

Mathematics

Robin Adams

September 22, 2023

Contents

1	Primitive Terms and Axioms	7
1.1	Primitive Terms	7
1.2	Axioms	7
1.3	Consequences of the Axioms	8
1.3.1	Definitions	8
1.3.2	The Empty Set	8
1.3.3	The Singleton	9
1.3.4	Subsets	9
1.4	Composition	10
1.5	Axioms Part Two	10
1.6	Cartesian Product	11
1.7	Quotient Sets	11
1.8	Partitions	11
2	Category Theory	13
2.1	Categories	13
2.1.1	Sections and Retractions	14
2.1.2	Isomorphisms	14
2.1.3	Initial Objects	15
2.1.4	Terminal Objects	15
2.1.5	Zero Objects	15
2.1.6	Triads	16
2.1.7	Cotriads	16
2.1.8	Pullbacks	16
2.1.9	Pushouts	18
2.1.10	Subcategories	19
2.1.11	Opposite Category	19
2.1.12	Groupoids	19
2.1.13	Concrete Categories	20
2.1.14	Power of Categories	20
2.1.15	Arrow Category	20
2.1.16	Slice Category	20
2.2	Functors	22
2.3	Natural Transformations	24

2.4	Bifunctors	24
3	Monoid Theory	25
4	Group Theory	27
5	Ring Theory	29
6	Linear Algebra	31
7	Topology	33
7.1	Topological Spaces	33
7.1.1	Subspaces	35
7.1.2	Topological Disjoint Union	35
7.1.3	Product Topology	35
7.1.4	Bases	35
7.1.5	Subbases	35
7.1.6	Countability Axioms	35
7.2	Continuous Functions	36
7.3	Convergence	37
7.4	Connected Spaces	37
7.5	Hausdorff Spaces	38
7.6	Separable Spaces	38
7.7	Sequential Compactness	38
7.8	Compactness	38
7.9	Quotient Spaces	39
7.10	Gluing	40
7.11	Metric Spaces	41
7.12	Complete Metric Spaces	41
7.13	Manifolds	42
8	Homotopy Theory	43
8.1	Homotopies	43
8.2	Homotopy Equivalence	43
9	Simplicial Complexes	45
9.1	Cell Decompositions	45
9.2	CW-complexes	45
10	Topological Groups	47
10.1	Continuous Actions	47
11	Topological Vector Spaces	49
11.1	Cauchy Sequences	49
11.2	Seminorms	50
11.3	Fréchet Spaces	50
11.4	Normed Spaces	50

CONTENTS

5

11.5 Inner Product Spaces	51
11.6 Banach Spaces	51
11.7 Hilbert Spaces	51
11.8 Locally Convex Spaces	52

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*. We write $A : \text{Set}$ for: A is a set.

For any set A , let there be *elements* of A . We write $a : \text{El}(A)$ for: a is an element of A .

For any sets A and B , let there be *functions* from A to B . We write $f : A \rightarrow B$ iff f is a function from A to B .

For any function $f : A \rightarrow B$ and element $a : \text{El}(A)$, let there be an element $f(a) : \text{El}(B)$, the *value* of the function f at the *argument* a .

1.2 Axioms

Axiom Schema 1.1 (Choice). *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x : \text{El}(X)$ and $y : \text{El}(Y)$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a : \text{El}(A)$, there exists $b : \text{El}(B)$ such that $P[A, B, a, b]$. Then there exists a function $f : A \rightarrow B$ such that $\forall a : \text{El}(A). P[A, B, a, f(a)]$.

Axiom 1.2 (Pairing). *For any sets A and B , there exists a set $A \times B$, the Cartesian product of A and B , and functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ such that, for all $a : \text{El}(A)$ and $b : \text{El}(B)$, there exists a unique $(a, b) : \text{El}(A \times B)$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Definition 1.3 (Injective). A function $f : A \rightarrow B$ is *injective* or an *injection* iff, for all $x, y : \text{El}(A)$, if $f(x) = f(y)$ then $x = y$.

Axiom Schema 1.4 (Separation). *For every property $P[X, x]$ where X is a set variable and $x : \text{El}(X)$, the following is an axiom:*

For every set A , there exists a set $S = \{x : \text{El}(A) \mid P[A, x]\}$ and an injection $i : S \rightarrow A$ such that, for all $x : \text{El}(A)$, we have

$$(\exists y : S. i(y) = x) \Leftrightarrow P[A, x] .$$

Axiom 1.5 (Infinity). *There exists a set \mathbb{N} , an element $0 : \text{El}(\mathbb{N})$, and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that:*

- $\forall n : \text{El}(\mathbb{N}). s(n) \neq 0$
- $\forall m, n : \text{El}(\mathbb{N}). s(m) = s(n) \Rightarrow m = n.$

1.3 Consequences of the Axioms

1.3.1 Definitions

Definition 1.6. Let $f, g : A \rightarrow B$. We say f and g are *equal*, $f = g$, iff $\forall x : \text{El}(A). f(x) = g(x)$.

Definition 1.7 (Surjective). A function $f : A \rightarrow B$ is *surjective* iff, for all $y : \text{El}(B)$, there exists $x : \text{El}(A)$ such that $f(x) = y$.

Definition 1.8 (Bijective). A function $f : A \rightarrow B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$ ".

Definition 1.9 (Composition). Given $f : A \rightarrow B$ and $g : B \rightarrow C$, let $g \circ f$ be the function such that $\forall a : \text{El}(A). (g \circ f)(a) = g(f(a))$.

1.3.2 The Empty Set

Theorem 1.10. *There exists a set which has no elements.*

PROOF:

<1>1. PICK a set A

PROOF: By the Axiom of Infinity, a set exists.

<1>2. LET: $S = \{x : \text{El}(A) \mid \perp\}$ with injection $i : S \rightarrow A$

PROOF: Axiom of Separation.

<1>3. S has no elements.

□

Theorem 1.11. *If E and E' have no elements then $E \approx E'$.*

PROOF:

$\langle 1 \rangle 1$. LET: E and E' have no elements.

$\langle 1 \rangle 2$. PICK a function $F : E \rightarrow E'$.

PROOF: Axiom of Choice since vacuously $\forall x : \text{El}(E) . \exists y : \text{El}(E') . \top$.

$\langle 1 \rangle 3$. F is injective.

PROOF: Vacuously, for all $x, y : \text{El}(E)$, if $F(x) = F(y)$ then $x = y$.

$\langle 1 \rangle 4$. F is surjective.

PROOF: Vacuously, for all $y : \text{El}(E')$, there exists $x : \text{El}(E)$ such that $F(x) = y$.

□

Definition 1.12 (Empty Set). The *empty set* \emptyset is the set with no elements.

1.3.3 The Singleton

Theorem 1.13. *There exists a set that has exactly one element.*

PROOF:

$\langle 1 \rangle 1$. PICK a set A that has an element.

PROOF: By the Axiom of Infinity, there exists a set that has an element.

$\langle 1 \rangle 2$. PICK $a : \text{El}(A)$

$\langle 1 \rangle 3$. PICK a set S and injection $i : S \rightarrow A$ such that, for all $x : \text{El}(A)$, there exists $s : \text{El}(S)$ such that $s = x$ if and only if $x = a$

$\langle 1 \rangle 4$. S has exactly one element.

