we have $\kappa_\alpha^{-1}(U)$ is open in X_α .
 - ⟨2⟩4. $U = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(U) \in \mathcal{T}_c$

□

Theorem 10.1.30. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The coproduct topology is the unique topology on $\coprod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : \coprod_{\alpha \in A} X_\alpha \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.*

PROOF:

- ⟨1⟩1. LET: $X = \coprod_{\alpha \in A} X_\alpha$
- ⟨1⟩2. LET: \mathcal{T}_c be the coproduct topology.
- ⟨1⟩3. For every topological space Z and function $f : (X, \mathcal{T}_c) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.
 - ⟨2⟩1. LET: Z be a topological space.
 - ⟨2⟩2. LET: $f : X \rightarrow Z$
 - ⟨2⟩3. If f is continuous then $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.

PROOF: Because the composite of two continuous functions is continuous.
 - ⟨2⟩4. If $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous then f is continuous.
 - ⟨3⟩1. ASSUME: $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.
 - ⟨3⟩2. LET: U be open in Z
 - ⟨3⟩3. For all $\alpha \in A$ we have $\kappa_\alpha^{-1}(f^{-1}(U))$ is open in X_α
 - ⟨3⟩4. $f^{-1}(U) = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(f^{-1}(U))$
 - ⟨3⟩5. $f^{-1}(U)$ is open in X
- ⟨1⟩4. For any topology \mathcal{T} on X , if for every topological space Z and function $f : (X, \mathcal{T}) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous, then $\mathcal{T} = \mathcal{T}_c$.
 - ⟨2⟩1. LET: \mathcal{T} be a topology on X .
 - ⟨2⟩2. ASSUME: For every topological space Z and function $f : (X, \mathcal{T}) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.
 - ⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_c$
 - ⟨3⟩1. For all $\alpha \in A$ we have $\kappa_\alpha : X_\alpha \rightarrow (X, \mathcal{T})$ is continuous.

PROOF: From ⟨2⟩1 since id_X is continuous.
 - ⟨3⟩2. $\mathcal{T} \subseteq \mathcal{T}_c$

PROOF: Proposition 10.1.29.
 - ⟨2⟩4. $\mathcal{T}_c \subseteq \mathcal{T}$

- ⟨3⟩1. LET: $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}_c)$ be the identity function.
- ⟨3⟩2. $f \circ \kappa_\alpha$ is continuous for all α .
- ⟨3⟩3. f is continuous.
- PROOF: ⟨2⟩1
- ⟨3⟩4. $\mathcal{T}_c \subseteq \mathcal{T}$

□

10.1.3 Product Topology

Definition 10.1.31 (Product Topology). Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

Proposition 10.1.32. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The product topology on $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis $\mathcal{B} = \{\prod_{\alpha \in A} U_\alpha : \text{for all } \alpha \in A, U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in A\}$.

PROOF:

- ⟨1⟩1. \mathcal{B} is a basis for a topology.
- ⟨1⟩2. LET: \mathcal{T} be the topology generated by \mathcal{B} .
- ⟨1⟩3. LET: \mathcal{T}_p be the product topology.
- ⟨1⟩4. $\mathcal{T} \subseteq \mathcal{T}_p$
 - ⟨2⟩1. LET: $B \in \mathcal{B}$
 - ⟨2⟩2. LET: $B = \prod_{\alpha \in A} U_\alpha$ with each U_α open in X_α and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$
 - ⟨2⟩3. $B = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$
 - ⟨2⟩4. $B \in \mathcal{T}_p$
- ⟨1⟩5. $\mathcal{T}_p \subseteq \mathcal{T}$
 - ⟨2⟩1. For every $\alpha \in A$ we have π_α is continuous.

PROOF: Since $\pi^{-1}(U)$ is open for every U open in X_α .

□

Theorem 10.1.33. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Then the product topology on $\prod_{\alpha \in A} X_\alpha$ is the unique topology such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f : Z \rightarrow X_\alpha$ is continuous.

PROOF:

- ⟨1⟩1. If we give $\prod_{\alpha \in A} X_\alpha$ the product topology, then for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
- ⟨2⟩1. Give $\prod_{\alpha \in A} X_\alpha$ the product topology.
- ⟨2⟩2. LET: Z be a topological space.
- ⟨2⟩3. LET: $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$
- ⟨2⟩4. If f is continuous then, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
- PROOF: Since the composite of two continuous functions is continuous.
- ⟨2⟩5. If, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then f is continuous.

- ⟨3⟩1. ASSUME: For all $\alpha \in A$ we have $\pi_\alpha \circ f$ is continuous.
 ⟨3⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with U_α open in X_α such that $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \dots, \alpha_n$.
 ⟨3⟩3. For all α we have $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ is open in Z .
 ⟨3⟩4. $f^{-1}(\prod_\alpha U_\alpha)$ is open in Z .
 PROOF: Since $f^{-1}(\prod_\alpha U_\alpha) = f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \dots \cap f^{-1}(\pi_{\alpha_n}^{-1}(U_{\alpha_n}))$.
 ⟨1⟩2. If \mathcal{T} is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then \mathcal{T} is the product topology.
 ⟨2⟩1. ASSUME: \mathcal{T} is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
 ⟨2⟩2. LET: \mathcal{T}_p be the product topology.
 ⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_p$
 ⟨3⟩1. LET: $Z = (\prod_\alpha X_\alpha, \mathcal{T}_p)$
 ⟨3⟩2. LET: $f : Z \rightarrow \prod_\alpha X_\alpha$ be the identity function
 ⟨3⟩3. For all α we have $\pi_\alpha \circ f$ is continuous.
 ⟨3⟩4. f is continuous.
 PROOF: ⟨2⟩1
 ⟨3⟩5. Every set open in \mathcal{T} is open in \mathcal{T}_p
 ⟨2⟩4. $\mathcal{T}_p \subseteq \mathcal{T}$
 ⟨3⟩1. $\text{id}_{\prod_\alpha X_\alpha}$ is continuous.
 ⟨3⟩2. For all α we have π_α is continuous.
 PROOF: ⟨2⟩1
 ⟨3⟩3. $\mathcal{T}_p \subseteq \mathcal{T}$
 PROOF: Since \mathcal{T}_p is the coarsest topology such that every π_α is continuous.

