Encyclopaedia of Mathematics and Physics

Robin Adams

Contents

Part I

Set Theory

Chapter 1

Foundations

1.1 The Theory of Semicategories

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B, and call A the *domain* of f and B the *codomain*.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $g \circ f : A \to C$, the *composite* of g and f.

Axiom 1.1 (Associativity)**.** *Given functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \ .$$

1.1.1 Identity Functions

Definition 1.2 (Identity Function)**.** Let A be a set. An *identity function* on A is a function $i : A \to A$ such that:

- For any set B and function $f : B \to A$, we have $i \circ f = f$.

- For any set B and function $f : A \to B$, we have $f \circ i = f$.

Proposition 1.3. *A set has at most one identity function.*

PROOF:
$\langle 1 \rangle 1$. LET: A be a set.
$\langle 1 \rangle 2$. LET: $i, j : A \to A$ be identity functions.
$\langle 1 \rangle 3$. $i = j$
PROOF: If i and j both satisfy the conditions then $i = i \circ j = j$.
$\square$

1.1.2 Monomorphisms and Epimorphisms

Definition 1.4 (Monomorphism)**.** We say a function $f : A \to B$ is a *monomorphism*, and write $f : A \rightarrowtail B$, iff, for any set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

Definition 1.5 (Epimorphism)**.** We say a function $f : A \to B$ is a *epimorphism*, and write $f : A \twoheadrightarrow B$, iff, for any set X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

1.2 The Theory of Categories

1.2.1 Minimalist Presentation

Axiom 1.6 (Identity Functions)**.** *Every set has an identity function.*

1.2.2 Practical Presentation

For any set A, let there be a function $\mathrm{id}_A : A \to A$.

Axiom 1.7 (Left Unit Law)**.** *For any function $f : A \to B$, we have $\mathrm{id}_B \circ f = f$.*

Axiom 1.8 (Right Unit Law)**.** *For any function $f : A \to B$, we have $f \circ \mathrm{id}_A = f$.*

1.2.3 Sections and Retractions

Definition 1.9 (Section, Retraction)**.** Let $r : A \to B$ and $s : B \to A$. We say that r is a *retraction* of s, and s is a *section* of r.

1.2.4 Bijections

Definition 1.10 (Bijection)**.** We say a function $f : A \to B$ is *bijective* or a *bijection*, and write $f : A \approx B$, iff there exists a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

1.2.5 Terminal Set

Definition 1.11 (Terminal Set)**.** A set T is *terminal* iff, for any set X, there is exactly one function $X \to T$.

Axiom 1.12 (Terminal Set)**.** *There exists a terminal set.*

Proposition 1.13. *For any terminal sets T and T', there is a unique bijection $T \approx T'$.*

Proof:
$\langle 1 \rangle 1.$ Let: i be the unique function $T \to T'$.
$\langle 1 \rangle 2.$ Let: j be the unique function $T' \to T$.
$\langle 1 \rangle 3.$ $i \circ j = \mathrm{id}_{T'}$

PROOF: Since there is only one function $T' \to T'$.

$\langle 1 \rangle 4.$ $j \circ i = \mathrm{id}_T$

PROOF: Since there is only one function $T \to T$.

□

Definition 1.14 (Terminal Set)**.** We denote the terminal set by 1.

Definition 1.15 (Element)**.** An *element* of a set A is a function $1 \to A$. We write $a \in A$ for $a : 1 \to A$. Given $a \in A$ and $f : A \to B$, we write $f(a)$ for $f \circ a$.

Axiom 1.16 (Extensionality)**.** *Let $f, g : A \to B$. Assume that, for all $a \in A$, if $f(a) = g(a)$ then $f = g$.*

Definition 1.17 (Injective)**.** We say a function $f : A \to B$ is *injective* or an *injection*, and we write $f : A \rightarrowtail B$, iff, for any $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

Definition 1.18 (Surjective)**.** We say a function $f : A \to B$ is *surjective* or a *surjection*, and we write $f : A \twoheadrightarrow B$, iff, for any $y \in B$, there exists $x \in A$ such that $f(x) = y$.

1.2.6 Empty Set

Axiom 1.19 (Empty Set)**.** *There exists a set with no elements.*

1.2.7 Products

Definition 1.20 (Product)**.** Let A, B and P be sets, and $\pi_1 : P \to A$, $\pi_2 : P \to B$. Then we say that (P, π_1, π_2) is a *product* of A and B iff, for any set X and functions $f : X \to A$ and $g : X \to B$, there exists a unique function $h : X \to A \times B$ such that

$$\pi_1 \circ h = f, \qquad \pi_2 \circ h = g \ .$$

Axiom 1.21 (Products)**.** *Any two sets have a product.*

Proposition 1.22. *If (P, p_1, p_2) and (Q, q_1, q_2) are products of A and B, then there exists a unique bijection $\phi : P \approx Q$ such that $q_1 \circ \phi = p_1$ and $q_2 \circ \phi = p_2$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $\phi : P \to Q$ be the unique function such that $q_1 \circ \phi = p_1$ and $q_2 \circ \phi = p_2$.

$\langle 1 \rangle 2.$ LET: $\phi^{-1} : Q \to P$ be the unique function such that $p_1 \circ \phi = q_1$ and $p_2 \circ \phi = q_2$.

$\langle 1 \rangle 3.$ $\phi \circ \phi^{-1} = \mathrm{id}_Q$

PROOF: Each is the unique $x : Q \to Q$ such that $q_1 \circ x = q_1$ and $q_2 \circ x = q_2$.

$\langle 1 \rangle 4.$ $\phi^{-1} \circ \phi = \mathrm{id}_P$

PROOF: Each is the unique $x : P \to P$ such that $p_1 \circ x = p_1$ and $p_2 \circ x = p_2$.

□

Definition 1.23. Given sets A and B, we write $A \times B$ for the product of A and B, with projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$. Given functions $f : A \to B$ and $g : A \to C$, we write $\langle f, g \rangle$ for the unique function $A \to B \times C$ such that

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g \ .$$

Definition 1.24. Given $f : A \to B$ and $g : C \to D$, we define $f \times g : A \times C \to B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle \ .$$

1.2.8 Function Sets

Definition 1.25 (Function Set)**.** Let A, B and F be sets, and let $\epsilon : F \times A \to B$. Then we say that F and ϵ form the *function set* from A to B, with ϵ the *evaluation function*, iff, for any set X and function $f : X \times A \to B$, there exists a unique function $g : X \to F$ such that

$$\epsilon \circ (g \times \mathrm{id}_A) = f \ .$$

Axiom 1.26 (Function Sets)**.** *Any two sets have a function set.*

Proposition 1.27. *Let $(F, \epsilon : F \times A \to B)$ and $(G, e : G \times A \to B)$ be function sets from A to B. Then there exists a unique bijection $\phi : F \approx G$ such that $e \circ (\phi \times \mathrm{id}_A) = \epsilon$.*

Definition 1.28. Given sets A and B, we write B^A for the function set from A to B, and $\epsilon : B^A \times A \to B$ for the evaluation function. Given $f : X \times A \to B$, we write λf for the unique function $X \to B^A$ such that

$$\epsilon \circ (\lambda f \times \mathrm{id}_A) = f \ .$$

1.2.9 Inverse Images

Definition 1.29 (Inverse Image)**.** Let A, B and I be sets. Let $f : A \to B$, $b \in B$ and $i : I \to A$. Then we say that I and i form the *inverse image* of b under f iff:

- $f \circ i = b \circ !_I$
- For any set X and function $j : X \to A$, if $f \circ j = b \circ !_X$, then there exists a unique $\overline{j} : X \to I$ such that $i \circ \overline{j} = j$.

Axiom 1.30 (Inverse Images)**.** *Given any sets A and B, function $f : A \to B$, and element $b \in B$, there exists an inverse image of b under f.*

Proposition 1.31. *If $(I, i : I \to A)$ and $(J, j : J \to A)$ are inverse images of $b \in B$ under $f : A \to B$, then there exists a unique isomorphism $\phi : I \approx J$ such that $j \circ \phi = i$.*

Definition 1.32. Let $f : A \to B$ and $b \in B$. We write $f^{-1}(b)$ and $i_{f,b} : f^{-1}(b) \to A$ for the inverse image of b under f.

1.2.10 Subset Classifiers

Definition 1.33 (Subset Classifier)**.** Let Ω be a set and $\top \in \Omega$. Then we say $(\Omega, \top)$ form a *subset classifier* iff, for any sets A and X and injection $j : A \rightarrowtail X$, there exists a unique $\chi : X \to \Omega$ such that (A, j) is the inverse image of $\top$ under χ.

Axiom 1.34 (Subset Classifier)**.** *There exists a subset classifier.*

Proposition 1.35. *If $(\Omega, \top)$ and $(\Omega', \top')$ are subset classifiers, then there exists a unique bijection $\phi : \Omega \approx \Omega'$ such that $\phi(\top) = \top'$.*

Definition 1.36. We write 2 and $\top \in 2$ for the subset classifier.

1.2.11 Natural Number Sets

Definition 1.37 (Natural Number Set)**.** Let N be a set, $z \in N$ and $s : N \to N$. Then we say (N, z, s) is a *natural number set* iff, for any set X, element $a \in X$ and function $f : X \to X$, there exists a unique $r : N \to X$ such that

$$r(z) = a, \qquad f \circ r = r \circ s \ .$$

Axiom 1.38 (Infinity)**.** *There exists a natural number set.*

Proposition 1.39. *If (N, z, s) and (N', z', s') are natural number sets, then there exists a unique bijection $\phi : N \approx N'$ such that $\phi(z) = z'$ and $s' \circ \phi = \phi \circ s$.*

Definition 1.40. We write $\mathbb{N}$, $0 \in \mathbb{N}$ and $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ for the natural number set.

1.2.12 The Axiom of Choice

Definition 1.41 (Axiom of Choice)**.** Every surjection is a retraction.

Chapter 2

Set Theory

Proposition 2.1. *Every infinite subset of a countably infinite set is countable.*

PROOF:
$\langle 1 \rangle 1$. LET: $i : A \hookrightarrow \mathbb{N}$ be an infinite subset of $\mathbb{N}$.
$\langle 1 \rangle 2$. Define $j : \mathbb{N} \to A$ by: $j(k)$ is the element such that $i(j(k))$ is least such that $i(j(k)) \notin \{i(j(0)), \ldots, i(j(k-1))\}$.
$\langle 1 \rangle 3$. j is a bijection.
□

Proposition 2.2. *A countable union of countable sets is countable.*

PROOF:
$\langle 1 \rangle 1$. LET: (A_n) be a sequence of countable sets.
$\langle 1 \rangle 2$. For $n \in \mathbb{N}$, PICK an enumeration $(e_{nm})_m$ of A_n.
$\langle 1 \rangle 3$. LET: (p_k) be the following enumeration of $\mathbb{N} \times \mathbb{N}$:

$$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots$$

$\langle 1 \rangle 4$. $(e_{\pi_1(p_k)\pi_2(p_k)})_k$ is an enumeration of $\bigcup_n A_n$.
□

Theorem 2.3. $2^{\mathbb{N}}$ *is uncountable.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $f : \mathbb{N} \approx 2^{\mathbb{N}}$
$\langle 1 \rangle 2$. LET: $S = \{n \in \mathbb{N} : n \notin f(n)\}$
$\langle 1 \rangle 3$. For all n, we have $n \in S \Leftrightarrow n \notin f(n)$
$\langle 1 \rangle 4$. For all n we have $S \neq f(n)$.
$\langle 1 \rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 1$.
□

Chapter 3

Relations

Definition 3.1 (Antisymmetric)**.** A relation R on a set A is *antisymmetric* iff, whenever xRy and yRx, then $x = y$.

Definition 3.2 (Transitive)**.** A relation R on a type A is *transitive* iff, whenever xRy and yRz, then xRz.

Chapter 4

Order Theory

Definition 4.1 (Linear Order)**.** A *linear order* on a set A is a binary relation $\leq$ on A that is transitive, antisymmetric and:

$$\forall x, y \in A. x \leq y \vee y \leq x \ .$$

A *linearly ordered set* is a pair $(A, \leq)$ where A is a set and $\leq$ is a binary relation on A.

We write $x < y$ for $x \leq y$ and $x \neq y$.

Definition 4.2 (Upper Bound)**.** Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is an *upper bound* in E iff $\forall x \in E. x \leq u$. We say E is *bounded above* iff it has an upper bound.

The *up-set* of E, denoted $E \uparrow$, is the set of upper bounds of E.

Definition 4.3 (Lower Bound)**.** Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then u is an *lower bound* in E iff $\forall x \in E. l \leq x$. We say E is *bounded below* iff it has a lower bound.

The *down-set* of E, denoted $E \downarrow$, is the set of lower bounds of E.

Definition 4.4 (Supremum)**.** Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is the *least upper bound* or *supremum* of E iff u is an upper bound for E and, for any upper bound u' for E, we have $u \leq u'$.

Definition 4.5 (Infimum)**.** Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then l is the *greatest lower bound* or *infimum* of E iff l is a lower bound for E and, for any lower bound l' for E, we have $l' \leq l$.

Definition 4.6 (Least Upper Bound Property)**.** A linearly ordered set S has the *least upper bound property* iff every nonempty subset of S that is bounded above has a least upper bound.

Proposition 4.7. *Let S be a linearly ordered set and $E \subseteq S$.*

1. *If $E \downarrow$ has a supremum l, then l is the infimum of E.*

2. If $E\uparrow$ has an infimum u, then U is the supremum of E.

PROOF:
⟨1⟩1. If $E\downarrow$ has a supremum l, then l is the infimum of E.
⟨2⟩1. l is a lower bound for E.
⟨3⟩1. LET: $x \in E$
⟨3⟩2. x is an upper bound for $E\downarrow$.
PROOF: For all $y \in E\downarrow$ we have $y \leq x$.
⟨3⟩3. $l \leq x$
⟨2⟩2. For any lower bound l' for E, we have $l' \leq l$.
PROOF: Since l is an upper bound for $E\downarrow$.
⟨1⟩2. If $E\uparrow$ has an infimum u, then u is the supremum of E.
PROOF: Dual.
□

Corollary 4.7.1. *A linearly ordered set has the least upper bound property if and only if every nonempty set bounded below has an infimum.*

Definition 4.8 (Closed Downwards)**.** Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed downwards* iff, whenever $x \in E$ and $y < x$, then $y \in E$.

Definition 4.9 (Closed Upwards)**.** Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed upwards* iff, whenever $x \in E$ and $x < y$, then $y \in E$.

Definition 4.10 (Greatest)**.** Let S be a linearly ordered set and $u \in S$. Then u is *greatest* in S iff $\forall x \in S. x \leq u$.

Definition 4.11 (Least)**.** Let S be a linearly ordered set and $l \in S$. Then l is *least* in S iff $\forall x \in S. l \leq x$.

Proposition 4.12. *Let $\leq$ be a linear order on a set S and $E \subseteq S$. Then $\leq \cap E^2$ is a linear order on E.*

PROOF: Easy. □

Given a linearly ordered set $(S, \leq)$ and $E \subseteq S$, we write just E for the linearly ordered set $(E, \leq \cap E^2)$.

Definition 4.13 (Lexicographic Order)**.** Let A and B be linearly ordered sets. The *lexicographic order* or *dictionary order* on $A \times B$ is the order defined by

$$(a, b) \leq (a', b') \Leftrightarrow a = a' \vee (a < a' \wedge b \leq b') \ .$$

Proposition 4.14. *The lexicographic order is a linear order.*

Chapter 5

Field Theory

Definition 5.1 (Field)**.** A *field* F consists of a set F, two operations $+, \cdot : F^2 \to F$ and an element $0 \in F$ such that:

- $+$ is commutative.
- $+$ is associative.
- $\forall x \in F. x + 0 = x$
- $\forall x \in F. \exists y \in F. x + y = 0$
- $\cdot$ is commutative.
- $\cdot$ is associative.
- There exists $1 \in F$ such that $1 \neq 0$ and $\forall x \in F. x1 = x$ and $\forall x \in F. x \neq 0 \Rightarrow \exists y \in F. xy = 1$
- *Distributive Law* $\forall x, y, z \in F. x(y + z) = xy + xz$

Proposition 5.2. *In any field F, the element 0 is the unique element such that $\forall x \in F. x + 0 = x$.*

Proof: If 0 and $0'$ both have this property then $0 = 0 + 0' = 0'$. □

Proposition 5.3. *In any field F, given $x \in F$, there is a unique $y \in F$ such that $x + y = 0$.*

Proof: If $x + y = x + y' = 0$ then

$$\begin{aligned} y &= y + 0 \\ &= y + x + y' \\ &= 0 + y' \\ &= y' \end{aligned}$$

□

Definition 5.4. Let F be a field. Let $x \in F$. We denote by $-x$ the unique element of F such that $x + (-x) = 0$.

Given $x, y \in F$, we write $x - y$ for $x + (-y)$.

Proposition 5.5. *In any field F, if $x + y = x + z$ then $y = z$.*

PROOF: If $x + y = x + z$ we have

$$\begin{aligned} -x + x + y &= -x + x + z \\ \therefore 0 + y &= 0 + z \\ \therefore y &= z \end{aligned}$$

∎

Proposition 5.6. *In any field F, we have $-(-x) = x$.*

PROOF: Since $x + (-x) = 0$. ∎

Proposition 5.7. *In any field F, the element 1 such that $\forall x \in F . x1 = x$ is unique.*

PROOF: If 1 and $1'$ both have this property then $1 = 1 \cdot 1' = 1'$. ∎

Proposition 5.8. *In any field F, given $x \in F$ with $x \neq 0$, the element y such that $xy = 1$ is unique.*

PROOF: If y and y' both have this property then we have

$$\begin{aligned} y &= y1 \\ &= yxy' \\ &= 1y' \\ &= y' \end{aligned}$$

∎

Definition 5.9. In any field F, if $x \neq 0$, we write x^{-1} for the unique element such that $xx^{-1} = 1$.

We write x/y for xy^{-1}.