□

Theorem 1.14. *If A and B both have exactly one element then $A \approx B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B both have exactly one element a and b respectively.

$\langle 1 \rangle 2$. LET: $F : A \rightarrow B$ be the function such that, for all $x : \text{El}(A)$, we have
 $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$. F is a bijection.

□

Definition 1.15 (Singleton). Let 1 be the set that has exactly one element. Let $*$ be its element.

1.3.4 Subsets

Definition 1.16 (Subset). A *subset* of a set A consists of a set S and an injection $i : S \rightarrow A$. We write $(S, i) : \text{Sub}(A)$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Definition 1.17 (Membership). Given $(S, i) : \text{Sub}(A)$ and $a : \text{El}(A)$, we write $a \in S$ for $\exists s : \text{El}(S) . i(s) = a$.

1.4 Composition

Definition 1.18 (Composite). Let $\phi : A \rightarrowtail B$ and $\psi : B \rightarrowtail C$. The *composite* $\psi \circ \phi : A \rightarrowtail C$ is the relation such that $a(\psi \circ \phi)c$ iff there exists b such that $a\phi b$ and $b\psi c$.

Definition 1.19 (Identity). For any set A , the *identity* function $\text{id}_A : A \rightarrow A$ is the function defined by $\text{id}_A(a) = a$.

Theorem 1.20. *Composition of relations is associative, and the identity function is an identity for composition. The composite of functions is a function. The composite of injective functions is injective. The composite of surjective functions is surjective. The composite of bijections is a bijection. A function $f : A \rightarrow B$ is a bijection iff there exists a function $f^{-1} : B \rightarrow A$ such that $f^{-1}f = \text{id}_A$ and $ff^{-1} = \text{id}_B$, in which case f^{-1} is unique.*

1.5 Axioms Part Two

Axiom 1.21 (Power Set). *For any set A , there exists a set $\mathcal{P}A$, the power set of A , and a relation $\in : A \rightarrowtail \mathcal{P}A$, called membership, such that, for any subset S of A , there exists a unique $\bar{S} \in \mathcal{P}A$ such that, for all $x \in A$, we have $x \in \bar{S}$ if and only if $x \in S$.*

We usually write just S for \bar{S} .

Axiom Schema 1.22 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A , there exists a set B , a function $p : B \rightarrow A$, a set Y and a relation $M : B \rightarrowtail Y$ such that:

- $\forall b \in B. P[A, \{y \in Y : bMy\}, p(b)]$
- *For all $a \in A$, if $\exists Y. P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 1.23 (Universe). Let $E : U \rightarrowtail X$ be a relation. Let us say that a set A is *small* iff there exists $u \in U$ such that $A \approx \{x \in X : uEx\}$.

Then (U, X, E) form a *universe* if and only if:

- \mathbb{N} is U -small.
- For any U -small sets A and B and relation $R : A \rightarrowtail B$, the tabulation of R is U -small.
- If A is U -small then so is $\mathcal{P}A$
- Let $f : A \rightarrow B$ be a function. If B is U -small and $f^{-1}(b)$ is U -small for all $b \in B$, then A is U -small.
- If $p : B \twoheadrightarrow A$ is a surjective function such that A is U -small, then there exists a U -small set C , a surjection $q : C \twoheadrightarrow A$, and a function $f : C \rightarrow B$ such that $q = pf$.

Axiom 1.24 (Universe). *There exists a universe.*

Let $E : U \multimap X$ be a universe. We shall say a set is *small* iff it is U -small, and *large* otherwise.

1.6 Cartesian Product

Definition 1.25 (Cartesian Product). Let A and B be sets. The *Cartesian product* of A and B , $A \times B$, is the tabulation of the relation $A \multimap B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

1.7 Quotient Sets

Proposition 1.26. *Let \sim be an equivalence relation on X . Then there exists a set X/\sim , the quotient set of X with respect to \sim , and a surjective function $\pi : X \rightarrow X/\sim$, the canonical projection, such that, for all $x, y : \text{El}(X)$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \rightarrow Q$ is another quotient with respect to \sim , then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

1.8 Partitions

Definition 1.27 (Partition). A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X .

Chapter 2

Category Theory

2.1 Categories

Definition 2.1. A *category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \text{Ob}(\mathcal{C})$.
- for any objects X and Y , a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y . We write $f : X \rightarrow Y$ for $f \in \mathcal{C}[X, Y]$.
- for any objects X, Y and Z , a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \rightarrow \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \rightarrow Y, g : Y \rightarrow Z$ and $h : Z \rightarrow W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X , there exists a morphism $\text{id}_X : X \rightarrow X$, the *identity morphism* on X , such that:
 - for any object Y and morphism $f : Y \rightarrow X$ we have $\text{id}_X \circ f = f$
 - for any object Y and morphism $f : X \rightarrow Y$ we have $f \circ \text{id}_X = f$

We write the composite of morphism f_1, \dots, f_n as $f_n \circ \dots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 2.2. Let **Set** be the category of small sets and functions.

Proposition 2.3. *The identity morphism on an object is unique.*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{C} be a category.

$\langle 1 \rangle 2$. LET: $A \in \mathcal{C}$

$\langle 1 \rangle 3$. LET: $i, j : A \rightarrow A$ be identity morphisms on A .

$\langle 1 \rangle 4$. $i = j$

PROOF:

$$\begin{aligned} i &= i \circ j & (j \text{ is an identity on } A) \\ &= j & (i \text{ is an identity on } A) \end{aligned}$$

□

Definition 2.4. Given $f : A \rightarrow B$ and an object C , define the function $f^* : \mathcal{C}[B, C] \rightarrow \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 2.5. Given $f : A \rightarrow B$ and an object C , define the function $f_* : \mathcal{C}[C, A] \rightarrow \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

2.1.1 Sections and Retractions

Definition 2.6 (Section, Retraction). Let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $rs = \text{id}_B$.

Proposition 2.7. Let $f : A \rightarrow B$ and $r, s : B \rightarrow A$. If r is a retraction of f and s is a section of f then $r = s$.

PROOF:

$$\begin{aligned} r &= r \text{id}_B & (\text{Unit Law}) \\ &= rfs & (s \text{ is a section of } f) \\ &= \text{id}_A s & (r \text{ is a retraction of } f) \\ &= s & (\text{Unit Law}) \end{aligned} \quad \square$$

2.1.2 Isomorphisms

Definition 2.8 (Isomorphism). A morphism $f : A \rightarrow B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$ that is both a retraction and section of f .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 2.9. The inverse of an isomorphism is unique.

PROOF: From Proposition 2.7. □

Proposition 2.10. If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.

PROOF: Since $ff^{-1} = \text{id}_B$ and $f^{-1}f = \text{id}_A$. □

Isomorphism.

Define the opposite category.

Slice categories

Definition 2.11. Let \mathcal{C} be a category and $B \in \mathcal{C}$. The category \mathcal{C}_B^B of objects *over and under* B is the category with:

- objects all triples (X, u, p) such that $u : B \rightarrow X$ and $p : X \rightarrow B$

- morphisms $f : (X, u, p) \rightarrow (Y, u', p')$ all morphisms $f : X \rightarrow Y$ such that $fu = u'$ and $p'f = p$.

