Encyclopaedia of Mathematics and Physics

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Chapter 1

Set Theory

Proposition 1.1. *Every infinite subset of a countably infinite set is countable.*

PROOF:
$\langle 1 \rangle 1$. LET: $i : A \hookrightarrow \mathbb{N}$ be an infinite subset of $\mathbb{N}$.
$\langle 1 \rangle 2$. Define $j : \mathbb{N} \to A$ by: $j(k)$ is the element such that $i(j(k))$ is least such that $i(j(k)) \notin \{i(j(0)), \ldots, i(j(k-1))\}$.
$\langle 1 \rangle 3$. j is a bijection.
□

Proposition 1.2. *A countable union of countable sets is countable.*

PROOF:
$\langle 1 \rangle 1$. LET: (A_n) be a sequence of countable sets.
$\langle 1 \rangle 2$. For $n \in \mathbb{N}$, PICK an enumeration $(e_{nm})_m$ of A_n.
$\langle 1 \rangle 3$. LET: (p_k) be the following enumeration of $\mathbb{N} \times \mathbb{N}$:
$$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots$$
$\langle 1 \rangle 4$. $(e_{\pi_1(p_k)\pi_2(p_k)})_k$ is an enumeration of $\bigcup_n A_n$.
□

Theorem 1.3. $2^{\mathbb{N}}$ *is uncountable.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $f : \mathbb{N} \approx 2^{\mathbb{N}}$
$\langle 1 \rangle 2$. LET: $S = \{n \in \mathbb{N} : n \notin f(n)\}$
$\langle 1 \rangle 3$. For all n, we have $n \in S \Leftrightarrow n \notin f(n)$
$\langle 1 \rangle 4$. For all n we have $S \neq f(n)$.
$\langle 1 \rangle 5$. Q.E.D.
    PROOF: This contradicts $\langle 1 \rangle 1$.
□

Chapter 2

Relations

Definition 2.1 (Antisymmetric). A relation R on a set A is *antisymmetric* iff, whenever xRy and yRx, then $x = y$.

Definition 2.2 (Transitive). A relation R on a type A is *transitive* iff, whenever xRy and yRz, then xRz.

Chapter 3

Order Theory

Definition 3.1 (Linear Order). A *linear order* on a set A is a binary relation $\leq$ on A that is transitive, antisymmetric and:

$$\forall x, y \in A.x \leq y \vee y \leq x \ .$$

A *linearly ordered set* is a pair $(A, \leq)$ where A is a set and $\leq$ is a binary relation on A.

We write $x < y$ for $x \leq y$ and $x \neq y$.

Definition 3.2 (Upper Bound). Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is an *upper bound* in E iff $\forall x \in E.x \leq u$. We say E is *bounded above* iff it has an upper bound.

The *up-set* of E, denoted $E \uparrow$, is the set of upper bounds of E.

Definition 3.3 (Lower Bound). Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then u is an *lower bound* in E iff $\forall x \in E.l \leq x$. We say E is *bounded below* iff it has a lower bound.

The *down-set* of E, denoted $E \downarrow$, is the set of lower bounds of E.

Definition 3.4 (Supremum). Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is the *least upper bound* or *supremum* of E iff u is an upper bound for E and, for any upper bound u' for E, we have $u \leq u'$.

Definition 3.5 (Infimum). Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then l is the *greatest lower bound* or *infimum* of E iff l is a lower bound for E and, for any lower bound l' for E, we have $l' \leq l$.

Definition 3.6 (Least Upper Bound Property). A linearly ordered set S has the *least upper bound property* iff every nonempty subset of S that is bounded above has a least upper bound.

Proposition 3.7. *Let S be a linearly ordered set and $E \subseteq S$.*

1. If $E \downarrow$ has a supremum l, then l is the infimum of E.

  2. If $E \uparrow$ has an infimum u, then U is the supremum of E.

PROOF:

$\langle 1 \rangle 1$. If $E \downarrow$ has a supremum l, then l is the infimum of E.

   $\langle 2 \rangle 1$. l is a lower bound for E.

      $\langle 3 \rangle 1$. LET: $x \in E$

      $\langle 3 \rangle 2$. x is an upper bound for $E \downarrow$.

         PROOF: For all $y \in E \downarrow$ we have $y \leq x$.

      $\langle 3 \rangle 3$. $l \leq x$

   $\langle 2 \rangle 2$. For any lower bound l' for E, we have $l' \leq l$.

      PROOF: Since l is an upper bound for $E \downarrow$.

$\langle 1 \rangle 2$. If $E \uparrow$ has an infimum u, then u is the supremum of E.

   PROOF: Dual.

☐

Corollary 3.7.1. *A linearly ordered set has the least upper bound property if and only if every nonempty set bounded below has an infimum.*

Definition 3.8 (Closed Downwards)**.** Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed downwards* iff, whenever $x \in E$ and $y < x$, then $y \in E$.

Definition 3.9 (Closed Upwards)**.** Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed upwards* iff, whenever $x \in E$ and $x < y$, then $y \in E$.

Definition 3.10 (Greatest)**.** Let S be a linearly ordered set and $u \in S$. Then u is *greatest* in S iff $\forall x \in S.x \leq u$.

Definition 3.11 (Least)**.** Let S be a linearly ordered set and $l \in S$. Then l is *least* in S iff $\forall x \in S.l \leq x$.

Proposition 3.12. *Let $\leq$ be a linear order on a set S and $E \subseteq S$. Then $\leq \cap E^2$ is a linear order on E.*

PROOF: Easy. ☐

   Given a linearly ordered set $(S, \leq)$ and $E \subseteq S$, we write just E for the linearly ordered set $(E, \leq \cap E^2)$.

Definition 3.13 (Lexicographic Order)**.** Let A and B be linearly ordered sets. The *lexicographic order* or *dictionary order* on $A \times B$ is the order defined by

$$(a, b) \leq (a', b') \Leftrightarrow a = a' \vee (a < a' \wedge b \leq b') \ .$$

Proposition 3.14. *The lexicographic order is a linear order.*

Chapter 4

Field Theory

Definition 4.1 (Field). A *field* F consists of a set F, two operations $+, \cdot :$ $F^2 \to F$ and an element $0 \in F$ such that:

- $+$ is commutative.

- $+$ is associative.

- $\forall x \in F.x + 0 = x$

- $\forall x \in F.\exists y \in F.x + y = 0$

- $\cdot$ is commutative.

- $\cdot$ is associative.

- There exists $1 \in F$ such that $1 \neq 0$ and $\forall x \in F.x1 = x$ and $\forall x \in F.x \neq 0 \Rightarrow \exists y \in F.xy = 1$

- *Distributive Law* $\forall x, y, z \in F.x(y + z) = xy + xz$

Proposition 4.2. *In any field F, the element 0 is the unique element such that* $\forall x \in F.x + 0 = x$.

PROOF: If 0 and $0'$ both have this property then $0 = 0 + 0' = 0'$. $\square$

Proposition 4.3. *In any field F, given $x \in F$, there is a unique $y \in F$ such that $x + y = 0$.*

PROOF: If $x + y = x + y' = 0$ then
$$
\begin{aligned}
y &= y + 0 \\
&= y + x + y' \\
&= 0 + y' \\
&= y'
\end{aligned}
$$
$\square$

Definition 4.4. Let F be a field. Let $x \in F$. We denote by $-x$ the unique element of F such that $x + (-x) = 0$.

Given $x, y \in F$, we write $x - y$ for $x + (-y)$.

Proposition 4.5. *In any field F, if $x + y = x + z$ then $y = z$.*

PROOF: If $x + y = x + z$ we have
$$-x + x + y = -x + x + z$$
$$\therefore 0 + y = 0 + z$$
$$\therefore y = z \qquad \qquad \square$$

Proposition 4.6. *In any field F, we have $-(-x) = x$.*

PROOF: Since $x + (-x) = 0$. $\square$

Proposition 4.7. *In any field F, the element 1 such that $\forall x \in F.x1 = x$ is unique.*

PROOF: If 1 and $1'$ both have this property then $1 = 1 \cdot 1' = 1'$. $\square$

Proposition 4.8. *In any field F, given $x \in F$ with $x \neq 0$, the element y such that $xy = 1$ is unique.*

PROOF: If y and y' both have this property then we have
$$y = y1$$
$$= yxy'$$
$$= 1y'$$
$$= y' \qquad \qquad \square$$

Definition 4.9. In any field F, if $x \neq 0$, we write x^{-1} for the unique element such that $xx^{-1} = 1$.

We write x/y for xy^{-1}.

Proposition 4.10. *In any field F, if $xy = xz$ and $x \neq 0$ then $y = z$.*

PROOF:
$$y = 1y$$
$$= x^{-1}xy$$
$$= x^{-1}xz$$
$$= 1z$$
$$= z \qquad \qquad \square$$

Proposition 4.11. *In any field F, if $x \neq 0$ then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.*

PROOF: Since $xx^{-1} = 1$. $\square$

Proposition 4.12. *In any field F, we have $x0 = 0$.*

PROOF:

$$x0 + 0 = x0$$
$$= x(0 + 0)$$
$$= x0 + x0$$
$$\therefore 0 = x0 \qquad \square$$

Proposition 4.13. *In any field F, if $xy = 0$ then $x = 0$ or $y = 0$.*

PROOF: If $xy = 0$ and $x \neq 0$ then we have $y = x^{-1}xy = x^{-1}0 = 0$. $\square$

Proposition 4.14. *In any field F, we have $(-x)y = -(xy)$.*

PROOF:

$$xy + (-x)y = (x + (-x))y$$
$$= 0y$$
$$= 0 \qquad \text{(Proposition 4.12)}\square$$

Corollary 4.14.1. *In any field F, we have $(-x)(-y) = xy$.*

PROOF:

$$(-x)(-y) = -(x(-y))$$
$$= -(-(xy))$$
$$= xy \qquad \text{(Proposition 4.6)}\square$$

Proposition 4.15. *Let K be a field. Let $a, b \in K$. If $a^2 = b^2$ then $a = b$ or $a = -b$.*

PROOF:

$$a^2 - b^2 = 0$$
$$\therefore (a - b)(a + b) = 0$$

Hence either $a - b = 0$ or $a + b = 0$, and the conclusion follows. $\square$

4.1 Ordered Fields

Definition 4.16 (Ordered Field). An *ordered field F* consists of a field F and a linear order $\leq$ on F such that:

- For all $x, y, z \in F$, if $y < z$ then $x + y < x + z$

- For all $x, y \in F$, if $x > 0$ and $y > 0$ then $xy > 0$.

We call x *positive* iff $x > 0$ and *negative* iff $x < 0$.

Example 4.17. $\mathbb{Q}$ is an ordered field.

Proposition 4.18. *In any ordered field, if x is positive then $-x$ is negative.*

PROOF: If $x > 0$ then $0 = x + (-x) > 0 = (-x) = -x$. $\square$

Proposition 4.19. *In any ordered field, if $y < z$ and x is positive then $xy < xz$.*

PROOF: If $y < z$ then we have

$$0 < z - y$$
$$\therefore 0 < x(z - y)$$
$$= xz - xy$$
$$\therefore xy < xz \qquad \square$$

Proposition 4.20. *In any ordered field, if $y < z$ and x is negative then $xy > xz$.*

PROOF:
$\langle 1 \rangle 1.$ $-x$ is positive.
$\langle 1 \rangle 2.$ $(-x)y < (-x)z$
$\langle 1 \rangle 3.$ $-(xy) < -(xz)$
$\langle 1 \rangle 4.$ $xz < xy$
$\square$

Proposition 4.21. *In any ordered field, if $x \neq 0$ then $x^2 > 0$.*

PROOF:
$\langle 1 \rangle 1.$ If $x > 0$ then $x^2 > 0$.
    PROOF: Proposition 4.19.
$\langle 1 \rangle 2.$ If $x < 0$ then $x^2 > 0$.
    PROOF: Proposition 4.20.
$\square$

Corollary 4.21.1. *In any ordered field, we have $1 > 0$.*

Proposition 4.22. *In any ordered field, if x is positive then x^{-1} is positive.*

PROOF: If $x^{-1} < 0$ then we would have $1 = xx^{-1} < x0 = 0$ contradicting Corollary 4.21.1. $\square$

Proposition 4.23. *In any ordered field, if $0 < x < y$ then $y^{-1} < x^{-1}$.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: $0 < x < y$
$\langle 1 \rangle 2.$ x^{-1} and y^{-1} are positive.
    PROOF: Proposition 4.22.
$\langle 1 \rangle 3.$ $xy^{-1} < yy^{-1} = 1$
$\langle 1 \rangle 4.$ $y^{-1} = x^{-1}xy^{-1} < x^{-1}1 = x^{-1}$
$\square$

Lemma 4.24. *Let K be an ordered field. Let $b \in K$ with $b > 1$. Let n be a positive integer. Then*

$$b^n - 1 \geq n(b - 1)$$

PROOF:

$$b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + \cdots + 1)$$
$$\geq (b - 1)(1 + 1 + \cdots + 1)$$
$$= n(b - 1) \qquad \square$$

Chapter 5

Real Analysis

5.1 Construction of the Real Numbers

Definition 5.1 (Cut). A *cut* is a subset α of $\mathbb{Q}$ such that:

- $\emptyset \neq \alpha \neq \mathbb{Q}$

- α is closed downwards.

- α has no greatest element.

In this section, we write R for the set of all cuts.

Proposition 5.2. *R is linearly ordered by $\subseteq$.*

PROOF: The only difficult part is to prove that, for any cuts α and β, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.
$\langle 1 \rangle 1$. ASSUME: $\alpha \nsubseteq \beta$
     PROVE: $\beta \subseteq \alpha$
$\langle 1 \rangle 2$. PICK $q \in \alpha$ such that $q \notin \beta$
$\langle 1 \rangle 3$. LET: $r \in \beta$
$\langle 1 \rangle 4$. $q \nless r$
$\langle 1 \rangle 5$. $r < q$
$\langle 1 \rangle 6$. $r \in \alpha$
$\square$

Proposition 5.3. *R has the least upper bound property.*

PROOF:
$\langle 1 \rangle 1$. LET: $E \subseteq R$ be nonempty and bounded above.
$\langle 1 \rangle 2$. LET: $s = \bigcup E$
     PROVE: s is a cut.
$\langle 1 \rangle 3$. $\emptyset \neq s$
  PROOF: Since E is nonempty and every element of E is nonempty.
$\langle 1 \rangle 4$. $s \neq \mathbb{Q}$

$\langle 2 \rangle$1. PICK an upper bound u for E.
$\langle 2 \rangle$2. PICK $q \notin u$
   PROVE: $q \notin s$
$\langle 2 \rangle$3. $\forall \alpha \in E . \alpha \subseteq u$
$\langle 2 \rangle$4. $s \subseteq u$
$\langle 2 \rangle$5. $q \notin s$
$\langle 1 \rangle$5. s is closed downwards.
   $\langle 2 \rangle$1. LET: $q \in s$ and $r < q$.
   $\langle 2 \rangle$2. PICK $\alpha \in E$ such that $q \in \alpha$.
   $\langle 2 \rangle$3. $r \in \alpha$
   $\langle 2 \rangle$4. $r \in s$
$\langle 1 \rangle$6. s has no greatest element.
   $\langle 2 \rangle$1. LET: $q \in s$
   $\langle 2 \rangle$2. PICK $\alpha \in E$ such that $q \in \alpha$.
   $\langle 2 \rangle$3. PICK $r \in \alpha$ such that $q < r$.
   $\langle 2 \rangle$4. $r \in s$
☐

Definition 5.4 (Addition). Given cuts α and β, we define

$$\alpha + \beta = \{ q + r : q \in \alpha, r \in \beta \} \ .$$

Proposition 5.5. *Given cuts α and β, we have $\alpha + \beta$ is a cut.*

PROOF:
$\langle 1 \rangle$1. $\alpha + \beta$ is nonempty.
   PROOF: Since α and β are nonempty.
$\langle 1 \rangle$2. $\alpha + \beta \neq \mathbb{Q}$
   $\langle 2 \rangle$1. PICK $q \in \mathbb{Q} - \alpha$ and $r \in \mathbb{Q} - \beta$.
      PROVE: $q + r \notin \alpha + \beta$
   $\langle 2 \rangle$2. ASSUME: for a contradiction $q + r \in \alpha + \beta$.
   $\langle 2 \rangle$3. PICK $x \in \alpha$ and $y \in \beta$ such that $q + r = x + y$
   $\langle 2 \rangle$4. $x < q$
   $\langle 2 \rangle$5. $y < r$
   $\langle 2 \rangle$6. $x + y < q + r$
   $\langle 2 \rangle$7. Q.E.D.
      PROOF: This is a contradiction.
$\langle 1 \rangle$3. $\alpha + \beta$ is closed downwards.
   $\langle 2 \rangle$1. LET: $q \in \alpha$, $r \in \beta$ and $x < q + r$
   $\langle 2 \rangle$2. $x - q < r$
   $\langle 2 \rangle$3. $x - q \in \beta$
   $\langle 2 \rangle$4. $x \in \alpha + \beta$
$\langle 1 \rangle$4. $\alpha + \beta$ has no greatest element.
   $\langle 2 \rangle$1. LET: $q \in \alpha$ and $r \in \beta$.
      PROVE: $q + r$ is not greatest in $\alpha + \beta$.
   $\langle 2 \rangle$2. PICK $q' \in \alpha$ with $q < q'$ and $r' \in \beta$ with $r < r'$.
   $\langle 2 \rangle$3. $q + r < q' + r' \in \alpha + \beta$

▯

Proposition 5.6. *Addition is commutative and associative on R.*

PROOF: Immediate from definitions and the fact that addition is commutative and associative on $\mathbb{Q}$. ▯

Definition 5.7. For any $q \in \mathbb{Q}$, let $q^* = \{r \in \mathbb{Q} : r < q\}$.

Proposition 5.8. *For any $q \in \mathbb{Q}$, we have q^* is a cut.*

PROOF:
$\langle 1 \rangle 1.$ $q^* \neq \emptyset$
  PROOF: Since $q - 1 \in q^*$.
$\langle 1 \rangle 2.$ $q^* \neq \mathbb{Q}$
  PROOF: Since $q \notin q^*$.
$\langle 1 \rangle 3.$ q^* is closed downwards.
  PROOF: Immediate from definition.
$\langle 1 \rangle 4.$ q^* has no greatest element.
  PROOF: For all $r \in q^*$ we have $r < (q + r)/2 \in q^*$.
▯

Proposition 5.9. *For any cut α we have $\alpha + 0^* = \alpha$.*

PROOF:
$\langle 1 \rangle 1.$ $\alpha + 0^* \subseteq \alpha$
  $\langle 2 \rangle 1.$ LET: $q \in \alpha$ and $r \in 0^*$
      PROVE: $q + r \in \alpha$
  $\langle 2 \rangle 2.$ $r < 0$
  $\langle 2 \rangle 3.$ $q + r < q$
  $\langle 2 \rangle 4.$ $q + r \in \alpha$
$\langle 1 \rangle 2.$ $\alpha \subseteq \alpha + 0^*$
  $\langle 2 \rangle 1.$ LET: $q \in \alpha$
  $\langle 2 \rangle 2.$ PICK $r \in \alpha$ such that $q < r$
  $\langle 2 \rangle 3.$ $q = r + (q - r) \in \alpha + 0^*$
▯

Proposition 5.10. *For any cut α, there exists a cut β such that $\alpha + \beta = 0$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\beta = \{p \in \mathbb{Q} : \exists r > 0. -p - r \notin \alpha\}$
$\langle 1 \rangle 2.$ β is a cut.
  $\langle 2 \rangle 1.$ $\beta \neq \emptyset$
    $\langle 3 \rangle 1.$ PICK $q \notin \alpha$
    $\langle 3 \rangle 2.$ $-q - 1 \in \beta$
  $\langle 2 \rangle 2.$ $\beta \neq \mathbb{Q}$
    $\langle 3 \rangle 1.$ PICK $q \in \alpha$
        PROVE: $-q \notin \beta$
    $\langle 3 \rangle 2.$ ASSUME: for a contradiction $-q \in \beta$

    $\langle 3 \rangle 3$. PICK $r > 0$ such that $q - r \notin \alpha$
    $\langle 3 \rangle 4$. $q - r < q$
    $\langle 3 \rangle 5$. Q.E.D.
      PROOF: This contradicts the fact that α is closed downwards.
  $\langle 2 \rangle 3$. β is closed downwards.
    $\langle 3 \rangle 1$. LET: $p \in \beta$ and $q < p$.
    $\langle 3 \rangle 2$. PICK $r > 0$ such that $-p - r \notin \alpha$
    $\langle 3 \rangle 3$. $-p - r < -q - r$
    $\langle 3 \rangle 4$. $-q - r \notin \alpha$
    $\langle 3 \rangle 5$. $q \in \beta$
  $\langle 2 \rangle 4$. β has no greatest element.
    $\langle 3 \rangle 1$. LET: $p \in \beta$
    $\langle 3 \rangle 2$. PICK $r > 0$ such that $-p - r \notin \alpha$
    $\langle 3 \rangle 3$. $-(p + r/2) - r/2 \notin \alpha$
    $\langle 3 \rangle 4$. $p + r/2 \in \beta$
$\langle 1 \rangle 3$. $\alpha + \beta \subseteq 0^*$
  $\langle 2 \rangle 1$. LET: $p \in \alpha$ and $q \in \beta$.
  $\langle 2 \rangle 2$. PICK $r > 0$ such that $-q - r \notin \alpha$.
  $\langle 2 \rangle 3$. $p < -q - r$
  $\langle 2 \rangle 4$. $p + q < -r$
  $\langle 2 \rangle 5$. $p + q < 0$
  $\langle 2 \rangle 6$. $p + q \in 0^*$
$\langle 1 \rangle 4$. $0^* \subseteq \alpha + \beta$
  $\langle 2 \rangle 1$. LET: $v \in 0^*$
  $\langle 2 \rangle 2$. LET: $w = -v/2$
  $\langle 2 \rangle 3$. $w > 0$
  $\langle 2 \rangle 4$. PICK an integer n such that $nw \in \alpha$ and $(n + 1)w \notin \alpha$.
  $\langle 2 \rangle 5$. LET: $p = -(n + 2)w$
  $\langle 2 \rangle 6$. $p \in \beta$
  $\langle 2 \rangle 7$. $v = nw + p$
  $\langle 2 \rangle 8$. $v \in \alpha + \beta$
$\Box$

Proposition 5.11. *Given $\alpha, \beta, \gamma \in R$, if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

PROOF:
$\langle 1 \rangle 1$. $\alpha + \beta \subseteq \alpha + \gamma$
  PROOF: Immediate from definitions.
$\langle 1 \rangle 2$. $\alpha + \beta \neq \alpha + \gamma$
  PROOF: If $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$ by cancellation.
$\Box$

Definition 5.12. Given cuts α and β, define $\alpha\beta$ by:

$$\alpha\beta = \begin{cases} \{p \in \mathbb{Q} : \exists r \in \alpha.\exists s \in \beta(p \leq rs \wedge r > 0 \wedge s > 0\} & \text{if } \alpha > 0^* \text{ and } \beta > 0^* \\ (-\alpha)(-\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^* \\ -((-\alpha)\beta) & \text{if } \alpha < 0^* \text{ and } \beta > 0^* \\ -(\alpha(-\beta)) & \text{if } \alpha > 0^* \text{ and } \beta < 0^* \\ 0^* & \text{if } \alpha = 0^* \text{ or } \beta = 0^* \end{cases}$$

Proposition 5.13. *For any cuts α and β, we have $\alpha\beta$ is a cut.*

PROOF:
$\langle 1 \rangle 1$. If $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta$ is a cut.
　$\langle 2 \rangle 1$. $\alpha\beta \neq \emptyset$
　　$\langle 3 \rangle 1$. PICK $q \in \alpha$ and $r \in \beta$ such that $q, r \notin 0^*$
　　$\langle 3 \rangle 2$. ASSUME: w.l.o.g. $0 < q$ and $0 < r$.
　　　PROOF: Since α and β have no greatest element.
　　$\langle 3 \rangle 3$. $qr \in \alpha\beta$
　$\langle 2 \rangle 2$. $\alpha\beta \neq \mathbb{Q}$
　　$\langle 3 \rangle 1$. PICK $r \notin \alpha$ and $s \notin \beta$
　　　PROVE: $rs \notin \alpha\beta$
　　$\langle 3 \rangle 2$. ASSUME: for a contradiction $rs \in \alpha\beta$.
　　$\langle 3 \rangle 3$. PICK $r' \in \alpha$ and $s' \in \beta$ such that $rs \leq r's'$ and $r' > 0$ and $s' > 0$.
　　$\langle 3 \rangle 4$. $r' < r$ and $s' < s$
　　$\langle 3 \rangle 5$. $r's' < rs$
　　$\langle 3 \rangle 6$. Q.E.D.
　　　PROOF: This is a contradiction.
　$\langle 2 \rangle 3$. $\alpha\beta$ is closed downwards.
　　$\langle 3 \rangle 1$. LET: $p \in \alpha\beta$ and $p' < p$
　　$\langle 3 \rangle 2$. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$
　　$\langle 3 \rangle 3$. $p' \leq rs$
　　$\langle 3 \rangle 4$. $p' \in \alpha\beta$
　$\langle 2 \rangle 4$. $\alpha\beta$ has no greatest element.
　　$\langle 3 \rangle 1$. LET: $p \in \alpha\beta$
　　$\langle 3 \rangle 2$. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$.
　　$\langle 3 \rangle 3$. PICK $r' \in \alpha$ and $s' \in \beta$ with $r < r'$ and $s < s'$.
　　$\langle 3 \rangle 4$. $p < r's' \in \alpha\beta$
$\langle 1 \rangle 2$. For any cuts α and β, we have $\alpha\beta$ is a cut.
　PROOF: Since if α is a cut then $-\alpha$ is a cut.
□

Proposition 5.14. *For any cuts α and β we have $\alpha\beta = \beta\alpha$.*

PROOF: Easy from the definitions. □

Proposition 5.15. *For any cuts α, β and γ we have*

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma \ .$$

Proof:

⟨1⟩1. Case: α, β and γ are all positive.

  Proof: In this case $\alpha(\beta\gamma) = (\alpha\beta)\gamma = \{p \in \mathbb{Q} : \exists r \in \alpha.\exists s \in \beta.\exists t \in \gamma.(p \leq rst \wedge r > 0 \wedge s > 0 \wedge t > 0)\}$.

⟨1⟩2. Case: One of α, β or γ is 0^*.

  Proof: Then $\alpha(\beta\gamma) = (\alpha\beta)\gamma = 0^*$.

⟨1⟩3. Case: α and β are positive, γ is negative.

  Proof:

$$
\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\
&= -(\alpha(\beta(-\gamma))) \\
&= -((\alpha\beta)(-\gamma)) && (\langle 1\rangle 1) \\
&= (\alpha\beta)\gamma
\end{aligned}
$$

⟨1⟩4. Case: α is positive, β is negative, γ is positive.

  Proof:

$$
\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-((-\beta)\gamma)) \\
&= -(\alpha((-\beta)\gamma)) \\
&= -((\alpha(-\beta))\gamma) && (\langle 1\rangle 1) \\
&= (-(\alpha(-\beta)))\gamma \\
&= (\alpha\beta)\gamma
\end{aligned}
$$

⟨1⟩5. Case: α is positive, β and γ are negative.

  Proof:

$$
\begin{aligned}
\alpha(\beta\gamma) &= \alpha((-\beta)(-\gamma)) \\
&= (\alpha(-\beta))(-\gamma) && (\langle 1\rangle 1) \\
&= (-(\alpha\beta))(-\gamma) \\
&= (\alpha\beta)\gamma
\end{aligned}
$$

⟨1⟩6. Case: α is negative, β and γ are positive.

  Proof: Similar to ⟨1⟩3.

⟨1⟩7. Case: α is negative, β is positive, γ is negative.

  Proof:

$$
\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\
&= (-\alpha)(\beta(-\gamma)) \\
&= ((-\alpha)\beta)(-\gamma) && (\langle 1\rangle 1) \\
&= (-(\alpha\beta))(-\gamma) \\
&= (\alpha\beta)\gamma
\end{aligned}
$$

⟨1⟩8. Case: α and β are negative, γ is positive.

  Proof: Similar to ⟨1⟩5.

⟨1⟩9. Case: α, β and γ are all negative.

PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-(-\beta)(-\gamma)) \\
&= -((-\alpha)((-\beta)(-\gamma))) \\
&= -(((-\alpha)(-\beta))(-\gamma)) \qquad\qquad (\langle 1\rangle 1) \\
&= -((\alpha\beta)(-\gamma)) \\
&= (\alpha\beta)\gamma
\end{aligned}$$

☐

Proposition 5.16. *For any cut α we have $\alpha 1^* = \alpha$.*

PROOF:
$\langle 1\rangle 1$. CASE: α is positive.
 $\langle 2\rangle 1$. $\alpha 1^* \subseteq \alpha$
 $\langle 2\rangle 2$. $\alpha \subseteq \alpha 1^*$
$\langle 1\rangle 2$. CASE: $\alpha = 0^*$
$\langle 1\rangle 3$. CASE: α is negative.
☐

Theorem 5.17. *There exists an ordered field with the least upper bound property.*

Proposition 5.18. *There is no rational p such that $p^2 = 2$.*

PROOF:
$\langle 1\rangle 1$. ASSUME: for a contradiction $p^2 = 2$.
$\langle 1\rangle 2$. PICK integers m, n not both even such that $p = m/n$.
$\langle 1\rangle 3$. $m^2 = 2n^2$
$\langle 1\rangle 4$. m is even.
$\langle 1\rangle 5$. PICK an integer k such that $m = 2k$.
$\langle 1\rangle 6$. $4k^2 = 2n^2$
$\langle 1\rangle 7$. $2k^2 = n^2$
$\langle 1\rangle 8$. n is even.
$\langle 1\rangle 9$. Q.E.D.
 PROOF: $\langle 1\rangle 2$, $\langle 1\rangle 4$ and $\langle 1\rangle 8$ form a contradiction.
☐

Theorem 5.19. *Any two complete ordered fields are isomorphic.*

Definition 5.20. Let $\mathbb{R}$ be the complete ordered field. We call its elements *real numbers*.

5.2 Properties of the Real Numbers

Theorem 5.21. $\mathbb{Q}$ *is a subfield of $\mathbb{R}$.*

Theorem 5.22 (Archimedean Property)**.** *Let $x, y \in \mathbb{R}$ with $x > 0$. There exists a positive integer n such that $nx > y$.*

PROOF:

$\langle 1 \rangle$1. LET: $A = \{nx : n \in \mathbb{Z}^+\}$

$\langle 1 \rangle$2. ASSUME: for a contradiction there is no positive integer n such that $nx > y$.

$\langle 1 \rangle$3. y is an upper bound for A.

$\langle 1 \rangle$4. LET: $\alpha = \sup A$

$\langle 1 \rangle$5. $\alpha - x$ is not an upper bound for A.

$\langle 1 \rangle$6. PICK a positive integer m such that $\alpha - x < mx$

$\langle 1 \rangle$7. $\alpha < (m+1)x \in A$

$\langle 1 \rangle$8. Q.E.D.

    PROOF: This contradicts $\langle 1 \rangle$4.

$\square$

Theorem 5.23. $\mathbb{Q}$ *is dense in* $\mathbb{R}$.

PROOF:

$\langle 1 \rangle$1. LET: $x, y \in \mathbb{R}$ with $x < y$

$\langle 1 \rangle$2. PICK a positive integer n such that
$$n(y - x) > 1 \ .$$
    PROOF: Archimedean property.

$\langle 1 \rangle$3. PICK a positive integer m_1 such that $m_1 > nx$

    PROOF: Archimedean property.

$\langle 1 \rangle$4. PICK a positive integer m_2 such that $m_2 > -nx$

    PROOF: Archimedean property.

$\langle 1 \rangle$5. $-m_2 < nx < m_1$

$\langle 1 \rangle$6. LET: m be the integer such that
$$m - 1 \leq nx < m \ .$$

$\langle 1 \rangle$7. $nx < m \leq 1 + nx < ny$

$\langle 1 \rangle$8. $x < m/n < y$

$\square$

Theorem 5.24. *For every real number* $x > 0$ *and positive integer* n, *there exists a unique positive real number* y *such that* $y^n = x$.

PROOF:

$\langle 1 \rangle$1. There exists a real $y > 0$ such that $y^n = x$.

    $\langle 2 \rangle$1. LET: $E = \{t \in \mathbb{R}^+ : t^n < x\}$

    $\langle 2 \rangle$2. LET: $y = \sup E$

        $\langle 3 \rangle$1. $E \neq \emptyset$

            $\langle 4 \rangle$1. LET: $t = x/(x+1)$

            $\langle 4 \rangle$2. $0 < t < 1$

            $\langle 4 \rangle$3. $t^n < t < x$

            $\langle 4 \rangle$4. $t \in E$

        $\langle 3 \rangle$2. $x + 1$ is an upper bound for E.

            $\langle 4 \rangle$1. LET: $t > x + 1$

            $\langle 4 \rangle$2. $t^n > t > x$

            $\langle 4 \rangle$3. $t \notin E$

$\langle 2\rangle 3.$ $y^n = x$

$\quad\langle 3\rangle 1.$ $y^n \not< x$

$\qquad\langle 4\rangle 1.$ ASSUME: for a contradiction $y^n < x$.

$\qquad\langle 4\rangle 2.$ PICK h such that $0 < h < 1$ and
$$h < \frac{x - y^n}{n(y+1)^{n-1}} \quad .$$

$\qquad\langle 4\rangle 3.$ $(y+h)^n - y^n < x - y^n$

$\qquad\quad$ PROOF:
$$(y+h)^n - y^n = ((y+h) - y)\sum_{i=0}^{n-1}(y+h)^{n-1-i}y^i$$
$$= h\sum_{i=0}^{n-1}(y+h)^{n-1-i}y^i$$
$$\leq hn(y+h)^{n-1}$$
$$\leq hn(y+1)^{n-1}$$
$$< x - y^n$$

$\qquad\langle 4\rangle 4.$ $(y+h)^n < x$

$\qquad\langle 4\rangle 5.$ $y + h \in E$

$\qquad\langle 4\rangle 6.$ Q.E.D.

$\qquad\quad$ PROOF: This contradicts the fact that y is an upper bound for E.

$\quad\langle 3\rangle 2.$ $y^n \not> x$

$\qquad\langle 4\rangle 1.$ ASSUME: for a contradiction $y^n > x$

$\qquad\langle 4\rangle 2.$ LET:
$$k = \frac{y^n - x}{ny^{n-1}}$$

$\qquad\langle 4\rangle 3.$ $0 < k < y$

$\qquad\langle 4\rangle 4.$ $y - k$ is an upper bound for E.

$\qquad\quad\langle 5\rangle 1.$ LET: $t \geq y - k$

$\qquad\quad\langle 5\rangle 2.$ $y^n - t^n \leq y^n - x$

$\qquad\qquad$ PROOF:
$$y^n - t^n \leq y^n - (y-k)^n$$
$$= (y - (y-k))\sum_{i=0}^{n-1}y^{n-i}(y-k)^i$$
$$= k\sum_{i=0}^{n-1}y^{n-i}(y-k)^i$$
$$\leq kny^{n-1}$$
$$= y^n - x$$

$\qquad\quad\langle 5\rangle 3.$ $t^n \geq x$

$\qquad\quad\langle 5\rangle 4.$ $t \notin E$

$\qquad\langle 4\rangle 5.$ Q.E.D.

$\qquad\quad$ PROOF: This contradicts the fact that y is the least upper bound of E.

$\langle 1\rangle 2.$ If y and y' are positive reals with $y^n = y'^n$ then $y = y'$.

PROOF: Since the function that sends y to y^n is strictly monotone.
☐

Definition 5.25 (nth Root)**.** Given any real number $x > 0$ and positive integer n, the *nth root* of x, denoted $x^{1/n}$, is the unique positive real such that

$$(x^{1/n})^n = x \ .$$

We write $\sqrt{x}$ for $x^{1/2}$.

Proposition 5.26. *Let a and b be positive real numbers and n a positive integer. Then*

$$(ab)^{1/n} = a^{1/n}b^{1/n} \ .$$

PROOF: Since $(a^{1/n}b^{1/n})^n = ab$. ☐

Lemma 5.27. *Let b be a real number with $b > 1$. Let n be a positive integer. Then*

$$b - 1 \geq n(b^{1/n} - 1) \ .$$

PROOF: From Lemma 4.24. ☐

Lemma 5.28. *Let b and t be real numbers with $b > 1$ and $t > 1$. For any positive integer n, if $n > \frac{b-1}{t-1}$ then $b^{1/n} < t$.*

PROOF:
$$b - 1 \geq n(b^{1/n} - 1)$$
$$\therefore \frac{b-1}{n} \geq b^{1/n} - 1$$
$$\therefore t - 1 > b^{1/n} - 1$$
$$\therefore t > b^{1/n} \qquad\qquad\qquad ☐$$

Lemma 5.29. *Let b be a real number with $b > 0$. Let m, n, p, q be integers with $n > 0$ and $q > 0$. Assume $m/n = p/q$. Then*

$$(b^m)^{1/n} = (b^p)^{1/q} \ .$$

PROOF:
⟨1⟩1. $(b^m)^{1/n} = (b^{1/n})^m$
  PROOF:
$$((b^{1/n})^m)^n = ((b^{1/n})^n)^m$$
$$= b^m$$

⟨1⟩2. $((b^m)^{1/n})^q = b^p$
  PROOF:
$$((b^m)^{1/n})^q = (b^{1/n})^{mq}$$
$$= (b^{1/n})^{np}$$
$$= b^p$$

☐

Definition 5.30. For a a positive real and q a rational number, we may therefore define a^q by

$$a^{m/n} = (a^m)^{1/n}$$

for m and n integers with $n > 0$.

Proposition 5.31. *Let a be a positive real and r, s rational numbers. Then*

$$a^{r+s} = a^r a^s \ .$$

PROOF:

$$\begin{aligned}
a^{m/n+p/q} &= a^{(mq+np)/nq} \\
&= (a^{mq+np})^{1/nq} \\
&= (a^{mq})^{1/nq}(a^{np})^{1/nq} \\
&= a^{m/n}a^{p/q} \qquad\qquad\qquad \square
\end{aligned}$$

Proposition 5.32. *Let $b > 1$ be a real number and q a rational number. Then*

$$b^q = \sup\{b^t : t \in \mathbb{Q}, t \le q\}$$

PROOF: It is the greatest element of this set. $\square$

Definition 5.33. Let $b > 1$ be a real number and x a real number. Then

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t \le x\} \ .$$

Lemma 5.34. *Let b, w and y be real numbers with $b > 1$. Assume $b^w < y$. Then there exists a positive integer n such that $b^{w+1/n} < y$.*

PROOF:
⟨1⟩1. LET: $t = yb^{-w}$
⟨1⟩2. PICK a positive integer n such that $n > \frac{b-1}{t-1}$.
⟨1⟩3. $b^{1/n} < t$
   PROOF: Lemma 5.28.
⟨1⟩4. $b^{w+1/n} < y$
$\square$

Lemma 5.35. *Let b, w and y be real numbers with $b > 1$. Assume $b^w > y$. Then there exists a positive integer n such that $b^{w-1/n} < y$.*

PROOF:
⟨1⟩1. LET: $t = b^w/y$
⟨1⟩2. PICK a positive integer n such that $n > \frac{b-1}{t-1}$
⟨1⟩3. $b^{1/n} < t$
   PROOF: Lemma 5.28.
⟨1⟩4. $y < b^{w-1/n}$
$\square$

Proposition 5.36. *For b and x real numbers with b > 1 we have*

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t < x\} \ .$$

PROOF:
$\langle 1 \rangle 1.$ b^x is an upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
$\langle 1 \rangle 2.$ LET: u be any upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
    PROVE: $b^x \le u$
$\langle 1 \rangle 3.$ LET: q be a rational number with $q \le x$.
    PROVE: $b^q \le u$
$\langle 1 \rangle 4.$ ASSUME: for a contradiction $b^q > u$.
$\langle 1 \rangle 5.$ PICK a positive integer n such that $b^{q-1/n} > u$.
  PROOF: Lemma 5.35.
$\langle 1 \rangle 6.$ $b^{q-1/n} \le u$
  PROOF: $\langle 1 \rangle 2$
$\langle 1 \rangle 7.$ Q.E.D.
  PROOF: This contradicts $\langle 1 \rangle 4$.
□

Lemma 5.37. *Let A be a set of positive real numbers with supremum $a > 0$ and B a set of positive real numbers with supremum $b > 0$. Then ab is the supremum of $\{xy : x \in A, y \in B\}$.*

PROOF:
$\langle 1 \rangle 1.$ For all $x \in A$ and $y \in B$ we have $xy \le ab$.
$\langle 1 \rangle 2.$ If u is any upper bound for $\{xy : x \in A, y \in B\}$ then $ab \le u$.
  $\langle 2 \rangle 1.$ LET: u be an upper bound for $\{xy : x \in A, y \in B\}$.
  $\langle 2 \rangle 2.$ For all $x \in A$ we have u/x is an upper bound for B.
  $\langle 2 \rangle 3.$ For all $x \in A$ we have $b \le u/x$
  $\langle 2 \rangle 4.$ For all $x \in A$ we have $x \le u/b$
  $\langle 2 \rangle 5.$ $a \le u/b$
  $\langle 2 \rangle 6.$ $ab \le u$
□

Proposition 5.38. *Let $b, x, y \in \mathbb{R}$ with $b > 1$. Then*

$$b^{x+y} = b^x b^y \ .$$

PROOF:
$\langle 1 \rangle 1.$ For any rational number $q < x + y$, there exist rational numbers $r < x$ and $s < y$ such that $q = r + s$.
  $\langle 2 \rangle 1.$ $q - x < y$
  $\langle 2 \rangle 2.$ PICK a rational t such that $q - x < t < y$
  $\langle 2 \rangle 3.$ $q = t + (q - t)$ and $t < y$, $q - t < x$
$\langle 1 \rangle 2.$ $b^x b^y = b^{x+y}$

PROOF:

$$b^x b^y = \sup\{b^q b^r : q, r \in \mathbb{Q}, q < x, r < y\}$$
$$= \sup\{b^{q+r} : q, r \in \mathbb{Q}, q < x, r < y\}$$
$$= \sup\{b^q : q \in \mathbb{Q}, q < x + y\}$$
$$= b^{x+y}$$

□

5.2.1 Logarithms

Proposition 5.39. *Let b and y be real numbers with $b > 1$ and $y > 0$. There exists a unique real x such that $b^x = y$.*

PROOF:
$\langle 1 \rangle 1$. LET: $x = \sup\{w : b^w < y\}$
      PROVE: $b^x = y$
  $\langle 2 \rangle 1$. $\{w : b^w < y\} \neq \emptyset$
    PROOF: It contains 0.
  $\langle 2 \rangle 2$. $\{w : b^w < y\}$ is bounded above.
    $\langle 3 \rangle 1$. LET: n be the least integer such that
$$n \geq \frac{y - 1}{b - 1}$$
      PROOF: Archimedean property.
    $\langle 3 \rangle 2$. LET: w be a real number with $b^w < y$
        PROVE: $w < n$
    $\langle 3 \rangle 3$. $b^w < n(b - 1) + 1$
    $\langle 3 \rangle 4$. $b^w < b^n$
    $\langle 3 \rangle 5$. $w < n$
$\langle 1 \rangle 2$. $b^x \leq y$
  $\langle 2 \rangle 1$. ASSUME: for a contradiction $b^x > y$
  $\langle 2 \rangle 2$. PICK a positive integer n such that $b^{x-1/n} > y$
    PROOF: Lemma 5.35.
  $\langle 2 \rangle 3$. PICK w such that $x - 1/n < w$ and $b^w < y$
    PROOF: Since $x - 1/n$ is not an upper bound for $\{w : b^w < y\}$.
  $\langle 2 \rangle 4$. $b^{x-1/n} < y$
  $\langle 2 \rangle 5$. Q.E.D.
    PROOF: This contradicts $\langle 2 \rangle 2$.
$\langle 1 \rangle 3$. $b^x \geq y$
  $\langle 2 \rangle 1$. ASSUME: for a contradiction $b^x < y$.
  $\langle 2 \rangle 2$. PICK a positive integer n such that $b^{x+1/n} < y$.
  $\langle 2 \rangle 3$. $x + 1/n \leq x$
  $\langle 2 \rangle 4$. Q.E.D.
    PROOF: This is a contradiction.

□

Definition 5.40 (Logarithm)**.** Let b and y be real numbers with $b > 1$ and $y > 0$. The *logarithm* of y to *base* b, denoted $\log_b y$, is the unique real number

such that

$$b^{\log_b y} = y \ .$$

5.2.2 Intervals

Definition 5.41 (Intervals). Let $a, b \in \mathbb{R}$.

The *open interval* (a, b) is $\{x \in \mathbb{R} : a < x < b\}$.

The *closed interval* $[a, b]$ is $\{x \in \mathbb{R} : a \leq x \leq b\}$.

The *half-open intervals* $[a, b)$ and $(a, b]$ are defined by

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

Proposition 5.42. *Let (I_n) be a sequence of closed intervals with $I_0 \supseteq I_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} I_n$ is nonempty.*

PROOF:

$\langle 1 \rangle 1$. LET: $I_n = [a_n, b_n]$

$\langle 1 \rangle 2$. LET: $x = \sup_n a_n$

   PROOF: $\{a_n : n \in \mathbb{N}\}$ is bounded above by b_0.

$\langle 1 \rangle 3$. $x \in \bigcap_{n=0}^{\infty} I_n$

   PROOF: For all n we have $a_n \leq x \leq b_n$ since b_n is an upper bound for $\{a_n : n \in \mathbb{N}\}$.

☐

Definition 5.43 (k-cell). Let k be a positive integer. A *k-cell* is a subset of $\mathbb{R}^k$ of the form

$$\{\vec{x} \in \mathbb{R}^k : \forall i = 1, \ldots, k . a_i \leq x_i \leq b_i\}$$

for some real numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$ with $a_i \leq b_i$ for each i.

Proposition 5.44. *Let (I_n) be a sequence of k-cells such that $I_0 \supseteq I_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$.*

PROOF:

$\langle 1 \rangle 1$. LET: $I_n = J_{n1} \times \cdots \times J_{nk}$ where each J_{ni} is a closed interval.

$\langle 1 \rangle 2$. For $i = 1, \ldots, k$, PICK $a_i \in \bigcap_{n=0}^{\infty} J_{ni}$.

$\langle 1 \rangle 3$. $(a_1, \ldots, a_k) \in \bigcap_{n=0}^{\infty} I_n$

☐

5.2.3 The Cantor Set

Definition 5.45 (Cantor Set). Define a sequence E_n of unions of intervals as follows:

- $E_0 = [0, 1]$

- E_{n+1} is formed from E_n by replacing every interval $[a, b]$ with $[a, (2a+b)/3]$ and $[(a + 2b)/3, b]$.

The *Cantor set* is $\bigcap_{n=0}^{\infty} E_n$.

5.3   The Extended Real Number System

Definition 5.46 (Extended Real Number System)**.** The *extended real number system* is the set $\mathbb{R} \cup \{+\infty, -\infty\}$.

We extend the ordering $\leq$ to the extended reals by defining

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

We extend $+$, $\cdot$ and $/$ to partial operations on the extended real by defining:

$$
\begin{aligned}
x + (+\infty) &= +\infty && (x \in \mathbb{R}) \\
x + (-\infty) &= -\infty && (x \in \mathbb{R}) \\
(+\infty) + x &= +\infty && (x \in \mathbb{R}) \\
(+\infty) + (+\infty) &\text{ is undefined} \\
(+\infty) + (-\infty) &\text{ is undefined} \\
(-\infty) + x &= -\infty && (x \in \mathbb{R}) \\
(-\infty) + (+\infty) &\text{ is undefined} \\
(-\infty) + (-\infty) &\text{ is undefined} \\
x \cdot (+\infty) &= +\infty && (x \in \mathbb{R}) \\
x \cdot (-\infty) &= -\infty && (x \in \mathbb{R}) \\
(+\infty) \cdot x &= +\infty && (x \in \mathbb{R}) \\
(+\infty) \cdot (+\infty) &\text{ is undefined} \\
(+\infty) \cdot (-\infty) &\text{ is undefined} \\
(-\infty) \cdot x &= -\infty && (x \in \mathbb{R}) \\
(-\infty) \cdot (+\infty) &\text{ is undefined} \\
(-\infty) \cdot (-\infty) &\text{ is undefined} \\
x / (+\infty) &= 0 && (x \in \mathbb{R}) \\
x / (-\infty) &= 0 && (x \in \mathbb{R}) \\
(+\infty) / x &\text{ is undefined} && (x \in \mathbb{R}) \\
(+\infty) / (+\infty) &\text{ is undefined} \\
(+\infty) / (-\infty) &\text{ is undefined} \\
(-\infty) / x &\text{ is undefined} && (x \in \mathbb{R}) \\
(-\infty) / (+\infty) &\text{ is undefined} \\
(-\infty) / (-\infty) &\text{ is undefined}
\end{aligned}
$$

Chapter 6

Complex Analysis

Definition 6.1 (Complex Numbers)**.** A *complex number* is a pair of real numbers. We write $\mathbb{C}$ for the set of complex numbers.

Define $+$ and $\cdot$ on $\mathbb{C}$ by:

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac - bd, ad + bc)$$

Theorem 6.2. *The complex numbers form a field.*

Theorem 6.3. *The function that maps a to $(a, 0)$ is an embedding of $\mathbb{R}$ in $\mathbb{C}$.*

Definition 6.4.
$$i = (0, 1)$$

Lemma 6.5.
$$(a, b) = a + ib$$

PROOF: Since $(a, 0) + (0, 1)(b, 0) = (a, b)$. $\square$

Lemma 6.6.
$$i^2 = -1$$

PROOF: Immediate from definitions. $\square$

Corollary 6.6.1. *There is no linear order on $\mathbb{C}$ that makes $\mathbb{C}$ into an ordered field.*

Definition 6.7 (Complex Conjugate)**.** For any complex number z, the *complex conjugate* $\overline{z}$ is defined by

$$\overline{a + ib} = a - ib \qquad (a, b \in \mathbb{R}) \ .$$

Definition 6.8 (Real Part)**.** For any complex number z, the *real part* of z, denoted $\mathrm{Re}(z)$, is defined by

$$\mathrm{Re}(a + ib) = a \qquad (a, b \in \mathbb{R}) \ .$$

Definition 6.9 (Imaginary Part)**.** For any complex number z, the *imaginar part* of z, denoted $\text{Im}(z)$, is defined by

$$\text{Im}(a + ib) = b \qquad (a, b \in \mathbb{R}) \ .$$

Theorem 6.10. *For all* $z, w \in \mathbb{C}$ *we have*

$$\overline{z + w} = \overline{z} + \overline{w} \ .$$

PROOF:
$$
\begin{aligned}
\overline{(a + ib) + (c + id)} &= \overline{(a + c) + i(b + d)} \\
&= (a + c) - i(b + d) \\
&= (a - ib) + (c - id) \\
&= \overline{a + ib} + \overline{c + id}
\end{aligned}
$$
$\square$

Theorem 6.11. *For all* $z, w \in \mathbb{C}$ *we have*

$$\overline{zw} = \overline{z} \cdot \overline{w} \ .$$

PROOF:
$$
\begin{aligned}
\overline{(a + ib)(c + id)} &= \overline{(ac - bd) + i(ad + bc)} \\
&= (ac - bd) - i(ad + bc) \\
&= (a - ib)(c - id) \\
&= \overline{a + ib} \cdot \overline{c + id}
\end{aligned}
$$
$\square$

Theorem 6.12. *For all* $z \in \mathbb{C}$ *we have*

$$\text{Re}(z) = \frac{1}{2}(z + \overline{z}) \ .$$

PROOF:
$$
\begin{aligned}
(a + ib) + \overline{a + ib} &= (a + ib) + (a - ib) \\
&= 2a \\
&= 2\,\text{Re}(a + ib)
\end{aligned}
$$
$\square$

Theorem 6.13. *For all* $z \in \mathbb{C}$ *we have*

$$\text{Im}(z) = \frac{1}{2i}(z - \overline{z}) \ .$$

PROOF:
$$
\begin{aligned}
(a + ib) - \overline{a + ib} &= (a + ib) - (a - ib) \\
&= 2ib \\
&= 2i\,\text{Im}(a + ib)
\end{aligned}
$$
$\square$

Theorem 6.14. *For all* $z \in \mathbb{C}$ *we have* $z\overline{z}$ *is a non-negative real.*

PROOF:
$$(a + ib)(\overline{a + ib}) = (a + ib)(a - ib)$$
$$= a^2 + b^2 \qquad \qquad \Box$$

Theorem 6.15. *For any $z \in \mathbb{C}$, if $z\overline{z} = 0$ then $z = 0$.*

PROOF: Let $z = a + ib$. Then $z\overline{z} = a^2 + b^2 = 0$ iff $a = b = 0$. $\Box$

Definition 6.16 (Absolute Value). For $z \in \mathbb{C}$, the *absolute value* of z is

$$|z| = (z\overline{z})^{1/2} \ .$$

Proposition 6.17. *For x a non-negative real we have $|x| = x$.*

PROOF: Since $|x| = \sqrt{x^2} = x$. $\Box$

Proposition 6.18. *For x a negative real we have $|x| = -x$.*

PROOF: Since $|x| = \sqrt{x^2} = -x$. $\Box$

Theorem 6.19. *For any complex number z we have $|z| \geq 0$.*

PROOF: Immediate from definition. $\Box$

Theorem 6.20. *For any complex number z, if $|z| = 0$ then $z = 0$.*

PROOF: From Theorem 6.15. $\Box$

Theorem 6.21. *For any complex number z we have*

$$|\overline{z}| = |z| \ .$$

PROOF: Immediate from definitions. $\Box$

Theorem 6.22. *For any complex numbers z and w we have*

$$|zw| = |z||w| \ .$$

PROOF:
$$|zw| = \sqrt{zw\overline{zw}}$$
$$= \sqrt{z\overline{z}}\sqrt{w\overline{w}} \qquad \text{(Proposition 5.26)}$$
$$= |z||w| \qquad \qquad \Box$$

Theorem 6.23. *For any complex number z we have*

$$|\operatorname{Re} z| \leq |z|$$

PROOF: Let $z = a + ib$. Then
$$|\operatorname{Re} z| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} \ . \Box$$

Theorem 6.24. *For any complex numbers z and w we have*

$$|z + w| \leq |z| + |w| \ .$$

Proof:

$$|z + w|^2 = (z + w)(\overline{z} + \overline{w})$$
$$= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w}$$
$$= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 \qquad \text{(Theorem 6.12)}$$
$$\leq |z|^2 + 2|z\overline{w}| + |w|^2 \qquad \text{(Theorem 6.23)}$$
$$= |z|^2 + 2|z||w| + |w|^2 \qquad \text{(Theorem 6.22)}$$
$$= (|z| + |w|)^2 \qquad\qquad\qquad \Box$$

Theorem 6.25 (Schwarz Inequality). *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be complex numbers. Then*

$$\left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 \ .$$

Proof:
$\langle 1 \rangle 1$. Let: $A = \sum_{j=1}^{n} |a_j|^2$
$\langle 1 \rangle 2$. Let: $B = \sum_{j=1}^{n} |b_j|^2$
$\langle 1 \rangle 3$. Let: $C = \sum_{j=1}^{n} a_j \overline{b_j}$
$\langle 1 \rangle 4$. Assume: w.l.o.g. $B > 0$
   Proof: If $B = 0$ then $b_1 = \cdots = b_n = 0$ and both sides of the inequality are 0.
$\langle 1 \rangle 5$. $\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = B(AB - |C|^2)$
   Proof:
$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = \sum_{j=1}^{n}(Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j})$$
$$= B^2 \sum_{j=1}^{n} |a_j|^2 - B\overline{C} \sum_{j=1}^{n} a_j \overline{b_j} - BC \sum_{j=1}^{n} \overline{a_j} b_j + |C|^2 \sum_{j=1}^{n} |b_j|^2$$
$$= B^2 A - 2B|C|^2 + B|C|^2$$
$$= B(AB - |C|^2)$$
$\langle 1 \rangle 6$. $B(AB - |C|^2) \geq 0$
$\langle 1 \rangle 7$. $AB \geq |C|^2$
$\Box$

Proposition 6.26. *For any non-zero complex number w, there are exactly two complex numbers z such that $z^2 = w$.*

Proof:
$\langle 1 \rangle 1$. There are at most two complex numbers z such that $z^2 = w$.
   Proof: Proposition 4.15.
$\langle 1 \rangle 2$. There are at least two complex numbers z such that $z^2 = w$.
   $\langle 2 \rangle 1$. Let: $w = u + iv$
   $\langle 2 \rangle 2$. Let: $a = \sqrt{\frac{|w| + u}{2}}$
   $\langle 2 \rangle 3$. Let: $b = \sqrt{\frac{|w| - u}{2}}$

$\langle 2\rangle 4$. CASE: $v \geq 0$
   $\langle 3\rangle 1$. LET: $z = a + ib$
   $\langle 3\rangle 2$. $z^2 = w$
      PROOF:

$$\begin{aligned}
z^2 &= (a + ib)^2 \\
&= a^2 - b^2 + 2iab \\
&= u + i\sqrt{|w|^2 - u^2} \\
&= u + iv \\
&= w
\end{aligned}$$

   $\langle 3\rangle 3$. $(-z)^2 = w$
$\langle 2\rangle 5$. CASE: $v \leq 0$
   $\langle 3\rangle 1$. LET: $z = a - ib$
   $\langle 3\rangle 2$. $z^2 = w$
      PROOF:

$$\begin{aligned}
z^2 &= (a - ib)^2 \\
&= a^2 - b^2 - 2iab \\
&= u - i\sqrt{|w|^2 - u^2} \\
&= u - i|v| \\
&= w
\end{aligned}$$

   $\langle 3\rangle 3$. $(-z)^2 = w$

☐

6.1 Algebraic Numbers

Definition 6.27 (Algebraic)**.** A complex number z is *algebraic* iff there exist integers a_0, a_1, ..., a_n not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \ ;$$

otherwise, it is *transcendental.*

Proposition 6.28. *The set of algebraic numbers is countable.*

PROOF: There are countably many finite sequences of integers $(a_0, a_1, \ldots, a_n)$, and for each one, there are only finitely many complex numbers z such that $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$. ☐

Part I

Linear Algebra

Chapter 7

Vector Spaces

7.1 Convex Sets

Definition 7.1 (Convex). Let $E \subseteq \mathbb{R}^k$. Then E is *convex* iff, for all $\vec{x}, \vec{y} \in E$ and $\lambda \in (0, 1)$,
$$\lambda \vec{x} + (1 - \lambda)\vec{y} \in E \ .$$

Proposition 7.2. *Every k-cell is convex.*

PROOF:
⟨1⟩1. LET: $C = \{\vec{x} \in \mathbb{R}^k : \forall i.a_i \leq x_i \leq b_i\}$ be a k-cell.
⟨1⟩2. LET: $\vec{x}, \vec{y} \in C$ and $\lambda \in (0, 1)$.
      PROVE: $\lambda \vec{x} + (1 - \lambda)\vec{y} \in C$
⟨1⟩3. For each i we have $a_i \leq \lambda x_i + (1 - \lambda)y_i \leq b_i$
   PROOF: Since $\lambda a_1 + (1 - \lambda)a_i \leq \lambda x_i + (1 - \lambda)y_i \leq \lambda b_i + (1 - \lambda)b_i$.
▮

Chapter 8

Real Inner Product Spaces

Definition 8.1 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^k$, define the *inner product $\vec{x} \cdot \vec{y}$*
by
$$(x_1, \ldots, x_k) \cdot (y_1, \ldots, y_k) = x_1 y_1 + \cdots + x_k y_k \ .$$

Definition 8.2 (Norm)**.** Define the *norm* of a vector $\vec{x} \in \mathbb{R}^k$ by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} \ .$$

Proposition 8.3.
$$\|\vec{x}\| \geq 0$$

PROOF: Immediate from the definition. $\square$

Proposition 8.4. *If* $\|\vec{x}\| = 0$ *then* $\vec{x} = \vec{0}$.

PROOF: If $\|\vec{x}\| = 0$ then $x_1^2 + \cdots + x_n^2 = 0$ so $x_1 = \cdots = x_n = 0$. $\square$

Proposition 8.5. *For* $\alpha \in \mathbb{R}$ *and* $\vec{x} \in \mathbb{R}^k$,

$$\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| \ .$$

PROOF: Easy. $\square$

Proposition 8.6. *For* $\vec{x}, \vec{y} \in \mathbb{R}^k$, *we have*

$$\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\| \|\vec{y}\| \ .$$

PROOF: By the Schwarz inequality. $\square$

Proposition 8.7. *For* $\vec{x}, \vec{y} \in \mathbb{R}^k$ *we have*

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \ .$$

PROOF:
$$\begin{aligned}
\|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
&= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 \qquad \text{(Proposition 8.6)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 \qquad\qquad\qquad\qquad \square
\end{aligned}$$

Corollary 8.7.1. *For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ we have*

$$\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \ .$$

8.1   Balls

Definition 8.8 (Closed Ball). Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *closed ball* with *centre* $\vec{x}$ and *radius* r is

$$\{y \in \mathbb{R}^k : \|y - x\| \leq r\} \ .$$

Proposition 8.9. *Every closed ball is convex.*

PROOF:
$\langle 1 \rangle 1.$ LET: B be the closed ball with center $\vec{a}$ and radius r.
$\langle 1 \rangle 2.$ LET: $\vec{x}, \vec{y} \in B$
$\langle 1 \rangle 3.$ LET: $\lambda \in (0, 1)$
$\langle 1 \rangle 4.$ $\lambda \vec{x} + (1 - \lambda)\vec{y} \in B$
   PROOF:
$$\begin{aligned}
\|\lambda \vec{x} + (1 - \lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1 - \lambda)(\vec{y} - \vec{a})\| \\
&= \lambda \|\vec{x} - \vec{a}\| + (1 - \lambda)\|\vec{y} - \vec{a}\| \\
&\leq \lambda r + (1 - \lambda)r \\
&= r \qquad\qquad\qquad\qquad\qquad \square
\end{aligned}$$
$\square$

Chapter 9

Complex Inner Product Spaces

Definition 9.1 (Inner Product). Let V be a complex vector space. An *inner product* on V is a function $\langle\ ,\ \rangle : V^2 \to \mathbb{C}$ such that, for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

- $\langle y, x \rangle = \overline{\langle x, y \rangle}$

- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

- $\langle \alpha x, y \rangle = \overline{\alpha} \langle x, y \rangle$

- $\langle x, x \rangle \geq 0$

- If $\langle x, x \rangle = 0$ then $x = 0$.

An *inner product space* consists of a complex vector space V and an inner product on V.

Definition 9.2 (Norm). Let V be an inner product space and $x \in V$. The *norm* of x is

$$\|x\| = \sqrt{\langle x, x \rangle}\ .$$

Proposition 9.3. *An inner product space is a metric space under*

$$d(x, y) = \|x - y\|\ .$$

Definition 9.4 (Bounded). Let V_1 and V_2 be inner product spaces and $T : V_1 \to V_2$ a linear transformation. Then T is *bounded* iff $\{\|T(x)\| : \|x\| = 1\}$ is bounded above.

Proposition 9.5. *Every linear transformation between finite dimensional inner product spaces is bounded.*

Definition 9.6 (Outer Product). Let V be an inner product space and $|\psi\rangle, |\phi\rangle \in V$. The *outer product* of $|\psi\rangle$ and $|\phi\rangle$ is

$$|\psi\rangle \langle\phi| : V \to V\ .$$

9.1 Hilbert Spaces

Definition 9.7 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Theorem 9.8 (Completeness Relation)**.** *Let $\mathcal{H}$ be a Hilbert space. Let $\{|e_n\rangle\}_{n\in\mathbb{N}}$ be a countable orthonormal basis for $\mathcal{H}$. Then*

$$\sum_{n=0}^{\infty} |e_n\rangle \langle e_n| = I \ .$$

PROOF:
$\langle 1 \rangle 1.$ LET: $|\psi\rangle \in \mathcal{H}$
$\langle 1 \rangle 2.$ LET: $|\psi\rangle = \sum_{n=0}^{\infty} \alpha_n |e_n\rangle$
$\langle 1 \rangle 3.$ $\sum_{n=0}^{\infty} \langle e_n|\phi\rangle |e_n\rangle = |\psi\rangle$
  PROOF:

$$\sum_{n=0}^{\infty} \langle e_n|\phi\rangle |e_n\rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_m \langle e_n|e_m\rangle |e_n\rangle$$
$$= \sum_{n=0}^{\infty} \alpha_n |e_n\rangle$$
$$= |\psi\rangle \qquad\qquad \square$$

$\square$

Definition 9.9 (Separable)**.** A Hilbert space is *separable* iff it has a countable dense orthonormal basis.

Chapter 10

Lie Algebras

Definition 10.1 (Lie Algebra)**.** Let K be a field. A *Lie algebra* $\mathcal{L}$ over K consists of a vector space $\mathcal{L}$ over K and an operation

$$[\ ,\ ] : \mathcal{L}^2 \to \mathcal{L}\ ,$$

the *Lie bracket* or *commutator*, such that, for all $x, y, z \in \mathcal{L}$ and $\alpha \in K$:

$$[x + y, z] = [x, z] + [y, z]$$
$$[x, y + z] = [x, y] + [x, z]$$
$$[\alpha x, y] = \alpha[x, y]$$
$$[x, x] = 0$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{(Jacobi identity)}$$

Lemma 10.2. *If K has characteristic 0 then the condition $[x, x] = 0$ can be replaced with $[x, y] = -[y, x]$.*

Proposition 10.3. *The commutator is determind by its values on any basis for $\mathcal{L}$.*

Example 10.4. $\mathbb{R}^3$ with the cross product is a real Lie algebra.

Example 10.5. For any $n \geq 0$, we have $GL(n, K)$ is a Lie algebra over K under

$$[A, B] = AB - BA\ .$$

Definition 10.6 (Linear Lie Algebra)**.** A *linear Lie algebra* over K is a Lie algebra over K that is a subalgebra of $GL(n, K)$ for some n.

Example 10.7 (Special Linear Algebra)**.** The *special Linear algebra $SL(n, \mathbb{R}) =$* $\{A \in GL(n, \mathbb{R}) : \text{tr} = 0\}$ is a real linear Lie algebra.

Example 10.8 (Orthogonal Lie Algebra)**.** The *orthogonal Lie algebra $SO(n, \mathbb{R}) =$* $\{A \in GL(n, \mathbb{R}) : A \text{ is skew-symmetric}\}$ is a real linear Lie algebra.

Example 10.9. Let $u(n)$ be the set of all skew-Hermitian $n \times n$-matrices as a real Lie algebra.

Let $su(n) = u(n) \cap SL(n, \mathbb{R})$.

Proposition 10.10. $SU(2)$ *is spanned by the* Pauli matrices

$$\sigma_x = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_y = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which satisfy

$$[\sigma_x, \sigma_y] = \sigma_z$$
$$[\sigma_y, \sigma_z] = \sigma_x$$
$$[\sigma_z, \sigma_x] = \sigma_y$$

10.1   Lie Algebar Homomorphisms

Definition 10.11 (Homomorphism)**.** Let L_1 and L_2 be Lie algebras over the same field. A *Lie algebra homomorphism* $\phi : L_1 \to L_2$ is a linear transformation such that

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in L_1$.

Lemma 10.12. *Every bijective Lie algebra homomorphism is an isomorphism.*

Definition 10.13 (Representation)**.** Let L be a real (complex) Lie algebra. A *representation* of L is a Lie algebra homomorphism $L \to GL(n, \mathbb{R})$ $(GL(n, \mathbb{C}))$ for some n.

Example 10.14. The linear transformation $\mathbb{R}^3 \to su(2)$ defined by

$$i \mapsto \sigma_x, j \mapsto \sigma_y, k \mapsto \sigma_z$$

is a representation of $\mathbb{R}^3$.

Part II

Topology

Chapter 11

Metric Spaces

Definition 11.1 (Metric). A *metric* on a set X is a function $d : X^2 \to \mathbb{R}$ such that, for all $x, y, z \in X$:

- $d(x, y) \geq 0$

- $d(x, y) = 0$ iff $x = y$

- $d(x, y) = d(y, x)$

- **Triangle Inequality** $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* X consists of a set X and a metric on X.

Example 11.2. $\mathbb{R}^k$ is a metric space under $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$. The triangle inequality is Corollary 8.7.1.

Example 11.3. For any set X, the *discrete* metric on X is defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Proposition 11.4. *Let* (X, d) *be a metric space and* Y *a subset of* X. *Then* $d \restriction Y^2$ *is a metric on* Y.

PROOF: Easy. □

11.1   Balls

Definition 11.5 (Open Ball). Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *open ball* with *centre* $\vec{x}$ and *radius* r is

$$\{y \in \mathbb{R}^k : \|y - x\| < r\} \ .$$

Proposition 11.6. *Every open ball in* $\mathbb{R}^k$ *is convex.*

PROOF:

⟨1⟩1. LET: B be the open ball with center $\vec{a}$ and radius r.

⟨1⟩2. LET: $\vec{x}, \vec{y} \in B$

⟨1⟩3. LET: $\lambda \in (0, 1)$

⟨1⟩4. $\lambda \vec{x} + (1 - \lambda)\vec{y} \in B$

PROOF:

$$\begin{aligned} \|\lambda \vec{x} + (1 - \lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1 - \lambda)(\vec{y} - \vec{a})\| \\ &= \lambda\|\vec{x} - \vec{a}\| + (1 - \lambda)\|\vec{y} - \vec{a}\| \\ &< \lambda r + (1 - \lambda)r \\ &= r \qquad\qquad \square \end{aligned}$$

□

11.2   Limit Points

Definition 11.7 (Limit Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is a *limit point* of E iff every open ball with centre p contains a point of E other than p.

Proposition 11.8. *Let X be a metric space. Let $E \subseteq X$. Let p be a limit point of E. Then every neighbourhood of p contains infinitely many points of E.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction N is a neighbourhood of p that contains only finitely many points $q_1, \ldots, q_n$ of $E - \{p\}$.

⟨1⟩2. LET: $r = \min(q_1, \ldots, q_n)$

⟨1⟩3. LET: B be the open ball with centre p and radius r.

⟨1⟩4. B is a neighbourhood of p that contains no points of E other than p.

□

Corollary 11.8.1. *A finite set has no limit points.*

Definition 11.9 (Isolated Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *isolated point* of E iff $p \in E$ and p is not a limit point of E.

11.3   Closed Sets

Definition 11.10 (Closed Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *closed* iff every limit point of E is a member of E.

11.4   Interior Points

Definition 11.11 (Interior Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *interior point* of E iff there exists an open ball B with centre p such that $B \subseteq E$.

Definition 11.12 (Interior)**.** The *interior* of a set E, denoted E°, is the set of all its interior points.

Proposition 11.13. *The interior of E is the largest open set that is included in E.*

PROOF:
$\langle 1 \rangle 1$. LET: I be the interior of E.
$\langle 1 \rangle 2$. I is open.
   $\langle 2 \rangle 1$. LET: $p \in I$
   $\langle 2 \rangle 2$. PICK an open ball B with centre p such that $B \subseteq E$.
   $\langle 2 \rangle 3$. $B \subseteq I$
      $\langle 3 \rangle 1$. LET: $q \in B$
      $\langle 3 \rangle 2$. There exists an open ball B' with centre q such that $B' \subseteq B$.
      $\langle 3 \rangle 3$. There exists an open ball B' with centre q such that $B' \subseteq E$.
      $\langle 3 \rangle 4$. $q \in I$
$\langle 1 \rangle 3$. If J is any open set and $J \subseteq E$ then $J \subseteq I$.
   $\langle 2 \rangle 1$. LET: J be an open set.
   $\langle 2 \rangle 2$. ASSUME: $J \subseteq E$
   $\langle 2 \rangle 3$. For all $p \in J$, there exists an open ball B with centre p such that $B \subseteq J$.
   $\langle 2 \rangle 4$. For all $p \in J$, there exists an open ball B with centre p such that $B \subseteq E$.
   $\langle 2 \rangle 5$. $p \in I$
☐

11.5 Open Sets

Definition 11.14 (Open Sets)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *open* iff every point in E is an interior point of E.

Proposition 11.15. *Every open ball is open.*

PROOF:
$\langle 1 \rangle 1$. LET: B be an open ball with centre c and radius r.
$\langle 1 \rangle 2$. LET: $x \in B$
$\langle 1 \rangle 3$. LET: $\epsilon = r - d(x, c)$
$\langle 1 \rangle 4$. LET: B' be the open ball with centre x and radius ϵ.
     PROVE: $B' \subseteq B$
$\langle 1 \rangle 5$. LET: $y \in B'$
$\langle 1 \rangle 6$. $d(y, c) < r$
   PROOF:

$$
\begin{aligned}
d(y, c) &\leq d(y, x) + d(x, c) && \text{(Triangle Inequality)} \\
&< \epsilon + d(x, c) && (\langle 1 \rangle 5) \\
&= r && (\langle 1 \rangle 3)
\end{aligned}
$$

☐

Proposition 11.16. *A set is open if and only if its complement is closed.*

PROOF:

$\langle 1 \rangle$1. LET: $E \subseteq X$

$\langle 1 \rangle$2. If E is open then $X - E$ is closed.

$\quad$ $\langle 2 \rangle$1. ASSUME: E is open.

$\quad$ $\langle 2 \rangle$2. LET: p be a limit point of $X - E$.

$\quad\quad$ PROVE: $p \in X - E$

$\quad$ $\langle 2 \rangle$3. ASSUME: for a contradiction $p \in E$.

$\quad$ $\langle 2 \rangle$4. PICK an open ball B with centre p such that $B \subseteq E$.

$\quad$ $\langle 2 \rangle$5. B contains a point of $X - E$.

$\quad\quad$ PROOF: $\langle 2 \rangle$2

$\quad$ $\langle 2 \rangle$6. Q.E.D.

$\quad\quad$ PROOF: This contradicts $\langle 2 \rangle$4.

$\langle 1 \rangle$3. If $X - E$ is closed then E is open.

$\quad$ $\langle 2 \rangle$1. ASSUME: $X - E$ is closed.

$\quad$ $\langle 2 \rangle$2. LET: $p \in E$

$\quad$ $\langle 2 \rangle$3. ASSUME: for a contradiction no open ball with centre p is a subset of $\quad\quad\quad$ E.

$\quad$ $\langle 2 \rangle$4. Every open ball with centre p intersects $X - E$.

$\quad$ $\langle 2 \rangle$5. p is a limit point of $X - E$.

$\quad$ $\langle 2 \rangle$6. $p \in X - E$

$\quad\quad$ PROOF: $\langle 2 \rangle$1

$\quad$ $\langle 2 \rangle$7. Q.E.D.

$\quad\quad$ PROOF: This contradicts $\langle 2 \rangle$2.

▢

Corollary 11.16.1. *A set is closed if and only if its complement is open.*

Proposition 11.17. *The union of a set of open sets is open.*

PROOF:

$\langle 1 \rangle$1. LET: $\mathcal{U}$ be a set of open sets.

$\langle 1 \rangle$2. LET: $p \in \bigcup \mathcal{U}$

$\langle 1 \rangle$3. PICK $U \in \mathcal{U}$ such that $p \in U$.

$\langle 1 \rangle$4. PICK an open ball B with centre p such that $B \subseteq U$.

$\langle 1 \rangle$5. $B \subseteq \bigcup \mathcal{U}$

▢

Corollary 11.17.1. *The intersection of a set of closed sets is closed.*

Proposition 11.18. *The intersection of two open sets is open.*

PROOF:

$\langle 1 \rangle$1. LET: U and V be open.

$\langle 1 \rangle$2. LET: $p \in U \cap V$

$\langle 1 \rangle$3. PICK open balls B_1 and B_2 with centre p such that $B_1 \subseteq U$ and $B_2 \subseteq V$.

$\langle 1 \rangle$4. ASSUME: w.l.o.g. the radius of B_1 is $\leq$ the radius of B_2.

$\langle 1 \rangle$5. $B_1 \subseteq U \cap V$

▢

Corollary 11.18.1. *The union of two closed sets is closed.*

Example 11.19. The intersection of a set of open sets is not necessarily open.

For every positive integer n, we have $(-1/n, 1/n)$ is open in $\mathbb{R}$, but $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$ is not open.

Theorem 11.20. *Let X be a metric space. Let $Y \subseteq X$ and $E \subseteq Y$. Then E is open in Y if and only if there exists an open subset G of X such that $E = G \cap Y$.*

PROOF:
$\langle 1 \rangle 1$. If E is open in Y then there exists an open subset G of X such that $E = G \cap Y$.
  $\langle 2 \rangle 1$. ASSUME: E is open in Y.
  $\langle 2 \rangle 2$. For $p \in E$, PICK $r_p > 0$ such that the open ball in Y with centre p and radius r_p is included in E.
  $\langle 2 \rangle 3$. For $p \in E$,
      LET: V_p be the open ball in X with centre p and radius r_p.
  $\langle 2 \rangle 4$. LET: $G = \bigcup_{p \in E} V_p$
  $\langle 2 \rangle 5$. G is open in Y.
    PROOF: Proposition 11.17.
  $\langle 2 \rangle 6$. $E = G \cap Y$
    $\langle 3 \rangle 1$. $E \subseteq G \cap Y$
      $\langle 4 \rangle 1$. LET: $p \in E$
      $\langle 4 \rangle 2$. $p \in V_p$
      $\langle 4 \rangle 3$. $p \in G$
    $\langle 3 \rangle 2$. $G \cap Y \subseteq E$
      $\langle 4 \rangle 1$. LET: $x \in G \cap Y$
      $\langle 4 \rangle 2$. PICK $p \in E$ such that $x \in V_p$
      $\langle 4 \rangle 3$. $d(x, p) < r_p$
      $\langle 4 \rangle 4$. $x \in E$
$\langle 1 \rangle 2$. For any open subset G of X, we have $G \cap Y$ is open in Y.
  $\langle 2 \rangle 1$. LET: G be an open subset of X.
  $\langle 2 \rangle 2$. LET: $p \in G \cap Y$
  $\langle 2 \rangle 3$. PICK $r > 0$ such that the open ball in X with centre p and radius r is included in G.
  $\langle 2 \rangle 4$. The open ball in Y with centre p and radius r is included in $G \cap Y$.
□

11.6 Perfect Sets

Definition 11.21 (Perfect Set). Let X be a metric space. Let $E \subseteq X$. Then E is *perfect* iff E is closed and every point in E is a limit point of E.

11.7 Bounded Sets

Definition 11.22 (Bounded Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *bounded* iff there exists a real number M and $q \in X$ such that, for all $p \in E$, we have $d(p, q) < M$.

Definition 11.23 (Diameter)**.** Let X be a metric space and $E \subseteq X$ be bounded. Then the *diameter* of E is $\sup\{d(x, y) : x, y \in E\}$.

Proposition 11.24. *Let X be a metric space. Let $E \subseteq X$ be bounded. Then $\overline{E}$ is bounded and*

$$\operatorname{diam} \overline{E} = \operatorname{diam} E \ .$$

Proof:
⟨1⟩1. $\operatorname{diam} E$ is an upper bound for $\{d(x, y) : x, y \in \overline{E}\}$.
  ⟨2⟩1. Let: $x, y \in \overline{E}$
  ⟨2⟩2. For all $\epsilon > 0$ we have $d(x, y) < \operatorname{diam} E + \epsilon$.
    ⟨3⟩1. Let: $\epsilon > 0$
    ⟨3⟩2. Pick $x', y' \in E$ such that $d(x', x) < \epsilon/2$ and $d(y', y) < \epsilon/2$
    ⟨3⟩3. $d(x', y') < \operatorname{diam} E$
    ⟨3⟩4. $d(x, y) < \operatorname{diam} E + \epsilon$
  ⟨2⟩3. $d(x, y) \leq \operatorname{diam} E$
⟨1⟩2. $\operatorname{diam} \overline{E}$ is an upper bound for $\{d(x, y) : x, y \in E\}$.
  Proof: This follows since $E \subseteq \overline{E}$.
□

11.8 Dense Sets

Definition 11.25 (Dense Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *dense* iff every point of X is either a limit point of E or a point of E, or both.

11.9 Closure

Definition 11.26 (Closure)**.** Let X be a metric space. Let $E \subseteq X$. Then the *closure* of E, denoted $\overline{E}$, is the union of E and the set of limit points of E.

Proposition 11.27. *$\overline{E}$ is the smallest closed set that includes E.*

Proof:
⟨1⟩1. $\overline{E}$ is closed.
  ⟨2⟩1. Let: p be a limit point of $\overline{E}$.
  ⟨2⟩2. Assume: $p \notin E$
    Prove: p is a limit point of E.
  ⟨2⟩3. Let: B be the open ball with centre p and radius r.
    Prove: B intersects E.
  ⟨2⟩4. Pick a point $q \in B \cap \overline{E}$.
  ⟨2⟩5. Pick an open ball B' with centre q such that $B' \subseteq B$.

$\langle 2 \rangle$6. PICK a point $r \in E \cap B'$
$\langle 2 \rangle$7. $r \in E \cap B$
$\langle 1 \rangle$2. If C is closed and $E \subseteq C$ then $\overline{E} \subseteq C$.
   $\langle 2 \rangle$1. ASSUME: C is closed.
   $\langle 2 \rangle$2. ASSUME: $E \subseteq C$
   $\langle 2 \rangle$3. LET: $p \in \overline{E}$
   $\langle 2 \rangle$4. ASSUME: for a contradiction $p \notin C$
   $\langle 2 \rangle$5. p is a limit point of C.
      $\langle 3 \rangle$1. LET: B be an open ball with centre p.
      $\langle 3 \rangle$2. B intersects E.
      $\langle 3 \rangle$3. B intersects C.
      $\langle 3 \rangle$4. B intersects C in a point other than p.
        PROOF: $\langle 2 \rangle$3
   $\langle 2 \rangle$6. Q.E.D.
     PROOF: This contradicts $\langle 2 \rangle$1.
☐

Corollary 11.27.1. *E is closed if and only if $E = \overline{E}$.*

Theorem 11.28. *Let E be a nonempty set of real numbers bounded above. Then* $\sup E \in \overline{E}$.

PROOF:
$\langle 1 \rangle$1. ASSUME: $\sup E \notin E$
     PROVE: $\sup E$ is a limit point of E.
$\langle 1 \rangle$2. LET: B be an open ball with centre $\sup E$ and radius r.
$\langle 1 \rangle$3. There exists $x \in E$ such that $x > \sup E - r$.
$\langle 1 \rangle$4. E intersects B in a point other than p.
☐

Proposition 11.29.
$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
$\langle 1 \rangle$1. $\overline{A} \cup \overline{B}$ is a closed set that includes $A \cup B$.
$\langle 1 \rangle$2. If C is a closed set that includes $A \cup B$ then $\overline{A} \cup \overline{B} \subseteq C$.
☐

Example 11.30. It is not true in general. that $\overline{\bigcup \mathcal{A}} = \bigcup_{A \in \mathcal{A}} \overline{A}$.
   In $\mathbb{R}$, let $\mathcal{A} = \{\{1/n\} : n \in \mathbb{Z}^+\}$. Then

$$\overline{\bigcup \mathcal{A}} = \{1/n : n \in \mathbb{Z}^+\} \cup \{0\}$$
$$\bigcup_{A \in \mathcal{A}} \overline{A} = \{1/n : n \in \mathbb{Z}^+\}$$

Proposition 11.31.
$$X - E^\circ = \overline{X - E}$$

PROOF:

$$p \in X - E^\circ \Leftrightarrow p \notin E^\circ$$
$$\Leftrightarrow \forall B \text{ an open ball with centre } p. B \not\subseteq E$$
$$\Leftrightarrow \forall B \text{ an open ball with centre } p. B \text{ intersects} X - E$$
$$\Leftrightarrow p \in \overline{X - E} \qquad\qquad\qquad\qquad \Box$$

11.10   Compact Sets

Definition 11.32 (Open Cover). Let X be a metric space. Let $E \subseteq X$. An *open cover* of E is a set $\mathcal{U}$ of open sets such that $E \subseteq \bigcup \mathcal{U}$.

Definition 11.33 (Compact Set). Let X be a metric space. Let $K \subseteq X$. Then K is *compact* iff every open cover of K includes a finite subcover.

Proposition 11.34. *Every finite set is compact.*

PROOF: Easy. $\Box$

Theorem 11.35. *Let X be a metric space. Let $Y \subseteq X$ and $K \subseteq Y$. Then K is compact in Y if and only if K is compact in X.*

PROOF:
$\langle 1\rangle 1.$ If K is compact in Y then K is compact in X.
   $\langle 2\rangle 1.$ ASSUME: K is compact in Y.
   $\langle 2\rangle 2.$ LET: $\mathcal{U}$ be an open cover of K in X.
   $\langle 2\rangle 3.$ $\{U \cap Y : U \in \mathcal{U}\}$ is an open cover of K in Y.
   $\langle 2\rangle 4.$ PICK a finite subcover $\{U_1 \cap Y, \ldots, U_n \cap Y\}$
   $\langle 2\rangle 5.$ $\{U_1, \ldots, U_n\}$ is a finite subset of $\mathcal{U}$ that is an open cover of K is X.
$\langle 1\rangle 2.$ If K is compact in X then K is compact in Y.
   $\langle 2\rangle 1.$ ASSUME: K is compact in X.
   $\langle 2\rangle 2.$ LET: $\mathcal{U}$ be an open cover of K in Y.
   $\langle 2\rangle 3.$ $\{U \text{ open in } X : U \cap Y \in \mathcal{U}\}$ is an open cover of K in X.
   $\langle 2\rangle 4.$ PICK a finite subcover $\{U_1, \ldots, U_n\}$.
   $\langle 2\rangle 5.$ $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ is a subset of $\mathcal{U}$ that is an open cover of E in Y.
$\Box$

Proposition 11.36. *Every compact set is closed.*

PROOF:
$\langle 1\rangle 1.$ LET: E be compact.
$\langle 1\rangle 2.$ LET: $p \in X - E$
       PROVE: There exists an open ball with centre p that is a subset of $X - E$.
$\langle 1\rangle 3.$ For all $q \in E$, there exist disjoint open balls B with centre q and B' with centre p.
$\langle 1\rangle 4.$ The set of open balls B such that there exists a disjoint open ball B' with centre p is an open cover of E.
$\langle 1\rangle 5.$ PICK a finite subcover $\{B_1, \ldots, B_n\}$.

$\langle 1 \rangle 6.$ For $i = 1, \ldots, n$, PICK an open ball B_i' with centre p such that $B_i \cap B_i' = \emptyset$.
$\langle 1 \rangle 7.$ $B_1' \cap \cdots \cap B_n'$ is an open ball with centre p that is a subset of $X - E$.
$\square$

Proposition 11.37. *Every closed subset of a compact set is compact.*

PROOF:
$\langle 1 \rangle 1.$ LET: E be compact and $C \subseteq E$ be closed.
$\langle 1 \rangle 2.$ LET: $\mathcal{U}$ be an open cover of C.
$\langle 1 \rangle 3.$ $\mathcal{U} \cup \{X - C\}$ is an open cover of E.
$\langle 1 \rangle 4.$ PICK a finite subcover $\{U_1, \ldots, U_n\}$ or $\{U_1, \ldots, U_n, X - C\}$.
$\langle 1 \rangle 5.$ $\{U_1, \ldots, U_n\}$ covers C.
$\square$

Corollary 11.37.1. *The intersection of a compact set and a closed set is compact.*

Proposition 11.38. *Let $\mathcal{K}$ be a nonempty set of compact sets. If every nonempty finite subset of $\mathcal{K}$ has nonempty intersection, then $\bigcap \mathcal{K}$ is nonempty.*

PROOF:
$\langle 1 \rangle 1.$ PICK $K \in \mathcal{K}$
$\langle 1 \rangle 2.$ ASSUME: $\bigcap \mathcal{K} = \emptyset$
$\langle 1 \rangle 3.$ $\{X - K' : K' \in \mathcal{K}\}$ is an open cover of K.
$\langle 1 \rangle 4.$ PICK a finite subcover $\{X - K_1, \ldots, X - K_n\}$.
$\langle 1 \rangle 5.$ There exists $p \in K \cap K_1 \cap \cdots \cap K_n$
$\langle 1 \rangle 6.$ Q.E.D.
   PROOF: $\langle 1 \rangle 4$ and $\langle 1 \rangle 5$ form a contradiction.
$\square$

Corollary 11.38.1. *Let (K_n) be a sequence of nonempty compact sets such that $K_0 \supseteq K_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.*

Theorem 11.39. *Let X be a metric space and $E \subseteq X$. Then E is compact if and only if every infinite subset of E has a limit point in E.*

PROOF:
$\langle 1 \rangle 1.$ If E is compact then every infinite subset of E has a limit point in E.
   $\langle 2 \rangle 1.$ ASSUME: E is compact.
   $\langle 2 \rangle 2.$ LET: $A \subseteq E$ be infinite.
   $\langle 2 \rangle 3.$ ASSUME: for a contradiction E has no limit point in K.
   $\langle 2 \rangle 4.$ For all $p \in K$, there exists an open ball B with centre p such that B does not intersect E outside p.
   $\langle 2 \rangle 5.$ The set of open balls that intersect E in at most one point is an open cover for K.
   $\langle 2 \rangle 6.$ PICK a finite subcover $B_1, \ldots, B_n$.
   $\langle 2 \rangle 7.$ E has at most n points.
   $\langle 2 \rangle 8.$ Q.E.D.
     PROOF: This contradicts the fact that E is finite.

$\langle 1 \rangle 2$. If every infinite subset of K has a limit point in K then K is compact.

  $\langle 2 \rangle 1$. ASSUME: Every infinite subset of K has a limit point in K.

  $\langle 2 \rangle 2$. LET: $\mathcal{U}$ be an open cover of K.

  $\langle 2 \rangle 3$. ASSUME: w.l.o.g. $\mathcal{U}$ is countable.

   PROOF: We may replace $\mathcal{U}$ with the set of all open balls B with centres in $\mathbb{Q}^2$ and rational radius such that there exists $U \in \mathcal{U}$ such that $B \subseteq U$.

  $\langle 2 \rangle 4$. PICK an enumeration $\mathcal{U} = \{G_n : n \in \mathbb{N}\}$.

  $\langle 2 \rangle 5$. For $n \in \mathbb{N}$,
       LET: $F_n = \bigcup_{i=0}^{n} G_n$.

  $\langle 2 \rangle 6$. For all $n \in \mathbb{N}$, we have $K - F_n \neq \emptyset$.

   PROOF: Since $\{G_0, \ldots, G_n\}$ does not cover K.

  $\langle 2 \rangle 7$. $\bigcap_{n=0}^{\infty} F_n = \emptyset$

   PROOF: Since $\{G_n : n \in \mathbb{N}\}$ covers K.

  $\langle 2 \rangle 8$. For $n \in \mathbb{N}$, PICK $a_n \in K - F_n$

  $\langle 2 \rangle 9$. LET: $E = \{a_n : n \in \mathbb{N}\}$

  $\langle 2 \rangle 10$. E is infinite.

   $\langle 3 \rangle 1$. LET: $n \in \mathbb{N}$
        PROVE: there exists m such that $a_m \notin \{a_0, a_1, \ldots, a_n\}$.

   $\langle 3 \rangle 2$. For $i = 0, \ldots, n$, PICK k_i such that $a_i \in G_{k_i}$.

   $\langle 3 \rangle 3$. LET: $m = \max(k_0, \ldots, k_n)$

   $\langle 3 \rangle 4$. ASSUME: for a contradiction $a_m = a_i$ for some $i = 0, \ldots, n$

   $\langle 3 \rangle 5$. $a_i \in G_{k_i}$

   $\langle 3 \rangle 6$. $a_i \notin F_m$

   $\langle 3 \rangle 7$. Q.E.D.
      PROOF: This is a contradiction since $k_i \leq m$.

  $\langle 2 \rangle 11$. PICK a limit point l for E in K.

   PROOF: From $\langle 2 \rangle 1$.

  $\langle 2 \rangle 12$. PICK n such that $l \in G_n$.

  $\langle 2 \rangle 13$. PICK an open ball B with centre l such that $B \subseteq G_n$

  $\langle 2 \rangle 14$. $B \cap E$ is infinite.

   PROOF: Proposition 11.8.

  $\langle 2 \rangle 15$. PICK $m \geq$ 'n such that $a_m \in B$.

  $\langle 2 \rangle 16$. $a_m \in G_n$

  $\langle 2 \rangle 17$. Q.E.D.

   PROOF: This is a contradiction since $a_m \notin F_m$.

$\square$

Theorem 11.40 (Heine-Borel)**.** *Let $E \subseteq \mathbb{R}^k$. Then E is compact if and only if it is closed and bounded.*

PROOF:

$\langle 1 \rangle 1$. If E is compact then E is closed.

  PROOF: Proposition 11.36.

$\langle 1 \rangle 2$. If E is compact then E is bounded.

  PROOF: Otherwise $\{(-N, N)^k : N \in \mathbb{Z}^+\}$ would be an open cover of E with no finite subcover.

$\langle 1 \rangle 3$. If E is closed and bounded then E is compact.

$\langle 2 \rangle 1$. ASSUME: E is closed and bounded.

$\langle 2 \rangle 2$. PICK $\vec{c}$ and M such that $\forall \vec{x} \in E. \lVert \vec{x} - \vec{c} \rVert < M$.

$\langle 2 \rangle 3$. $E \subseteq \prod_{i=1}^{k} [c_i - M, c_i + M]$

$\langle 2 \rangle 4$. E is compact.

  PROOF: Proposition 11.37.

☐

Corollary 11.40.1 (Weierstrass's Theorem). *Every bounded infinite subset of* $\mathbb{R}^k$ *has a limit point.*

PROOF: It is a bounded infinite subset of some k-cell and therefore has a limit point in that k-cell. ☐

Example 11.41. It is not true that, in any metric space, a set is compact if and only if it is closed and bounded.

   In $\mathbb{Q}$, the set $\{p \in \mathbb{Q} : 2 < p^2 < 3\}$ is closed and bounded but not compact.

Theorem 11.42. *Every nonempty perfect set in* $\mathbb{R}^k$ *is uncountable.*

PROOF:

$\langle 1 \rangle 1$. LET: P be a nonempty perfect set in $\mathbb{R}^k$.

$\langle 1 \rangle 2$. P is infinite.

  PROOF: Corollary 11.8.1.

$\langle 1 \rangle 3$. ASSUME: for a contradiction P is countable.

$\langle 1 \rangle 4$. PICK an enumeration $P = \{x_n : n \in \mathbb{N}\}$.

$\langle 1 \rangle 5$. PICK a sequence (V_n) of open balls such that, for all n, we have $\overline{V_{n+1}} \subseteq V_n$ and $x_n \notin \overline{V_{n+1}}$ and $V_n \cap P \neq \emptyset$

  $\langle 2 \rangle 1$. ASSUME: as induction hypothesis we have picked V_0, ..., V_{n-1} that satisfy these conditions.

  $\langle 2 \rangle 2$. PICK $p \in P \cap V_n$ such that $p \neq x_n$

    PROOF: We cannot have $P \cap V_n = \{x_n\}$ because then V_n would be a neighbourhood of x_n that only intersects P at x_n.

  $\langle 2 \rangle 3$. PICK an open ball B with centre p such that $B \subseteq V_n \cap P - \{x_n\}$

  $\langle 2 \rangle 4$. LET: V_{n+1} be the open ball with centre p and half the radius of B.

  $\langle 2 \rangle 5$. $\overline{V_{n+1}} \subseteq V_n$

    PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq V_n$.

  $\langle 2 \rangle 6$. $x_n \notin \overline{V_{n+1}}$

    PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq P - \{x_n\}$.

  $\langle 2 \rangle 7$. $V_{n+1} \cap P \neq \emptyset$

    PROOF: Since $p \in V_{n+1} \cap P$.

$\langle 1 \rangle 6$. For $n \in \mathbb{N}$,

    LET: $K_n = \overline{V_n} \cap P$.

$\langle 1 \rangle 7$. For all $n \in \mathbb{N}$, K_n is compact.

  PROOF: By the Heine-Borel Theorem.

$\langle 1 \rangle 8$. $\bigcap_{n=0}^{\infty} K_n \cap P = \emptyset$

  PROOF: Since for each n we have $x_n \notin K_{n+1}$.

$\langle 1 \rangle 9$. $\bigcap_{n=0}^{\infty} K_n = \emptyset$

  PROOF: Since $\bigcap_{n=0}^{\infty} K_n \subseteq P$.

⟨1⟩10. Q.E.D.
    PROOF: This contradicts Proposition 11.38.
☐

Corollary 11.42.1. *For any $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a, b]$ is uncountable.*

Corollary 11.42.2. $\mathbb{R}$ *is uncountable.*

Corollary 11.42.3. *The set of transcendental numbers is uncountable.*

PROOF: Since the set of algebraic numbers is countable. ☐

Example 11.43. The Cantor set is a perfect set in $\mathbb{R}$ that does not include any open interval.

PROOF:
⟨1⟩1. LET: (E_n) be the sequence of unions of closed intervals from the definition
            of the Cantor set, and C be the Cantor set.
⟨1⟩2. $C \neq \emptyset$
    PROOF: Since $0 \in C$.
⟨1⟩3. C is closed.
    PROOF: Each E_n is closed and C is their intersection.
⟨1⟩4. Every point of C is a limit point of C.
    ⟨2⟩1. LET: $p \in C$
    ⟨2⟩2. LET: B be an open ball with centre p and radius r.
    ⟨2⟩3. PICK n such that each of the intervals that make up E_n has length
        $< r/2$.
    ⟨2⟩4. LET: I be the interval in E_n that contains p.
    ⟨2⟩5. $I \subseteq B$
    ⟨2⟩6. The endpoint of I that is not p is in $P \cap B$.
⟨1⟩5. C does not include any open interval.
    ⟨2⟩1. LET: (α, β) be any open interval.
    ⟨2⟩2. PICK m such that $3^{-m} < (\beta - \alpha)/6$
    ⟨2⟩3. PICK k such that $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq (\alpha, \beta)$
    ⟨2⟩4. $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq P$
    ⟨2⟩5. $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \cap E_m = \emptyset$
    ⟨2⟩6. Q.E.D.
      PROOF: This is a contradiction.
☐

Corollary 11.43.1. *The Cantor set is uncountable.*

Proposition 11.44. *Let X be a metric space. Let (K_n) be a sequence of compact sets in X such that $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$. Assume $\operatorname{diam} K_n \to 0$ as $n \to \infty$. Then $\bigcap_{n=0}^{\infty} K_n$ is a singleton.*

PROOF:
⟨1⟩1. $\bigcap_n K_n \neq \emptyset$

PROOF: Corollary 11.38.1.

$\langle 1 \rangle 2.$ $\bigcap_n K_n$ has no more than one point.

   $\langle 2 \rangle 1.$ ASSUME: for a contradiction $a, b \in \bigcap_n K_n$ with $a \neq b$.

   $\langle 2 \rangle 2.$ LET: $\epsilon = d(a, b)$

   $\langle 2 \rangle 3.$ PICK n such that diam $K_n < \epsilon$

   $\langle 2 \rangle 4.$ $a, b \in K_n$

   $\langle 2 \rangle 5.$ Q.E.D.

     PROOF: This is a contradiction.

$\square$

11.11   Connected Sets

Definition 11.45 (Separated)**.** Let X be a metric space. Let $A, B \subseteq X$. Then A and B are *separated* iff $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Proposition 11.46. *Any two disjoint open sets are separated.*

PROOF:

$\langle 1 \rangle 1.$ LET: A and B be disjoint open sets.

$\langle 1 \rangle 2.$ ASSUME: for a contradiction $p \in \overline{A} \cap B$.

$\langle 1 \rangle 3.$ B is a neighbourhood of p.

$\langle 1 \rangle 4.$ B intersects A.

$\square$

Definition 11.47 (Connected)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *connected* iff E is not the union of two nonempty separated sets.

Theorem 11.48. *A subset E of the real line is connected if and only if it is convex.*

PROOF:

$\langle 1 \rangle 1.$ If E is connected then E is convex.

   $\langle 2 \rangle 1.$ ASSUME: E is connected.

   $\langle 2 \rangle 2.$ LET: $x, y \in E$

   $\langle 2 \rangle 3.$ LET: $z \in (x, y)$

   $\langle 2 \rangle 4.$ $z \in E$

     PROOF: Otherwise $E \cap (-\infty, z)$ and $E \cap (z, +\infty)$ would be a separation of E.

$\langle 1 \rangle 2.$ If E is convex then E is connected.

   $\langle 2 \rangle 1.$ ASSUME: E is convex.

   $\langle 2 \rangle 2.$ ASSUME: for a contradiction $E = A \cup B$ where A and B are nonempty and separated.

   $\langle 2 \rangle 3.$ PICK $a \in A$ and $b \in B$.

   $\langle 2 \rangle 4.$ ASSUME: w.l.o.g. $a < b$

   $\langle 2 \rangle 5.$ LET: $z = \sup(A \cap [a, b])$

   $\langle 2 \rangle 6.$ $z \in \overline{A}$

   $\langle 2 \rangle 7.$ $z \notin B$

$\langle 2 \rangle 8.$ $z < b$
$\langle 2 \rangle 9.$ CASE: $z \in A$
  $\langle 3 \rangle 1.$ $z \notin \overline{B}$
  $\langle 3 \rangle 2.$ PICK $z_1 \in (z, b)$ such that $z_1 \notin B$
  $\langle 3 \rangle 3.$ $a < z_1 < b$
  $\langle 3 \rangle 4.$ $z_1 \notin E$
    PROOF: We have $z_1 \notin A$ from $\langle 2 \rangle 5$ since $z_1 \in [a, b]$ and $z_1 > z$, and
    $z_1 \notin B$ from $\langle 3 \rangle 2$.
  $\langle 3 \rangle 5.$ Q.E.D.
    PROOF: This contradicts $\langle 2 \rangle 1$.
$\langle 2 \rangle 10.$ CASE: $z \notin A$
  PROOF: Then $a < z < b$ and $z \notin E$ contradicting $\langle 2 \rangle 1$.
□

Proposition 11.49. *Every connected metric space with more than one point is uncountable.*

PROOF:
$\langle 1 \rangle 1.$ LET: X be a connected metric space with more than one points.
$\langle 1 \rangle 2.$ PICK distinct points $p, q \in X$.
$\langle 1 \rangle 3.$ LET: $\epsilon = d(p, q)$
$\langle 1 \rangle 4.$ For every $r \in (0, \epsilon)$, there exists a point $x \in X$ such that $d(p, x) = r$.
  PROOF: Otherwise $\{x \in X : d(p, x) < r\}$ and $\{x \in X : d(p, x) > r\}$ would
  form a separation of X.
□

Proposition 11.50. *The closure of a connected set is connected.*

PROOF:
$\langle 1 \rangle 1.$ LET: X be a metric space.
$\langle 1 \rangle 2.$ LET: E be a connected subspace of X.
$\langle 1 \rangle 3.$ ASSUME: for a contradiction A and B form a separation of $\overline{E}$
    PROVE: $A \cap E$ and $B \cap E$ form a separation of E.
$\langle 1 \rangle 4.$ $A \cap E \neq \emptyset$
  $\langle 2 \rangle 1.$ ASSUME: for a contradiction $A \cap E = \emptyset$
  $\langle 2 \rangle 2.$ $E \subseteq B$
  $\langle 2 \rangle 3.$ $\overline{E} \subseteq \overline{B}$
  $\langle 2 \rangle 4.$ $A \subseteq \overline{B}$
  $\langle 2 \rangle 5.$ $A \cap \overline{B} = A \neq \emptyset$
  $\langle 2 \rangle 6.$ Q.E.D.
    PROOF: This contradicts $\langle 1 \rangle 3$.
$\langle 1 \rangle 5.$ $B \cap E \neq \emptyset$
  PROOF: Similar.
$\langle 1 \rangle 6.$ $\overline{A \cap E} \cap B \cap E = \emptyset$
  PROOF: Since $\overline{A \cap E} \cap B \cap E \subseteq \overline{A} \cap B$.
$\langle 1 \rangle 7.$ $A \cap E \cap \overline{B \cap E} = \emptyset$
  PROOF: Similar.

☐

Example 11.51. The interior of a connected set is not necessarily connected.

Two touching discs in $\mathbb{R}^2$ form a connected set but the interior is disconnected.

Proposition 11.52. *Every convex set in $\mathbb{R}^k$ is connected.*

PROOF:
$\langle 1 \rangle 1.$ LET: E be a convex set in $\mathbb{R}^k$.
$\langle 1 \rangle 2.$ ASSUME: for a contradiction A and B form a separation of E.
$\langle 1 \rangle 3.$ PICK $\vec{a} \in A$ and $\vec{b} \in B$.
$\langle 1 \rangle 4.$ Define $p : [0,1] \to \mathbb{R}^k$ by $p(t) = (1-t)\vec{a} + t\vec{b}$.
$\langle 1 \rangle 5.$ $p^{-1}(A)$ and $p^{-1}(B)$ are separated sets in $\mathbb{R}$.
$\langle 1 \rangle 6.$ PICK $x \in [0,1]$ such that $x \notin p^{-1}(A)$ and $x \notin p^{-1}(B)$.
   PROOF: There exists such an x since $[0,1]$ is connected.
$\langle 1 \rangle 7.$ $p(x) \in E$
   PROOF: Since E is convex.
$\langle 1 \rangle 8.$ $p(x) \notin A \cup B$
$\langle 1 \rangle 9.$ Q.E.D.
   PROOF: This contradicts $\langle 1 \rangle 2$.
☐

11.12 Separable Spaces

Definition 11.53 (Separable)**.** A metric space is *separable* iff it has a countable dense subset.

Example 11.54. $\mathbb{R}^k$ is separable since $\mathbb{Q}^k$ is dense.

Proposition 11.55. *Every compact metric space is separable.*

PROOF:
$\langle 1 \rangle 1.$ LET: X be a compact metric space.
$\langle 1 \rangle 2.$ For $n \in \mathbb{Z}^+$, pick finitely many points $a_{n1}, \ldots, a_{nr_n}$ such that $\{B(a_{ni}, 1/n) : 1 \le i \le r_n\}$ covers X.
   PROOF: Since $\{B(x, 1/n) : x \in X\}$ covers X.
$\langle 1 \rangle 3.$ $\{a_{ni} : n \in \mathbb{Z}^+, 1 \le i \le r_n\}$ is dense.
   $\langle 2 \rangle 1.$ LET: U be an open set and $p \in U$.
   $\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$.
   $\langle 2 \rangle 3.$ PICK n such that $1/n < \epsilon$.
   $\langle 2 \rangle 4.$ PICK i such that $p \in B(a_{ni}, 1/n)$
   $\langle 2 \rangle 5.$ $a_{ni} \in U$
☐

11.13   Bases

Definition 11.56 (Basis)**.** A *basis* for a metric space X is a set $\mathcal{B}$ of open sets such that, for every open set U and point $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.

Proposition 11.57. *Every separable metric space has a countable basis.*

PROOF:
⟨1⟩1. LET: X be a separable metric space.
⟨1⟩2. PICK a countable dense set D in X.
⟨1⟩3. LET: $\mathcal{B} = \{B(p, \epsilon) : p \in D, \epsilon \in \mathbb{Q}^+\}$
       PROVE: $\mathcal{B}$ is a basis.
⟨1⟩4. LET: U be an open set in X and $p \in U$
⟨1⟩5. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$
⟨1⟩6. PICK $q \in B(p, \epsilon) \cap D$
⟨1⟩7. PICK a rational δ such that $d(p, q) < \delta < \epsilon$.
⟨1⟩8. $B(q, \delta) \in \mathcal{B}$ and $B(q, \delta) \subseteq U$.
□

11.14   Condensation Points

Definition 11.58 (Condensation Point)**.** Let X be a metric space, $p \in X$ and $E \subseteq X$. Then p is a *condensation point* of E iff every neighbourhood of p contains uncountably many points in E.

Proposition 11.59. *Let X be a metric space. Let $E \subseteq X$. Let P be the set of condensation points of E. Then P is perfect.*

PROOF:
⟨1⟩1. P is closed.
    ⟨2⟩1. LET: $p \in X - P$
    ⟨2⟩2. PICK a neighbourhood U of p that contains only countably many points
          of E.
    ⟨2⟩3. For every $x \in U$, we have that U is a neighbourhood of x that contains
          only countably many points of E.
    ⟨2⟩4. $p \in U \subseteq X - P$
⟨1⟩2. Every point in P is a limit point of P.
    PROOF: Immediate from definitions.
□

Proposition 11.60. *Let X be a metric space with a countable basis. Let $E \subseteq X$ be uncountable. Let P be the set of condensation points of E. Then $E - P$ is countable.*

PROOF:
⟨1⟩1. PICK a countable basis $\mathcal{B}$ for X.
⟨1⟩2. LET: $W = \bigcup\{B \in \mathcal{B} : E \cap B \text{ is countable}\}$

$\langle 1 \rangle 3.$ $P = X - W$

   $\langle 2 \rangle 1.$ $P \subseteq X - W$

      $\langle 3 \rangle 1.$ ASSUME: for a contradiction $p \in P \cap W$

      $\langle 3 \rangle 2.$ PICK $B \in \mathcal{B}$ such that $p \in B$ and $E \cap B$ is countable.

      $\langle 3 \rangle 3.$ $E \cap B$ is uncountable.

      $\langle 3 \rangle 4.$ Q.E.D.

         PROOF: This is a contradiction.

   $\langle 2 \rangle 2.$ $X - W \subseteq P$

      $\langle 3 \rangle 1.$ LET: $p \in X - W$

      $\langle 3 \rangle 2.$ LET: U be a neighbourhood of p.

      $\langle 3 \rangle 3.$ PICK $B \in \mathcal{B}$ such that $p \in B \subseteq U$.

      $\langle 3 \rangle 4.$ $E \cap B$ is uncountable.

         PROOF: Since $p \notin W$.

      $\langle 3 \rangle 5.$ $E \cap W$ is uncountable.

$\langle 1 \rangle 4.$ $E - P = E \cap W$

$\langle 1 \rangle 5.$ $E - P$ is countable.

$\square$

Corollary 11.60.1. *Every closed subset of a metric space with a countable basis is the union of a perfect set and a countable set.*

PROOF:

$\langle 1 \rangle 1.$ LET: X be a metric space with a countable basis.

$\langle 1 \rangle 2.$ LET: E be a closed subset of X.

$\langle 1 \rangle 3.$ LET: P be the set of condensation points of E.

$\langle 1 \rangle 4.$ $E - P$ is countable.

   PROOF: Proposition 11.60.

$\langle 1 \rangle 5.$ $P \cap E$ is perfect.

   $\langle 2 \rangle 1.$ $P \cap E$ is closed.

      PROOF: Proposition 11.59.

   $\langle 2 \rangle 2.$ Every point in $P \cap E$ is a limit point of $P \cap E$.

      $\langle 3 \rangle 1.$ LET: $l \in P \cap E$

      $\langle 3 \rangle 2.$ LET: U be a neighbourhood of l.

      $\langle 3 \rangle 3.$ PICK $x \in P \cap U$

      $\langle 3 \rangle 4.$ U is a neighbourhood of x.

      $\langle 3 \rangle 5.$ U contains uncountably many points of E.

      $\langle 3 \rangle 6.$ U intersects $P \cap E$

         PROOF: It cannot be that every point in U and E is not in P since $E - P$ is countable.

$\square$

Corollary 11.60.2. *Let X be a metric space with a countable basis. Then every countable set in X has an isolated point.*

Chapter 12

Convergence

Definition 12.1 (Converge)**.** Let X be a metric space. Let (p_n) be a sequence in X and $l \in X$. Then we say (p_n) *converges* to the *limit l*, and write

$$p_n \to l \text{ as } n \to \infty \ ,$$

iff for every $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon$.

We say (p_n) *diverges* iff it does not converge to any limit.

Proposition 12.2. *A sequence has at most one limit.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $p_n \to l$ and $p_n \to m$ as $n \to \infty$.
$\langle 1 \rangle 2$. ASSUME: for a contradiction $l \neq m$.
$\langle 1 \rangle 3$. LET: $\epsilon = d(l, m)/2$
$\langle 1 \rangle 4$. There exists N such that $\forall n \geq N . d(p_n, l) < \epsilon$ and $d(p_n, m) < \epsilon$
$\langle 1 \rangle 5$. $d(l, m) < 2\epsilon$
$\langle 1 \rangle 6$. Q.E.D.
   PROOF: This is a contradiction.
□

Proposition 12.3. *Every convergent sequence is bounded.*

PROOF:
$\langle 1 \rangle 1$. LET: $p_n \to l$ as $n \to \infty$
$\langle 1 \rangle 2$. PICK N such that $\forall n \geq N . d(p_n, l) < 1$
$\langle 1 \rangle 3$. LET: $M = \max(d(p_0, l), \ldots, d(p_{N-1}, l), 1)$
$\langle 1 \rangle 4$. For all n, we have $d(p_n, l) \leq M$.
□

Proposition 12.4. *If l is a limit point of E, then there exists a sequence in E that converges to l.*

PROOF:

⟨1⟩1. For $n \in \mathbb{Z}^+$, PICK a point $a_n \in E$ such that $d(a_n, l) < 1/n$.
  PROOF: Since $B(l, 1/n)$ intersects E.
⟨1⟩2. $a_n \to l$ as $n \to \infty$.
☐

Corollary 12.4.1. *Every sequence in a compact metric space has a convergent subsequence.*

PROOF: By Theorem 11.39. ☐

Proposition 12.5. *Assume $s_n \to s$ and $t_n \to t$ in $\mathbb{C}$. Then $s_n + t_n \to s + t$.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N such that, for all $n \geq N$, we have $d(s_n, s) < \epsilon/2$ and $d(t_n, t) < \epsilon/2$.
⟨1⟩3. For all $n \geq N$ we have $d(s_n + t_n, s + t) < \epsilon$.
☐

Lemma 12.6. *If $s_n \to s$ as $n \to \infty$ in $\mathbb{C}$, and $c \in \mathbb{C}$, then $cs_n \to cs$ as $n \to \infty$.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. ASSUME: w.l.o.g. $c \neq 0$
⟨1⟩3. PICK N such that $\forall n \geq N.|s_n - s| < \epsilon/|c|$.
⟨1⟩4. $\forall n \geq N.|cs_n - cs| < \epsilon$
☐

Proposition 12.7. *If $s_n \to s$ and $t_n \to t$ in $\mathbb{C}$ then $s_n t_n \to st$.*

PROOF:
⟨1⟩1. $(s_n - s)(t_n - t) \to 0$ as $n \to \infty$
  ⟨2⟩1. LET: $\epsilon > 0$
  ⟨2⟩2. PICK N such that, for all $n \geq N$, we have $|s_n - s| < \sqrt{\epsilon}$ and $|t_n - t| < \sqrt{\epsilon}$.
  ⟨2⟩3. For all $n \geq N$ we have $|(s_n - s)(t_n - t)| < \epsilon$
⟨1⟩2. $s_n t_n - st \to 0$ as $n \to \infty$
  PROOF:
$$s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$$
$$\to 0 \qquad\qquad\qquad\qquad \text{as } n \to \infty$$
☐

Proposition 12.8. *If $s_n \to s$ as $n \to \infty$ in $\mathbb{C}$, and every s_n and s is nonzero, then $1/s_n \to 1/s$ as $n \to \infty$.*

PROOF:
⟨1⟩1. PICK m such that, for all $n \geq m$, we have $|s_n - s| < \frac{1}{2}|s|$.
⟨1⟩2. $\forall n \geq m.|s_n| > \frac{1}{2}|s|$
⟨1⟩3. LET: $\epsilon > 0$
⟨1⟩4. PICK $N > m$ such that, for all $n \geq N$, we have
$$|s_n - s| < \frac{1}{2}|s|^2 \epsilon \ .$$

$\langle 1 \rangle 5$. For all $n \geq N$, we have

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| < \epsilon \ .$$

PROOF:

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \frac{|s_n - s|}{|s_n||s|}$$

$$< \frac{|s|^2 \epsilon}{2|s_n||s|}$$

$$= \frac{|s|\epsilon}{2|s_n|}$$

$$< \epsilon$$

□

Theorem 12.9. *Let $(\vec{x_n})$ be a sequence in $\mathbb{R}^k$ and $\vec{l} \in \mathbb{R}^k$. Then $\vec{x_n} \to \vec{l}$ as $n \to \infty$ iff, for $i = 1, \ldots, k$, we have $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. If $\vec{x_n} \to \vec{l}$ then $\pi_i(\vec{x_n}) \to \pi_i(l)$.
   $\langle 2 \rangle 1$. $\|\vec{x_n} - \vec{l}\| \to 0$ as $n \to \infty$.
   $\langle 2 \rangle 2$. $\sqrt{\sum_{i=1}^{k}(\pi_i(\vec{x_n}) - \pi_i(l))^2} \to 0$ as $n \to \infty$.
   $\langle 2 \rangle 3$. $\sum_{i=1}^{k}(\pi_i(\vec{x_n}) - \pi_i(l))^2 \to 0$ as $n \to \infty$.
   $\langle 2 \rangle 4$. $(\pi_i(\vec{x_n}) - \pi_i(l))^2 \to 0$ as $n \to \infty$
   $\langle 2 \rangle 5$. $\pi_i(\vec{x_n}) - \pi_i(l) \to 0$ as $n \to \infty$.
$\langle 1 \rangle 2$. If $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ for every i then $\vec{x_n} \to l$.
   $\langle 2 \rangle 1$. ASSUME: $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ for every i.
   $\langle 2 \rangle 2$. $\vec{x_n} \to \vec{l}$
      PROOF:

$$\|\vec{x_n} - \vec{l}\|^2 = \sum_{i=1}^{k}(\pi_i(\vec{x_n}) - \pi_i(\vec{l}))^2$$

$$\to 0$$

□

Corollary 12.9.1. *If $\vec{x_n} \to \vec{x}$ and $\vec{y_n} \to \vec{y}$ in $\mathbb{R}^k$, then $\vec{x_n} + \vec{y_n} \to \vec{x} + \vec{y}$.*

Corollary 12.9.2. *If $\beta_n \to \beta$ in $\mathbb{R}$ and $\vec{x_n} \to \vec{l}$ in $\mathbb{R}^k$, then $\beta_n \vec{x_n} \to \beta\vec{l}$.*

Proposition 12.10. *If $\vec{x_n} \to \vec{x}$ and $\vec{y_n} \to \vec{y}$ in $\mathbb{R}^k$, then $\vec{x_n} \cdot \vec{y_n} \to \vec{x} \cdot \vec{y}$.*

PROOF:

$$\vec{x_n} \cdot \vec{y_n} = \sum_{i=1}^{k} \pi_i(\vec{x_n})\pi_i(\vec{y_n})$$

$$\to \sum_{i=1}^{k} \pi_i(\vec{x})\pi_i(\vec{y})$$

$$= \vec{x} \cdot \vec{y} \qquad\qquad □$$

Proposition 12.11. *Let (p_n) be a sequence in the metric space X. The set E^* of all limits of convergent subsequences is a closed set.*

Proof:

$\langle 1 \rangle 1$. Assume: w.l.o.g. $\{p_n : n \in \mathbb{N}\}$ is infinite.

$\langle 1 \rangle 2$. Let: q be a limit point of E^*.

  Prove: $q \in E^*$

$\langle 1 \rangle 3$. Pick an integer n_0 such that $q \neq p_{n_0}$.

$\langle 1 \rangle 4$. Extend a strictly increasing sequence of integers (n_i) such that, for all i, we have $d(q, p_{n_i}) \leq 2^i d(q, p_{n_0})$.

  $\langle 2 \rangle 1$. Assume: as induction hypothesis we have picked $n_0 < n_1 < \cdots < n_i$ such that, for $0 \leq j \leq i$, we have $d(q, p_{n_j}) \leq 2^j d(q, p_{n_0})$.

  $\langle 2 \rangle 2$. Pick $x \in E^*$ such that $d(x, q) < 2^{-(i+2)} \delta$

  $\langle 2 \rangle 3$. There exists a subsequence of (p_n) that converges to x.

  $\langle 2 \rangle 4$. There exists $n_{i+1} > n_i$ such that $d(p_{n_{i+1}}, x) < 2^{-(i+2)} \delta$.

  $\langle 2 \rangle 5$. $d(p_{n_{i+1}}, q) < 2^{-(i+1)} \delta$

$\langle 1 \rangle 5$. $p_{n_i} \to q$ as $i \to \infty$.

$\langle 1 \rangle 6$. $q \in E^*$

$\square$

Theorem 12.12. *Every monotonically increasing sequence in $\mathbb{R}$ that is bounded above converges to its supremum.*

Proof:

$\langle 1 \rangle 1$. Let: (s_n) be a monotonically increasing sequence with supremum s.

$\langle 1 \rangle 2$. Let: $\epsilon > 0$

$\langle 1 \rangle 3$. Pick S such that $|s_N - s| < \epsilon$

$\langle 1 \rangle 4$. For all $n \geq N$, we have $s - \epsilon < s - s_N \leq s - s_n \leq s$.

$\langle 1 \rangle 5$. $\forall n \geq N.|s_n - s| < \epsilon$

$\square$

Theorem 12.13. *Every monotonically decreasing sequence in $\mathbb{R}$ that is bounded below converges to its infimum.*

Proof: Similar. $\square$

Proposition 12.14 (Sandwich Theorem). *Let (a_n), (b_n) and (c_n) be sequences of real numbers and $l \in \mathbb{R}$. Assume $\forall n.a_n \leq b_n \leq c_n$ and $a_n \to l$ and $c_n \to l$. Then $b_n \to l$.*

Proof:

$\langle 1 \rangle 1$. Let: $\epsilon > 0$

$\langle 1 \rangle 2$. Pick N such that, for all $n \geq N$, we have $|a_n - l| < \epsilon$ and $|c_n - l| < \epsilon$.

$\langle 1 \rangle 3$. $\forall n \geq N.|b_n - l| < \epsilon$

$\square$

Theorem 12.15. *For any real $p > 0$ we have*

$$\frac{1}{(n+1)^p} \to 0$$

as $n \to \infty$.

PROOF:

⟨1⟩1. LET: $\epsilon > 0$

⟨1⟩2. PICK N such that $N > (1/\epsilon)^{1/p}$.

⟨1⟩3. LET: $n \geq N$

⟨1⟩4. $1/n^p < \epsilon$

☐

Theorem 12.16. *For any real $p > 0$ we have*

$$p^{\frac{1}{n+1}} \to 1$$

as $n \to \infty$.

PROOF:

⟨1⟩1. CASE: $p > 1$

  ⟨2⟩1. For $n \in \mathbb{N}$

    LET: $x_n = p^{\frac{1}{n+1}} - 1$.

  ⟨2⟩2. $\forall n \in \mathbb{N}.x_n > 0$

  ⟨2⟩3. $\forall n \in \mathbb{N}$.

$$1 + (n+1)x_n \leq p \ .$$

  PROOF: Since $1 + (n+1)x_n \leq (1+x_n)^{n+1}$ by the Binomial Theorem.

  ⟨2⟩4. $\forall n \in \mathbb{N}$.

$$0 < x_n \leq \frac{p-1}{n+1} \ .$$

  ⟨2⟩5. $x_n \to 0$ as $n \to \infty$.

    PROOF: Sandwich Theorem.

⟨1⟩2. CASE: $p = 1$

  PROOF: Trivial.

⟨1⟩3. CASE: $p < 1$

  PROOF: Then $p^{1/(n+1)} = 1/((1/p)^{1/(n+1)}) \to 1/1 = 1$ by ⟨1⟩1.

☐

Theorem 12.17.

$$(n+1)^{1/(n+1)} \to 1 \ as \ n \to \infty$$

PROOF:

⟨1⟩1. For $n \in \mathbb{N}$,

    LET: $x_n = (n+1)^{1/(n+1)} - 1$.

⟨1⟩2. $\forall n \in \mathbb{N}.x_n \geq 0$

⟨1⟩3. $\forall n \in \mathbb{N}$

$$n + 1 \geq \frac{n(n+1)}{2}x_n^2 \ .$$

  PROOF: Since $(1+x_n)^{n+1} \geq \frac{n(n+1)}{2}x_n^2$ by the Binomial Theorem.

⟨1⟩4. $\forall n \geq 1$

$$0 \leq x_n \leq \sqrt{\frac{2}{n}}$$

⟨1⟩5. $x_n \to 0$ as $n \to \infty$.

  PROOF: Sandwich Theorem.

☐

Theorem 12.18. *Let p and α be real numbers with $p > 0$. Then*

$$\frac{n^{\alpha}}{(1+p)^n} \to 0 \ as \ n \to \infty \ .$$

PROOF:

⟨1⟩1. PICK a positive integer k such that $k > \alpha$.
   PROOF: Archimedean Property.

⟨1⟩2. $\forall n > 2k$

$$(1+p)^n > \frac{n^k p^k}{2^k k!} \ .$$

$$(1+p)^n > \binom{n}{k} p^k \qquad\qquad \text{(Binomial Theorem)}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$$

$$> \frac{n^k p^k}{2^k k!} \qquad\qquad (n > 2k \text{ so if } n-k < i \le n \text{ then } i > n/2)$$

⟨1⟩3. $\forall n > 2k$

$$0 < \frac{n^{\alpha}}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k} \ .$$

⟨1⟩4. $n^{\alpha-k} \to 0$ as $n \to \infty$
   PROOF: Theorem 12.15.

⟨1⟩5. $\frac{n^{\alpha}}{(1+p)^n} \to 0$ as $n \to \infty$.
   PROOF: Sandwich Theorem.

☐

Corollary 12.18.1. *For any real number x with $|x| < 1$ we have $x^n \to 0$ as $n \to \infty$.*

PROOF: Taking $\alpha = 0$. ☐

12.1   Cauchy Sequences

Definition 12.19 (Cauchy Sequence). Let (p_n) be a sequence in the metric space X. Then (p_n) is a *Cauchy sequence* iff, for every $\epsilon > 0$, there exists N such that, for all $m, n \ge N$, we have $d(p_m, p_n) < \epsilon$.

Proposition 12.20. *Let (p_n) be a sequence in the metric space X and let $E_N = \{p_n : n \ge N\}$ for all N. Then (p_n) is a Cauchy sequence if and only if $\operatorname{diam} E_N \to 0$ as $N \to \infty$.*

PROOF: Immediate from definitions. ☐

Theorem 12.21. *Every convergent sequence is Cauchy.*

PROOF:

$\langle 1 \rangle 1$. LET: (p_n) be a convergent sequence with limit l.
$\langle 1 \rangle 2$. LET: $\epsilon > 0$
$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon/2$
$\langle 1 \rangle 4$. $\forall m, n \geq N.d(p_m, p_n) < \epsilon$
□

12.2 Complete Metric Spaces

Definition 12.22 (Complete Metric Space)**.** A metric space is *complete* iff every Cauchy sequence converges.

Theorem 12.23. *Every compact metric space is complete.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a compact metric space.
$\langle 1 \rangle 2$. LET: (p_n) be a Cauchy sequence in X.
$\langle 1 \rangle 3$. For $N \in \mathbb{N}$,
　　 LET: $E_N = \{p_n : n \geq N\}$.
$\langle 1 \rangle 4$. diam $\overline{E_N} \to 0$ as $N \to \infty$.
$\langle 1 \rangle 5$. For all N, every $\overline{E_N}$ is compact.
　 PROOF: Proposition 11.37.
$\langle 1 \rangle 6$. For all N we have $\overline{E_N} \supseteq \overline{E_{N+1}}$.
$\langle 1 \rangle 7$. LET: l be the unique point in $\bigcap_{N=0}^{\infty} \overline{E_N}$
　　 PROVE: $p_n \to l$ as $n \to \infty$.
　 PROOF: Proposition 11.44.
$\langle 1 \rangle 8$. LET: $\epsilon > 0$
$\langle 1 \rangle 9$. PICK N_0 such that $\forall N \geq N_0.$ diam $\overline{E_N} < \epsilon$.
$\langle 1 \rangle 10$. $\forall q \in E_N.d(l, q) < \epsilon$
$\langle 1 \rangle 11$. $\forall n \geq N.d(l, p_n) < \epsilon$
□

Corollary 12.23.1. *Let X be a metric space. If every closed bounded set in X is compact, then X is complete.*

PROOF:
$\langle 1 \rangle 1$. LET: S be a Cauchy sequence in X.
$\langle 1 \rangle 2$. S is bounded.
$\langle 1 \rangle 3$. $\overline{S}$ is closed and bounded.
$\langle 1 \rangle 4$. $\overline{S}$ is compact.
$\langle 1 \rangle 5$. S is a Cauchy sequence in $\overline{S}$.
$\langle 1 \rangle 6$. S converges.
□

Corollary 12.23.2. *For every natural number k, we have $\mathbb{R}^k$ is complete.*

Corollary 12.23.3. *Every closed subspace of a complete metric space is complete.*

12.3   Divergent Sequences

Definition 12.24. Let (s_n) be a sequence in $\mathbb{R}$. Then we say s_n *diverges to* $+\infty$, and write

$$s_n \to +\infty \text{ as } n \to \infty \ ,$$

iff for every real number M, there exists an integer N such that

$$\forall n \geq N.s_n \geq M \ .$$

We say s_n *diverges to* $-\infty$, and write

$$s_n \to -\infty \text{ as } n \to \infty \ ,$$

iff for every real number M, there exists an integer N such that

$$\forall n \geq N.s_n \leq M \ .$$

Definition 12.25 (Limit Supremum, Limit Infimum)**.** Let (s_n) be a sequence in $\mathbb{R}$. Let E be the set of all $l \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that there exists a subsequence of (s_n) that converges to l.

The *limit supremum* of (s_n), denoted

$$\limsup_{n\to\infty} s_n \ ,$$

is the supremum of E in the extended reals.

The *limit infimum* of (s_n), denoted

$$\liminf_{n\to\infty} s_n \ ,$$

is the infimum of E in the extended reals.

PROOF: The set E is always nonempty because: if (s_n) is unbounded above then $+\infty \in E$; if it is unbounded below then $-\infty \in E$; and if it is bounded above and below then there is a real number in E by Corollary 12.4.1. $\square$

Theorem 12.26. *Let (s_n) be a sequence in $\mathbb{R}$. Then there exists a subsequence of (s_n) that converges or diverges to $\limsup_{n\to\infty} s_n$*

PROOF:
$\langle 1 \rangle 1$. CASE: $\limsup_n s_n = +\infty$
  PROOF: (s_n) is unbounded above and so has a subsequence that diverges to $+\infty$.
$\langle 1 \rangle 2$. CASE: $\limsup_n s_n \in \mathbb{R}$
  PROOF: Then $\limsup s_n$ is in the set of limits of subsequences of (s_n) by Proposition 12.11.
$\langle 1 \rangle 3$. CASE: $\limsup_n s_n = -\infty$
  PROOF: (s_n) is unbounded below and so has a subsequence that diverges to $-\infty$.

☐

Theorem 12.27. *Let (s_n) be a sequence in $\mathbb{R}$. Then there exists a subsequence of (s_n) that converges or diverges to $\liminf_{n \to \infty} s_n$*

PROOF: Similar. ☐

Theorem 12.28. *Let (s_n) be a sequence in $\mathbb{R}$. If $x > \limsup_n s_n$, then there exists N such that $\forall n \geq N.s_n < x$.*

PROOF: If not, we could choose a subsequence of (s_n) that converges to a value $\geq x$, contradicting the definition of $\limsup_n s_n$. ☐

Theorem 12.29. *Let (s_n) be a sequence in $\mathbb{R}$. If $x < \liminf_n s_n$, then there exists N such that $\forall n \geq N.s_n > x$.*

PROOF: Similar. ☐

Theorem 12.30. *Let (s_n) be a sequence in $\mathbb{R}$. Let s^* be an extended real such that:*

- *There exists a subsequence of (s_n) that converges or diverges to s^*.*

- *For any $x > s^*$, there exists N such that $\forall n \geq N.s_n < x$.*

Then $s^* = \limsup_n s_n$.

PROOF:
⟨1⟩1. LET: E be the set of subsequential limits of (s_n).
⟨1⟩2. s^* is an upper bound for E.
　⟨2⟩1. LET: $x \in E$
　⟨2⟩2. ASSUME: for a contradiction $x > s^*$.
　⟨2⟩3. $s^* \in \mathbb{R}$
　⟨2⟩4. LET: $y = x$ if $x \in \mathbb{R}$, or $s^* + 1$ if $x = +\infty$
　⟨2⟩5. There exists N such that $\forall n \geq N.s_n < y$.
　⟨2⟩6. Q.E.D.
　　PROOF: This contradicts the fact that some subsequence of (s_n) converges or diverges to x.
⟨1⟩3. If u is an upper bound for E then $s^* \leq u$.
☐

Theorem 12.31. *Let (s_n) be a sequence in $\mathbb{R}$. Let s^* be an extended real such that:*

- *There exists a subsequence of (s_n) that converges or diverges to s^*.*

- *For any $x < s^*$, there exists N such that $\forall n \geq N.s_n > x$.*

Then $s^* = \liminf_n s_n$.

PROOF: Similar. ☐

Proposition 12.32. *Let (s_n) be a sequence of real numbers and $l \in \mathbb{R}$. Then (s_n) converges to l iff $\limsup_n s_n = \liminf_n s_n = l$.*

PROOF:

$\langle 1 \rangle 1$. If (s_n) converges to l then $\limsup_n s_n = \liminf_n s_n = l$.
   PROOF: If (s_n) converges to l then every subsequence of (s_n) converges to l.

$\langle 1 \rangle 2$. If $\limsup_n s_n = \liminf_n s_n = l$ then (s_n) converges to l.
   $\langle 2 \rangle 1$. ASSUME: $\limsup_n s_n = \liminf_n s_n = l$
   $\langle 2 \rangle 2$. For all $\epsilon > 0$, there exists N such that $\forall n \geq N. l - \epsilon < s_n < l + \epsilon$.
     PROOF: Theorem 12.30 and 12.31.
   $\langle 2 \rangle 3$. $s_n \to l$ as $n \to \infty$.

□

Theorem 12.33. *Let (s_n) and (t_n) be sequences of real numbers and $N \in \mathbb{N}$. Assume $\forall n \geq N. s_n \leq t_n$. Then*

$$\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n \ \ .$$

PROOF:

$\langle 1 \rangle 1$. For any subsequence (t_{n_r}) of (t_n) that converges or diverges to $\pm \infty$, we
     have $\liminf_n s_n \leq \lim_r t_{n_r}$
   $\langle 2 \rangle 1$. LET: (t_{n_r}) be a subsequence of (t_n) with limit l.
   $\langle 2 \rangle 2$. PICK m such that a subsequence of (s_{n_r}) has limit m.
   $\langle 2 \rangle 3$. $\forall r. s_{n_r} \leq t_{n_r}$
   $\langle 2 \rangle 4$. $m \leq l$
   $\langle 2 \rangle 5$. $\liminf_n s_n \leq l$

$\langle 1 \rangle 2$. $\liminf_n s_n \leq \liminf_n t_n$

□

Theorem 12.34. *Let (s_n) and (t_n) be sequences of real numbers and $N \in \mathbb{N}$. Assume $\forall n \geq N. s_n \leq t_n$. Then*

$$\limsup_{n \to \infty} s_n \leq \limsup_{n \to \infty} t_n \ \ .$$

PROOF: Similar. □

Part III

More Algebra

Chapter 13

Lie Groups

Definition 13.1 (Lie Group). A *Lie group G* is a group G that is also an analytic differentiable manifold such that the group operation and inverse operation are analytic.

A *homomorphism of Lie groups* is a group homomorphism that is an analytic function.

Lemma 13.2. *Every bijective Lie group homomorphism is an isomorphism.*

Definition 13.3 (Unitary Group). The *unitary group $U(n)$* is the Lie group of all $n \times n$ unitary matrices.

Definition 13.4 (Special Unitary Group). The *special unitary group $SU(n)$* is the Lie group of all $n \times n$ unitary matrices with determinant 1.

Definition 13.5 (Lie Subgroup). Let G be a Lie group. A *Lie subgroup* of G is a subgroup that is also an analytic submanifold of G.

Example 13.6. $U(n)$ and $SU(n)$ are Lie subgroups of $GL(n, \mathbb{C})$.