Proposition 5.10. *In any field F, if $xy = xz$ and $x \neq 0$ then $y = z$.*

PROOF:

$$\begin{aligned} y &= 1y \\ &= x^{-1}xy \\ &= x^{-1}xz \\ &= 1z \\ &= z \end{aligned}$$

∎

Proposition 5.11. *In any field F, if $x \neq 0$ then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.*

PROOF: Since $xx^{-1} = 1$. ∎

Proposition 5.12. *In any field F, we have $x0 = 0$.*

PROOF:
$$\begin{aligned} x0 + 0 &= x0 \\ &= x(0+0) \\ &= x0 + x0 \\ \therefore 0 &= x0 \end{aligned} \qquad \square$$

Proposition 5.13. *In any field F, if $xy = 0$ then $x = 0$ or $y = 0$.*

PROOF: If $xy = 0$ and $x \neq 0$ then we have $y = x^{-1}xy = x^{-1}0 = 0$. $\square$

Proposition 5.14. *In any field F, we have $(-x)y = -(xy)$.*

PROOF:
$$\begin{aligned} xy + (-x)y &= (x + (-x))y \\ &= 0y \\ &= 0 && \text{(Proposition 5.12)}\square \end{aligned}$$

Corollary 5.14.1. *In any field F, we have $(-x)(-y) = xy$.*

PROOF:
$$\begin{aligned} (-x)(-y) &= -(x(-y)) \\ &= -(-(xy)) \\ &= xy && \text{(Proposition 5.6)}\square \end{aligned}$$

Proposition 5.15. *Let K be a field. Let $a, b \in K$. If $a^2 = b^2$ then $a = b$ or $a = -b$.*

PROOF:
$$\begin{aligned} a^2 - b^2 &= 0 \\ \therefore (a-b)(a+b) &= 0 \end{aligned}$$
Hence either $a - b = 0$ or $a + b = 0$, and the conclusion follows. $\square$

5.1 Ordered Fields

Definition 5.16 (Ordered Field)**.** An *ordered field F* consists of a field F and a linear order $\leq$ on F such that:

- For all $x, y, z \in F$, if $y < z$ then $x + y < x + z$
- For all $x, y \in F$, if $x > 0$ and $y > 0$ then $xy > 0$.

We call x *positive* iff $x > 0$ and *negative* iff $x < 0$.

Example 5.17. $\mathbb{Q}$ is an ordered field.

Proposition 5.18. *In any ordered field, if x is positive then $-x$ is negative.*

PROOF: If $x > 0$ then $0 = x + (-x) > 0 = (-x) = -x$. $\square$

Proposition 5.19. *In any ordered field, if $y < z$ and x is positive then $xy < xz$.*

PROOF: If $y < z$ then we have

$$\begin{aligned} 0 &< z - y \\ \therefore 0 &< x(z - y) \\ &= xz - xy \\ \therefore xy &< xz \end{aligned}$$

∎

Proposition 5.20. *In any ordered field, if $y < z$ and x is negative then $xy > xz$.*

PROOF:
⟨1⟩1. $-x$ is positive.
⟨1⟩2. $(-x)y < (-x)z$
⟨1⟩3. $-(xy) < -(xz)$
⟨1⟩4. $xz < xy$
∎

Proposition 5.21. *In any ordered field, if $x \neq 0$ then $x^2 > 0$.*

PROOF:
⟨1⟩1. If $x > 0$ then $x^2 > 0$.
PROOF: Proposition 5.19.
⟨1⟩2. If $x < 0$ then $x^2 > 0$.
PROOF: Proposition 5.20.
∎

Corollary 5.21.1. *In any ordered field, we have $1 > 0$.*

Proposition 5.22. *In any ordered field, if x is positive then x^{-1} is positive.*

PROOF: If $x^{-1} < 0$ then we would have $1 = xx^{-1} < x0 = 0$ contradicting Corollary 5.21.1. ∎

Proposition 5.23. *In any ordered field, if $0 < x < y$ then $y^{-1} < x^{-1}$.*

PROOF:
⟨1⟩1. ASSUME: $0 < x < y$
⟨1⟩2. x^{-1} and y^{-1} are positive.
PROOF: Proposition 5.22.
⟨1⟩3. $xy^{-1} < yy^{-1} = 1$
⟨1⟩4. $y^{-1} = x^{-1}xy^{-1} < x^{-1}1 = x^{-1}$
∎

Lemma 5.24. *Let K be an ordered field. Let $b \in K$ with $b > 1$. Let n be a positive integer. Then*

$$b^n - 1 \geq n(b - 1)$$

PROOF:

$$\begin{aligned} b^n - 1 &= (b-1)(b^{n-1} + b^{n-2} + \cdots + 1) \\ &\geq (b-1)(1 + 1 + \cdots + 1) \\ &= n(b-1) \end{aligned}$$

∎

Chapter 6

Real Analysis

6.1 Construction of the Real Numbers

Definition 6.1 (Cut)**.** A *cut* is a subset α of $\mathbb{Q}$ such that:

- $\emptyset \neq \alpha \neq \mathbb{Q}$
- α is closed downwards.
- α has no greatest element.

In this section, we write R for the set of all cuts.

Proposition 6.2. *R is linearly ordered by $\subseteq$.*

PROOF: The only difficult part is to prove that, for any cuts α and β, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.
$\langle 1 \rangle 1$. ASSUME: $\alpha \nsubseteq \beta$
PROVE: $\beta \subseteq \alpha$
$\langle 1 \rangle 2$. PICK $q \in \alpha$ such that $q \notin \beta$
$\langle 1 \rangle 3$. LET: $r \in \beta$
$\langle 1 \rangle 4$. $q \nless r$
$\langle 1 \rangle 5$. $r < q$
$\langle 1 \rangle 6$. $r \in \alpha$
$\square$

Proposition 6.3. *R has the least upper bound property.*

PROOF:
$\langle 1 \rangle 1$. LET: $E \subseteq R$ be nonempty and bounded above.
$\langle 1 \rangle 2$. LET: $s = \bigcup E$
PROVE: s is a cut.
$\langle 1 \rangle 3$. $\emptyset \neq s$
PROOF: Since E is nonempty and every element of E is nonempty.
$\langle 1 \rangle 4$. $s \neq \mathbb{Q}$

⟨2⟩1. Pick an upper bound u for E.
⟨2⟩2. Pick $q \notin u$
Prove: $q \notin s$
⟨2⟩3. $\forall \alpha \in E . \alpha \subseteq u$
⟨2⟩4. $s \subseteq u$
⟨2⟩5. $q \notin s$
⟨1⟩5. s is closed downwards.
⟨2⟩1. Let: $q \in s$ and $r < q$.
⟨2⟩2. Pick $\alpha \in E$ such that $q \in \alpha$.
⟨2⟩3. $r \in \alpha$
⟨2⟩4. $r \in s$
⟨1⟩6. s has no greatest element.
⟨2⟩1. Let: $q \in s$
⟨2⟩2. Pick $\alpha \in E$ such that $q \in \alpha$.
⟨2⟩3. Pick $r \in \alpha$ such that $q < r$.
⟨2⟩4. $r \in s$
□

Definition 6.4 (Addition)**.** Given cuts α and β, we define

$$\alpha + \beta = \{q + r : q \in \alpha, r \in \beta\} \ .$$

Proposition 6.5. *Given cuts α and β, we have $\alpha + \beta$ is a cut.*

Proof:
⟨1⟩1. $\alpha + \beta$ is nonempty.
Proof: Since α and β are nonempty.
⟨1⟩2. $\alpha + \beta \neq \mathbb{Q}$
⟨2⟩1. Pick $q \in \mathbb{Q} - \alpha$ and $r \in \mathbb{Q} - \beta$.
Prove: $q + r \notin \alpha + \beta$
⟨2⟩2. Assume: for a contradiction $q + r \in \alpha + \beta$.
⟨2⟩3. Pick $x \in \alpha$ and $y \in \beta$ such that $q + r = x + y$
⟨2⟩4. $x < q$
⟨2⟩5. $y < r$
⟨2⟩6. $x + y < q + r$
⟨2⟩7. Q.E.D.
Proof: This is a contradiction.
⟨1⟩3. $\alpha + \beta$ is closed downwards.
⟨2⟩1. Let: $q \in \alpha$, $r \in \beta$ and $x < q + r$
⟨2⟩2. $x - q < r$
⟨2⟩3. $x - q \in \beta$
⟨2⟩4. $x \in \alpha + \beta$
⟨1⟩4. $\alpha + \beta$ has no greatest element.
⟨2⟩1. Let: $q \in \alpha$ and $r \in \beta$.
Prove: $q + r$ is not greatest in $\alpha + \beta$.
⟨2⟩2. Pick $q' \in \alpha$ with $q < q'$ and $r' \in \beta$ with $r < r'$.
⟨2⟩3. $q + r < q' + r' \in \alpha + \beta$

$\square$

Proposition 6.6. *Addition is commutative and associative on R.*

PROOF: Immediate from definitions and the fact that addition is commutative and associative on $\mathbb{Q}$. $\square$

Definition 6.7. For any $q \in \mathbb{Q}$, let $q^* = \{r \in \mathbb{Q} : r < q\}$.

Proposition 6.8. *For any $q \in \mathbb{Q}$, we have q^* is a cut.*

PROOF:
$\langle 1 \rangle 1.$ $q^* \neq \emptyset$
  PROOF: Since $q - 1 \in q^*$.
$\langle 1 \rangle 2.$ $q^* \neq \mathbb{Q}$
  PROOF: Since $q \notin q^*$.
$\langle 1 \rangle 3.$ q^* is closed downwards.
  PROOF: Immediate from definition.
$\langle 1 \rangle 4.$ q^* has no greatest element.
  PROOF: For all $r \in q^*$ we have $r < (q + r)/2 \in q^*$.
$\square$

Proposition 6.9. *For any cut α we have $\alpha + 0^* = \alpha$.*

PROOF:
$\langle 1 \rangle 1.$ $\alpha + 0^* \subseteq \alpha$
  $\langle 2 \rangle 1.$ LET: $q \in \alpha$ and $r \in 0^*$
    PROVE: $q + r \in \alpha$
  $\langle 2 \rangle 2.$ $r < 0$
  $\langle 2 \rangle 3.$ $q + r < q$
  $\langle 2 \rangle 4.$ $q + r \in \alpha$
$\langle 1 \rangle 2.$ $\alpha \subseteq \alpha + 0^*$
  $\langle 2 \rangle 1.$ LET: $q \in \alpha$
  $\langle 2 \rangle 2.$ PICK $r \in \alpha$ such that $q < r$
  $\langle 2 \rangle 3.$ $q = r + (q - r) \in \alpha + 0^*$
$\square$

Proposition 6.10. *For any cut α, there exists a cut β such that $\alpha + \beta = 0$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\beta = \{p \in \mathbb{Q} : \exists r > 0. -p - r \notin \alpha\}$
$\langle 1 \rangle 2.$ β is a cut.
  $\langle 2 \rangle 1.$ $\beta \neq \emptyset$
    $\langle 3 \rangle 1.$ PICK $q \notin \alpha$
    $\langle 3 \rangle 2.$ $-q - 1 \in \beta$
  $\langle 2 \rangle 2.$ $\beta \neq \mathbb{Q}$
    $\langle 3 \rangle 1.$ PICK $q \in \alpha$
      PROVE: $-q \notin \beta$
    $\langle 3 \rangle 2.$ ASSUME: for a contradiction $-q \in \beta$

⟨3⟩3. Pick $r > 0$ such that $q - r \notin \alpha$
⟨3⟩4. $q - r < q$
⟨3⟩5. Q.E.D.
Proof: This contradicts the fact that α is closed downwards.
⟨2⟩3. β is closed downwards.
⟨3⟩1. Let: $p \in \beta$ and $q < p$.
⟨3⟩2. Pick $r > 0$ such that $-p - r \notin \alpha$
⟨3⟩3. $-p - r < -q - r$
⟨3⟩4. $-q - r \notin \alpha$
⟨3⟩5. $q \in \beta$
⟨2⟩4. β has no greatest element.
⟨3⟩1. Let: $p \in \beta$
⟨3⟩2. Pick $r > 0$ such that $-p - r \notin \alpha$
⟨3⟩3. $-(p + r/2) - r/2 \notin \alpha$
⟨3⟩4. $p + r/2 \in \beta$
⟨1⟩3. $\alpha + \beta \subseteq 0^*$
⟨2⟩1. Let: $p \in \alpha$ and $q \in \beta$.
⟨2⟩2. Pick $r > 0$ such that $-q - r \notin \alpha$.
⟨2⟩3. $p < -q - r$
⟨2⟩4. $p + q < -r$
⟨2⟩5. $p + q < 0$
⟨2⟩6. $p + q \in 0^*$
⟨1⟩4. $0^* \subseteq \alpha + \beta$
⟨2⟩1. Let: $v \in 0^*$
⟨2⟩2. Let: $w = -v/2$
⟨2⟩3. $w > 0$
⟨2⟩4. Pick an integer n such that $nw \in \alpha$ and $(n + 1)w \notin \alpha$.
⟨2⟩5. Let: $p = -(n + 2)w$
⟨2⟩6. $p \in \beta$
⟨2⟩7. $v = nw + p$
⟨2⟩8. $v \in \alpha + \beta$
□

Proposition 6.11. *Given $\alpha, \beta, \gamma \in R$, if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

Proof:
⟨1⟩1. $\alpha + \beta \subseteq \alpha + \gamma$
Proof: Immediate from definitions.
⟨1⟩2. $\alpha + \beta \neq \alpha + \gamma$
Proof: If $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$ by cancellation.
□

Definition 6.12. Given cuts α and β, define $\alpha\beta$ by:

$$\alpha\beta = \begin{cases} \{p \in \mathbb{Q} : \exists r \in \alpha . \exists s \in \beta (p \leq rs \wedge r > 0 \wedge s > 0\} & \text{if } \alpha > 0^* \text{ and } \beta > 0^* \\ (-\alpha)(-\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^* \\ -((-\alpha)\beta) & \text{if } \alpha < 0^* \text{ and } \beta > 0^* \\ -(\alpha(-\beta)) & \text{if } \alpha > 0^* \text{ and } \beta < 0^* \\ 0^* & \text{if } \alpha = 0^* \text{ or } \beta = 0^* \end{cases}$$

Proposition 6.13. *For any cuts α and β, we have $\alpha\beta$ is a cut.*

PROOF:
⟨1⟩1. If $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta$ is a cut.
  ⟨2⟩1. $\alpha\beta \neq \emptyset$
    ⟨3⟩1. PICK $q \in \alpha$ and $r \in \beta$ such that $q, r \notin 0^*$
    ⟨3⟩2. ASSUME: w.l.o.g. $0 < q$ and $0 < r$.
      PROOF: Since α and β have no greatest element.
    ⟨3⟩3. $qr \in \alpha\beta$
  ⟨2⟩2. $\alpha\beta \neq \mathbb{Q}$
    ⟨3⟩1. PICK $r \notin \alpha$ and $s \notin \beta$
      PROVE: $rs \notin \alpha\beta$
    ⟨3⟩2. ASSUME: for a contradiction $rs \in \alpha\beta$.
    ⟨3⟩3. PICK $r' \in \alpha$ and $s' \in \beta$ such that $rs \leq r's'$ and $r' > 0$ and $s' > 0$.
    ⟨3⟩4. $r' < r$ and $s' < s$
    ⟨3⟩5. $r's' < rs$
    ⟨3⟩6. Q.E.D.
      PROOF: This is a contradiction.
  ⟨2⟩3. $\alpha\beta$ is closed downwards.
    ⟨3⟩1. LET: $p \in \alpha\beta$ and $p' < p$
    ⟨3⟩2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$
    ⟨3⟩3. $p' \leq rs$
    ⟨3⟩4. $p' \in \alpha\beta$
  ⟨2⟩4. $\alpha\beta$ has no greatest element.
    ⟨3⟩1. LET: $p \in \alpha\beta$
    ⟨3⟩2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$.
    ⟨3⟩3. PICK $r' \in \alpha$ and $s' \in \beta$ with $r < r'$ and $s < s'$.
    ⟨3⟩4. $p < r's' \in \alpha\beta$
⟨1⟩2. For any cuts α and β, we have $\alpha\beta$ is a cut.
  PROOF: Since if α is a cut then $-\alpha$ is a cut.
□

Proposition 6.14. *For any cuts α and β we have $\alpha\beta = \beta\alpha$.*

PROOF: Easy from the definitions. □

Proposition 6.15. *For any cuts α, β and γ we have*

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma \quad .$$

PROOF:
$\langle 1 \rangle 1$. CASE: α, β and γ are all positive.
PROOF: In this case $\alpha(\beta\gamma) = (\alpha\beta)\gamma = \{p \in \mathbb{Q} : \exists r \in \alpha . \exists s \in \beta . \exists t \in \gamma . (p \leq rst \wedge r > 0 \wedge s > 0 \wedge t > 0)\}$.
$\langle 1 \rangle 2$. CASE: One of α, β or γ is 0^*.
PROOF: Then $\alpha(\beta\gamma) = (\alpha\beta)\gamma = 0^*$.
$\langle 1 \rangle 3$. CASE: α and β are positive, γ is negative.
PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\ &= -(\alpha(\beta(-\gamma))) \\ &= -((\alpha\beta)(-\gamma)) && (\langle 1 \rangle 1) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 4$. CASE: α is positive, β is negative, γ is positive.
PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-((-\beta)\gamma)) \\ &= -(\alpha((-\beta)\gamma)) \\ &= -((\alpha(-\beta))\gamma) && (\langle 1 \rangle 1) \\ &= (-(\alpha(-\beta)))\gamma \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 5$. CASE: α is positive, β and γ are negative.
PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha((-\beta)(-\gamma)) \\ &= (\alpha(-\beta))(-\gamma) && (\langle 1 \rangle 1) \\ &= (-(\alpha\beta))(-\gamma) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 6$. CASE: α is negative, β and γ are positive.
PROOF: Similar to $\langle 1 \rangle 3$.
$\langle 1 \rangle 7$. CASE: α is negative, β is positive, γ is negative.
PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\ &= (-\alpha)(\beta(-\gamma)) \\ &= ((-\alpha)\beta)(-\gamma) && (\langle 1 \rangle 1) \\ &= (-(\alpha\beta))(-\gamma) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 8$. CASE: α and β are negative, γ is positive.
PROOF: Similar to $\langle 1 \rangle 5$.
$\langle 1 \rangle 9$. CASE: α, β and γ are all negative.

PROOF:

$$\begin{aligned}
\alpha(\beta\gamma) &= \alpha(-(-\beta)(-\gamma)) \\
&= -((-\alpha)((-\beta)(-\gamma))) \\
&= -(((-\alpha)(-\beta))(-\gamma)) && (\langle 1\rangle 1) \\
&= -((\alpha\beta)(-\gamma)) \\
&= (\alpha\beta)\gamma
\end{aligned}$$

□

Proposition 6.16. *For any cut α we have $\alpha 1^* = \alpha$.*

PROOF:
$\langle 1\rangle 1$. CASE: α is positive.
  $\langle 2\rangle 1$. $\alpha 1^* \subseteq \alpha$
  $\langle 2\rangle 2$. $\alpha \subseteq \alpha 1^*$
$\langle 1\rangle 2$. CASE: $\alpha = 0^*$
$\langle 1\rangle 3$. CASE: α is negative.
□

Theorem 6.17. *There exists an ordered field with the least upper bound property.*

Proposition 6.18. *There is no rational p such that $p^2 = 2$.*

PROOF:
$\langle 1\rangle 1$. ASSUME: for a contradiction $p^2 = 2$.
$\langle 1\rangle 2$. PICK integers m, n not both even such that $p = m/n$.
$\langle 1\rangle 3$. $m^2 = 2n^2$
$\langle 1\rangle 4$. m is even.
$\langle 1\rangle 5$. PICK an integer k such that $m = 2k$.
$\langle 1\rangle 6$. $4k^2 = 2n^2$
$\langle 1\rangle 7$. $2k^2 = n^2$
$\langle 1\rangle 8$. n is even.
$\langle 1\rangle 9$. Q.E.D.
  PROOF: $\langle 1\rangle 2$, $\langle 1\rangle 4$ and $\langle 1\rangle 8$ form a contradiction.
□

Theorem 6.19. *Any two complete ordered fields are isomorphic.*

Definition 6.20. Let $\mathbb{R}$ be the complete ordered field. We call its elements *real numbers.*

6.2 Properties of the Real Numbers

Theorem 6.21. *$\mathbb{Q}$ is a subfield of $\mathbb{R}$.*

Theorem 6.22 (Archimedean Property)**.** *Let $x, y \in \mathbb{R}$ with $x > 0$. There exists a positive integer n such that $nx > y$.*

PROOF:
$\langle 1\rangle 1$. LET: $A = \{nx : n \in \mathbb{Z}^+\}$
$\langle 1\rangle 2$. ASSUME: for a contradiction there is no positive integer n such that $nx > y$.
$\langle 1\rangle 3$. y is an upper bound for A.
$\langle 1\rangle 4$. LET: $\alpha = \sup A$
$\langle 1\rangle 5$. $\alpha - x$ is not an upper bound for A.
$\langle 1\rangle 6$. PICK a positive integer m such that $\alpha - x < mx$
$\langle 1\rangle 7$. $\alpha < (m+1)x \in A$
$\langle 1\rangle 8$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 4$.
□

Theorem 6.23. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

PROOF:
$\langle 1\rangle 1$. LET: $x, y \in \mathbb{R}$ with $x < y$
$\langle 1\rangle 2$. PICK a positive integer n such that
$$n(y - x) > 1 \ .$$
PROOF: Archimedean property.
$\langle 1\rangle 3$. PICK a positive integer m_1 such that $m_1 > nx$
PROOF: Archimedean property.
$\langle 1\rangle 4$. PICK a positive integer m_2 such that $m_2 > -nx$
PROOF: Archimedean property.
$\langle 1\rangle 5$. $-m_2 < nx < m_1$
$\langle 1\rangle 6$. LET: m be the integer such that
$$m - 1 \leq nx < m \ .$$
$\langle 1\rangle 7$. $nx < m \leq 1 + nx < ny$
$\langle 1\rangle 8$. $x < m/n < y$
□

Theorem 6.24. *For every real number $x > 0$ and positive integer n, there exists a unique positive real number y such that $y^n = x$.*

PROOF:
$\langle 1\rangle 1$. There exists a real $y > 0$ such that $y^n = x$.
$\langle 2\rangle 1$. LET: $E = \{t \in \mathbb{R}^+ : t^n < x\}$
$\langle 2\rangle 2$. LET: $y = \sup E$
$\langle 3\rangle 1$. $E \neq \emptyset$
$\langle 4\rangle 1$. LET: $t = x/(x+1)$
$\langle 4\rangle 2$. $0 < t < 1$
$\langle 4\rangle 3$. $t^n < t < x$
$\langle 4\rangle 4$. $t \in E$
$\langle 3\rangle 2$. $x + 1$ is an upper bound for E.
$\langle 4\rangle 1$. LET: $t > x + 1$
$\langle 4\rangle 2$. $t^n > t > x$
$\langle 4\rangle 3$. $t \notin E$

⟨2⟩3. $y^n = x$

⟨3⟩1. $y^n \nless x$

⟨4⟩1. ASSUME: for a contradiction $y^n < x$.

⟨4⟩2. PICK h such that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y+1)^{n-1}} \ .$$

⟨4⟩3. $(y+h)^n - y^n < x - y^n$

PROOF:

$$\begin{aligned}(y+h)^n - y^n &= ((y+h) - y)\sum_{i=0}^{n-1}(y+h)^{n-1-i}y^i \\ &= h\sum_{i=0}^{n-1}(y+h)^{n-1-i}y^i \\ &\leq hn(y+h)^{n-1} \\ &\leq hn(y+1)^{n-1} \\ &< x - y^n\end{aligned}$$

⟨4⟩4. $(y+h)^n < x$

⟨4⟩5. $y + h \in E$

⟨4⟩6. Q.E.D.

PROOF: This contradicts the fact that y is an upper bound for E.

⟨3⟩2. $y^n \ngtr x$

⟨4⟩1. ASSUME: for a contradiction $y^n > x$

⟨4⟩2. LET:

$$k = \frac{y^n - x}{ny^{n-1}}$$

⟨4⟩3. $0 < k < y$

⟨4⟩4. $y - k$ is an upper bound for E.

⟨5⟩1. LET: $t \geq y - k$

⟨5⟩2. $y^n - t^n \leq y^n - x$

PROOF:

$$\begin{aligned}y^n - t^n &\leq y^n - (y-k)^n \\ &= (y - (y-k))\sum_{i=0}^{n-1} y^{n-i}(y-k)^i \\ &= k\sum_{i=0}^{n-1} y^{n-i}(y-k)^i \\ &\leq kny^{n-1} \\ &= y^n - x\end{aligned}$$

⟨5⟩3. $t^n \geq x$

⟨5⟩4. $t \notin E$

⟨4⟩5. Q.E.D.

PROOF: This contradicts the fact that y is the least upper bound of E.

⟨1⟩2. If y and y' are positive reals with $y^n = y'^n$ then $y = y'$.

PROOF: Since the function that sends y to y^n is strictly monotone.
∎

Definition 6.25 (nth Root)**.** Given any real number $x > 0$ and positive integer n, the *nth root* of x, denoted $x^{1/n}$, is the unique positive real such that

$$(x^{1/n})^n = x \ .$$

We write $\sqrt{x}$ for $x^{1/2}$.

Proposition 6.26. *Let a and b be positive real numbers and n a positive integer. Then*

$$(ab)^{1/n} = a^{1/n} b^{1/n} \ .$$

PROOF: Since $(a^{1/n} b^{1/n})^n = ab$. ∎

Lemma 6.27. *Let b be a real number with $b > 1$. Let n be a positive integer. Then*

$$b - 1 \geq n(b^{1/n} - 1) \ .$$

PROOF: From Lemma 5.24. ∎

Lemma 6.28. *Let b and t be real numbers with $b > 1$ and $t > 1$. For any positive integer n, if $n > \frac{b-1}{t-1}$ then $b^{1/n} < t$.*

PROOF:

$$\begin{aligned} b - 1 &\geq n(b^{1/n} - 1) \\ \therefore \frac{b-1}{n} &\geq b^{1/n} - 1 \\ \therefore t - 1 &> b^{1/n} - 1 \\ \therefore t &> b^{1/n} \end{aligned}$$

∎

Lemma 6.29. *Let b be a real number with $b > 0$. Let m, n, p, q be integers with $n > 0$ and $q > 0$. Assume $m/n = p/q$. Then*

$$(b^m)^{1/n} = (b^p)^{1/q} \ .$$

PROOF:
⟨1⟩1. $(b^m)^{1/n} = (b^{1/n})^m$
PROOF:

$$\begin{aligned} ((b^{1/n})^m)^n &= ((b^{1/n})^n)^m \\ &= b^m \end{aligned}$$

⟨1⟩2. $((b^m)^{1/n})^q = b^p$
PROOF:

$$\begin{aligned} ((b^m)^{1/n})^q &= (b^{1/n})^{mq} \\ &= (b^{1/n})^{np} \\ &= b^p \end{aligned}$$

∎

Definition 6.30. For a a positive real and q a rational number, we may therefore define a^q by

$$a^{m/n} = (a^m)^{1/n}$$

for m and n integers with $n > 0$.

Proposition 6.31. *Let a be a positive real and r, s rational numbers. Then*

$$a^{r+s} = a^r a^s \ .$$

PROOF:

$$\begin{aligned} a^{m/n+p/q} &= a^{(mq+np)/nq} \\ &= (a^{mq+np})^{1/nq} \\ &= (a^{mq})^{1/nq}(a^{np})^{1/nq} \\ &= a^{m/n}a^{p/q} \end{aligned}$$ ∎

Proposition 6.32. *Let $b > 1$ be a real number and q a rational number. Then*

$$b^q = \sup\{b^t : t \in \mathbb{Q}, t \leq q\}$$

PROOF: It is the greatest element of this set. ∎

Definition 6.33. Let $b > 1$ be a real number and x a real number. Then

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t \leq x\} \ .$$

Lemma 6.34. *Let b, w and y be real numbers with $b > 1$. Assume $b^w < y$. Then there exists a positive integer n such that $b^{w+1/n} < y$.*

PROOF:
⟨1⟩1. LET: $t = yb^{-w}$
⟨1⟩2. PICK a positive integer n such that $n > \frac{b-1}{t-1}$.
⟨1⟩3. $b^{1/n} < t$
  PROOF: Lemma 6.28.
⟨1⟩4. $b^{w+1/n} < y$
∎

Lemma 6.35. *Let b, w and y be real numbers with $b > 1$. Assume $b^w > y$. Then there exists a positive integer n such that $b^{w-1/n} < y$.*

PROOF:
⟨1⟩1. LET: $t = b^w/y$
⟨1⟩2. PICK a positive integer n such that $n > \frac{b-1}{t-1}$
⟨1⟩3. $b^{1/n} < t$
  PROOF: Lemma 6.28.
⟨1⟩4. $y < b^{w-1/n}$
∎

Proposition 6.36. *For b and x real numbers with $b > 1$ we have*

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t < x\} \ .$$

PROOF:
⟨1⟩1. b^x is an upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
⟨1⟩2. LET: u be any upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
PROVE: $b^x \leq u$
⟨1⟩3. LET: q be a rational number with $q \leq x$.
PROVE: $b^q \leq u$
⟨1⟩4. ASSUME: for a contradiction $b^q > u$.
⟨1⟩5. PICK a positive integer n such that $b^{q-1/n} > u$.
PROOF: Lemma 6.35.
⟨1⟩6. $b^{q-1/n} \leq u$
PROOF: ⟨1⟩2
⟨1⟩7. Q.E.D.
PROOF: This contradicts ⟨1⟩4.
□

Lemma 6.37. *Let A be a set of positive real numbers with supremum $a > 0$ and B a set of positive real numbers with supremum $b > 0$. Then ab is the supremum of $\{xy : x \in A, y \in B\}$.*

PROOF:
⟨1⟩1. For all $x \in A$ and $y \in B$ we have $xy \leq ab$.
⟨1⟩2. If u is any upper bound for $\{xy : x \in A, y \in B\}$ then $ab \leq u$.
⟨2⟩1. LET: u be an upper bound for $\{xy : x \in A, y \in B\}$.
⟨2⟩2. For all $x \in A$ we have u/x is an upper bound for B.
⟨2⟩3. For all $x \in A$ we have $b \leq u/x$
⟨2⟩4. For all $x \in A$ we have $x \leq u/b$
⟨2⟩5. $a \leq u/b$
⟨2⟩6. $ab \leq u$
□

Proposition 6.38. *Let $b, x, y \in \mathbb{R}$ with $b > 1$. Then*

$$b^{x+y} = b^x b^y \ .$$

PROOF:
⟨1⟩1. For any rational number $q < x + y$, there exist rational numbers $r < x$ and $s < y$ such that $q = r + s$.
⟨2⟩1. $q - x < y$
⟨2⟩2. PICK a rational t such that $q - x < t < y$
⟨2⟩3. $q = t + (q - t)$ and $t < y$, $q - t < x$
⟨1⟩2. $b^x b^y = b^{x+y}$

PROOF:

$$\begin{aligned} b^x b^y &= \sup\{b^q b^r : q, r \in \mathbb{Q}, q < x, r < y\} \\ &= \sup\{b^{q+r} : q, r \in \mathbb{Q}, q < x, r < y\} \\ &= \sup\{b^q : q \in \mathbb{Q}, q < x + y\} \\ &= b^{x+y} \end{aligned}$$

□

6.2.1 Logarithms

Proposition 6.39. *Let b and y be real numbers with $b > 1$ and $y > 0$. There exists a unique real x such that $b^x = y$.*

PROOF:
⟨1⟩1. LET: $x = \sup\{w : b^w < y\}$
PROVE: $b^x = y$
⟨2⟩1. $\{w : b^w < y\} \neq \emptyset$
PROOF: It contains 0.
⟨2⟩2. $\{w : b^w < y\}$ is bounded above.
⟨3⟩1. LET: n be the least integer such that

$$n \geq \frac{y-1}{b-1}$$

PROOF: Archimedean property.
⟨3⟩2. LET: w be a real number with $b^w < y$
PROVE: $w < n$
⟨3⟩3. $b^w < n(b-1) + 1$
⟨3⟩4. $b^w < b^n$
⟨3⟩5. $w < n$
⟨1⟩2. $b^x \leq y$
⟨2⟩1. ASSUME: for a contradiction $b^x > y$
⟨2⟩2. PICK a positive integer n such that $b^{x-1/n} > y$
PROOF: Lemma 6.35.
⟨2⟩3. PICK w such that $x - 1/n < w$ and $b^w < y$
PROOF: Since $x - 1/n$ is not an upper bound for $\{w : b^w < y\}$.
⟨2⟩4. $b^{x-1/n} < y$
⟨2⟩5. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
⟨1⟩3. $b^x \geq y$
⟨2⟩1. ASSUME: for a contradiction $b^x < y$.
⟨2⟩2. PICK a positive integer n such that $b^{x+1/n} < y$.
⟨2⟩3. $x + 1/n \leq x$
⟨2⟩4. Q.E.D.
PROOF: This is a contradiction.

□

Definition 6.40 (Logarithm)**.** Let b and y be real numbers with $b > 1$ and $y > 0$. The *logarithm* of y to *base* b, denoted $\log_b y$, is the unique real number such that

$$b^{\log_b y} = y \ .$$

6.2.2 Intervals

Definition 6.41 (Intervals)**.** Let $a, b \in \mathbb{R}$.

The *open interval* (a, b) is $\{x \in \mathbb{R} : a < x < b\}$.

The *closed interval* $[a, b]$ is $\{x \in \mathbb{R} : a \leq x \leq b\}$.

The *half-open intervals* $[a, b)$ and $(a, b]$ are defined by

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

Definition 6.42 (k-cell)**.** Let k be a positive integer. A *k-cell* is a subset of $\mathbb{R}^k$ of the form

$$\{\vec{x} \in \mathbb{R}^k : \forall i = 1, \ldots, k. a_i \leq x_i \leq b_i\}$$

for some real numbers a_1, …, a_k, b_1, …, b_k with $a_i \leq b_i$ for each i.

6.2.3 The Cantor Set

Definition 6.43 (Cantor Set)**.** Define a sequence E_n of unions of intervals as follows:

- $E_0 = [0, 1]$
- E_{n+1} is formed from E_n by replacing every interval $[a, b]$ with $[a, (2a+b)/3]$ and $[(a + 2b)/3, b]$.

The *Cantor set* is $\bigcap_{n=0}^{\infty} E_n$.

6.3 The Extended Real Number System

Definition 6.44 (Extended Real Number System)**.** The *extended real number system* is the set $\mathbb{R} \cup \{+\infty, -\infty\}$.

We extend the ordering $\leq$ to the extended reals by defining

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

We extend $+$, $\cdot$ and $/$ to partial operations on the extended real by defining:

$$
\begin{array}{rl}
x + (+\infty) = +\infty & (x \in \mathbb{R}) \\
x + (-\infty) = -\infty & (x \in \mathbb{R}) \\
(+\infty) + x = +\infty & (x \in \mathbb{R}) \\
(+\infty) + (+\infty) \text{ is undefined} & \\
(+\infty) + (-\infty) \text{ is undefined} & \\
(-\infty) + x = -\infty & (x \in \mathbb{R}) \\
(-\infty) + (+\infty) \text{ is undefined} & \\
(-\infty) + (-\infty) \text{ is undefined} & \\
x \cdot (+\infty) = +\infty & (x \in \mathbb{R}) \\
x \cdot (-\infty) = -\infty & (x \in \mathbb{R}) \\
(+\infty) \cdot x = +\infty & (x \in \mathbb{R}) \\
(+\infty) \cdot (+\infty) \text{ is undefined} & \\
(+\infty) \cdot (-\infty) \text{ is undefined} & \\
(-\infty) \cdot x = -\infty & (x \in \mathbb{R}) \\
(-\infty) \cdot (+\infty) \text{ is undefined} & \\
(-\infty) \cdot (-\infty) \text{ is undefined} & \\
x/(+\infty) = 0 & (x \in \mathbb{R}) \\
x/(-\infty) = 0 & (x \in \mathbb{R}) \\
(+\infty)/x \text{ is undefined} & (x \in \mathbb{R}) \\
(+\infty)/(+\infty) \text{ is undefined} & \\
(+\infty)/(-\infty) \text{ is undefined} & \\
(-\infty)/x \text{ is undefined} & (x \in \mathbb{R}) \\
(-\infty)/(+\infty) \text{ is undefined} & \\
(-\infty)/(-\infty) \text{ is undefined} &
\end{array}
$$

Chapter 7

Complex Analysis

Definition 7.1 (Complex Numbers)**.** A *complex number* is a pair of real numbers. We write $\mathbb{C}$ for the set of complex numbers.