Proposition 2.12.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B) \backslash \text{id}_B \cong (\mathcal{C} \backslash B) / \text{id}_B$$

$(B, \text{id}_B, \text{id}_B)$ is the zero object in \mathcal{C}_B^B .

2.1.3 Initial Objects

Definition 2.13 (Initial Object). An object I is *initial* iff, for any object X , there exists exactly one morphism $I \rightarrow X$.

Proposition 2.14. *The empty set is initial in Set.*

PROOF: For any set A , the nowhere-defined function is the unique function $\emptyset \rightarrow A$. \square

Proposition 2.15. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : I \rightarrow I'$ be the unique morphism $I \rightarrow I'$.

$\langle 1 \rangle 2$. LET: $i^{-1} : I' \rightarrow I$ be the unique morphism $I' \rightarrow I$.

$\langle 1 \rangle 3$. $ii^{-1} = \text{id}_{I'}$

PROOF: There is only one morphism $I' \rightarrow I'$.

$\langle 1 \rangle 4$. $i^{-1}i = \text{id}_I$

PROOF: There is only one morphism $I \rightarrow I$.

\square

2.1.4 Terminal Objects

Definition 2.16 (Terminal Object). An object T is *terminal* iff, for any object X , there exists exactly one morphism $X \rightarrow T$.

Proposition 2.17. *1 is terminal in Set.*

PROOF: For any set A , the constant function to $*$ is the only function $A \rightarrow 1$. \square

2.1.5 Zero Objects

Definition 2.18 (Zero Object). An object Z is a *zero object* iff it is an initial object and a terminal object.

Definition 2.19 (Zero Morphism). Let \mathcal{C} be a category with a zero object Z . Let $A, B \in \mathcal{C}$. The *zero morphism* $A \rightarrow B$ is the unique morphism $A \rightarrow Z \rightarrow B$.

Proposition 2.20. *There is no zero object in Set.*

PROOF: Since $\emptyset \not\approx 1$. \square

2.1.6 Triads

Definition 2.21 (Triad). Let \mathcal{C} be a category. A *triad* consists of objects X , Y , M and morphisms $\alpha : X \rightarrow M$, $\beta : Y \rightarrow M$. We call M the *codomain* of the triad.

2.1.7 Cotriads

Definition 2.22 (Cotriad). Let \mathcal{C} be a category. A *cotriad* consists of objects X , Y , W and morphisms $\xi : W \rightarrow X$, $\eta : W \rightarrow Y$. We call W the *domain* of the triad.

2.1.8 Pullbacks

Definition 2.23 (Pullback). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

is a *pullback* iff, for every object Z and morphism $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$.

In this case we also say that η is the *pullback* of β along α .

Proposition 2.24. *The pullback of an isomorphism is an isomorphism.*

PROOF:

$\langle 1 \rangle 1$. LET:

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback diagram.

$\langle 1 \rangle 2$. ASSUME: β is an isomorphism.

$\langle 1 \rangle 3$. LET: ξ^{-1} be the unique morphism $X \rightarrow W$ such that $\xi \xi^{-1} = \text{id}_X$ and $\eta \xi^{-1} = \beta^{-1} \alpha$.

PROOF: This exists since $\alpha \text{id}_X = \beta \beta^{-1} \alpha = \alpha$.

$\langle 1 \rangle 4$. $\xi^{-1} \xi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\xi x = \xi$ and $\eta x = \eta$.

□

Proposition 2.25. *Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in \mathcal{C}/A . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w : A \rightarrow W$ be the unique morphism such that $\xi w = x$ and $\eta w = y$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ is the pullback of β and α in \mathcal{C}/A .

PROOF:

$\langle 1 \rangle 1$. LET: $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 2$. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle 3$. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle 4$. $hz = w$

$\langle 2 \rangle 1$. $\xi h z = \xi w$

PROOF:

$$\xi h z = f z \quad (\langle 1 \rangle 3)$$

$$= x \quad (\langle 1 \rangle 2)$$

$$= \xi w$$

$\langle 2 \rangle 2$. $\eta h z = \eta w$

PROOF: Similar.

$\langle 1 \rangle 5$. $h : (Z, z) \rightarrow (W, w)$

□

Proposition 2.26. *Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in \mathcal{C}/A . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w = x\xi : W \rightarrow A$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ form a pullback of α and β in \mathcal{C}/A .

PROOF:

$\langle 1 \rangle 1$. $\eta : (W, w) \rightarrow (Y, y)$

PROOF:

$$y\eta = m\beta\eta$$

$$= m\alpha\xi$$

$$= x\xi$$

$$= w$$

$\langle 1 \rangle 2$. LET: $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 3$. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle 4$. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle 5$. $h : (Z, z) \rightarrow (W, w)$

PROOF:

$$\begin{aligned} wh &= x\xi h \\ &= xf && (\langle 1 \rangle 4) \\ &= z && (\langle 1 \rangle 3) \end{aligned}$$

□

2.1.9 Pushouts

Definition 2.27 (Pushout). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

is a *pushout* iff, for every object Z and morphism $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ such that $f\xi = g\eta$, there exists a unique $h : M \rightarrow Z$ such that $h\alpha = f$ and $h\beta = g$.

We also say that β is the *pushout* of ξ along η .

Proposition 2.28. *The diagram above is a pushout in \mathcal{C} iff it is a pullback in \mathcal{C}^{op} .*

PROOF: Immediate from definitions. □

Proposition 2.29. *The pushout of an isomorphism is an isomorphism.*

PROOF: Dual to Proposition 2.24. □

Proposition 2.30. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m := \alpha x : A \rightarrow M$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in $\mathcal{C} \setminus A$.

PROOF: Dual to Proposition 2.26. □

Proposition 2.31. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in \mathcal{C}/A . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m : M \rightarrow A$ be the unique morphism such that $m\alpha = x$ and $m\beta = y$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in $\mathcal{C} \backslash A$.

PROOF: Dual to Proposition 2.25. \square

2.1.10 Subcategories

Definition 2.32 (Subcategory). A *subcategory* \mathcal{C}' of a category \mathcal{C} consists of:

- a subset $\text{Ob}(\mathcal{C}')$ of \mathcal{C}
- for all $A, B \in \text{Ob}(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \text{Ob}(\mathcal{C}')$, we have $\text{id}_A \in \mathcal{C}'[A, A]$
- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \text{Ob}(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

2.1.11 Opposite Category

Definition 2.33 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- $\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$
- Given $f \in \mathcal{C}^{\text{op}}[A, B]$ and $g \in \mathcal{C}^{\text{op}}[B, C]$, their composite in \mathcal{C}^{op} is $f \circ g$, where \circ is composition in \mathcal{C} .

Proposition 2.34. *An object is initial in \mathcal{C} iff it is terminal in \mathcal{C}^{op} .*

PROOF: Immediate from definitions. \square

Proposition 2.35. *An object is terminal in \mathcal{C} iff it is initial in \mathcal{C}^{op} .*

PROOF: Immediate from definitions. \square

Corollary 2.35.1. *If T and T' are terminal objects in \mathcal{C} then there exists a unique isomorphism $T \cong T'$.*

2.1.12 Groupoids

Definition 2.36 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

2.1.13 Concrete Categories

Definition 2.37 (Concrete Category). A *concrete category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*
- for any object $A \in \text{Ob}(\mathcal{C})$, a set $|A|$
- for any objects $A, B \in \text{Ob}(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$
- for any object A we have $\text{id}_{|A|} \in \mathcal{C}[A, A]$.

2.1.14 Power of Categories

Definition 2.38. Let \mathcal{C} be a category and J a set. The category \mathcal{C}^J is the category with:

- objects all J -indexed families of objects of \mathcal{C}
- morphisms $\{X_j\}_{j \in J} \rightarrow \{Y_j\}_{j \in J}$ all families $\{f_j\}_{j \in J}$ where $f_j : X_j \rightarrow Y_j$

2.1.15 Arrow Category

Definition 2.39 (Arrow Category). Let \mathcal{C} be a category. The *arrow category* $\mathcal{C}^{\rightarrow}$ is the category with:

- objects all triples (A, B, f) where $f : A \rightarrow B$ in \mathcal{C}
- morphisms $(A, B, f) \rightarrow (C, D, g)$ all pairs $(u : A \rightarrow C, v : B \rightarrow D)$ such that $vf = gu$.

2.1.16 Slice Category

Definition 2.40 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category under A*, $\mathcal{C} \backslash A$, is the category with:

- objects all pairs (B, f) where $B \in \mathcal{C}$ and $f : A \rightarrow B$
- morphisms $(B, f) \rightarrow (C, g)$ are morphisms $u : B \rightarrow C$ such that $uf = g$.

We identify this with the subcategory of $\mathcal{C}^{\rightarrow}$ formed by mapping (B, f) to (A, B, f) and u to (id_A, u) .

Proposition 2.41. *If $s : (B, f) \rightarrow (C, g)$ in $\mathcal{C} \backslash A$, then any retraction of s in \mathcal{C} is a retraction of s in $\mathcal{C} \backslash A$.*

PROOF:

$\langle 1 \rangle 1$. LET: $r : C \rightarrow B$ be a retraction of s in \mathcal{C} .

$\langle 1 \rangle 2. rg = f$

PROOF: $rg = rsf = f$.

$\langle 1 \rangle 3. r : (C, g) \rightarrow (B, f)$ in $\mathcal{C} \backslash A$

$\langle 1 \rangle 4. rs = \text{id}_{(B, f)}$

PROOF: Because composition is inherited from \mathcal{C} .

□

Proposition 2.42. id_A is the initial object in $\mathcal{C} \backslash A$.

PROOF: For any $(B, f) \in \mathcal{C} \backslash A$, we have f is the only morphism $A \rightarrow B$ such that $f \text{id}_A = f$. □

Proposition 2.43. If A is terminal in \mathcal{C} then id_A is the zero object in $\mathcal{C} \backslash A$.

PROOF: For any $(B, f) \in \mathcal{C} \backslash A$, the unique morphism $! : B \rightarrow A$ is the unique morphism such that $!f = \text{id}_A$. □

Definition 2.44 (Pointed Sets). The category of pointed sets is $\mathbf{Set} \backslash 1$.

Definition 2.45. Let \mathcal{C} be a category and $A \in \mathcal{C}$. The slice category over A , \mathcal{C}/A , is the category with:

- objects all pairs (B, f) with $f : B \rightarrow A$
- morphisms $u : (B, f) \rightarrow (C, g)$ all morphisms $u : B \rightarrow C$ such that $gu = f$.

Proposition 2.46. Let $u : (B, f) \rightarrow (C, g) : \mathcal{C}/A$. Any section of u in \mathcal{C} is a section of u in \mathcal{C}/A .

PROOF: Dual to Proposition 2.41. □

Proposition 2.47. id_A is terminal in \mathcal{C}/A .

PROOF: Dual to Proposition 2.42. □

Proposition 2.48. If A is initial in \mathcal{C} then id_A is the zero object in \mathcal{C}/A .

PROOF: Dual to Proposition 2.43. □

Definition 2.49. Let $A \in \mathcal{C}$. The category of objects over and under A , written \mathcal{C}_A^A , is the category with:

- objects all triples (X, u, p) where $u : A \rightarrow X$, $p : X \rightarrow A$ and $pu = \text{id}_A$
- morphism $f : (X, u, p) \rightarrow (Y, v, q)$ all morphisms $f : X \rightarrow Y$ such that $fu = v$ and $qf = p$

Proposition 2.50. $(A, \text{id}_A, \text{id}_A)$ is the zero object in \mathcal{C}_A^A .

PROOF: For any object (X, u, p) , we have p is the unique morphism $(X, u, p) \rightarrow (A, \text{id}_A, \text{id}_A)$, and u is the unique morphism $(A, \text{id}_A, \text{id}_A) \rightarrow (X, u, p)$. □

2.2 Functors

Definition 2.51 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- a function $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism $f : A \rightarrow B$ in \mathcal{C} , a morphism $Ff : FA \rightarrow FB$ in \mathcal{D}

such that:

- for all $A : \text{El}(\text{Ob}(\mathcal{C}))$ we have $F\text{id}_A = \text{id}_{FA}$
- for any morphism $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathcal{C} , we have $F(g \circ f) = Fg \circ Ff$

Definition 2.52 (Identity Functor). For any category \mathcal{C} , the *identity* functor on \mathcal{C} is the functor $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$\begin{aligned} I_{\mathcal{C}}A &:= A & (A \in \mathcal{C}) \\ I_{\mathcal{C}}f &:= f & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

Proposition 2.53. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $r : A \rightarrow B$ is a retraction of $s : B \rightarrow A$ in \mathcal{C} then Fr is a retraction of Fs .

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned} \quad \square$$

Corollary 2.53.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $\phi : A \cong B$ is an isomorphism in \mathcal{C} then $F\phi : FA \cong FB$ is an isomorphism in \mathcal{D} with $(F\phi)^{-1} = F\phi^{-1}$.

Definition 2.54 (Composition of Functors). Given functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$, the *composite* functor $GF : \mathcal{C} \rightarrow \mathcal{E}$ is defined by

$$\begin{aligned} (GF)A &= G(FA) & (A \in \mathcal{C}) \\ (GF)f &= G(Ff) & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

Definition 2.55 (Category of Categories). Let **Cat** be the category of small categories and functors.

Definition 2.56 (Isomorphism of Categories). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F is an *isomorphism of categories* iff there exists a functor $F^{-1} : \mathcal{D} \rightarrow \mathcal{C}$, the *inverse* of F , such that $FF^{-1} = I_{\mathcal{D}}$ and $F^{-1}F = I_{\mathcal{C}}$.

Categories \mathcal{C} and \mathcal{D} are *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, iff there exists an isomorphism between them.

Proposition 2.57. If A is initial in \mathcal{C} then $\mathcal{C} \setminus A \cong \mathcal{C}$.

PROOF:

$\langle 1 \rangle 1$. Define $F : \mathcal{C} \backslash A \rightarrow \mathcal{C}$ by

$$F(B, f) = B$$

$$F(u : (B, f) \rightarrow (C, g)) = u$$

$\langle 1 \rangle 2$. Define $G : \mathcal{C} \rightarrow \mathcal{C} \backslash A$ by

$$GB = (B, !_B)$$

where $!_B$ is the unique morphism $A \rightarrow B$

$$G(u : B \rightarrow C) = u : (B, !_B) \rightarrow (C, !_C)$$

$\langle 1 \rangle 3$. $FG = \text{id}_{\mathcal{C}}$

$\langle 1 \rangle 4$. $GF = \text{id}_{\mathcal{C} \backslash A}$

PROOF: Since $GF(B, f) = (B, !_B) = (B, f)$ because the morphism $A \rightarrow B$ is unique.

□

Proposition 2.58. *If A is terminal in \mathcal{C} then $\mathcal{C}/A \cong \mathcal{C}$.*

PROOF: Dual. □

Proposition 2.59.

$$\mathcal{C}_A^A \cong (\mathcal{C}/A) \backslash (A, \text{id}_A) \cong (\mathcal{C} \backslash A) / (A, \text{id}_A)$$

PROOF:

$\langle 1 \rangle 1$. Define a functor $F : \mathcal{C}_A^A \rightarrow (\mathcal{C}/A) \backslash (A, \text{id}_A)$.

$\langle 2 \rangle 1$. Given $A \xrightarrow{u} X \xrightarrow{p} A$ in \mathcal{C}_A^A , let $F(X, u, p) = ((X, p), u)$

$\langle 2 \rangle 2$. Given $f : (A \xrightarrow{u} X \xrightarrow{p} A) \rightarrow (A \xrightarrow{v} Y \xrightarrow{q} A)$, let $Ff = f$.

$\langle 1 \rangle 2$. Define a functor $G : (\mathcal{C}/A) \backslash (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.

$\langle 1 \rangle 3$. Define a functor $H : \mathcal{C}_A^A \rightarrow (\mathcal{C} \backslash A) / (A, \text{id}_A)$.

$\langle 1 \rangle 4$. Define a functor $K : (\mathcal{C} \backslash A) / (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.

□

Natural transformation.

Pullback

Pushout

Product

Coproduct

Adjunction

Definition 2.60 (Forgetful Functor). For any concrete category \mathcal{C} , define the *forgetful* functor $U : \mathcal{C} \rightarrow \mathbf{Set}$ by:

$$UA = |A|$$

$$Uf = f$$

Definition 2.61 (Switching Functor). For any category \mathcal{C} , define the *switching* functor $T : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ by

$$T(A, B) = (B, A)$$

$$T(f, g) = (g, f)$$

2.3 Natural Transformations

Definition 2.62 (Natural Transformation). Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$. A *natural transformation* $\tau : F \Rightarrow G$ is a family of morphisms $\{\tau_X : FX \rightarrow GX\}_{X \in \mathcal{C}}$ such that, for every morphism $f : X \rightarrow Y : \mathcal{C}$, we have $Gf \circ \tau_X = \tau_Y \circ Ff$.

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \tau_X \downarrow & & \downarrow \tau_Y \\ GX & \xrightarrow{Gf} & GY \end{array}$$

Definition 2.63 (Natural Isomorphism). A natural transformation $\tau : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$ is a *natural isomorphism*, $\tau : F \cong G$, iff for all $X \in \mathcal{C}$, τ_X is an isomorphism $FX \cong GX$.

Functors F and G are *naturally isomorphic*, $F \cong G$, iff there exists a natural isomorphism between them.

Definition 2.64 (Inverse). Let $\tau : F \cong G$. The *inverse* natural isomorphism $\tau^{-1} : G \cong F$ is defined by $(\tau^{-1})_X = \tau_X^{-1}$.

2.4 Bifunctors

Definition 2.65 (Commutative). A bifunctor $\square : \mathcal{C}^2 \rightarrow \mathcal{C}$ is *commutative* iff $\square \cong \square \circ T$, where $T : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ is the swap functor.

Definition 2.66 (Associative). A bifunctor \square is *associative* iff $\square \circ (\square \times \text{id}) \cong \square \circ (\text{id} \times \square)$.

Product and coproduct are commutative and associative.

Chapter 3

Monoid Theory

Definition 3.1 (Monoid). A *monoid* is a category with one object.

Definition 3.2. Let \mathcal{C} be a category and $X \in \mathcal{C}$. The monoid $\text{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \rightarrow X$ under composition.

Proposition 3.3. *For any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and $X \in \mathcal{C}$, we have that $F : \text{End}_{\mathcal{C}}(X) \rightarrow \text{End}_{\mathcal{D}}(FX)$ is a monoid homomorphism.*

PROOF: Since $F\text{id}_X = \text{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$. \square

Chapter 4

Group Theory

Definition 4.1. Let **Grp** be the category of small groups and group homomorphisms.

Definition 4.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G .

Proposition 4.3. *The trivial group is a zero object in **Grp**.*

PROOF: Easy. \square

The zero morphism $G \rightarrow H$ maps every element in G to e .

Definition 4.4. Let \mathcal{C} be a category and $X \in \mathcal{C}$. We write $\text{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Proposition 4.5. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $X \in \mathcal{C}$. Then $F : \text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(FX)$ is a group homomorphism.*

PROOF: Since $F\text{id}_X = \text{id}_{FX}$, $F(g \circ f) = Fg \circ Ff$, and $Ff^{-1} = (Ff)^{-1}$. \square

Chapter 5

Ring Theory

Definition 5.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Chapter 6

Linear Algebra

Definition 6.1. For any field K , let \mathbf{Vect}_K be the concrete category of small vector spaces over K and linear transformations.

Dual space functor $\mathbf{Vect}_K^{\mathrm{op}} \rightarrow \mathbf{Vect}_K$.

Chapter 7

Topology

7.1 Topological Spaces

Definition 7.1 (Topological Space). Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say (X, \mathcal{O}) is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call \mathcal{O} the *topology* of the topological space, and call its elements *open sets*. We shall often write X for the topological space (X, \mathcal{O}) .

Definition 7.2 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 7.3. *A set B is open if and only if $X - B$ is closed.*

Proposition 7.4. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that \mathcal{C} is the set of closed sets if and only if:*

- For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$
- For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.
- $\emptyset \in \mathcal{C}$

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Definition 7.5 (Neighbourhood). Let X be a topological space, $x \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x , and x is an *interior* point of U , iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 7.6. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 7.7. *Let X be a set and $\mathcal{N} : X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that, for all $x \in X$, we have \mathcal{N}_x is the set of neighbourhoods of x , if and only if:*

- *For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$*
- *For all $x \in X$ we have $X \in \mathcal{N}_x$*
- *For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$*
- *For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$*
- *For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M. M \in \mathcal{N}_y$.*

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$.

Definition 7.8 (Exterior Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x .

Definition 7.9 (Boundary Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B .

Definition 7.10 (Interior). Let X be a topological space and $B \subseteq X$. The *interior* of B , B° , is the set of all interior points of B .

Proposition 7.11. *The interior of B is the union of all the open sets included in B .*

Definition 7.12 (Closure). Let X be a topological space and $B \subseteq X$. The *closure* of B , \overline{B} , is the set of all points that are not exterior points of B .

Proposition 7.13. *The closure of B is the intersection of all the closed sets that include B .*

Proposition 7.14. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 7.15 (Kuratowski Closure Axioms). *Let X be a set and $- : \mathcal{P}X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} such that, for all $B \subseteq X$, \overline{B} is the closure of B , if and only if:*

- $\overline{\emptyset} = \emptyset$
- *For all $A \subseteq X$ we have $A \subseteq \overline{A}$*
- *For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$*
- *For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

In this case, \mathcal{O} is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

7.1.1 Subspaces

Definition 7.16 (Subspace). Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

Example 7.17. The *unit sphere* S^2 is $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a subspace of \mathbb{R}^3 .

7.1.2 Topological Disjoint Union

Definition 7.18. Let X and Y be topological spaces. The *disjoint union* is $X + Y$ where $U \subseteq X + Y$ is open if and only if $\kappa_1^{-1}(U)$ is open in X and $\kappa_2^{-1}(U)$ is open in Y .

7.1.3 Product Topology

Definition 7.19 (Product Topology). Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

7.1.4 Bases

Definition 7.20 (Basis). Let X be a topological space. A *basis* for the topology on X is a set of open sets \mathcal{B} such that every open set is the union of a subset of \mathcal{B} .

7.1.5 Subbases

Definition 7.21 (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of \mathcal{S} .

Definition 7.22 (Space with Basepoint). A *space with basepoint* is a pair (X, x) where X is a topological space and $x \in \text{El}(X)$.

7.1.6 Countability Axioms

Definition 7.23 (Neighbourhood Basis). Let X be a topological space and $x_0 \in \text{El}(X)$. A *neighbourhood basis* of x_0 is a set \mathcal{U} of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of \mathcal{U} .

Definition 7.24 (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

Definition 7.25 (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

\mathbb{R}^n is second countable.

An uncountable discrete space is first countable but not second countable.

Proposition 7.26. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.*

PROOF:

$\langle 1 \rangle 1$. For all $\lambda : \text{El}(\Lambda)$, PICK U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.

$\langle 1 \rangle 2$. For all $\lambda : \text{El}(\Lambda)$, PICK $x_\lambda \in U_\lambda$.

$\langle 1 \rangle 3$. ASSUME: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda \in \Lambda}$.

$\langle 1 \rangle 4$. PICK $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$

$\langle 1 \rangle 5$. There is no $U \in B$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

□

7.2 Continuous Functions

Definition 7.27 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* iff, for every open set V in Y , the inverse image $f^{-1}(V)$ is open in X .

Proposition 7.28. 1. id_X is continuous

2. The composite of two continuous functions is continuous.

3. If $f : X \rightarrow Y$ is continuous and $X_0 \subseteq X$ then $f|_{X_0} : X_0 \rightarrow Y$ is continuous.

4. If $f : X + Y \rightarrow Z$, then f is continuous iff $f \circ \kappa_1 : X \rightarrow Z$ and $f \circ \kappa_2 : Y \rightarrow Z$ are continuous.

5. If $f : Z \rightarrow X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Definition 7.29 (Homeomorphism). Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Definition 7.30 (Retraction). Let X be a topological space and A a subspace of X . A continuous function $\rho : X \rightarrow A$ is a *retraction* iff $\rho|_A = \text{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 7.31. Let **Top** be the category of small topological spaces and continuous functions.

Forgetful functor **Top** \rightarrow **Set**.

Basepoint preserving continuous functor.

7.3 Convergence

Definition 7.32 (Convergence). Let X be a topological space. Let (x_n) be a sequence in X . A point $a \in \text{El}(X)$ is a *limit* of the sequence iff, for every neighbourhood U of a , there exists n_0 such that $\forall n \geq n_0. x_n \in U$.

Convergence in a product space is pointwise convergence.

If $f : X \rightarrow Y$ is continuous and $x_n \rightarrow l$ in X then $f(x_n) \rightarrow f(l)$ in Y .

Example 7.33. The converse does not hold.

Let X be the set of all continuous functions $[0, 1] \rightarrow [-1, 1]$ under the product topology. Let $i : X \rightarrow L^2([0, 1])$ be the inclusion.

If $f_n \rightarrow f$ then $i(f_n) \rightarrow i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K. \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0, 1]} U_\lambda$ where $U_\lambda = [-1, 1]$ except for $\lambda = \lambda_1, \dots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \dots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 7.34. *The converse does hold for first countable spaces. If $f : X \rightarrow Y$ where X is first countable, and Y is a topological space, and whenever $x_n \rightarrow x$ then $f(x_n) \rightarrow f(x)$, then f is continuous.*

7.4 Connected Spaces

Definition 7.35 (Connected). A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 7.36. *The continuous image of a connected space is connected.*

Proposition 7.37. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are connected, then X is connected.*

Proposition 7.38. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 7.39 (Path-connected). A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \rightarrow X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 7.40. *The continuous image of a path connected space is path connected.*

Proposition 7.41. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are path connected, then X is path connected.*

Proposition 7.42. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

7.5 Hausdorff Spaces

Definition 7.43 (Hausdorff). A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 7.44. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 7.45. 1. *Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

Proposition 7.46. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \rightarrow B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X , then $f = g$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.

$\langle 1 \rangle 2$. PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.

$\langle 1 \rangle 3$. PICK $x \in f^{-1}(U) \cap g^{-1}(V)$

$\langle 1 \rangle 4$. $f(x) = g(x) \in U \cap V$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 7.47. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$ be uniformly continuous. Let \hat{X} and \hat{Y} be the completions of X and Y . Then f extends uniquely to a continuous map $\hat{X} \rightarrow \hat{Y}$.*

PROOF: The extension maps $\lim_{n \rightarrow \infty} x_n$ to $\lim_{n \rightarrow \infty} f(x_n)$. □

7.6 Separable Spaces

Definition 7.48 (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

7.7 Sequential Compactness

Definition 7.49 (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

7.8 Compactness

Definition 7.50 (Compact). A topological space is *compact* iff every open cover has a finite subcover.

Proposition 7.51. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P . Then $X \in P$.*

PROOF:

- $\langle 1 \rangle 1$. P is an open cover of X
 $\langle 1 \rangle 2$. PICK a finite subcover $U_1, \dots, U_n \in P$
 $\langle 1 \rangle 3$. $X = U_1 \cup \dots \cup U_n \in P$
 \square

Corollary 7.51.1. *Let f be a compact space and $f : X \rightarrow \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. \square

Proposition 7.52. *The continuous image of a compact space is compact.*

Proposition 7.53. *A closed subspace of a compact space is compact.*

Proposition 7.54. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. X and Y are compact.
2. $X + Y$ is compact.
3. $X \times Y$ is compact.

Proposition 7.55. *A compact subspace of a Hausdorff space is closed.*

Proposition 7.56. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 7.57. *A first countable compact space is sequentially compact.*

7.9 Quotient Spaces

Definition 7.58 (Quotient Space). Let X be a topological space and \sim an equivalence relation on X . The *quotient topology* on X/\sim is defined by: $U : \text{El}(\mathcal{P}X)$ is open in X/\sim if and only if $\pi^{-1}(U)$ is open in X .

Proposition 7.59. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $f : X/\sim \rightarrow Y$. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proposition 7.60. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $\phi : Y \rightarrow X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \rightarrow X$ such that $\pi \circ \Phi = \phi|U$. Then ϕ is continuous.

Proposition 7.61. *A quotient of a connected space is connected.*

Proposition 7.62. *A quotient of a path connected space is path connected.*

Proposition 7.63. *Let X be a topological space and \sim an equivalence relation on X . If X/\sim is Hausdorff then every equivalence class of \sim is closed in X .*

Definition 7.64. Let X be a topological space and $A_1, \dots, A_r \subseteq X$. Then $X/A_1, \dots, A_r$ is the quotient space of X with respect to \sim where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 7.65 (Cone). Let X be a topological space. The *cone over X* is the space $(X \times [0, 1])/(X \times \{1\})$.

Definition 7.66 (Suspension). Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 7.67 (Wedge Product). Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 7.68 (Smash Product). Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 7.69. $D^n/S^{n-1} \cong S^n$

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : D^n/S^{n-1} \rightarrow S^n$ be the function induced by the map $D^n \rightarrow S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.

$\langle 1 \rangle 2$. ϕ is a bijection.

$\langle 1 \rangle 3$. ϕ is a homeomorphism.

PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.

□

7.10 Gluing

Definition 7.70 (Gluing). Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \rightarrow Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where \sim is the equivalence relation generated by $x \sim \phi(x)$ for all $x : \text{El}(X)$.

Proposition 7.71. *Y is a subspace of $Y \cup_\phi X$.*

Definition 7.72. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0, 1])/\alpha$ is the quotient space of $X \times [0, 1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x : \text{El}(X)$.

Definition 7.73 (Möbius Strip). The *Möbius strip* is $([-1, 1] \times [0, 1])/\alpha$ where $\alpha(x) = -x$.

Definition 7.74 (Klein Bottle). The *Klein bottle* is $(S^1 \times [0, 1])/\alpha$ where $\alpha(z) = \bar{z}$.

Proposition 7.75. Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\text{id}_M} M \cong K$.

PROOF:

$\langle 1 \rangle 1$. LET: $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{i\theta/2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-i\theta/2}, t)$.

$\langle 1 \rangle 2$. f induces a bijection $M \cup_{\text{id}_M} M \approx K$

$\langle 1 \rangle 3$. f is a homeomorphism.

□

7.11 Metric Spaces

Definition 7.76 (Metric Space). Let X be a set and $d : X^2 \rightarrow \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geq 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d) . We often write X for the metric space (X, d) .

Definition 7.77 (Topology of a Metric Space). Let (X, d) be a metric space. The topology *induced* by the metric d is defined by: for $V \subseteq X$, we have V is open if and only if, for all $x \in V$, there exists $\epsilon > 0$ such that $\{y \in X : d(x, y) < \epsilon\} \subseteq V$.

Definition 7.78 (Metrizable). A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 7.79. Every metrizable space is Hausdorff.

Every metrizable space is first countable.

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

7.12 Complete Metric Spaces

Definition 7.80 (Complete). A metric space is *complete* iff every Cauchy sequence converges.

Example 7.81. \mathbb{R} is complete.

Proposition 7.82. *The product of two complete metric spaces is complete.*

Proposition 7.83. *Every compact metric space is complete.*

Proposition 7.84. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 7.85 (Completion). Let X be a metric space. A *completion* of X is a complete metric space \hat{X} and injection $i : X \rightarrow \hat{X}$ such that:

- The metric on X is the restriction of the metric on \hat{X}
- X is dense in \hat{X} .

Proposition 7.86. *Let $i_1 : X \rightarrow Y_1$ and $i_2 : X \rightarrow Y_2$ be completions of X . Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

PROOF: Define $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$. \square

Theorem 7.87. *Every metric space has a completion.*

PROOF: Let \hat{X} be the set of Cauchy sequences in X quotiented by \sim where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \rightarrow 0$. \square

7.13 Manifolds

Definition 7.88 (Manifold). An *n -dimensional manifold* is a second countable Hausdorff space locally homeomorphic to \mathbb{R}^n .

Chapter 8

Homotopy Theory

8.1 Homotopies

Definition 8.1 (Homotopy). Let X and Y be topological spaces. Let $f, g : X \rightarrow Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0, 1] \rightarrow Y$ such that

- $\forall x : \text{El}(X) . h(x, 0) = f(x)$
- $\forall x : \text{El}(X) . h(x, 1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \rightarrow Y$.

Proposition 8.2. Let $f, f' : X \rightarrow Y$ and $g, g' : Y \rightarrow Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.

Definition 8.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

Definition 8.4. A functor $F : \mathbf{Top} \rightarrow \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \rightarrow Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

8.2 Homotopy Equivalence

Definition 8.5 (Homotopy Equivalence). Let X and Y be topological spaces. A *homotopy equivalence* between X and Y , $f : X \simeq Y$, is a continuous function $f : X \rightarrow Y$ such that there exists a continuous function $g : Y \rightarrow X$, the *homotopy inverse* to f , such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.

Definition 8.6 (Contractible). A topological space X is *contractible* iff $X \simeq 1$.

Example 8.7. \mathbb{R}^n is contractible.

Example 8.8. D^n is contractible.

Definition 8.9 (Deformation Retract). Let X be a topological space and A a subspace of X . A retraction $\rho : X \rightarrow A$ is a *deformation retraction* iff $i \circ \rho \simeq \text{id}_X$, where i is the inclusion $A \hookrightarrow X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 8.10 (Strong Deformation Retract). Let X be a topological space and A a subspace of X . A *strong deformation retraction* $\rho : X \rightarrow A$ is a continuous function such that there exists a homotopy $h : X \times [0, 1] \rightarrow X$ between $i \circ \rho$ and id_X such that, for all $a : \text{El}(X)$ and $t : \text{El}([0, 1])$, we have $h(a, t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 8.11. $\{0\}$ is a strong deformation retract of \mathbb{R}^n and of D^n .

Example 8.12. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 8.13. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 8.14. For any topological space X , the singleton consisting of the vertex is a strong deformation retract of the cone over X .

Chapter 9

Simplicial Complexes

Definition 9.1 (Simplex). A k -dimensional simplex or k -simplex in \mathbb{R}^n is the convex hull $s(x_0, \dots, x_k)$ of $k + 1$ points in general position.

Definition 9.2 (Face). A *sub-simplex* or *face* of $s(x_0, \dots, x_k)$ is the convex hull of a subset of $\{x_0, \dots, x_k\}$.

Definition 9.3 (Simplicial Complex). A *simplicial complex* in \mathbb{R}^n is a set K of simplices such that:

- for every simplex s in K , every face of s is in K .
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2 .
- K is locally finite, i.e. every point of \mathbb{R}^n has a neighbourhood that only intersects finitely many elements of K .

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of \mathbb{R}^n .

9.1 Cell Decompositions

Definition 9.4 (n -cell). An n -cell is a topological space homeomorphic to \mathbb{R}^n .

Definition 9.5 (Cell Decomposition). Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n -cells.

Definition 9.6 (n -skeleton). Given a cell decomposition of X , the n -skeleton X^n is the union of all the cells of dimension $\leq n$.

9.2 CW-complexes

Definition 9.7 (CW-Complex). A *CW-complex* consists of a topological space X and a cell decomposition \mathcal{E} of X such that:

1. *Characteristic Maps* For every n -cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \rightarrow X$ such that $\Phi_e((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in \mathcal{E} of dimension $< n$.
2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have \bar{e} intersects only finitely many other cells in \mathcal{E} .
3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \bar{e}$ is closed.

Proposition 9.8. *If a cell decomposition \mathcal{E} satisfies the Characteristic Maps axiom, then for every n -cell $e \in \mathcal{E}$ we have $\bar{e} = \Phi_e(D^n)$. Therefore \bar{e} is compact and $\bar{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

PROOF:

$\langle 1 \rangle 1.$ $e \subseteq \Phi_e(D^n) \subseteq \bar{e}$

PROOF:

$$\begin{aligned}
 e &= \Phi_e((D^n)^\circ) \\
 &\subseteq \Phi_e(D^n) \\
 &= \Phi_e(\overline{(D^n)^\circ}) \\
 &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
 &= \bar{e}
 \end{aligned}$$

$\langle 1 \rangle 2.$ $\Phi_e(D^n)$ is compact.

PROOF: Because D^n is compact.

$\langle 1 \rangle 3.$ $\Phi_e(D^n)$ is closed.

$\langle 1 \rangle 4.$ $\Phi_e(D^n) = \bar{e}$

□

Chapter 10

Topological Groups

Definition 10.1 (Topological Group). A *topological group* is a group G with a topology such that the function $G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.

Example 10.2. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 10.3. Any subgroup of a topological group is a topological group under the subspace topology.

Definition 10.4 (Homogeneous Space). A *homogeneous space* is a topological space of the form G/H , where G is a topological group and H is a normal subgroup of G , under the quotient topology.

Proposition 10.5. Let G be a topological group and H a normal subgroup of G . Then G/H is Hausdorff if and only if H is closed.

PROOF: See Bourbaki, N., General Topology. III.12 \square

10.1 Continuous Actions

Definition 10.6 (Continuous Action). Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \rightarrow X$ such that:

- $\forall x : \text{El}(X) . ex = x$
- $\forall g, h : \text{El}(G) . \forall x : \text{El}(X) . g(hx) = (gh)x$

A G -space consists of a topological space X and a continuous action of G on X .

Definition 10.7 (Orbit). Let X be a G -space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 10.8. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:

$\langle 1 \rangle 1$. LET: $f_3 : S^2/SO(2) \rightarrow [-1, 1]$ be the function induced by $\pi_3 : S^2 \rightarrow [-1, 1]$

$\langle 1 \rangle 2$. f_3 is bijective.

$\langle 1 \rangle 3$. $S^2/SO(2)$ is compact.

PROOF: It is the continuous image of S^2 which is compact.

$\langle 1 \rangle 4$. $[-1, 1]$ is Hausdorff.

$\langle 1 \rangle 5$. f_3 is a homeomorphism.

□

Definition 10.9 (Stabilizer). Let X be a G -space and $x \in X$. The *stabilizer* of x is $G_x := \{g : \text{El}(G) \mid gx = x\}$.

Proposition 10.10. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx .*

PROOF:

$\langle 1 \rangle 1$. If $gG_x = hG_x$ then $gx = hx$.

$\langle 2 \rangle 1$. ASSUME: $gG_x = hG_x$

$\langle 2 \rangle 2$. $g^{-1}h \in G_x$

$\langle 2 \rangle 3$. $g^{-1}hx = x$

$\langle 2 \rangle 4$. $gx = hx$

$\langle 1 \rangle 2$. If $gx = hx$ then $gG_x = hG_x$.

PROOF: Similar.

$\langle 1 \rangle 3$. The function is continuous.

PROOF: Proposition 7.59.

□

Chapter 11

Topological Vector Spaces

Definition 11.1 (Topological Vector Space). Let K be either \mathbb{R} or \mathbb{C} . A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Subtraction is a continuous function $E^2 \rightarrow E$
- Multiplication is a continuous function $K \times E \rightarrow E$

Proposition 11.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. \square

Theorem 11.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 \square

Proposition 11.4. *Let E be a topological vector space and E_0 a subspace of E . Then $\overline{E_0}$ is a subspace of E .*

Definition 11.5. Let E be a topological vector space. The topological space associated with E is $E/\overline{\{0\}}$.

11.1 Cauchy Sequences

Definition 11.6 (Cauchy Sequence). Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geq n_0, x_n - x_m \in U$.

Definition 11.7 (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

11.2 Seminorms

Definition 11.8 (Seminorm). Let E be a vector space over K . A *seminorm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ such that:

1. $\forall x : \text{El}(E) . \|x\| \geq 0$
2. $\forall \alpha : \text{El}(K) . \forall x : \text{El}(E) . \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y : \text{El}(E) . \|x + y\| \leq \|x\| + \|y\|$

Example 11.9. The function that maps (x_1, \dots, x_n) to $|x_i|$ is a seminorm on \mathbb{R}^n .

Definition 11.10. Let E be a vector space over K . Let Λ be a set of seminorms on E . The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x : \text{El}(E)$.

Proposition 11.11. E is a topological vector space under this topology. It is Hausdorff iff, for all $x : \text{El}(E)$, if $\forall \lambda \in \Lambda . \lambda(x) = 0$ then $x = 0$.

11.3 Fréchet Spaces

Definition 11.12 (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 11.13. Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$. Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 11.14 (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

11.4 Normed Spaces

Definition 11.15 (Normed Space). Let E be a vector space over K . A *norm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ is a seminorm such that, $\forall x \in E . \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 11.16. If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E .

Proposition 11.17. *Let $\| \cdot \|$ be a seminorm on the vector space E . Then $\| \cdot \|$ defines a norm on $E/\{0\}$.*

Proposition 11.18. *Let E and F be normed spaces. Any continuous linear map $E \rightarrow F$ is uniformly continuous.*

Definition 11.19. For $p \geq 1$, let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\{0\}.$$

11.5 Inner Product Spaces

Proposition 11.20. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E .*

11.6 Banach Spaces

Definition 11.21 (Banach Space). A *Banach space* is a complete normed space.

Example 11.22. For any topological space X , the set $C(X)$ of bounded continuous functions $X \rightarrow \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 11.23. *The completion of a normed space is a Banach space.*

Proposition 11.24. *Let E and F be normed spaces. Let $f : E \rightarrow F$ be a continuous linear map. Then the extension to the completions $\hat{E} \rightarrow \hat{F}$ is linear.*

Proposition 11.25. $L^p(\mathbb{R}^n)$ is a Banach space.

Proposition 11.26. $C(\mathbb{R})$ is first countable but not second countable.

PROOF: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n . Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.

It is first countable because it is metrizable. \square

11.7 Hilbert Spaces

Definition 11.27 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Example 11.28. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \rightarrow \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

Proposition 11.29. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

11.8 Locally Convex Spaces

Definition 11.30 (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 11.31. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Proposition 11.32. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Example 11.33. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \rightarrow \mathbb{R}$ is continuous as a map $E' \rightarrow \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 11.34 (Thom Space). Let E be a vector bundle with a Riemannian metric, $DE = \{x : \text{El}(E) \mid \|x\| \leq 1\}$ its disc bundle and $SE := \{v : \text{El}(E) \mid \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE .