Mathematics

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Chapter 1

Sets and Classes

1.1 Classes

Our language is the language of first-order logic with equality over one primitive binary predicate $\in$. We call all the objects we reason about *sets*. When $a \in b$, we say a is a *member* or *element* of b, or b *contains* a. We write $b \ni a$ for $a \in b$, and $a \notin b$ for $\neg(a \in b)$. We write $\forall x \in a.\phi$ as an abbreviation for $\forall x(x \in a \rightarrow \phi)$, and $\exists x \in a.\phi$ as an abbreviation for $\exists x(x \in a \wedge \phi)$.

We shall speak informally of *classes* as an abbreviation for talking about predicates. A *class* is determined by a unary predicate $\phi[x]$ (possibly with parameters). We write $\{x \mid \phi[x]\}$ or $\{x : \phi[x]\}$ for the class determined by $\phi[x]$. We write 'a is an element of $\{x \mid \phi[x]\}$' or '$a \in \{x \mid \phi[x]\}$' for $\phi[a]$.

We write $\{t[x_1, \ldots, x_n] \mid P[x_1, \ldots, x_n]\}$ for

$$\{y \mid \exists x_1, \ldots, x_n(y = t[x_1, \ldots, x_n] \wedge P[x_1, \ldots, x_n])\} \ .$$

We say two classes $\mathbf{A}$ and $\mathbf{B}$ are *equal*, and write $\mathbf{A} = \mathbf{B}$, iff $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$.

Proposition Schema 1.1.1. *For any class* $\mathbf{A}$*, the following is a theorem.*

$$\mathbf{A} = \mathbf{A}$$

PROOF: We have $\forall x(x \in \mathbf{A} \Leftrightarrow x \in \mathbf{A})$. $\square$

Proposition Schema 1.1.2. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, the following is a theorem.*

If $\mathbf{A} = \mathbf{B}$ *then* $\mathbf{B} = \mathbf{A}$.

PROOF: If $\forall x(x \in \mathbf{A} \Leftrightarrow x \in \mathbf{B})$ then $\forall x(x \in \mathbf{B} \Leftrightarrow x \in \mathbf{A})$. $\square$

Proposition Schema 1.1.3. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, the following is a theorem.*

If $\mathbf{A} = \mathbf{B}$ *and* $\mathbf{B} = \mathbf{C}$ *then* $\mathbf{A} = \mathbf{C}$.

PROOF: If $\forall x(x \in \mathbf{A} \Leftrightarrow x \in \mathbf{B})$ and $\forall x(x \in \mathbf{B} \Leftrightarrow x \in \mathbf{C})$ then $\forall x(x \in \mathbf{A} \Leftrightarrow x \in \mathbf{C})$. $\square$

1.1.1 Subclasses

Definition 1.1.4 (Subclass)**.** We say a class $\mathbf{A}$ is a *subclass* of $\mathbf{B}$, or $\mathbf{B}$ is a *superclass* of $\mathbf{A}$, or $\mathbf{B}$ *includes* $\mathbf{A}$, and write $\mathbf{A} \subseteq \mathbf{B}$ or $\mathbf{B} \supseteq \mathbf{A}$, iff every element of $\mathbf{A}$ is an element of $\mathbf{B}$. Otherwise we write $\mathbf{A} \nsubseteq \mathbf{B}$ or $\mathbf{B} \nsupseteq \mathbf{A}$.

We say $\mathbf{A}$ is a *proper* subclass of $\mathbf{B}$, $\mathbf{B}$ is a *proper* superclass of $\mathbf{A}$, or $\mathbf{B}$ *properly* includes $\mathbf{A}$, and write $\mathbf{A} \subsetneq \mathbf{B}$ or $\mathbf{B} \supsetneq \mathbf{A}$, iff $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$.

Proposition Schema 1.1.5. *For any class* $\mathbf{A}$*, the following is a theorem.*

$$\mathbf{A} \subseteq \mathbf{A}$$

Proof: Every element of $\mathbf{A}$ is an element of $\mathbf{A}$. □

Proposition Schema 1.1.6. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, the following is a theorem.*

If $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{A}$ *then* $\mathbf{A} = \mathbf{B}$*.*

Proof: If every element of $\mathbf{A}$ is an element of $\mathbf{B}$, and every element of $\mathbf{B}$ is an element of $\mathbf{A}$, then $\mathbf{A}$ and $\mathbf{B}$ have exactly the same elements. □

Proposition Schema 1.1.7. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, the following is a theorem.*

If $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{C}$ *then* $\mathbf{A} \subseteq \mathbf{C}$*.*

Proof: If every element of $\mathbf{A}$ is an element of $\mathbf{B}$ and every element of $\mathbf{B}$ is an element of $\mathbf{C}$ then every element of $\mathbf{A}$ is an element of $\mathbf{C}$. □

1.1.2 Constructions of Classes

Definition 1.1.8 (Empty Class)**.** The *empty class* $\emptyset$ is $\{x \mid \bot\}$. Every other class is *nonempty.*

Definition 1.1.9 (Universal Class)**.** The *universal class* $\mathbf{V}$ is $\{x \mid \top\}$.

Definition 1.1.10 (Enumeration)**.** Given objects a_1, …, a_n, we define the class $\{a_1, \ldots, a_n\}$ to be the class $\{x \mid x = a_1 \vee \cdots \vee x = a_n\}$.

Definition 1.1.11 (Intersection)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *intersection* $\mathbf{A} \cap \mathbf{B}$ is $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$.

Definition 1.1.12 (Union)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *union* $\mathbf{A} \cup \mathbf{B}$ is $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$.

Definition 1.1.13 (Relative Complement)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be classes. The *relative complement* of $\mathbf{B}$ in $\mathbf{A}$ is the class $\mathbf{A} - \mathbf{B} := \{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

Definition 1.1.14 (Symmetric Difference)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *symmetric difference* is the class $\mathbf{A} + \mathbf{B} := (\mathbf{A} - \mathbf{B}) \cup (\mathbf{B} - \mathbf{A})$.

Definition 1.1.15 (Pairwise disjoint)**.** Let $\mathbf{A}$ be a class. We say the elements of $\mathbf{A}$ are *pairwise disjoint* iff, for all $x, y \in \mathbf{A}$, if $x \cap y \neq \emptyset$ then $x = y$.

1.2 Sets and the Axiom of Extensionality

Definition 1.2.1 (Axiom of Extensionality)**.** The *Axiom of Extensionality* is the statement: if two sets have exactly the same members, then they are equal.

$$\forall x, y(\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y) \ .$$

When working in a theory with the Axiom of Extensionality, we may identify a set a with the class $\{x \mid x \in a\}$. Our use of the symbols $\in$ and $=$ is consistent. We say a class $\mathbf{A}$ *is a set* iff there exists a set a such that $a = \mathbf{A}$; that is, $\{x \mid \phi[x]\}$ is a set iff $\exists a \forall x(x \in a \leftrightarrow \phi[x])$. Otherwise, $\mathbf{A}$ is a *proper class.*

Definition 1.2.2 (Subset)**.** If A is a set and $A \subseteq \mathbf{B}$, we say A is a *subset* of $\mathbf{B}$.

Definition 1.2.3 (Union)**.** The *union* of a class $\mathbf{A}$ is $\{x \mid \exists X \in \mathbf{A}.x \in X\}$. We write $\bigcup_{P(x)} t(x)$ for $\bigcup\{t(x) \mid P(x)\}$.

Definition 1.2.4 (Intersection)**.** The *intersection* of a class $\mathbf{A}$ is $\{x \mid \forall X \in \mathbf{A}.x \in X\}$. We write $\bigcap_{P(x)} t(x)$ for $\bigcap\{t(x) \mid P(x)\}$.

Definition 1.2.5 (Power Class)**.** For any class $\mathbf{A}$, the *power class* $\mathcal{P}\mathbf{A}$ is $\{X \mid X \subseteq \mathbf{A}\}$.

1.3 The Other Axioms

Definition 1.3.1 (Pairing Axiom)**.** The *Pairing Axiom* is the statement: for any sets a and b, the class $\{a, b\}$ is a set.

$$\forall a \forall b \exists c \forall x(x \in c \Leftrightarrow x = a \vee x = b)$$

Definition 1.3.2 (Union Axiom)**.** The *Union Axiom* is the statement: for any set A, the class $\bigcup A$ is a set.

$$\forall A \exists B \forall x(x \in B \Leftrightarrow \exists y(y \in A \wedge x \in y))$$

Definition 1.3.3 (Comprehension Axiom Scheme)**.** The *Comprehension Axiom Scheme* is the set of sentences of the form, for any class $\mathbf{A}$: If $\mathbf{A}$ is a subclass of a set then $\mathbf{A}$ is a set.

That is, for any property $P[x, y_1, \ldots, y_n]$:

For any sets a_1, …, a_n and B, the class $\{x \in B \mid P[x, a_1, \ldots, a_n]\}$ is a set.

$$\forall a_1, \ldots, a_n, B.\exists C.\forall x(x \in C \Leftrightarrow x \in B \wedge P[x, a_1, \ldots, a_n])$$

Definition 1.3.4 (Replacement Axiom Scheme)**.** The *Replacement Axiom Scheme* is the set of sentences of the form, for some property $P[x, y, z_1, \ldots, z_n]$:

For any sets a_1, …, a_n, B, assume for all $x \in B$ there exists at most one y such that $P[x, y, a_1, \ldots, a_n]$. Then $\{y \mid \exists x \in B.P[x, y, a_1, \ldots, a_n]$ is a set.

$$\forall a_1, \ldots, a_n, B(\forall x \in B.\forall y, y'(P[x, y, a_1, \ldots, a_n] \wedge P[x, y', a_1, \ldots, a_n] \Rightarrow y = y') \Rightarrow$$
$$\exists C \forall y(y \in C \Leftrightarrow \exists x \in B.P[x, y, a_1, \ldots, a_n]))$$

Definition 1.3.5 (Power Set Axiom)**.** The *Power Set Axiom* is the statement: the power class of a set is a set.

$$\forall A \exists B \forall x (x \in B \Leftrightarrow \forall y (y \in x \Rightarrow y \in A))$$

Definition 1.3.6 (Axiom of Infinity)**.** The *Axiom of Infinity* is the statement: there exists a set I such that $\emptyset \in I$ and $\forall x \in I . x \cup \{x\} \in I$.

$$\exists I (\exists e \in I . \forall x . x \notin e \wedge \forall x \in I . \exists y \in I . \forall z (z \in y \Leftrightarrow z \in x \vee z = x))$$

Definition 1.3.7 (Axiom of Choice)**.** The *Axiom of Choice* is the statement: For any set A of pairwise disjoint, nonempty sets, there exists a set C such that, for all $x \in A$, we have $x \cap C$ has exactly one element.

$$\begin{aligned}
&\forall A (\forall x \in A . \exists y y \in x \wedge \\
&\forall x, y \in A . \forall z (z \in x \wedge z \in y \Rightarrow x = y) \Rightarrow \\
&\exists C . \forall x \in A . \exists y \forall z (z \in x \wedge z \in C \Leftrightarrow z = y))
\end{aligned}$$

Definition 1.3.8 (Axiom of Regularity)**.** The *Axiom of Regularity* is the statement: for any A, if A has a member, then there exists $m \in A$ such that $m \cap A = \emptyset$.

$$\forall A (\exists x . x \in A \Rightarrow \exists m \in A . \neg \exists x (x \in m \wedge x \in A))$$

Definition 1.3.9 (Zermelo Set Theory)**.** *Zermelo set theory* is the theory whose axioms are:

- Extensionality
- Pairing
- Union
- Comprehension
- Power Set
- Infinity
- Choice
- Regularity

We label theorems with Z when they are provable in Zermelo set theory.

Definition 1.3.10 (Zermelo-Fraenkel Set Theory)**.** *Zermelo-Fraenkel set theory* is the theory whose axioms are:

- Extensionality
- Union

- Replacement
- Power Set
- Infinity
- Choice
- Regularity

We label theorems with ZFC when they are provable in Zermelo-Fraenkel set theory.

We label a theorem with FOL if it can be proved in first-order logic, i.e. from no axioms.

1.4 ZFC Extends Z

Proposition 1.4.1 (Z,ZFC)**.** *The empty class $\emptyset$ is a set.*

PROOF: Immediate from the Axiom of Infinity. $\square$

Proposition 1.4.2 (ZFC)**.** *The Axiom of Pairing is a theorem of ZFC.*

PROOF:
$\langle 1\rangle 1$. LET: a, b be sets.
$\langle 1\rangle 2$. LET: $P(x, y)$ be the predicate $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$.
$\langle 1\rangle 3$. For all $x \in \mathcal{P}\mathcal{P}\emptyset$, there exists at most one y such that $P(x, y)$.
  $\langle 2\rangle 1$. LET: $x \in \mathcal{P}\mathcal{P}\emptyset$
  $\langle 2\rangle 2$. LET: y and y' be sets.
  $\langle 2\rangle 3$. ASSUME: $P(x, y)$ and $P(x, y')$
  $\langle 2\rangle 4$. $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$
    PROOF: From $\langle 2\rangle 3$.
  $\langle 2\rangle 5$. $(x = \emptyset \wedge y' = a) \vee (x = \mathcal{P}\emptyset \wedge y' = b)$
    PROOF: From $\langle 2\rangle 3$.
  $\langle 2\rangle 6$. $\emptyset \neq \mathcal{P}\emptyset$
    PROOF: Since $\emptyset \in \mathcal{P}\emptyset$ and $\emptyset \notin \emptyset$.
  $\langle 2\rangle 7$. $y = y'$
$\langle 1\rangle 4$. LET: A be the set $\{y \mid \exists x \in \mathcal{P}\mathcal{P}\emptyset . P(x, y)\}$.
$\langle 1\rangle 5$. $A = \{a, b\}$
$\square$

Proposition Schema 1.4.3 (ZFC)**.** *Every instance of the Comprehension Axiom Scheme is a theorem of ZFC.*

PROOF:
$\langle 1\rangle 1$. LET: $P(x)$ be a predicate.
$\langle 1\rangle 2$. LET: A be a set.
$\langle 1\rangle 3$. LET: $Q(x, y)$ be the predicate $P(x) \wedge y = x$.

$\langle 1\rangle 4.$ For all $x \in A$, there exists at most one y such that $Q(x, y)$.
  $\langle 2\rangle 1.$ LET: $x \in A$
  $\langle 2\rangle 2.$ LET: y and y' be sets.
  $\langle 2\rangle 3.$ ASSUME: $Q(x, y)$ and $Q(x, y')$
  $\langle 2\rangle 4.$ $x \in A \wedge P(x) \wedge y = x \wedge y' = x$
    PROOF: From $\langle 2\rangle 3$.
  $\langle 2\rangle 5.$ $y = y'$
    PROOF: From $\langle 2\rangle 4$.
$\langle 1\rangle 5.$ LET: B be the set $\{y \mid \exists x \in A.Q(x, y)\}$
  PROOF: This is a set by an Axiom of Replacement and $\langle 1\rangle 4$.
$\langle 1\rangle 6.$ $B = \{y \in A \mid P(y)\}$
  PROOF:

$$\begin{aligned} y \in B &\Leftrightarrow \exists x \in A.Q(x, y) && (\langle 1\rangle 5) \\ &\Leftrightarrow \exists x \in A(P(x) \wedge y = x) && (\langle 1\rangle 3) \\ &\Leftrightarrow P(y) \end{aligned}$$

□

Corollary Schema 1.4.3.1 (ZFC)**.** *Every axiom of Z is a theorem of ZFC.*

It follows that every theorem of Z is a theorem of ZFC.

1.5 Consequences of the Axioms

Proposition 1.5.1 (Z)**.** *The union of two sets is a set.*

PROOF: Because $A \cup B = \bigcup\{A, B\}$. □

Proposition Schema 1.5.2 (Z)**.** *For any number n, the following is a theorem:*
For any sets $a_1, \ldots, a_n$, the class $\{a_1, \ldots, a_n\} = \{x \mid x = a_1 \vee \cdots \vee x = a_n\}$ is a set.

PROOF: The case $n = 1$ follows from Pairing since $\{a\} = \{a, a\}$.
If we have proved the theorem for n we have $\{a_1, \ldots, a_n, a_{n+1}\} = \{a_1, \ldots, a_n\} \cup \{a_{n+1}\}$. □

Proposition 1.5.3 (Z)**.** *No set is a member of itself.*

PROOF:
$\langle 1\rangle 1.$ LET: x be any set.
$\langle 1\rangle 2.$ PICK $m \in \{x\}$ such that $m \cap \{x\} = \emptyset$.
  PROOF: Axiom of Regularity.
$\langle 1\rangle 3.$ $m = x$
$\langle 1\rangle 4.$ $x \cap \{x\} = \emptyset$
$\langle 1\rangle 5.$ $x \notin x$
□

Corollary 1.5.3.1 (Z)**.** *The universal class* $\mathbf{V}$ *is a proper class.*

PROOF: If $\mathbf{V}$ is a set then $\mathbf{V} \in \mathbf{V}$, contradicting the Proposition. □

Proposition 1.5.4 (Z). *There are no sets a and b such that $a \in b$ and $b \in a$.*

PROOF:
$\langle 1 \rangle 1$. LET: a and b be any sets.
$\langle 1 \rangle 2$. PICK $m \in \{a, b\}$ such that $m \cap \{a, b\} = \emptyset$
$\langle 1 \rangle 3$. CASE: $m = a$
PROOF: Then $b \notin a$.
$\langle 1 \rangle 4$. CASE: $m = b$
PROOF: Then $a \notin b$.
□

Proposition 1.5.5 (Z). *The intersection of a set and a class is a set.*

PROOF: Immediate from Comprehension. □

Proposition 1.5.6 (Z). *The relative complement of a class in a set is a set.*

[Z]

PROOF: Immediate from Comprehension. □

Corollary 1.5.6.1 (Z). *The symmetric difference of two sets is a set.*

Proposition 1.5.7 (Z). *The intersection of a nonempty class is a set.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathbf{A}$ be a nonempty class.
$\langle 1 \rangle 2$. PICK $B \in \mathbf{A}$
$\langle 1 \rangle 3$. $\bigcap \mathbf{A} \subseteq B$
$\langle 1 \rangle 4$. $\bigcap \mathbf{A}$ is a set.
PROOF: By Comprehension.
□

Proposition Schema 1.5.8 (FOL). *For any classes $\mathbf{A}$ and $\mathbf{B}$, the following is a theorem:*
If $\mathbf{A} \subseteq \mathbf{B}$ then $\mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathbf{B}$.

PROOF: Every subset of $\mathbf{A}$ is a subset of $\mathbf{B}$. □

Proposition Schema 1.5.9 (FOL). *For any classes $\mathbf{A}$ and $\mathbf{B}$, the following is a theorem:*
If $\mathbf{A} \subseteq \mathbf{B}$ then $\bigcup \mathbf{A} \subseteq \bigcup \mathbf{B}$.

PROOF: If $x \in X \in \mathbf{A}$ then $x \in X \in \mathbf{B}$. □

Proposition Schema 1.5.10 (Z). *For any class $\mathbf{A}$, the following is a theorem:*

$$\mathbf{A} = \bigcup \mathcal{P}\mathbf{A}$$

PROOF:

$\langle 1\rangle 1.$ $\mathbf{A} \subseteq \bigcup \mathcal{P}\mathbf{A}$
  PROOF: For all $x \in \mathbf{A}$ we have $x \in \{x\} \in \mathcal{P}\mathbf{A}$.
$\langle 1\rangle 2.$ $\bigcup \mathcal{P}\mathbf{A} \subseteq \mathbf{A}$
  $\langle 2\rangle 1.$ LET: $x \in \bigcup \mathcal{P}\mathbf{A}$
  $\langle 2\rangle 2.$ PICK $X \in \mathcal{P}\mathbf{A}$ such that $x \in X$
  $\langle 2\rangle 3.$ $X \subseteq \mathbf{A}$
  $\langle 2\rangle 4.$ $x \in \mathbf{A}$
□

1.6 Transitive Classes

Definition 1.6.1 (Transitive Class)**.** A class $\mathbf{A}$ is a *transitive class* iff whenever $x \in y \in \mathbf{A}$ then $x \in \mathbf{A}$.

Proposition Schema 1.6.2 (FOL)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
The following are equivalent.

1. $\mathbf{A}$ *is a transitive class.*
2. $\bigcup \mathbf{A} \subseteq \mathbf{A}$
3. *Every element of* $\mathbf{A}$ *is a subset of* $\mathbf{A}$.
4. $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$

PROOF: Immediate from definitions. □

Proposition Schema 1.6.3 (FOL)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
If $\mathbf{A}$ *is a transitive class then* $\bigcup \mathbf{A}$ *is a transitive class.*

PROOF:
$\langle 1\rangle 1.$ ASSUME: $\mathbf{A}$ is a transitive class.
$\langle 1\rangle 2.$ LET: $x \in y \in \bigcup \mathbf{A}$
$\langle 1\rangle 3.$ $y \in \mathbf{A}$
  PROOF: Since $\bigcup \mathbf{A} \subseteq \mathbf{A}$ by Proposition 1.6.2.
$\langle 1\rangle 4.$ $x \in \bigcup \mathbf{A}$
□

Proposition Schema 1.6.4 (Z)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
We have $\mathbf{A}$ *is a transitive class if and only if* $\mathcal{P}\mathbf{A}$ *is a transitive class.*

PROOF:
$\langle 1\rangle 1.$ If $\mathbf{A}$ is a transitive class then $\mathcal{P}\mathbf{A}$ is a transitive class.
  $\langle 2\rangle 1.$ ASSUME: $\mathbf{A}$ is a transitive class.
  $\langle 2\rangle 2.$ $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$
    PROOF: Proposition 1.6.2.
  $\langle 2\rangle 3.$ $\mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathcal{P}\mathbf{A}$

PROOF: Proposition 1.5.8.
⟨2⟩4. $\mathcal{P}\mathbf{A}$ is a transitive class.
PROOF: Proposition 1.6.2.
⟨1⟩2. If $\mathcal{P}\mathbf{A}$ is a transitive class then $\mathbf{A}$ is a transitive class.
⟨2⟩1. ASSUME: $\mathcal{P}\mathbf{A}$ is a transitive class.
⟨2⟩2. $\bigcup \mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$
PROOF: Proposition 1.6.2.
⟨2⟩3. $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$
PROOF: Proposition 1.5.10.
⟨2⟩4. $\mathbf{A}$ is a transitive class.
PROOF: Proposition 1.6.2.
□

Proposition Schema 1.6.5 (FOL)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
If every member of $\mathbf{A}$ *is a transitive set then* $\bigcup \mathbf{A}$ *is a transitive class.*

PROOF:
⟨1⟩1. ASSUME: Every member of $\mathbf{A}$ is a transitive set.
⟨1⟩2. LET: $x \in y \in \bigcup \mathbf{A}$
⟨1⟩3. PICK $A \in \mathbf{A}$ such that $y \in A$.
⟨1⟩4. $x \in A$
PROOF: Since A is a transitive set.
⟨1⟩5. $x \in \bigcup \mathbf{A}$
□

Proposition Schema 1.6.6 (FOL)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
If every member of $\mathbf{A}$ *is a transitive set then* $\bigcap \mathbf{A}$ *is a transitive class.*

PROOF:
⟨1⟩1. ASSUME: Every member of $\mathbf{A}$ is a transitive set.
⟨1⟩2. LET: $x \in y \in \bigcap \mathbf{A}$
PROVE: $x \in \bigcap \mathbf{A}$
⟨1⟩3. LET: $A \in \mathbf{A}$
⟨1⟩4. $y \in A$
⟨1⟩5. $x \in A$
PROOF: Since A is a transitive set.
□

Chapter 2

Relations

2.1 Ordered Pairs

Definition 2.1.1 (Ordered Pair)**.** For any sets a and b, the *ordered pair* (a, b) is defined to be $\{\{a\}, \{a, b\}\}$.

Theorem 2.1.2 (Z)**.** *For any sets a, b, c, d, we have $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.*

PROOF:
⟨1⟩1. If $(a, b) = (c, d)$ then $a = c$ and $b = d$.
  ⟨2⟩1. ASSUME: $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$
  ⟨2⟩2. $\bigcap\{\{a\}, \{a, b\}\} = \bigcap\{\{c\}, \{c, d\}\}$
  ⟨2⟩3. $\{a\} = \{c\}$
  ⟨2⟩4. $a = c$
  ⟨2⟩5. $\bigcup\{\{a\}, \{a, b\}\} = \bigcup\{\{c\}, \{c, d\}\}$
  ⟨2⟩6. $\{a, b\} = \{c, d\}$
  ⟨2⟩7. $b = c$ or $b = d$
  ⟨2⟩8. $a = d$ or $b = d$
  ⟨2⟩9. If $b = c$ and $a = d$ then $b = d$
    PROOF: By ⟨2⟩4.
  ⟨2⟩10. $b = d$
    PROOF: From ⟨2⟩7, ⟨2⟩8, ⟨2⟩9.
⟨1⟩2. If $a = c$ and $b = d$ then $(a, b) = (c, d)$.
  PROOF: First-order logic.
□

Definition 2.1.3 (Cartesian Product)**.** The *Cartesian product* of classes $\mathbf{A}$ and $\mathbf{B}$ is the class $\mathbf{A} \times \mathbf{B} := \{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$.

Proposition 2.1.4 (Z)**.** *For any sets A and B, the class $A \times B$ is a set.*

PROOF: It is a subset of $\mathcal{PP}(A \cup B)$. □

Proposition Schema 2.1.5 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, the following is a theorem:*

$$\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$$

PROOF:

$$\begin{aligned} (x, y) \in \mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) &\Leftrightarrow x \in \mathbf{A} \wedge (y \in \mathbf{B} \vee y \in \mathbf{C}) \\ &\Leftrightarrow (x \in \mathbf{A} \wedge y \in \mathbf{B}) \vee (x \in \mathbf{A} \wedge y \in \mathbf{C}) \\ &\Leftrightarrow (x, y) \in (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C}) \end{aligned}$$

□

Proposition Schema 2.1.6 (Z)**.** *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, the following is a theorem:*

If $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ *and* $\mathbf{A}$ *is nonempty then* $\mathbf{B} = \mathbf{C}$*.*

PROOF:

⟨1⟩1. PICK $a \in \mathbf{A}$

⟨1⟩2. For all x we have $x \in \mathbf{B}$ iff $x \in \mathbf{C}$.

PROOF:

$$\begin{aligned} x \in \mathbf{B} &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{B} \\ &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{C} \\ &\Leftrightarrow x \in \mathbf{C} \end{aligned}$$

□

Proposition Schema 2.1.7 (Z)**.** *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, the following is a theorem:*

$$\mathbf{A} \times \bigcup \mathbf{B} = \{(a, b) \mid \exists Y \in \mathbf{B}.(a \in \mathbf{A} \wedge b \in Y)\}$$

PROOF:

$$\begin{aligned} (x, y) \in A \times \bigcup \mathbf{B} &\Leftrightarrow x \in A \wedge \exists Y \in \mathbf{B}.y \in Y \\ &\Leftrightarrow \exists Y \in \mathbf{B}(x \in A \wedge y \in Y) \end{aligned}$$

□

2.2 Relations

Definition 2.2.1 (Relation)**.** A *relation* $\mathbf{R}$ between classes $\mathbf{A}$ and $\mathbf{B}$ is a subclass of $\mathbf{A} \times \mathbf{B}$.

A *(binary) relation on* $\mathbf{A}$ is a relation between $\mathbf{A}$ and $\mathbf{A}$.

We write $x\mathbf{R}y$ for $(x, y) \in \mathbf{R}$.

2.2.1 Identity Functions

Definition 2.2.2 (Identity Function)**.** For any class $\mathbf{A}$, the *identity function* or *diagonal relation* $\mathrm{id}_{\mathbf{A}}$ on $\mathbf{A}$ is

$$\mathrm{id}_{\mathbf{A}} := \{(x, x) \mid x \in \mathbf{A}\} \ .$$

2.2.2 Inverses

Definition 2.2.3 (Inverse)**.** The *inverse* of a relation $\mathbf{R}$ between $\mathbf{A}$ and $\mathbf{B}$ is the relation $\mathbf{R}^{-1}$ between $\mathbf{B}$ and $\mathbf{A}$ defined by

$$b\mathbf{R}^{-1}a \Leftrightarrow a\mathbf{R}b \enspace .$$

Proposition Schema 2.2.4 (Z)**.** *For any classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{R}$, the following is a theorem:*

If $\mathbf{R}$ is a relation between $\mathbf{A}$ and $\mathbf{B}$, we have $(\mathbf{R}^{-1})^{-1} = \mathbf{R}$.

PROOF:

$$\begin{aligned} x(\mathbf{R}^{-1})^{-1}y &\Leftrightarrow y\mathbf{R}^{-1}x \\ &\Leftrightarrow x\mathbf{R}y \end{aligned} \qquad \square$$

2.2.3 Composition

Definition 2.2.5 (Composition)**.** Let $\mathbf{R}$ be a relation between $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{S}$ be a relation between $\mathbf{B}$ and $\mathbf{C}$. The *composition* $\mathbf{S} \circ \mathbf{R}$ is the relation between $\mathbf{A}$ and $\mathbf{C}$ defined by

$$a(\mathbf{S} \circ \mathbf{R})c \Leftrightarrow \exists b(a\mathbf{R}b \wedge b\mathbf{S}c) \enspace .$$

Proposition Schema 2.2.6 (Z)**.** *For any classes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{S}$, the following is a theorem:*

If $\mathbf{R}$ is a relation between $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{S}$ is a relation between $\mathbf{B}$ and $\mathbf{C}$, then

$$(\mathbf{S} \circ \mathbf{R})^{-1} = \mathbf{R}^{-1} \circ \mathbf{S}^{-1} \enspace .$$

PROOF:

$$\begin{aligned} z(\mathbf{S} \circ \mathbf{R})^{-1}x &\Leftrightarrow x(\mathbf{S} \circ \mathbf{R})z \\ &\Leftrightarrow \exists y.(x\mathbf{R}y \wedge y\mathbf{S}z) \\ &\Leftrightarrow \exists y.(y\mathbf{R}^{-1}x \wedge z\mathbf{S}^{-1}y) \\ &\Leftrightarrow z(\mathbf{R}^{-1} \circ \mathbf{S}^{-1})x \end{aligned} \qquad \square$$

2.2.4 Properties of Relaitons

Definition 2.2.7 (Reflexive)**.** Let $\mathbf{R}$ be a binary relation on $\mathbf{A}$. Then $\mathbf{R}$ is *reflexive* on $\mathbf{A}$ iff $\forall x \in \mathbf{A}.(x, x) \in \mathbf{R}$.

Proposition Schema 2.2.8 (Z)**.** *For any classes $\mathbf{A}$ and $\mathbf{R}$, the following is a theorem:*

If $\mathbf{R}$ is a reflexive relation on $\mathbf{A}$ then so is $\mathbf{R}^{-1}$.

PROOF:

$\langle 1 \rangle 1.$ LET: $x \in \mathbf{A}$

$\langle 1 \rangle 2.$ $x\mathbf{R}x$

PROOF: Since $\mathbf{R}$ is reflexive.

⟨1⟩3. $x\mathbf{R}^{-1}x$
□

Definition 2.2.9 (Irreflexive). A relation $\mathbf{R}$ is *irreflexive* iff there is no x such that $(x, x) \in \mathbf{R}$.

Definition 2.2.10 (Symmetric). A relation $\mathbf{R}$ is *symmetric* iff, whenever $x\mathbf{R}y$, then $y\mathbf{R}x$.

Definition 2.2.11 (Antisymmetric). A relation $\mathbf{R}$ is *antisymmetric* iff, whenever $x\mathbf{R}y$ and $y\mathbf{R}x$, then $x = y$.

Proposition Schema 2.2.12 (Z). *For any classes $\mathbf{A}$ and $\mathbf{R}$, the following is a theorem:*

If $\mathbf{R}$ is an antisymmetric relation on $\mathbf{A}$ then so is $\mathbf{R}^{-1}$.

Proof:
⟨1⟩1. Assume: $x\mathbf{R}^{-1}y$ and $y\mathbf{R}^{-1}x$
⟨1⟩2. $y\mathbf{R}x$ and $x\mathbf{R}y$
⟨1⟩3. $x = y$
Proof: Since $\mathbf{R}$ is antisymmetric.
□

Definition 2.2.13 (Transitive). A relation $\mathbf{R}$ is *transitive* iff, whenever $x\mathbf{R}y$ and $y\mathbf{R}z$, then $x\mathbf{R}z$.

Proposition Schema 2.2.14 (Z). *For any classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{R}$, the following is a theorem:*

If $\mathbf{R}$ is a transitive relation between $\mathbf{A}$ and $\mathbf{B}$ then $\mathbf{R}^{-1}$ is transitive.

Proof:
⟨1⟩1. Assume: $(x, y), (y, z) \in \mathbf{R}^{-1}$
⟨1⟩2. $(y, x), (z, y) \in \mathbf{R}$
⟨1⟩3. $(z, x) \in \mathbf{R}$
⟨1⟩4. $(x, z) \in \mathbf{R}^{-1}$
□

Proposition 2.2.15 (Z). *For any relation R on a set A, there exists a smallest transitive relation on A that includes R.*

Proof: The relation is $\bigcap\{S \in \mathcal{P}A^2 \mid R \subseteq S, S \text{ is transitive}\}$. □

Definition 2.2.16 (Transitive Closure). For any relation R on a set A, the *transitive closure* of R is the smallest transitive relation that includes R.

Definition 2.2.17 (Minimal). Let $\mathbf{R}$ be a relation on $\mathbf{A}$. An element $m \in \mathbf{A}$ is *minimal* iff there is no $x \in \mathbf{A}$ such that $x\mathbf{R}m$.

Definition 2.2.18 (Maximal). Let $\mathbf{R}$ be a relation on $\mathbf{A}$. An element $m \in \mathbf{A}$ is *maximal* iff there is no $x \in \mathbf{A}$ such that $m\mathbf{R}x$.

2.3 n-ary Relations

Definition Schema 2.3.1. For any sets $a_1, \ldots, a_n$, define the *ordered n-tuple* $(a_1, \ldots, a_n)$ by

$$(a_1) := a_1$$
$$(a_1, \ldots, a_n, a_{n+1}) = ((a_1, \ldots, a_n), a_{n+1})$$

Definition Schema 2.3.2. An *n-ary relation on* $\mathbf{A}$ is a class of ordered n-tuples all of whose components are in $\mathbf{A}$.

2.4 Well Founded Relations

Definition 2.4.1 (Well Founded)**.** A relation $\mathbf{R}$ on a class $\mathbf{A}$ is *well founded* iff:

- for all $a \in A$, the class $\{x \in \mathbf{A} \mid x\mathbf{R}a\}$ is a set;
- every nonempty subset of $\mathbf{A}$ has an $\mathbf{R}$-minimal element.

Proposition 2.4.2 (Z)**.** *For any class* $\mathbf{A}$*, the relation* $\{(x, y) \in \mathbf{A}^2 \mid x \in y\}$ *is well founded.*

Proof:
⟨1⟩1. For all $a \in \mathbf{A}$, the class $\{x \in \mathbf{A} \mid x \in a\}$ is a set.
  Proof: It is a subclass of a.
⟨1⟩2. Every nonempty subset of $\mathbf{A}$ has an $\in$-minimal element.
  ⟨2⟩1. Let: C be a nonempty subset of $\mathbf{A}$
  ⟨2⟩2. Pick $m \in C$ such that $m \cap C = \emptyset$
    Proof: Axiom of Regularity.
  ⟨2⟩3. m is $\in$-minimal in C.
□

Proposition Schema 2.4.3 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{R}$*, the following is a theorem:*

Assume $\mathbf{R}$ *is a well founded relation on* $\mathbf{A}$ *and* $\mathbf{B} \subseteq \mathbf{A}$ *is nonempty. Then* $\mathbf{B}$ *has an* $\mathbf{R}$*-minimal element.*

Proof:
⟨1⟩1. Pick $b \in \mathbf{B}$
⟨1⟩2. Let: $S = \{x \in \mathbf{B} \mid x\mathbf{R}b\}$
  Proof: S is a set because it is a subclass of $\{x \in \mathbf{A} \mid x\mathbf{R}b\}$.
⟨1⟩3. Case: $S = \emptyset$
  Proof: In this case b is an $\mathbf{R}$-minimal element of $\mathbf{B}$.
⟨1⟩4. Case: $S \neq \emptyset$
  Proof: In this cases S has an $\mathbf{R}$-minimal element, which is an $\mathbf{R}$-minimal element of $\mathbf{B}$.
□

Theorem Schema 2.4.4 (Transfinite Induction Principle (Z))**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{R}$*, the following is a theorem:*

Assume $\mathbf{R}$ *is a well founded relation on* $\mathbf{A}$ *and* $\mathbf{B} \subseteq \mathbf{A}$*. Assume that, for all* $t \in \mathbf{A}$*,*

$$\{x \in \mathbf{A} \mid x\mathbf{R}t\} \subseteq \mathbf{B} \Rightarrow t \in \mathbf{B} \enspace .$$

Then $\mathbf{B} = \mathbf{A}$*.*

Proof:
$\langle 1\rangle 1$. Assume: for a contradiction $\mathbf{B} \neq \mathbf{A}$
$\langle 1\rangle 2$. Pick an $\mathbf{R}$-minimal element m of $\mathbf{A} - \mathbf{B}$.
Proof: Proposition 2.4.3.
$\langle 1\rangle 3$. $\{x \in \mathbf{A} \mid x\mathbf{R}m\} \subseteq \mathbf{B}$
Proof: By minimality of m.
$\langle 1\rangle 4$. $m \in \mathbf{B}$
$\langle 1\rangle 5$. Q.E.D.
Proof: This is a contradiction.
□

Theorem 2.4.5 (Z)**.** *The transitive closure of a well founded relation on a set is well founded.*

Proof:
$\langle 1\rangle 1$. Let: R be a well founded relation on the set A.
$\langle 1\rangle 2$. Let: R^t be the transitive closure of R.
$\langle 1\rangle 3$. For any $x, y \in A$, if xR^ty then there exists $z \in A$ such that zRy.
Proof: $\{(x, y) \in A^2 \mid \exists z \in A.zRy\}$ is a transitive relation on A that includes R.
$\langle 1\rangle 4$. Let: B be a nonempty subset of A.
$\langle 1\rangle 5$. Pick an R-minimal element b of B.
$\langle 1\rangle 6$. b is R^t-minimal in B.
Proof: If there exists x such that xR^tb then there exists z such that zRb by $\langle 1\rangle 3$.
□

Definition 2.4.6 (Initial Segment)**.** Let $\mathbf{R}$ be a relation on $\mathbf{A}$ and $a \in \mathbf{A}$. The *initial segment* up to a is

$$\operatorname{seg} a := \{x \in \mathbf{A} \mid x\mathbf{R}a\} \enspace .$$

Theorem Schema 2.4.7 (Transfinite Recursion Theorem Schema (ZFC))**.** *For any classes* $\mathbf{A}$*,* $\mathbf{R}$ *and any property* $G[x, y, z]$*, there exists a class* $\mathbf{F}$ *such that, for any class* $\mathbf{F}'$ *the following is a theorem:*

Assume that $\mathbf{R}$ *is a well-founded relation on* $\mathbf{A}$*. Assume that, for any* f *and* t*, there exists a unique* z *such that* $G[f, t, z]$*. Then* $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ *such that, for all* $t \in \mathbf{A}$*, we have* $\mathbf{F} \restriction \operatorname{seg} t$ *is a set and*

$$G[\mathbf{F} \restriction \operatorname{seg} t, t, \mathbf{F}(t)] \enspace .$$

If $\mathbf{F}' : \mathbf{A} \to \mathbf{V}$ *satisfies that, for all* $t \in \mathbf{A}$*, we have* $\mathbf{F}' \restriction \operatorname{seg} t$ *is a set and* $G[\mathbf{F}' \restriction \operatorname{seg} t, t, \mathbf{F}'(t)]$*, then* $\mathbf{F}' = \mathbf{F}$*.*

PROOF:
⟨1⟩1. For B a subset of $\mathbf{A}$, let us say a function $v : B \to \mathbf{V}$ is *acceptable* iff, for all $x \in B$, we have $\operatorname{seg} x \subseteq B$ and $G[v \restriction \operatorname{seg} x, x, v(x)]$
⟨1⟩2. LET: $\mathbf{K}$ be the class of all acceptable functions.
⟨1⟩3. LET: $\mathbf{F} = \bigcup \mathbf{K}$
⟨1⟩4. For all $B, C \subseteq \mathbf{A}$, given $v_1 : B \to \mathbf{V}$ and $v_2 : C \to \mathbf{V}$ acceptable and $x \in B \cap C$, we have $v_1(x) = v_2(x)$
⟨2⟩1. ASSUME: as transfinite induction hypothesis $\forall y \mathbf{R} x . y \in B \cap C \Rightarrow v_1(y) = v_2(y)$
⟨2⟩2. $v_1 \restriction \operatorname{seg} x = v_2 \restriction \operatorname{seg} x$
⟨2⟩3. $G[v_1 \restriction \operatorname{seg} x, x, v_1(x)]$
⟨2⟩4. $G[v_2 \restriction \operatorname{seg} x, x, v_2(x)]$
⟨2⟩5. $v_1(x) = v_2(x)$
⟨1⟩5. $\mathbf{F}$ is a function.
⟨2⟩1. ASSUME: $(x, y), (x, z) \in \mathbf{F}$
⟨2⟩2. PICK acceptable $v_1 : B \to \mathbf{V}$ and $v_2 : C \to \mathbf{V}$ such that $v_1(x) = y$ and $v_2(x) = z$
⟨2⟩3. $y = z$
PROOF: By ⟨1⟩4.
⟨1⟩6. For all $t \in \operatorname{dom} \mathbf{F}$, we have $\mathbf{F} \restriction \operatorname{seg} t$ is a set and $G[\mathbf{F} \restriction \operatorname{seg} t, t, \mathbf{F}(t)]$
⟨2⟩1. LET: $t \in \operatorname{dom} \mathbf{F}$
⟨2⟩2. PICK an acceptable $v : A \to \mathbf{V}$ such that $t \in A$
⟨2⟩3. For all $y \mathbf{R} x$ we have $v(y) = \mathbf{F}(y)$
⟨2⟩4. $\mathbf{F} \restriction \operatorname{seg} x = v \restriction \operatorname{seg} x$
⟨2⟩5. $G[v \restriction \operatorname{seg} x, x, v(x)]$
⟨2⟩6. $G[\mathbf{F} \restriction \operatorname{seg} x, x, \mathbf{F}(x)]$
⟨1⟩7. $\operatorname{dom} \mathbf{F} = \mathbf{A}$
⟨2⟩1. LET: $x \in \mathbf{A}$
⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall y \mathbf{R} x . y \in \mathbf{A}$
⟨2⟩3. ASSUME: for a contradiction $x \notin \operatorname{dom} \mathbf{F}$
⟨2⟩4. $\mathbf{F} \restriction \operatorname{seg} x$ is a set
PROOF: Axiom of Replacement.
⟨2⟩5. $\mathbf{F} \restriction \operatorname{seg} x$ is acceptable
⟨2⟩6. LET: y be the unique object such that $G[\mathbf{F} \restriction \operatorname{seg} x, x, y]$
⟨2⟩7. $\mathbf{F} \restriction \operatorname{seg} x \cup \{(x, y)\}$ is acceptable
⟨2⟩8. $x \in \operatorname{dom} \mathbf{F}$
⟨2⟩9. Q.E.D.
PROOF: This is a contradiction.
⟨1⟩8. If $\mathbf{F}' : \mathbf{A} \to \mathbf{V}$ satisfies the theorem, then $\mathbf{F}' = \mathbf{F}$.
⟨2⟩1. LET: $x \in \mathbf{A}$
PROVE: $\mathbf{F}'(x) = \mathbf{F}(x)$
⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall y \mathbf{R} x . \mathbf{F}'(y) = \mathbf{F}(y)$
⟨2⟩3. $\mathbf{F} \restriction x = \mathbf{F}' \restriction x$
⟨2⟩4. $G[\mathbf{F} \restriction x, x, \mathbf{F}(x)]$
⟨2⟩5. $G[\mathbf{F}' \restriction x, x, \mathbf{F}'(x)]$
⟨2⟩6. $\mathbf{F}(x) = \mathbf{F}'(x)$

□

Chapter 3

Functions

3.1 Functions

Definition 3.1.1 (Function)**.** A *function* from $\mathbf{A}$ to $\mathbf{B}$ is a relation $\mathbf{F}$ between $\mathbf{A}$ and $\mathbf{B}$ such that, for all $x \in \mathbf{A}$, there is only one y such that $x\mathbf{F}y$. We denote this y by $\mathbf{F}(x)$.

A *binary operation* on a class $\mathbf{A}$ is a function $\mathbf{A}^2 \to \mathbf{A}$.

Definition 3.1.2 (Closed)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{A}$ be a function and $\mathbf{B} \subseteq \mathbf{A}$. Then $\mathbf{B}$ is *closed* under $\mathbf{F}$ iff $\forall x \in \mathbf{B}.\mathbf{F}(x) \in \mathbf{B}$.

Proposition 3.1.3 (Z)**.** *For any class $\mathbf{A}$, the following is a theorem:*

$$\mathrm{id}_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$$

Proof: For all $x \in \mathbf{A}$, the only y such that $(x, y) \in \mathrm{id}_{\mathbf{A}}$ is $y = x$. □

Proposition Schema 3.1.4 (Z)**.** *For any classes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{F}$ and $\mathbf{G}$, the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{G} : \mathbf{B} \to \mathbf{C}$. Then $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \to \mathbf{C}$ and, for all $x \in \mathbf{A}$, we have

$$(\mathbf{G} \circ \mathbf{F})(x) = \mathbf{G}(\mathbf{F}(x)) \ .$$

Proof:
$\langle 1 \rangle 1.$ $\forall x \in \mathbf{A}.(x, \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G} \circ \mathbf{F})$
  Proof: Because $(x, \mathbf{F}(x)) \in \mathbf{F}$ and $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$.
$\langle 1 \rangle 2.$ If $(x, z) \in \mathbf{F} \circ \mathbf{G}$ then $z = \mathbf{G}(\mathbf{F}(x))$
  $\langle 2 \rangle 1.$ Pick $y \in \mathbf{B}$ such that $x\mathbf{F}y$ and $y\mathbf{G}z$
  $\langle 2 \rangle 2.$ $y = \mathbf{F}(x)$
  $\langle 2 \rangle 3.$ $z = \mathbf{G}(y)$
  $\langle 2 \rangle 4.$ $z = \mathbf{G}(\mathbf{F}(x))$
□

Proposition 3.1.5 (Z)**.** *For any set A there exists a function $F : \mathcal{P}A - \{\emptyset\} \to A$ (a* choice function *for A) such that, for every nonempty $B \subseteq A$, we have $F(B) \in B$.*

PROOF:
$\langle 1 \rangle 1.$ LET: A be a set.
$\langle 1 \rangle 2.$ LET: $\mathcal{A} = \{\{B\} \times B \mid B \in \mathcal{P}A - \{\emptyset\}\}$
$\langle 1 \rangle 3.$ Every member of $\mathcal{A}$ is nonempty.
$\langle 1 \rangle 4.$ Any two distinct members of $\mathcal{A}$ are disjoint.
$\langle 1 \rangle 5.$ PICK a set C such that, for all $X \in \mathcal{A}$, we have $C \cap X$ is a singleton.
  PROOF: Axiom of Choice.
$\langle 1 \rangle 6.$ LET: $F = C \cap \bigcup \mathcal{A}$
$\langle 1 \rangle 7.$ $F : \mathcal{P}A - \{\emptyset\} \to A$
  $\langle 2 \rangle 1.$ F is a function.
    $\langle 3 \rangle 1.$ LET: $(B, b), (B, b') \in F$
    $\langle 3 \rangle 2.$ $(B, b), (B, b') \in \{B\} \times B$
      PROOF: Since $(B, b), (B, b') \in \bigcup \mathcal{A}$.
    $\langle 3 \rangle 3.$ $(B, b), (B, b') \in C \cap (\{B\} \times B)$
    $\langle 3 \rangle 4.$ $(B, b) = (B, b')$
      PROOF: From $\langle 1 \rangle 5$.
    $\langle 3 \rangle 5.$ $b = b'$
  $\langle 2 \rangle 2.$ $\operatorname{dom} F = \mathcal{P}A - \{\emptyset\}$
    PROOF:

$$\begin{aligned}
& B \in \operatorname{dom} F \\
\Leftrightarrow & \exists b.(B, b) \in F \\
\Leftrightarrow & \exists b.((B, b) \in \bigcup \mathcal{A} \wedge (B, b) \in C) \\
\Leftrightarrow & \exists b. \exists B' \in \mathcal{P}A - \{\emptyset\}.((B, b) \in \{B'\} \times B' \wedge (B, b) \in C) \\
\Leftrightarrow & B \in \mathcal{P}A - \{\emptyset\} \wedge \exists b \in B.(B, b) \in C \\
\Leftrightarrow & B \in \mathcal{P}A - \{\emptyset\} && (\langle 1 \rangle 5)
\end{aligned}$$

  $\langle 2 \rangle 3.$ $\operatorname{ran} F \subseteq A$
$\langle 1 \rangle 8.$ For every nonempty $B \subseteq A$ we have $F(B) \in B$
□

Proposition 3.1.6 (Z)**.** *For any relation R between A and B, there exists a function $H : A \to B$ such that $H \subseteq R$ (i.e. $\forall x \in A.xRH(x)$).*

PROOF:
$\langle 1 \rangle 1.$ LET: R be a relation between A and B.
$\langle 1 \rangle 2.$ PICK a choice function G for B.
$\langle 1 \rangle 3.$ Define $H : A \to B$ by $H(x) = G(\{y \mid xRy\})$
$\langle 1 \rangle 4.$ $H \subseteq R$
□

3.1.1 Injective Functions

Definition 3.1.7 (Injective)**.** A function $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ is *one-to-one*, *injective* or an *injection*, $\mathbf{F} : \mathbf{A} \rightarrowtail \mathbf{B}$, iff, for all $x, y \in \mathbf{A}$, if $\mathbf{F}(x) = \mathbf{F}(y)$, then $x = y$.

Proposition 3.1.8 (Z)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
$\mathrm{id}_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$ *is injective.*

Proof: If $\mathrm{id}_{\mathbf{A}}(x) = \mathrm{id}_{\mathbf{A}}(y)$ then immediately $x = y$. ∎

Proposition Schema 3.1.9 (Z)**.** *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{F}$, $\mathbf{G}$*, the following is a theorem:*
Assume $\mathbf{F} : \mathbf{A} \rightarrowtail \mathbf{B}$ *and* $\mathbf{G} : \mathbf{B} \rightarrowtail \mathbf{C}$*. Then* $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \rightarrowtail \mathbf{C}$.

Proof:
⟨1⟩1. Let: $x, y \in \mathbf{A}$
⟨1⟩2. Assume: $(\mathbf{G} \circ \mathbf{F})(x) = (\mathbf{G} \circ \mathbf{F})(y)$
⟨1⟩3. $\mathbf{G}(\mathbf{F}(x)) = \mathbf{G}(\mathbf{F}(y))$
⟨1⟩4. $\mathbf{F}(x) = \mathbf{F}(y)$
Proof: Since $\mathbf{G}$ is injective.
⟨1⟩5. $x = y$
Proof: Since $\mathbf{F}$ is injective.
∎

Proposition 3.1.10 (Z)**.** *Let* $F : A \to B$ *where* A *is nonempty. There exists* $G : B \to A$ *(a* left inverse*) such that* $G \circ F = \mathrm{id}_A$ *if and only if* F *is one-to-one.*

Proof:
⟨1⟩1. If there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$ then F is one-to-one.
⟨2⟩1. Assume: $G : B \to A$ and $G \circ F = I_A$
⟨2⟩2. Let: $x, y \in A$
⟨2⟩3. Assume: $F(x) = F(y)$
⟨2⟩4. $x = y$
Proof: $x = G(F(x)) = G(F(y)) = y$
⟨1⟩2. If F is one-to-one then there exists $G : B \to A$ such that $G \circ F = I_A$.
⟨2⟩1. Assume: F is one-to-one.
⟨2⟩2. Pick $a \in A$
⟨2⟩3. Let: $G : B \to A$ be the function defined by: $G(b)$ is the (unique) $x \in A$ such that $F(x) = b$ if there exists such an x, $G(b) = a$ otherwise.
⟨2⟩4. For all $x \in A$ we have $G(F(x)) = x$.
∎

3.1.2 Surjective Functions

Definition 3.1.11 (Surjective)**.** Let $F : A \to B$. We say that F is *surjective*, or maps A *onto* B, and write $F : A \twoheadrightarrow B$, iff for all $y \in B$ there exists $x \in A$ such that $F(x) = y$.

Proposition Schema 3.1.12 (Z)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
$\mathrm{id}_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$ *is surjective.*

PROOF: For any $y \in \mathbf{A}$ we have $\mathrm{id}_{\mathbf{A}}(y) = y$. □

Proposition Schema 3.1.13 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{F}$ *and* $\mathbf{G}$*, the following is a theorem:*
If $\mathbf{F} : \mathbf{A} \twoheadrightarrow \mathbf{B}$ *and* $\mathbf{G} : \mathbf{B} \twoheadrightarrow \mathbf{C}$*, then* $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \twoheadrightarrow \mathbf{C}$*.*

PROOF:
⟨1⟩1. LET: $c \in \mathbf{C}$
⟨1⟩2. PICK $b \in \mathbf{B}$ such that $\mathbf{G}(b) = c$.
⟨1⟩3. PICK $a \in \mathbf{A}$ such that $\mathbf{F}(a) = b$.
⟨1⟩4. $(\mathbf{G} \circ \mathbf{F})(a) = c$
□

Proposition 3.1.14 (Z)**.** *Let* $F : A \to B$*. There exists* $H : B \to A$ *(a* right inverse*) such that* $F \circ H = \mathrm{id}_B$ *if and only if* F *maps* A *onto* B*.*

PROOF:
⟨1⟩1. If F has a right inverse then F is surjective.
⟨2⟩1. ASSUME: F has a right inverse $H : B \to A$.
⟨2⟩2. LET: $y \in B$
⟨2⟩3. $F(H(y)) = y$
⟨2⟩4. There exists $x \in A$ such that $F(x) = y$
⟨1⟩2. If F is surjective then F has a right inverse.
⟨2⟩1. ASSUME: F is surjective.
⟨2⟩2. PICK a function $H : B \to A$ such that $H \subseteq F^{-1}$
PROOF: Proposition 3.1.6.
⟨2⟩3. $F \circ H = \mathrm{id}_B$
⟨3⟩1. LET: $y \in B$
⟨3⟩2. $(y, H(y)) \in F^{-1}$
⟨3⟩3. $F(H(y)) = y$
□

3.1.3 Bijections

Definition 3.1.15 (Bijection)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$. Then $\mathbf{F}$ is *bijective* or a *bijection*, $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$, iff it is injective and surjective.

Proposition Schema 3.1.16 (Z)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
The identity function $\mathrm{id}_{\mathbf{A}} : \mathbf{A} \approx \mathbf{A}$ *is a bijection.*

PROOF: Proposition 3.1.8 and 3.1.12. □

Proposition Schema 3.1.17 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{F}$*, the following is a theorem:*
If $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$ *then* $\mathbf{F}^{-1} : \mathbf{B} \approx \mathbf{A}$*.*

PROOF:
⟨1⟩1. $\mathbf{F}^{-1} : \mathbf{B} \to \mathbf{A}$
⟨2⟩1. LET: $b \in \mathbf{B}$

⟨2⟩2. PICK $a \in \mathbf{A}$ such that $\mathbf{F}(a) = b$.
PROOF: Since $\mathbf{F}$ is surjective.
⟨2⟩3. $(b, a) \in \mathbf{F}^{-1}$
⟨2⟩4. If $(b, a') \in \mathbf{F}^{-1}$ then $a' = a$.
⟨3⟩1. LET: $a' \in \mathbf{A}$ such that $(b, a') \in \mathbf{F}^{-1}$
⟨3⟩2. $\mathbf{F}(a') = \mathbf{F}(a)$
⟨3⟩3. $a' = a$
PROOF: Since $\mathbf{F}$ is injective.
⟨1⟩2. $\mathbf{F}^{-1}$ is injective.
⟨2⟩1. LET: $x, y \in \mathbf{B}$
⟨2⟩2. ASSUME: $\mathbf{F}^{-1}(x) = \mathbf{F}^{-1}(y)$
⟨2⟩3. $x = y$
PROOF: $x = \mathbf{F}(\mathbf{F}^{-1}(x)) = \mathbf{F}(\mathbf{F}^{-1}(y)) = y$.
⟨1⟩3. $\mathbf{F}^{-1}$ is surjective.
PROOF: For all $a \in \mathbf{A}$ we have $\mathbf{F}^{-1}(\mathbf{F}(a)) = a$.
□

Proposition Schema 3.1.18 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{F}$ *and* $\mathbf{G}$*, the following is a theorem:*
If $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$ *and* $\mathbf{G} : \mathbf{B} \approx \mathbf{C}$ *then* $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \approx \mathbf{C}$.

PROOF: Propositions 3.1.9 and 3.1.13. □

3.1.4 Restrictions

Definition 3.1.19 (Restriction)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$. Let $\mathbf{C} \subseteq \mathbf{A}$. The *restriction* of $\mathbf{F}$ to $\mathbf{C}$, denoted $\mathbf{F} \upharpoonright \mathbf{C}$, is the function

$$\begin{aligned} \mathbf{F} \upharpoonright \mathbf{C} &: \mathbf{C} \to \mathbf{B} \\ (\mathbf{F} \upharpoonright \mathbf{C})(x) &= \mathbf{F}(x) \qquad\qquad (x \in \mathbf{C}) \end{aligned}$$

3.1.5 Images

Definition 3.1.20 (Image)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{C} \subseteq \mathbf{A}$. The *image* of $\mathbf{C}$ under $\mathbf{F}$ is the class

$$\mathbf{F}(\mathbf{C}) := \{\mathbf{F}(x) \mid x \in \mathbf{C}\} \ .$$

Proposition Schema 3.1.21 (Z)**.** *For any classes* $\mathbf{F}$*,* $\mathbf{A}$ *and* $\mathbf{B}$*, the following is a theorem.*
If $\mathbf{F} : \mathbf{A} \to \mathbf{B}$*, then for any subset* $S \subseteq \mathbf{A}$*, the class* $\mathbf{F}(S)$ *is a set.*

PROOF: By an Axiom of Replacement. □

Proposition Schema 3.1.22 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$ *and* $\mathbf{F}$*, the following is a theorem:*
Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C} \subseteq \mathcal{P}\mathbf{A}$*. Then*

$$\mathbf{F}\left(\bigcup \mathbf{C}\right) = \{y \mid \exists X \in \mathbf{C}. y \in \mathbf{F}(X)\}$$

PROOF:

$$\begin{aligned} y \in \mathbf{F}\left(\bigcup \mathbf{C}\right) &\Leftrightarrow \exists x \in \bigcup \mathbf{C}. y = \mathbf{F}(x) \\ &\Leftrightarrow \exists x. \exists X. X \in \mathbf{C} \land x \in X \land y = \mathbf{F}(x) \\ &\Leftrightarrow \exists X \in \mathbf{C}. y \in \mathbf{F}(X) \end{aligned}$$

∎

Proposition Schema 3.1.23 (Z)**.** *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ *and* $\mathbf{F}$, *the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$. *Then*

$$\mathbf{F}(\mathbf{C} \cup \mathbf{D}) = \mathbf{F}(\mathbf{C}) \cup \mathbf{F}(\mathbf{D}) \ .$$

PROOF:

$$\begin{aligned} y \in \mathbf{F}(\mathbf{C} \cup \mathbf{D}) &\Leftrightarrow \exists x \in \mathbf{C} \cup \mathbf{D}. y = \mathbf{F}(x) \\ &\Leftrightarrow \exists x \in \mathbf{C}. y = \mathbf{F}(x) \lor \exists x \in \mathbf{D}. y = \mathbf{F}(x) \\ &\Leftrightarrow y \in \mathbf{F}(\mathbf{C}) \cup \mathbf{F}(\mathbf{D}) \end{aligned}$$

∎

Proposition 3.1.24 (Z)**.** *For any classes* $\mathbf{F}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *and* $\mathbf{D}$, *the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$. *Then*

$$\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B}) \ .$$

Equality holds if $\mathbf{F}$ *is injective.*

PROOF:

$\langle 1 \rangle 1.$ $\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
  $\langle 2 \rangle 1.$ LET: $y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$
  $\langle 2 \rangle 2.$ PICK $x \in \mathbf{A} \cap \mathbf{B}$ such that $y = \mathbf{F}(x)$
  $\langle 2 \rangle 3.$ $y \in \mathbf{F}(\mathbf{A})$
    PROOF: Since $x \in \mathbf{A}$.
  $\langle 2 \rangle 4.$ $y \in \mathbf{F}(\mathbf{B})$
    PROOF: Since $x \in \mathbf{B}$.
$\langle 1 \rangle 2.$ If $\mathbf{F}$ is injective then $\mathbf{F}(\mathbf{A} \cap \mathbf{B}) = \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$.
  $\langle 2 \rangle 1.$ ASSUME: $\mathbf{F}$ is injective.
  $\langle 2 \rangle 2.$ LET: $y \in \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
  $\langle 2 \rangle 3.$ PICK $x \in \mathbf{A}$ such that $y = \mathbf{F}(x)$
  $\langle 2 \rangle 4.$ PICK $x' \in \mathbf{B}$ such that $y = \mathbf{F}(x')$
  $\langle 2 \rangle 5.$ $x = x'$
    PROOF: $\langle 2 \rangle 1$
  $\langle 2 \rangle 6.$ $x \in \mathbf{A} \cap \mathbf{B}$
  $\langle 2 \rangle 7.$ $y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$
∎

Proposition Schema 3.1.25 (Z)**.** *For any classes* $\mathbf{F}$, $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$, *the following is a theorem:*

Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C} \subseteq \mathcal{P}\mathbf{A}$. *Then*

$$\mathbf{F}\left(\bigcap \mathbf{C}\right) \subseteq \bigcap\{\mathbf{F}(X) \mid X \in \mathbf{A}\} \enspace .$$

Equality holds if $\mathbf{F}$ *is injective and* $\mathbf{A}$ *is nonempty.*

PROOF:
$\langle 1 \rangle 1.$ $\mathbf{F}\left(\bigcap \mathbf{A}\right) \subseteq \bigcap\{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
 $\langle 2 \rangle 1.$ LET: $y \in \mathbf{F}(\bigcap \mathbf{A})$
 $\langle 2 \rangle 2.$ PICK $x \in \bigcap \mathbf{A}$ such that $y = \mathbf{F}(x)$
 $\langle 2 \rangle 3.$ LET: $X \in \mathbf{A}$
 $\langle 2 \rangle 4.$ $x \in X$
 $\langle 2 \rangle 5.$ $y \in \mathbf{F}(X)$
$\langle 1 \rangle 2.$ If $\mathbf{F}$ is injective then $\mathbf{F}\left(\bigcap \mathbf{A}\right) = \bigcap\{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
 $\langle 2 \rangle 1.$ ASSUME: $\mathbf{F}$ is injective.
 $\langle 2 \rangle 2.$ ASSUME: $\mathbf{A}$ is nonempty.
 $\langle 2 \rangle 3.$ LET: $y \in \bigcap\{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
 $\langle 2 \rangle 4.$ PICK $X_0 \in \mathbf{A}$
 $\langle 2 \rangle 5.$ PICK $x \in X_0$ such that $(x, y) \in \mathbf{F}$
 $\langle 2 \rangle 6.$ $x \in \bigcap \mathbf{A}$
  $\langle 3 \rangle 1.$ LET: $X \in \mathbf{A}$
  $\langle 3 \rangle 2.$ PICK $x' \in X$ such that $(x', y) \in \mathbf{F}$.
  $\langle 3 \rangle 3.$ $x = x'$
   PROOF: $\langle 2 \rangle 1$
  $\langle 3 \rangle 4.$ $x \in X$
 $\langle 2 \rangle 7.$ $y \in \mathbf{F}(\bigcap \mathbf{A})$
□

Proposition 3.1.26 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{D}$ *and* $\mathbf{F}$*, the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$*. Then*

$$\mathbf{F}(\mathbf{C}) - \mathbf{F}(\mathbf{D}) \subseteq \mathbf{F}(\mathbf{C} - \mathbf{D}) \enspace .$$

Equality holds if $\mathbf{F}$ *is injective.*

PROOF:
$\langle 1 \rangle 1.$ $\mathbf{F}(\mathbf{C}) - \mathbf{F}(\mathbf{D}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
 $\langle 2 \rangle 1.$ LET: $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
 $\langle 2 \rangle 2.$ PICK $x \in \mathbf{A}$ such that $y = \mathbf{F}(x)$
 $\langle 2 \rangle 3.$ $x \notin \mathbf{B}$
 $\langle 2 \rangle 4.$ $x \in \mathbf{A} - \mathbf{B}$
 $\langle 2 \rangle 5.$ $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
$\langle 1 \rangle 2.$ If $\mathbf{F}$ is injective then $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) = \mathbf{F}(\mathbf{A} - \mathbf{B})$
 $\langle 2 \rangle 1.$ ASSUME: $\mathbf{F}$ is injective.
 $\langle 2 \rangle 2.$ LET: $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
 $\langle 2 \rangle 3.$ PICK $x \in \mathbf{A} - \mathbf{B}$ such that $y = \mathbf{F}(x)$
 $\langle 2 \rangle 4.$ $y \in \mathbf{F}(\mathbf{A})$
 $\langle 2 \rangle 5.$ $y \notin \mathbf{F}(\mathbf{B})$

⟨3⟩1. ASSUME: for a contradiction $y \in \mathbf{F}(\mathbf{B})$
⟨3⟩2. PICK $x' \in \mathbf{B}$ such that $y = \mathbf{F}(x')$
⟨3⟩3. $x = x'$
PROOF: ⟨2⟩1
⟨3⟩4. $x \in \mathbf{B}$
⟨3⟩5. Q.E.D.
PROOF: This contradicts ⟨2⟩3.
□

3.1.6 Inverse Images

Definition 3.1.27 (Inverse Image)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{C} \subseteq \mathbf{B}$. Then the *inverse image* of $\mathbf{C}$ under $\mathbf{F}$ is

$$\mathbf{F}^{-1}(\mathbf{C}) = \{x \in \mathbf{A} \mid \mathbf{F}(x) \in \mathbf{C}\} \enspace .$$

Proposition Schema 3.1.28 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$ *and* $\mathbf{F}$*, the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C} \subseteq \mathcal{P}\mathbf{B}$*. Then*

$$\mathbf{F}^{-1}\left(\bigcap \mathbf{C}\right) = \bigcap \{\mathbf{F}^{-1}(X) \mid X \in \mathbf{C}\} \enspace .$$

PROOF:

$$\begin{aligned} x \in \mathbf{F}^{-1}\left(\bigcap \mathbf{C}\right) &\Leftrightarrow \mathbf{F}(x) \in \bigcap \mathbf{C} \\ &\Leftrightarrow \forall X \in \mathbf{C}. \mathbf{F}(x) \in X \\ &\Leftrightarrow \forall X \in \mathbf{C}. x \in \mathbf{F}^{-1}(X) \end{aligned}$$ □

Proposition Schema 3.1.29 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{D}$ *and* $\mathbf{F}$*, the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C}, \mathbf{D} \subseteq \mathbf{B}$*. Then*

$$\mathbf{F}^{-1}(\mathbf{C} - \mathbf{D}) = \mathbf{F}^{-1}(\mathbf{C}) - \mathbf{F}^{-1}(\mathbf{D}) \enspace .$$

PROOF:

$$\begin{aligned} x \in \mathbf{F}^{-1}(\mathbf{C} - \mathbf{D}) &\Leftrightarrow \mathbf{F}(x) \in \mathbf{C} - \mathbf{D} \\ &\Leftrightarrow \mathbf{F}(x) \in \mathbf{C} \wedge \mathbf{F}(x) \notin \mathbf{D} \\ &\Leftrightarrow x \in \mathbf{F}^{-1}(\mathbf{C}) \wedge x \in \mathbf{F}^{-1}(\mathbf{D}) \\ &\Leftrightarrow x \in \mathbf{F}^{-1}(\mathbf{C}) - \mathbf{F}^{-1}(\mathbf{D}) \end{aligned}$$ □

3.1.7 Function Sets

Proposition 3.1.30 (ZFC)**.** *For any classes* $\mathbf{B}$ *and* $\mathbf{F}$*, the following is a theorem:*

Let A *be a set. If* $\mathbf{F} : A \to \mathbf{B}$ *then* $\mathbf{F}$ *is a set.*

PROOF: By an Axiom of Replacement, we have $R = \{\mathbf{F}(x) \mid x \in A\}$ is a set. Hence $\mathbf{F}$ is a set since $\mathbf{F} \subseteq A \times R$. □

Definition 3.1.31 (Dependent Product Class)**.** Let I be a set and let $\mathbf{H}(i)$ be a class for all $i \in I$. We write $\prod_{i \in I} \mathbf{H}(i)$ for the class of all functions $f : I \to \bigcup_{i \in I} \mathbf{H}(i)$ such that $\forall i \in I.f(i) \in \mathbf{H}(i)$.

We write $\mathbf{B}^I$ for $\prod_{i \in I} \mathbf{B}$ where $\mathbf{B}$ does not depend on I.

Proposition Schema 3.1.32 (ZFC)**.** *Let I be a set. Let $H(i)$ be a set for every $i \in I$. Then $\prod_{i \in I} \mathbf{H}(i)$ is a set.*

PROOF:
$\langle 1 \rangle 1.$ $\{\mathbf{H}(i) \mid i \in I\}$ is a set.
PROOF: By an Axiom of Replacement.
$\langle 1 \rangle 2.$ $\bigcup_{i \in I} \mathbf{H}(i)$ is a set.
$\langle 1 \rangle 3.$ $\prod_{i \in I} \mathbf{H}(i)$ is a set.
PROOF: It is a subset of $\mathcal{P}\left(I \times \bigcup_{i \in I} \mathbf{H}(i)\right)$.
□

Proposition 3.1.33 (Z)**.** *Let I be a set. Let $H(i)$ be a set for all $i \in I$. If $\forall i \in I.H(i) \neq \emptyset$ then $\prod_{i \in I} H(i) \neq \emptyset$.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: $\forall i \in I.H(i) \neq \emptyset$
$\langle 1 \rangle 2.$ LET: $R = \{(i, x) \mid i \in I, x \in H(i)\}$
$\langle 1 \rangle 3.$ PICK a function $f : I \to \bigcup_{i \in I} H(i)$ such that $f \subseteq R$
PROOF: Proposition 3.1.6.
$\langle 1 \rangle 4.$ $f \in \prod_{i \in I} H(i)$
□

3.2 Equinumerosity

Definition 3.2.1 (Equinumerous)**.** Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between A and B.

Theorem 3.2.2 (Cantor 1873 (Z))**.** *No set is equinumerous to its power set.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: for a contradiction $f : A \approx \mathcal{P}A$
$\langle 1 \rangle 2.$ LET: $S = \{x \in A \mid x \notin f(x)\}$
$\langle 1 \rangle 3.$ PICK $a \in A$ such that $f(a) = S$
$\langle 1 \rangle 4.$ $a \in S$ if and only if $a \notin S$
$\langle 1 \rangle 5.$ Q.E.D.
PROOF: This is a contradiction.
□

3.3 Domination

Definition 3.3.1 (Dominate)**.** A set A is *dominated* by a set B, $A \preccurlyeq B$, iff there exists an injection $A \to B$.

Proposition 3.3.2 (Z)**.** *Given sets A and B, if $A \neq \emptyset$ or $B = \emptyset$, then we have $A \preccurlyeq B$ iff there exists a surjective function $B \to A$.*

Proof:
⟨1⟩1. If $A \preccurlyeq B$ and $A \neq \emptyset$ then there exists a surjective function $B \to A$.
  ⟨2⟩1. Assume: $f : A \to B$ be injective.
  ⟨2⟩2. Pick $a \in A$
  ⟨2⟩3. Let: $g : B \to A$ be the function defined by $g(b) = f^{-1}(b)$ if $b \in \operatorname{ran} f$, and $g(b) = a$ otherwise.
  ⟨2⟩4. g is surjective.
⟨1⟩2. If there exists a surjective function $B \to A$ then $A \preccurlyeq B$.
  ⟨2⟩1. Assume: there exists a surjective function $g : B \to A$
  ⟨2⟩2. $\forall a \in A . \exists b \in B . g(b) = a$
  ⟨2⟩3. Choose a function $f : A \to B$ such that $\forall a \in A . g(f(a)) = a$
  ⟨2⟩4. f is injective.
□

Chapter 4

Equivalence Relations

Definition 4.0.1 (Equivalence Relation)**.** An *equivalence relation* on a class $\mathbf{A}$ is a binary relation on $\mathbf{A}$ that is reflexive, symmetric and transitive.

Proposition 4.0.2 (Z)**.** *Equinumerosity is an equivalence relation on the class of all sets.*

PROOF: Propositions 3.1.16, 3.1.17, 3.1.18. □

Definition 4.0.3 (Respects)**.** Let $\mathbf{R}$ be an equivalence relation on $\mathbf{A}$ and $\mathbf{F} : \mathbf{A} \to \mathbf{B}$. Then $\mathbf{F}$ *respects* $\mathbf{A}$ iff, whenever $(x, y) \in \mathbf{R}$, then $\mathbf{F}(x) = \mathbf{F}(y)$.

Definition 4.0.4 (Equivalence Class)**.** Let $\mathbf{R}$ be an equivalence relation on $\mathbf{A}$ and $a \in \mathbf{A}$. The *equivalence class* of *a modulo* $\mathbf{R}$ is

$$[a]_{\mathbf{R}} := \{x \mid a\mathbf{R}x\} \ .$$

Proposition Schema 4.0.5 (Z)**.** *For any classes* $\mathbf{A}$ *and* $\mathbf{R}$*, the following is a theorem.*

Assume $\mathbf{R}$ *be an equivalence relation on* $\mathbf{A}$*. Let* $a, b \in \mathbf{A}$*. Then* $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ *if and only if* $a\mathbf{R}b$*.*

PROOF:
⟨1⟩1. If $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ then $a\mathbf{R}b$.
  ⟨2⟩1. ASSUME: $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$
  ⟨2⟩2. $b\mathbf{R}b$
    PROOF: Reflexivity
  ⟨2⟩3. $b \in [b]_{\mathbf{R}}$
  ⟨2⟩4. $b \in [a]_{\mathbf{R}}$
  ⟨2⟩5. $a\mathbf{R}b$
⟨1⟩2. If $a\mathbf{R}b$ then $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$.
  ⟨2⟩1. For all $x, y \in \mathbf{A}$, if $x\mathbf{R}y$ then $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$
    ⟨3⟩1. LET: $x, y \in \mathbf{A}$
    ⟨3⟩2. ASSUME: $x\mathbf{R}y$

⟨3⟩3. LET: $t \in [y]_{\mathbf{R}}$
⟨3⟩4. $y\mathbf{R}t$
⟨3⟩5. $x\mathbf{R}t$
PROOF: Transitivity, ⟨3⟩2, ⟨3⟩4.
⟨3⟩6. $t \in [x]_{\mathbf{R}}$
⟨2⟩2. ASSUME: $a\mathbf{R}b$
⟨2⟩3. $[b]_{\mathbf{R}} \subseteq [a]_{\mathbf{R}}$
PROOF: ⟨2⟩1, ⟨2⟩2.
⟨2⟩4. $b\mathbf{R}a$
PROOF: Symmetry, ⟨2⟩2.
⟨2⟩5. $[a]_{\mathbf{R}} \subseteq [b]_{\mathbf{R}}$
PROOF: ⟨2⟩1, ⟨2⟩4.
⟨2⟩6. $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$
PROOF: ⟨2⟩3, ⟨2⟩5.
□

Definition 4.0.6 (Partition)**.** A *partition* Π of a set A is a set of nonempty subsets of A that is disjoint and exhaustive, i.e.

1. no two different sets in Π have any common elements, and
2. each element of A is in some set in Π.

Definition 4.0.7. Let R be an equivalence relation on a set A. The *quotient set* A/R is the set of all equivalence classes.

Theorem 4.0.8 (Z)**.** *Let A be a set and $\mathbf{B}$ a class. Let R be an equivalence relation on A and $F : A \to \mathbf{B}$. Then F respects R if and only if there exists $\hat{F} : A/R \to \mathbf{B}$ such that*

$$\forall a \in A.\hat{F}([a]_R) = F(a) \ .$$

In this case, $\hat{F}$ is unique.

PROOF:
⟨1⟩1. If F respects R then there exists $\hat{F} : A/R \to \mathbf{B}$ such that $\forall a \in A.\hat{F}([a]_R) = F(a)$.
⟨2⟩1. ASSUME: F respects R.
⟨2⟩2. LET: $\hat{F} = \{([a]_R, F(a)) \mid a \in A\}$
⟨2⟩3. $\hat{F}$ is a function.
⟨3⟩1. ASSUME: $a, a' \in A$ and $[a]_R = [a']_R$
PROVE: $F(a) = F(a')$
⟨3⟩2. $(a, a') \in R$
PROOF: Proposition 4.0.5.
⟨3⟩3. $F(a) = F(a')$
PROOF: ⟨2⟩1
⟨2⟩4. $\operatorname{dom} \hat{F} = A/R$
⟨2⟩5. $\operatorname{ran} \hat{F} \subseteq \mathbf{B}$

⟨2⟩6. $\forall a \in A.\hat{F}([a]_R) = F(a)$

⟨1⟩2. If there exists $\hat{F} : A/R \to \mathbf{B}$ such that $\forall a \in A.\hat{F}([a]_R) = F(a)$ then F respects R.

⟨2⟩1. Assume: $\hat{F} : A/R \to \mathbf{B}$ and $\forall a \in A.\hat{F}([a]_R) = F(a)$

⟨2⟩2. Let: $a, a' \in A$

⟨2⟩3. Assume: $(a, a') \in R$

⟨2⟩4. $[a]_R = [a']_R$

Proof: Proposition 4.0.5.

⟨2⟩5. $F(a) = F(a')$

Proof: ⟨2⟩1

⟨1⟩3. If $G, H : A/R \to \mathbf{B}$ and $\forall a \in A.G([a]_R) = H([a]_R)$ then $G = H$.

□

Proposition 4.0.9 (Z)**.** *Let R be an equivalence relation on a set A. Then A/R is a partition of A.*

Proof:

⟨1⟩1. Every member of A/R is nonempty.

Proof: Since $a \in [a]_R$ by reflexivity.

⟨1⟩2. No two different sets in A/R have any common elements.

⟨2⟩1. Let: $[a]_R, [b]_R \in A/R$

⟨2⟩2. Let: $c \in [a]_R \cap [b]_R$

Prove: $[a]_R = [b]_R$

⟨2⟩3. aRc

Proof: ⟨2⟩2

⟨2⟩4. bRc

Proof: ⟨2⟩2

⟨2⟩5. cRb

Proof: Symmetry, ⟨2⟩4

⟨2⟩6. aRb

Proof: Transitivity, ⟨2⟩3, ⟨2⟩5

⟨2⟩7. $[a]_R = [b]_R$

Proof: Proposition 4.0.5, ⟨2⟩6

⟨1⟩3. Each element of A is in some set in A/R.

Proof: Since $a \in [a]_R$ by reflexivity.

□

Proposition 4.0.10 (Z)**.** *For any partition P of a set A, there exists a unique equivalence relation R on A such that $A/R = P$, namely xRy iff $\exists X \in P(x \in X \wedge y \in X)$.*

Proof: Easy. □

Definition 4.0.11 (Natural Map)**.** Let A be a set and R an equivalence relation on A. The *natural map* $A \to A/R$ is the function that maps $a \in A$ to $[a]_R$.

Chapter 5

Ordering Relations

5.1 Partial Orders

Definition 5.1.1 (Partial Ordering)**.** Let $\mathbf{A}$ be a class. A *partial ordering* on $\mathbf{A}$ is a relation $\mathbf{R}$ on $\mathbf{A}$ that is reflexive, antisymmetric and transitive.

We often write $\leq$ for a partial ordering, and then write $x < y$ for $x \leq y \wedge x \neq y$.

Proposition Schema 5.1.2. *For any classes* $\mathbf{A}$ *and* $\mathbf{R}$*, the following is a theorem:*

If $\mathbf{R}$ *is a partial order on* $\mathbf{A}$ *then so is* $\mathbf{R}^{-1}$*.*

PROOF:
⟨1⟩1. $\mathbf{R}^{-1}$ is reflexive.
PROOF: Proposition 2.2.8.
⟨1⟩2. $\mathbf{R}^{-1}$ is antisymmetric.
PROOF: Proposition 2.2.12.
⟨1⟩3. $\mathbf{R}^{-1}$ is transitive.
⟨2⟩1. ASSUME: $x\mathbf{R}^{-1}y$ and $y\mathbf{R}^{-1}z$
⟨2⟩2. $y\mathbf{R}x$ and $z\mathbf{R}y$
⟨2⟩3. $z\mathbf{R}x$
PROOF: Since $\mathbf{R}$ is transitive.
⟨2⟩4. $x\mathbf{R}^{-1}z$
□

Proposition Schema 5.1.3. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{F}$ *and* $\mathbf{R}$*, the following is a theorem:*

Assume $\mathbf{R}$ *is a partial order on* $\mathbf{B}$ *and* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *is injective. Define* $\mathbf{S}$ *on* $\mathbf{A}$ *by* $x\mathbf{S}y$ *iff* $\mathbf{F}(x)\mathbf{R}\mathbf{F}(y)$*. Then* $\mathbf{S}$ *is a partial order on* $\mathbf{A}$*.*

PROOF:
⟨1⟩1. $\mathbf{S}$ is reflexive.
PROOF: For any $x \in \mathbf{A}$ we have $\mathbf{F}(x)\mathbf{R}\mathbf{F}(x)$.

$\langle 1\rangle 2.$ **S** is antisymmetric.
  $\langle 2\rangle 1.$ LET: $x, y \in \mathbf{A}$
  $\langle 2\rangle 2.$ ASSUME: $x\mathbf{S}y$ and $y\mathbf{S}x$
  $\langle 2\rangle 3.$ $\mathbf{F}(x)\mathbf{R}\mathbf{F}(y)$ and $\mathbf{F}(y)\mathbf{R}\mathbf{F}(x)$
  $\langle 2\rangle 4.$ $\mathbf{F}(x) = \mathbf{F}(y)$
    PROOF: **R** is antisymmetric.
  $\langle 2\rangle 5.$ $x = y$
$\langle 1\rangle 3.$ **S** is transitive.
□

Corollary Schema 5.1.3.1. *For any classes* **A**, **B** *and* **R**, *the following is a theorem:*

Assume **R** *be a partial order on* **A** *and* $\mathbf{B} \subseteq \mathbf{A}$. *Then* $\mathbf{R} \cap \mathbf{B}^2$ *is a partial order on* **B**.

Definition 5.1.4 (Partially Ordered Set)**.** A *partially ordered set* or *poset* is a pair $(A, \leq)$ where A is a set and $\leq$ is a partial ordering on A. We often write just A for $(A, \leq)$.

If $(A, \leq)$ is a poset and $B \subseteq A$ we write just B for the poset $(B, \leq \cap B^2)$.

Definition 5.1.5 (Strictly Monotone)**.** Let $(A, <_A)$ and $(B, <_B)$ be posets. A function $f : A \to B$ is *strictly monotone* iff, whenever $x <_A y$, then $f(x) <_B f(y)$.

Definition 5.1.6 (Least)**.** Let $\leq$ be a partial order on **A**. An element $m \in \mathbf{A}$ is *least* iff for all $x \in \mathbf{A}$ we have $m \leq x$.

Proposition 5.1.7. *A partial order has at most one least element.*

PROOF: If m and m' are least then $m \leq m'$ and $m' \leq m$, so $m = m'$. □

Definition 5.1.8 (Greatst)**.** Let $\leq$ be a partial order on **A**. An element $m \in \mathbf{A}$ is *greatest* iff for all $x \in A$ we have $x \leq m$.

Proposition 5.1.9. *A poset has at most one greatest element.*

PROOF: If m and m' are greatest then $m \leq m'$ and $m' \leq m$, so $m = m'$. □

Definition 5.1.10 (Upper Bound)**.** Let $\leq$ be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Let $u \in \mathbf{A}$. Then u is an *upper bound* for **B** iff $\forall x \in \mathbf{B}. x \leq u$.

Definition 5.1.11 (Lower Bound)**.** Let $\leq$ be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Let $l \in \mathbf{A}$. Then l is a *lower bound* for **B** iff $\forall x \in \mathbf{B}. l \leq x$.

Definition 5.1.12 (Bounded Above)**.** Let $\leq$ be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Then **B** is *bounded above* iff it has an upper bound.

Definition 5.1.13 (Bounded Below)**.** Let $\leq$ be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Then **B** is *bounded below* iff it has a lower bound.

Definition 5.1.14 (Least Upper Bound)**.** Let $\leq$ be a partial ordering on $\mathbf{A}$ and $\mathbf{B} \subseteq \mathbf{A}$. Let $s \in \mathbf{A}$. Then s is the *least upper bound* or *supremum* of $\mathbf{B}$ iff s is an upper bound for $\mathbf{B}$ and, for every upper bound u for $\mathbf{B}$, we have $s \leq u$.

Definition 5.1.15 (Greatest Lower Bound)**.** Let $\leq$ be a partial ordering on $\mathbf{A}$ and $\mathbf{B} \subseteq \mathbf{A}$. Let $i \in \mathbf{A}$. Then i is the *greatest lower bound* or *infimum* of $\mathbf{B}$ iff i is a lower bound for $\mathbf{B}$ and, for every lower bound l for $\mathbf{B}$, we have $i \leq l$.

Definition 5.1.16 (Complete)**.** A partial order is *complete* iff every nonempty subset bounded above has a supremum, and every nonempty subset bounded below has an infimum.

Definition 5.1.17 (Order Isomorphism)**.** Let A and B be posets. An *order isomorphism* between A and B, $f : A \cong B$, is a bijection $f : A \approx B$ such that f and f^{-1} are monotone.

Theorem 5.1.18 (Knaster Fixed-Point Theorem)**.** *Let A be a complete poset with a greatest and least element. Let $\phi : A \to A$ be monotone. Then there exists $a \in A$ such that $\phi(a) = a$.*

Proof:
$\langle 1 \rangle 1.$ Let: $B = \{x \in A \mid x \leq \phi(x)\}$
$\langle 1 \rangle 2.$ Let: $a = \sup B$
  Proof: B is nonempty because the least element of A is in B, and it is bounded above by the greatest element of A.
$\langle 1 \rangle 3.$ For all $b \in B$ we have $b \leq \phi(a)$
  $\langle 2 \rangle 1.$ Let: $b \in B$
  $\langle 2 \rangle 2.$ $b \leq \phi(b)$
  $\langle 2 \rangle 3.$ $b \leq a$
  $\langle 2 \rangle 4.$ $\phi(b) \leq \phi(a)$
  $\langle 2 \rangle 5.$ $b \leq \phi(a)$
$\langle 1 \rangle 4.$ $a \leq \phi(a)$
$\langle 1 \rangle 5.$ $\phi(a) \leq \phi(\phi(a))$
$\langle 1 \rangle 6.$ $\phi(a) \in B$
$\langle 1 \rangle 7.$ $\phi(a) \leq a$
$\langle 1 \rangle 8.$ $\phi(a) = a$
$\square$

Definition 5.1.19 (Dense)**.** Let $\leq$ be a partial order on $\mathbf{A}$ and $\mathbf{B} \subseteq \mathbf{A}$. Then $\mathbf{B}$ is *dense* iff, for all $x, y \in \mathbf{A}$, if $x < y$ then there exists $z \in \mathbf{B}$ such that $x < z < y$.

Proposition 5.1.20. *Let A be a complete poset with no least element. Let $B \subseteq A$ be dense. Let $\theta : A \to A$ be a monotone map that is the identity on B. Then $\theta = \mathrm{id}_A$.*

Proof:
$\langle 1 \rangle 1.$ Let: $a \in A$
    Prove: $\theta(a) = a$

$\langle 1\rangle 2.$ LET: $S(a) = \{b \in B \mid b < a\}$
$\langle 1\rangle 3.$ $S(a)$ is nonempty and bounded above.
  $\langle 2\rangle 1.$ $S(a)$ is nonempty.
    $\langle 3\rangle 1.$ PICK $a_1 < a$
      PROOF: Since a is not least.
    $\langle 3\rangle 2.$ There exists $b \in B$ such that $a_1 < b < a$.
  $\langle 2\rangle 2.$ $S(a)$ is bounded above by a.
$\langle 1\rangle 4.$ $\sup S(a) \leq a$
$\langle 1\rangle 5.$ $\sup S(a) = a$
  $\langle 2\rangle 1.$ ASSUME: for a contradiction $\sup S(a) < a$
  $\langle 2\rangle 2.$ PICK $b \in B$ such that $\sup S(a) < b < a$
  $\langle 2\rangle 3.$ $b \in S(a)$
  $\langle 2\rangle 4.$ Q.E.D.
    PROOF: This contradicts the fact that $\sup S(a) < b$.
$\langle 1\rangle 6.$ For all $b \in S(a)$ we have $b \leq \theta(a)$
  $\langle 2\rangle 1.$ LET: $b \in S(a)$
  $\langle 2\rangle 2.$ $b < a$
  $\langle 2\rangle 3.$ $\theta(b) \leq \theta(a)$
  $\langle 2\rangle 4.$ $b \leq \theta(a)$
    PROOF: $\theta(b) = b$
$\langle 1\rangle 7.$ $a \leq \theta(a)$
  PROOF: Since $a = \sup S(a)$ and $\theta(a)$ is an upper bound for $S(a)$.
$\langle 1\rangle 8.$ $a \not< \theta(a)$
  $\langle 2\rangle 1.$ ASSUME: for a contradiction $a < \theta(a)$.
  $\langle 2\rangle 2.$ PICK $b \in B$ such that $a < b < \theta(a)$
  $\langle 2\rangle 3.$ $\theta(a) \leq \theta(b) = b$
  $\langle 2\rangle 4.$ Q.E.D.
    PROOF: This contradicts the fact that $b < \theta(a)$.
$\langle 1\rangle 9.$ $\theta(a) = a$
$\square$

Theorem 5.1.21. *Let A and P be complete posets with no least or greatest element. Let B be dense in A and Q be dense in P. Every order isomorphism $\phi : B \cong Q$ extends uniquely to an order isomorphism $A \cong P$.*

PROOF:
$\langle 1\rangle 1.$ For $a \in A$, let $S(a) = \{b \in B \mid b < a\}$.
$\langle 1\rangle 2.$ Define $\overline{\phi} : A \to P$ by $\overline{\phi}(a) = \sup \phi(S(a))$.
  $\langle 2\rangle 1.$ $\phi(S(a))$ is nonempty.
    $\langle 3\rangle 1.$ PICK $a_1 < a$
      PROOF: Since a is not least.
    $\langle 3\rangle 2.$ PICK $b \in B$ such that $a_1 < b < a$.
    $\langle 3\rangle 3.$ $\phi(b) \in \phi(S(a))$
  $\langle 2\rangle 2.$ $\phi(S(a))$ is bounded above.
    $\langle 3\rangle 1.$ PICK $a_2 > a$
      PROOF: Since a is not greatest.
    $\langle 3\rangle 2.$ PICK $b \in B$ such that $a < b < a_2$

⟨3⟩3. $\phi(b)$ is an upper bound for $\phi(S(a))$.

⟨1⟩3. $\overline{\phi}$ is monotone.

PROOF: If $a \leq a'$ then $S(a) \subseteq S(a')$ and so $\overline{\phi}(a) \leq \overline{\phi}(a')$.

⟨1⟩4. $\overline{\phi}$ extends ϕ.

⟨2⟩1. LET: $b \in B$

PROVE: $\phi(b) = \sup \phi(S(b))$

⟨2⟩2. $\phi(b)$ is an upper bound for $\phi(S(b))$

⟨2⟩3. LET: u be any upper bound for $\phi(S(b))$

PROVE: $\phi(b) \leq u$

⟨2⟩4. ASSUME: for a contradiction $u < \phi(b)$

⟨2⟩5. PICK $q \in Q$ such that $u < q < \phi(b)$

⟨2⟩6. PICK $b' \in B$ such that $\phi(b') = q$

⟨2⟩7. $b' < b$

⟨2⟩8. $b' \in S(b)$

⟨2⟩9. $q = \phi(b') \leq u$

⟨2⟩10. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩5. LET: $\psi = \phi^{-1}$

⟨1⟩6. LET: $\overline{\psi} : P \to A$ be the function $\overline{\psi}(p) = \sup\{\psi(q) \mid q \in Q, q < p\}$

⟨1⟩7. $\overline{\psi}$ is monotone and extends ψ

PROOF: Similar.

⟨1⟩8. $\overline{\psi} \circ \overline{\phi} : A \to A$ is monotone and the identity on B.

⟨1⟩9. $\overline{\psi} \circ \overline{\phi} = \mathrm{id}_A$

PROOF: Proposition 5.1.20.

⟨1⟩10. $\overline{\phi} \circ \overline{\psi} = \mathrm{id}_B$

PROOF: Proposition 5.1.20.

⟨1⟩11. If $\phi^* : A \cong P$ is any order isomorphism that extends ϕ then $\phi^* = \overline{\phi}$.

⟨2⟩1. LET: $a \in A$

PROVE: $\phi^*(a) = \sup \phi(S(a))$

⟨2⟩2. $\phi^*(a)$ is an upper bound for $\phi(S(a))$

⟨2⟩3. LET: u be any upper bound for $\phi(S(a))$

PROVE: $\phi^*(a) \leq u$

⟨2⟩4. ASSUME: for a contradiction $u < \phi^*(a)$

⟨2⟩5. PICK $q \in Q$ such that $u < q < \phi^*(a)$

⟨2⟩6. PICK $b \in B$ such that $q = \phi(b)$

⟨2⟩7. $b < a$

⟨2⟩8. $b \in S(a)$

⟨2⟩9. $q = \phi(b) \leq u$

⟨2⟩10. Q.E.D.

PROOF: This is a contradiction.

□

Definition 5.1.22 (Initial Segment)**.** Let $\leq$ be a partial order on $\mathbf{A}$ and $t \in A$. The *initial segment* up to t is the class

$$\operatorname{seg} t := \{x \in \mathbf{A} \mid x < t\} \ .$$

Definition 5.1.23 (Lexicographic Ordering)**.** Let $\mathbf{R}$ be a partial order on $\mathbf{A}$ and $\mathbf{S}$ a partial order on $\mathbf{B}$. The *lexicographic ordering* $\leq$ on $\mathbf{A} \times \mathbf{B}$ is defined by:

$$(a, b) \leq (a', b') \Leftrightarrow (a\mathbf{R}a' \wedge a \neq a') \vee (a = a' \wedge b\mathbf{S}b') \ .$$

Proposition Schema 5.1.24. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{R}$ *and* $\mathbf{S}$*, the following is a theorem:*

If $\mathbf{R}$ *is a partial order on* $\mathbf{A}$ *and* $\mathbf{S}$ *is a partial order on* $\mathbf{B}$ *then the lexicographic ordering on* $\mathbf{A} \times \mathbf{B}$ *is a partial order.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\leq$ be the lexicographic ordering on $\mathbf{A} \times \mathbf{B}$
$\langle 1 \rangle 2.$ $\leq$ is reflexive.
  PROOF: For any $a \in \mathbf{A}$ and $b \in \mathbf{B}$ we have $a = a$ and $b\mathbf{S}b$, so $(a, b) \leq (a, b)$.
$\langle 1 \rangle 3.$ $\leq$ is antisymmetric.
  $\langle 2 \rangle 1.$ ASSUME: $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$
  $\langle 2 \rangle 2.$ $(a\mathbf{R}a' \wedge a \neq a') \vee (a = a' \wedge b\mathbf{S}b')$
  $\langle 2 \rangle 3.$ $(a'\mathbf{R}a \wedge a' \neq a) \vee (a' = a \wedge b\mathbf{S}b')$
  $\langle 2 \rangle 4.$ CASE: $a = a'$
    PROOF: Then $b\mathbf{S}b'$ and $b'\mathbf{S}b$ hence $b = b'$ and $(a, b) = (a', b')$.
  $\langle 2 \rangle 5.$ CASE: $a \neq a'$
    PROOF: Then $a\mathbf{R}a'$ and $a'\mathbf{R}a$ hence $a = a'$ which is a contradiction.
$\langle 1 \rangle 4.$ $\leq$ is transitive.
  $\langle 2 \rangle 1.$ ASSUME: $(a_1, b_1) \leq (a_2, b_2) \leq (a_3, b_3)$
  $\langle 2 \rangle 2.$ $(a_1\mathbf{R}a_2 \wedge a_1 \neq a_2) \vee (a_1 = a_2 \wedge b_1\mathbf{S}b_2)$
  $\langle 2 \rangle 3.$ $(a_2\mathbf{R}a_3 \wedge a_2 \neq a_3) \vee (a_2 = a_3 \wedge b_2\mathbf{S}b_3)$
  $\langle 2 \rangle 4.$ CASE: $a_1\mathbf{R}a_2, a_1 \neq a_2, a_2\mathbf{R}a_3, a_2 \neq a_3$
    $\langle 3 \rangle 1.$ $a_1\mathbf{R}a_3$
      PROOF: Since $\mathbf{R}$ is transitive.
    $\langle 3 \rangle 2.$ $a_1 \neq a_3$
      PROOF: If $a_1 = a_3$ then $a_1\mathbf{R}a_2$ and $a_2\mathbf{R}a_1$ so $a_1 = a_2$ which is a contradiction.
  $\langle 2 \rangle 5.$ CASE: $a_1\mathbf{R}a_2, a_1 \neq a_2, a_2 = a_3, b_2\mathbf{S}b_3$
    PROOF: Then $a_1\mathbf{R}a_3$ and $a_1 \neq a_3$.
  $\langle 2 \rangle 6.$ CASE: $a_1 = a_2, b_1\mathbf{S}b_2, a_2\mathbf{R}a_3, a_2 \neq a_3$
    PROOF: Then $a_1\mathbf{R}a_3$ and $a_1 \neq a_3$.
  $\langle 2 \rangle 7.$ CASE: $a_1 = a_2, b_1\mathbf{S}b_2, a_2 = a_3, b_2\mathbf{S}b_3$
    PROOF: Then $a_1 = a_3$ and $b_1\mathbf{S}b_3$.

$\square$

5.2 Linear Orders

Definition 5.2.1 (Linear Ordering)**.** Let $\mathbf{A}$ be a class. A *linear ordering* or *total ordering* on $\mathbf{A}$ is a partial ordering $\leq$ on $\mathbf{A}$ that is *total*, i.e.

$$\forall x, y \in \mathbf{A}. x \leq y \vee y \leq x$$

We often use the symbol $<$ for a linear ordering, and then write $x < y$ for $(x, y) \in <$.

Proposition Schema 5.2.2 (Trichotomy)**.** *For any classes* $\mathbf{A}$ *and* $\leq$*, the following is a theorem:*

Assume $\leq$ *be a linear ordering on* $\mathbf{A}$*. For any* $x, y \in \mathbf{A}$*, exactly one of* $x < y$*,* $x = y$*,* $y < x$ *holds.*

PROOF: Immediate from definitions. □

Proposition Schema 5.2.3. *For any classes* $\mathbf{A}$ *and* $<$*, the following is a theorem:*

Let $<$ *be a transitive relation on* $\mathbf{A}$ *that satisfies trichotomy. Define* $\leq$ *on* $\mathbf{A}$ *by* $x \leq y$ *iff* $x < y$ *or* $x = y$*. Then* $\leq$ *is a linear ordering on* $\mathbf{A}$ *and* $x < y$ *iff* $x \leq y$ *and* $x \neq y$*.*

PROOF:
⟨1⟩1. $\leq$ is reflexive.
  PROOF: By definition we have $\forall x \in \mathbf{A}. x \leq x$.
⟨1⟩2. $\leq$ is antisymmetric.
  ⟨2⟩1. ASSUME: $x \leq y$ and $y \leq x$
  ⟨2⟩2. $x < y$ or $x = y$
  ⟨2⟩3. $y < x$ or $y = x$
  ⟨2⟩4. We cannot have $x < y$ and $y < x$
    PROOF: Trichotomy.
  ⟨2⟩5. $x = y$
⟨1⟩3. $\leq$ is transitive.
  ⟨2⟩1. ASSUME: $x \leq y$ and $y \leq z$
  ⟨2⟩2. $x < y$ or $x = y$
  ⟨2⟩3. $y < z$ or $y = z$
  ⟨2⟩4. CASE: $x < y$ and $y < z$
    PROOF: Then $x < z$ by transitivity, so $x \leq z$.
  ⟨2⟩5. CASE: $x = y$
    PROOF: Then we have $y \leq z$ and so $x \leq z$.
  ⟨2⟩6. CASE: $y = z$
    PROOF: Then we have $x \leq y$ and so $x \leq z$.
⟨1⟩4. $\leq$ is total.
  PROOF: Immediate from trichotomy.
□

Proposition Schema 5.2.4. *For any classes* $\mathbf{A}$ *and* $\mathbf{R}$*, the following is a theorem:*

If $\mathbf{R}$ *is a linear ordering on* $\mathbf{A}$ *then* $\mathbf{R}^{-1}$ *is also a linear ordering on* $\mathbf{A}$*.*

PROOF:
⟨1⟩1. $\mathbf{R}^{-1}$ is a partial order on $\mathbf{A}$.
  PROOF: Proposition 5.1.2.
⟨1⟩2. $\mathbf{R}^{-1}$ is total.

⟨2⟩1. LET: $x, y \in \mathbf{A}$
⟨2⟩2. $x\mathbf{R}y$ or $y\mathbf{R}x$.
⟨2⟩3. $y\mathbf{R}^{-1}x$ or $x\mathbf{R}^{-1}y$.
□

Proposition Schema 5.2.5. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{F}$, $\mathbf{R}$, $\mathbf{S}$, *the following is a theorem:*

Assume $\mathbf{R}$ *is a linear order on* $\mathbf{A}$, $\mathbf{S}$ *is a partial order on* $\mathbf{B}$, *and* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$. *If* $\mathbf{F}$ *is strictly monotone then it is injective.*

PROOF:
⟨1⟩1. LET: $x, y \in \mathbf{A}$
⟨1⟩2. ASSUME: $x \neq y$
PROVE: $\mathbf{F}(x) \neq \mathbf{F}(y)$
⟨1⟩3. ASSUME: w.l.o.g. $x\mathbf{R}y$
PROOF: $\mathbf{R}$ is total.
⟨1⟩4. $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$ and $\mathbf{F}(x) \neq \mathbf{F}(y)$
PROOF: $\mathbf{F}$ is strictly monotone.
□

Proposition Schema 5.2.6. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\leq$, $\preccurlyeq$ *and* $\mathbf{F}$, *the following is a theorem:*

Assume $\leq$ *is a linear order on* $\mathbf{A}$ *and* $\preccurlyeq$ *is a linear order on* $\mathbf{B}$. *Assume* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{F}$ *is strictly monotone. For all* $x, y \in \mathbf{A}$, *if* $\mathbf{F}(x) \prec \mathbf{F}(y)$ *then* $x < y$.

PROOF:
⟨1⟩1. $\mathbf{F}(x) \neq \mathbf{F}(y)$ and $\mathbf{F}(y) \not\prec \mathbf{F}(x)$
PROOF: Trichotomy.
⟨1⟩2. $x \neq y$ and $y \not< x$
PROOF: $\mathbf{F}$ is strictly monotone.
⟨1⟩3. $x < y$
PROOF: Trichotomy.
□

Corollary Schema 5.2.6.1. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\leq$, $\preccurlyeq$ *and* $\mathbf{F}$, *the following is a theorem:*

Assume $\leq$ *is a linear order on* $\mathbf{A}$ *and* $\preccurlyeq$ *is a linear order on* $\mathbf{B}$. *Assume* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{F}$ *is strictly monotone. Then* $\mathbf{F}$ *is an order isomorphism.*

Proposition Schema 5.2.7. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{F}$ *and* $\mathbf{S}$, *the following is a theorem:*

Assume $\mathbf{S}$ *is a linear order on* $\mathbf{B}$ *and* $\mathbf{F} : \mathbf{A} \rightarrowtail \mathbf{B}$. *Define* $\mathbf{R}$ *on* $\mathbf{A}$ *by* $x\mathbf{R}y$ *if and only if* $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$. *Then* $\mathbf{R}$ *is a linear order on* $\mathbf{A}$.

PROOF:
⟨1⟩1. $\mathbf{R}$ is a partial order on $\mathbf{A}$.
PROOF: Proposition 5.1.3.

$\langle 1 \rangle 2$. $\mathbf{R}$ is total.

PROOF: For all $x, y \in \mathbf{A}$ we have $\mathbf{F}(x)\mathbf{SF}(y)$ or $\mathbf{F}(y)\mathbf{SF}(x)$.

□

Corollary Schema 5.2.7.1. *For any classes* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{R}$, *the following is a theorem:*

Assume $\mathbf{R}$ *be a linear order on* $\mathbf{A}$ *and* $\mathbf{B} \subseteq \mathbf{A}$. *Then* $\mathbf{R} \cap \mathbf{B}^2$ *is a linear order on* $\mathbf{B}$.

Proposition Schema 5.2.8. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{R}$ *and* $\mathbf{S}$, *the following is a theorem:*

Assume $\mathbf{R}$ *is a linear order on* $\mathbf{A}$ *and* $\mathbf{S}$ *is a linear order on* $\mathbf{B}$. *Then the lexicographic ordering is a linear order on* $\mathbf{A} \times \mathbf{B}$.

PROOF:

$\langle 1 \rangle 1$. LET: $\leq$ be the lexicographic order on $\mathbf{A} \times \mathbf{B}$

$\langle 1 \rangle 2$. $\leq$ is a partial order.

PROOF: Proposition 5.1.24.

$\langle 1 \rangle 3$. $\leq$ is total.

$\langle 2 \rangle 1$. LET: $a, a' \in \mathbf{A}$ and $b, b' \in \mathbf{B}$

$\langle 2 \rangle 2$. CASE: $a\mathbf{R}a'$ and $a \neq a'$

PROOF: Then $(a, b) \leq (a', b')$.

$\langle 2 \rangle 3$. CASE: $a = a'$

PROOF: We have $b\mathbf{S}b'$ or $b'\mathbf{S}b$, so $(a, b) \leq (a', b')$ or $(a', b') \leq (a, b)$.

$\langle 2 \rangle 4$. CASE: $a'\mathbf{R}a$ and $a \neq a'$

PROOF: Then $(a', b') \leq (a, b)$.

□

5.3 Well Orderings

Definition 5.3.1 (Well Ordering)**.** A *well ordering* on a class $\mathbf{A}$ is a well-founded linear ordering on $\mathbf{A}$.

Proposition Schema 5.3.2. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{F}$ *and* $\mathbf{S}$, *the following is a theorem:*

Assume $\mathbf{S}$ *well orders* $\mathbf{B}$ *and* $\mathbf{F} : \mathbf{A} \rightarrowtail \mathbf{B}$. *Define* $\mathbf{R}$ *on* $\mathbf{A}$ *by* $x\mathbf{R}y$ *if and only if* $\mathbf{F}(x)\mathbf{SF}(y)$. *Then* $\mathbf{R}$ *well orders* $\mathbf{A}$.

PROOF:

$\langle 1 \rangle 1$. $\mathbf{R}$ linearly orders $\mathbf{A}$.

PROOF: Proposition 5.2.7.

$\langle 1 \rangle 2$. For all $t \in \mathbf{A}$ we have $\{x \in \mathbf{A} \mid x\mathbf{R}t \wedge x \neq t\}$ is a set.

$\langle 2 \rangle 1$. LET: $t \in \mathbf{A}$

$\langle 2 \rangle 2$. LET: $S = \{y \in \mathbf{B} \mid y\mathbf{SF}(t) \wedge y \neq \mathbf{F}(t)\}$

$\langle 2 \rangle 3$. LET: $P(x, y)$ be the property $\mathbf{F}(y) = x$

$\langle 2 \rangle 4$. For all $x \in S$ there exists at most one y such that $P(x, y)$

PROOF: $\mathbf{F}$ is injective.

⟨2⟩5. LET: $T = \{y \mid \exists x \in S.P(x, y)\}$
PROOF: Axiom of Replacement.
⟨2⟩6. $T = \{x \in \mathbf{A} \mid x\mathbf{R}t \wedge x \neq t\}$
⟨1⟩3. Every nonempty subset of $\mathbf{A}$ has a least element.
⟨2⟩1. LET: S be a nonempty subset of $\mathbf{A}$.
⟨2⟩2. LET: y be the least element of $\mathbf{F}(S)$.
⟨2⟩3. PICK $x \in S$ such that $\mathbf{F}(x) = y$.
⟨2⟩4. x is least in S.
□

Corollary Schema 5.3.2.1. *For any classes* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{R}$, *the following is a theorem:*

If $\mathbf{R}$ *well orders* $\mathbf{B}$ *and* $\mathbf{A} \subseteq \mathbf{B}$ *then* $\mathbf{R} \cap \mathbf{A}^2$ *well orders* $\mathbf{A}$.

Proposition 5.3.3. *For any well ordered sets* A *and* B, *the lexicographic order well orders* $A \times B$.

PROOF:
⟨1⟩1. $A \times B$ is linearly ordered.
PROOF: Proposition 5.2.8.
⟨1⟩2. Every nonempty subset of $A \times B$ has a least element.
⟨2⟩1. LET: S be a nonempty subset of $A \times B$.
⟨2⟩2. LET: a be the least element of $\{x \in A \mid \exists y \in B.(x, y) \in S\}$.
⟨2⟩3. LET: b be the least element of $\{y \in B \mid (a, y) \in S\}$.
⟨2⟩4. (a, b) is least in S.
□

Definition 5.3.4 (End Extension)**.** Let A and B be well ordered sets. Then B is an *end extension* of A iff $A \subseteq B$ and:

- Whenever $x, y \in A$ then $x \leq_A y$ iff $x \leq_B y$.
- Whenever $x \in A$ and $y \in B - A$ then $x < y$.

Theorem 5.3.5. *Let* $\leq$ *be a linear ordering on* A. *Assume that, for any* $B \subseteq A$ *such that* $\forall t \in A. \operatorname{seg} t \subseteq B \Rightarrow t \in B$, *we have* $B = A$. *Then* $\leq$ *is a well ordering on* A.

PROOF:
⟨1⟩1. LET: $C \subseteq A$ be nonempty.
⟨1⟩2. LET: $B = \{t \in A \mid \forall x \in C.t < x\}$
⟨1⟩3. $B \cap C = \emptyset$
⟨1⟩4. $B \neq A$
⟨1⟩5. PICK $t \in A$ such that $\operatorname{seg} t \subseteq B$ and $t \notin B$
⟨1⟩6. t is least in C.
□

Proposition Schema 5.3.6. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{F}$, $\mathbf{G}$, $\leq$ *and* $\preccurlyeq$, *the following is a theorem:*

Assume $\leq$ *well orders* $\mathbf{A}$ *and* $\preccurlyeq$ *well orders* $\mathbf{B}$. *Assume* $\mathbf{F}$ *and* $\mathbf{G}$ *are order isomorphisms between* $\mathbf{A}$ *and* $\mathbf{B}$. *Then* $\mathbf{F} = \mathbf{G}$.

PROOF:
⟨1⟩1. For all $x \in \mathbf{A}$, if $\forall t < x . \mathbf{F}(t) = \mathbf{G}(t)$, then $\mathbf{F}(x) = \mathbf{G}(x)$
⟨2⟩1. LET: $x \in \mathbf{A}$
⟨2⟩2. ASSUME: $\forall t < x . \mathbf{F}(t) = \mathbf{G}(t)$
⟨2⟩3. $\mathbf{F}(\operatorname{seg} x) = \mathbf{G}(\operatorname{seg} x)$
⟨2⟩4. $\mathbf{F}(x)$ is the least element of $\mathbf{B} - \mathbf{F}(\operatorname{seg} x)$
⟨2⟩5. $\mathbf{G}(x)$ is the least element of $\mathbf{B} - \mathbf{G}(\operatorname{seg} x)$
⟨2⟩6. $\mathbf{F}(x) = \mathbf{G}(x)$
⟨1⟩2. $\forall x \in \mathbf{A} . \mathbf{F}(x) = \mathbf{G}(x)$
PROOF: Transfinite induction.
□

Theorem 5.3.7. *Let A and B be well ordered sets. Then one of the following holds: $A \cong B$; there exists $b \in B$ such that $A \cong \operatorname{seg} b$; there exists $a \in A$ such that $\operatorname{seg} a \cong B$.*

PROOF:
⟨1⟩1. PICK e that is not in A or B.
⟨1⟩2. LET: $F : A \to B \cup \{e\}$ be the function defined by transfinite recursion thus:

$$F(t) = \begin{cases} \text{the least element of } B - F(\operatorname{seg} t) & \text{if } B - F(\operatorname{seg} t) \neq \emptyset \\ e & \text{if } B - F(\operatorname{seg} t) = \emptyset \end{cases}$$

⟨1⟩3. CASE: $e \in \operatorname{ran} F$
⟨2⟩1. LET: t be least such that $F(t) = e$
⟨2⟩2. $F \restriction \operatorname{seg} t : \operatorname{seg} t \cong B$
⟨1⟩4. CASE: $\operatorname{ran} F = B$
PROOF: We have $F : A \cong B$
⟨1⟩5. CASE: $\operatorname{ran} F \subsetneq B$
⟨2⟩1. LET: b be the least element of $B - \operatorname{ran} F$
⟨2⟩2. $F : A \cong \operatorname{seg} b$
□

Chapter 6

Ordinal Numbers

6.1 Ordinals

Definition 6.1.1 (Ordinal Number)**.** An *ordinal (number)* is a transitive set α that is *well-ordered by* $\in$; that is, such that $\{(x,y) \in \alpha^2 \mid x \in y \vee x = y\}$ well orders α.

Given $x, y \in \alpha$, we write $x < y$ iff $x \in y$, and $x \leq y$ iff $x \in y$ or $x = y$.

Let **On** be the class of ordinal numbers. For $\alpha, \beta \in$ **On**, we write $\alpha < \beta$ iff $\alpha \in \beta$, and $\alpha \leq \beta$ iff $\alpha < \beta$ or $\alpha = \beta$.

Proposition 6.1.2. *For any ordinal numbers α and β, if $\alpha \cong \beta$ then $\alpha = \beta$.*

Proof:
⟨1⟩1. Let: $f : \alpha \cong \beta$
⟨1⟩2. For all $x \in \alpha$, if $\forall t < x.f(t) = t$ then $f(x) = x$
  ⟨2⟩1. $f(x) \subseteq x$
    ⟨3⟩1. Let: $y \in f(x)$
    ⟨3⟩2. $y \in \beta$
    ⟨3⟩3. Pick $t \in \alpha$ such that $f(t) = y$
      Proof: f is surjective.
    ⟨3⟩4. $f(t) \in f(x)$
    ⟨3⟩5. $t \in x$
      Proof: Since f is an order isomorphism.
    ⟨3⟩6. $f(t) = t$
      Proof: Induction hypothesis.
    ⟨3⟩7. $y = t$
    ⟨3⟩8. $y \in x$
  ⟨2⟩2. $x \subseteq f(x)$
    ⟨3⟩1. Let: $t \in x$
    ⟨3⟩2. $f(t) \in f(x)$
    ⟨3⟩3. $f(t) = t$
    ⟨3⟩4. $t \in f(x)$

$\langle 1 \rangle 3.$ $\forall x \in \alpha . f(x) = x$
PROOF: Transfinite induction.
$\langle 1 \rangle 4.$ $\alpha = \beta$
PROOF: Since $\beta = \{f(t) \mid t \in \alpha\} = \{t \mid t \in \alpha\} = \alpha$.
□

Theorem 6.1.3. *Every well-ordered set is isomorphic to a unique ordinal.*

PROOF:
$\langle 1 \rangle 1.$ For any well-ordered set A, there exists an ordinal α such that $A \cong \alpha$.
$\langle 2 \rangle 1.$ LET: A be a well-ordered set.
$\langle 2 \rangle 2.$ Define the function E on A by transfinite recursion thus:

$$E(t) = \{E(x) \mid x < t\} \qquad (t \in A) \ .$$

$\langle 2 \rangle 3.$ LET: $\alpha = \{E(x) \mid x \in A\}$
$\langle 2 \rangle 4.$ α is an ordinal.
$\langle 3 \rangle 1.$ α is a transitive set.
$\langle 4 \rangle 1.$ LET: $x \in y \in \alpha$
$\langle 4 \rangle 2.$ PICK $t \in A$ such that $y = E(t)$
$\langle 4 \rangle 3.$ $x \in E(t) = \{E(s) \mid s < t\}$
$\langle 4 \rangle 4.$ PICK $s < t$ such that $x = E(s)$
$\langle 4 \rangle 5.$ $x \in \alpha$
$\langle 3 \rangle 2.$ α is well-ordered by $\in$.
$\langle 4 \rangle 1.$ LET: $<= \{(x, y) \in \alpha \mid x \in y\}$
$\langle 4 \rangle 2.$ $<$ is transitive.
$\langle 5 \rangle 1.$ LET: $x, y, z \in \alpha$ with $x \in y \in z$
$\langle 5 \rangle 2.$ PICK $t \in A$ such that $z = E(t)$
$\langle 5 \rangle 3.$ PICK $s \in A$ such that $s < t$ and $y = E(s)$
$\langle 5 \rangle 4.$ PICK $r \in A$ such that $r < s$ and $x = E(r)$
$\langle 5 \rangle 5.$ $r < t$
$\langle 5 \rangle 6.$ $x \in z$
$\langle 4 \rangle 3.$ $<$ satisfies trichotomy.
$\langle 5 \rangle 1.$ LET: $x, y \in \alpha$
$\langle 5 \rangle 2.$ PICK $s, t \in A$ such that $E(s) = x$ and $E(t) = y$
$\langle 5 \rangle 3.$ Exactly one of $s < t$, $s = t$, $t < s$ holds.
$\langle 5 \rangle 4.$ CASE: $s < t$
$\langle 6 \rangle 1.$ $x \in y$
$\langle 6 \rangle 2.$ $x \neq y$ and $y \notin x$
PROOF: Axiom of Regularity.
$\langle 5 \rangle 5.$ CASE: $s = t$
$\langle 6 \rangle 1.$ $x = y$
$\langle 6 \rangle 2.$ $x \notin y$ and $y \notin x$
PROOF: Axiom of Regularity.
$\langle 5 \rangle 6.$ CASE: $t < s$
PROOF: Similar to $\langle 5 \rangle 4$.
$\langle 4 \rangle 4.$ $\leq$ is a linear order on α.
PROOF: Proposition 5.2.3.
$\langle 4 \rangle 5.$ Every nonempty subset of α has a least element.

$\langle 5 \rangle 1$. LET: S be a nonempty subset of α
$\langle 5 \rangle 2$. LET: $T = \{x \in A \mid E(x) \in S\}$
$\langle 5 \rangle 3$. LET: t be the least element of T.
PROVE: $E(t)$ is least in S
$\langle 5 \rangle 4$. LET: $y \in S$
$\langle 5 \rangle 5$. PICK $s \in T$ such that $E(s) = y$
$\langle 5 \rangle 6$. $t \leq s$
$\langle 5 \rangle 7$. $x \leq y$
$\langle 2 \rangle 5$. E is surjective.
PROOF: By definition of α.
$\langle 2 \rangle 6$. E is strictly monotone.
PROOF: If $s < t$ then $E(s) \in E(t)$ by definition of $E(t)$.
$\langle 2 \rangle 7$. Q.E.D.
PROOF: Corollary 5.2.6.1.
$\langle 1 \rangle 2$. For any ordinals α and β, if $\alpha \cong \beta$ then $\alpha = \beta$.
PROOF: Proposition 6.1.2.
□

Proposition 6.1.4. *The class* **On** *is a transitive class. That is, every element of an ordinal is an ordinal.*

PROOF:
$\langle 1 \rangle 1$. LET: α be an ordinal.
$\langle 1 \rangle 2$. LET: $\beta \in \alpha$
$\langle 1 \rangle 3$. β is a transitive set.
$\langle 2 \rangle 1$. LET: $x \in y \in \beta$
$\langle 2 \rangle 2$. $y \in \alpha$
PROOF: α is transitive.
$\langle 2 \rangle 3$. $x \in \alpha$
PROOF: α is transitive.
$\langle 2 \rangle 4$. $x \in \beta$
PROOF: Since $\{(x, y) \in \alpha^2 \mid x \in y\}$ is transitive.
$\langle 1 \rangle 4$. β is well ordered by $\in$.
PROOF: By Corollary 5.3.2.1.
□

Proposition 6.1.5. *Given two ordinal numbers α, β, exactly one of $\alpha \in \beta$, $\alpha = \beta$, $\beta \in \alpha$ holds.*

PROOF:
$\langle 1 \rangle 1$. At most one holds.
PROOF: Since every ordinal is a transitive set and we never have $\alpha \in \alpha$.
$\langle 1 \rangle 2$. At least one holds.
$\langle 2 \rangle 1$. Either $\alpha \cong \beta$ or $\exists t \in \beta . \alpha \cong \operatorname{seg} t$ or $\exists t \in \alpha . \operatorname{seg} t \cong \beta$.
$\langle 2 \rangle 2$. CASE: $\alpha \cong \beta$
PROOF: Then $\alpha = \beta$ by Proposition 6.1.2.
$\langle 2 \rangle 3$. CASE: There exists $t \in \beta$ such that $\alpha \cong \operatorname{seg} t$

$\langle 3 \rangle 1.$ t is an ordinal number.
PROOF: Proposition 6.1.4.
$\langle 3 \rangle 2.$ $t = \operatorname{seg} t$
$\langle 4 \rangle 1.$ $t \subseteq \operatorname{seg} t$
$\langle 5 \rangle 1.$ LET: $s \in t$
$\langle 5 \rangle 2.$ $s \in \beta$
PROOF: β is a transitive set.
$\langle 5 \rangle 3.$ $s \in \operatorname{seg} t$
$\langle 4 \rangle 2.$ $\operatorname{seg} t \subseteq t$
PROOF: Immediate from definitions.
$\langle 3 \rangle 3.$ $\alpha = t$
PROOF: Proposition 6.1.2.
$\langle 3 \rangle 4.$ $\alpha \in \beta$
$\langle 2 \rangle 4.$ CASE: There exists $t \in \alpha$ such that $\operatorname{seg} t \cong \beta$
PROOF: $\beta \in \alpha$ similarly.
□

Proposition 6.1.6. *Any nonempty set S of ordinal numbers has a least element.*

PROOF:
$\langle 1 \rangle 1.$ PICK $\beta \in S$
$\langle 1 \rangle 2.$ CASE: $\beta \cap S = \emptyset$
PROOF: Then β is least in S.
$\langle 1 \rangle 3.$ CASE: $\beta \cap S \neq \emptyset$
PROOF: The least element of $\beta \cap S$ is least in S.
□

Theorem 6.1.7. *The class* **On** *is well ordered by* $\in$.

PROOF:
$\langle 1 \rangle 1.$ LET: $\mathbf{E} = \{(x, y) \in \mathbf{On}^2 \mid x \in y\}$
$\langle 1 \rangle 2.$ $\mathbf{E}$ is transitive.
PROOF: If $\alpha \in \beta \in \gamma$ then $\alpha \in \gamma$ because every ordinal is a transitive set.
$\langle 1 \rangle 3.$ $\mathbf{E}$ satisfies trichotomy.
PROOF: Proposition 6.1.5.
$\langle 1 \rangle 4.$ $\mathbf{E}$ linearly orders $\mathbf{On}$.
PROOF: Proposition 5.2.3.
$\langle 1 \rangle 5.$ For any $\alpha \in \mathbf{On}$, the class $\{\beta \in \mathbf{On} \mid \beta \mathbf{E} \alpha\}$ is a set.
PROOF: It is equal to α.
$\langle 1 \rangle 6.$ Every nonempty subset of $\mathbf{On}$ has an $\mathbf{E}$-least element.
PROOF: Proposition 6.1.6.
□

Corollary 6.1.7.1 (Burali-Forti Paradox)**.** *The class* **On** *is a proper class.*

PROOF: If it were a set, it would be a transitive set well-ordered by $\in$, and hence a member of itself, contradicting the Axiom of Regularity.

Proposition 6.1.8. *Any transitive set of ordinal numbers is an ordinal number.*

PROOF: It is well-ordered by $\in$ by Corollary 5.3.2.1 and Theorem 6.1.7. ∎

Proposition 6.1.9. *$\emptyset$ is an ordinal number.*

PROOF: Vacuously, it is a transitive set well-ordered by $\in$.

Definition 6.1.10. We define $0 = \emptyset$.

Proposition 6.1.11. *If A is a set of ordinal numbers then $\bigcup A$ is an ordinal number.*

PROOF:
$\langle 1 \rangle 1$. $\bigcup A$ is a transitive set.
  PROOF: Proposition 1.6.3.
$\langle 1 \rangle 2$. $\bigcup A$ is a set of ordinals.
  PROOF: Proposition 6.1.4.
∎

Corollary 6.1.11.1. *The poset* **On** *is complete.*

PROOF: For any nonempty set A of ordinals, $\bigcup A$ is its supremum. ∎

Proposition 6.1.12. *Let α be an ordinal and $S \subseteq \alpha$. Then S is well-ordered by $\in$ and the ordinal of $(S, \in)$ is $\leq \alpha$.*

PROOF:
$\langle 1 \rangle 1$. S is well ordered by $\in$.
$\langle 1 \rangle 2$. LET: β be the ordinal of $(S, \in)$
$\langle 1 \rangle 3$. LET: $E : S \approx \beta$ be the unique isomorphism.
$\langle 1 \rangle 4$. $\forall \gamma \in S . E(\gamma) \leq \gamma$
  $\langle 2 \rangle 1$. LET: $\gamma \in S$
  $\langle 2 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall \delta < \gamma . E(\delta) \leq \delta$
  $\langle 2 \rangle 3$. $E(\gamma)$ is the least element of β that is greater than $E(\delta)$ for all $\delta < \gamma$
  $\langle 2 \rangle 4$. γ is greater than $E(\delta)$ for all $\delta < \gamma$
  $\langle 2 \rangle 5$. $E(\gamma) \leq \gamma$
$\langle 1 \rangle 5$. $\beta \leq \alpha$
  $\langle 2 \rangle 1$. $\forall \gamma < \beta . \gamma < \alpha$
    $\langle 3 \rangle 1$. LET: $\gamma < \beta$
    $\langle 3 \rangle 2$. PICK $\delta \in S$ such that $E(\delta) = \gamma$
    $\langle 3 \rangle 3$. $\gamma = E(\delta) \leq \delta < \alpha$
∎

Proposition 6.1.13. *Let α be a set. Then the following are equivalent.*

1. *α is an ordinal.*
2. *α is a transitive set and, for all $x, y \in \alpha$, either $x = y$ or $x \in y$ or $y \in x$.*
3. *α is a transitive set of transitive sets.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  PROOF: Immediate from definitions.
⟨1⟩2. $2 \Rightarrow 3$
  ⟨2⟩1. ASSUME: α is a transitive set and, for all $x, y \in \alpha$, either $x = y$ or $x \in y$ or $y \in x$
  ⟨2⟩2. LET: $z \in \alpha$
    PROVE: z is transitive.
  ⟨2⟩3. LET: $x \in y \in z$
  ⟨2⟩4. $y \in \alpha$
  ⟨2⟩5. $x \in \alpha$
  ⟨2⟩6. Either $x = z$ or $x \in z$ or $z \in x$
  ⟨2⟩7. $x \neq z$
    PROOF: We cannot have $x \in y \in x$ by the Axiom of Regularity.
  ⟨2⟩8. $z \notin x$
    PROOF: We cannot have $x \in y \in z \in x$ by the Axiom of Regularity.
⟨1⟩3. $3 \Rightarrow 1$
  ⟨2⟩1. LET: x be a transitive set of transitive sets.
  ⟨2⟩2. ASSUME: as $\in$-induction hypothesis that, for all $y \in x$, if y is a transitive set of transitive sets then y is a transitive set of ordinals.
  ⟨2⟩3. Every element of x is an ordinal.
    ⟨3⟩1. LET: $y \in x$
    ⟨3⟩2. y is transitive.
    ⟨3⟩3. Every element of y is transitive.
      PROOF: Since every element of y is an element of x, because x is transitive.
    ⟨3⟩4. y is an ordinal.
      PROOF: ⟨2⟩2
  ⟨2⟩4. Q.E.D.
    PROOF: Proposition 6.1.8.

□

Lemma 6.1.14. *Let A and B be well-ordered sets. If B is an end extension of A then the ordinal of A is $\leq$ the ordinal of B.*

PROOF:
⟨1⟩1. LET: α be the ordinal of A and β the ordinal of B.
⟨1⟩2. LET: $E_A : A \cong \alpha$ and $E_B : B \cong \beta$ be the canonical isomorphisms.
⟨1⟩3. $\forall a \in A. E_A(a) = E_B(a)$
  ⟨2⟩1. LET: $a \in A$
  ⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall x < a. E_A(x) = E_B(x)$
  ⟨2⟩3. $E_A(a)$ is the least ordinal that is greater than $E_A(x)$ for all $x < a$
  ⟨2⟩4. $E_B(a)$ is the least ordinal that is greater than $E_B(x)$ for all $x < b$
  ⟨2⟩5. $\{x \in A \mid x <_A a\} = \{x \in B \mid x <_B a\}$
  ⟨2⟩6. $E_A(a) = E_B(a)$
⟨1⟩4. $\alpha \subseteq \beta$
⟨1⟩5. $\alpha \leq \beta$

□

Lemma 6.1.15. *Let $\mathcal{C}$ be a set of well ordered sets such that, for any $A, B \in \mathcal{C}$, we have that one of A and B is an end extension of the other. Let $W = \bigcup \mathcal{C}$ under $x \leq y$ iff there exists $A \in W$ such that $x, y \in A$ and $x \leq y$. Then W is a well ordered set whose ordinal is the supremum of the ordinals of the members of $\mathcal{C}$.*

PROOF:
⟨1⟩1. $\leq$ is reflexive on W.
  ⟨2⟩1. LET: $x \in W$
  ⟨2⟩2. PICK $A \in W$ such that $x \in A$.
  ⟨2⟩3. $x \leq x$
⟨1⟩2. $\leq$ is antisymmetric on W.
  ⟨2⟩1. LET: $x, y \in W$
  ⟨2⟩2. ASSUME: $x \leq y$ and $y \leq x$
  ⟨2⟩3. PICK $A \in W$ such that $x, y \in A$ and $x \leq_A y$, and $B \in W$ such that $x, y \in B$ and $y \leq_B x$
  ⟨2⟩4. ASSUME: w.l.o.g. B is an end extension of A
  ⟨2⟩5. $x \leq_B y$ and $y \leq_B x$
  ⟨2⟩6. $x = y$
⟨1⟩3. $\leq$ is transitive on W.
  ⟨2⟩1. ASSUME: $x \leq y \leq z$
  ⟨2⟩2. PICK $A, B \in W$ such that $x \leq_A y$ and $y \leq_B z$
  ⟨2⟩3. CASE: A is an end extension of B.
    ⟨3⟩1. $x \leq_A y$ and $y \leq_A z$
    ⟨3⟩2. $x \leq_A z$
    ⟨3⟩3. $x \leq z$
  ⟨2⟩4. CASE: B is an end extension of A.
    PROOF: Similar.
⟨1⟩4. $\leq$ is total on W.
  ⟨2⟩1. LET: $x, y \in W$
  ⟨2⟩2. PICK $A, B \in \mathcal{C}$ such that $x \in A$ and $y \in B$
  ⟨2⟩3. ASSUME: w.l.o.g. B is an end extension of A
  ⟨2⟩4. $x \leq_B y$ or $y \leq_B x$
  ⟨2⟩5. $x \leq_W y$ or $y \leq_W x$
⟨1⟩5. Every nonempty subset of W has a least element.
  ⟨2⟩1. LET: S be a nonempty subset of W
  ⟨2⟩2. PICK $s \in S$
  ⟨2⟩3. PICK $A \in \mathcal{C}$ such that $s \in A$
  ⟨2⟩4. LET: a be the $\leq_A$-least element of $S \cap A$
    PROVE: a is least in S
  ⟨2⟩5. LET: $x \in S$
    PROVE: $a \leq x$
  ⟨2⟩6. PICK $B \in \mathcal{C}$ such that $x \in B$
  ⟨2⟩7. CASE: A is an end extension of B
    ⟨3⟩1. $a \leq_A x$

⟨3⟩2. $a \leq x$
⟨2⟩8. CASE: B is an end extension of A
⟨3⟩1. CASE: $x \in A$
⟨4⟩1. $a \leq_A x$
⟨4⟩2. $a \leq x$
⟨3⟩2. CASE: $x \in B - A$
⟨4⟩1. $a \leq_B x$
⟨4⟩2. $a \leq x$
⟨1⟩6. For all $A \in \mathcal{C}$, W is an end extension of A.
⟨2⟩1. For all $x, y \in A$, we have $x \leq_A y$ if and only if $x \leq_W y$
⟨3⟩1. LET: $x, y \in A$
⟨3⟩2. If $x \leq_A y$ then $x \leq_W y$
PROOF: Immediate from definitions.
⟨3⟩3. If $x \leq_W y$ then $x \leq_A y$
⟨4⟩1. ASSUME: $x \leq_W y$
⟨4⟩2. PICK $B \in \mathcal{C}$ such that $x \leq_B y$
⟨4⟩3. CASE: A is an end extension of B
PROOF: Then $x \leq_A y$.
⟨4⟩4. CASE: B is an end extension of A
PROOF: Then $x \leq_A y$.
⟨2⟩2. For all $x \in A$ and $y \in W - A$ we have $x < y$
⟨3⟩1. LET: $x \in A$ and $y \in W - A$
⟨3⟩2. PICK $B \in \mathcal{C}$ such that $y \in B$
⟨3⟩3. B is an end extension of A
⟨3⟩4. $x <_B y$
⟨3⟩5. $x <_W y$
⟨1⟩7. For all $A \in \mathcal{C}$, the ordinal of A is $\leq$ the ordinal of W.
PROOF: Lemma 6.1.14.
⟨1⟩8. For any ordinal α, if for all $A \in \mathcal{C}$ the ordinal of A is $\leq \alpha$, then the ordinal of W is $\leq \alpha$.
⟨2⟩1. LET: α be an ordinal.
⟨2⟩2. ASSUME: for all $A \in \mathcal{C}$, the ordinal of A is $\leq \alpha$
⟨2⟩3. LET: β be the ordinal of W
⟨2⟩4. LET: $E : W \approx \beta$ be the canonical isomorphism.
⟨2⟩5. ASSUME: for a contradiction $\alpha < \beta$
⟨2⟩6. LET: $a \in W$ be the element with $E(a) = \alpha$
⟨2⟩7. PICK $A \in \mathcal{C}$ such that $a \in A$
⟨2⟩8. LET: γ be the ordinal of A and $E_A : A \cong \gamma$ be the canonical isomorphism.
⟨2⟩9. For all $x \in A$ we have $E_A(x) = E(x)$
PROOF: Transfinite induction on x.
⟨2⟩10. $E_A(a) = \alpha$
⟨2⟩11. $\alpha < \gamma$
⟨2⟩12. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
□

6.2 Successors

Definition 6.2.1 (Successor)**.** The *successor* of a set a is the set $a^+ := a \cup \{a\}$.

Proposition 6.2.2. *A set a is a transitive set if and only if*

$$\bigcup(a^+) = a \ .$$

PROOF:
⟨1⟩1. If a is a transitive set then $\bigcup(a^+) = a$.
⟨2⟩1. ASSUME: a is a transitive set.
⟨2⟩2. $\bigcup(a^+) \subseteq a$
⟨3⟩1. LET: $x \in \bigcup(a^+)$
PROVE: $x \in a$
⟨3⟩2. PICK $y \in a^+$ such that $x \in y$.
⟨3⟩3. $y \in a$ or $y = a$.
⟨3⟩4. CASE: $y \in a$
PROOF: Then $x \in a$ because a is a transitive set.
⟨3⟩5. CASE: $y = a$
PROOF: Then $x \in a$ immediately.
⟨2⟩3. $a \subseteq \bigcup(a^+)$
PROOF: Since $a \in a^+$.
⟨1⟩2. If $\bigcup(a^+) = a$ then a is a transitive set.
⟨2⟩1. ASSUME: $\bigcup(a^+) = a$
⟨2⟩2. $\bigcup a \subseteq a$
PROOF:

$$\begin{aligned} \bigcup a &\subseteq \bigcup(a^+) && \text{(Proposition 1.5.9)} \\ &= a && (\langle 2\rangle 1) \end{aligned}$$

⟨2⟩3. a is a transitive set.
PROOF: Proposition 1.6.2.
□

Proposition 6.2.3. *For any set a, we have a is a transitive set if and only if a^+ is a transitive set.*

PROOF:
⟨1⟩1. If a is a transitive set then a^+ is a transitiveset.
PROOF: If a is a transitive set then $\bigcup(a^+) = a \subseteq a^+$ by Proposition 6.2.2 and so a^+ is a transitive set.
⟨1⟩2. If a^+ is a transitive set then a is a transitive set.
⟨2⟩1. ASSUME: a^+ is a transitive set.
⟨2⟩2. LET: $x \in y \in a$
⟨2⟩3. $x \in y \in a^+$
⟨2⟩4. $x \in a^+$
PROOF: ⟨2⟩1
⟨2⟩5. $x \neq a$
PROOF: From ⟨2⟩2 and the Axiom of Regularity.

⟨2⟩6. $x \in a$

□

Definition 6.2.4. We write 0 for $\emptyset$, 1 for $\emptyset^+$, 2 for $\emptyset^{++}$, etc.

Proposition 6.2.5. *For any set A we have $\mathcal{P}A \approx 2^A$.*

PROOF: The function $H : \mathcal{P}A \to 2^A$ defined by $H(S)(a) = \{\emptyset\}$ if $a \in S$ and $\emptyset$ if $a \notin S$ is a bijection. □

Proposition 6.2.6. *For any ordinal number α we have α^+ is an ordinal number.*

PROOF:
⟨1⟩1. α^+ is a transitive set.
  PROOF: Proposition 6.2.3.
⟨1⟩2. α^+ is well-ordered by $\in$.
  ⟨2⟩1. For all $x, y, z \in \alpha^+$, if $x \in y \in z$ then $x \in z$
    ⟨3⟩1. CASE: $z = \alpha$
      PROOF: Then $x \in \alpha$ since α is a transitive set.
    ⟨3⟩2. CASE: $z \in \alpha$
      PROOF: Then $x \in z$ since α is well-ordered by $\in$.
  ⟨2⟩2. For all $x, y \in \alpha^+$ we have $x \in y$ or $x = y$ or $y \in x$
    ⟨3⟩1. CASE: $x, y \in \alpha$
      PROOF: The result follows because α is well-ordered by $\in$.
    ⟨3⟩2. CASE: $x \in \alpha$, $y = \alpha$
      PROOF: Then $x \in y$.
    ⟨3⟩3. CASE: $x = \alpha$, $y \in \alpha$
      PROOF: Then $y \in x$.
    ⟨3⟩4. CASE: $x = \alpha$, $y = \alpha$
      PROOF: Then $x = y$.
  ⟨2⟩3. Every nonempty subset of α^+ has an $\in$-least element.
    ⟨3⟩1. LET: $S \subseteq \alpha^+$ be nonempty
    ⟨3⟩2. CASE: $S = \{\alpha\}$
      PROOF: α is least in S.
    ⟨3⟩3. CASE: $S \neq \{\alpha\}$
      ⟨4⟩1. $S - \{\alpha\}$ is a nonempty subset of α
      ⟨4⟩2. LET: β be least in $S - \{\alpha\}$
      ⟨4⟩3. β is least in S.

□

Proposition 6.2.7. *For ordinals α and β, if $\alpha^+ = \beta^+$ then $\alpha = \beta$.*

PROOF: If $\alpha^+ = \beta^+$ then

$$\begin{aligned} \alpha &= \bigcup(\alpha^+) && \text{(Proposition 6.2.2)} \\ &= \bigcup(\beta^+) \\ &= \beta && \text{(Proposition 6.2.2)} \end{aligned}$$

Proposition 6.2.8. *For ordinals α and β, we have $\alpha < \beta$ if and only if $\alpha^+ < \beta^+$.*

Proof:

$$\begin{aligned} \alpha < \beta &\Leftrightarrow \alpha^+ \leq \beta \\ &\Leftrightarrow \alpha^+ < \beta^+ \end{aligned} \quad \square$$

Definition 6.2.9 (Successor Ordinal)**.** An ordinal α is a *successor ordinal* iff $\alpha = \beta^+$ for some β.

Definition 6.2.10 (Limit Ordinal)**.** A *limit ordinal* is an ordinal that is neither 0 nor a successor ordinal.

Proposition 6.2.11. *If λ is a limit ordinal and $\beta < \lambda$ then $\beta^+ < \lambda$.*

Proof: Since $\beta^+ \leq \lambda$ and $\beta^+ \neq \lambda$. $\square$

6.3 The Well-Ordering Theorem and Zorn's Lemma

Theorem 6.3.1 (Hartogs)**.** *For any set A, there exists an ordinal not dominated by A.*

Proof:
$\langle 1 \rangle 1$. Let: α be the class of all ordinals β such that $\beta \preccurlyeq A$
Prove: α is a set.
$\langle 1 \rangle 2$. Let: $W = \{(B, R) \mid B \subseteq A, R \text{ is a well ordering on } B\}$
$\langle 1 \rangle 3$. α is the class of the ordinals of the elements of W.
$\langle 2 \rangle 1$. For all $(B, R) \in W$, the ordinal of (B, R) is in α.
$\langle 3 \rangle 1$. Let: $(B, R) \in W$
$\langle 3 \rangle 2$. Let: β be the ordinal of (B, R)
$\langle 3 \rangle 3$. Let: $E : B \cong \beta$ be the canonical isomorphism.
$\langle 3 \rangle 4$. Let: $i : B \hookrightarrow A$ be the inclusion
$\langle 3 \rangle 5$. $i \circ E^{-1}$ is an injection $\beta \to A$
$\langle 3 \rangle 6$. $\beta \in \alpha$
$\langle 2 \rangle 2$. For all $\beta \in \alpha$, there exists $(B, R) \in W$ such that β is the ordinal number of (B, R).
$\langle 3 \rangle 1$. Let: $\beta \in \alpha$
$\langle 3 \rangle 2$. Pick an injection $f : \beta \to A$
$\langle 3 \rangle 3$. Define $\leq$ on $\operatorname{ran} f$ by $f(x) \leq f(y)$ iff $x \leq y$
$\langle 3 \rangle 4$. $(\operatorname{ran} f, \leq) \in W$
$\langle 3 \rangle 5$. β is the ordinal number of $(\operatorname{ran} f, \leq)$
$\langle 1 \rangle 4$. α is a set.
Proof: By an Axiom of Replacement.
$\langle 1 \rangle 5$. α is an ordinal.
Proof: It is a transitive set of ordinals.
$\langle 1 \rangle 6$. $\alpha \not\preccurlyeq A$
Proof: Since $\alpha \notin \alpha$.

□

Theorem 6.3.2 (Numeration Theorem)**.** *Every set is equinumerous with some ordinal.*

PROOF:
$\langle 1\rangle 1.$ LET: A be any set.
$\langle 1\rangle 2.$ PICK an ordinal α not dominated by A.
$\langle 1\rangle 3.$ PICK a choice function G for A.
$\langle 1\rangle 4.$ PICK $e \notin A$
$\langle 1\rangle 5.$ LET: $F : \alpha \to A \cup \{e\}$ by transfinite recursion:

$$F(\gamma) = \begin{cases} G(A - F(\{\delta \mid \delta < \gamma\}) & \text{if } A - F(\{\delta \mid \delta < \gamma\}) \neq \emptyset \\ e & \text{if } A - F(\{\delta \mid \delta < \gamma\}) = \emptyset \end{cases}$$

$\langle 1\rangle 6.$ $e \in \operatorname{ran} F$
  $\langle 2\rangle 1.$ ASSUME: for a contradiction $e \notin \operatorname{ran} F$
  $\langle 2\rangle 2.$ F is an injection $\alpha \to A$.
    $\langle 3\rangle 1.$ LET: $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$
      PROVE: $F(\beta) \neq F(\gamma)$
    $\langle 3\rangle 2.$ ASSUME: w.l.o.g. $\beta < \gamma$
    $\langle 3\rangle 3.$ $F(\gamma) \in A - F(\{\delta \mid \delta < \gamma\})$
    $\langle 3\rangle 4.$ $F(\gamma) \notin F(\{\delta \mid \delta < \gamma\})$
    $\langle 3\rangle 5.$ $F(\gamma) \neq F(\beta)$
  $\langle 2\rangle 3.$ Q.E.D.
    PROOF: This contradicts $\langle 1\rangle 2$.
$\langle 1\rangle 7.$ LET: δ be least such that $F(\delta) = e$
$\langle 1\rangle 8.$ $F \upharpoonright \delta : \delta \approx A$

Theorem 6.3.3 (Well-Ordering Theorem)**.** *Any set can be well ordered.*

PROOF:
$\langle 1\rangle 1.$ PICK an ordinal δ and a bijection $F : A \approx \delta$
$\langle 1\rangle 2.$ Define $\leq$ on A by $F(x) \leq F(y)$ iff $x \leq y$ for $x, y \in \delta$
$\langle 1\rangle 3.$ $\leq$ is a well ordering on A.
□

Theorem 6.3.4 (Zorn's Lemma)**.** *Let $\mathcal{A}$ be a set such that, for every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$. Then $\mathcal{A}$ has a maximal element.*

PROOF:
$\langle 1\rangle 1.$ PICK a well ordering $<$ on $\mathcal{A}$.
$\langle 1\rangle 2.$ LET: $F : \mathcal{A} \to 2$ be the function defined by transfinite recursion by:

$$F(A) = \begin{cases} 1 & \text{if } A \text{ includes every set } B < A \text{ for which } F(B) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\langle 1\rangle 3.$ LET: $\mathcal{C} = \{A \in \mathcal{A} \mid F(A) = 1\}$
  PROVE: $\bigcup \mathcal{C}$ is a maximal element of $\mathcal{A}$
$\langle 1\rangle 4.$ For all $A \in \mathcal{A}$, we have $A \in \mathcal{C}$ iff $\forall B < A. B \in \mathcal{C} \Rightarrow B \subseteq A$
$\langle 1\rangle 5.$ $\mathcal{C}$ is a chain.

$\langle 2 \rangle 1.$ LET: $A, A' \in \mathcal{C}$
$\langle 2 \rangle 2.$ ASSUME: w.l.o.g. $A \leq A'$
$\langle 2 \rangle 3.$ $A \subseteq A'$
PROOF: By $\langle 1 \rangle 4$
$\langle 1 \rangle 6.$ $\bigcup \mathcal{C} \in \mathcal{A}$
$\langle 1 \rangle 7.$ $\bigcup \mathcal{C}$ is maximal in $\mathcal{A}$.
$\langle 2 \rangle 1.$ LET: $A \in \mathcal{A}$ and $\bigcup \mathcal{C} \subseteq A$
$\langle 2 \rangle 2.$ $A \in \mathcal{C}$
PROOF: By $\langle 1 \rangle 4$ since $\forall B \in \mathcal{C}. B \subseteq A$.
$\langle 2 \rangle 3.$ $A \subseteq \bigcup \mathcal{C}$
$\langle 2 \rangle 4.$ $A = \bigcup \mathcal{C}$
□

Proposition 6.3.5 (Teichmüller-Tukey Lemma)**.** *Let $\mathcal{A}$ be a nonempty set such that, for every B, we have $B \in \mathcal{A}$ if and only if every finite subset of B is a member of $\mathcal{A}$. Then $\mathcal{A}$ has a maximal element.*

PROOF:
$\langle 1 \rangle 1.$ For every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$
$\langle 2 \rangle 1.$ LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain.
$\langle 2 \rangle 2.$ Every finite subset of $\bigcup \mathcal{B}$ is a member of $\mathcal{A}$.
$\langle 3 \rangle 1.$ LET: C be a finite subset of $\bigcup \mathcal{B}$.
$\langle 3 \rangle 2.$ PICK $B \in \mathcal{B}$ such that $C \subseteq B$.
$\langle 3 \rangle 3.$ $B \in \mathcal{A}$
$\langle 3 \rangle 4.$ Every finite subset of B is in $\mathcal{A}$.
$\langle 3 \rangle 5.$ $C \in \mathcal{A}$
$\langle 2 \rangle 3.$ $\bigcup \mathcal{B} \in \mathcal{A}$.
$\langle 1 \rangle 2.$ Q.E.D.
PROOF: Zorn's lemma.
□

Theorem Schema 6.3.6. *For any class $\mathbf{A}$, there exists a class $\mathbf{F}$ such that the following is a theorem:*
If $\mathbf{A}$ is a proper class of ordinals, then $\mathbf{F} : \mathbf{On} \to \mathbf{A}$ is an order isomorphism.

PROOF:
$\langle 1 \rangle 1.$ Define $\mathbf{F} : \mathbf{On} \to \mathbf{A}$ by transfinite recursion as follows: $\mathbf{F}(\alpha)$ is the least element of $\mathbf{A}$ that is different from $\mathbf{F}(\beta)$ for all $\beta < \alpha$.
$\langle 1 \rangle 2.$ For all $\alpha, \beta \in \mathbf{On}$, if $\alpha < \beta$ then $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$
PROOF: We have $\mathbf{F}(\alpha) \neq \mathbf{F}(\beta)$ by the definition of $\mathbf{F}(\beta)$, and $\mathbf{F}(\beta) \not< \mathbf{F}(\alpha)$ by the leastness of $\mathbf{F}(\alpha)$.
$\langle 1 \rangle 3.$ $\mathbf{F}$ is surjective.
$\langle 2 \rangle 1.$ LET: $\alpha \in \mathbf{A}$
$\langle 2 \rangle 2.$ ASSUME: as transfinite induction hypothesis $\forall \beta \in \mathbf{A}$, if $\beta < \alpha$ then there exists γ such that $\beta = \mathbf{F}(\gamma)$.
$\langle 2 \rangle 3.$ LET: $\gamma = \{\delta \in \mathbf{On} \mid \mathbf{F}(\delta) < \alpha\}$
$\langle 2 \rangle 4.$ γ is a set.

PROOF: Axiom of Replacement applied to α.
⟨2⟩5. γ is a transitive set.
PROOF: If $\mathbf{F}(\delta) < \alpha$ and $\epsilon < \delta$ then $\mathbf{F}(\epsilon) < \alpha$ by ⟨1⟩2.
⟨2⟩6. γ is an ordinal.
PROOF: Proposition 6.1.8.
⟨2⟩7. $\mathbf{F}(\gamma) = \alpha$
⟨3⟩1. $\mathbf{F}(\gamma)$ is the least element of $\mathbf{A}$ different from $\mathbf{F}(\delta)$ for all $\delta < \gamma$
⟨3⟩2. $\mathbf{F}(\gamma)$ is the least element of $\mathbf{A}$ different from x for all $x \in \mathbf{A}$ with $x < \alpha$
⟨3⟩3. $\mathbf{F}(\gamma) = \alpha$
□

6.4 Ordinal Operations

Definition 6.4.1 (Ordinal Operation)**.** An *ordinal operation* is a function $\mathbf{On} \to \mathbf{On}$.

Definition 6.4.2 (Continuous)**.** An ordinal operation $\mathbf{T} : \mathbf{On} \to \mathbf{On}$ is *continuous* iff, for every limit ordinal λ, we have $\mathbf{T}(\lambda) = \bigcup_{\alpha<\lambda} \mathbf{T}(\alpha)$.

Definition 6.4.3 (Normal)**.** An ordinal operation is *normal* iff it is continuous and strictly monotone.

Proposition Schema 6.4.4. *For any class $\mathbf{T}$, the following is a theorem.*

If $\mathbf{T}$ is a continuous ordinal operation and $\forall \gamma . \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$, then $\mathbf{T}$ is normal.

PROOF:
⟨1⟩1. LET: $P[\beta]$ be the property $\forall \gamma < \beta . \mathbf{T}(\gamma) < \mathbf{T}(\beta)$
⟨1⟩2. $P[0]$
PROOF: Vacuous.
⟨1⟩3. For any ordinal γ, if $P[\gamma]$ then $P[\gamma^+]$
⟨2⟩1. ASSUME: $P[\gamma]$
⟨2⟩2. LET: $\delta < \gamma^+$
⟨2⟩3. CASE: $\delta < \gamma$
PROOF: Then $\mathbf{T}(\delta) < \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$.
⟨2⟩4. CASE: $\delta = \gamma$
PROOF: Then $\mathbf{T}(\delta) = \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$.
⟨1⟩4. For any limit ordinal λ, if $\forall \gamma < \lambda . P[\gamma]$ then $P[\lambda]$.
⟨2⟩1. ASSUME: $\forall \gamma < \lambda . P[\gamma]$
⟨2⟩2. LET: $\delta < \lambda$
⟨2⟩3. $\mathbf{T}(\delta) < \mathbf{T}(\lambda)$
PROOF:

$$\begin{aligned}\mathbf{T}(\delta) &< \mathbf{T}(\delta^+) \\ &\leq \bigcup_{\epsilon<\lambda} \mathbf{T}(\epsilon) \\ &= \mathbf{T}(\lambda)\end{aligned}$$

∎

Proposition Schema 6.4.5. *For any class* $\mathbf{T}$*, the following is a theorem:*

Assume $\mathbf{T}$ *is a normal ordinal operation. For every ordinal* α*, we have* $\alpha \leq \mathbf{T}(\alpha)$*.*

PROOF:
$\langle 1 \rangle 1$. LET: γ be an ordinal.
$\langle 1 \rangle 2$. ASSUME: as induction hypothesis $\forall \delta < \gamma . \mathbf{T}(\delta) \geq \delta$
$\langle 1 \rangle 3$. For all $\delta < \gamma$ we have $\delta < \mathbf{T}(\gamma)$
  PROOF: $\mathbf{T}$ is strictly monotone.
$\langle 1 \rangle 4$. $\gamma \leq \mathbf{T}(\gamma)$
∎

Proposition Schema 6.4.6. *For any class* $\mathbf{T}$*, the following is a theorem:*

Assume $\mathbf{T}$ *is a normal ordinal operation. For any ordinal* $\beta \geq \mathbf{T}(0)$*, there exists a greatest ordinal* γ *such that* $\mathbf{T}(\gamma) \leq \beta$*.*

PROOF:
$\langle 1 \rangle 1$. There exists γ such that $\mathbf{T}(\gamma) > \beta$
  $\langle 2 \rangle 1$. For all γ we have $\mathbf{T}(\gamma) \geq \gamma$
    PROOF: Proposition 6.4.5.
  $\langle 2 \rangle 2$. $\mathbf{T}(\beta^+) > \beta$
$\langle 1 \rangle 2$. LET: δ be least such that $\mathbf{T}(\delta) > \beta$
$\langle 1 \rangle 3$. δ is a successor ordinal.
  $\langle 2 \rangle 1$. $\delta \neq 0$
    PROOF: Since $\mathbf{T}(0) \leq \beta$.
  $\langle 2 \rangle 2$. δ is not a limit ordinal.
    $\langle 3 \rangle 1$. ASSUME: for a contradiction δ is a limit ordinal.
    $\langle 3 \rangle 2$. $\beta < \bigcup_{\epsilon < \delta} \mathbf{T}(\epsilon)$
      PROOF: $\mathbf{T}$ is continuous.
    $\langle 3 \rangle 3$. There exists $\epsilon < \delta$ such that $\beta < \mathbf{T}(\epsilon)$
    $\langle 3 \rangle 4$. Q.E.D.
      PROOF: This contradicts the minimality of δ.
$\langle 1 \rangle 4$. LET: $\delta = \gamma^+$
$\langle 1 \rangle 5$. γ is greatest such that $\mathbf{T}(\gamma) \leq \beta$
∎

Theorem Schema 6.4.7. *For any class* $\mathbf{T}$*, the following is a theorem:*

Assume that $\mathbf{T}$ *is a normal ordinal operation. For any nonempty set of ordinals* S*, we have*

$$\mathbf{T}(\sup S) = \sup_{\alpha \in S} \mathbf{T}(\alpha) \ .$$

PROOF:
$\langle 1 \rangle 1$. $\forall \alpha \in S . \mathbf{T}(\alpha) \leq \mathbf{T}(\sup S)$
  PROOF: Since $\mathbf{T}$ is monotone.
$\langle 1 \rangle 2$. For any ordinal β, if $\forall \alpha \in S . \mathbf{T}(\alpha) \leq \beta$, then $\mathbf{T}(\sup S) \leq \beta$

⟨2⟩1. LET: β be an ordinal.
⟨2⟩2. LET: $\gamma = \sup S$
⟨2⟩3. ASSUME: $\forall \alpha \in S.\mathbf{T}(\alpha) \leq \beta$
⟨2⟩4. CASE: γ is 0 or a successor ordinal
PROOF: Then we must have $\gamma \in S$ so $\mathbf{T}(\gamma) \leq \beta$ from ⟨2⟩3.
⟨2⟩5. CASE: γ is a limit ordinal
⟨3⟩1. $\mathbf{T}(\gamma) = \sup_{\alpha<\gamma} \mathbf{T}(\alpha)$
PROOF: $\mathbf{T}$ is continuous.
⟨3⟩2. ASSUME: for a contradiction $\beta < \mathbf{T}(\gamma)$
⟨3⟩3. PICK $\alpha < \gamma$ such that $\beta < \mathbf{T}(\alpha)$
PROOF: ⟨3⟩1, ⟨3⟩2
⟨3⟩4. PICK $\alpha' \in S$ such that $\alpha < \alpha'$
PROOF: ⟨2⟩2, ⟨3⟩3
⟨3⟩5. $\beta < \mathbf{T}(\alpha') \leq \beta$
PROOF: $\mathbf{T}$ is strictly monotone, ⟨3⟩3, ⟨3⟩4, ⟨2⟩3.
⟨3⟩6. Q.E.D.
PROOF: This is a contradiction.

□

Proposition 6.4.8. *For any classes* $\mathbf{A}$ *and* $\mathbf{T}$*, the following is a theorem:*

Assume $\mathbf{A}$ *is a proper class of ordinals such that, for every set* $S \subseteq \mathbf{A}$*, we have* $\bigcup S \in \mathbf{A}$*. Assume* $\mathbf{T}$ *is the unique order isomorphism* $\mathbf{On} \cong \mathbf{A}$*. Then* $\mathbf{T}$ *is normal.*

PROOF:
⟨1⟩1. $\mathbf{T}$ is strictly monotone.
PROOF: Since it is an order isomorphism.
⟨1⟩2. $\mathbf{T}$ is continuous.
⟨2⟩1. LET: λ be a limit ordinal.
⟨2⟩2. $\mathbf{T}'(\lambda)$ is the least member of $\mathbf{A}$ that is greater than $\mathbf{T}'(\alpha)$ for all $\alpha < \lambda$
⟨2⟩3. $\mathbf{T}'(\lambda) = \sup_{\alpha<\lambda} \mathbf{T}'(\alpha)$

□

Proposition Schema 6.4.9. *For any class* $\mathbf{T}$*, the following is a theorem:*

If $\mathbf{T}$ *is a normal ordinal operation, then for any limit ordinal* λ*, we have* $\mathbf{T}(\lambda)$ *is a limit ordinal.*

PROOF:
⟨1⟩1. $\mathbf{T}(\lambda) \neq 0$
PROOF: Since $0 \leq \mathbf{T}(0) < \mathbf{T}(\lambda)$.
⟨1⟩2. $\mathbf{T}(\lambda)$ is not a successor ordinal.
⟨2⟩1. ASSUME: for a contradiction $\mathbf{T}(\lambda) = \alpha^+$
⟨2⟩2. $\alpha < \mathbf{T}(\lambda) = \sup_{\beta<\lambda} \mathbf{T}(\beta)$
⟨2⟩3. PICK $\beta < \lambda$ such that $\alpha < \mathbf{T}(\beta)$
⟨2⟩4. $\alpha^+ \leq \mathbf{T}(\beta) < \mathbf{T}(\lambda)$
⟨2⟩5. Q.E.D.
PROOF: This is a contradiction.

□

6.5 Ordinal Arithmetic

6.5.1 Addition

Definition 6.5.1. Let A and B be disjoint well-ordered sets. The *concatenation* of A and B is the set $A \cup B$ under the relation:

- if $a, a' \in A$ then $a \leq a'$ iff $a \leq a'$ in A
- if $b, b' \in B$ then $b \leq b'$ iff $b \leq b'$ in B
- if $a \in A$ and $b \in B$ then $a \leq b$ and $b \nleq a$.

Proposition 6.5.2. *If A and B are disjoint well-ordered sets, then their concatenation is well-ordered.*

PROOF:
$\langle 1 \rangle 1$. $\leq$ is reflexive.
PROOF: For all $a \in A$ we have $a \leq a$, and for all $b \in B$ we have $b \leq b$.
$\langle 1 \rangle 2$. $\leq$ is antisymmetric.
$\langle 2 \rangle 1$. ASSUME: $x \leq y \leq x$
$\langle 2 \rangle 2$. CASE: $x, y \in A$
PROOF: Then $x = y$ since the order on A is antisymmetric.
$\langle 2 \rangle 3$. CASE: $x \in A$ and $y \in B$
PROOF: This is impossible as it would imply $y \nleq x$.
$\langle 2 \rangle 4$. CASE: $x \in B$ and $y \in A$
PROOF: This is impossible as it would imply $x \nleq y$.
$\langle 2 \rangle 5$. CASE: $x, y \in B$
PROOF: Then $x = y$ since the order on B is antisymmetric.
$\langle 1 \rangle 3$. $\leq$ is transitive.
$\langle 2 \rangle 1$. ASSUME: $x \leq y \leq z$
$\langle 2 \rangle 2$. CASE: $x, z \in A$
PROOF: In this case $y \in A$ since $y \leq z$, and so $x \leq z$ since the order on A is transitive.
$\langle 2 \rangle 3$. CASE: $x \in A$ and $z \in B$
PROOF: Then $x \leq z$ immediately.
$\langle 2 \rangle 4$. CASE: $x \in B$ and $z \in A$
PROOF: This is impossible because we have $y \notin A$ since $x \leq y$ and $y \notin B$ since $y \leq z$.
$\langle 2 \rangle 5$. CASE: $x, z \in B$
PROOF: In this case $y \in B$ since $x \leq y$, and so $x \leq z$ since the order on B is transitive.
$\langle 1 \rangle 4$. $\leq$ is total.
$\langle 2 \rangle 1$. LET: $x, y \in A \cup B$
$\langle 2 \rangle 2$. CASE: $x, y \in A$
PROOF: Then $x \leq y$ or $y \leq x$ because the order on A is total.
$\langle 2 \rangle 3$. CASE: $x \in A$ and $y \in B$
PROOF: Then $x \leq y$.

$\langle 2\rangle 4$. CASE: $x \in B$ and $y \in A$
PROOF: Then $y \leq x$.
$\langle 2\rangle 5$. CASE: $x, y \in B$
PROOF: Then $x \leq y$ or $y \leq x$ because the order on B is total.
$\langle 1\rangle 5$. Every nonempty subset of $A \cup B$ has a least element.
$\langle 2\rangle 1$. LET: S be a nonempty subset of $A \cup B$
$\langle 2\rangle 2$. CASE: $S \cap A = \emptyset$
PROOF: Then $S \subseteq B$ and so S has a least element.
$\langle 2\rangle 3$. CASE: $S \cap A \neq \emptyset$
PROOF: The least element of $S \cap A$ is the least element of S.
□

Definition 6.5.3 (Ordinal Addition)**.** Let α and β be ordinal numbers. Then $\alpha + \beta$ is the ordinal number of the concatenation of A and B, where A is any well ordered set with ordinal α and B is any well ordered set with ordinal β.

Theorem 6.5.4 (Associative Law for Addition)**.** *For any ordinals ρ, σ and τ, we have*

$$\rho + (\sigma + \tau) = (\rho + \sigma) + \tau \ .$$

PROOF: Given disjoint well ordered sets A, B and C, the concatenation of A with (the concatenation of B and C) is the same as the concatenation of (the concatenation of A and B) and C. □

Theorem 6.5.5. *For any ordinal ρ we have*

$$\rho + 0 = 0 + \rho = \rho \ .$$

PROOF: For any well ordered set A, the concatenation of A with $\emptyset$ is A, and the concatenation of $\emptyset$ with A is A. □

Theorem 6.5.6. *For any ordinal α we have $\alpha + 1 = \alpha^+$.*

PROOF: Since α^+ is the concatenation of α and $\{\alpha\}$. □

Theorem 6.5.7. *For any ordinal α, the operation that maps β to $\alpha + \beta$ is normal.*

PROOF:
$\langle 1\rangle 1$. For any limit ordinal λ, we have $\alpha + \lambda = \sup_{\beta < \lambda}(\alpha + \beta)$.
$\langle 2\rangle 1$. LET: λ be a limit ordinal.
$\langle 2\rangle 2$. $(\{0\} \times \alpha) \cup (\{1\} \times \lambda) = \bigcup_{\beta \in \lambda}((\{0\} \times \alpha) \cup (\{1\} \times \beta))$, where the order on the right hand side is as in Lemma 6.1.15.
PROOF:

$$\begin{aligned}
(\{0\} \times \alpha) \cup (\{1\} \times \lambda) &= (\{0\} \times \alpha) \cup (\{1\} \times \bigcup_{\beta < \lambda} \beta) \\
&= (\{0\} \times \alpha) \cup \bigcup_{\beta < \lambda} (\{1\} \times \beta) \\
&= \bigcup_{\beta < \lambda} ((\{0\} \times \alpha) \cup (\{1\} \times \beta))
\end{aligned}$$

$\langle 1 \rangle$2. For any ordinal β we have $\alpha + \beta < \alpha + \beta^+$
  PROOF: Since $\alpha + \beta^+ = \alpha + \beta + 1 = (\alpha + \beta)^+$.
$\square$

Corollary 6.5.7.1. *For any ordinals α, β, γ, we have $\beta < \gamma$ if and only if $\alpha + \beta < \alpha + \gamma$.*

Corollary 6.5.7.2 (Left Cancellation for Addition)**.** *For any ordinals α, β and γ, if $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$.*

Theorem 6.5.8. *For any ordinals α, β, γ, if $\beta \leq \gamma$ then $\beta + \alpha \leq \gamma + \alpha$.*

PROOF: Transfinite induction on α. $\square$

Theorem 6.5.9 (Subtraction Theorem)**.** *Let α and β be ordinals with $\alpha \leq \beta$. Then there exists a unique ordinal δ such that $\alpha + \delta = \beta$.*

PROOF:
$\langle 1 \rangle$1. For all ordinals α and β with $\alpha \leq \beta$, there exists δ such that $\alpha + \delta = \beta$
  $\langle 2 \rangle$1. LET: α and β be ordinals with $\alpha \leq \beta$
  $\langle 2 \rangle$2. LET: δ be the greatest ordinal such that $\alpha + \delta \leq \beta$
    PROOF: Proposition 6.4.6.
  $\langle 2 \rangle$3. $\alpha + \delta = \beta$
    PROOF: If $\alpha + \delta < \beta$ then $\alpha + \delta + 1 \leq \beta$ contradicting the greatestness of δ.
$\langle 1 \rangle$2. Q.E.D.
  PROOF: Uniqueness follows from the Left Cancellation Law.
$\square$

6.5.2 Multiplication

Definition 6.5.10 (Ordinal Multiplication)**.** Let α and β be ordinal numbers. Then $\alpha\beta$ is the ordinal number of $A \times B$ under the lexicographic order, where A is any well ordered set with ordinal α and B is any well ordered set with ordinal β.

This is well defined by Proposition 5.3.3.

Theorem 6.5.11 (Associative Law)**.** *For any ordinals ρ, σ and τ, we have*

$$\rho(\sigma\tau) = (\rho\sigma)\tau \quad .$$

PROOF: Let A, B and C be well ordered sets with ordinals ρ, σ and τ. Then both $\rho(\sigma\tau)$ and $(\rho\sigma)\tau$ are the ordinal of $A \times B \times C$ under
$(a, b, c) \leq (a', b', c') \Leftrightarrow a \leq a' \vee (a = a' \wedge b \leq b') \vee (a = a' \wedge b = b' \wedge c \leq c')$. $\square$

Theorem 6.5.12 (Left Distributive Law)**.** *For any ordinals ρ, σ and τ, we have*

$$\rho(\sigma + \tau) = \rho\sigma + \rho\tau$$

PROOF: Let A, B and C be well ordered sets with ordinals ρ, σ and τ and with $B \cap C = \emptyset$. Then both $\rho(\sigma + \tau)$ and $\rho\sigma + \rho\tau$ are the ordinal of $A \times (B \cup C)$ under the lexicographic ordering. □

Theorem 6.5.13. *For any ordinal ρ we have $\rho 0 = 0\rho = 0$.*

PROOF: For any well ordered set A we have $A \times \emptyset = \emptyset \times A = \emptyset$. □

Theorem 6.5.14. *For any ordinal ρ we have $\rho 1 = 1\rho = \rho$.*

PROOF: Easy. □

Theorem 6.5.15. *For any ordinals ρ and σ, if $\rho\sigma = 0$ then $\rho = 0$ or $\sigma = 0$.*

PROOF: If $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$. □

Theorem 6.5.16. *For any non-zero ordinal α, the operation that maps β to $\alpha\beta$ is normal.*

PROOF:
⟨1⟩1. For any limit ordinal λ, we have $\alpha\lambda = \bigcup_{\beta<\lambda} \alpha\beta$
  ⟨2⟩1. LET: λ be a limit ordinal
  ⟨2⟩2. $\alpha \times \lambda = \bigcup_{\beta<\lambda}(\alpha \times \beta)$ as well-ordered sets
⟨1⟩2. For any ordinal β we have $\alpha\beta < \alpha\beta^+$
  PROOF: $\alpha\beta^+ = \alpha\beta + \alpha > \alpha\beta$
□

Corollary 6.5.16.1. *For any ordinals α, β, γ, if $\alpha \neq 0$ then $\beta < \gamma$ if and only if $\alpha\beta < \alpha\gamma$.*

Corollary 6.5.16.2 (Left Cancellation for Multiplication)**.** *For any ordinals α, β, γ, if $\alpha \neq 0$ and $\alpha\beta = \alpha\gamma$ then $\beta = \gamma$.*

Theorem 6.5.17. *For any ordinals α, β and γ, if $\beta \leq \gamma$ then $\beta\alpha \leq \gamma\alpha$.*

PROOF: Transfinite induction on α. □

Theorem 6.5.18 (Division Theorem)**.** *Let α and δ be ordinal numbers with $\delta \neq 0$. Then there exist unique ordinals β and γ with $\gamma < \delta$ and*

$$\alpha = \delta\beta + \gamma \ .$$

PROOF:
⟨1⟩1. For any ordinal numbers α and δ with $\delta \neq 0$, there exist ordinals β and γ such that $\gamma < \delta$ and $\alpha = \delta\beta + \gamma$
  ⟨2⟩1. LET: α and δ be ordinals with $\delta \neq 0$
  ⟨2⟩2. LET: β be the greatest ordinal such that $\delta\beta \leq \alpha$
    PROOF: Proposition 6.4.6.
  ⟨2⟩3. There exists an ordinal γ such that $\alpha = \delta\beta + \gamma$
    PROOF: Subtraction Theorem

⟨1⟩2. For any ordinals δ, β, β', γ, γ', if $\delta\beta+\gamma = \delta\beta'+\gamma'$ and $\delta \neq 0$ and $\gamma, \gamma' < \delta$ then $\beta = \beta'$ and $\gamma = \gamma'$

⟨2⟩1. LET: δ, β, β', γ, γ' be ordinals.

⟨2⟩2. ASSUME: $\delta \neq 0$ and $\delta\beta + \gamma = \delta\beta' + \gamma'$

⟨2⟩3. $\beta = \beta'$

⟨3⟩1. $\beta \not< \beta'$

PROOF: If $\beta < \beta'$ then

$$\begin{aligned}\delta\beta' + \gamma' &\geq \delta\beta' \\ &\geq \delta(\beta + 1) \\ &= \delta\beta + \delta \\ &> \delta\beta + \gamma\end{aligned}$$

⟨3⟩2. $\beta' \not< \beta$

PROOF: Similar.

⟨2⟩4. $\gamma = \gamma'$

PROOF: By Cancellation.

□

6.5.3 Exponentiation

Definition 6.5.19. Given ordinals α and β, define the ordinal α^β as follows:

$$\begin{aligned}0^\alpha &:= 0 && (\alpha > 0) \\ \alpha^0 &:= 1 \\ \alpha^{\beta^+} &:= \alpha^\beta \alpha && (\alpha > 0) \\ \alpha^\lambda &:= \sup_{\beta < \lambda} \alpha^\beta && (\alpha > 0, \lambda \text{ a limit ordinal})\end{aligned}$$

Theorem 6.5.20. *Let α be an ordinal ≥ 2. The operation that maps β to α^β is normal.*

PROOF:

⟨1⟩1. For λ a limit ordinal we have $\alpha^\lambda = \sup_{\beta<\lambda} \alpha^\beta$

PROOF: By definition.

⟨1⟩2. For any ordinal β we have $\alpha^\beta < \alpha^{\beta^+}$

PROOF: We have $\alpha^{\beta^+} = \alpha^\beta \alpha > \alpha^\beta$ by Theorem 6.5.16 since $\alpha > 1$ and $\alpha^\beta \neq 0$.

□

Corollary 6.5.20.1. *For any ordinals α, β, γ, if $\alpha \geq 2$ then $\beta < \gamma$ if and only if $\alpha^\beta < \alpha^\gamma$.*

Corollary 6.5.20.2 (Cancellation for Exponentiation)**.** *For any ordinals α, β, γ, if $\alpha \geq 2$ and $\alpha^\beta = \alpha^\gamma$ then $\beta = \gamma$.*

Theorem 6.5.21. *For any ordinals α, β and γ, if $\beta \leq \gamma$ then $\beta^\alpha \leq \gamma^\alpha$.*

PROOF: Transfinite induction on α.

Theorem 6.5.22 (Logarithm Theorem)**.** *Let α and β be ordinal numbers with $\alpha \neq 0$ and $\beta > 1$. Then there exist unique ordinals γ, δ and ρ such that*

$$\alpha = \beta^\gamma \delta + \rho, \qquad 0 \neq \delta < \beta, \qquad \rho < \beta^\gamma \enspace .$$

PROOF:
⟨1⟩1. For any ordinals α and β with $\alpha \neq 0$ and $\beta > 1$, there exist ordinals γ, δ, ρ such that

$$\alpha = \beta^\gamma \delta + \rho, \qquad 0 \neq \delta < \beta, \qquad \rho < \beta^\gamma \enspace .$$

⟨2⟩1. LET: α and β be ordinals with $\alpha \neq 0$ and $\beta > 1$.
⟨2⟩2. LET: γ be the greatest ordinal such that $\beta^\gamma \leq \alpha$.
PROOF: Proposition 6.4.6.
⟨2⟩3. LET: δ and ρ be the unique ordinals with $\rho < \beta^\gamma$ such that $\alpha = \beta^\gamma \delta + \rho$.
PROOF: By the Division Theorem.
⟨2⟩4. $\delta \neq 0$
PROOF: If $\delta = 0$ then $\alpha = \beta^\gamma 0 + \rho = \rho < \beta^\gamma \leq \alpha$ which is a contradiction.
⟨2⟩5. $\delta < \beta$
PROOF: If $\beta \leq \delta$ then $\alpha \geq \beta^\gamma \delta \geq \beta^\gamma \beta = \beta^{\gamma+1}$, contradicting the greatestness of γ.
⟨1⟩2. If $\beta^\gamma \delta + \rho = \beta^{\gamma'} \delta' + \rho'$ with $\beta > 1$, $0 \neq \delta < \beta$, $0 \neq \delta' < \beta$, $\rho < \beta^\gamma$ and $\rho' < \beta^{\gamma'}$, then $\gamma = \gamma'$, $\delta = \delta'$ and $\rho = \rho'$.
⟨2⟩1. LET: $\alpha = \beta^\gamma \delta + \rho = \beta^{\gamma'} \delta' + \rho'$
⟨2⟩2. $\beta^\gamma \leq \alpha < \beta^{\gamma+1}$
⟨2⟩3. $\beta^{\gamma'} \leq \alpha < \beta^{\gamma'+1}$
⟨2⟩4. $\beta^\gamma < \beta^{\gamma'+1}$ and $\beta^{\gamma'} < \beta^{\gamma+1}$
⟨2⟩5. $\gamma < \gamma' + 1$ and $\gamma' < \gamma + 1$
⟨2⟩6. $\gamma = \gamma'$
⟨2⟩7. $\delta = \delta'$ and $\rho = \rho'$
PROOF: By the Division Theorem.
□

Theorem 6.5.23. *For any ordinal numbers α, β, γ, we have*

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma \enspace .$$

PROOF:
⟨1⟩1. LET: $P[\gamma]$ be the property: for any ordinals α and β we have $\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$
⟨1⟩2. $P[0]$
PROOF:

$$\begin{aligned} \alpha^{\beta+0} &= \alpha^\beta \\ &= \alpha^\beta 1 \\ &= \alpha^\beta \alpha^0 \end{aligned}$$

⟨1⟩3. For all γ, if $P[\gamma]$ then $P[\gamma + 1]$
PROOF:

$$\begin{aligned} \alpha^{\beta+\gamma+1} &= \alpha^{\beta+\gamma} \alpha \\ &= \alpha^\beta \alpha^\gamma \alpha && \text{(induction hypothesis)} \\ &= \alpha^\beta \alpha^{\gamma+1} \end{aligned}$$

⟨1⟩4. For any limit ordinal λ, if $\forall \gamma < \lambda . P[\gamma]$ then $P[\lambda]$.
⟨2⟩1. LET: λ be a limit ordinal
⟨2⟩2. ASSUME: $\forall \gamma < \lambda . P[\gamma]$
⟨2⟩3. LET: α and β be any ordinals.
⟨2⟩4. CASE: $\alpha = 0$
PROOF: We have $\alpha^{\beta+\lambda} = \alpha^\beta \alpha^\lambda = 0$.
⟨2⟩5. CASE: $\alpha = 1$
PROOF: We have $\alpha^{\beta+\lambda} = \alpha^\beta \alpha^\lambda = 1$.
⟨2⟩6. CASE: $\alpha > 1$
PROOF:
$$\begin{aligned} \alpha^{\beta+\lambda} &= \alpha^{\sup_{\gamma<\lambda}(\beta+\gamma)} \\ &= \sup_{\gamma<\lambda} \alpha^{\beta+\gamma} && \text{(Theorem 6.4.7)} \\ &= \sup_{\gamma<\lambda} \alpha^\beta \alpha^\gamma && (\langle 2\rangle 2) \\ &= \alpha^\beta \sup_{\gamma<\lambda} \alpha^\gamma && \text{(Theorem 6.4.7)} \\ &= \alpha^\beta \alpha^\lambda \end{aligned}$$
□

Theorem 6.5.24. *For any ordinal numbers α, β and γ, we have*
$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma} \ .$$

PROOF:
⟨1⟩1. LET: $P[\gamma]$ be the property: For any ordinals α and β, we have $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$
⟨1⟩2. $P[0]$
PROOF:
$$\begin{aligned} (\alpha^\beta)^0 &= 1 \\ &= \alpha^{\beta 0} \end{aligned}$$
⟨1⟩3. $\forall \gamma \in \mathbf{On}. P[\gamma] \Rightarrow P[\gamma+1]$
PROOF:
$$\begin{aligned} (\alpha^\beta)^{\gamma+1} &= (\alpha^\beta)^\gamma \alpha^\beta \\ &= \alpha^{\beta\gamma} \alpha^\beta \\ &= \alpha^{\beta\gamma+\beta} \\ &= \alpha^{\beta(\gamma+1)} \end{aligned}$$
⟨1⟩4. For any limit ordinal λ, if $\forall \gamma < \lambda . P[\gamma]$ then $P[\lambda]$.
⟨2⟩1. LET: λ be a limit ordinal.
⟨2⟩2. ASSUME: $\forall \gamma < \lambda . P[\gamma]$
⟨2⟩3. LET: α and β be any ordinals.
⟨2⟩4. CASE: $\alpha = 0$ and $\beta = 0$

PROOF:

$$\begin{aligned}(0^\beta)^\lambda &= 1^\lambda \\ &= 1 \\ &= 0^0 \\ &= 0^{0\lambda}\end{aligned}$$

⟨2⟩5. CASE: $\alpha = 0$ and $\beta \neq 0$
PROOF: $(0^\beta)^\lambda = 0^{\beta\lambda} = 0$.

⟨2⟩6. CASE: $\alpha = 1$
PROOF: $(1^\beta)^\lambda = 1^{\beta\lambda} = 1$

⟨2⟩7. CASE: $\alpha > 1$
PROOF:

$$\begin{aligned}(\alpha^\beta)^\lambda &= \sup_{\gamma<\lambda}(\alpha^\beta)^\gamma \\ &= \sup_{\gamma<\lambda}\alpha^{\beta\gamma} \\ &= \alpha^{\sup_{\gamma<\lambda}\beta\gamma} \\ &= \alpha^{\beta\lambda}\end{aligned}$$

∎

Chapter 7

Cardinal Numbers

7.1 Cardinal Numbers

Definition 7.1.1 (Cardinality)**.** For any set A, the *cardinality* or *cardinal number* $|A|$ of A is the least ordinal equinumerous with A.

Let **Card** be the class of all cardinal numbers.

Proposition 7.1.2. *For any sets A and B, we have $A \approx B$ iff $|A| = |B|$.*

PROOF: Easy. ∎

Definition 7.1.3 (Addition)**.** Given cardinal numbers κ and λ, we define $\kappa + \lambda$ to be $|A \cup B|$ where A and B are disjoint sets of cardinality κ and λ respectively.

We prove this is well-defined.

PROOF:
$\langle 1 \rangle 1$. ASSUME: $A \approx A'$, $B \approx B'$, and $A \cap B = A' \cap B' = \emptyset$
$\langle 1 \rangle 2$. PICK bijections $f : A \approx A'$ and $g : B \approx B'$
$\langle 1 \rangle 3$. The function $A \cup B \to A' \cup B'$ that maps $a \in A$ to $f(a)$ and $b \in B$ to $g(b)$ is a bijection.
∎

Proposition 7.1.4. *For any cardinal number κ, we have $\kappa + 0 = \kappa$.*

PROOF: Let A and B be disjoint sets of cardinality κ and 0. Then $B = \emptyset$ so $A \cup B = A$ and so $|A \cup B| = \kappa$. ∎

Theorem 7.1.5 (Associative Law for Addition)**.** *For any cardinal numbers κ, λ, μ we have $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$.*

PROOF: Since $A \cup (B \cup C) = (A \cup B) \cup C$. ∎

Proposition 7.1.6. *For any cardinal numbers κ and λ we have $\kappa + \lambda = \lambda + \kappa$.*

PROOF: Since $A \cup B = B \cup A$. ∎

Definition 7.1.7 (Multiplication)**.** For κ and λ cardinal numbers, we define $\kappa\lambda$ to be the cardinal number of $A \times B$, where $|A| = \kappa$ and $|B| = \lambda$.

We prove this is well-defined.

PROOF: If $f : A \approx A'$ and $g : B \approx B'$ then the function that maps (a, b) to $(f(a), g(b))$ is a bijection $A \times B \approx A' \times B'$. □

Proposition 7.1.8. *For any cardinal number κ we have $\kappa \cdot 0 = 0$.*

PROOF: Since $A \times \emptyset = \emptyset$. □

Proposition 7.1.9. *For any cardinal number κ we have $\kappa \cdot 1 = \kappa$.*

PROOF: The function that maps (a, e) to a is a bijection $A \times \{e\} \approx A$. □

Theorem 7.1.10 (Distributive Law)**.** *For any cardinal numbers κ, λ and μ, we have $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$.*

PROOF: Since $A \times (B \cup C) = (A \times B) \cup (A \times C)$. □

Theorem 7.1.11 (Associative Law for Multiplication)**.** *For any cardinal numbers κ, λ and μ, we have $\kappa(\lambda\mu) = (\kappa\lambda)\mu$.*

PROOF: Since $A \times (B \times C) \approx (A \times B) \times C$. □

Theorem 7.1.12 (Commutative Law for Multiplication)**.** *For any cardinal numbers κ and λ, we have $\kappa\lambda = \lambda\kappa$.*

PROOF: Since $A \times B \approx B \times A$. □

Theorem 7.1.13. *For any cardinal numbers κ and λ, if $\kappa\lambda = 0$ then $\kappa = 0$ or $\lambda = 0$.*

PROOF: if $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$. □

Definition 7.1.14 (Exponentiation)**.** Given cardinal numbers κ and λ, we define κ^λ to be $|A^B|$, where $|A| = \kappa$ and $|B| = \lambda$.

We prove this is well-defined.

PROOF:If $f : A \approx A'$ and $g : B \approx B'$, then the function that maps $h : B \to A$ to $f \circ h \circ g^{-1}$ is a bijection $A^B \approx A'^{B'}$. □

Proposition 7.1.15. *For any cardinal numbers κ, λ and μ,*

$$\kappa^{\lambda+\mu} = (\kappa^\lambda)^\mu$$

PROOF: The function that maps $f : A \times B \to C$ to $\lambda a \in A.\lambda b \in B.f(a, b)$ is a bijection $A^{B \times C} \approx (A^B)^C$. □

Proposition 7.1.16. *For any cardinal numbers κ, λ and μ,*

$$(\kappa\lambda)^\mu = \kappa^\mu\lambda^\mu \quad .$$

PROOF: The function $f : A^C \times B^C \to (A \times B)^C$ with $f(g,h)(c) = (g(c), h(c))$ is a bijection. □

Proposition 7.1.17. *For any cardinal numbers κ, λ and μ, we have*

$$\kappa^{\lambda+\mu} = \kappa^\lambda \kappa^\mu \enspace .$$

PROOF: If $B \cap C = \emptyset$, then $f : A^B \times A^C \to A^{B \cup C}$ given by $f(g,h)(b) = g(b)$ and $f(g,h)(c) = h(c)$ is a bijection. □

Proposition 7.1.18. *For any cardinal number κ, we have $\kappa^0 = 1$.*

PROOF: For any set A, we have $A^\emptyset = \{\emptyset\}$. □

Proposition 7.1.19. *For any cardinal number κ, we have $\kappa^1 = \kappa$.*

PROOF: For any sets A and B, if $B = \{b\}$ then the function $f : A \to A^B$ with $f(a)(b) = a$ is a bijection. □

Proposition 7.1.20. *For any non-zero cardinal number κ we have $0^\kappa = 0$.*

PROOF: If A is nonempty then there is no function $A \to \emptyset$. □

Proposition 7.1.21. *For any set A we have $|\mathcal{P}A| = 2^{|A|}$.*

PROOF: The function $f : \mathcal{P}A \to 2^A$ where $f(X)(a) = 0$ if $a \notin X$ and $f(X)(a) = 1$ if $a \in X$. □

Corollary 7.1.21.1. *For any cardinal number κ we have $\kappa < 2^\kappa$.*

PROOF: By Cantor's Theorem. □

7.2 Ordering on Cardinal Numbers

Definition 7.2.1. Given cardinal numbers κ and λ, we have $\kappa \leq \lambda$ iff $A \preccurlyeq B$, where $|A| = \kappa$ and $|B| = \lambda$.

PROOF:
$\langle 1 \rangle 1$. LET: $|A| = \kappa$ and $|B| = \lambda$
$\langle 1 \rangle 2$. PICK bijections $f : A \approx \kappa$ and $g : B \approx \lambda$
$\langle 1 \rangle 3$. If $\kappa \leq \lambda$ then $A \preccurlyeq B$
  PROOF: Let $i : \kappa \hookrightarrow \lambda$ be the inclusion. Then $g^{-1} \circ i \circ f$ is an injection $A \to B$.
$\langle 1 \rangle 4$. If $A \preccurlyeq B$ then $\kappa \leq \lambda$
  $\langle 2 \rangle 1$. ASSUME: $A \preccurlyeq B$
  $\langle 2 \rangle 2$. PICK an injection $h : A \rightarrowtail B$
  $\langle 2 \rangle 3$. $g(h(A)) \subseteq B$ is well-ordered by $\in$
  $\langle 2 \rangle 4$. LET: γ be the ordinal number of $(g(h(A)), \in)$
  $\langle 2 \rangle 5$. $\gamma \leq \lambda$
    PROOF: Proposition 6.1.12.
  $\langle 2 \rangle 6$. $\kappa \leq \gamma$

PROOF: By the leastness of κ, since A is equinumerous with γ.
⟨2⟩7. $\kappa \leq \lambda$
□

Corollary 7.2.1.1. *There is no largest cardinal number.*

Proposition 7.2.2. *For any cardinal numbers κ, λ, μ, if $\kappa \leq \lambda$ then $\kappa + \mu \leq \lambda + \mu$.*

PROOF: If $f : A \to B$ is injective, and $A \cap C = B \cap C = \emptyset$, then the function $A \cup C \to B \cup C$ that maps a to $f(a)$ and maps c to c is an injection. □

Proposition 7.2.3. *For any cardinal numbers κ, λ, μ, if $\kappa \leq \lambda$ then $\kappa\mu \leq \lambda\mu$.*

PROOF: If $f : A \to B$ is injective, then the function $A \times C \to B \times C$ that maps (a, c) to $(f(a), c)$ is injective. □

Proposition 7.2.4. *For any cardinal numbers κ, λ, μ, if $\kappa \leq \lambda$ then $\kappa^\mu \leq \lambda^\mu$.*

PROOF: Given an injection $f : A \to B$, the function that maps $A^C \to B^C$ that maps g to $f \circ g$ is an injection. □

Proposition 7.2.5. *For any cardinal numbers κ, λ, μ, if $\kappa \leq \lambda$ and μ and κ are not both 0, then $\mu^\kappa \leq \mu^\lambda$.*

PROOF:
⟨1⟩1. LET: A, B and C be sets with A and C not both empty.
⟨1⟩2. LET: $f : A \to B$ be an injection.
PROVE: $C^A \preccurlyeq C^B$
⟨1⟩3. CASE: $C = \emptyset$
PROOF: Then $A \neq \emptyset$ so $C^A = \emptyset \preccurlyeq C^B$.
⟨1⟩4. CASE: $C \neq \emptyset$
⟨2⟩1. PICK $c \in C$
⟨2⟩2. LET: $g : C^A \to C^B$ be the function $g(h)(f(a)) = h(a)$, $g(h)(b) = c$ if $b \notin f(A)$
⟨2⟩3. g is an injection.
□

Proposition 7.2.6. *Let $\mathcal{A}$ be a set such that $\forall X \in \mathcal{A}.|X| \leq \kappa$. Then*

$$\left|\bigcup \mathcal{A}\right| \leq |\mathcal{A}|\kappa \ .$$

PROOF:
⟨1⟩1. For $X \in \mathcal{A}$, choose a surjection $f_X : \kappa \to X$.
⟨1⟩2. Define $g : \mathcal{A} \times \kappa \to \bigcup \mathcal{A}$ by $g(X, \alpha) = f_X(\alpha)$
⟨1⟩3. g is surjective.
□

Lemma 7.2.7. *The union of a set of cardinal numbers is a cardinal number.*

PROOF:

$\langle 1 \rangle 1$. Let: A be a set of cardinal numbers.
Prove: $\bigcup A$ is the smallest ordinal equinumerous with $\bigcup A$
$\langle 1 \rangle 2$. Let: $\alpha < \bigcup A$
Prove: $\alpha \not\approx \bigcup A$
$\langle 1 \rangle 3$. Pick $\kappa \in A$ such that $\alpha < \kappa$
$\langle 1 \rangle 4$. $\alpha \prec \kappa$
$\langle 1 \rangle 5$. $\alpha \prec \bigcup A$
$\square$

Chapter 8

Natural Numbers

8.1 Inductive Sets

Definition 8.1.1 (Inductive)**.** A set I is *inductive* iff $0 \in I$ and $\forall x \in I.x^+ \in I$.

Definition 8.1.2 (Natural Number)**.** A *natural number* is a set that belongs to every inductive set.

Theorem 8.1.3. *The class* $\mathbb{N}$ *of natural numbers is a set.*

PROOF:
$\langle 1 \rangle 1.$ PICK an inductive set I.
  PROOF: Axiom of Infinity.
$\langle 1 \rangle 2.$ $\mathbb{N} \subseteq I$
∎

Theorem 8.1.4. $\mathbb{N}$ *is inductive, and is a subset of every other inductive set.*

PROOF:
$\langle 1 \rangle 1.$ $\mathbb{N}$ is inductive.
  $\langle 2 \rangle 1.$ $0 \in \mathbb{N}$
    PROOF: Since 0 is a member of every inductive set.
  $\langle 2 \rangle 2.$ $\forall n \in \mathbb{N}.n^+ \in \mathbb{N}$
    $\langle 3 \rangle 1.$ LET: $n \in \mathbb{N}$
    $\langle 3 \rangle 2.$ LET: I be any inductive set.
        PROVE: $n^+ \in I$
    $\langle 3 \rangle 3.$ $n \in I$
      PROOF: $\langle 3 \rangle 1$, $\langle 3 \rangle 2$
    $\langle 3 \rangle 4.$ $n^+ \in I$
      PROOF: $\langle 3 \rangle 2$, $\langle 3 \rangle 3$
$\langle 1 \rangle 2.$ $\mathbb{N}$ is a subset of every inductive set.
  PROOF: Immediate from definitions.
∎

Corollary 8.1.4.1 (Induction Principle for $\mathbb{N}$)**.** *Any inductive subset of* $\mathbb{N}$ *coincides with* $\mathbb{N}$.

Theorem 8.1.5. *Every natural number except 0 is the successor of some natural number.*

Proof: Trivially by induction. □

Proposition 8.1.6. *Every natural number is an ordinal.*

Proof: By induction. □

Proposition 8.1.7. $\mathbb{N}$ *is a transitive set.*

Proof:
$\langle 1 \rangle 1.$ $0 \subseteq \mathbb{N}$
$\langle 1 \rangle 2.$ $\forall n \in \mathbb{N}. n \subseteq \mathbb{N} \Rightarrow n^+ \subseteq \mathbb{N}$
$\langle 1 \rangle 3.$ $\forall n \in \mathbb{N}. n \subseteq \mathbb{N}$
  Proof: From $\langle 1 \rangle 1$ and $\langle 1 \rangle 2$ by induction.
□

Corollary 8.1.7.1. $\mathbb{N}$ *is an ordinal.*

Definition 8.1.8. We define $\omega = \mathbb{N}$.

Proposition 8.1.9 (Dependent Choice)**.** *Let A be a nonempty set and R a relation on A such that $\forall x \in A. \exists y \in A.(y, x) \in R$. Then there exists a function $f : \mathbb{N} \to A$ such that $\forall n \in \mathbb{N}.(f(n+1), f(n)) \in R$.*

Proof:
$\langle 1 \rangle 1.$ Pick a choice function F for A.
$\langle 1 \rangle 2.$ Pick $a \in A$
$\langle 1 \rangle 3.$ Define $f : \mathbb{N} \to A$ by $f(0) = a$ and $f(n+1) = F(\{y \in A \mid (y, f(n)) \in R\})$.
□

Theorem Schema 8.1.10. *For any classes* $\mathbf{A}$ *and* $\mathbf{R}$*, the following is a theorem:*

Assume $\mathbf{R}$ is a relation on $\mathbf{A}$ and, for all $a \in \mathbf{A}$, the class $\{x \in \mathbf{A} \mid x\mathbf{R}a\}$ is a set. Then $\mathbf{R}$ is well founded if and only if there does not exist a function $f : \mathbb{N} \to \mathbf{A}$ such that $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$.

Proof:
$\langle 1 \rangle 1.$ If there exists a function $f : \mathbb{N} \to \mathbf{A}$ such that $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$ then $\mathbf{R}$ is not well founded.
  Proof: $f(\mathbb{N})$ is a nonempty subset of $\mathbf{A}$ with no $\mathbf{R}$-minimal element.
$\langle 1 \rangle 2.$ If $\mathbf{R}$ is not well founded then there exists a function $f : \mathbb{N} \to \mathbf{A}$ such that $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$.
  $\langle 2 \rangle 1.$ Assume: $\mathbf{R}$ is not well founded.
  $\langle 2 \rangle 2.$ Pick a nonempty subset $B \subseteq \mathbf{A}$ that has no $\mathbf{R}$-minimal element.
  $\langle 2 \rangle 3.$ $\forall x \in B. \exists y \in B. y\mathbf{R}x$

⟨2⟩4. Choose a function $g : B \to B$ such that $\forall x \in B.g(x)\mathbf{R}x$
⟨2⟩5. Pick $b \in B$
⟨2⟩6. Define $f : \mathbb{N} \to \mathbf{A}$ recursively by $f(0) = b$ and $\forall n \in \mathbb{N}.f(n+1) = g(f(n))$
⟨2⟩7. $\forall n \in \mathbb{N}.f(n+1)\mathbf{R}f(n)$
□

8.2 Cardinality

Definition 8.2.1 (Finite)**.** A set is *finite* iff it is equinumerous to some natural number; otherwise it is *infinite*.

Theorem 8.2.2 (Pigeonhole Principle)**.** *No natural number is equinumerous to a proper subset of itself.*

Proof:
⟨1⟩1. Let: $P(n)$ be the property: any one-to-one function $n \to n$ is surjective.
⟨1⟩2. $P(0)$
Proof: The only function $0 \to 0$ is injective.
⟨1⟩3. For every natural number n, if $P(n)$ then $P(n+1)$.
⟨2⟩1. Assume: $P(n)$
⟨2⟩2. Let: f be a one-to-one function $n+1 \to n+1$
⟨2⟩3. $f \upharpoonright n$ is a one-to-one function $n \to n+1$
⟨2⟩4. Case: $n \notin ranf$
⟨3⟩1. $f \upharpoonright n : n \to n$
⟨3⟩2. $\mathrm{ran}(f \upharpoonright n) = n$
⟨3⟩3. $f(n) = n$
Proof: ⟨2⟩1.
⟨3⟩4. $\mathrm{ran}\, f = n+1$
⟨2⟩5. Case: $n \in \mathrm{ran}\, f$
⟨3⟩1. Pick $p \in n$ such that $f(p) = n$
⟨3⟩2. Let: $\hat{f} : n \to n$ be the function

$$\hat{f}(p) = f(n)$$
$$\hat{f}(x) = f(x) \qquad (x \neq p)$$

⟨3⟩3. $\hat{f}$ is one-to-one
⟨3⟩4. $\mathrm{ran}\, \hat{f} = n$
Proof: ⟨2⟩1
⟨3⟩5. $\mathrm{ran}\, f = n+1$
⟨1⟩4. For every natural number n, $P(n)$.
□

Corollary 8.2.2.1. *No finite set is equinumerous to a proper subset of itself.*

Corollary 8.2.2.2. *Every natural number is a cardinal number.*

Proof: For any natural number n, we have that n is the least ordinal such that $n \approx n$. □

Corollary 8.2.2.3. *$\mathbb{N}$ is a cardinal number.*

Corollary 8.2.2.4. *$\mathbb{N}$ is infinite.*

PROOF: The function that maps n to $n+1$ is a bijection between $\mathbb{N}$ and $\mathbb{N}-\{0\}$. □

Corollary 8.2.2.5. *If C is a proper subset of a natural number n, then there exists $m < n$ such that $C \approx m$.*

PROOF: By Proposition 6.1.12. □

Corollary 8.2.2.6. *Any subset of a finite set is finite.*

Proposition 8.2.3. *For any natural numbers m and n we have $m+n$ (cardinal addition) is a natural number.*

PROOF: Induction on n. □

Corollary 8.2.3.1. *The union of two finite sets is finite.*

Corollary 8.2.3.2. *The union of a finite set of finite sets is finite.*

PROOF: By induction on the number of elements. □

Proposition 8.2.4. *For natural numbers m and n, the cardinal sum $m+n$ is equal to the ordinal sum $m+n$.*

PROOF: Induction on n. □

Proposition 8.2.5. *For any natural numbers m and n, we have mn (cardinal multiplication) is a natural number.*

Corollary 8.2.5.1. *If A and B are finite sets then $A \times B$ is finite.*

Proposition 8.2.6. *For natural numbers m and n, the cardinal product mn is equal to the ordinal product mn.*

PROOF: Induction on n. □

Proposition 8.2.7. *For any natural numbers m and n we have m^n (cardinal exponentiation) is a natural number.*

PROOF: Induction on n.

Corollary 8.2.7.1. *If A and B are finite sets then A^B are finite.*

Proposition 8.2.8. *For natural numbers m and n, the cardinal exponentiation m^n and the ordinal exponentiation m^n agree.*

PROOF: Induction on n. □

Proposition 8.2.9. $\mathbb{N}^2 \approx \mathbb{N}$

PROOF: The function $J : \mathbb{N}^2 \to \mathbb{N}$ defined by $J(m, n) = ((m+n)^2 + 3m + n)/2$ is a bijection. ∎

Proposition 8.2.10. *For any infinite cardinal κ we have $\aleph_0 \leq \kappa$.*

PROOF:
⟨1⟩1. LET: A be an infinite set.
PROVE: $\mathbb{N} \preccurlyeq A$
⟨1⟩2. PICK a choice function F for A.
⟨1⟩3. Define $h : \mathbb{N} \to \{X \in \mathcal{P}A \mid X \text{ is finite}\}$ by

$$h(0) = \emptyset$$
$$h(n+1) = h(n) \cup \{F(A - \{h(m) \mid m < n\})\}$$

⟨1⟩4. Define $g : \mathbb{N} \to A$ by $g(n) = F(A - \{h(m) \mid m < n\})$
⟨1⟩5. g is injective.
PROOF: If $m < n$ then $g(m) \neq g(n)$.
∎

Theorem Schema 8.2.11 (König's Lemma)**.** *For any classes* $\mathbf{A}$ *and* $\mathbf{R}$*, the following is a theorem:*

Assume $\mathbf{R}$ *is a well founded relation on* $\mathbf{A}$ *such that, for all* $y \in \mathbf{A}$*, the class* $\{x \in \mathbf{A} \mid x\mathbf{R}y\}$ *is a finite set. Let* $\mathbf{R}^t$ *be the transitive closure of* $\mathbf{R}$*. Then,for all* $y \in \mathbf{A}$*, the class* $\{x \in \mathbf{A} \mid x\mathbf{R}^t y\}$ *is a finite set.*

PROOF:
⟨1⟩1. LET: $y \in \mathbf{A}$
⟨1⟩2. ASSUME: as transfinite induction hypothesis $\forall x \mathbf{R} y. \{z \in \mathbf{A} \mid z\mathbf{R}^t x\}$ is a finite set.
⟨1⟩3. $\{x \mid x\mathbf{R}^t y\} = \bigcup_{x\mathbf{R}y}(\{x\} \cup \{z \mid z\mathbf{R}^t x\}$
⟨1⟩4. $\{x \mid x\mathbf{R}^t y\}$ is finite.
PROOF: Corollary 8.2.3.2.
∎

8.3 Countable Sets

Definition 8.3.1 (Countable)**.** A set A is *countable* iff $|A| \leq \aleph_0$.

Theorem 8.3.2. *The union of a countable set of countable sets is countable.*

PROOF: Proposition 7.2.6. ∎

8.4 Arithmetic

Definition 8.4.1 (Even)**.** A natural number n is *even* iff there exists $m \in \mathbb{N}$ such that $n = 2m$.

Definition 8.4.2 (Odd)**.** A natural number n is *odd* iff there exists $p \in \mathbb{N}$ such that $n = 2p + 1$.

Proposition 8.4.3. *Every natural number is either even or odd.*

Proof:
$\langle 1 \rangle 1.$ 0 is even.
Proof: $0 = 2 \times 0$.
$\langle 1 \rangle 2.$ For any natural number n, if n is either even or odd then n^+ is either even or odd.
Proof:
$\langle 2 \rangle 1.$ Let: $n \in \mathbb{N}$
$\langle 2 \rangle 2.$ If n is even then n^+ is odd.
Proof: If $n = 2p$ then $n^+ = 2p + 1$.
$\langle 2 \rangle 3.$ If n is odd then n^+ is even.
Proof: If $n = 2p + 1$ then $n^+ = 2(p + 1)$.
□

Proposition 8.4.4. *No natural number is both even and odd.*

Proof:
$\langle 1 \rangle 1.$ 0 is not odd.
Proof: For any p we have $2p + 1 = (2p)^+ \neq 0$.
$\langle 1 \rangle 2.$ For any natural number n, if n is not both even and odd, then n^+ is not both even and odd.
$\langle 2 \rangle 1.$ Let: n be a natural number.
$\langle 2 \rangle 2.$ If n^+ is even then n is odd.
$\langle 3 \rangle 1.$ Assume: n^+ is even.
$\langle 3 \rangle 2.$ Pick p such that $n^+ = 2p$
$\langle 3 \rangle 3.$ $p \neq 0$
Proof: Since $n^+ \neq 0$.
$\langle 3 \rangle 4.$ Pick q such that $p = q^+$
Proof: Theorem 8.1.5.
$\langle 3 \rangle 5.$ $n^+ = 2q + 2$
Proof: $\langle 3 \rangle 2$, $\langle 3 \rangle 4$.
$\langle 3 \rangle 6.$ $n = 2q + 1$
Proof: Proposition 6.2.7, $\langle 3 \rangle 5$
$\langle 3 \rangle 7.$ n is odd.
$\langle 2 \rangle 3.$ If n^+ is odd then n is even.
$\langle 3 \rangle 1.$ Assume: n^+ is odd.
$\langle 3 \rangle 2.$ Pick p such that $n^+ = 2p + 1$
$\langle 3 \rangle 3.$ $n = 2p$
Proof: Proposition 6.2.7, $\langle 3 \rangle 2$
$\langle 3 \rangle 4.$ n is even.
□

Proposition 8.4.5. *Let m, n, p, q be natural numbres. Assume $m + n = p + q$. Then $m < p$ if and only if $q < n$.*

Proof:
$\langle 1 \rangle 1.$ If $m < p$ then $q < n$.

PROOF: If $m < p$ and $n \leq q$ then $m + n < p + q$.
⟨1⟩2. If $q < n$ then $m < p$.
PROOF: Similar.
∎

Proposition 8.4.6. *Let m, n, p and q be natural numbers. Assume $n < m$ and $q < p$. Then*

$$mq + np < mp + nq \ .$$

PROOF:
⟨1⟩1. PICK positive natural numbers a and b such that $m = n + a$ and $p = q + b$.
⟨1⟩2. $mp + nq > mq + np$
PROOF:

$$\begin{aligned} mp + nq &= (n + a)(q + b) + nq \\ &= 2nq + nb + aq + ab \\ mq + np &= (n + a)q + n(q + b) \\ &= 2nq + aq + nb \\ \therefore mp + nq &= mq + np + ab \\ &> mq + np \end{aligned}$$

∎

8.5 Sequences

Definition 8.5.1 (Sequence)**.** Let A be a set. A *finite sequence* in A is a function $a : n \to A$ for some natural number n; we write it as $(a(0), a(1), \ldots, a(n-1))$. An *(infinite) sequence* in A is a function $\mathbb{N} \to A$.

We write A^* for the set of all finite sequences in A.

Proposition 8.5.2. *If A is countable then A^* is countable.*

PROOF: For any n, the set A^n is countable, and A^* is equinumerous with $\bigcup_n A^n$.
∎

8.6 Transitive Closure of a Set

Proposition 8.6.1. *For any set A, there exists a unique transitive set C such that:*

- $A \subseteq C$
- *For any transitive set X, if $A \subseteq X$ then $C \subseteq X$*

PROOF:
⟨1⟩1. Define a function $F : \mathbb{N} \to \mathbf{V}$ by

$$F(0) = A$$
$$F(n + 1) = A \cup \bigcup (F(0) \cup \cdots \cup F(n))$$

$\langle 1\rangle 2$. For all $n \in \mathbb{N}$ and $a \in F(n)$ we have $a \subseteq F(n+1)$
PROOF: $a \in F(0) \cup \cdots \cup F(n)$ so $a \subseteq \bigcup(F(0) \cup \cdots \cup F(n)) \subseteq F(n+1)$.
$\langle 1\rangle 3$. LET: $C = \bigcup_{n \in \mathbb{N}} F(n)$
$\langle 1\rangle 4$. C is transitive.
$\langle 2\rangle 1$. LET: $x \in y \in C$
$\langle 2\rangle 2$. PICK $n \in \mathbb{N}$ such that $y \in F(n)$
$\langle 2\rangle 3$. $y \subseteq F(n+1)$
PROOF: $\langle 1\rangle 2$
$\langle 2\rangle 4$. $x \in F(n+1)$
$\langle 2\rangle 5$. $x \in C$
$\langle 1\rangle 5$. $A \subseteq C$
PROOF: Since $F(0) = A$.
$\langle 1\rangle 6$. For any transitive set X, if $A \subseteq X$ then $C \subseteq X$
$\langle 2\rangle 1$. LET: X be a transitive set
$\langle 2\rangle 2$. ASSUME: $A \subseteq X$
$\langle 2\rangle 3$. For all $n \in \mathbb{N}$ we have $F(n) \subseteq X$.
$\langle 3\rangle 1$. $F(0) \subseteq X$
PROOF: $\langle 2\rangle 2$
$\langle 3\rangle 2$. For all $n \in \mathbb{N}$, if $F(n) \subseteq X$, then $F(n+1) \subseteq X$.
$\langle 4\rangle 1$. LET: $n \in \mathbb{N}$
$\langle 4\rangle 2$. ASSUME: $\forall m < n . F(m) \subseteq X$
$\langle 4\rangle 3$. $F(0) \cup \cdots \cup F(n) \subseteq X$
$\langle 4\rangle 4$. $\bigcup(F(0) \cup \cdots \cup F(n)) \subseteq X$
PROOF: Since X is transitive.
$\langle 4\rangle 5$. $F(n+1) \subseteq X$
$\langle 2\rangle 4$. $C \subseteq X$
$\langle 1\rangle 7$. Let D be a transitive set such that $A \subseteq D$ and, for any transitive set X, if $A \subseteq X$ then $D \subseteq X$. Then $D = C$.
PROOF: We have $C \subseteq D$ and $D \subseteq C$.
$\square$

8.7 The Veblen Fixed Point Theorem

Theorem Schema 8.7.1 (Veblen Fixed Point Theorem). *For any class* $\mathbf{T}$, *the following is a theorem:*

Assume $\mathbf{T}$ *is a normal ordinal operation. For every ordinal* β, *there exists* $\gamma \geq \beta$ *such that* $\mathbf{T}(\gamma) = \gamma$.

PROOF:
$\langle 1\rangle 1$. LET: β be an ordinal.
$\langle 1\rangle 2$. ASSUME: w.l.o.g. $\beta < \mathbf{T}(\beta)$
PROOF: We have $\beta \leq \mathbf{T}(\beta)$ by Proposition 6.4.5, and if $\beta = \mathbf{T}(\beta)$ we take $\gamma := \beta$.

$\langle 1\rangle 3.$ Define $f : \mathbb{N} \to \mathbf{On}$ by recursion thus:

$$f(0) = \beta$$
$$f(n^+) = \mathbf{T}(f(n))$$

$\langle 1\rangle 4.$ LET: $\gamma = \sup_{n\in\mathbb{N}} f(n)$

$\langle 1\rangle 5.$ $\beta \leq \gamma$

PROOF: Since $\beta = f(0)$.

$\langle 1\rangle 6.$ $\mathbf{T}(\gamma) = \gamma$

$\langle 2\rangle 1.$ $\mathbf{T}(\gamma) \leq \gamma$

PROOF:

$$\begin{aligned} \mathbf{T}(\gamma) &= \sup_{n\in\mathbb{N}} \mathbf{T}(f(n)) && \text{(Theorem 6.4.7)} \\ &= \sup_{n\in\mathbb{N}} f(n^+) && (\langle 1\rangle 3) \\ &\leq \sup_{n\in\mathbb{N}} f(n) \\ &= \gamma \end{aligned}$$

$\langle 2\rangle 2.$ $\gamma \leq \mathbf{T}(\gamma)$

PROOF:Proposition 6.4.5.

□

Definition 8.7.2 (Derived Operation)**.** Let $\mathbf{T}$ be a normal ordinal operation. The *derived* operation $\mathbf{T}' : \mathbf{On} \to \mathbf{V}$ is the unique order isomorphism between $\mathbf{On}$ and the fixed points of $\mathbf{T}$.

Proposition Schema 8.7.3. *For any class* $\mathbf{T}$*, the following is a theorem:*

If $\mathbf{T}$ *is a normal ordinal operation, then the derived operation is normal.*

PROOF:

$\langle 1\rangle 1.$ For any set S of fixed points of $\mathbf{T}$, we have $\bigcup S$ is a fixed point of $\mathbf{T}$

$\langle 2\rangle 1.$ LET: S be a set of fixed points of $\mathbf{T}$.

$\langle 2\rangle 2.$ $\mathbf{T}(\sup S) = \sup S$

PROOF:

$$\begin{aligned} \mathbf{T}(\sup S) &= \sup_{\alpha\in S} \mathbf{T}(\alpha) && \text{(Theorem 6.4.7)} \\ &= \sup_{\alpha\in S} \alpha && (\langle 2\rangle 1) \\ &= \sup S \end{aligned}$$

$\langle 1\rangle 2.$ Q.E.D.

PROOF: Proposition 6.4.8.

□

8.8 Cantor Normal Form

Theorem 8.8.1. *For any ordinal* α*, there exist a unique sequence of nonzero natural numbers* $(n_1, \ldots, n_k)$ *and sequence of ordinals* $(\gamma_1, \ldots, \gamma_k)$ *such that*

$$\gamma_k < \gamma_{k-1} < \cdots < \gamma_1$$

and

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_k} n_k \ .$$

PROOF:

$\langle 1 \rangle 1$. For any ordinal α, there exist a sequence of nonzero natural numbers $(n_1, \ldots, n_k)$ and sequence of ordinals $(\gamma_1, \ldots, \gamma_k)$ such that

$$\gamma_k < \gamma_{k-1} < \cdots < \gamma_1$$

and

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_k} n_k \ .$$

$\langle 2 \rangle 1$. LET: α be an ordinal

$\langle 2 \rangle 2$. ASSUME: as an induction hypothesis that, for all $\beta < \alpha$, the theorem holds.

$\langle 2 \rangle 3$. ASSUME: w.l.o.g. $\alpha \neq 0$

$\langle 2 \rangle 4$. LET: γ_1, n_1, ρ_1 be the unique ordinals such that $0 \neq n_1 < \omega$, $\rho_1 < \omega^{\gamma_1}$, and $\alpha = \omega^{\gamma_1} n_1 + \rho_1$

$\langle 2 \rangle 5$. LET: $(\gamma_2, \ldots, \gamma_k)$ and $(n_2, \ldots, n_k)$ be sequences such that $\gamma_k < \gamma_{k-1} < \cdots < \gamma_2$ and $\rho_1 = \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_k} n_k$

$\langle 2 \rangle 6$. $\gamma_2 < \gamma_1$

PROOF: Since $\omega^{\gamma_2} \leq \rho_1 < \omega^{\gamma_1}$

$\langle 1 \rangle 2$. If

$$\gamma_k < \gamma_{k-1} < \cdots < \gamma_1$$
$$\gamma'_k < \gamma'_{k-1} < \cdots < \gamma'_1$$

and

$$\omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_k} n_k = \omega^{\gamma'_1} n'_1 + \omega^{\gamma'_2} n'_2 + \cdots + \omega^{\gamma'_k} n'_k$$

then $\gamma_i = \gamma'_i$ for all i and $n_i = n'_i$ for all i

PROOF: Prove by induction on i using the Logarithm Theorem.

□

Definition 8.8.2 (Cantor Normal Form)**.** For any ordinal α, the *Cantor normal form* of α is the expression $\alpha = \omega^{\gamma_1} n_1 + \cdots + \omega^{\gamma_k} n_k$ such that n_1, …, n_k are nonzero natural numbers and $\gamma_k < \gamma_{k-1} < \cdots < \gamma_1$.

Chapter 9

Infinite Cardinals

9.1 Alephs

Definition 9.1.1 (Aleph)**.** Let $\aleph$ be the unique order isomorphism between **On** and the class of infinite cardinals.

Proposition 9.1.2. *The operation* $\aleph$ *is normal.*

PROOF: Proposition 6.4.8 and Lemma 7.2.7. $\square$

Definition 9.1.3 (Continuum Hypothesis)**.** The *continuum hypothesis* is the statement that $\aleph_1 = 2^{\aleph_0}$.

Definition 9.1.4 (Generalised Continuum Hypothesis)**.** The *generalised continuum hypothesis* is the statement that, for all α, $\aleph_{\alpha^+} = 2^{\aleph_\alpha}$.

9.2 Beths

Definition 9.2.1 (Beth)**.** Define the operation $\beth : \mathbf{On} \rightarrow \mathbf{Card}$ by transfinite recursion as follows:

$$\begin{aligned} \beth_0 &:= \aleph_0 \\ \beth_{\alpha^+} &:= 2^{\beth_\alpha} \\ \beth_\lambda &:= \bigcup_{\alpha<\lambda} \beth_\alpha && (\lambda \text{ a limit ordinal}) \end{aligned}$$

Proposition 9.2.2. $\beth$ *is a normal operation.*

PROOF: It is continuous by definition, and $\beth_\alpha < \beth_{\alpha^+}$ by Cantor's Theorem. $\square$

Proposition 9.2.3. *The continuum hypothesis is equivalent to the statement* $\beth_1 = \aleph_1$.

The generalised continuum hypothesis is equivalent to the statement $\beth = \alpha$.

PROOF: Immediate from definitions. $\square$

Chapter 10

The Cumulative Hierarchy

Definition 10.0.1 (Cumulative Hierarchy)**.** Define the function $V : \mathbf{On} \to \mathbf{V}$ by transfinite recursion thus:

$$V_\alpha = \bigcup_{\beta < \alpha} \mathcal{P} V_\beta$$

Proposition 10.0.2. *For all $\alpha \in \mathbf{On}$, V_α is a transitive set.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\alpha \in \mathbf{On}$
$\langle 1 \rangle 2.$ ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha . V_\beta$ is a transitive set.
$\langle 1 \rangle 3.$ For all $\beta < \alpha$, $\mathcal{P} V_\beta$ is a transitive set.
PROOF: Proposition 1.6.4.
$\langle 1 \rangle 4.$ V_α is a transitive set.
PROOF: Proposition 1.6.3.
$\square$

Proposition 10.0.3. *For any ordinals α and β, if $\beta < \alpha$ then $V_\beta \subseteq V_\alpha$.*

PROOF: Since $V_\beta \in \mathcal{P} V_\beta \subseteq V_\alpha$ and V_α is a transitive set. $\square$

Theorem 10.0.4.

1. $V_0 = \emptyset$
2. $\forall \alpha \in \mathbf{On} . V_{\alpha^+} = \mathcal{P} V_\alpha$
3. *For any limit ordinal λ, $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$.*

PROOF:
$\langle 1 \rangle 1.$ $V_0 = \emptyset$
PROOF: Immediate from definition.
$\langle 1 \rangle 2.$ $\forall \alpha \in \mathbf{On} . V_{\alpha^+} = \mathcal{P} V_\alpha$
PROOF:

⟨2⟩1. LET: $\alpha \in \mathbf{On}$

⟨2⟩2. For all $\beta < \alpha$ we have $\mathcal{P}V_\beta \subseteq \mathcal{P}V_\alpha$

PROOF: Propositions 1.5.8 and 10.0.3.

⟨2⟩3. $V_{\alpha^+} = \mathcal{P}V_\alpha$

$$\begin{aligned} V_{\alpha^+} &= \bigcup_{\beta<\alpha^+} \mathcal{P}V_\beta \\ &= \bigcup_{\beta<\alpha} \mathcal{P}V_\beta \cup \mathcal{P}V_\alpha \\ &\mathcal{P}V_\alpha \end{aligned}$$

□

⟨1⟩3. For any limit ordinal λ, $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$

PROOF:

⟨2⟩1. $V_\lambda \subseteq \bigcup_{\alpha<\lambda} V_\alpha$

PROOF:

$$\begin{aligned} V_\lambda &= \bigcup_{\alpha<\lambda} \mathcal{P}V_\alpha \\ &= \bigcup_{\alpha<\lambda} V_{\alpha^+} && (\langle 1\rangle 2) \\ &\subseteq \bigcup_{\alpha<\lambda} V_\alpha \end{aligned}$$

⟨2⟩2. $\bigcup_{\alpha<\lambda} V_\alpha \subseteq V_\lambda$

PROOF: Proposition 10.0.3.

□

Proposition 10.0.5. *For every set A, there exists an ordinal α such that $A \in V_\alpha$.*

PROOF:

⟨1⟩1. Let us say a set A is *grounded* iff there exists an ordinal α such that $A \in V_\alpha$.

⟨1⟩2. For any set A, if every element of A is grounded, then A is grounded.

⟨2⟩1. LET: A be a set.

⟨2⟩2. $S = \{\alpha \mid \exists a \in A.\alpha \text{ is the least ordinal such that } a \in V_\alpha\}$

PROOF: S is a set by an Axiom of Replacement.

⟨2⟩3. LET: $\beta = \sup S$

⟨2⟩4. $A \subseteq V_\beta$

⟨3⟩1. LET: $a \in A$

⟨3⟩2. LET: α be the least ordinal such that $a \in V_\beta$

⟨3⟩3. $\alpha \in S$

⟨3⟩4. $\alpha \leq \beta$

⟨3⟩5. $a \in V_\beta$

⟨2⟩5. $A \in V_{\beta^+}$

⟨1⟩3. ASSUME: for a contradiction there exists an ungrounded set.

⟨1⟩4. PICK a transitive set B that has an ungrounded member.

PROOF: Pick a transitive set c, and take B to be the transitive closure of $\{c\}$.

$\langle 1\rangle 5$. LET: $A = \{x \in B \mid x \text{ is ungrounded}\}$
$\langle 1\rangle 6$. PICK $m \in A$ such that $m \cap A = \emptyset$
PROOF: Axiom of Regularity.
$\langle 1\rangle 7$. Every member of m is grounded.
$\langle 2\rangle 1$. ASSUME: for a contradiction $x \in m$ is ungrounded.
$\langle 2\rangle 2$. $x \in B$
PROOF: Since B is transitive ($\langle 1\rangle 4$).
$\langle 2\rangle 3$. $x \in A$
PROOF: $\langle 1\rangle 5$
$\langle 2\rangle 4$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 6$.
$\langle 1\rangle 8$. m is grounded.
PROOF: $\langle 1\rangle 2$
$\langle 1\rangle 9$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 6$.
□

Definition 10.0.6 (Rank)**.** The *rank* of a set A is the least ordinal α such that $A \in V_{\alpha^+}$.

Proposition 10.0.7. *For any set A we have*

$$\operatorname{rank} A = \bigcup_{a \in A} (\operatorname{rank} a)^+$$

PROOF:
$\langle 1\rangle 1$. LET: $\alpha = \bigcup_{a \in A} (\operatorname{rank} a)^+$
$\langle 1\rangle 2$. $A \subseteq V_\alpha$
$\langle 2\rangle 1$. LET: $a \in A$
$\langle 2\rangle 2$. $a \in V_{(\operatorname{rank} a)^+}$
$\langle 2\rangle 3$. $a \in V_\alpha$
$\langle 1\rangle 3$. $A \in V_{\alpha^+}$
$\langle 1\rangle 4$. If $A \subseteq V_\beta$ then $\alpha \leq \beta$
$\langle 2\rangle 1$. ASSUME: $A \subseteq V_\beta$
$\langle 2\rangle 2$. For all $a \in A$ we have $(\operatorname{rank} a)^+ \leq \beta$
PROOF: Since $a \in V_\beta$.
$\langle 2\rangle 3$. $\alpha \leq \beta$
□

Corollary 10.0.7.1. *For any sets a and b, if $a \in b$ then* $\operatorname{rank} a < \operatorname{rank} b$.

Proposition 10.0.8. *For any ordinal number α we have* $\operatorname{rank} \alpha = \alpha$.

PROOF:
$\langle 1\rangle 1$. LET: α be an ordinal.
$\langle 1\rangle 2$. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha. \operatorname{rank} \beta = \beta$
$\langle 1\rangle 3$. $\operatorname{rank} \alpha = \bigcup_{\beta < \alpha} \beta^+$

PROOF:

$$\begin{aligned}\operatorname{rank}\alpha &= \bigcup_{\beta<\alpha}(\operatorname{rank}\beta)^+ \\ &= \bigcup_{\beta<\alpha}\beta^+\end{aligned}$$

⟨1⟩4. $\bigcup_{\beta<\alpha}\beta^+ \leq \alpha$

PROOF: Since for all $\beta < \alpha$ we have $\beta^+ \leq \alpha$.

⟨1⟩5. $\alpha \leq \bigcup_{\beta<\alpha}\beta^+$

⟨2⟩1. LET: $\gamma = \bigcup_{\beta<\alpha}\beta^+$

⟨2⟩2. ASSUME: for a contradiction $\gamma < \alpha$

⟨2⟩3. $\gamma^+ \leq \bigcup_{\beta<\alpha}\beta^+ = \gamma$

⟨2⟩4. Q.E.D.

PROOF: This is a contradiction.

□

Chapter 11

Models of Set Theory

Definition 11.0.1 (Relativization)**.** Let σ be a sentence in the language of set theory and $\mathbf{M}$ a class. The *relativization* of σ to $\mathbf{M}$ is the sentence $\sigma^{\mathbf{M}}$ formed by replacing every quantifier $\forall x$ with $\forall x \in \mathbf{M}$, and $\exists x$ with $\exists x \in \mathbf{M}$.

We write '$\mathbf{M}$ is a model of σ' for the sentence $\sigma^{\mathbf{M}}$.

Theorem Schema 11.0.2. *For any class* $\mathbf{M}$*, the following is a theorem:*

If $\mathbf{M}$ *is a transitive class, then* $\mathbf{M}$ *is a model of the Axiom of Extensionality.*

Proof:
$\langle 1 \rangle 1.$ Assume: $\mathbf{M}$ is a transitive class.
Prove: $\forall x, y \in \mathbf{M}(\forall z \in \mathbf{M}(z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$
$\langle 1 \rangle 2.$ Let: $x, y \in \mathbf{M}$
$\langle 1 \rangle 3.$ Assume: $\forall z \in \mathbf{M}(z \in x \Leftrightarrow z \in y)$
$\langle 1 \rangle 4.$ $\forall z(z \in x \Leftrightarrow z \in y)$
Proof: Since $z \in x \Rightarrow z \in \mathbf{M}$ and $z \in y \Rightarrow z \in \mathbf{M}$ by $\langle 1 \rangle 1$.
$\langle 1 \rangle 5.$ $x = y$
□

Theorem 11.0.3. *If* α *is a non-zero ordinal then* V_α *is a model of the statement: The empty class is a set.*

Proof:
$\langle 1 \rangle 1.$ Let: $\alpha \neq 0$
Prove: $\exists x \in V_\alpha . \forall y \in V_\alpha . y \notin x$
$\langle 1 \rangle 2.$ $\emptyset \in V_\alpha$
$\langle 1 \rangle 3.$ $\forall y \in V_\alpha . y \notin \emptyset$
□

Theorem 11.0.4. *For any limit ordinal* λ*, we have* V_λ *is a model of the statement: for any sets* a *and* b*, the class* $\{a, b\}$ *is a set.*

Proof:
$\langle 1 \rangle 1.$ Let: λ be a limit ordinal.

PROVE: $\forall a, b \in V_\lambda . \exists c \in V_\lambda . \forall x \in V_\lambda (x \in c \Leftrightarrow x = a \vee x = b)$
$\langle 1 \rangle 2.$ LET: $a, b \in V_\lambda$
$\langle 1 \rangle 3.$ PICK $\alpha, \beta < \lambda$ such that $a \in V_\alpha$ and $b \in V_\beta$
$\langle 1 \rangle 4.$ ASSUME: w.l.o.g. $\alpha \leq \beta$
$\langle 1 \rangle 5.$ $a, b \in V_\beta$
$\langle 1 \rangle 6.$ $\{a, b\} \in V_{\beta+1}$
$\langle 1 \rangle 7.$ $\{a, b\} \in V_\lambda$
$\langle 1 \rangle 8.$ $\forall x \in V_\lambda (x \in \{a, b\} \Leftrightarrow x = a \vee x = b)$
$\square$

Theorem 11.0.5. *For any ordinal α, we have V_α is a model of the Union Axiom.*

PROOF:
$\langle 1 \rangle 1.$ LET: α be an ordinal.
PROVE: $\forall a \in V_\alpha . \exists b \in V_\alpha . \forall x \in V_\alpha (x \in b \Leftrightarrow \exists y \in V_\alpha (x \in y \wedge y \in a))$
$\langle 1 \rangle 2.$ LET: $a \in V_\alpha$
$\langle 1 \rangle 3.$ PICK $\beta < \alpha$ such that $a \subseteq V_\beta$
$\langle 1 \rangle 4.$ $\bigcup a \subseteq V_\beta$
PROOF: V_β is a transitive set.
$\langle 1 \rangle 5.$ $\bigcup a \in V_\alpha$
$\langle 1 \rangle 6.$ $\forall x \in V_\alpha (x \in \bigcup a \Leftrightarrow \exists y \in V_\alpha (x \in y \wedge y \in a))$
PROOF: V_α is a transitive set.
$\square$

Theorem 11.0.6. *For any limit ordinal λ, we have V_λ is a model of the Power Set Axiom.*

PROOF:
$\langle 1 \rangle 1.$ LET: λ be a limit ordinal.
PROVE: $\forall a \in V_\lambda . \exists b \in V_\lambda . \forall x \in V_\lambda (x \in b \Leftrightarrow \forall y \in V_\lambda (y \in x \Rightarrow y \in a))$
$\langle 1 \rangle 2.$ LET: $a \in V_\lambda$
$\langle 1 \rangle 3.$ PICK $\alpha < \lambda$ such that $a \in V_\alpha$
$\langle 1 \rangle 4.$ $\mathcal{P} a \in V_{\alpha+1}$
$\langle 1 \rangle 5.$ $\mathcal{P} a \in V_\lambda$
$\langle 1 \rangle 6.$ $\forall x \in V_\lambda (x \in \mathcal{P} a \Leftrightarrow \forall y \in V_\lambda (y \in x \Rightarrow y \in a))$
$\square$

Theorem Schema 11.0.7. *For any property $P[x, y_1, \ldots, y_n]$, the following is a theorem:*

For any ordinal α, the set V_α is a model of the statement: for any sets a_1, $\ldots$, a_n, B, the class $\{x \in B \mid P[x, a_1, \ldots, a_n]\}$ is a set.

PROOF:
$\langle 1 \rangle 1.$ LET: α be an ordinal.
$\langle 1 \rangle 2.$ LET: $a_1, \ldots, a_n, B \in V_\alpha$
$\langle 1 \rangle 3.$ LET: $C = \{x \in B \mid P[x, a_1, \ldots, a_n]^{V_\alpha}\}$
$\langle 1 \rangle 4.$ $C \in V_\alpha$

⟨1⟩5. $\forall x \in V_\alpha (x \in C \Leftrightarrow x \in B \wedge P[x, a_1, \ldots, a_n]^{V_\alpha})$
□

Theorem 11.0.8. *For any ordinal $\alpha > \omega$, we have: V_α is a model of the Axiom of Infinity.*

PROOF:
⟨1⟩1. LET: $\alpha > \omega$
⟨1⟩2. $\mathbb{N} \in V_\alpha$
⟨1⟩3. $\exists e \in V_\alpha (e \in \mathbb{N} \wedge \forall x \in V_\alpha . x \notin e)$
⟨1⟩4. $\forall x \in V_\alpha (x \in \mathbb{N} \Rightarrow \exists y \in V_\alpha \forall z \in V_\alpha (z \in y \Leftrightarrow z \in x \vee z = x))$
□

Theorem 11.0.9. *For any ordinal α, we have V_α is a model of the Axiom of Choice.*

PROOF:
⟨1⟩1. LET: α be an ordinal.
⟨1⟩2. LET: $A \in V_\alpha$
⟨1⟩3. ASSUME: $\forall x \in V_\alpha (x \in A \Rightarrow \exists y \in V_\alpha . y \in A)$
⟨1⟩4. ASSUME: $\forall x, y, z \in V_\alpha (x \in A \wedge y \in A \wedge z \in x \wedge z \in y \Rightarrow x = y)$
⟨1⟩5. A is a set of pairwise disjoint nonempty sets.
⟨1⟩6. PICK c such that, for all $x \in A$, $x \cap c = \emptyset$
⟨1⟩7. $c \cap \bigcup A \in V_\alpha$
⟨1⟩8. $\forall x \in V_\alpha (x \in A \Rightarrow \exists y \in V_\alpha \forall z \in V_\alpha (z \in x \wedge z \in c \cap \bigcup A \Leftrightarrow z = y))$
□

Theorem 11.0.10. *For any ordinal α, we have V_α is a model of the Axiom of Regularity.*

PROOF:
⟨1⟩1. LET: α be an ordinal.
⟨1⟩2. LET: $A \in V_\alpha$
⟨1⟩3. ASSUME: $\exists x \in V_\alpha . x \in A$
⟨1⟩4. PICK $m \in A$ of least rank.
⟨1⟩5. $m \in V_\alpha$
⟨1⟩6. $\neg \exists x \in V_\alpha (x \in m \wedge x \in A)$
□

Theorem Schema 11.0.11. *For any axiom α of Zermelo set theory, the following is a theorem:*

For any limit ordinal $\lambda > \omega$, we have V_λ is a model of α.

PROOF: Theorems 11.0.2, 11.0.3, 11.0.4, 11.0.5, 11.0.6, 11.0.7, 11.0.8, 11.0.9, 11.0.10. □

Corollary Schema 11.0.11.1. *for any axiom α of Zermelo set theory, the following is a theorem:*

$V_{\omega 2}$ is a model of α.

Chapter 12

Group Theory

12.1 Groups

Definition 12.1.1 (Group)**.** A *group* G consists of a set G and a function $\cdot : G^2 \to G$ such that:

1. $\cdot$ is associative
2. There exists $e \in G$ such that $\forall x \in G.xe = x$ and $\forall x \in G.\exists y \in G.xy = e$.

Proposition 12.1.2. *The inverse of an element in a group is unique.*

Proof:
⟨1⟩1. Assume: b and b' are inverses of a.
⟨1⟩2. $b = b'$
Proof:

$$\begin{aligned} b &= be \\ &= bab' \\ &= eb' \\ &= b' \end{aligned}$$

$\square$

Definition 12.1.3. We write x^{-1} for the inverse of x.

Proposition 12.1.4. *In any group, if $ab = ac$ then $b = c$.*

Proof:

$$\begin{aligned} b &= eb \\ &= a^{-1}ab \\ &= a^{-1}ac \\ &= ec \\ &= c \end{aligned} \qquad \square$$

12.2 Abelian Groups

Definition 12.2.1 (Abelian group)**.** An *Abelian group* is a group whose multiplication is commutative.

We may say we are writing an Abelian group *additively*, meaning we write $a + b$ for ab, 0 for e and $-a$ for a^{-1}. In this case we write $a - b$ for ab^{-1}.

Chapter 13

Ring Theory

13.1 Rings

Definition 13.1.1 (Commutative Ring)**.** A *commutative ring* consists of a set R and two binary operations $+$, $\cdot$ on R such that:

- D is an Abelian group under $+$. Let us write 0 for its identity element.
- $\cdot$ is commutative and associative, and distributes over $+$.
- $\cdot$ has an identity element 1 that is different from 0.

Proposition 13.1.2. *In any commutative ring,* $0x = 0$.

PROOF:

$$\begin{aligned} (0+0)x &= 0x \\ \therefore 0x + 0x &= 0x + 0 \\ \therefore 0x &= 0 && \text{(Proposition 12.1.4)}\square \end{aligned}$$

Proposition 13.1.3. *In any commutative ring,* $(-a)b = -(ab)$.

PROOF:

$$\begin{aligned} ab + (-a)b &= (a + (-a))b \\ &= 0b \\ &= 0 && \text{(Proposition 13.1.2)}\square \end{aligned}$$

13.2 Ordered Rings

Definition 13.2.1 (Ordered Commutative Ring)**.** An *ordered commutative ring* consists of a commutative ring R with a linear order $<$ on R such that:

- for all $x, y, z \in R$, we have $x < y$ if and only if $x + z < y + z$.

- for all $x, y, z \in R$, if $0 < z$ then we have $x < y$ if and only if $xz < yz$.

Proposition 13.2.2. *In any ordered commutative ring,* $0 < 1$.

PROOF: If $1 < 0$ then we have $0 < -1$ and so $0 < (-1)(-1) = 1$, which is a contradiction. □

Proposition 13.2.3. *The ordering on an ordered commutative ring is dense; that is, if $x < y$ then there exists z such that $x < z < y$.*

PROOF: Take $z = (x + y)/2$. □

13.3 Integral Domains

Definition 13.3.1 (Integral Domain)**.** An *integral domain* is a commutative ring such that, for all $a, b \in D$, if $ab = 0$ then $a = 0$ or $b = 0$.

Proposition 13.3.2. *In any integral domain, if $ab = ac$ and $a \neq 0$ then $b = c$.*

PROOF: We have $a(b - c) = 0$ and $a \neq 0$ so $b - c = 0$ hence $b = c$. □

Definition 13.3.3 (Ordered Integral Domain)**.** An *ordered integral domain* is an ordered commutative ring that is an integral domain.

Chapter 14

Field Theory

14.1 Fields

Definition 14.1.1 (Field)**.** A *field* F is a commutative ring such that $0 \neq 1$ and, for all $x \in F$, if $x \neq 0$ then there exists $y \in F$ such that $xy = 1$.

Proposition 14.1.2. *Every field is an integral domain.*

PROOF: If $ab = 0$ and $a \neq 0$ then $b = a^{-1}ab = 0$. □

Proposition 14.1.3. *In any field F, we have $F - \{0\}$ is an Abelian group under multiplication.*

PROOF: Immediate from the definition. □

Definition 14.1.4 (Field of Fractions)**.** Let D be an integral domain. The *field of fractions* of D is the quotient set $F = (D \times (D - \{0\}))/\sim$ where

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

under

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)]$$
$$[(a, b)][(c, d)] = [(ac, bd)]$$

We prove this is a field.

PROOF:
⟨1⟩1. $\sim$ is an equivalence relation on $D \times (D - \{0\})$.
  PROOF:
  ⟨2⟩1. $\sim$ is reflexive.
    PROOF: We always have $ab = ba$.
  ⟨2⟩2. $\sim$ is symmetric.
    PROOF: If $ad = bc$ then $cb = da$.

⟨2⟩3. $\sim$ is transitive.
⟨3⟩1. ASSUME: $(a, b) \sim (c, d) \sim (e, f)$
⟨3⟩2. $ad = bc$ and $cf = de$
⟨3⟩3. $adf = bde$
PROOF: $adf = bcf = bde$
⟨3⟩4. $af = be$
PROOF: Proposition 13.3.2.
□
⟨1⟩2. Addition is well-defined.
PROOF:
⟨2⟩1. If $b \neq 0$ and $d \neq 0$ then $bd \neq 0$.
PROOF: Since D is an integral domain.
⟨2⟩2. If $ab' = a'b$ and $cd' = c'd$ then $(ad + bc)b'd' = (a'd' + b'c')bd$.
PROOF:

$$\begin{aligned}(ad + bc)b'd' &= ab'dd' + bb'cd' \\ &= a'bdd' + bb'c'd \\ &= (a'd' + b'c')bd\end{aligned}$$

□
⟨1⟩3. Multiplication is well-defined.
PROOF:
⟨2⟩1. If $b \neq 0$ and $d \neq 0$ then $bd \neq 0$.
PROOF: Since D is an integral domain.
⟨2⟩2. If $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$ then $[(ac, bd)] = [(a'c', b'd')]$.
PROOF: If $ab' = a'b$ and $cd' = c'd$ then $acb'd' = a'c'bd$.
□
⟨1⟩4. Addition is commutative.
PROOF: $[(a, b)] + [(c, d)] = [(ad + bc, bd)] = [(cb + da, db)] = [(c, d)] + [(a, b)]$ □
⟨1⟩5. Addition is associative.
PROOF:

$$\begin{aligned}[(a, b)] + ([(c, d)] + [(e, f)]) &= [(a, b)] + [(cf + de, df)] \\ &= [(adf + bcf + bde, bdf)] \\ &= [(ad + bc, bd)] + [(e, f)] \\ &= ([(a, b)] + [(c, d)]) + [(e, f)]\end{aligned}$$ □

⟨1⟩6. For any $x \in F$ we have $x + [(0, 1)] = x$
PROOF: $[(a, b)] + [(0, 1)] = [(a \cdot 1 + b \cdot 0, b \cdot 1)] = [(a, b)]$ □
⟨1⟩7. For any $x \in F$, there exists $y \in F$ such that $x + y = [(0, 1)]$.
PROOF: $[(a, b)] + [(-a, b)] = [(ab - ab, b^2)] = [(0, b^2)] = [(0, 1)]$ □
⟨1⟩8. Multiplication is commutative.
PROOF: $[(a, b)][(c, d)] = [(c, d)][(a, b)] = [(ac, bd)]$. □
⟨1⟩9. Multiplication is assocative.
PROOF: $[(a, b)]([(c, d)][(e, f)]) = ([(a, b)][(c, d)])[(e, f)] = [(ace, bdf)]$. □
⟨1⟩10. For any $x \in F$ we have $x[(1, 1)] = x$
PROOF: $[(a, b)][(1, 1)] = [(a, b)]$ □
⟨1⟩11. For any non-zero $x \in F$, there exists $y \in F$ such that $xy = [(1, 1)]$.

PROOF:
⟨2⟩1. LET: $[(a, b)] \in \mathbb{Q}$
⟨2⟩2. ASSUME: $[(a, b)] \neq [(0, 1)]$
⟨2⟩3. $a \neq 0$
⟨2⟩4. $[(a, b)][(b, a)] = [(1, 1)]$
□
□

Definition 14.1.5. For any field F, let $N(F)$ be the intersection of all the subsets $S \subseteq F$ such that $1 \in S$ and $\forall x \in S.x + 1 \in S$.

Definition 14.1.6 (Characteristic Zero)**.** A field F has *characteristic 0* iff $0 \notin N(F)$.

Proposition 14.1.7. *In a field F with characteristic 0, the function $n : \mathbb{N} \to N(F)$ defined by*

$$n(0) = 1$$
$$n(x + 1) = n(x) + 1$$

is a bijection.

PROOF:
⟨1⟩1. n is injective.
⟨2⟩1. ASSUME: for a contradiction $n(i) = n(j)$ with $i \neq j$
⟨2⟩2. ASSUME: w.l.o.g. $i < j$
⟨2⟩3. $n(j - i) = 0$
⟨2⟩4. Q.E.D.
PROOF: This contradicts the fact that F has characteristic 0.
⟨1⟩2. n is surjective.
PROOF: Since $\operatorname{ran} n$ is a subset of F that includes 1 and is closed under +1.
□

Definition 14.1.8. In any field F, let

$$I(F) = N(F) \cup \{0\} \cup \{-x \mid x \in N(F)\}$$

Definition 14.1.9. In any field F, let

$$Q(F) = \{x/y \mid x, y \in I(F), y \neq 0\}$$

Proposition 14.1.10. *$Q(F)$ is the smallest subfield of F.*

PROOF: $Q(F)$ is closed under + and $\cdot$, and any subset of F closed under + and $\cdot$ that contains 0 and 1 must include $Q(F)$. □

Theorem 14.1.11. *Let F and G be fields of characteristic 0. Then there exists a unique field isomorphism between $Q(F)$ and $Q(G)$.*

PROOF:

$\langle 1\rangle 1$. LET: $\phi : N(F) \to N(G)$ be the unique function such that $\phi(1) = 1$ and $\forall x \in N(F).\phi(x+1) = \phi(x) + 1$.
$\langle 1\rangle 2$. ϕ is a bijection.
PROOF: Similar to Proposition 14.1.7.
$\langle 1\rangle 3$. $\forall x, y \in N(F).\phi(x+y) = \phi(x) + \phi(y)$
PROOF: Induction on y.
$\langle 1\rangle 4$. $\forall x, y \in N(F).\phi(xy) = \phi(x)\phi(y)$
PROOF: Induction on y.
$\langle 1\rangle 5$. Extend ϕ to a bijection $I(F) \cong I(G)$ such that $\forall x, y \in I(F).\phi(x+y) = \phi(x) + \phi(y)$ and $\forall x, y \in I(F).\phi(xy) = \phi(x)\phi(y)$
$\langle 2\rangle 1$. Define $\phi(0) = 0$ and $\phi(-x) = -\phi(x)$ for $x \in N(F)$
$\langle 3\rangle 1$. $0 \notin N(F)$
$\langle 3\rangle 2$. For all $x \in N(F)$ we have $-x \notin N(F)$
PROOF: Then we would have $x + -x = 0 \in N(F)$.
$\langle 3\rangle 3$. For all $x \in N(F)$ we have $-x \neq 0$
$\langle 2\rangle 2$. For all $x, y \in I(F)$ we have $\phi(x+y) = \phi(x) + \phi(y)$
PROOF: Case analysis on x and y.
$\langle 2\rangle 3$. For all $x, y \in I(F)$ we have $\phi(xy) = \phi(x)\phi(y)$
PROOF: Case analysis on x and y.
$\langle 1\rangle 6$. Extend ϕ to a bijection $Q(F) \cong Q(G)$ such that $\forall x, y \in Q(F).\phi(x+y) = \phi(x) + \phi(y)$ and $\forall x, y \in Q(F).\phi(xy) = \phi(x)\phi(y)$
$\langle 2\rangle 1$. Define $\phi(x/y) = \phi(x)/\phi(y)$
$\langle 1\rangle 7$. ϕ is unique.
$\langle 2\rangle 1$. LET: θ satisfy the theorem.
$\langle 2\rangle 2$. For all $x \in N(F)$ we have $\theta(x) = \phi(x)$
$\langle 2\rangle 3$. For all $x \in I(F)$ we have $\theta(x) = \phi(x)$
$\langle 2\rangle 4$. For all $x \in Q(F)$ we have $\theta(x) = \phi(x)$
$\square$

14.2 Ordered Fields

Definition 14.2.1 (Ordered Field)**.** An *ordered field* is an ordered commutative ring that is a field.

Proposition 14.2.2. *Every ordered field F has characteristic 0.*

PROOF: We have $0 < n$ for all $n \in N(F)$. $\square$

Proposition 14.2.3. *Let F be a field of characteristic 0. Then there exists a unique relation $<$ on $Q(F)$ that makes $Q(F)$ into an ordered field.*

PROOF: Easy. $\square$

Corollary 14.2.3.1. *Let F and G be ordered fields. Let ϕ be the unique field isomorphism between $Q(F)$ and $Q(G)$. Then ϕ is an ordered field isomorphism.*

Definition 14.2.4 (Archimedean)**.** An ordered field F is *Archimedean* iff

$$\forall x \in F.\exists n \in N(F).n > x \ .$$

Proposition 14.2.5. *Let F be an Archimedean ordered field. Let $x, y \in F$ with $x > 0$. Then there exists $n \in N(F)$ such that $nx > y$.*

PROOF: Pick $n > y/x$. □

Proposition 14.2.6. *Let F be an Archimedean ordered field. For all $x, y \in F$, if $x < y$, then there exists $r \in Q(F)$ such that $x < r < y$.*

PROOF:
⟨1⟩1. CASE: $x > 0$
  ⟨2⟩1. PICK $n \in N(F)$ such that $n(y - x) > 1$
    PROOF: Proposition 14.2.5.
  ⟨2⟩2. $ny > 1 + nx$
  ⟨2⟩3. LET: m be the least element of $N(F)$ such that $m > nx$.
  ⟨2⟩4. $m - 1 \leq nx$
  ⟨2⟩5. $nx < m < ny$
  ⟨2⟩6. $x < m/n < y$
⟨1⟩2. CASE: $x \leq 0$
  ⟨2⟩1. PICK $k \in N(F)$ such that $k > -x$
  ⟨2⟩2. $0 < x + k < y + k$
  ⟨2⟩3. PICK $r \in Q(F)$ such that $x + k < r < y + k$
    PROOF: ⟨1⟩1
  ⟨2⟩4. $x < r - k < y$

Definition 14.2.7 (Complete)**.** An ordered field F is *complete* iff every nonempty subset of F bounded above has a least upper bound.

Proposition 14.2.8. *Every complete ordered field is Archimedean.*

PROOF:
⟨1⟩1. LET: F be a complete ordered field.
⟨1⟩2. LET: $x \in F$
⟨1⟩3. ASSUME: for a contradiction there is no member of $N(F)$ greater than x.
⟨1⟩4. x is an upper bound for $N(F)$.
⟨1⟩5. LET: $y = \sup N(F)$
⟨1⟩6. PICK $n \in N(F)$ such that $y - 1 < n$
⟨1⟩7. $y < n + 1$
⟨1⟩8. Q.E.D.
  PROOF: This is a contradiction.
□

Proposition 14.2.9. *Let F be a complete ordered field and $a \in F$ be nonnegative. Then there exists $b \in F$ such that $b^2 = a$.*

PROOF:
⟨1⟩1. LET: $B = \{x \in F \mid 0 \leq x \leq 1 + a\}$
⟨1⟩2. LET: $\phi : B \to B$ be the function

$$\phi(x) = x + \frac{1}{2(1+a)}(a - x^2) \ .$$

⟨1⟩3. ϕ is strictly monotone.
  ⟨2⟩1. LET: $0 \leq x < y \leq 1 + a$
  ⟨2⟩2. $1 - \frac{x+y}{2(1+a)} > 0$
  ⟨2⟩3. $\phi(y) - \phi(x) = (y - x)(1 - \frac{x+y}{2(1+a)}) > 0$
  ⟨2⟩4. $\phi(x) < \phi(y)$
⟨1⟩4. PICK $b \in B$ such that $\phi(b) = b$.
  PROOF: Knaster Fixed-Point Theorem.
⟨1⟩5. $b^2 = a$
□

Theorem 14.2.10 (Uniqueness of the Complete Ordered Field)**.** *If F and G are complete ordered fields, then there exists a unique bijection $\phi : F \cong G$ such that, for all $x, y \in F$,*

$$\phi(x + y) = \phi(x) + \phi(y)$$
$$\phi(xy) = \phi(x)\phi(y)$$

This bijection also satisfies: for all $x, y \in F$,

$$x < y \Leftrightarrow \phi(x) < \phi(y) \ .$$

PROOF:
⟨1⟩1. PICK a bijection $\phi : Q(F) \cong Q(G)$ such that, for all $x, y \in Q(F)$,
$$\phi(x + y) = \phi(x) + \phi(y)$$
$$\phi(xy) = \phi(x)\phi(y)$$
$$x < y \Leftrightarrow \phi(x) < \phi(y)$$
  PROOF: Corollary 14.2.3.1.
⟨1⟩2. $Q(F)$ intersects every interval in F.
  PROOF: Proposition 14.2.6.
⟨1⟩3. $Q(G)$ intersects every interval in G.
  PROOF: Proposition 14.2.6.
⟨1⟩4. PICK an order isomorphism $\psi : F \cong G$ that extends ϕ.
  PROOF: Theorem 5.1.21.
⟨1⟩5. $\forall x, y \in F. \psi(x + y) = \psi(x) + \psi(y)$
  ⟨2⟩1. LET: $x, y \in F$
  ⟨2⟩2. $\psi(x) + \psi(y) \not< \psi(x + y)$
    ⟨3⟩1. ASSUME: for a contradiction $\psi(x) + \psi(y) < \psi(x + y)$
    ⟨3⟩2. PICK $r' \in Q(G)$ such that $\psi(x) < r' < \psi(x + y) - \psi(y)$
    ⟨3⟩3. PICK $s' \in Q(G)$ such that $\psi(y) < s' < \psi(x + y) - r'$
    ⟨3⟩4. $r' + s' < \psi(x + y)$
    ⟨3⟩5. PICK $r, s \in Q(F)$ such that $\phi(r) = r'$ and $\phi(s) = s'$
    ⟨3⟩6. $\phi(r + s) = r' + s'$
    ⟨3⟩7. $\psi(x) < \psi(r)$
    ⟨3⟩8. $\psi(y) < \psi(s)$
    ⟨3⟩9. $\psi(x + y) > \psi(r + s)$
    ⟨3⟩10. $x < r$

$\langle 3\rangle$11. $y < s$
$\langle 3\rangle$12. $x + y > r + s$
$\langle 3\rangle$13. Q.E.D.
PROOF: This is a contradiction.
$\langle 2\rangle$3. $\psi(x+y) \not< \psi(x) + \psi(y)$
PROOF: Similar.
$\langle 1\rangle$6. $\forall x, y \in F. \psi(xy) = \psi(x)\psi(y)$
$\langle 2\rangle$1. LET: $x, y \in F$
$\langle 2\rangle$2. CASE: x and y are positive.
$\langle 3\rangle$1. $\psi(x)\psi(y) \not< \psi(xy)$
$\langle 4\rangle$1. ASSUME: for a contradiction $\psi(x)\psi(y) < \psi(xy)$
$\langle 4\rangle$2. PICK $r' \in Q(G)$ such that $\psi(x) < r' < \psi(xy)/\psi(y)$
$\langle 4\rangle$3. PICK $s' \in Q(G)$ such that $\psi(y) < s' < \psi(xy)/r'$
$\langle 4\rangle$4. $r's' < \psi(xy)$
$\langle 4\rangle$5. PICK $r, s \in Q(F)$ such that $\phi(r) = r'$ and $\phi(s) = s'$
$\langle 4\rangle$6. $\phi(rs) = r's'$
$\langle 4\rangle$7. $x < r$, $y < s$ and $rs < xy$
$\langle 4\rangle$8. Q.E.D.
PROOF: This is a contradiction.
$\langle 3\rangle$2. $\psi(xy) \not< \psi(x)\psi(y)$
PROOF: Similar.
$\langle 2\rangle$3. CASE: x and y are not both positive.
PROOF: Follows from $\langle 2\rangle$2 since $\psi(-x) = -\psi(x)$ by $\langle 1\rangle$5.
$\langle 1\rangle$7. For any field isomorphism $\theta : F \cong G$, we have $\theta = \psi$.
$\langle 2\rangle$1. $\theta \restriction Q(F) = \phi$
PROOF: Theorem 14.1.11.
$\langle 2\rangle$2. θ is strictly monotone.
$\langle 3\rangle$1. LET: $x, y \in F$ with $x < y$
$\langle 3\rangle$2. $y - x > 0$
$\langle 3\rangle$3. PICK $z \in F$ such that $z^2 = y - x$
$\langle 3\rangle$4. $\theta(z)^2 = \theta(y) - \theta(x)$
$\langle 3\rangle$5. $\theta(y) - \theta(x) > 0$
$\langle 3\rangle$6. $\theta(x) < \theta(y)$
$\langle 2\rangle$3. $\theta = \psi$
PROOF: By the uniqueness of ψ.
$\square$

Chapter 15

Number Systems

15.1 The Integers

Definition 15.1.1. The set of *integers* $\mathbb{Z}$ is the quotient set $\mathbb{N}^2/\sim$, where $(m,n) \sim (p,q)$ iff $m+q = n+p$.

We prove$\sim$ is an equivalence relation on $\mathbb{N}^2$.

PROOF:
⟨1⟩1. $\sim$ is reflexive.
  PROOF: For all $m, n \in \mathbb{N}$ we have $m+n = n+m$.
⟨1⟩2. $\sim$ is symmetric.
  PROOF: If $m+q = n+p$ then $p+n = q+m$.
⟨1⟩3. $\sim$ is transitive.
  ⟨2⟩1. ASSUME: $(m,n) \sim (p,q) \sim (r,s)$
  ⟨2⟩2. $m+q = n+p$ and $p+s = q+r$
  ⟨2⟩3. $m+q+s = n+q+r$
  ⟨2⟩4. $m+s = n+r$
    PROOF: By cancellation.
□

Definition 15.1.2 (Addition)**.** Define *addition* $+$ on $\mathbb{Z}$ by $[(m,n)] + [(p,q)] = [(m+p, n+q)]$.

We prove this is well-defined.

PROOF: If $m+n' = n+m'$ and $p+q' = q+p'$ then $m+p+n'+q' = n+q+m'+p'$.
□

Proposition 15.1.3. *Addition on $\mathbb{Z}$ is commutative.*

PROOF: $[(m,n)]+[(p,q)] = [(m+p, n+q)] = [(p+m, q+n)] = [(p,q)]+[(m,n)]$.
□

Proposition 15.1.4. *Addition on $\mathbb{Z}$ is associative.*

Proof: $[(m,n)] + ([(p,q)] + [(r,s)]) = [(m+p+r, n+q+s)] = ([(m,n)] + [(p,q)]) + [(r,s)]$. □

Proposition 15.1.5. *Given natural numbers m and n, we have $[(m,0)] = [(n,0)]$ iff $m = n$.*

Proof: Immediate from definitions. □

Definition 15.1.6. We identify any natural number n with the integer $[(n,0)]$.

Proposition 15.1.7. *Addition on integers agrees with addition on natural numbers.*

Proof: Since $[(m,0)] + [(n,0)] = [(m+n,0)]$. □

Proposition 15.1.8. *For all $a \in \mathbb{Z}$ we have $a + 0 = a$.*

Proof: $[(m,n)] + [(0,0)] = [(m+0, n+0)] = [(m,n)]$. □

Proposition 15.1.9. *For all $a \in \mathbb{Z}$, there exists $b \in \mathbb{Z}$ such that $a + b = 0$.*

Proof: $[(m,n)] + [(n,m)] = [(m+n, m+n)] = [(0,0)]$ □

Proposition 15.1.10. *The integers form an Abelian group under addition.*

Proof: Proposition 15.1.3, 15.1.4, 15.1.8, 15.1.9. □

Definition 15.1.11. Define multiplication $\cdot$ on $\mathbb{Z}$ by: $[(m,n)][(p,q)] = [(mp+nq, mq+np)]$.

We prove this is well defined.

Proof:

$\langle 1 \rangle 1.$ Assume: $m + n' = n + m'$ and $p + q' = q + p'$
Prove: $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$

$\langle 1 \rangle 2.$ $mp + n'p = np + m'p$

$\langle 1 \rangle 3.$ $nq + m'q = mq + n'q$

$\langle 1 \rangle 4.$ $m'p + m'q' = m'q + m'p'$

$\langle 1 \rangle 5.$ $n'q + n'p' = n'p + n'q'$

$\langle 1 \rangle 6.$ $mp + n'p + nq + m'q + m'p + m'q' + n'q + n'p' = np + m'p + mq + n'q + m'q + m'p' + n'p + n'q'$

$\langle 1 \rangle 7.$ $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$
Proof: By cancellation.

□

Proposition 15.1.12. *Multiplication on integers agrees with multiplication on natural numbers.*

Proof: Since $[(m,0)][(n,0)] = [(mn+0, m0+n0)] = [(mn,0)]$. □

Proposition 15.1.13. *Multiplication on $\mathbb{Z}$ is commutative.*

Proof: $[(m,n)][(p,q)] = [(mp+nq, mq+np)] = [(pm+qn, pn+qm)] = [(p,q)][(m,n)]$. □

Proposition 15.1.14. *Multiplication on $\mathbb{Z}$ is associative.*

PROOF:

$$\begin{aligned}[(m,n)]([(p,q)][(r,s)]) &= [(m,n)][(pr+qs, ps+qr)] \\ &= [(mpr+mqs+nps+nqr, mps+mqr+npr+nqs)] \\ &= [(mp+nq, mq+np)][(r,s)] \\ &= ([(m,n)][(p,q)])[(r,s)]\end{aligned}$$

□

Proposition 15.1.15. *Multiplication distributes over addition.*

PROOF:

$$\begin{aligned}[(m,n)]([(p,q)]+[(r,s)]) &= [(m,n)][(p+r, q+s)] \\ &= [(mp+mr+nq+ns, np+nr+mq+ms)] \\ [(m,n)][(p,q)]+[(m,n)][(r,s)] &= [(mp+nq, mq+np)]+[(mr+ns, ms+nr)] \\ &= [(mp+nq+mr+ns, mq+np+ms+nr)]\end{aligned}$$

□

Proposition 15.1.16. *For any integer a we have $a1 = a$.*

PROOF: Since $[(m,n)][(1,0)] = [(m1+n0, m0+n1)] = [(m,n)]$. □

Proposition 15.1.17. *For any integeres a and b, if $ab = 0$ then $a = 0$ or $b = 0$.*

PROOF:
⟨1⟩1. ASSUME: $[(m,n)][(p,q)] = [(0,0)]$
⟨1⟩2. $mp+nq = mq+np$
⟨1⟩3. ASSUME: $[(m,n)] \neq [(0,0)]$
⟨1⟩4. $m \neq n$
PROVE: $p = q$
⟨1⟩5. CASE: $m < n$
⟨2⟩1. $p \not< q$
PROOF: If $p < q$ then $mq+np < mp+nq$ by Proposition 8.4.6.
⟨2⟩2. $q \not< p$
PROOF: If $q < p$ then $mp+nq < mq+np$ by Proposition 8.4.6.
⟨2⟩3. $p = q$
PROOF: By trichotomy.
⟨1⟩6. CASE: $n < m$
PROOF: Similar.
□

Proposition 15.1.18. *The integers $\mathbb{Z}$ form an integral domain.*

PROOF: Propositions 15.1.13, 15.1.14, 15.1.15, 15.1.16, 15.1.17, 15.1.10. □

Definition 15.1.19. Define $<$ on $\mathbb{Z}$ by $[(m,n)] < [(p,q)]$ if and only if $m+q < n+p$.

We prove this is well-defined.

PROOF:

⟨1⟩1. ASSUME: $m + n' = n + m'$ and $p + q' = q + p'$.
PROVE: $m + q < n + p$ if and only if $m' + q' < n' + p'$

⟨1⟩2. $m + q < n + p$ if and only if $m' + q' < n' + p'$

PROOF:

$$\begin{aligned}
m + q < n + p &\Leftrightarrow m + n' + q < n + n' + p && \text{(Corollary 6.5.7.1)} \\
&\Leftrightarrow m' + n + q < n + n' + p \\
&\Leftrightarrow m' + q < n' + p && \text{(Corollary 6.5.7.1)} \\
&\Leftrightarrow m' + q + p' < n' + p + p' && \text{(Corollary 6.5.7.1)} \\
&\Leftrightarrow m' + q' + p < n' + p + p' \\
&\Leftrightarrow m' + q' < n' + p' && \text{(Corollary 6.5.7.1)} \square
\end{aligned}$$

Proposition 15.1.20. *The ordering on the integers agrees with the ordering on the natural numbers.*

PROOF: We have $[(m, 0)] < [(n, 0)]$ iff $m < n$. $\square$

Proposition 15.1.21. *$<$ is a linear order on $\mathbb{Z}$.*

PROOF:

⟨1⟩1. $<$ is irreflexive.
PROOF: We never have $m + n < m + n$.

⟨1⟩2. $<$ is transitive.

⟨2⟩1. ASSUME: $[(m, n)] < [(p, q)] < [(r, s)]$

⟨2⟩2. $m + q < n + p$ and $p + s < q + r$

⟨2⟩3. $m + q + s < n + q + r$
PROOF: $m + q + s < n + p + s < n + q + r$

⟨2⟩4. $m + s < n + r$
PROOF: Corollary 6.5.7.1.

⟨1⟩3. $<$ is total.
PROOF: Given natural numbers m, n, p and q, either $m + q < n + p$, or $m + q = n + p$, or $n + p < m + q$.

$\square$

Definition 15.1.22 (Positive)**.** An integer a is *positive* iff $a > 0$.

Theorem 15.1.23. *For any integers a, b and c, we have $a < b$ if and only if $a + c < b + c$.*

PROOF:

⟨1⟩1. If $a < b$ then $a + c < b + c$.

⟨2⟩1. LET: $a = [(m, n)]$, $b = [(p, q)]$ and $c = [(r, s)]$.

⟨2⟩2. ASSUME: $a < b$

⟨2⟩3. $m + q < n + p$

⟨2⟩4. $m + r + q + s < n + r + p + s$

⟨2⟩5. $[(m + r, n + s)] < [(p + r, q + s)]$

⟨2⟩6. $a + c < b + c$

$\langle 1\rangle 2$. If $a + c < b + c$ then $a < b$.
PROOF: From $\langle 1\rangle 1$ and Proposition 5.2.6.
□

Proposition 15.1.24. *Let a, b and c be integers. If $0 < c$, then $a < b$ if and only if $ac < bc$.*

PROOF:
$\langle 1\rangle 1$. LET: $c = [(r, s)]$
$\langle 1\rangle 2$. ASSUME: $0 < c$
$\langle 1\rangle 3$. $s < r$
$\langle 1\rangle 4$. For all integers a and b, if $a < b$ then $ac < bc$
$\langle 2\rangle 1$. LET: $a = [(m, n)]$, $b = [(p, q)]$.
$\langle 2\rangle 2$. ASSUME: $a < b$
$\langle 2\rangle 3$. $m + q < n + p$
$\langle 2\rangle 4$. $(m + q)r + (p + n)s < (m + q)s + (p + n)r$
PROOF: Proposition 8.4.6, $\langle 1\rangle 3$, $\langle 2\rangle 3$.
$\langle 2\rangle 5$. $mr + ns + ps + qr < ms + nr + pr + qs$
$\langle 2\rangle 6$. $[(mr + ns, ms + nr)] < [(pr + qs, ps + qr)]$
$\langle 2\rangle 7$. $ac < bc$
$\langle 1\rangle 5$. For all integers a and b, if $ac < bc$ then $a < b$
PROOF: From $\langle 1\rangle 4$ and Proposition 5.2.6.
□

Proposition 15.1.25. *Let a be a positive integer. For any integer b, there exists $k \in \mathbb{N}$ such that $b < ak$.*

PROOF:
$\langle 1\rangle 1$. CASE: $b \leq 0$
PROOF: Take $k = 1$.
$\langle 1\rangle 2$. CASE: $b > 0$
PROOF: Take $k = b + 1$.
□

15.2 The Rationals

Definition 15.2.1 (Rational Numbers)**.** The set $\mathbb{Q}$ of *rational numbers* is the field of fractions over the integers.

Proposition 15.2.2. *For any integers a and b, we have $[(a, 1)] = [(b, 1)]$ iff $a = b$.*

PROOF: Immediate from definitions. □

Henceforth we identify any integer a with the rational number $[(a, 1)]$.

Proposition 15.2.3. *Addition on the rationals agrees with addition on the integers.*

Proof: $[(a, 1)] + [(b, 1)] = [(a \cdot 1 + b \cdot 1, 1 \cdot 1)] = [(a + b, 1)]$. □

Proposition 15.2.4. *Multiplication on the rationals agrees with multiplication on the integers.*

Proof: $[(a, 1)][(b, 1)] = [(ab, 1)]$ □

Definition 15.2.5. Define the ordering $<$ on the rationals by: if b and d are positive, then $[(a, b)] < [(c, d)]$ iff $ad < bc$.

We prove this is well-defined.

Proof:
$\langle 1\rangle 1$. For any rational q, there exist integers a, b with b positive such that $q = [(a, b)]$.
  Proof: Since $[(a, b)] = [(-a, -b)]$, and if $b \neq 0$ then one of b and $-b$ is positive.
$\langle 1\rangle 2$. If b, b', d and d' are positive, $[(a, b)] = [(a', b')]$, and $[(c, d)] = [(c', d')]$, then $ad < bc$ iff $a'd' < b'c'$.
  Proof:
  $\langle 2\rangle 1$. If $ad < bc$ then $a'd' < b'c'$.
    $\langle 3\rangle 1$. Assume: $ad < bc$
    $\langle 3\rangle 2$. $ab'd < bb'c$
    $\langle 3\rangle 3$. $a'bd < bb'c$
    $\langle 3\rangle 4$. $a'd < b'c$
    $\langle 3\rangle 5$. $a'dd' < b'cd'$
    $\langle 3\rangle 6$. $a'dd' < b'c'd$
    $\langle 3\rangle 7$. $a'd' < b'c'$
  $\langle 2\rangle 2$. If $a'd' < b'c'$ then $ad < bc$.
    Proof: Similar.
□

Proposition 15.2.6. *The ordering on the rationals agrees with the ordering on the integers.*

Proof: We have $[(a, 1)] < [(b, 1)]$ if and only if $a < b$. □

Proposition 15.2.7. *The relation $<$ is a linear ordering on $\mathbb{Q}$.*

Proof:
$\langle 1\rangle 1$. $<$ is irreflexive.
  Proof: We never have $ab < ab$.
$\langle 1\rangle 2$. $<$ is transitive.
  $\langle 2\rangle 1$. Assume: $[(a, b)] < [(c, d)] < [(e, f)]$ where b, d and f are positive.
  $\langle 2\rangle 2$. $ad < bc$ and $cf < de$
  $\langle 2\rangle 3$. $adf < bde$
    Proof: $adf < bcf < bde$
  $\langle 2\rangle 4$. $af < be$
$\langle 1\rangle 3$. $<$ is total.
  Proof: For any integers a, b, c, d, we have $ad < bc$ or $ad = bc$ or $bc < ad$.

$\square$

Proposition 15.2.8. *For any rationals r, s and t, we have $r < s$ if and only if $r + t < s + t$.*

PROOF:
⟨1⟩1. LET: a, b, c, d, e, f be integers with b, d and f positive.
⟨1⟩2. $[(a,b)] + [(e,f)] < [(c,d)] + [(e,f)]$ if and only if $[(a,b)] < [(c,d)]$.
PROOF:

$$\begin{aligned}
[(a,b)] + [(e,f)] < [(c,d)] + [(e,f)] &\Leftrightarrow [(af+be, bf)] < [(cf+de, df)] \\
&\Leftrightarrow (af+be)df < (cf+de)bf \\
&\Leftrightarrow afdf + bedf < cfbf + debf \\
&\Leftrightarrow afdf < cfbf \\
&\Leftrightarrow ad < bc \\
&\Leftrightarrow [(a,b)] < [(c,d)]
\end{aligned}$$

$\square$

Corollary 15.2.8.1. *For any rational r, we have $r < 0$ if and only if $0 < -r$.*

Definition 15.2.9 (Absolute Value)**.** For any rational r, the *absolute value* of r is defined by

$$|r| := \begin{cases} -r & \text{if } 0 < -r \\ r & \text{otherwise} \end{cases}$$

Proposition 15.2.10. *For any rationals r, s and t, if t is positive then $r < s$ iff $rt < st$.*

PROOF:
⟨1⟩1. LET: $r = [(a,b)]$, $s = [(c,d)]$ and $t = [(e,f)]$ where b, d and f are positive.
⟨1⟩2. ASSUME: $0 < t$
⟨1⟩3. $e > 0$
⟨1⟩4. $rt < st$ iff $r < s$
PROOF:

$$\begin{aligned}
rt < st &\Leftrightarrow [(ae, bf)] < [(ce, df)] \\
&\Leftrightarrow aedf < cebf \\
&\Leftrightarrow ad < bc \\
&\Leftrightarrow r < s
\end{aligned}$$

$\square$

Corollary 15.2.10.1. *The rationals form an ordered field.*

Proposition 15.2.11. *Let p be a positive rational. For any rational number r, there exists $k \in \mathbb{N}$ such that $r < pk$.*

PROOF:
⟨1⟩1. LET: $p = a/b$ and $r = c/d$ where a, b and d are positive.

$\langle 1\rangle 2.$ Pick $k \in \mathbb{N}$ such that $bc < adk$
Proof: Proposition 15.1.25.
$\langle 1\rangle 3.$ $r < pk$
□

Proposition 15.2.12. $\mathbb{Q} \approx \mathbb{N}$

Proof: Arrange the rationals in order $0/1$, $1/1$, $1/2$, $0/2$, $-1/2$, $-1/1$, $-2/1$, $-2/2$, $-2/3$, $-1/3$, $0/3$, $1/3$, $2/3$, etc. then remove all duplicates. □

15.3 The Real Numbers

Definition 15.3.1 (Cauchy Sequence)**.** A *Cauchy sequence* is a sequence (q_n) of rationals such that, for every positive rational ϵ, there exists $k \in \mathbb{N}$ such that $\forall m, n > k. |q_m - q_n| < \epsilon$.

Definition 15.3.2 (Dedekind Cut)**.** A *Dedekind cut* is a set $x \subseteq \mathbb{Q}$ such that:

1. $\emptyset \neq x \neq \mathbb{Q}$
2. x is closed downwards.
3. x has no greatest member.

The set $\mathbb{R}$ of *real numbers* is the set of Dedekind cuts.

Proposition 15.3.3. *For any rational q, we have $\{r \in \mathbb{Q} \mid r < q\} \in \mathbb{R}$.*

Proof:
$\langle 1\rangle 1.$ Let: $q \in \mathbb{Q}$
$\langle 1\rangle 2.$ Let: $q \downarrow = \{r \mid r < q\}$
$\langle 1\rangle 3.$ $q \downarrow \neq \emptyset$
Proof: We have $q - 1 \in q \downarrow$.
$\langle 1\rangle 4.$ $q \downarrow \neq \mathbb{Q}$
Proof: Since $q \notin q \downarrow$.
$\langle 1\rangle 5.$ $q \downarrow$ is closed downwards.
Proof: Trivial.
$\langle 1\rangle 6.$ $q \downarrow$ has no greatest element.
Proof: For all $r \in q \downarrow$ we have $r < (q + r)/2 \in q \downarrow$.
□

Proposition 15.3.4. *For rationals q and r, we have $q = r$ if and only if $\{s \in \mathbb{Q} \mid s < q\} = \{s \in \mathbb{Q} \mid s < r\}$.*

Proof:
$\langle 1\rangle 1.$ Let: $q \downarrow = \{s \in \mathbb{Q} \mid s < q\}$
$\langle 1\rangle 2.$ Let: $r \downarrow = \{s \in \mathbb{Q} \mid s < r\}$
$\langle 1\rangle 3.$ If $q = r$ then $q \downarrow = r \downarrow$
Proof: Trivial.

$\langle 1\rangle 4$. If $q < r$ then $q\downarrow \neq r\downarrow$
  PROOF: We have $q \in r\downarrow$ and $q \notin q\downarrow$.
$\langle 1\rangle 5$. If $r < q$ then $q\downarrow \neq r\downarrow$
  PROOF: We have $r \in q\downarrow$ and $q \notin q\downarrow$.
□

Henceforth we identify a rational q with the real number $\{r \in \mathbb{Q} \mid r < q\}$.

Definition 15.3.5. Define the ordering $<$ on $\mathbb{R}$ by: $x < y$ iff $x \subsetneq y$.

Proposition 15.3.6. *The ordering on the reals agrees with the ordering on the rationals.*

PROOF:
$\langle 1\rangle 1$. LET: $q, r \in \mathbb{Q}$
$\langle 1\rangle 2$. LET: $q\downarrow = \{s \in \mathbb{Q} \mid s < q\}$.
$\langle 1\rangle 3$. LET: $r\downarrow = \{s \in \mathbb{Q} \mid s < r\}$.
    PROVE: $q < r$ iff $q\downarrow \subsetneq r\downarrow$
$\langle 1\rangle 4$. If $q < r$ then $q\downarrow \subsetneq r\downarrow$
  $\langle 2\rangle 1$. ASSUME: $q < r$
  $\langle 2\rangle 2$. $q\downarrow \subseteq r\downarrow$
    PROOF: If $s < q$ then $s < r$.
  $\langle 2\rangle 3$. $q\downarrow \neq r\downarrow$
    PROOF: Proposition 15.3.4.
$\langle 1\rangle 5$. If $q\downarrow \subsetneq r\downarrow$ then $q < r$
  $\langle 2\rangle 1$. ASSUME: $q\downarrow \subsetneq r\downarrow$
  $\langle 2\rangle 2$. PICK $s \in r\downarrow$ such that $s \notin q\downarrow$
  $\langle 2\rangle 3$. $q \leq s < r$
□

Proposition 15.3.7. *The ordering $<$ is a linear ordering on $\mathbb{R}$.*

PROOF:
$\langle 1\rangle 1$. $<$ is irreflexive.
  PROOF: No set is a proper subset of itself.
$\langle 1\rangle 2$. $<$ is transitive.
  PROOF: Since the relationship $\subsetneq$ is transitive on the class of all sets.
$\langle 1\rangle 3$. $<$ is total.
  $\langle 2\rangle 1$. LET: x, y be Dedekind cuts.
  $\langle 2\rangle 2$. ASSUME: $x \nsubseteq y$
    PROVE: $y \subsetneq x$
  $\langle 2\rangle 3$. PICK $q \in x$ such that $q \notin y$
  $\langle 2\rangle 4$. LET: $r \in y$
    PROVE: $r \in x$
  $\langle 2\rangle 5$. $q \nleq r$
    PROOF: Since y is closed downwards.
  $\langle 2\rangle 6$. $r < q$
  $\langle 2\rangle 7$. $r \in x$

PROOF: Since x is closed downwards.
□

Proposition 15.3.8. *Any bounded nonempty subset of* $\mathbb{R}$ *has a least upper bound.*

PROOF:
⟨1⟩1. LET: A be a bounded nonempty subset of $\mathbb{R}$.
⟨1⟩2. $\bigcup A$ is a Dedekind cut.
  ⟨2⟩1. $\bigcup A \neq \emptyset$
    ⟨3⟩1. PICK$x \in A$
    ⟨3⟩2. PICK $q \in x$
    ⟨3⟩3. $q \in \bigcup A$
  ⟨2⟩2. $\bigcup A \neq \mathbb{Q}$
    ⟨3⟩1. PICK an upper bound u for A
    ⟨3⟩2. PICK $q \notin u$
      PROVE: $q \notin \bigcup A$
    ⟨3⟩3. ASSUME: for a contradiction $q \in \bigcup A$
    ⟨3⟩4. PICK $x \in A$ such that $q \in x$
    ⟨3⟩5. $x \leq u$
    ⟨3⟩6. $q \in u$
    ⟨3⟩7. Q.E.D.
      PROOF: This is a contradiction.
  ⟨2⟩3. $\bigcup A$ is closed downwards.
    ⟨3⟩1. LET: $q \in \bigcup A$ and $r < q$
    ⟨3⟩2. PICK $x \in A$ such that $q \in x$
    ⟨3⟩3. $r \in x$
    ⟨3⟩4. $r \in \bigcup A$
  ⟨2⟩4. $\bigcup A$ has no greatest element.
    ⟨3⟩1. LET: $q \in \bigcup A$
    ⟨3⟩2. PICK $x \in A$ such that $q \in x$
    ⟨3⟩3. PICK $r \in x$ such that $q < r$
    ⟨3⟩4. $r \in \bigcup A$
⟨1⟩3. $\bigcup A$ is an upper bound for A.
  PROOF: For all $x \in A$ we have $x \subseteq \bigcup A$.
⟨1⟩4. For any upper bound u for $\bigcup A$ we have $\bigcup A \leq u$.
  PROOF: If $\forall x \in A . x \subseteq u$ we have $\bigcup A \subseteq u$.
□

Definition 15.3.9 (Addition)**.** Define *addition* $+$ on the reals by

$$x + y := \{q + r \mid q \in x, r \in y\} \ .$$

We prove this is well-defined.

PROOF:
⟨1⟩1. LET: $x, y \in \mathbb{R}$
  PROVE: $X + y$ is a Dedekind cut.

⟨1⟩2. $x + y \neq \emptyset$
  PROOF: Pick $q \in x$ and $r \in y$; then $q + r \in x + y$.
⟨1⟩3. $x + y \neq \mathbb{Q}$
  ⟨2⟩1. PICK $q \notin x$ and $r \notin y$
    PROVE: $q + r \notin x + y$
  ⟨2⟩2. ASSUME: for a contradiction $q + r \in x + y$
  ⟨2⟩3. PICK $q' \in x$ and $r' \in y$ such that $q + r = q' + r'$
  ⟨2⟩4. $q' < q$ and $r' < r$
  ⟨2⟩5. $q' + r' < q + r$
  ⟨2⟩6. Q.E.D.
    PROOF: This is a contradiction.
⟨1⟩4. $x + y$ is closed downwards.
  ⟨2⟩1. LET: $q \in x$ and $r \in y$
  ⟨2⟩2. LET: $s < q + r$
    PROVE: $s \in x + y$
  ⟨2⟩3. $s - r < q$
  ⟨2⟩4. $s - r \in x$
  ⟨2⟩5. $s = (s - r) + r \in x + y$
⟨1⟩5. $x + y$ has no greatest element.
  ⟨2⟩1. LET: $q \in x$ and $r \in y$
    PROVE: There exists $s \in x + y$ such that $q + r < s$
  ⟨2⟩2. PICK $q' \in x$ and $r' \in y$ such that $q < q'$ and $r < r'$
  ⟨2⟩3. $q + r < q' + r' \in x + y$

□

Proposition 15.3.10. *Addition on the reals agrees with addition on the rationals.*

PROOF:
⟨1⟩1. LET: $q, r \in \mathbb{Q}$
⟨1⟩2. $q \downarrow + r \downarrow \subseteq (q + r) \downarrow$
  PROOF: If $s_1 < q$ and $s_2 < r$ then $s_1 + s_2 < q + r$.
⟨1⟩3. $(q + r) \downarrow \subseteq q \downarrow + r \downarrow$
  ⟨2⟩1. LET: $s < q + r$
  ⟨2⟩2. $s - r < q$
  ⟨2⟩3. PICK t such that $s - r < t < q$
  ⟨2⟩4. $s - t < r$
  ⟨2⟩5. $s = t + (s - t) \in q \downarrow + r \downarrow$

□

Proposition 15.3.11. *Addition is associative.*

PROOF:

$$\begin{aligned} x + (y + z) &= \{q + r \mid q \in x, r \in y + z\} \\ &= \{q + s_1 + s_2 \mid q \in x, s_1 \in y, s_2 \in z\} \\ &= \{r + s_2 \mid r \in x + y, s_2 \in z\} \\ &= (x + y) + z \end{aligned}$$

□

Proposition 15.3.12. *Addition is commutative.*

PROOF:
$$\begin{aligned} x + y &= \{q + r \mid q \in x, r \in y\} \\ &= \{r + q \mid r \in y, q \in x\} \\ &= y + x \end{aligned}$$
∎

Proposition 15.3.13. *For any $x \in \mathbb{R}$ we have $x + 0 = x$.*

PROOF:
⟨1⟩1. $x + 0 \subseteq x$
  PROOF: If $q \in x$ and $r < 0$ then $q + r < q$ so $q + r \in x$.
⟨1⟩2. $x \subseteq x + 0$
  ⟨2⟩1. LET: $q \in x$
  ⟨2⟩2. PICK $r \in x$ such that $q < r$.
    PROOF: x has no greatest element.
  ⟨2⟩3. $q - r < 0$
  ⟨2⟩4. $q = r + (q - r) \in x + 0$
∎

Definition 15.3.14. For $x \in \mathbb{R}$, define $-x := \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$.

Proposition 15.3.15. *For all $x \in \mathbb{R}$ we have $-x \in \mathbb{R}$.*

PROOF:
⟨1⟩1. LET: $x \in \mathbb{R}$
⟨1⟩2. $-x \neq \emptyset$
  ⟨2⟩1. PICK $s \notin x$
  ⟨2⟩2. $-s - 1 \in -x$
⟨1⟩3. $-x \neq \mathbb{Q}$
  ⟨2⟩1. PICK $s \in x$
    PROVE: $-s \notin -x$
  ⟨2⟩2. ASSUME: for a contradiction $-s \in -x$
  ⟨2⟩3. PICK $r > -s$ such that $-r \notin x$
  ⟨2⟩4. $-r < s$
  ⟨2⟩5. Q.E.D.
    PROOF: This contradicts the fact that x is closed downwards.
⟨1⟩4. $-x$ is closed downwards.
  PROOF: Immediate from definition.
⟨1⟩5. $-x$ has no greatest element.
  ⟨2⟩1. LET: $q \in -x$
  ⟨2⟩2. PICK $r > q$ such that $-r \notin x$
  ⟨2⟩3. PICK s such that $q < s < r$
  ⟨2⟩4. $s \in -x$
∎

Lemma 15.3.16. *Let p be a positive rational number. For any real number x, there exists a rational $q \in x$ such that $p + q \notin x$.*

PROOF:
$\langle 1\rangle 1$. PICK $q_0 \in x$
$\langle 1\rangle 2$. There exists $k \in \mathbb{N}$ such that $q_0 + kp \notin x$
$\langle 2\rangle 1$. PICK $q_1 \notin x$
$\langle 2\rangle 2$. PICK $k \in \mathbb{N}$ such that $q_1 - q_0 < pk$
PROOF: Proposition 15.2.11.
$\langle 2\rangle 3$. $q_1 < q_0 + kp$
$\langle 2\rangle 4$. $q_0 + kp \notin x$
$\langle 1\rangle 3$. LET: k be the least natural number such that $q_0 + kp \notin x$
$\langle 1\rangle 4$. $k \neq 0$
PROOF: $\langle 1\rangle 1$
$\langle 1\rangle 5$. LET: $q = q_0 + (k-1)p$
$\langle 1\rangle 6$. $q \in x$ and $q + p \notin x$.
□

Proposition 15.3.17. *For every real x we have $x + (-x) = 0$.*

PROOF:
$\langle 1\rangle 1$. LET: x be a real number.
$\langle 1\rangle 2$. $x + (-x) \subseteq 0$
$\langle 2\rangle 1$. LET: $q_1 \in x$ and $q_2 \in -x$
$\langle 2\rangle 2$. PICK $r > q_2$ such that $-r \notin x$
$\langle 2\rangle 3$. $q_1 < -r$
$\langle 2\rangle 4$. $r < -q_1$
$\langle 2\rangle 5$. $q_2 < -q_1$
$\langle 2\rangle 6$. $q_1 + q_2 < 0$
$\langle 1\rangle 3$. $0 \subseteq x + (-x)$
$\langle 2\rangle 1$. LET: $p < 0$
$\langle 2\rangle 2$. $0 < -p$
$\langle 2\rangle 3$. PICK $q \in x$ such that $q - p/2 \notin x$
PROOF: Lemma 15.3.16.
$\langle 2\rangle 4$. LET: $s = p/2 - q$
$\langle 2\rangle 5$. $-s \notin x$
$\langle 2\rangle 6$. $p - q < s$
$\langle 2\rangle 7$. $p - q \in -x$
$\langle 2\rangle 8$. $p \in x + (-x)$
□

Corollary 15.3.17.1. *The reals form an Abelian group under addition.*

Proposition 15.3.18. *For any reals x, y and z, we have $x < y$ if and only if $x + z < y + z$.*

PROOF:
$\langle 1\rangle 1$. $\forall x, y, z \in \mathbb{R}. x \leq y \Rightarrow x + z \leq y + z$
$\langle 2\rangle 1$. LET: $x, y, z \in \mathbb{R}$
$\langle 2\rangle 2$. ASSUME: $x \leq y$
$\langle 2\rangle 3$. For all $q \in x$ and $r \in z$ we have $q + r \in y + z$

$\langle 1\rangle 2.$ $\forall x, y, z \in \mathbb{R}. x + z = y + z \Leftrightarrow x = y$
PROOF: Proposition 12.1.4.
$\langle 1\rangle 3.$ $\forall x, y, z \in \mathbb{R}. x < y \Rightarrow x + z < y + z$
$\langle 1\rangle 4.$ Q.E.D.
PROOF: Proposition 5.2.6.
□

Definition 15.3.19 (Absolute Value)**.** The *absolute value* of a real number x is defined to be

$$|x| = \begin{cases} x & \text{if } 0 \leq x \\ -x & \text{if } x < 0 \end{cases}$$

Definition 15.3.20 (Multiplication)**.** Define *multiplication* $\cdot$ on $\mathbb{R}$ as follows:

- If x and y are non-negative then

$$xy = 0 \cup \{rs \mid 0 \leq r \in x \wedge 0 \leq s \in y\} \ .$$

- If x and y are both negative then

$$xy = (-x)(-y) \ .$$

- If one of x and y is negative and one is non-negative then

$$xy = -(|x||y|) \ .$$

We prove this is well-defined.

PROOF:
$\langle 1\rangle 1.$ LET: x and y be non-negative reals.
PROVE: xy is real.
$\langle 1\rangle 2.$ $xy \neq \emptyset$
PROOF: Since $-1 \in xy$.
$\langle 1\rangle 3.$ $xy \neq \mathbb{Q}$
$\langle 2\rangle 1.$ PICK $r \notin x$ and $s \notin y$
PROVE: $rs \notin xy$
$\langle 2\rangle 2.$ $0 \leq r$ and $0 \leq s$
PROOF: Since $0 \subseteq x$ and $0 \subseteq y$.
$\langle 2\rangle 3.$ ASSUME: for a contradiction $rs \in xy$
$\langle 2\rangle 4.$ PICK r' and s' such that $0 \leq r' \in x$, $0 \leq s' \in y$ and $rs = r's'$
$\langle 2\rangle 5.$ $r' < r$
$\langle 2\rangle 6.$ $s' < s$
$\langle 2\rangle 7.$ $r's' < rs$
$\langle 2\rangle 8.$ Q.E.D.
PROOF: This is a contradiction.
$\langle 1\rangle 4.$ xy is closed downwards.
$\langle 2\rangle 1.$ LET: $q \in xy$ and $r < q$

⟨2⟩2. CASE: $q \in 0$
PROOF: Then $r < q < 0$ so $r \in xy$
⟨2⟩3. CASE: $q = s_1 s_2$ where $0 \leq s_1 \in x$ and $0 \leq s_2 \in y$
⟨3⟩1. ASSUME: w.l.o.g. $0 \leq r$
⟨3⟩2. $0 < s_1$ and $0 < s_2$
⟨3⟩3. $r/s_2 < s_1$
⟨3⟩4. $r/s_2 \in x$
⟨3⟩5. $r = (r/s_2)s_2 \in xy$
⟨1⟩5. xy has no greatest element.
⟨2⟩1. LET: $q \in xy$
⟨2⟩2. CASE: $q \in 0$
PROOF: $q < q/2 \in 0$
⟨2⟩3. CASE: $q = rs$ where $0 \leq r \in x$ and $0 \leq s \in y$
⟨3⟩1. PICK r' and s' with $r < r' \in x$ and $s < s' \in y$
⟨3⟩2. $q < r's' \in xy$
□

Proposition 15.3.21. *Multiplication is commutative.*

PROOF: Immediate from definition. □

Proposition 15.3.22. *Multiplication is associative.*

PROOF:
⟨1⟩1. For non-negative reals x, y and z, we have $x(yz) = (xy)z$
PROOF: It computes to $0 \cup \{qrs \mid 0 \leq q \in x, 0 \leq r \in y, 0 \leq s \in z\}$.
⟨1⟩2. For all reals x, y and z, we have $x(yz) = (xy)z$
PROOF: It is equal to $|x||y||z|$ if an even number of them are negative, and $-(|x||y||z|)$ otherwise.
□

Proposition 15.3.23. *Multiplication distributes over addition.*

PROOF:
⟨1⟩1. For all non-negative reals x, y and z, we have $x(y + z) = xy + xz$
⟨2⟩1. LET: x, y and z be non-negative reals.
⟨2⟩2. $x(y + z) \subseteq xy + xz$
⟨3⟩1. LET: $q \in x(y + z)$
⟨3⟩2. CASE: $q < 0$
PROOF: Then we have $q/2 \in xy$ and $q/2 \in xz$ so $q \in xy + xz$.
⟨3⟩3. CASE: $q = rs$ where $0 \leq r \in x$ and $0 \leq s \in y + z$
⟨4⟩1. PICK $s_1 \in y$ and $s_2 \in z$ such that $s = s_1 + s_2$
⟨4⟩2. $rs_1 \in xy$
PROOF: If $s_1 < 0$ then $rs_1 < 0$ so $rs_1 \in xy$. If $0 \leq s_1$ then we also have $rs_1 \in xy$.
⟨4⟩3. $rs_2 \in xz$
PROOF: Similar.
⟨4⟩4. $q \in xy + xz$

PROOF: Since $q = rs_1 + rs_2$.

$\langle 2\rangle 3.$ $xy + xz \subseteq x(y + z)$

$\langle 3\rangle 1.$ LET: $q \in xy$ and $r \in xz$.

PROVE: $q + r \in x(y + z)$

$\langle 3\rangle 2.$ CASE: $q < 0$ and $r < 0$

PROOF: Then $q + r < 0$ so $q + r \in x(y + z)$.

$\langle 3\rangle 3.$ CASE: $q < 0$ and $r = r_1 r_2$ where $0 \leq r_1 \in x$ and $0 \leq r_2 \in z$

$\langle 4\rangle 1.$ $q + r < r$

$\langle 4\rangle 2.$ $q + r \in xz$

$\langle 4\rangle 3.$ ASSUME: w.l.o.g. $0 \leq q + r$

PROOF: Otherwise $q + r \in x(y + z)$ immediately.

$\langle 4\rangle 4.$ PICK s_1, s_2 with $0 \leq s_1 \in x$, $0 \leq s_2 \in y$ and $q + r = s_1 s_2$

$\langle 4\rangle 5.$ $s_2 \in y + z$

PROOF: Since $0 \in z$ so $s_2 = s_2 + 0 \in y + z$.

$\langle 4\rangle 6.$ $q + r \in x(y + z)$

$\langle 3\rangle 4.$ CASE: $q = q_1 q_2$ where $0 \leq q_1 \in x$ and $0 \leq q_2 \in y$ and $r < 0$

PROOF: Similar.

$\langle 3\rangle 5.$ CASE: $q = q_1 q_2$ where $0 \leq q_1 \in x$ and $0 \leq q_2 \in y$ and $r = r_1 r_2$ where $0 \leq r_1 \in x$ and $0 \leq r_2 \in z$

$\langle 4\rangle 1.$ ASSUME: w.l.o.g. $q_1 \leq r_1$

$\langle 4\rangle 2.$ $q + r \leq r_1(q_2 + r_2) \in x(y + z)$

$\langle 1\rangle 2.$ For any negative real x and non-negative reals y and z, we have $x(y+z) = xy + xz$

PROOF:

$$\begin{aligned} x(y + z) = -(-x)(y + z) &= -((-x)y + (-x)z) && (\langle 1\rangle 1) \\ &= -((-x)y) - ((-x)z) \\ &= xy + xz \end{aligned}$$

$\langle 1\rangle 3.$ For any non-negative real x and reals y and z with one negative and one non-negative, we have $x(y + z) = xy + xz$

$\langle 2\rangle 1.$ ASSUME: w.l.o.g. y is negative and z is non-negative.

$\langle 2\rangle 2.$ CASE: $0 \leq y + z$

PROOF:

$$\begin{aligned} xy + xz &= xy + x(-y + y + z) \\ &= -(x(-y)) + x(-y + y + z) \\ &= -(x(-y)) + x(-y) + x(y + z) && (\langle 1\rangle 1) \\ &= x(y + z) \end{aligned}$$

$\langle 2\rangle 3.$ CASE: $y + z < 0$

$\langle 3\rangle 1.$ $-y - z > 0$

$\langle 3\rangle 2.$ $-y = z - y - z$

$\langle 3\rangle 3.$ $xy + xz = x(y + z)$

PROOF:

$$\begin{aligned} xy + xz &= -(x(-y)) + xz \\ &= -(x(z - y - z)) + xz \\ &= -(xz + x(-y - z)) + xz && (\langle 1\rangle 1) \\ &= -xy - x(-y - z) + xz \\ &= -x(-y - z) \\ &= x(y + z) \end{aligned}$$

$\langle 1\rangle 4$. For any non-negative real x and negative reals y and z, we have $x(y+z) = xy + xz$

PROOF:

$$\begin{aligned} x(y + z) &= -x(-y - z) \\ &= -(x(-y) + x(-z)) && (\langle 1\rangle 1) \\ &= -x(-y) - x(-z) \\ &= xy + xz \end{aligned}$$

$\langle 1\rangle 5$. For any negative real x and reals y and z with one negative and one non-negative, we have $x(y + z) = xy + xz$

$\langle 2\rangle 1$. ASSUME: w.l.o.g. y is negative and z is non-negative.

$\langle 2\rangle 2$. CASE: $0 \leq y + z$

PROOF:

$$\begin{aligned} x(y + z) &= -((-x)(y + z)) \\ &= -((-x)y + (-x)z) && (\langle 1\rangle 3) \\ &= -((-x)y) - ((-x)z) \\ &= (-x)(-y) - ((-x)z) \\ &= xy + xz \end{aligned}$$

$\langle 2\rangle 3$. CASE: $y + z < 0$

PROOF:

$$\begin{aligned} x(y + z) &= (-x)(-y - z) \\ &= (-x)(-y) + (-x)(-z) && (\langle 1\rangle 3) \\ &= xy + xz \end{aligned}$$

$\langle 1\rangle 6$. For any negative reals x, y and z, we have $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned} x(y + z) &= (-x)(-y - z) \\ &= (-x)(-y) + (-x)(-z) && (\langle 1\rangle 1) \\ &= xy + xz \end{aligned}$$

□

Proposition 15.3.24. *For any real x we have $x1 = x$.*

PROOF:

$\langle 1\rangle 1$. CASE: $0 \leq x$

$\langle 2\rangle 1$. $x1 \subseteq x$

$\langle 3\rangle 1$. LET: $q \in x1$

$\langle 3\rangle 2$. CASE: $q < 0$
PROOF: Then $q \in x$ because $0 \leq x$.
$\langle 3\rangle 3$. $q = rs$ where $0 \leq r \in x$ and $0 \leq s < 1$
PROOF: Then $q < r$ so $q \in x$.
$\langle 2\rangle 2$. $x \subseteq x1$
$\langle 3\rangle 1$. LET: $q \in x$
$\langle 3\rangle 2$. ASSUME: w.l.o.g. $0 \leq q$
$\langle 3\rangle 3$. PICK r such that $q < r \in x$
$\langle 3\rangle 4$. $0 \leq q/r < 1$
$\langle 3\rangle 5$. $q = r(q/r) \in x1$
$\langle 1\rangle 2$. CASE: $x < 0$
PROOF:

$$\begin{aligned} x1 &= -((-x)1) \\ &= -(-x) && (\langle 1\rangle 1) \\ &= x \end{aligned}$$

□

Lemma 15.3.25. *Let $x \in \mathbb{R}$ and c be a positive rational. Then there exists $a \in x$ and a non-least rational upper bound b for x such that $b - a = c$.*

PROOF:
$\langle 1\rangle 1$. PICK $a_1 \in x$ such that if x has a rational supremum s then $a_1 > s - c$
$\langle 1\rangle 2$. There exists a natural number n such that $a_1 + nc$ is an upper bound for x.
$\langle 2\rangle 1$. PICK a non-least upper bound b_1 for x.
$\langle 2\rangle 2$. PICK a natural number n such that $nc > b_1 - a_1$
PROOF: Proposition 15.2.11.
$\langle 2\rangle 3$. $a_1 + nc > b_1$
$\langle 2\rangle 4$. $a_1 + nc$ is an upper bound for x.
$\langle 1\rangle 3$. LET: k be the least natural number such that $a_1 + kc$ is an upper bound for x.
$\langle 1\rangle 4$. $a_1 + (k-1)c \in x$
$\langle 1\rangle 5$. $a_1 + kc$ is not the supremum of x.
$\langle 2\rangle 1$. ASSUME: for a contradiction $a_1 + kc$ is the supremum of x.
$\langle 2\rangle 2$. $a_1 > a_1 + (k-1)c$
PROOF: $\langle 1\rangle 1$
$\langle 2\rangle 3$. Q.E.D.
PROOF: This is a contradiction.
$\langle 1\rangle 6$. LET: $a = a_1 + (k-1)c$
$\langle 1\rangle 7$. LET: $b = a_1 + kc$
$\langle 1\rangle 8$. $b - a = c$
□

Proposition 15.3.26. *For any non-zero real x, there exists a real y such that $xy = 1$.*

PROOF:

⟨1⟩1. CASE: $0 < x$
⟨2⟩1. LET: $y = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{u^{-1} \mid u$ is an upper bound for x but not the supremum of $x\}$
⟨2⟩2. y is a real number.
⟨3⟩1. $y \neq \emptyset$
PROOF: Since $0 \in y$.
⟨3⟩2. $y \neq \mathbb{Q}$
⟨4⟩1. PICK $q \in x$ such that $0 < q$
⟨4⟩2. $q^{-1} \notin y$
⟨3⟩3. y is closed downwards.
⟨4⟩1. LET: $q \in y$ and $r < q$
PROVE: $r \in y$
⟨4⟩2. ASSUME: w.l.o.g. $0 < r$
⟨4⟩3. q^{-1} is a non-least upper bound for x.
⟨4⟩4. $q^{-1} < r^{-1}$
⟨4⟩5. r^{-1} is a non-least upper bound for x.
⟨4⟩6. $r \in y$
⟨3⟩4. y has no greatest element.
⟨4⟩1. LET: $q \in y$
PROVE: There exists $r \in y$ such that $q < r$
⟨4⟩2. CASE: $q \leq 0$
⟨5⟩1. PICK a non-least upper bound u for x.
⟨5⟩2. $q < u^{-1} \in x$
⟨4⟩3. CASE: $q = u^{-1}$ where u is a non-least upper bound for x.
⟨5⟩1. PICK a non-least upper bound v with $v < u$
⟨5⟩2. $u^{-1} < v^{-1} \in y$
⟨2⟩3. $0 < y$
⟨2⟩4. $xy \subseteq 1$
⟨3⟩1. LET: $q \in xy$
⟨3⟩2. ASSUME: w.l.o.g. $0 < q$
⟨3⟩3. PICK $0 < r \in x$ and $0 < s \in y$ such that $q = rs$
⟨3⟩4. s^{-1} is a non-least upper bound for x
⟨3⟩5. $r < s^{-1}$
⟨3⟩6. $rs < 1$
⟨2⟩5. $1 \subseteq xy$
⟨3⟩1. LET: $q < 1$
PROVE: $q \in xy$
⟨3⟩2. ASSUME: w.l.o.g. $0 < q$
⟨3⟩3. PICK a_1 with $0 < a_1 \in x$
⟨3⟩4. $(1-q)a_1 > 0$
⟨3⟩5. PICK $a \in x$ and a non-least upper bound w of x such that $w - a = (1-q)a_1$
PROOF: Lemma 15.3.25.
⟨3⟩6. $w - a < (1-q)w$
⟨3⟩7. $qw < a$
⟨3⟩8. $w < a/q$
⟨3⟩9. a/q is a non-least upper bound for x

$\langle 3\rangle 10.$ $q/a \in y$
$\langle 3\rangle 11.$ $q \in xy$
$\langle 1\rangle 2.$ CASE: $x < 0$
$\langle 2\rangle 1.$ PICK y such that $(-x)y = 1$
PROOF: $\langle 1\rangle 1$
$\langle 2\rangle 2.$ $x(-y) = 1$
□

Proposition 15.3.27. *For real numbers x, y and z, if $0 < z$ then $x < y$ if and only if $xz < yz$.*

PROOF:
$\langle 1\rangle 1.$ For any real numbers x, y and z, if $0 < z$ and $x < y$ then $xz < yz$
$\langle 2\rangle 1.$ LET: x, y and z be real numbers.
$\langle 2\rangle 2.$ ASSUME: $0 < z$ and $x < y$.
$\langle 2\rangle 3.$ $y = x + (y - x)$
$\langle 2\rangle 4.$ $y - x > 0$
$\langle 2\rangle 5.$ $(y - x)z > 0$
$\langle 2\rangle 6.$ $yz > xz$
PROOF:

$$\begin{aligned} yz &= (x + (y - x))z \\ &= xz + (y - x)z \\ &> xz \end{aligned}$$

$\langle 1\rangle 2.$ For any real numbers x, y and z, if $0 < z$ and $xz < yz$ then $x < y$
PROOF: Proposition 5.2.6.
□

Corollary 15.3.27.1. *The real numbers form a complete ordered field.*

Proposition 15.3.28.

$$(0, 1) \approx \mathbb{R}$$

PROOF: The function $f(x) = (2x - 1)/(x - x^2)$ is a bijection between $(0, 1)$ and $\mathbb{R}$. □

Proposition 15.3.29.

$$|\mathbb{R}| = 2^{\aleph_0}$$

PROOF:
$\langle 1\rangle 1.$ $(0, 1) \preccurlyeq 2^{\mathbb{N}}$
PROOF: The function H where $H(x)(n)$ is the nth binary digit of the binary expansion of x is an injection.
$\langle 1\rangle 2.$ $2^{\mathbb{N}} \preccurlyeq \mathbb{R}$
PROOF: Map f to the real number in $[0, 1/9]$ whose $n + 1$st decimal digit is $f(n)$.
□

Proposition 15.3.30. *The set of algebraic numbers is countable.*

PROOF: There are countably many integer polynomials, each with finitely many roots. $\square$

Corollary 15.3.30.1. *There are uncountably many transcendental numbers.*

Proposition 15.3.31. *Let A be a set of disks in the plane, no two of which intersect. Then A is countable.*

PROOF: Every circle includes a point with rational coordinates. Define $f : \{q \in \mathbb{Q}^2 \mid \exists C \in A.q \in C\} \to A$ by $f(q) = C$ iff $q \in C$. Then f is surjective. $\square$

Proposition 15.3.32. *There exists an uncountable set of circles in the plane that do not intersect.*

PROOF: The set of all circles with origin O is uncountable. $\square$

Chapter 16

Complex Analysis

Definition 16.0.1. For $p \geq 1$, let l^p be the set of all sequences of complex numbers (x_n) such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

Proposition 16.0.2. *If* $(x_n), (y_n) \in l^p$ *then* $(x_n + y_n) \in l^p$.

PROOF:
⟨1⟩1. LET: $(x_n), (y_n) \in l^p$
⟨1⟩2. $\sum_{n=1}^{\infty} |x_n + y_n|^p \leq 2^p (\sum_{n=1}^{\infty} |x_n|^p + \sum_{n=1}^{\infty} |y_n|^p)$
PROOF:
⟨2⟩1. For all $n \in \mathbb{N}$ we have $|x_n + y_n|^p \leq 2^p(|x_n|^p + |y_n|^p)$.
PROOF:

$$\begin{aligned} |x_n + y_n|^p &\leq (|x_n| + |y_n|)^p && \text{(Triangle Inequality)} \\ &\leq (2\max(|x_n|, |y_n|))^p \\ &\leq 2^p(|x_n|^p + |y_n|^p) \end{aligned}$$

□

Theorem 16.0.3 (Hölder's Inequality)**.** *Let* p *and* q *be reals such that* $p > 1$, $q > 1$ *and* $1/p + 1/q = 1$. *Let* $(x_n) \in l^p$ *and* $(y_n) \in l^q$. *Then*

$$\sum_n |x_n y_n| \leq \left(\sum_n |x_n|^p \right)^{1/p} \left(\sum_n |y_n|^q \right)^{1/q}$$

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. neither (x_n) nor (y_n) are all zero.
⟨1⟩2. For $0 \leq x \leq 1$ we have

$$x^{1/p} \leq \frac{1}{p}x + \frac{1}{q} \ .$$

⟨2⟩1. LET: $f(x) = x/p + 1/q - x^{1/p}$
⟨2⟩2. $f'(x) = 1/p(1 - x^{(1-p)/p})$
⟨2⟩3. $f'(x) \geq 0$ for all $x \in [0, 1]$
⟨2⟩4. f is a monotonically decreasing function on $[0, 1]$

⟨2⟩5. $f(0) = 1/q$
⟨2⟩6. $f(1) = 0$
⟨2⟩7. $f(x) \geq 0$ for all $x \in [0, 1]$
⟨1⟩3. For any $a, b \geq 0$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad .$$

⟨2⟩1. Case: $a^p \leq b^q$
⟨3⟩1. $ab^{-q/p} \leq \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q}$
Proof: Substituting $x = a^p/b^q$ in ⟨1⟩2.
⟨3⟩2. $ab^{1-q} \leq \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q}$
Proof: From ⟨3⟩1 since $1 - q = -q/p$.
⟨3⟩3. $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$
Proof: Multiplying ⟨3⟩2 by b^q.
⟨2⟩2. Case: $b^q \leq a^p$
Proof: Similar.
⟨1⟩4. For any integers $1 \leq j \leq n$, we have

$$\frac{|x_j|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}} \frac{|y_j|}{\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}} \leq \frac{1}{p}\frac{|x_j|^p}{\sum_{k=1}^n |x_k|^p} + \frac{1}{q}\frac{|y_j|^q}{\sum_{k=1}^n |y_k|^q}$$

Proof: From ⟨1⟩3 substituting

$$a = \frac{|x_j|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}} \text{ and } b = \frac{|y_j|}{\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}}$$

⟨1⟩5. For any positive integer n we have

$$\frac{\sum_{k=1}^n |x_k||y_k|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \left(\sum_{k=1}^n |y_k|^q\right)^{1/q}} \leq 1$$

Proof:

$$\begin{aligned}\frac{\sum_{j=1}^n |x_j||y_j|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \left(\sum_{k=1}^n |y_k|^q\right)^{1/q}} &\leq \frac{1}{p} + \frac{1}{q} \quad \text{(Summing ⟨1⟩4 from } j = 1 \text{ to } n\text{)} \\ &= 1\end{aligned}$$

⟨1⟩6.

$$\sum_n |x_n y_n| \leq \left(\sum_n |x_n|^p\right)^{1/p} \left(\sum_n |y_n|^q\right)^{1/q}$$

Proof: Taking the limit $n \to \infty$ in ⟨1⟩5.
□

Theorem 16.0.4 (Minkowski's Inequality)**.** *Let $p \geq 1$. Let $(x_n), (y_n) \in l^p$. Then*

$$\left(\sum_{n=1}^\infty |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^\infty |y_n|^p\right)^{1/p}$$

Proof:
⟨1⟩1. Case: $p = 1$
Proof: This is just the Triangle Inequality.
⟨1⟩2. Case: $p > 1$
⟨2⟩1. Let: $q = p/(p - 1)$

⟨2⟩2.
$$\sum_{n=1}^{\infty} |x_n + y_n|^p \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q}$$

PROOF:

⟨3⟩1. $(|x_n + y_n|^{p-1}) \in l^q$

PROOF:
$$\begin{aligned} \sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q} &= \sum_{n=1}^{\infty} |x_n + y_n|^p && (\langle 2\rangle 2) \\ &< \infty && (\text{Proposition 16.0.2}) \end{aligned}$$

⟨3⟩2. Q.E.D.

PROOF:
$$\begin{aligned} \sum_{n=1}^{\infty} |x_n + y_n|^p &= \sum_{n=1}^{\infty} |x_n + y_n||x_n + y_n|^{p-1} \\ &\leq \sum_{n=1}^{\infty} |x_n||x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n||x_n + y_n|^{p-1} \\ &\leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \\ &\quad + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} && (\text{Hölder's Inequality, } \langle 2\rangle 2) \end{aligned}$$

⟨2⟩3.
$$\sum_{n=1}^{\infty} |x_n + y_n|^p \leq \left\{ \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \right\} \left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{1/q}$$

⟨3⟩1. $q(p-1) = p$

PROOF: ⟨2⟩2

⟨3⟩2. Q.E.D.

PROOF: From ⟨2⟩2, ⟨3⟩1.

□

Part I

Linear Algebra

Chapter 17

Vector Spaces

17.1 Vector Spaces

Definition 17.1.1 (Vector Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *vector space* over K is a triple $(V, +, \cdot)$ such that:

- V is a nonempty set, whose elemnts are called *vectors*;
- $+ : V^2 \to V$
- $\cdot : K \times V \to V$

such that the following hold for all $u, v, w \in V$ and $\alpha, \beta \in K$:

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. For every $u, v \in V$ there exists $w \in V$ such that $u + w = v$
4. $\alpha(\beta v) = (\alpha\beta)v$
5. $(\alpha + \beta)v = \alpha v + \beta v$
6. $\alpha(u + v) = \alpha u + \alpha v$
7. $1v = v$

Elements of K are called *scalars*.

We write *real vector space* for 'vector space over $\mathbb{R}$', and *complex vector space* for 'vector space over $\mathbb{C}$'.

Proposition 17.1.2. *Let K be either $\mathbb{R}$ and $\mathbb{C}$. The set $\{0\}$ is a vector space over K under the unique functions $+ : \{0\}^2 \to \{0\}$, $\cdot : K \times \{0\} \to \{0\}$.*

Proof: Each axiom holds trivially because $x = y$ holds for all $x, y \in \{0\}$. □

Proposition 17.1.3. *The set $\mathbb{R}$ is a real vector space under real addition and real multiplication.*

PROOF: TODO — after we have proved these facts about $\mathbb{R}$. ☐

Proposition 17.1.4. *The set $\mathbb{C}$ is a real vector space under complex addition and complex multiplication.*

PROOF: TODO

Proposition 17.1.5. *The set $\mathbb{C}$ is a complex vector space under complex addition and complex multiplication.*

PROOF: TODO

Proposition 17.1.6. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let $\{V_i\}_{i \in I}$ be a family of vector spaces over K. Then $\prod_{i \in I} V_i$ is a vector space over K under the operations given by*

$$\begin{aligned} \{x_i\}_{i \in I} + \{y_i\}_{i \in I} &= \{x_i + y_i\}_{i \in I} \\ \alpha\{x_i\}_{i \in I} &= \{\alpha x_i\}_{i \in I} \end{aligned}$$

PROOF: Each axiom follows from the corresponding axiom in V_i. ☐☐

Corollary 17.1.6.1. *Let V be a vector space over K. For any set I, we have V^I is a vector space over K.*

Corollary 17.1.6.2. *Let $n \in \mathbb{Z}_+$. Then $\mathbb{R}^n$ is a real vector space, and $\mathbb{C}^n$ is both a real and a complex vector space, under*

$$\begin{aligned} (x_1, \ldots, x_n) + (y_1, \ldots, y_n) &= (x_1 + y_1, \ldots, x_n + y_n) \\ \lambda(x_1, \ldots, x_n) &= (\lambda x_1, \ldots, \lambda x_n) \end{aligned}$$

Proposition 17.1.7. *Let V be a vector space over K. Then there exists a unique $0 \in V$ such that, for all $v \in V$, we have $v + 0 = v$.*

PROOF:
$\langle 1 \rangle 1$. There exists $0 \in V$ such that $\forall v \in V. v + 0 = v$
  $\langle 2 \rangle 1$. Pick $v \in V$
  $\langle 2 \rangle 2$. Pick $0 \in V$ such that $v + 0 = v$
    PROOF: Axiom 3.
  $\langle 2 \rangle 3$. For all $u \in V$, we have $u + 0 = u$
    $\langle 3 \rangle 1$. LET: $u \in V$
    $\langle 3 \rangle 2$. PICK $u' \in V$ such that $v + u' = u$
      PROOF: Axiom 3.
    $\langle 3 \rangle 3$. $u + 0 = u$

$$\begin{aligned} u + 0 &= v + u' + 0 && (\langle 3 \rangle 2) \\ &= v + u' && (\langle 2 \rangle 2) \\ &= u && (\langle 3 \rangle 2) \end{aligned}$$

⟨1⟩2. If $0, 0' \in V$ are such that $\forall v \in V.v + 0 = v$ and $\forall v \in V.v + 0' = v$, then $0 = 0'$.
⟨2⟩1. LET: $0, 0' \in V$
⟨2⟩2. ASSUME: $\forall v \in V.v + 0 = v$
⟨2⟩3. ASSUME: $\forall v \in V.v + 0' = v$
⟨2⟩4. $0 = 0'$

$$\begin{aligned} 0 &= 0 + 0' && (\langle 2\rangle 2) \\ &= 0' && (\langle 2\rangle 3) \end{aligned}$$

□

Proposition 17.1.8. *Let V be a vector space. For any $v \in V$, there exists a unique $-v \in V$ such that $v + (-v) = 0$.*

PROOF:
⟨1⟩1. LET: $v \in V$
⟨1⟩2. There exists $-v \in V$ such that $v + (-v) = u$
PROOF: Axiom 3.
⟨1⟩3. If $v + x = 0$ and $v + y = 0$ then $x = y$
⟨2⟩1. ASSUME: $v + x = 0$
⟨2⟩2. ASSUME: $v + y = 0$
⟨2⟩3. $x = y$
PROOF:

$$\begin{aligned} x &= x + 0 && \text{(Proposition 17.1.7)} \\ &= x + v + y && (\langle 2\rangle 2) \\ &= 0 + y && (\langle 2\rangle 1) \\ &= y && \text{(Proposition 17.1.7)} \end{aligned}$$

□

Proposition 17.1.9. *Let V be a vector space. For any $u, v \in V$, there exists a unique $u - v \in V$ such that $v + (u - v) = u$, namely $u - v = u + (-v)$.*

PROOF:
⟨1⟩1. LET: $u, v \in V$
⟨1⟩2. $v + (u + (-v)) = u$
PROOF:

$$\begin{aligned} v + u + (-v) &= u + 0 && \text{(Proposition 17.1.8)} \\ &= u && \text{(Proposition 17.1.7)} \end{aligned}$$

⟨1⟩3. For all $x \in V$, if $v + x = u$ then $x = u + (-v)$.
⟨2⟩1. LET: $x \in V$
⟨2⟩2. ASSUME: $v + x = u$
⟨2⟩3. $x = u + (-v)$
PROOF:

$$\begin{aligned} u + (-v) &= v + x + (-v) && (\langle 2\rangle 2) \\ &= x + 0 && \text{(Proposition 17.1.8)} \\ &= x && \text{(Proposition 17.1.7)} \end{aligned}$$

∎

Proposition 17.1.10. *Let V be a vector space over K. Let $u, v, w \in V$. If $u + v = u + w$ then $v = w$.*

Proof:
⟨1⟩1. Assume: $u + v = u + w$
⟨1⟩2. $v = w$
Proof:

$$\begin{aligned} v &= v + 0 && \text{(Proposition 17.1.7)} \\ &= v + u + (-u) && \text{(Proposition 17.1.8)} \\ &= w + u + (-u) && (\langle 1\rangle 1) \\ &= w + 0 && \text{(Proposition 17.1.8)} \\ &= w && \text{(Proposition 17.1.7)} \end{aligned}$$

Proposition 17.1.11. *Let V be a vector space over K. Let $\lambda \in K$. Then $\lambda 0 = 0$.*

Proof:
⟨1⟩1. $\lambda 0 + \lambda 0 = \lambda 0 + 0$
Proof:

$$\begin{aligned} \lambda 0 + \lambda 0 &= \lambda(0 + 0) && \text{(Axiom 6)} \\ &= \lambda 0 && \text{(Proposition 17.1.7)} \end{aligned}$$

⟨1⟩2. $\lambda 0 = 0$
Proof: Proposition 17.1.10.
∎

Proposition 17.1.12. *Let V be a vector space over K. Let $\lambda \in K$ and $v \in V$. If $\lambda v = 0$ then $\lambda = 0$ or $v = 0$.*

Proof:
⟨1⟩1. Assume: $\lambda \neq 0$
⟨1⟩2. Assume: $\lambda v = 0$
⟨1⟩3. $v = 0$
Proof:

$$\begin{aligned} v &= 1v && \text{(Axiom 7)} \\ &= \lambda^{-1}\lambda v && \\ &= \lambda^{-1} 0 && (\langle 1\rangle 2) \\ &= 0 && \end{aligned}$$

∎

Proposition 17.1.13. *Let V be a vector space over K. For all $v \in V$ we have $0v = 0$.*

Proof:
⟨1⟩1. $0v + 0 = 0v + 0v$

$$
\begin{aligned}
0v + 0 &= 0v && \text{(Proposition 17.1.7)} \\
&= (0+0)v \\
&= 0v + 0v && \text{(Axiom 5)}
\end{aligned}
$$

$\langle 1\rangle 2.$ $0v = 0$
PROOF: Proposition 17.1.10, $\langle 1\rangle 1$.
□

Proposition 17.1.14. *Let V be a vector space over K. Let $v \in V$. Then $(-1)v = -v$.*

PROOF:
$\langle 1\rangle 1.$ $v + (-1)v = 0$
PROOF:

$$
\begin{aligned}
v + (-1)v &= 1v + (-1)v && \text{(Axiom 7)} \\
&= (1 + (-1))v && \text{(Axiom 5)} \\
&= 0v \\
&= 0 && \text{(Proposition 17.1.13)}
\end{aligned}
$$

$\langle 1\rangle 2.$ Q.E.D.
PROOF: Proposition 17.1.8.
□

17.2 Subspaces

Definition 17.2.1 (Subspace)**.** Let V be a vector space over K and $U \subseteq V$. Then U is a *subspace* of V iff $\forall \alpha, \beta \in K. \forall u, v \in U. \alpha u + \beta v \in U$. It is a *proper* subspace iff in addition $U \neq V$.

Proposition 17.2.2. *Let V be a vector space over K and U a subspace of V. Then U is a vector space over K under the restrictions of the operations of V.*

PROOF: Each of the axioms follows from the corresponding axiom in V. For axiom 3, we have if $u, v \in U$ then $v - u = 1v + (-1)u \in U$. □

Proposition 17.2.3. *Every vector space is a subspace of itself.*

PROOF: Trivial. □

Proposition 17.2.4. *Let Ω be a subset of $\mathbb{R}^N$. Let $\mathcal{C}(\Omega)$ be the set of all continuous functions $\Omega \to \mathbb{C}$. Then $\mathcal{C}(\Omega)$ is a subspace of $\mathbb{C}^\Omega$.*

PROOF: If $f, g : \Omega \to \mathbb{C}$ are continuous then so is $\alpha f + \beta g$. □

Proposition 17.2.5. *Let Ω be an open set in $\mathbb{R}^N$. Let $\mathcal{C}^k(\Omega)$ be the set of all continuous functions $\Omega \to \mathbb{C}$ with continuous partial derivatives of order k. Then $\mathcal{C}^k(\Omega)$ is a subspace of $\mathbb{C}^\Omega$.*

PROOF: If $f, g : \Omega \to \mathbb{C}$ have continuous partial derivatives of order k then so does $\alpha f + \beta g$. $\square$

Proposition 17.2.6. *Let Ω be an open set in $\mathbb{R}^N$. Let $\mathcal{C}^\infty(\Omega)$ be the set of all infinitely differentiable functions $\Omega \to \mathbb{C}$. Then $\mathcal{C}^\infty(\Omega)$ is a subspace of $\mathbb{C}^\Omega$.*

PROOF: If $f, g : \Omega \to \mathbb{C}$ are infinitely differentiable then so is $\alpha f + \beta g$. $\square$

Proposition 17.2.7. *Let Ω be an open set in $\mathbb{R}^N$. Let $\mathcal{P}(\Omega)$ be the set of all polynomials in N variables considered as functions $\Omega \to \mathbb{C}$. Then $\mathcal{P}(\Omega)$ is a subspace of $\mathbb{C}^\Omega$.*

PROOF: If $f, g : \Omega \to \mathbb{C}$ are polynomials in N variables then so is $\alpha f + \beta g$. $\square$

Proposition 17.2.8. *Let V be a vector space and U_1, U_2 subspaces of V. If $U_1 \subseteq U_2$ then U_1 is a subspace of U_2.*

PROOF: Trivial. $\square$

Proposition 17.2.9. *Let V be a vector space over K . The intersection of a set of subspaces of V is a subspace of V.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{U}$ be a set of subspaces of V.
$\langle 1 \rangle 2$. LET: $u, v \in \bigcap \mathcal{U}$ and $\lambda, \mu \in K$
$\langle 1 \rangle 3$. $\lambda u + \mu v \in \bigcap \mathcal{U}$
  $\langle 2 \rangle 1$. LET: $U \in \mathcal{U}$
  $\langle 2 \rangle 2$. $u, v \in U$
    PROOF: $\langle 1 \rangle 2$, $\langle 2 \rangle 1$.
  $\langle 2 \rangle 3$. $\lambda u + \beta v \in U$
    PROOF: $\langle 1 \rangle 1$, $\langle 1 \rangle 2$, $\langle 2 \rangle 1$, $\langle 2 \rangle 2$.
$\square$

Proposition 17.2.10. *The set of all bounded complex sequences is a proper subspace of $\mathbb{C}^{\mathbb{N}}$.*

PROOF: If (x_n) and (y_n) are bounded then so is $(\lambda x_n + \mu y_n)$. $\square$

Proposition 17.2.11. *The set of all convergent complex sequences is a proper subspace of the space of all bounded complex sequences.*

PROOF: If (x_n) and (y_n) converge then so does $(\lambda x_n + \mu y_n)$. $\square$

Proposition 17.2.12. *The set l^p of all sequences (x_n) in $\mathbb{C}$ such that $\sum_n |x_n|^p < \infty$ is a subspace of $\mathbb{C}^{\mathbb{N}}$.*

PROOF: It is closed under addition by Proposition 16.0.2, and it is easy to see that it is closed under scalar multiplication. $\square$

17.3 Linear Independence and Bases

Definition 17.3.1 (Linear Combination)**.** Let V be a vector space over K. Let $v, v_1, \ldots, v_n \in V$. Then v is a *linear combination* of $v_1, \ldots, v_n$ iff there exist scalars $\lambda_1, \ldots, \lambda_n \in K$ such that

$$v = \lambda_1 v_1 + \cdots + \lambda_n v_n \ .$$

Definition 17.3.2 (Linearly Independent)**.** Let V be a vector space over K. Let $A \subseteq V$. Then A is *linearly independent* iff, for all $\lambda_1, \ldots, \lambda_n \in K$ and $v_1, \ldots, v_n \in A$, if $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ then $\lambda_1 = \cdots = \lambda_n = 0$.

Definition 17.3.3 (Span)**.** Let V be a vector space over K and $A \subseteq V$. The *span* of A, or the subspace of V *spanned* by A, is the set of all linear combinations of vectors in A.

Proposition 17.3.4. *Let V be a vector space over K and $A \subseteq V$. Then* $\operatorname{span} A$ *is a subspace of V.*

PROOF: Given $\alpha, \beta \in K$ and $\lambda_1 u_1 + \cdots + \lambda_m u_m, \mu_1 v_1 + \cdots + \mu_n v_n \in \operatorname{span} A$, we have

$$\begin{aligned}
&\alpha(\lambda_1 u_1 + \cdots + \lambda_m u_m) + \beta(\mu_1 v_1 + \cdots + \mu_n v_n) \\
&= \alpha\lambda_1 u_1 + \cdots + \alpha\lambda_m u_m + \beta\mu_1 v_1 + \cdots + \beta\mu_n v_n \\
&\in \operatorname{span} A
\end{aligned} \qquad \square$$

Definition 17.3.5 (Basis)**.** Let V be a vector space over K and $B \subseteq V$. Then B is a *basis* for V iff B is linearly independent and $\operatorname{span} B = V$.

Definition 17.3.6 (Finite Dimensional)**.** A vector space is *finite dimensional* iff there exists a finite basis; otherwise it is *infinite dimensional.*

Proposition 17.3.7. *In a finite dimensional space, any two bases have the same size.*

TODO

Definition 17.3.8 (Dimension)**.** The *dimension* of a finite dimensional vector space V, $\dim V$, is the number of vectors in any basis.

Proposition 17.3.9. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Then K^n as a vector space over K has dimension n.*

PROOF: The vectors with one component 1 and all other components 0 form a basis. $\square$

Proposition 17.3.10. *As a real vector space, $\mathbb{C}^n$ has dimension $2n$.*

PROOF: The vectors with one component either 1 or i and all other components 0 form a basis. $\square$

Proposition 17.3.11. *Let Ω be a nonempty open set in $\mathbb{R}^n$. The space $\mathcal{C}(\Omega)$ is infinite dimensional.*

PROOF: Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}$ be the first projection. The functions 1, $\pi_1(x)$, $\pi_1(x)^2$, $\pi_1(x)^3$, ... form an infinite linearly independent set in $\mathcal{C}(\Omega)$. □

Proposition 17.3.12. *The spaces $\mathcal{C}^k(\mathbb{R}^n)$ and $\mathcal{C}^\infty(\mathbb{R}^n)$ are infinite dimensional.*

PROOF: The monomials 1, x, x^2, ... form an infinite linearly independent set. □

17.4 Linear Mappings

Definition 17.4.1 (Kernel)**.** Let U and V be vector spaces and $T : U \to V$. The *kernel* of T is

$$\ker T := \{u \in U \mid T(u) = 0\} \ .$$

Definition 17.4.2 (Linear Mapping)**.** Let U and V be vector spaces over K. A function $L : U \to V$ is a *linear mapping* iff $\forall x, y \in U. \forall \alpha, \beta \in K. L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$.

Proposition 17.4.3. *Let U and V be vector spaces over K. The set of linear mappings from U to V is a subspace of V^U.*

17.5 Eigenvalues and Eigenvectors

Definition 17.5.1 (Eigenvalue and Eigenvector)**.** Let V be a vector space over K. Let $A : V \to V$ be a linear transformation. Let $v \in V$ and $\lambda \in K$. Then v is an *eigenvector* of A with *eigenvalue* λ iff $A(v) = \lambda v$.

Chapter 18

Normed Spaces

Definition 18.0.1 (Norm)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. A *norm* on V is a function $\| \ \| : V \to \mathbb{R}$ such that, for all $u, v \in V$ and $\lambda \in K$:

1. If $\|v\| = 0$ then $v = 0$.

2. $\|\lambda v\| = |\lambda| \|v\|$

3. (*Triangle Inequality*) $\|u + v\| \leq \|u\| + \|v\|$

A *normed space* over K is a pair $(V, \| \ \|)$ where V is a vector space over K and $\| \ \|$ is a norm on V.

Proposition 18.0.2. *In a normed space,* $\|0\| = 0$.

Proof: $\|0\| = |0|\|0\| = 0$ by Axiom 2. □

Proposition 18.0.3. *Let V be a normed vector space over K. For all $v \in V$ we have $\|v\| \geq 0$.*

Proof:

$$\begin{aligned} 0 &= \|0\| && \text{(Proposition 18.0.2)} \\ &= \|v - v\| \\ &\leq \|v\| + \| - v\| && \text{(Triangle Inequality)} \\ &= 2\|v\| && \text{(Axiom 2)} \end{aligned}$$

Proposition 18.0.4. *Let V be a normed space. Let $u, v \in V$. Then*

$$|\|u\| - \|v\|| \leq \|u - v\| \ .$$

PROOF:

$$\begin{aligned}
\|u\| &\leq \|u - v\| + \|v\| && \text{(Triangle Inequality)} \\
\therefore \|u\| - \|v\| &\leq \|u - v\| \\
\|v\| &\leq \|v - u\| + \|u\| && \text{(Triangle Inequality)} \\
&= \|u - v\| + \|u\| && \text{(Axiom 2)} \\
\therefore \|v\| - \|u\| &\leq \|u - v\|
\end{aligned}$$

Definition 18.0.5 (Euclidean Norm)**.** The *Euclidean norm* on K^n is defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2} \ .$$

Proposition 18.0.6. *The Euclidean norm on K^n is a norm.*

PROOF:
$\langle 1 \rangle 1$. If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$
  PROOF: If $\sqrt{|x_1|^2 + \cdots + |x_n|^2} = 0$ then $x_1 = \cdots = x_n = 0$.
$\langle 1 \rangle 2$. $\|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$
  PROOF:

$$\begin{aligned}
\|\lambda\vec{x}\| &\sqrt{|\lambda x_1|^2 + \cdots + |\lambda x_n|^2} \\
&= \sqrt{|\lambda|^2|x_1|^2 + \cdots + |\lambda|^2|x_n|^2} \\
&= |\lambda|\sqrt{|x_1|^2 + \cdots + |x_n|^2} \\
&= |\lambda|\|\vec{x}\|
\end{aligned}$$

$\langle 1 \rangle 3$. $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
  PROOF:

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 &= |u_1 + v_1|^2 + \cdots + |u_n + v_n|^2 \\
&= |u_1|^2 + \cdots + |u_n|^2 + |v_1|^2 + \cdots + |v_n|^2 \\
&+ 2|u_1||v_1| + \cdots + 2|u_n||v_n| \\
&\leq \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2|u_1 v_1 + \cdots + u_n v_n| \\
&\leq \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\| && \text{(Cauchy-Schwarz)} \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2
\end{aligned}$$

□

Corollary 18.0.6.1. *The absolute value function $|\ |$ is a norm on K.*

Proposition 18.0.7. *The function $\|\vec{x}\| = |x_1| + \cdots + |x_n|$ is a norm on $\mathbb{C}^n$.*

PROOF:
$\langle 1 \rangle 1$. If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$
  PROOF: If $|x_1| + \cdots + |x_n| = 0$ then $x_1 = \cdots = x_n = 0$.
$\langle 1 \rangle 2$. $\|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$

Proof:

$$\begin{aligned} \|\lambda\vec{x}\| |\lambda x_1| + \cdots + |\lambda x_n| \\ &= |\lambda|(|x_1| + \cdots + |x_n|) \\ &= |\lambda|\|\vec{x}\| \end{aligned}$$

⟨1⟩3. $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
Proof:

$$\begin{aligned} \|\vec{u}+\vec{v}\|^2 &= |u_1+v_1| + \cdots + |u_n+v_n| \\ &\leq |u_1| + |v_1| + \cdots + |u_n| + |v_n| \\ &= \|\vec{u}\| + \|\vec{v}\| \end{aligned}$$

□

Proposition 18.0.8. *The function* $\|\vec{x}\| = \max(|x_1|, \ldots, |x_n|)$ *is a norm on* $\mathbb{C}^n$*.*

Proof:
⟨1⟩1. If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$
Proof: If $\max(|x_1|, \ldots, |x|n|) = 0$ then $x_1 = \cdots = x_n = 0$.
⟨1⟩2. $\|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$
Proof:

$$\begin{aligned} \|\lambda\vec{x}\| &= \max(|\lambda x_1|, \ldots, |\lambda x_n|) \\ &= |\lambda| \max(|x_1|, \ldots, |x_n|) \\ &= |\lambda|\|\vec{x}\| \end{aligned}$$

⟨1⟩3. $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
Proof:

$$\begin{aligned} \|\vec{u}+\vec{v}\| &= \max(|u_1+v_1|, \ldots, |u_n+v_n|) \\ &\leq \max(|u_1| + |v_1|, \ldots, |u_n| + |v_n|) \\ &\leq \max(|u_1|, \ldots, |u_n|) + \max(|v_1|, \ldots, |v_n|) \end{aligned}$$

□

Definition 18.0.9 (Uniform Convergence Norm)**.** Let Ω be a closed bounded subset of $\mathbb{R}^n$. The *uniform convergence norm* on $\mathcal{C}(\Omega)$ is the function defined by $\|f\| = \max_{x\in\Omega} |f(x)|$.

Proposition 18.0.10. *Let* Ω *be a closed bounded subset of* $\mathbb{R}^n$*. The uniform convergence norm is a norm on* $\mathcal{C}(\Omega)$*.*

Proof:
⟨1⟩1. If $\|f\| = 0$ then $f = 0$
Proof: If $\max_x |f(x)| = 0$ then $f(x) = 0$ for all x.
⟨1⟩2. $\|\lambda f\| = |\lambda|\|f\|$
Proof:

$$\begin{aligned} \|\lambda f\| &= \max_x |\lambda f(x)| \\ &= |\lambda| \max_x |f(x)| \\ &= |\lambda|\|f\| \end{aligned}$$

⟨1⟩3. $\|f+g\| \leq \|f\| + \|g\|$
PROOF:

$$\begin{aligned}\|f+g\| &= \max_x |f(x)+g(x)| \\ &\leq \max_x (|f(x)| + |g(x)|) \\ &\leq \max_x |f(x)| + \max_x |g(x)| \\ &= \|f\| + \|g\|\end{aligned}$$

□

Proposition 18.0.11. *Let $p \geq 1$. The function $\|(z_n)\| = \left(\sum_{n=1}^{\infty} |z_n|^p\right)^{1/p}$ is a norm on l^p.*

PROOF:
⟨1⟩1. If $\|(z_n)\| = 0$ then $(z_n) = (0)$
PROOF: If $(\sum_n |z_n|^p)^{1/p} = 0$ then $\sum_n |z_n|^p = 0$ so $|z_n|^p = 0$ for all n, and so $z_n = 0$ for all n.
⟨1⟩2. $\|(\lambda z_n)\| = |\lambda| \|(z_n)\|$
PROOF:

$$\begin{aligned}\|(\lambda z_n)\| &= \left(\sum_n |\lambda z_n|^p\right)^{1/p} \\ &= |\lambda| \left(\sum_n |z_n|^p\right)^{1/p} \\ &= |\lambda| \, |(z_n)|\end{aligned}$$

⟨1⟩3. The triangle inequality holds.
PROOF: This is Minkowski's Inequality. □

Proposition 18.0.12. *Let V be a normed space and U a vector subspace of V. Then U is a normed space under the restriction of the norm to U.*

PROOF: Each axiom follows from the fact it holds in V. □

Proposition 18.0.13. *Let V be a normed space over K. Let $x_1, \ldots, x_n$ be linearly independent elements of V. Then there exists a real number $c > 0$ such that, for all $\alpha_1, \ldots, \alpha_n \in K$, we have*

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|) \ .$$

PROOF:
⟨1⟩1. Define $f : K^n \to \mathbb{R}$ by

$$f(\alpha_1, \ldots, \alpha_n) = \|\alpha_1 x_1 + \cdots + \alpha_n x_n\|$$

⟨1⟩2. f is continuous.
⟨2⟩1. LET: $(\alpha_1, \ldots, \alpha_n) \in K^n$ and $\epsilon > 0$
⟨2⟩2. LET: $\delta = \epsilon / (\|x_1\| + \cdots + \|x_n\|)$
PROOF: $x_1, \ldots, x_n$ are not all zero because they are linearly independent.
⟨2⟩3. LET: $(\beta_1, \ldots, \beta_n)$ with $|\alpha_i - \beta_i| < \delta$ for all i

⟨2⟩4. $\|(\alpha_1 x_1 + \cdots + \alpha_n x_n) - (\beta_1 x_1 + \beta_n x_n)\| < \epsilon$

PROOF:

$$\begin{aligned} & \|(\alpha_1 x_1 + \cdots + \alpha_n x_n) - (\beta_1 x_1 + \beta_n x_n)\| & \\ \leq & |\alpha_1 - \beta_1|\|x_1\| + \cdots + |\alpha_n - \beta_n|\|x_n\| & \text{(Axioms 2 and 3)} \\ < & \delta(\|x_1\| + \cdots + \|x_n\|) & (\langle 2\rangle 3) \\ = & \epsilon & (\langle 2\rangle 2) \end{aligned}$$

⟨1⟩3. PICK $(\beta_1, \ldots, \beta_n) \in \{(\beta_1, \ldots, \beta_n) \in K^n \mid |\beta_1| + \cdots + |\beta_n| = 1\}$ at which f attains its minimum.

PROOF: Extreme Value Theorem.

⟨1⟩4. Let $c = f(\beta_1, \ldots, \beta_n)$

⟨1⟩5. $c > 0$

PROOF: Linear independence.

⟨1⟩6. LET: $\alpha_1, \ldots, \alpha_n \in K$

⟨1⟩7. $\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|)$

⟨2⟩1. ASSUME: w.l.o.g. $\alpha_1 \ \ldots,\ \alpha_n$ are not all zero.

⟨2⟩2. LET: $\beta_i = \alpha_i/(|\alpha_1| + \cdots + |\alpha_n|)$ for $i = 1, \ldots, n$

⟨2⟩3. $|\beta_1| + \cdots + |\beta_n| = 1$

⟨2⟩4. $f(\beta_1, \ldots, \beta_n) \geq c$

⟨2⟩5. Q.E.D.

PROOF: Multiply both sides by $|\alpha_1| + \cdots + |\alpha_n|$.

□

Proposition 18.0.14. *Let V be a normed space over K. Define $d : V^2 \to \mathbb{R}$ by $d(x, y) = \|x - y\|$. Then d is a metric on V.*

PROOF:

⟨1⟩1. For all $x, y \in V$ we have $d(x, y) \geq 0$

PROOF: Proposition 18.0.3.

⟨1⟩2. For all $x, y \in V$ we have $d(x, y) = 0$ iff $x = y$

⟨2⟩1. If $d(x, y) = 0$ then $x = y$

PROOF: Axiom 1.

⟨2⟩2. If $x = y$ then $d(x, y) = 0$

PROOF: Proposition 18.0.2.

⟨1⟩3. $\forall x, y \in V. d(x, y) = d(y, x)$

PROOF: By Axiom 2.

⟨1⟩4. $\forall x, y, z \in V. d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Axiom 3.

□

Henceforth we identify any normed space with this metric space.

18.1 Convergence

Proposition 18.1.1. *Let V be a normed space over K. Let (x_n) be a sequence in V and $l \in V$. Then $x_n \to l$ as $n \to \infty$ in V if and only if $\|x_n - l\| \to 0$ as $n \to \infty$ in $\mathbb{R}$.*

PROOF: Immediate from definitions. □

Proposition 18.1.2. *In a normed space, a sequence has at most one limit.*

PROOF:
⟨1⟩1. LET: V be a vector space over K.
⟨1⟩2. ASSUME: $x_n \to l$ and $x_n \to m$ as $n \to \infty$.
⟨1⟩3. ASSUME: for a contradiction $l \neq m$
⟨1⟩4. LET: $\epsilon = \|l - m\|/2$
⟨1⟩5. PICK N such that $\forall n \geq N. \|x_n - l\| < \epsilon$ and $\forall n \geq N. \|x_n - m\| < \epsilon$
PROOF: ⟨1⟩2, ⟨1⟩4
⟨1⟩6. $\|l - m\| < \|l - m\|$
PROOF:

$$\begin{aligned} \|l - m\| &\leq \|x_N - l\| + \|x_N - m\| && \text{(Triangle Inequality)} \\ &< 2\epsilon && (\langle 1\rangle 5) \\ &= \|l - m\| && (\langle 1\rangle 4 \end{aligned}$$

⟨1⟩7. Q.E.D.
PROOF: This is a contradiction.
□

Definition 18.1.3 (Bounded)**.** Let V be a normed space over K. A sequence (x_n) in V is *bounded* iff there exists B such that $\forall n \leq N. \|x_n\| < B$.

Proposition 18.1.4. *Every convergent sequence is bounded.*

PROOF:
⟨1⟩1. LET: $x_n \to l$ as $n \to \infty$
⟨1⟩2. PICK N such that $\forall n \geq N. \|x_n - l\| < 1$
⟨1⟩3. LET: $B = \max(\|x_1\|, \|x_2\|, \ldots, \|x_{N-1}\|, \|l\| + 1)$
⟨1⟩4. LET: $n \in \mathbb{N}$
⟨1⟩5. $\|x_n\| \leq B$
⟨2⟩1. CASE: $n < N$
PROOF: $\|x_n\| \leq B$ from ⟨1⟩3.
⟨2⟩2. CASE: $n \geq N$
PROOF:

$$\begin{aligned} \|x_n\| &\leq \|l\| + \|x_n - l\| && \text{(Triangle Inequality)} \\ &< \|l\| + 1 && (\langle 1\rangle 2) \\ &\leq B && (\langle 1\rangle 3) \end{aligned}$$

□

Proposition 18.1.5. *Let V be a normed space over K. If $x_n \to l$ as $n \to \infty$ in V, and $\lambda_n \to \lambda$ as $n \to \infty$ in K, then $\lambda_n x_n \to \lambda l$ as $n \to \infty$.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. LET: $x_n \to l$ as $n \to \infty$
⟨1⟩3. LET: $\lambda_n \to \lambda$ as $n \to \infty$

⟨1⟩4. LET: $\epsilon > 0$
⟨1⟩5. PICK N such that, for all $n \geq N$, we have $\|x_n - l\| < \epsilon/2|\lambda|$ and $|\lambda_n - \lambda| < \sqrt{\epsilon/2}$ and $\|x_n\| < \sqrt{\epsilon/2}$
⟨1⟩6. LET: $n \geq N$
⟨1⟩7. $\|\lambda_n x_n - \lambda l\| < \epsilon$
PROOF:

$$\begin{aligned}
\|\lambda_n x_n - \lambda l\| &\leq \|\lambda_n x_n - \lambda x_n\| + \|\lambda x_n - \lambda l\| && \text{(Triangle Inequality)} \\
&= |\lambda_n - \lambda| \|x_n\| + |\lambda| \|x_n - l\| && \text{(Axiom 2)} \\
&< \sqrt{\epsilon/2}\sqrt{\epsilon/2} + |\lambda| \epsilon/2|\lambda| && (\langle 1\rangle 5) \\
&= \epsilon
\end{aligned}$$

□

Proposition 18.1.6. *Let V be a normed space over K. If $x_n \to l$ and $y_n \to m$ as $n \to \infty$, then $x_n + y_n \to l + m$ as $n \to \infty$.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N such that, for all $n \geq N$, we have $\|x_n - l\| < \epsilon/2$ and $\|y_n - m\| < \epsilon/2$
⟨1⟩3. LET: $n \geq N$
⟨1⟩4. $\|(x_n + y_n) - (l + m)\| < \epsilon$
PROOF:

$$\begin{aligned}
\|(x_n + y_n) - (l + m)\| &\leq \|x_n - l\| + \|y_n - m\| && \text{(Triangle Inequality)} \\
&< \epsilon/2 + \epsilon/2 && (\langle 1\rangle 2) \\
&= \epsilon
\end{aligned}$$

□

Definition 18.1.7 (Uniform Convergence)**.** Let Ω be a closed bounded subset of $\mathbb{R}^n$. Let (f_n) be a sequence in $\mathcal{C}(\Omega)$ and $f \in \mathcal{C}(\Omega)$. Then (f_n) *converges uniformly* to f iff, for every $\epsilon > 0$, there exists N such that $\forall x \in \Omega. \forall n \geq N. |f_n(x) - f(x)| < \epsilon$.

Proposition 18.1.8. *Let Ω be a closed bounded subset of $\mathbb{R}^n$. Let (f_n) be a sequence in $\mathcal{C}(\Omega)$ and $f \in \mathcal{C}(\Omega)$. Then (f_n)* converges uniformly *to f iff f_n converges to f under the uniform convergence norm.*

PROOF:

$$\begin{aligned}
&(f_n) \text{ converges to } f \text{ under the uniform convergence norm} \\
\Leftrightarrow &\forall \epsilon > 0. \exists N. \forall n \geq N. \|f_n - f\| < \epsilon \\
\Leftrightarrow &\forall \epsilon > 0. \exists N. \forall n \geq N. \forall x \in X. |f_n(x) - f(x)| < \epsilon
\end{aligned}$$

□

Definition 18.1.9 (Pointwise Convergence)**.** Let (f_n) be a sequence in $\mathcal{C}([0,1])$ and $f \in \mathcal{C}([0,1])$. Then (f_n) *converges pointwise* to f iff, for all $t \in [0,1]$, we have $|f_n(t) - f(t)| \to 0$ as $n \to \infty$.

Proposition 18.1.10. *There is no norm n on $\mathcal{C}([0,1])$ such that, for every sequence (f_n) and function f in $\mathcal{C}([0,1])$, we have (f_n) converges pointwise to f if and only if (f_n) converges to f under n.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $\| \ \|$ is a norm on $\mathcal{C}([0,1])$ such that, for every sequence (f_n) and function f in $\mathcal{C}([0,1])$, we have (f_n) converges pointwise to f if and only if (f_n) converges to f under $\| \ \|$.
⟨1⟩2. For $n \in \mathbb{Z}_+$, define $g_n \in \mathcal{C}([0,1])$ by

$$g_n(t) = \begin{cases} 2^n t & \text{if } 0 \leq t \leq 2^{-n} \\ 2 - 2^n t & \text{if } 2^{-n} \leq t \leq 2^{1-n} \\ 0 & \text{if } 2^{1-n} \leq t \leq 1 \end{cases}$$

⟨1⟩3. For all n, $\|g_n\| \neq 0$
PROOF: Axiom 1.
⟨1⟩4. For $n \in \mathbb{Z}_+$, define $f_n \in \mathcal{C}([0,1])$ by $f_n = g_n / \|g_n\|$
⟨1⟩5. For all n, $\|f_n\| = 1$
PROOF: Axiom 2.
⟨1⟩6. (f_n) does not converge under $\| \ \|$
⟨1⟩7. (f_n) converges pointwise to 0.
⟨1⟩8. This is a contradiction.
□

Definition 18.1.11 (Equivalence of Norms)**.** Let $\| \ \|_1$ and $\| \ \|_2$ be two norms on the same vector space V. Then the norms are *equivalent* if and only if, for any sequence (x_n) in V and $l \in V$, we have that (x_n) converges to l under $\| \ \|_1$ if and only if (x_n) converges to l under $\| \ \|_2$.

Theorem 18.1.12. *Let $\| \ \|_1$ and $\| \ \|_2$ be two norms on the same vector space E over K. Then $\| \ \|_1$ and $\| \ \|_2$ are equivalent if and only if there exist positive real numbers α and β such that, for all $x \in E$,*

$$\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1 \ .$$

PROOF:
⟨1⟩1. If $\| \ \|_1$ and $\| \ \|_2$ are equivalent then there exist positive real numbers α and β such that, for all $x \in E$, $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$.
⟨2⟩1. ASSUME: $\| \ \|_1$ and $\| \ \|_2$ are equivalent.
⟨2⟩2. There exists $\alpha > 0$ such that, for all $x \in E$, we have $\alpha\|x\|_1 \leq \|x\|_2$
⟨3⟩1. ASSUME: for a contradiction there is no $\alpha > 0$ such that, for all $x \in E$, we have $\alpha\|x\|_1 \leq \|x\|_2$.
⟨3⟩2. For all $n \in \mathbb{Z}_+$, PICK $x_n \in E$ such that $1/n\|x_n\|_1 > \|x\|_2$
⟨3⟩3. For all $n \in \mathbb{Z}_+$,
LET:

$$y_n = \frac{1}{\sqrt{n}} \frac{x_n}{\|x_n\|_2}$$

⟨3⟩4. (y_n) converges to 0 under $\| \ \|_2$
⟨3⟩5. (y_n) converges to 0 under $\| \ \|_1$
⟨3⟩6. For all $n \in \mathbb{Z}_+$, we have $\|y_n\| > \sqrt{n}$
⟨3⟩7. This is a contradiction.
⟨2⟩3. There exists $\beta > 0$ such that, for all $x \in E$, we have $\|x\|_2 \leq \beta\|x\|_1$
PROOF: Similar.

⟨1⟩2. If there exist positive real numbers α and β such that, for all $x \in E$, $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$. then $\| \ \|_1$ and $\| \ \|_2$ are equivalent.
⟨2⟩1. ASSUME: α and β are positive reals with $\forall x \in E.\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$.
⟨2⟩2. Let (x_n) be a sequence in E and $l \in E$
⟨2⟩3. If (x_n) converges to l under $\| \ \|_1$ then (x_n) converges to l under $\| \ \|_2$.
⟨3⟩1. ASSUME: (x_n) converges to l under $\| \ \|_1$
⟨3⟩2. LET: $\epsilon > 0$
⟨3⟩3. PICK N such that $\forall n \geq N.\|x_n - l\|_1 < \epsilon/\beta$
⟨3⟩4. $\forall n \geq N.\|x_n - l\|_2 < \epsilon$
⟨2⟩4. If (x_n) converges to l under $\| \ \|_2$ then (x_n) converges to l under $\| \ \|_1$.
PROOF: Similar.
□

Theorem 18.1.13. *Any two norms on a finite dimensional vector space are equivalent.*

PROOF:
⟨1⟩1. LET: V be a finite dimensional vector space over K.
⟨1⟩2. ASSUME: w.l.o.g. $\dim V > 0$
⟨1⟩3. PICK a basis $\{e_1, \ldots, e_n\}$ for V.
⟨1⟩4. LET: $\| \ \|_0 : V \to \mathbb{R}$ be the function: $\|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 = |\alpha_1| + \cdots + |\alpha_n|$.
⟨1⟩5. $\| \ \|_0$ is a norm.
⟨2⟩1. If $\|v\|_0 = 0$ then $v = 0$
PROOF: If $|\alpha_1| + \cdots + |\alpha_n| = 0$ then $\alpha_1 = \cdots = \alpha_n = 0$ so $\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$.
⟨2⟩2. $\|\lambda v\|_0 = |\lambda|\|v\|_0$
PROOF:
$$\begin{aligned}
\|\lambda(\alpha_1 e_1 + \cdots + \alpha_n e_n)\|_0 &= \|\lambda\alpha_1 e_1 + \cdots + \lambda\alpha_n e_n\|_0 \\
&= |\lambda\alpha_1| + \cdots + |\lambda\alpha_n| && (\langle 1\rangle 4) \\
&= |\lambda|(|\alpha_1| + \cdots + |\alpha_n|) \\
&= |\lambda|\|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 && (\langle 1\rangle 4)
\end{aligned}$$
⟨2⟩3. $\|u + v\|_0 \leq \|u\|_0 + \|v\|_0$
PROOF:
$$\begin{aligned}
\|(\alpha_1 e_1 + \cdots + \alpha_n e_n) + (\beta_1 e_1 + \cdots + \beta_n e_n)\| &= |\alpha_1 + \beta_1| + \cdots + |\alpha_n + \beta_n| \\
&\leq |\alpha_1| + \cdots + |\alpha_n| + |\beta_1| + \cdots + |\beta_n| \\
&= \|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 + \|\beta_1 e_1 + \cdots + \beta_n e_n\|_0
\end{aligned}$$
⟨1⟩6. Any norm on V is equivalent to $\| \ \|_0$.
⟨2⟩1. LET: $\| \ \|$ be any norm on V.
⟨2⟩2. PICK $\alpha > 0$ such that, for all $\alpha_1, \ldots, \alpha_n \in K$, we have $\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \geq \alpha(|\alpha_1| + \cdots + |\alpha_n|)$
PROOF: Proposition 18.0.13, ⟨2⟩1, ⟨1⟩3.
⟨2⟩3. LET: $\beta = \max(\|e_1\|, \ldots, \|e_n\|)$
⟨2⟩4. $\beta > 0$
PROOF: e_1, ..., e_n cannot all be zero by ⟨1⟩3.

$\langle 2\rangle 5.$ For all $x \in V$ we have $\alpha\|x\|_0 \leq \|x\| \leq \beta\|x\|_0$
$\langle 3\rangle 1.$ LET: $x \in V$
$\langle 3\rangle 2.$ $\alpha\|x\|_0 \leq \|x\|$
PROOF: $\langle 1\rangle 3$, $\langle 1\rangle 4$, $\langle 2\rangle 2$.
$\langle 3\rangle 3.$ $\|x\| \leq \beta\|x\|_0$
$\langle 4\rangle 1.$ LET: $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$
$\langle 4\rangle 2.$ Q.E.D.
PROOF:

$$\begin{aligned}
\|x\| &= \|\alpha_1 e_1 + \cdots + \alpha_n e_n\| && (\langle 4\rangle 1) \\
&\leq |\alpha_1|\|e_1\| + \cdots + |\alpha_n|\|e_n\| && (\langle 2\rangle 1) \\
&\leq \beta(|\alpha_1| + \cdots + |\alpha_n|) && (\langle 2\rangle 3) \\
&= \beta\|x\|_0 && (\langle 1\rangle 4)
\end{aligned}$$

$\langle 2\rangle 6.$ Q.E.D.
PROOF: Theorem 18.1.12, $\langle 1\rangle 5$, $\langle 2\rangle 1$, $\langle 2\rangle 2$, $\langle 2\rangle 4$, $\langle 2\rangle 5$.
□

Definition 18.1.14 (Open Ball)**.** Let V be a normed space over K. Let $x \in V$. Let $r > 0$. The *open ball* with *centre* x and *radius* r is

$$B(x, r) := \{y \in V \mid \|y - x\| < r\} \ .$$

Definition 18.1.15 (Closed Ball)**.** Let V be a normed space over K. Let $x \in V$. Let $r > 0$. The *closed ball* with *centre* x and *radius* r is

$$\overline{B}(x, r) := \{y \in V \mid \|y - x\| \leq r\} \ .$$

Definition 18.1.16 (Sphere)**.** Let V be a normed space over K. Let $x \in V$. Let $r > 0$. The *sphere* with *centre* x and *radius* r is

$$S(x, r) := \{y \in V \mid \|y - x\| = r\} \ .$$

Definition 18.1.17 (Open Set)**.** Let V be a normed space over K. A set $S \subseteq V$ is *open* iff, for all $x \in S$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S$.

Proposition 18.1.18. *Equivalent norms define the same set of open sets.*

PROOF:
$\langle 1\rangle 1.$ LET: V be a normed space over K.
$\langle 1\rangle 2.$ LET: $\| \ \|_1$ and $\| \ \|_2$ be equivalent norms on V.
$\langle 1\rangle 3.$ PICK reals $\alpha, \beta > 0$ such that, for all $x \in V$, we have $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$
$\langle 1\rangle 4.$ LET: $S \subseteq V$
$\langle 1\rangle 5.$ If S is open under $\| \ \|_1$ then S is open under $\| \ \|_2$.
$\langle 2\rangle 1.$ ASSUME: S is open under $\| \ \|_1$.
$\langle 2\rangle 2.$ LET: $x \in S$
$\langle 2\rangle 3.$ PICK $\epsilon > 0$ such that $\{y \in V \mid \|x - y\|_1 < \epsilon\} \subseteq S$.
$\langle 2\rangle 4.$ LET: $\delta = \alpha\epsilon$

⟨2⟩5. $\{y \in V \mid \|x - y\|_2 < \delta\} \subseteq S$
⟨1⟩6. If S is open under $\| \; \|_2$ then S is open under $\| \; \|_1$.
PROOF: Similar.
□

Proposition 18.1.19. *Every open ball is open.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. LET: $c \in V$ and $r > 0$
PROVE: $B(c, r)$ is open.
⟨1⟩3. LET: $x \in B(c, r)$
⟨1⟩4. LET: $\epsilon = r - \|x - c\|$
PROVE: $B(x, \epsilon) \subseteq B(c, r)$
⟨1⟩5. LET: $y \in B(x, \epsilon)$
PROVE: $y \in B(c, r)$
⟨1⟩6. $\|y - c\| < r$
PROOF:

$$\begin{aligned} \|y - c\| &\leq \|y - x\| + \|x - c\| && \text{(Triangle Inequality)} \\ &< \epsilon + \|x - c\| && (\langle 1\rangle 5) \\ &= r && (\langle 1\rangle 4) \end{aligned}$$

□

Proposition 18.1.20. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega . g(x) < f(x)\}$ is open.*

PROOF:
⟨1⟩1. LET: $g \in U$
⟨1⟩2. LET: $\epsilon = \max_{x \in \Omega}(f(x) - g(x))$
PROVE: $B(g, \epsilon) \subseteq S$
⟨1⟩3. $\epsilon > 0$
⟨1⟩4. LET: $h \in B(g, \epsilon/2)$
PROVE: $h \in S$
⟨1⟩5. LET: $x \in \Omega$
⟨1⟩6. $h(x) < f(x)$
PROOF:

$$\begin{aligned} h(x) &\leq g(x) + \epsilon/2 && (\langle 1\rangle 4) \\ &< g(x) + \epsilon && (\langle 1\rangle 3) \\ &\leq f(x) && (\langle 1\rangle 2) \end{aligned}$$

□

Proposition 18.1.21. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega . g(x) > f(x)\}$ is open.*

PROOF: Given $g \in U$, let $\epsilon = \max_x (g(x) - f(x))/2$. Then $B(g, \epsilon) \subseteq U$. □

Proposition 18.1.22. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$ be such that $f(x) > 0$ for all $x \in \Omega$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega . |g(x)| < f(x)\}$ is open.*

Proof: Given $g \in U$, let $\epsilon = \max_x(f(x) - |g(x)|)/2$. Then $B(g, \epsilon) \subseteq U$. □

Proposition 18.1.23. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$ be such that $f(x) > 0$ for all $x \in \Omega$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega. |g(x)| > f(x)\}$ is open.*

Proof: Given $g \in U$, let $\epsilon = \max_x(|g(x)| - f(x))/2$. Then $B(g, \epsilon) \subseteq U$. □

Proposition 18.1.24. *The union of a set of open sets is open.*

Proof:
⟨1⟩1. Let: V be a normed space over K.
⟨1⟩2. Let: $\mathcal{U}$ be a set of open sets in V.
⟨1⟩3. Let: $x \in \bigcup \mathcal{U}$
⟨1⟩4. Pick $U \in \mathcal{U}$ such that $x \in U$.
⟨1⟩5. Pick $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
⟨1⟩6. $B(x, \epsilon) \subseteq \bigcup \mathcal{U}$
□

Proposition 18.1.25. *The intersection of two open sets is open.*

Proof:
⟨1⟩1. Let: V be a normed space over K.
⟨1⟩2. Let: U_1 and U_2 be open sets in V.
⟨1⟩3. Let: $x \in U_1 \cap U_2$
⟨1⟩4. Pick $\epsilon_1 > 0$ such that $B(x, \epsilon_1) \subseteq U_1$
⟨1⟩5. Pick $\epsilon_2 > 0$ such that $B(x, \epsilon_2) \subseteq U_2$
⟨1⟩6. Let: $\epsilon = \min(\epsilon_1, \epsilon_2)$
⟨1⟩7. $B(x, \epsilon) \subseteq U_1 \cap U_2$
□

Proposition 18.1.26. *In any normed space, $\emptyset$ is open.*

Proof: Vacuous. □

Proposition 18.1.27. *In any normed space V, the whole space V is open.*

Proof: For any $x \in V$ we have $B(x, 1) \subseteq V$. □

Definition 18.1.28 (Closed Set)**.** Let V be a normed space over K. A set $S \subseteq V$ is *closed* iff $V - S$ is open.

Proposition 18.1.29. *Every closed ball is closed.*

Proof:
⟨1⟩1. Let: V be a normed space over K.
⟨1⟩2. Let: $c \in V$ and $r > 0$
Prove: $\overline{B}(c, r)$ is closed.
⟨1⟩3. Let: $x \in V - \overline{B}(c, r)$
⟨1⟩4. Let: $\epsilon = \|x - c\| - r$
Prove: $B(x, \epsilon) \subseteq V - \overline{B}(c, r)$

⟨1⟩5. $\epsilon > 0$
PROOF: Since $\|x - c\| > r$ by ⟨1⟩3.
⟨1⟩6. LET: $y \in B(x, \epsilon)$
⟨1⟩7. $\|y - c\| > r$
PROOF:
$$\begin{aligned}
\|y - c\| &\geq \|x - c\| - \|x - y\| && \text{(Triangle Inequality)} \\
&> \|x - c\| - \epsilon && (\langle 1\rangle 6) \\
&= r && (\langle 1\rangle 4)
\end{aligned}$$
□

Proposition 18.1.30. *The intersection of a set of closed sets is closed.*

PROOF: From Proposition 18.1.24. □

Proposition 18.1.31. *The union of two closed sets is closed.*

PROOF: From Proposition 18.1.25. □

Proposition 18.1.32. *Every sphere is closed.*

PROOF: $S(c, r) = \overline{B}(c, r) - B(c, r)$. □

Proposition 18.1.33. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega . g(x) \leq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega . g(x) > f(x)\}$. □

Proposition 18.1.34. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega . g(x) \geq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega . g(x) < f(x)\}$. □

Proposition 18.1.35. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega . |g(x)| \leq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega . |g(x)| > f(x)\}$. □

Proposition 18.1.36. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega . |g(x)| \geq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega . |g(x)| < f(x)\}$. □

Proposition 18.1.37. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $x_0 \in \Omega$ and $\lambda \in \mathbb{C}$. Then $C = \{g \in \mathcal{C}(\Omega) \mid g(x_0) = \lambda\}$ is closed.*

PROOF: Given $g \in \mathcal{C}(\Omega) - C$, let $\epsilon = |g(x_0) - \lambda|/2$. Then $B(g, \epsilon) \subseteq \mathcal{C}(\Omega) - C$. □

Proposition 18.1.38. *In any normed space V, we have $\emptyset$ is closed.*

PROOF: Since $V - \emptyset = V$ is open. □

Proposition 18.1.39. *In any normed space V, the whole space V is closed.*

PROOF: Since $V - V = \emptyset$ is open. □

Theorem 18.1.40. *Let V be a normed space over K. Let S be a subset of V. Then S is closed if and only if, for any sequence (x_n) in S, if $x_n \to l$ as $n \to \infty$ then $l \in S$.*

PROOF:
⟨1⟩1. If S is closed then, for any sequence (x_n) in S, if $x_n \to l$ as $n \to \infty$ then $l \in S$.
⟨2⟩1. ASSUME: S is closed.
⟨2⟩2. LET: (x_n) be a sequence in S.
⟨2⟩3. ASSUME: $x_n \to l$ as $n \to \infty$.
⟨2⟩4. ASSUME: for a contradiction $l \notin S$.
⟨2⟩5. PICK $\epsilon > 0$ such that $B(l, \epsilon) \subseteq V - S$
⟨2⟩6. PICK N such that $\forall n \geq N . x_n \in B(l, \epsilon)$
⟨2⟩7. $x_N \in V - S$
⟨2⟩8. This contradicts ⟨2⟩2.
⟨1⟩2. If, for any sequence (x_n) in S, if $x_n \to l$ as $n \to \infty$ then $l \in S$, then S is closed.
⟨2⟩1. ASSUME: for any sequence (x_n) in S, if $x_n \to l$ as $n \to \infty$ then $l \in S$.
⟨2⟩2. LET: $x \in V - S$
⟨2⟩3. ASSUME: for a contradiction there is no $\epsilon > 0$ such that $B(x, \epsilon) \subseteq V - S$.
⟨2⟩4. For $n \in \mathbb{Z}_+$, PICK $x_n \in B(x, 1/n) \cap S$
⟨2⟩5. $x_n \to x$ as $n \to \infty$
⟨2⟩6. $x \in S$
⟨2⟩7. This contradicts ⟨2⟩2.
□

Definition 18.1.41 (Closure)**.** Let V be a normed space over K. Let S be a subset of V. The *closure* of S, $\mathrm{cl}\, S$, is the intersection of the set of closed sets that include S.

Proposition 18.1.42. *Let V be a normed space over K. Let S be a subset of V. Then the closure of S is the smallest closed set that includes S.*

PROOF: Proposition 18.1.30. □

Theorem 18.1.43. *Let V be a normed space over K. Let S be a subset of V. Then*

$$\mathrm{cl}\, S = \{l \in V \mid \exists \text{ a sequence } (x_n) \text{ in } S . x_n \to l \text{ as } n \to \infty\} \enspace .$$

PROOF:
⟨1⟩1. For all $l \in \mathrm{cl}\, S$, there exists a sequence (x_n) in S such that $x_n \to l$ as $n \to \infty$.
⟨2⟩1. LET: $l \in \mathrm{cl}\, S$
⟨2⟩2. For $n \in \mathbb{Z}_+$, pick $x_n \in B(l, 1/n) \cap S$
PROOF: There must be such an x_n otherwise $S - B(l, 1/n)$ would be a smaller closed set that includes S.

⟨2⟩3. $x_n \to l$ as $n \to \infty$
⟨1⟩2. For any sequence (x_n) in S, if $x_n \to l$ as $n \to \infty$ then $l \in \mathrm{cl}\, S$.
PROOF: Theorem 18.1.40.
□

Definition 18.1.44 (Dense)**.** Let V be a normed space over K. Let $S \subseteq V$. Then S is *dense* if and only if $\mathrm{cl}\, S = V$.

Theorem 18.1.45 (Weierstrass Approximation Theorem)**.** *Let a and b be real numbers with $a < b$. In $\mathcal{C}([a, b])$, the set of polynomials is dense.*

PROOF:TODO

Proposition 18.1.46. *Let $p \geq 1$. The set of all sequences that have only finitely many non-zero terms is dense in l^p.*

PROOF:
⟨1⟩1. LET: $(z_n) \in l^p$
⟨1⟩2. LET: $\epsilon > 0$
PROVE: There exists a sequence (x_n) with only finitely many non-zero terms such that $\left(\sum_{n=1}^{\infty} |z_n - x_n|^p\right)^{1/p} < \epsilon$
⟨1⟩3. PICK N such that $|\sum_{n=1}^{\infty} |z_n|^p - \sum_{n=1}^{N} |z_n|^p| < \epsilon^p$
⟨1⟩4. LET: (x_n) be the sequence that agrees with (z_n) up to term N, and then zeros after that.
⟨1⟩5. $\left(\sum_{n=1}^{\infty} |z_n - x_n|^p\right)^{1/p} < \epsilon$
PROOF:
$$\begin{aligned} \left(\sum_{n=1}^{\infty} |z_n - x_n|^p\right)^{1/p} &= \left(\sum_{n=N+1}^{\infty} |z_n|^p\right)^{1/p} && (\langle 1\rangle 4) \\ &< \epsilon && (\langle 1\rangle 2) \end{aligned}$$
□

Theorem 18.1.47. *Let V be a normed space over K. Let $S \subseteq V$. Then the following are equivalent.*

1. *S is dense.*
2. *For all $l \in V$, there exists a sequence (x_n) in S such that $x_n \to l$ as $n \to \infty$.*
3. *Every nonempty open subset of V intersects S.*

PROOF:
⟨1⟩1. $1 \Leftrightarrow 2$
PROOF: Theorem 18.1.43.
⟨1⟩2. $1 \Rightarrow 3$
⟨2⟩1. ASSUME: S is dense.
⟨2⟩2. LET: U be a nonempty open subset of V.
⟨2⟩3. $X - U$ does not include S.

PROOF: Lest we have $\mathrm{cl}\, S \subseteq X - U$.
⟨2⟩4. U intersects S.
⟨1⟩3. $3 \Rightarrow 1$
⟨2⟩1. ASSUME: Every nonempty subset of V intersects S.
⟨2⟩2. Every closed proper subset of V does not include S.
⟨2⟩3. $\mathrm{cl}\, S = V$
□

Definition 18.1.48 (Compact)**.** Let V be a normed space over K and $S \subseteq V$. Then S is *compact* if and only if every sequence in S has a convergent subsequence whose limit is in S.

Proposition 18.1.49. *In K^n, a set is compact if and only if it is bounded and closed.*

PROOF: TODO

Definition 18.1.50 (Bounded)**.** Let V be a normed space over K and $S \subseteq V$. Then S is *bounded* iff there exists $r > 0$ such that $V \subseteq B(0, r)$.

Theorem 18.1.51. *Every compact set is closed and bounded.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. LET: $S \subseteq V$ be compact.
⟨1⟩3. S is closed.
⟨2⟩1. LET: (x_n) be a sequence in S that converges to l
⟨2⟩2. PICK a sequence (x_{n_r}) that converges to $x \in S$
PROOF: ⟨1⟩2, ⟨2⟩1
⟨2⟩3. $x_{n_r} \to l$ as $n \to \infty$
PROOF: ⟨2⟩1, ⟨2⟩2
⟨2⟩4. $l = x$
PROOF: Proposition 18.1.2.
⟨2⟩5. $l \in S$
PROOF: ⟨2⟩2, ⟨2⟩4
⟨2⟩6. Q.E.D.
PROOF: Theorem 18.1.40.
⟨1⟩4. S is bounded.
⟨2⟩1. ASSUME: for a contradiction S is unbounded.
⟨2⟩2. For $n \in \mathbb{Z}_+$, PICK $x_n \in S - B(0, n)$
⟨2⟩3. PICK a convergent subsequence (x_{n_r}) that converges to l, say.
⟨2⟩4. PICK $N \in \mathbb{Z}_+$ such that $\|l\| < N$
⟨2⟩5. PICK r such that $n_r > N$ and $\|x_{n_r} - l\| < N - \|l\|$
⟨2⟩6. $\|x_{n_r}\| < N < n_r$
⟨2⟩7. This contradicts ⟨2⟩2.
□

Proposition 18.1.52. *In $\mathcal{C}([0, 1])$, the closed ball $\overline{B}(0, 1)$ is closed and bounded but not compact.*

PROOF: The sequence of functions (x^n) is in $\overline{B}(0,1)$ but has no convergent subsequence. ∎

Theorem 18.1.53 (Riesz's Lemma)**.** *Let V be a normed vector space over K. Let X be a closed proper subspace of V. Let $0 < \epsilon < 1$. Then there exists $x \in V$ such that $\|x\| = 1$ and $\forall y \in X. \|x - y\| \geq \epsilon$.*

PROOF:
⟨1⟩1. PICK $z \in V - X$
⟨1⟩2. LET: $d = \inf_{x \in X} \|z - x\|$
⟨1⟩3. $d > 0$
PROOF: Since X is closed, there exists $e > 0$ such that $B(z, d) \subseteq V - X$ and hence $\|z - x\| \geq d$ for all $x \in X$.
⟨1⟩4. PICK $x_0 \in X$ such that $d \leq \|z - x_0\| \leq d/\epsilon$
PROOF: One exists since d/ϵ is not a lower bound for $\{\|z - x\| \mid x \in X\}$.
⟨1⟩5. LET: $x = (z - x_0)/\|z - x_0\|$
⟨1⟩6. LET: $y \in X$
⟨1⟩7. $\|x - y\| \geq \epsilon$
PROOF:

$$\begin{aligned}
\|x - y\| &= \left\| \frac{z - x_0}{\|z - x_0\|} - y \right\| && (\langle 1\rangle 5) \\
&= \frac{1}{\|z - x_0\|} \|z - (x_0 + \|z - x_0\| y)\| \\
&\geq \frac{1}{\|z - x_0\|} d && (\langle 1\rangle 2) \\
&\geq \epsilon && (\langle 1\rangle 4)
\end{aligned}$$

∎

Theorem 18.1.54. *Let V be a normed space over K. Then V is finite dimensional if and only if $\overline{B}(0,1)$ is compact.*

PROOF:
⟨1⟩1. If V is finite dimensional then $\overline{B}(0,1)$ is compact.
⟨2⟩1. ASSUME: V is finite dimensional.
⟨2⟩2. PICK a basis $\{e_1, \ldots, e_n\}$
⟨2⟩3. ASSUME: w.l.o.g. $\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| = |\alpha_1| + \cdots + |\alpha_n|$
⟨2⟩4. LET: $(\alpha_{k1} e_1 + \cdots + \alpha_{kn} e_n)$ be a sequence in $\overline{B}(0,1)$
⟨2⟩5. PICK a convergent subsequence $(\alpha_{k_r 1})$ of (α_{k1}), a convergent subsequence $(\alpha_{k'_r 2})$ of $(\alpha_{k_r 2})$, ..., a convergent subsequence $(\alpha_{k''_r n})$.
⟨2⟩6. $(\alpha_{k''_r 1} e_1 + \cdots + \alpha_{k''_r n})$ converges.
⟨1⟩2. If V is infinite dimensional then $\overline{B}(0,1)$ is not compact.
⟨2⟩1. ASSUME: V is infinite dimensional.
⟨2⟩2. Choose a sequence (x_n) with $\|x_n\| = 1$ and $\|x_m - x_n\| \geq 1/2$ for $m \neq n$
⟨3⟩1. ASSUME: $x_1, \ldots, x_n$ satisfy $\|x_i\| = 1$ and $\|x_i - x_j\| \geq 1/2$ for $i \neq j$
⟨3⟩2. PICK $x_{n+1} \in V$ such that $\|x_{n+1}\| = 1$ and for all $y \in \operatorname{span}\{x_1, \ldots, x_n\}$ we have $\|x_{n+1} - y\| \geq 1/2$

⟨4⟩1. $\text{span}\{x_1, \ldots, x_n\}$ is closed.
⟨5⟩1. LET: $S = \text{span}\{x_1, \ldots, x_n\}$
⟨5⟩2. LET: (a_n) be a sequence in S that converges to $a \in V$
PROVE: $a \in S$
⟨5⟩3. (a_n) is a Cauchy sequence in V.
⟨5⟩4. (a_n) is a Cauchy sequence in S.
⟨5⟩5. A finite dimensional normed space is a Banach space.
⟨5⟩6. S is complete.
⟨5⟩7. $a \in S$
⟨4⟩2. $\text{span}\{x_1, \ldots, x_n\}$ is a proper subspace of V.
PROOF: ⟨2⟩1
⟨4⟩3. Q.E.D.
PROOF: Riesz's Lemma.
⟨2⟩3. ASSUME: for a contradiction (x_{n_r}) is a subsequence that converges to l
⟨2⟩4. For all $r \in \mathbb{N}$, we have $\|x_{n_r} - l\| + \|x_{n_{r+1}} - l\| \geq 1/2$
⟨2⟩5. This is a contradiction.
□

Proposition 18.1.55. *Let V be a normed space. The closure of a subspace of V is a subspace.*

PROOF:
⟨1⟩1. LET: U be a subspace of V
⟨1⟩2. LET: $x, y \in \text{cl}\, U$ and $\alpha, \beta \in K$
⟨1⟩3. PICK sequences (x_n), (y_n) in U that converge to x and y respectively.
⟨1⟩4. $\alpha x_n + \beta y_n \to \alpha x + \beta y$ as $n \to \infty$
⟨1⟩5. $\alpha x + \beta y \in \text{cl}\, U$
□

18.2 Continuous Linear Mappings

Definition 18.2.1 (Continuous)**.** Let U and V be normed spaces. Let $f : U \to V$ and $x \in U$. Then f is *continuous at* x iff, for any sequence (x_n) in U, if $x_n \to x$ as $n \to \infty$ then $f(x_n) \to f(x)$ as $n \to \infty$. The function f is *continuous* iff f is continuous at every point.

Proposition 18.2.2. *Let V be a normed space. Then the norm is a continuous function $V \to \mathbb{R}$.*

PROOF: From Proposition 18.0.4. □

Proposition 18.2.3. *Let U and V be normed space. Let $f : U \to V$. Then the following are equivalent.*

1. *f is continuous.*
2. *For any open set S in V, we have $f^{-1}(S)$ is open in U.*

3. *For any closed set C in V, we have $f^{-1}(C)$ is closed in U.*

Proposition 18.2.4. *Let U and V be normed spaces over K. Let $T : U \to V$ be a linear transformation. If T is continuous at some point, then it is continuous.*

PROOF:
⟨1⟩1. ASSUME: T is continuous at u_0
⟨1⟩2. LET: $x_n \to l$ as $n \to \infty$ in U
⟨1⟩3. $x_n - l + u_0 \to u_0$ as $n \to \infty$.
⟨1⟩4. $T(x_n - l + u_0) \to T(u_0)$ as $n \to \infty$.
⟨1⟩5. $T(x_n) - T(l) + T(u_0) \to T(u_0)$ as $n \to \infty$.
⟨1⟩6. $T(x_n) \to T(l)$ as $n \to \infty$.
□

Definition 18.2.5 (Bounded)**.** Let U and V be normed spaces over K. Let $T : U \to V$ be a linear transformation. Then T is *bounded* iff there exists $\alpha > 0$ such that, for all $x \in U$, we have $\|T(x)\| \leq \alpha\|x\|$.

Theorem 18.2.6. *Let U and V be normed spaces over K. Let $T : U \to V$ be a linear transformation. Then T is continuous if and only if it is bounded.*

PROOF:
⟨1⟩1. If T is continuous then T is bounded.
  ⟨2⟩1. ASSUME: T is not bounded.
  ⟨2⟩2. For $n \in \mathbb{Z}_+$, PICK $x_n \in U$ such that $\|T(x_n)\| > n\|x_n\|$.
  ⟨2⟩3. For $n \in \mathbb{Z}_+$,
    LET:
$$y_n = \frac{x_n}{n\|x_n\|}$$
  ⟨2⟩4. $y_n \to 0$ as $n \to \infty$
  ⟨2⟩5. $T(y_n) \not\to 0$ as $n \to \infty$
  ⟨2⟩6. T is not continuous.
⟨1⟩2. If T is bounded then T is continuous.
  ⟨2⟩1. ASSUME: T is bounded.
  ⟨2⟩2. PICK $\alpha > 0$ such that $\forall x \in U. \|T(x)\| \leq \alpha\|x\|$.
  ⟨2⟩3. T is continuous at 0.
    ⟨3⟩1. LET: (x_n) be a sequence that converges to 0 in U
    ⟨3⟩2. $T(x_n) \to 0$ as $n \to \infty$
      PROOF:
$$\begin{aligned} \|T(x_n)\| &\leq \alpha\|x_n\| && (\langle 2\rangle 2) \\ &\to 0 && \text{as } n \to \infty \end{aligned}$$
  ⟨2⟩4. T is continuous.
    PROOF: Proposition 18.2.4.
□

Proposition 18.2.7. *Let U and V be normed spaces over K where U is finite dimensional. Let $T : U \to V$ be a linear transformation. Then T is bounded.*

PROOF:

⟨1⟩1. Pick a basis $\{e_1, \ldots, e_n\}$ of unit vectors for U.
⟨1⟩2. Let: $M = \max(\|T(e_1)\|, \ldots, \|T(e_n)\|)$
⟨1⟩3. Pick $C > 0$ such that, for all $\alpha_1, \ldots, \alpha_n \in K$, we have $|\alpha_1| + \cdots + |\alpha_n| \leq C\|\alpha_1 e_1 + \cdots + \alpha_n e_n\|$
Proof: Theorem 18.1.13.
⟨1⟩4. Let: $x \in U$
Prove: $\|T(x)\| \leq CM\|x\|$
⟨1⟩5. Let: $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$
⟨1⟩6. $\|T(x)\| \leq CM\|x\|$
Proof:

$$\begin{aligned} \|T(x)\| &= \|\alpha_1 T(e_1) + \cdots + \alpha_n T(e_n)\| && (T \text{ linear}) \\ &\leq |\alpha_1|\|T(e_1)\| + \cdots + |\alpha_n|\|T(e_n)\| && (\text{Triangle inequality}) \\ &\leq M(|\alpha_1| + \cdots + |\alpha_n|) && (\langle 1\rangle 2) \\ &\leq CM\|x\| && (\langle 1\rangle 3) \end{aligned}$$

□

Corollary 18.2.7.1. *Let U and V be normed spaces over K where U is finite dimensional. Let $T : U \to V$ be a linear transformation. Then T is continuous.*

Proposition 18.2.8. *Let U and V be normed spaces over K. Let $T : U \to V$ be a linear transformation. If T is continuous, then T is uniformly continuous.*

Proof:
⟨1⟩1. Assume: T is continuous
⟨1⟩2. Pick $B > 0$ such that $\forall x \in U. \|T(x)\| \leq B\|x\|$
⟨1⟩3. Let: $\epsilon > 0$
⟨1⟩4. Let: $\delta = \epsilon / B$
⟨1⟩5. Let: $x, y \in U$
⟨1⟩6. Assume: $\|x - y\| < \delta$
⟨1⟩7. $\|T(x) - T(y)\| < \epsilon$
Proof:

$$\begin{aligned} \|T(x) - T(y)\| &= \|T(x - y)\| && (T \text{ linear}) \\ &\leq B\|x - y\| && (\langle 1\rangle 2) \\ &< B\delta && (\langle 1\rangle 6) \\ &= \epsilon && (\langle 1\rangle 4) \end{aligned}$$

□

Proposition 18.2.9. *Let U and V be normed spaces over K. The set $\mathcal{B}(U, V)$ of all bounded linear maps from U to V forms a subspace of the space of all linear maps from U to V.*

Proof:
⟨1⟩1. Let: $S, T : U \to V$ be bounded linear maps and $\alpha, \beta \in K$.
Prove: $\alpha S + \beta T$ is bounded.
⟨1⟩2. Pick $B, C > 0$ such that $\forall x \in U. \|S(x)\| \leq B\|x\|$ and $\|T(x)\| \leq C\|x\|$
⟨1⟩3. $\forall x \in U. \|(\alpha S + \beta T)(x)\| \leq (|\alpha| B + |\beta| C)\|x\|$

□

Proposition 18.2.10. *Let U and V be normed spaces over K. Define the* operator norm $\| \ \|$ *on $\mathcal{B}(U,V)$ by $\|T\| := \sup\{\|T(x)\| \mid x \in U, \|x\| = 1\}$. Then $\| \ \|$ is a norm on $\mathcal{B}(U,V)$.*

Proof:
⟨1⟩1. For all $T \in \mathcal{B}(U,V)$, the set $\{\|T(x)\| \mid x \in U, \|x\| = 1\}$ is bounded above.
⟨2⟩1. Let: $T \in \mathcal{B}(U,V)$
⟨2⟩2. Pick B such that $\forall x \in U.\|T(x)\| \leq B\|x\|$.
⟨2⟩3. B is an upper bound for $\{\|T(x)\| \mid x \in U, \|x\| = 1\}$.
⟨1⟩2. If $\|T\| = 0$ then $T = 0$.
⟨2⟩1. Assume: $\|T\| = 0$
⟨2⟩2. Let: $x \in U$
Prove: $T(x) = 0$
⟨2⟩3. Assume: w.l.o.g. $\|x\| \neq 0$
⟨2⟩4. Let: $y = x/\|x\|$
⟨2⟩5. $\|y\| = 1$
⟨2⟩6. $\|T(y)\| = 0$
⟨2⟩7. $T(y) = 0$
⟨2⟩8. $T(x) = 0$
⟨1⟩3. For all $\lambda \in K$ and $T \in \mathcal{B}(U,V)$, we have $\|\lambda T\| = |\lambda|\|T\|$
⟨2⟩1. Let: $\lambda \in K$ and $T \in \mathcal{B}(U,V)$
⟨2⟩2. $\|\lambda T\| = |\lambda|\|T\|$
Proof:

$$\begin{aligned}\|\lambda T\| &= \sup\{\|\lambda T(x)\| \mid x \in U, \|x\| = 1\} \\ &= \sup\{|\lambda|\|T(x)\| \mid x \in U, \|x\| = 1\} \\ &= |\lambda| \sup\{\|T(x)\| \mid x \in U, \|x\| = 1\} \\ &= |\lambda|\|T\|\end{aligned}$$

⟨1⟩4. For all $S, T \in \mathcal{B}(U,V)$, we have $\|S + T\| \leq \|S\| + \|T\|$.
⟨2⟩1. Let: $S, T \in \mathcal{B}(U,V)$
⟨2⟩2. $\|S + T\| \leq \|S\| + \|T\|$
Proof:

$$\begin{aligned}\|S + T\| &= \sup\{\|S(x) + T(x)\| \mid x \in U, \|x\| = 1\} \\ &\leq \sup\{\|S(x)\| + \|T(x)\| \mid x \in U, \|x\| = 1\} \\ &\leq \sup\{\|S(x)\| \mid x \in U, \|x\| = 1\} + \sup\{\|T(x)\| \mid x \in U, \|x\| = 1\} \\ &= \|S\| + \|T\|\end{aligned}$$

□

Proposition 18.2.11. *Let U and V be normed spaces. Let $T \in \mathcal{B}(U,V)$. Then $\|T\|$ is the least number such that $\forall u \in U.\|T(u)\| \leq \|T\|\|u\|$.*

Proof:
⟨1⟩1. $\forall u \in U.\|T(u)\| \leq \|T\|\|u\|$
⟨2⟩1. Let: $u \in U$
⟨2⟩2. Let: $v = u/\|u\|$

⟨2⟩3. $\|T(v)\| \leq \|T\|$
⟨2⟩4. $\|T(u)\| \leq \|T\|\|u\|$
⟨1⟩2. If α satisfies $\forall u \in U.\|T(u)\| \leq \alpha\|u\|$, then $\|T\| \leq \alpha$
⟨2⟩1. Assume: $\forall u \in U.\|T(u)\| \leq \alpha\|u\|$
⟨2⟩2. For all $x \in U$, if $\|x\| = 1$ then $\|T(x)\| \leq \alpha$
⟨2⟩3. $\|T\| \leq \alpha$
□

Proposition 18.2.12. *Let V be a normed space. Then id_V is a bounded linear function $V \to V$, and $\|\mathrm{id}_V\| = 1$.*

Proposition 18.2.13. *Let U and V be normed spaces. The constant zero function $U \to V$ is a bounded linear transformation with norm 0.*

Proposition 18.2.14. *Let $N \in \mathbb{N}$. Let $T : \mathbb{C}^N \to \mathbb{C}^N$ be a linear transformation with matrix $A = (a_{ij})$. Then T is bounded and*

$$\|T\| \leq \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}|^2} \ .$$

Definition 18.2.15 (Uniform Convergence)**.** Let U and V be normed spaces. Let (T_n) be a sequence in $\mathcal{B}(U,V)$ and $T \in \mathcal{B}(U,V)$. Then (T_n) *converges uniformly* to T iff (T_n) converges to T under the standard norm defined above.

Theorem 18.2.16. *Let U and V be normed spaces. Let $T : U \to V$ be a continuous linear function. Then $\ker T$ is a closed subspace of U.*

Proof:
⟨1⟩1. $\ker T$ is a subspace of U
Proof: If $x, y \in \ker T$ and $\alpha, \beta \in K$ then $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) = 0$.
⟨1⟩2. $\ker T$ is closed.
Proof: Let (x_n) be a sequence in $\ker T$ and $x_n \to l$. Then $T(l) = \lim_{n\to\infty} T(x_n) = 0$.
□

Theorem 18.2.17. *Let U and V be normed spaces. Let W be a closed subspace of U and $T : W \to V$ be a continuous linear mapping. Then the graph $G = \{(x, T(x)) \mid x \in W\}$ is closed in $U \times V$.*

Proof:
⟨1⟩1. Assume: w.l.o.g. $T \neq 0$
⟨1⟩2. Let: $(x, y) \in (U \times V) - G$, i.e. $y \neq T(x)$
⟨1⟩3. Let: $\epsilon = \|y - T(x)\| > 0$
⟨1⟩4. Let: $x' \in W$ with $\|x - x'\| < \epsilon/3\|T\|$
⟨1⟩5. Let: $y' \in V$ with $\|y - y'\| < \epsilon/3$
⟨1⟩6. $y' \neq T(x')$

PROOF:

$$\begin{aligned}\|y' - T(x')\| &\geq \|y - T(x)\| - \|y - y'\| - \|T(x) - T(x')\| \\ &> \epsilon/3 \\ &> 0\end{aligned}$$

∎

Theorem 18.2.18 (Diagonal Theorem)**.** *Let E be a normed space over K. Let (x_{ij}) be an infinite matrix of elements of V. If:*

1. *For all $j \in \mathbb{Z}_+$, we have $x_{ij} \to 0$ as $i \to \infty$;*
2. *Every increasing sequence of positive integers (p_j) has a subsequence (p_{j_r}) such that*

$$\sum_{s=1}^{\infty} x_{p_{j_r} p_{j_s}} \to 0 \text{ as } r \to \infty$$

then $x_{ii} \to 0$ as $i \to \infty$.

PROOF:
⟨1⟩1. ASSUME: for a contradiction $x_{ii} \not\to 0$ as $i \to \infty$
⟨1⟩2. PICK $\epsilon > 0$ such that, for all N, there exists $n \geq N$ such that $\|x_{nn}\| \geq \epsilon$
⟨1⟩3. PICK an increasing sequence of integers (p_j) such that $\|x_{p_j p_j}\| \geq \epsilon$ for all j.
⟨1⟩4. PICK a subsequence (q_i) such that $\sum_{j=1}^{\infty} x_{q_i q_j} \to 0$ as $i \to \infty$
⟨1⟩5. For all i, we have $x_{q_i q_j} \to 0$ as $j \to \infty$
⟨1⟩6. For all j, we have $x_{q_i q_j} \to 0$ as $i \to \infty$
⟨1⟩7. Define a subsequence (r_n) of (q_i) by $r_1 = q_1$ and, for all n, r_{n+1} is the first entry such that $r_{n+1} > r_n$, $\|x_{q_i r_n}\| < \epsilon/2^{n+1}$ for all $q_i \geq r_{n+1}$, and $\|x_{r_j r_{n+1}}\| < \epsilon/2^{n+2}$ for $j = 1, \ldots, n$.
⟨1⟩8. $\|x_{r_i r_j}\| < \epsilon/2^{j+1}$ for all i, j such that $i \neq j$
⟨1⟩9. PICK a subsequence (s_j) of (r_j) such that $\sum_{j=1}^{\infty} x_{s_i s_j} \to 0$ as $i \to \infty$
⟨1⟩10. For all i we have $\|\sum_{j=1}^{\infty} x_{s_i s_j}\| \geq \epsilon/2$

PROOF:

$$\begin{aligned}\left\|\sum_{j=1}^{\infty} x_{s_i s_j}\right\| &= \left\|x_{s_i s_i} + \sum_{i \neq j} x_{s_i s_j}\right\| \\ &\geq \left|\|x_{s_i s_i}\| - \left\|\sum_{i \neq j} x_{s_i s_j}\right\|\right| && \text{(Proposition 18.0.4)} \\ &\geq \left|\|x_{s_i s_i}\| - \sum_{i \neq j} \|x_{s_i s_j}\|\right| \\ &\geq \epsilon/2 && (\langle 1\rangle 2,\ \langle 1\rangle 8)\end{aligned}$$

⟨1⟩11. Q.E.D.

PROOF: ⟨1⟩9 and ⟨1⟩10 form a contradiction.

∎

18.3 Banach Spaces

Definition 18.3.1 (Cauchy Sequence)**.** Let V be a normed space over K. A *Cauchy sequence* is a sequence of points (x_n) such that, for every $\epsilon > 0$, there exists N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$.

Theorem 18.3.2. *Let V be a normed space over K. Let (x_n) be a sequence in V. The following are equivalent.*

1. *(x_n) is Cauchy.*
2. *For every pair of increasing sequences of positive integers (p_n) and (q_n), we have $\|x_{p_n} - x_{q_n}\| \to 0$ as $n \to \infty$.*
3. *For every increasing sequence of positive integers (p_n), we have $\|x_{p_n} - x_n\| \to 0$ as $n \to \infty$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
⟨2⟩1. ASSUME: (x_n) is Cauchy.
⟨2⟩2. LET: (p_n) and (q_n) are increasing sequences of positive integers.
⟨2⟩3. LET: $\epsilon > 0$
⟨2⟩4. PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$
⟨2⟩5. $\forall n \geq N. \|x_{p_n} - x_{q_n}\| < \epsilon$
PROOF: Since $p_n, q_n \geq n \geq N$.
⟨1⟩2. $2 \Rightarrow 3$
PROOF: Trivial.
⟨1⟩3. $2 \Rightarrow 1$
⟨2⟩1. ASSUME: (x_n) is not Cauchy
⟨2⟩2. Pick $\epsilon > 0$ such that, for every $N \in \mathbb{Z}_+$, there exist $m_N, n_N \geq N$ such that $\|x_{m_N} - x_{n_N}\| \geq \epsilon$
⟨2⟩3. ASSUME: w.l.o.g. (m_N) and (n_N) are increasing sequences.
⟨2⟩4. $\|x_{m_N} - x_{n_N}\| \not\to 0$ as $N \to \infty$.
⟨1⟩4. $3 \Rightarrow 2$
⟨2⟩1. ASSUME: 3
⟨2⟩2. LET: (p_n) and (q_n) be increasing sequences of positive integers.
⟨2⟩3. LET: $\epsilon > 0$
⟨2⟩4. PICK N such that $\forall n \geq N. \|x_{p_n} - x_n\| < \epsilon/2$ and $\forall n \geq N. \|x_{q_n} - x_n\| < \epsilon/2$
⟨2⟩5. $\forall n \geq N. \|x_{p_n} - x_{q_n}\| < \epsilon$
□

Proposition 18.3.3. *Every convergent sequence is Cauchy.*

PROOF:
⟨1⟩1. LET: $x_n \to l$ as $n \to \infty$.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK N such that $\forall n \geq N. \|x_n - l\| < \epsilon/2$

⟨1⟩4. For all $m, n \geq N$ we have $\|x_m - x_n\| < \epsilon$.
□

Proposition 18.3.4. *In* $\mathcal{P}([0,1])$*, let*

$$P_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} \quad .$$

Then (P_n) *is Cauchy but does not converge.*

PROOF: It converges to e^x in $\mathcal{C}([0,1])$, therefore it is Cauchy in $\mathcal{C}([0,1])$, hence Cauchy in $\mathcal{P}([0,1])$. Since $e^x \notin \mathcal{P}([0,1])$, it does not converge in that space. □

Proposition 18.3.5. *Let* V *be a normed space over* K*. Let* (x_n) *be a Cauchy sequence in* V*. Then* $(\|x_n\|)$ *converges in* $\mathbb{R}$*.*

PROOF:
⟨1⟩1. $(\|x_n\|)$ is Cauchy.
  ⟨2⟩1. LET: $\epsilon > 0$
  ⟨2⟩2. PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$
  ⟨2⟩3. $\forall m, n \geq N. |\|x_m\| - \|x_n\|| < \epsilon$
    PROOF: Proposition 18.0.4.
⟨1⟩2. Q.E.D.
  PROOF: Since every Cauchy sequence in $\mathbb{R}$ converges.
□

Proposition 18.3.6. *Every Cauchy sequence is bounded.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. LET: (x_n) be a Cauchy sequence in V.
⟨1⟩3. PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < 1$.
⟨1⟩4. LET: $B = \max(\|x_1\|, \ldots, \|x_{N-1}\|, \|x_N\| + 1$
⟨1⟩5. $\forall n. \|x_n\| \leq B$
□

Definition 18.3.7 (Banach Space)**.** A normed space V over K is *complete* or a *Banach space* iff every Cauchy sequence converges.

Proposition 18.3.8. l^2 *is complete.*

PROOF:
⟨1⟩1. LET: (a_n) be a Cauchy sequence in l^2 where $a_n = (\alpha_{n1}, \alpha_{n2}, \ldots)$.
⟨1⟩2. For all $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}_+$ such that $\forall m, n \geq n_0. \sum_{k=1}^{\infty} |\alpha_{mk} - \alpha_{nk}|^2 < \epsilon^2$.
⟨1⟩3. For every $k \in \mathbb{Z}_+$ and $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}_+$ such that $\forall m, n \geq n_0. |\alpha_{mk} - \alpha_{nk}| < \epsilon$.
⟨1⟩4. For every $k \in \mathbb{Z}_+$, (α_{nk}) is Cauchy in $\mathbb{C}$.
⟨1⟩5. For every $k \in \mathbb{Z}_+$, (α_{nk}) converges in $\mathbb{C}$.
⟨1⟩6. For $k \in \mathbb{Z}_+$,

LET: $\alpha_k = \lim_{n\to\infty} \alpha_{nk}$
⟨1⟩7. Let a be the sequence (α_k)
⟨1⟩8. For all $\epsilon > 0$, there exists n_0 such that $\forall n \geq n_0. \sum_{k=1}^{\infty} |\alpha_k - \alpha_{nk}|^2 \leq \epsilon^2$.
PROOF: Letting $m \to \infty$ in ⟨1⟩2.
⟨1⟩9. $a \in l^2$
⟨2⟩1. PICK n_0 such that $\forall n \geq n_0. \sum_{k=1}^{\infty} |\alpha_k - \alpha_{nk}|^2 \leq 1$
PROOF: ⟨1⟩8
⟨2⟩2. $(\alpha_k - \alpha_{n_0 k}) \in l^2$
⟨2⟩3. $(\alpha_{n_0 k}) \in l^2$
PROOF: By ⟨1⟩1 since the sequence is a_{n_0}.
⟨2⟩4. $(\alpha_k) \in l^2$
PROOF: Proposition 16.0.2.
⟨1⟩10. $a_n \to a$ as $n \to \infty$
PROOF: By ⟨1⟩8 since $\|a - a_n\| = \sqrt{\sum_{k=1}^{\infty} |\alpha_k - \alpha_{nk}|^2}$.
□

Proposition 18.3.9. *Let a and b be real numbers with $a < b$. The space $\mathcal{C}([a,b])$ is complete.*

PROOF:
⟨1⟩1. LET: $X = [a,b]$
⟨1⟩2. LET: (f_n) be a Cauchy sequence in $\mathcal{C}([a,b])$.
⟨1⟩3. For all $\epsilon > 0$, there exists n_0 such that $\forall n, m \geq n_0. \|f_n - f_m\| < \epsilon$.
⟨1⟩4. For all $\epsilon > 0$, there exists n_0 such that $\forall n, m \geq n_0. \forall x \in X. |f_n(x) - f_m(x)| < \epsilon$.
⟨1⟩5. For all $x \in [a,b]$, $(f_n(x))$ is Cauchy.
⟨1⟩6. Define $f : [a,b] \to \mathbb{C}$ by $f(x) = \lim_{n\to\infty} f_n(x)$.
⟨1⟩7. For all $\epsilon > 0$, there exists n_0 such that $\forall n \geq n_0. \forall x \in X. |f_n(x) - f(x)| < \epsilon$
PROOF: Letting $m \to \infty$ in ⟨1⟩4.
⟨1⟩8. f is continuous
⟨2⟩1. LET: $x_0 \in X$
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK n_0 such that $\forall n \geq n_0. \forall x \in X. |f_n(x) - f(x)| < \epsilon/3$
PROOF: By ⟨1⟩7.
⟨2⟩4. PICK $\delta > 0$ such that $\forall x \in X. |x - x_0| < \delta \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \epsilon/3$
PROOF: Since f_{n_0} is continuous.
⟨2⟩5. LET: $x \in X$
⟨2⟩6. ASSUME: $|x - x_0| < \delta$
⟨2⟩7. $|f(x) - f(x_0)| < \epsilon$
PROOF:
$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| && \text{(Triangle Inequality)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 2\rangle 3,\ \langle 2\rangle 4) \\ &= \epsilon \end{aligned}$$
⟨1⟩9. (f_n) converges to f uniformly.
PROOF: From ⟨1⟩7
□

Definition 18.3.10 (Series)**.** Let V be a normed space over K. A *convergent series* in V is a sequence (x_n) in V such that $(x_1 + \cdots + x_n)$ is a convergent sequence, in which case we write $\sum_{n=1}^{\infty} x_n$ for its limit.

Definition 18.3.11 (Absolutely Convergent Series)**.** Let V be a normed space over K. An *absolutely convergent series* in V is a sequence (x_n) such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Proposition 18.3.12. *In $\mathcal{P}([0,1])$, the series $\sum_{n=0}^{\infty} x^n/n!$ is absolutely convergent but not convergent.*

Proof: Proposition 18.3.4. □

Theorem 18.3.13. *A normed space is complete if and only if every absolutely convergent series is convergent.*

Proof:
⟨1⟩1. Let: V be a normed space over K.
⟨1⟩2. If V is complete then every absolutely convergent series is convergent.
⟨2⟩1. Assume: V is complete.
⟨2⟩2. Let: $\sum_{n=1}^{\infty} a_n$ be absolutely convergent in V.
⟨2⟩3. For $n \in \mathbb{Z}_+$,
Let: $s_n = \sum_{k=1}^{n} a_k$
.
⟨2⟩4. (s_n) is Cauchy.
⟨3⟩1. Let: $\epsilon > 0$
⟨3⟩2. Pick k such that $\sum_{n=k+1}^{\infty} \|a_n\| < \epsilon$
⟨3⟩3. Let: $m > n > k$
⟨3⟩4. $\|s_m - s_n\| < \epsilon$
Proof:

$$\begin{aligned}
\|s_m - s_n\| &= \left\| \sum_{i=n+1}^{m} a_i \right\| && (\langle 2\rangle 3,\ \langle 3\rangle 3) \\
&\leq \sum_{i=s+1}^{m} \|a_i\| && \text{(Triangle inequality)} \\
&\leq \sum_{i=k+1}^{\infty} \|a_i\| \\
&< \epsilon && (\langle 3\rangle 2,\ \langle 3\rangle 3)
\end{aligned}$$

⟨2⟩5. (s_n) converges.
⟨1⟩3. If every absolutely convergent series is convergent then V is complete.
⟨2⟩1. Assume: Every absolutely convergent series in V is convergent.
⟨2⟩2. Let: (a_n) be a Cauchy sequence in V.
⟨2⟩3. Pick a strictly increasing sequence of positive integers (p_n) such that $\forall k. \forall m, n \geq p_k. \|x_m - x_n\| < 2^{-k}$.
⟨2⟩4. $\sum_{k=1}^{\infty} (x_{p_{k+1}} - x_{p_k})$ is absolutely convergent.

PROOF:
$$\sum_{k=1}^{\infty} \|x_{p_{k+1}} - x_{p_k}\| < \sum_{k=1}^{\infty} 2^{-k} \quad (\langle 2\rangle 3)$$
$$< \infty$$

$\langle 2\rangle 5$. $\sum_{k=1}^{\infty}(x_{p_{k+1}} - x_{p_k})$ is convergent.
PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 5$
$\langle 2\rangle 6$. LET: $s = \sum_{k=1}^{\infty}(x_{p_{k+1}} - x_{p_k})$
$\langle 2\rangle 7$. $x_{p_k} \to s + x_{p_1}$ as $k \to \infty$.
$\langle 3\rangle 1$. $\sum_{i=1}^{k}(x_{p_{i+1}} - x_{p_i}) \to s$ as $k \to \infty$
PROOF: $\langle 2\rangle 6$
$\langle 3\rangle 2$. $x_{p_{k+1}} - x_{p_1} \to s$ as $k \to \infty$
$\langle 2\rangle 8$. $x_n \to s + x_{p_1}$ as $k \to \infty$.
PROOF:
$\langle 3\rangle 1$. LET: $\epsilon > 0$
$\langle 3\rangle 2$. PICK N such that $\forall k \geq N. \|x_{p_k} - (s + x_{p_1})\| < \epsilon/2$ and $\forall m, n \geq N. \|x_m - x_n\| < \epsilon/2$
PROOF: $\langle 2\rangle 2$, $\langle 2\rangle 7$
$\langle 3\rangle 3$. $\forall n \geq N. \|x_n - (s + x_{p_1})\| < \epsilon$
$\square$

Theorem 18.3.14. *A closed vector subspace of a Banach space is a Banach space.*

PROOF:
$\langle 1\rangle 1$. LET: V be a Banach space over K.
$\langle 1\rangle 2$. LET: U be a closed vector subspace of V.
$\langle 1\rangle 3$. LET: (a_n) be a Cauchy sequence in U.
$\langle 1\rangle 4$. (a_n) is a Cauchy sequence in V.
$\langle 1\rangle 5$. LET: $l = \lim_{n\to\infty} a_n$
$\langle 1\rangle 6$. $l \in U$
PROOF: Theorem 18.1.40.
$\langle 1\rangle 7$. $a_n \to l$ as $n \to \infty$ in U.
$\square$

Definition 18.3.15 (Completion)**.** Let V be a normed space over K. A *completion* of V consists of a normed space W over K and an injection $\phi : V \to W$ such that:

1. $\forall x, y \in V. \forall \alpha, \beta \in K. \phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$
2. $\forall x \in V. \|\phi(x)\| = \|x\|$
3. $\phi(V)$ is dense in W.
4. W is complete.

Proposition 18.3.16. *Every normed space has a completion.*

PROOF:

$\langle 1\rangle 1$. LET: V be a normed space over K.
$\langle 1\rangle 2$. Let us say two Cauchy sequences (x_n), (y_n) ore *equivalent* iff $x_n - y_n \to 0$ as $n \to \infty$.
$\langle 1\rangle 3$. Equivalence is an equivalence relation on the set of Cauchy sequences.
$\langle 1\rangle 4$. LET: W be the set of Cauchy sequences in V quotiented by equivalence.
$\langle 1\rangle 5$. Define addition and multiplication on W by

$$[(x_n)] + [(y_n)] = [(x_n + y_n)]$$
$$\lambda[(x_n)] = [(\lambda x_n)]$$

$\langle 1\rangle 6$. Define a norm on W by $\|[(x_n)]\| = \lim_{n\to\infty} \|x_n\|$
$\langle 1\rangle 7$. Define $\phi : V \to W$ by $\phi(v) = [(v)]$.
$\langle 1\rangle 8$. ϕ is injective.
$\langle 1\rangle 9$. $\forall x, y \in V.\forall \alpha, \beta \in K.\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$
$\langle 1\rangle 10$. $\forall x \in V.\|\phi(x)\| = \|x\|$
$\langle 1\rangle 11$. $\phi(V)$ is dense in W.
$\langle 2\rangle 1$. LET: $[(a_n)] \in W$ and $\epsilon > 0$.
PROVE: $B([(a_n)], \epsilon)$ intersects $\phi(V)$.
$\langle 2\rangle 2$. PICK N such that $\forall m, n \ge N.\|a_m - a_n\| < \epsilon/2$
$\langle 2\rangle 3$. $\phi(a_N) \in B([(a_n)], \epsilon)$
PROOF:

$$\begin{aligned}\|[(a_n)] - \phi(a_N)\| &= \lim_{n\to\infty} \|a_n - a_N\| \\ &\le \epsilon/2 && (\langle 2\rangle 2) \\ &< \epsilon\end{aligned}$$

$\langle 1\rangle 12$. W is complete.
$\langle 2\rangle 1$. LET: (X_n) be a Cauchy sequence in W.
$\langle 2\rangle 2$. For $n \in \mathbb{Z}_+$, PICK $x_n \in V$ such that

$$\|\phi(x_n) - X_n\| < 1/n \ .$$

$\langle 2\rangle 3$. (x_n) is Cauchy in V.
$\langle 3\rangle 1$. LET: $\epsilon > 0$
$\langle 3\rangle 2$. PICK N such that $\forall m, n \ge N.\|X_n - X_m\| < \epsilon/3$ and $1/N < \epsilon/3$
$\langle 3\rangle 3$. LET: $m, n \ge N$
$\langle 3\rangle 4$. $\|x_m - x_n\| < \epsilon$
PROOF:

$$\begin{aligned}\|x_m - x_n\| &= \|\phi(x_m) - \phi(x_n)\| \\ &\le \|\phi(x_m) - X_m\| + \|X_m - X_n\| + \|X_n - \phi(x_n)\| \\ &< \|X_m - X_n\| + 1/m + 1/n \\ &< \epsilon\end{aligned}$$

$\langle 2\rangle 4$. LET: $X = [(x_n)]$
$\langle 2\rangle 5$. $X_n \to X$ as $n \to \infty$
PROOF:

$$\begin{aligned}\|X_n - X\| &\le \|X_n - \phi(x_n)\| + \|\phi(x_n) - X\| \\ &< \|\phi(x_n) - X\| + 1/n \\ &\to 0 && \text{as } n \to \infty\end{aligned}$$

$\square$

Proposition 18.3.17. *Let U be a normed space and V a Banach space. Then $\mathcal{B}(U,V)$ is a Banach space.*

PROOF:
⟨1⟩1. LET: (T_n) be a Cauchy sequence in $\mathcal{B}(U,V)$
⟨1⟩2. For all $u \in U$, $(T_n(u))$ is a Cauchy sequence in V.
⟨2⟩1. LET: $u \in U$
⟨2⟩2. LET: $\epsilon > 0$
PROVE: $\exists N. \forall m,n \geq N. \|T_m(u) - T_n(u)\| < \epsilon$
⟨2⟩3. ASSUME: w.l.o.g. $u \neq 0$
⟨2⟩4. PICK N such that $\forall m,n \geq N. \|T_m - T_n\| < \epsilon / \|u\|$
⟨2⟩5. LET: $m,n \geq N$
⟨2⟩6. $\|T_m(u) - T_n(u)\| < \epsilon$
PROOF:

$$\begin{aligned} \|T_m(u) - T_n(u)\| &\leq \|T_m - T_n\|\|u\| && \text{(Proposition 18.2.11)} \\ &< \epsilon \end{aligned}$$

⟨1⟩3. Define $T : U \to V$ by $T(u) = \lim_{n\to\infty} T_n(u)$
⟨1⟩4. $T \in \mathcal{B}(U,V)$
⟨2⟩1. For all $x,y \in U$ and $\alpha, \beta \in K$ we have $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$
⟨3⟩1. LET: $x, y \in U$
⟨3⟩2. LET: $\alpha, \beta \in K$
⟨3⟩3. $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$
PROOF:

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_{n\to\infty} T_n(\alpha x + \beta y) \\ &= \lim_{n\to\infty} (\alpha T_n(x) + \beta T_n(y)) \\ &= \alpha T(x) + \beta T(y) \end{aligned}$$

⟨2⟩2. T is bounded.
⟨3⟩1. PICK N such that $\forall n \geq N. \|T_n - T\| < 1$
⟨3⟩2. PICK $B > 0$ such that $\forall x \in U. \|T_N(x)\| \leq B\|x\|$
⟨3⟩3. LET: $x \in U$
⟨3⟩4. $\|T(x)\| \leq (B+1)\|x\|$
PROOF:

$$\begin{aligned} \|T(x)\| &\leq \|T_N(x) - T(x)\| + \|T(x)\| && \text{(Triangle inequality)} \\ &\leq \|T_N - T\|\|x\| + \|T\|\|x\| && \text{(Proposition 18.2.11)} \\ &< \|x\| + B\|x\| && (\langle 3\rangle 1,\ \langle 3\rangle 2) \\ &= (B+1)\|x\| \end{aligned}$$

⟨1⟩5. $T_n \to T$ as $n \to \infty$
⟨2⟩1. LET: $\epsilon > 0$
⟨2⟩2. PICK N such that $\forall m,n \geq N. \|T_m - T_n\| < \epsilon/2$
⟨2⟩3. LET: $n \geq N$
PROVE: $\|T_n - T\| < \epsilon$
⟨2⟩4. LET: $x \in U$ with $\|x\| = 1$
⟨2⟩5. $\|T_n(x) - T(x)\| \leq \epsilon/2$
PROOF: Let $n \to \infty$ in ⟨2⟩2.

□

Corollary 18.3.17.1. *For any normed space V over K, the space $\mathcal{B}(V, K)$ is a Banach space.*

Theorem 18.3.18. *Let U be a normed space and V a Banach space. Let W be a subspace of U. Let $T : W \to V$ be a continuous linear transformation. Then T has a unique extension to a continuous linear transformation $\mathrm{cl}\, W \to V$.*

PROOF:
⟨1⟩1. Define $S : \mathrm{cl}\, W \to V$ by: $S(x) = \lim_{n\to\infty} T(x_n)$, where (x_n) is any sequence in W that converges to x.
  ⟨2⟩1. For all $x \in \mathrm{cl}\, W$, there exists a sequence (x_n) in W that converges to x.
  PROOF: Theorem 18.1.43.
  ⟨2⟩2. If (x_n) is a Cauchy sequence in W then $(T(x_n))$ is Cauchy in V.
    ⟨3⟩1. ASSUME: w.l.o.g. $T \neq 0$
    ⟨3⟩2. LET: (x_n) be a Cauchy sequence in W.
    ⟨3⟩3. PICK $B > 0$ such that $\forall x \in W. \|T(x)\| \leq B\|x\|$
    ⟨3⟩4. LET: $\epsilon > 0$
    ⟨3⟩5. PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon/\|T\|$
    ⟨3⟩6. LET: $m, n \geq N$
    ⟨3⟩7. $\|T(x_m) - T(x_n)\| < \epsilon$
  ⟨2⟩3. If (x_n) and (y_n) are sequences in W that converge to the same element in $\mathrm{cl}\, W$ then $(T(x_n))$ and $(T(y_n))$ have the same limit in V.
    ⟨3⟩1. ASSUME: w.l.o.g. $T \neq 0$
    ⟨3⟩2. ASSUME: $x_n \to l$ and $y_n \to l$ as $n \to \infty$
    ⟨3⟩3. LET: $T(x_n) \to a$ and $T(y_n) \to b$ as $n \to \infty$
    ⟨3⟩4. ASSUME: for a contradiction $a \neq b$
    ⟨3⟩5. LET: $\epsilon = \|a - b\|$
    ⟨3⟩6. PICK N such that $\forall n \geq N. \|x_n - l\| < \epsilon/3\|T\|$ and $\forall n \geq N. \|y_n - l\| < \epsilon/3\|T\|$
    ⟨3⟩7. $\forall n \geq N. \|T(x_n) - T(y_n)\| < 2\epsilon/3$
    ⟨3⟩8. $\|a - b\| \leq 2\epsilon/3$
    ⟨3⟩9. This contradicts ⟨3⟩5.
⟨1⟩2. S extends T
  ⟨2⟩1. LET: $w \in W$
  ⟨2⟩2. $w \to w$ as $n \to \infty$
  ⟨2⟩3. $T(w) \to T(w)$ as $n \to \infty$
  ⟨2⟩4. $S(w) = T(w)$
⟨1⟩3. S is bounded.
  ⟨2⟩1. LET: $x \in \mathrm{cl}\, W$
  PROVE: $\|S(x)\| \leq \|T\|\|x\|$
  ⟨2⟩2. PICK a sequence (x_n) in W that converges to x.
  ⟨2⟩3. $\|T(x_n)\| \leq \|T\|\|x_n\|$ for all n.
  ⟨2⟩4. $\|S(x)\| \leq \|T\|\|x\|$
  PROOF: Taking the limit as $n \to \infty$.
⟨1⟩4. S is linear.

⟨2⟩1. LET: $x, y \in \operatorname{cl} W$ and $\alpha, \beta \in K$
⟨2⟩2. PICK sequences (x_n) and (y_n) in W that converge to x and y.
⟨2⟩3. $T(\alpha x_n + \beta y_n) = \alpha T(x_n) + \beta T(y_n)$ for all n.
⟨2⟩4. $S(\alpha x + \beta y) = \alpha S(x) + \beta S(y)$
PROOF: Taking the limit as $n \to \infty$.
⟨1⟩5. S is unique.
⟨2⟩1. LET: S' be a continuous linear extension of S defined on $\operatorname{cl} W$.
⟨2⟩2. LET: $x \in W$
PROVE: $S(x) = S'(x)$
⟨2⟩3. PICK a sequence (x_n) in W that converges to x.
⟨2⟩4. $T(x_n) = S'(x_n) \to S'(x)$ as $n \to \infty$
⟨2⟩5. $S'(x) = S(x)$
□

Corollary 18.3.18.1. *Let U be a normed space and V a Banach space. Let W be a dense subspace of U. Let $T : W \to V$ be a continuous linear transformation. Then T has a unique extension to a continuous linear transformation $U \to V$.*

Definition 18.3.19 (Functional)**.** Let V be a normed space over K. A *functional* on V is a bounded linear mapping $V \to K$. The *dual space* of V is the space $\mathcal{B}(V, K)$ of all functionals.

Theorem 18.3.20 (Banach-Steinhaus Theorem)**.** *Let $\mathcal{T}$ be a family of bounded linear mappings from a Banach space X into a normed space Y. If, for every $x \in X$, there exists a constant M_x such that $\forall T \in \mathcal{T}. \|T(x)\| \leq M_x$, then there exists a constant $M > 0$ such that $\forall T \in \mathcal{T}. \|T\| \leq M$.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction no such M exists.
⟨1⟩2. For $n \in \mathbb{Z}_+$, PICK $T_n \in \mathcal{T}$ such that $\|T_n\| > n2^n$.
⟨1⟩3. For $n \in \mathbb{Z}_+$, PICK $x_n \in X$ such that $\|x_n\| = 1$ and $\|T_n(x_n)\| > n2^n$.
⟨1⟩4. For $n \in \mathbb{Z}_+$,

$$\left\| \frac{1}{n} T_n \left(\frac{x_n}{2^n} \right) \right\| > 1 \ .$$

⟨1⟩5. For $i, j \in \mathbb{Z}_+$,
LET: $y_{ij} = \frac{1}{i} T_i(\frac{x_j}{2^j})$.
⟨1⟩6. For all $j \in \mathbb{Z}_+$, $y_{ij} \to 0$ as $i \to \infty$
⟨2⟩1. LET: $j \in \mathbb{Z}_+$
⟨2⟩2. PICK M such that $\forall T \in \mathcal{T}. \|T(x_j/2^j)\| \leq M$
⟨2⟩3. For all i, $\|y_{ij}\| \leq M/i$
⟨1⟩7. For any increasing sequence of positive integers (p_i), we have $\sum_{j=1}^{\infty} y_{p_i p_j} \to 0$ as $i \to \infty$
⟨2⟩1. LET: (p_i) be an increasing sequence of positive integers.
⟨2⟩2. LET: $z = \sum_{j=1}^{\infty} x_{p_j}/2^{p_j}$
PROOF: This converges by Theorem 18.3.13.
⟨2⟩3. PICK C such that $\forall T \in \mathcal{T}. \|T(z)\| \leq C$
⟨2⟩4. For all i, $\|\sum_{j=1}^{\infty} y_{p_i p_j}\| \leq C/p_i$.

Proof:

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} y_{p_i p_j} \right\| &= \left\| \sum_{j=1}^{\infty} \frac{1}{p_i} T_{p_i} \left(\frac{x_{p_j}}{2^{p_j}} \right) \right\| && (\langle 1 \rangle 5) \\
&= \frac{1}{p_i} \left\| T_{p_i} \left(\sum_{j=1}^{\infty} \frac{x_{p_j}}{2^{p_j}} \right) \right\| && (T_{p_i} \text{ continuous}) \\
&= \frac{1}{p_i} \| T_{p_i}(z) \| && (\langle 2 \rangle 2) \\
&\leq \frac{C}{p_i} && (\langle 2 \rangle 3)
\end{aligned}$$

$\langle 2 \rangle 5$. $\sum_{j=1}^{\infty} y_{p_i p_j} \to 0$ as $i \to \infty$

$\langle 1 \rangle 8$. $y_{ii} \to 0$ as $i \to \infty$

Proof: Diagonal Theorem, $\langle 1 \rangle 6$, $\langle 1 \rangle 7$.

$\langle 1 \rangle 9$. Q.E.D.

Proof: $\langle 1 \rangle 4$ and $\langle 1 \rangle 8$ form a contradiction.

□

18.4 Contraction Mappings

Definition 18.4.1 (Contraction Mapping)**.** Let E be a normed space over K. Let $A \subseteq E$. A function $f : A \to E$ is a *contraction (mapping)* iff there exists a real α such that $0 < \alpha < 1$ and

$$\forall x, y \in A. \| f(x) - f(y) \| \leq \alpha \| x - y \| \ .$$

Proposition 18.4.2. *Contraction mappings are uniformly continuous.*

Proof:

$\langle 1 \rangle 1$. Let: E be a normed space over K.

$\langle 1 \rangle 2$. Let: $A \subseteq E$

$\langle 1 \rangle 3$. Let: $f : A \to E$ be a contraction mapping.

$\langle 1 \rangle 4$. Pick α such that $0 < \alpha < 1$ and $\forall x, y \in A. \| f(x) - f(y) \| \leq \alpha \| x - y \|$.

$\langle 1 \rangle 5$. Let: $\epsilon > 0$

$\langle 1 \rangle 6$. Let: $\delta = \epsilon / \alpha$

$\langle 1 \rangle 7$. For all $x, y \in A$, if $\| x - y \| < \delta$ then $\| f(x) - f(y) \| < \epsilon$.

□

Theorem 18.4.3 (Banach Fixed Point Theorem)**.** *Let E be a Banach space over K. Let F be a nonempty closed subset of E. Let $f : F \to F$ be a contraction mapping. Then there exists a unique $z \in F$ such that $f(z) = z$.*

Proof:

$\langle 1 \rangle 1$. Pick α such that $0 < \alpha < 1$ and

$$\forall x, y \in F. \| f(x) - f(y) \| \leq \alpha \| x - y \| \ .$$

$\langle 1 \rangle 2$. Pick $x_0 \in F$

$\langle 1\rangle 3.$ For $n \in \mathbb{Z}_+$,
LET: $x_n = f^n(x_0)$.
$\langle 1\rangle 4.$ (x_n) is a Cauchy sequence.
$\langle 2\rangle 1.$ For all $n \in \mathbb{Z}_+$ we have $\|x_{n+1} - x_n\| \leq \alpha^n \|x_1 - x_0\|$.
$\langle 2\rangle 2.$ For all $m, n \in \mathbb{Z}_+$ with $m < n$ we have $\|x_n - x_m\| < \alpha^m \|x_1 - x_0\|/(1-\alpha)$.
PROOF:

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\| && \text{(Triangle inequality)} \\ &\leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m)\|x_1 - x_0\| && (\langle 2\rangle 1) \\ &< \frac{\|x_1 - x_0\|}{1-\alpha}\alpha^m \end{aligned}$$

$\langle 2\rangle 3.$ LET: $\epsilon > 0$
$\langle 2\rangle 4.$ PICK N such that $\alpha^N \|x_1 - x_0\|/(1-\alpha) < \epsilon$
$\langle 2\rangle 5.$ For all $m, n \geq N$, we have $\|x_n - x_m\| < \epsilon$
$\langle 1\rangle 5.$ LET: $z = \lim_{n\to\infty} x_n$
$\langle 1\rangle 6.$ $f(z) = z$
PROOF:

$$\begin{aligned} f(z) &= f\left(\lim_{n\to\infty} x_n\right) \\ &= \lim_{n\to\infty} f(x_n) && \text{(Proposition 18.4.2)} \\ &= \lim_{n\to\infty} x_{n+1} \\ &= z \end{aligned}$$

$\langle 1\rangle 7.$ For any $w \in F$, if $f(w) = w$ then $w = z$.
$\langle 2\rangle 1.$ LET: $w \in F$
$\langle 2\rangle 2.$ ASSUME: $f(w) = w$
$\langle 2\rangle 3.$ $\|z - w\| \leq \alpha \|z - w\|$
PROOF: $\|z - w\| = \|f(z) - f(w)\| \leq \alpha \|z - w\|$
$\langle 2\rangle 4.$ $\|z - w\| = 0$
$\langle 2\rangle 5.$ $z = w$
$\square$

Chapter 19

Inner Product Spaces

Definition 19.0.1 (Inner Product Space)**.** Let E be a complex vector space. An *inner product* on E is a function $\langle\ ,\ \rangle : E^2 \to \mathbb{C}$ such that, for all $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$, we have:

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$
2. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$
3. $\langle x, x\rangle \geq 0$
4. If $\langle x, x\rangle = 0$ then $x = 0$

An *inner product space* consists of a complex vector space V and an inner product on V.

Proposition 19.0.2. *Let E be an inner product space. For any $x \in E$, we have $\langle x, x\rangle$ is real.*

Proof: Since $\langle x, x\rangle = \overline{\langle x, x\rangle}$. □

Proposition 19.0.3.

$$\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle$$

Proposition 19.0.4.

$$\langle 0, y\rangle = \langle x, 0\rangle = 0$$

Proposition 19.0.5. *The function $\langle \vec{x}, \vec{y}\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$ is an inner product on $\mathbb{C}^n$.*

Proposition 19.0.6. *The function $\langle (x_n), (y_n)\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$ is an inner product on l^2.*

Proposition 19.0.7. *The function $\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$ is an inner product on $\mathcal{C}([a, b])$.*

Proposition 19.0.8. *Let $p > 1$ and $\Omega \subseteq \mathbb{R}^N$. Let $L^p(\Omega)$ be the set of all functions $f : \Omega \to \mathbb{C}$ such that $|f|^p$ is Lebesgue integrable.*

The function $\langle f, g \rangle = \int_\Omega f(x)\overline{g(x)}dx$ is an inner product on $L^2(\Omega)$.

Proposition 19.0.9. *Let E_1 and E_2 be inner product spaces. Then the function $\langle (e_1, e_2), (e'_1, e'_2) \rangle = \langle e_1, e'_1 \rangle + \langle e_2, e'_2 \rangle$ is an inner product on $E_1 \times E_2$.*

Definition 19.0.10 (Norm)**.** In an inner product space, define $\|x\| = \sqrt{\langle x, x \rangle}$.

Proposition 19.0.11 (Schwarz's Inequality)**.** *In any inner product space,*

$$|\langle x, y \rangle| \leq \|x\|\|y\| \enspace .$$

Equality holds iff x and y are linearly dependent.

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. $y \neq 0$
⟨1⟩2. $|\langle x, y \rangle| \leq \|x\|\|y\|$
  ⟨2⟩1. For all $\alpha \in \mathbb{C}$ we have $\langle x, x \rangle + \overline{\alpha}\langle x, y \rangle + \alpha\langle y, x \rangle + |\alpha|^2\langle y, y \rangle$
    PROOF: The right-hand side is $\langle x + \alpha y, x + \alpha y \rangle$.
  ⟨2⟩2. $\langle x, x \rangle\langle y, y \rangle - |\langle x, y \rangle|^2 \geq 0$
    PROOF: Taking $\alpha = -\langle x, x \rangle / \langle y, y \rangle$ in ⟨2⟩1.
⟨1⟩3. If $|\langle x, y \rangle| = \|x\|\|y\|$ then x and y are linearly dependent.
  ⟨2⟩1. ASSUME: $|\langle x, y \rangle| = \|x\|\|y\|$
  ⟨2⟩2. $\langle x, y \rangle\langle y, x \rangle = \langle x, x \rangle\langle y, y \rangle$
  ⟨2⟩3. $\langle y, y \rangle x - \langle x, x \rangle y = 0$
    PROOF:
$$\begin{aligned}\langle \langle y, y \rangle x - \langle x, y \rangle y, \langle y, y \rangle x - \langle x, y \rangle y \rangle &= \langle y, y \rangle^2 \langle x, x \rangle - \langle y, y \rangle\langle y, x \rangle\langle x, y \rangle - \langle x, y \rangle\langle y, y \rangle\langle y, x \rangle + \langle x, y \rangle\langle y, x \\ &= 0\end{aligned}$$
⟨1⟩4. If x and y are linearly dependent then $|\langle x, y \rangle| = \|x\|\|y\|$
  ⟨2⟩1. ASSUME: x and y are linearly dependent.
  ⟨2⟩2. LET: $y = \alpha x$
  ⟨2⟩3. $|\langle x, y \rangle| = \|x\|\|y\|$
    PROOF:
$$\begin{aligned}|\langle x, y \rangle| &= |\langle x, \alpha x \rangle| \\ &= |\alpha||\langle x, x \rangle| \\ &= |\alpha|\|x\|^2 \\ &= \|x\|\|\alpha x\| \\ &= \|x\|\|y\|\end{aligned}$$

□

Corollary 19.0.11.1 (Triangle Inequality)**.** *In any inner product space,*

$$\|x + y\| \leq \|x\| + \|y\|$$

PROOF:

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\
&\leq \langle x,x\rangle + 2|\langle x,y\rangle| + \langle y,y\rangle \\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 && \text{(Schwarz's Inequality)} \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

□

Corollary 19.0.11.2. *The norm in an inner product space is a norm.*

Theorem 19.0.12 (Parallelogram Law)**.** *In any inner product space,*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

PROOF:
⟨1⟩1. $\|x+y\|^2 = \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2$
⟨1⟩2. $\|x-y\|^2 = \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2$
⟨1⟩3. Q.E.D.
PROOF: Add ⟨1⟩1 and ⟨1⟩2.
□

Proposition 19.0.13. *Let E be a normed space over $\mathbb{C}$. Then there exists an inner product on E that induces the norm of E iff E satisfies the Parallelogram Law.*

PROOF: If E satisfies the parallelogram law, define

$$\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2) \enspace .$$

Definition 19.0.14 (Orthogonal)**.** Vectors x and y in an inner product space are *orthogonal*, $x \perp y$, iff $\langle x,y\rangle = 0$.

Theorem 19.0.15 (Pythagorean Formula)**.** *If x and y are orthogonal then*

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \enspace .$$

Definition 19.0.16 (Weak Convergence)**.** Let E be an inner product space. Let (x_n) be a sequence in E and $l \in E$. Then (x_n) *weakly converges* to l, $x_n \stackrel{w}{\to} l$ as $n \to \infty$, iff $\forall y \in E. \langle x_n, y\rangle \to \langle l, y\rangle$ as $n \to \infty$.

Proposition 19.0.17. *In any inner product space E, the inner product $\langle\ ,\ \rangle : E^2 \to \mathbb{C}$ is continuous.*

PROOF:
⟨1⟩1. LET: $x_n \to x$ and $y_n \to y$ in E.
⟨1⟩2. $\langle x_n, y_n\rangle \to \langle x,y\rangle$
PROOF:

$$\begin{aligned}
|\langle x_n,y_n\rangle - \langle x,y\rangle| &\leq |\langle x_n,y_n\rangle - \langle x_n,y\rangle| + |\langle x_n,y\rangle - \langle x,y\rangle| \\
&= |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\
&\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| && \text{(Schwarz's Inequality)} \\
&\to 0
\end{aligned}$$

using the fact that (x_n) is bounded.

∎

Theorem 19.0.18. *$x_n \to l$ if and only if $x_n \stackrel{w}{\to} l$ and $\|x_n\| \to \|x\|$.*

PROOF:
⟨1⟩1. If $x_n \to l$ then $x_n \stackrel{w}{\to} l$ and $\|x_n\| \to \|l\|$.
PROOF: Easy using the fact that the inner product is continuous.
⟨1⟩2. If $x_n \stackrel{w}{\to} l$ and $\|x_n\| \to \|l\|$ then $x_n \to l$.
⟨2⟩1. ASSUME: $x_n \stackrel{w}{\to} l$ and $\|x_n\| \to \|l\|$
⟨2⟩2. $\langle x_n, l\rangle \to \|l\|^2$
⟨2⟩3. $\|x_n - l\| \to 0$
PROOF:

$$\begin{aligned} \|x_n - l\|^2 &= \langle x_n - l, x_n - l\rangle \\ &= \langle x_n, x_n\rangle - \langle x_n, l\rangle - \langle l, x_n\rangle + \langle l, l\rangle \\ &= \|x_n\|^2 - \langle x_n, l\rangle - \overline{\langle x_n, l\rangle} + \|l\|^2 \\ &\to \|l\|^2 - 2\|l\|^2 + \|l\|^2 \\ &= 0 \end{aligned}$$

∎

Theorem 19.0.19. *Let S be a subset of an inner product space E such that $\operatorname{span} S$ is dense in E. If (x_n) is a bounded sequence in E and, for all $y \in S$, we have $\langle x_n, y\rangle \to \langle x, y\rangle$ then $x_n \stackrel{w}{\to} x$.*

PROOF:
⟨1⟩1. For all $y \in \operatorname{span} S$, we have $\langle x_n, y\rangle \to \langle x, y\rangle$
⟨1⟩2. LET: $z \in E$
PROVE: $\langle x_n, z\rangle \to \langle x, z\rangle$
⟨1⟩3. LET: $\epsilon > 0$
PROVE: There exists n_0 such that $\forall n \geq n_0 . |\langle x_n, z\rangle - \langle x, z\rangle| < \epsilon$
⟨1⟩4. PICK $M > 0$ such that $\|x\| \leq M$ and $\forall n \in \mathbb{Z}_+ . \|x_n\| \leq M$.
⟨1⟩5. PICK $y_0 \in \operatorname{span} S$ such that $\|z - y_0\| < \epsilon/3M$
⟨1⟩6. PICK $n_0 \in \mathbb{Z}_+$ such that, for all $n \geq n_0$, we have $|\langle x_n, y_0\rangle - \langle x, y_0\rangle| < \epsilon/3$
⟨1⟩7. LET: $n \geq n_0$
⟨1⟩8. $|\langle x_n, z\rangle - \langle x, z\rangle| < \epsilon$
PROOF:

$$\begin{aligned} |\langle x_n, z\rangle - \langle x, z\rangle| &\leq |\langle x_n, z\rangle - \langle x_n, y_0\rangle| + |\langle x_n, y_0\rangle - \langle x, y_0\rangle| + |\langle x, y_0\rangle - \langle x, z\rangle| \\ &< \|x_n\|\|z - y_0\| + \epsilon/3 + \|x\|\|y_0 - z\| \\ &< M(\epsilon/3M) + \epsilon/3 + M(\epsilon/3M) \\ &= \epsilon \end{aligned}$$

∎

19.1 Orthonormal Bases

Definition 19.1.1 (Orthogonal)**.** Let V be an inner product space and $S \subseteq V$. Then S is *orthogonal* iff any two distinct elements of S are orthogonal.

Definition 19.1.2 (Orthonormal)**.** Let V be an inner product space and $S \subseteq V$. Then S is *orthonormal* iff it is orthogonal and $\forall x \in S. \|x\| = 1$.

Proposition 19.1.3. *Orthonormal sets are linearly independent.*

PROOF:
$\langle 1 \rangle$1. LET: S be orthonormal
$\langle 1 \rangle$2. ASSUME: $\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$ where $e_1, \ldots, e_n \in S$
$\langle 1 \rangle$3. $|\alpha_1|^2 + \cdots |\alpha_n|^2 = 0$
PROOF:

$$\begin{aligned} 0 &= \sum_{m=1}^{n} \langle 0, \alpha_m e_m \rangle \\ &= \sum_{m=1}^{n} \langle \sum_{k=1}^{n} \alpha_k e_k, \alpha_m e_m \rangle \\ &= \sum_{m=1}^{n} \sum_{k=1}^{n} \alpha_k \overline{\alpha_m} \langle e_k, e_m \rangle \\ &= \sum_{k=1}^{n} |\alpha_k|^2 \end{aligned}$$

$\langle 1 \rangle$4. $\alpha_1 = \cdots = \alpha_n = 0$
$\square$

Proposition 19.1.4. *In l^2, let e_n be the sequence whose nth element is 1 and whose other elements are 0. Then $\{e_n \mid n \in \mathbb{Z}_+\}$ is orthonormal.*

Proposition 19.1.5. *In $L^2([-\pi, \pi])$, let $\phi_n(x) = e^{inx}/\sqrt{2\pi}$ for $n \in \mathbb{Z}$. Then $\{\phi_n \mid n \in \mathbb{Z}\}$ is orthonormal.*

Definition 19.1.6 (Legendre Polynomials)**.** The *Legendre polynomials* $P_n \in \mathbb{Q}[x]$ for $n \in \mathbb{N}$ are defined by

$$\begin{aligned} P_0 &= 1 \\ P_n &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \end{aligned}$$

Proposition 19.1.7. *Let P_n be the nth Legendre polynomial. Then $\{P_n \mid n \in \mathbb{N}\}$ is orthogonal in $L^2([-1, 1])$.*

Definition 19.1.8 (Hermite Polynomial)**.** The *Hermite polynomials* $H_n \in \mathbb{R}[x]$ for $n \in \mathbb{N}$ are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad .$$

Proposition 19.1.9. *Let H_n be the nth Hermite polynomial. Then $\{e^{-x^2/2} H_n(x) \mid n \in \mathbb{N}\}$ is orthogonal in $L^2(\mathbb{R})$.*

Theorem 19.1.10. *Let V be an inner product space. If $x_1, \dots, x_n \in V$ are orthogonal then*

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2 \ .$$

Theorem 19.1.11 (Bessel's Equality)**.** *Let V be an inner product space. Let $x_1, \dots, x_n \in V$ be orthonormal. Let $x \in V$. Then*

$$\left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \ .$$

Proof:

$$\begin{aligned}
\left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 &= \left\langle x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k, x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\rangle \\
&= \langle x, x \rangle - \left\langle x, \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\rangle - \left\langle \sum_{k=1}^{n} \langle x, x_k \rangle x_k, x \right\rangle \\
&+ \left\langle \sum_{k=1}^{n} \langle x, x_k \rangle x_k, \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\rangle \\
&= \langle x, x \rangle - 2 \sum_{k=1}^{n} \langle x, x_k \rangle \langle x_k, x \rangle + \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, x_i \rangle \langle x_j, x \rangle \langle x_i, x_j \rangle \\
&= \|x\|^2 - 2 \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 + \sum_{i=1}^{n} \langle x, x_i \rangle \langle x_i, x \rangle \\
&= \|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2
\end{aligned}$$

□

Corollary 19.1.11.1 (Bessel's Inequality)**.** *Let V be an inner product space. Let $x_1, \dots, x_n \in V$ be orthonormal. Let $x \in E$. Then*

$$\sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \le \|x\|^2 \ .$$

Corollary 19.1.11.2. *Orthonormal sequences are weakly convergent to 0.*

Proof: Let (x_n) be an orthonormal sequence. Taking the limit in Bessel's inequality we have $\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \le \|x\|^2 < \infty$ and so $\langle x, x_k \rangle \to 0$ as $k \to \infty$. □

Corollary 19.1.11.3 (Generalized Fourier Series)**.** *Let V be an inner product space. Let (e_n) be an orthonormal sequence in V. For any $x \in V$, the* generalized Fourier series *of x is*

$$\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \ ,$$

and $\langle x, e_n \rangle$ *is called the* n*th* generalized Fourier coefficient *of* x *with respect to* (e_n). *We have* $(\langle x, e_n \rangle e_n)_n \in l^2$.

Definition 19.1.12 (Complete Orthonormal Sequence)**.** Let E be an inner product space. Let (x_n) be an orthonormal sequence in E. Then (x_n) is *complete* iff, for all $x \in E$, we have

$$\sum_{n=1}^{\infty} \langle x, x_n \rangle x_n = x \quad .$$

Chapter 20

Hilbert Spaces

Definition 20.0.1 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Proposition 20.0.2. *For $n \in \mathbb{N}$, $\mathbb{C}^n$ is a Hilbert space.*

Proposition 20.0.3. *l^2 is a Hilbert space.*

Proposition 20.0.4. *$L^2(\mathbb{R})$ is a Hilbert space.*

Proposition 20.0.5. *$L^2([a,b])$ is a Hilbert space.*

Proposition 20.0.6. *Let ρ be a measurable function on $[a,b]$ such that $\rho(x) > 0$ almost everywhere. Let $L^{2\rho}([a,b])$ be the set of all measurable functions $f : [a,b] \to \mathbb{C}$ such that*

$$\int_a^b |f(x)|^2 \rho(x) dx < \infty \ .$$

Define an inner product on $L^{2\rho}([a,b])$ by

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\rho(x) dx \ .$$

Then $L^{2\rho}([a,b])$ is a Hilbert space.

Proposition 20.0.7. *Let m and N be positive integers. Let Ω be an open set in $\mathbb{R}^N$. Let $\tilde{H}^m(\Omega)$ be the set of all $f \in \mathcal{C}^m(\Omega)$ such that, for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$ with $|\alpha| := \alpha_1 + \cdots + \alpha_N \leq m$, we have*

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} \in L^2(\Omega) \ .$$

Define an inner product on $\tilde{H}^m(\Omega)$ by

$$\langle f, g \rangle := \int_\Omega \sum_\alpha D^\alpha f \overline{D^\alpha g} \ .$$

Let $H^m(\Omega)$ be the completion of $\tilde{H}^m(\Omega)$. Then $H^m(\Omega)$ is a Hilbert space.

Theorem 20.0.8. *Weakly convergent sequences in a Hilbert space are bounded.*

Proof:
⟨1⟩1. Let: H be a Hilbert space.
⟨1⟩2. Let: (x_n) be a weakly convergent sequence in H.
⟨1⟩3. For $n \in \mathbb{Z}_+$,
Let: $f_n : H \to \mathbb{C}$, $f_n(x) = \langle x, x_n \rangle$
⟨1⟩4. For $n \in \mathbb{Z}_+$, f_n is a bounded linear functional.
⟨1⟩5. For every $x \in H$, the sequence $(f_n(x))$ is bounded.
Proof: Since it converges.
⟨1⟩6. Pick $M > 0$ such that, for all $n \in \mathbb{Z}_+$, we have $\|f_n\| \leq M$.
Proof: Banach-Steinhaus Theorem, ⟨1⟩4, ⟨1⟩5.
⟨1⟩7. $\forall n \in \mathbb{Z}_+ . \|f_n\| = \|x_n\|$
⟨2⟩1. Let: $n \in \mathbb{Z}_+$
⟨2⟩2. $\|f_n\| \leq \|x_n\|$
Proof: Since for all $x \in H$ we have $|f_n(x)| = |\langle x, x_n \rangle| \leq \|x\| \|x_n\|$ by Schwarz's Inequality.
⟨2⟩3. $\|x_n\| \leq \|f_n\|$
Proof: Since $\|x_n\|^2 = |\langle x_n, x_n \rangle| = |f_n(x_n)| \leq \|f_n\| \|x_n\|$.
⟨1⟩8. $\forall n \in \mathbb{Z}_+ . \|x_n\| \leq M$
Proof: ⟨1⟩6, ⟨1⟩7
□

Theorem 20.0.9. *Let H be a Hilbert space. Let (x_n) be an orthonormal sequence in H and let (α_n) be a sequence of complex numbers. Then the series $\sum_{n=1}^\infty \alpha_n x_n$ converges in H if and only if $\sum_{n=1}^\infty |\alpha_n|$ converges in $\mathbb{R}$, in which case*

$$\left\| \sum_{n=1}^\infty \alpha_n x_n \right\|^2 = \sum_{n=1}^\infty |\alpha_n|^2 \ .$$

Proof:
⟨1⟩1. For $m > k > 0$ we have

$$\left\| \sum_{n=k}^m \alpha_n x_n \right\|^2 = \sum_{n=k}^m |\alpha_n|^2 \ .$$

Proof: Theorem 19.1.10.
⟨1⟩2. If $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$ then $\sum_{n=1}^\infty \alpha_n x_n$ converges.
⟨2⟩1. Assume: $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$
⟨2⟩2. $\left(\sum_{n=1}^m \alpha_n x_n\right)_m$ is Cauchy.
Proof: From ⟨1⟩1.
⟨2⟩3. $\sum_{n=1}^\infty \alpha_n x_n$ converges.
⟨1⟩3. If $\sum_{n=1}^\infty \alpha_n x_n$ converges then $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$.
Proof: From ⟨1⟩1.
⟨1⟩4. If $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$ then

$$\left\| \sum_{n=1}^\infty \alpha_n x_n \right\|^2 = \sum_{n=1}^\infty |\alpha_n|^2 \ .$$

PROOF: From $\langle 1\rangle 1$.
∎

Proposition 20.0.10. *Every complete orthonormal sequence in a Hilbert space is a basis.*

PROOF:
$\langle 1\rangle 1$. LET: E be an inner product space.
$\langle 1\rangle 2$. LET: (e_n) be a complete orthonormal sequence in E.
$\langle 1\rangle 3$. For all $x \in E$, there exists a sequence (α_n) in $\mathbb{C}$ such that $x = \sum_n \alpha_n e_n$.
PROOF: Immediate from $\langle 1\rangle 2$.
$\langle 1\rangle 4$. If $\sum_n \alpha_n e_n = \sum_n \beta_n e_n$ then $\alpha_n = \beta_n$ for all n.
$\langle 2\rangle 1$. LET: $x = \sum_n \alpha_n e_n = \sum_n \beta_n e_n$
$\langle 2\rangle 2$. $\sum_n |\alpha_n - \beta_n|^2 = 0$
PROOF:

$$\begin{aligned}
0 &= \|x - x\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n e_n - \sum_{n=1}^{\infty} \beta_n e_n \right\|^2 && (\langle 2\rangle 1) \\
&= \left\| \sum_{n=1}^{\infty} (\alpha_n - \beta_n) e_n \right\|^2 \\
&= \sum_{n=1}^{\infty} |\alpha_n - \beta_n|^2 && \text{(Theorem 20.0.9)}
\end{aligned}$$

$\langle 2\rangle 3$. $\alpha_n = \beta_n$ for all n.
∎

Theorem 20.0.11. *An orthonormal sequence (x_n) in a Hilbert space H is complete if and only if, for all $x \in H$, if $\forall n. \langle x, x_n\rangle = 0$ then $x = 0$.*

PROOF:
$\langle 1\rangle 1$. If (x_n) is complete then, for all $x \in H$, if $\forall n.\langle x, x_n\rangle = 0$ then $x = 0$.
$\langle 2\rangle 1$. ASSUME: (x_n) is complete.
$\langle 2\rangle 2$. LET: $x \in H$
$\langle 2\rangle 3$. ASSUME: $\forall n.\langle x, x_n\rangle = 0$
$\langle 2\rangle 4$. $x = \sum_{n=1}^{\infty} \langle x, x_n\rangle x_n = 0$
$\langle 1\rangle 2$. If, for all $x \in H$, if $\forall n.\langle x, x_n\rangle = 0$ then $x = 0$, then (x_n) is complete.
$\langle 2\rangle 1$. ASSUME: For all $x \in H$, if $\forall n.\langle x, x_n\rangle = 0$, then $x = 0$.
$\langle 2\rangle 2$. LET: $y = x - \sum_{n=1}^{\infty} \langle x, x_n\rangle x_n$
$\langle 2\rangle 3$. For all n, $\langle y, x_n\rangle = 0$
$\langle 3\rangle 1$. LET: $n \in \mathbb{Z}_+$
$\langle 3\rangle 2$. $\langle y, x_n\rangle = 0$

PROOF:

$$
\begin{aligned}
\langle y, x_n \rangle &= \left\langle x - \sum_{m=1}^{\infty} \langle x, x_m \rangle x_m, x_n \right\rangle \\
&= \langle x, x_n \rangle - \sum_{m=1}^{\infty} \langle x, x_m \rangle \langle x_m, x_n \rangle \\
&= \langle x, x_n \rangle - \langle x, x_n \rangle \\
&= 0
\end{aligned}
$$

⟨2⟩4. $y = 0$
⟨2⟩5. $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$
□

Theorem 20.0.12 (Parseval's Formula)**.** *Let H be a Hilbert space. Let (x_n) be an orthonormal sequence in H. Then (x_n) is complete if and only if, for all $x \in H$,*

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \quad .$$

PROOF:
⟨1⟩1. If (x_n) is complete then for all $x \in H$ we have $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$.
⟨2⟩1. ASSUME: (x_n) is complete.
⟨2⟩2. LET: $x \in H$
⟨2⟩3. $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$
PROOF:

$$
\begin{aligned}
\|x\|^2 &= \left\| \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n \right\|^2 && (\langle 2 \rangle 1) \\
&= \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 && (\text{Theorem 20.0.9})
\end{aligned}
$$

⟨1⟩2. If, for all $x \in H$, we have $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$, then (x_n) is complete.
⟨2⟩1. ASSUME: For all $x \in H$, we have $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$
⟨2⟩2. LET: $x \in H$
⟨2⟩3. $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$
□

Proposition 20.0.13. *For $n \in \mathbb{Z}$, let $\pi_n(x) = e^{inx}/\sqrt{2\pi}$. Then $\{\pi_n \mid n \in \mathbb{Z}\}$ is a complete orthonormal set in $L^2([-\pi, \pi])$.*

TODO

Proposition 20.0.14. $B = \{1/\sqrt{2\pi}\} \cup \{\cos nx/\sqrt{\pi} \mid n \in \mathbb{Z}_+\} \cup \{\sin nx/\sqrt{\pi} \mid n \in \mathbb{Z}_+\}$ *is a complete orthonormal set in $L^2([-\pi, \pi])$.*

PROOF:
⟨1⟩1. For all $f \in B$ we have $\|f\| = 1$
⟨2⟩1. $\|1/\sqrt{2\pi}\| = 1$

Proof:

$$\begin{aligned}\|1/\sqrt{2\pi}\| &= \int_{-\pi}^{\pi} dx/2\pi \\ &= 1\end{aligned}$$

⟨2⟩2. For all $n \in \mathbb{Z}_+$ we have $\|\cos nx/\sqrt{\pi}\| = 1$

Proof:

$$\begin{aligned}\|\cos nx/\sqrt{\pi}\| &= 1/\pi \int_{-\pi}^{\pi} \cos^2 nx\,dx \\ &= 1/2\pi \int_{-\pi}^{\pi} (\cos 2nx + 1)dx \\ &= 1/2\pi \left[1/2n \sin 2nx + x\right]_{-\pi}^{\pi} \\ &= (1/2\pi)(2\pi) \\ &= 1\end{aligned}$$

⟨2⟩3. For all $n \in \mathbb{Z}_+$ we have $\|\sin nx/\sqrt{\pi}\| = 1$

Proof:

$$\begin{aligned}\|\sin nx/\sqrt{\pi}\| &= 1/\pi \int_{-\pi}^{\pi} \sin^2 nx\,dx \\ &= -1/2\pi \int_{-\pi}^{\pi} (\cos 2nx - 1)dx \\ &= -1/2\pi \left[1/2n \sin 2nx - x\right]_{-\pi}^{\pi} \\ &= (-1/2\pi)(-2\pi) \\ &= 1\end{aligned}$$

⟨1⟩2. For all $f, g \in B$ with $f \neq g$ we have $\langle f, g\rangle = 0$

⟨2⟩1. $\langle 1, \cos nx\rangle = 0$

Proof:

$$\begin{aligned}\int_{-\pi}^{\pi} \cos nx\,dx &= [1/n \sin nx]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

⟨2⟩2. $\langle 1, \sin nx\rangle = 0$

Proof:

$$\begin{aligned}\int_{-\pi}^{\pi} \sin nx\,dx &= [-1/n \cos nx]_{-\pi}^{\pi} \\ &= -1/n \cos n\pi + 1/n \cos n\pi \\ &= 0\end{aligned}$$

⟨2⟩3. If $m \neq n$ then $\langle \cos mx, \cos nx\rangle = 0$

Proof:

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx \cos nx\,dx &= 1/2 \int_{-\pi}^{\pi} (\cos(n+m)x - \cos(n-m)x)dx \\ &= 1/2 \left[\frac{1}{n+m} \sin(n+m)x - \frac{1}{n-m} \sin(n-m)x\right]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

⟨2⟩4. $\langle \cos mx, \sin nx\rangle = 0$

PROOF:

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx \sin nx \, dx &= 1/2 \int_{-\pi}^{\pi} (\sin(n+m)x - \sin(n-m)x) dx \\ &= 1/2 \left[-\frac{1}{n+m} \cos(n+m)x + \frac{1}{n-m} \cos(n-m)x \right]_{-\pi}^{\pi} \\ &= 0 \qquad \text{(cos is odd)}\end{aligned}$$

⟨2⟩5. If $m \neq n$ then $\langle \sin mx, \sin nx \rangle = 0$

PROOF:

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= 1/2 \int_{-\pi}^{\pi} (\cos(n-m)x - \cos(n+m)x) dx \\ &= 1/2 \left[\frac{1}{n-m} \sin(n-m)x - \frac{1}{n+m} \sin(n+m)x \right]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

⟨1⟩3. For all $f \in L^2([-\pi, \pi])$, if $\forall g \in B. \langle f, g \rangle = 0$ then $f = 0$

⟨2⟩1. LET: $f \in L^2([-\pi, \pi])$

⟨2⟩2. ASSUME: $\forall g \in B. \langle f, g \rangle = 0$

⟨2⟩3. For all $n \in \mathbb{Z}$, $\langle f, e^{inx} \rangle = 0$

PROOF: Since $e^{inx} = \cos nx + i \sin nx$.

⟨2⟩4. $f = 0$

PROOF: From Proposition 20.0.13.

□

Proposition 20.0.15. $\{\frac{1}{\sqrt{\pi}}\} \cup \{\sqrt{\frac{2}{\pi}} \cos nx \mid n \in \mathbb{Z}_+\}$ *is a complete orthonormal set in* $L^2([0, \pi])$.

Proposition 20.0.16. $\{\sqrt{\frac{2}{\pi}} \sin nx \mid n \in \mathbb{Z}_+\}$ *is a complete orthonormal set in* $L^2([0, \pi])$.

Definition 20.0.17 (Signum)**.** The *signum* function $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathrm{sgn}\, x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Definition 20.0.18 (Rademacher Functions)**.** The *Rademarcher functions* $R : \mathbb{N} \times [0, 1] \to \{-1, 0, 1\}$ are defined by

$$R(m, x) = \mathrm{sgn}(\sin(2^m \pi x)) \ .$$

Proposition 20.0.19. *The Rademacher functios* $\{R(m, -) \mid m \in \mathbb{N}\}$ *are orthonormal in* $L^2([0, 1])$.

PROOF:

⟨1⟩1. $\forall m \in \mathbb{N}. \|R(m, -)\| = 1$

PROOF: $\int_0^1 \mathrm{sgn}(\sin(2^m \pi x))^2 dx = 1$ since the integrand is 1 except for finitely many points in $[0, 1]$.

⟨1⟩2. Given natural numbers $m \neq n$, we have $\langle R(m,-), R(n,-)\rangle = 0$

⟨2⟩1. Given reals a, b and a natural number m, we have $\int_a^b R(m,x)dx = 0$ whenever $2^m(b-a)$ is an even integer.

PROOF: If $m > 0$, or if $m = 0$ and $b - a$ is an even integer, then the regions where $R(m,x) = 1$ are isometric with the regions where $R(m,x) = -1$.

⟨2⟩2. LET: m and n be natural numbers with $n < m$.

⟨2⟩3. $\langle R(m,-), R(n,-)\rangle = 0$

PROOF:

$$\begin{aligned}\int_0^1 R(m,x)R(n,x)dx &= \sum_{k=1}^{2^n} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} R(m,x)R(n,x)dx \\ &= \sum_{k=1}^{2^n} (-i)^{k+1} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} R(m,x)dx \\ &= 0 && (\langle 2\rangle 1, 2^m\left(\frac{k}{2^n} - \frac{k-1}{2^n}\right) = 2^{m-n} \text{ is an even integer})\end{aligned}$$

∎

Proposition 20.0.20. *The set of Rademacher functions is not complete.*

PROOF:

⟨1⟩1. Define $f : [0,1] \to \mathbb{C}$ by $f(x) = 0$ if $0 \leq x < 1/4$, $f(x) = 1$ if $1/4 \leq x \leq 3/4$, $f(x) = 0$ if $3/4 < x \leq 1$.

⟨1⟩2. $f \in L^2([0,1])$

⟨1⟩3. $\langle R(0,-), f\rangle = 1/2$

⟨1⟩4. $\langle R(m,-), f\rangle = 0$ for $m \geq 1$

⟨1⟩5. $f \neq 1/2R(0,-)$

∎

Definition 20.0.21 (Walsh Functions)**.** Define the *Walsh functions* $W : \mathbb{N} \times [0,1] \to \{-1,0,1\}$ as follows. Given $m \in \mathbb{N}$, let $m = \sum_{k=1}^n 2^{k-1}a_k$ where each a_k is either 0 or 1. Then

$$W(m,x) = \prod_{k=1}^n R(k,x)^{a_k} \quad .$$

Proposition 20.0.22. *The set of Walsh functions* $\{W(m,-) \mid m \in \mathbb{N}\}$ *is a compete orthonormal set.*

TODO