Define $+$ and $\cdot$ on $\mathbb{C}$ by:

$$\begin{aligned}(a,b)+(c,d) &= (a+c,b+d)\\ (a,b)(c,d) &= (ac-bd,ad+bc)\end{aligned}$$

Theorem 7.2. *The complex numbers form a field.*

Theorem 7.3. *The function that maps a to $(a,0)$ is an embedding of $\mathbb{R}$ in $\mathbb{C}$.*

Definition 7.4.

$$i = (0,1)$$

Lemma 7.5.

$$(a,b) = a+ib$$

Proof: Since $(a,0)+(0,1)(b,0) = (a,b)$. □

Lemma 7.6.

$$i^2 = -1$$

Proof: Immediate from definitions. □

Corollary 7.6.1. *There is no linear order on $\mathbb{C}$ that makes $\mathbb{C}$ into an ordered field.*

Definition 7.7 (Complex Conjugate)**.** For any complex number z, the *complex conjugate* $\overline{z}$ is defined by

$$\overline{a+ib} = a-ib \qquad (a,b\in\mathbb{R})\ .$$

Definition 7.8 (Real Part)**.** For any complex number z, the *real part* of z, denoted $\mathrm{Re}(z)$, is defined by

$$\mathrm{Re}(a+ib) = a \qquad (a,b\in\mathbb{R})\ .$$

Definition 7.9 (Imaginary Part)**.** For any complex number z, the *imaginar part* of z, denoted $\mathrm{Im}(z)$, is defined by

$$\mathrm{Im}(a+ib) = b \qquad (a, b \in \mathbb{R}) \ .$$

Theorem 7.10. *For all $z, w \in \mathbb{C}$ we have*

$$\overline{z+w} = \overline{z} + \overline{w} \ .$$

Proof:

$$\begin{aligned}
\overline{(a+ib)+(c+id)} &= \overline{(a+c)+i(b+d)} \\
&= (a+c) - i(b+d) \\
&= (a-ib) + (c-id) \\
&= \overline{a+ib} + \overline{c+id}
\end{aligned}$$

□

Theorem 7.11. *For all $z, w \in \mathbb{C}$ we have*

$$\overline{zw} = \overline{z} \cdot \overline{w} \ .$$

Proof:

$$\begin{aligned}
\overline{(a+ib)(c+id)} &= \overline{(ac-bd)+i(ad+bc)} \\
&= (ac-bd) - i(ad+bc) \\
&= (a-ib)(c-id) \\
&= \overline{a+ib} \cdot \overline{c+id}
\end{aligned}$$

□

Theorem 7.12. *For all $z \in \mathbb{C}$ we have*

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}) \ .$$

Proof:

$$\begin{aligned}
(a+ib) + \overline{a+ib} &= (a+ib) + (a-ib) \\
&= 2a \\
&= 2\,\mathrm{Re}(a+ib)
\end{aligned}$$

□

Theorem 7.13. *For all $z \in \mathbb{C}$ we have*

$$\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z}) \ .$$

Proof:

$$\begin{aligned}
(a+ib) - \overline{a+ib} &= (a+ib) - (a-ib) \\
&= 2ib \\
&= 2i\,\mathrm{Im}(a+ib)
\end{aligned}$$

□

Theorem 7.14. *For all $z \in \mathbb{C}$ we have $z\overline{z}$ is a non-negative real.*

PROOF:

$$\begin{aligned}(a+ib)(\overline{a+ib}) &= (a+ib)(a-ib)\\ &= a^2+b^2\end{aligned}$$ □

Theorem 7.15. *For any $z \in \mathbb{C}$, if $z\overline{z} = 0$ then $z = 0$.*

PROOF: Let $z = a + ib$. Then $z\overline{z} = a^2 + b^2 = 0$ iff $a = b = 0$. □

Definition 7.16 (Absolute Value)**.** For $z \in \mathbb{C}$, the *absolute value* of z is

$$|z| = (z\overline{z})^{1/2} \quad .$$

Proposition 7.17. *For x a non-negative real we have $|x| = x$.*

PROOF: Since $|x| = \sqrt{x^2} = x$. □

Proposition 7.18. *For x a negative real we have $|x| = -x$.*

PROOF: Since $|x| = \sqrt{x^2} = -x$. □

Theorem 7.19. *For any complex number z we have $|z| \geq 0$.*

PROOF: Immediate from definition. □

Theorem 7.20. *For any complex number z, if $|z| = 0$ then $z = 0$.*

PROOF: From Theorem 7.15. □

Theorem 7.21. *For any complex number z we have*

$$|\overline{z}| = |z| \quad .$$

PROOF: Immediate from definitions. □

Theorem 7.22. *For any complex numbers z and w we have*

$$|zw| = |z||w| \quad .$$

PROOF:

$$\begin{aligned}|zw| &= \sqrt{zw\overline{zw}}\\ &= \sqrt{z\overline{z}}\sqrt{w\overline{w}} && \text{(Proposition 6.26)}\\ &= |z||w|\end{aligned}$$ □

Theorem 7.23. *For any complex number z we have*

$$|\operatorname{Re} z| \leq |z|$$

PROOF: Let $z = a + ib$. Then

$$|\operatorname{Re} z| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} \quad .$$ □

Theorem 7.24. *For any complex numbers z and w we have*

$$|z + w| \leq |z| + |w| \quad .$$

PROOF:

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \\
&= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 && \text{(Theorem 7.12)} \\
&\leq |z|^2 + 2|z\overline{w}| + |w|^2 && \text{(Theorem 7.23)} \\
&= |z|^2 + 2|z||w| + |w|^2 && \text{(Theorem 7.22)} \\
&= (|z|+|w|)^2 && \square
\end{aligned}$$

Theorem 7.25 (Schwarz Inequality)**.** *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be complex numbers. Then*

$$\left|\sum_{j=1}^n a_j\overline{b_j}\right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2 \enspace .$$

PROOF:
⟨1⟩1. LET: $A = \sum_{j=1}^n |a_j|^2$
⟨1⟩2. LET: $B = \sum_{j=1}^n |b_j|^2$
⟨1⟩3. LET: $C = \sum_{j=1}^n a_j\overline{b_j}$
⟨1⟩4. ASSUME: w.l.o.g. $B > 0$
PROOF: If $B = 0$ then $b_1 = \cdots = b_n = 0$ and both sides of the inequality are 0.
⟨1⟩5. $\sum_{j=1}^n |Ba_j - Cb_j|^2 = B(AB - |C|^2)$
PROOF:

$$\begin{aligned}
\sum_{j=1}^n |Ba_j - Cb_j|^2 &= \sum_{j=1}^n (Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j}) \\
&= B^2\sum_{j=1}^n |a_j|^2 - B\overline{C}\sum_{j=1}^n a_j\overline{b_j} - BC\sum_{j=1}^n \overline{a_j}b_j + |C|^2\sum_{j=1}^n |b_j|^2 \\
&= B^2A - 2B|C|^2 + B|C|^2 \\
&= B(AB - |C|^2)
\end{aligned}$$

⟨1⟩6. $B(AB - |C|^2) \geq 0$
⟨1⟩7. $AB \geq |C|^2$
$\square$

Proposition 7.26. *For any non-zero complex number w, there are exactly two complex numbers z such that $z^2 = w$.*

PROOF:
⟨1⟩1. There are at most two complex numbers z such that $z^2 = w$.
PROOF: Proposition 5.15.
⟨1⟩2. There are at least two complex numbers z such that $z^2 = w$.
⟨2⟩1. LET: $w = u + iv$
⟨2⟩2. LET: $a = \sqrt{\frac{|w|+u}{2}}$
⟨2⟩3. LET: $b = \sqrt{\frac{|w|-u}{2}}$

⟨2⟩4. CASE: $v \geq 0$
⟨3⟩1. LET: $z = a + ib$
⟨3⟩2. $z^2 = w$
PROOF:

$$\begin{aligned} z^2 &= (a+ib)^2 \\ &= a^2 - b^2 + 2iab \\ &= u + i\sqrt{|w|^2 - u^2} \\ &= u + iv \\ &= w \end{aligned}$$

⟨3⟩3. $(-z)^2 = w$
⟨2⟩5. CASE: $v \leq 0$
⟨3⟩1. LET: $z = a - ib$
⟨3⟩2. $z^2 = w$
PROOF:

$$\begin{aligned} z^2 &= (a-ib)^2 \\ &= a^2 - b^2 - 2iab \\ &= u - i\sqrt{|w|^2 - u^2} \\ &= u - i|v| \\ &= w \end{aligned}$$

⟨3⟩3. $(-z)^2 = w$
□

7.1 Algebraic Numbers

Definition 7.27 (Algebraic)**.** A complex number z is *algebraic* iff there exist integers a_0, a_1, …, a_n not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \ ;$$

otherwise, it is *transcendental*.

Proposition 7.28. *The set of algebraic numbers is countable.*

PROOF: There are countably many finite sequences of integers $(a_0, a_1, \ldots, a_n)$, and for each one, there are only finitely many complex numbers z such that $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$. □

Part II

Linear Algebra

Chapter 8

Vector Spaces

8.1 Convex Sets

Definition 8.1 (Convex)**.** Let $E \subseteq \mathbb{R}^k$. Then E is *convex* iff, for all $\vec{x}, \vec{y} \in E$ and $\lambda \in (0,1)$,

$$\lambda \vec{x} + (1-\lambda)\vec{y} \in E \enspace .$$

Proposition 8.2. *Every k-cell is convex.*

PROOF:
$\langle 1 \rangle 1$. LET: $C = \{\vec{x} \in \mathbb{R}^k : \forall i. a_i \leq x_i \leq b_i\}$ be a k-cell.
$\langle 1 \rangle 2$. LET: $\vec{x}, \vec{y} \in C$ and $\lambda \in (0,1)$.
PROVE: $\lambda \vec{x} + (1-\lambda)\vec{y} \in C$
$\langle 1 \rangle 3$. For each i we have $a_i \leq \lambda x_i + (1-\lambda) y_i \leq b_i$
PROOF: Since $\lambda a_1 + (1-\lambda) a_i \leq \lambda x_i + (1-\lambda) y_i \leq \lambda b_i + (1-\lambda) b_i$.
□

8.2 Linear Transformations

Definition 8.3 (Norm)**.** For $A \in \mathbf{Vect}_{\mathbb{R}}[\mathbb{R}^n, \mathbb{R}^m]$, define the *norm* of A to be

$$\|A\| := \{\|A\vec{x}\| : \vec{x} \in \mathbb{R}^n, \|\vec{x}\| = 1\} \enspace .$$

We prove that this always exists.

PROOF: Since for $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1^2 + \cdots + x_n^{\cdot} = 1$ we have

$$\begin{aligned} \|A(x_1, \ldots, x_n)\| &= \|\sum_{i=1}^n x_i A\vec{e_i}\| \\ &\leq \sum_{i=1}^n |x_i| \|A\vec{e_i}\| \\ &\leq \sum_{i=1}^n \|A\vec{e_i}\| \end{aligned}$$

□

Chapter 9

Real Inner Product Spaces

Definition 9.1 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^k$, define the *inner product* $\vec{x} \cdot \vec{y}$ by

$$(x_1, \ldots, x_k) \cdot (y_1, \ldots, y_k) = x_1 y_1 + \cdots + x_k y_k \quad .$$

Definition 9.2 (Norm)**.** Define the *norm* of a vector $\vec{x} \in \mathbb{R}^k$ by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} \quad .$$

Proposition 9.3.

$$\|\vec{x}\| \geq 0$$

PROOF: Immediate from the definition. □

Proposition 9.4. *If* $\|\vec{x}\| = 0$ *then* $\vec{x} = \vec{0}$.

PROOF: If $\|\vec{x}\| = 0$ then $x_1^2 + \cdots + x_n^2 = 0$ so $x_1 = \cdots = x_n = 0$. □

Proposition 9.5. *For* $\alpha \in \mathbb{R}$ *and* $\vec{x} \in \mathbb{R}^k$,

$$\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| \quad .$$

PROOF: Easy. □

Proposition 9.6. *For* $\vec{x}, \vec{y} \in \mathbb{R}^k$, *we have*

$$\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\| \|\vec{y}\| \quad .$$

PROOF: By the Schwarz inequality. □

Proposition 9.7. *For* $\vec{x}, \vec{y} \in \mathbb{R}^k$ *we have*

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \quad .$$

PROOF:

$$
\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \\
&= \vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Proposition 9.6)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square
\end{aligned}
$$

Corollary 9.7.1. *For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ we have*

$$\|\vec{x}-\vec{z}\| \leq \|\vec{x}-\vec{y}\| + \|\vec{y}-\vec{z}\| \enspace .$$

Definition 9.8 (Bounded Function)**.** Let E be a set. Let $f : E \to \mathbb{R}^k$. Then f is *bounded* iff $f(E)$ is bounded.

9.1 Balls

Definition 9.9 (Closed Ball)**.** Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *closed ball* with *centre* $\vec{x}$ and *radius* r is

$$\{y \in \mathbb{R}^k : \|y - x\| \leq r\} \enspace .$$

Proposition 9.10. *Every closed ball is convex.*

PROOF:
⟨1⟩1. LET: B be the closed ball with center $\vec{a}$ and radius r.
⟨1⟩2. LET: $\vec{x}, \vec{y} \in B$
⟨1⟩3. LET: $\lambda \in (0,1)$
⟨1⟩4. $\lambda\vec{x} + (1-\lambda)\vec{y} \in B$
PROOF:

$$
\begin{aligned}
\|\lambda\vec{x} + (1-\lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x}-\vec{a}) + (1-\lambda)(\vec{y}-\vec{a})\| \\
&= \lambda\|\vec{x}-\vec{a}\| + (1-\lambda)\|\vec{y}-\vec{a}\| \\
&\leq \lambda r + (1-\lambda) r \\
&= r && \square
\end{aligned}
$$

$\square$

Chapter 10

Complex Inner Product Spaces

Definition 10.1 (Inner Product)**.** Let V be a complex vector space. An *inner product* on V is a function $\langle\ ,\ \rangle : V^2 \to \mathbb{C}$ such that, for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

- $\langle y, x\rangle = \overline{\langle x, y\rangle}$
- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
- $\langle \alpha x, y\rangle = \overline{\alpha}\langle x, y\rangle$
- $\langle x, x\rangle \geq 0$
- If $\langle x, x\rangle = 0$ then $x = 0$.

An *inner product space* consists of a complex vector space V and an inner product on V.

Definition 10.2 (Norm)**.** Let V be an inner product space and $x \in V$. The *norm* of x is

$$\|x\| = \sqrt{\langle x, x\rangle}\ .$$

Proposition 10.3. *An inner product space is a metric space under*

$$d(x, y) = \|x - y\|\ .$$

Definition 10.4 (Bounded)**.** Let V_1 and V_2 be inner product spaces and $T : V_1 \to V_2$ a linear transformation. Then T is *bounded* iff $\{\|T(x)\| : \|x\| = 1\}$ is bounded above.

Proposition 10.5. *Every linear transformation between finite dimensional inner product spaces is bounded.*

Definition 10.6 (Outer Product)**.** Let V be an inner product space and $|\psi\rangle, |\phi\rangle \in V$. The *outer product* of $|\psi\rangle$ and $|\phi\rangle$ is

$$|\psi\rangle\langle\phi| : V \to V\ .$$

10.1 Hilbert Spaces

Definition 10.7 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Theorem 10.8 (Completeness Relation)**.** *Let $\mathcal{H}$ be a Hilbert space. Let $\{|e_n\rangle\}_{n\in\mathbb{N}}$ be a countable orthonormal basis for $\mathcal{H}$. Then*

$$\sum_{n=0}^{\infty} |e_n\rangle \langle e_n| = I \ .$$

PROOF:
$\langle 1\rangle$1. LET: $|\psi\rangle \in \mathcal{H}$
$\langle 1\rangle$2. LET: $|\psi\rangle = \sum_{n=0}^{\infty} \alpha_n |e_n\rangle$
$\langle 1\rangle$3. $\sum_{n=0}^{\infty} \langle e_n|\phi\rangle |e_n\rangle = |\psi\rangle$
PROOF:

$$\begin{aligned}\sum_{n=0}^{\infty} \langle e_n|\phi\rangle |e_n\rangle &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \alpha_m \langle e_n|e_m\rangle |e_n\rangle \\ &= \sum_{n=0}^{\infty} \alpha_n |e_n\rangle \\ &= |\psi\rangle\end{aligned}$$

□

□

Definition 10.9 (Separable)**.** A Hilbert space is *separable* iff it has a countable dense orthonormal basis.

Chapter 11

Lie Algebras

Definition 11.1 (Lie Algebra)**.** Let K be a field. A *Lie algebra* $\mathcal{L}$ over K consists of a vector space $\mathcal{L}$ over K and an operation

$$[\ ,\ ] : \mathcal{L}^2 \to \mathcal{L}\ ,$$

the *Lie bracket* or *commutator*, such that, for all $x, y, z \in \mathcal{L}$ and $\alpha \in K$:

$$\begin{aligned} [x+y, z] &= [x, z] + [y, z] \\ [x, y+z] &= [x, y] + [x, z] \\ [\alpha x, y] &= \alpha[x, y] \\ [x, x] &= 0 \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \qquad \text{(Jacobi identity)} \end{aligned}$$

Lemma 11.2. *If K has characteristic 0 then the condition $[x, x] = 0$ can be replaced with $[x, y] = -[y, x]$.*

Proposition 11.3. *The commutator is determind by its values on any basis for $\mathcal{L}$.*

Example 11.4. $\mathbb{R}^3$ with the cross product is a real Lie algebra.

Example 11.5. For any $n \geq 0$, we have $GL(n, K)$ is a Lie algebra over K under

$$[A, B] = AB - BA\ .$$

Definition 11.6 (Linear Lie Algebra)**.** A *linear Lie algebra* over K is a Lie algebra over K that is a subalgebra of $GL(n, K)$ for some n.

Example 11.7 (Special Linear Algebra)**.** The *special Linear algebra* $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \mathrm{tr} = 0\}$ is a real linear Lie algebra.

Example 11.8 (Orthogonal Lie Algebra)**.** The *orthogonal Lie algebra* $SO(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A \text{ is skew-symmetric}\}$ is a real linear Lie algebra.

Example 11.9. Let $u(n)$ be the set of all skew-Hermitian $n \times n$-matrices as a real Lie algebra.

Let $su(n) = u(n) \cap SL(n, \mathbb{R})$.

Proposition 11.10. *$SU(2)$ is spanned by the* Pauli matrices

$$\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which satisfy

$$\begin{aligned} [\sigma_x, \sigma_y] &= \sigma_z \\ [\sigma_y, \sigma_z] &= \sigma_x \\ [\sigma_z, \sigma_x] &= \sigma_y \end{aligned}$$

11.1 Lie Algebar Homomorphisms

Definition 11.11 (Homomorphism)**.** Let L_1 and L_2 be Lie algebras over the same field. A *Lie algebra homomorphism* $\phi : L_1 \to L_2$ is a linear transformation such that

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in L_1$.

Lemma 11.12. *Every bijective Lie algebra homomorphism is an isomorphism.*

Definition 11.13 (Representation)**.** Let L be a real (complex) Lie algebra. A *representation* of L is a Lie algebra homomorphism $L \to GL(n, \mathbb{R})$ $(GL(n, \mathbb{C}))$ for some n.

Example 11.14. The linear transformation $\mathbb{R}^3 \to su(2)$ defined by

$$i \mapsto \sigma_x, j \mapsto \sigma_y, k \mapsto \sigma_z$$

is a representation of $\mathbb{R}^3$.

Part III

Topology

Chapter 12

Metric Spaces

Definition 12.1 (Metric)**.** A *metric* on a set X is a function $d : X^2 \to \mathbb{R}$ such that, for all $x, y, z \in X$:

- $d(x, y) \geq 0$
- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- **Triangle Inequality** $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* X consists of a set X and a metric on X.

Example 12.2. $\mathbb{R}^k$ is a metric space under $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$. The triangle inequality is Corollary 9.7.1.

Example 12.3. For any set X, the *discrete* metric on X is defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Proposition 12.4. *Let (X, d) be a metric space and Y a subset of X. Then $d \restriction Y^2$ is a metric on Y.*

Proof: Easy. □

12.1 Balls

Definition 12.5 (Open Ball)**.** Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *open ball* with *centre* $\vec{x}$ and *radius* r is

$$\{y \in \mathbb{R}^k : \|y - x\| < r\} \enspace .$$

Proposition 12.6. *Every open ball in $\mathbb{R}^k$ is convex.*

PROOF:
⟨1⟩1. LET: B be the open ball with center $\vec{a}$ and radius r.
⟨1⟩2. LET: $\vec{x}, \vec{y} \in B$
⟨1⟩3. LET: $\lambda \in (0, 1)$
⟨1⟩4. $\lambda\vec{x} + (1-\lambda)\vec{y} \in B$
PROOF:

$$\begin{aligned}\|\lambda\vec{x} + (1-\lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1-\lambda)(\vec{y} - \vec{a})\| \\ &= \lambda\|\vec{x} - \vec{a}\| + (1-\lambda)\|\vec{y} - \vec{a}\| \\ &< \lambda r + (1-\lambda)r \\ &= r\end{aligned}$$ □

□

12.2 Limit Points

Definition 12.7 (Limit Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is a *limit point* of E iff every open ball with centre p contains a point of E other than p.

Proposition 12.8. *Let X be a metric space. Let $E \subseteq X$. Let p be a limit point of E. Then every neighbourhood of p contains infinitely many points of E.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction N is a neighbourhood of p that contains only finitely many points $q_1, \ldots, q_n$ of $E - \{p\}$.
⟨1⟩2. LET: $r = \min(q_1, \ldots, q_n)$
⟨1⟩3. LET: B be the open ball with centre p and radius r.
⟨1⟩4. B is a neighbourhood of p that contains no points of E other than p.
□

Corollary 12.8.1. *A finite set has no limit points.*

Definition 12.9 (Isolated Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *isolated point* of E iff $p \in E$ and p is not a limit point of E.

12.3 Closed Sets

Definition 12.10 (Closed Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *closed* iff every limit point of E is a member of E.

12.4 Interior Points

Definition 12.11 (Interior Point)**.** Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *interior point* of E iff there exists an open ball B with centre p such that $B \subseteq E$.

Definition 12.12 (Interior)**.** The *interior* of a set E, denoted E°, is the set of all its interior points.

Proposition 12.13. *The interior of E is the largest open set that is included in E.*

PROOF:
$\langle 1 \rangle 1$. LET: I be the interior of E.
$\langle 1 \rangle 2$. I is open.
  $\langle 2 \rangle 1$. LET: $p \in I$
  $\langle 2 \rangle 2$. PICK an open ball B with centre p such that $B \subseteq E$.
  $\langle 2 \rangle 3$. $B \subseteq I$
    $\langle 3 \rangle 1$. LET: $q \in B$
    $\langle 3 \rangle 2$. There exists an open ball B' with centre q such that $B' \subseteq B$.
    $\langle 3 \rangle 3$. There exists an open ball B' with centre q such that $B' \subseteq E$.
    $\langle 3 \rangle 4$. $q \in I$
$\langle 1 \rangle 3$. If J is any open set and $J \subseteq E$ then $J \subseteq I$.
  $\langle 2 \rangle 1$. LET: J be an open set.
  $\langle 2 \rangle 2$. ASSUME: $J \subseteq E$
  $\langle 2 \rangle 3$. For all $p \in J$, there exists an open ball B with centre p such that $B \subseteq J$.
  $\langle 2 \rangle 4$. For all $p \in J$, there exists an open ball B with centre p such that $B \subseteq E$.
  $\langle 2 \rangle 5$. $p \in I$
$\square$

12.5 Open Sets

Definition 12.14 (Open Sets)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *open* iff every point in E is an interior point of E.

Proposition 12.15. *Every open ball is open.*

PROOF:
$\langle 1 \rangle 1$. LET: B be an open ball with centre c and radius r.
$\langle 1 \rangle 2$. LET: $x \in B$
$\langle 1 \rangle 3$. LET: $\epsilon = r - d(x, c)$
$\langle 1 \rangle 4$. LET: B' be the open ball with centre x and radius ϵ.
  PROVE: $B' \subseteq B$
$\langle 1 \rangle 5$. LET: $y \in B'$
$\langle 1 \rangle 6$. $d(y, c) < r$
  PROOF:
$$\begin{aligned} d(y,c) &\leq d(y,x) + d(x,c) && \text{(Triangle Inequality)} \\ &< \epsilon + d(x,c) && (\langle 1 \rangle 5) \\ &= r && (\langle 1 \rangle 3) \end{aligned}$$
$\square$

Proposition 12.16. *A set is open if and only if its complement is closed.*

PROOF:
⟨1⟩1. LET: $E \subseteq X$
⟨1⟩2. If E is open then $X - E$ is closed.
⟨2⟩1. ASSUME: E is open.
⟨2⟩2. LET: p be a limit point of $X - E$.
PROVE: $p \in X - E$
⟨2⟩3. ASSUME: for a contradiction $p \in E$.
⟨2⟩4. PICK an open ball B with centre p such that $B \subseteq E$.
⟨2⟩5. B contains a point of $X - E$.
PROOF: ⟨2⟩2
⟨2⟩6. Q.E.D.
PROOF: This contradicts ⟨2⟩4.
⟨1⟩3. If $X - E$ is closed then E is open.
⟨2⟩1. ASSUME: $X - E$ is closed.
⟨2⟩2. LET: $p \in E$
⟨2⟩3. ASSUME: for a contradiction no open ball with centre p is a subset of E.
⟨2⟩4. Every open ball with centre p intersects $X - E$.
⟨2⟩5. p is a limit point of $X - E$.
⟨2⟩6. $p \in X - E$
PROOF: ⟨2⟩1
⟨2⟩7. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
□

Corollary 12.16.1. *A set is closed if and only if its complement is open.*

Proposition 12.17. *The union of a set of open sets is open.*

PROOF:
⟨1⟩1. LET: $\mathcal{U}$ be a set of open sets.
⟨1⟩2. LET: $p \in \bigcup \mathcal{U}$
⟨1⟩3. PICK $U \in \mathcal{U}$ such that $p \in U$.
⟨1⟩4. PICK an open ball B with centre p such that $B \subseteq U$.
⟨1⟩5. $B \subseteq \bigcup \mathcal{U}$
□

Corollary 12.17.1. *The intersection of a set of closed sets is closed.*

Proposition 12.18. *The intersection of two open sets is open.*

PROOF:
⟨1⟩1. LET: U and V be open.
⟨1⟩2. LET: $p \in U \cap V$
⟨1⟩3. PICK open balls B_1 and B_2 with centre p such that $B_1 \subseteq U$ and $B_2 \subseteq V$.
⟨1⟩4. ASSUME: w.l.o.g. the radius of B_1 is $\leq$ the radius of B_2.
⟨1⟩5. $B_1 \subseteq U \cap V$
□

Corollary 12.18.1. *The union of two closed sets is closed.*

Example 12.19. The intersection of a set of open sets is not necessarily open.

For every positive integer n, we have $(-1/n, 1/n)$ is open in $\mathbb{R}$, but $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$ is not open.

Theorem 12.20. *Let X be a metric space. Let $Y \subseteq X$ and $E \subseteq Y$. Then E is open in Y if and only if there exists an open subset G of X such that $E = G \cap Y$.*

PROOF:
$\langle 1 \rangle 1$. If E is open in Y then there exists an open subset G of X such that $E = G \cap Y$.
  $\langle 2 \rangle 1$. ASSUME: E is open in Y.
  $\langle 2 \rangle 2$. For $p \in E$, PICK $r_p > 0$ such that the open ball in Y with centre p and radius r_p is included in E.
  $\langle 2 \rangle 3$. For $p \in E$,
    LET: V_p be the open ball in X with centre p and radius r_p.
  $\langle 2 \rangle 4$. LET: $G = \bigcup_{p \in E} V_p$
  $\langle 2 \rangle 5$. G is open in Y.
    PROOF: Proposition 12.17.
  $\langle 2 \rangle 6$. $E = G \cap Y$
    $\langle 3 \rangle 1$. $E \subseteq G \cap Y$
      $\langle 4 \rangle 1$. LET: $p \in E$
      $\langle 4 \rangle 2$. $p \in V_p$
      $\langle 4 \rangle 3$. $p \in G$
    $\langle 3 \rangle 2$. $G \cap Y \subseteq E$
      $\langle 4 \rangle 1$. LET: $x \in G \cap Y$
      $\langle 4 \rangle 2$. PICK $p \in E$ such that $x \in V_p$
      $\langle 4 \rangle 3$. $d(x, p) < r_p$
      $\langle 4 \rangle 4$. $x \in E$
$\langle 1 \rangle 2$. For any open subset G of X, we have $G \cap Y$ is open in Y.
  $\langle 2 \rangle 1$. LET: G be an open subset of X.
  $\langle 2 \rangle 2$. LET: $p \in G \cap Y$
  $\langle 2 \rangle 3$. PICK $r > 0$ such that the open ball in X with centre p and radius r is included in G.
  $\langle 2 \rangle 4$. The open ball in Y with centre p and radius r is included in $G \cap Y$.
$\square$

12.6 Perfect Sets

Definition 12.21 (Perfect Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *perfect* iff E is closed and every point in E is a limit point of E.

12.7 Bounded Sets

Definition 12.22 (Bounded Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *bounded* iff there exists a real number M and $q \in X$ such that, for all $p \in E$, we have $d(p, q) < M$.

Definition 12.23 (Diameter)**.** Let X be a metric space and $E \subseteq X$ be bounded. Then the *diameter* of E is $\sup\{d(x, y) : x, y \in E\}$.

Proposition 12.24. *Let X be a metric space. Let $E \subseteq X$ be bounded. Then $\overline{E}$ is bounded and*

$$\operatorname{diam} \overline{E} = \operatorname{diam} E \ .$$

PROOF:
⟨1⟩1. $\operatorname{diam} E$ is an upper bound for $\{d(x, y) : x, y \in \overline{E}\}$.
  ⟨2⟩1. LET: $x, y \in \overline{E}$
  ⟨2⟩2. For all $\epsilon > 0$ we have $d(x, y) < \operatorname{diam} E + \epsilon$.
    ⟨3⟩1. LET: $\epsilon > 0$
    ⟨3⟩2. PICK $x', y' \in E$ such that $d(x', x) < \epsilon/2$ and $d(y', y) < \epsilon/2$
    ⟨3⟩3. $d(x', y') < \operatorname{diam} E$
    ⟨3⟩4. $d(x, y) < \operatorname{diam} E + \epsilon$
  ⟨2⟩3. $d(x, y) \leq \operatorname{diam} E$
⟨1⟩2. $\operatorname{diam} \overline{E}$ is an upper bound for $\{d(x, y) : x, y \in E\}$.
  PROOF: This follows since $E \subseteq \overline{E}$.
□

12.8 Dense Sets

Definition 12.25 (Dense Set)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *dense* iff every point of X is either a limit point of E or a point of E, or both.

12.9 Closure

Definition 12.26 (Closure)**.** Let X be a metric space. Let $E \subseteq X$. Then the *closure* of E, denoted $\overline{E}$, is the union of E and the set of limit points of E.

Proposition 12.27. *$\overline{E}$ is the smallest closed set that includes E.*

PROOF:
⟨1⟩1. $\overline{E}$ is closed.
  ⟨2⟩1. LET: p be a limit point of $\overline{E}$.
  ⟨2⟩2. ASSUME: $p \notin E$
    PROVE: p is a limit point of E.
  ⟨2⟩3. LET: B be the open ball with centre p and radius r.
    PROVE: B intersects E.
  ⟨2⟩4. PICK a point $q \in B \cap \overline{E}$.
  ⟨2⟩5. PICK an open ball B' with centre q such that $B' \subseteq B$.

⟨2⟩6. Pick a point $r \in E \cap B'$
⟨2⟩7. $r \in E \cap B$
⟨1⟩2. If C is closed and $E \subseteq C$ then $\overline{E} \subseteq C$.
⟨2⟩1. Assume: C is closed.
⟨2⟩2. Assume: $E \subseteq C$
⟨2⟩3. Let: $p \in \overline{E}$
⟨2⟩4. Assume: for a contradiction $p \notin C$
⟨2⟩5. p is a limit point of C.
⟨3⟩1. Let: B be an open ball with centre p.
⟨3⟩2. B intersects E.
⟨3⟩3. B intersects C.
⟨3⟩4. B intersects C in a point other than p.
Proof: ⟨2⟩3
⟨2⟩6. Q.E.D.
Proof: This contradicts ⟨2⟩1.
□

Corollary 12.27.1. *E is closed if and only if $E = \overline{E}$.*

Theorem 12.28. *Let E be a nonempty set of real numbers bounded above. Then $\sup E \in \overline{E}$.*

Proof:
⟨1⟩1. Assume: $\sup E \notin E$
Prove: $\sup E$ is a limit point of E.
⟨1⟩2. Let: B be an open ball with centre $\sup E$ and radius r.
⟨1⟩3. There exists $x \in E$ such that $x > \sup E - r$.
⟨1⟩4. E intersects B in a point other than p.
□

Proposition 12.29.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proof:
⟨1⟩1. $\overline{A} \cup \overline{B}$ is a closed set that includes $A \cup B$.
⟨1⟩2. If C is a closed set that includes $A \cup B$ then $\overline{A} \cup \overline{B} \subseteq C$.
□

Example 12.30. It is not true in general. that $\overline{\bigcup \mathcal{A}} = \bigcup_{A \in \mathcal{A}} \overline{A}$.

In $\mathbb{R}$, let $\mathcal{A} = \{\{1/n\} : n \in \mathbb{Z}^+\}$. Then

$$\overline{\bigcup \mathcal{A}} = \{1/n : n \in \mathbb{Z}^+\} \cup \{0\}$$

$$\bigcup_{A \in \mathcal{A}} \overline{A} = \{1/n : n \in \mathbb{Z}^+\}$$

Proposition 12.31.

$$X - E^\circ = \overline{X - E}$$

PROOF:

$$\begin{aligned} p \in X - E^\circ &\Leftrightarrow p \notin E^\circ \\ &\Leftrightarrow \forall B \text{ an open ball with centre } p . B \nsubseteq E \\ &\Leftrightarrow \forall B \text{ an open ball with centre } p . B \text{ intersects} X - E \\ &\Leftrightarrow p \in \overline{X - E} \end{aligned}$$

∎

12.10 Compact Sets

Definition 12.32 (Open Cover)**.** Let X be a metric space. Let $E \subseteq X$. An *open cover* of E is a set $\mathcal{U}$ of open sets such that $E \subseteq \bigcup \mathcal{U}$.

Definition 12.33 (Compact Set)**.** Let X be a metric space. Let $K \subseteq X$. Then K is *compact* iff every open cover of K includes a finite subcover.

Proposition 12.34. *Every finite set is compact.*

PROOF: Easy. ∎

Theorem 12.35. *Let X be a metric space. Let $Y \subseteq X$ and $K \subseteq Y$. Then K is compact in Y if and only if K is compact in X.*

PROOF:
⟨1⟩1. If K is compact in Y then K is compact in X.
  ⟨2⟩1. ASSUME: K is compact in Y.
  ⟨2⟩2. LET: $\mathcal{U}$ be an open cover of K in X.
  ⟨2⟩3. $\{U \cap Y : U \in \mathcal{U}\}$ is an open cover of K in Y.
  ⟨2⟩4. PICK a finite subcover $\{U_1 \cap Y, \ldots, U_n \cap Y\}$
  ⟨2⟩5. $\{U_1, \ldots, U_n\}$ is a finite subset of $\mathcal{U}$ that is an open cover of K is X.
⟨1⟩2. If K is compact in X then K is compact in Y.
  ⟨2⟩1. ASSUME: K is compact in X.
  ⟨2⟩2. LET: $\mathcal{U}$ be an open cover of K in Y.
  ⟨2⟩3. $\{U \text{ open in } X : U \cap Y \in \mathcal{U}\}$ is an open cover of K in X.
  ⟨2⟩4. PICK a finite subcover $\{U_1, \ldots, U_n\}$.
  ⟨2⟩5. $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ is a subset of $\mathcal{U}$ that is an open cover of E in Y.
∎

Proposition 12.36. *Every compact set is closed.*

PROOF:
⟨1⟩1. LET: E be compact.
⟨1⟩2. LET: $p \in X - E$
  PROVE: There exists an open ball with centre p that is a subset of $X - E$.
⟨1⟩3. For all $q \in E$, there exist disjoint open balls B with centre q and B' with centre p.
⟨1⟩4. The set of open balls B such that there exists a disjoint open ball B' with centre p is an open cover of E.
⟨1⟩5. PICK a finite subcover $\{B_1, \ldots, B_n\}$.

⟨1⟩6. For $i = 1, \ldots, n$, PICK an open ball B'_i with centre p such that $B_i \cap B'_i = \emptyset$.
⟨1⟩7. $B'_1 \cap \cdots \cap B'_n$ is an open ball with centre p that is a subset of $X - E$.
□

Proposition 12.37. *Every closed subset of a compact set is compact.*

PROOF:
⟨1⟩1. LET: E be compact and $C \subseteq E$ be closed.
⟨1⟩2. LET: $\mathcal{U}$ be an open cover of C.
⟨1⟩3. $\mathcal{U} \cup \{X - C\}$ is an open cover of E.
⟨1⟩4. PICK a finite subcover $\{U_1, \ldots, U_n\}$ or $\{U_1, \ldots, U_n, X - C\}$.
⟨1⟩5. $\{U_1, \ldots, U_n\}$ covers C.
□

Corollary 12.37.1. *The intersection of a compact set and a closed set is compact.*

Proposition 12.38. *Let $\mathcal{K}$ be a nonempty set of compact sets. If every nonempty finite subset of $\mathcal{K}$ has nonempty intersection, then $\bigcap \mathcal{K}$ is nonempty.*

PROOF:
⟨1⟩1. PICK $K \in \mathcal{K}$
⟨1⟩2. ASSUME: $\bigcap \mathcal{K} = \emptyset$
⟨1⟩3. $\{X - K' : K' \in \mathcal{K}\}$ is an open cover of K.
⟨1⟩4. PICK a finite subcover $\{X - K_1, \ldots, X - K_n\}$.
⟨1⟩5. There exists $p \in K \cap K_1 \cap \cdots \cap K_n$
⟨1⟩6. Q.E.D.
PROOF: ⟨1⟩4 and ⟨1⟩5 form a contradiction.
□

Corollary 12.38.1. *Let (K_n) be a sequence of nonempty compact sets such that $K_0 \supseteq K_1 \supseteq \cdots$. Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.*

Theorem 12.39. *Let X be a metric space and $E \subseteq X$. Then E is compact if and only if every infinite subset of E has a limit point in E.*

PROOF:
⟨1⟩1. If E is compact then every infinite subset of E has a limit point in E.
⟨2⟩1. ASSUME: E is compact.
⟨2⟩2. LET: $A \subseteq E$ be infinite.
⟨2⟩3. ASSUME: for a contradiction E has no limit point in K.
⟨2⟩4. For all $p \in K$, there exists an open ball B with centre p such that B does not intersect E outside p.
⟨2⟩5. The set of open balls that intersect E in at most one point is an open cover for K.
⟨2⟩6. PICK a finite subcover $B_1, \ldots, B_n$.
⟨2⟩7. E has at most n points.
⟨2⟩8. Q.E.D.
PROOF: This contradicts the fact that E is finite.

⟨1⟩2. If every infinite subset of K has a limit point in K then K is compact.
⟨2⟩1. ASSUME: Every infinite subset of K has a limit point in K.
⟨2⟩2. LET: $\mathcal{U}$ be an open cover of K.
⟨2⟩3. ASSUME: w.l.o.g. $\mathcal{U}$ is countable.
PROOF: We may replace $\mathcal{U}$ with the set of all open balls B with centres in $\mathbb{Q}^2$ and rational radius such that there exists $U \in \mathcal{U}$ such that $B \subseteq U$.
⟨2⟩4. PICK an enumeration $\mathcal{U} = \{G_n : n \in \mathbb{N}\}$.
⟨2⟩5. For $n \in \mathbb{N}$,
LET: $F_n = \bigcup_{i=0}^{n} G_n$.
⟨2⟩6. For all $n \in \mathbb{N}$, we have $K - F_n \neq \emptyset$.
PROOF: Since $\{G_0, \ldots, G_n\}$ does not cover K.
⟨2⟩7. $\bigcap_{n=0}^{\infty} F_n = \emptyset$
PROOF: Since $\{G_n : n \in \mathbb{N}\}$ covers K.
⟨2⟩8. For $n \in \mathbb{N}$, PICK $a_n \in K - F_n$
⟨2⟩9. LET: $E = \{a_n : n \in \mathbb{N}\}$
⟨2⟩10. E is infinite.
⟨3⟩1. LET: $n \in \mathbb{N}$
PROVE: there exists m such that $a_m \notin \{a_0, a_1, \ldots, a_n\}$.
⟨3⟩2. For $i = 0, \ldots, n$, PICK k_i such that $a_i \in G_{k_i}$.
⟨3⟩3. LET: $m = \max(k_0, \ldots, k_n)$
⟨3⟩4. ASSUME: for a contradiction $a_m = a_i$ for some $i = 0, \ldots, n$
⟨3⟩5. $a_i \in G_{k_i}$
⟨3⟩6. $a_i \notin F_m$
⟨3⟩7. Q.E.D.
PROOF: This is a contradiction since $k_i \leq m$.
⟨2⟩11. PICK a limit point l for E in K.
PROOF: From ⟨2⟩1.
⟨2⟩12. PICK n such that $l \in G_n$.
⟨2⟩13. PICK an open ball B with centre l such that $B \subseteq G_n$
⟨2⟩14. $B \cap E$ is infinite.
PROOF: Proposition 12.8.
⟨2⟩15. PICK $m \geq$ ‘n such that $a_m \in B$.
⟨2⟩16. $a_m \in G_n$
⟨2⟩17. Q.E.D.
PROOF: This is a contradiction since $a_m \notin F_m$.
□

Theorem 12.40 (Heine-Borel). *Let $E \subseteq \mathbb{R}^k$. Then E is compact if and only if it is closed and bounded.*

PROOF:
⟨1⟩1. If E is compact then E is closed.
PROOF: Proposition 12.36.
⟨1⟩2. If E is compact then E is bounded.
PROOF: Otherwise $\{(-N, N)^k : N \in \mathbb{Z}^+\}$ would be an open cover of E with no finite subcover.
⟨1⟩3. If E is closed and bounded then E is compact.

⟨2⟩1. ASSUME: E is closed and bounded.
⟨2⟩2. PICK $\vec{c}$ and M such that $\forall \vec{x} \in E. \|\vec{x} - \vec{c}\| < M$.
⟨2⟩3. $E \subseteq \prod_{i=1}^{k} [c_i - M, c_i + M]$
⟨2⟩4. E is compact.
PROOF: Proposition 12.37.
□

Corollary 12.40.1 (Weierstrass's Theorem)**.** *Every bounded infinite subset of* $\mathbb{R}^k$ *has a limit point.*

PROOF: It is a bounded infinite subset of some k-cell and therefore has a limit point in that k-cell. □

Example 12.41. It is not true that, in any metric space, a set is compact if and only if it is closed and bounded.

In $\mathbb{Q}$, the set $\{p \in \mathbb{Q} : 2 < p^2 < 3\}$ is closed and bounded but not compact.

Theorem 12.42. *Every nonempty perfect set in* $\mathbb{R}^k$ *is uncountable.*

PROOF:
⟨1⟩1. LET: P be a nonempty perfect set in $\mathbb{R}^k$.
⟨1⟩2. P is infinite.
PROOF: Corollary 12.8.1.
⟨1⟩3. ASSUME: for a contradiction P is countable.
⟨1⟩4. PICK an enumeration $P = \{x_n : n \in \mathbb{N}\}$.
⟨1⟩5. PICK a sequence (V_n) of open balls such that, for all n, we have $\overline{V_{n+1}} \subseteq V_n$ and $x_n \notin \overline{V_{n+1}}$ and $V_n \cap P \neq \emptyset$
⟨2⟩1. ASSUME: as induction hypothesis we have picked $V_0, \ldots, V_{n-1}$ that satisfy these conditions.
⟨2⟩2. PICK $p \in P \cap V_n$ such that $p \neq x_n$
PROOF: We cannot have $P \cap V_n = \{x_n\}$ because then V_n would be a neighbourhood of x_n that only intersects P at x_n.
⟨2⟩3. PICK an open ball B with centre p such that $B \subseteq V_n \cap P - \{x_n\}$
⟨2⟩4. LET: V_{n+1} be the open ball with centre p and half the radius of B.
⟨2⟩5. $\overline{V_{n+1}} \subseteq V_n$
PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq V_n$.
⟨2⟩6. $x_n \notin \overline{V_{n+1}}$
PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq P - \{x_n\}$.
⟨2⟩7. $V_{n+1} \cap P \neq \emptyset$
PROOF: Since $p \in V_{n+1} \cap P$.
⟨1⟩6. For $n \in \mathbb{N}$,
LET: $K_n = \overline{V_n} \cap P$.
⟨1⟩7. For all $n \in \mathbb{N}$, K_n is compact.
PROOF: By the Heine-Borel Theorem.
⟨1⟩8. $\bigcap_{n=0}^{\infty} K_n \cap P = \emptyset$
PROOF: Since for each n we have $x_n \notin K_{n+1}$.
⟨1⟩9. $\bigcap_{n=0}^{\infty} K_n = \emptyset$
PROOF: Since $\bigcap_{n=0}^{\infty} K_n \subseteq P$.

⟨1⟩10. Q.E.D.
PROOF: This contradicts Proposition 12.38.
□

Corollary 12.42.1. *For any $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a, b]$ is uncountable.*

Corollary 12.42.2. *$\mathbb{R}$ is uncountable.*

Corollary 12.42.3. *The set of transcendental numbers is uncountable.*

PROOF: Since the set of algebraic numbers is countable. □

Example 12.43. The Cantor set is a perfect set in $\mathbb{R}$ that does not include any open interval.

PROOF:
⟨1⟩1. LET: (E_n) be the sequence of unions of closed intervals from the definition of the Cantor set, and C be the Cantor set.
⟨1⟩2. $C \neq \emptyset$
PROOF: Since $0 \in C$.
⟨1⟩3. C is closed.
PROOF: Each E_n is closed and C is their intersection.
⟨1⟩4. Every point of C is a limit point of C.
⟨2⟩1. LET: $p \in C$
⟨2⟩2. LET: B be an open ball with centre p and radius r.
⟨2⟩3. PICK n such that each of the intervals that make up E_n has length $< r/2$.
⟨2⟩4. LET: I be the interval in E_n that contains p.
⟨2⟩5. $I \subseteq B$
⟨2⟩6. The endpoint of I that is not p is in $P \cap B$.
⟨1⟩5. C does not include any open interval.
⟨2⟩1. LET: (α, β) be any open interval.
⟨2⟩2. PICK m such that $3^{-m} < (\beta - \alpha)/6$
⟨2⟩3. PICK k such that $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq (\alpha, \beta)$
⟨2⟩4. $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subseteq P$
⟨2⟩5. $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \cap E_m = \emptyset$
⟨2⟩6. Q.E.D.
PROOF: This is a contradiction.
□

Corollary 12.43.1. *The Cantor set is uncountable.*

Proposition 12.44. *Let X be a metric space. Let (K_n) be a sequence of compact sets in X such that $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$. Assume $\operatorname{diam} K_n \to 0$ as $n \to \infty$. Then $\bigcap_{n=0}^{\infty} K_n$ is a singleton.*

PROOF:
⟨1⟩1. $\bigcap_n K_n \neq \emptyset$

PROOF: Corollary 12.38.1.
⟨1⟩2. $\bigcap_n K_n$ has no more than one point.
⟨2⟩1. ASSUME: for a contradiction $a, b \in \bigcap_n K_n$ with $a \neq b$.
⟨2⟩2. LET: $\epsilon = d(a, b)$
⟨2⟩3. PICK n such that $\operatorname{diam} K_n < \epsilon$
⟨2⟩4. $a, b \in K_n$
⟨2⟩5. Q.E.D.
PROOF: This is a contradiction.
□

12.11 Connected Sets

Definition 12.45 (Separated)**.** Let X be a metric space. Let $A, B \subseteq X$. Then A and B are *separated* iff $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Proposition 12.46. *Any two disjoint open sets are separated.*

PROOF:
⟨1⟩1. LET: A and B be disjoint open sets.
⟨1⟩2. ASSUME: for a contradiction $p \in \overline{A} \cap B$.
⟨1⟩3. B is a neighbourhood of p.
⟨1⟩4. B intersects A.
□

Definition 12.47 (Connected)**.** Let X be a metric space. Let $E \subseteq X$. Then E is *connected* iff E is not the union of two nonempty separated sets.

Theorem 12.48. *A subset E of the real line is connected if and only if it is convex.*

PROOF:
⟨1⟩1. If E is connected then E is convex.
⟨2⟩1. ASSUME: E is connected.
⟨2⟩2. LET: $x, y \in E$
⟨2⟩3. LET: $z \in (x, y)$
⟨2⟩4. $z \in E$
PROOF: Otherwise $E \cap (-\infty, z)$ and $E \cap (z, +\infty)$ would be a separation of E.
⟨1⟩2. If E is convex then E is connected.
⟨2⟩1. ASSUME: E is convex.
⟨2⟩2. ASSUME: for a contradiction $E = A \cup B$ where A and B are nonempty and separated.
⟨2⟩3. PICK $a \in A$ and $b \in B$.
⟨2⟩4. ASSUME: w.l.o.g. $a < b$
⟨2⟩5. LET: $z = \sup(A \cap [a, b])$
⟨2⟩6. $z \in \overline{A}$
⟨2⟩7. $z \notin B$

⟨2⟩8. $z < b$
⟨2⟩9. CASE: $z \in A$
⟨3⟩1. $z \notin \overline{B}$
⟨3⟩2. PICK $z_1 \in (z, b)$ such that $z_1 \notin B$
⟨3⟩3. $a < z_1 < b$
⟨3⟩4. $z_1 \notin E$
PROOF: We have $z_1 \notin A$ from ⟨2⟩5 since $z_1 \in [a, b]$ and $z_1 > z$, and $z_1 \notin B$ from ⟨3⟩2.
⟨3⟩5. Q.E.D.
PROOF: This contradicts ⟨2⟩1.
⟨2⟩10. CASE: $z \notin A$
PROOF: Then $a < z < b$ and $z \notin E$ contradicting ⟨2⟩1.
□

Proposition 12.49. *Every connected metric space with more than one point is uncountable.*

PROOF:
⟨1⟩1. LET: X be a connected metric space with more than one points.
⟨1⟩2. PICK distinct points $p, q \in X$.
⟨1⟩3. LET: $\epsilon = d(p, q)$
⟨1⟩4. For every $r \in (0, \epsilon)$, there exists a point $x \in X$ such that $d(p, x) = r$.
PROOF: Otherwise $\{x \in X : d(p, x) < r\}$ and $\{x \in X : d(p, x) > r\}$ would form a separation of X.
□

Proposition 12.50. *The closure of a connected set is connected.*

PROOF:
⟨1⟩1. LET: X be a metric space.
⟨1⟩2. LET: E be a connected subspace of X.
⟨1⟩3. ASSUME: for a contradiction A and B form a separation of $\overline{E}$
PROVE: $A \cap E$ and $B \cap E$ form a separation of E.
⟨1⟩4. $A \cap E \neq \emptyset$
⟨2⟩1. ASSUME: for a contradiction $A \cap E = \emptyset$
⟨2⟩2. $E \subseteq B$
⟨2⟩3. $\overline{E} \subseteq \overline{B}$
⟨2⟩4. $A \subseteq \overline{B}$
⟨2⟩5. $A \cap \overline{B} = A \neq \emptyset$
⟨2⟩6. Q.E.D.
PROOF: This contradicts ⟨1⟩3.
⟨1⟩5. $B \cap E \neq \emptyset$
PROOF: Similar.
⟨1⟩6. $\overline{A \cap E} \cap B \cap E = \emptyset$
PROOF: Since $\overline{A \cap E} \cap B \cap E \subseteq \overline{A} \cap B$.
⟨1⟩7. $A \cap E \cap \overline{B \cap E} = \emptyset$
PROOF: Similar.

□

Example 12.51. The interior of a connected set is not necessarily connected.

Two touching discs in $\mathbb{R}^2$ form a connected set but the interior is disconnected.

Proposition 12.52. *Every convex set in $\mathbb{R}^k$ is connected.*

PROOF:
⟨1⟩1. LET: E be a convex set in $\mathbb{R}^k$.
⟨1⟩2. ASSUME: for a contradiction A and B form a separation of E.
⟨1⟩3. PICK $\vec{a} \in A$ and $\vec{b} \in B$.
⟨1⟩4. Define $p : [0,1] \to \mathbb{R}^k$ by $p(t) = (1-t)\vec{a} + t\vec{b}$.
⟨1⟩5. $p^{-1}(A)$ and $p^{-1}(B)$ are separated sets in $\mathbb{R}$.
⟨1⟩6. PICK $x \in [0,1]$ such that $x \notin p^{-1}(A)$ and $x \notin p^{-1}(B)$.
  PROOF: There exists such an x since $[0,1]$ is connected.
⟨1⟩7. $p(x) \in E$
  PROOF: Since E is convex.
⟨1⟩8. $p(x) \notin A \cup B$
⟨1⟩9. Q.E.D.
  PROOF: This contradicts ⟨1⟩2.
□

12.12 Separable Spaces

Definition 12.53 (Separable)**.** A metric space is *separable* iff it has a countable dense subset.

Example 12.54. $\mathbb{R}^k$ is separable since $\mathbb{Q}^k$ is dense.

Proposition 12.55. *Every compact metric space is separable.*

PROOF:
⟨1⟩1. LET: X be a compact metric space.
⟨1⟩2. For $n \in \mathbb{Z}^+$, pick finitely many points $a_{n1}, \ldots, a_{nr_n}$ such that $\{B(a_{ni}, 1/n) : 1 \leq i \leq r_n\}$ covers X.
  PROOF: Since $\{B(x, 1/n) : x \in X\}$ covers X.
⟨1⟩3. $\{a_{ni} : n \in \mathbb{Z}^+, 1 \leq i \leq r_n\}$ is dense.
  ⟨2⟩1. LET: U be an open set and $p \in U$.
  ⟨2⟩2. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$.
  ⟨2⟩3. PICK n such that $1/n < \epsilon$.
  ⟨2⟩4. PICK i such that $p \in B(a_{ni}, 1/n)$
  ⟨2⟩5. $a_{ni} \in U$
□

12.13 Bases

Definition 12.56 (Basis)**.** A *basis* for a metric space X is a set $\mathcal{B}$ of open sets such that, for every open set U and point $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.

Proposition 12.57. *Every separable metric space has a countable basis.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a separable metric space.
$\langle 1 \rangle 2$. PICK a countable dense set D in X.
$\langle 1 \rangle 3$. LET: $\mathcal{B} = \{B(p, \epsilon) : p \in D, \epsilon \in \mathbb{Q}^+\}$
PROVE: $\mathcal{B}$ is a basis.
$\langle 1 \rangle 4$. LET: U be an open set in X and $p \in U$
$\langle 1 \rangle 5$. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$
$\langle 1 \rangle 6$. PICK $q \in B(p, \epsilon) \cap D$
$\langle 1 \rangle 7$. PICK a rational δ such that $d(p, q) < \delta < \epsilon$.
$\langle 1 \rangle 8$. $B(q, \delta) \in \mathcal{B}$ and $B(q, \delta) \subseteq U$.
$\square$

12.14 Condensation Points

Definition 12.58 (Condensation Point)**.** Let X be a metric space, $p \in X$ and $E \subseteq X$. Then p is a *condensation point* of E iff every neighbourhood of p contains uncountably many points in E.

