

# Summary of Halmos' Naive Set Theory

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# Chapter 1

## Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*,  $\in$ . If  $x \in A$  we say that  $x$  *belongs to*  $A$ ,  $x$  is an *element* of  $A$ , or  $x$  is *contained in*  $A$ . If this does not hold we write  $x \notin A$ .

**Axiom 1.1** (Axiom of Extensionality). *Two sets are equal if and only if they have the same elements.*

**Axiom 1.2** (Axiom of Comprehension, Aussonderungsaxiom). *To every set  $A$  and to every condition  $S(x)$  there corresponds a set  $B$  whose elements are exactly those elements  $x$  of  $A$  for which  $S(x)$  holds.*

**Axiom 1.3.** *A set exists.*

**Axiom 1.4** (Axiom of Pairing). *For any two sets, there exists a set that they both belong to.*

**Axiom 1.5** (Union Axiom). *For every set  $A$ , there exists a set that contains all the elements that belong to at least one element of  $A$ .*

## Chapter 2

# The Subset Relation

**Definition 2.1** (Subset). Let  $A$  and  $B$  be sets. We say that  $A$  is a *subset* of  $B$ , or  $B$  *includes*  $A$ , and write  $A \subseteq B$  or  $B \supseteq A$ , iff every element of  $A$  is an element of  $B$ .

**Theorem 2.2.** *For any set  $A$ , we have  $A \subseteq A$ .*

PROOF: Every element of  $A$  is an element of  $A$ .  $\square$

**Theorem 2.3.** *For any sets  $A$ ,  $B$  and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ .*

PROOF: If every element of  $A$  is an element of  $B$ , and every element of  $B$  is an element of  $C$ , then every element of  $A$  is an element of  $C$ .  $\square$

**Theorem 2.4.** *For any sets  $A$  and  $B$ , if  $A \subseteq B$  and  $B \subseteq A$  then  $A = B$ .*

PROOF: If every element of  $A$  is an element of  $B$ , and every element of  $B$  is an element of  $A$ , then  $A$  and  $B$  have the same elements, and therefore are equal by the Axiom of Extensionality.  $\square$

**Definition 2.5** (Proper Subset). Let  $A$  and  $B$  be sets. We say that  $A$  is a *proper subset* of  $B$ , or  $B$  *properly includes*  $A$ , and write  $A \subsetneq B$  or  $B \supsetneq A$ , iff  $A \subseteq B$  and  $A \neq B$ .

## Chapter 3

# Comprehension Notation

**Definition 3.1.** Given a set  $A$  and a condition  $S(x)$ , we write  $\{x \in A : S(x)\}$  for the set whose elements are exactly those elements  $x$  of  $A$  for which  $S(x)$  holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality.  $\square$

**Theorem 3.2.** *There is no set that contains every set.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $A$  be a set.

PROVE: There exists a set  $B$  such that  $B \notin A$ .

$\langle 1 \rangle 2.$  LET:  $B = \{x \in A : x \notin x\}$

$\langle 1 \rangle 3.$  If  $B \in A$  then we have  $B \in B$  if and only if  $B \notin B$ .

$\langle 1 \rangle 4.$   $B \notin A$

$\square$

## Chapter 4

# Unordered Pairs

**Theorem 4.1.** *There exists a set with no elements.*

PROOF: Pick a set  $A$  by Axiom 1.3. Then the set  $\{x \in A : x \neq x\}$  has no elements.  $\square$

**Definition 4.2** (Empty Set). The *empty set*  $\emptyset$  is the set with no elements.

**Theorem 4.3.** *For any set  $A$  we have  $\emptyset \subset A$ .*

PROOF: Vacuous.  $\square$

**Definition 4.4** ((Unordered) Pair). For any sets  $a$  and  $b$ , the *(unordered) pair*  $\{a, b\}$  is the set whose elements are just  $a$  and  $b$ .

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality.  $\square$

**Definition 4.5** (Singleton). For any set  $a$ , the *singleton*  $\{a\}$  is defined to be  $\{a, a\}$ .

## Chapter 5

# Unions and Intersections

**Definition 5.1** (Union). For any set  $\mathcal{C}$ , the *union* of  $\mathcal{C}$ ,  $\bigcup \mathcal{C}$ , is the set whose elements are the elements of the elements of  $\mathcal{C}$ .

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality.  $\square$

**Proposition 5.2.**

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of  $\emptyset$ .  $\square$

**Proposition 5.3.** For any set  $A$ , we have  $\bigcup \{A\} = A$ .

PROOF: For any  $x$ , we have  $x$  is an element of an element of  $\{A\}$  if and only if  $x$  is an element of  $A$ .  $\square$

**Definition 5.4.** We write  $A \cup B$  for  $\bigcup \{A, B\}$ .

**Proposition 5.5.** For any set  $A$ , we have  $A \cup \emptyset = A$ .

PROOF:  $x \in A \cup \emptyset$  iff  $x \in A$  or  $x \in \emptyset$ , iff  $x \in A$ .  $\square$

**Proposition 5.6** (Commutativity). For any sets  $A$  and  $B$ , we have  $A \cup B = B \cup A$ .

PROOF:  $x \in A \cup B$  iff  $x \in A$  or  $x \in B$ , iff  $x \in B$  or  $x \in A$ , iff  $x \in B \cup A$ .  $\square$

**Proposition 5.7** (Associativity). For any sets  $A$ ,  $B$  and  $C$ , we have  $A \cup (B \cup C) = (A \cup B) \cup C$ .

PROOF: Each is the set of all  $x$  such that  $x \in A$  or  $x \in B$  or  $x \in C$ .  $\square$

**Proposition 5.8** (Idempotence). For any set  $A$ , we have  $A \cup A = A$ .

PROOF:  $x \in A$  or  $x \in A$  is equivalent to  $x \in A$ .  $\square$

**Proposition 5.9.** *For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A \cup B = B$ .*

PROOF: For any  $x$ , the statement "if  $x \in A$  then  $x \in B$ " is equivalent to " $x \in A$  or  $x \in B$  if and only if  $x \in B$ ".  $\square$

**Proposition 5.10.** *For any sets  $a$  and  $b$ , we have  $\{a\} \cup \{b\} = \{a, b\}$ .*

PROOF: Immediate from definitions.  $\square$

**Definition 5.11.** Given sets  $a$ ,  $b$  and  $c$ , let

$$\{a, b, c\} := \{a\} \cup \{b\} \cup \{c\} .$$