□

Example 10.1.34. It is not true that, for any function $f : \prod_{\alpha \in A} X_\alpha \rightarrow Y$, if f is continuous in every variable separately then f is continuous.

Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

Then f is continuous in x and in y , but is not continuous.

Proposition 10.1.35. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology on $\prod_{i \in I} Y_i$ as a subspace of $\prod_{i \in I} X_i$.

PROOF:

- ⟨1⟩1. Given $\prod_{i \in I} Y_i$ the subspace topology.
 ⟨1⟩2. LET: $\iota : \prod_{i \in I} Y_i$ be the inclusion.
 ⟨1⟩3. LET: Z be any topological space.
 ⟨1⟩4. LET: $f : Z \rightarrow \prod_{i \in I} Y_i$

⟨1⟩5. f is continuous if and only if, for all $i \in I$, we have $\pi_i \circ f$ is continuous.

PROOF:

f is continuous $\Leftrightarrow \iota \circ f : Z \rightarrow \prod_{i \in I} X_i$ is continuous (Theorem 10.1.27)

$\Leftrightarrow \forall i \in I. \pi_i \circ \iota \circ f : Z \rightarrow X_i$ is continuous (Theorem 10.1.33)

$\Leftrightarrow \forall i \in I. \iota_i \circ \pi_i \circ f : Z \rightarrow X_i$ is continuous

$\Leftrightarrow \forall i \in I. \pi_i \circ f : Z \rightarrow Y_i$ is continuous (Theorem 10.1.27)

where ι_i is the inclusion $Y_i \rightarrow X_i$.

□

10.1.4 Bases

Definition 10.1.36 (Basis). Let X be a topological space. A *basis* for the topology on X is a set of open sets \mathcal{B} such that every open set is the union of a subset of \mathcal{B} . The elements of \mathcal{B} are called *basic open neighbourhoods* of their elements.

Definition 10.1.37 (Order Topology). Let X be a linearly ordered set. The *order topology* on X is the topology generated by the open interval (a, b) as well as the open rays $(a, +\infty)$ and $(-\infty, b)$ for $a, b \in X$.

Definition 10.1.38 (Lower Limit Topology). The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology generated by the basis consisting of all half-open intervals $[a, b)$.

Proposition 10.1.39. Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is a basis for a topology on X if and only if:

1. $\bigcup \mathcal{B} = X$
2. For all $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

In this case, the topology is unique and is the set of all unions of subsets of \mathcal{B} . We call it the topology generated by \mathcal{B} .

10.1.5 Subbases

Definition 10.1.40 (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of \mathcal{S} .

Definition 10.1.41 (Space with Basepoint). A *space with basepoint* is a pair (X, x) where X is a topological space and $x \in \text{El}(X)$.

10.1.6 Countability Axioms

Definition 10.1.42 (Neighbourhood Basis). Let X be a topological space and $x_0 \in \text{El}(X)$. A *neighbourhood basis* of x_0 is a set \mathcal{U} of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of \mathcal{U} .

Definition 10.1.43 (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

Definition 10.1.44 (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

\mathbb{R}^n is second countable.

An uncountable discrete space is first countable but not second countable.

Proposition 10.1.45. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.

PROOF:

$\langle 1 \rangle 1$. For all $\lambda : \text{El}(\Lambda)$, PICK U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.

$\langle 1 \rangle 2$. For all $\lambda : \text{El}(\Lambda)$, PICK $x_\lambda \in U_\lambda$.

$\langle 1 \rangle 3$. ASSUME: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda \in \Lambda}$.

$\langle 1 \rangle 4$. PICK $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$

$\langle 1 \rangle 5$. There is no $U \in B$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

□

10.2 Continuous Functions

Definition 10.2.1 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* iff, for every open set V in Y , the inverse image $f^{-1}(V)$ is open in X .

Proposition 10.2.2. 1. id_X is continuous

2. The composite of two continuous functions is continuous.

3. If $f : X \rightarrow Y$ is continuous and $X_0 \subseteq X$ then $f|_{X_0} : X_0 \rightarrow Y$ is continuous.

4. If $f : X + Y \rightarrow Z$, then f is continuous iff $f \circ \kappa_1 : X \rightarrow Z$ and $f \circ \kappa_2 : Y \rightarrow Z$ are continuous.

5. If $f : Z \rightarrow X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Proposition 10.2.3. Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{B} be a basis for Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF: Since every element of \mathcal{B} is open in Y .

$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X , then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

$\langle 2 \rangle 2$. LET: U be open in Y .

$\langle 2 \rangle 3$. LET: $x \in f^{-1}(U)$

$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq U$.

$\langle 2 \rangle 5$. $x \in f^{-1}(B) \subseteq f^{-1}(U)$

□

Definition 10.2.4 (Homeomorphism). Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Definition 10.2.5 (Retraction). Let X be a topological space and A a subspace of X . A continuous function $\rho : X \rightarrow A$ is a *retraction* iff $\rho|_A = \text{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 10.2.6. Let **Top** be the category of small topological spaces and continuous functions.

Proposition 10.2.7. \emptyset is initial in **Top**.

Proposition 10.2.8. 1 is terminal in **Top**.

Forgetful functor **Top** \rightarrow **Set**.

Basepoint preserving continuous functor.

Proposition 10.2.9. Let (X, \mathcal{T}) be a topological space. Let S be the Sierpiński two-point space. Define $\Phi : \mathcal{T} \rightarrow \mathbf{Top}[X, S]$ by $\Phi(U)(x) = 1$ iff $x \in U$. Then Φ is a bijection.

PROOF:

$\langle 1 \rangle 1$. For all $U \in \mathcal{T}$ we have $\Phi(U)$ is continuous.

$\langle 2 \rangle 1$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 2$. $\Phi(U)(\{1\})$ is open.

PROOF: Since $\Phi(U)(\{1\}) = U$.

$\langle 1 \rangle 2$. Φ is injective.

PROOF: If $\Phi(U) = \Phi(V)$ then we have $\forall x (x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$.

$\langle 1 \rangle 3$. Φ is surjective.

PROOF: Given $f : X \rightarrow S$ continuous we have $\Phi(f^{-1}(1)) = f$.

□

10.2.1 Paths

Definition 10.2.10 (Path). A *path* in a topological space X is a continuous function $[0, 1] \rightarrow X$.

10.2.2 Loops

Definition 10.2.11 (Loop). A *loop* in a topological space X is a path $\alpha : [0, 1] \rightarrow X$ such that $\alpha(0) = \alpha(1)$.

10.3 Convergence

Definition 10.3.1 (Convergence). Let X be a topological space. Let (x_n) be a sequence in X . A point $a \in \text{El}(X)$ is a *limit* of the sequence iff, for every neighbourhood U of a , there exists n_0 such that $\forall n \geq n_0, x_n \in U$.

Convergence in a product space is pointwise convergence.

If $f : X \rightarrow Y$ is continuous and $x_n \rightarrow l$ in X then $f(x_n) \rightarrow f(l)$ in Y .

Example 10.3.2. The converse does not hold.

Let X be the set of all continuous functions $[0, 1] \rightarrow [-1, 1]$ under the product topology. Let $i : X \rightarrow L^2([0, 1])$ be the inclusion.

If $f_n \rightarrow f$ then $i(f_n) \rightarrow i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K_\epsilon, \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0, 1]} U_\lambda$ where $U_\lambda = [-1, 1]$ except for $\lambda = \lambda_1, \dots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \dots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 10.3.3. *The converse does hold for first countable spaces. If $f : X \rightarrow Y$ where X is first countable, and Y is a topological space, and whenever $x_n \rightarrow x$ then $f(x_n) \rightarrow f(x)$, then f is continuous.*

10.4 Subspaces

Definition 10.4.1 (Subspace). Let X be a topological space, Y a set, and $f : Y \rightarrow X$. The *subspace topology* on Y induced by f is $\mathcal{T} = \{i^{-1}(U) : U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF: Since $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$.

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: Since $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.

$\langle 1 \rangle 3$. $Y \in \mathcal{T}$

PROOF: Since $Y = f^{-1}(X)$.

□

Proposition 10.4.2. *Let X be a topological space, Y a set and $f : Y \rightarrow X$ a function. Then the subspace topology on Y is the coarsest topology such that f is continuous.*

PROOF: Immediate from definition. □

10.5 Embedding

Definition 10.5.1 (Embedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *embedding* iff f is injective and the topology on X is the subspace induced by f .

10.6 Quotient Spaces

Definition 10.6.1 (Quotient Topology). Let X be a topological space, S a set, and $\pi : X \twoheadrightarrow S$ be a surjection. The *quotient topology* on S induced by π is $\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.

PROOF: Since $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}$.

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

PROOF: Since $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$.

$\langle 1 \rangle 3$. $X \in \mathcal{T}$

PROOF: Since $X = \pi^{-1}(Y)$.

□

Proposition 10.6.2. Let X be a topological space, S a set and $\pi : X \twoheadrightarrow S$ a surjection. Then the quotient topology on S is the finest topology such that π is continuous.

PROOF: Immediate from definitions. □

Definition 10.6.3 (Quotient Map). Let X and S be topological spaces and $\pi : X \rightarrow S$. Then π is a *quotient map* iff π is surjective and the topology on S is the quotient topology induced by π .

Theorem 10.6.4. Let X be a topological space, let S be a set, and let $\pi : X \twoheadrightarrow S$ be surjective. Then the quotient topology on S is the unique topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.

PROOF:

$\langle 1 \rangle 1$. If S is given the quotient topology, then for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.

$\langle 2 \rangle 1$. Give S the quotient topology.

$\langle 2 \rangle 2$. LET: Z be a topological space.

$\langle 2 \rangle 3$. LET: $f : S \rightarrow Z$

$\langle 2 \rangle 4$. If f is continuous then $f \circ \pi$ is continuous.

PROOF: The composite of two continuous functions is continuous.

$\langle 2 \rangle 5$. If $f \circ \pi$ is continuous then f is continuous.

- 1

Proposition 10.6.5. *Let Z be a topological space. Define $\pi : [0, 1] \rightarrow S^1$ by $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$. Given any continuous function $f : S^1 \rightarrow Z$, we have $f \circ \pi$ is a loop in Z . This defines a bijection between $\mathbf{Top}[S^1, Z]$ and the set of loops in Z .*

Definition 10.6.6 (Projective Space). The *projective space* \mathbb{RP}^n is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by \sim where $x \sim \lambda x$ for all $x \in \mathbb{R}^{n+1} - \{0\}$ and $\lambda \in \mathbb{R}$.

Definition 10.6.8 (Möbius Band). The *Möbius band* is the quotient of $[0, 1]^2$ by \sim where $(0, y) \sim (1, 1 - y)$.

Proposition 10.6.10. \mathbb{RP}^2 is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (1 - x, 1)$ and $(0, y) \sim (1, 1 - y)$.

PROOF:TODO

Example 10.6.11. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and $\{Y_i\}_{i \in I}$ a family of sets. Let $q_i : X_i \rightarrow Y_i$ be a surjective function for all $i \in I$. Give each Y_i the quotient topology. It is not true in general that the product topology on $\prod_{i \in I} Y_i$ is the same as the quotient topology induced by $\prod_{i \in I} q_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$.

PROOF:

- ⟨1⟩1. LET: $X^* = \mathbb{R} - \mathbb{Z}_+ + \{b\}$ be the quotient space obtained from \mathbb{R} by identifying the subset \mathbb{Z}_+ to the point b .
 ⟨1⟩2. LET: $p : \mathbb{R} \rightarrow X^*$ be the quotient map.
 PROVE: $p \times \text{id}_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$ is not a quotient map.
 ⟨1⟩3. For $n \in \mathbb{Z}_+$,
 LET: $c_n = \sqrt{2}/n$
 ⟨1⟩4. For $n \in \mathbb{Z}_+$,
 LET: $U_n = \{(x, y) \in \mathbb{Q} \times \mathbb{R} : n - 1/4 < x < n + 1/4 \text{ and } ((y > x + c_n - n \text{ and } y > -x + c_n + n) \text{ or } (y < x + c_n - n \text{ and } y < -x + c_n + n))\}$
 ⟨1⟩5. For all $n \in \mathbb{Z}_+$, U_n is open in $\mathbb{R} \times \mathbb{Q}$
 ⟨1⟩6. For all $n \in \mathbb{Z}_+$ we have $\{n\} \times \mathbb{Q} \subseteq U_n$
 ⟨1⟩7. LET: $U = \bigcup_{n \in \mathbb{Z}_+} U_n$
 ⟨1⟩8. U is open in $\mathbb{R} \times \mathbb{Q}$.
 ⟨1⟩9. U is saturated with respect to $p \times \text{id}_{\mathbb{Q}}$.
 ⟨1⟩10. LET: $U' = (p \times \text{id}_{\mathbb{Q}})(U)$
 ⟨1⟩11. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$.

10.7 Connected Spaces

Definition 10.7.1 (Connected). A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 10.7.2. *The continuous image of a connected space is connected.*

Proposition 10.7.3. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are connected, then X is connected.*

Proposition 10.7.4. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 10.7.5 (Path-connected). A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \rightarrow X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 10.7.6. *The continuous image of a path connected space is path connected.*

Proposition 10.7.7. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are path connected, then X is path connected.*

Proposition 10.7.8. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

10.8 Hausdorff Spaces

Definition 10.8.1 (Hausdorff). A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 10.8.2. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 10.8.3. 1. *Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

Proposition 10.8.4. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \rightarrow B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X , then $f = g$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.

$\langle 1 \rangle 2$. PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.

$\langle 1 \rangle 3$. PICK $x \in f^{-1}(U) \cap g^{-1}(V)$

$\langle 1 \rangle 4$. $f(x) = g(x) \in U \cap V$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 10.8.5. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$ be uniformly continuous. Let \hat{X} and \hat{Y} be the completions of X and Y . Then f extends uniquely to a continuous map $\hat{X} \rightarrow \hat{Y}$.*

PROOF: The extension maps $\lim_{n \rightarrow \infty} x_n$ to $\lim_{n \rightarrow \infty} f(x_n)$. □

10.9 Separable Spaces

Definition 10.9.1 (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

10.10 Sequential Compactness

Definition 10.10.1 (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

10.11 Compactness

Definition 10.11.1 (Compact). A topological space is *compact* iff every open cover has a finite subcover.

Proposition 10.11.2. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P . Then $X \in P$.*

PROOF:

- $\langle 1 \rangle 1$. P is an open cover of X
 $\langle 1 \rangle 2$. PICK a finite subcover $U_1, \dots, U_n \in P$
 $\langle 1 \rangle 3$. $X = U_1 \cup \dots \cup U_n \in P$
 \square

Corollary 10.11.2.1. *Let f be a compact space and $f : X \rightarrow \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. \square

Proposition 10.11.3. *The continuous image of a compact space is compact.*

Proposition 10.11.4. *A closed subspace of a compact space is compact.*

Proposition 10.11.5. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. X and Y are compact.
2. $X + Y$ is compact.
3. $X \times Y$ is compact.

Proposition 10.11.6. *A compact subspace of a Hausdorff space is closed.*

Proposition 10.11.7. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 10.11.8. *A first countable compact space is sequentially compact.*

10.12 Quotient Spaces

Definition 10.12.1 (Quotient Space). Let X be a topological space and \sim an equivalence relation on X . The *quotient topology* on X/\sim is defined by: $U : \text{El}(\mathcal{P}X)$ is open in X/\sim if and only if $\pi^{-1}(U)$ is open in X .

Proposition 10.12.2. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $f : X/\sim \rightarrow Y$. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proposition 10.12.3. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $\phi : Y \rightarrow X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \rightarrow X$ such that $\pi \circ \Phi = \phi|U$. Then ϕ is continuous.

Proposition 10.12.4. *A quotient of a connected space is connected.*

Proposition 10.12.5. *A quotient of a path connected space is path connected.*

Proposition 10.12.6. *Let X be a topological space and \sim an equivalence relation on X . If X/\sim is Hausdorff then every equivalence class of \sim is closed in X .*

Definition 10.12.7. Let X be a topological space and $A_1, \dots, A_r \subseteq X$. Then $X/A_1, \dots, A_r$ is the quotient space of X with respect to \sim where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 10.12.8 (Cone). Let X be a topological space. The *cone over X* is the space $(X \times [0, 1])/(X \times \{1\})$.

Definition 10.12.9 (Suspension). Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 10.12.10 (Wedge Product). Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 10.12.11 (Smash Product). Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 10.12.12. $D^n/S^{n-1} \cong S^n$

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : D^n/S^{n-1} \rightarrow S^n$ be the function induced by the map $D^n \rightarrow S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.

$\langle 1 \rangle 2$. ϕ is a bijection.

$\langle 1 \rangle 3$. ϕ is a homeomorphism.

PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.

□

10.13 Gluing

Definition 10.13.1 (Gluing). Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \rightarrow Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where \sim is the equivalence relation generated by $x \sim \phi(x)$ for all $x : \text{El}(X)$.

Proposition 10.13.2. *Y is a subspace of $Y \cup_\phi X$.*

Definition 10.13.3. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0, 1])/\alpha$ is the quotient space of $X \times [0, 1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x : \text{El}(X)$.

Definition 10.13.4 (Möbius Strip). The *Möbius strip* is $([-1, 1] \times [0, 1])/\alpha$ where $\alpha(x) = -x$.

Definition 10.13.5 (Klein Bottle). The *Klein bottle* is $(S^1 \times [0, 1])/\alpha$ where $\alpha(z) = \bar{z}$.

Proposition 10.13.6. Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\text{id}_M} M \cong K$.

PROOF:

$\langle 1 \rangle 1$. LET: $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{\pi i \theta/2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-\pi i \theta/2}, t)$.

$\langle 1 \rangle 2$. f induces a bijection $M \cup_{\text{id}_M} M \approx K$

$\langle 1 \rangle 3$. f is a homeomorphism.

□

10.14 Metric Spaces

Definition 10.14.1 (Metric Space). Let X be a set and $d : X^2 \rightarrow \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geq 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d) . We often write X for the metric space (X, d) .

Definition 10.14.2 (Ball). Let X be a metric space. Let $x \in X$ and $r > 0$. The *ball* with *centre* x and *radius* r is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} .$$

Definition 10.14.3 (Metric Topology). Let (X, d) be a metric space. The *metric topology* on X is the topology generated by the basis consisting of the balls.

Definition 10.14.4 (Metrisable). A topological space is *metrisable* iff there exists a metric that induces its topology.

Proposition 10.14.5. Every metrisable space is Hausdorff.

Every metrisable space is first countable.

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

10.14.1 Products

Definition 10.14.6 (Euclidean Metric). Let X and Y be metric spaces. The *Euclidean metric* on $X \times Y$ is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}.$$

We write $X \times Y$ for the set $X \times Y$ under this metric.

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1.$ $d((x_1, y_1), (x_2, y_2)) \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2.$ $d((x_1, y_1), (x_2, y_2)) = 0$ iff $(x_1, y_1) = (x_2, y_2)$

PROOF: $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = 0$ iff $d(x_1, x_2) = d(y_1, y_2) = 0$ iff $x_1 = x_2$ and $y_1 = y_2$.

$\langle 1 \rangle 3.$ $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$

PROOF: Since $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = \sqrt{d(x_2, x_1)^2 + d(y_2, y_1)^2}$.

$\langle 1 \rangle 4.$ The triangle inequality holds.

PROOF:

$$\begin{aligned} & (d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)))^2 \\ &= d((x_1, y_1), (x_2, y_2))^2 + 2d((x_1, y_1), (x_2, y_2))d((x_2, y_2), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3))^2 \\ &= d(x_1, x_2)^2 + d(y_1, y_2)^2 + 2\sqrt{(d(x_1, x_2)^2 + d(y_1, y_2)^2)(d(x_2, x_3)^2 + d(y_2, y_3)^2)} + d(x_2, x_3)^2 + d(y_2, y_3)^2 \\ &\geq d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(y_1, y_2)^2 + d(y_2, y_3)^2 + 2(d(x_1, x_2)d(x_2, x_3) + d(y_1, y_2)d(y_2, y_3)) \\ &\quad \text{(Cauchy-Schwarz)} \\ &= (d(x_1, x_2) + d(x_2, x_3))^2 + (d(y_1, y_2) + d(y_2, y_3))^2 \\ &\geq d(x_1, x_3)^2 + d(y_1, y_3)^2 \\ &= d((x_1, y_1), (x_3, y_3))^2 \end{aligned}$$

□

Proposition 10.14.7. *Let X and Y be metric spaces. The Euclidean metric on $X \times Y$ induces the product topology on $X \times Y$.*

PROOF:

$\langle 1 \rangle 1.$ Every open ball is open in the product topology.

$\langle 2 \rangle 1.$ LET: $(x, y) \in B((a, b), \epsilon)$

PROVE: $B(x, \sqrt{\epsilon}) \times B(y, \sqrt{\epsilon}) \subseteq B((a, b), \epsilon)$

$\langle 2 \rangle 2.$ LET: $x' \in B(x, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$ and $y' \in B(y, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$

PROVE: $d((x', y'), (a, b)) < \epsilon$

$\langle 2 \rangle 3.$ $d((x', y'), (x, y)) < \epsilon - d((x, y), (a, b))$

PROOF:

$$\begin{aligned} d((x', y'), (x, y)) &= \sqrt{d(x', x)^2 + d(y', y)^2} \\ &< \sqrt{(\epsilon - d((x, y), (a, b)))^2/2 + (\epsilon - d((x, y), (a, b)))^2/2} \\ &= \epsilon - d((x, y), (a, b)) \end{aligned}$$

$\langle 2 \rangle 4.$ $d((x', y'), (a, b)) < \epsilon$

PROOF:

$$d((x', y'), (a, b)) \leq d((x', y'), (x, y)) + d((x, y), (a, b)) \quad (\text{Triangle Inequality})$$

$$< \epsilon \quad (\langle 2 \rangle 3)$$

$\langle 1 \rangle 2.$ If U is open in X and V is open in Y then $U \times V$ is open under the Euclidean metric.

$\langle 2 \rangle 1.$ LET: $(x, y) \in U \times V$

$\langle 2 \rangle 2.$ PICK $\delta, \epsilon > 0$ such that $B(x, \delta) \subseteq U$ and $B(y, \epsilon) \subseteq V$

PROVE: $(B((x, y), \min(\delta, \epsilon))) \subseteq U \times V$

$\langle 2 \rangle 3.$ LET: $(x', y') \in B((x, y), \min(\delta, \epsilon))$

$\langle 2 \rangle 4.$ $d(x', x) < \delta$

$\langle 3 \rangle 1.$ $d((x', y'), (x, y)) < \min(\delta, \epsilon)$

$\langle 3 \rangle 2.$ $d(x', x)^2 + d(y', y)^2 < \delta^2$

$\langle 3 \rangle 3.$ $d(x', x)^2 < \delta^2$

$\langle 2 \rangle 5.$ $d(y', y) < \epsilon$

PROOF: Similar.

$\langle 2 \rangle 6.$ $(x', y') \in U \times V$

□

10.15 Complete Metric Spaces

Definition 10.15.1 (Complete). A metric space is *complete* iff every Cauchy sequence converges.

Example 10.15.2. \mathbb{R} is complete.

Proposition 10.15.3. *The product of two complete metric spaces is complete.*

Proposition 10.15.4. *Every compact metric space is complete.*

Proposition 10.15.5. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 10.15.6 (Completion). Let X be a metric space. A *completion* of X is a complete metric space \hat{X} and injection $i : X \rightarrow \hat{X}$ such that:

- The metric on X is the restriction of the metric on \hat{X}
- X is dense in \hat{X} .

Proposition 10.15.7. *Let $i_1 : X \rightarrow Y_1$ and $i_2 : X \rightarrow Y_2$ be completions of X . Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

PROOF: Define $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$. □

Theorem 10.15.8. *Every metric space has a completion.*

PROOF: Let \hat{X} be the set of Cauchy sequences in X quotiented by \sim where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \rightarrow 0$. □

10.16 Manifolds

Definition 10.16.1 (Manifold). An *n-dimensional manifold* is a second countable Hausdorff space locally homeomorphic to \mathbb{R}^n .

Chapter 11

Homotopy Theory

11.1 Homotopies

Definition 11.1.1 (Homotopy). Let X and Y be topological spaces. Let $f, g : X \rightarrow Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0, 1] \rightarrow Y$ such that

- $\forall x : \text{El}(X) . h(x, 0) = f(x)$
- $\forall x : \text{El}(X) . h(x, 1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \rightarrow Y$.

Proposition 11.1.2. Let $f, f' : X \rightarrow Y$ and $g, g' : Y \rightarrow Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.

Definition 11.1.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

A *homotopy functor* is a functor $\mathbf{Top} \rightarrow \mathcal{C}$ that factors through the canonical functor $\mathbf{Top} \rightarrow \mathbf{HTop}$.

Definition 11.1.4. A functor $F : \mathbf{Top} \rightarrow \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \rightarrow Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

11.2 Homotopy Equivalence

Definition 11.2.1 (Homotopy Equivalence). Let X and Y be topological spaces. A *homotopy equivalence* between X and Y , $f : X \simeq Y$, is a continuous function $f : X \rightarrow Y$ such that there exists a continuous function $g : Y \rightarrow X$, the *homotopy inverse* to f , such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.

Definition 11.2.2 (Contractible). A topological space X is *contractible* iff $X \simeq 1$.

Example 11.2.3. \mathbb{R}^n is contractible.

Example 11.2.4. D^n is contractible.

Definition 11.2.5 (Deformation Retract). Let X be a topological space and A a subspace of X . A retraction $\rho : X \rightarrow A$ is a *deformation retraction* iff $i \circ \rho \simeq \text{id}_X$, where i is the inclusion $A \hookrightarrow X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 11.2.6 (Strong Deformation Retract). Let X be a topological space and A a subspace of X . A *strong deformation retraction* $\rho : X \rightarrow A$ is a continuous function such that there exists a homotopy $h : X \times [0, 1] \rightarrow X$ between $i \circ \rho$ and id_X such that, for all $a : \text{El}(X)$ and $t : \text{El}([0, 1])$, we have $h(a, t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 11.2.7. $\{0\}$ is a strong deformation retract of \mathbb{R}^n and of D^n .

Example 11.2.8. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 11.2.9. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 11.2.10. For any topological space X , the singleton consisting of the vertex is a strong deformation retract of the cone over X .

Chapter 12

Simplicial Complexes

Definition 12.0.1 (Simplex). A k -dimensional simplex or k -simplex in \mathbb{R}^n is the convex hull $s(x_0, \dots, x_k)$ of $k + 1$ points in general position.

Definition 12.0.2 (Face). A *sub-simplex* or *face* of $s(x_0, \dots, x_k)$ is the convex hull of a subset of $\{x_0, \dots, x_k\}$.

Definition 12.0.3 (Simplicial Complex). A *simplicial complex* in \mathbb{R}^n is a set K of simplices such that:

- for every simplex s in K , every face of s is in K .
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2 .
- K is locally finite, i.e. every point of \mathbb{R}^n has a neighbourhood that only intersects finitely many elements of K .

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of \mathbb{R}^n .

12.1 Cell Decompositions

Definition 12.1.1 (n -cell). An n -cell is a topological space homeomorphic to \mathbb{R}^n .

Definition 12.1.2 (Cell Decomposition). Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n -cells.

Definition 12.1.3 (n -skeleton). Given a cell decomposition of X , the n -skeleton X^n is the union of all the cells of dimension $\leq n$.

12.2 CW-complexes

Definition 12.2.1 (CW-Complex). A *CW-complex* consists of a topological space X and a cell decomposition \mathcal{E} of X such that:

1. *Characteristic Maps* For every n -cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \rightarrow X$ such that $\Phi_e((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in \mathcal{E} of dimension $< n$.
2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have \bar{e} intersects only finitely many other cells in \mathcal{E} .
3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \bar{e}$ is closed.

Proposition 12.2.2. *If a cell decomposition \mathcal{E} satisfies the Characteristic Maps axiom, then for every n -cell $e \in \mathcal{E}$ we have $\bar{e} = \Phi_e(D^n)$. Therefore \bar{e} is compact and $\bar{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

PROOF:

$\langle 1 \rangle 1.$ $e \subseteq \Phi_e(D^n) \subseteq \bar{e}$

PROOF:

$$\begin{aligned}
 e &= \Phi_e((D^n)^\circ) \\
 &\subseteq \Phi_e(D^n) \\
 &= \Phi_e(\overline{(D^n)^\circ}) \\
 &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
 &= \bar{e}
 \end{aligned}$$

$\langle 1 \rangle 2.$ $\Phi_e(D^n)$ is compact.

PROOF: Because D^n is compact.

$\langle 1 \rangle 3.$ $\Phi_e(D^n)$ is closed.

$\langle 1 \rangle 4.$ $\Phi_e(D^n) = \bar{e}$

□

Chapter 13

Topological Groups

Definition 13.0.1 (Topological Group). A *topological group* is a group G with a topology such that the function $G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.

Example 13.0.2. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 13.0.3. Any subgroup of a topological group is a topological group under the subspace topology.

Definition 13.0.4 (Homogeneous Space). A *homogeneous space* is a topological space of the form G/H , where G is a topological group and H is a normal subgroup of G , under the quotient topology.

Proposition 13.0.5. Let G be a topological group and H a normal subgroup of G . Then G/H is Hausdorff if and only if H is closed.

PROOF: See Bourbaki, N., General Topology. III.12 \square

13.1 Continuous Actions

Definition 13.1.1 (Continuous Action). Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \rightarrow X$ such that:

- $\forall x : \text{El}(X) . ex = x$
- $\forall g, h : \text{El}(G) . \forall x : \text{El}(X) . g(hx) = (gh)x$

A G -space consists of a topological space X and a continuous action of G on X .

Definition 13.1.2 (Orbit). Let X be a G -space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 13.1.3. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:

$\langle 1 \rangle 1$. LET: $f_3 : S^2/SO(2) \rightarrow [-1, 1]$ be the function induced by $\pi_3 : S^2 \rightarrow [-1, 1]$

$\langle 1 \rangle 2$. f_3 is bijective.

$\langle 1 \rangle 3$. $S^2/SO(2)$ is compact.

PROOF: It is the continuous image of S^2 which is compact.

$\langle 1 \rangle 4$. $[-1, 1]$ is Hausdorff.

$\langle 1 \rangle 5$. f_3 is a homeomorphism.

□

Definition 13.1.4 (Stabilizer). Let X be a G -space and $x \in X$. The *stabilizer* of x is $G_x := \{g \in G \mid gx = x\}$.

Proposition 13.1.5. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx .*

PROOF:

$\langle 1 \rangle 1$. If $gG_x = hG_x$ then $gx = hx$.

$\langle 2 \rangle 1$. ASSUME: $gG_x = hG_x$

$\langle 2 \rangle 2$. $g^{-1}h \in G_x$

$\langle 2 \rangle 3$. $g^{-1}hx = x$

$\langle 2 \rangle 4$. $gx = hx$

$\langle 1 \rangle 2$. If $gx = hx$ then $gG_x = hG_x$.

PROOF: Similar.

$\langle 1 \rangle 3$. The function is continuous.

PROOF: Proposition 10.12.2.

□

Chapter 14

Topological Vector Spaces

Definition 14.0.1 (Topological Vector Space). Let K be either \mathbb{R} or \mathbb{C} . A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Subtraction is a continuous function $E^2 \rightarrow E$
- Multiplication is a continuous function $K \times E \rightarrow E$

Proposition 14.0.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. \square

Theorem 14.0.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 \square

Proposition 14.0.4. *Let E be a topological vector space and E_0 a subspace of E . Then $\overline{E_0}$ is a subspace of E .*

Definition 14.0.5. Let E be a topological vector space. The topological space associated with E is $E/\overline{\{0\}}$.

14.1 Cauchy Sequences

Definition 14.1.1 (Cauchy Sequence). Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geq n_0, x_n - x_m \in U$.

Definition 14.1.2 (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

14.2 Seminorms

Definition 14.2.1 (Seminorm). Let E be a vector space over K . A *seminorm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ such that:

1. $\forall x : \text{El}(E) . \|x\| \geq 0$
2. $\forall \alpha : \text{El}(K) . \forall x : \text{El}(E) . \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y : \text{El}(E) . \|x + y\| \leq \|x\| + \|y\|$

Example 14.2.2. The function that maps (x_1, \dots, x_n) to $|x_i|$ is a seminorm on \mathbb{R}^n .

Definition 14.2.3. Let E be a vector space over K . Let Λ be a set of seminorms on E . The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x : \text{El}(E)$.

Proposition 14.2.4. E is a topological vector space under this topology. It is Hausdorff iff, for all $x : \text{El}(E)$, if $\forall \lambda \in \Lambda . \lambda(x) = 0$ then $x = 0$.

14.3 Fréchet Spaces

Definition 14.3.1 (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 14.3.2. Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$. Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 14.3.3 (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

14.4 Normed Spaces

Definition 14.4.1 (Normed Space). Let E be a vector space over K . A *norm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ is a seminorm such that, $\forall x \in E . \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 14.4.2. If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E .

Definition 14.4.3 (p -norm). For any $p \geq 1$, the p -norm on \mathbb{R}^n is defined by

$$\|\vec{x}\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1$. For all $\vec{x} \in \mathbb{R}^n$ we have $\|\vec{x}\|_p \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. For all $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ we have $\|\alpha \vec{x}\|_p = |\alpha| \|\vec{x}\|_p$

PROOF:

$$\begin{aligned} \|\alpha(x_1, \dots, x_n)\| &= \|(\alpha x_1, \dots, \alpha x_n)\| \\ &= \left(\sum_{i=1}^n (\alpha x_i)^p \right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|\vec{x}\|_p \end{aligned}$$

$\langle 1 \rangle 3$. The triangle inequality holds.

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\ &= \sum_{i=1}^n |x_i + y_i| |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^n (|x_i| + |y_i|) |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} \\ &\leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} \quad (\text{Hölder's Inequality}) \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p) \|\vec{x} + \vec{y}\|_p^{p-1} \end{aligned}$$

Assuming w.l.o.g. $\|\vec{x} + \vec{y}\|_p^{p-1} \neq 0$ (using ??) we have $\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$.

$\langle 1 \rangle 4$. For any $\vec{x} \in \mathbb{R}^n$, we have $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.

PROOF: $\sum_{i=1}^n x_i^p = 0$ iff $x_1 = \dots = x_n = 0$.

□

Definition 14.4.4 (Sup-norm). The *sup-norm* on \mathbb{R}^n is defined by

$$\|(x_1, \dots, x_n)\|_\infty := \max(|x_1|, \dots, |x_n|) .$$

Proposition 14.4.5. *The 2-norm on \mathbb{R}^n induces the standard metric.*

PROOF: Immediate from definitions. \square

Definition 14.4.6. For $p \geq 1$, the normed space l_p is the set of all sequences (x_n) in \mathbb{R} such that $\sum_{n=1}^\infty x_n^p$ converges, under

$$\|(x_n)\|_p := \left(\sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} .$$

Proposition 14.4.7. *The spaces l_p for $p \geq 1$ are all homeomorphic.*

PROOF: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. <http://dx.doi.org/10.1007/BF01075865>.

Definition 14.4.8. Let l_∞ be the set of all bounded sequences in \mathbb{R} under

$$\|(x_n)\| := \sup_n |x_n|$$

Proposition 14.4.9. *For all $p \geq 1$ we have l_p is not homeomorphic to l_∞ .*

Proposition 14.4.10. *Let $\| \cdot \|$ be a seminorm on the vector space E . Then $\| \cdot \|$ defines a norm on $E/\{0\}$.*

Proposition 14.4.11. *Let E and F be normed spaces. Any continuous linear map $E \rightarrow F$ is uniformly continuous.*

Definition 14.4.12. For $p \geq 1$. let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\{0\} .$$

14.5 Inner Product Spaces

Proposition 14.5.1. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E .*

14.6 Banach Spaces

Definition 14.6.1 (Banach Space). A *Banach space* is a complete normed space.

Example 14.6.2. For any topological space X , the set $C(X)$ of bounded continuous functions $X \rightarrow \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 14.6.3. *The completion of a normed space is a Banach space.*

Proposition 14.6.4. *Let E and F be normed spaces. Let $f : E \rightarrow F$ be a continuous linear map. Then the extension to the completions $\hat{E} \rightarrow \hat{F}$ is linear.*

Proposition 14.6.5. $L^p(\mathbb{R}^n)$ is a Banach space.

Proposition 14.6.6. $C(\mathbb{R})$ is first countable but not second countable.

PROOF: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n . Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.

It is first countable because it is metrizable. \square

14.7 Hilbert Spaces

Definition 14.7.1 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Example 14.7.2. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \rightarrow \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

Proposition 14.7.3. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

14.8 Locally Convex Spaces

Definition 14.8.1 (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 14.8.2. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Proposition 14.8.3. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Example 14.8.4. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \rightarrow \mathbb{R}$ is continuous as a map $E' \rightarrow \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonné, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 14.8.5 (Thom Space). Let E be a vector bundle with a Riemannian metric, $DE = \{x : \text{El}(E) \mid \|x\| \leq 1\}$ its disc bundle and $SE := \{v : \text{El}(E) \mid \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE .