

Mathematics

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Contents

I	Set Theory	11
1	Primitive Terms and Axioms	13
1.1	Primitive Terms	13
1.2	Injections, Surjections and Bijections	13
1.3	Axioms	14
2	Sets and Functions	17
2.1	Composition	17
2.2	Injections	17
2.3	Surjections	19
2.4	Bijections	20
2.5	Domination	21
2.6	Identity Function	22
	2.6.1 Injections, Surjections, Bijections	22
	2.6.2 Composition	22
2.7	The Empty Set	24
2.8	The Singleton	24
	2.8.1 Injections	24
2.9	The Set Two	24
2.10	Subsets	25
2.11	Power Set	26
2.12	Saturated Set	26
2.13	Union	27
	2.13.1 Intersection	27
	2.13.2 Direct Image	27
2.14	Inverse Image	27
	2.14.1 Saturated Sets	28
2.15	Relations	28
	2.15.1 Equivalence Relations	29
2.16	Power Set	29
	2.16.1 Partitions	29
2.17	Cartesian Product	29
2.18	Quotient Sets	29
2.19	Partitions	30

2.20	Disjoint Union	30
2.21	Natural Numbers	30
2.22	Finite and Infinite Sets	31
2.23	Countable Sets	32
2.24	Fixed Points	34
2.25	Finite Intersection Property	34
3	Relations	35
4	Order Theory	37
4.1	Strict Partial Orders	37
4.1.1	Linear Orders	37
4.1.2	Sets of Finite Type	39
4.2	Linear Continua	39
4.3	Well Orders	40
II	Category Theory	45
5	Category Theory	47
5.1	Categories	47
5.1.1	Monomorphisms	51
5.1.2	Epimorphisms	51
5.1.3	Sections and Retractions	52
5.1.4	Isomorphisms	52
5.1.5	Initial Objects	53
5.1.6	Terminal Objects	53
5.1.7	Zero Objects	54
5.1.8	Triads	54
5.1.9	Cotriads	54
5.1.10	Pullbacks	54
5.1.11	Pushouts	57
5.1.12	Subcategories	60
5.1.13	Opposite Category	60
5.1.14	Groupoids	61
5.1.15	Concrete Categories	61
5.1.16	Power of Categories	61
5.1.17	Arrow Category	61
5.1.18	Slice Category	61
5.2	Functors	64
5.3	Natural Transformations	67
5.4	Bifunctors	68
5.5	Functor Categories	69

<i>CONTENTS</i>	5
III Number Systems	71
6 The Real Numbers	73
6.1 Subtraction	75
6.2 The Ordered Square	81
6.3 Punctured Euclidean Space	82
6.4 Topologist's Sine Curve	82
6.5 The Long Line	82
7 Integers and Rationals	83
7.1 Positive Integers	83
7.1.1 Exponentiation	84
7.2 Integers	85
7.3 Rational Numbers	87
7.4 Algebraic Numbers	88
IV Algebra	89
8 Monoid Theory	91
9 Group Theory	93
9.1 Category of Small Groups	93
10 Ring Theory	95
11 Field Theory	97
12 Linear Algebra	99
12.1 Inner Product Spaces	100
V Analysis	103
13 Real Analysis	105
13.1 Hermite Polynomials	105
13.2 Fourier Transforms	106
VI Topology	107
14 Topological Spaces	109
14.1 Topological Spaces	109
14.2 Closed Sets	110
14.3 Neighbourhoods	111
14.4 Interior	112
14.5 Closure	112

14.6	Bases	113
14.7	Order Topology	114
14.7.1	Subspaces	116
14.7.2	Product Topology	117
14.8	Subbases	117
14.9	Neighbourhood Bases	118
14.10	First Countable Spaces	118
14.11	Second Countable Spaces	118
14.12	Interior	119
14.13	Closure	119
14.13.1	Bases	120
14.13.2	Subspaces	120
14.13.3	Product Topology	121
14.13.4	Interior	121
14.14	Boundary	122
14.15	Limit Points	123
14.16	Isolated Points	124
15	Continuous Functions	125
15.0.1	Order Topology	129
15.0.2	Paths	130
15.0.3	Loops	130
15.1	Convergence	130
15.1.1	Closure	131
15.1.2	Continuous Functions	131
15.1.3	Infinite Series	131
15.2	Strong Continuity	131
15.3	Subspaces	132
15.3.1	Product Topology	134
15.4	Embedding	135
15.5	Open Maps	135
15.5.1	Subspaces	136
15.6	Locally Finite	136
15.7	Closed Maps	137
15.8	Product Topology	137
15.8.1	Closed Sets	137
15.8.2	Closure	139
15.8.3	Convergence	139
15.9	Topological Disjoint Union	140
15.10	Quotient Spaces	142
15.10.1	Quotient Maps	143
15.11	Box Topology	147
15.11.1	Bases	148
15.11.2	Subspaces	148
15.11.3	Closure	148
15.12	Separations	149

15.13	Connected Spaces	149
15.13.1	The Real Numbers	150
15.13.2	The Indiscrete Topology	150
15.13.3	The Cofinite Topology	150
15.13.4	Finer and Coarser	150
15.13.5	Boundary	150
15.13.6	Continuous Functions	151
15.13.7	Subspaces	151
15.13.8	Order Topology	153
15.13.9	Product Topology	154
15.13.10	Quotient Spaces	156
15.14	T_1 Spaces	156
15.14.1	Limit Points	157
15.15	Hausdorff Spaces	157
15.15.1	Product Topology	159
15.15.2	Box Topology	159
15.15.3	T_1 Spaces	159
15.16	Separable Spaces	160
15.17	Sequential Compactness	160
15.18	Compactness	160
15.19	Gluing	161
15.20	Homogeneous Spaces	162
15.21	Regular Spaces	162
15.22	Totally Disconnected Spaces	162
15.23	Path Connected Spaces	162
15.23.1	The Ordered Square	162
15.23.2	Punctured Euclidean Space	162
15.23.3	The Topologist's Sine Curve	163
15.23.4	The Long Line	163
15.23.5	Continuous Functions	163
15.23.6	Subspaces	164
15.23.7	Product Topology	164
15.23.8	Connected Spaces	164
15.24	Locally Homeomorphic	165
15.24.1	The Long Line	165
15.25	Components	166
15.26	Path Components	167
15.27	Weak Local Connectedness	168
15.28	Local Connectedness	169
15.29	Local Path Connectedness	171
15.30	Quasicomponents	173
15.31	Compact Spaces	174
15.32	Perfect Maps	181

16 Metric Spaces	183
16.1 Metric Spaces	183
16.1.1 Balls	183
16.1.2 Subspaces	189
16.1.3 Convergence	189
16.1.4 Continuous Functions	190
16.1.5 First Countable Spaces	192
16.1.6 Hausdorff Spaces	192
16.1.7 Bounded Sets	193
16.1.8 Uniform Convergence	193
16.1.9 Standard Bounded Metric	196
16.1.10 Product Spaces	196
16.2 Uniform Metric	197
16.2.1 Products	201
16.2.2 Connected Spaces	202
16.3 Isometric Embeddings	203
16.4 Lebesgue Numbers	203
16.5 Uniform Continuity	204
16.6 Complete Metric Spaces	205
16.7 Manifolds	205
17 Homotopy Theory	207
17.1 Homotopies	207
17.2 Homotopy Equivalence	207
18 Simplicial Complexes	209
18.1 Cell Decompositions	209
18.2 CW-complexes	209
19 Topological Groups	211
19.1 Topological Groups	211
19.1.1 Subgroups	212
19.1.2 Left Cosets	213
19.1.3 Homogeneous Spaces	215
19.2 Symmetric Neighbourhoods	215
19.3 Continuous Actions	217
20 Topological Vector Spaces	219
20.1 Cauchy Sequences	219
20.2 Seminorms	220
20.3 Fréchet Spaces	220
20.4 Normed Spaces	220
20.5 Unit Ball	226
20.6 Unit Sphere	227
20.7 Inner Product Spaces	227
20.8 Banach Spaces	228

20.9 Hilbert Spaces	228
20.10 Locally Convex Spaces	229

VII Probability Theory 231

21 Discrete Random Variables 233

22 Continuous Random Variables 235

VIII Quantum Theory 237

23 The Postulates of Quantum Mechanics 239

23.1 Observables as a Random Variable	241
---	-----

23.2 Compatible Observables	242
---------------------------------------	-----

24 The Wave Function 245

24.1 The Schrödinger Equation	245
---	-----

24.2 Statistical Interpretation	246
---	-----

24.3 Momentum	247
-------------------------	-----

24.4 The Time-Independent Schrödinger Equation	249
--	-----

24.5 The Quantum Harmonic Oscillator	251
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24.6 The Free Particle	255
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Part I

Set Theory

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*.

For any set A , let there be *elements* of A . We write $a \in A$ for: a is an element of A .

For any sets A and B , let there be a set B^A , whose elements are called *functions* from A to B . We write $f : A \rightarrow B$ for $f \in B^A$.

For any function $f : A \rightarrow B$ and element $a \in A$, let there be an element $f(a) \in B$, the *value* of the function f at the *argument* a .

1.2 Injections, Surjections and Bijections

Definition 1.2.1 (Injective). We say a function $f : A \rightarrow B$ is *injective* or an *injection*, and write $f : A \rightarrowtail B$, iff, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

We say a set A is *dominated* by a set B , and write $A \leqslant B$, iff there exists an injection from A to B .

Definition 1.2.2 (Surjective). We say a function $f : A \rightarrow B$ is *surjective* or a *surjection*, and write $f : A \twoheadrightarrow B$, iff, for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Definition 1.2.3 (Bijective). We say a function $f : A \rightarrow B$ is *bijective* or a *bijection*, and write $f : A \approx B$, iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

We say the set A is *strictly* dominated by the set B , and write $A < B$, iff $A \leqslant B$ and $A \not\approx B$.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define “Let C be the set such that $P(C)$ ”.

1.3 Axioms

Axiom Schema 1.3.1 (Choice). *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x \in X$ and $y \in Y$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a \in A$, there exists $b \in B$ such that $P[A, B, a, b]$. Then there exists a function $f : A \rightarrow B$ such that $\forall a \in A. P[A, B, a, f(a)]$.

Axiom 1.3.2 (Extensionality). *Let $f, g : A \rightarrow B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Axiom 1.3.3 (Pairing). *For any sets A and B , there exists a set $A \times B$, the Cartesian product of A and B , and functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ such that, for all $a \in A$ and $b \in B$, there exists a unique $(a, b) \in A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Axiom Schema 1.3.4 (Separation). *For every property $P[X, x]$ where X is a set variable and $x \in X$, the following is an axiom:*

For every set A , there exists a set $S = \{x \in A : P[A, x]\}$ and an injection $i : S \rightarrow A$ such that, for all $x \in A$, we have

$$(\exists y \in S. i(y) = x) \Leftrightarrow P[A, x] .$$

Axiom 1.3.5 (Infinity). *There exists a set \mathbb{N} , an element $0 \in \mathbb{N}$, and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that:*

- $\forall n \in \mathbb{N}. s(n) \neq 0$
- $\forall m, n \in \mathbb{N}. s(m) = s(n) \Rightarrow m = n$.

Axiom Schema 1.3.6 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A , there exist sets B and Y and functions $p : B \rightarrow A$, and $m : B \times Y \Rightarrow \mathbb{N}$ such that:

- m is injective.
- $\forall b \in B. P[A, \{y \in Y : m(b, y) = 0\}, p(b)]$
- For all $a \in A$, if $\exists Y. P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.

Definition 1.3.7 (Composition). Let $f : A \rightarrow B$ and $g : B \rightarrow C$. The composite $g \circ f : A \rightarrow C$ is the function such that, for all $a \in A$, we have

$$(g \circ f)(a) = g(f(a)) .$$

Axiom 1.3.8 (Universe). *There exists a set E , a set U and a function $el : E \rightarrow U$ such that the following holds.*

Let us say that a set A is small iff there exists $u \in U$ such that $A \approx \{e \in E : el(e) = u\}$.

- \mathbb{N} is small.
- For any U -small sets A and B , the set B^A is small.
- For any U -small sets A and B , the set $A \times B$ is small.
- Let $f : A \rightarrow B$ be a function. If B is small and $\{a \in A : f(a) = b\}$ is small for all $b \in B$, then A is small.
- If $p : B \twoheadrightarrow A$ is a surjective function such that A is small, then there exists a U -small set C , a surjection $q : C \twoheadrightarrow A$, and a function $f : C \rightarrow B$ such that $q = p \circ f$.

Chapter 2

Sets and Functions

2.1 Composition

Proposition 2.1.1. *Given functions $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f .$$

PROOF:

$\langle 1 \rangle 1$. For all $x \in A$ we have $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$.

$\langle 2 \rangle 1$. LET: $x \in A$

$\langle 2 \rangle 2$. $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$

PROOF:

$$\begin{aligned} (h \circ (g \circ f))(x) &= h((g \circ f)(x)) && \text{(Definition of composition)} \\ &= h(g(f(x))) && \text{(Definition of composition)} \\ &= (h \circ g)(f(x)) && \text{(Definition of composition)} \\ &= ((h \circ g) \circ f)(x) && \text{(Definition of composition)} \end{aligned}$$

$\langle 1 \rangle 2$. Q.E.D.

PROOF: By the Axiom of Extensionality.

□

2.2 Injections

Proposition 2.2.1. *The composite of injective functions is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: A , B and C be sets.

$\langle 1 \rangle 2$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 3$. LET: $g : B \rightarrow C$

$\langle 1 \rangle 4$. ASSUME: g is injective.

$\langle 1 \rangle 5$. ASSUME: f is injective.

$\langle 1 \rangle 6$. LET: $x, y \in A$

$\langle 1 \rangle 7$. ASSUME: $(g \circ f)(x) = (g \circ f)(y)$

PROVE: $x = y$

$\langle 1 \rangle 8$. $g(f(x)) = g(f(y))$

PROOF:

$$g(f(x)) = (g \circ f)(x) \quad (\text{definition of composition})$$

$$= (g \circ f)(y) \quad (\langle 1 \rangle 7)$$

$$= g(f(y)) \quad (\text{definition of composition})$$

$\langle 1 \rangle 9$. $f(x) = f(y)$

PROOF: $\langle 1 \rangle 4, \langle 1 \rangle 8$

$\langle 1 \rangle 10$. $x = y$

PROOF: $\langle 1 \rangle 5, \langle 1 \rangle 9$

□

Proposition 2.2.2. *For functions $f : A \rightarrow B$ and $g : B \rightarrow C$, if $g \circ f$ is injective then f is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: A, B and C be sets.

$\langle 1 \rangle 2$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 3$. LET: $g : B \rightarrow C$

$\langle 1 \rangle 4$. ASSUME: $g \circ f$ is injective.

$\langle 1 \rangle 5$. LET: $x, y \in A$

$\langle 1 \rangle 6$. ASSUME: $f(x) = f(y)$

$\langle 1 \rangle 7$. $(g \circ f)(x) = (g \circ f)(y)$

PROOF:

$$(g \circ f)(x) = g(f(x)) \quad (\text{definition of composition})$$

$$= g(f(y)) \quad (\langle 1 \rangle 6)$$

$$= (g \circ f)(y) \quad (\text{definition of composition})$$

$\langle 1 \rangle 8$. $x = y$

PROOF: $\langle 1 \rangle 4, \langle 1 \rangle 7$

□

Proposition 2.2.3. *Let $f : A \rightarrow B$ be injective. For every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: f is injective.

$\langle 1 \rangle 2$. LET: X be a set.

$\langle 1 \rangle 3$. LET: $x, y : X \rightarrow A$

$\langle 1 \rangle 4$. ASSUME: $f \circ x = f \circ y$

$\langle 1 \rangle 5$. $\forall t \in X. x(t) = y(t)$

$\langle 2 \rangle 1$. LET: $t \in X$

$\langle 2 \rangle 2$. $f(x(t)) = f(y(t))$

PROOF:

$$\begin{aligned} f(x(t)) &= (f \circ x)(t) && \text{(definition of composition)} \\ &= (f \circ y)(t) && (\langle 1 \rangle 4) \\ &= f(y(t)) && \text{(definition of composition)} \end{aligned}$$

$\langle 2 \rangle 3$. $x(t) = y(t)$

PROOF: $\langle 1 \rangle 1$, $\langle 2 \rangle 2$

$\langle 1 \rangle 6$. $x = y$

PROOF: Axiom of Extensionality, $\langle 1 \rangle 5$

□

We will prove the converse as Proposition 2.8.4.

2.3 Surjections

Proposition 2.3.1. *The composite of surjective functions is surjective.*

PROOF:

$\langle 1 \rangle 1$. LET: A , B and C be sets.

$\langle 1 \rangle 2$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$

$\langle 1 \rangle 3$. ASSUME: g is surjective.

$\langle 1 \rangle 4$. ASSUME: f is surjective.

$\langle 1 \rangle 5$. LET: $c \in C$

$\langle 1 \rangle 6$. PICK $b \in B$ such that $g(b) = c$.

PROOF: $\langle 1 \rangle 3$

$\langle 1 \rangle 7$. PICK $a \in A$ such that $f(a) = b$.

PROOF: $\langle 1 \rangle 4$

$\langle 1 \rangle 8$. $(g \circ f)(a) = c$

PROOF:

$$\begin{aligned} (g \circ f)(a) &= g(f(a)) && \text{(definition of composition)} \\ &= g(b) && (\langle 1 \rangle 7) \\ &= c && (\langle 1 \rangle 6) \end{aligned}$$

□

Proposition 2.3.2. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is surjective then g is surjective.*

PROOF:

$\langle 1 \rangle 1$. LET: A , B and C be sets.

$\langle 1 \rangle 2$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$.

$\langle 1 \rangle 3$. ASSUME: $g \circ f$ is surjective.

$\langle 1 \rangle 4$. LET: $c \in C$

$\langle 1 \rangle 5$. PICK $a \in A$ such that $(g \circ f)(a) = c$

PROOF: $\langle 1 \rangle 3$

$\langle 1 \rangle 6$. $g(f(a)) = c$

PROOF: From $\langle 1 \rangle 5$ and the definition of composition.

$\langle 1 \rangle 7$. Q.E.D.

PROOF: There exists $b \in B$ such that $g(b) = c$, namely $b = f(a)$.
 \square

Proposition 2.3.3. *Let $f : A \rightarrow B$ be a surjection. For any set X and functions $x, y : B \rightarrow X$, if $x \circ f = y \circ f$ then $x = y$.*

PROOF:

- $\langle 1 \rangle 1$. LET: $b \in B$
- $\langle 1 \rangle 2$. PICK $a \in A$ such that $f(a) = b$
- $\langle 1 \rangle 3$. $x(f(a)) = y(f(a))$
- $\langle 1 \rangle 4$. $x(b) = y(b)$
- $\langle 1 \rangle 5$. Q.E.D.

PROOF: Axiom of Extensionality.
 \square

We will prove the converse as Proposition 2.9.2.

2.4 Bijections

Proposition 2.4.1. *The composite of bijections is a bijection.*

PROOF:

- $\langle 1 \rangle 1$. LET: A, B and C be sets.
- $\langle 1 \rangle 2$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$
- $\langle 1 \rangle 3$. ASSUME: g is bijective.
- $\langle 1 \rangle 4$. ASSUME: f is bijective.
- $\langle 1 \rangle 5$. g is injective.
 PROOF: From $\langle 1 \rangle 3$.
- $\langle 1 \rangle 6$. g is surjective.
 PROOF: From $\langle 1 \rangle 3$.
- $\langle 1 \rangle 7$. f is injective.
 PROOF: From $\langle 1 \rangle 4$.
- $\langle 1 \rangle 8$. f is surjective.
 PROOF: From $\langle 1 \rangle 4$.
- $\langle 1 \rangle 9$. $g \circ f$ is injective.
 PROOF: Proposition 2.2.1, $\langle 1 \rangle 5$, $\langle 1 \rangle 7$.
- $\langle 1 \rangle 10$. $g \circ f$ is surjective.
 PROOF: Proposition 2.3.1, $\langle 1 \rangle 6$, $\langle 1 \rangle 8$.
- $\langle 1 \rangle 11$. $g \circ f$ is bijective.
 PROOF: $\langle 1 \rangle 9$, $\langle 1 \rangle 10$

\square

Proposition 2.4.2.

$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps f to $(\pi_1 \circ f, \pi_2 \circ f)$ is a bijection. \square

Proposition 2.4.3.

$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function Φ such that $\Phi(f)(c)(b) = f(b, c)$ is a bijection. \square

2.5 Domination

Definition 2.5.1 (Dominate). Let A and B be sets. We say that B *dominates* A , and write $A \leqslant B$, iff there exists an injective function $A \rightarrow B$.

Theorem 2.5.2 (Schroeder-Bernstein). *Let A and B be sets. If $A \leqslant B$ and $B \leqslant A$ then $A \approx B$.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ and $g : B \rightarrow A$ be injections.

$\langle 1 \rangle 2$. Define the subsets A_n of A by

$$\begin{aligned} A_0 &:= A - g(B) \\ A_{n+1} &:= g(f(A_n)) \end{aligned}$$

$\langle 1 \rangle 3$. Define $h : A \rightarrow B$ by

$$h(x) = \begin{cases} f(x) & \text{if } \exists n. x \in A_n \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

$\langle 1 \rangle 4$. h is injective.

$\langle 2 \rangle 1$. LET: $x, y \in A$

$\langle 2 \rangle 2$. ASSUME: $h(x) = h(y)$

$\langle 2 \rangle 3$. CASE: $x \in A_m$ and $y \in A_n$.

PROOF: Then $f(x) = f(y)$ so $x = y$ since f is injective.

$\langle 2 \rangle 4$. CASE: $x \in A_m$ and there is no y such that $y \in A_n$.

$\langle 3 \rangle 1$. $f(x) = g^{-1}(y)$

$\langle 3 \rangle 2$. $y = g(f(x))$

$\langle 3 \rangle 3$. $y \in A_{m+1}$

$\langle 3 \rangle 4$. Q.E.D.

PROOF: This is a contradiction.

$\langle 2 \rangle 5$. CASE: $y \in A_n$ and there is no m such that $x \in A_m$.

PROOF: Similar.

$\langle 2 \rangle 6$. CASE: There is no m such that $x \in A_m$ and there is no n such that $y \in A_n$.

PROOF: Then $g^{-1}(x) = g^{-1}(y)$ and so $x = y$.

$\langle 1 \rangle 5$. h is surjective.

$\langle 2 \rangle 1$. LET: $y \in B$

$\langle 2 \rangle 2$. CASE: $g(y) \in A_n$

$\langle 3 \rangle 1$. $n \neq 0$

$\langle 3 \rangle 2$. PICK $x \in A_{n-1}$ such that $g(y) = g(f(x))$

$\langle 3 \rangle 3$. $y = f(x)$

$\langle 3 \rangle 4$. $y = h(x)$

$\langle 2 \rangle 3$. CASE: There is no n such that $g(y) \in A_n$.

PROOF: Then $h(g(y)) = y$.

\square

2.6 Identity Function

Definition 2.6.1 (Identity). For any set A , the *identity* function $\text{id}_A : A \rightarrow A$ is the function defined by $\text{id}_A(a) = a$.

2.6.1 Injections, Surjections, Bijections

Proposition 2.6.2. For any set A , the identity function id_A is a bijection.

PROOF:

$\langle 1 \rangle 1$. LET: A be a set.

$\langle 1 \rangle 2$. id_A is injective.

PROOF: If $\text{id}_A(x) = \text{id}_A(y)$ then $x = y$.

$\langle 1 \rangle 3$. id_A is surjective.

PROOF: For any $y \in A$, there exists $x \in A$ such that $\text{id}_A(x) = y$, namely $x = y$.

□

2.6.2 Composition

Proposition 2.6.3. Let $f : A \rightarrow B$. Then $\text{id}_B \circ f = f = f \circ \text{id}_A$.

PROOF: Each is the function that maps a to $f(a)$. □

Proposition 2.6.4. Let $f : A \rightarrow B$.

1. If there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$ then f is injective.

2. If f is injective and A is nonempty, then there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$.

PROOF:

$\langle 1 \rangle 1$. If there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$ then f is injective.

PROOF: If $f(x) = f(y)$ then $x = g(f(x)) = g(f(y)) = y$.

$\langle 1 \rangle 2$. If f is injective and A is nonempty, then there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$.

$\langle 2 \rangle 1$. ASSUME: f is injective and A is nonempty.

$\langle 2 \rangle 2$. PICK $a \in A$

$\langle 2 \rangle 3$. Choose a function $g : B \rightarrow A$ such that $f(g(x)) = x$ if there exists $y \in A$ such that $f(y) = x$, otherwise $g(x) = a$.

$\langle 2 \rangle 4$. LET: $x \in A$

PROVE: $g(f(x)) = x$

$\langle 2 \rangle 5$. $f(g(f(x))) = f(x)$

$\langle 2 \rangle 6$. $g(f(x)) = x$

□

Proposition 2.6.5. Let $f : A \rightarrow B$. Then f is surjective if and only if there exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$.

PROOF:

⟨1⟩1. If f is surjective then there exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$.

⟨2⟩1. ASSUME: f is surjective.

⟨2⟩2. PICK $g : B \rightarrow A$ such that, for all $b \in B$, we have $f(g(b)) = b$.

PROOF: Axiom of Choice.

⟨2⟩3. $f \circ g = \text{id}_B$.

⟨1⟩2. If there exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$ then f is surjective.

⟨2⟩1. LET: $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$

⟨2⟩2. LET: X be a set.

⟨2⟩3. LET: $h, k : B \rightarrow X$

⟨2⟩4. ASSUME: $h \circ f = k \circ f$

⟨2⟩5. $h = k$

PROOF: $h = h \circ f \circ g = k \circ f \circ g = k$

□

Corollary 2.6.5.1. *Let A and B be sets.*

1. *If there exists a surjective function $A \rightarrow B$ then there exists an injective function $B \rightarrow A$.*
2. *If there exists an injective function $A \rightarrow B$ and A is nonempty then there exists a surjective function $B \rightarrow A$.*

Proposition 2.6.6. *Let $f : A \rightarrow B$. Then f is bijective if and only if there exists a function $f^{-1} : B \rightarrow A$, the inverse of f , such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$, in which case the inverse is unique.*

PROOF:

⟨1⟩1. If f is bijective then there exists $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$.

⟨2⟩1. ASSUME: f is bijective.

⟨2⟩2. PICK $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$

PROOF: Proposition 2.9.2.

⟨2⟩3. $f \circ g \circ f = f$

⟨2⟩4. $g \circ f = \text{id}_A$

PROOF: Proposition 2.2.3.

⟨1⟩2. If there exists $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$, then f is bijective.

⟨2⟩1. LET: $f^{-1} : B \rightarrow A$ satisfy $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$

⟨2⟩2. f is injective.

PROOF: If $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$.

⟨2⟩3. f is surjective.

PROOF: Proposition 2.9.2.

⟨1⟩3. If $g, h : B \rightarrow A$ satisfy $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$ and $f \circ h = \text{id}_B$ and $h \circ f = \text{id}_A$ then $g = h$.

PROOF: We have $g = g \circ f \circ h = h$.

□

2.7 The Empty Set

Theorem 2.7.1. *There exists a set which has no elements.*

PROOF: Take $\{x \in \mathbb{N} : \perp\}$. \square

Theorem 2.7.2. *If E and E' have no elements then $E \approx E'$.*

PROOF:

$\langle 1 \rangle 1$. LET: E and E' have no elements.

$\langle 1 \rangle 2$. PICK a function $F : E \rightarrow E'$.

PROOF: Axiom of Choice since vacuously $\forall x \in E. \exists y \in E'. \top$.

$\langle 1 \rangle 3$. F is injective.

PROOF: Vacuously, for all $x, y \in E$, if $F(x) = F(y)$ then $x = y$.

$\langle 1 \rangle 4$. F is surjective.

PROOF: Vacuously, for all $y \in E'$, there exists $x \in E$ such that $F(x) = y$.

\square

Definition 2.7.3 (Empty Set). The *empty set* \emptyset is the set with no elements.

2.8 The Singleton

Theorem 2.8.1. *There exists a set that has exactly one element.*

PROOF: The set $\{x \in \mathbb{N} : x = 0\}$ has exactly one element. \square

Theorem 2.8.2. *If A and B both have exactly one element then $A \approx B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B both have exactly one element a and b respectively.

$\langle 1 \rangle 2$. LET: $F : A \rightarrow B$ be the function such that, for all $x \in A$, we have
 $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$. F is a bijection.

\square

Definition 2.8.3 (Singleton). Let 1 be the set that has exactly one element.
 Let $*$ be its element.

2.8.1 Injections

Proposition 2.8.4. *Let $f : A \rightarrow B$. Assume that, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$. Then f is injective.*

PROOF: Take $X = 1$. \square

2.9 The Set Two

Definition 2.9.1 (The Set Two). Let $2 = \{x \in \mathbb{N} : x = 0 \vee x = 1\}$.

Proposition 2.9.2. *Let $f : A \rightarrow B$. Assume that, for any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$. Then f is surjective.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: For any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$.

$\langle 1 \rangle 2$. LET: $b \in B$

$\langle 1 \rangle 3$. LET: $h : B \rightarrow 2$ be the function that maps everything to 1.

$\langle 1 \rangle 4$. LET: $k : B \rightarrow 2$ be the function that maps b to 0 and everything else to 1.

$\langle 1 \rangle 5$. $h \neq k$

$\langle 1 \rangle 6$. $h \circ f \neq k \circ f$

$\langle 1 \rangle 7$. PICK $a \in A$ such that $h(f(a)) \neq k(f(a))$

$\langle 1 \rangle 8$. $f(a) = b$

□

2.10 Subsets

Definition 2.10.1 (Subset). A *subset* of a set A consists of a set S and an injection $i : S \rightarrow A$. We write $(S, i) \subseteq A$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Proposition 2.10.2. *For any subset (S, i) of A we have $(S, i) = (S, i)$.*

PROOF: We have $\text{id}_S : S \approx S$ and $i \circ \text{id}_S = i$.

Proposition 2.10.3. *If $(S, i) = (T, j)$ then $(T, j) = (S, i)$.*

PROOF: If $\phi : S \approx T$ and $j \circ \phi = i$ then $\phi^{-1} : T \approx S$ and $i \circ \phi^{-1} = j$. □

Proposition 2.10.4. *If $(R, i) = (S, j)$ and $(S, j) = (T, k)$ then $(R, i) = (T, k)$.*

PROOF: If $\phi : R \approx S$ and $j \circ \phi = i$, and $\psi : S \approx T$ and $k \circ \psi = j$, then $\psi \circ \phi : R \approx T$ and $k \circ \psi \circ \phi = i$. □

Definition 2.10.5 (Membership). Given $(S, i) \subseteq A$ and $a \in A$, we write $a \in (S, i)$ for $\exists s \in S. i(s) = a$.

Proposition 2.10.6. *If $a \in (S, i)$ and $(S, i) = (T, j)$ then $a \in (T, j)$.*

PROOF: If $i(s) = a$ then $j(\phi(s)) = a$. □

Definition 2.10.7 (Union). Given subsets S and T of A , the *union* is the subset $\{x \in A : x \in S \vee x \in T\}$.

Definition 2.10.8 (Intersection). Given subsets S and T of A , the *intersection* is the subset $\{x \in A : x \in S \wedge x \in T\}$.

Proposition 2.10.9 (Distributive Law).

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

Proposition 2.10.10 (Distributive Law).

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

Definition 2.10.11. Given a set A , we write \emptyset for the subset $(\emptyset, !)$ where $!$ is the unique function $\emptyset \rightarrow A$.

Proposition 2.10.12.

$$S \cup \emptyset = S$$

Proposition 2.10.13.

$$S \cap \emptyset = \emptyset$$

Definition 2.10.14 (Inclusion). Given subsets (S, i) and (T, j) of a set A , we write $(S, i) \subseteq (T, j)$ iff there exists $f : S \rightarrow T$ such that $j \circ f = i$.

Proposition 2.10.15.

$$\emptyset \subseteq S$$

Definition 2.10.16 (Disjoint). Subsets S and T of A are *disjoint* iff $S \cap T = \emptyset$.

Definition 2.10.17 (Difference). Given subsets S and T of A , the *difference* of S and T is $S - T = \{x \in A : x \in S \wedge x \notin T\}$.

Proposition 2.10.18 (De Morgan's Law).

$$R - (S \cup T) = (R - S) \cap (R - T)$$

Proposition 2.10.19 (De Morgan's Law).

$$R - (S \cap T) = (R - S) \cup (R - T)$$

2.11 Power Set

Definition 2.11.1 (Power Set). The *power set* of a set A is

$$\mathcal{P}A := 2^A$$

Definition 2.11.2 (Cover). Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then \mathcal{A} is a *cover* of X , or *covers* X , iff $\bigcup \mathcal{A} = X$.

Given a subset Y of X and $\mathcal{A} \subseteq \mathcal{P}X$, we say \mathcal{A} *covers* Y iff $Y \subseteq \bigcup \mathcal{A}$.

2.12 Saturated Set

Definition 2.12.1 (Saturated). Let A and B be sets. Let $f : A \rightarrow B$ be surjective. Let $C \subseteq A$. Then C is *saturated* with respect to f iff, for all $x \in C$ and $y \in A$, if $f(x) = f(y)$ then $y \in C$.

2.13 Union

Definition 2.13.1 (Union). Given $\mathcal{A} \in \mathcal{PPX}$, its *union* is

$$\bigcup \mathcal{A} := \{x \in X : \exists S \in \mathcal{A}. x \in S\} \in \mathcal{PX} .$$

2.13.1 Intersection

Definition 2.13.2 (Intersection). Given $\mathcal{A} \in \mathcal{PPX}$, its *intersection* is

$$\bigcap \mathcal{A} := \{x \in X : \forall S \in \mathcal{A}. x \in S\} \in \mathcal{PX} .$$

2.13.2 Direct Image

Definition 2.13.3 (Direct Image). Let $f : A \rightarrow B$. Let S be a subset of A . The *(direct) image* of S under f is the subset of B given by

$$f(S) := \{f(a) : a \in S\} .$$

Proposition 2.13.4.

1. If $S \subseteq T$ then $f(S) \subseteq f(T)$
2. $f(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f(S)$

Example 2.13.5. It is not true in general that $f(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f(S)$. Take f to be the only function $\{0, 1\} \rightarrow \{0\}$, and $\mathcal{S} = \{\{0\}, \{1\}\}$. Then $f(\bigcap \mathcal{S}) = \emptyset$ but $\bigcap_{S \in \mathcal{S}} f(S) = \{0\}$.

Example 2.13.6. It is not true in general that $f(S - T) = f(S) - f(T)$. Take f to be the only function $\{0, 1\} \rightarrow \{0\}$, $S = \{0\}$ and $T = \{1\}$. Then $f(S - T) = \{0\}$ but $f(S) - f(T) = \emptyset$.

2.14 Inverse Image

Definition 2.14.1 (Inverse Image). Let $f : A \rightarrow B$. Let S be a subset of B . The *inverse image* or *preimage* of S under f is the subset of A given by

$$f^{-1}(S) := \{x \in A : f(x) \in S\} .$$

Proposition 2.14.2. 1. If $S \subseteq T$ then $f^{-1}(S) \subseteq f^{-1}(T)$

2. $f^{-1}(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f^{-1}(S)$
3. $f^{-1}(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f^{-1}(S)$
4. $f^{-1}(S - T) = f^{-1}(S) - f^{-1}(T)$
5. $S \subseteq f^{-1}(f(S))$. Equality holds if f is injective.
6. $f(f^{-1}(T)) \subseteq T$. Equality holds if f is surjective.
7. $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$

2.14.1 Saturated Sets

Proposition 2.14.3. *Let A and B be sets. Let $f : A \rightarrow B$ be surjective. Let $C \subseteq A$. Then C is saturated if and only if there exists $D \subseteq B$ such that $C = f^{-1}(D)$.*

PROOF:

$\langle 1 \rangle 1$. If C is saturated then there exists $D \subseteq B$ such that $C = f^{-1}(D)$.

$\langle 2 \rangle 1$. ASSUME: C is saturated.

$\langle 2 \rangle 2$. LET: $D = f(C)$

$\langle 2 \rangle 3$. $C \subseteq f^{-1}(D)$

$\langle 3 \rangle 1$. LET: $x \in C$

$\langle 3 \rangle 2$. $f(x) \in D$

PROOF: $\langle 2 \rangle 2$

$\langle 3 \rangle 3$. $x \in f^{-1}(D)$

$\langle 2 \rangle 4$. $f^{-1}(D) \subseteq C$

$\langle 3 \rangle 1$. LET: $x \in f^{-1}(D)$

$\langle 3 \rangle 2$. $f(x) \in D$

$\langle 3 \rangle 3$. PICK $y \in C$ such that $f(x) = f(y)$

PROOF: $\langle 2 \rangle 2$

$\langle 3 \rangle 4$. $x \in C$

PROOF: $\langle 2 \rangle 1$

$\langle 1 \rangle 2$. If there exists $D \subseteq B$ such that $C = f^{-1}(D)$ then C is saturated.

$\langle 2 \rangle 1$. LET: $D \subseteq B$ be such that $C = f^{-1}(D)$.

$\langle 2 \rangle 2$. LET: $x \in C$ and $y \in A$

$\langle 2 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 4$. $f(x) \in D$

$\langle 2 \rangle 5$. $f(y) \in D$

$\langle 2 \rangle 6$. $y \in C$

□

2.15 Relations

Definition 2.15.1 (Relation). Let A and B be sets. A *relation* R between A and B , $R : A \rightarrow B$, is a subset of $A \times B$.

Given $a \in A$ and $b \in B$, we write aRb for $(a, b) \in R$.

A relation *on* a set A is a relation between A and A .

Definition 2.15.2 (Reflexive). A relation R on a set A is *reflexive* iff $\forall a \in A. aRa$.

Definition 2.15.3 (Symmetric). A relation R on a set A is *symmetric* iff, whenever xRy , then yRx .

Definition 2.15.4 (Transitive). A relation R on a set A is *transitive* iff, whenever xRy and yRz , then xRz .

2.15.1 Equivalence Relations

Definition 2.15.5 (Equivalence Relation). A relation R on a set A is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Definition 2.15.6 (Equivalence Class). Let R be an equivalence relation on a set A and $a \in A$. The *equivalence class* of a with respect to R is

$$\{x \in A : xRa\} .$$

Proposition 2.15.7. *Two equivalence classes are either disjoint or equal.*

2.16 Power Set

Definition 2.16.1 (Power Set). The *power set* of a set A is $\mathcal{P}A := 2^A$.

Given $S \in \mathcal{P}A$ and $a \in A$, we write $a \in S$ for $S(a) = 1$.

Definition 2.16.2 (Pairwise Disjoint). Let $P \subseteq \mathcal{P}A$. We say the members of P are *pairwise disjoint* iff, for all $S, T \in P$, if $S \neq T$ then $S \cap T = \emptyset$.

2.16.1 Partitions

Definition 2.16.3 (Partition). Let A be a set. A *partition* of A is a set $P \in \mathcal{P}\mathcal{P}A$ such that:

- $\bigcup P = A$
- Every member of P is nonempty.
- The members of P are pairwise disjoint.

2.17 Cartesian Product

Definition 2.17.1 (Cartesian Product). Let A and B be sets. The *Cartesian product* of A and B , $A \times B$, is the tabulation of the relation $A \looparrowright B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

2.18 Quotient Sets

Proposition 2.18.1. *Let \sim be an equivalence relation on X . Then there exists a set X/\sim , the quotient set of X with respect to \sim , and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the canonical projection, such that, for all $x, y \in X$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to \sim , then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

2.19 Partitions

Definition 2.19.1 (Partition). A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X .

2.20 Disjoint Union

Theorem 2.20.1. For any sets A and B , there exists a set $A + B$, the disjoint union of A and B , and functions $\kappa_1 : A \rightarrow A + B$ and $\kappa_2 : B \rightarrow A + B$, the injections, such that, for every set X and functions $f : A \rightarrow X$ and $g : B \rightarrow X$, there exists a unique function $[f, g] : A + B \rightarrow X$ such that $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$.

PROOF:

$\langle 1 \rangle 1$. LET: $A + B := \{p \in \mathcal{P}A \times \mathcal{P}B : \exists a \in A. p = (\{a\}, \emptyset) \vee \exists b \in B. p = (\emptyset, \{b\})\}$

Definition 2.20.2 (Restriction). Let $f : A \rightarrow B$ and let (S, i) be a subset of A . The *restriction* of f to S is the function $f \upharpoonright S : S \rightarrow B$ defined by $f \upharpoonright S = f \circ i$.

2.21 Natural Numbers

Theorem 2.21.1 (Principle of Recursive Definition). Let A be a set. Let F be the set of all functions $\{m \in \mathbb{N} : m < n\} \rightarrow A$ for some n . Let $\rho : F \rightarrow A$. Then there exists a unique $g : \mathbb{N} \rightarrow A$ such that, for all $n \in \mathbb{N}$, we have

$$g(n) = \rho(g \upharpoonright \{m \in \mathbb{N} : m < n\}) .$$

PROOF:

$\langle 1 \rangle 1$. Given a subset $B \subseteq \mathbb{N}$, let us say that a function $g : B \rightarrow A$ is *acceptable* iff, for all $n \in B$, we have

$$\forall m < n. m \in B$$

and

$$g(n) = \rho(g \upharpoonright \{m \in \mathbb{N} : m < n\}) .$$

$\langle 1 \rangle 2$. For all $n \in \mathbb{N}$, there exists an acceptable function $\{m \in \mathbb{N} : m < n\} \rightarrow A$.

$\langle 2 \rangle 1$. LET: $P[n]$ be the property: There exists an acceptable function $\{m \in \mathbb{N} : m < n\} \rightarrow A$.

$\langle 2 \rangle 2$. $P[0]$

PROOF: The unique function $\emptyset \rightarrow A$ is acceptable.

$\langle 2 \rangle 3$. For any natural number n , if $P[n]$ then $P[n + 1]$.

$\langle 3 \rangle 1$. ASSUME: $P[n]$

$\langle 3 \rangle 2$. PICK an acceptable $f : \{m \in \mathbb{N} : m < n\} \rightarrow A$.

$\langle 3 \rangle 3$. LET: $g : \{m \in \mathbb{N} : m < n + 1\} \rightarrow A$ be the function

$$g(m) = \begin{cases} f(m) & \text{if } m < n \\ \rho(f) & \text{if } m = n \end{cases}$$

$\langle 3 \rangle 4$. g is acceptable.

- ⟨1⟩3. If $g : B \rightarrow A$ and $h : C \rightarrow A$ are acceptable, then g and h agree on $B \cap C$.
 ⟨1⟩4. Define $g : \mathbb{N} \rightarrow A$ by: $g(n) = a$ iff there exists an acceptable $h : \{m \in \mathbb{N} : m < n + 1\}$ such that $h(n) = a$.
 ⟨1⟩5. g is acceptable.
 ⟨1⟩6. If $g' : \mathbb{N} \rightarrow A$ is acceptable then $g' = g$.
 □

2.22 Finite and Infinite Sets

Definition 2.22.1 (Finite). A set A is *finite* iff there exists $n \in \mathbb{N}$ such that $A \approx \{m \in \mathbb{N} : m < n\}$. In this case, we say A has *cardinality* n .

Proposition 2.22.2. Let $n \in \mathbb{N}$. Let A be a set. Let $a_0 \in A$. Then $A \approx \{m \in \mathbb{N} : m < n + 1\}$ if and only if $A - \{a_0\} \approx \{m \in \mathbb{N} : m < n\}$.

Theorem 2.22.3. Let A be a set. Suppose that $A \approx \{m \in \mathbb{N} : m < n\}$. Let B be a proper subset of A . Then $B \not\approx \{m \in \mathbb{N} : m < n\}$ but there exists $m < n$ such that $B \approx \{k \in \mathbb{N} : k < m\}$.

PROOF:

- ⟨1⟩1. LET: $P[n]$ be the property: for every set A , if $A \approx \{m \in \mathbb{N} : m < n\}$, then for every proper subset B of A , we have $B \not\approx \{m \in \mathbb{N} : m < n\}$ but there exists $m < n$ such that $B \approx \{k \in \mathbb{N} : k < m\}$.
 ⟨1⟩2. $P[0]$
 PROOF: If $A \approx \{m \in \mathbb{N} : m < 0\}$ then A is empty and so has no proper subset.
 ⟨1⟩3. For every natural number n , if $P[n]$ then $P[n + 1]$.
 ⟨2⟩1. LET: n be a natural number.
 ⟨2⟩2. ASSUME: $P[n]$
 ⟨2⟩3. LET: A be a set.
 ⟨2⟩4. ASSUME: $A \approx \{m \in \mathbb{N} : m < n + 1\}$
 ⟨2⟩5. LET: B be a proper subset of A .
 ⟨2⟩6. CASE: $B = \emptyset$
 PROOF: Then $B \not\approx \{m \in \mathbb{N} : m < n + 1\}$ but $B \approx \{k \in \mathbb{N} : k < 0\}$.
 ⟨2⟩7. CASE: $B \neq \emptyset$
 ⟨3⟩1. PICK $b_0 \in B$
 ⟨3⟩2. $A - \{b_0\} \approx \{m \in \mathbb{N} : m < n\}$
 ⟨3⟩3. $B - \{b_0\}$ is a proper subset of $A - \{b_0\}$
 ⟨3⟩4. $B - \{b_0\} \not\approx \{m \in \mathbb{N} : m < n\}$
 ⟨3⟩5. $B \approx \{m \in \mathbb{N} : m < n + 1\}$
 ⟨3⟩6. PICK $m < n$ such that $B - \{b_0\} \approx \{k \in \mathbb{N} : k < m\}$
 ⟨3⟩7. $m + 1 < n + 1$
 ⟨3⟩8. $B \approx \{k \in \mathbb{N} : k < m + 1\}$
 □

Corollary 2.22.3.1. If A is finite then there is no bijection between A and a proper subset of A .

Corollary 2.22.3.2. \mathbb{N} is infinite.

Corollary 2.22.3.3. The cardinality of a finite set is unique.

Corollary 2.22.3.4. A subset of a finite set is finite.

Corollary 2.22.3.5. If A is finite and B is a proper subset of A then $|B| < |A|$.

Corollary 2.22.3.6. Let A be a set. Then the following are equivalent:

1. A is finite.
2. There exists a surjection from an initial segment of \mathbb{N} onto A .
3. There exists an injection from A to an initial segment of \mathbb{N} .

Corollary 2.22.3.7. A finite union of finite sets is finite.

Corollary 2.22.3.8. A finite Cartesian product of finite sets is finite.

Theorem 2.22.4. Let A be a set. The following are equivalent:

1. There exists an injective function $\mathbb{N} \rightarrow A$.
2. There exists a bijection between A and a proper subset of A .
3. A is infinite.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ LET: $f : \mathbb{N} \rightarrow A$ be injective.

$\langle 2 \rangle 2.$ LET: $s : \mathbb{N} \approx \mathbb{N} - \{0\}$ be the function $s(n) = n + 1$.

$\langle 2 \rangle 3.$ $f \circ s \circ f^{-1} : A \approx A - \{f(0)\}$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: Corollary 2.22.3.1.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: Choose a function $f : \mathbb{N} \rightarrow A$ such that $f(n) \in A - \{f(m) : m < n\}$ for all n .

□

2.23 Countable Sets

Definition 2.23.1 (Countable). A set A is *countably infinite* iff $A \approx \mathbb{N}$.

Proposition 2.23.2. $\mathbb{N} \times \mathbb{N}$ is countably infinite.

PROOF: Define $f : \mathbb{N} \times \mathbb{N} \approx \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x\}$ by

$$f(x, y) = (x + y, y)$$

Define $g : \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x\} \approx \mathbb{N}$ by

$$g(x, y) = x(x - 1)/2 + y. \quad \square$$

Proposition 2.23.3. *Every infinite subset of \mathbb{N} is countably infinite.*

PROOF:

$\langle 1 \rangle 1$. LET: C be an infinite subset of \mathbb{N}

$\langle 1 \rangle 2$. Define $h : \mathbb{Z} \rightarrow C$ by recursion thus: $h(n)$ is the smallest element of $C - \{h(m) : m < n\}$.

$\langle 1 \rangle 3$. h is injective.

PROOF: If $m < n$ then $h(m) \neq h(n)$ because $h(n) \in C - \{h(m) : m < n\}$.

$\langle 1 \rangle 4$. h is surjective.

$\langle 2 \rangle 1$. For all $n \in \mathbb{N}$ we have $n \leq h(n)$.

$\langle 2 \rangle 2$. LET: $c \in C$

$\langle 2 \rangle 3$. $c \leq h(c)$

$\langle 2 \rangle 4$. LET: n be least such that $c \leq h(n)$

$\langle 2 \rangle 5$. $c \in C - \{h(m) : m < n\}$

$\langle 2 \rangle 6$. $h(n) \leq c$

$\langle 2 \rangle 7$. $h(n) = c$

□

Definition 2.23.4 (Countable). A set is *countable* iff it is either finite or countably infinite; otherwise it is *uncountable*.

Proposition 2.23.5. *Let B be a nonempty set. Then the following are equivalent.*

1. B is countable.
2. There exists a surjection $\mathbb{N} \twoheadrightarrow B$.
3. There exists an injection $B \hookrightarrow \mathbb{N}$.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: B is countable.

$\langle 2 \rangle 2$. CASE: B is finite.

$\langle 3 \rangle 1$. PICK a natural number n and bijection $f : \{m \in \mathbb{N} : m < n\} \approx B$

$\langle 3 \rangle 2$. PICK $b \in B$

$\langle 3 \rangle 3$. Extend f to a surjection $g : \mathbb{N} \twoheadrightarrow B$ by setting $g(m) = b$ for $m \geq n$.

$\langle 2 \rangle 3$. CASE: B is countably infinite.

PROOF: Then there exists a bijection $\mathbb{N} \approx B$.

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

PROOF: Given a surjection $f : \mathbb{N} \twoheadrightarrow B$, define $g : B \hookrightarrow \mathbb{N}$ by $g(b)$ is the smallest number such that $f(g(b)) = b$.

$\langle 1 \rangle 3$. $3 \Rightarrow 1$

$\langle 2 \rangle 1$. LET: $f : B \hookrightarrow \mathbb{N}$ be injective.

$\langle 2 \rangle 2$. $f(B)$ is countable.

$\langle 2 \rangle 3$. $B \approx f(B)$

$\langle 2 \rangle 4$. B is countable.

□

Corollary 2.23.5.1. *A subset of a countable set is countable.*

Corollary 2.23.5.2. $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

PROOF: The function that maps (m, n) to $2^m 3^n$ is injective. \square

Corollary 2.23.5.3. *The Cartesian product of two countable sets is countable.*

Theorem 2.23.6. *A countable union of countable sets is countable.*

PROOF:

$\langle 1 \rangle 1$. LET: A be a set.

$\langle 1 \rangle 2$. LET: $\mathcal{B} \subseteq \mathcal{P}A$ be a countable set of countable sets such that $\bigcup \mathcal{B} = A$

$\langle 1 \rangle 3$. PICK a surjection $B : \mathbb{N} \rightarrow \mathcal{B}$

$\langle 1 \rangle 4$. ASSUME: w.l.o.g. each $B(n)$ is nonempty.

$\langle 1 \rangle 5$. For $n \in \mathbb{N}$, PICK a surjective function $g_n : \mathbb{N} \rightarrow B(n)$

$\langle 1 \rangle 6$. LET: $h : \mathbb{N} \times \mathbb{N} \rightarrow A$ be the function $h(m, n) = g_m(n)$

$\langle 1 \rangle 7$. h is surjective.

\square

Theorem 2.23.7. $2^{\mathbb{N}}$ *is uncountable.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : \mathbb{N} \rightarrow 2^{\mathbb{N}}$

PROVE: f is not surjective.

$\langle 1 \rangle 2$. Define $g : \mathbb{N} \rightarrow 2$ by $g(n) = 1 - f(n)(n)$.

$\langle 1 \rangle 3$. For all $n \in \mathbb{N}$ we have $g(n) \neq f(n)(n)$.

$\langle 1 \rangle 4$. For all $n \in \mathbb{N}$ we have $g \neq f(n)$.

\square

Theorem 2.23.8. *For any set A , there is no surjective function $A \rightarrow \mathcal{P}A$.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow \mathcal{P}A$

$\langle 1 \rangle 2$. LET: $S = \{x \in A : x \notin f(x)\}$

$\langle 1 \rangle 3$. For all $a \in A$ we have $S \neq f(a)$

PROOF: We have $a \in S$ if and only if $a \notin f(a)$.

\square

Corollary 2.23.8.1. *For any set A , there is no injective function $\mathcal{P}A \rightarrow A$.*

2.24 Fixed Points

Definition 2.24.1 (Fixed Point). Let A be a set and $f : A \rightarrow A$. A *fixed point* of f is an element $a \in A$ such that $f(a) = a$.

2.25 Finite Intersection Property

Definition 2.25.1 (Finite Intersection Property). Let X be a set. Let $\mathcal{C} \subseteq \mathcal{P}X$. Then \mathcal{C} has the *finite intersection property* iff every finite nonempty subset of \mathcal{C} has nonempty intersection.

Chapter 3

Relations

Definition 3.0.1 (Reflexive). A relation $R \subseteq A \times A$ is *reflexive* iff, for all $a \in A$, we have $(a, a) \in R$.

Definition 3.0.2 (Antisymmetric). A relation $R \subseteq A \times A$ is *antisymmetric* iff, for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Definition 3.0.3 (Transitive). A relation $R \subseteq A \times A$ is *transitive* iff, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Definition 3.0.4 (Partial Order). A *partial order* on a set A is a relation on A that is reflexive, antisymmetric and transitive.

We say (A, \leq) is a *partially ordered set* or *poset* iff \leq is a partial order on A .

Definition 3.0.5 (Greatest). Let A be a poset and $a \in A$. Then a is the *greatest* element iff $\forall x \in A. x \leq a$.

Definition 3.0.6 (Least). Let A be a poset and $a \in A$. Then a is the *least* element iff $\forall x \in A. a \leq x$.

Definition 3.0.7 (Upper Bound). Let A be a poset, $S \subseteq A$, and $u \in A$. Then u is an *upper bound* for S iff $\forall x \in S. x \leq u$. We say S is *bounded above* iff it has an upper bound.

Definition 3.0.8 (Lower Bound). Let A be a poset, $S \subseteq A$, and $l \in A$. Then l is a *lower bound* for S iff $\forall x \in S. l \leq x$. We say S is *bounded below* iff it has a lower bound.

Definition 3.0.9 (Supremum). Let A be a poset, $S \subseteq A$ and $s \in A$. Then s is the *supremum* or *least upper bound* for S iff s is the least element in the sub-poset of upper bounds for A .

Definition 3.0.10 (Infimum). Let A be a poset, $S \subseteq A$ and $i \in A$. Then i is the *infimum* or *greatest lower bound* for S iff i is the greatest element in the sub-poset of lower bounds for A .

Definition 3.0.11 (Least Upper Bound Property). A poset A has the *least upper bound property* iff every nonempty subset of A that is bounded above has a least upper bound.

Proposition 3.0.12. *Let A be a poset. Then A has the least upper bound property if and only if every nonempty subset of A that is bounded below has a greatest lower bound.*

PROOF:

$\langle 1 \rangle 1$. If A has the least upper bound property then every subset of A that is bounded below has a greatest lower bound.

$\langle 2 \rangle 1$. ASSUME: A has the least upper bound property.

$\langle 2 \rangle 2$. LET: $S \subseteq A$ be nonempty and bounded below.

$\langle 2 \rangle 3$. LET: L be the set of lower bounds of S .

$\langle 2 \rangle 4$. L is nonempty.

PROOF: Because S is bounded below.

$\langle 2 \rangle 5$. L is bounded above.

PROOF: Pick an element $s \in S$. Then s is an upper bound for L .

$\langle 2 \rangle 6$. LET: s be the supremum of L .

$\langle 2 \rangle 7$. s is the greatest lower bound of S .

$\langle 3 \rangle 1$. s is a lower bound of S .

$\langle 4 \rangle 1$. LET: $x \in S$

$\langle 4 \rangle 2$. x is an upper bound for L .

$\langle 4 \rangle 3$. $s \leq x$

$\langle 3 \rangle 2$. For any lower bound l of S we have $l \leq s$.

PROOF: Immediate from $\langle 2 \rangle 6$.

$\langle 1 \rangle 2$. If every subset of A that is bounded below has a greatest lower bound, then A has the least upper bound property.

PROOF: Dual.

□

Chapter 4

Order Theory

4.1 Strict Partial Orders

Definition 4.1.1 (Strict Partial Order). A *strict partial order* on a set A is a relation on A that is irreflexive and transitive.

Proposition 4.1.2. 1. If \leq is a partial order on A then $<$ is a strict partial order on A , where $x < y$ iff $x \leq y \wedge x \neq y$.

2. If $<$ is a strict partial order on A then \leq is a partial order on A , where $x \leq y$ iff $x < y \vee x = y$.

3. These two relations are inverses of one another.

4.1.1 Linear Orders

Definition 4.1.3 (Linear Order). A *linear order* on a set A is a partial order \leq on A such that, for all $x, y \in A$, we have $x \leq y$ or $y \leq x$.

A *linearly ordered set* is a pair (X, \leq) such that X is a set and \leq is a linear order on X .

Definition 4.1.4 (Open Interval). Let X be a linearly ordered set and $a, b \in X$. The *open interval* (a, b) is the set

$$\{x \in X : a < x < b\} .$$

Definition 4.1.5 (Immediate Predecessor, Immediate Successor). Let X be a linearly ordered set and $a, b \in X$. Then b is the (*immediate*) *successor* of a , and a is the (*immediate*) *predecessor* of b , iff $a < b$ and there is no x such that $a < x < b$.

Definition 4.1.6 (Dictionary Order). Let A and B be linearly ordered sets. The *dictionary order* on $A \times B$ is the order defined by

$$(a, b) < (a', b') \Leftrightarrow a < a' \vee (a = a' \wedge b < b') .$$

Theorem 4.1.7 (Maximum Principle). *Every poset has a maximal linearly ordered subset.*

PROOF:

⟨1⟩1. LET: (A, \leq) be a poset.

⟨1⟩2. PICK a well ordering \preceq of A .

PROOF: Well Ordering Theorem.

⟨1⟩3. LET: $h : A \rightarrow 2$ be the function defined by \preceq -recursion thus:

$$h(a) = \begin{cases} 1 & \text{if } a \text{ is } \leq\text{-comparable with every } b < a \text{ such that } h(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

⟨1⟩4. LET: $B = \{x \in A : h(x) = 1\}$

PROVE: B is a maximal subset linearly ordered by \leq .

⟨1⟩5. B is linearly ordered by \leq .

⟨2⟩1. LET: $x, y \in B$

⟨2⟩2. ASSUME: w.l.o.g. $x \preceq y$

⟨2⟩3. y is \leq -comparable with x

⟨1⟩6. For any subset $C \subseteq A$ linearly ordered by \leq , if $B \subseteq C$ then $B = C$.

⟨2⟩1. LET: $x \in C$

⟨2⟩2. x is comparable with every $y \preceq x$ such that $h(y) = 1$

⟨2⟩3. $x \in B$

□

Theorem 4.1.8 (Zorn's Lemma). *Let A be a poset. If every linearly ordered subset of A is bounded above, then A has a maximal element.*

PROOF:

⟨1⟩1. PICK a maximal linearly ordered subset B of A .

PROOF: Maximal Principle

⟨1⟩2. PICK an upper bound c for B .

PROVE: c is maximal.

⟨1⟩3. LET: $x \in A$

⟨1⟩4. ASSUME: $c \leq x$

PROVE: $x = c$

⟨1⟩5. x is an upper bound for B .

⟨1⟩6. $x \in B$

PROOF: By the maximality of B , since $B \cup \{x\}$ is linearly ordered.

⟨1⟩7. $x \leq c$

PROOF: ⟨1⟩2

⟨1⟩8. $x = c$

□

Corollary 4.1.8.1 (Kuratowski's Lemma). *Let $\mathcal{A} \subseteq \mathcal{P}X$. Suppose that, for every subset $\mathcal{B} \subseteq \mathcal{A}$ that is linearly ordered by inclusion, we have $\bigcup \mathcal{B} \in \mathcal{A}$. Then \mathcal{A} has a maximal element.*

Definition 4.1.9 (Closed Interval). Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *closed interval* $[a, b]$ is

$$[a, b] := \{x \in X : a \leq x \leq b\} .$$

Definition 4.1.10 (Half-Open Interval). Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *half-open intervals* $(a, b]$ and $[a, b)$ are defined by

$$\begin{aligned}(a, b] &:= \{x \in X : a < x \leq b\} \\ [a, b) &:= \{x \in X : a \leq x < b\}\end{aligned}$$

Definition 4.1.11 (Open Ray). Let X be a linearly ordered set and $a \in X$. The *open rays* $(a, +\infty)$ and $(-\infty, a)$ are defined by:

$$\begin{aligned}(a, +\infty) &:= \{x \in X : a < x\} \\ (-\infty, a) &:= \{x \in X : x < a\}\end{aligned}$$

Definition 4.1.12 (Closed Ray). Let X be a linearly ordered set and $a \in X$. The *closed rays* $[a, +\infty)$ and $(-\infty, a]$ are defined by:

$$\begin{aligned}[a, +\infty) &:= \{x \in X : a \leq x\} \\ (-\infty, a] &:= \{x \in X : x \leq a\}\end{aligned}$$

Definition 4.1.13 (Convex). Let X be a linearly ordered set and $Y \subseteq X$. Then Y is *convex* iff, for all $a, b \in Y$ and $c \in X$, if $a < c < b$ then $c \in Y$.

4.1.2 Sets of Finite Type

Definition 4.1.14 (Finite Type). Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then \mathcal{A} is of *finite type* if and only if, for any $B \subseteq X$, we have $B \in \mathcal{A}$ if and only if every finite subset of B is in \mathcal{A} .

Proposition 4.1.15 (Tukey's Lemma). *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$. If \mathcal{A} is of finite type, then \mathcal{A} has a maximal element.*

PROOF:

$\langle 1 \rangle 1$. For every subset $\mathcal{B} \subseteq \mathcal{A}$ that is linearly ordered by inclusion, we have $\bigcup \mathcal{B} \in \mathcal{A}$.

$\langle 2 \rangle 1$. LET: $\mathcal{B} \subseteq \mathcal{A}$

$\langle 2 \rangle 2$. ASSUME: \mathcal{B} is linearly ordered by inclusion.

$\langle 2 \rangle 3$. Every finite subset of $\bigcup \mathcal{B}$ is in \mathcal{A}

$\langle 2 \rangle 4$. $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Kuratowski's Lemma.

□

4.2 Linear Continua

Definition 4.2.1 (Linear Continuum). A *linear continuum* is a linearly ordered set with more than one element that is dense and has the least upper bound property.

Proposition 4.2.2. *Every convex subset of a linear continuum with more than one element is a linear continuum.*

PROOF: Easy. \square

Corollary 4.2.2.1. *Every interval and ray in a linear continuum is a linear continuum.*

4.3 Well Orders

Definition 4.3.1 (Well Ordered Set). A *well ordered set* is a linearly ordered set such that every nonempty subset has a least element.

Proposition 4.3.2. *Any subset of a well ordered set is well ordered.*

Proposition 4.3.3. *The product of two well ordered sets is well ordered under the dictionary order.*

Theorem 4.3.4 (Well Ordering Theorem). *Every set has a well ordering.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a set.

$\langle 1 \rangle 2$. PICK a choice function $c : \mathcal{P}X - \{\emptyset\} \rightarrow X$

$\langle 1 \rangle 3$. Define a *tower* to be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well ordering of T , and

$$\forall x \in T. x = c(X - \{y \in T : y < x\}) .$$

$\langle 1 \rangle 4$. Given two towers, either they are equal or one is a section of the other.

$\langle 2 \rangle 1$. LET: $(T_1, <_1)$ and $(T_2, <_2)$ be towers.

$\langle 2 \rangle 2$. ASSUME: w.l.o.g. there exists a strictly monotone function $h : T_1 \rightarrow T_2$

$\langle 2 \rangle 3$. $h(T_1)$ is either T_2 or a section of T_2

PROOF: Proposition 4.3.11.

$\langle 2 \rangle 4$. $\forall x \in T_1. h(x) = x$

$\langle 3 \rangle 1$. LET: $x \in T_1$

$\langle 3 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall y < x. h(y) = y$

$\langle 3 \rangle 3$. $h(x)$ is the least element of $T_2 - \{h(y) \in T_1 : y < x\}$

$\langle 3 \rangle 4$. $h(x)$ is the least element of $T_2 - \{y \in T_1 : y < x\}$

PROOF: $\langle 3 \rangle 2$

$\langle 3 \rangle 5$. $h(x) = x$

PROOF:

$$h(x) = c(X - \{y \in T_2 : y < h(x)\}) \quad (\langle 1 \rangle 3)$$

$$= c(X - \{y \in T_2 : y < x\}) \quad (\langle 3 \rangle 4)$$

$$= c(X - \{y \in T_1 : y < x\}) \quad (\langle 3 \rangle 2)$$

$$= x \quad (\langle 1 \rangle 3)$$

$\langle 1 \rangle 5$. If $(T, <)$ is a tower and $T \neq X$, then there exists a tower of which $(T, <)$ is a section.

PROOF: Let $T_1 = T \cup \{c(T)\}$ and $<_1$ be the extension of $<$ such that $x < c(T)$ for all $x \in T$.

- $\langle 1 \rangle 6$. LET: $\mathbf{T} = \bigcup \{T : \exists R. (T, R) \text{ is a tower}\}$ and $\mathbf{R} = \bigcup \{R : \exists T. (T, R) \text{ is a tower}\}$
 $\langle 1 \rangle 7$. (\mathbf{T}, \mathbf{R}) is a tower.
 $\langle 2 \rangle 1$. \mathbf{R} is irreflexive.
 PROOF: Since for every tower $(T, <)$ we have $<$ is irreflexive.
 $\langle 2 \rangle 2$. \mathbf{R} is transitive.
 $\langle 3 \rangle 1$. ASSUME: $x\mathbf{R}y$ and $y\mathbf{R}z$
 $\langle 3 \rangle 2$. PICK towers $(T_1, <_1)$ and $(T_2, <_2)$ such that $x <_1 y$ and $y <_2 z$
 $\langle 3 \rangle 3$. ASSUME: w.l.o.g. $(T_1, <_1)$ is either $(T_2, <_2)$ or a section of $(T_2, <_2)$
 $\langle 3 \rangle 4$. $x <_2 y <_2 z$
 $\langle 3 \rangle 5$. $x <_2 z$
 $\langle 3 \rangle 6$. $x\mathbf{R}z$
 $\langle 2 \rangle 3$. For all $x, y \in \mathbf{T}$, either $x\mathbf{R}y$ or $x = y$ or $y\mathbf{R}x$
 PROOF: There exists a tower that has both x and y .
 $\langle 2 \rangle 4$. Every nonempty subset of \mathbf{T} has an \mathbf{R} -least element.
 $\langle 3 \rangle 1$. LET: $A \subseteq \mathbf{T}$ be nonempty.
 $\langle 3 \rangle 2$. PICK $a \in A$
 $\langle 3 \rangle 3$. PICK a tower $(T, <)$ such that $a \in T$.
 $\langle 3 \rangle 4$. LET: b be the $<$ -least element of $A \cap T$
 PROVE: b is \mathbf{R} -least in A .
 $\langle 3 \rangle 5$. LET: $x \in A$
 $\langle 3 \rangle 6$. Etc.
 $\langle 2 \rangle 5$. $\forall x \in \mathbf{T}. x = c(X - \{y \in \mathbf{T} : y\mathbf{R}x\})$
 $\langle 1 \rangle 8$. $\mathbf{T} = X$
 $\langle 1 \rangle 9$. \mathbf{R} is a well ordering of X .
 \square

Proposition 4.3.5. *There exists a well-ordered set with a largest element Ω such that $(-\infty, \Omega)$ is uncountable but, for all $\alpha < \Omega$, we have $(-\infty, \alpha)$ is countable.*

PROOF:

- $\langle 1 \rangle 1$. PICK an uncountable well ordered set B .
 $\langle 1 \rangle 2$. LET: $C = 2 \times B$ under the dictionary order.
 $\langle 1 \rangle 3$. LET: Ω be the least element of C such that $(-\infty, \Omega)$ is uncountable.
 $\langle 1 \rangle 4$. LET: $A = (-\infty, \Omega]$
 $\langle 1 \rangle 5$. A is a well ordered set with largest element Ω such that $(-\infty, \Omega)$ is uncountable but, for all $\alpha < \Omega$, we have $(-\infty, \alpha)$ is countable.
 \square

Proposition 4.3.6. *Every well ordered set has the least upper bound property.*

PROOF: For any subset that is bounded above, the set of upper bounds is nonempty, hence has a least element. \square

Proposition 4.3.7. *In a well ordered set, every element that is not greatest has a successor.*

PROOF: If a is not greatest, then $\{x : x > a\}$ is nonempty, hence has a least element. \square

Theorem 4.3.8 (Transfinite Induction). *Let J be a well ordered set. Let $S \subseteq J$. Assume that, for every $\alpha \in J$, if $\forall x < \alpha. x \in S$ then $\alpha \in S$. Then $S = J$.*

PROOF: Otherwise $J - S$ would be a nonempty subset of J with no least element. \square

Proposition 4.3.9. *Let I be a well ordered set. Let $\{A_i\}_{i \in I}$ be a family of well ordered sets. Define $<$ on $\coprod_{i \in I} A_i$ by: $\kappa_i(a) < \kappa_j(b)$ iff either $i < j$, or $i = j$ and $a < b$ in A_i . Then $<$ well orders $\coprod_{i \in I} A_i$.*

PROOF: Easy. \square

Theorem 4.3.10 (Principle of Transfinite Recursion). *Let J be a well ordered set. Let C be a set. Let \mathcal{F} be the set of all functions from a section of J into C . Let $\rho : \mathcal{F} \rightarrow C$. Then there exists a unique function $h : J \rightarrow C$ such that, for all $\alpha \in J$, we have*

$$h(\alpha) = \rho(h \upharpoonright (-\infty, \alpha)) .$$

PROOF:

$\langle 1 \rangle 1$. For a function h mapping either a section of J or all of J into C , let us say h is *acceptable* iff, for all $x \in \text{dom } h$, we have $(-\infty, x) \subseteq \text{dom } h$ and $h(x) = \rho(h \upharpoonright (-\infty, x))$.

$\langle 1 \rangle 2$. If h and k are acceptable functions then $h(x) = k(x)$ for all x in both domains.

$\langle 2 \rangle 1$. LET: $x \in J$

$\langle 2 \rangle 2$. ASSUME: as transfinite induction hypothesis that, for all $y < x$ and any acceptable functions h and k with $y \in \text{dom } h \cap \text{dom } k$, we have $h(y) = k(y)$

$\langle 2 \rangle 3$. LET: h and k be acceptable functions with $x \in \text{dom } h \cap \text{dom } k$

$\langle 2 \rangle 4$. $h \upharpoonright (-\infty, x) = k \upharpoonright (-\infty, x)$

PROOF: By $\langle 2 \rangle 2$.

$\langle 2 \rangle 5$. $h(x) = k(x)$

PROOF: By $\langle 2 \rangle 3$, each is the least element of the set in $\langle 2 \rangle 4$.

$\langle 1 \rangle 3$. For $\alpha \in J$, if there exists an acceptable function $(-\infty, \alpha) \rightarrow C$, then there exists an acceptable function $(-\infty, \alpha] \rightarrow C$.

$\langle 2 \rangle 1$. LET: $\alpha \in J$

$\langle 2 \rangle 2$. LET: $f : (-\infty, \alpha) \rightarrow C$ be acceptable.

$\langle 2 \rangle 3$. LET: $g : (-\infty, \alpha] \rightarrow C$ be the function given by

$$g(x) = \begin{cases} f(x) & \text{if } x < \alpha \\ \rho(f) & \text{if } x = \alpha \end{cases}$$

$\langle 2 \rangle 4$. g is acceptable.

$\langle 1 \rangle 4$. Let $K \subseteq J$. Assume that, for all $\alpha \in K$, there exists an acceptable function $(-\infty, \alpha) \rightarrow C$. Then there exists an acceptable function $\bigcup_{\alpha \in K} (-\infty, \alpha) \rightarrow C$.

$\langle 2 \rangle 1$. Define $f : \bigcup_{\alpha \in K} (-\infty, \alpha) \rightarrow C$ by: $f(x) = y$ iff there exists $\alpha \in K$ and $g : (-\infty, \alpha) \rightarrow C$ acceptable such that $g(x) = y$.

$\langle 1 \rangle 5$. For every $\beta \in J$, there exists an acceptable function $(-\infty, \beta) \rightarrow C$

- ⟨2⟩1. LET: $\beta \in J$
- ⟨2⟩2. ASSUME: as transfinite induction hypothesis that, for all $\alpha < \beta$, there exists an acceptable function $(-\infty, \alpha) \rightarrow C$
- ⟨2⟩3. CASE: β has a predecessor
 - ⟨3⟩1. LET: α be the predecessor of β .
 - ⟨3⟩2. There exists an acceptable function $(-\infty, \alpha) \rightarrow C$.
 - ⟨3⟩3. There exists an acceptable function $(-\infty, \beta) \rightarrow C$.
 PROOF: By ⟨1⟩3 since $(-\infty, \beta) = (-\infty, \alpha]$.
- ⟨2⟩4. CASE: β has no predecessor.
 PROOF: The result follows by ⟨1⟩4 since $(-\infty, \beta) = \bigcup_{\alpha < \beta} (-\infty, \alpha)$.
- ⟨1⟩6. There exists an acceptable function $J \rightarrow C$.
 - ⟨2⟩1. CASE: J has a greatest element.
 - ⟨3⟩1. LET: g be greatest.
 - ⟨3⟩2. There exists an acceptable function $(-\infty, g) \rightarrow C$.
 PROOF: ⟨1⟩5
 - ⟨3⟩3. There exists an acceptable function $J \rightarrow C$.
 PROOF: By ⟨1⟩3 since $J = (-\infty, g]$.
 - ⟨2⟩2. CASE: J has no greatest element.
 PROOF: By ⟨1⟩4 since $J = \bigcup_{\alpha \in J} (-\infty, \alpha)$.

□

Corollary 4.3.10.1 (Cardinal Comparability). *Let A and B be sets. Then either $A \leq B$ or $B \leq A$.*

PROOF: Choose well orderings of A and B . Then either there exists a surjection $A \twoheadrightarrow B$, or there exists an injective function $h : A \rightarrow B$ defined by transfinite recursion by $h(x)$ is the least element of $B - h((-\infty, x))$. □

Proposition 4.3.11. *Let J and E be well ordered sets. Let $h : J \rightarrow E$. Then the following are equivalent.*

1. h is strictly monotone and $h(J)$ is either E or a section of E .
2. For all $\alpha \in J$, we have $h(\alpha)$ is the least element of $E - h((-\infty, \alpha))$.

PROOF:

- ⟨1⟩1. $1 \Rightarrow 2$
 - ⟨2⟩1. ASSUME: 1
 - ⟨2⟩2. $h(J)$ is closed downwards.
 - ⟨2⟩3. LET: $\alpha \in J$
 - ⟨2⟩4. $h(\alpha) \in E - h((-\infty, \alpha))$
 PROOF: If $\beta < \alpha$ then $h(\beta) < h(\alpha)$.
 - ⟨2⟩5. For all $y \in E - h((-\infty, \alpha))$ we have $h(\alpha) \leq y$
 - ⟨3⟩1. ASSUME: for a contradiction $y < h(\alpha)$
 - ⟨3⟩2. $y \in h(J)$
 - ⟨3⟩3. PICK $\beta \in J$ such that $h(\beta) = y$
 - ⟨3⟩4. $h(\beta) < h(\alpha)$
 - ⟨3⟩5. $\beta < \alpha$

⟨3⟩6. Q.E.D.

PROOF: This contradicts the fact that $y \notin h((-\infty, \alpha))$.

⟨1⟩2. $2 \Rightarrow 1$

⟨2⟩1. ASSUME: 2

⟨2⟩2. h is strictly monotone.

⟨3⟩1. LET: $\alpha, \beta \in J$ with $\alpha < \beta$

⟨3⟩2. $h(\alpha) \neq h(\beta)$

PROOF: Because $h(\beta) \in E - h((-\infty, \beta))$.

⟨3⟩3. $h(\alpha) \leq h(\beta)$

PROOF: Because $h(\alpha)$ is least in $E - h((-\infty, \alpha))$.

⟨3⟩4. $h(\alpha) < h(\beta)$

⟨2⟩3. $h(J)$ is either E or a section of E .

⟨3⟩1. ASSUME: $h(J) \neq E$

⟨3⟩2. LET: e be least in $E - h(J)$

PROVE: $h(J) = (-\infty, e)$

⟨3⟩3. $h(J) \subseteq (-\infty, e)$

⟨4⟩1. LET: $\alpha \in J$

⟨4⟩2. $h(\alpha) \neq e$

PROOF: $e \notin h(J)$

⟨4⟩3. $h(\alpha) \leq e$

PROOF: Since $h(\alpha)$ is least in $E - h((-\infty, \alpha))$.

⟨4⟩4. $h(\alpha) < e$

⟨3⟩4. $(-\infty, e) \subseteq h(J)$

PROOF: If $e' < e$ then $e' \in h(J)$ by leastness of e .

□

Part II

Category Theory

Chapter 5

Category Theory

5.1 Categories

Definition 5.1.1. A *category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \text{Ob}(\mathcal{C})$.
- for any objects X and Y , a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y . We write $f : X \rightarrow Y$ for $f \in \mathcal{C}[X, Y]$.
- for any objects X, Y and Z , a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \rightarrow \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \rightarrow Y, g : Y \rightarrow Z$ and $h : Z \rightarrow W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X , there exists a morphism $\text{id}_X : X \rightarrow X$, the *identity morphism* on X , such that:
 - for any object Y and morphism $f : Y \rightarrow X$ we have $\text{id}_X \circ f = f$
 - for any object Y and morphism $f : X \rightarrow Y$ we have $f \circ \text{id}_X = f$

We write the composite of morphism f_1, \dots, f_n as $f_n \circ \dots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 5.1.2. Let **Set** be the category of small sets and functions.

Definition 5.1.3. Let **LPos** be the category of linearly ordered sets and monotone functions.

Proposition 5.1.4. Any finite linearly ordered set is isomorphic to $\{m \in \mathbb{N} : m < n\}$ for some n .

PROOF:

$\langle 1 \rangle$ 1. Every finite nonempty linearly ordered set has a greatest element.

- $\langle 2 \rangle 1$. LET: $P[n]$ be the property: for any linearly ordered set A , if there exists a bijection $A \approx \{m \in \mathbb{N} : m < n\}$ and A is nonempty then A has a greatest element.
- $\langle 2 \rangle 2$. $P[0]$
 PROOF: Vacuous.
- $\langle 2 \rangle 3$. $\forall n \in \mathbb{N}. P[n] \Rightarrow P[n+1]$
- $\langle 3 \rangle 1$. LET: $n \in \mathbb{N}$
- $\langle 3 \rangle 2$. ASSUME: $P[n]$
- $\langle 3 \rangle 3$. LET: A be a nonempty linearly ordered set.
- $\langle 3 \rangle 4$. LET: $f : A \approx \{m \in \mathbb{N} : m < n+1\}$
- $\langle 3 \rangle 5$. LET: $a = f^{-1}(n)$
- $\langle 3 \rangle 6$. $f \upharpoonright (A - \{a\}) : A - \{a\} \approx \{m \in \mathbb{N} : m < n\}$
- $\langle 3 \rangle 7$. ASSUME: w.l.o.g. a is not greatest in A .
- $\langle 3 \rangle 8$. LET: b be greatest in $A - \{a\}$
 PROOF: $\langle 3 \rangle 2$
- $\langle 3 \rangle 9$. b is greatest in A .
- $\langle 1 \rangle 2$. LET: $P[n]$ be the property: for any linearly ordered set A , if there exists a bijection $A \approx \{m \in \mathbb{N} : m < n\}$ then there exists an isomorphism in **LPos** $A \cong \{m \in \mathbb{N} : m < n\}$.
- $\langle 1 \rangle 3$. $P[0]$
 PROOF: If there exists a bijection $A \approx \emptyset$ then A is empty and so the unique function $A \rightarrow \emptyset$ is an order isomorphism.
- $\langle 1 \rangle 4$. For every natural number n , if $P[n]$ then $P[n+1]$.
- $\langle 2 \rangle 1$. LET: n be a natural number.
- $\langle 2 \rangle 2$. ASSUME: $P[n]$
- $\langle 2 \rangle 3$. LET: A be a linearly ordered set.
- $\langle 2 \rangle 4$. ASSUME: A has $n+1$ elements.
- $\langle 2 \rangle 5$. LET: a be the greatest element in A .
- $\langle 2 \rangle 6$. LET: $f : A - \{a\} \cong \{m \in \mathbb{N} : m < n\}$ be an order isomorphism.
 PROOF: $\langle 2 \rangle 2$
- $\langle 2 \rangle 7$. Define $g : A \rightarrow \{m \in \mathbb{N} : m < n+1\}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ n & \text{if } x = a \end{cases}$$
- $\langle 2 \rangle 8$. g is an order isomorphism.
- $\langle 1 \rangle 5$. $\forall n \in \mathbb{N}. P[n]$
 \square

Corollary 5.1.4.1. *Any finite linearly ordered set is well ordered.*

Proposition 5.1.5. *Let J and E be well ordered sets. Suppose there is a strictly monotone map $J \rightarrow E$. Then J is isomorphic either to E or a section of E .*

PROOF:

- $\langle 1 \rangle 1$. LET: $k : J \rightarrow E$ be strictly monotone.
- $\langle 1 \rangle 2$. ASSUME: w.l.o.g. E is nonempty.
- $\langle 1 \rangle 3$. PICK $e_0 \in E$

⟨1⟩4. LET: $h : J \rightarrow E$ be the function defined by transfinite recursion thus:

$$h(\alpha) = \begin{cases} \text{the least element in } E - h((-\infty, \alpha)) & \text{if } h((-\infty, \alpha)) \neq E \\ e_0 & \text{if } h((-\infty, \alpha)) = E \end{cases}$$

⟨1⟩5. $\forall \alpha \in J, h(\alpha) \leq k(\alpha)$

⟨2⟩1. LET: $\alpha \in J$

⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha, h(\beta) \leq k(\beta)$.

⟨2⟩3. $\forall \beta < \alpha, h(\beta) < k(\alpha)$

⟨2⟩4. $h((-\infty, \alpha)) \neq E$

⟨2⟩5. $h(\alpha)$ is the least element in $E - h((-\infty, \alpha))$.

⟨2⟩6. $k(\alpha) \in E - h((-\infty, \alpha))$

⟨2⟩7. $h(\alpha) \leq k(\alpha)$

⟨1⟩6. $\forall \alpha \in J, h((-\infty, \alpha)) \neq E$

PROOF: For $\beta < \alpha$ we have $h(\beta) \leq k(\beta) < k(\alpha)$ so $k(\alpha) \notin h((-\infty, \alpha))$.

⟨1⟩7. For all $\alpha \in J$, we have $h(\alpha)$ is the least element of $E - h((-\infty, \alpha))$.

⟨1⟩8. h is strictly monotone and $h(J)$ is either E or a section of E .

PROOF: Proposition 4.3.11.

□

Proposition 5.1.6. *If A and B are well ordered sets, then exactly one of the following conditions hold: $A \cong B$, or A is isomorphic to a section of B , or B is isomorphic to a section of A .*

PROOF:

⟨1⟩1. At least one of the conditions holds.

⟨2⟩1. B is isomorphic to either $A + B$ or a section of $A + B$.

⟨2⟩2. CASE: $B \cong A + B$

⟨3⟩1. LET: ϕ be the isomorphism $B \cong A + B$

⟨3⟩2. LET: b_0 be the least element in B .

⟨3⟩3. A is isomorphic to the section $(-\infty, \phi^{-1}(\kappa_2(b_0)))$ of B .

⟨2⟩3. CASE: $a \in A$ and $B \cong (-\infty, \kappa_1(a))$

PROOF: Then B is isomorphic to the section $(-\infty, a)$ of A .

⟨2⟩4. CASE: $b \in B$ and $\phi : B \cong (-\infty, \kappa_2(b))$

⟨3⟩1. CASE: b is least in B .

PROOF: Then $A \cong B$.

⟨3⟩2. CASE: b is not least in B .

⟨4⟩1. LET: b_0 be least in B .

⟨4⟩2. A is isomorphic to the section $(-\infty, \phi^{-1}(\kappa_2(b_0)))$ of B .

⟨1⟩2. At most one of the conditions holds.

PROOF: Since a well ordered set cannot be isomorphic to a section of itself.

□

Theorem 5.1.7. *There exists a well ordered set, unique up to order isomorphism, that is uncountable but such that every section is countable.*

PROOF:

⟨1⟩1. There exists a well ordered set that is uncountable but such that every section is countable.

- $\langle 2 \rangle 1$. PICK a well ordered set A with an element $\Omega \in A$ such that $(-\infty, \Omega)$ is uncountable but $\forall \alpha < \Omega. (-\infty, \alpha)$ is countable.
 $\langle 2 \rangle 2$. LET: $(-\infty, \Omega)$ is uncountable but every section is countable.
 $\langle 1 \rangle 2$. If A and B are uncountable well ordered sets such that every section is countable, then $A \cong B$.

PROOF: Since it cannot be that one of A and B is isomorphic to a section of the other.

□

Definition 5.1.8 (Minimal Uncountable Well Ordered Set). The *minimal uncountable well ordered set* Ω is the well ordered set that is uncountable but such that every section is countable.

We write $\bar{\Omega}$ for the well ordered set $\Omega \cup \{\Omega\}$ where Ω is greatest.

Proposition 5.1.9. *Every countable subset of Ω is bounded above.*

PROOF:

- $\langle 1 \rangle 1$. LET: A be a countable subset of Ω .
 $\langle 1 \rangle 2$. For all $a \in A$ we have $(-\infty, a)$ is countable.
 $\langle 1 \rangle 3$. $\bigcup_{a \in A} (-\infty, a)$ is countable.
 $\langle 1 \rangle 4$. $\bigcup_{a \in A} (-\infty, a) \neq \Omega$
 $\langle 1 \rangle 5$. PICK $x \in \Omega - \bigcup_{a \in A} (-\infty, a)$
 $\langle 1 \rangle 6$. x is an upper bound for A .

□

Proposition 5.1.10. *Ω has no greatest element.*

PROOF: For any $\alpha \in \Omega$ we have $(-\infty, \alpha]$ is countable and hence not the whole of Ω . □

Proposition 5.1.11. *There are uncountably many elements of Ω that have no predecessor.*

PROOF:

- $\langle 1 \rangle 1$. LET: A be the set of all elements of Ω that have no predecessor.
 $\langle 1 \rangle 2$. LET: $f : A \times \mathbb{N} \rightarrow \Omega$ be the function that maps (a, n) to the n th successor of a .
 $\langle 1 \rangle 3$. f is surjective.
 $\langle 2 \rangle 1$. ASSUME: for a contradiction $x \in \Omega$ and there is no element $a \in A$ and $n \in \mathbb{N}$ such that x is the n th successor of a .
 $\langle 2 \rangle 2$. LET: x_n be the n th predecessor of x for $n \in \mathbb{N}$.
 $\langle 2 \rangle 3$. $\{x_n : n \in \mathbb{N}\}$ is a nonempty subset of Ω with no least element.
 $\langle 1 \rangle 4$. $A \times \mathbb{N}$ is uncountable.
 $\langle 1 \rangle 5$. A is uncountable.

□

Definition 5.1.12. We identify a poset (A, \leq) with the category with:

- set of objects A

- for $a, b \in A$, the set of homomorphisms is $\{x \in 1 : a \leq b\}$

Proposition 5.1.13. *A category is a poset iff, for any two objects, there exists at most one morphism between them.*

Proposition 5.1.14. *The identity morphism on an object is unique.*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{C} be a category.

$\langle 1 \rangle 2$. LET: $A \in \mathcal{C}$

$\langle 1 \rangle 3$. LET: $i, j : A \rightarrow A$ be identity morphisms on A .

$\langle 1 \rangle 4$. $i = j$

PROOF:

$$\begin{aligned} i &= i \circ j & (j \text{ is an identity on } A) \\ &= j & (i \text{ is an identity on } A) \end{aligned}$$

□

Proposition 5.1.15. *Let A be a linearly ordered set. Then A is well ordered if and only if it does not contain a subset of order type \mathbb{N}^{op} .*

PROOF:

$\langle 1 \rangle 1$. If A is well ordered then it does not contain a subset of order type \mathbb{N}^{op} .

PROOF: A subset of order type \mathbb{N}^{op} would be a subset with no least element.

$\langle 1 \rangle 2$. If A is not well ordered then it contains a subset of order type \mathbb{N}^{op} .

$\langle 2 \rangle 1$. ASSUME: A is not well ordered.

$\langle 2 \rangle 2$. PICK a nonempty subset S with no least element.

$\langle 2 \rangle 3$. PICK $a_0 \in S$

$\langle 2 \rangle 4$. Extend to a sequence (a_n) in S such that $a_{n+1} < a_n$ for all n .

$\langle 2 \rangle 5$. $\{a_n : n \in \mathbb{N}\}$ has order type \mathbb{N}^{op} .

□

Corollary 5.1.15.1. *Let A be a linearly ordered set. If every countable subset of A is well ordered, then A is well ordered.*

Definition 5.1.16. Given $f : A \rightarrow B$ and an object C , define the function $f^* : \mathcal{C}[B, C] \rightarrow \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 5.1.17. Given $f : A \rightarrow B$ and an object C , define the function $f_* : \mathcal{C}[C, A] \rightarrow \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

5.1.1 Monomorphisms

Definition 5.1.18 (Monomorphism). Let $f : A \rightarrow B$. Then f is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for any object X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

5.1.2 Epimorphisms

Definition 5.1.19 (Epimorphism). Let $f : A \rightarrow B$. Then f is *epic* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for any object X and functions $x, y : B \rightarrow X$, if $x \circ f = y \circ f$ then $x = y$.

5.1.3 Sections and Retractions

Definition 5.1.20 (Section, Retraction). Let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $rs = \text{id}_B$.

Proposition 5.1.21. *Let $f : A \rightarrow B$ and $r, s : B \rightarrow A$. If r is a retraction of f and s is a section of f then $r = s$.*

PROOF:

$$\begin{aligned}
 r &= r \text{id}_B && \text{(Unit Law)} \\
 &= rfs && (s \text{ is a section of } f) \\
 &= \text{id}_A s && (r \text{ is a retraction of } f) \\
 &= s && \text{(Unit Law)} \square
 \end{aligned}$$

Proposition 5.1.22. *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $s : B \rightarrow A$ be a section of $r : A \rightarrow B$.

$\langle 1 \rangle 2$. LET: X be an object and $x, y : X \rightarrow B$

$\langle 1 \rangle 3$. ASSUME: $s \circ x = s \circ y$

$\langle 1 \rangle 4$. $x = y$

PROOF: $x = r \circ s \circ x = r \circ s \circ y = y$.

\square

Proposition 5.1.23. *Every retraction is epic.*

PROOF: Dual. \square

5.1.4 Isomorphisms

Definition 5.1.24 (Isomorphism). A morphism $f : A \rightarrow B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$ that is both a retraction and section of f .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 5.1.25. *The inverse of an isomorphism is unique.*

PROOF: From Proposition 5.1.21. \square

Proposition 5.1.26. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $ff^{-1} = \text{id}_B$ and $f^{-1}f = \text{id}_A$. \square

Isomorphism.

Define the opposite category.

Slice categories

Definition 5.1.27. Let \mathcal{C} be a category and $B \in \mathcal{C}$. The category \mathcal{C}_B^B of objects *over and under* B is the category with:

- objects all triples (X, u, p) such that $u : B \rightarrow X$ and $p : X \rightarrow B$
- morphisms $f : (X, u, p) \rightarrow (Y, u', p')$ all morphisms $f : X \rightarrow Y$ such that $fu = u'$ and $p'f = p$.

Proposition 5.1.28.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B) \backslash \text{id}_B \cong (\mathcal{C} \backslash B) / \text{id}_B$$

$(B, \text{id}_B, \text{id}_B)$ is the zero object in \mathcal{C}_B^B .

5.1.5 Initial Objects

Definition 5.1.29 (Initial Object). An object I is *initial* iff, for any object X , there exists exactly one morphism $I \rightarrow X$.

Proposition 5.1.30. *The empty set is initial in Set.*

PROOF: For any set A , the nowhere-defined function is the unique function $\emptyset \rightarrow A$. \square

Proposition 5.1.31. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : I \rightarrow I'$ be the unique morphism $I \rightarrow I'$.

$\langle 1 \rangle 2$. LET: $i^{-1} : I' \rightarrow I$ be the unique morphism $I' \rightarrow I$.

$\langle 1 \rangle 3$. $ii^{-1} = \text{id}_{I'}$

PROOF: There is only one morphism $I' \rightarrow I'$.

$\langle 1 \rangle 4$. $i^{-1}i = \text{id}_I$

PROOF: There is only one morphism $I \rightarrow I$.

\square

5.1.6 Terminal Objects

Definition 5.1.32 (Terminal Object). An object T is *terminal* iff, for any object X , there exists exactly one morphism $X \rightarrow T$.

Proposition 5.1.33. *1 is terminal in Set.*

PROOF: For any set A , the constant function to $*$ is the only function $A \rightarrow 1$. \square

Proposition 5.1.34. *If T and T' are terminal objects, then there exists a unique isomorphism $T \cong T'$.*

PROOF: Dual to Proposition 5.1.31. \square

5.1.7 Zero Objects

Definition 5.1.35 (Zero Object). An object Z is a *zero object* iff it is an initial object and a terminal object.

Definition 5.1.36 (Zero Morphism). Let \mathcal{C} be a category with a zero object Z . Let $A, B \in \mathcal{C}$. The *zero morphism* $A \rightarrow B$ is the unique morphism $A \rightarrow Z \rightarrow B$.

Proposition 5.1.37. *There is no zero object in **Set**.*

PROOF: Since $\emptyset \not\approx 1$. \square

5.1.8 Triads

Definition 5.1.38 (Triad). Let \mathcal{C} be a category. A *triad* consists of objects X, Y, M and morphisms $\alpha : X \rightarrow M, \beta : Y \rightarrow M$. We call M the *codomain* of the triad.

5.1.9 Cotriads

Definition 5.1.39 (Cotriad). Let \mathcal{C} be a category. A *cotriad* consists of objects X, Y, W and morphisms $\xi : W \rightarrow X, \eta : W \rightarrow Y$. We call W the *domain* of the triad.

5.1.10 Pullbacks

Definition 5.1.40 (Pullback). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

is a *pullback* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$.

In this case we also say that η is the *pullback* of β along α .

Proposition 5.1.41. *If $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$ form a pullback of $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$, and $\xi' : W' \rightarrow X$ and $\eta' : W' \rightarrow Y$ also form the pullback of α and β , then there exists a unique isomorphism $\phi : W \cong W'$ such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : W \rightarrow W'$ be the unique morphism such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.

$\langle 1 \rangle 2$. LET: $\phi^{-1} : W' \rightarrow W$ be the unique morphism such that $\eta\phi^{-1} = \eta'$ and $\xi\phi^{-1} = \xi'$.

$\langle 1 \rangle 3$. $\phi\phi^{-1} = \text{id}_{W'}$

PROOF: Each is the unique $x : W' \rightarrow W'$ such that $\eta'x = \eta'$ and $\xi'x = \xi'$.

$\langle 1 \rangle 4$. $\phi^{-1}\phi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\eta x = \eta$ and $\xi x = \xi$.

□

Proposition 5.1.42. *For any morphism $h : A \rightarrow B$, the following diagram is a pullback diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: Z be an object.

$\langle 1 \rangle 2$. LET: $f : Z \rightarrow B$ and $g : Z \rightarrow A$ satisfy $\text{id}_B f = hg$

$\langle 1 \rangle 3$. $g : Z \rightarrow A$ is the unique morphism such that $\text{id}_A g = g$ and $hg = f$.

□

Proposition 5.1.43. *The pullback of an isomorphism is an isomorphism.*

PROOF:

$\langle 1 \rangle 1$. LET:

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback diagram.

$\langle 1 \rangle 2$. ASSUME: β is an isomorphism.

$\langle 1 \rangle 3$. LET: ξ^{-1} be the unique morphism $X \rightarrow W$ such that $\xi\xi^{-1} = \text{id}_X$ and $\eta\xi^{-1} = \beta^{-1}\alpha$.

PROOF: This exists since $\alpha\text{id}_X = \beta\beta^{-1}\alpha = \alpha$.

$\langle 1 \rangle 4$. $\xi^{-1}\xi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\xi x = \xi$ and $\eta x = \eta$.

□

Proposition 5.1.44. *Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w : A \rightarrow W$ be the unique morphism such that $\xi w = x$ and $\eta w = y$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ is the pullback of β and α in $\mathcal{C} \setminus A$.

PROOF:

$\langle 1 \rangle 1$. LET: $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 2$. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle 3$. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle 4$. $hz = w$

$\langle 2 \rangle 1$. $\xi hz = \xi w$

PROOF:

$$\xi hz = fz \quad (\langle 1 \rangle 3)$$

$$= x \quad (\langle 1 \rangle 2)$$

$$= \xi w$$

$\langle 2 \rangle 2$. $\eta hz = \eta w$

PROOF: Similar.

$\langle 1 \rangle 5$. $h : (Z, z) \rightarrow (W, w)$

□

Proposition 5.1.45. Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in \mathcal{C}/A . Let

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w = x\xi : W \rightarrow A$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ form a pullback of α and β in \mathcal{C}/A .

PROOF:

$\langle 1 \rangle 1$. $\eta : (W, w) \rightarrow (Y, y)$

PROOF:

$$y\eta = m\beta\eta$$

$$= m\alpha\xi$$

$$= x\xi$$

$$= w$$

$\langle 1 \rangle 2$. LET: $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 3$. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle 4$. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle 5$. $h : (Z, z) \rightarrow (W, w)$

PROOF:

$$wh = x\xi h$$

$$= xf \quad (\langle 1 \rangle 4)$$

$$= z \quad (\langle 1 \rangle 3)$$

□

Proposition 5.1.46. In **Set**, let $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$. Let $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$ with inclusion $i : W \rightarrow X \times Y$. Let $\xi = \pi_1 i : W \rightarrow X$ and $\eta = \pi_2 i : W \rightarrow Y$. Then ξ and η form the pullback of α and β .

PROOF:

$\langle 1 \rangle 1.$ $\alpha\xi = \beta\eta$

PROOF: For $w \in W$, if $i(w) = (x, y)$ then $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$.

$\langle 1 \rangle 2.$ For every set Z and functions $f : Z \rightarrow X$, $g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$

PROOF: For $z \in Z$, let $h(z)$ be the unique element of W such that $i(h(z)) = (f(z), g(z))$.

□

Pullback lemma

5.1.11 Pushouts

Definition 5.1.47 (Pushout). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array} \quad (5.1)$$

is a *pushout* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ such that $f\xi = g\eta$, there exists a unique $h : M \rightarrow Z$ such that $h\alpha = f$ and $h\beta = g$.

We also say that β is the *pushout* of ξ along η .

Proposition 5.1.48. If $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$ form a pushout of $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$, and $\alpha' : X \rightarrow M'$ and $\beta' : Y \rightarrow M'$ also form a pushout of ξ and η , then there exists a unique isomorphism $\phi : M \cong M'$ such that $\phi\alpha = \alpha'$ and $\phi\beta = \beta'$.

PROOF: Dual to Proposition 5.1.41. □

Proposition 5.1.49. For any morphism $h : A \rightarrow B$, the following diagram is a pushout diagram.

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF: Dual to Proposition 5.1.42.

Proposition 5.1.50. The diagram (5.1) is a pushout in \mathcal{C} iff it is a pullback in \mathcal{C}^{op} .

PROOF: Immediate from definitions. □

Proposition 5.1.51. The pushout of an isomorphism is an isomorphism.

PROOF: Dual to Proposition 5.1.43. \square

Proposition 5.1.52. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m := \alpha x : A \rightarrow M$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in $\mathcal{C} \setminus A$.

PROOF: Dual to Proposition 5.1.45. \square

Proposition 5.1.53. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in \mathcal{C}/A . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m : M \rightarrow A$ be the unique morphism such that $m\alpha = x$ and $m\beta = y$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in \mathcal{C}/A .

PROOF: Dual to Proposition 5.1.44. \square

Proposition 5.1.54. *Set has pushouts.*

PROOF:

$\langle 1 \rangle 1$. LET: $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$.

$\langle 1 \rangle 2$. LET: \sim be the equivalence relation on $X + Y$ generated by $\xi(w) \sim \eta(w)$ for all $w \in W$

$\langle 1 \rangle 3$. LET: $M = (X + Y)/\sim$ with canonical projection $\pi : X + Y \twoheadrightarrow M$.

$\langle 1 \rangle 4$. LET: $\alpha = \pi \circ \kappa_1 : X \rightarrow M$

$\langle 1 \rangle 5$. LET: $\beta = \pi \circ \kappa_2 : Y \rightarrow M$

$\langle 1 \rangle 6$. LET: Z be any set, $f : X \rightarrow Z$ and $g : Y \rightarrow Z$.

$\langle 1 \rangle 7$. ASSUME: $f\xi = g\eta$

$\langle 1 \rangle 8$. LET: $h : X + Y \rightarrow Z$ be the function defined by $h(x) = f(x)$ and $h(y) = g(y)$ for $x \in X$ and $y \in Y$

$\langle 1 \rangle 9$. h respects \sim

PROOF: For $w \in W$ we have

$$h(\xi(w)) = f(\xi(w)) \quad (\langle 1 \rangle 8)$$

$$= g(\eta(w)) \quad (\langle 1 \rangle 7)$$

$$= h(\eta(w)) \quad (\langle 1 \rangle 8)$$

$\langle 1 \rangle 10$. LET: $\bar{h} : M \rightarrow Z$ be the induced function.

$\langle 1 \rangle 11$. $\bar{h}\alpha = f$

PROOF:

$$\begin{aligned}\bar{h}(\alpha(x)) &= \bar{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x)\end{aligned}$$

$\langle 1 \rangle 12.$ $\bar{h}\beta = g$

PROOF: Similar.

$\langle 1 \rangle 13.$ For all $k : M \rightarrow Z$, if $k\alpha = f$ and $k\beta = g$ then $k = \bar{h}$.

PROOF:

$$\begin{aligned}k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\ &= f(x) \\ k(\pi(\kappa_2(y))) &= k(\beta(y)) \\ &= g(y) \\ \therefore k \circ \pi &= h \\ \therefore k &= \bar{h}\end{aligned}$$

□

Definition 5.1.55. Let $u : A \rightarrow X$ be an injection. The *pointed set obtained from X by collapsing (A, u)* , denoted $X/(A, u)$, is the pushout

$$\begin{array}{ccc} A & \xrightarrow{\quad} & 1 \\ \downarrow u & & \downarrow * \\ X & \xrightarrow{\quad} & X/(A, u) \end{array}$$

Proposition 5.1.56. In \mathbf{Set}_* , any two morphisms $1 \rightarrow X$ and $1 \rightarrow Y$ have a pushout.

PROOF: The pushout of $a : (1, *) \rightarrow (X, x)$ and $b : (1, *) \rightarrow (Y, y)$ is $(X+Y/\sim, x)$ where \sim is the equivalence relation generated by $x \sim y$. □

Definition 5.1.57 (Wedge). The *wedge* of pointed sets X and Y , $X \vee Y$, is the pushout of the unique morphism $1 \rightarrow X$ and $1 \rightarrow Y$.

Definition 5.1.58 (Smash). Let X and Y be pointed sets. Let $\xi : X \vee Y \rightarrow X$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc} 1 & \xrightarrow{\quad} & X & & \\ \downarrow & & \downarrow & \searrow & \\ Y & \xrightarrow{\quad} & X \vee Y & \xrightarrow{\xi} & X \\ & \searrow 0 & & & \end{array}$$

Let $\eta : X \vee Y \rightarrow Y$ be the unique morphism such that the following diagram

commutes.



Let $\zeta = \langle \xi, \eta \rangle : X \vee Y \rightarrow X \times Y$. The *smash* of X and Y , $X \wedge Y$, is the result of collapsing $X \times Y$ with respect to ζ .

Pushout lemma

5.1.12 Subcategories

Definition 5.1.59 (Subcategory). A *subcategory* \mathcal{C}' of a category \mathcal{C} consists of:

- a subset $\text{Ob}(\mathcal{C}')$ of \mathcal{C}
- for all $A, B \in \text{Ob}(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \text{Ob}(\mathcal{C}')$, we have $\text{id}_A \in \mathcal{C}'[A, A]$
- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \text{Ob}(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

5.1.13 Opposite Category

Definition 5.1.60 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- $\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$
- Given $f \in \mathcal{C}^{\text{op}}[A, B]$ and $g \in \mathcal{C}^{\text{op}}[B, C]$, their composite in \mathcal{C}^{op} is $f \circ g$, where \circ is composition in \mathcal{C} .

Proposition 5.1.61. An object is *initial* in \mathcal{C} iff it is *terminal* in \mathcal{C}^{op} .

PROOF: Immediate from definitions. \square

Proposition 5.1.62. An object is *terminal* in \mathcal{C} iff it is *initial* in \mathcal{C}^{op} .

PROOF: Immediate from definitions. \square

Corollary 5.1.62.1. If T and T' are terminal objects in \mathcal{C} then there exists a unique isomorphism $T \cong T'$.

5.1.14 Groupoids

Definition 5.1.63 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

5.1.15 Concrete Categories

Definition 5.1.64 (Concrete Category). A *concrete category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*
- for any object $A \in \text{Ob}(\mathcal{C})$, a set $|A|$
- for any objects $A, B \in \text{Ob}(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$
- for any object A we have $\text{id}_{|A|} \in \mathcal{C}[A, A]$.

5.1.16 Power of Categories

Definition 5.1.65. Let \mathcal{C} be a category and J a set. The category \mathcal{C}^J is the category with:

- objects all J -indexed families of objects of \mathcal{C}
- morphisms $\{X_j\}_{j \in J} \rightarrow \{Y_j\}_{j \in J}$ all families $\{f_j\}_{j \in J}$ where $f_j : X_j \rightarrow Y_j$

5.1.17 Arrow Category

Definition 5.1.66 (Arrow Category). Let \mathcal{C} be a category. The *arrow category* \mathcal{C}^\rightarrow is the category with:

- objects all triples (A, B, f) where $f : A \rightarrow B$ in \mathcal{C}
- morphisms $(A, B, f) \rightarrow (C, D, g)$ all pairs $(u : A \rightarrow C, v : B \rightarrow D)$ such that $vf = gu$.

5.1.18 Slice Category

Definition 5.1.67 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category under A* , $\mathcal{C}_{\backslash A}$, is the category with:

- objects all pairs (B, f) where $B \in \mathcal{C}$ and $f : A \rightarrow B$
- morphisms $(B, f) \rightarrow (C, g)$ are morphisms $u : B \rightarrow C$ such that $uf = g$.

We identify this with the subcategory of \mathcal{C}^\rightarrow formed by mapping (B, f) to (A, B, f) and u to (id_A, u) .

Proposition 5.1.68. *If $s : (B, f) \rightarrow (C, g)$ in $\mathcal{C} \setminus A$, then any retraction of s in \mathcal{C} is a retraction of s in $\mathcal{C} \setminus A$.*

PROOF:

$\langle 1 \rangle 1$. LET: $r : C \rightarrow B$ be a retraction of s in \mathcal{C} .

$\langle 1 \rangle 2$. $rg = f$

PROOF: $rg = rsf = f$.

$\langle 1 \rangle 3$. $r : (C, g) \rightarrow (B, f)$ in $\mathcal{C} \setminus A$

$\langle 1 \rangle 4$. $rs = \text{id}_{(B, f)}$

PROOF: Because composition is inherited from \mathcal{C} .

□

Proposition 5.1.69. id_A is the initial object in $\mathcal{C} \setminus A$.

PROOF: For any $(B, f) \in \mathcal{C} \setminus A$, we have f is the only morphism $A \rightarrow B$ such that $f\text{id}_A = f$. □

Proposition 5.1.70. *If A is terminal in \mathcal{C} then id_A is the zero object in $\mathcal{C} \setminus A$.*

PROOF: For any $(B, f) \in \mathcal{C} \setminus A$, the unique morphism $! : B \rightarrow A$ is the unique morphism such that $!f = \text{id}_A$. □

Definition 5.1.71 (Pointed Sets). The *category of pointed sets* is $\mathbf{Set} \setminus 1$.

Definition 5.1.72. Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category over A* , \mathcal{C}/A , is the category with:

- objects all pairs (B, f) with $f : B \rightarrow A$
- morphisms $u : (B, f) \rightarrow (C, g)$ all morphisms $u : B \rightarrow C$ such that $gu = f$.

Proposition 5.1.73. *Let $u : (B, f) \rightarrow (C, g) : \mathcal{C}/A$. Any section of u in \mathcal{C} is a section of u in \mathcal{C}/A .*

PROOF: Dual to Proposition 5.1.68. □

Proposition 5.1.74. id_A is terminal in \mathcal{C}/A .

PROOF: Dual to Proposition 5.1.69. □

Proposition 5.1.75. *If A is initial in \mathcal{C} then id_A is the zero object in \mathcal{C}/A .*

PROOF: Dual to Proposition 5.1.70. □

Definition 5.1.76. Let $A \in \mathcal{C}$. The category of objects *over and under* A , written \mathcal{C}_A^A , is the category with:

- objects all triples (X, u, p) where $u : A \rightarrow X$, $p : X \rightarrow A$ and $pu = \text{id}_A$
- morphism $f : (X, u, p) \rightarrow (Y, v, q)$ all morphisms $f : X \rightarrow Y$ such that $fu = v$ and $qf = p$

Proposition 5.1.77. $(A, \text{id}_A, \text{id}_A)$ is the zero object in \mathcal{C}_A^A .

PROOF: For any object (X, u, p) , we have p is the unique morphism $(X, u, p) \rightarrow (A, \text{id}_A, \text{id}_A)$, and u is the unique morphism $(A, \text{id}_A, \text{id}_A) \rightarrow (X, u, p)$. \square

Definition 5.1.78 (Fibre Collapsing). Let B be a set. Let $u : (A, a) \rightarrow (X, x)$ in \mathbf{Set}/B . Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow u & & \downarrow j \\ X & \xrightarrow{i} & C \end{array}$$

Let $c : C \rightarrow B$ be the unique morphism such that $cj = \text{id}_B$ and $ci = x$. Then $(C, j, c) \in \mathbf{Set}_B^B$ is called the set over and under B obtained from X by *fibre collapsing* with respect to u . If (A, u) is a subset of X , we denote this set over and under B by $X/_B(A, u)$.

Definition 5.1.79 (Fibre Wedge). Let B be a small set. Let $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$. The *fibre wedge* of X and Y is the pushout of u_X and u_Y :

$$\begin{array}{ccc} B & \xrightarrow{u_X} & X \\ \downarrow u_Y & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \end{array}$$

Definition 5.1.80 (Fibre Smash). Let $X, Y \in \mathbf{Set}_B^B$. Let $\xi : X \vee_B Y \rightarrow X$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \xi & \\ & & X \end{array}$$

0

Let $\eta : X \vee_B Y \rightarrow Y$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \eta & \\ & & Y \end{array}$$

0

Let $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \rightarrow X \times Y$. The *fibre smash* of X and Y , $X \wedge_B Y$, is the result of collapsing $X \times Y$ with respect to ζ .

Proposition 5.1.81. *Set has products and coproducts.*

Proposition 5.1.82. *Let \mathcal{C} be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in \mathcal{C} and $Z \in \mathcal{C}$. Let $\coprod_{\alpha \in I} X_\alpha$ be the coproduct of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[\coprod_{\alpha \in I} X_\alpha, Z] \approx \prod_{\alpha \in I} \mathcal{C}[X_\alpha, Z] .$$

Proposition 5.1.83. *Let \mathcal{C} be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in \mathcal{C} and $Z \in \mathcal{C}$. Let $\prod_{\alpha \in I} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[Z, \prod_{\alpha \in I} X_\alpha] \approx \prod_{\alpha \in I} \mathcal{C}[Z, X_\alpha] .$$

Proposition 5.1.84. *A product in \mathcal{C} constitutes a product in \mathcal{C}/A .*

Proposition 5.1.85. *A coproduct in \mathcal{C} constitutes a product in \mathcal{C}/A .*

5.2 Functors

Definition 5.2.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- a function $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism $f : A \rightarrow B$ in \mathcal{C} , a morphism $Ff : FA \rightarrow FB$ in \mathcal{D}

such that:

- for all $A \in \text{Ob}(\mathcal{C})$ we have $F\text{id}_A = \text{id}_{FA}$
- for any morphism $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathcal{C} , we have $F(g \circ f) = Fg \circ Ff$

Proposition 5.2.2. *Functors preserve isomorphisms.*

PROOF:

$\langle 1 \rangle 1$. LET: $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

$\langle 1 \rangle 2$. LET: $f : A \cong B$ in \mathcal{C}

$\langle 1 \rangle 3$. $Ff^{-1} \circ Ff = \text{id}_{FA}$

PROOF:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\text{id}_A \\ &= \text{id}_{FA} \end{aligned}$$

$\langle 1 \rangle 4$. $Ff \circ Ff^{-1} = \text{id}_{FB}$

PROOF:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

Definition 5.2.3 (Identity Functor). For any category \mathcal{C} , the *identity* functor on \mathcal{C} is the functor $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$\begin{aligned} I_{\mathcal{C}}A &:= A & (A \in \mathcal{C}) \\ I_{\mathcal{C}}f &:= f & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

Proposition 5.2.4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $r : A \rightarrow B$ is a retraction of $s : B \rightarrow A$ in \mathcal{C} then Fr is a retraction of Fs .

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

Corollary 5.2.4.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $\phi : A \cong B$ is an isomorphism in \mathcal{C} then $F\phi : FA \cong FB$ is an isomorphism in \mathcal{D} with $(F\phi)^{-1} = F\phi^{-1}$.

Definition 5.2.5 (Composition of Functors). Given functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$, the *composite* functor $GF : \mathcal{C} \rightarrow \mathcal{E}$ is defined by

$$\begin{aligned} (GF)A &= G(FA) & (A \in \mathcal{C}) \\ (GF)f &= G(Ff) & (f : A \rightarrow B : \mathcal{C}) \end{aligned}$$

Definition 5.2.6 (Category of Categories). Let **Cat** be the category of small categories and functors.

Definition 5.2.7 (Isomorphism of Categories). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F is an *isomorphism of categories* iff there exists a functor $F^{-1} : \mathcal{D} \rightarrow \mathcal{C}$, the *inverse* of F , such that $FF^{-1} = I_{\mathcal{D}}$ and $F^{-1}F = I_{\mathcal{C}}$.

Categories \mathcal{C} and \mathcal{D} are *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, iff there exists an isomorphism between them.

Proposition 5.2.8. If A is initial in \mathcal{C} then $\mathcal{C} \setminus A \cong \mathcal{C}$.

PROOF:

⟨1⟩1. Define $F : \mathcal{C} \setminus A \rightarrow \mathcal{C}$ by

$$F(B, f) = B$$

$$F(u : (B, f) \rightarrow (C, g)) = u$$

⟨1⟩2. Define $G : \mathcal{C} \rightarrow \mathcal{C} \setminus A$ by

$$GB = (B, !_B)$$

where $!_B$ is the unique morphism $A \rightarrow B$

$$G(u : B \rightarrow C) = u : (B, !_B) \rightarrow (C, !_C)$$

⟨1⟩3. $FG = \text{id}_{\mathcal{C}}$

⟨1⟩4. $GF = \text{id}_{\mathcal{C} \setminus A}$

PROOF: Since $GF(B, f) = (B, !_B) = (B, f)$ because the morphism $A \rightarrow B$ is unique.

□

Proposition 5.2.9. *If A is terminal in \mathcal{C} then $\mathcal{C}/A \cong \mathcal{C}$.*

PROOF: Dual. \square

Proposition 5.2.10.

$$\mathcal{C}_A^A \cong (\mathcal{C}/A) \backslash (A, \text{id}_A) \cong (\mathcal{C} \backslash A) / (A, \text{id}_A)$$

PROOF:

- $\langle 1 \rangle 1$. Define a functor $F : \mathcal{C}_A^A \rightarrow (\mathcal{C}/A) \backslash (A, \text{id}_A)$.
 $\langle 2 \rangle 1$. Given $A \xrightarrow{u} X \xrightarrow{p} A$ in \mathcal{C}_A^A , let $F(X, u, p) = ((X, p), u)$
 $\langle 2 \rangle 2$. Given $f : (A \xrightarrow{u} X \xrightarrow{p} A) \rightarrow (A \xrightarrow{v} Y \xrightarrow{q} A)$, let $Ff = f$.
 $\langle 1 \rangle 2$. Define a functor $G : (\mathcal{C}/A) \backslash (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.
 $\langle 1 \rangle 3$. Define a functor $H : \mathcal{C}_A^A \rightarrow (\mathcal{C} \backslash A) / (A, \text{id}_A)$.
 $\langle 1 \rangle 4$. Define a functor $K : (\mathcal{C} \backslash A) / (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.
 \square

Definition 5.2.11 (Forgetful Functor). For any concrete category \mathcal{C} , define the *forgetful* functor $U : \mathcal{C} \rightarrow \mathbf{Set}$ by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

Definition 5.2.12 (Switching Functor). For any category \mathcal{C} , define the *switching* functor $T : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

Definition 5.2.13 (Reduction). Let $\Phi : \mathbf{Set} \rightarrow \mathbf{Set}$ be a functor. The *reduction* of Φ is the functor $\Phi^* : \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ defined by: $\Phi^*(X, a)$ is the collapse of $\Phi(X)$ with respect to $\Phi(a) : \Phi(1) \rightarrow \Phi(X)$.

Definition 5.2.14. Extend the wedge \vee to a functor $\mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ by defining, given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, then $f \vee g$ is the unique morphism that makes the following diagram commute.

$$\begin{array}{ccccc} 1 & \longrightarrow & X & & \\ \downarrow & & \downarrow & \searrow f & \\ Y & \longrightarrow & X \vee Y & & X' \\ & \searrow g & \searrow f \vee g & & \downarrow \\ & & Y' & \longrightarrow & X' \vee Y' \end{array}$$

Definition 5.2.15. Extend smash to a functor $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ as follows. Given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, let $f \wedge g : X \wedge Y \rightarrow X' \wedge Y'$ be the

unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc}
 X \vee Y & \xrightarrow{\quad} & 1 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 X \times Y & \xrightarrow{\quad} & X \wedge Y & & \\
 & \searrow & \downarrow & \searrow & \\
 & & X' \vee Y' & \xrightarrow{\quad} & 1 \\
 f \times g \swarrow & & \downarrow & \searrow & \\
 & & X' \times Y' & \xrightarrow{\quad} & X' \wedge Y'
 \end{array}$$

Definition 5.2.16 (Reduction). Let B be a small set. Let $\Phi_B : \mathbf{Set}/B \rightarrow \mathbf{Set}/B$ be a functor. The *reduction* of Φ_B is the functor $\Phi_B^B : \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ defined as follows.

For $(X, u : B \rightarrow X, p : X \rightarrow B) \in \mathbf{Set}_B^B$, let $\Phi_B^B(X)$ be the set over and under B obtained from $\Phi_B(X)$ by collapsing with respect to $\Phi_B(u) : \Phi_B(B) \rightarrow \Phi_B(X)$.

Definition 5.2.17. Extend \vee_B to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$.

Definition 5.2.18. Extend \wedge_B to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$.

Definition 5.2.19 (Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *faithful* iff, for any objects $A, B \in \mathcal{C}$ and morphisms $f, g : A \rightarrow B : \mathcal{C}$, if $Ff = Fg$ then $f = g$.

Definition 5.2.20 (Full). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *full* iff, for any objects $A, B \in \mathcal{C}$ and morphism $g : FA \rightarrow FB : \mathcal{D}$, there exists $f : A \rightarrow B : \mathcal{C}$ such that $Ff = g$.

Definition 5.2.21 (Fully Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *fully faithful* iff it is full and faithful.

Definition 5.2.22 (Full Embedding). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *full embedding* iff it is fully faithful and injective on objects.

5.3 Natural Transformations

Definition 5.3.1 (Natural Transformation). Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$. A *natural transformation* $\tau : F \Rightarrow G$ is a family of morphisms $\{\tau_X : FX \rightarrow GX\}_{X \in \mathcal{C}}$ such that, for every morphism $f : X \rightarrow Y : \mathcal{C}$, we have $Gf \circ \tau_X = \tau_Y \circ Ff$.

$$\begin{array}{ccc}
 FX & \xrightarrow{Ff} & FY \\
 \tau_X \downarrow & & \downarrow \tau_Y \\
 GX & \xrightarrow{Gf} & GY
 \end{array}$$

Definition 5.3.2 (Natural Isomorphism). A natural transformation $\tau : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$ is a *natural isomorphism*, $\tau : F \cong G$, iff for all $X \in \mathcal{C}$, τ_X is an isomorphism $FX \cong GX$.

Functors F and G are *naturally isomorphic*, $F \cong G$, iff there exists a natural isomorphism between them.

Definition 5.3.3 (Inverse). Let $\tau : F \cong G$. The *inverse* natural isomorphism $\tau^{-1} : G \cong F$ is defined by $(\tau^{-1})_X = \tau_X^{-1}$.

5.4 Bifunctors

Definition 5.4.1 (Commutative). A bifunctor $\square : \mathcal{C}^2 \rightarrow \mathcal{C}$ is *commutative* iff $\square \cong \square \circ T$, where $T : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ is the swap functor.

Proposition 5.4.2. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is commutative.

PROOF: Since the pushout of f and g is the pushout of g and f . \square

Proposition 5.4.3. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is commutative.

PROOF: In the diagram defining $X \wedge Y$, construct the isomorphism between the version with X and Y and the version with Y with X for every object. \square

Proposition 5.4.4. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is commutative.

Proposition 5.4.5. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is commutative.

Definition 5.4.6 (Associative). A bifunctor \square is *associative* iff $\square \circ (\square \times \text{id}) \cong \square \circ (\text{id} \times \square)$.

Proposition 5.4.7. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is associative.

PROOF: Since $X \vee (Y \vee Z)$ and $(X \vee Y) \vee Z$ are both the pushout of the unique morphisms $1 \rightarrow X$, $1 \rightarrow Y$ and $1 \rightarrow Z$. \square

Proposition 5.4.8. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is associative.

PROOF: Draw isomorphisms between the diagrams for $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$. \square

Product and coproduct are commutative and associative.

Proposition 5.4.9. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is associative.

Proposition 5.4.10. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is associative.

Proposition 5.4.11. Let \mathcal{C} be a category with binary coproducts. Let $\square : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ be a bifunctor. Then \square distributes over $+$ iff the canonical morphism

$$(X \square Z) + (Y \square Z) \rightarrow (X + Y) \square Z$$

is an isomorphism for all X, Y, Z .

Proposition 5.4.12. *In a category with binary products and binary coproducts, then \times distributes over $+$.*

Proposition 5.4.13. *In $\mathbf{Set}/*$, we have \times does not distribute over \vee .*

Proposition 5.4.14. *In $\mathbf{Set}/*$, we have \wedge distributes over \vee .*

Proposition 5.4.15. *In \mathbf{Set}/B , we have \times_B distributes over $+_B$.*

Proposition 5.4.16. *In \mathbf{Set}/B^B , we have \wedge_B distributes over \vee_B .*

5.5 Functor Categories

Definition 5.5.1 (Functor Category). Given categories \mathcal{C} and \mathcal{D} , define the *functor category* $\mathcal{C}^{\mathcal{D}}$ to be the category with objects the functors from \mathcal{D} to \mathcal{C} and morphisms the natural transformations.

Definition 5.5.2 (Yoneda Embedding). Let \mathcal{C} be a category. The *Yoneda embedding* $Y : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ is the functor that maps an object A to $\mathcal{C}[-, A]$ and morphisms similarly.

Theorem 5.5.3 (Yoneda Lemma). *Let \mathcal{C} be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\text{op}}}[\mathcal{C}[-, X], F] \cong FX$$

that maps $\tau : \mathcal{C}[-, X] \Rightarrow F$ to $\tau_X(\text{id}_X)$.

PROOF:

$\langle 1 \rangle 1$. ϕ is natural in X .

PROOF:

$\langle 2 \rangle 1$. LET: $f : X \rightarrow Y : \mathcal{C}$

$\langle 2 \rangle 2$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$. $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$

PROOF:

$$\begin{aligned} \phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\text{id}_Y \circ f) \\ &= \tau_Y(f) \\ &= \tau_Y(f \circ \text{id}_X) \\ &= Ff(\tau_X(\text{id}_X)) && (\tau \text{ natural}) \\ &= Ff(\phi(\tau)) \end{aligned}$$

$\langle 1 \rangle 2$. ϕ is natural in F .

$\langle 2 \rangle 1$. LET: $\alpha : F \Rightarrow G : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$

$\langle 2 \rangle 2$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$. $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$

PROOF: $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\text{id}_X)) = \alpha_X(\phi(\tau))$

$\langle 1 \rangle 3$. Each ϕ_{XF} is injective.

$\langle 2 \rangle 1$. LET: $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 2$. ASSUME: $\phi(\sigma) = \phi(\tau)$

$\langle 2 \rangle 3$. LET: $f : Y \rightarrow X$

$\langle 2 \rangle 4$. $\sigma_Y(f) = \tau_Y(f)$

PROOF:

$$\begin{aligned}
 \sigma_Y(f) &= \sigma_Y(\text{id}_X \circ f) \\
 &= Ff(\sigma_X(\text{id}_X)) && (\sigma \text{ is natural}) \\
 &= Ff(\tau_X(\text{id}_X)) && (\langle 2 \rangle 2) \\
 &= \tau_Y(\text{id}_X \circ f) && (\tau \text{ is natural}) \\
 &= \tau_Y(f)
 \end{aligned}$$

$\langle 1 \rangle 4$. Each ϕ_{XF} is surjective.

$\langle 2 \rangle 1$. LET: $X \in \mathcal{C}$ and $F : \mathcal{C} \rightarrow \mathcal{D}$

$\langle 2 \rangle 2$. LET: $a \in FX$

$\langle 2 \rangle 3$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$ be given by $\tau_Y(g) = Fg(a)$ for $g : Y \rightarrow X$

$\langle 2 \rangle 4$. τ is natural.

$\langle 3 \rangle 1$. LET: $h : Y \rightarrow Z : \mathcal{C}$

PROVE: $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \text{id}_X]$

$\langle 3 \rangle 2$. LET: $g : Z \rightarrow X$

$\langle 3 \rangle 3$. $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$

PROOF:

$$\begin{aligned}
 \tau_Y(g \circ h) &= F(g \circ h)(a) \\
 &= Fh(Fg(a)) \\
 &= Fh(\tau_Z(g))
 \end{aligned}$$

$\langle 2 \rangle 5$. $\phi(\tau) = a$

PROOF:

$$\begin{aligned}
 \phi_X(\tau) &= \tau_X(\text{id}_X) \\
 &= F\text{id}_X(a) \\
 &= a
 \end{aligned}$$

□

Corollary 5.5.3.1. *The Yoneda embedding is fully faithful.*

Corollary 5.5.3.2. *Given objects A and B in \mathcal{C} , we have $A \cong B$ if and only if $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$.*

Part III

Number Systems

Chapter 6

The Real Numbers

Theorem 6.0.1. *The following hold in the real numbers:*

1. $x + (y + z) = (x + y) + z$
2. $x(yz) = (xy)z$
3. $x + y = y + x$
4. $xy = yx$
5. $x + 0 = x$
6. $x1 = x$
7. $x + (-x) = 0$
8. If $x \neq 0$ then $x \cdot (1/x) = 1$
9. $x(y + z) = xy + xz$
10. If $x > y$ then $x + z > y + z$.
11. If $x > y$ and $z > 0$ then $xz > yz$.
12. \mathbb{R} has the least upper bound property.
13. If $x < y$ then there exists z such that $x < z < y$.

Definition 6.0.2. Given real numbers x and y with $y \neq 0$, we write x/y for xy^{-1} .

Theorem 6.0.3. *For any real numbers x and y , if $x + y = x$ then $y = 0$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 2.$ ASSUME: $x + y = x$

$\langle 1 \rangle 3.$ $y = 0$

PROOF:

$$\begin{aligned}
y &= y + 0 && \text{(Definition of zero)} \\
&= y + (x + (-x)) && \text{(Definition of } -x) \\
&= (y + x) + (-x) && \text{(Associativity of Addition)} \\
&= (x + y) + (-x) && \text{(Commutativity of Addition)} \\
&= x + (-x) && (\langle 1 \rangle 2) \\
&= 0 && \text{(Definition of } -x)
\end{aligned}$$

□

Theorem 6.0.4.

$$\forall x \in \mathbb{R}. 0x = 0$$

PROOF:

 $\langle 1 \rangle 1$. LET: $x \in \mathbb{R}$ $\langle 1 \rangle 2$. $xx + 0x = xx$

PROOF:

$$\begin{aligned}
xx + 0x &= (x + 0)x && \text{(Distributive Law)} \\
&= xx && \text{(Definition of 0)}
\end{aligned}$$

 $\langle 1 \rangle 3$. $0x = 0$ PROOF: Theorem 6.0.3, $\langle 1 \rangle 2$.

□

Theorem 6.0.5.

$$-0 = 0$$

PROOF: Since $0 + 0 = 0$. □**Theorem 6.0.6.**

$$\forall x \in \mathbb{R}. -(-x) = x$$

PROOF: Since $-x + x = 0$. □**Theorem 6.0.7.**

$$\forall x, y \in \mathbb{R}. x(-y) = -(xy)$$

PROOF:

$$\begin{aligned}
x(-y) + xy &= x((-y) + y) && \text{(Distributive Law)} \\
&= x0 && \text{(Definition of } -y) \\
&= 0 && \text{(Theorem 6.0.4) } \square
\end{aligned}$$

Theorem 6.0.8.

$$\forall x \in \mathbb{R}. (-1)x = -x$$

PROOF:

$$\begin{aligned}
(-1)x &= -(1 \cdot x) && \text{(Theorem 6.0.7)} \\
&= -x && \text{(Definition of 1) } \square
\end{aligned}$$

Proposition 6.0.9. *Let X be a linearly ordered set. Let $a, b, c \in X$ with $a < b < c$. Then $[a, c] \cong [0, 1]$ if and only if $[a, b] \cong [0, 1]$ and $[b, c] \cong [0, 1]$.*

PROOF:

$\langle 1 \rangle 1$. For all $x \in (0, 1)$ we have $[0, x] \cong [0, 1]$.

PROOF: The function that maps t to t/x is an order isomorphism.

$\langle 1 \rangle 2$. For all $x \in (0, 1)$ we have $[x, 1] \cong [0, 1]$.

PROOF: The function that maps t to $(t - x)/(1 - x)$ is an order isomorphism.

$\langle 1 \rangle 3$. We have $[0, 2] \cong [0, 1]$.

PROOF: The function that maps t to $t/2$ is an order isomorphism.

□

Proposition 6.0.10. *Let X be a linearly ordered set. Let (a_n) be a strictly increasing sequence in X . Let b be its supremum. Then $[a_0, b] \cong [0, 1]$ if and only if, for all n , we have $[a_n, a_{n+1}] \cong [0, 1]$.*

PROOF:

$\langle 1 \rangle 1$. For all $x, y \in [0, 1]$ with $x < y$ we have $[x, y] \cong [0, 1]$.

PROOF: The function that maps t to $(t - x)/(y - x)$ is an order isomorphism.

$\langle 1 \rangle 2$. We have $[0, 1] \cong [0, +\infty)$.

PROOF: The function that maps t to $1/(1 - t) - 1$ is an order isomorphism.

□

6.1 Subtraction

Definition 6.1.1 (Subtraction). We write $x - y$ for $x + (-y)$.

Theorem 6.1.2.

$$\forall x, y, z \in \mathbb{R}. x(y - z) = xy - xz$$

PROOF:

$$\begin{aligned} x(y - z) &= x(y + (-z)) && \text{(Definition of subtraction)} \\ &= xy + x(-z) && \text{(Distributive Law)} \\ &= xy + (-(xz)) && \text{(Theorem 6.0.7)} \\ &= xy - xz && \text{(Definition of subtraction)} \quad \square \end{aligned}$$

Theorem 6.1.3.

$$\forall x, y \in \mathbb{R}. -(x + y) = -x - y$$

PROOF:

$$\begin{aligned} -(x + y) &= (-1)(x + y) && \text{(Theorem 6.0.8)} \\ &= (-1)x + (-1)y && \text{(Distributive Law)} \\ &= -x + (-y) && \text{(Theorem 6.0.8)} \\ &= -x - y && \text{(Definition of subtraction)} \quad \square \end{aligned}$$

Theorem 6.1.4.

$$\forall x, y \in \mathbb{R}. -(x - y) = -x + y$$

PROOF:

$$\begin{aligned}
 -(x - y) &= -(x + (-y)) && \text{(Definition of subtraction)} \\
 &= -x - (-y) && \text{(Theorem 6.1.3)} \\
 &= -x + (-(-y)) && \text{(Definition of subtraction)} \\
 &= -x + y && \text{(Theorem 6.0.6)} \quad \square
 \end{aligned}$$

Definition 6.1.5 (Reciprocal). Given $x \in \mathbb{R}$ with $x \neq 0$, the *reciprocal* of x , $1/x$, is the unique real number such that $x \cdot 1/x = 1$.

Theorem 6.1.6. For any real numbers x and y , if $x \neq 0$ and $xy = x$ then $y = 1$.

PROOF:

- $\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$
 $\langle 1 \rangle 2$. ASSUME: $x \neq 0$
 $\langle 1 \rangle 3$. ASSUME: $xy = x$
 $\langle 1 \rangle 4$. $y = 1$

PROOF:

$$\begin{aligned}
 y &= y1 && \text{(Definition of 1)} \\
 &= y(x \cdot 1/x) && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\
 &= (yx)1/x && \text{(Associativity of Multiplication)} \\
 &= (xy)1/x && \text{(Commutativity of Multiplication)} \\
 &= x \cdot 1/x && (\langle 1 \rangle 3) \\
 &= 1 && \text{(Definition of } 1/x, \langle 1 \rangle 2)
 \end{aligned}$$

\square

Definition 6.1.7 (Quotient). Given real numbers x and y with $y \neq 0$, the *quotient* x/y is defined by

$$x/y = x \cdot 1/y .$$

Theorem 6.1.8. For any real number x , if $x \neq 0$ then $x/x = 1$.

PROOF: Immediate from definitions. \square

Theorem 6.1.9.

$$\forall x \in \mathbb{R}. x/1 = x$$

PROOF:

- $\langle 1 \rangle 1$. LET: $x \in \mathbb{R}$
 $\langle 1 \rangle 2$. $1/1 = 1$
 PROOF: Since $1 \cdot 1 = 1$.
 $\langle 1 \rangle 3$. $x/1 = x$
 PROOF: Since $x/1 = x \cdot 1/1 = x \cdot 1 = x$.

\square

Theorem 6.1.10. For any real numbers x and y , if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 2$. ASSUME: $xy = 0$ and $x \neq 0$

PROVE: $y = 0$

$\langle 1 \rangle 3$. $y = 0$

PROOF:

$$\begin{aligned} y &= 1y && \text{(Definition of 1)} \\ &= (1/x)xy && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\ &= (1/x)0 && (\langle 1 \rangle 2) \\ &= 0 && \text{(Theorem 6.0.4)} \end{aligned}$$

□

Theorem 6.1.11. For any real numbers y and z , if $y \neq 0$ and $z \neq 0$ then $(1/y)(1/z) = 1/(yz)$.

PROOF: Since $yz(1/y)(1/z) = 1 \cdot 1 = 1$. □

Corollary 6.1.11.1. For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have $(x/y)(z/w) = (xz)/(yw)$.

Theorem 6.1.12. For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + yz}{yw}$$

PROOF:

$$\begin{aligned} yw \left(\frac{x}{y} + \frac{z}{w} \right) &= yw \frac{x}{y} + yw \frac{z}{w} \\ &= wx + yz \end{aligned} \quad \square$$

Theorem 6.1.13. For any real number x , if $x \neq 0$ then $1/x \neq 0$.

PROOF: Since $x \cdot 1/x = 1 \neq 0$. □

Theorem 6.1.14. For any real numbers w, z , if $w \neq 0 \neq z$ then $1/(w/z) = z/w$.

PROOF: Since $(z/w)(w/z) = (wz)/(wz) = 1$. □

Theorem 6.1.15. For any real numbers a, x and y , if $y \neq 0$ then $(ax)/y = a(x/y)$

PROOF: Since $ya(x/y) = ax$. □

Theorem 6.1.16. For any real numbers x and y , if $y \neq 0$ then $(-x)/y = x/(-y) = -(x/y)$.

PROOF:

$\langle 1 \rangle 1$. $(-x)/y = -(x/y)$

PROOF: Take $a = -1$ in Theorem 6.1.15.

$\langle 1 \rangle 2$. $x/(-y) = -(x/y)$

PROOF: Since $(-y)(-(x/y)) = y(x/y) = x$.
 \square

Theorem 6.1.17. *For any real numbers x, y, z and w , if $x > y$ and $w > z$ then $x + w > y + z$.*

PROOF: We have $y + z < x + z < x + w$ by Monotonicity of Addition twice. \square

Corollary 6.1.17.1. *For any real numbers x and y , if $x > 0$ and $y > 0$ then $x + y > 0$.*

Theorem 6.1.18. *For any real numbers x and y , if $x > 0$ and $y > 0$ then $xy > 0$.*

PROOF:

$$\begin{aligned} xy &> 0y && \text{(Monotonicity of Multiplication)} \\ &= 0 && \text{(Theorem 6.0.4)} \quad \square \end{aligned}$$

Theorem 6.1.19. *For any real number x , we have $x > 0$ iff $-x < 0$.*

PROOF:

$\langle 1 \rangle 1$. If $0 < x$ then $-x < 0$

PROOF: By Monotonicity of Addition adding $-x$ to both sides.

$\langle 1 \rangle 2$. If $-x < 0$ then $0 < x$

PROOF: By Monotonicity of Addition adding x to both sides.

\square

Theorem 6.1.20. *For any real numbers x and y , we have $x > y$ iff $-x < -y$.*

PROOF:

$\langle 1 \rangle 1$. If $y < x$ then $-x < -y$.

PROOF: By Monotonicity of Addition adding $-x - y$ to both sides.

$\langle 1 \rangle 2$. If $-x < -y$ then $y < x$.

PROOF: By Monotonicity of Addition adding $x + y$ to both sides.

\square

Theorem 6.1.21. *For any real numbers x, y and z , if $x > y$ and $z < 0$ then $xz < yz$.*

PROOF:

$\langle 1 \rangle 1$. LET: x, y and z be real numbers.

$\langle 1 \rangle 2$. ASSUME: $x > y$

$\langle 1 \rangle 3$. ASSUME: $z < 0$

$\langle 1 \rangle 4$. $-z > 0$

PROOF: Theorem 6.1.19, $\langle 1 \rangle 3$.

$\langle 1 \rangle 5$. $x(-z) > y(-z)$

PROOF: $\langle 1 \rangle 2$, $\langle 1 \rangle 4$, Monotonicity of Multiplication.

$\langle 1 \rangle 6$. $-(xz) > -(yz)$

PROOF: Theorem 6.0.7, $\langle 1 \rangle 5$.

$\langle 1 \rangle 7. \quad xz < yz$

PROOF: Theorem 6.1.19, $\langle 1 \rangle 6$.

□

Theorem 6.1.22. *For any real number x , if $x \neq 0$ then $xx > 0$.*

PROOF:

$\langle 1 \rangle 1$. If $x > 0$ then $xx > 0$

PROOF: By Monotonicity of Multiplication.

$\langle 1 \rangle 2$. If $x < 0$ then $xx > 0$

PROOF: Theorem 6.1.21.

□

Theorem 6.1.23.

$$0 < 1$$

PROOF: By Theorem 6.1.22 since $1 = 1 \cdot 1$. □

Definition 6.1.24 (Positive). A real number x is *positive* iff $x > 0$.

We write \mathbb{R}_+ for the set of positive reals.

Theorem 6.1.25. *For any real numbers x and y , we have xy is positive if and only if x and y are both positive or both negative.*

PROOF: By the Monotonicity of Multiplication and Theorem 6.1.21. □

Corollary 6.1.25.1. *For any real number x , if $x > 0$ then $1/x > 0$.*

PROOF: Since $x \cdot 1/x = 1$ is positive. □

Theorem 6.1.26. *For any real numbers x and y , if $x > y > 0$ then $1/x < 1/y$.*

PROOF: If $1/y \leq 1/x$ then $1 < 1$ by Monotonicity of Multiplication. □

Theorem 6.1.27. *For any real numbers x and y , if $x < y$ then $x < (x+y)/2 < y$.*

PROOF: We have $2x < x+y$ and $x+y < 2y$ by Monotonicity of Addition, hence $x < (x+y)/2 < y$ by Monotonicity of Multiplication since $1/2 > 0$. □

Corollary 6.1.27.1. \mathbb{R} is a linear continuum.

Definition 6.1.28 (Negative). A real number x is *negative* iff $x < 0$.

We write $\overline{\mathbb{R}_+}$ for the set of nonnegative reals.

Theorem 6.1.29. *For every positive real number a , there exists a unique positive real \sqrt{a} such that $\sqrt{a}^2 = a$.*

PROOF:

$\langle 1 \rangle 1$. LET: a be a positive real.

$\langle 1 \rangle 2$. For any real numbers x and h , if $0 \leq h < 1$, then

$$(x+h)^2 < x^2 + h(2x+1) .$$

⟨2⟩1. LET: x and h be real numbers.

⟨2⟩2. ASSUME: $0 \leq h < 1$

⟨2⟩3. $(x + h)^2 < x^2 + h(2x + 1)$

PROOF:

$$\begin{aligned} (x + h)^2 &= x^2 + 2hx + h^2 \\ &< x^2 + 2hx + h & (\langle 2 \rangle 2) \\ &= x^2 + h(2x + 1) \end{aligned}$$

⟨1⟩3. For any real numbers x and h , if $h > 0$ then

$$(x - h)^2 > x^2 - 2hx .$$

⟨2⟩1. LET: x and h be real numbers.

⟨2⟩2. ASSUME: $h > 0$

⟨2⟩3. $(x - h)^2 > x^2 - 2hx$

PROOF:

$$\begin{aligned} (x - h)^2 &= x^2 - 2hx + h^2 \\ &> x^2 - 2hx & (\langle 2 \rangle 2) \end{aligned}$$

⟨1⟩4. For any positive real x , if $x^2 < a$ then there exists $h > 0$ such that
 $(x + h)^2 < a$.

⟨2⟩1. LET: x be a positive real.

⟨2⟩2. ASSUME: $x^2 < a$

⟨2⟩3. LET: $h = \min((a - x^2)/(2x + 1), 1/2)$

⟨2⟩4. $0 < h < 1$

⟨2⟩5. $(x + h)^2 < a$

PROOF:

$$\begin{aligned} (x + h)^2 &< x^2 + h(2x + 1) & (\langle 1 \rangle 2) \\ &\leq a \end{aligned}$$

⟨1⟩5. For any positive real x , if $x^2 > a$ then there exists $h > 0$ such that
 $(x - h)^2 > a$.

⟨2⟩1. LET: x be a positive real.

⟨2⟩2. ASSUME: $x^2 > a$

⟨2⟩3. LET: $h = (x^2 - a)/2x$

⟨2⟩4. $h > 0$

⟨2⟩5. $(x - h)^2 > a$

PROOF:

$$\begin{aligned} (x - h)^2 &> x^2 - 2hx \\ &= a & (\langle 2 \rangle 3) \end{aligned}$$

⟨1⟩6. LET: $B = \{x \in \mathbb{R} : x^2 < a\}$

⟨1⟩7. B is bounded above.

PROOF: If $a \geq 1$ then a is an upper bound. If $a < 1$ then 1 is an upper bound.

⟨1⟩8. B contains at least one positive real.

PROOF: If $a \geq 1$ then $1 \in B$. If $a < 1$ then $a \in B$.

⟨1⟩9. LET: $b = \sup B$

⟨1⟩10. $b^2 = a$

⟨2⟩1. $b^2 \geq a$

⟨3⟩1. ASSUME: for a contradiction $b^2 < a$

- $\langle 3 \rangle 2$. PICK $h > 0$ such that $(b + h)^2 < a$
 PROOF: $\langle 1 \rangle 4$
 $\langle 3 \rangle 3$. $b + h \in B$
 $\langle 3 \rangle 4$. Q.E.D.
 PROOF: This contradicts $\langle 1 \rangle 9$.
 $\langle 2 \rangle 2$. $b^2 \leq a$
 $\langle 3 \rangle 1$. ASSUME: for a contradiction $b^2 > a$
 $\langle 3 \rangle 2$. PICK $h > 0$ such that $(b - h)^2 > a$
 PROOF: $\langle 1 \rangle 5$
 $\langle 3 \rangle 3$. PICK $x \in B$ such that $b - h < x$
 PROOF: $\langle 1 \rangle 9$
 $\langle 3 \rangle 4$. $(b - h)^2 < x^2 < a$
 $\langle 3 \rangle 5$. Q.E.D.
 PROOF: This contradicts $\langle 3 \rangle 2$
 $\langle 1 \rangle 11$. For any positive reals b and c , if $b^2 = c^2$ then $b = c$.
 $\langle 2 \rangle 1$. LET: b and c be positive reals.
 $\langle 2 \rangle 2$. ASSUME: $b^2 = c^2$
 $\langle 2 \rangle 3$. $b^2 - c^2 = 0$
 $\langle 2 \rangle 4$. $(b - c)(b + c) = 0$
 $\langle 2 \rangle 5$. $b - c = 0$ or $b + c = 0$
 $\langle 2 \rangle 6$. $b + c \neq 0$
 PROOF: Since $b + c > 0$
 $\langle 2 \rangle 7$. $b - c = 0$
 $\langle 2 \rangle 8$. $b = c$
 \square

Theorem 6.1.30. *The set of real numbers is uncountable.*

Definition 6.1.31. We write \mathbb{R}^ω for the set of sequences in \mathbb{R}^ω that are eventually zero.

Definition 6.1.32 (Hilbert Cube). The *Hilbert cube* is $\prod_{n=0}^\infty [0, 1/(n+1)]$.

6.2 The Ordered Square

Definition 6.2.1 (Ordered Square). The *ordered square* I_o^2 is the set $[0, 1]^2$ under the dictionary order.

Proposition 6.2.2. *The ordered square is a linear continuum.*

PROOF:

- $\langle 1 \rangle 1$. I_o^2 has the least upper bound property.
 $\langle 2 \rangle 1$. LET: S be a nonempty subset of I_o^2 .
 $\langle 2 \rangle 2$. LET: a be the supremum of $\pi_1(S)$
 $\langle 2 \rangle 3$. CASE: $a \in \pi_1(S)$
 $\langle 3 \rangle 1$. LET: b be the supremum of $\{y \in [0, 1] : (a, y) \in S\}$
 $\langle 3 \rangle 2$. (a, b) is the supremum of S .

$\langle 2 \rangle 4$. CASE: $a \notin \pi_1(S)$

PROOF: $(a, 0)$ is the supremum of S .

$\langle 1 \rangle 2$. I_o^2 is dense.

$\langle 2 \rangle 1$. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$

PROVE: There exists $(x_3, y_3) \in I_o^2$ such that $(x_1, y_1) < (x_3, y_3) < (x_2, y_2)$

$\langle 2 \rangle 2$. CASE: $x_1 < x_2$

$\langle 3 \rangle 1$. PICK x_3 such that $x_1 < x_3 < x_2$

$\langle 3 \rangle 2$. $(x_1, y_1) < (x_3, 0) < (x_2, y_2)$

$\langle 2 \rangle 3$. CASE: $x_1 = x_2$ and $y_1 < y_2$

$\langle 3 \rangle 1$. PICK y_3 such that $y_1 < y_3 < y_2$

$\langle 3 \rangle 2$. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

6.3 Punctured Euclidean Space

Definition 6.3.1 (Punctured Euclidean Space). Let n be a positive integer. The *punctured Euclidean space* is $\mathbb{R}^n - \{\vec{0}\}$.

6.4 Topologist's Sine Curve

Definition 6.4.1 (Topologist's Sine Curve). The *topologist's sine curve* is

$$(\{0\} \times [-1, 1]) \cup \{(x, \sin 1/x) : 0 < x \leq 1\}.$$

6.5 The Long Line

Definition 6.5.1 (Long Line). The *long line* is $S_\Omega \times [0, 1]$ in the dictionary order.

Proposition 6.5.2. For any $a \in S_\Omega$ with $a \neq 0$ we have $[(0, 0), (a, 0)) \cong [0, 1]$.

PROOF: By transfinite induction on a using Propositions 6.0.9 and 6.0.10. □

Chapter 7

Integers and Rationals

7.1 Positive Integers

Definition 7.1.1 (Inductive). A set of real numbers A is *inductive* iff $1 \in A$ and $\forall x \in A. x + 1 \in A$.

Definition 7.1.2 (Positive Integer). The set \mathbb{Z}_+ of *positive integers* is the intersection of the set of inductive sets.

Proposition 7.1.3. *Every positive integer is positive.*

PROOF: The set of positive reals is inductive. \square

Proposition 7.1.4. *1 is the least element of \mathbb{Z}_+ .*

PROOF: Since $\{x \in \mathbb{R} : x \geq 1\}$ is inductive. \square

Proposition 7.1.5. *\mathbb{Z}_+ is inductive.*

PROOF: 1 is an element of every inductive set, and for all $x \in \mathbb{R}$, if x is an element of every inductive set then so is $x + 1$. \square

Theorem 7.1.6 (Principle of Induction). *If A is an inductive set of positive integers then $A = \mathbb{Z}_+$.*

PROOF: Immediate from definitions. \square

Theorem 7.1.7 (Well-Ordering Property). *\mathbb{Z}_+ is well ordered.*

PROOF: Construct the obvious order isomorphism $\omega \cong \mathbb{Z}_+$. \square

Theorem 7.1.8 (Archimedean Ordering Property). *The set \mathbb{Z}_+ is unbounded above.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction \mathbb{Z}_+ is bounded above.

⟨1⟩2. LET:

$$s = \sup \mathbb{Z}_+$$

⟨1⟩3. PICK $n \in \mathbb{Z}_+$ such that $s - 1 < n$

⟨1⟩4. $s < n + 1$

⟨1⟩5. Q.E.D.

PROOF: ⟨1⟩2 and ⟨1⟩4 form a contradiction.

□

7.1.1 Exponentiation

Definition 7.1.9. For a a real number and n a positive integer, define the real number a^n recursively as follows:

$$\begin{aligned} a^1 &= a \\ a^{n+1} &= a^n a \end{aligned}$$

Theorem 7.1.10. For all $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}_+$, we have

$$a^n a^m = a^{n+m}$$

PROOF:

⟨1⟩1. LET: $P(m)$ be the property $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. a^n a^m = a^{n+m}$

⟨1⟩2. $P(1)$

PROOF: $a^n a^1 = a^n a = a^{n+1}$.

⟨1⟩3. $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$

⟨2⟩1. LET: m be a positive integer.

⟨2⟩2. ASSUME: $P(m)$

⟨2⟩3. LET: $a \in \mathbb{R}$

⟨2⟩4. LET: $n \in \mathbb{Z}_+$

⟨2⟩5. $a^n a^{m+1} = a^{n+m+1}$

PROOF:

$$\begin{aligned} a^n a^{m+1} &= a^n a^m a \\ &= a^{n+m} a && (\langle 2 \rangle 2) \\ &= a^{n+m+1} \end{aligned}$$

⟨1⟩4. Q.E.D.

PROOF: By induction.

□

Theorem 7.1.11. For all $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}_+$,

$$(a^n)^m = a^{nm}.$$

PROOF:

⟨1⟩1. LET: $P(m)$ be the property $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. (a^n)^m = a^{nm}$.

⟨1⟩2. $P(1)$

PROOF: $(a^n)^1 = a^n = a^{n \cdot 1}$

⟨1⟩3. $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$

PROOF:

$$\begin{aligned} (a^n)^{m+1} &= (a^n)^m a^n \\ &= a^{nm} a^n \\ &= a^{nm+n} && \text{(Theorem 7.1.10)} \\ &= a^{n(m+1)} \end{aligned}$$

□

Theorem 7.1.12. *For any real numbers a and b and positive integer m ,*

$$a^m b^m = (ab)^m .$$

PROOF: Induction on m . □

7.2 Integers

Definition 7.2.1 (Integer). The set \mathbb{Z} of *integers* is

$$\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \{-x : x \in \mathbb{Z}_+\} .$$

Proposition 7.2.2. *The sum, difference and product of two integers is an integer.*

PROOF: Easy. □

Example 7.2.3. $1/2$ is not an integer.

Proposition 7.2.4. *For any integer n , there is no integer a such that $n < a < n+1$.*

PROOF:

⟨1⟩1. For any positive integer n , there is no integer a such that $n < a < n+1$.

⟨2⟩1. There is no integer a such that $1 < a < 2$.

⟨3⟩1. There is no positive integer a such that $1 < a < 2$.

⟨4⟩1. We do not have $1 < 1 < 2$.

⟨4⟩2. For any positive integer n , we do not have $1 < n+1 < 2$.

PROOF: Since $n \geq 1$ so $n+1 \geq 2$.

⟨3⟩2. We do not have $1 < 0 < 2$.

⟨3⟩3. For any positive integer a , we do not have $1 < -a < 2$.

PROOF: Since $-a < 0 < 1$.

⟨2⟩2. For any positive integer n , if there is no integer a such that $n < a < n+1$, then there is no integer a such that $n+1 < a < n+2$.

PROOF: If $n+1 < a < n+2$ then $n < a-1 < n+1$.

⟨1⟩2. There is no integer a such that $0 < a < 1$.

PROOF: If $0 < a < 1$ then $1 < a+1 < 2$.

⟨1⟩3. For any positive integer n , there is no integer a such that $-n < a < -n+1$.

PROOF: If $-n < a < -n+1$ then $n-1 < -a < n$.

□

Theorem 7.2.5. *Every nonempty subset of \mathbb{Z} bounded above has a largest element.*

PROOF:

⟨1⟩1. LET: S be a nonempty subset of \mathbb{Z} bounded above.

⟨1⟩2. LET: u be an upper bound for S .

⟨1⟩3. PICK an integer $n > u$

PROOF: Archimedean property.

⟨1⟩4. LET: k be the least positive integer such that $n - k \in S$.

⟨2⟩1. PICK $m \in S$

⟨2⟩2. $n - m$ is a positive integer.

⟨2⟩3. There exists a positive integer k such that $n - k \in S$.

⟨1⟩5. $n - k$ is the greatest element in S .

⟨2⟩1. LET: $m \in S$

⟨2⟩2. $n - m \geq k$

⟨2⟩3. $m \leq n - k$

□

Theorem 7.2.6. *For any real number x , if x is not an integer then there exists a unique integer n such that $n < x < n + 1$.*

PROOF:

⟨1⟩1. $\{n \in \mathbb{Z} : n < x\}$ is a nonempty set of integers bounded above.

⟨2⟩1. PICK $m > -x$

PROOF: Archimedean property.

⟨2⟩2. $-m < x$

⟨2⟩3. $\{n \in \mathbb{Z} : n < x\}$ is nonempty.

⟨1⟩2. LET: n be the greatest integer such that $n < x$

⟨1⟩3. $x < n + 1$

⟨1⟩4. If n' is an integer with $n' < x < n' + 1$ then $n' = n$.

PROOF: We have $n' < n + 1$ so $n' \leq n$, and $n < n' + 1$ so $n \leq n'$.

□

Definition 7.2.7 (Even). An integer n is *even* iff $n/2$ is an integer; otherwise, n is *odd*.

Theorem 7.2.8. *If the integer m is odd then there exists an integer n such that $m = 2n + 1$.*

PROOF:

⟨1⟩1. LET: n be the integer such that $n < m/2 < n + 1$

PROOF: Theorem 7.2.6.

⟨1⟩2. $2n < m < 2n + 2$

⟨1⟩3. $m = 2n + 1$

□

Theorem 7.2.9. *The product of two odd integers is odd.*

PROOF: $(2m + 1)(2n + 1) = 2(2mn + m + n) + 1$. \square

Corollary 7.2.9.1. *If p is an odd integer and n is a positive integer then p^n is an odd integer.*

Definition 7.2.10 (Exponentiation). Extend the definition of exponentiation so a^n is defined for:

- all real numbers a and non-negative integers n
- all non-zero real numbers a and integers n

as follows:

$$\begin{aligned} a^0 &= 1 \\ a^{-n} &= 1/a^n \end{aligned} \quad (n \text{ a positive integer})$$

Theorem 7.2.11 (Laws of Exponents). *For all non-zero reals a and b and integers m and n ,*

$$\begin{aligned} a^n a^m &= a^{n+m} \\ (a^n)^m &= a^{nm} \\ a^m b^m &= (ab)^m \end{aligned}$$

PROOF: Easy. \square

Theorem 7.2.12. \mathbb{Z} is countable.

PROOF: The function that maps an integer n to $2n$ if $n \geq 0$ and $-1 - 2n$ if $n < 0$ is a bijection $\mathbb{Z} \approx \mathbb{N}$. \square

7.3 Rational Numbers

Definition 7.3.1 (Rational Number). The set \mathbb{Q} of *rational numbers* is the set of all real numbers that are the quotient of two integers. A real that is not rational is *irrational*.

Theorem 7.3.2. $\sqrt{2}$ is irrational.

PROOF:

- $\langle 1 \rangle$ 1. For any positive rational a , there exist positive integers m and n not both even such that $a = m/n$.
- $\langle 2 \rangle$ 1. LET: a be a positive rational.
- $\langle 2 \rangle$ 2. LET: n be the least positive integer such that na is a positive integer.
- $\langle 2 \rangle$ 3. LET: $m = na$
- $\langle 2 \rangle$ 4. ASSUME: for a contradiction m and n are both even.
- $\langle 2 \rangle$ 5. $m/2 = (n/2)a$
- $\langle 2 \rangle$ 6. Q.E.D.

PROOF: This contradicts the leastness of n ($\langle 2 \rangle 2$).

$\langle 1 \rangle 2$. ASSUME: for a contradiction $\sqrt{2}$ is rational.

$\langle 1 \rangle 3$. PICK positive integers m and n not both even such that $\sqrt{2} = m/n$.

$\langle 1 \rangle 4$. $m^2 = 2n^2$

$\langle 1 \rangle 5$. m^2 is even.

$\langle 1 \rangle 6$. m is even.

PROOF: Theorem 7.2.9.

$\langle 1 \rangle 7$. LET: $k = m/2$

$\langle 1 \rangle 8$. $4k^2 = 2n^2$

$\langle 1 \rangle 9$. $n^2 = 2k^2$

$\langle 1 \rangle 10$. n^2 is even.

$\langle 1 \rangle 11$. n is even.

PROOF: Theorem 7.2.9.

$\langle 1 \rangle 12$. Q.E.D.

PROOF: $\langle 1 \rangle 3$, $\langle 1 \rangle 6$ and $\langle 1 \rangle 11$ form a contradiction.

□

Theorem 7.3.3. \mathbb{Q} is countably infinite.

PROOF: The function $\mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$ that maps (m, n) to $m/(n+1)$ is a surjection.

□

7.4 Algebraic Numbers

Definition 7.4.1 (Algebraic Number). A real number r is *algebraic* iff there exists a natural number n and rational numbers a_0, a_1, \dots, a_{n-1} such that

$$r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0$$

Otherwise, r is *transcendental*.

Proposition 7.4.2. The set of algebraic numbers is countably infinite.

PROOF: There are countably many finite sequences of rational numbers, and each corresponding polynomial has only finitely many roots. □

Corollary 7.4.2.1. The set of transcendental numbers is uncountable.

Part IV

Algebra

Chapter 8

Monoid Theory

Definition 8.0.1 (Monoid). A *monoid* is a category with one object.

Definition 8.0.2. Let \mathcal{C} be a category and $X \in \mathcal{C}$. The monoid $\text{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \rightarrow X$ under composition.

Proposition 8.0.3. *For any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and $X \in \mathcal{C}$, we have that $F : \text{End}_{\mathcal{C}}(X) \rightarrow \text{End}_{\mathcal{D}}(FX)$ is a monoid homomorphism.*

PROOF: Since $F\text{id}_X = \text{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$. \square

Chapter 9

Group Theory

9.1 Category of Small Groups

Definition 9.1.1. Let **Grp** be the category of small groups and group homomorphisms.

Definition 9.1.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G .

Proposition 9.1.3. *The trivial group is a zero object in **Grp**.*

PROOF: Easy. \square

The zero morphism $G \rightarrow H$ maps every element in G to e .

Definition 9.1.4. Let \mathcal{C} be a category and $X \in \mathcal{C}$. We write $\text{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Proposition 9.1.5. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $X \in \mathcal{C}$. Then $F : \text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(FX)$ is a group homomorphism.*

PROOF: Since $F \text{id}_X = \text{id}_{FX}$, $F(g \circ f) = Fg \circ Ff$, and $Ff^{-1} = (Ff)^{-1}$. \square

Proposition 9.1.6. **Grp** has products.

Definition 9.1.7 (Free Product). The product of a family of groups in **Grp** is called the *free product*.

Proposition 9.1.8. **Ab** has products given by direct sums.

Definition 9.1.9 (Left Coset). Let G be a group and H a subgroup of G . The *left cosets* of H are the sets of the form

$$xH := \{xh : h \in H\}$$

We write G/H for the set of left cosets of H in G .

Proposition 9.1.10. *Let G be a group and H a subgroup of G . Then G/H is a partition of G .*

PROOF:

$\langle 1 \rangle 1$. $\bigcup (G/H) = G$

PROOF: Since $x = xe$ and so $x \in xH$.

$\langle 1 \rangle 2$. Any two distinct left cosets of H are disjoint.

PROOF: Since if $z \in xH$ and $z \in yH$ then $xH = yH = zH$.

□

Definition 9.1.11. Let G be a group. Let A and B be subsets of G . Then

$$AB := \{ab : a \in A, b \in B\} .$$

Definition 9.1.12. Let G be a group. Let A be a subset of G . Then

$$A^{-1} := \{a^{-1} : a \in A\} .$$

Chapter 10

Ring Theory

Definition 10.0.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Definition 10.0.2 (Spectrum). Let R be a commutative ring. The *spectrum* of R , $\text{spec } R$, is the set of all prime ideals of R .

Definition 10.0.3 (Zariski Topology). Let R be a commutative ring. The *Zariski topology* on $\text{spec } R$ is the topology where the closed sets are the sets of the form

$$VE := \{p \in \text{spec } R : E \subseteq p\}$$

for any $E \in \mathcal{P}R$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$

$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{A} \in \mathcal{C}$

$\langle 2 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{C}$

$\langle 2 \rangle 2$. LET: $E = \bigcup \{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$

PROVE: $VE = \bigcap \mathcal{A}$

$\langle 2 \rangle 3$. For all $p \in \text{spec } R$, if $E \subseteq p$ then $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 1$. LET: $p \in \text{spec } R$

$\langle 3 \rangle 2$. ASSUME: $E \subseteq p$

$\langle 3 \rangle 3$. LET: $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$

$\langle 3 \rangle 4$. $E' \subseteq E$

$\langle 3 \rangle 5$. $E' \subseteq p$

$\langle 3 \rangle 6$. $p \in VE'$

$\langle 2 \rangle 4$. For all $p \in \text{spec } R$, if $p \in \bigcap \mathcal{A}$ then $E \subseteq p$

$\langle 3 \rangle 1$. LET: $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 2$. For all $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$ we have $E' \subseteq p$

$\langle 3 \rangle 3$. $E \subseteq p$

$\langle 1 \rangle 3$. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

PROOF: Since $VE \cup VE' = V(E \cap E')$

$\langle 1 \rangle 4. \emptyset \in \mathcal{C}$

$\langle 2 \rangle 1. VR = \emptyset$

PROOF: If $p \in VR$ then $R \subseteq p$ contradicting the fact that p is a prime ideal.

□

Definition 10.0.4. For any ring R , let $R - \mathbf{Mod}$ be the category of small R -modules and R -module homomorphisms.

Proposition 10.0.5. $R - \mathbf{Mod}$ has products and coproducts.

Chapter 11

Field Theory

Proposition 11.0.1. *Field does not have binary products.*

PROOF: There cannot be a field K with field homomorphisms $K \rightarrow \mathbb{Z}_2$ and $K \rightarrow \mathbb{Z}_3$, because its characteristic would be both 2 and 3. \square

Chapter 12

Linear Algebra

Definition 12.0.1 (Span). Let V be a vector space and $A \subseteq V$. The *span* of A is the set of all linear combinations of elements of A .

Definition 12.0.2 (Independent). Let V be a vector space and $A \subseteq V$. Then A is *linearly independent* iff, whenever

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

where $v_1, \dots, v_n \in A$, then

$$\alpha_1 = \cdots = \alpha_n = 0 \text{ .}$$

Proposition 12.0.3. *Let V be a vector space, $A \subseteq V$ and $v \in V$. If A is linearly independent and $v \notin \text{span } A$, then $A \cup \{v\}$ is independent.*

PROOF:

$\langle 1 \rangle 1$. LET: $\alpha_1 v_1 + \cdots + \alpha_n v_n + \beta v = 0$ where $v_1, \dots, v_n \in A$

$\langle 1 \rangle 2$. $\beta = 0$

PROOF: Otherwise $v = (\alpha_1/\beta)v_1 + \cdots + (\alpha_n/\beta)v_n \in \text{span } A$.

$\langle 1 \rangle 3$. $\alpha_1 = \cdots = \alpha_n = 0$

PROOF: Since A is linearly independent.

□

Theorem 12.0.4. *Every vector space has a basis.*

PROOF:

$\langle 1 \rangle 1$. LET: V be a vector space.

$\langle 1 \rangle 2$. PICK a maximal linearly independent set \mathcal{B} .

PROOF: By Tukey's Lemma.

$\langle 1 \rangle 3$. $\text{span } \mathcal{B} = V$

PROOF: Proposition 12.0.3.

□

Definition 12.0.5. For any field K , we write \mathbf{Vect}_K for $K - \mathbf{Mod}$.

Dual space functor $\mathbf{Vect}_K^{\text{op}} \rightarrow \mathbf{Vect}_K$.

Definition 12.0.6 (Invariant). Let $T : V \rightarrow V$ be a linear operator and S a subspace of V . Then S is *invariant* under T iff $T(S) \subseteq S$.

12.1 Inner Product Spaces

Definition 12.1.1 (Inner Product). Let V be a complex vector space. An *inner product* on V is a function $\langle \cdot | \cdot \rangle : V^2 \rightarrow \mathbb{C}$ such that, for all $|\phi\rangle, |\psi\rangle, |\psi_1\rangle, |\psi_2\rangle \in V$ and $c_1, c_2 \in \mathbb{C}$,

1. $\langle \phi | c_1\psi_1 + c_2\psi_2 \rangle = c_1\langle \phi | \psi_1 \rangle + c_2\langle \phi | \psi_2 \rangle$
2. $\langle \psi | \phi \rangle = \overline{\langle \phi | \psi \rangle}$
3. $\langle \psi | \psi \rangle \geq 0$
4. If $\langle \psi | \psi \rangle = 0$ then $|\psi\rangle = 0$.

An *inner product space* is a complex vector space with an inner product.

Example 12.1.2. The function $\langle \cdot | \cdot \rangle : (\mathbb{C}^n)^2 \rightarrow \mathbb{C}$ defined by

$$\langle (a_1, \dots, a_n) | (b_1, \dots, b_n) \rangle = \overline{a_1}b_1 + \dots + \overline{a_n}b_n$$

is an inner product on \mathbb{C}^n .

Definition 12.1.3 (Normalised). Let V be an inner product space and $|\psi\rangle \in V$. Then $|\psi\rangle$ is *normalised* iff $\langle \psi | \psi \rangle = 1$.

Definition 12.1.4 (Orthogonal). Let V be an inner product space and $|\phi\rangle, |\psi\rangle \in V$. Then $|\phi\rangle$ and $|\psi\rangle$ are *orthogonal* iff $\langle \phi | \psi \rangle = 0$.

Definition 12.1.5 (Orthonormal). Let V be an inner product space and $S \subseteq V$. Then S is *orthonormal* iff:

- Every element of S is normalised.
- Any two distinct elements of S are orthogonal.

Definition 12.1.6 (Hermitian). Let V be an inner product space. A linear operator $T : V \rightarrow V$ is *Hermitian* iff

$$\langle \psi | T | \phi \rangle = \overline{\langle \phi | T | \psi \rangle}$$

for all $|\phi\rangle, |\psi\rangle \in V$.

Proposition 12.1.7. *Every eigenvalue of a Hermitian operator is real.*

PROOF:

$\langle 1 \rangle 1$. LET: $T : V \rightarrow V$ be Hermitian.

$\langle 1 \rangle 2$. LET: $|\phi\rangle$ be an eigenvector with eigenvalue α .

$$\langle 1 \rangle 3. \langle \phi | T | \phi \rangle = \alpha \langle \phi | \phi \rangle$$

$$\langle 1 \rangle 4. \langle \phi | T | \phi \rangle = \bar{\alpha} \langle \phi | \phi \rangle$$

PROOF:

$$\begin{aligned} \langle \phi | T | \phi \rangle &= \overline{\langle \phi | T | \phi \rangle} \\ &= \bar{\alpha} \overline{\langle \phi | \phi \rangle} \\ &= \bar{\alpha} \langle \phi | \phi \rangle \end{aligned}$$

$$\langle 1 \rangle 5. \alpha = \bar{\alpha}$$

□

Definition 12.1.8 (Hermitian Conjugate). Let V be an inner product space and $T : V \rightarrow V$ a linear operator. The *Hermitian conjugate* $T^\dagger : V \rightarrow V$ is the linear operator such that, for all $|\phi\rangle, |\psi\rangle \in V$, we have

$$\langle \phi | T^\dagger | \psi \rangle = \overline{\langle \psi | T | \phi \rangle}.$$

Proposition 12.1.9.

$$(cT)^\dagger = \bar{c}T^\dagger$$

PROOF:

$$\begin{aligned} \langle \phi | \bar{c}T^\dagger | \psi \rangle &= \bar{c} \langle \phi | T^\dagger | \psi \rangle \\ &= \bar{c} \overline{\langle \psi | T | \phi \rangle} \\ &= \overline{\langle \psi | cT | \phi \rangle} \end{aligned} \quad \square$$

Proposition 12.1.10.

$$(ST)^\dagger = T^\dagger S^\dagger$$

PROOF:

$$\begin{aligned} \langle \phi | T^\dagger S^\dagger | \psi \rangle &= \overline{\langle S^\dagger | \psi \rangle \langle T | \phi \rangle} \\ &= \langle T | \phi \rangle \langle S^\dagger | \psi \rangle \\ &= \overline{\langle \psi | ST | \phi \rangle} \end{aligned} \quad \square$$

Proposition 12.1.11. T is Hermitian iff $T = T^\dagger$.

PROOF: Immediate from definitions. □

Proposition 12.1.12. If M is the matrix of T then \overline{M}^T is the matrix of T^\dagger .

Definition 12.1.13 (Unitary). Let V be an inner product space and $T : V \rightarrow V$ a linear operator. Then T is *unitary* iff $T^\dagger T = I$.

Proposition 12.1.14. If α is an eigenvalue of a unitary operator then $|\alpha| = 1$.

PROOF:

$\langle 1 \rangle 1.$ LET: $T : V \rightarrow V$ be unitary.

$\langle 1 \rangle 2.$ LET: $|\phi\rangle$ be an eigenstate with eigenvalue α .

$$\langle 1 \rangle 3. |\alpha|^2 \langle \phi | \phi \rangle = \langle \phi | \phi \rangle$$

PROOF:

$$\begin{aligned}
 \langle \phi | \phi \rangle &= \langle \phi | T^\dagger T | \phi \rangle \\
 &= \alpha \langle \phi | T^\dagger | \phi \rangle \\
 &= \alpha \overline{\langle \phi | T | \phi \rangle} \\
 &= \alpha \overline{\alpha \langle \phi | \phi \rangle} \\
 &= |\alpha|^2 \langle \phi | \phi \rangle
 \end{aligned}$$

$\langle 1 | 4 \rangle$. $|\alpha| = 1$

□

Definition 12.1.15 (Bra Vector). Let V be an inner product space and $|\phi\rangle \in V$. The *bra vector* or *Hermitian conjugate* $\langle \phi |$ is the linear functional $\langle \phi | : V \rightarrow \mathbb{C}$ that maps $|\psi\rangle$ to $\langle \phi | \psi \rangle$.

Proposition 12.1.16. *The hermitian conjugate of $c|\phi\rangle$ is $\bar{c}\langle \phi |$.*

PROOF: Since $\langle c|\phi\rangle|\psi\rangle = \bar{c}\langle \phi|\psi\rangle$. □

Proposition 12.1.17. *The Hermitian conjugate of $T|\phi\rangle$ is $\langle \phi | T^\dagger$, the linear functional that maps $|\psi\rangle$ to $\langle \phi | T^\dagger |\psi\rangle$.*

PROOF:

$$\begin{aligned}
 \langle T|\phi\rangle|\psi\rangle &= \overline{\langle \psi | T | \phi \rangle} \\
 &= \langle \phi | T^\dagger | \psi \rangle
 \end{aligned}$$

□

Definition 12.1.18 (Outer Product). Let $|\phi\rangle, |\psi\rangle \in V$. The *outer product* $|\psi\rangle\langle \phi | : V \rightarrow \mathbb{C}$ is the linear functional defined by

$$(|\psi\rangle\langle \phi |)|\chi\rangle = \langle \phi | \chi \rangle |\psi\rangle .$$

Proposition 12.1.19. *Let $\{|\psi_1\rangle, |\psi_2\rangle, \dots\}$ be a countable orthonormal basis. Then*

$$\sum_{n=1}^{\infty} |\psi_n\rangle\langle \psi_n| = I .$$

Part V

Analysis

Chapter 13

Real Analysis

13.1 Hermite Polynomials

Definition 13.1.1 (Hermite Polynomials). For $n \in \mathbb{N}$, define a sequence of natural numbers a_n, a_{n-2}, \dots by

$$a_n = 2$$
$$a_{k-2} = -\frac{k(k-1)}{2(n-k+2)}a_k$$

with the final entry being a_1^n if n is odd and a_0 if n is even.

The n th *Hermite polynomial* H_n is the polynomial

$$H_n(x) = \sum_k a_k x^k .$$

Example 13.1.2.

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x$$
$$H_4(x) = 16x^4 - 48x^2 + 12$$
$$H_5(x) = 32x^5 - 160x^3 + 120x$$

Proposition 13.1.3.

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Proposition 13.1.4 (Rodrigues Formula).

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Proposition 13.1.5.

$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x)$$

Proposition 13.1.6.

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) = e^{-z^2 + 2zx}$$

13.2 Fourier Transforms

Definition 13.2.1 (Fourier Transform). The *Fourier transform* of a function f is

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-ixt} dt .$$

Definition 13.2.2 (Fourier Transform). The *inverse Fourier transform* of a function F is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(t) e^{ixt} dt .$$

Theorem 13.2.3 (Plancherel's Theorem). *F is the Fourier transform of f if and only if f is the inverse Fourier transform of F .*

Part VI

Topology

Chapter 14

Topological Spaces

14.1 Topological Spaces

Definition 14.1.1 (Topological Space). Let X be a set and $\mathcal{T} \subseteq \mathcal{P}X$. Then we say (X, \mathcal{T}) is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For any $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.
- $X \in \mathcal{T}$

We call \mathcal{T} the *topology* of the topological space, and call its elements *open* sets. We shall often write X for the topological space (X, \mathcal{T}) .

Example 14.1.2 (Discrete Topology). For any set X , the power set $\mathcal{P}X$ is called the *discrete* topology on X .

Example 14.1.3 (Indiscrete Topology). For any set X , the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Example 14.1.4 (Cofinite Topology). For any set X , the *cofinite* topology is $\mathcal{T} = \{\emptyset\} \cup \{X - U : U \subseteq X \text{ is finite}\}$.

We prove this is a topology.

Example 14.1.5 (Cocountable Topology). For any set X , the *cocountable* topology is $\{X - U : U \subseteq X \text{ is countable}\}$.

Example 14.1.6 (Sierpiński Two-Point Space). The *Sierpiński two-point space* is $\{0, 1\}$ under the topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Proposition 14.1.7. Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.

PROOF:

$\langle 1 \rangle 1$. If U is open then, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.

PROOF: Take $V = U$.

$\langle 1 \rangle 2$. If, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$, then U is open.

PROOF: Since then U is the union of all the open subsets of U .

□

Proposition 14.1.8. *The intersection of a set of topologies on a set X is a topology on X .*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{T} be a set of topologies on X .

$\langle 1 \rangle 2$. For all $\mathcal{U} \subseteq \bigcap \mathcal{T}$ we have $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$.

$\langle 2 \rangle 1$. LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$

$\langle 2 \rangle 2$. LET: $T \in \mathcal{T}$

$\langle 2 \rangle 3$. $\mathcal{U} \subseteq T$

$\langle 2 \rangle 4$. $\bigcup \mathcal{U} \in T$

$\langle 1 \rangle 3$. For all $U, V \in \bigcap \mathcal{T}$ we have $U \cap V \in \bigcap \mathcal{T}$.

$\langle 2 \rangle 1$. LET: $U, V \in \bigcap \mathcal{T}$

$\langle 2 \rangle 2$. LET: $T \in \mathcal{T}$

$\langle 2 \rangle 3$. $U, V \in T$

$\langle 2 \rangle 4$. $U \cap V \in T$

$\langle 1 \rangle 4$. $X \in \bigcap \mathcal{T}$.

□

Definition 14.1.9 (Finer, Coarser). Let \mathcal{T} and \mathcal{T}' be topologies on the set X . Then \mathcal{T} is *coarser*, *smaller* or *weaker* than \mathcal{T}' , or \mathcal{T}' is *finer*, *larger* or *stronger* than \mathcal{T} , iff $\mathcal{T} \subseteq \mathcal{T}'$.

14.2 Closed Sets

Definition 14.2.1 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 14.2.2. *A set B is open if and only if $X - B$ is closed.*

PROOF: We have B is open iff $X - (X - B)$ is open iff $X - B$ is closed. □

Theorem 14.2.3. *Let X be a set. Let $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology on X such that \mathcal{C} is the set of closed sets if and only if:*

$$1. \emptyset \in \mathcal{C}$$

$$2. \forall \mathcal{A} \subseteq \mathcal{C}. \bigcap \mathcal{A} \in \mathcal{C}$$

$$3. \forall C, D \in \mathcal{C}. C \cup D \in \mathcal{C}$$

In this case, the topology is unique, and is $\{X - C : C \in \mathcal{C}\}$.

PROOF:

$\langle 1 \rangle 1$. In any topology on X we have \emptyset is closed.

PROOF: Since $X - \emptyset = X$ is open.

$\langle 1 \rangle 2$. In any topology on X , the intersection of a set \mathcal{A} of closed sets is closed.

PROOF: Since $X - \bigcap \mathcal{A} = \bigcup_{A \in \mathcal{A}} (X - A)$ is open.

$\langle 1 \rangle 3$. In any topology on X , the union of two closed sets is closed.

PROOF: For any closed sets C and D , we have $X - (C \cup D) = (X - C) \cap (X - D)$ is open.

$\langle 1 \rangle 4$. If \mathcal{C} is a set satisfying 1–3, then $\{X - C : C \in \mathcal{C}\}$ is a topology on X with respect to which \mathcal{C} is the set of closed sets.

$\langle 2 \rangle 1$. LET: \mathcal{C} be a set satisfying 1–3.

$\langle 2 \rangle 2$. LET: $\mathcal{T} = \{X - C : C \in \mathcal{C}\}$

$\langle 2 \rangle 3$. For all $U \in \mathcal{T}$ we have $X - U \in \mathcal{C}$.

$\langle 2 \rangle 4$. \mathcal{T} is a topology on X .

$\langle 3 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.

$\langle 4 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 4 \rangle 2$. For all $U \in \mathcal{U}$ we have $X - U \in \mathcal{C}$.

$\langle 4 \rangle 3$. $X - \bigcup \mathcal{U} \in \mathcal{C}$

PROOF:

$$\begin{aligned} X - \bigcup \mathcal{U} &= \bigcap_{U \in \mathcal{U}} (X - U) \\ &\in \mathcal{C} \end{aligned}$$

$\langle 4 \rangle 4$. $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 3 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

$\langle 3 \rangle 3$. $X \in \mathcal{T}$

$\langle 2 \rangle 5$. For any set C we have $C \in \mathcal{C}$ iff C is closed with respect to \mathcal{T} .

$\langle 1 \rangle 5$. If \mathcal{T} is any topology on X then $\mathcal{T} = \{X - C : C \text{ is closed in } \mathcal{T}\}$.

PROOF: Proposition 14.2.2.

□

14.3 Neighbourhoods

Definition 14.3.1 (Neighbourhood). Let X be a topological space, $x \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x , and x is an *interior* point of U , iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 14.3.2. *A set B is open if and only if it is a neighbourhood of each of its points.*

PROOF: This is Proposition 14.1.7. □

Proposition 14.3.3. *Let X be a set and $\mathcal{N} : X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that, for all $x \in X$, we have \mathcal{N}_x is the set of neighbourhoods of x , if and only if:*

- For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$

- For all $x \in X$ we have $X \in \mathcal{N}_x$
- For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$
- For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$
- For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M. M \in \mathcal{N}_y$.

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$.

PROOF: Straightforward. \square

14.4 Interior

Definition 14.4.1 (Interior). The interior of B is the union of all the open sets included in B .

14.5 Closure

Definition 14.5.1 (Closure). Let X be a topological space and $B \subseteq X$. The closure of B , \overline{B} , is the intersection of all the closed sets that include B .

Proposition 14.5.2. A set C is closed if and only if $C = \overline{C}$.

PROOF: Easy. \square

Corollary 14.5.2.1. A set B is open iff $X - B = \overline{X - B}$.

Proposition 14.5.3 (Kuratowski Closure Axioms). Let X be a set and $- : \mathcal{P}X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} such that, for all $B \subseteq X$, \overline{B} is the closure of B , if and only if:

- $\overline{\emptyset} = \emptyset$
- For all $A \subseteq X$ we have $A \subseteq \overline{A}$
- For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$
- For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$

In this case, \mathcal{O} is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

PROOF: Straightforward. \square

14.6 Bases

Definition 14.6.1 (Basis). Let X be a topological space. A *basis* for the topology on X is a set of open sets \mathcal{B} such that every open set is the union of a subset of \mathcal{B} . The elements of \mathcal{B} are called *basic open neighbourhoods* of their elements.

Example 14.6.2. Let X be a set. The set of all one-element subsets of X is a basis for the discrete topology on X .

Proposition 14.6.3. Let X be a topological space. Let \mathcal{B} be a basis for the topology on X . Then the topology on X is the coarsest topology that includes \mathcal{B} .

Proposition 14.6.4. Let X and Y be topological spaces. Let \mathcal{B} be a basis for the topology on X and \mathcal{C} a basis for the topology on Y . Then

$$\{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Theorem 14.6.5. There are infinitely many primes.

Furstenberg's proof:

PROOF:

$\langle 1 \rangle 1$. For $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$,

LET: $S(a, b) := \{an + b : n \in \mathbb{N}\}$

$\langle 1 \rangle 2$. LET: \mathcal{T} be the topology generated by the basis $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$

$\langle 2 \rangle 1$. For every $n \in \mathbb{Z}$, there exist a, b such that $n \in S(a, b)$.

PROOF: $n \in S(n, 0)$

$\langle 2 \rangle 2$. If $n \in S(a_1, b_1) \cap S(a_2, b_2)$ then there exist a_3, b_3 such that $n \in S(a_3, b_3) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 1$. LET: $d = \text{lcm}(a_1, a_2)$

PROVE: $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 2$. LET: $d = a_1 k = a_2 l$

$\langle 3 \rangle 3$. LET: $n = a_1 c + b_1 = a_2 d + b_2$

$\langle 3 \rangle 4$. LET: $z \in \mathbb{Z}$

PROVE: $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$

$\langle 3 \rangle 5$. $dz + n \in S(a_1, b_1)$

PROOF:

$$\begin{aligned} dz + n &= a_1 kz + a_1 c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

$\langle 3 \rangle 6$. $dz + n \in S(a_2, b_2)$

PROOF: Similar.

$\langle 1 \rangle 3$. For all $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$ we have $S(a, b)$ is closed.

$\langle 2 \rangle 1$. LET: $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$

$\langle 2 \rangle 2$. LET: $n \in \mathbb{Z} - S(a, b)$

$\langle 2 \rangle 3$. $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$

$\langle 3 \rangle 1$. LET: $x \in S(a, n)$

- $\langle 3 \rangle 2$. ASSUME: for a contradiction $x \in S(a, b)$
 $\langle 3 \rangle 3$. PICK m such that $x = am + b$
 $\langle 3 \rangle 4$. PICK l such that $x = al + n$
 $\langle 3 \rangle 5$. $n = a(m - l) + b$
 $\langle 3 \rangle 6$. $n \in S(a, b)$
 $\langle 3 \rangle 7$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 2$.

$\langle 1 \rangle 4$.

$$\mathbb{Z} - \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$$

PROOF: Since every integer except 1 and -1 is divisible by a prime.

- $\langle 1 \rangle 5$. No nonempty finite set is open.
 $\langle 2 \rangle 1$. LET: U be a nonempty open set
 $\langle 2 \rangle 2$. PICK $n \in U$
 $\langle 2 \rangle 3$. There exist a, b such that $n \in S(a, b) \subseteq U$
 $\langle 2 \rangle 4$. U is infinite.
 $\langle 1 \rangle 6$. $\mathbb{Z} - \{1, -1\}$ is not closed.
 $\langle 1 \rangle 7$. $\bigcup_{p \text{ prime}} S(p, 0)$ is not closed.
 $\langle 1 \rangle 8$. The union of finitely many closed sets is closed.
 $\langle 1 \rangle 9$. There are infinitely many primes.

□

14.7 Order Topology

Definition 14.7.1 (Order Topology). Let X be a linearly ordered set. The *order topology* on X is the topology generated by the open interval (a, b) as well as the open rays $(a, +\infty)$ and $(-\infty, b)$ for $a, b \in X$.

The *standard topology* on \mathbb{R} is the order topology.

Proposition 14.7.2. *Let X be a linearly ordered set. Then the order topology is generated by the basis consisting of:*

- all open intervals (a, b)
- all intervals of the form $[\perp, b)$ where \perp is the least element of X , if any
- all intervals of the form $(a, \top]$ where \top is the greatest element of X , if any.

Proposition 14.7.3. *Let X be a linearly ordered set. The open rays in X form a subbasis for the order topology.*

Definition 14.7.4 (Lower Limit Topology). The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology on \mathbb{R} generated by the basis consisting of all half-open intervals $[a, b)$.

We write \mathbb{R}_l for \mathbb{R} under the lower limit topology.

Definition 14.7.5 (K -topology). Let $K = \{1/n : n \in \mathbb{Z}_+\}$. The K -topology on \mathbb{R} is the topology generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) - K$.

We write \mathbb{R}_K for \mathbb{R} under the K -topology.

Proposition 14.7.6. *Let X be a linearly ordered set under the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology.*

PROOF:

$\langle 1 \rangle 1$. The order topology is coarser than the subspace topology.

$\langle 2 \rangle 1$. For all $a \in Y$, the open ray $\{y \in Y : a < y\}$ is open in the subspace topology.

PROOF: It is $(a, +\infty) \cap Y$.

$\langle 2 \rangle 2$. For all $a \in Y$, the open ray $\{y \in Y : y < a\}$ is open in the subspace topology.

PROOF: It is $(-\infty, a) \cap Y$.

$\langle 1 \rangle 2$. The subspace topology is coarser than the order topology.

$\langle 2 \rangle 1$. For all $a \in X$, the set $(-\infty, a) \cap Y$ is open in the order topology.

$\langle 3 \rangle 1$. CASE: $a \in Y$

PROOF: Then $(-\infty, a) \cap Y = \{y \in Y : y < a\}$ is an open ray in Y .

$\langle 3 \rangle 2$. CASE: a is an upper bound for Y

PROOF: Then $(-\infty, a) \cap Y = Y$.

$\langle 3 \rangle 3$. CASE: a is a lower bound for Y

PROOF: Then $(-\infty, a) \cap Y = \emptyset$.

$\langle 3 \rangle 4$. Q.E.D.

PROOF: These are the only three cases because Y is convex.

$\langle 2 \rangle 2$. For all $a \in X$, the set $(a, +\infty) \cap Y$ is open in the order topology.

PROOF: Similar.

□

Example 14.7.7. We cannot remove the hypothesis that the set Y is convex.

Let $X = \mathbb{R}$ and $Y = [0, 1) \cup \{2\}$. Then $\{2\}$ is open in the subspace topology but not in the order topology on Y .

Proposition 14.7.8. *Let X be a topological space. Let \mathcal{B} be a basis for the topology on X and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proposition 14.7.9. *Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Assume that, for every open set U and element $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then \mathcal{B} is a basis for the topology on X .*

Proposition 14.7.10. *Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is a basis for a topology on X if and only if:*

1. $\bigcup \mathcal{B} = X$

2. For all $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

In this case, the topology is unique and is the set of all unions of subsets of \mathcal{B} . We call it the topology generated by \mathcal{B} .

Proposition 14.7.11. *Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' , respectively, on X . Then \mathcal{T}' is finer than \mathcal{T} if and only if, for every $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Corollary 14.7.11.1. *The topologies of \mathbb{R}_l and \mathbb{R}_K are strictly finer than the standard topology on \mathbb{R} but are not comparable to one another.*

Proposition 14.7.12. *In a linearly ordered set under the order topology, every closed interval and closed ray is closed.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a linearly ordered set under the order topology.

$\langle 1 \rangle 2$. Every closed interval in X is closed.

PROOF: Since $X - [a, b] = (-\infty, a) \cup (b, +\infty)$.

$\langle 1 \rangle 3$. Every closed ray in X is closed.

PROOF: Since $X - [a, +\infty) = (-\infty, a)$ and $X - (-\infty, a] = (a, +\infty)$.

□

14.7.1 Subspaces

Proposition 14.7.13. *Let X be a topological space. Let Y be a subspace of X . Let \mathcal{B} be a basis for the topology on X . Then $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for the topology on Y .*

PROOF:

$\langle 1 \rangle 1$. For all $B \in \mathcal{B}$ we have $B \cap Y$ is open in Y .

PROOF: Since B is open in X .

$\langle 1 \rangle 2$. For any open set V in Y and $y \in V$, there exists $B \in \mathcal{B}$ such that $y \in B \cap Y \subseteq V$.

$\langle 2 \rangle 1$. LET: V be open in Y .

$\langle 2 \rangle 2$. LET: $y \in V$

$\langle 2 \rangle 3$. PICK U open in X such that $V = U \cap Y$.

$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$.

$\langle 2 \rangle 5$. $y \in B \cap Y \subseteq V$

□

Proposition 14.7.14. *Let X be a topological space and Y a subspace of X . Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set B in X such that $A = B \cap Y$.*

PROOF:

A is closed in $Y \Leftrightarrow Y - A$ is open in Y

$\Leftrightarrow \exists U$ open in $X. Y - A = U \cap Y$

$\Leftrightarrow \exists C$ closed in $X. Y - A = Y - C$

$\Leftrightarrow \exists C$ closed in $X. A = Y \cap C$

□

14.7.2 Product Topology

Proposition 14.7.15. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. For all $i \in I$, let \mathcal{B}_i be a basis for the topology on X_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i : \text{for finitely many } i \in I \text{ we have } B_i \in \mathcal{B}_i, \text{ and } B_i = X_i \text{ for all other } i\}$ is a basis for the product topology on $\prod_{i \in I} X_i$.*

PROOF:

$\langle 1 \rangle 1$. Every $B \in \mathcal{B}$ is open in the product topology.

PROOF: Since every element of \mathcal{B}_i is open in X_i .

$\langle 1 \rangle 2$. For any open set U in the product topology and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 1$. LET: U be a set open in the box topology.

$\langle 2 \rangle 2$. LET: $x \in U$

$\langle 2 \rangle 3$. PICK a family $\{U_i\}_{i \in I}$ where U_i is open in X_i for $i = i_1, \dots, i_n$, and $U_i = X_i$ for all other i , such that $x \in \prod_{i \in I} U_i \subseteq U$

$\langle 2 \rangle 4$. For $i = i_1, \dots, i_n$, choose $B_i \in \mathcal{B}_i$ such that $x_i \in B_i \subseteq U_i$. Let $B_i = X_i$ for all other i .

$\langle 2 \rangle 5$. $\prod_{i \in I} B_i \in \mathcal{B}$

$\langle 2 \rangle 6$. $x \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} U_i \subseteq U$

□

14.8 Subbases

Definition 14.8.1 (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a set \mathcal{S} of open sets such that every open set is a union of finite intersections of \mathcal{S} .

Proposition 14.8.2. *Let X be a set and $\mathcal{S} \subseteq X$. Then \mathcal{S} is a subbasis for a topology on X if and only if $\bigcup \mathcal{S} = X$, in which case the topology is unique and is the set of all unions of finite intersections of elements of \mathcal{S} .*

Proposition 14.8.3. *Let X be a topological space. Let \mathcal{S} be a subbasis for the topology on X . Then the topology on X is the coarsest topology that includes \mathcal{S} .*

Proposition 14.8.4. *Let X and Y be topological spaces. Then*

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

PROOF:

$\langle 1 \rangle 1$. Every element of \mathcal{S} is open.

PROOF: Since $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$.

$\langle 1 \rangle 2$. Every open set is a union of finite intersections of elements of \mathcal{S} .

PROOF: Since, for U open in X and V open in Y , we have $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$.

□

Definition 14.8.5 (Space with Basepoint). A *space with basepoint* is a pair (X, x) where X is a topological space and $x \in X$.

14.9 Neighbourhood Bases

Definition 14.9.1 (Neighbourhood Basis). Let X be a topological space and $x_0 \in X$. A *neighbourhood basis* of x_0 is a set \mathcal{U} of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of \mathcal{U} .

14.10 First Countable Spaces

Definition 14.10.1 (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

Proposition 14.10.2. \mathbb{R}_l is first countable.

PROOF: For any $x \in \mathbb{R}$ we have $\{[x, x + 1/n) : n \in \mathbb{Z}_+\}$ is a countable local basis.
□

Proposition 14.10.3. The ordered square is first countable.

PROOF:

⟨1⟩1. Every point (a, b) with $0 < b < 1$ has a countable local basis.

PROOF: The set of all intervals $((a, q), (a, r))$ where q and r are rational and $0 \leq q < b < r \leq 1$ is a countable local basis.

⟨1⟩2. Every point $(a, 0)$ has a countable local basis with $a > 0$.

PROOF: The set of all intervals $((q, 0), (a, r))$ where q and r are rational with $0 \leq q < a$ and $0 < r \leq 1$ is a countable local basis.

⟨1⟩3. Every point $(a, 1)$ has a countable local basis with $a < 1$.

PROOF: The set of all intervals $((a, q), (r, 1))$ with q and r rational and $0 \leq q < 1, a < r \leq 1$ is a countable local basis.

⟨1⟩4. $(0, 0)$ has a countable local basis.

PROOF: The set of all intervals $[(0, 0), (0, r))$ with r rational and $0 < r \leq 1$ is a countable local basis.

⟨1⟩5. $(1, 1)$ has a countable local basis.

PROOF: The set of all intervals $((1, q), (1, 1])$ with q rational and $0 \leq q < 1$ is a countable local basis.
□

14.11 Second Countable Spaces

Definition 14.11.1 (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

\mathbb{R}^n is second countable.

An uncountable discrete space is first countable but not second countable.

Proposition 14.11.2. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.*

PROOF:

$\langle 1 \rangle 1$. For all $\lambda \in \Lambda$, PICK U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.

$\langle 1 \rangle 2$. For all $\lambda \in \Lambda$, PICK $x_\lambda \in U_\lambda$.

$\langle 1 \rangle 3$. ASSUME: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda \in \Lambda}$.

$\langle 1 \rangle 4$. PICK $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$

$\langle 1 \rangle 5$. There is no $U \in B$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 14.11.3. *The long line cannot be embedded in \mathbb{R}^n for any n .*

PROOF: Since the long line is not second countable but \mathbb{R}^n is. □

14.12 Interior

Definition 14.12.1 (Interior). Let X be a topological space. Let $A \subseteq X$. The *interior* of A , A° , is the union of all the open sets included in A .

14.13 Closure

Definition 14.13.1 (Closure). Let X be a topological space. Let $A \subseteq X$. The *closure* of A , \bar{A} , is the intersection of all the closed sets that include A .

Proposition 14.13.2. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \bar{A}$ if and only if every open set that contains x intersects A .*

PROOF:

$x \in \bar{A} \Leftrightarrow$ for every closed set C , if $A \subseteq C$ then $x \in C$

\Leftrightarrow for every open set U , if $A \subseteq X - U$ then $x \in X - U$

\Leftrightarrow for every open set U , if $A \cap U = \emptyset$ then $x \notin U$

\Leftrightarrow for every open set U , if $x \in U$ then A intersects U □

Proposition 14.13.3. *Let X be a topological space. Let $A \subseteq B \subseteq X$. Then $\bar{A} \subseteq \bar{B}$.*

PROOF: Since every closed set that includes B is a closed set that includes A . □

Proposition 14.13.4. *Let X be a topological space. Let $A, B \subseteq X$. Then $\overline{A \cup B} = \bar{A} \cup \bar{B}$.*

PROOF:

$\langle 1 \rangle 1$. $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$

PROOF: Since $\overline{A \cup B}$ is a closed set that includes $A \cup B$.

$\langle 1 \rangle 2$. $\overline{A \cup B} \subseteq \overline{A \cup B}$

PROOF: Since $A \subseteq \overline{A \cup B}$ and $B \subseteq \overline{A \cup B}$ by Proposition 14.13.3.

□

Proposition 14.13.5. *Let X be a topological space. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then*

$$\bigcup \{\overline{A} : A \in \mathcal{A}\} \subseteq \overline{\bigcup \mathcal{A}}.$$

PROOF: For all $A \in \mathcal{A}$ we have $\overline{A} \subseteq \overline{\bigcup \mathcal{A}}$ by Proposition 14.13.3. □

Example 14.13.6. The converse does not always hold. In \mathbb{R} , let $\mathcal{A} = \{\{x\} : 0 < x < 1\}$. Then $\bigcup \{\overline{A} : A \in \mathcal{A}\} = (0, 1)$ but $\overline{\bigcup \mathcal{A}} = [0, 1]$.

Proposition 14.13.7. *Let X be a topological space. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then $\overline{\bigcap \mathcal{A}} \subseteq \bigcap \{\overline{A} : A \in \mathcal{A}\}$.*

PROOF: Since $\overline{\bigcap \mathcal{A}} \subseteq \overline{A}$ for all $A \in \mathcal{A}$ by Proposition 14.13.3. □

Example 14.13.8. The converse does not always hold. In \mathbb{R} , if A is the set of all rational numbers and B is the set of all irrational numbers then $\bigcap A \cap B = \emptyset$ but $\bigcap A \cap \bigcap B = \mathbb{R}$.

14.13.1 Bases

Proposition 14.13.9. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let \mathcal{B} be a basis for the topology on X . Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .*

PROOF:

$\langle 1 \rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

PROOF: Proposition 14.13.2 since every element of \mathcal{B} is open.

$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A , then $x \in \overline{A}$.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

$\langle 2 \rangle 2$. LET: U be an open set that contains x .

$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 4$. B intersects A .

PROOF: $\langle 2 \rangle 1$

$\langle 2 \rangle 5$. U intersects A .

□

14.13.2 Subspaces

Proposition 14.13.10. *Let X be a topological space. Let Y be a subspace of X . Let $A \subseteq Y$. Let \overline{A} be the closure of A in X . Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF:

⟨1⟩1. $\overline{A} \cap Y$ is the closed in Y .

PROOF: Since \overline{A} is closed in X .

⟨1⟩2. For any closed set B in Y , if $A \subseteq B$ then $\overline{A} \cap Y \subseteq B$.

⟨2⟩1. LET: B be closed in Y .

⟨2⟩2. ASSUME: $A \subseteq B$

⟨2⟩3. PICK C closed in X such that $B = C \cap Y$.

⟨2⟩4. $A \subseteq C$

⟨2⟩5. $\overline{A} \subseteq C$

⟨2⟩6. $\overline{A} \cap Y \subseteq B$

□

14.13.3 Product Topology

Proposition 14.13.11. *Let X and Y be topological spaces. Let $A \subseteq X$ and $B \subseteq Y$. Then $\overline{A \times B} = \overline{A} \times \overline{B}$.*

PROOF:

⟨1⟩1. $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$

PROOF: Since $\overline{A \times B}$ is a closed set that includes $A \times B$ by Proposition 15.8.2.

⟨1⟩2. $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$

⟨2⟩1. LET: $x \in \overline{A}$ and $y \in \overline{B}$.

⟨2⟩2. LET: U be an open set that contains (x, y) .

⟨2⟩3. PICK open sets V in X and W in Y such that $(x, y) \in V \times W \subseteq U$.

⟨2⟩4. V intersects A and W intersects B .

⟨2⟩5. U intersects $A \times B$.

□

14.13.4 Interior

Proposition 14.13.12. *Let X be a topological space and $A \subseteq X$. Then*

$$X - A^\circ = \overline{X - A}$$

PROOF:

$$\begin{aligned} X - A^\circ &= X - \bigcup \{U \text{ open in } X : U \subseteq A\} \\ &= \bigcap \{X - U : U \text{ open in } X, U \subseteq A\} \quad (\text{De Morgan's Law}) \\ &= \bigcap \{C : C \text{ closed in } X, X - A \subseteq C\} \\ &= \overline{X - A} \end{aligned} \quad \square$$

Proposition 14.13.13. *Let X be a topological space and $A \subseteq X$. Then*

$$X - \overline{A} = (X - A)^\circ$$

PROOF: Dual. □

14.14 Boundary

Definition 14.14.1 (Boundary). Let X be a topological space. Let $A \subseteq X$. The *boundary* of A is

$$\partial A := \overline{A} \cap \overline{X - A}.$$

Proposition 14.14.2. Let X be a topological space. Let $A \subseteq X$. Then

$$A^\circ \cap \partial A = \emptyset.$$

PROOF:

- $\langle 1 \rangle 1.$ $A^\circ \subseteq A$
- $\langle 1 \rangle 2.$ $X - A \subseteq X - A^\circ$
- $\langle 1 \rangle 3.$ $\overline{X - A} \subseteq X - A^\circ$
- $\langle 1 \rangle 4.$ $\partial A \subseteq X - A^\circ$

□

Proposition 14.14.3. Let X be a topological space. Let $A \subseteq X$. Then

$$\overline{A} = A^\circ \cup \partial A$$

- $\langle 1 \rangle 1.$ $A^\circ \subseteq \overline{A}$

PROOF: Since $A^\circ \subseteq A \subseteq \overline{A}$.

- $\langle 1 \rangle 2.$ $\partial A \subseteq \overline{A}$

PROOF: Definition of ∂A .

- $\langle 1 \rangle 3.$ $\overline{A} \subseteq A^\circ \cup \partial A$

- $\langle 2 \rangle 1.$ LET: $x \in \overline{A}$

- $\langle 2 \rangle 2.$ ASSUME: $x \notin A^\circ$

PROVE: $x \in \partial A$

- $\langle 2 \rangle 3.$ $x \in \overline{X - A}$

PROOF: Since $\overline{X - A} = X - A^\circ$.

- $\langle 2 \rangle 4.$ $x \in \partial A$

PROOF: Since $\partial A = \overline{A} \cap \overline{X - A}$.

□

Proposition 14.14.4. Let X be a topological space. Let $A \subseteq X$. Then $\partial A = \emptyset$ if and only if A is both open and closed.

PROOF:

- $\langle 1 \rangle 1.$ If $\partial A = \emptyset$ then A is open and closed.

- $\langle 2 \rangle 1.$ ASSUME: $\partial A = \emptyset$

- $\langle 2 \rangle 2.$ $\overline{A} = A^\circ$

PROOF: Proposition 14.14.3.

- $\langle 2 \rangle 3.$ $\overline{A} = A = A^\circ$

- $\langle 1 \rangle 2.$ If A is open and closed then $\partial A = \emptyset$.

PROOF: If A is open and closed then

$$\begin{aligned} \partial A &= \overline{A} \cap \overline{X - A} \\ &= \overline{A} \cap (X - A^\circ) \\ &= A \cap (X - A) \\ &= \emptyset \end{aligned}$$

□

Proposition 14.14.5. *Let X be a topological space. Let $U \subseteq X$. Then U is open if and only if $\partial U = \overline{U} - U$.*

PROOF:

⟨1⟩1. If U is open then $\partial U = \overline{U} - U$

PROOF: If U is open then

$$\begin{aligned}\partial U &= \overline{U} \cap \overline{X - U} \\ &= \overline{U} \cap (X - U^\circ) \\ &= \overline{U} - U^\circ \\ &= \overline{U} - U\end{aligned}$$

⟨1⟩2. If $\partial U = \overline{U} - U$ then U is open.

⟨2⟩1. ASSUME: $\partial U = \overline{U} - U$

⟨2⟩2. $\overline{U} - U^\circ = \overline{U} - U$

⟨2⟩3. $U \subseteq U^\circ$

⟨2⟩4. $U = U^\circ$

□

14.15 Limit Points

Definition 14.15.1 (Limit Point). Let X be a topological space, $x \in X$ and $A \subseteq X$. Then x is a *limit point*, *cluster point* or *point of accumulation* of A iff every neighbourhood of x intersects $A - \{x\}$.

Proposition 14.15.2. *Let X be a topological space. Let $A \subseteq X$. Let A' be the set of limit points of A . Then*

$$\overline{A} = A \cup A'$$

PROOF:

⟨1⟩1. $\overline{A} \subseteq A \cup A'$

⟨2⟩1. LET: $x \in \overline{A}$

⟨2⟩2. ASSUME: $x \notin A$

PROVE: $x \in A'$

⟨2⟩3. LET: U be a neighbourhood of x .

⟨2⟩4. PICK $y \in U \cap A$

PROOF: Proposition 14.13.2.

⟨2⟩5. $y \neq x$

⟨1⟩2. $A \subseteq \overline{A}$

PROOF: Immediate from the definition of \overline{A} .

⟨1⟩3. $A' \subseteq \overline{A}$

PROOF: From Proposition 14.13.2.

□

Corollary 14.15.2.1. *A set is closed if and only if it contains all its limit points.*

14.16 Isolated Points

Definition 14.16.1 (Isolated Point). Let X be a topological space. Let $a \in X$. Then a is an *isolated point* iff $\{a\}$ is open.

Chapter 15

Continuous Functions

Definition 15.0.1 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* iff, for every open set V in Y , the inverse image $f^{-1}(V)$ is open in X .

Proposition 15.0.2. *The composite of two continuous functions is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous.

$\langle 1 \rangle 2$. LET: U be open in Z .

$\langle 1 \rangle 3$. $g^{-1}(U)$ is open in Y .

$\langle 1 \rangle 4$. $\inf f(g^{-1}(U))$ is open in X .

□

Proposition 15.0.3. 1. id_X is continuous

2. If $f : X \rightarrow Y$ is continuous and $X_0 \subseteq X$ then $f|_{X_0} : X_0 \rightarrow Y$ is continuous.

3. If $f : X + Y \rightarrow Z$, then f is continuous iff $f \circ \kappa_1 : X \rightarrow Z$ and $f \circ \kappa_2 : Y \rightarrow Z$ are continuous.

4. If $f : Z \rightarrow X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Proposition 15.0.4. Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then the following are equivalent.

1. f is continuous.

2. For all $A \subseteq X$ we have $f(\overline{A}) \subseteq \overline{f(A)}$.

3. For every closed B in Y , we have $f^{-1}(B)$ is closed in X .

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: f is continuous.

- $\langle 2 \rangle 2$. LET: $A \subseteq X$
 $\langle 2 \rangle 3$. LET: $x \in \bar{A}$
 PROVE: $f(x) \in \overline{f(A)}$
 $\langle 2 \rangle 4$. LET: V be a neighbourhood of $f(x)$.
 PROVE: V intersects $f(A)$.
 $\langle 2 \rangle 5$. $f^{-1}(V)$ is a neighbourhood of x .
 $\langle 2 \rangle 6$. PICK $y \in f^{-1}(V) \cap A$
 $\langle 2 \rangle 7$. $f(y) \in V \cap f(A)$
 $\langle 1 \rangle 2$. $2 \Rightarrow 3$
 $\langle 2 \rangle 1$. ASSUME: 2
 $\langle 2 \rangle 2$. LET: B be closed in Y
 $\langle 2 \rangle 3$. LET: $A = f^{-1}(B)$
 PROVE: $\bar{A} = A$
 $\langle 2 \rangle 4$. $f(A) \subseteq B$
 $\langle 2 \rangle 5$. $\bar{A} \subseteq A$
 $\langle 3 \rangle 1$. LET: $x \in \bar{A}$
 $\langle 3 \rangle 2$. $f(x) \in B$
 PROOF:

$$f(x) \in f(\bar{A})$$

$$\subseteq \overline{f(A)} \quad (\langle 2 \rangle 1)$$

$$\subseteq \bar{B} \quad (\langle 2 \rangle 4)$$

$$= B \quad (\langle 2 \rangle 2)$$
 $\langle 1 \rangle 3$. $3 \Rightarrow 1$
 $\langle 2 \rangle 1$. ASSUME: 3
 $\langle 2 \rangle 2$. LET: V be open in Y .
 $\langle 2 \rangle 3$. $f^{-1}(Y - V)$ is closed in X .
 $\langle 2 \rangle 4$. $X - f^{-1}(V)$ is closed in X .
 $\langle 2 \rangle 5$. $f^{-1}(V)$ is open in X .

□

Proposition 15.0.5. *Let X and Y be topological spaces. Any constant function $X \rightarrow Y$ is continuous.*

PROOF:

- $\langle 1 \rangle 1$. LET: $b \in Y$
 $\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$ be the constant function with value b .
 $\langle 1 \rangle 3$. LET: $V \subseteq Y$ be open.
 $\langle 1 \rangle 4$. $f^{-1}(V)$ is either \emptyset or X .
 $\langle 1 \rangle 5$. $f^{-1}(V)$ is open.

□

Proposition 15.0.6. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{B} be a basis for Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .*

PROOF:

⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF: Since every element of \mathcal{B} is open in Y .

⟨1⟩2. If, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X , then f is continuous.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

⟨2⟩2. LET: U be open in Y .

⟨2⟩3. LET: $x \in f^{-1}(U)$

⟨2⟩4. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq U$.

⟨2⟩5. $x \in f^{-1}(B) \subseteq f^{-1}(U)$

□

Proposition 15.0.7. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{S} be a subbasis for the topology on Y . Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .*

PROOF:

⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

PROOF: Immediate from definitions.

⟨1⟩2. If, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X , then f is continuous.

⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

⟨2⟩2. For all $V_1, \dots, V_n \in \mathcal{S}$ we have $f^{-1}(V_1 \cap \dots \cap V_n)$ is open in X .

PROOF: Since $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$.

⟨2⟩3. Q.E.D.

PROOF: By Proposition 15.0.6 since the set of all finite intersections of elements of \mathcal{S} forms a basis for the topology on Y .

□

Proposition 15.0.8. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Then f is continuous if and only if, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.*

PROOF:

⟨1⟩1. If f is continuous then, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.

⟨2⟩1. ASSUME: f is continuous.

⟨2⟩2. LET: $x \in \mathbb{R}$

⟨2⟩3. LET: $\epsilon > 0$

⟨2⟩4. $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open in X .

⟨2⟩5. PICK a, b such that $x \in (a, b) \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$.

⟨2⟩6. LET: $\delta = \min(x - a, b - x)$

⟨2⟩7. LET: $y \in \mathbb{R}$

⟨2⟩8. ASSUME: $|y - x| < \delta$

⟨2⟩9. $y \in (a, b)$

⟨2⟩10. $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$

⟨2⟩11. $|f(y) - f(x)| < \epsilon$

⟨1⟩2. If, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$, then f is continuous.

⟨2⟩1. ASSUME: For all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.

- ⟨2⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.
 - ⟨3⟩1. LET: $a \in \mathbb{R}$
 - ⟨3⟩2. LET: $x \in f^{-1}((a, +\infty))$
 - ⟨3⟩3. LET: $\epsilon = f(x) - a$
 - ⟨3⟩4. PICK $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$
 - ⟨3⟩5. $x \in (x - \delta, x + \delta) \subseteq f^{-1}((a, +\infty))$
- ⟨2⟩3. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.
 - PROOF: Similar.
- ⟨2⟩4. Q.E.D.
 - PROOF: Proposition 15.0.8.

□

Definition 15.0.9 (Continuity at a Point). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let $a \in X$. Then f is *continuous at a* iff, for every neighbourhood V of $f(a)$, there exists a neighbourhood U of a such that $f(U) \subseteq V$.

Proposition 15.0.10. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is continuous if and only if f is continuous at every point in X .*

- ⟨1⟩1. If f is continuous then f is continuous at every point in X .
 - ⟨2⟩1. ASSUME: f is continuous.
 - ⟨2⟩2. LET: $a \in X$
 - ⟨2⟩3. LET: V be a neighbourhood of $f(a)$
 - ⟨2⟩4. LET: $U = f^{-1}(V)$
 - ⟨2⟩5. U is a neighbourhood of a .
 - ⟨2⟩6. $f(U) \subseteq V$
- ⟨1⟩2. If f is continuous at every point in X then f is continuous.
 - ⟨2⟩1. ASSUME: f is continuous at every point in X .
 - ⟨2⟩2. LET: V be open in Y .
 - ⟨2⟩3. LET: $x \in f^{-1}(V)$
 - ⟨2⟩4. V is a neighbourhood of $f(x)$
 - ⟨2⟩5. PICK a neighbourhood U of x such that $f(U) \subseteq V$
 - ⟨2⟩6. $x \in U \subseteq f^{-1}(V)$

□

Definition 15.0.11 (Homeomorphism). Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Proposition 15.0.12. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is a homeomorphism iff f is bijective and, for all $U \subseteq X$, we have $f(U)$ is open if and only if U is open.*

PROOF: Immediate from definitions. □

Definition 15.0.13 (Topological Property). A property P of topological spaces is a *topological* property iff, for any topological spaces X and Y , if $P[X]$ and $X \cong Y$ then $P[Y]$.

Definition 15.0.14 (Retraction). Let X be a topological space and A a subspace of X . A continuous function $\rho : X \rightarrow A$ is a *retraction* iff $\rho|_A = \text{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 15.0.15. Let **Top** be the category of small topological spaces and continuous functions.

Proposition 15.0.16. \emptyset is initial in **Top**.

Proposition 15.0.17. 1 is terminal in **Top**.

Forgetful functor **Top** \rightarrow **Set**.

Basepoint preserving continuous functor.

Proposition 15.0.18. Let (X, \mathcal{T}) be a topological space. Let S be the Sierpiński two-point space. Define $\Phi : \mathcal{T} \rightarrow \mathbf{Top}[X, S]$ by $\Phi(U)(x) = 1$ iff $x \in U$. Then Φ is a bijection.

PROOF:

$\langle 1 \rangle 1$. For all $U \in \mathcal{T}$ we have $\Phi(U)$ is continuous.

$\langle 2 \rangle 1$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 2$. $\Phi(U)(\{1\})$ is open.

PROOF: Since $\Phi(U)(\{1\}) = U$.

$\langle 1 \rangle 2$. Φ is injective.

PROOF: If $\Phi(U) = \Phi(V)$ then we have $\forall x (x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$.

$\langle 1 \rangle 3$. Φ is surjective.

PROOF: Given $f : X \rightarrow S$ continuous we have $\Phi(f^{-1}(1)) = f$.

□

15.0.1 Order Topology

Proposition 15.0.19. Let X and Y be linearly ordered sets under the order topology. Let $f : X \rightarrow Y$ be strictly monotone and surjective. Then f is a homeomorphism.

PROOF:

$\langle 1 \rangle 1$. f is continuous.

$\langle 2 \rangle 1$. For all $b \in Y$ we have $f^{-1}((b, +\infty))$ is open in X .

$\langle 3 \rangle 1$. LET: $b \in Y$

$\langle 3 \rangle 2$. LET: a be the element of X such that $f(a) = b$.

$\langle 3 \rangle 3$. $f^{-1}((b, +\infty)) = (a, +\infty)$

$\langle 2 \rangle 2$. For all $b \in Y$ we have $f^{-1}((-\infty, b))$ is open in X .

PROOF: Similar.

$\langle 1 \rangle 2$. f^{-1} is continuous.

PROOF: Similar.

□

Corollary 15.0.19.1. For n a positive integer, the n th root function $\overline{\mathbb{R}_+} \rightarrow \overline{\mathbb{R}_+}$ is continuous.

15.0.2 Paths

Definition 15.0.20 (Path). A *path* in a topological space X is a continuous function $[0, 1] \rightarrow X$.

Definition 15.0.21 (Constant Path). Let X be a topological space and $a \in X$. The *constant* path at a is the path $p : [0, 1] \rightarrow X$ with $p(t) = a$ for all $t \in [0, 1]$.

Definition 15.0.22 (Reverse Path). Let X be a topological space and $p : [0, 1] \rightarrow X$. The *reverse* of p is the path $q : [0, 1] \rightarrow X$ with $q(t) = p(1 - t)$ for all $t \in [0, 1]$.

Definition 15.0.23 (Concatenation). Let X be a topological space and $p, q : [0, 1] \rightarrow X$ be paths in X with $p(1) = q(0)$. The *concatenation* of p and q is the path $r : [0, 1] \rightarrow X$ with

$$r(t) = \begin{cases} p(2t) & \text{if } 0 \leq t \leq 1/2 \\ q(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

15.0.3 Loops

Definition 15.0.24 (Loop). A *loop* in a topological space X is a path $\alpha : [0, 1] \rightarrow X$ such that $\alpha(0) = \alpha(1)$.

15.1 Convergence

Definition 15.1.1 (Convergence). Let X be a topological space. Let (x_n) be a sequence in X . A point $a \in X$ is a *limit* of the sequence iff, for every neighbourhood U of a , there exists n_0 such that $\forall n \geq n_0, x_n \in U$.

Proposition 15.1.2. If $f : X \rightarrow Y$ is continuous and $x_n \rightarrow l$ in X then $f(x_n) \rightarrow f(l)$ in Y .

Example 15.1.3. The converse does not hold.

Let X be the set of all continuous functions $[0, 1] \rightarrow [-1, 1]$ under the product topology. Let $i : X \rightarrow L^2([0, 1])$ be the inclusion.

If $f_n \rightarrow f$ then $i(f_n) \rightarrow i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K, \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0, 1]} U_\lambda$ where $U_\lambda = [-1, 1]$ except for $\lambda = \lambda_1, \dots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \dots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 15.1.4. The converse does hold for first countable spaces. If $f : X \rightarrow Y$ where X is first countable, and Y is a topological space, and whenever $x_n \rightarrow x$ then $f(x_n) \rightarrow f(x)$, then f is continuous.

Proposition 15.1.5. If (s_n) is an increasing sequence of real numbers bounded above, then (s_n) converges.

PROOF:

$\langle 1 \rangle 1$. LET: s be the supremum of $\{s_n : n \in \mathbb{N}\}$.

PROVE: $s_n \rightarrow s$ as $n \rightarrow \infty$.

$\langle 1 \rangle 2$. LET: $\epsilon > 0$

$\langle 1 \rangle 3$. PICK N such that $s_N > s - \epsilon$.

$\langle 1 \rangle 4$. $\forall n \geq N. s - \epsilon \leq s_n \leq s$

$\langle 1 \rangle 5$. $\forall n \geq N. |s_n - s| < \epsilon$

□

15.1.1 Closure

Proposition 15.1.6. *Let X be a topological space. Let $A \subseteq X$. Let (a_n) be a sequence in A and $l \in X$. If $a_n \rightarrow l$ as $n \rightarrow \infty$, then $l \in \overline{A}$.*

PROOF:

$\langle 1 \rangle 1$. LET: U be a neighbourhood of l .

$\langle 1 \rangle 2$. PICK N such that $\forall n \in N. a_n \in U$

$\langle 1 \rangle 3$. $a_N \in A \cap U$

□

15.1.2 Continuous Functions

Proposition 15.1.7. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$ be continuous. Let $x_n \rightarrow x$ as $n \rightarrow \infty$ in X . Then $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$ in Y .*

PROOF:

$\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(x)$.

$\langle 1 \rangle 2$. PICK N such that $\forall n \geq N. x_n \in f^{-1}(V)$

$\langle 1 \rangle 3$. $\forall n \geq N. f(x_n) \in V$

□

15.1.3 Infinite Series

Definition 15.1.8 (Series). Let (a_n) be a sequence of real numbers. We say that the infinite series $\sum_{n=0}^{\infty} a_n$ *converges* to s , and write

$$\sum_{n=0}^{\infty} a_n = s$$

iff $\sum_{n=0}^N a_n \rightarrow s$ as $N \rightarrow \infty$.

15.2 Strong Continuity

Definition 15.2.1 (Strong Continuity). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is *strongly continuous* iff, for every $V \subseteq Y$, we have V is open in Y if and only if $f^{-1}(V)$ is open in X .

Proposition 15.2.2. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is strongly continuous if and only if, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X .*

PROOF:

$$\begin{aligned} f \text{ is continuous} &\Leftrightarrow \forall V \subseteq Y (V \text{ is open in } Y \Leftrightarrow f^{-1}(V) \text{ is open in } X) \\ &\Leftrightarrow \forall C \subseteq Y (Y - C \text{ is open in } Y \Leftrightarrow f^{-1}(Y - C) \text{ is open in } X) \\ &\Leftrightarrow \forall C \subseteq Y (C \text{ is closed in } Y \Leftrightarrow f^{-1}(C) \text{ is closed in } X) \quad \square \end{aligned}$$

15.3 Subspaces

Definition 15.3.1 (Subspace). Let X be a topological space, Y a set, and $f : Y \rightarrow X$. The *subspace topology* on Y induced by f is $\mathcal{T} = \{f^{-1}(U) : U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

- $\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
 PROOF: Since $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$.
 - $\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
 PROOF: Since $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.
 - $\langle 1 \rangle 3$. $Y \in \mathcal{T}$
 PROOF: Since $Y = f^{-1}(X)$.
- \square

Proposition 15.3.2. *Let X be a topological space, Y a set and $f : Y \rightarrow X$ a function. Then the subspace topology on Y is the coarsest topology such that f is continuous.*

PROOF: Immediate from definition. \square

Proposition 15.3.3 (Local Formulation of Continuity). *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{U} be a set of open subspaces of X such that $X = \bigcup \mathcal{U}$. If $f|_U : U \rightarrow Y$ is continuous for all $U \in \mathcal{U}$, then f is continuous.*

PROOF:

- $\langle 1 \rangle 1$. LET: $x \in X$
 PROVE: f is continuous at x .
 - $\langle 1 \rangle 2$. LET: V be a neighbourhood of $f(x)$.
 - $\langle 1 \rangle 3$. PICK $U \in \mathcal{U}$ such that $x \in U$.
 - $\langle 1 \rangle 4$. PICK W open in U such that $x \in W$ and $f(W) \subseteq V$.
 - $\langle 1 \rangle 5$. W is open in X .
- \square

Theorem 15.3.4. *Let X be a topological space and (Y, i) a subset of X . Then the subspace topology on Y is the unique topology such that, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f : Z \rightarrow X$ is continuous.*

PROOF:

- ⟨1⟩1. If we give Y the subspace topology then, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.
- ⟨2⟩1. Given Y the subspace topology.
- ⟨2⟩2. LET: Z be a topological space.
- ⟨2⟩3. LET: $f : Z \rightarrow Y$
- ⟨2⟩4. If f is continuous then $i \circ f$ is continuous.
PROOF: Since i is continuous.
- ⟨2⟩5. If $i \circ f$ is continuous then f is continuous.
- ⟨3⟩1. ASSUME: $i \circ f$ is continuous.
- ⟨3⟩2. LET: U be open in Y .
- ⟨3⟩3. $f^{-1}(i^{-1}(i(U)))$ is open in Z .
- ⟨3⟩4. $f^{-1}(U)$ is open in Z .
- ⟨1⟩2. If, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.
- ⟨2⟩1. ASSUME: For every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.
- ⟨2⟩2. i is continuous.
- ⟨2⟩3. For every open set U in X , we have $i^{-1}(X)$ is open in Y
- ⟨2⟩4. LET: Z be the set Y under the subspace topology and $f : Z \rightarrow Y$ the identity function.
- ⟨2⟩5. $i \circ f$ is continuous.
- ⟨2⟩6. f is continuous.
- ⟨2⟩7. Every set open in Y is open in Z .

□

Proposition 15.3.5. *Let X be a topological space, Y a subspace of X and $U \subseteq Y$. If Y is open in X and U is open in Y then U is open in X .*

PROOF:

- ⟨1⟩1. PICK V open in X such that $U = V \cap Y$
- ⟨1⟩2. U is open in X .

PROOF: It is the intersection of two open sets in X .

□

Proposition 15.3.6. *Let X be a topological space. Let Y be a subspace of X . Let $C \subseteq Y$. If Y is closed in X and C is closed in Y then C is closed in X .*

PROOF: Similar. □

Proposition 15.3.7. *Let Y be a subspace of X and $A \subseteq Y$. Then the subspace topology on A as a subspace of Y is the same as the subspace topology on A as a subspace of X .*

PROOF:

- ⟨1⟩1. LET: \mathcal{T}_Y be the subspace topology on A as a subspace of Y .
- ⟨1⟩2. LET: \mathcal{T}_X be the subspace topology on A as a subspace of X .

- ⟨1⟩3. LET: $U \subseteq A$
 ⟨1⟩4. $U \in \mathcal{T}_Y \Leftrightarrow U \in \mathcal{T}_X$

PROOF:

$$\begin{aligned}
 U \in \mathcal{T}_Y &\Leftrightarrow \exists V \text{ open in } Y. U = V \cap A \\
 &\Leftrightarrow \exists V. \exists W \text{ open in } X. (V = Y \cap W \wedge U = V \cap A) \\
 &\Leftrightarrow \exists W \text{ open in } X. U = Y \cap W \cap A \\
 &\Leftrightarrow \exists W \text{ open in } X. U = W \cap A \\
 &\Leftrightarrow U \in \mathcal{T}_X
 \end{aligned}$$

□

Proposition 15.3.8. *Let X be a topological space. Let \mathcal{B} be a basis for the topology on X . Let $Y \subseteq X$. Then $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the topology on Y .*

PROOF:

- ⟨1⟩1. Every element of \mathcal{B}' is open.
 PROOF: For all $B \in \mathcal{B}$, we have B is open in X , so $B \cap Y$ is open in Y .
 ⟨1⟩2. For any open set V in Y and $y \in V$, there exists $B' \in \mathcal{B}'$ such that
 $y \in B' \subseteq V$
 ⟨2⟩1. LET: V be open in Y .
 ⟨2⟩2. LET: $y \in V$
 ⟨2⟩3. PICK U open in X such that $V = U \cap Y$.
 ⟨2⟩4. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$
 ⟨2⟩5. $B \cap Y \in \mathcal{B}'$ and $y \in B \cap Y \subseteq V$

□

Proposition 15.3.9. *Let X be a topological space and Y a subspace of X . Let $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X .*

PROOF:

- ⟨1⟩1. PICK C closed in X such that $A = C \cap Y$.
 ⟨1⟩2. A is closed in X .
 PROOF: It is the intersection of two closed sets in X .

□

15.3.1 Product Topology

Proposition 15.3.10. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology on $\prod_{i \in I} Y_i$ as a subspace of $\prod_{i \in I} X_i$.*

PROOF:

- ⟨1⟩1. Given $\prod_{i \in I} Y_i$ the subspace topology.
 ⟨1⟩2. LET: $\iota : \prod_{i \in I} Y_i$ be the inclusion.
 ⟨1⟩3. LET: Z be any topological space.
 ⟨1⟩4. LET: $f : Z \rightarrow \prod_{i \in I} Y_i$
 ⟨1⟩5. f is continuous if and only if, for all $i \in I$, we have $\pi_i \circ f$ is continuous.

PROOF:

f is continuous $\Leftrightarrow \iota \circ f : Z \rightarrow \prod_{i \in I} X_i$ is continuous (Theorem 15.3.4)

$\Leftrightarrow \forall i \in I. \pi_i \circ \iota \circ f : Z \rightarrow X_i$ is continuous (Theorem 15.8.4)

$\Leftrightarrow \forall i \in I. \iota_i \circ \pi_i \circ f : Z \rightarrow X_i$ is continuous

$\Leftrightarrow \forall i \in I. \pi_i \circ f : Z \rightarrow Y_i$ is continuous (Theorem 15.3.4)

where ι_i is the inclusion $Y_i \rightarrow X_i$.

□

15.4 Embedding

Definition 15.4.1 (Embedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *embedding* iff f is injective and the topology on X is the subspace induced by f .

Proposition 15.4.2. *Every embedding is continuous.*

PROOF: Theorem 15.3.4. □

Proposition 15.4.3. *Let X and Y be topological spaces. Let $b \in Y$. The function $\kappa : X \rightarrow X \times Y$ that maps x to (x, b) is an embedding.*

PROOF:

⟨1⟩1. For all U open in X , we have $U = \kappa^{-1}(V)$ for some V open in $X \times Y$.

PROOF: Take $V = U \times Y$.

⟨1⟩2. For all V open in $X \times Y$ we have $\kappa^{-1}(V)$ is open in X .

PROOF: Since $\pi_1 \circ \kappa = \text{id}_X$ and $\pi_2 \circ \kappa$ (which is the constant function with value b) are both continuous, hence κ is continuous.

□

15.5 Open Maps

Definition 15.5.1 (Open Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *open map* iff, for all U open in X , we have $f(U)$ is open in Y .

Proposition 15.5.2. *Let X and Y be topological spaces. The projections $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are open maps.*

PROOF:

⟨1⟩1. π_1 is an open map.

⟨2⟩1. LET: U be open in $X \times Y$.

⟨2⟩2. LET: $x \in \pi_1(U)$

⟨2⟩3. PICK y such that $(x, y) \in U$

⟨2⟩4. PICK V and W open in X and Y respectively such that $(x, y) \in V \times W \subseteq U$

- $\langle 2 \rangle 5.$ $x \in V \subseteq \pi_1(U)$
 $\langle 1 \rangle 2.$ π_2 is an open map.

PROOF: Similar.

□

15.5.1 Subspaces

Proposition 15.5.3. *Let X and Y be topological spaces. Let $p : X \rightarrow Y$ be an open map. Let A be an open set in X . Then $p \upharpoonright A : A \rightarrow p(A)$ is an open map.*

PROOF:

- $\langle 1 \rangle 1.$ LET: U be open in A .
 $\langle 1 \rangle 2.$ U is open in X .

PROOF: Proposition 15.3.5.

- $\langle 1 \rangle 3.$ $p(U)$ is open in Y .
 $\langle 1 \rangle 4.$ $p(U)$ is open in $p(A)$.

PROOF: Since $p(U) = p(U) \cap p(A)$.

□

15.6 Locally Finite

Definition 15.6.1 (Locally Finite). Let X be a topological space. Let $\{A_i\}_{i \in I}$ be a family of subsets of X . Then $\{A_i\}_{i \in I}$ is *locally finite* iff, for every $x \in X$, there exist only finitely many $i \in I$ such that $x \in A_i$.

Theorem 15.6.2 (Pasting Lemma). *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let $\{A_i\}_{i \in I}$ be a locally finite family of closed subspaces of X such that $X = \bigcup_{i \in I} A_i$. If $f \upharpoonright A_i : A_i \rightarrow Y$ is continuous for all $i \in I$, then f is continuous.*

PROOF:

- $\langle 1 \rangle 1.$ LET: B be closed in Y .
 $\langle 1 \rangle 2.$ LET: $A = f^{-1}(B)$
 PROVE: A is closed in X .
 $\langle 1 \rangle 3.$ $A = \bigcup_{i \in I} f \upharpoonright A_i^{-1}(B)$
 $\langle 1 \rangle 4.$ LET: $x \in X - A$
 PROVE: There exists a neighbourhood U' of x such that $U' \subseteq X - A$.
 $\langle 1 \rangle 5.$ PICK a neighbourhood U of x such that U intersects A_i for only finitely many $i \in I$.
 $\langle 1 \rangle 6.$ LET: i_1, \dots, i_n be the elements of I such that U intersects A_{i_1}, \dots, A_{i_n} .
 $\langle 1 \rangle 7.$ For $j = 1, \dots, n$,
 LET: $S_j = f \upharpoonright A_{i_j}^{-1}(B)$
 $\langle 1 \rangle 8.$ For $j = 1, \dots, n$, we have S_j is closed in X .
 $\langle 1 \rangle 9.$ For $j = 1, \dots, n$, we have $x \notin S_j$.
 $\langle 1 \rangle 10.$ LET: $U' = U \cap \bigcap_{j=1}^n (X - S_j)$
 $\langle 1 \rangle 11.$ U' is a neighbourhood of x .

⟨1⟩12. $U' \subseteq X - A$

□

15.7 Closed Maps

Definition 15.7.1 (Closed Map). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is a *closed map* iff, for every closed set C in X , we have $f(C)$ is closed in Y .

15.8 Product Topology

Definition 15.8.1 (Product Topology). Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

15.8.1 Closed Sets

Proposition 15.8.2. Let X and Y be topological spaces. Let A be a closed set in X and B a closed set in Y . Then $A \times B$ is closed in $X \times Y$.

PROOF: Since $(X \times Y) - (A \times B) = ((X - A) \times Y) \cup (X \times (Y - B))$. □

Proposition 15.8.3. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The product topology on $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis $\mathcal{B} = \{\prod_{\alpha \in A} U_\alpha : \text{for all } \alpha \in A, U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in A\}$.

PROOF:

⟨1⟩1. \mathcal{B} is a basis for a topology.

⟨1⟩2. LET: \mathcal{T} be the topology generated by \mathcal{B} .

⟨1⟩3. LET: \mathcal{T}_p be the product topology.

⟨1⟩4. $\mathcal{T} \subseteq \mathcal{T}_p$

⟨2⟩1. LET: $B \in \mathcal{B}$

⟨2⟩2. LET: $B = \prod_{\alpha \in A} U_\alpha$ with each U_α open in X_α and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$

⟨2⟩3. $B = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$

⟨2⟩4. $B \in \mathcal{T}_p$

⟨1⟩5. $\mathcal{T}_p \subseteq \mathcal{T}$

⟨2⟩1. For every $\alpha \in A$ we have π_α is continuous.

PROOF: Since $\pi^{-1}(U)$ is open for every U open in X_α .

□

Theorem 15.8.4. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Then the product topology on $\prod_{\alpha \in A} X_\alpha$ is the unique topology such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f : Z \rightarrow X_\alpha$ is continuous.

PROOF:

- (1)1. If we give $\prod_{\alpha \in A} X_\alpha$ the product topology, then for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
 (2)1. Give $\prod_{\alpha \in A} X_\alpha$ the product topology.
 (2)2. LET: Z be a topological space.
 (2)3. LET: $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$
 (2)4. If f is continuous then, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
 PROOF: Since the composite of two continuous functions is continuous.
 (2)5. If, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then f is continuous.
 (3)1. ASSUME: For all $\alpha \in A$ we have $\pi_\alpha \circ f$ is continuous.
 (3)2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with U_α open in X_α such that $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \dots, \alpha_n$.
 (3)3. For all α we have $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ is open in Z .
 (3)4. $f^{-1}(\prod_\alpha U_\alpha)$ is open in Z
 PROOF: Since $f^{-1}(\prod_\alpha U_\alpha) = f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \dots \cap f^{-1}(\pi_{\alpha_n}^{-1}(U_{\alpha_n}))$.
 (1)2. If \mathcal{T} is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then \mathcal{T} is the product topology.
 (2)1. ASSUME: \mathcal{T} is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
 (2)2. LET: \mathcal{T}_p be the product topology.
 (2)3. $\mathcal{T} \subseteq \mathcal{T}_p$
 (3)1. LET: $Z = (\prod_\alpha X_\alpha, \mathcal{T}_p)$
 (3)2. LET: $f : Z \rightarrow \prod_\alpha X_\alpha$ be the identity function
 (3)3. For all α we have $\pi_\alpha \circ f$ is continuous.
 (3)4. f is continuous.
 PROOF: (2)1
 (3)5. Every set open in \mathcal{T} is open in \mathcal{T}_p
 (2)4. $\mathcal{T}_p \subseteq \mathcal{T}$
 (3)1. $\text{id}_{\prod_\alpha X_\alpha}$ is continuous.
 (3)2. For all α we have π_α is continuous.
 PROOF: (2)1
 (3)3. $\mathcal{T}_p \subseteq \mathcal{T}$
 PROOF: Since \mathcal{T}_p is the coarsest topology such that every π_α is continuous.

□

Example 15.8.5. It is not true that, for any function $f : \prod_{\alpha \in A} X_\alpha \rightarrow Y$, if f is continuous in every variable separately then f is continuous.

Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

Then f is continuous in x and in y , but is not continuous.

Proposition 15.8.6. *Let $\{X_i\}_{i \in I}$ be a nonempty family of topological spaces. The product topology on $\prod_{i \in I} X_i$ is the topology generated by the subbasis $\{\pi_i^{-1}(U) : i \in I, U \text{ is open in } X_i\}$.*

PROOF:

$\langle 1 \rangle 1.$ $\{\pi_i^{-1}(U) : i \in I, U \text{ is open in } X_i\}$ is a subbasis for a topology on $\prod_{i \in I} X_i$.

$\langle 2 \rangle 1.$ PICK $i_0 \in I$

$\langle 2 \rangle 2.$ $\prod_{i \in I} X_i = \pi_{i_0}^{-1}(X_{i_0})$

$\langle 1 \rangle 2.$ The topology generated by this subbasis is the product topology.

PROOF: Since the basis in Proposition 15.8.3 is the set of all finite intersections of elements of this subbasis.

□

15.8.2 Closure

Proposition 15.8.7. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

$\langle 1 \rangle 1.$ $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

$\langle 2 \rangle 1.$ LET: $x \in \prod_{i \in I} \overline{A_i}$

$\langle 2 \rangle 2.$ For any family $\{U_i\}_{i \in I}$ where each U_i is open in X_i , and $U_i = X_i$ for all but finitely many $i \in I$, if $x \in \prod_{i \in I} U_i$ then $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$.

$\langle 3 \rangle 1.$ LET: $\{U_i\}_{i \in I}$ be a family where each U_i is open in X_i , and $U_i = X_i$ for all but finitely many i .

$\langle 3 \rangle 2.$ ASSUME: $x \in \prod_{i \in I} \overline{A_i}$

$\langle 3 \rangle 3.$ For all $i \in I$ we have U_i intersects A_i

PROOF: Since $\pi_i(x) \in \overline{A_i}$ and U_i is a neighbourhood of $\pi_i(x)$.

$\langle 3 \rangle 4.$ $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$

$\langle 2 \rangle 3.$ $x \in \overline{\prod_{i \in I} A_i}$

PROOF: Proposition 14.13.9.

$\langle 1 \rangle 2.$ $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

$\langle 2 \rangle 1.$ LET: $x \in \overline{\prod_{i \in I} A_i}$

$\langle 2 \rangle 2.$ LET: $i \in I$

PROVE: $\pi_i(x) \in \overline{A_i}$

$\langle 2 \rangle 3.$ LET: U be a neighbourhood of $\pi_i(x)$ in X_i

$\langle 2 \rangle 4.$ $\pi_i^{-1}(U)$ is a neighbourhood of x in $\prod_{i \in I} X_i$

$\langle 2 \rangle 5.$ PICK $y \in \pi_i^{-1}(U) \cap \prod_{i \in I} A_i$

$\langle 2 \rangle 6.$ $\pi_i(y) \in U \cap A_i$

□

15.8.3 Convergence

Proposition 15.8.8. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (x_n) be a sequence of points in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $x_n \rightarrow l$ as $n \rightarrow \infty$ if and*

only if, for all $i \in I$, we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.

PROOF:

$\langle 1 \rangle 1$. If $x_n \rightarrow l$ as $n \rightarrow \infty$ then, for all $i \in I$, we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.

PROOF: Proposition 15.1.2.

$\langle 1 \rangle 2$. If, for all $i \in I$, we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$, then $x_n \rightarrow l$ as $n \rightarrow \infty$.

$\langle 2 \rangle 1$. ASSUME: For all $i \in I$ we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.

$\langle 2 \rangle 2$. LET: U be a neighbourhood of l .

$\langle 2 \rangle 3$. PICK $i_1, \dots, i_n \in I$ and open sets U_j in X_{i_j} for $j = 1, \dots, n$ such that $l \in \pi_{i_1}^{-1}(U_1) \cap \dots \cap \pi_{i_n}^{-1}(U_n) \subseteq U$

$\langle 2 \rangle 4$. For $j = 1, \dots, n$ we have $\pi_{i_j}(l) \in U_j$

$\langle 2 \rangle 5$. PICK N such that, for all $m \geq N$, we have $\pi_{i_j}(x_m) \in U_j$

$\langle 2 \rangle 6$. $\forall m \geq N. x_m \in U$

□

15.9 Topological Disjoint Union

Definition 15.9.1 (Coproduct Topology). Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The *coproduct topology* on $\coprod_{\alpha \in A} X_\alpha$ is

$$\mathcal{T} = \left\{ \coprod_{\alpha \in A} U_\alpha : \{U_\alpha\}_{\alpha \in A} \text{ is a family with } U_\alpha \text{ open in } X_\alpha \text{ for all } \alpha \right\}.$$

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF:

$$\bigcup_{i \in I} \coprod_{\alpha \in A} U_{i\alpha} = \coprod_{\alpha \in A} \bigcup_{i \in I} U_{i\alpha}$$

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF:

$$\coprod_{\alpha \in A} U_\alpha \cap \coprod_{\alpha \in A} V_\alpha = \coprod_{\alpha \in A} (U_\alpha \cap V_\alpha)$$

$\langle 1 \rangle 3$. $\coprod_{\alpha \in A} X_\alpha \in \mathcal{T}$

PROOF: Since every X_α is open in X_α .

□

Proposition 15.9.2. The coproduct topology is the finest topology on $\coprod_{\alpha \in A} X_\alpha$ such that every injection $\kappa_\alpha : X_\alpha \rightarrow \coprod_{\alpha \in A} X_\alpha$ is continuous.

PROOF:

$\langle 1 \rangle 1$. LET: $P = \coprod_{\alpha \in A} X_\alpha$

$\langle 1 \rangle 2$. LET: \mathcal{T}_c be the coproduct topology.

$\langle 1 \rangle 3$. LET: \mathcal{T} be any topology on P

$\langle 1 \rangle 4$. For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T}_c)$ is continuous.

- ⟨2⟩1. LET: $\alpha \in A$
 - ⟨2⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with each U_α open in X_α .
 - ⟨2⟩3. For all $\alpha \in A$, we have $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha)$ is open in X_α .
 PROOF: Since $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha) = U_\alpha$.
 - ⟨1⟩5. If, for all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$ is continuous, then $\mathcal{T} \subseteq \mathcal{T}_c$.
 - ⟨2⟩1. ASSUME: For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$ is continuous.
 - ⟨2⟩2. LET: $U \in \mathcal{T}$
 - ⟨2⟩3. For all $\alpha \in A$, we have $\kappa_\alpha^{-1}(U)$ is open in X_α .
 - ⟨2⟩4. $U = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(U) \in \mathcal{T}_c$
-

Theorem 15.9.3. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The coproduct topology is the unique topology on $\coprod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : \coprod_{\alpha \in A} X_\alpha \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.*

PROOF:

- ⟨1⟩1. LET: $X = \coprod_{\alpha \in A} X_\alpha$
- ⟨1⟩2. LET: \mathcal{T}_c be the coproduct topology.
- ⟨1⟩3. For every topological space Z and function $f : (X, \mathcal{T}_c) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.
- ⟨2⟩1. LET: Z be a topological space.
- ⟨2⟩2. LET: $f : X \rightarrow Z$
- ⟨2⟩3. If f is continuous then $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.
 PROOF: Because the composite of two continuous functions is continuous.
- ⟨2⟩4. If $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous then f is continuous.
 - ⟨3⟩1. ASSUME: $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.
 - ⟨3⟩2. LET: U be open in Z
 - ⟨3⟩3. For all $\alpha \in A$ we have $\kappa_\alpha^{-1}(f^{-1}(U))$ is open in X_α
 - ⟨3⟩4. $f^{-1}(U) = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(f^{-1}(U))$
 - ⟨3⟩5. $f^{-1}(U)$ is open in X
- ⟨1⟩4. For any topology \mathcal{T} on X , if for every topological space Z and function $f : (X, \mathcal{T}) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous, then $\mathcal{T} = \mathcal{T}_c$.
- ⟨2⟩1. LET: \mathcal{T} be a topology on X .
- ⟨2⟩2. ASSUME: For every topological space Z and function $f : (X, \mathcal{T}) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.
- ⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_c$
 - ⟨3⟩1. For all $\alpha \in A$ we have $\kappa_\alpha : X_\alpha \rightarrow (X, \mathcal{T})$ is continuous.
 PROOF: From ⟨2⟩1 since id_X is continuous.
 - ⟨3⟩2. $\mathcal{T} \subseteq \mathcal{T}_c$
 PROOF: Proposition 15.9.2.
- ⟨2⟩4. $\mathcal{T}_c \subseteq \mathcal{T}$
 - ⟨3⟩1. LET: $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}_c)$ be the identity function.
 - ⟨3⟩2. $f \circ \kappa_\alpha$ is continuous for all α .

⟨3⟩3. f is continuous.

PROOF: ⟨2⟩1

⟨3⟩4. $\mathcal{T}_c \subseteq \mathcal{T}$

□

15.10 Quotient Spaces

Definition 15.10.1 (Quotient Topology). Let X be a topological space, S a set, and $\pi : X \twoheadrightarrow S$ be a surjection. The *quotient topology* on S induced by π is $\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

⟨1⟩1. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.

PROOF: Since $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}$.

⟨1⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

PROOF: Since $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$.

⟨1⟩3. $X \in \mathcal{T}$

PROOF: Since $X = \pi^{-1}(Y)$.

□

Proposition 15.10.2. *Let X be a topological space, S a set and $\pi : X \twoheadrightarrow S$ a surjection. Then the quotient topology on S is the finest topology such that π is continuous.*

PROOF: Immediate from definitions. □

Theorem 15.10.3. *Let X be a topological space, let S be a set, and let $\pi : X \twoheadrightarrow S$ be surjective. Then the quotient topology on S is the unique topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.*

PROOF:

⟨1⟩1. If S is given the quotient topology, then for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.

⟨2⟩1. Give S the quotient topology.

⟨2⟩2. LET: Z be a topological space.

⟨2⟩3. LET: $f : S \rightarrow Z$

⟨2⟩4. If f is continuous then $f \circ \pi$ is continuous.

PROOF: The composite of two continuous functions is continuous.

⟨2⟩5. If $f \circ \pi$ is continuous then f is continuous.

⟨3⟩1. ASSUME: $f \circ \pi$ is continuous.

⟨3⟩2. LET: U be open in Z .

⟨3⟩3. $\pi^{-1}(f^{-1}(U))$ is open in X .

⟨3⟩4. $f^{-1}(U)$ is open in S .

- $\langle 1 \rangle 2$. If S is given a topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous, then that topology is the quotient topology.
 $\langle 2 \rangle 1$. Give S a topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.
 $\langle 2 \rangle 2$. LET: $U \subseteq S$
 $\langle 2 \rangle 3$. If $\pi^{-1}(U)$ is open in X then U is open in S .
 $\langle 3 \rangle 1$. LET: Z be S under the quotient topology induced by π .
 $\langle 3 \rangle 2$. LET: $f : S \rightarrow Z$ be the identity function.
 $\langle 3 \rangle 3$. $f \circ \pi$ is continuous.
 $\langle 3 \rangle 4$. f is continuous.
PROOF: $\langle 2 \rangle 1$
 $\langle 3 \rangle 5$. U is open in Z .
 $\langle 3 \rangle 6$. U is open in X .
 $\langle 2 \rangle 4$. If U is open in S then $\pi^{-1}(U)$ is open in X .
PROOF: Since π is continuous (taking $Z = S$ and $f = \text{id}_S$ in $\langle 2 \rangle 1$).

□

15.10.1 Quotient Maps

Definition 15.10.4 (Quotient Map). Let X and S be topological spaces and $\pi : X \rightarrow S$. Then π is a *quotient map* iff π is surjective and the topology on S is the quotient topology induced by π .

Proposition 15.10.5. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is a quotient map if and only if f is surjective and strongly continuous.*

PROOF: Immediate from definition. □

Proposition 15.10.6. *Let X and Y be topological spaces. Let $p : X \rightarrow Y$ be surjective. Then the following are equivalent.*

1. p is a quotient map.
2. p is continuous and maps saturated open sets to open sets.
3. p is continuous and maps saturated closed sets to closed sets.

PROOF:

- $\langle 1 \rangle 1$. $1 \Rightarrow 2$
 $\langle 2 \rangle 1$. ASSUME: p is a quotient map.
 $\langle 2 \rangle 2$. p is continuous.
 $\langle 2 \rangle 3$. p maps saturated open sets to open sets.
 $\langle 3 \rangle 1$. LET: $U \subseteq X$ be a saturated open set.
 $\langle 3 \rangle 2$. $p^{-1}(p(U)) = U$
 $\langle 3 \rangle 3$. $p^{-1}(p(U))$ is open in X .
 $\langle 3 \rangle 4$. $p(U)$ is open in Y .
 $\langle 1 \rangle 2$. $2 \Rightarrow 3$

- ⟨2⟩1. ASSUME: p is continuous and maps saturated open sets to open sets.
- ⟨2⟩2. LET: C be a saturated closed set in X .
- ⟨2⟩3. $X - C$ is a saturated open set.
- ⟨2⟩4. $Y - p(C)$ is open.
- ⟨2⟩5. $p(C)$ is closed.
- ⟨1⟩3. $3 \Rightarrow 1$
- ⟨2⟩1. ASSUME: p is continuous and maps closed sets to closed sets.
- ⟨2⟩2. LET: $C \subseteq Y$
- ⟨2⟩3. ASSUME: $p^{-1}(C)$ is closed in X .
PROVE: C is closed in Y .
- ⟨2⟩4. $p^{-1}(C)$ is saturated.
- ⟨2⟩5. $p(p^{-1}(C))$ is closed.
- ⟨2⟩6. C is closed.

□

Corollary 15.10.6.1. *Let X and Y be topological spaces. Let $p : X \rightarrow Y$ be continuous and surjective. If p is either an open map or a closed map, then p is a quotient map.*

Example 15.10.7. The converse does not hold.

Let $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \vee y = 0\}$. Then the first projection $\pi_1 : A \rightarrow \mathbb{R}$ is a quotient map that is neither an open map nor a closed map.

PROOF:

- ⟨1⟩1. π_1 is a quotient map.
- ⟨2⟩1. LET: $U \subseteq \mathbb{R}$
- ⟨2⟩2. If U is open then $\pi_1^{-1}(U)$ is open.
PROOF: Since $\pi_1^{-1}(U) = (U \times \mathbb{R}) \cap A$.
- ⟨2⟩3. If $\pi_1^{-1}(U)$ is open then U is open.
- ⟨3⟩1. ASSUME: $\pi_1^{-1}(U)$ is open.
- ⟨3⟩2. LET: $x \in U$
- ⟨3⟩3. $(x, 0) \in \pi_1^{-1}(U)$
- ⟨3⟩4. PICK open neighbourhoods V of x and W of 0 such that $V \times W \subseteq \pi_1^{-1}(U)$
- ⟨3⟩5. $V \subseteq U$
PROOF: For all $x' \in V$ we have $(x', 0) \in V \times W \subseteq \pi_1^{-1}(U)$.
- ⟨1⟩2. π_1 is not an open map.
PROOF: $\pi_1(((-1, 1) \times (1, 2)) \cap A) = [0, 1)$ which is not open in \mathbb{R} .
- ⟨1⟩3. π_1 is not a closed map.
PROOF: $\pi_1(\{(x, 1/x) \in \mathbb{R}^2 : x > 0\}) = (0, +\infty)$ is not closed in \mathbb{R} .

□

Corollary 15.10.7.1. *Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be families of topological spaces and $p_i : X_i \rightarrow Y_i$ for all $i \in I$.*

- 1. *If every p_i is an open quotient map, then $\prod_{i \in I} p_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$ is an open quotient map.*

2. If every p_i is a closed quotient map, then $\prod_{i \in I} p_i : \prod_{i \in I} X_i \twoheadrightarrow \prod_{i \in I} Y_i$ is a closed quotient map.

Example 15.10.8. The product of two quotient maps is not necessarily a quotient map.

Let Y be the quotient space of \mathbb{R}_K obtained by collapsing the set K to a point. Let $p : \mathbb{R}_K \twoheadrightarrow Y$ be the quotient map. Then $q \times q : \mathbb{R}_K^2 \rightarrow Y^2$ is not a quotient map.

PROOF:

$\langle 1 \rangle 1$. LET: $\Delta = \{(y, y) : y \in Y\}$

$\langle 1 \rangle 2$. Y is not Hausdorff.

$\langle 2 \rangle 1$. LET: $*_K \in Y$ be the point such that $q(K) = \{*_K\}$

$\langle 2 \rangle 2$. ASSUME: for a contradiction U and V are disjoint neighbourhoods of 0 and $*_K$

$\langle 2 \rangle 3$. $q^{-1}(U)$ and $q^{-1}(V)$ are disjoint open sets with $0 \in q^{-1}(U)$ and $K \subseteq q^{-1}(V)$

$\langle 2 \rangle 4$. Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 3$. Δ is not closed in Y^2 .

$\langle 1 \rangle 4$. $(q \times q)^{-1}(\Delta)$ is closed in \mathbb{R}_K^2 .

PROOF: It is $\{(x, x) : x \in \mathbb{R}\} \cup K^2$.

□

Proposition 15.10.9. Let $\pi : X \twoheadrightarrow S$ be a quotient map. Let Z be a topological space. Let $f : X \rightarrow Z$ be continuous. Then there exists a continuous map $g : S \rightarrow Z$ such that $f = g \circ \pi$ if and only if, for all $s \in S$, we have f is constant on $\pi^{-1}(s)$.

PROOF: From Theorem 15.10.3. □

Proposition 15.10.10. Let Z be a topological space. Define $\pi : [0, 1] \rightarrow S^1$ by $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$. Given any continuous function $f : S^1 \rightarrow Z$, we have $f \circ \pi$ is a loop in Z . This defines a bijection between $\mathbf{Top}[S^1, Z]$ and the set of loops in Z .

PROOF: Since π is a quotient map. □

Definition 15.10.11 (Projective Space). The *projective space* \mathbb{RP}^n is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by \sim where $x \sim \lambda x$ for all $x \in \mathbb{R}^{n+1} - \{0\}$ and $\lambda \in \mathbb{R}$.

Definition 15.10.12 (Torus). The *torus* T is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$.

Definition 15.10.13 (Möbius Band). The *Möbius band* is the quotient of $[0, 1]^2$ by \sim where $(0, y) \sim (1, 1 - y)$.

Definition 15.10.14 (Klein Bottle). The *Klein bottle* is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$.

Proposition 15.10.15. \mathbb{RP}^2 is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (1 - x, 1)$ and $(0, y) \sim (1, 1 - y)$.

PROOF: TODO

Example 15.10.16. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and $\{Y_i\}_{i \in I}$ a family of sets. Let $q_i : X_i \twoheadrightarrow Y_i$ be a surjective function for all $i \in I$. Give each Y_i the quotient topology. It is not true in general that the product topology on $\prod_{i \in I} Y_i$ is the same as the quotient topology induced by $\prod_{i \in I} q_i : \prod_{i \in I} X_i \twoheadrightarrow \prod_{i \in I} Y_i$.

PROOF:

$\langle 1 \rangle 1$. LET: $X^* = \mathbb{R} - \mathbb{Z}_+ + \{b\}$ be the quotient space obtained from \mathbb{R} by identifying the subset \mathbb{Z}_+ to the point b .

$\langle 1 \rangle 2$. LET: $p : \mathbb{R} \rightarrow X^*$ be the quotient map.

PROVE: $p \times \text{id}_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$ is not a quotient map.

$\langle 1 \rangle 3$. For $n \in \mathbb{Z}_+$,

LET: $c_n = \sqrt{2}/n$

$\langle 1 \rangle 4$. For $n \in \mathbb{Z}_+$,

LET: $U_n = \{(x, y) \in \mathbb{Q} \times \mathbb{R} : n - 1/4 < x < n + 1/4 \text{ and } ((y > x + c_n - n \text{ and } y > -x + c_n + n) \text{ or } (y < x + c_n - n \text{ and } y < -x + c_n + n))\}$

$\langle 1 \rangle 5$. For all $n \in \mathbb{Z}_+$, U_n is open in $\mathbb{R} \times \mathbb{Q}$

$\langle 1 \rangle 6$. For all $n \in \mathbb{Z}_+$ we have $\{n\} \times \mathbb{Q} \subseteq U_n$

$\langle 1 \rangle 7$. LET: $U = \bigcup_{n \in \mathbb{Z}_+} U_n$

$\langle 1 \rangle 8$. U is open in $\mathbb{R} \times \mathbb{Q}$.

$\langle 1 \rangle 9$. U is saturated with respect to $p \times \text{id}_{\mathbb{Q}}$.

$\langle 1 \rangle 10$. LET: $U' = (p \times \text{id}_{\mathbb{Q}})(U)$

$\langle 1 \rangle 11$. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$.

Proposition 15.10.17. Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $\phi : Y \rightarrow X/\sim$.

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \rightarrow X$ such that $\pi \circ \Phi = \phi|_U$. Then ϕ is continuous.

Proposition 15.10.18. Let X be a topological space and \sim an equivalence relation on X . If X/\sim is Hausdorff then every equivalence class of \sim is closed in X .

Definition 15.10.19. Let X be a topological space and $A_1, \dots, A_r \subseteq X$. Then $X/A_1, \dots, A_r$ is the quotient space of X with respect to \sim where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 15.10.20 (Cone). Let X be a topological space. The cone over X is the space $(X \times [0, 1])/(X \times \{1\})$.

Definition 15.10.21 (Suspension). Let X be a topological space. The suspension of X is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 15.10.22 (Wedge Product). Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 15.10.23 (Smash Product). Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 15.10.24. $D^n/S^{n-1} \cong S^n$

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : D^n/S^{n-1} \rightarrow S^n$ be the function induced by the map $D^n \rightarrow S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.

$\langle 1 \rangle 2$. ϕ is a bijection.

$\langle 1 \rangle 3$. ϕ is a homeomorphism.

PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.

□

15.11 Box Topology

Definition 15.11.1 (Box Topology). Let $\{X_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $X = \prod_{i \in I} X_i$ is the topology generated by the basis $\mathcal{B} = \{\prod_{i \in I} U_i : \{U_i\}_{i \in I} \text{ is a family with each } U_i \text{ an open set in } X_i\}$.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. $\bigcup \mathcal{B} = X$

PROOF: Since $\prod_{i \in I} X_i \in \mathcal{B}$.

$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

$\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$

$\langle 2 \rangle 2$. LET: $x \in B_1 \cap B_2$

$\langle 2 \rangle 3$. PICK a family $\{U_i\}_{i \in I}$ such that $B_1 = \prod_{i \in I} U_i$.

$\langle 2 \rangle 4$. PICK a family $\{V_i\}_{i \in I}$ such that $B_2 = \prod_{i \in I} V_i$.

$\langle 2 \rangle 5$. LET: $B_3 = \prod_{i \in I} (U_i \cap V_i)$

$\langle 2 \rangle 6$. $x \in B_3 \subseteq B_1 \cap B_2$

□

Proposition 15.11.2. *The box topology is finer than the product topology.*

PROOF: Immediate from definitions. □

Proposition 15.11.3. *On a finite family of topological spaces, the box topology and the product topology are the same.*

PROOF: Immediate from definitions. □

Proposition 15.11.4. *The box topology is strictly finer than the product topology on the Hilbert cube.*

PROOF: The set $\prod_{n=0}^{\infty} (0, 1/(n+1)^2)$ is open in the box topology but not in the product topology. □

15.11.1 Bases

Proposition 15.11.5. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. For all $i \in I$, let \mathcal{B}_i be a basis for the topology on X_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i : \forall i \in I, B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} X_i$.*

PROOF:

$\langle 1 \rangle 1$. For every family $\{B_i\}_{i \in I}$ where $\forall i \in I, B_i \in \mathcal{B}_i$, we have $\prod_{i \in I} B_i$ is open in the box topology.

PROOF: Since each B_i is open in X_i .

$\langle 1 \rangle 2$. For any open set U in the box topology and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 1$. LET: U be a set open in the box topology.

$\langle 2 \rangle 2$. LET: $x \in U$

$\langle 2 \rangle 3$. PICK a family $\{U_i\}_{i \in I}$ where each U_i is open in X_i such that $x \in \prod_{i \in I} U_i \subseteq U$

$\langle 2 \rangle 4$. For $i \in I$, choose $B_i \in \mathcal{B}_i$ such that $x_i \in B_i \subseteq U_i$.

$\langle 2 \rangle 5$. $\prod_{i \in I} B_i \in \mathcal{B}$

$\langle 2 \rangle 6$. $x \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} U_i \subseteq U$

□

15.11.2 Subspaces

Proposition 15.11.6. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the box topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology that $\prod_{i \in I} Y_i$ inherits as a subspace of $\prod_{i \in I} X_i$ under the box topology.*

PROOF: A basis for the box topology is

$$\begin{aligned} & \left\{ \prod_{i \in I} V_i : V_i \text{ open in } Y_i \right\} \\ &= \left\{ \prod_{i \in I} (U_i \cap Y_i) : U_i \text{ open in } X_i \right\} \\ &= \left\{ \prod_{i \in I} U_i \cap \prod_{i \in I} Y_i : U_i \text{ open in } X_i \right\} \end{aligned}$$

which is a basis for the subspace topology by Proposition 14.7.13. □

15.11.3 Closure

Proposition 15.11.7. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

$\langle 1 \rangle 1$. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

- $\langle 2 \rangle 1$. LET: $x \in \prod_{i \in I} \overline{A_i}$
 $\langle 2 \rangle 2$. For any family $\{U_i\}_{i \in I}$ where each U_i is open in X_i , if $x \in \prod_{i \in I} U_i$ then $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$.
 $\langle 3 \rangle 1$. LET: $\{U_i\}_{i \in I}$ be a family where each U_i is open in X_i .
 $\langle 3 \rangle 2$. ASSUME: $x \in \prod_{i \in I} A_i$
 $\langle 3 \rangle 3$. For all $i \in I$ we have U_i intersects A_i
 PROOF: Since $\pi_i(x) \in A_i$ and U_i is a neighbourhood of $\pi_i(x)$.
 $\langle 3 \rangle 4$. $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$
 $\langle 2 \rangle 3$. $x \in \overline{\prod_{i \in I} A_i}$
 PROOF: Proposition 14.13.9.
 $\langle 1 \rangle 2$. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
 $\langle 2 \rangle 1$. LET: $x \in \overline{\prod_{i \in I} A_i}$
 $\langle 2 \rangle 2$. LET: $i \in I$
 PROVE: $\pi_i(x) \in \overline{A_i}$
 $\langle 2 \rangle 3$. LET: U be a neighbourhood of $\pi_i(x)$ in X_i
 $\langle 2 \rangle 4$. $\pi_i^{-1}(U)$ is a neighbourhood of x in $\prod_{i \in I} X_i$
 $\langle 2 \rangle 5$. PICK $y \in \pi_i^{-1}(U) \cap \prod_{i \in I} A_i$
 $\langle 2 \rangle 6$. $\pi_i(y) \in U \cap A_i$

□

15.12 Separations

Definition 15.12.1 (Separation). Let X be a topological space. A *separation* of X is a pair (U, V) of disjoint nonempty open subsets in X such that $U \cup V = X$.

Subspaces

Proposition 15.12.2. Let X be a topological space and Y a subspace of X . Then a separation of Y is a pair (A, B) of disjoint nonempty subsets of Y , neither of which contains a limit point of the other, such that $A \cup B = Y$.

PROOF: Since the following are equivalent:

- Neither of A and B contains a limit point of the other.
- A contains all its own limit points in Y , and B contains all its own limit points in Y .
- A and B are closed in Y .

□

15.13 Connected Spaces

Definition 15.13.1 (Connected). A topological space is *connected* iff it has no separation.

15.13.1 The Real Numbers

Example 15.13.2. The space \mathbb{R}_l is disconnected. The sets $(-\infty, 0)$ and $[0, +\infty)$ form a separation.

15.13.2 The Indiscrete Topology

Example 15.13.3. Any indiscrete space is connected.

15.13.3 The Cofinite Topology

Example 15.13.4. Any infinite set under the cofinite topology is connected.

PROOF:

$\langle 1 \rangle 1$. LET: X be an infinite set under the cofinite topology.

$\langle 1 \rangle 2$. ASSUME: for a contradiction (C, D) is a separation of X .

$\langle 1 \rangle 3$. $X = (X - C) \cup (X - D) \cup (C \cap D)$

$\langle 1 \rangle 4$. Q.E.D.

PROOF: This is a contradiction since X is infinite, $X - C$ and $X - D$ are finite, and $C \cap D = \emptyset$.

□

Example 15.13.5. The rationals are disconnected. For any irrational a , we have $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of \mathbb{Q} .

Example 15.13.6. \mathbb{R}^ω under the box topology is not connected. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 15.13.7. *A topological space X is connected if and only if the only sets that are both open and closed are \emptyset and X .*

PROOF: Since (U, V) is a separation of X iff U is both open and closed and $V = X - U$. □

15.13.4 Finer and Coarser

Proposition 15.13.8. *Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . Assume $\mathcal{T} \subseteq \mathcal{T}'$. If \mathcal{T}' is connected then \mathcal{T} is connected.*

PROOF: If (C, D) is a separation of (X, \mathcal{T}) then it is a separation of (X, \mathcal{T}') . □

15.13.5 Boundary

Proposition 15.13.9. *Let X be a topological space. Let $A \subseteq X$. Let C be a connected subspace of X . If C intersects A and $X - A$ then C intersects ∂A .*

PROOF: Otherwise $(C \cap \overline{A}, C \cap \overline{X - A})$ would be a separation of C . □

15.13.6 Continuous Functions

Proposition 15.13.10. *The continuous image of a connected space is connected.*

PROOF:

- ⟨1⟩1. LET: X and Y be topological spaces.
- ⟨1⟩2. LET: $f : X \rightarrow Y$ be a surjective continuous function.
- ⟨1⟩3. LET: (C, D) be a separation of Y .
- ⟨1⟩4. $(f^{-1}(C), f^{-1}(D))$ is a separation of X .

□

15.13.7 Subspaces

Proposition 15.13.11. *Let X be a topological space. Let (C, D) be a separation of X . Let Y be a connected subspace of X . Then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $(Y \cap C, Y \cap D)$ would be a separation of Y . □

Proposition 15.13.12. *Let X be a topological space. Let \mathcal{A} be a set of connected subspaces of X and B a connected subspace of X . Assume that, for all $A \in \mathcal{A}$, we have $A \cap B \neq \emptyset$. Then $\bigcup \mathcal{A} \cup B$ is connected.*

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction (C, D) is a separation of $\bigcup \mathcal{A} \cup B$.
- ⟨1⟩2. ASSUME: w.l.o.g. $B \subseteq C$
PROOF: Proposition 15.13.11.
- ⟨1⟩3. For all $A \in \mathcal{A}$ we have $A \subseteq C$
PROOF: Proposition 15.13.11.
- ⟨1⟩4. $D = \emptyset$
- ⟨1⟩5. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 15.13.13. *Let X be a topological space. Let A be a connected subspace of X . Let B be a subspace of X . If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction (C, D) is a separation of B .
- ⟨1⟩2. ASSUME: w.l.o.g. $A \subseteq C$
PROOF: Proposition 15.13.11.
- ⟨1⟩3. $\overline{A} \subseteq \overline{C}$
- ⟨1⟩4. $\overline{C} \cap D = \emptyset$
- ⟨1⟩5. $B \cap D = \emptyset$
- ⟨1⟩6. Q.E.D.

PROOF: This is a contradiction.

□

Corollary 15.13.13.1. *The topologist's sine curve is connected.*

PROOF: The set $\{(x, \sin 1/x) : 0 < x \leq 1\}$ is connected, since it is the continuous image of the connected set $(0, 1]$. The topologist's sine curve is its closure, hence connected by Proposition 15.13.13. \square

Proposition 15.13.14. *Let X be a topological space. Let (A_n) be a sequence of connected subspaces of X such that, for all n , we have $A_n \cap A_{n+1} \neq \emptyset$. Then $\bigcup_n A_n$ is connected.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction (C, D) is a separation of $\bigcup_n A_n$

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $A_0 \subseteq C$

PROOF: Proposition 15.13.11.

$\langle 1 \rangle 3$. $\forall n. A_n \subseteq C$

$\langle 2 \rangle 1$. ASSUME: as induction hypothesis $A_n \subseteq C$

$\langle 2 \rangle 2$. PICK $x \in A_n \cap A_{n+1}$

$\langle 2 \rangle 3$. $x \in C$

$\langle 2 \rangle 4$. $A_{n+1} \subseteq C$

PROOF: Proposition 15.13.11.

$\langle 1 \rangle 4$. $\bigcup_n A_n \subseteq C$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

\square

Proposition 15.13.15. *Let X be a connected topological space. Let $Y \subseteq X$ be connected. Let (A, B) be a separation of $X - Y$. Then $Y \cup A$ and $Y \cup B$ are connected.*

PROOF:

$\langle 1 \rangle 1$. $Y \cup A$ is connected.

$\langle 2 \rangle 1$. ASSUME: for a contradiction (C, D) is a separation of $Y \cup A$

$\langle 2 \rangle 2$. ASSUME: w.l.o.g. $Y \subseteq C$

$\langle 2 \rangle 3$. PICK C' and D' open in X such that $C = C' \cap (Y \cup A)$ and $D = D' \cap (Y \cup A)$

$\langle 2 \rangle 4$. $D = D' \cap A$

$\langle 2 \rangle 5$. $C' \cap D' \cap A = \emptyset$

$\langle 2 \rangle 6$. $A \subseteq C' \cup D'$

$\langle 2 \rangle 7$. PICK A' and B' open in X such that $A = A' - Y$ and $B = B' - Y$

$\langle 2 \rangle 8$. $A' \cap B' \subseteq Y$

$\langle 2 \rangle 9$. $X - Y \subseteq A' \cup B'$

$\langle 2 \rangle 10$. $A' \subseteq C' \cup D'$

$\langle 2 \rangle 11$. $(D' \cap A', B' \cup C')$ is a separation of X .

$\langle 1 \rangle 2$. $Y \cup B$ is connected.

PROOF: Similar.

\square

15.13.8 Order Topology

Proposition 15.13.16. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if X is a linear continuum.*

PROOF:

$\langle 1 \rangle 1$. If L is a linear continuum then L is connected.

$\langle 2 \rangle 1$. LET: L be a linear continuum.

$\langle 2 \rangle 2$. ASSUME: for a contradiction (A, B) is a separation of L .

$\langle 2 \rangle 3$. PICK $a \in A$ and $b \in B$.

$\langle 2 \rangle 4$. ASSUME: w.l.o.g. $a < b$

$\langle 2 \rangle 5$. LET: $c = \sup\{x \in A : x < b\}$

$\langle 2 \rangle 6$. $c \notin A$

$\langle 3 \rangle 1$. ASSUME: for a contradiction $c \in A$.

$\langle 3 \rangle 2$. PICK $e > c$ such that $[c, e) \subseteq A$.

$\langle 3 \rangle 3$. PICK z such that $c < z < e$.

$\langle 3 \rangle 4$. $z \in A$

$\langle 3 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 5$.

$\langle 2 \rangle 7$. $c \notin B$

$\langle 3 \rangle 1$. ASSUME: for a contradiction $c \in B$.

$\langle 3 \rangle 2$. PICK $d < c$ such that $(d, c] \subseteq B$.

$\langle 3 \rangle 3$. PICK z such that $d < z < c$

$\langle 3 \rangle 4$. z is an upper bound for $\{x \in A : x < b\}$

$\langle 3 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 5$.

$\langle 2 \rangle 8$. Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 2$. If L is connected then L is a linear continuum.

$\langle 2 \rangle 1$. ASSUME: L is connected.

$\langle 2 \rangle 2$. L is dense.

$\langle 3 \rangle 1$. LET: $a, b \in L$ with $a < b$.

$\langle 3 \rangle 2$. ASSUME: for a contradiction there is no c such that $a < c < b$.

$\langle 3 \rangle 3$. $((-\infty, b), (a, +\infty))$ is a separation of L .

$\langle 2 \rangle 3$. L has the least upper bound property.

$\langle 3 \rangle 1$. ASSUME: for a contradiction $S \subseteq L$ is a nonempty set bounded above with no least upper bound.

$\langle 3 \rangle 2$. LET: $S \uparrow$ be the set of upper bounds for S .

$\langle 3 \rangle 3$. LET: $S \uparrow \downarrow$ be the set of lower bounds for $S \uparrow$.

PROVE: $(S \uparrow \downarrow, S \uparrow)$ is a separation of L .

$\langle 3 \rangle 4$. $S \uparrow \neq \emptyset$

PROOF: Since S is bounded above.

$\langle 3 \rangle 5$. $S \uparrow \downarrow \neq \emptyset$

PROOF: Since $\emptyset \neq S \subseteq S \uparrow \downarrow$.

$\langle 3 \rangle 6$. $S \uparrow$ is open.

$\langle 4 \rangle 1$. LET: $u \in S \uparrow$

$\langle 4 \rangle 2$. PICK $v \in S \uparrow$ such that $v < u$

PROOF: Since u is not the least upper bound for S .

$\langle 4 \rangle 3$. $u \in (v, +\infty) \subseteq S \uparrow$

$\langle 3 \rangle 7$. $S \uparrow \downarrow$ is open.

$\langle 4 \rangle 1$. LET: $l \in S \uparrow \downarrow$

$\langle 4 \rangle 2$. $l \notin S \uparrow$

PROOF: Since l is not the least upper bound for S .

$\langle 4 \rangle 3$. PICK $s \in S$ such that $l < s$

$\langle 4 \rangle 4$. $l \in (-\infty, s) \subseteq S \uparrow \downarrow$

$\langle 3 \rangle 8$. $S \uparrow \cap S \uparrow \downarrow \neq \emptyset$

PROOF: An element of both would be a least upper bound for S .

$\langle 3 \rangle 9$. $S \uparrow \cup S \uparrow \downarrow = L$

$\langle 4 \rangle 1$. LET: $x \in L$

$\langle 4 \rangle 2$. ASSUME: $x \notin S \uparrow$

$\langle 4 \rangle 3$. There exists $s \in S$ such that $x < s$.

$\langle 4 \rangle 4$. $\forall u \in S \uparrow . x < u$

$\langle 4 \rangle 5$. $x \in S \uparrow \downarrow$

□

Theorem 15.13.17 (Intermediate Value Theorem). *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \rightarrow Y$ be continuous. Let $a, b \in X$ and $r \in Y$. If $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.*

PROOF: Otherwise $\{x \in X : f(x) < r\}$ and $\{x \in X : f(x) > r\}$ would form a separation of X . □

Corollary 15.13.17.1. *Every continuous function $[0, 1] \rightarrow [0, 1]$ has a fixed point.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : [0, 1] \rightarrow [0, 1]$ be continuous.

$\langle 1 \rangle 2$. LET: $g : [0, 1] \rightarrow [-1, 1]$ be the function $g(x) = f(x) - x$.

$\langle 1 \rangle 3$. $g(0) \geq 0$

$\langle 1 \rangle 4$. $g(1) \leq 0$

$\langle 1 \rangle 5$. There exists $x \in [0, 1]$ such that $g(x) = 0$.

PROOF: Intermediate Value Theorem.

$\langle 1 \rangle 6$. There exists $x \in [0, 1]$ such that $f(x) = x$.

□

15.13.9 Product Topology

Proposition 15.13.18. *The product of a family of connected spaces is connected.*

PROOF:

$\langle 1 \rangle 1$. The product of two connected spaces is connected.

PROOF:

$\langle 2 \rangle 1$. LET: X and Y be connected topological spaces.

- <2>2. ASSUME: w.l.o.g. X and Y are nonempty.
 <2>3. PICK $(a, b) \in X \times Y$
 <2>4. $X \times \{b\}$ is connected.
 PROOF: It is homeomorphic to X .
 <2>5. For all $x \in X$ we have $\{x\} \times Y$ is connected.
 PROOF: It is homeomorphic to Y .
 <2>6. For all $x \in X$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.
 PROOF: Proposition 15.13.12.
 <2>7. $X \cup Y$ is connected.
 PROOF: Proposition 15.13.12 since $X \cup Y = \bigcup_{x \in X} ((X \times \{b\}) \cup (\{x\} \times Y))$
 and the subspaces all have the point (a, b) in common.
 <1>2. LET: $\{X_i\}_{i \in I}$ be a family of connected spaces.
 <1>3. LET: $X = \prod_{i \in I} X_i$
 <1>4. ASSUME: w.l.o.g. each X_i is nonempty.
 <1>5. PICK $a \in X$
 <1>6. For every finite $K \subseteq I$,
 LET: $X_K = \{x \in X : \forall i \notin K. \pi_i(x) = \pi_i(a)\}$
 <1>7. For every finite $K \subseteq I$, we have X_K is connected.
 PROOF: It is homeomorphic to $\prod_{i \in K} X_i$ which is connected by <1>1.
 <1>8. LET: $Y = \bigcup_{K \text{ a finite subset of } I} X_K$
 <1>9. Y is connected.
 PROOF: Proposition 15.13.12 since $a \in X_K$ for all K .
 <1>10. $X = \overline{Y}$
 <2>1. LET: $x \in X$
 <2>2. LET: U be a neighbourhood of x .
 PROVE: U intersects Y .
 <2>3. PICK a finite subset K of I and U_i open in each X_i such that $U_i = X_i$
 for all $i \notin K$, and $x \in \prod_i U_i \subseteq U$
 <2>4. LET: $y \in X$ be the point with $\pi_i(y) = \pi_i(x)$ for $i \in K$ and $\pi_i(y) = \pi_i(a)$
 for $i \notin K$
 <2>5. $y \in U \cap Y$
 <1>11. X is connected.
 PROOF: Proposition 15.13.13.
 □

Proposition 15.13.19. *Let X and Y be topological spaces. Let A be a proper subset of X and B a proper subset of Y . Then $(X \times Y) - (A \times B)$ is connected.*

PROOF:

- <1>1. PICK $x_0 \in X - A$
 <1>2. PICK $y_0 \in Y - B$
 <1>3. LET: $C = ((X - A) \times Y) \cup (X \times \{y_0\})$
 <1>4. LET: $D = (\{x_0\} \times Y) \cup (X \times (Y - B))$
 <1>5. C is connected.
 <2>1. $C = \bigcup_{x \in X - A} (\{x\} \times Y) \cup (X \times \{y_0\})$
 <2>2. For all $x \in X - A$ we have $\{x\} \times Y$ is connected.
 PROOF: It is homeomorphic to Y .

⟨2⟩3. $X \times \{y_0\}$ is connected.

PROOF: It is homeomorphic to X .

⟨2⟩4. For all $x \in X - A$ we have $(x, y_0) \in (\{x\} \times Y) \cap (X \times \{y_0\})$

⟨2⟩5. C is connected.

PROOF: Proposition 15.13.12.

⟨1⟩6. D is connected.

PROOF: Similar.

⟨1⟩7. $(X \times Y) - (A \times B) = C \cup D$

⟨1⟩8. $(X \times Y) - (A \times B)$ is connected.

PROOF: Proposition 15.13.12 since $(x_0, y_0) \in C \cap D$.

□

15.13.10 Quotient Spaces

Proposition 15.13.20. *A quotient of a connected space is connected.*

PROOF:

⟨1⟩1. LET: $p : X \twoheadrightarrow Y$ be a quotient map.

⟨1⟩2. If (C, D) is a separation of Y then $(p^{-1}(C), p^{-1}(D))$ is a separation of X .

□

Proposition 15.13.21. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Assume that Y is connected, for all $y \in Y$, we have $p^{-1}(y)$ is connected. Then X is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction (A, B) is a separation of X .

⟨1⟩2. For all $y \in Y$, either $p^{-1}(y) \subseteq A$ or $p^{-1}(y) \subseteq B$.

⟨1⟩3. $(\{y \in Y : p^{-1}(y) \subseteq A\}, \{y \in Y : p^{-1}(y) \subseteq B\})$ form a separation of Y .

⟨1⟩4. Q.E.D.

PROOF: This is a contradiction.

□

15.14 T_1 Spaces

Definition 15.14.1 (T_1). A topological space is T_1 iff every one-point set is closed.

Proposition 15.14.2. *A topological space is T_1 iff every finite set is closed.*

PROOF: Since the union of finitely many closed sets is closed. □

Proposition 15.14.3. *Let X be a topological space. Then X is T_1 if and only if, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x .*

PROOF:

⟨1⟩1. If X is T_1 then, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x .

- ⟨2⟩1. ASSUME: X is T_1 .
 - ⟨2⟩2. LET: $x, y \in X$
 - ⟨2⟩3. ASSUME: $x \neq y$
 - ⟨2⟩4. $X - \{y\}$ is a neighbourhood of x that does not contain y .
 - ⟨2⟩5. $X - \{x\}$ is a neighbourhood of y that does not contain x .
 - ⟨1⟩2. If, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x , then X is T_1 .
 - ⟨2⟩1. ASSUME: For all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x .
 - ⟨2⟩2. LET: $x \in X$
PROVE: $\{x\}$ is closed.
 - ⟨2⟩3. LET: $y \in X - \{x\}$
 - ⟨2⟩4. PICK a neighbourhood U of y that does not contain x .
 - ⟨2⟩5. $y \in U \subseteq X - \{x\}$
-

15.14.1 Limit Points

Proposition 15.14.4. *Let X be a T_1 space. Let $A \subseteq X$ and $l \in X$. Then l is a limit point of A if and only if every neighbourhood of l contains infinitely many points of A .*

PROOF:

- ⟨1⟩1. If l is a limit point of A then every neighbourhood of l contains infinitely many points of A .
- ⟨2⟩1. ASSUME: l is a limit point of A .
- ⟨2⟩2. LET: U be a neighbourhood of l .
- ⟨2⟩3. ASSUME: for a contradiction $U \cap A - \{l\}$ is finite.
- ⟨2⟩4. $U \cap A - \{l\}$ is closed.
- PROOF: Since X is T_1 .
- ⟨2⟩5. $U - (A - \{l\})$ is a neighbourhood of l .
- ⟨2⟩6. $U - (A - \{l\})$ intersects A .
- ⟨2⟩7. Q.E.D.
- ⟨1⟩2. If every neighbourhood of l contains infinitely many points of A then l is a limit point of A .

PROOF: Immediate from definitions.

□

15.15 Hausdorff Spaces

Definition 15.15.1 (Hausdorff). A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 15.15.2. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space.
- ⟨1⟩2. LET: (a_n) be a sequence in X and $l, m \in X$
- ⟨1⟩3. ASSUME: $a_n \rightarrow l$ and $a_n \rightarrow m$
- ⟨1⟩4. ASSUME: for a contradiction $l \neq m$
- ⟨1⟩5. PICK disjoint open sets U and V with $l \in U$ and $m \in V$
- ⟨1⟩6. PICK M, N such that $\forall n \geq M. a_n \in U$ and $\forall n \geq N. a_n \in V$
- ⟨1⟩7. $a_{\max(M, N)} \in U \cap V$
- ⟨1⟩8. Q.E.D.

PROOF: This contradicts the fact that $U \cap V = \emptyset$.

□

Example 15.15.3. We cannot weaken the hypothesis from being Hausdorff to being T_1 .

In the cofinite topology on any infinite set, every sequence converges to every point.

Proposition 15.15.4. *Any linearly ordered set is Hausdorff under the order topology.*

PROOF:

- ⟨1⟩1. LET: X be a linearly ordered set under the order topology.
- ⟨1⟩2. LET: $a, b \in X$ with $a \neq b$.
- ⟨1⟩3. ASSUME: w.l.o.g. $a < b$.
- ⟨1⟩4. CASE: There exists $c \in X$ such that $a < c < b$.
 - ⟨2⟩1. LET: $U = (-\infty, c)$
 - ⟨2⟩2. LET: $V = (c, +\infty)$
 - ⟨2⟩3. U and V are disjoint open sets with $a \in U$ and $b \in V$
- ⟨1⟩5. CASE: There is no $c \in X$ such that $a < c < b$.
 - ⟨2⟩1. LET: $U = (-\infty, b)$
 - ⟨2⟩2. LET: $V = (a, +\infty)$
 - ⟨2⟩3. U and V are disjoint open sets with $a \in U$ and $b \in V$

□

Proposition 15.15.5. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space.
- ⟨1⟩2. LET: Y be a subspace of X .
- ⟨1⟩3. LET: $a, b \in Y$ with $a \neq b$.
- ⟨1⟩4. PICK disjoint open sets U and V in X with $a \in U$ and $b \in V$.
- ⟨1⟩5. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y with $a \in U \cap Y$ and $b \in V \cap Y$.

□

Proposition 15.15.6. *The disjoint union of two Hausdorff spaces is Hausdorff.*

Proposition 15.15.7. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \rightarrow B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X , then $f = g$.*

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.
- ⟨1⟩2. PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.
- ⟨1⟩3. PICK $x \in f^{-1}(U) \cap g^{-1}(V)$
- ⟨1⟩4. $f(x) = g(x) \in U \cap V$
- ⟨1⟩5. Q.E.D.

PROOF: This is a contradiction.

□

15.15.1 Product Topology

Proposition 15.15.8. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

- ⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
- ⟨1⟩2. LET: $x, y \in \prod_{i \in I} X_i$ with $x \neq y$.
- ⟨1⟩3. PICK $i \in I$ such that $\pi_i(x) \neq \pi_i(y)$
- ⟨1⟩4. PICK disjoint open sets U and V in X_i such that $\pi_i(x) \in U$ and $\pi_i(y) \in V$.
- ⟨1⟩5. $x \in \pi_i^{-1}(U)$ and $y \in \pi_i^{-1}(V)$.

□

15.15.2 Box Topology

Proposition 15.15.9. *The box product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

- ⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
- ⟨1⟩2. LET: $x, y \in \prod_{i \in I} X_i$ with $x \neq y$.
- ⟨1⟩3. PICK $i \in I$ such that $\pi_i(x) \neq \pi_i(y)$
- ⟨1⟩4. PICK disjoint open sets U and V in X_i such that $\pi_i(x) \in U$ and $\pi_i(y) \in V$.
- ⟨1⟩5. $x \in \pi_i^{-1}(U)$ and $y \in \pi_i^{-1}(V)$.

□

15.15.3 T_1 Spaces

Proposition 15.15.10. *Every Hausdorff space is T_1 .*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space.
- ⟨1⟩2. LET: $a \in X$
PROVE: $X - \{a\}$ is open.
- ⟨1⟩3. LET: $x \in X - \{a\}$
- ⟨1⟩4. PICK disjoint open sets U and V with $a \in U$ and $x \in V$
- ⟨1⟩5. $x \in V \subseteq X - U \subseteq X - \{a\}$

□

Example 15.15.11. The converse does not hold. If X is an infinite set under the cofinite topology, then X is T_1 but not Hausdorff.

Proposition 15.15.12. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$ be uniformly continuous. Let \hat{X} and \hat{Y} be the completions of X and Y . Then f extends uniquely to a continuous map $\hat{X} \rightarrow \hat{Y}$.*

PROOF: The extension maps $\lim_{n \rightarrow \infty} x_n$ to $\lim_{n \rightarrow \infty} f(x_n)$. \square

Proposition 15.15.13. *Let X be a topological space. Then X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in X^2 .*

PROOF:

Δ is closed

$\Leftrightarrow X^2 - \Delta$ is open

$\Leftrightarrow \forall x, y \in X ((x, y) \notin \Delta \Rightarrow \exists V, W \text{ open in } X (x \in V \wedge y \in W \wedge V \times W \subseteq X^2 - \Delta))$

$\Leftrightarrow \forall x, y \in X (x \neq y \Rightarrow \exists V, W \text{ open in } X (x \in V \wedge y \in W \wedge V \cap W = \emptyset))$

$\Leftrightarrow X$ is Hausdorff \square

15.16 Separable Spaces

Definition 15.16.1 (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

15.17 Sequential Compactness

Definition 15.17.1 (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

15.18 Compactness

Definition 15.18.1 (Compact). A topological space is *compact* iff every open cover has a finite subcover.

Proposition 15.18.2. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P . Then $X \in P$.*

PROOF:

$\langle 1 \rangle 1.$ P is an open cover of X

$\langle 1 \rangle 2.$ PICK a finite subcover $U_1, \dots, U_n \in P$

$\langle 1 \rangle 3.$ $X = U_1 \cup \dots \cup U_n \in P$

\square

Corollary 15.18.2.1. *Let f be a compact space and $f : X \rightarrow \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. \square

Proposition 15.18.3. *The continuous image of a compact space is compact.*

Proposition 15.18.4. *A closed subspace of a compact space is compact.*

Proposition 15.18.5. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. X and Y are compact.
2. $X + Y$ is compact.
3. $X \times Y$ is compact.

Proposition 15.18.6. *A compact subspace of a Hausdorff space is closed.*

Proposition 15.18.7. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 15.18.8. *A first countable compact space is sequentially compact.*

15.19 Gluing

Definition 15.19.1 (Gluing). Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \rightarrow Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where \sim is the equivalence relation generated by $x \sim \phi(x)$ for all $x \in X_0$.

Proposition 15.19.2. *Y is a subspace of $Y \cup_\phi X$.*

Definition 15.19.3. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0, 1])/\alpha$ is the quotient space of $X \times [0, 1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x \in X$.

Definition 15.19.4 (Möbius Strip). The *Möbius strip* is $([-1, 1] \times [0, 1])/\alpha$ where $\alpha(x) = -x$.

Definition 15.19.5 (Klein Bottle). The *Klein bottle* is $(S^1 \times [0, 1])/\alpha$ where $\alpha(z) = \bar{z}$.

Proposition 15.19.6. *Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\text{id}_M} M \cong K$.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{\pi i \theta / 2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-\pi i \theta / 2}, t)$.

$\langle 1 \rangle 2$. f induces a bijection $M \cup_{\text{id}_M} M \approx K$

$\langle 1 \rangle 3$. f is a homeomorphism.

\square

15.20 Homogeneous Spaces

Definition 15.20.1 (Homogeneous). A topological space X is *homogeneous* iff, for any $x, y \in X$, there exists a homeomorphism $f : X \cong X$ such that $f(x) = y$.

15.21 Regular Spaces

Definition 15.21.1 (Regular). A topological space X is *regular* iff it is T_1 and, for every closed set A and point $x \notin A$, there exist disjoint open sets U and V with $A \subseteq U$ and $x \in V$.

15.22 Totally Disconnected Spaces

Definition 15.22.1 (Totally Disconnected). A topological space X is *totally disconnected* iff the only connected subspaces are the one-point subspaces.

Example 15.22.2. Every discrete space is totally disconnected.

Example 15.22.3. The rationals are totally disconnected.

15.23 Path Connected Spaces

Definition 15.23.1 (Path-connected). A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \rightarrow X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

15.23.1 The Ordered Square

Proposition 15.23.2. *The ordered square is not path connected.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [a, b] \rightarrow I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.

$\langle 1 \rangle 2$. p is surjective.

PROOF: Intermediate Value Theorem.

$\langle 1 \rangle 3$. For all $x \in [0, 1]$, the set $p^{-1}(\{x\} \times (0, 1))$ is a nonempty open set in $[0, 1]$.

$\langle 1 \rangle 4$. For all $x \in [0, 1]$ choose a rational $q_x \in p^{-1}(\{x\} \times (0, 1))$.

$\langle 1 \rangle 5$. The mapping that maps x to q_x is an injective function $[0, 1] \rightarrow \mathbb{Q}$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts the fact that $[0, 1]$ is uncountable and \mathbb{Q} is countable.

□

15.23.2 Punctured Euclidean Space

Proposition 15.23.3. *For $n > 1$, the punctured Euclidean space $\mathbb{R}^n - \{0\}$ is path connected.*

PROOF: Given points x and y , take the straight line from x to y if this does not pass through 0. Otherwise pick a point z not on this line, and take the two straight lines from x to z then from z to y . \square

15.23.3 The Topologist's Sine Curve

Proposition 15.23.4. *The topologist's sine curve is not path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) : 0 < x \leq 1\}$

$\langle 1 \rangle 2$. ASSUME: for a contradiction $p : [0, 1] \rightarrow \bar{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

$\langle 1 \rangle 3$. LET: b be the largest element of $p^{-1}(\{0\} \times [-1, 1])$

$\langle 1 \rangle 4$. For n a positive integer, choose t_n such that $b < t_n < ((n-1)b + 1)/n$ and $\pi_2(p(t_n)) = (-1)^n$

$\langle 1 \rangle 5$. $t_n \rightarrow b$ as $n \rightarrow \infty$

$\langle 1 \rangle 6$. $(p(t_n))$ does not converge.

$\langle 1 \rangle 7$. Q.E.D.

PROOF: This is a contradiction.

\square

15.23.4 The Long Line

Proposition 15.23.5. *The long line is path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $L = S_\Omega \times [0, 1)$ be the long line.

$\langle 1 \rangle 2$. LET: $(a, b), (c, d) \in L$

$\langle 1 \rangle 3$. PICK e such that $a < e$ and $c < e$

$\langle 1 \rangle 4$. $(a, b), (c, d) \in [(0, 0), (e, 0)) \cong [0, 1)$

PROOF: Using Proposition 6.5.2.

$\langle 1 \rangle 5$. There is a path from (a, b) to (c, d) .

\square

15.23.5 Continuous Functions

Proposition 15.23.6. *The continuous image of a path connected space is path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a path connected space and Y a topological space.

$\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$ be a surjective continuous function.

PROVE: Y is path connected.

$\langle 1 \rangle 3$. LET: $a, b \in Y$

$\langle 1 \rangle 4$. PICK $x, y \in X$ with $f(x) = a$ and $f(y) = b$.

$\langle 1 \rangle 5$. PICK a path $p : [0, 1] \rightarrow X$ from x to y .

$\langle 1 \rangle 6$. $f \circ p$ is a path from a to b .

\square

15.23.6 Subspaces

Proposition 15.23.7. *Let $\{X\}$ be a topological space. Let \mathcal{A} be a set of connected subspaces of X . If $\bigcap \mathcal{A} \neq \emptyset$ then $\bigcup \mathcal{A}$ is connected.*

PROOF:

- $\langle 1 \rangle 1$. PICK $a \in \bigcap \mathcal{A}$
- $\langle 1 \rangle 2$. PICK $x, y \in \bigcup \mathcal{A}$
- $\langle 1 \rangle 3$. PICK $A, B \in \mathcal{A}$ with $x \in A$ and $y \in B$.
- $\langle 1 \rangle 4$. PICK a path p from x to a in A , and a path q from a to y in B .
- $\langle 1 \rangle 5$. The concatenation of p and q is a path from x to y in $\bigcup \mathcal{A}$.

□

Proposition 15.23.8. *A quotient of a path connected space is path connected.*

15.23.7 Product Topology

Proposition 15.23.9. *The product of a family of path connected spaces is path connected.*

PROOF:

- $\langle 1 \rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of path connected spaces.
- $\langle 1 \rangle 2$. LET: $x, y \in \prod_{i \in I} X_i$
- $\langle 1 \rangle 3$. For $i \in I$, PICK a path $p_i : [0, 1] \rightarrow X_i$ from $\pi_i(x)$ to $\pi_i(y)$
- $\langle 1 \rangle 4$. $\lambda t \in [0, 1]. \lambda i \in I. p_i(t)$ is a path from x to y in $\prod_{i \in I} X_i$.

□

Proposition 15.23.10. *Let $A \subseteq \mathbb{R}^2$. If A is countable then $\mathbb{R}^2 - A$ is path connected.*

PROOF:

- $\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}^2 - A$
- $\langle 1 \rangle 2$. PICK two non-parallel lines L through x and L' through y that do not pass through any points in A .

PROOF: These exist since uncountably many lines pass through any point.

- $\langle 1 \rangle 3$. There exists a path from x to y that follows L from x to the point of intersection of L and L' , and then follows L' to y .

□

15.23.8 Connected Spaces

Proposition 15.23.11. *Every path connected space is connected.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be a path connected space.
- $\langle 1 \rangle 2$. ASSUME: for a contradiction (A, B) is a separation of X .
- $\langle 1 \rangle 3$. PICK $a \in A$ and $b \in B$
- $\langle 1 \rangle 4$. PICK a path $p : [0, 1] \rightarrow X$ from a to b .

⟨1⟩5. $(p^{-1}(A), p^{-1}(B))$ is a separation of $[0, 1]$.

⟨1⟩6. Q.E.D.

PROOF: This contradicts Proposition 15.13.16.

□

Corollary 15.23.11.1. *For $n > 1$, we have \mathbb{R}^n and \mathbb{R} are not homeomorphic.*

PROOF: Removing a point from \mathbb{R} gives a disconnected space. □

Proposition 15.23.12. *Every open connected subspace of \mathbb{R}^2 is path connected.*

PROOF:

⟨1⟩1. LET: U be an open connected subspace of \mathbb{R}^2 .

⟨1⟩2. ASSUME: w.l.o.g. $U \neq \emptyset$

⟨1⟩3. PICK $x_0 \in U$

⟨1⟩4. LET: $V = \{x \in U : \text{there exists a path from } x_0 \text{ to } x\}$

⟨1⟩5. V is open in U .

⟨2⟩1. LET: $x \in V$

⟨2⟩2. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$

⟨2⟩3. $B(x, \epsilon) \subseteq V$

PROOF: For all $y \in B(x, \epsilon)$, take a path from x_0 to x and then a straight line from x to y .

⟨1⟩6. V is closed in U .

⟨2⟩1. LET: $x \in U - V$

⟨2⟩2. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$

⟨2⟩3. $B(x, \epsilon) \subseteq U - V$

⟨3⟩1. LET: $y \in B(x, \epsilon)$

⟨3⟩2. There is a path from y to x .

⟨3⟩3. There is no path from x_0 to y .

⟨1⟩7. $V = U$

PROOF: U is connected.

□

15.24 Locally Homeomorphic

Definition 15.24.1. Let X and Y be topological spaces. Then X is *locally homeomorphic* to Y if and only if every point in X has a neighbourhood that is homeomorphic to an open set in Y .

15.24.1 The Long Line

Proposition 15.24.2. *The long line is locally homeomorphic to $[0, 1)$.*

PROOF: By Proposition 6.5.2. □

15.25 Components

Definition 15.25.1 ((Connected) Component). Let X be a topological space. Define the equivalence relation \sim on X by: $x \sim y$ iff there exists a connected $C \subseteq X$ such that $x \in C$ and $y \in C$. The *components* of X are the equivalence classes with respect to \sim .

We prove this is an equivalence relation.

PROOF:

$\langle 1 \rangle 1.$ \sim is reflexive.

PROOF: For any $x \in X$, we have $\{x\}$ is connected and $x \in \{x\}$, hence $x \sim x$.

$\langle 1 \rangle 2.$ \sim is symmetric.

PROOF: Immediate from definition.

$\langle 1 \rangle 3.$ \sim is transitive.

$\langle 2 \rangle 1.$ ASSUME: $x \sim y$ and $y \sim z$

$\langle 2 \rangle 2.$ PICK connected subspaces C and D of X with $x \in C$, $y \in C$, $y \in D$ and $z \in D$.

$\langle 2 \rangle 3.$ $C \cup D$ is connected.

PROOF: Proposition 15.13.12.

$\langle 2 \rangle 4.$ $x \in C \cup D$ and $z \in C \cup D$.

$\langle 2 \rangle 5.$ $x \sim z$

□

Example 15.25.2. The components of \mathbb{Q} are the singleton subsets.

Example 15.25.3. The components of \mathbb{R}_l are the singleton subsets.

Proposition 15.25.4. *Every component of a topological space is connected.*

PROOF:

$\langle 1 \rangle 1.$ LET: C be a component of the topological space X .

$\langle 1 \rangle 2.$ ASSUME: for a contradiction (A, B) is a separation of C .

$\langle 1 \rangle 3.$ PICK $a \in A$ and $b \in B$.

$\langle 1 \rangle 4.$ $a \sim b$

$\langle 1 \rangle 5.$ PICK a connected subspace D of X such that $a \in D$ and $b \in D$.

$\langle 1 \rangle 6.$ $D \subseteq C$

$\langle 1 \rangle 7.$ $(A \cap D, B \cap D)$ is a separation of D .

$\langle 1 \rangle 8.$ Q.E.D.

PROOF: This is a contradiction.

□

Proposition 15.25.5. *Let X be a topological space. Let A be a nonempty connected subspace of X . Then there exists a unique component C of X such that $A \subseteq C$.*

PROOF:

$\langle 1 \rangle 1.$ PICK $a \in A$

$\langle 1 \rangle 2.$ LET: C be the \sim -equivalence class of a .

$\langle 1 \rangle 3.$ $A \subseteq C$

PROOF: For all $x \in A$ we have $a \sim x$ hence $x \in C$.

$\langle 1 \rangle 4$. For any component C' , if $A \subseteq C'$ then $C' = C$.

PROOF: Since the components are pairwise disjoint.

□

Proposition 15.25.6. *Every component of a topological space is closed.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a topological space.

$\langle 1 \rangle 2$. LET: C be a component of X .

$\langle 1 \rangle 3$. \overline{C} is connected.

PROOF: Proposition 15.13.13.

$\langle 1 \rangle 4$. $\overline{C} \subseteq C$

PROOF: Proposition 15.25.5.

$\langle 1 \rangle 5$. $C = \overline{C}$

□

Corollary 15.25.6.1. *If a topological space has only finitely many components, then its components are open.*

15.26 Path Components

Definition 15.26.1 (Path Component). Let X be a topological space. Define the equivalence relation \sim on X by: $x \sim y$ iff there exists a path from x to y . The *path components* of X are the equivalence classes with respect to \sim .

We prove \sim is an equivalence relation.

PROOF:

$\langle 1 \rangle 1$. \sim is reflexive.

PROOF: For any $a \in X$ the constant path at a is a path from a to a .

$\langle 1 \rangle 2$. \sim is symmetric.

PROOF: If p is a path from a to b then the reverse of p is a path from b to a .

$\langle 1 \rangle 3$. \sim is transitive.

PROOF: If p is a path from a to b and q is a path from b to c then the concatenation of p and q is a path from a to c .

□

Example 15.26.2. The topologist's sine curve has two path components, namely $\{0\} \times [0, 1]$ (which is closed and not open) and $\{(x, \sin 1/x) : 0 < x \leq 1\}$ (which is open and not closed).

Proposition 15.26.3. *Every path component is path connected.*

PROOF: If x and y are in the same path component then $x \sim y$ so there is a path from x to y . □

Corollary 15.26.3.1. *Every path component is a subset of a component.*

Proposition 15.26.4. *Let X be a topological space. Let A be a nonempty path connected subspace of X . Then there exists a unique path component C of X such that $A \subseteq C$.*

PROOF:

$\langle 1 \rangle 1$. PICK $a \in A$

$\langle 1 \rangle 2$. LET: C be the path component of a .

$\langle 1 \rangle 3$. $A \subseteq C$

PROOF: For all $x \in A$ we have $a \sim x$ (because A is path connected) hence $x \in C$.

$\langle 1 \rangle 4$. For any path component C' , if $A \subseteq C'$ then $C = C'$.

PROOF: This holds because the path components are pairwise disjoint.

□

Example 15.26.5. In \mathbb{R}^ω under the box topology, \vec{x} and \vec{y} are in the same component if and only if $\vec{x} - \vec{y}$ is eventually zero.

PROOF:

$\langle 1 \rangle 1$. LET: B be the set of sequences that are eventually zero.

$\langle 1 \rangle 2$. B is connected.

PROOF: For $\vec{x} \in B$, the straight line path from 0 to \vec{x} is continuous.

$\langle 1 \rangle 3$. B is maximally connected.

PROOF: Since $(B, \mathbb{R}^\omega - B)$ form a separation of \mathbb{R}^ω .

$\langle 1 \rangle 4$. For all $\vec{y} \in \mathbb{R}^\omega$, the component that contains \vec{y} is $\{\vec{x} \in \mathbb{R}^\omega : \vec{x} - \vec{y} \text{ is eventually zero}\}$.

PROOF: Since the function that maps \vec{x} to $\vec{x} + \vec{y}$ is a homeomorphism of \mathbb{R}^ω with itself.

□

Example 15.26.6. The path components of I_o^2 are $\{\{x\} \times [0, 1] : 0 \leq x \leq 1\}$.

PROOF:

$\langle 1 \rangle 1$. For all $x \in [0, 1]$ we have $\{x\} \times [0, 1]$ is path connected.

PROOF: It is homeomorphic to $[0, 1]$.

$\langle 1 \rangle 2$. Given $x, y, s, t \in [0, 1]$ with $x \neq y$, there is no path from (x, s) to (y, t) .

$\langle 2 \rangle 1$. ASSUME: for a contradiction $p : [0, 1] \rightarrow I_o^2$ is a path from (x, s) to (y, t) .

$\langle 2 \rangle 2$. For z between x and y , PICK a rational $q_z \in [0, 1]$ such that $p(q_z) \in \{z\} \times [0, 1]$.

$\langle 2 \rangle 3$. $\{q_z : z \text{ is between } x \text{ and } y\}$ is an uncountable set of rationals.

$\langle 2 \rangle 4$. Q.E.D.

PROOF: This is a contradiction.

□

15.27 Weak Local Connectedness

Definition 15.27.1 (Weakly Locally Connected). Let X be a topological space and $x \in X$. Then X is *weakly locally connected* at x iff, for every neighbourhood

U of x , there exists a connected $Y \subseteq X$ and a neighbourhood V of x such that $V \subseteq Y \subseteq U$.

15.28 Local Connectedness

Definition 15.28.1 (Locally Connected). Let X be a topological space and $x \in X$. Then X is *locally connected* at x iff, for every neighbourhood U of x , there exists a connected neighbourhood V of x such that $V \subseteq U$.

The space X is *locally connected* iff it is locally connected at every point.

Example 15.28.2. Every interval and ray in the real line is connected and locally connected.

Example 15.28.3. The space $[-1, 0) \cup (0, 1]$ is locally connected but not connected.

Example 15.28.4. The topologist's sine curve is connected but not locally connected.

Example 15.28.5. The rationals \mathbb{Q} are neither connected nor locally connected.

Example 15.28.6. For n a positive integer, let $a_n = (1/n, 0)$. Let $p = (0, 0)$. Let the infinite broom X be the union of all the line segments joining (a_{n+1}, q) to $(a_n, 0)$ for n any positive integer and q any rational in $[0, 1/n]$. Then X is weakly locally connected at p but not locally connected at p .

PROOF:

- ⟨1⟩1. X is weakly locally connected at p .
- ⟨2⟩1. LET: U be any neighbourhood of p .
- ⟨2⟩2. PICK N such that, for all $n \geq N$ and every rational $q \in [0, 1/n]$, the line segment joining (a_{n+1}, q) to $(a_n, 0)$ is included in U .
- ⟨2⟩3. LET: Y be the union of all these line segments.
- ⟨2⟩4. Y is connected.
- ⟨2⟩5. LET: $V = B(p, 1/n) \cap X$
- ⟨2⟩6. $V \subseteq Y \subseteq U$
- ⟨1⟩2. X is not locally connected at p .
- ⟨2⟩1. LET: $U = B(p, 1/2) \cap X$
- ⟨2⟩2. LET: V be a neighbourhood of p with $V \subseteq U$
PROVE: V is disconnected.
- ⟨2⟩3. LET: n be least such that $(a_n, 0) \in V$
- ⟨2⟩4. $(a_{n-1}, 0) \notin V$
- ⟨2⟩5. Some part of a line segment joining some (a_n, q) to $(a_{n-1}, 0)$ is in V
- ⟨2⟩6. V is disconnected.

□

Theorem 15.28.7. Let X be a topological space. Then X is locally connected if and only if, for every open set U in X , every component of U is open in X .

PROOF:

- ⟨1⟩1. If X is locally connected then, for every open set U in X , every component of U is open in X .
 - ⟨2⟩1. ASSUME: X is locally connected.
 - ⟨2⟩2. LET: U be an open set in X .
 - ⟨2⟩3. LET: C be a component of U .
 - ⟨2⟩4. LET: $x \in C$
 - ⟨2⟩5. PICK a connected neighbourhood V of x in X such that $V \subseteq U$
 - ⟨2⟩6. $x \in V \subseteq C$
- ⟨1⟩2. If, for every open set U in X , every component of U is open in X , then X is locally connected.
 - ⟨2⟩1. ASSUME: For every open set U in X , every component of U is open in X .
 - ⟨2⟩2. LET: $x \in X$
 - ⟨2⟩3. LET: U be a neighbourhood of x .
 - ⟨2⟩4. LET: V be the component of U that contains x .
 - ⟨2⟩5. V is a connected neighbourhood of x and $V \subseteq U$.

□

Proposition 15.28.8. *The ordered square is locally connected.*

PROOF: Since every basic open set is connected because it is a linear continuum.

□

Example 15.28.9. Let T be the union of all line segments connecting a point $(q, 0)$ to $(0, 1)$ where $q \in [0, 1]$ is rational, and all line segments connecting a point $(q, 1)$ to $(1, 0)$ where $q \in [0, 1]$ is rational. Then T is path connected but is locally connected at no point.

Proposition 15.28.10. *If a topological space is weakly locally connected at every point then it is locally connected.*

PROOF:

- ⟨1⟩1. LET: X be a topological space that is weakly locally connected at every point.
- ⟨1⟩2. For every open set U in X , every component of U is open in X .
 - ⟨2⟩1. LET: U be an open set in X .
 - ⟨2⟩2. LET: C be a component of U .
 - ⟨2⟩3. For all $x \in C$, there exists a neighbourhood V of x such that $V \subseteq C$.
 - ⟨3⟩1. LET: $x \in C$
 - ⟨3⟩2. PICK a connected $Y \subseteq X$ and a neighbourhood V of x such that $V \subseteq Y \subseteq U$
 - PROOF: ⟨1⟩1
 - ⟨3⟩3. $Y \subseteq C$
 - PROOF: Proposition 15.25.5.
 - ⟨3⟩4. $V \subseteq C$
- ⟨2⟩4. C is open.
- PROOF: Proposition 14.1.7.

⟨1⟩3. X is locally connected.

PROOF: Theorem 15.28.7.

□

Proposition 15.28.11. *A quotient of a locally connected space is locally connected.*

PROOF:

⟨1⟩1. LET: X be a locally connected space.

⟨1⟩2. LET: $p : X \rightarrow Y$ be a quotient map.

⟨1⟩3. For every open set V in Y , every component of V is open in Y .

⟨2⟩1. LET: V be an open set in Y .

⟨2⟩2. LET: C be a component of V .

⟨2⟩3. $p^{-1}(C)$ is a union of components of $p^{-1}(V)$

⟨3⟩1. LET: $x \in p^{-1}(C)$

⟨3⟩2. LET: D be the component of $p^{-1}(V)$ that contains x .

PROVE: $D \subseteq p^{-1}(C)$

⟨3⟩3. $p(D)$ is connected.

PROOF: Proposition 15.13.10.

⟨3⟩4. $p(D) \subseteq C$

⟨3⟩5. $D \subseteq p^{-1}(C)$

⟨2⟩4. Every component of $p^{-1}(V)$ is open in X .

PROOF: Theorem 15.28.7.

⟨2⟩5. $p^{-1}(C)$ is open in X .

⟨2⟩6. C is open in Y .

⟨1⟩4. Y is locally connected.

PROOF: Theorem 15.28.7.

□

15.29 Local Path Connectedness

Definition 15.29.1 (Locally Path Connected). Let X be a topological space and $x \in X$. Then X is *locally path connected* at x iff, for every neighbourhood U of x , there exists a path connected neighbourhood V of x such that $V \subseteq U$.

The space X is *locally path connected* iff it is locally connected at every point.

Theorem 15.29.2. *Let X be a topological space. Then X is locally path connected if and only if, for every open set U in X , every path component of U is open in X .*

PROOF:

⟨1⟩1. If X is locally path connected then, for every open set U in X , every path component of U is open in X .

⟨2⟩1. ASSUME: X is locally path connected.

⟨2⟩2. LET: U be an open set in X .

⟨2⟩3. LET: C be a path component of U .

- ⟨2⟩4. LET: $x \in C$
- ⟨2⟩5. PICK a path connected neighbourhood V of x in X such that $V \subseteq U$
- ⟨2⟩6. $x \in V \subseteq C$
- ⟨1⟩2. If, for every open set U in X , every path component of U is open in X , then X is locally path connected.
- ⟨2⟩1. ASSUME: For every open set U in X , every path component of U is open in X .
- ⟨2⟩2. LET: $x \in X$
- ⟨2⟩3. LET: U be a neighbourhood of x .
- ⟨2⟩4. LET: V be the path component of U that contains x .
- ⟨2⟩5. V is a path connected neighbourhood of x and $V \subseteq U$.

□

Theorem 15.29.3. *In a locally path connected space, the components are the same as the path components.*

PROOF:

- ⟨1⟩1. LET: X be a locally path connected space.
- ⟨1⟩2. LET: P be a path component of X .
- ⟨1⟩3. LET: C be the component that includes P .
- PROVE: $P = C$
- ⟨1⟩4. LET: Q be the union of all the path components of C other than P .
- ⟨1⟩5. P and Q are open in C .
- PROOF: Theorem 15.29.2.
- ⟨1⟩6. $P \cup Q = C$ and $P \cap Q = \emptyset$
- ⟨1⟩7. $Q = \emptyset$
- PROOF: Otherwise (P, Q) would be a separation of C .
- ⟨1⟩8. $P = C$

□

Example 15.29.4. The converse does not hold. In \mathbb{Q} , the components are the same as the path components, namely the one-point sets, but \mathbb{Q} is not locally path connected.

Example 15.29.5. The ordered square is not locally path connected.

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction I_o^2 is locally path connected at $(0, 1)$.
- ⟨1⟩2. PICK a path connected neighbourhood U of $(0, 1)$.
- ⟨1⟩3. PICK $a > 0$ such that $[(0, 1), (a, 0)] \subseteq U$
- ⟨1⟩4. PICK a path $p : [0, 1] \rightarrow I_o^2$ from $(0, 1)$ to $(a, 0)$.
- ⟨1⟩5. For every $x \in (0, a)$, PICK a rational $q_x \in [0, 1]$ such that $q_x \in ((x, 0), (x, 1))$
- ⟨1⟩6. $\{q_x : x \in (0, a)\}$ is an uncountable set of rationals.
- ⟨1⟩7. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 15.29.6. *Every connected open subspace of a locally path connected space is path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a locally path connected space.

$\langle 1 \rangle 2$. LET: U be a connected open subspace.

$\langle 1 \rangle 3$. LET: P be a path component of U .

PROVE: $P = U$

$\langle 1 \rangle 4$. LET: Q be the union of the path components of U that are not P .

$\langle 1 \rangle 5$. P and Q are open.

PROOF: Theorem 15.29.2.

$\langle 1 \rangle 6$. $Q = \emptyset$

PROOF: Otherwise (P, Q) would be a separation of U .

$\langle 1 \rangle 7$. $P = U$

□

15.30 Quasicomponents

Definition 15.30.1 (Quasicomponent). Let X be a topological space. Define the equivalence relation \sim on X by: $x \sim y$ iff there is no separation (U, V) of X with $x \in U$ and $y \in V$. The *quasicomponents* of X are the equivalence classes with respect to \sim .

We prove this is an equivalence relation.

PROOF:

$\langle 1 \rangle 1$. \sim is reflexive.

PROOF: For any $x \in X$, there cannot exist a separation (U, V) of X with $x \in U$ and $x \in V$.

$\langle 1 \rangle 2$. \sim is symmetric.

PROOF: Immediate from definition.

$\langle 1 \rangle 3$. \sim is transitive.

$\langle 2 \rangle 1$. ASSUME: $x \sim y$ and $y \sim z$

$\langle 2 \rangle 2$. ASSUME: for a contradiction (U, V) is a separation of X with $x \in U$ and $z \in V$.

$\langle 2 \rangle 3$. $y \in U$ or $y \in V$

$\langle 2 \rangle 4$. $y \notin U$

PROOF: $y \in U$ would contradict the fact that $y \sim z$.

$\langle 2 \rangle 5$. $y \notin V$

PROOF: $y \in V$ would contradict the fact that $x \sim y$.

$\langle 2 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 15.30.2. *Every component of a topological space is a subset of a quasicomponent.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a topological space.

$\langle 1 \rangle 2$. LET: C be a component of X .

PROVE: $\forall x, y \in C. x \sim y$
 $\langle 1 \rangle 3$. LET: $x, y \in C$
 $\langle 1 \rangle 4$. ASSUME: for a contradiction (U, V) is a separation of X with $x \in U$ and $y \in V$
 $\langle 1 \rangle 5$. $(U \cap C, V \cap C)$ is a separation of C .
 $\langle 1 \rangle 6$. Q.E.D.
 PROOF: This contradicts the fact that C is connected (Proposition 15.25.4).
 \square

Proposition 15.30.3. *In a locally connected topological space, the components are the same as the quasicomponents.*

PROOF:
 $\langle 1 \rangle 1$. LET: X be a locally connected topological space.
 $\langle 1 \rangle 2$. LET: C be a component of X .
 $\langle 1 \rangle 3$. LET: Q be the quasicomponent that includes C .
 PROVE: $Q = C$
 $\langle 1 \rangle 4$. ASSUME: for a contradiction $C \neq Q$
 $\langle 1 \rangle 5$. PICK $c \in C$ and $d \in Q - C$
 $\langle 1 \rangle 6$. $(C, X - C)$ is a separation of X with $c \in C$ and $d \in X - C$.
 PROOF: Since the components of X are open (Theorem 15.28.7).
 $\langle 1 \rangle 7$. Q.E.D.
 PROOF: This contradicts the fact that $c \sim d$.
 \square

15.31 Compact Spaces

Definition 15.31.1 (Open Cover). Let X be a topological space. An *open cover* of X is a cover of X whose elements are open sets.

Definition 15.31.2 (Compact). A topological space is *compact* iff every open cover includes a finite subcover.

Example 15.31.3. The space \mathbb{R} is not compact, because the open cover $\{(n, n+2) : n \in \mathbb{Z}\}$ has no finite subcover.

Example 15.31.4. Every finite topological space is compact.

Example 15.31.5. Any set under the cofinite topology is compact.

Lemma 15.31.6. *Let X be a topological space and Y a subspace of X . Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection that covers Y .*

PROOF:
 $\langle 1 \rangle 1$. If Y is compact then every covering of Y by sets open in X contains a finite subcollection that covers Y .
 $\langle 2 \rangle 1$. ASSUME: Y is compact.

- ⟨2⟩2. LET: \mathcal{A} be a covering of Y by sets open in X .
 - ⟨2⟩3. $\{U \cap Y : U \in \mathcal{A}\}$ is an open covering of Y .
 - ⟨2⟩4. PICK a finite subcovering $\{U_1 \cap Y, \dots, U_n \cap Y\}$.
 - ⟨2⟩5. $\{U_1, \dots, U_n\}$ is a finite subcollection of \mathcal{A} that covers Y .
 - ⟨1⟩2. If every covering of Y by sets open in X contains a finite subcollection that covers Y then Y is compact.
 - ⟨2⟩1. ASSUME: Every covering of Y by sets open in X contains a finite subcollection that covers Y .
 - ⟨2⟩2. LET: \mathcal{A} be an open cover of Y .
 - ⟨2⟩3. $\{U \text{ open in } X : U \cap Y \in \mathcal{A}\}$ covers Y .
 - ⟨2⟩4. PICK a finite subcollection $\{U_1, \dots, U_n\}$ that covers Y .
 - ⟨2⟩5. $\{U_1 \cap Y, \dots, U_n \cap Y\}$ is a finite subcollection of \mathcal{A} that covers Y .
-

Theorem 15.31.7. *Every closed subspace of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET: X be a compact space.
- ⟨1⟩2. LET: C be a closed subspace of X .
- ⟨1⟩3. Every covering of C by sets open in X contains a finite subcollection that covers C .
- ⟨2⟩1. LET: \mathcal{A} be a covering of C by sets open in X .
- ⟨2⟩2. $\mathcal{A} \cup \{X - C\}$ is an open covering of X .
- ⟨2⟩3. PICK a finite subcover \mathcal{B} .
- ⟨2⟩4. $\mathcal{B} - \{X - C\}$ is a finite subcollection of \mathcal{A} that covers C .
- ⟨1⟩4. C is compact.

PROOF: Lemma 15.31.6.

□

Lemma 15.31.8. *Let X be a Hausdorff space. Let Y be a compact subspace. Let $x \in X - Y$. Then there exist disjoint open sets U and V such that $x \in U$ and $Y \subseteq V$.*

PROOF:

- ⟨1⟩1. For all $y \in Y$, there exist disjoint open sets U' and V' with $x \in U'$ and $y \in V'$.
 - ⟨1⟩2. $\{V' \text{ open in } X : \exists U' \text{ open in } X. U' \cap V' = \emptyset \wedge x \in U'\}$ is an cover of Y by sets open in X .
 - ⟨1⟩3. PICK a finite subcollection $\{V_1, \dots, V_n\}$ that covers Y .
 - ⟨1⟩4. For $i = 1, \dots, n$, PICK an open set U_i with $U_i \cap V_i = \emptyset$ and $x \in U_i$.
 - ⟨1⟩5. LET: $U = U_1 \cap \dots \cap U_n$ and $V = V_1 \cup \dots \cup V_n$.
 - ⟨1⟩6. U and V are disjoint open sets with $x \in U$ and $Y \subseteq V$.
-

Proposition 15.31.9. *Let X be a Hausdorff space. Let A and B be disjoint compact subspaces of X . Then there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.*

PROOF:

$\langle 1 \rangle 1$. For all $x \in A$, there exist disjoint open sets U' and V' with $x \in U'$ and $B \subseteq V'$.

PROOF: Lemma 15.31.8.

$\langle 1 \rangle 2$. $\{U' \text{ open in } X : \exists V' \text{ open in } X. U' \cap V' = \emptyset \wedge B \subseteq V'\}$ is a set of open sets that covers A .

$\langle 1 \rangle 3$. PICK a finite subset $\{U_1, \dots, U_n\}$ that covers A .

PROOF: Lemma 15.31.6.

$\langle 1 \rangle 4$. For $i = 1, \dots, n$, PICK V_i open in X such that $U_i \cap V_i = \emptyset$ and $B \subseteq V_i$.

$\langle 1 \rangle 5$. LET: $U = U_1 \cup \dots \cup U_n$

$\langle 1 \rangle 6$. LET: $V = V_1 \cap \dots \cap V_n$

$\langle 1 \rangle 7$. U and V are disjoint and open.

$\langle 1 \rangle 8$. $A \subseteq U$

$\langle 1 \rangle 9$. $B \subseteq V$

□

Theorem 15.31.10. *Every compact subspace of a Hausdorff space is closed.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a Hausdorff space.

$\langle 1 \rangle 2$. LET: Y be a compact subspace of X .

$\langle 1 \rangle 3$. For any $x \in X - Y$ there exists an open set U such that $x \in U \subseteq X - Y$.

PROOF: Lemma 15.31.8.

$\langle 1 \rangle 4$. $X - Y$ is open.

PROOF: Proposition 14.1.7.

$\langle 1 \rangle 5$. Y is closed.

□

Example 15.31.11. We cannot weaken the hypothesis from the space being Hausdorff to the space being T_1 .

Let X be any infinite set under the cofinite topology. Then X is T_1 . The closed sets are the finite sets and X , but every subspace is compact.

Theorem 15.31.12. *The continuous image of a compact space is compact.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a compact space and $f : X \rightarrow Y$ be a surjective continuous function.

$\langle 1 \rangle 2$. LET: \mathcal{V} be an open cover of Y .

$\langle 1 \rangle 3$. $\{f^{-1}(V) : V \in \mathcal{V}\}$ is an open cover of X .

$\langle 1 \rangle 4$. PICK a finite subcover $\{f^{-1}(V_1), \dots, f^{-1}(V_n)\}$.

$\langle 1 \rangle 5$. $\{V_1, \dots, V_n\}$ covers Y .

□

Proposition 15.31.13. *Let X be a compact space and Y a Hausdorff space. Every continuous function from X to Y is a closed map.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : X \rightarrow Y$ be continuous.

⟨1⟩2. LET: C be a closed set in X .

⟨1⟩3. C is compact.

PROOF: Theorem 15.31.7.

⟨1⟩4. $f(C)$ is compact.

PROOF: Theorem 15.31.12.

⟨1⟩5. $f(C)$ is closed in Y .

PROOF: Theorem 15.31.10.

□

Corollary 15.31.13.1. *Let X be a compact space and Y a Hausdorff space. Let $f : X \rightarrow Y$ be a continuous bijection. Then f is a homeomorphism.*

Corollary 15.31.13.2. *Let \mathcal{T} and \mathcal{T}' be two compact Hausdorff topologies on the same set X . Then either $\mathcal{T} = \mathcal{T}'$ or they are incompatible.*

Theorem 15.31.14. *Let X and Y be topological spaces. Let A be a compact subspace of X and B a compact subspace of Y . Let N be an open set that includes $A \times B$. Then there exists open sets U in X and V in Y such that*

$$A \times B \subseteq U \times V \subseteq N.$$

PROOF:

⟨1⟩1. For all $x \in A$, there exist open sets U in X and V in Y such that $x \in U$, $B \subseteq V$, and $U \times V \subseteq N$.

⟨2⟩1. LET: $x \in A$

⟨2⟩2. For all $y \in B$, there exist neighbourhoods U of x and V of y such that $U \times V \subseteq N$

⟨2⟩3. PICK open sets V_1, \dots, V_n that cover B such that, for $i = 1, \dots, n$, there exists a neighbourhood U_i of x such that $U_i \times V_i \subseteq N$

⟨2⟩4. LET: $U = U_1 \cap \dots \cap U_n$

⟨2⟩5. LET: $V = V_1 \cup \dots \cup V_n$

⟨2⟩6. U is open in X .

⟨2⟩7. V is open in Y .

⟨2⟩8. $x \in U$

⟨2⟩9. $B \subseteq V$

⟨2⟩10. $U \times V \subseteq N$

⟨1⟩2. PICK open sets U_1, \dots, U_n in X that cover A such that, for $i = 1, \dots, n$, there exists V_i open in Y such that $B \subseteq V_i$ and $U_i \times V_i \subseteq N$.

⟨1⟩3. LET: $U = U_1 \cup \dots \cup U_n$

⟨1⟩4. LET: $V = V_1 \cap \dots \cap V_n$

⟨1⟩5. U is open in X .

⟨1⟩6. V is open in Y .

⟨1⟩7. $A \times B \subseteq U \times V \subseteq N$

□

Corollary 15.31.14.1 (Tube Lemma). *Let X be a topological space and Y a compact space. Let $x_0 \in X$. Let N be an open set in $X \times Y$ that includes $\{x_0\} \times Y$. Then there exists a neighbourhood W of x_0 such that $W \times Y \subseteq N$.*

Theorem 15.31.15. *The product of two compact spaces is compact.*

PROOF:

- <1>1. LET: X and Y be compact spaces.
 <1>2. LET: \mathcal{A} be an open covering of $X \times Y$.
 <1>3. For all $x \in X$, there exists a neighbourhood W of x such that $W \times Y$ can be covered by finitely many elements of \mathcal{A} .
 <2>1. LET: $x \in X$
 <2>2. $\{x\} \times Y$ is compact.
 PROOF: It is homeomorphic to Y .
 <2>3. PICK a finite subcollection $\{A_1, \dots, A_n\}$ of \mathcal{A} that covers $\{x\} \times Y$.
 <2>4. There exists a neighbourhood W of x such that $W \times Y \subseteq A_1 \cup \dots \cup A_n$.
 PROOF: Tube Lemma
 <1>4. PICK finitely many open sets W_1, \dots, W_n that cover X such that each $W_i \times Y$ can be covered by finitely many elements of \mathcal{A} .
 <1>5. For $i = 1, \dots, n$, PICK a finite subset \mathcal{A}_i of \mathcal{A} that covers $W_i \times Y$.
 <1>6. $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_n$ covers $X \times Y$.
 □

Theorem 15.31.16. *Let X be a topological space. Then X is compact if and only if, for every set \mathcal{C} of compact sets, if \mathcal{C} has the finite intersection property then $\bigcap \mathcal{C} \neq \emptyset$.*

PROOF: The following are equivalent:

- X is compact.
- For every set \mathcal{A} of open sets, if $\bigcup \mathcal{A} = X$ then there is a finite subset of \mathcal{A} that covers X .
- For every set \mathcal{C} of closed sets, if $\bigcup_{C \in \mathcal{C}} (X - C) = X$ then there is a finite subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\bigcup_{C \in \mathcal{C}_0} (X - C) = X$.
- For every set \mathcal{C} of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\bigcap \mathcal{C}_0 = \emptyset$.
- For every set \mathcal{C} of closed sets, if \mathcal{C} has the finite intersection property then $\bigcap \mathcal{C} \neq \emptyset$.

□

Corollary 15.31.16.1. *Let X be a compact set. Let (C_n) be a sequence of nonempty closed sets such that $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$. Then $\bigcap_{n=0}^{\infty} C_n \neq \emptyset$.*

Proposition 15.31.17. *Let X be a topological space. Let Y and Z be compact subspaces of X . The $Y \cup Z$ is compact.*

PROOF:

- <1>1. LET: \mathcal{U} be a set of open sets that covers $Y \cup Z$.
 <1>2. PICK finite subsets \mathcal{U}_1 that covers Y and \mathcal{U}_2 that covers Z .

⟨1⟩3. $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset of \mathcal{U} that covers $Y \cup Z$.

□

Proposition 15.31.18. *Let X be a topological space and Y a compact space. Then the projection $\pi_1 : X \times Y \rightarrow X$ is a closed map.*

PROOF:

⟨1⟩1. LET: C be closed in $X \times Y$.

⟨1⟩2. LET: $x \in X - \pi_1(C)$

⟨1⟩3. $\{x\} \times Y \subseteq (X \times Y) - C$

⟨1⟩4. PICK a neighbourhood U of x such that $U \times Y \subseteq (X \times Y) - C$

PROOF: Tube Lemma.

⟨1⟩5. $x \in U \subseteq X - \pi_1(C)$

□

Theorem 15.31.19. *Let X be a topological space and Y a compact Hausdorff space. Let $f : X \rightarrow Y$. Then f is continuous if and only if the graph of f ,*

$$G_f = \{(x, f(x)) : x \in X\} ,$$

is closed in $X \times Y$.

PROOF:

⟨1⟩1. If f is continuous then G_f is closed in $X \times Y$.

⟨2⟩1. ASSUME: f is continuous.

⟨2⟩2. LET: $(x, y) \in (X \times Y) - G_f$

⟨2⟩3. PICK disjoint open neighbourhoods U of y and V of $f(x)$.

⟨2⟩4. $(x, y) \in (f^{-1}(V) \times U) \subseteq (X \times Y) - G_f$

⟨1⟩2. If G_f is closed in $X \times Y$ then f is continuous.

⟨2⟩1. ASSUME: G_f is closed in $X \times Y$.

⟨2⟩2. LET: $x_0 \in X$

⟨2⟩3. LET: V be a neighbourhood of $f(x_0)$.

⟨2⟩4. $G_f \cap (X \times (Y - V))$ is closed.

⟨2⟩5. $\pi_1(G_f \cap (X \times (Y - V)))$ is closed.

PROOF: Proposition 15.31.18.

⟨2⟩6. LET: $U = X - \pi_1(G_f \cap (X \times (Y - V)))$

⟨2⟩7. $x_0 \in U$

⟨2⟩8. $f(U) \subseteq V$

□

Theorem 15.31.20. *Let X be a compact Hausdorff space. Let \mathcal{A} be a set of closed connected subspaces of X that is linearly ordered under inclusion. Then $\bigcap \mathcal{A}$ is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction (C, D) is a separation of $\bigcap \mathcal{A}$.

⟨1⟩2. PICK disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$.

⟨2⟩1. C and D are closed in X .

PROOF: Proposition 15.3.6.

$\langle 2 \rangle 2$. C and D are compact.

PROOF: Theorem 15.31.7.

$\langle 2 \rangle 3$. Q.E.D.

PROOF: Proposition 15.31.9.

$\langle 1 \rangle 3$. $\bigcap_{A \in \mathcal{A}} (A - (U \cup V))$ is nonempty.

$\langle 2 \rangle 1$. $\{A - (U \cap V) : A \in \mathcal{A}\}$ has the finite intersection property.

$\langle 3 \rangle 1$. LET: $A_1, \dots, A_n \in \mathcal{A}$

$\langle 3 \rangle 2$. ASSUME: w.l.o.g. $A_1 \supseteq \dots \supseteq A_n$

PROOF: \mathcal{A} is linearly ordered by inclusion.

$\langle 3 \rangle 3$. $\bigcap_{i=1}^n (A_i - (U \cup V)) = A_n - (U \cup V)$

$\langle 3 \rangle 4$. $A_n - (U \cup V) \neq \emptyset$

PROOF: Otherwise $(A_n \cap U, A_n \cap V)$ would be a separation of A_n .

$\langle 2 \rangle 2$. Q.E.D.

PROOF: Theorem 15.31.16.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: This is a contradiction.

□

Theorem 15.31.21. *Let X be a linearly ordered set with the least upper bound property under the order topology. Every closed interval in X is compact.*

PROOF:

$\langle 1 \rangle 1$. LET: $a, b \in X$ with $a < b$.

$\langle 1 \rangle 2$. LET: \mathcal{A} be an open covering of $[a, b]$

$\langle 1 \rangle 3$. For any $x \in [a, b)$, there exists $y \in (x, b]$ such that $[x, y]$ can be covered by at most two elements of \mathcal{A} .

$\langle 2 \rangle 1$. LET: $x \in [a, b)$

$\langle 2 \rangle 2$. PICK $U \in \mathcal{A}$ with $x \in U$.

$\langle 2 \rangle 3$. PICK $y > x$ such that $[x, y] \subseteq U$.

$\langle 2 \rangle 4$. PICK $V \in \mathcal{A}$ such that $y \in V$.

$\langle 2 \rangle 5$. $[x, y]$ can be covered by U and V .

$\langle 1 \rangle 4$. LET: $C = \{y \in (a, b] : [a, y] \text{ can be covered by finitely many elements of } \mathcal{A}\}$

$\langle 1 \rangle 5$. $C \neq \emptyset$

PROOF: By $\langle 1 \rangle 3$, there exists $y \in (a, b]$ such that $[a, y]$ can be covered by at most two elements of \mathcal{A} .

$\langle 1 \rangle 6$. LET: $c = \sup C$

$\langle 1 \rangle 7$. $c \in C$

$\langle 2 \rangle 1$. PICK $U \in \mathcal{A}$ such that $c \in U$.

$\langle 2 \rangle 2$. PICK $y < c$ such that $(y, c] \subseteq U$.

$\langle 2 \rangle 3$. PICK $z \in C$ such that $y < z$

$\langle 2 \rangle 4$. PICK a finite $\mathcal{A}_0 \subseteq \mathcal{A}$ that covers $[a, z]$.

$\langle 2 \rangle 5$. $\mathcal{A}_0 \cup \{U\}$ covers $[a, c]$.

$\langle 2 \rangle 6$. $c \in C$

$\langle 1 \rangle 8$. $c = b$

$\langle 2 \rangle 1$. ASSUME: for a contradiction $c < b$.

$\langle 2 \rangle 2$. PICK $y > c$ such that $[c, y]$ can be covered by at most two elements of \mathcal{A} .

$\langle 2 \rangle 3$. $[a, c]$ can be covered by finitely many elements of \mathcal{A} .

PROOF: $\langle 1 \rangle 7$

$\langle 2 \rangle 4$. $[a, y]$ can be covered by finitely many elements of \mathcal{A} .

$\langle 2 \rangle 5$. $y \in C$

$\langle 2 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 9$. $[a, b]$ can be covered by finitely many elements of \mathcal{A} .

□

Corollary 15.31.21.1. *Every closed interval in \mathbb{R} is compact.*

Proposition 15.31.22. *A linearly ordered set that is compact under the order topology has a greatest and a least element.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a linearly ordered set that is compact under the order topology.

$\langle 1 \rangle 2$. X has a greatest element.

$\langle 2 \rangle 1$. ASSUME: for a contradiction X has no greatest element.

$\langle 2 \rangle 2$. The open rays $(-\infty, a)$ form an open cover of X .

$\langle 2 \rangle 3$. PICK a finite subcover $\{(-\infty, a_1), \dots, (-\infty, a_n)\}$ where $a_1 \leq \dots \leq a_n$

$\langle 2 \rangle 4$. $a_n < a_n$

$\langle 2 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 3$. X has a least element.

PROOF: Similar.

□

Corollary 15.31.22.1 (Extreme Value Theorem). *Let X be a compact space and Y a linearly ordered set in the order topology. Let $f : X \rightarrow Y$ be continuous. Then there exist $c, d \in X$ such that, for all $x \in X$, we have $f(c) \leq f(x) \leq f(d)$.*

PROOF: Since $f(X)$ is compact, and so has a greatest and least element. □

15.32 Perfect Maps

Definition 15.32.1 (Perfect Map). Let X and Y be topological spaces. A *perfect map* from X to Y is a closed continuous surjective map $p : X \rightarrow Y$ such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact.

Proposition 15.32.2. *Let X be a topological space and Y a compact space. If there exists a perfect map $X \rightarrow Y$, then X is compact.*

PROOF:

$\langle 1 \rangle 1$. LET: $p : X \rightarrow Y$ be a perfect map.

$\langle 1 \rangle 2$. For all $y \in Y$ and every open set U such that $p^{-1}(y) \subseteq U$, there exists a neighbourhood W of y such that $p^{-1}(W) \subseteq U$.

- ⟨2⟩1. LET: $y \in Y$
- ⟨2⟩2. LET: U be an open set such that $p^{-1}(y) \subseteq U$.
- ⟨2⟩3. LET: $W = Y - p(X - U)$
- ⟨2⟩4. W is open.
 - ⟨3⟩1. $X - U$ is closed.
 - PROOF: Proposition 14.2.2, ⟨2⟩2.
 - ⟨3⟩2. $p(X - U)$ is closed.
 - PROOF: Since p is a closed map (⟨1⟩1).
 - ⟨3⟩3. $Y - p(X - U)$ is open.
- ⟨2⟩5. $y \in W$
 - ⟨3⟩1. ASSUME: for a contradiction $y \in p(X - U)$
 - ⟨3⟩2. PICK $x \in X - U$ such that $p(x) = y$
 - ⟨3⟩3. $x \in p^{-1}(y)$
 - ⟨3⟩4. $x \in U$
 - PROOF: ⟨2⟩2
 - ⟨3⟩5. Q.E.D.
 - PROOF: This contradicts ⟨3⟩2.
- ⟨2⟩6. $p^{-1}(W) \subseteq U$
 - ⟨3⟩1. LET: $x \in p^{-1}(W)$
 - ⟨3⟩2. $p(x) \in W$
 - ⟨3⟩3. $p(x) \notin p(X - U)$
 - PROOF: ⟨2⟩3
 - ⟨3⟩4. $x \notin X - U$
 - ⟨3⟩5. $x \in U$
- ⟨1⟩3. LET: \mathcal{U} be an open cover of X .
- ⟨1⟩4. For every $y \in Y$, there exists a neighbourhood W of y such that $p^{-1}(W)$ can be covered by finitely many elements of \mathcal{U} .
 - ⟨2⟩1. LET: $y \in Y$
 - ⟨2⟩2. PICK finitely many sets $U_1, \dots, U_n \in \mathcal{U}$ that cover $p^{-1}(y)$
 - PROOF: Since $p^{-1}(y)$ is compact.
 - ⟨2⟩3. There exists a neighbourhood W of y such that $p^{-1}(W) \subseteq U_1 \cup \dots \cup U_n$.
 - PROOF: ⟨1⟩2
- ⟨1⟩5. PICK finitely many open sets W_1, \dots, W_n in Y such that each $p^{-1}(W_i)$ can be covered by finitely many elements of \mathcal{U} .
 - PROOF: Since Y is compact.
- ⟨1⟩6. For $i = 1, \dots, n$, PICK a finite subset \mathcal{U}_i of \mathcal{U} such that $p^{-1}(W_i) \subseteq \bigcup \mathcal{U}_i$.
- ⟨1⟩7. $\mathcal{U}_1 \cup \dots \cup \mathcal{U}_n$ covers X .

□

Chapter 16

Metric Spaces

16.1 Metric Spaces

Definition 16.1.1 (Metric Space). Let X be a set and $d : X^2 \rightarrow \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geq 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d) . We often write X for the metric space (X, d) .

Definition 16.1.2 (Discrete Metric). On any set X , define the *discrete* metric by $d(x, y) = 0$ if $x = y$, 1 if $x \neq y$.

Definition 16.1.3 (Standard Metric). The *standard metric* on \mathbb{R} is defined by $d(x, y) = |x - y|$.

Definition 16.1.4 (Distance to a Set). Let X be a metric space. Let $A \subseteq X$ be nonempty and $x \in X$. The *distance from x to A* is

$$d(x, A) = \inf_{a \in A} d(x, a) \ .$$

16.1.1 Balls

Definition 16.1.5 ((Open) Ball). Let X be a metric space. Let $x \in X$ and $r > 0$. The (*open*) *ball* with *centre* x and *radius* r is

$$B(x, r) = \{y \in X \mid d(x, y) < r\} \ .$$

Definition 16.1.6 (Closed Ball). Let X be a metric space. Let $x \in X$ and $r > 0$. The *closed ball* with centre x and radius r is

$$\overline{B(x, r)} = \{y \in X \mid d(x, y) < r\} .$$

Definition 16.1.7 (Metric Topology). Let (X, d) be a metric space. The *metric topology* on X is the topology generated by the basis consisting of the balls.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. Every point is a member of some ball.

PROOF: Since $x \in B(x, 1)$.

$\langle 1 \rangle 2$. If B_1 and B_2 are balls and $x \in B_1 \cap B_2$, then there exists a ball B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.

$\langle 2 \rangle 1$. LET: $x \in B(a, \epsilon_1) \cap B(b, \epsilon_2)$

$\langle 2 \rangle 2$. LET: $\epsilon = \min(\epsilon_1 - d(x, a), \epsilon_2 - d(x, b))$

PROVE: $x \in B(x, \epsilon) \subseteq B(a, \epsilon_1) \cap B(b, \epsilon_2)$

$\langle 2 \rangle 3$. $B(x, \epsilon) \subseteq B(a, \epsilon_1)$

$\langle 3 \rangle 1$. LET: $y \in B(x, \epsilon)$

$\langle 3 \rangle 2$. $d(y, a) < \epsilon_1$

PROOF:

$$d(y, a) \leq d(y, x) + d(x, a) \quad (\text{Triangle Inequality})$$

$$< \epsilon + d(x, a) \quad (\langle 3 \rangle 1)$$

$$\leq \epsilon_1 \quad (\langle 2 \rangle 2)$$

$\langle 2 \rangle 4$. $B(x, \epsilon) \subseteq B(b, \epsilon_2)$

PROOF: Similar.

□

Proposition 16.1.8. *The discrete metric on a set X induces the discrete topology.*

PROOF: Since $B(x, 1/2) = \{x\}$ for all $x \in X$. □

Proposition 16.1.9. *The standard metric on \mathbb{R} induces the standard topology.*

PROOF:

$\langle 1 \rangle 1$. Every ball is open in the standard topology.

PROOF: Since $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$.

$\langle 1 \rangle 2$. Every open ray is open in the metric topology.

PROOF: If $x \in (a, +\infty)$ then $x \in B(x, x - a) \subseteq (a, +\infty)$. Similarly for $(-\infty, a)$.

□

Proposition 16.1.10. *Multiplication is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

$\langle 1 \rangle 2$. LET: $\delta = \min(\epsilon/(|x| + |y| + 1), 1)$

$\langle 1 \rangle 3$. LET: $(x', y') \in \mathbb{R}^2$ with $\rho((x, y), (x', y')) < \delta$

$$\langle 1 \rangle 4. |x - x'|, |y - y'| < \delta$$

$$\langle 1 \rangle 5. |xy - x'y'| < \epsilon$$

PROOF:

$$\begin{aligned} |xy - x'y'| &= |xy - xy' + xy - x'y - xy + x'y + xy' - x'y'| \\ &\leq |xy - xy'| + |xy - x'y| + |xy - x'y - xy' + x'y'| = |x||y - y'| + |x - x'||y| + |x - x'||y - y'| \\ &< |x|\delta + |y|\delta + \delta^2 \end{aligned} \quad (\langle 1 \rangle 4)$$

$$\leq |x|\delta + |y|\delta + \delta \quad (\langle 1 \rangle 2)$$

$$= (|x| + |y| + 1)\delta$$

$$\leq \epsilon \quad (\langle 1 \rangle 2)$$

□

Corollary 16.1.10.1. *The unit circle S^1 is a closed subset of \mathbb{R}^2 .*

PROOF: The function f that maps (x, y) to $x^2 + y^2$ is continuous, and $S^1 = f^{-1}(\{1\})$. □

Corollary 16.1.10.2. *The unit ball B^2 is a closed subset of \mathbb{R}^2 .*

PROOF: The function f that maps (x, y) to $x^2 + y^2$ is continuous, and $B^2 = f^{-1}([0, 1])$. □

Proposition 16.1.11. *Let (a_n) and (b_n) be sequences of real numbers. Let $c, s, t \in \mathbb{R}$. Assume*

$$\sum_{n=0}^{\infty} a_n = s \text{ and } \sum_{n=0}^{\infty} b_n = t .$$

Then

$$\sum_{n=0}^{\infty} (ca_n + b_n) = cs + t .$$

PROOF:

$$\sum_{n=0}^N (ca_n + b_n) = c \sum_{n=0}^N a_n + \sum_{n=0}^N b_n \rightarrow cs + t \text{ as } n \rightarrow \infty \quad \square$$

Proposition 16.1.12 (Comparison Test). *Let (a_n) and (b_n) be sequences of real numbers. Assume $|a_n| \leq b_n$ for all n . Assume $\sum_{n=0}^{\infty} b_n$ converges. Then $\sum_{n=0}^{\infty} a_n$ converges.*

PROOF:

$\langle 1 \rangle 1$. For all n ,

$$\text{LET: } c_n = |a_n| + a_n$$

$\langle 1 \rangle 2$. $\sum_{n=0}^{\infty} |a_n|$ converges.

PROOF: Since $(\sum_{n=0}^N |a_n|)_N$ is an increasing sequence of real numbers bounded above by $\sum_{n=0}^{\infty} b_n$.

$\langle 1 \rangle 3$. $\sum_{n=0}^{\infty} c_n$ converges.

PROOF: Since $(\sum_{n=0}^N c_n)_N$ is an increasing sequence of real numbers bounded above by $2 \sum_{n=0}^{\infty} a_n$.

⟨1⟩4. $\sum_{n=0}^{\infty} a_n$ converges.

PROOF: Since $a_n = c_n - |a_n|$.

□

Proposition 16.1.13. *Let X be a metric space. Let $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:

⟨1⟩1. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

⟨2⟩1. ASSUME: U is open.

⟨2⟩2. LET: $x \in U$

⟨2⟩3. PICK a ball $B(a, \delta)$ such that $x \in B(a, \delta) \subseteq U$

⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$

PROVE: $B(x, \epsilon) \subseteq U$

⟨2⟩5. LET: $y \in B(x, \epsilon)$

⟨2⟩6. $y \in B(a, \delta)$

PROOF:

$$\begin{aligned} d(a, y) &\leq d(a, x) + d(x, y) && \text{(Triangle Inequality)} \\ &< d(a, x) + \epsilon && (\langle 2 \rangle 5) \\ &= \delta \end{aligned}$$

⟨2⟩7. $y \in U$

PROOF: ⟨2⟩3

⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.

PROOF: Immediate from definition of the metric topology.

□

Proposition 16.1.14. *Let X be a metric space. Let $a, b, c \in X$. Then*

$$|d(a, b) - d(a, c)| \leq d(b, c) .$$

PROOF:

⟨1⟩1. $d(a, b) - d(a, c) \leq d(b, c)$

PROOF: Triangle Inequality.

⟨1⟩2. $d(a, c) - d(a, b) \leq d(b, c)$

PROOF: Triangle Inequality.

□

Proposition 16.1.15. *Let (X, d) be a metric space. Then the metric topology on X is the coarsest topology such that $d : X^2 \rightarrow \mathbb{R}$ is continuous.*

PROOF:

⟨1⟩1. d is continuous with respect to the metric topology.

⟨2⟩1. LET: $(a, b) \in X^2$

⟨2⟩2. LET: V be a neighbourhood of $d(a, b)$.

⟨2⟩3. PICK $\epsilon > 0$ such that $(d(a, b) - \epsilon, d(a, b) + \epsilon) \subseteq V$.

⟨2⟩4. LET: $U = B(a, \epsilon/2) \times B(b, \epsilon/2)$

⟨2⟩5. LET: $(x, y) \in U$

⟨2⟩6. $|d(x, y) - d(a, b)| < \epsilon$

PROOF:

$$\begin{aligned} |d(x, y) - d(a, b)| &\leq |d(x, y) - d(a, y)| + |d(a, y) - d(a, b)| \\ &\leq d(a, x) + d(b, y) && \text{(Proposition 16.1.14)} \\ &< \epsilon \end{aligned}$$

$\langle 2 \rangle 7. d(x, y) \in V$

$\langle 1 \rangle 2.$ If \mathcal{T} is a topology on X with respect to which d is continuous then \mathcal{T} is finer than the metric topology.

$\langle 2 \rangle 1.$ LET: \mathcal{T} be a topology on X with respect to which d is continuous.

$\langle 2 \rangle 2.$ LET: $a \in X$ and $\epsilon > 0$.

PROVE: $B(a, \epsilon) \in \mathcal{T}$

$\langle 2 \rangle 3.$ LET: $x \in B(a, \epsilon)$

$\langle 2 \rangle 4.$ $(a, x) \in d^{-1}((0, \epsilon))$

$\langle 2 \rangle 5.$ PICK $U, V \in \mathcal{T}$ such that $(a, x) \in U \times V \subseteq d^{-1}((0, \epsilon))$

$\langle 2 \rangle 6.$ $x \in V \subseteq B(a, \epsilon)$

□

Proposition 16.1.16. Let d and d' be two metrics on the same set X . Let \mathcal{T} and \mathcal{T}' be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1.$ If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon).$$

$\langle 2 \rangle 1.$ ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 2.$ LET: $x \in X$ and $\epsilon > 0$

$\langle 2 \rangle 3.$ $x \in B_d(x, \epsilon) \in \mathcal{T}'$

$\langle 2 \rangle 4.$ There exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: Proposition 16.1.13.

$\langle 1 \rangle 2.$ If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$.

$\langle 2 \rangle 1.$ ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

$\langle 2 \rangle 2.$ LET: $U \in \mathcal{T}$

$\langle 2 \rangle 3.$ For all $x \in U$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$

$\langle 3 \rangle 1.$ LET: $x \in U$

$\langle 3 \rangle 2.$ PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 16.1.13.

$\langle 3 \rangle 3.$ PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

PROOF: $\langle 2 \rangle 1$

$\langle 3 \rangle 4.$ $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4.$ $U \in \mathcal{T}'$

PROOF: Proposition 16.1.13.

□

Definition 16.1.17 (Metrisable). A topological space is *metrisable* iff there exists a metric that induces its topology.

Proposition 16.1.18. \mathbb{R}^2 under the dictionary order is metrizable.

PROOF:

$\langle 1 \rangle 1$. LET: $d : (\mathbb{R}^2)^2 \rightarrow \mathbb{R}$ be defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} \min(|y_2 - y_1|, 1) & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

$\langle 1 \rangle 2$. d is a metric.

$\langle 2 \rangle 1$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) \geq 0$.

PROOF: Immediate from definition.

$\langle 2 \rangle 2$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) = 0$ iff $x = y$.

PROOF: Immediate from definition.

$\langle 2 \rangle 3$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) = d(y, x)$.

PROOF: Immediate from definition.

$\langle 2 \rangle 4$. For all $x, y, z \in \mathbb{R}^2$ we have $d(x, z) \leq d(x, y) + d(y, z)$.

PROOF: Easy.

$\langle 1 \rangle 3$. The metric topology induced by d is finer than the order topology.

$\langle 2 \rangle 1$. LET: $a, b \in \mathbb{R}^2$

$\langle 2 \rangle 2$. LET: $x \in (a, b)$

$\langle 2 \rangle 3$. CASE: $\pi_1(x) = \pi_1(a) = \pi_1(b)$

$\langle 3 \rangle 1$. LET: $\epsilon = \min(\pi_2(x) - \pi_2(a), \pi_2(b) - \pi_2(x))$

$\langle 3 \rangle 2$. $B(x, \epsilon) \subseteq (a, b)$

$\langle 2 \rangle 4$. CASE: $\pi_1(a) = \pi_1(x) < \pi_1(b)$

$\langle 3 \rangle 1$. LET: $\epsilon = \pi_2(x) - \pi_2(a)$

$\langle 3 \rangle 2$. $B(x, \epsilon) \subseteq (a, b)$

$\langle 2 \rangle 5$. CASE: $\pi_1(a) < \pi_1(x) = \pi_1(b)$

PROOF: Similar.

$\langle 2 \rangle 6$. CASE: $\pi_1(a) < \pi_1(x) < \pi_1(b)$

PROOF: Then $B(x, \epsilon) \subseteq (a, b)$.

$\langle 1 \rangle 4$. The order topology is finer than the metric topology.

PROOF: Since $B((a, b), \epsilon) = ((a, b - \epsilon), (a, b + \epsilon))$ if $\epsilon \leq 1$, and \mathbb{R}^2 if $\epsilon > 1$.

□

Proposition 16.1.19. Every metrizable space is first countable.

PROOF: For any point a , the set $\{B(a, 1/n) : n \in \mathbb{Z}_+\}$ is a local basis at a . □

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

Proposition 16.1.20. Every compact metric space is bounded.

PROOF:

$\langle 1 \rangle 1$. LET: X be a compact metric space.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. X is nonempty.

$\langle 1 \rangle 3$. PICK $a \in X$

$\langle 1 \rangle 4$. $\{B(a, n) : n \in \mathbb{Z}_+\}$ is an open cover of X .

$\langle 1 \rangle 5$. PICK a finite subcover $\{B(a, n_1), \dots, B(a, n_k)\}$.

$\langle 1 \rangle 6$. LET: $N = \max(n_1, \dots, n_k)$

⟨1⟩7. $X = B(a, N)$

□

Example 16.1.21. The converse does not hold. An infinite discrete space is bounded but not compact.

16.1.2 Subspaces

Proposition 16.1.22. *Let (X, d) be a metric space and $Y \subseteq X$. Then $d|Y^2$ is a metric on Y that induces the subspace topology.*

PROOF:

⟨1⟩1. LET: $d' = d|Y^2 : Y^2 \rightarrow \mathbb{R}$

⟨1⟩2. d' is a metric.

PROOF: Each of the axioms follows from the axiom in X .

⟨1⟩3. The metric topology induced by d' is finer than the subspace topology.

⟨2⟩1. LET: U be open in X

PROVE: $U \cap Y$ is open in the d' -topology.

⟨2⟩2. LET: $y \in U \cap Y$

⟨2⟩3. PICK $\epsilon > 0$ such that $B_d(y, \epsilon) \subseteq U$

⟨2⟩4. $B_{d'}(y, \epsilon) \subseteq U \cap Y$

⟨1⟩4. The subspace topology is finer than the metric topology induced by d' .

⟨2⟩1. LET: $y \in Y$ and $\epsilon > 0$

PROVE: $B_{d'}(y, \epsilon)$ is open in the subspace topology.

⟨2⟩2. $B_{d'}(y, \epsilon) = B_d(y, \epsilon) \cap Y$

□

16.1.3 Convergence

Proposition 16.1.23 (Sequence Lemma). *Let X be a metric space. Let $A \subseteq X$. Let $l \in \overline{A}$. Then there exists a sequence in A that converges to l .*

PROOF:

⟨1⟩1. For $n \in \mathbb{N}$, PICK $a_n \in B(l, 1/(n+1)) \cap A$.

⟨1⟩2. $a_n \rightarrow l$ as $n \rightarrow \infty$.

□

Corollary 16.1.23.1. \mathbb{R}^ω under the box topology is not first countable.

PROOF:

⟨1⟩1. LET: A be the set of all sequences of positive reals.

⟨1⟩2. $0 \in \overline{A}$

⟨1⟩3. LET: (a_n) be a sequence in A

PROVE: (a_n) does not converge to 0.

⟨1⟩4. For all $n \in \mathbb{N}$,

LET: $a_n = (x_{nm})$

⟨1⟩5. LET: $B' = \prod_{n=0}^{\infty} (-x_{nn}, x_{nn})$

⟨1⟩6. B' is open in the box topology.

- <1>7. $0 \in B'$
 <1>8. For all n we have $a_n \notin B'$
 \square

Corollary 16.1.23.2. *If J is an uncountable set then \mathbb{R}^J under the product topology is not first countable.*

PROOF:

- <1>1. LET: $A = \{x \in \mathbb{R}^J : \pi_j(x) = 1 \text{ for all but finitely many } j \in J\}$
 <1>2. $0 \in \overline{A}$
 <1>3. LET: (a_n) be a sequence in A .
 PROVE: (a_n) does not converge to 0.
 <1>4. For $n \in \mathbb{N}$,
 LET: $J_n = \{j \in J : \pi_j(a_n) \neq 1\}$
 <1>5. $\bigcup_{n \in \mathbb{N}} J_n$ is countable.
 <1>6. PICK $\beta \in J - \bigcup_{n \in \mathbb{N}} J_n$
 <1>7. $\forall n \in \mathbb{N}. \pi_\beta(a_n) = 1$
 <1>8. LET: $U = \pi_\beta^{-1}((-1, 1))$
 <1>9. $0 \in U$
 <1>10. $\forall n \in \mathbb{N}. a_n \notin U$
 <1>11. (a_n) does not converge to 0.
 \square

16.1.4 Continuous Functions

Proposition 16.1.24. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:

- <1>1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
 <2>1. ASSUME: f is continuous.
 <2>2. LET: $x \in X$
 <2>3. LET: $\epsilon > 0$
 <2>4. $x \in f^{-1}(B(f(x), \epsilon))$
 <2>5. There exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$.
 <1>2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.
 <2>1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
 <2>2. LET: V be open in Y
 <2>3. LET: $x \in f^{-1}(V)$
 <2>4. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
 <2>5. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
 <2>6. $B(x, \delta) \subseteq f^{-1}(V)$
 \square

Proposition 16.1.25. *Let X be a metrizable space and Y a topological space. Let $f : X \rightarrow Y$. Assume that, for every sequence (x_n) in X and $l \in X$, if $x_n \rightarrow l$ as $n \rightarrow \infty$ then $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$. Then f is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $A \subseteq X$

PROVE: $f(\overline{A}) \subseteq \overline{f(A)}$

$\langle 1 \rangle 2$. LET: $l \in \overline{A}$

PROVE: $f(l) \in \overline{f(A)}$

$\langle 1 \rangle 3$. PICK a sequence (x_n) in A such that $x_n \rightarrow l$ as $n \rightarrow \infty$.

$\langle 1 \rangle 4$. $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$.

$\langle 1 \rangle 5$. $f(l) \in \overline{f(A)}$

□

Proposition 16.1.26. *The function $i : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ that maps x to x^{-1} is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $a, b \in \mathbb{R}$ with $a < b$

PROVE: $i^{-1}((a, b))$ is open.

$\langle 1 \rangle 2$. CASE: $0 < a$

PROOF: $i^{-1}((a, b)) = (b^{-1}, a^{-1})$

$\langle 1 \rangle 3$. CASE: $a = 0$

PROOF: $i^{-1}((a, b)) = (b^{-1}, +\infty)$

$\langle 1 \rangle 4$. CASE: $a < 0 < b$

PROOF: $i^{-1}((a, b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$

$\langle 1 \rangle 5$. CASE: $b = 0$

PROOF: $i^{-1}((a, b)) = (-\infty, a^{-1})$

$\langle 1 \rangle 6$. CASE: $b < 0$

PROOF: $i^{-1}((a, b)) = (b^{-1}, a^{-1})$

□

Proposition 16.1.27. *Subtraction is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF: Since $a - b = a + (-1)b$ and both addition and multiplication are continuous. □

Proposition 16.1.28. *Division is a continuous function $\mathbb{R} \times (\mathbb{R} - \{0\}) \rightarrow \mathbb{R}$.*

PROOF: Since both multiplication and the function that maps x to x^{-1} are continuous. □

Proposition 16.1.29. *Let X be a metric space. Let $A \subseteq X$ be nonempty. The function $d(-, A) : X \rightarrow \mathbb{R}$ is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $x \in X$ and $\epsilon > 0$.

$\langle 1 \rangle 2$. LET: $\delta = \epsilon/2$

$\langle 1 \rangle 3$. LET: $y \in X$ with $d(x, y) < \delta$.

$\langle 1 \rangle 4.$ $d(x, A) < d(y, A) + \epsilon$

$\langle 2 \rangle 1.$ $\forall a \in A. d(x, a) < d(y, a) + \delta$

PROOF:

$$\begin{aligned} d(x, a) &\leq d(y, a) + d(x, y) && \text{(Triangle Inequality)} \\ &< d(y, a) + \delta && (\langle 1 \rangle 3) \end{aligned}$$

$\langle 2 \rangle 2.$ $\forall a \in A. d(x, A) < d(y, a) + \delta$

$\langle 2 \rangle 3.$ $\forall a \in A. d(x, A) - \delta < d(y, a)$

$\langle 2 \rangle 4.$ $d(x, A) - \delta \leq d(y, A)$

$\langle 2 \rangle 5.$ $d(x, A) \leq d(y, A) + \delta$

$\langle 2 \rangle 6.$ $d(x, A) < d(y, A) + \epsilon$

$\langle 1 \rangle 5.$ $d(y, A) < d(x, A) + \epsilon$

PROOF: Similar.

$\langle 1 \rangle 6.$ $|d(x, A) - d(y, A)| < \epsilon$

□

Proposition 16.1.30. *Addition is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

$\langle 1 \rangle 2.$ LET: $\delta = \epsilon/2$

$\langle 1 \rangle 3.$ LET: $(x', y') \in (x - \delta, x + \delta) \times (y - \delta, y + \delta)$

$\langle 1 \rangle 4.$ $|x - x'|, |y - y'| < \delta$

$\langle 1 \rangle 5.$ $|(x + y) - (x' + y')| < \epsilon$

PROOF:

$$\begin{aligned} |(x + y) - (x' + y')| &\leq |x - x'| + |y - y'| \\ &< \delta + \delta && (\langle 1 \rangle 4) \\ &= \epsilon && (\langle 1 \rangle 2) \end{aligned}$$

□

16.1.5 First Countable Spaces

Proposition 16.1.31. *Every metrizable space is first countable.*

PROOF: For any point x , the set $\{B(x, 1/n) : n \in \mathbb{Z}_+\}$ is a countable basis at x .

□

Corollary 16.1.31.1. \mathbb{R}^ω under the box topology is not metrizable.

Corollary 16.1.31.2. *If J is an uncountable set then \mathbb{R}^J under the product topology is not metrizable.*

16.1.6 Hausdorff Spaces

Proposition 16.1.32. *Every metric space is Hausdorff.*

PROOF:

$\langle 1 \rangle 1.$ LET: X be a metric space.

- <1>2. LET: $x, y \in X$ with $x \neq y$.
 <1>3. LET: $\epsilon = d(x, y)$
 <1>4. $B(x, \epsilon/2)$ and $B(y, \epsilon/2)$ are disjoint neighbourhoods of x and y .
 \square

16.1.7 Bounded Sets

Definition 16.1.33 (Bounded). Let X be a metric space. Let $A \subseteq X$. Then A is *bounded* iff there exists M such that $\forall x, y \in A. d(x, y) \leq M$. Its *diameter* is then defined to be

$$\text{diam } A := \sup\{d(x, y) : x, y \in A\} .$$

16.1.8 Uniform Convergence

Definition 16.1.34 (Uniform Convergence). Let X be a set and Y a metric space. Let (f_n) be a sequence of functions $X \rightarrow Y$, and $f : X \rightarrow Y$. Then (f_n) *converges uniformly* to f iff, for all $\epsilon > 0$, there exists N such that

$$\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon .$$

Example 16.1.35. For $n \in \mathbb{N}$ define $f_n : [0, 1] \rightarrow \mathbb{R}$ by $f_n(x) = x^n$. Define $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = 0$ for $x < 1$, $f(1) = 1$. Then f_n converges pointwise to f , but does not converge uniformly to f .

We prove that, for all N , there exists $n \geq N$ and $x \in [0, 1]$ such that $|x^n - f(x)| \geq 1/2$. Take $n = N$ and x to be the N th root of $3/4$.

Example 16.1.36. For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} .$$

Then for all $x \in \mathbb{R}$ we have $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$, but (f_n) does not converge uniformly to 0.

We prove that, for all N , there exists $n \geq N$ and $x \in \mathbb{R}$ such that $|f_n(x)| \geq 1/2$. Take $n = N$ and $x = 1/N$. We have $f_N(1/N) = 1$.

Theorem 16.1.37 (Uniform Limit Theorem). *Let X be a topological space and Y a metric space. Let (f_n) be a sequence of functions $X \rightarrow Y$, and $f : X \rightarrow Y$. If every f_n is continuous and (f_n) converges uniformly to f , then f is continuous.*

PROOF:

<1>1. LET: V be open in Y .

<1>2. LET: $x_0 \in f^{-1}(V)$

PROVE: There exists a neighbourhood U of x_0 such that $f(U) \subseteq V$.

<1>3. LET: $y_0 = f(x_0)$

<1>4. PICK $\epsilon > 0$ such that $B(y_0, \epsilon) \subseteq V$.

- $\langle 1 \rangle 5$. PICK N such that $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/3$.
 $\langle 1 \rangle 6$. PICK a neighbourhood U of x_0 such that $f_N(U) \subseteq B(f_N(x_0), \epsilon/3)$.
 PROVE: $f(U) \subseteq V$
 $\langle 1 \rangle 7$. LET: $y \in U$
 $\langle 1 \rangle 8$. $d(f(y), y_0) < \epsilon$
 PROOF:

$$\begin{aligned}
 d(f(y), y_0) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x_0)) + d(f_N(x_0), y_0) \\
 &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1 \rangle 5, \langle 1 \rangle 6)l \\
 &= \epsilon
 \end{aligned}$$

 $\langle 1 \rangle 9$. $f(y) \in V$
 PROOF: $\langle 1 \rangle 4$
 \square

Proposition 16.1.38. *Let X be a topological space. Let Y be a metric space. Let f_n be a sequence of functions $X \rightarrow Y$ and $f : X \rightarrow Y$. Let x_n be a sequence of points in X and $l \in X$. If f_n converges uniformly to f , x_n converges to l , and f is continuous, then $f_n(x_n)$ converges to $f(l)$.*

- PROOF:
 $\langle 1 \rangle 1$. f is continuous.
 $\langle 1 \rangle 2$. LET: $\epsilon > 0$
 $\langle 1 \rangle 3$. PICK $\delta > 0$ such that $\forall y \in X. d(y, l) < \delta \Rightarrow d(f(y), f(l)) < \epsilon/2$
 $\langle 1 \rangle 4$. PICK N such that $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/2$ and $\forall n \geq N. d(x_n, l) < \delta$
 $\langle 1 \rangle 5$. For all $n \geq N$ we have $d(f_n(x_n), f(l)) < \epsilon$
 PROOF:

$$\begin{aligned}
 d(f_n(x_n), f(l)) &\leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(l)) \\
 &< \epsilon/2 + \epsilon/2 \\
 &= \epsilon
 \end{aligned}$$

 \square

Theorem 16.1.39 (Weierstrass M -Test). *Let X be a set. Let (f_n) be a sequence of functions $X \rightarrow \mathbb{R}$. Let (M_n) be a sequence of real numbers. For $n \in \mathbb{N}$, let*

$$s_n(x) = \sum_{i=0}^n f_i(x) \quad .$$

Assume that $\forall n \in \mathbb{N}. \forall x \in X. |f_n(x)| \leq M_n$. Assume that $\sum_{n=0}^{\infty} M_n$ converges. Then (s_n) uniformly converges to s where $s(x) = \sum_{n=0}^{\infty} f_n(x)$.

- PROOF:
 $\langle 1 \rangle 1$. For all $x \in X$ we have $\sum_{n=0}^{\infty} f_n(x)$ converges.
 PROOF: By the Comparison Test.
 $\langle 1 \rangle 2$. For $n \in \mathbb{N}$,
 LET: $r_n = \sum_{i=n+1}^{\infty} M_i$.
 $\langle 1 \rangle 3$. For all $k, n \in \mathbb{N}$ and $x \in X$, if $k > n$ then $|s_k(x) - s_n(x)| \leq r_n$.

PROOF:

$$\begin{aligned}
 |s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\
 &\leq \sum_{i=n+1}^k |f_i(x)| \\
 &\leq \sum_{i=n+1}^k M_i \\
 &\leq \sum_{i=n+1}^{\infty} M_i \\
 &= r_n
 \end{aligned}$$

$\langle 1 \rangle 4$. For all $n \in \mathbb{N}$ we have $|s(x) - s_n(x)| \leq r_n$.

PROOF: Taking the limit $k \rightarrow \infty$ in $\langle 1 \rangle 3$.

$\langle 1 \rangle 5$. (s_n) converges uniformly to s .

PROOF: We have $\bar{\rho}(s_n, s) \leq r_n$ and so $\bar{\rho}(s_n, s) \rightarrow 0$ as $n \rightarrow \infty$ by the Sandwich Theorem.

□

Theorem 16.1.40. *Let X be a compact space. Let $f_n : X \rightarrow \mathbb{R}$ be a monotone increasing sequence of continuous functions, and $f : X \rightarrow \mathbb{R}$ be continuous. If f_n converges to f pointwise, then f_n converges to f uniformly.*

PROOF:

$\langle 1 \rangle 1$. LET: $\epsilon > 0$

$\langle 1 \rangle 2$. For all $x \in X$, there exists a neighbourhood U of x and an integer N such that, for all $y \in U$, we have $|f_N(y) - f(y)| < \epsilon$

$\langle 2 \rangle 1$. LET: $x \in X$

$\langle 2 \rangle 2$. PICK N such that $\forall n \geq N, |f_n(x) - f(x)| < \epsilon/3$.

$\langle 2 \rangle 3$. PICK neighbourhoods U_1, U_2 of x such that $\forall y \in U_1, |f_N(y) - f_N(x)| < \epsilon/3$ and $\forall y \in U_2, |f(y) - f(x)| < \epsilon/3$.

$\langle 2 \rangle 4$. LET: $U = U_1 \cap U_2$

$\langle 2 \rangle 5$. LET: $y \in U$

$\langle 2 \rangle 6$. $|f_N(y) - f(y)| < \epsilon$

PROOF:

$$\begin{aligned}
 |f_N(y) - f(y)| &\leq |f_N(y) - f_N(x)| + |f_N(x) - f(x)| + |f(x) - f(y)| \quad (\text{Triangle Inequality}) \\
 &< \epsilon/3 + \epsilon/3 + \epsilon/3 \quad (\langle 2 \rangle 2, \langle 2 \rangle 3) \\
 &= \epsilon
 \end{aligned}$$

$\langle 1 \rangle 3$. PICK open sets U_1, \dots, U_k that cover X such that, for all i , there exists N_i such that $\forall y \in U_i, |f_{N_i}(y) - f(y)| < \epsilon$

$\langle 1 \rangle 4$. LET: $N = \max(N_1, \dots, N_k)$

$\langle 1 \rangle 5$. For all $x \in X$ we have $|f_N(x) - f(x)| < \epsilon$

$\langle 1 \rangle 6$. For all $n \geq N$ and $x \in X$ we have $|f_n(x) - f(x)| < \epsilon$

□

Example 16.1.41. We cannot remove the requirement that (f_n) is monotone increasing.

For all $n \in \mathbb{Z}_+$, define $f_n : [0, 1] \rightarrow \mathbb{R}$ by

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} .$$

Then $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all x , but the convergence is not uniform.

Example 16.1.42. We cannot remove the requirement that X is compact.

For all $n \in \mathbb{Z}_+$, define $f_n : [0, 1) \rightarrow \mathbb{R}$ by $f_n(x) = -x^n$. Then (f_n) is monotone increasing and $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all x , but the convergence is not uniform.

16.1.9 Standard Bounded Metric

Definition 16.1.43 (Standard Bounded Metric). Let (X, d) be a metric space. The *standard bounded metric* corresponding to d is

$$\bar{d}(x, y) := \min(d(x, y), 1) .$$

Proposition 16.1.44. *The standard bounded metric associated with d induces the same topology as d .*

PROOF:

- $\langle 1 \rangle 1$. LET: (X, d) be a metric space.
- $\langle 1 \rangle 2$. Every d -ball is open under the topology induced by \bar{d} .
 - $\langle 2 \rangle 1$. LET: $a \in X$ and $\epsilon > 0$
 - $\langle 2 \rangle 2$. LET: $x \in B_d(a, \epsilon)$
 - $\langle 2 \rangle 3$. LET: $\delta = \min(\epsilon - d(a, x), 1/2)$
 - $\langle 2 \rangle 4$. $B_{\bar{d}}(x, \delta) \subseteq B_d(a, \epsilon)$
- $\langle 1 \rangle 3$. Every \bar{d} -ball is open under the topology induced by d .

PROOF: Since $B_{\bar{d}}(a, \epsilon) = B_d(a, \epsilon)$ if $\epsilon \leq 1$, and X if $\epsilon > 1$.

□

16.1.10 Product Spaces

Proposition 16.1.45. *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

- $\langle 1 \rangle 1$. LET: (X_n, d_n) be a sequence of metric spaces.
- $\langle 1 \rangle 2$. For $n \in \mathbb{N}$,
 - LET: \bar{d}_n be the standard bounded metric associated with d_n .
- $\langle 1 \rangle 3$. LET: $X = \prod_{n \in \mathbb{N}} X_n$
- $\langle 1 \rangle 4$. Define $D : X^2 \rightarrow \mathbb{R}$ by $D(x, y) = \sup_{n \in \mathbb{N}} \bar{d}_n(\pi_n(x), \pi_n(y)) / (n + 1)$.
- $\langle 1 \rangle 5$. D is a metric on X .
 - $\langle 2 \rangle 1$. For all $x, y \in X$ we have $D(x, y) \geq 0$.

- ⟨2⟩2. For all $x, y \in X$ we have $D(x, y) = 0$ iff $x = y$.
- ⟨2⟩3. For all $x, y \in X$ we have $D(x, y) = D(y, x)$.
- ⟨2⟩4. For all $x, y, z \in X$ we have $D(x, z) \leq D(x, y) + D(y, z)$.
- ⟨1⟩6. The product topology is finer than the metric topology induced by D .
 - ⟨2⟩1. LET: $a \in X$ and $\epsilon > 0$.
 - ⟨2⟩2. LET: $x \in B(a, \epsilon)$
 - ⟨2⟩3. LET: $\delta = \epsilon - D(a, x)$
 - ⟨2⟩4. PICK $N \in \mathbb{N}$ such that $1/(N+1) < \delta$
 - ⟨2⟩5. $x \in \prod_{n=0}^N B_{d_n}(\pi_n(a), n\delta) \times \prod_{n=N+1}^{\infty} B(a, \epsilon)$
- ⟨1⟩7. The metric topology induced by D is finer than the product topology.
 - ⟨2⟩1. LET: $n \in \mathbb{N}$ and U be an open set in X_n .
PROVE: $\pi_n^{-1}(U)$ is open in the metric topology.
 - ⟨2⟩2. LET: $x \in \pi_n^{-1}(U)$
 - ⟨2⟩3. PICK $\epsilon > 0$ such that $B_{d_n}(\pi_n(x), \epsilon) \subseteq U$
 - ⟨2⟩4. $B(x, \epsilon/(n+1)) \subseteq \pi_n^{-1}(U)$

□

Definition 16.1.46. For $n \geq 1$, the *unit ball* B^n is the closed ball $\overline{B(0, 1)}$ in \mathbb{R}^n under the Euclidean metric.

Theorem 16.1.47. Let n be a positive integer. Let A be a subspace of \mathbb{R}^n . Then A is compact if and only if it is closed and bounded.

PROOF:

- ⟨1⟩1. If A is compact then A is closed.
PROOF: Theorem 15.31.10.
- ⟨1⟩2. If A is compact then A is bounded.
PROOF: Proposition 16.1.20.
- ⟨1⟩3. If A is closed and bounded then A is compact.
 - ⟨2⟩1. ASSUME: A is closed and bounded.
 - ⟨2⟩2. PICK M such that $\forall \vec{x}, \vec{y} \in A. \rho(\vec{x}, \vec{y}) < M$.
 - ⟨2⟩3. ASSUME: w.l.o.g. $A \neq \emptyset$
 - ⟨2⟩4. PICK $\vec{a} \in A$
 - ⟨2⟩5. $A \subseteq \prod_{i=1}^n [a_i - M, a_i + M]$
 - ⟨2⟩6. A is compact.
PROOF: Theorem 15.31.7.

Corollary 16.1.47.1. For $n \geq 1$, the unit sphere S^{n-1} and the unit ball B^n are compact.

16.2 Uniform Metric

Definition 16.2.1 (Uniform Metric). Let J be a nonempty set. The *uniform metric* $\bar{\rho}$ on \mathbb{R}^J is defined by

$$\bar{\rho}(x, y) = \sup_{j \in J} \bar{d}(x_j, y_j)$$

where \bar{d} is the standard bounded metric associated with the standard metric on \mathbb{R} .

On \mathbb{R}^n we call the uniform metric the *square metric* and denote it by ρ .

The topology it induces is called the *uniform topology*.

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1$. For all $x, y \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, y) \geq 0$.

PROOF: Pick $j_0 \in J$. Then

$$\begin{aligned}\bar{\rho}(x, y) &= \sup_j \bar{d}(x_j, y_j) \\ &\geq \bar{d}(x_{j_0}, y_{j_0}) \\ &\geq 0\end{aligned}$$

$\langle 1 \rangle 2$. For all $x, y \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, y) = 0$ iff $x = y$.

PROOF:

$$\begin{aligned}\bar{\rho}(x, y) = 0 &\Leftrightarrow \sup_j \bar{d}(x_j, y_j) = 0 \\ &\Leftrightarrow \forall j. \bar{d}(x_j, y_j) = 0 \\ &\Leftrightarrow \forall j. x_j = y_j \\ &\Leftrightarrow x = y\end{aligned}$$

$\langle 1 \rangle 3$. For all $x, y \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, y) = \bar{\rho}(y, x)$.

PROOF:

$$\begin{aligned}\bar{\rho}(x, y) &= \sup_j \bar{d}(x_j, y_j) \\ &= \sup_j \bar{d}(y_j, x_j) \\ &= \bar{\rho}(y, x)\end{aligned}$$

$\langle 1 \rangle 4$. For all $x, y, z \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, z) \leq \bar{\rho}(x, y) + \bar{\rho}(y, z)$.

PROOF:

$$\begin{aligned}\bar{\rho}(x, z) &= \sup_j \bar{d}(x_j, z_j) \\ &\leq \sup_j (\bar{d}(x_j, y_j) + \bar{d}(y_j, z_j)) \\ &\leq \sup_j \bar{d}(x_j, y_j) + \sup_j \bar{d}(y_j, z_j) \\ &= \bar{\rho}(x, y) + \bar{\rho}(y, z)\end{aligned}$$

□

Proposition 16.2.2. *The uniform topology is finer than the product topology. It is strictly finer iff J is infinite.*

PROOF:

$\langle 1 \rangle 1$. The uniform topology is finer than the product topology.

$\langle 2 \rangle 1$. LET: U be open in \mathbb{R} and $j \in J$

PROVE: $\pi_j^{-1}(U)$ is open in the uniform topology.

$\langle 2 \rangle 2$. LET: $x \in \pi_j^{-1}(U)$

- ⟨2⟩3. $\pi_j(x) \in U$
- ⟨2⟩4. PICK $\epsilon > 0$ such that $B_{\bar{d}}(\pi_j(x), \epsilon) \subseteq U$
- ⟨2⟩5. $B_{\bar{p}}(x, \epsilon) \subseteq \pi_j^{-1}(U)$
- ⟨1⟩2. If J is finite then the uniform topology is equal to the product topology.
PROOF: In \mathbb{R}^n , the uniform topology is the square topology.
- ⟨1⟩3. If J is infinite then the uniform topology is not equal to the product topology.
PROOF: If J is infinite then $B(0, 1)$ is not open in the product topology.

□

Proposition 16.2.3. *The uniform topology is coarser than the box topology. It is strictly coarser iff J is infinite.*

PROOF:

- ⟨1⟩1. The uniform topology is coarser than the box topology.
- ⟨2⟩1. LET: U be open in the uniform topology.
PROVE: U is open in the box topology.
- ⟨2⟩2. LET: $x \in U$
- ⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
- ⟨2⟩4. $\prod_{j \in J} (x_j - \epsilon, x_j + \epsilon) \subseteq U$
- ⟨1⟩2. If J is finite then the uniform topology is equal to the box topology.
PROOF: On \mathbb{R}^n , the uniform metric is the square metric.
- ⟨1⟩3. If J is infinite then the uniform topology is not equal to the box topology.
- ⟨2⟩1. ASSUME: J is infinite.
- ⟨2⟩2. PICK a sequence (j_n) of distinct elements in J .
- ⟨2⟩3. LET: $U = \prod_j U_j$ where $J_{j_n} = (-1/(n+1), 1/(n+1))$ for $n \in \mathbb{N}$ and $J_j = (-1, 1)$ for all other j .
- ⟨2⟩4. U is not open in the uniform topology.

□

Proposition 16.2.4. *The uniform topology on \mathbb{R}^∞ is strictly finer than the product topology.*

PROOF: The set of all sequences $(x_n) \in \mathbb{R}^\infty$ such that $\forall n. |x_n| < 1$ is open in the uniform topology but not in the product topology. □

Proposition 16.2.5. *The uniform topology on \mathbb{R}^∞ is strictly coarser than the box topology.*

PROOF: The set of sequences $(x_n) \in \mathbb{R}^\infty$ such that $\forall n. |x_n| < 1/n$ is open in the box topology but not in the uniform topology. □

Proposition 16.2.6. *The uniform topology on the Hilbert cube is the same as the product topology.*

PROOF:

- ⟨1⟩1. LET: (x_n) be in the Hilbert cube H and $\epsilon > 0$.
PROVE: $B((x_n), \epsilon) \cap H$ is open in the product topology.
- ⟨1⟩2. PICK N such that $1/N < \epsilon$

$\langle 1 \rangle 3.$ $B((x_n), \epsilon) = (\prod_{n=0}^N (x_n - \epsilon, x_n + \epsilon) \times \prod_{n=N+1}^{\infty} [0, 1/(n+1)]) \cap H$
 \square

Corollary 16.2.6.1. *The uniform topology on the Hilbert cube is strictly finer than the box topology.*

Proposition 16.2.7. *Let X be a set and Y a metric space. Let (f_n) be a sequence of functions $X \rightarrow Y$, and $f : X \rightarrow Y$. Then (f_n) converges uniformly to f iff (f_n) converges to f in Y^X under the uniform topology.*

PROOF:

$\langle 1 \rangle 1.$ If (f_n) converges uniformly to f then (f_n) converges to f in Y^X under the uniform topology.

$\langle 2 \rangle 1.$ ASSUME: (f_n) converges uniformly to f .

$\langle 2 \rangle 2.$ LET: $\epsilon > 0$

$\langle 2 \rangle 3.$ PICK N such that $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/2$

$\langle 2 \rangle 4.$ $\forall n \geq N. \bar{\rho}(f_n, f) \leq \epsilon/2$

$\langle 2 \rangle 5.$ $\forall n \geq N. \bar{\rho}(f_n, f) < \epsilon$

$\langle 1 \rangle 2.$ If (f_n) converges to f in Y^X under the uniform topology then (f_n) converges uniformly to f .

$\langle 2 \rangle 1.$ ASSUME: (f_n) converges to f in Y^X under the uniform topology.

$\langle 2 \rangle 2.$ LET: $\epsilon > 0$

$\langle 2 \rangle 3.$ PICK N such that $\forall n \geq N. \bar{\rho}(f_n, f) < \epsilon$

$\langle 2 \rangle 4.$ $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon$

\square

Proposition 16.2.8. *In \mathbb{R}^ω under the uniform topology, \vec{x} and \vec{y} lie in the same component if and only if $\vec{x} - \vec{y}$ is bounded.*

PROOF:

$\langle 1 \rangle 1.$ The set of bounded sequences form a component of \mathbb{R}^ω .

$\langle 2 \rangle 1.$ LET: B be the set of bounded sequences.

$\langle 2 \rangle 2.$ B is connected.

$\langle 3 \rangle 1.$ LET: $\vec{x} \in B$

PROVE: The straight line path $p : [0, 1] \rightarrow \mathbb{R}^\omega$ from 0 to \vec{x} is continuous.

$\langle 3 \rangle 2.$ LET: $t \in [0, 1]$ and $\epsilon > 0$

$\langle 3 \rangle 3.$ PICK $B > 0$ such that $\forall n. |x_n| < B$

$\langle 3 \rangle 4.$ LET: $\delta = \epsilon/B$

$\langle 3 \rangle 5.$ LET: $s \in [0, 1]$ with $|s - t| < \delta$

$\langle 3 \rangle 6.$ For all n we have $|p(s)_n - p(t)_n| < \epsilon/2$

PROOF:

$$\begin{aligned} |p(s)_n - p(t)_n| &= |s - t| |x_n| \\ &< \delta B \\ &= \epsilon \end{aligned}$$

$\langle 3 \rangle 7.$ $\bar{\rho}(p(s), p(t)) \leq \epsilon/2$

$\langle 3 \rangle 8.$ $\bar{\rho}(p(s), p(t)) < \epsilon$

⟨2⟩3. B is maximally connected.

PROOF: Since $(B, \mathbb{R}^\omega - B)$ form a separation of \mathbb{R}^ω .

⟨1⟩2. For any $\vec{y} \in \mathbb{R}^\omega$, the component containing \vec{y} is $\{\vec{x} \in \mathbb{R}^\omega : \vec{x} - \vec{y} \text{ is bounded}\}$.

PROOF: Since the function that maps \vec{x} to $\vec{x} + \vec{y}$ is a homeomorphism between \mathbb{R}^ω and itself.

□

16.2.1 Products

Definition 16.2.9 (Euclidean Metric). Let X and Y be metric spaces. The *Euclidean metric* on $X \times Y$ is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}.$$

We write $X \times Y$ for the set $X \times Y$ under this metric.

We prove this is a metric.

PROOF:

⟨1⟩1. $d((x_1, y_1), (x_2, y_2)) \geq 0$

PROOF: Immediate from definition.

⟨1⟩2. $d((x_1, y_1), (x_2, y_2)) = 0$ iff $(x_1, y_1) = (x_2, y_2)$

PROOF: $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = 0$ iff $d(x_1, x_2) = d(y_1, y_2) = 0$ iff $x_1 = x_2$ and $y_1 = y_2$.

⟨1⟩3. $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$

PROOF: Since $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = \sqrt{d(x_2, x_1)^2 + d(y_2, y_1)^2}$.

⟨1⟩4. The triangle inequality holds.

PROOF:

$$\begin{aligned} & (d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)))^2 \\ &= d((x_1, y_1), (x_2, y_2))^2 + 2d((x_1, y_1), (x_2, y_2))d((x_2, y_2), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3))^2 \\ &= d(x_1, x_2)^2 + d(y_1, y_2)^2 + 2\sqrt{(d(x_1, x_2)^2 + d(y_1, y_2)^2)(d(x_2, x_3)^2 + d(y_2, y_3)^2)} + d(x_2, x_3)^2 + d(y_2, y_3)^2 \\ &\geq d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(y_1, y_2)^2 + d(y_2, y_3)^2 + 2(d(x_1, x_2)d(x_2, x_3) + d(y_1, y_2)d(y_2, y_3)) \\ &\quad (\text{Cauchy-Schwarz}) \\ &= (d(x_1, x_2) + d(x_2, x_3))^2 + (d(y_1, y_2) + d(y_2, y_3))^2 \\ &\geq d(x_1, x_3)^2 + d(y_1, y_3)^2 \\ &= d((x_1, y_1), (x_3, y_3))^2 \end{aligned}$$

□

Proposition 16.2.10. Let X and Y be metric spaces. The Euclidean metric on $X \times Y$ induces the product topology on $X \times Y$.

PROOF:

⟨1⟩1. Every open ball is open in the product topology.

⟨2⟩1. LET: $(x, y) \in B((a, b), \epsilon)$

PROVE: $B(x, \sqrt{\epsilon}) \times B(y, \sqrt{\epsilon}) \subseteq B((a, b), \epsilon)$

⟨2⟩2. LET: $x' \in B(x, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$ and $y' \in B(y, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$

- PROVE: $d((x', y'), (a, b)) < \epsilon$
- ⟨2⟩3. $d((x', y'), (x, y)) < \epsilon - d((x, y), (a, b))$
- PROOF:
- $$\begin{aligned} d((x', y'), (x, y)) &= \sqrt{d(x', x)^2 + d(y', y)^2} \\ &< \sqrt{(\epsilon - d((x, y), (a, b)))^2/2 + (\epsilon - d((x, y), (a, b)))^2/2} \\ &= \epsilon - d((x, y), (a, b)) \end{aligned}$$
- ⟨2⟩4. $d((x', y'), (a, b)) < \epsilon$
- PROOF:
- $$\begin{aligned} d((x', y'), (a, b)) &\leq d((x', y'), (x, y)) + d((x, y), (a, b)) \quad (\text{Triangle Inequality}) \\ &< \epsilon \end{aligned} \quad (\langle 2 \rangle 3)$$
- ⟨1⟩2. If U is open in X and V is open in Y then $U \times V$ is open under the Euclidean metric.
- ⟨2⟩1. LET: $(x, y) \in U \times V$
- ⟨2⟩2. PICK $\delta, \epsilon > 0$ such that $B(x, \delta) \subseteq U$ and $B(y, \epsilon) \subseteq V$
- PROVE: $(B((x, y), \min(\delta, \epsilon))) \subseteq U \times V$
- ⟨2⟩3. LET: $(x', y') \in B((x, y), \min(\delta, \epsilon))$
- ⟨2⟩4. $d(x', x) < \delta$
- ⟨3⟩1. $d((x', y'), (x, y)) < \min(\delta, \epsilon)$
- ⟨3⟩2. $d(x', x)^2 + d(y', y)^2 < \delta^2$
- ⟨3⟩3. $d(x', x)^2 < \delta^2$
- ⟨2⟩5. $d(y', y) < \epsilon$
- PROOF: Similar.
- ⟨2⟩6. $(x', y') \in U \times V$
-

Proposition 16.2.11. *The square metric on \mathbb{R}^n induces the product topology.*

- PROOF:
- ⟨1⟩1. LET: d be the Euclidean metric on \mathbb{R}^n and ρ the square metric.
- ⟨1⟩2. For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_d(x, \delta) \subseteq B_\rho(x, \epsilon)$
- PROOF: If $d(x, y) < \epsilon$ then $\rho(x, y) < \epsilon$.
- ⟨1⟩3. For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_\rho(x, \delta) \subseteq B_d(x, \epsilon)$
- PROOF: If $\rho(x, y) < \epsilon/\sqrt{n}$ then $d(x, y) < \epsilon$.
- ⟨1⟩4. d and ρ induce the same topology.
- PROOF: Proposition 16.1.16.
-

16.2.2 Connected Spaces

Example 16.2.12. The space \mathbb{R}^ω under the uniform topology is disconnected. The set of bounded sequences and the set of unbounded sequences form a separation.

16.3 Isometric Embeddings

Definition 16.3.1 (Isometric Embedding). Let X and Y be metric spaces. Let $f : X \rightarrow Y$. Then f is an *isometric embedding* of X in Y iff, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 16.3.2. *Every isometric embedding is an embedding.*

PROOF:

- ⟨1⟩1. LET: X and Y be metric spaces.
- ⟨1⟩2. LET: $f : X \rightarrow Y$ be an isometric embedding.
- ⟨1⟩3. f is injective.
- ⟨1⟩4. The subspace topology induced by f is finer than the metric topology.
 - ⟨2⟩1. LET: $x \in X$ and $\epsilon > 0$
 PROVE: $B(x, \epsilon)$ is open in the subspace topology.
 - ⟨2⟩2. $B(x, \epsilon) = f^{-1}(B(f(x), \epsilon))$
- ⟨1⟩5. The metric topology is finer than the subspace topology induced by f .
 - ⟨2⟩1. LET: V be open in Y
 PROVE: $f^{-1}(V)$ is open in X
 - ⟨2⟩2. LET: $x \in f^{-1}(V)$
 - ⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
 - ⟨2⟩4. $B(x, \epsilon) \subseteq f^{-1}(V)$

□

16.4 Lebesgue Numbers

Definition 16.4.1 (Lebesgue Number). Let X be a metric space. Let \mathcal{A} be an open covering of X . A *Lebesgue number* for \mathcal{A} is a real number $\delta > 0$ such that every bounded set with diameter $< \delta$ is included in some member of \mathcal{A} .

Lemma 16.4.2 (Lebesgue Number Lemma). *In a compact metric space, every open cover has a Lebesgue number.*

PROOF:

- ⟨1⟩1. LET: X be a compact metric space.
- ⟨1⟩2. LET: \mathcal{A} be an open cover of X .
- ⟨1⟩3. ASSUME: w.l.o.g. $X \notin \mathcal{A}$
- ⟨1⟩4. PICK $A_1, \dots, A_n \in \mathcal{A}$ that cover X .
- ⟨1⟩5. For $i = 1, \dots, n$,
 LET: $C_i = X - A_i$.
- ⟨1⟩6. LET: $f : X \rightarrow \mathbb{R}$ be the function

$$f(x) = \frac{1}{n} \sum_{i=1}^n d(x, C_i) .$$

PROOF: Each C_i is nonempty by ⟨1⟩3.

- ⟨1⟩7. $\forall x \in X. f(x) > 0$
- ⟨2⟩1. LET: $x \in X$

- $\langle 2 \rangle 2$. PICK i such that $x \in A_i$
 $\langle 2 \rangle 3$. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A_i$
 $\langle 2 \rangle 4$. $d(x, C_i) \geq \epsilon$
 $\langle 2 \rangle 5$. $f(x) \geq \epsilon/n$
 $\langle 1 \rangle 8$. LET: δ be the minimum value of $f(X)$.
 PROVE: δ is a Lebesgue number for \mathcal{A} .
 PROOF: $f(X)$ has a least element by the Extreme Value Theorem.
 $\langle 1 \rangle 9$. LET: $B \subseteq X$ have diameter $< \delta$
 $\langle 1 \rangle 10$. PICK $x_0 \in B$
 $\langle 1 \rangle 11$. $B \subseteq B(x_0, \delta)$
 $\langle 1 \rangle 12$. LET: m be such that $d(x_0, C_m) = \max(d(x_0, C_1), \dots, d(x_0, C_n))$
 $\langle 1 \rangle 13$. $d(x_0, C_m) \geq \delta$
 PROOF:

$$\delta \leq f(x_0) \quad (\langle 1 \rangle 8)$$

$$= \frac{1}{n} \sum_{i=1}^n d(x_0, C_i) \quad (\langle 1 \rangle 6)$$

$$\leq \frac{1}{n} \sum_{i=1}^n d(x_0, C_m) \quad (\langle 1 \rangle 12)$$

$$= d(x_0, C_m)$$

- $\langle 1 \rangle 14$. $B \subseteq A_m$
 PROOF: $\langle 1 \rangle 11, \langle 1 \rangle 13$
 \square

16.5 Uniform Continuity

Definition 16.5.1 (Uniformly Continuous). Let X and Y be metric spaces. Let $f : X \rightarrow Y$. Then f is *uniformly continuous* iff, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Theorem 16.5.2 (Uniform Continuity Theorem). *Every continuous function from a compact metric space to a metric space is uniformly continuous.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be a compact metric space.
 $\langle 1 \rangle 2$. LET: Y be a metric space.
 $\langle 1 \rangle 3$. LET: $f : X \rightarrow Y$ be continuous.
 $\langle 1 \rangle 4$. LET: $\epsilon > 0$
 $\langle 1 \rangle 5$. PICK a Lebesgue number δ for $\{f^{-1}(B(y, \epsilon/2)) : y \in Y\}$.
 $\langle 1 \rangle 6$. LET: $x, x' \in X$
 $\langle 1 \rangle 7$. ASSUME: $d(x, x') < \delta$
 $\langle 1 \rangle 8$. PICK $y \in Y$ such that $\{x, x'\} \subseteq f^{-1}(B(y, \epsilon/2))$
 $\langle 1 \rangle 9$. $d(f(x), f(x')) < \epsilon$

PROOF:

$$\begin{aligned}
 d(f(x), f(x')) &\leq d(f(x), y) + d(y, f(x')) && \text{(Triangle Inequality)} \\
 &< \epsilon/2 + \epsilon/2 && (\langle 1 \rangle 8) \\
 &= \epsilon
 \end{aligned}$$

□

16.6 Complete Metric Spaces

Definition 16.6.1 (Complete). A metric space is *complete* iff every Cauchy sequence converges.

Example 16.6.2. \mathbb{R} is complete.

Proposition 16.6.3. *The product of two complete metric spaces is complete.*

Proposition 16.6.4. *Every compact metric space is complete.*

Proposition 16.6.5. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 16.6.6 (Completion). Let X be a metric space. A *completion* of X is a complete metric space \hat{X} and injection $i : X \rightarrow \hat{X}$ such that:

- The metric on X is the restriction of the metric on \hat{X}
- X is dense in \hat{X} .

Proposition 16.6.7. *Let $i_1 : X \rightarrow Y_1$ and $i_2 : X \rightarrow Y_2$ be completions of X . Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

PROOF: Define $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$. □

Theorem 16.6.8. *Every metric space has a completion.*

PROOF: Let \hat{X} be the set of Cauchy sequences in X quotiented by \sim where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \rightarrow 0$. □

16.7 Manifolds

Definition 16.7.1 (Manifold). An *n -dimensional manifold* is a second countable Hausdorff space locally homeomorphic to \mathbb{R}^n .

Chapter 17

Homotopy Theory

17.1 Homotopies

Definition 17.1.1 (Homotopy). Let X and Y be topological spaces. Let $f, g : X \rightarrow Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0, 1] \rightarrow Y$ such that

- $\forall x \in X. h(x, 0) = f(x)$
- $\forall x \in X. h(x, 1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \rightarrow Y$.

Proposition 17.1.2. Let $f, f' : X \rightarrow Y$ and $g, g' : Y \rightarrow Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.

Definition 17.1.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

A *homotopy functor* is a functor $\mathbf{Top} \rightarrow \mathcal{C}$ that factors through the canonical functor $\mathbf{Top} \rightarrow \mathbf{HTop}$.

Definition 17.1.4. A functor $F : \mathbf{Top} \rightarrow \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \rightarrow Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

17.2 Homotopy Equivalence

Definition 17.2.1 (Homotopy Equivalence). Let X and Y be topological spaces. A *homotopy equivalence* between X and Y , $f : X \simeq Y$, is a continuous function $f : X \rightarrow Y$ such that there exists a continuous function $g : Y \rightarrow X$, the *homotopy inverse* to f , such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.

Definition 17.2.2 (Contractible). A topological space X is *contractible* iff $X \simeq 1$.

Example 17.2.3. \mathbb{R}^n is contractible.

Example 17.2.4. D^n is contractible.

Definition 17.2.5 (Deformation Retract). Let X be a topological space and A a subspace of X . A retraction $\rho : X \rightarrow A$ is a *deformation retraction* iff $i \circ \rho \simeq \text{id}_X$, where i is the inclusion $A \hookrightarrow X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 17.2.6 (Strong Deformation Retract). Let X be a topological space and A a subspace of X . A *strong deformation retraction* $\rho : X \rightarrow A$ is a continuous function such that there exists a homotopy $h : X \times [0, 1] \rightarrow X$ between $i \circ \rho$ and id_X such that, for all $a \in X$ and $t \in [0, 1]$, we have $h(a, t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 17.2.7. $\{0\}$ is a strong deformation retract of \mathbb{R}^n and of D^n .

Example 17.2.8. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 17.2.9. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 17.2.10. For any topological space X , the singleton consisting of the vertex is a strong deformation retract of the cone over X .

Chapter 18

Simplicial Complexes

Definition 18.0.1 (Simplex). A k -dimensional simplex or k -simplex in \mathbb{R}^n is the convex hull $s(x_0, \dots, x_k)$ of $k + 1$ points in general position.

Definition 18.0.2 (Face). A *sub-simplex* or *face* of $s(x_0, \dots, x_k)$ is the convex hull of a subset of $\{x_0, \dots, x_k\}$.

Definition 18.0.3 (Simplicial Complex). A *simplicial complex* in \mathbb{R}^n is a set K of simplices such that:

- for every simplex s in K , every face of s is in K .
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2 .
- K is locally finite, i.e. every point of \mathbb{R}^n has a neighbourhood that only intersects finitely many elements of K .

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of \mathbb{R}^n .

18.1 Cell Decompositions

Definition 18.1.1 (n -cell). An n -cell is a topological space homeomorphic to \mathbb{R}^n .

Definition 18.1.2 (Cell Decomposition). Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n -cells.

Definition 18.1.3 (n -skeleton). Given a cell decomposition of X , the n -skeleton X^n is the union of all the cells of dimension $\leq n$.

18.2 CW-complexes

Definition 18.2.1 (CW-Complex). A *CW-complex* consists of a topological space X and a cell decomposition \mathcal{E} of X such that:

1. *Characteristic Maps* For every n -cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \rightarrow X$ such that $\Phi_e((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in \mathcal{E} of dimension $< n$.
2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have \bar{e} intersects only finitely many other cells in \mathcal{E} .
3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \bar{e}$ is closed.

Proposition 18.2.2. *If a cell decomposition \mathcal{E} satisfies the Characteristic Maps axiom, then for every n -cell $e \in \mathcal{E}$ we have $\bar{e} = \Phi_e(D^n)$. Therefore \bar{e} is compact and $\bar{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

PROOF:

$\langle 1 \rangle 1.$ $e \subseteq \Phi_e(D^n) \subseteq \bar{e}$

PROOF:

$$\begin{aligned}
 e &= \Phi_e((D^n)^\circ) \\
 &\subseteq \Phi_e(D^n) \\
 &= \Phi_e(\overline{(D^n)^\circ}) \\
 &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
 &= \bar{e}
 \end{aligned}$$

$\langle 1 \rangle 2.$ $\Phi_e(D^n)$ is compact.

PROOF: Because D^n is compact.

$\langle 1 \rangle 3.$ $\Phi_e(D^n)$ is closed.

$\langle 1 \rangle 4.$ $\Phi_e(D^n) = \bar{e}$

□

Chapter 19

Topological Groups

19.1 Topological Groups

Definition 19.1.1 (Topological Group). A *topological group* is a group G with a topology such that the function $G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.

Example 19.1.2. \mathbb{Z} is a topological group under addition.

PROOF: The function that sends (x, y) to xy^{-1} is continuous because the topology on \mathbb{Z} is discrete. \square

Example 19.1.3. \mathbb{R} is a topological group under addition.

PROOF: From Propositions 16.1.30 and 16.1.10. \square

Example 19.1.4. \mathbb{R}_+ is a topological group under multiplication.

PROOF: From Propositions 16.1.10 and 16.1.26. \square

Example 19.1.5. S^1 as a subspace of \mathbb{C} is a topological group under multiplication.

PROOF:

$\langle 1 \rangle 1$. LET: $f : S^1 \rightarrow S^1$ be the function $f(x, y) = xy^{-1}$

$\langle 1 \rangle 2$. LET: U be an open set in S^1

PROVE: $f^{-1}(U)$ is open in $(S^1)^2$

$\langle 1 \rangle 3$. LET: $(x, y) \in f^{-1}(U)$

$\langle 1 \rangle 4$. $xy^{-1} \in U$

$\langle 1 \rangle 5$. LET: $x = e^{i\phi}$ and $y = e^{i\psi}$

$\langle 1 \rangle 6$. $xy^{-1} = e^{i(\phi-\psi)} \in U$

$\langle 1 \rangle 7$. PICK $\epsilon > 0$ such that, for all t , if $|\phi - \psi - t| < \epsilon$ then $e^{it} \in U$

$\langle 1 \rangle 8$. $(x, y) \in \{e^{it} : |\phi - t| < \epsilon/2\} \times \{e^{it} : |\psi - t| < \epsilon/2\} \subseteq f^{-1}(U)$

\square

Example 19.1.6. $GL(n, \mathbb{R})$ is a topological group considered as a subspace of \mathbb{R}^{n^2} .

PROOF: Since the calculations for matrix multiplication and inverse are compositions of continuous functions. \square

Example 19.1.7. $GL(n, \mathbb{C})$ is a topological group.

PROOF: Similar. \square

Proposition 19.1.8. *Let G be a group with a topology. Then G is a topological group if and only if the functions $m : G^2 \rightarrow G$ that sends (x, y) to xy and the function $i : G \rightarrow G$ that sends x to x^{-1} are continuous.*

PROOF:

$\langle 1 \rangle 1$. If G is a topological group then i is continuous.

PROOF: Since $x^{-1} = ex^{-1}$.

$\langle 1 \rangle 2$. If G is a topological group then m is continuous.

PROOF: Since $xy = x(y^{-1})^{-1}$.

$\langle 1 \rangle 3$. If m and i are continuous then G is a topological group.

PROOF: Since $xy^{-1} = m(x, i(y))$.

\square

Proposition 19.1.9. *Let G be a topological group. Let $\alpha \in G$. The function that maps x to αx is a homeomorphism between G and itself.*

PROOF:

$\langle 1 \rangle 1$. For any $\alpha \in G$, the function that maps x to αx is continuous.

PROOF: From the definition of topological group.

$\langle 1 \rangle 2$. For any $\alpha \in G$, the function that maps x to αx is a homeomorphism between G and itself.

PROOF: Its inverse is the function that maps x to $\alpha^{-1}x$.

\square

Corollary 19.1.9.1. *Every topological group is homogeneous.*

Proposition 19.1.10. *Let G be a topological group. Let $\alpha \in G$. The function that maps x to $x\alpha$ is a homeomorphism between G and itself.*

PROOF: Similar. \square

19.1.1 Subgroups

Proposition 19.1.11. *Any subgroup of a topological group is a topological group under the subspace topology.*

PROOF: Since the restriction of continuous functions is continuous. \square

Proposition 19.1.12. *Let G be a topological group and H a subgroup of G . Then \overline{H} is a topological group under the subspace topology.*

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \overline{H}$

PROVE: $xy^{-1} \in \overline{H}$

⟨1⟩2. LET: U be a neighbourhood of xy^{-1} .

PROVE: U intersects H .

⟨1⟩3. LET: $f : G^2 \rightarrow G$ be the function that maps (x, y) to xy^{-1} .

⟨1⟩4. $f^{-1}(U)$ is a neighbourhood of (x, y)

⟨1⟩5. PICK neighbourhoods V of x and W of y such that $V \times W \subseteq f^{-1}(U)$.

⟨1⟩6. PICK elements $x' \in V \cap H$ and $y' \in W \cap H$

⟨1⟩7. $x'y'^{-1} \in U \cap H$

□

Proposition 19.1.13. *Let G be a topological group. The component of G that contains e is a normal subgroup of G .*

PROOF:

⟨1⟩1. LET: C be the component that contains e .

⟨1⟩2. For all $x \in G$, we have Cx is the component of G that contains x .

PROOF: Since right multiplication by x is a homeomorphism between G and itself.

⟨1⟩3. C is a subgroup of G .

⟨2⟩1. LET: $g, h \in C$

⟨2⟩2. $C = Ch$

PROOF: ⟨1⟩2

⟨2⟩3. PICK $x \in C$ such that $xh = g$

⟨2⟩4. $x = gh^{-1}$

⟨2⟩5. $gh^{-1} \in C$

⟨1⟩4. C is a normal subgroup of G .

⟨2⟩1. LET: $g \in G$ and $h \in C$.

PROVE: $ghg^{-1} \in C$

⟨2⟩2. $C = Ch^{-1}$

⟨2⟩3. $Cg = Ch^{-1}g$

⟨2⟩4. $g \in Ch^{-1}g$

⟨2⟩5. PICK $x \in C$ such that $g = xh^{-1}g$

⟨2⟩6. $x = ghg^{-1}$

⟨2⟩7. $ghg^{-1} \in C$

□

19.1.2 Left Cosets

Proposition 19.1.14. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. Let $\alpha \in G$. Define $f_\alpha : G/H \rightarrow G/H$ by*

$$f_\alpha(xH) = \alpha xH .$$

Then f_α is a homeomorphism.

PROOF:

⟨1⟩1. For all $\alpha \in G$ we have f_α is well defined.

- 1

Proposition 19.1.15. *Let G be a T_1 topological group and H a closed subgroup of G . Then G/H is T_1 .*

$\langle 1 \rangle 1$. LET: U be open in G .
 $\langle 1 \rangle 2$. $\forall h \in H. Uh$ is open in G .
 PROOF: Since the function that maps g to gh is an automorphism of G .
 $\langle 1 \rangle 3$. UH is open in G
 PROOF: It is $\bigcup_{h \in H} Uh$.

⟨1⟩4. $UH = \pi^{-1}(\pi(U))$

PROOF:

$$\begin{aligned}\pi^{-1}(\pi(U)) &= \{x \in G : \exists y \in U. xH = yH\} \\ &= \{x \in G : \exists y \in U. x^{-1}y \in H\} \\ &= \{x \in G : \exists y \in U. \exists h \in H. y^{-1}x = h\} \\ &= \{x \in G : \exists y \in U. \exists h \in H. x = yh\} \\ &= UH\end{aligned}$$

⟨1⟩5. $\pi^{-1}(\pi(U))$ is open in G .

⟨1⟩6. $\pi(U)$ is open in G/H .

□

Proposition 19.1.17. *Let G be a topological group. Let H be a normal subgroup of G . Then G/H is a topological group.*

PROOF:

⟨1⟩1. LET: $f : G^2 \rightarrow G$ be the map $f(x, y) = xy^{-1}$

⟨1⟩2. LET: $g : (G/H)^2 \rightarrow G/H$ be the map $g(xH, yH) = xy^{-1}H$

⟨1⟩3. $g \circ (\pi \times \pi) = \pi \circ f : G^2 \rightarrow G/H$

⟨1⟩4. $g \circ (\pi \times \pi)$ is continuous.

PROOF: Since π and f are continuous.

⟨1⟩5. π is an open quotient map.

PROOF: Proposition 19.1.16.

⟨1⟩6. $\pi \times \pi$ is an open quotient map.

PROOF: Corollary 15.10.7.1.

⟨1⟩7. g is continuous.

PROOF: Theorem 15.10.3.

□

19.1.3 Homogeneous Spaces

Definition 19.1.18 (Homogeneous Space). A *homogeneous space* is a topological space of the form G/H , where G is a topological group and H is a normal subgroup of G , under the quotient topology.

Proposition 19.1.19. *Let G be a topological group and H a normal subgroup of G . Then G/H is Hausdorff if and only if H is closed.*

PROOF: See Bourbaki, N., General Topology. III.12 □

19.2 Symmetric Neighbourhoods

Definition 19.2.1 (Symmetric Neighbourhood). Let G be a topological group. Let V be a neighbourhood of e . Then V is *symmetric* iff $V = V^{-1}$.

Proposition 19.2.2. *Let G be a topological group. Let U be a neighbourhood of e . Then there exists a symmetric neighbourhood V of e such that $VV \subseteq U$.*

PROOF:

- ⟨1⟩1. PICK a neighbourhood V' of e such that $V'V' \subseteq U$.
- ⟨2⟩1. LET: $m : G^2 \rightarrow G$ be the function $m(x, y) = xy$
- ⟨2⟩2. $m^{-1}(U)$ is open in G^2
- ⟨2⟩3. $(e, e) \in m^{-1}(U)$
- ⟨2⟩4. PICK neighbourhoods V_1, V_2 of e such that $V_1 \times V_2 \subseteq m^{-1}(U)$
- ⟨2⟩5. LET: $V' = V_1 \cap V_2$
- ⟨1⟩2. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
- ⟨2⟩1. LET: $f : G^2 \rightarrow G$ be the function $m(x, y) = xy^{-1}$
- ⟨2⟩2. $f^{-1}(V')$ is open in G^2
- ⟨2⟩3. $(e, e) \in m^{-1}(V')$
- ⟨2⟩4. PICK neighbourhoods W_1, W_2 of e such that $W_1 \times W_2 \subseteq f^{-1}(V')$
- ⟨2⟩5. LET: $W = W_1 \cap W_2$
- ⟨1⟩3. LET: $V = WW^{-1}$
- ⟨1⟩4. V is a neighbourhood of e .
- ⟨1⟩5. V is symmetric.
- ⟨1⟩6. $VV \subseteq U$

□

Proposition 19.2.3. *Every T_1 topological group is regular.*

PROOF:

- ⟨1⟩1. LET: G be a T_1 topological group.
- ⟨1⟩2. LET: A be a closed set in G and $x \in G - A$.
- ⟨1⟩3. $G - Ax^{-1}$ is a neighbourhood of e .
- ⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq G - Ax^{-1}$.
- ⟨1⟩5. LET: $U = VA$ and $U' = Vx$
- ⟨1⟩6. U and U' are disjoint open sets with $A \subseteq U$ and $x \in U'$.

□

Proposition 19.2.4. *Let G be a T_1 topological group. Let H be a closed subgroup of G . Then G/H is regular.*

PROOF:

- ⟨1⟩1. LET: A be a closed set in G/H and $xH \in G/H - A$.
- ⟨1⟩2. $G - \pi^{-1}(A)x^{-1}$ is a neighbourhood of e .
- ⟨1⟩3. PICK a symmetric neighbourhood V of e such that $VV \subseteq G - \pi^{-1}(A)x^{-1}$.
- ⟨1⟩4. LET: $U = \pi(V)A$ and $U' = \pi(V)(xH)$.
- ⟨1⟩5. U and U' are disjoint open sets with $A \subseteq U$ and $xH \in U'$
- ⟨2⟩1. ASSUME: for a contradiction $U \cap U' \neq \emptyset$.
- ⟨2⟩2. PICK $v_1, v_2 \in V$ and $a \in G$ such that $aH \in A$ and $v_1aH = v_2xH$.
- ⟨2⟩3. $a^{-1}v_1^{-1}v_2x \in H$
- ⟨2⟩4. $v_1^{-1}v_2 \in \pi^{-1}(A)x^{-1}$
- ⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩3.

□

Proposition 19.2.5. *Let G be a topological group. Let A and B be subspaces of G . If A is closed and B is compact, then AB is closed.*

PROOF:

$\langle 1 \rangle 1$. For all $c \in G - AB$, there exists a neighbourhood W of c such that $WB^{-1} \cap A = \emptyset$.

$\langle 2 \rangle 1$. LET: $c \in G - AB$

$\langle 2 \rangle 2$. LET: $\phi : G^2 \rightarrow G$ be the function $\phi(x, y) = xy^{-1}$

$\langle 2 \rangle 3$. $\{c\} \times B \subseteq \phi^{-1}(G - A)$

$\langle 2 \rangle 4$. PICK a neighbourhood W of c such that $W \times B \subseteq \phi^{-1}(G - A)$

PROOF: Tube Lemma.

$\langle 2 \rangle 5$. $WB^{-1} \cap A = \emptyset$

$\langle 1 \rangle 2$. For all $c \in G - AB$, there exists a neighbourhood W of c such that $W \subseteq G - AB$.

□

Corollary 19.2.5.1. *Let G be a topological group. Let H be a compact subgroup of G . Let $p : G \rightarrow G/H$ be the quotient map. Then p is a perfect map.*

PROOF: The only thing remaining to prove is that, for all $gH \in G/H$, we have $p^{-1}(gH)$ is compact. This holds because $p^{-1}(gH) = gH$ is homeomorphic to H . □

Corollary 19.2.5.2. *Let G be a topological group. Let H be a compact subgroup of G . If G/H is compact then G is compact.*

19.3 Continuous Actions

Definition 19.3.1 (Continuous Action). Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \rightarrow X$ such that:

- $\forall x \in X. ex = x$
- $\forall g, h \in G. \forall x \in X. g(hx) = (gh)x$

A G -space consists of a topological space X and a continuous action of G on X .

Definition 19.3.2 (Orbit). Let X be a G -space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 19.3.3. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:

$\langle 1 \rangle 1$. LET: $f_3 : S^2/SO(2) \rightarrow [-1, 1]$ be the function induced by $\pi_3 : S^2 \rightarrow [-1, 1]$

$\langle 1 \rangle 2$. f_3 is bijective.

$\langle 1 \rangle 3$. $S^2/SO(2)$ is compact.

PROOF: It is the continuous image of S^2 which is compact.

$\langle 1 \rangle 4$. $[-1, 1]$ is Hausdorff.

$\langle 1 \rangle 5$. f_3 is a homeomorphism.

□

Definition 19.3.4 (Stabilizer). Let X be a G -space and $x \in X$. The *stabilizer* of x is $G_x := \{g \in G : gx = x\}$.

Proposition 19.3.5. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx .*

PROOF:

$\langle 1 \rangle 1$. If $gG_x = hG_x$ then $gx = hx$.

$\langle 2 \rangle 1$. ASSUME: $gG_x = hG_x$

$\langle 2 \rangle 2$. $g^{-1}h \in G_x$

$\langle 2 \rangle 3$. $g^{-1}hx = x$

$\langle 2 \rangle 4$. $gx = hx$

$\langle 1 \rangle 2$. If $gx = hx$ then $gG_x = hG_x$.

PROOF: Similar.

$\langle 1 \rangle 3$. The function is continuous.

PROOF: Theorem 15.10.3.

□

Chapter 20

Topological Vector Spaces

Definition 20.0.1 (Topological Vector Space). Let K be either \mathbb{R} or \mathbb{C} . A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Subtraction is a continuous function $E^2 \rightarrow E$
- Multiplication is a continuous function $K \times E \rightarrow E$

Proposition 20.0.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. \square

Theorem 20.0.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 \square

Proposition 20.0.4. *Let E be a topological vector space and E_0 a subspace of E . Then $\overline{E_0}$ is a subspace of E .*

Definition 20.0.5. Let E be a topological vector space. The topological space associated with E is $E/\overline{\{0\}}$.

20.1 Cauchy Sequences

Definition 20.1.1 (Cauchy Sequence). Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geq n_0, x_n - x_m \in U$.

Definition 20.1.2 (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

20.2 Seminorms

Definition 20.2.1 (Seminorm). Let E be a vector space over K . A *seminorm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ such that:

1. $\forall x \in E, \|x\| \geq 0$
2. $\forall \alpha \in K, \forall x \in E, \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y \in E, \|x + y\| \leq \|x\| + \|y\|$

Example 20.2.2. The function that maps (x_1, \dots, x_n) to $|x_i|$ is a seminorm on \mathbb{R}^n .

Definition 20.2.3. Let E be a vector space over K . Let Λ be a set of seminorms on E . The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x \in E$.

Proposition 20.2.4. E is a topological vector space under this topology. It is Hausdorff iff, for all $x \in E$, if $\forall \lambda \in \Lambda, \lambda(x) = 0$ then $x = 0$.

20.3 Fréchet Spaces

Definition 20.3.1 (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 20.3.2. Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$. Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 20.3.3 (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

20.4 Normed Spaces

Definition 20.4.1 (Normed Space). Let E be a vector space over K . A *norm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ is a seminorm such that, $\forall x \in E, \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 20.4.2. If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E .

Definition 20.4.3 (p -norm). For any $p \geq 1$, the p -norm on \mathbb{R}^n is defined by

$$\|\vec{x}\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1$. For all $\vec{x} \in \mathbb{R}^n$ we have $\|\vec{x}\|_p \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. For all $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ we have $\|\alpha \vec{x}\|_p = |\alpha| \|\vec{x}\|_p$

PROOF:

$$\begin{aligned} \|\alpha(x_1, \dots, x_n)\| &= \|(\alpha x_1, \dots, \alpha x_n)\| \\ &= \left(\sum_{i=1}^n (\alpha x_i)^p \right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|\vec{x}\|_p \end{aligned}$$

$\langle 1 \rangle 3$. The triangle inequality holds.

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\ &= \sum_{i=1}^n |x_i + y_i| |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^n (|x_i| + |y_i|) |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} \\ &\leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} \quad (\text{Hölder's Inequality}) \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p) \|\vec{x} + \vec{y}\|_p^{p-1} \end{aligned}$$

Assuming w.l.o.g. $\|\vec{x} + \vec{y}\|_p^{p-1} \neq 0$ (using ??) we have $\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$.

$\langle 1 \rangle 4$. For any $\vec{x} \in \mathbb{R}^n$, we have $\|\vec{x}\|_p = 0$ iff $\vec{x} = \vec{0}$.

PROOF: $\sum_{i=1}^n x_i^p = 0$ iff $x_1 = \dots = x_n = 0$.

□

Proposition 20.4.4. The p -norm on \mathbb{R}^n induces the product topology.

PROOF:

⟨1⟩1. LET: d be the metric induced by the p -norm and ρ the square metric on \mathbb{R}^n .

⟨1⟩2. The metric topology is finer than the product topology.

⟨2⟩1. LET: $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$

⟨2⟩2. LET: $\delta = \epsilon/n^{\frac{1}{p}}$

PROVE: $B_\rho(\vec{x}, \delta) \subseteq B_d(\vec{x}, \epsilon)$

⟨2⟩3. LET: $\vec{y} \in B_\rho(\vec{x}, \delta)$

⟨2⟩4. $\forall i. |x_i - y_i| < \delta$

⟨2⟩5. $d(\vec{x}, \vec{y}) < \epsilon$

PROOF:

$$\begin{aligned} d(\vec{x}, \vec{y}) &= \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}} \\ &< \left(\sum_{i=1}^n \delta^p \right)^{\frac{1}{p}} && (\langle 2 \rangle 4) \\ &= n^{\frac{1}{p}} \delta \\ &= \epsilon && (\langle 2 \rangle 2) \end{aligned}$$

⟨1⟩3. The product topology is finer than the metric topology.

⟨2⟩1. LET: $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$

⟨2⟩2. LET: $\vec{y} \in B_d(\vec{x}, \epsilon)$

⟨2⟩3. $d(\vec{x}, \vec{y}) < \epsilon$

⟨2⟩4. $\sum_{i=1}^n |x_i - y_i|^p < \epsilon^p$

⟨2⟩5. $\forall i. |x_i - y_i|^p < \epsilon^p$

⟨2⟩6. $\forall i. |x_i - y_i| < \epsilon$

⟨2⟩7. $\rho(\vec{x}, \vec{y}) < \epsilon$

□

Definition 20.4.5 (Sup-norm). The *sup-norm* on \mathbb{R}^n is defined by

$$\|(x_1, \dots, x_n)\|_\infty := \max(|x_1|, \dots, |x_n|) .$$

Proposition 20.4.6. The 2-norm on \mathbb{R}^n induces the standard metric.

PROOF: Immediate from definitions. □

Definition 20.4.7. For $p \geq 1$, the normed space l_p is the set of all sequences (x_n) in \mathbb{R} such that $\sum_{n=1}^\infty x_n^p$ converges, under

$$\|(x_n)\|_p := \left(\sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} .$$

Proposition 20.4.8. The spaces l_p for $p \geq 1$ are all homeomorphic.

PROOF: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. <http://dx.doi.org/10.1007/BF01075865>.

Proposition 20.4.9. *The metric topology on l_2 is strictly finer than the uniform topology.*

PROOF:

⟨1⟩1. LET: d be the metric induced by the l^2 -norm and $\bar{\rho}$ the uniform topology.

⟨1⟩2. The metric topology is finer than the uniform topology.

⟨2⟩1. LET: $x \in l_2$

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. LET: $\delta = \epsilon/2$

⟨2⟩4. LET: $y \in B_d(x, \delta)$

⟨2⟩5. $\sum_{n=0}^{\infty} (x_n - y_n)^2 < \delta^2$

⟨2⟩6. $\forall n. (x_n - y_n)^2 < \delta^2$

⟨2⟩7. $\forall n. |x_n - y_n| < \delta$

⟨2⟩8. $\forall n. \bar{d}(x_n, y_n) < \delta$

⟨2⟩9. $\bar{\rho}(x, y) \leq \delta$

⟨2⟩10. $\bar{\rho}(x, y) < \epsilon$

⟨2⟩11. $y \in B_{\bar{\rho}}(x, \epsilon)$

⟨1⟩3. The metric topology is not the same as the uniform topology.

⟨2⟩1. ASSUME: for a contradiction $B_d(0, 1)$ is open in the uniform topology.

⟨2⟩2. PICK $\epsilon > 0$ such that $B_{\bar{\rho}}(0, \epsilon) \subseteq B_d(0, 1)$

⟨2⟩3. PICK an integer N such that $1/N < \epsilon^2/4$

⟨2⟩4. LET: (x_n) be the sequence with $x_n = \epsilon/2$ for $n < N$ and $x_n = 0$ for $n \geq N$

⟨2⟩5. $(x_n) \in l_2$

⟨2⟩6. $(x_n) \in B_{\bar{\rho}}(0, \epsilon)$

PROOF: Since $\bar{\rho}((x_n), 0) = \epsilon/2$.

⟨2⟩7. $d((x_n), 0) > 1$

PROOF:

$$\begin{aligned} d((x_n), 0)^2 &= \sum_{n=0}^{\infty} x_n^2 \\ &= N\epsilon^2/4 \\ &> 1 \end{aligned}$$

□

Proposition 20.4.10. *The metric topology on l_2 is strictly coarser than the box topology.*

PROOF:

⟨1⟩1. The box topology is finer than the metric topology.

⟨2⟩1. LET: $(x_n) \in l_2$ and $\epsilon > 0$.

⟨2⟩2. LET: $(y_n) \in B((x_n), \epsilon)$

⟨2⟩3. PICK a sequence of real numbers (δ_n) such that $\sum_{n=0}^{\infty} \delta_n^2 < (\epsilon - d((x_n), (y_n)))^2$

⟨2⟩4. LET: $U = \prod_n (y_n - \delta_n, y_n + \delta_n)$

PROVE: $U \subseteq B((x_n), \epsilon)$

⟨2⟩5. LET: $(z_n) \in U$

⟨2⟩6. $d((z_n), (y_n)) < \epsilon - d((x_n), (y_n))$

PROOF:

$$\begin{aligned} d((z_n), (y_n))^2 &= \sum_{n=0}^{\infty} (z_n - y_n)^2 \\ &< \sum_{n=0}^{\infty} \delta_n^2 \\ &< (\epsilon - d((x_n), (y_n)))^2 \end{aligned}$$

$$\langle 2 \rangle 7. d((z_n), (x_n)) < \epsilon$$

$\langle 1 \rangle 2.$ The box topology is not equal to the metric topology.

$\langle 2 \rangle 1.$ LET: $U = \prod_n (-1/n, 1/n)$

$\langle 2 \rangle 2.$ ASSUME: for a contradiction U is open in the metric topology.

$\langle 2 \rangle 3.$ PICK $\epsilon > 0$ such that $B(0, \epsilon) \subseteq U$

$\langle 2 \rangle 4.$ PICK N such that $1/N < \epsilon/2$.

$\langle 2 \rangle 5.$ LET: (x_n) be the sequence with $x_N = \epsilon/2$ and $x_n = 0$ for all other n .

$\langle 2 \rangle 6.$ $d((x_n), 0) = \epsilon/2$

$\langle 2 \rangle 7.$ $(x_n) \notin U$

□

Proposition 20.4.11. *The l^2 -topology on \mathbb{R}^∞ is strictly finer than the uniform topology.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: for a contradiction $B_d(0, 1) \cap \mathbb{R}^\infty$ is open in the uniform topology.

$\langle 1 \rangle 2.$ PICK $\epsilon > 0$ such that $B_{\bar{\rho}}(0, \epsilon) \cap \mathbb{R}^\infty \subseteq B_d(0, 1) \cap \mathbb{R}^\infty$

$\langle 1 \rangle 3.$ PICK an integer N such that $1/N < \epsilon^2/4$

$\langle 1 \rangle 4.$ LET: (x_n) be the sequence with $x_n = \epsilon/2$ for $n < N$ and $x_n = 0$ for $n \geq N$

$\langle 1 \rangle 5.$ $(x_n) \in \mathbb{R}^\infty$

$\langle 1 \rangle 6.$ $(x_n) \in B_{\bar{\rho}}(0, \epsilon)$

PROOF: Since $\bar{\rho}((x_n), 0) = \epsilon/2$.

$\langle 1 \rangle 7.$ $d((x_n), 0) > 1$

PROOF:

$$\begin{aligned} d((x_n), 0)^2 &= \sum_{n=0}^{\infty} x_n^2 \\ &= N\epsilon^2/4 \\ &> 1 \end{aligned}$$

□

Proposition 20.4.12. *The l^2 -topology on \mathbb{R}^∞ is strictly coarser than the box topology.*

$\langle 1 \rangle 1.$ LET: $U = \prod_n (-1/n, 1/n) \cap \mathbb{R}^\infty$

$\langle 1 \rangle 2.$ ASSUME: for a contradiction U is open in the metric topology.

$\langle 1 \rangle 3.$ PICK $\epsilon > 0$ such that $B(0, \epsilon) \cap \mathbb{R}^\infty \subseteq U \cap \mathbb{R}^\infty$

$\langle 1 \rangle 4.$ PICK N such that $1/N < \epsilon/2$.

⟨1⟩5. LET: (x_n) be the sequence with $x_N = \epsilon/2$ and $x_n = 0$ for all other n .

⟨1⟩6. $d((x_n), 0) = \epsilon/2$

⟨1⟩7. $(x_n) \notin U$

□

Proposition 20.4.13. *The l^2 -topology on the Hilbert cube the same as the product topology.*

PROOF:

⟨1⟩1. For every $(x_n) \in H$ and $\epsilon > 0$, there exists a neighbourhood U of (x_n) in the product topology such that $U \subseteq B((x_n), \epsilon)$.

⟨2⟩1. LET: $(x_n) \in H$

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. PICK N such that $\sum_{i=N+1}^{\infty} 1/i^2 < \epsilon^2/2$

⟨2⟩4. LET: $B' = (\prod_{i=0}^N (x_i - \epsilon/\sqrt{2N}, x_i + \epsilon/\sqrt{2N}) \times \prod_{i=N+1}^{\infty} [0, 1/(i+1)]) \cap H$

PROVE: $B' \subseteq B((x_n), \epsilon)$

⟨2⟩5. LET: $(y_n) \in B'$

⟨2⟩6. $d((x_n), (y_n)) < \epsilon$

PROOF:

$$\begin{aligned} d((x_n), (y_n))^2 &= \sum_{i=0}^{\infty} |x_n - y_n|^2 \\ &< \sum_{i=0}^N \epsilon^2/2N + \sum_{i=N+1}^{\infty} 1/(i+1)1/(i+1)^2 \\ &< \epsilon^2/2 + \epsilon^2/2 \\ &= \epsilon^2 \end{aligned}$$

⟨1⟩2. The product topology is finer than the l^2 -topology.

⟨2⟩1. LET: $(x_n) \in H$ and $\epsilon > 0$

PROVE: $B((x_n), \epsilon) \cap H$ is open in the product topology.

⟨2⟩2. LET: $(y_n) \in B((x_n), \epsilon)$

⟨2⟩3. PICK a neighbourhood U of (y_n) in the product topology such that

$U \subseteq B((y_n), \epsilon - d((x_n), (y_n)))$

⟨2⟩4. $U \subseteq B((x_n), \epsilon)$

□

Definition 20.4.14. Let l_{∞} be the set of all bounded sequences in \mathbb{R} under

$$\|(x_n)\| := \sup_n |x_n|$$

Proposition 20.4.15. *For all $p \geq 1$ we have l_p is not homeomorphic to l_{∞} .*

Proposition 20.4.16. *Let $\| \cdot \|$ be a seminorm on the vector space E . Then $\| \cdot \|$ defines a norm on $E/\{0\}$.*

Proposition 20.4.17. *Let E and F be normed spaces. Any continuous linear map $E \rightarrow F$ is uniformly continuous.*

Definition 20.4.18. For $p \geq 1$, let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n) / \{0\}.$$

20.5 Unit Ball

Proposition 20.5.1. *Let n be a positive integer. Every open ball $B(\vec{x}, \epsilon)$ in \mathbb{R}^n is path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $\vec{y}, \vec{z} \in B(\vec{x}, \epsilon)$

$\langle 1 \rangle 2$. LET: $\vec{p} : [0, 1] \rightarrow B(\vec{x}, \epsilon)$ be the path $\vec{p}(t) = (1-t)\vec{y} + t\vec{z}$.

$\langle 2 \rangle 1$. LET: $t \in [0, 1]$

PROVE: $\vec{p}(t) \in B(\vec{x}, \epsilon)$

$\langle 2 \rangle 2$. $d(\vec{p}(t), \vec{x}) < \epsilon$

PROOF:

$$\begin{aligned} d(\vec{p}(t), \vec{x}) &= \|(1-t)\vec{y} + t\vec{z} - \vec{x}\| \\ &= \|(1-t)(\vec{y} - \vec{x}) + t(\vec{z} - \vec{x})\| \\ &\leq (1-t)\|\vec{y} - \vec{x}\| + t\|\vec{z} - \vec{x}\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. \vec{p} is a path from \vec{x} to \vec{y} .

□

Proposition 20.5.2. *Let n be a positive integer. Every closed ball $B(\vec{x}, \epsilon)$ in \mathbb{R}^n is path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $\vec{y}, \vec{z} \in \overline{B(\vec{x}, \epsilon)}$

$\langle 1 \rangle 2$. LET: $\vec{p} : [0, 1] \rightarrow \overline{B(\vec{x}, \epsilon)}$ be the path $\vec{p}(t) = (1-t)\vec{y} + t\vec{z}$.

$\langle 2 \rangle 1$. LET: $t \in [0, 1]$

PROVE: $\vec{p}(t) \in \overline{B(\vec{x}, \epsilon)}$

$\langle 2 \rangle 2$. $d(\vec{p}(t), \vec{x}) \leq \epsilon$

PROOF:

$$\begin{aligned} d(\vec{p}(t), \vec{x}) &= \|(1-t)\vec{y} + t\vec{z} - \vec{x}\| \\ &= \|(1-t)(\vec{y} - \vec{x}) + t(\vec{z} - \vec{x})\| \\ &\leq (1-t)\|\vec{y} - \vec{x}\| + t\|\vec{z} - \vec{x}\| \\ &\leq (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. \vec{p} is a path from \vec{x} to \vec{y} .

□

20.6 Unit Sphere

Definition 20.6.1 (Unit Sphere). Let n be a positive integer. The *unit sphere* S^{n-1} is

$$S^{n-1} := \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = 1\} .$$

Proposition 20.6.2. For $n > 1$, the unit sphere S^{n-1} is path connected.

PROOF: The map $g : \mathbb{R}^n - \{\vec{0}\} \rightarrow S^{n-1}$ defined by $g(\vec{x}) = \vec{x}/\|\vec{x}\|$ is continuous and surjective. Hence S^{n-1} is the continuous image of a path connected space. \square

20.7 Inner Product Spaces

Definition 20.7.1 (Inner Product). Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n .$$

Proposition 20.7.2.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF:

$$\begin{aligned} \vec{x} \cdot (\vec{y} + \vec{z}) &= x_1(y_1 + z_1) + \cdots + x_n(y_n + z_n) \\ &= x_1 y_1 + x_1 z_1 + \cdots + x_n y_n + x_n z_n \\ &= \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z} \end{aligned} \quad \square$$

Proposition 20.7.3. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\| .$$

PROOF:

- $\langle 1 \rangle 1$. ASSUME: w.l.o.g. $\vec{x} \neq \vec{0} \neq \vec{y}$
- $\langle 1 \rangle 2$. LET: $a = 1/\|\vec{x}\|$
- $\langle 1 \rangle 3$. LET: $b = 1/\|\vec{y}\|$
- $\langle 1 \rangle 4$. $\|a\vec{x} + b\vec{y}\| \geq 0$
- $\langle 1 \rangle 5$. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
- $\langle 1 \rangle 6$. $ab\vec{x} \cdot \vec{y} \geq -1$
- $\langle 1 \rangle 7$. $\|a\vec{x} - b\vec{y}\| \geq 0$
- $\langle 1 \rangle 8$. $ab\vec{x} \cdot \vec{y} \leq 1$
- $\langle 1 \rangle 9$. $|\vec{x} \cdot \vec{y}| \leq 1/ab$

\square

Proposition 20.7.4. Let $(x_n), (y_n)$ be sequences of real numbers. If $\sum_{n=0}^{\infty} x_n^2$ and $\sum_{n=0}^{\infty} y_n^2$ converge then $\sum_{n=0}^{\infty} |x_n y_n|$ converges.

PROOF:

$$\begin{aligned} \sum_{n=0}^N |x_n y_n| &\leq \sqrt{\sum_{n=0}^N x_n^2 \sum_{n=0}^N y_n^2} && \text{(Proposition 20.7.3)} \\ &\leq \sqrt{\sum_{n=0}^{\infty} x_n^2 \sum_{n=0}^{\infty} y_n^2} && \square \end{aligned}$$

Proposition 20.7.5. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E .*

20.8 Banach Spaces

Definition 20.8.1 (Banach Space). A *Banach space* is a complete normed space.

Example 20.8.2. For any topological space X , the set $C(X)$ of bounded continuous functions $X \rightarrow \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 20.8.3. *The completion of a normed space is a Banach space.*

Proposition 20.8.4. *Let E and F be normed spaces. Let $f : E \rightarrow F$ be a continuous linear map. Then the extension to the completions $\hat{E} \rightarrow \hat{F}$ is linear.*

Proposition 20.8.5. *$L^p(\mathbb{R}^n)$ is a Banach space.*

Proposition 20.8.6. *$C(\mathbb{R})$ is first countable but not second countable.*

PROOF: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n . Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.

It is first countable because it is metrizable. \square

20.9 Hilbert Spaces

Definition 20.9.1 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Example 20.9.2. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \rightarrow \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

Proposition 20.9.3. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

20.10 Locally Convex Spaces

Definition 20.10.1 (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 20.10.2. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Proposition 20.10.3. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Example 20.10.4. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \rightarrow \mathbb{R}$ is continuous as a map $E' \rightarrow \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonné, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 20.10.5 (Thom Space). Let E be a vector bundle with a Riemannian metric, $DE = \{x \in E : \|x\| \leq 1\}$ its disc bundle and $SE := \{v \in E : \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE .

Part VII

Probability Theory

Chapter 21

Discrete Random Variables

Definition 21.0.1 (Discrete random variable). Let Ω be a countable set. A *discrete random variable* X that takes values in Ω is a function

$$X : \Omega \rightarrow [0, 1]$$

such that

$$\sum_{a \in \Omega} X(a) = 1 \ .$$

We write $P(X = a)$ for $X(a)$, and call this the *probability* that X takes value a .

Definition 21.0.2 (Expected Value). Let X be a discrete random variable that takes values in a real vector space Ω . The *expected value* of X is

$$\langle X \rangle = \sum_{a \in \Omega} P(X = a)a \ .$$

Definition 21.0.3 (Variance). The *variance* of X is

$$\langle (X - \langle X \rangle)^2 \rangle \ .$$

Proposition 21.0.4. *The variance of X is $\langle X^2 \rangle - \langle X \rangle^2$.*

PROOF:

$$\begin{aligned} \langle (X - \langle X \rangle)^2 \rangle &= \sum_{a \in \Omega} (a - \langle X \rangle)^2 P(X = a) \\ &= \sum_{a \in \Omega} (a^2 - 2a\langle X \rangle + \langle X \rangle^2) P(X = a) \\ &= \sum_{a \in \Omega} a^2 P(X = a) - 2\langle X \rangle \sum_{a \in \Omega} a P(X = a) + \langle X \rangle^2 \sum_{a \in \Omega} P(X = a) \\ &= \langle X^2 \rangle - 2\langle X \rangle^2 + \langle X \rangle^2 \\ &= \langle X^2 \rangle - \langle X \rangle^2 \end{aligned}$$

□

Corollary 21.0.4.1.

$$\langle X^2 \rangle \geq \langle X \rangle^2$$

PROOF: For all $a \in \Omega$ we have $(a - \langle X \rangle)^2 \geq 0$, so the variance of X must be ≥ 0 . \square

Definition 21.0.5 (Standard Deviation). The *standard deviation* of X , denoted σ_X , is the square root of the variance.

Chapter 22

Continuous Random Variables

Definition 22.0.1 (Continuous random variable). A *continuous random variable* X that takes values in \mathbb{R} is an integrable function

$$\rho : \mathbb{R} \rightarrow [0, 1]$$

such that

$$\int \rho = 1 \quad .$$

Given a measurable set $S \subseteq \mathbb{R}$, *probability* that X takes a value in S is

$$\int_S \rho \quad .$$

Example 22.0.2. A *Gaussian distribution* is

$$\rho(x) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2}$$

for some λ and a .

Definition 22.0.3 (Expected Value). Let X be a discrete random variable that takes values in a real vector space Ω . The *expected value* of X is

$$\langle X \rangle = \int x \rho(x) dx$$

Example 22.0.4. The Gaussian distribution

$$\rho(x) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2}$$

has expected value a .

Definition 22.0.5 (Variance). The *variance* of X is

$$\langle (X - \langle X \rangle)^2 \rangle .$$

Proposition 22.0.6. The variance of X is $\langle X^2 \rangle - \langle X \rangle^2$.

PROOF:

$$\begin{aligned} \langle (X - \langle X \rangle)^2 \rangle &= \int (x - \langle X \rangle)^2 \rho(x) dx \\ &= \int (x^2 - 2x\langle X \rangle + \langle X \rangle^2) \rho(x) dx \\ &= \int x^2 \rho(x) dx - 2\langle X \rangle \int x \rho(x) dx + \langle X \rangle^2 \int \rho(x) dx \\ &= \langle X^2 \rangle - 2\langle X \rangle^2 + \langle X \rangle^2 \\ &= \langle X^2 \rangle - \langle X \rangle^2 \end{aligned} \quad \square$$

Corollary 22.0.6.1.

$$\langle X^2 \rangle \geq \langle X \rangle^2$$

PROOF: For all $x \in \mathbb{R}$ we have $(x - \langle X \rangle)^2 \geq 0$, so the variance of X must be ≥ 0 . \square

Example 22.0.7. The Gaussian distribution

$$\rho(x) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2}$$

has variance $\frac{1}{2\lambda}$.

Definition 22.0.8 (Standard Deviation). The *standard deviation* of X , denoted σ_X , is the square root of the variance.

Example 22.0.9. The Gaussian distribution

$$\rho(x) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2}$$

has standard deviation $1/\sqrt{2\lambda}$.

Part VIII

Quantum Theory

Chapter 23

The Postulates of Quantum Mechanics

Axiom 23.0.1 (Principle of Superposition). *For any physical system S , there exists a complex inner product space V such that the states of S correspond to the one-dimensional subspaces of V . We call the elements of this subspace the state vectors that correspond to the state.*

Definition 23.0.2 (Observable). An *observable* of a physical system is a physical quantity that can be measured by an experiment whose value is a real number.

Definition 23.0.3 (Eigenstate). Let S be a physical system. Let E be an observable of S . Let $\alpha \in \mathbb{R}$. A state s of S is an *eigenstate* of E with *eigenvalue* α iff the probability is 1 that, if we measure E when the system is in state s , then the outcome will be α .

Axiom 23.0.4. *Let E be an observable of a physical system S . Let $\alpha \in \mathbb{R}$. The state vectors corresponding to the eigenstates of E form a subspace of the Hilbert space associated with S , called the eigenspace of α .*

Axiom 23.0.5 (Projection Postulate). *Let S be a physical system with associated state space V . Let E be an observable of S . Let V_α be the eigenspace of V with eigenvalue α . Let $P_\alpha : V \rightarrow V_\alpha$ be the projection operator. If S is in a state with normalised state vector $|\psi\rangle$, and we measure E on S , then the probability of getting the value α is*

$$p_E(\alpha|\psi) = \frac{\langle\psi|P_\alpha|\psi\rangle}{\langle\psi|\psi\rangle}$$

and after the experiment, the state of the system is $P_\alpha|\psi\rangle$.

Proposition 23.0.6. *Let E be an observable of a physical system S . Then eigenstates of E with different eigenvalues are orthogonal.*

PROOF:

- $\langle 1 \rangle 1$. LET: $|\psi\rangle$ a normalised eigenstate with eigenvalue β and $\alpha \neq \beta$.
 $\langle 1 \rangle 2$. The probability of measuring α if the system is in state $|\psi\rangle$ is 0.
 $\langle 1 \rangle 3$. $\langle \psi | P_\alpha | \psi \rangle = 0$
 $\langle 1 \rangle 4$. $P_\alpha |\psi\rangle = 0$
 $\langle 1 \rangle 5$. $|\psi\rangle$ is orthogonal to every eigenvector with eigenvalue α .
 \square

Proposition 23.0.7. *Let E be an observable of a physical system S with state space V . Assume E has countably many outcomes $\alpha_1, \alpha_2, \dots$ and, for any state vector $|\psi\rangle$, we have*

$$\sum_{n=1}^{\infty} P_{\alpha_n} |\psi\rangle \text{ converges.}$$

Then the eigenvectors of E span V .

PROOF:

- $\langle 1 \rangle 1$. LET: $|\psi\rangle \in V$
 $\langle 1 \rangle 2$. LET: $|\psi'\rangle = |\psi\rangle - \sum_{n=1}^{\infty} P_{\alpha_n} |\psi\rangle$
 $\langle 1 \rangle 3$. ASSUME: for a contradiction $|\psi'\rangle \neq 0$
 $\langle 1 \rangle 4$. LET: $|\phi\rangle = |\psi'\rangle / \|\psi'\rangle\|$
 $\langle 1 \rangle 5$. For all n we have $P_{\alpha_n} |\phi\rangle = 0$

PROOF:

$$\begin{aligned}
 P_{\alpha_n} |\psi'\rangle &= P_{\alpha_n} |\psi\rangle - \sum_{m=1}^{\infty} P_{\alpha_n} P_{\alpha_m} |\psi\rangle \\
 &= P_{\alpha_n} |\psi\rangle - P_{\alpha_n} |\psi\rangle \quad (\text{Proposition 23.0.6}) \\
 &= 0
 \end{aligned}$$

- $\langle 1 \rangle 6$. For all n , the probability of measuring α_n if the system is in state $|\phi\rangle$ is 0.
 $\langle 1 \rangle 7$. Q.E.D.

PROOF: This is a contradiction.

\square

Definition 23.0.8. For any observable E of a physical system with state space V , pick an orthonormal basis \mathcal{B} of normalised eigenvectors of E . The operator that represents E is $\hat{E} : V \rightarrow V$ where, for $|\phi\rangle \in \mathcal{B}$, if α is the eigenvalue of $|\phi\rangle$ then

$$\hat{E} |\phi\rangle = \alpha |\phi\rangle .$$

Proposition 23.0.9. *For any observable E , the operator \hat{E} is Hermitian.*

PROOF:

- $\langle 1 \rangle 1$. LET: $\hat{\phi} = \sum_n c_n \hat{\psi}_n$ and $\hat{\psi} = \sum_n d_n \hat{\psi}_n$ be any two state vectors, where the $\hat{\psi}_n$ are the basis vectors with eigenvalues α_n .
 $\langle 1 \rangle 2$. $\langle \phi | \hat{E} | \psi \rangle = \overline{\langle \psi | \hat{E} | \phi \rangle}$

PROOF:

$$\begin{aligned}
 \langle \phi | \hat{E} | \psi \rangle &= \sum_m \sum_n \overline{c_m} d_n \langle \psi_m | \hat{E} | \psi_n \rangle \\
 &= \sum_m \sum_n \overline{c_m} d_n \alpha_n \langle \psi_m | \psi_n \rangle \\
 &= \sum_n \overline{c_n} d_n \alpha_n \\
 \langle \psi | \hat{E} | \phi \rangle &= \sum_n \overline{d_n} c_n \alpha_n \quad \text{similarly}
 \end{aligned}$$

□

Definition 23.0.10. Let E be an observable and $f : \mathbb{R} \rightarrow \mathbb{R}$. Then $f(E)$ is the observable that consists of performing the experiment to obtain the value of E and then applying f to it.

Proposition 23.0.11. $\widehat{E^n} = \hat{E}^n$

PROOF: Since if $|\phi\rangle$ is an eigenstate with eigenvalue α then $\hat{E}^n |\phi\rangle = \alpha^n |\phi\rangle$. □

Corollary 23.0.11.1. For any real polynomial p we have $\widehat{p(E)} = p(\hat{E})$.

23.1 Observables as a Random Variable

Definition 23.1.1 (Uncertainty). Let E be an observable. The *uncertainty* of E in a given state, ΔE , is the standard deviation of the value of E when measured.

Proposition 23.1.2. Let $|\psi\rangle$ be a normalised state. Let E be an observable. Then the expected value of E in state $|\psi\rangle$ is

$$\langle E \rangle = \langle \psi | \hat{E} | \psi \rangle$$

PROOF:

⟨1⟩1. PICK an orthonormal basis \mathcal{B} of eigenstates for E .

⟨1⟩2. LET: $\psi = \sum_i c_i |\psi_i\rangle$ where each $|\psi_i\rangle \in \mathcal{B}$ has eigenvalue α_i .

⟨1⟩3. $\langle E \rangle = \langle \psi | \hat{E} | \psi \rangle$

PROOF:

$$\begin{aligned}
 \langle E \rangle &= \sum_{\alpha} \alpha \langle \psi | P_{\alpha} | \psi \rangle \\
 &= \sum_{\alpha} \alpha \sum_i |c_i|^2 \langle \psi_i | P_{\alpha} | \psi_i \rangle \\
 &= \sum_i |c_i|^2 \sum_{\alpha} \alpha \langle \psi_i | P_{\alpha} | \psi_i \rangle \\
 &= \sum_i |c_i|^2 \alpha_i \\
 &= \langle \psi | \hat{E} | \psi \rangle
 \end{aligned}$$

□

Corollary 23.1.2.1. *Let $|\psi\rangle$ be a normalised state. Let E be an observable. Then the variance of E in state $|\psi\rangle$ is*

$$(\Delta E)^2 = \langle \psi | \hat{E}^2 | \psi \rangle - \langle \psi | \hat{E} | \psi \rangle^2$$

23.2 Compatible Observables

Definition 23.2.1 (Compatible). Two observables A and B on a physical system are *compatible* iff, when we measure A then B then A , the second measurement of A always yields the same value as the first.

Proposition 23.2.2. *Let A and B be observables of a physical system S with state space V . Then the following are equivalent.*

1. A and B are compatible.
2. $\hat{A}\hat{B} = \hat{B}\hat{A}$
3. There exists a basis for V whose elements are eigenvectors of both A and B .

PROOF:

⟨1⟩1. $1 \Rightarrow 3$

⟨2⟩1. ASSUME: A and B are compatible.

⟨2⟩2. PICK orthonormal bases \mathcal{B}_A and \mathcal{B}_B whose elements are eigenvectors of \hat{A} and of \hat{B} respectively.

⟨2⟩3. For $|\psi\rangle \in \mathcal{B}_A$,

LET: $|\psi\rangle = c_{\psi 1} |\phi_{\psi 1}\rangle + \cdots + c_{\psi n_\psi} |\phi_{\psi n_\psi}\rangle$ where each $c_{\psi i}$ is nonzero and each $|\phi_{\psi i}\rangle \in \mathcal{B}_B$

⟨2⟩4. LET: α be the \hat{A} -eigenvalue of $|\psi\rangle$

⟨2⟩5. Each $|\phi_{\psi i}\rangle$ is an eigenvalue of \hat{A} with eigenvalue α .

PROOF: There is a nonzero probability that, if we perform A then B , we will obtain value α for A and then be in state $|\phi_{\psi i}\rangle$. If we perform A in this state, the value is certain to be α .

⟨2⟩6. $\{|\phi_{\psi i}\rangle : \psi \in \mathcal{B}_A, 1 \leq i \leq n_\psi\}$ is a basis consisting of eigenvectors of both \hat{A} and \hat{B} .

⟨1⟩2. $3 \Rightarrow 2$

⟨2⟩1. ASSUME: \mathcal{B} is a basis for V whose elements are eigenvectors of both A and B

⟨2⟩2. For all $|\phi\rangle \in \mathcal{B}$ we have $\hat{A}\hat{B}|\phi\rangle = \hat{B}\hat{A}|\phi\rangle$

⟨2⟩3. For all $|\phi\rangle$ we have $\hat{A}\hat{B}|\phi\rangle = \hat{B}\hat{A}|\phi\rangle$

⟨1⟩3. $2 \Rightarrow 1$

⟨2⟩1. ASSUME: $\hat{A}\hat{B} = \hat{B}\hat{A}$

⟨2⟩2. ASSUME: We perform A and obtain the value α , leaving the system in state $|\phi\rangle$.

- $\langle 2 \rangle 3.$ $|\phi\rangle$ is an eigenvector of \hat{A} with eigenvalue α .
 $\langle 2 \rangle 4.$ $\hat{A}\hat{B}|\phi\rangle = \alpha$

$$\hat{B}|\phi\rangle$$

PROOF:

$$\begin{aligned}
 \hat{A}\hat{B}|\phi\rangle &= \hat{B}\hat{A}|\phi\rangle \\
 &= \alpha\hat{B}|\phi\rangle
 \end{aligned}$$

- $\langle 2 \rangle 5.$ $\hat{B}|\phi\rangle$ is an eigenvector of A with eigenvalue α .
 $\langle 2 \rangle 6.$ If we perform B then A , we are certain to get the value α .

□

Definition 23.2.3 (Complete). A set \mathcal{E} of compatible observables is *complete* iff, for any states $|\phi\rangle$ and $|\psi\rangle$, if for all $E \in \mathcal{E}$ we have that $|\phi\rangle$ and $|\psi\rangle$ are eigenvectors of E with the same eigenvalue, then $|\phi\rangle = |\psi\rangle$.

Chapter 24

The Wave Function

24.1 The Schrödinger Equation

Definition 24.1.1 (Planck's constant). *Planck's constant* is

$$h = 6.62607015 \times 10^{-34} \text{ Js} .$$

Definition 24.1.2 (Reduced Planck's constant). The *reduced Planck's constant* is

$$\hbar = h/2\pi .$$

Consider a particle of mass m moving in one dimension under a force given by a potential energy function $V(x, t) : \mathbb{R} \times [0, +\infty) \rightarrow [0, +\infty]$. Associated with the particle is a *wave function* $\Psi(x, t) : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{C}$ that is differentiable in t , twice differentiable in x , satisfies the (*time-dependent*) *Schrödinger equation*: for all x and t ,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x, t)\Psi(x, t)$$

and satisfies

$$\int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx = 1 .$$

Proposition 24.1.3.

$$\frac{\partial}{\partial t} |\Psi(x, t)|^2 = \frac{i\hbar}{2m} \frac{\partial}{\partial x} \left(\Psi^*(x, t) \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi(x, t) \right)$$

PROOF:

$\langle 1 \rangle 1.$

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V(x, t)\Psi(x, t)$$

PROOF: Schrödinger equation.

$\langle 1 \rangle 2.$

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V(x, t) \Psi(x, t)^*$$

PROOF: Taking complex conjugates in $\langle 1 \rangle 1.$

$\langle 1 \rangle 3.$

$$\frac{\partial}{\partial t} |\Psi(x, t)|^2 = \frac{i\hbar}{2m} \frac{\partial}{\partial x} \left(\Psi^*(x, t) \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi(x, t) \right)$$

PROOF:

$$\begin{aligned} \frac{\partial}{\partial t} |\Psi(x, t)|^2 &= \frac{\partial}{\partial t} (\Psi(x, t)^* \Psi(x, t)) \\ &= \Psi^* \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t} \Psi \\ &= \frac{i\hbar}{2m} \left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2} \Psi \right) \quad (\langle 1 \rangle 1, \langle 1 \rangle 2) \\ &= \frac{i\hbar}{2m} \frac{\partial}{\partial x} \left(\Psi^*(x, t) \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi(x, t) \right) \end{aligned}$$

□

Proposition 24.1.4. For all $t \in [0, +\infty)$ we have

$$\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 1 \quad .$$

PROOF:

$\langle 1 \rangle 1.$

$$\frac{d}{dt} \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 0$$

PROOF:

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} |\Psi(x, t)|^2 dx \\ &= \frac{i\hbar}{2m} \left[\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right]_{-\infty}^{\infty} \quad (\text{Proposition 24.1.3}) \\ &= 0 \quad (\Psi \rightarrow 0 \text{ as } x \rightarrow \pm\infty) \end{aligned}$$

$\langle 2 \rangle 1.$ $\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx$ is constant.

□

24.2 Statistical Interpretation

Born's statistical interpretation of the wave function:

The *position* on the particle at time t is a random variable x with probability density function $|\Psi(x, t)|^2$ at time t .

Proposition 24.2.1. The expected value of position is

$$\langle x \rangle = \int_{-\infty}^{+\infty} x |\Psi(x, t)|^2 dx = \int_{-\infty}^{+\infty} \Psi(x, t)^* x \Psi(x, t) dx$$

PROOF: Immediate from definitions. \square

24.3 Momentum

Associated with any *observable* quantity Q is a linear operator

$$\hat{Q} : \mathcal{C}(\mathbb{R}, \mathbb{C}) \rightarrow \mathcal{C}(\mathbb{R}, \mathbb{C}) \ .$$

The *expected value* of Q at time t is then

$$\langle Q \rangle = \int_{-\infty}^{+\infty} \Psi(x, t)^* \hat{Q}(\lambda x, \Psi(x, t)) dx \ .$$

Position x is represented by the operator \hat{x} , multiplication by x .

Momentum p is represented by the operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$.

Kinetic energy T is represented by

$$\hat{T} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \ .$$

Proposition 24.3.1 (Ehrenfest's Theorem). 1.

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt}$$

2.

$$\frac{d\langle p \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle$$

PROOF:

$$\begin{aligned}
m \frac{d\langle x \rangle}{dt} &= m \frac{d}{dt} \int_{-\infty}^{+\infty} x |\Psi|^2 dx \\
&= m \int_{-\infty}^{+\infty} x \frac{\partial}{\partial t} |\Psi|^2 dx \\
&= \frac{i\hbar}{2} \int_{-\infty}^{+\infty} x \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) dx && \text{(Proposition 24.1.3)} \\
&= -\frac{i\hbar}{2} \int_{-\infty}^{+\infty} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) dx \\
&\quad + \left[x \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \right]_{-\infty}^{+\infty} && \text{(integrating by parts)} \\
&= -\frac{i\hbar}{2} \int_{-\infty}^{+\infty} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) dx \\
&= -i\hbar \int_{-\infty}^{+\infty} \Psi^* \frac{\partial \Psi}{\partial x} dx && \text{(integrating by parts)} \\
&= \langle p \rangle \\
\frac{d\langle p \rangle}{dt} &= -i\hbar \frac{d}{dt} \int_{-\infty}^{+\infty} \Psi(x, t)^* \frac{\partial \Psi}{\partial x} dx \\
&= -i\hbar \int_{-\infty}^{+\infty} \frac{\partial}{\partial t} \left(\Psi(x, t)^* \frac{\partial \Psi}{\partial x} \right) dx \\
&= -i\hbar \int_{-\infty}^{+\infty} \left(\Psi^* \frac{\partial^2 \Psi}{\partial t \partial x} + \frac{\partial \Psi^*}{\partial t} \frac{\partial \Psi}{\partial x} \right) dx \\
&\quad \frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V \Psi && \text{(Schrödinger equation)} \\
\therefore \frac{\partial^2 \Psi}{\partial x \partial t} &= \frac{i\hbar}{2m} \frac{\partial^3 \Psi}{\partial x^3} - \frac{i}{\hbar} \frac{\partial V}{\partial x} \Psi - \frac{i}{\hbar} V \frac{\partial \Psi}{\partial x} \\
\frac{\partial \Psi^*}{\partial t} &= -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V \Psi^* \\
\therefore \Psi^* \frac{\partial^2 \Psi}{\partial t \partial x} + \frac{\partial \Psi^*}{\partial t} \frac{\partial \Psi}{\partial x} &= \frac{i\hbar}{2m} \Psi^* \frac{\partial^3 \Psi}{\partial x^3} - \frac{i}{\hbar} \frac{\partial V}{\partial x} \Psi^* \Psi - \frac{i}{\hbar} V \Psi^* \frac{\partial \Psi}{\partial x} \\
&\quad - \frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} \frac{\partial \Psi}{\partial x} + \frac{i}{\hbar} V \Psi^* \frac{\partial \Psi}{\partial x} \\
\therefore \frac{d\langle p \rangle}{dt} &= \frac{\hbar^2}{2m} \int_{-\infty}^{+\infty} \left(\Psi^* \frac{\partial^3 \Psi}{\partial x^3} - \frac{\partial^2 \Psi^*}{\partial x^2} \frac{\partial \Psi}{\partial x} \right) dx \\
&\quad + \int_{-\infty}^{+\infty} \Psi^* \left(-\frac{\partial V}{\partial x} \right) \Psi dx \\
&= -\frac{\hbar^2}{2m} \int_{-\infty}^{+\infty} \left(\frac{\partial \Psi^*}{\partial x} \frac{\partial^2 \Psi}{\partial x^3} + \frac{\partial^2 \Psi^*}{\partial x^2} \frac{\partial \Psi}{\partial x} \right) dx \\
&\quad + \left[\Psi^* \frac{\partial^2 \Psi}{\partial x^2} \right]_{-\infty}^{+\infty} + \left\langle -\frac{\partial V}{\partial x} \right\rangle \\
&= -\frac{\hbar^2}{2m} \left[\frac{\partial \Psi^*}{\partial x} \frac{\partial \Psi}{\partial x} \right]_{-\infty}^{+\infty} + \left\langle \bullet - \frac{\partial V}{\partial x} \right\rangle \\
&= \left\langle -\frac{\partial V}{\partial x} \right\rangle
\end{aligned}$$

Proposition 24.3.2 (Canonical Commutation Relation).

$$[\hat{x}, \hat{p}] = i\hbar$$

PROOF:

$$\begin{aligned} [\hat{x}, \hat{p}]\psi &= -i\hbar x \frac{d\psi}{dx} + i\hbar \frac{d}{dx}(x\psi) \\ &= -i\hbar \left(x \frac{d\psi}{dx} - x \frac{d\psi}{dx} - \psi \right) \\ &= i\hbar \psi \end{aligned} \quad \square$$

24.4 The Time-Independent Schrödinger Equation

Definition 24.4.1 (Hamiltonian). Assume that the potential V does not vary with t . The *Hamiltonian* or *total energy* H is the quantity with operator

$$\begin{aligned} \hat{H} &= \frac{\hat{p}^2}{2m} + V(t)\hat{I} \\ &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)\hat{I} \end{aligned}$$

Definition 24.4.2 (Time-independent Schrödinger equation). Assume that the potential V does not vary with t . Let $E \geq 0$. The *time-independent Schrödinger equation* with *energy* E is

$$\hat{H}\psi = E\psi$$

i.e.

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x) \quad .$$

Proposition 24.4.3. Let $\psi : \mathbb{R} \rightarrow \mathbb{C}$. Then

$$\Psi(x, t) = \psi(x)e^{-\frac{iEt}{\hbar}}$$

is a solution to the time-dependent Schrödinger equation iff ψ is a solution to the time-independent Schrödinger equation.

PROOF:

$$\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} &= i\hbar \psi \left(-\frac{iE}{\hbar} \right) e^{-\frac{iEt}{\hbar}} \\ &= E\psi e^{-\frac{iEt}{\hbar}} \\ -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi &= e^{-\frac{iEt}{\hbar}} \left(-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi \right) \end{aligned}$$

and these are equal iff the time-independent equation holds. \square

Proposition 24.4.4 (Solutions to the Time-Independent Equation are Stationary States). *Let the wave function of the particle be*

$$\Psi(x, t) = \psi(x)e^{-\frac{iEt}{\hbar}}$$

For any quantity Q , the expectation value $\langle Q \rangle$ is constant in t .

PROOF:

$$\begin{aligned}\langle Q \rangle &= \int_{-\infty}^{+\infty} \Psi^*(x, t) \hat{Q}(\lambda x. \Psi(x, t))(x) dx \\ &= \int_{-\infty}^{+\infty} \psi^*(x) e^{\frac{iEt}{\hbar}} \hat{Q}(\lambda x. \psi(x) e^{-\frac{iEt}{\hbar}}) dx \\ &= \int_{-\infty}^{+\infty} \psi^*(x) \hat{Q}(\psi)(x) dx\end{aligned}$$

since \hat{Q} is linear. \square

Corollary 24.4.4.1. *If the wave function is given by $\psi(x)e^{-\frac{iEt}{\hbar}}$ then $\langle p \rangle = 0$.*

Proposition 24.4.5. *If the wave function is given by $\psi(x)e^{-\frac{iEt}{\hbar}}$ then $\langle H \rangle = E$ and $\sigma_H = 0$.*

PROOF:

$$\begin{aligned}\langle H \rangle &= \int \psi^* \hat{H} \psi dx \\ &= \int \psi^* E \psi dx \quad (\text{time-independent Schrödinger equation}) \\ &= E \int |\psi|^2 dx \\ &= E \\ \langle H^2 \rangle &= \int \psi^* \hat{H}^2 \psi dx \\ &= E^2 \int \psi^* \psi dx \\ &= E^2 \\ \therefore \sigma_H^2 &= \langle H^2 \rangle - \langle H \rangle^2 \\ &= 0\end{aligned} \quad \square$$

Example 24.4.6 (The Infinite Square Well). *The infinite square well with size a ($a > 0$) is a particle moving under the potential*

$$V(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a, \\ \infty & \text{otherwise} \end{cases}$$

The normalizable solutions to the time-independent equation are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x$$

with associated energy

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} .$$

24.5 The Quantum Harmonic Oscillator

The *quantum harmonic oscillator* with frequency ω is a particle of mass m moving under the potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 .$$

Proposition 24.5.1. *The Hamiltonian operator for the quantum harmonic oscillator is*

$$\hat{H} = \frac{1}{2m} (\hat{p}^2 + (m\omega x)^2) .$$

PROOF: Immediate from definitions. \square

Definition 24.5.2 (Ladder Operators). The *raising operator* \hat{a}_+ is

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}} (-i\hat{p} + m\omega\hat{x})$$

The *lowering operator* \hat{a}_- is

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}} (i\hat{p} + m\omega\hat{x})$$

Together, these are called the *ladder operators*.

Proposition 24.5.3.

$$[\hat{a}_-, \hat{a}_+] = 1$$

Proposition 24.5.4.

$$\hat{H} = \hbar\omega(\hat{a}_-\hat{a}_+ - \frac{1}{2}) = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})$$

Proposition 24.5.5. *If ψ is a solution to the time-independent Schrödinger equation with energy E , then $\hat{a}_+\psi$ is a solution with energy $E + \hbar\omega$, and $\hat{a}_-\psi$ is a solution with energy $E - \hbar\omega$.*

Proposition 24.5.6. *For any integrable functions $f, g : \mathbb{R} \rightarrow \mathbb{C}$,*

$$\int_{-\infty}^{+\infty} f^*(\hat{a}_\pm g) dx = \int_{-\infty}^{+\infty} (\hat{a}_\mp f)^* g dx$$

PROOF:

$$\begin{aligned}
 \int_{-\infty}^{+\infty} f^*(\hat{a}_{\pm}g)dx &= \frac{1}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{+\infty} f^*(\mp\hbar\frac{d}{dx} + m\omega x)g dx \\
 &= \frac{1}{\sqrt{2\hbar m\omega}} \left[\mp \int f^* \frac{dg}{dx} dx + \int m\omega f^* x g dx \right] \\
 &= \frac{1}{\sqrt{2\hbar m\omega}} \left[\pm \int \frac{df^*}{dx} g dx + \int m\omega f^* x g dx \right] \quad (\text{integrating by parts}) \\
 &= \frac{1}{\sqrt{2\hbar m\omega}} \int ((\pm\hbar\frac{d}{dx} + m\omega x)f^*)g dx \\
 &= \int (a_{\mp}pf)^* g dx \quad \square
 \end{aligned}$$

Proposition 24.5.7. *The normalized solutions to the time-independent Schrödinger equation are*

$$\psi_n(x) = \frac{1}{\sqrt{n!}} (\hat{a}_+)^n \psi_0$$

with energies

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega$$

where

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}.$$

PROOF:

$\langle 1 \rangle 1.$ $\psi(x) = e^{-\frac{m\omega}{2\hbar}x^2}$ is a solution with energy $\frac{1}{2}\hbar m\omega$.

PROOF:

$$\begin{aligned}
 \frac{d\psi}{dx} &= -\frac{m\omega}{\hbar}x\psi \\
 \therefore \frac{d^2\psi}{dx^2} &= -\frac{m\omega}{\hbar} \left(x \frac{d\psi}{dx} + \psi \right) \\
 &= \frac{m^2\omega^2}{\hbar^2}x^2\psi - \frac{m\omega}{\hbar}\psi \\
 \therefore \hat{H}\psi &= -\frac{\hbar^2}{2m} \frac{m^2\omega^2}{\hbar^2}x^2\psi - \frac{\hbar^2}{2m} \frac{m\omega}{\hbar}\psi + \frac{1}{2}m\omega^2x^2\psi \\
 &= \frac{1}{2}\hbar m\omega\psi
 \end{aligned}$$

$\langle 1 \rangle 2.$ For this ψ we have $\int |\psi|^2 dx = \sqrt{\frac{\pi\hbar}{m\omega}}$.

PROOF:

$$\begin{aligned}
 \int |\psi|^2 dx &= \int e^{-\frac{m\omega}{\hbar}x^2} dx \\
 &= \sqrt{\frac{\pi\hbar}{m\omega}}
 \end{aligned}$$

$\langle 1 \rangle 3.$ ψ_0 is a normalized solution.

$\langle 1 \rangle 4.$ For all n we have $(\hat{a}_+)^n \psi_0$ is a solution with energy E_n .

PROOF: Proposition 24.5.5.

⟨1⟩5. $\hat{a}_- \hat{a}_+ \chi_n = (n+1)\chi - n$

PROOF: Since from Proposition 24.5.4 we have

$$\hbar\omega \hat{a}_- \hat{a}_+ \chi_n - \frac{1}{2}\chi_n = (n + \frac{1}{2})\hbar\omega \chi_n$$

⟨1⟩6. For all n , we have $\int |(\hat{a}_+)^n \psi_0|^2 dx = n!$.

PROOF:

⟨2⟩1. LET: $\chi_n = (\hat{a}_+)^n \psi_0$

⟨2⟩2. ASSUME: as induction hypothesis $\int |\chi_n|^2 dx = n!$.

⟨2⟩3. $\int |\chi_{n+1}|^2 dx = (n+1)!$

PROOF:

$$\begin{aligned} \int |\chi_{n+1}|^2 dx &= \int (\hat{a}_+ \chi_n)^* (\hat{a}_+ \chi_n) dx \\ &= \int \chi_n^* (\hat{a}_- \hat{a}_+ \chi_n) dx && \text{(Proposition 24.5.6)} \\ &= (n+1) \int \chi_n^* \chi_n dx && (\langle 1 \rangle 5) \\ &= (n+1)n! && (\langle 1 \rangle 3) \\ &= (n+1)! \end{aligned}$$

⟨1⟩7. For all n , ψ_n is a normalized solution.

⟨1⟩8. For all $n > 0$, $\hat{a}_- \psi_n = \sqrt{n} \psi_{n-1}$

PROOF: Using ⟨1⟩5.

⟨1⟩9. For any non-zero solution ψ , if $\hat{a}_- \psi$ has energy ≤ 0 then ψ is a constant multiplied by ψ_0 .

PROOF:

⟨2⟩1. ASSUME: $\hat{a}_- \psi$ has energy ≤ 0

⟨2⟩2. $\hat{a}_- \psi = 0$

⟨2⟩3.

$$\hbar \frac{d\psi}{dx} + m\omega x \psi = 0$$

⟨2⟩4.

$$\frac{1}{\psi} \frac{d\psi}{dx} = -\frac{m\omega}{\hbar} x$$

⟨2⟩5. $\ln \psi = -\frac{m\omega}{2\hbar} x^2$ plus a constant.

⟨2⟩6. $\psi = e^{-\frac{m\omega}{2\hbar} x^2}$ multiplied by a constant.

⟨1⟩10. For any solution ψ with energy > 0 , there exists n such that ψ is a constant multiplied by ψ_n .

⟨2⟩1. LET: n be least such that $(\hat{a}_-)^{n+1} \psi$ has non-positive energy.

⟨2⟩2. $(\hat{a}_-)^n \psi$ is a constant multiplied by ψ_0 .

⟨2⟩3. ψ is a constant multiplied by $(\hat{a}_+)^n \psi_0$.

⟨2⟩4. ψ is a constant multiplied by ψ_n .

□

Definition 24.5.8. We call ψ_0 the *ground state* of the quantum harmonic oscillator, and the other ψ_n the *excited states*.

Proposition 24.5.9.

$$x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$$

PROOF: Straightforward calculation. \square

Proposition 24.5.10.

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$$

PROOF: Straightforward calculation. \square

Proposition 24.5.11. *If $m \neq n$ then*

$$\int \psi_m^* \psi_n dx = 0$$

PROOF: We have

$$\int \psi_m^* \hat{a}_+ \hat{a}_- \psi_n dx = n \int \psi_m^* \psi_n dx$$

and

$$\begin{aligned} \int \psi_m^* \hat{a}_+ \hat{a}_- \psi_n dx &= \int \hat{a}_+ \hat{a}_- \psi_m^* \psi_n dx && \text{(Proposition 24.5.6)} \\ &= m \int \psi_m^* \psi_n dx \end{aligned}$$

Therefore either $m = n$ or $\int \psi_m^* \psi_n dx = 0$. \square

Proposition 24.5.12. *For the n th excited state of the quantum harmonic oscillator we have:*

$$\begin{aligned} \langle x \rangle &= 0 \\ \langle p \rangle &= 0 \\ \sigma_x &= \sqrt{\left(n + \frac{1}{2}\right) \frac{\hbar}{m\omega}} \\ \sigma_p &= \sqrt{\left(n + \frac{1}{2}\right) \hbar m\omega} \\ \langle T \rangle &= \frac{1}{2} \hbar \omega \left(n + \frac{1}{2}\right) \\ \langle V \rangle &= \frac{1}{2} \hbar \omega \left(n + \frac{1}{2}\right) \end{aligned}$$

PROOF: These follow from

$$\begin{aligned}
 \langle x^2 \rangle &= \int_{-\infty}^{+\infty} \psi_n^* x^2 \psi_n dx \\
 &= \frac{\hbar}{2m\omega} \int_{-\infty}^{+\infty} \psi_n^* [\hat{a}_+^2 + \hat{a}_+ \hat{a}_- + \hat{a}_- \hat{a}_+ + \hat{a}_-^2] \psi_n dx \\
 &= \frac{\hbar}{2m\omega} \left[\int_{-\infty}^{+\infty} a \psi_n^* \psi_{n+2} dx + n \int \psi_n^* \psi_n dx + (n+1) \int \psi_n^* \psi_n dx + \int b \psi_n^* \psi_{n-2} dx \right] \\
 &= \frac{\hbar}{2m\omega} (2n+1)
 \end{aligned}$$

and $\langle p^2 \rangle = \frac{\hbar m \omega}{2} (2n+1)$ similar. \square

Proposition 24.5.13. *The n th excited state is*

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\frac{\xi^2}{2}}$$

where H_n is the n th Hermite polynomial and

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x .$$

PROOF: From Proposition 24.5.7 by induction. \square

24.6 The Free Particle

Proposition 24.6.1. *The solutions to the time-independent Schrödinger equation with $V = 0$ are*

$$A e^{i(kx - \frac{\hbar k^2}{2m} t)}$$

Proposition 24.6.2. *The normalized solutions to the time-dependent Schrödinger equation with $V = 0$ are the wave packets*

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)} dk$$

where $\phi : \mathbb{R} \rightarrow \mathbb{C}$ is a function such that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) dk = 1 .$$

Proposition 24.6.3. *Let a particle moving under potential $V = 0$ have initial wave function $\Psi(x, 0)$. Then we have*

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)} dk$$

where

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x, 0) e^{-ikx} dx .$$

PROOF: We have $\phi(k)$ is the inverse Fourier transform of $\Phi(x, t)$ hence of $\Phi(x, 0)$.
 \square