Proposition 12.59. *Let X be a metric space. Let $E \subseteq X$. Let P be the set of condensation points of E. Then P is perfect.*

PROOF:
$\langle 1 \rangle 1$. P is closed.
  $\langle 2 \rangle 1$. LET: $p \in X - P$
  $\langle 2 \rangle 2$. PICK a neighbourhood U of p that contains only countably many points of E.
  $\langle 2 \rangle 3$. For every $x \in U$, we have that U is a neighbourhood of x that contains only countably many points of E.
  $\langle 2 \rangle 4$. $p \in U \subseteq X - P$
$\langle 1 \rangle 2$. Every point in P is a limit point of P.
  PROOF: Immediate from definitions.
$\square$

Proposition 12.60. *Let X be a metric space with a countable basis. Let $E \subseteq X$ be uncountable. Let P be the set of condensation points of E. Then $E - P$ is countable.*

PROOF:
$\langle 1 \rangle 1$. PICK a countable basis $\mathcal{B}$ for X.
$\langle 1 \rangle 2$. LET: $W = \bigcup\{B \in \mathcal{B} : E \cap B \text{ is countable}\}$

$\langle 1\rangle 3.$ $P = X - W$
$\langle 2\rangle 1.$ $P \subseteq X - W$
$\langle 3\rangle 1.$ Assume: for a contradiction $p \in P \cap W$
$\langle 3\rangle 2.$ Pick $B \in \mathcal{B}$ such that $p \in B$ and $E \cap B$ is countable.
$\langle 3\rangle 3.$ $E \cap B$ is uncountable.
$\langle 3\rangle 4.$ Q.E.D.
Proof: This is a contradiction.
$\langle 2\rangle 2.$ $X - W \subseteq P$
$\langle 3\rangle 1.$ Let: $p \in X - W$
$\langle 3\rangle 2.$ Let: U be a neighbourhood of p.
$\langle 3\rangle 3.$ Pick $B \in \mathcal{B}$ such that $p \in B \subseteq U$.
$\langle 3\rangle 4.$ $E \cap B$ is uncountable.
Proof: Since $p \notin W$.
$\langle 3\rangle 5.$ $E \cap W$ is uncountable.
$\langle 1\rangle 4.$ $E - P = E \cap W$
$\langle 1\rangle 5.$ $E - P$ is countable.
□

Corollary 12.60.1. *Every closed subset of a metric space with a countable basis is the union of a perfect set and a countable set.*

Proof:
$\langle 1\rangle 1.$ Let: X be a metric space with a countable basis.
$\langle 1\rangle 2.$ Let: E be a closed subset of X.
$\langle 1\rangle 3.$ Let: P be the set of condensation points of E.
$\langle 1\rangle 4.$ $E - P$ is countable.
Proof: Proposition 12.60.
$\langle 1\rangle 5.$ $P \cap E$ is perfect.
$\langle 2\rangle 1.$ $P \cap E$ is closed.
Proof: Proposition 12.59.
$\langle 2\rangle 2.$ Every point in $P \cap E$ is a limit point of $P \cap E$.
$\langle 3\rangle 1.$ Let: $l \in P \cap E$
$\langle 3\rangle 2.$ Let: U be a neighbourhood of l.
$\langle 3\rangle 3.$ Pick $x \in P \cap U$
$\langle 3\rangle 4.$ U is a neighbourhood of x.
$\langle 3\rangle 5.$ U contains uncountably many points of E.
$\langle 3\rangle 6.$ U intersects $P \cap E$
Proof: It cannot be that every point in U and E is not in P since $E - P$ is countable.
□

Corollary 12.60.2. *Let X be a metric space with a countable basis. Then every countable set in X has an isolated point.*

Chapter 13

Convergence

Definition 13.1 (Converge)**.** Let X be a metric space. Let (p_n) be a sequence in X and $l \in X$. Then we say (p_n) *converges* to the *limit* l, and write

$$p_n \to l \text{ as } n \to \infty \enspace ,$$

iff for every $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon$.

We say (p_n) *diverges* iff it does not converge to any limit.

Proposition 13.2. *A sequence has at most one limit.*

PROOF:
$\langle 1 \rangle$1. ASSUME: $p_n \to l$ and $p_n \to m$ as $n \to \infty$.
$\langle 1 \rangle$2. ASSUME: for a contradiction $l \neq m$.
$\langle 1 \rangle$3. LET: $\epsilon = d(l, m)/2$
$\langle 1 \rangle$4. There exists N such that $\forall n \geq N. d(p_n, l) < \epsilon$ and $d(p_n, m) < \epsilon$
$\langle 1 \rangle$5. $d(l, m) < 2\epsilon$
$\langle 1 \rangle$6. Q.E.D.
PROOF: This is a contradiction.
□

Proposition 13.3. *Every convergent sequence is bounded.*

PROOF:
$\langle 1 \rangle$1. LET: $p_n \to l$ as $n \to \infty$
$\langle 1 \rangle$2. PICK N such that $\forall n \geq N. d(p_n, l) < 1$
$\langle 1 \rangle$3. LET: $M = \max(d(p_0, l), \ldots, d(p_{N-1}, l), 1)$
$\langle 1 \rangle$4. For all n, we have $d(p_n, l) \leq M$.
□

Proposition 13.4. *If l is a limit point of E, then there exists a sequence in E that converges to l.*

PROOF:

$\langle 1 \rangle 1$. For $n \in \mathbb{Z}^+$, Pick a point $a_n \in E$ such that $d(a_n, l) < 1/n$.
Proof: Since $B(l, 1/n)$ intersects E.
$\langle 1 \rangle 2$. $a_n \to l$ as $n \to \infty$.
□

Corollary 13.4.1. *Every sequence in a compact metric space has a convergent subsequence.*

Proof: By Theorem 12.39. □

Proposition 13.5. *Assume $s_n \to s$ and $t_n \to t$ in $\mathbb{R}^k$. Then $s_n + t_n \to s + t$.*

Proof:
$\langle 1 \rangle 1$. Let: $\epsilon > 0$
$\langle 1 \rangle 2$. Pick N such that, for all $n \geq N$, we have $\|s_n - s\| < \epsilon/2$ and $\|t_n - t\| < \epsilon/2$.
$\langle 1 \rangle 3$. For all $n \geq N$ we have $\|(s_n + t_n) - (s + t)\| < \epsilon$.
Proof: Since $\|(s_n + t_n) - (s + t)\| \leq \|s_n - s\| + \|t_n - t\|$.
□

Lemma 13.6. *If $s_n \to s$ as $n \to \infty$ in $\mathbb{C}$, and $c \in \mathbb{C}$, then $cs_n \to cs$ as $n \to \infty$.*

Proof:
$\langle 1 \rangle 1$. Let: $\epsilon > 0$
$\langle 1 \rangle 2$. Assume: w.l.o.g. $c \neq 0$
$\langle 1 \rangle 3$. Pick N such that $\forall n \geq N . |s_n - s| < \epsilon / |c|$.
$\langle 1 \rangle 4$. $\forall n \geq N . |cs_n - cs| < \epsilon$
□

Proposition 13.7. *If $s_n \to s$ and $t_n \to t$ in $\mathbb{C}$ then $s_n t_n \to st$.*

Proof:
$\langle 1 \rangle 1$. $(s_n - s)(t_n - t) \to 0$ as $n \to \infty$
$\langle 2 \rangle 1$. Let: $\epsilon > 0$
$\langle 2 \rangle 2$. Pick N such that, for all $n \geq N$, we have $|s_n - s| < \sqrt{\epsilon}$ and $|t_n - t| < \sqrt{\epsilon}$.
$\langle 2 \rangle 3$. For all $n \geq N$ we have $|(s_n - s)(t_n - t)| < \epsilon$
$\langle 1 \rangle 2$. $s_n t_n - st \to 0$ as $n \to \infty$
Proof:

$$\begin{aligned} s_n t_n - st &= (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s) \\ &\to 0 & \text{as } n \to \infty \end{aligned}$$

□

Proposition 13.8. *If $s_n \to s$ as $n \to \infty$ in $\mathbb{C}$, and every s_n and s is nonzero, then $1/s_n \to 1/s$ as $n \to \infty$.*

Proof:
$\langle 1 \rangle 1$. Pick m such that, for all $n \geq m$, we have $|s_n - s| < \frac{1}{2}|s|$.
$\langle 1 \rangle 2$. $\forall n \geq m . |s_n| > \frac{1}{2}|s|$
$\langle 1 \rangle 3$. Let: $\epsilon > 0$

⟨1⟩4. Pick $N > m$ such that, for all $n \geq N$, we have

$$|s_n - s| < \frac{1}{2}|s|^2\epsilon \ .$$

⟨1⟩5. For all $n \geq N$, we have

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| < \epsilon \ .$$

Proof:

$$\begin{aligned}\left|\frac{1}{s_n} - \frac{1}{s}\right| &= \frac{|s_n - s|}{|s_n||s|} \\ &< \frac{|s|^2\epsilon}{2|s_n||s|} \\ &= \frac{|s|\epsilon}{2|s_n|} \\ &< \epsilon\end{aligned}$$

□

Theorem 13.9. *Let $(\vec{x_n})$ be a sequence in $\mathbb{R}^k$ and $\vec{l} \in \mathbb{R}^k$. Then $\vec{x_n} \to \vec{l}$ as $n \to \infty$ iff, for $i = 1, \ldots, k$, we have $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ as $n \to \infty$.*

Proof:
⟨1⟩1. If $\vec{x_n} \to \vec{l}$ then $\pi_i(\vec{x_n}) \to \pi_i(l)$.
⟨2⟩1. $\|\vec{x_n} - \vec{l}\| \to 0$ as $n \to \infty$.
⟨2⟩2. $\sqrt{\sum_{i=1}^k (\pi_i(\vec{x_n}) - \pi_i(l))^2} \to 0$ as $n \to \infty$.
⟨2⟩3. $\sum_{i=1}^k (\pi_i(\vec{x_n}) - \pi_i(l))^2 \to 0$ as $n \to \infty$.
⟨2⟩4. $(\pi_i(\vec{x_n}) - \pi_i(l))^2 \to 0$ as $n \to \infty$
⟨2⟩5. $\pi_i(\vec{x_n}) - \pi_i(l) \to 0$ as $n \to \infty$.
⟨1⟩2. If $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ for every i then $\vec{x_n} \to l$.
⟨2⟩1. Assume: $\pi_i(\vec{x_n}) \to \pi_i(\vec{l})$ for every i.
⟨2⟩2. $\vec{x_n} \to \vec{l}$
Proof:

$$\begin{aligned}\|\vec{x_n} - \vec{l}\|^2 &= \sum_{i=1}^k (\pi_i(\vec{x_n}) - \pi_i(\vec{l}))^2 \\ &\to 0\end{aligned}$$

□

Corollary 13.9.1. *If $\beta_n \to \beta$ in $\mathbb{R}$ and $\vec{x_n} \to \vec{l}$ in $\mathbb{R}^k$, then $\beta_n\vec{x_n} \to \beta\vec{l}$.*

Proposition 13.10. *If $\vec{x_n} \to \vec{x}$ and $\vec{y_n} \to \vec{y}$ in $\mathbb{R}^k$, then $\vec{x_n} \cdot \vec{y_n} \to \vec{x} \cdot \vec{y}$.*

PROOF:

$$\begin{aligned}\vec{x_n} \cdot \vec{y_n} &= \sum_{i=1}^{k} \pi_i(\vec{x_n})\pi_i(\vec{y_n}) \\ &\to \sum_{i=1}^{k} \pi_i(\vec{x})\pi_i(\vec{y}) \\ &= \vec{x} \cdot \vec{y}\end{aligned}$$

□

Proposition 13.11. *Let (p_n) be a sequence in the metric space X. The set E^* of all limits of convergent subsequences is a closed set.*

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. $\{p_n : n \in \mathbb{N}\}$ is infinite.
⟨1⟩2. LET: q be a limit point of E^*.
PROVE: $q \in E^*$
⟨1⟩3. PICK an integer n_0 such that $q \neq p_{n_0}$.
⟨1⟩4. Extend a strictly increasing sequence of integers (n_i) such that, for all i, we have $d(q, p_{n_i}) \leq 2^i d(q, p_{n_0})$.
⟨2⟩1. ASSUME: as induction hypothesis we have picked $n_0 < n_1 < \cdots < n_i$ such that, for $0 \leq j \leq i$, we have $d(q, p_{n_j}) \leq 2^j d(q, p_{n_0})$.
⟨2⟩2. PICK $x \in E^*$ such that $d(x, q) < 2^{-(i+2)}\delta$
⟨2⟩3. There exists a subsequence of (p_n) that converges to x.
⟨2⟩4. There exists $n_{i+1} > n_i$ such that $d(p_{n_{i+1}}, x) < 2^{-(i+2)}\delta$.
⟨2⟩5. $d(p_{n_{i+1}}, q) < 2^{-(i+1)}\delta$
⟨1⟩5. $p_{n_i} \to q$ as $i \to \infty$.
⟨1⟩6. $q \in E^*$
□

Theorem 13.12. *Every monotonically increasing sequence in $\mathbb{R}$ that is bounded above converges to its supremum.*

PROOF:
⟨1⟩1. LET: (s_n) be a monotonically increasing sequence with supremum s.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK S such that $|s_N - s| < \epsilon$
⟨1⟩4. For all $n \geq N$, we have $s - \epsilon < s - s_N \leq s - s_n \leq s$.
⟨1⟩5. $\forall n \geq N. |s_n - s| < \epsilon$
□

Theorem 13.13. *Every monotonically decreasing sequence in $\mathbb{R}$ that is bounded below converges to its infimum.*

PROOF: Similar. □

Proposition 13.14 (Sandwich Theorem)**.** *Let (a_n), (b_n) and (c_n) be sequences of real numbers and $l \in \mathbb{R}$. Assume $\forall n. a_n \leq b_n \leq c_n$ and $a_n \to l$ and $c_n \to l$. Then $b_n \to l$.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N such that, for all $n \geq N$, we have $|a_n - l| < \epsilon$ and $|c_n - l| < \epsilon$.
⟨1⟩3. $\forall n \geq N . |b_n - l| < \epsilon$
□

Theorem 13.15. *For any real $p > 0$ we have*

$$\frac{1}{(n+1)^p} \to 0$$

as $n \to \infty$.

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N such that $N > (1/\epsilon)^{1/p}$.
⟨1⟩3. LET: $n \geq N$
⟨1⟩4. $1/n^p < \epsilon$
□

Theorem 13.16. *For any real $p > 0$ we have*

$$p^{\frac{1}{n+1}} \to 1$$

as $n \to \infty$.

PROOF:
⟨1⟩1. CASE: $p > 1$
⟨2⟩1. For $n \in \mathbb{N}$
LET: $x_n = p^{\frac{1}{n+1}} - 1$.
⟨2⟩2. $\forall n \in \mathbb{N} . x_n > 0$
⟨2⟩3. $\forall n \in \mathbb{N}$.

$$1 + (n+1)x_n \leq p \ .$$

PROOF: Since $1 + (n+1)x_n \leq (1 + x_n)^{n+1}$ by the Binomial Theorem.
⟨2⟩4. $\forall n \in \mathbb{N}$.

$$0 < x_n \leq \frac{p-1}{n+1} \ .$$

⟨2⟩5. $x_n \to 0$ as $n \to \infty$.
PROOF: Sandwich Theorem.
⟨1⟩2. CASE: $p = 1$
PROOF: Trivial.
⟨1⟩3. CASE: $p < 1$
PROOF: Then $p^{1/(n+1)} = 1/((1/p)^{1/(n+1)}) \to 1/1 = 1$ by ⟨1⟩1.
□

Theorem 13.17.

$$(n+1)^{1/(n+1)} \to 1 \text{ as } n \to \infty$$

PROOF:

⟨1⟩1. For $n \in \mathbb{N}$,
LET: $x_n = (n+1)^{1/(n+1)} - 1$.
⟨1⟩2. $\forall n \in \mathbb{N}. x_n \geq 0$
⟨1⟩3. $\forall n \in \mathbb{N}$

$$n + 1 \geq \frac{n(n+1)}{2} x_n^2 \quad .$$

PROOF: Since $(1 + x_n)^{n+1} \geq \frac{n(n+1)}{2} x_n^2$ by the Binomial Theorem.
⟨1⟩4. $\forall n \geq 1$

$$0 \leq x_n \leq \sqrt{\frac{2}{n}}$$

⟨1⟩5. $x_n \to 0$ as $n \to \infty$.
PROOF: Sandwich Theorem.
□

Theorem 13.18. *Let p and α be real numbers with $p > 0$. Then*

$$\frac{n^\alpha}{(1+p)^n} \to 0 \text{ as } n \to \infty \quad .$$

PROOF:
⟨1⟩1. PICK a positive integer k such that $k > \alpha$.
PROOF: Archimedean Property.
⟨1⟩2. $\forall n > 2k$

$$(1+p)^n > \frac{n^k p^k}{2^k k!} \quad .$$

$$\begin{aligned} (1+p)^n &> \binom{n}{k} p^k && \text{(Binomial Theorem)} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k!} p^k \\ &> \frac{n^k p^k}{2^k k!} && (n > 2k \text{ so if } n - k < i \leq n \text{ then } i > n/2) \end{aligned}$$

⟨1⟩3. $\forall n > 2k$

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k} \quad .$$

⟨1⟩4. $n^{\alpha-k} \to 0$ as $n \to \infty$
PROOF: Theorem 13.15.
⟨1⟩5. $\frac{n^\alpha}{(1+p)^n} \to 0$ as $n \to \infty$.
PROOF: Sandwich Theorem.
□

Corollary 13.18.1. *For any real number x with $|x| < 1$ we have $x^n \to 0$ as $n \to \infty$.*

PROOF: Taking $\alpha = 0$. □

13.1 Cauchy Sequences

Definition 13.19 (Cauchy Sequence)**.** Let (p_n) be a sequence in the metric space X. Then (p_n) is a *Cauchy sequence* iff, for every $\epsilon > 0$, there exists N such that, for all $m, n \geq N$, we have $d(p_m, p_n) < \epsilon$.

Proposition 13.20. *Let (p_n) be a sequence in the metric space X and let $E_N = \{p_n : n \geq N\}$ for all N. Then (p_n) is a Cauchy sequence if and only if* $\operatorname{diam} E_N \to 0$ *as $N \to \infty$.*

PROOF: Immediate from definitions. □

Theorem 13.21. *Every convergent sequence is Cauchy.*

PROOF:
⟨1⟩1. LET: (p_n) be a convergent sequence with limit l.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK N such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon/2$
⟨1⟩4. $\forall m, n \geq N . d(p_m, p_n) < \epsilon$
□

13.2 Complete Metric Spaces

Definition 13.22 (Complete Metric Space)**.** A metric space is *complete* iff every Cauchy sequence converges.

Theorem 13.23. *Every compact metric space is complete.*

PROOF:
⟨1⟩1. LET: X be a compact metric space.
⟨1⟩2. LET: (p_n) be a Cauchy sequence in X.
⟨1⟩3. For $N \in \mathbb{N}$,
LET: $E_N = \{p_n : n \geq N\}$.
⟨1⟩4. $\operatorname{diam} \overline{E_N} \to 0$ as $N \to \infty$.
⟨1⟩5. For all N, every $\overline{E_N}$ is compact.
PROOF: Proposition 12.37.
⟨1⟩6. For all N we have $\overline{E_N} \supseteq \overline{E_{N+1}}$.
⟨1⟩7. LET: l be the unique point in $\bigcap_{N=0}^{\infty} \overline{E_N}$
PROVE: $p_n \to l$ as $n \to \infty$.
PROOF: Proposition 12.44.
⟨1⟩8. LET: $\epsilon > 0$
⟨1⟩9. PICK N_0 such that $\forall N \geq N_0 . \operatorname{diam} \overline{E_N} < \epsilon$.
⟨1⟩10. $\forall q \in E_N . d(l, q) < \epsilon$
⟨1⟩11. $\forall n \geq N . d(l, p_n) < \epsilon$
□

Corollary 13.23.1. *Let X be a metric space. If every closed bounded set in X is compact, then X is complete.*

PROOF:
⟨1⟩1. LET: S be a Cauchy sequence in X.
⟨1⟩2. S is bounded.
⟨1⟩3. $\overline{S}$ is closed and bounded.
⟨1⟩4. $\overline{S}$ is compact.
⟨1⟩5. S is a Cauchy sequence in $\overline{S}$.
⟨1⟩6. S converges.
□

Corollary 13.23.2. *For every natural number k, we have $\mathbb{R}^k$ is complete.*

Corollary 13.23.3. *Every closed subspace of a complete metric space is complete.*

Proposition 13.24. *Let X be a complete metric space. Let (E_n) be a sequence of nonempty closed bounded sets in X with*

$$E_0 \supseteq E_1 \supseteq \cdots$$

and $\operatorname{diam} E_n \to 0$ *as $n \to \infty$. Then $\bigcap_{n=0}^{\infty} E_n$ consists of exactly one point.*

PROOF:
⟨1⟩1. LET: $K = \bigcap_{n=0}^{\infty} E_n$
⟨1⟩2. K has at least one point.
  ⟨2⟩1. For each n, PICK $a_n \in E_n$
  ⟨2⟩2. (a_n) is Cauchy.
    ⟨3⟩1. LET: $\epsilon > 0$
    ⟨3⟩2. PICK N such that $\forall n \geq N. \operatorname{diam} E_n < \epsilon$
    ⟨3⟩3. $\forall m, n \geq N. d(a_m, a_n) < \epsilon$
  ⟨2⟩3. LET: $l = \lim_{n\to\infty} a_n$
  ⟨2⟩4. $l \in K$
    ⟨3⟩1. LET: $n \in \mathbb{N}$
    ⟨3⟩2. For all $m \geq n$ we have $a_m \in E_n$
    ⟨3⟩3. $l \in E_n$
⟨1⟩3. K has at most one point.
  ⟨2⟩1. ASSUME: for a contradiction $a, b \in K$ such that $a \neq b$
  ⟨2⟩2. PICK n such that $\operatorname{diam} E_n < d(a, b)$
  ⟨2⟩3. $a, b \in E_n$
  ⟨2⟩4. Q.E.D.
    PROOF: This is a contradiction.
□

Theorem 13.25 (Baire's Theorem)**.** *Let X be a complete metric space. Let (G_n) be a sequence of dense open subsets of X. Then $\bigcap_{n=0}^{\infty} G_n$ is not empty.*

PROOF:
⟨1⟩1. PICK a sequence (E_n) of open balls such that $E_0 \supseteq E_1 \supseteq \cdots$ and $\operatorname{diam} E_n \leq 1/2^n$ and $\overline{E_n} \subseteq G_n$.

$\langle 2\rangle 1$. Assume: as induction hypothesis we have chosen $E_0, \ldots, E_n$ with centres $c_0, \ldots, c_n$.
$\langle 2\rangle 2$. Pick $x \in E_n \cap G_{n+1}$
$\langle 2\rangle 3$. Pick $0 < \epsilon \leq 1/2^{n+2}$ such that $B(x, \epsilon) \subseteq E_n \cap G_{n+1}$
$\langle 2\rangle 4$. Let: $E_{n+1} = B(x, \epsilon/2)$
$\langle 2\rangle 5$. $E_{n+1} \subseteq E_n$
$\langle 2\rangle 6$. $\operatorname{diam} \overline{E_{n+1}} \leq 1/2^{n+1}$
$\langle 2\rangle 7$. $\overline{E_{n+1}} \subseteq G_{n+1}$
$\langle 1\rangle 2$. Let: $\bigcap_{n=0}^{\infty} \overline{E_n} = \{p\}$
Proof: Proposition 13.24.
$\langle 1\rangle 3$. $p \in \bigcap_{n=0}^{\infty} G_n$
□

13.3 Divergent Sequences

Definition 13.26. Let (s_n) be a sequence in $\mathbb{R}$. Then we say s_n *diverges to* $+\infty$, and write

$$s_n \to +\infty \text{ as } n \to \infty \quad ,$$

iff for every real number M, there exists an integer N such that

$$\forall n \geq N. s_n \geq M \quad .$$

We say s_n *diverges to* $-\infty$, and write

$$s_n \to -\infty \text{ as } n \to \infty \quad ,$$

iff for every real number M, there exists an integer N such that

$$\forall n \geq N. s_n \leq M \quad .$$

Definition 13.27 (Limit Supremum, Limit Infimum)**.** Let (s_n) be a sequence in $\mathbb{R}$. Let E be the set of all $l \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that there exists a subsequence of (s_n) that converges to l.

The *limit supremum* of (s_n), denoted

$$\limsup_{n \to \infty} s_n \quad ,$$

is the supremum of E in the extended reals.

The *limit infimum* of (s_n), denoted

$$\liminf_{n \to \infty} s_n \quad ,$$

is the infimum of E in the extended reals.

Proof: The set E is always nonempty because: if (s_n) is unbounded above then $+\infty \in E$; if it is unbounded below then $-\infty \in E$; and if it is bounded above and below then there is a real number in E by Corollary 13.4.1. □

Theorem 13.28. *Let (s_n) be a sequence in $\mathbb{R}$. Then there exists a subsequence of (s_n) that converges or diverges to $\limsup_{n\to\infty} s_n$*

PROOF:
⟨1⟩1. CASE: $\limsup_n s_n = +\infty$
PROOF: (s_n) is unbounded above and so has a subsequence that diverges to $+\infty$.
⟨1⟩2. CASE: $\limsup_n s_n \in \mathbb{R}$
PROOF: Then $\limsup s_n$ is in the set of limits of subsequences of (s_n) by Proposition 13.11.
⟨1⟩3. CASE: $\limsup_n s_n = -\infty$
PROOF: (s_n) is unbounded below and so has a subsequence that diverges to $-\infty$.
□

Theorem 13.29. *Let (s_n) be a sequence in $\mathbb{R}$. Then there exists a subsequence of (s_n) that converges or diverges to $\liminf_{n\to\infty} s_n$*

PROOF: Similar. □

Theorem 13.30. *Let (s_n) be a sequence in $\mathbb{R}$. If $x > \limsup_n s_n$, then there exists N such that $\forall n \geq N. s_n < x$.*

PROOF: If not, we could choose a subsequence of (s_n) that converges to a value $\geq x$, contradicting the definition of $\limsup_n s_n$. □

Theorem 13.31. *Let (s_n) be a sequence in $\mathbb{R}$. If $x < \liminf_n s_n$, then there exists N such that $\forall n \geq N. s_n > x$.*

PROOF: Similar. □

Theorem 13.32. *Let (s_n) be a sequence in $\mathbb{R}$. Let s^* be an extended real such that:*

- *There exists a subsequence of (s_n) that converges or diverges to s^*.*
- *For any $x > s^*$, there exists N such that $\forall n \geq N. s_n < x$.*

Then $s^* = \limsup_n s_n$.

PROOF:
⟨1⟩1. LET: E be the set of subsequential limits of (s_n).
⟨1⟩2. s^* is an upper bound for E.
⟨2⟩1. LET: $x \in E$
⟨2⟩2. ASSUME: for a contradiction $x > s^*$.
⟨2⟩3. $s^* \in \mathbb{R}$
⟨2⟩4. LET: $y = x$ if $x \in \mathbb{R}$, or $s^* + 1$ if $x = +\infty$
⟨2⟩5. There exists N such that $\forall n \geq N. s_n < y$.
⟨2⟩6. Q.E.D.
PROOF: This contradicts the fact that some subsequence of (s_n) converges or diverges to x.

$\langle 1 \rangle 3$. If u is an upper bound for E then $s^* \leq u$.
$\square$

Theorem 13.33. *Let (s_n) be a sequence in $\mathbb{R}$. Let s^* be an extended real such that:*

- *There exists a subsequence of (s_n) that converges or diverges to s^*.*
- *For any $x < s^*$, there exists N such that $\forall n \geq N. s_n > x$.*

Then $s^* = \liminf_n s_n$.

PROOF: Similar. $\square$

Proposition 13.34. *Let (s_n) be a sequence of real numbers and $l \in \mathbb{R}$. Then (s_n) converges to l iff $\limsup_n s_n = \liminf_n s_n = l$.*

PROOF:
$\langle 1 \rangle 1$. If (s_n) converges to l then $\limsup_n s_n = \liminf_n s_n = l$.
  PROOF: If (s_n) converges to l then every subsequence of (s_n) converges to l.
$\langle 1 \rangle 2$. If $\limsup_n s_n = \liminf_n s_n = l$ then (s_n) converges to l.
  $\langle 2 \rangle 1$. ASSUME: $\limsup_n s_n = \liminf_n s_n = l$
  $\langle 2 \rangle 2$. For all $\epsilon > 0$, there exists N such that $\forall n \geq N. l - \epsilon < s_n < l + \epsilon$.
    PROOF: Theorem 13.32 and 13.33.
  $\langle 2 \rangle 3$. $s_n \to l$ as $n \to \infty$.
$\square$

Theorem 13.35. *Let (s_n) and (t_n) be sequences of real numbers and $N \in \mathbb{N}$. Assume $\forall n \geq N. s_n \leq t_n$. Then*

$$\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n \quad .$$

PROOF:
$\langle 1 \rangle 1$. For any subsequence (t_{n_r}) of (t_n) that converges or diverges to $\pm\infty$, we have $\liminf_n s_n \leq \lim_r t_{n_r}$
  $\langle 2 \rangle 1$. LET: (t_{n_r}) be a subsequence of (t_n) with limit l.
  $\langle 2 \rangle 2$. PICK m such that a subsequence of (s_{n_r}) has limit m.
  $\langle 2 \rangle 3$. $\forall r. s_{n_r} \leq t_{n_r}$
  $\langle 2 \rangle 4$. $m \leq l$
  $\langle 2 \rangle 5$. $\liminf_n s_n \leq l$
$\langle 1 \rangle 2$. $\liminf_n s_n \leq \liminf_n t_n$
$\square$

Theorem 13.36. *Let (s_n) and (t_n) be sequences of real numbers and $N \in \mathbb{N}$. Assume $\forall n \geq N. s_n \leq t_n$. Then*

$$\limsup_{n \to \infty} s_n \leq \limsup_{n \to \infty} t_n \quad .$$

PROOF: Similar. $\square$

Theorem 13.37. *For any sequence (c_n) of positive real numbers, we have*

$$\limsup_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n} \ .$$

PROOF:
⟨1⟩1. LET: $\alpha = \limsup_n c_{n+1}/c_n$
⟨1⟩2. ASSUME: w.l.o.g. $\alpha < +\infty$
⟨1⟩3. For all $\beta > \alpha$ we have $\limsup_{n\to\infty} c_n^{1/n} \leq \beta$.
⟨2⟩1. LET: $\beta > \alpha$
⟨2⟩2. PICK N such that, for all $n \geq N$, we have

$$\frac{c_{n+1}}{c_n} \leq \beta \ .$$

PROOF: Theorem 13.30.
⟨2⟩3. For all $k \geq 0$ we have

$$c_{N+k+1} \leq \beta c_{N+k} \ .$$

⟨2⟩4. For all $n \geq N$ we have

$$c_n \leq c_N \beta^{-N} \beta^n \ .$$

PROOF: Induction on n.
⟨2⟩5. For all $n \geq N$ we have

$$c_n^{1/n} \leq (c_N \beta^{-N})^{1/n} \beta \ .$$

⟨2⟩6.

$$\limsup_{n\to\infty} c_n^{1/n} \leq \beta$$

PROOF:

$$\begin{aligned} \limsup_{n\to\infty} c_n^{1/n} &\leq \limsup_{n\to\infty} (c_N \beta^{-N})^{1/n} \beta && \text{(Theorem 13.36)} \\ &= \beta && \text{(Theorem 13.16)} \end{aligned}$$

⟨1⟩4.

$$\limsup_{n\to\infty} c_n^{1/n} \leq \alpha$$

□

Theorem 13.38. *For any sequence (c_n) of positive real numbers, we have*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} c_n^{1/n} \ .$$

PROOF: Similar. □

Proposition 13.39. *Let (a_n) and (b_n) be sequences of reals. Assume that it is not the case that one of $\limsup_n a_n$, $\limsup_n b_n$ is $+\infty$ and the other is $-\infty$. Then*

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \ .$$

13.4 Infinite Series

Definition 13.40. Let (a_n) be a sequence in $\mathbb{R}^k$ and $s \in \mathbb{R}^k$. We say the *infinite series* $\sum_{n=0}^{\infty} a_n$ *converges* to s, and write

$$\sum_{n=0}^{\infty} a_n = s$$

iff

$$\sum_{n=0}^{N} a_n \to s \text{ as } N \to \infty \ .$$

If $(\sum_{n=0}^{N} a_n)$ diverges, we say the infinite series $\sum_{n=0}^{\infty} a_n$ diverges.

Theorem 13.41. *Let (a_n) be a sequence in $\mathbb{R}^k$. Then $\sum_{n=0}^{\infty} a_n$ converges if and only if, for all $\epsilon > 0$, there exists N such that, for all $m, n \geq N$,*

$$\left\| \sum_{i=m}^{n} a_i \right\| \leq \epsilon \ .$$

Proof: This is what it means for $(\sum_{i=0}^{n} a_i)$ to be a Cauchy sequence. □

Corollary 13.41.1. *If $\sum_{n=0}^{\infty} a_n$ converges then $a_n \to 0$ as $n \to \infty$.*

Theorem 13.42. *A series of nonnegative reals converges if and only if its partial sums form a bounded sequence.*

Proof: Its partial sums are a monotonically increasing sequence, and so converge if and only if they are bounded above. □

Theorem 13.43 (Comparison Test)**.** *Let (a_n) be a sequence in $\mathbb{R}^k$ and (c_n) a sequence of real numbers. If there exists N such that $\forall n \geq N. \|a_n\| \leq c_n$, and if $\sum_n c_n$ converges, then $\sum_n a_n$ converges.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. Pick N such that $\forall n \geq N. \|a_n\| \leq c_n$ and $\forall m, n \geq N. \sum_{k=m}^{n} c_k < \epsilon$.
⟨1⟩3. $\forall m, n \geq N. \|\sum_{k=m}^{n} a_k\| \leq \epsilon$
□

Corollary 13.43.1. *Let (a_n) and (d_n) be sequences of real numbers. If there exists N such that $\forall n \geq N. a_n \geq d_n \geq 0$, and if $\sum_n d_n$ diverges, then $\sum_n a_n$ diverges.*

Theorem 13.44 (Geometric Series)**.** *For x a real number with $0 \leq x < 1$ we have*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \ .$$

PROOF:Since $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x} \to \frac{1}{1-x}$ as $n \to \infty$. □

Theorem 13.45. *For x a real number with $x \geq 1$ we have $\sum_{n=0}^{\infty} x^n$ diverges.*

PROOF: If $x = 1$ then $\sum_{n=0}^{N} x^n = N + 1$. If $x > 1$ then $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$. Both of these sequences diverge. □

Theorem 13.46. *Let (a_n) be a monotonically decreasing sequence of nonnegative real numbers. Then $\sum_n a_n$ converges if and only if $\sum_n 2^n a_{2^n}$ converges.*

PROOF:
⟨1⟩1. For $N \in \mathbb{N}$,
LET: $s_N = \sum_{n=0}^{N} a_n$.
⟨1⟩2. For $N \in \mathbb{N}$,
LET: $t_N = \sum_{n=0}^{N} 2^n a_{2^n}$.
⟨1⟩3. For natural number N and k with $N < 2^k$ we have $s_N \leq a_0 + t_{k-1}$.
PROOF:

$$\begin{aligned} s_N &\leq \sum_{n=0}^{2^k-1} a_n \\ &= a_0 + \sum_{i=0}^{k-1} \sum_{n=2^i} 2^{i+1} - 1 a_n \\ &\leq a_0 + \sum_{i=0}^{k-1} 2^i a_{2^i} \\ &= a_0 + t_{k-1} \end{aligned}$$

⟨1⟩4. For natural number N and k with $N > 2^k$ we have $t_k < 2s_N$.
PROOF:

$$\begin{aligned} s_N &\geq \sum_{n=1}^{2^k} a_n \\ &\geq \sum_{i=0}^{k} \sum_{n=2^i+1} 2^{i+1} a_n \\ &\geq \sum_{i=0}^{k} 2^i a_{2^{i+1}} \\ &= (1/2) t_k \end{aligned}$$

⟨1⟩5. (s_N) converges if and only if (t_k) converges.
□

Theorem 13.47. *If p is a real number with $p > 1$ then $\sum_n 1/n^p$ converges.*

PROOF: Since

$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}$$

which converges since $2^{1-p} < 1$. □

Theorem 13.48. *If p is a real number with $p \leq 1$ then $\sum_n 1/n^p$ diverges.*

PROOF: If $p \leq 0$ then $1/n^p$ does not converge to 0.
If $0 < p \leq 1$ we have
$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}$$
which diverges since $2^{1-p} \geq 1$. ∎

Theorem 13.49. *Let p be a real number. The series*
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$
converges if and only if $p > 1$.

PROOF:
$$\begin{aligned} 2^k \frac{1}{2^k(\ln 2^k)^p} &= \frac{1}{(k \ln 2)^p} \\ &= \frac{1}{(\ln 2)^p} \cdot \frac{1}{k^p} \end{aligned}$$
and this series converges iff $\sum_k \frac{1}{k^p}$ converges iff $p > 1$. ∎

Theorem 13.50 (Root Test)**.** *Let $(a_n)_{n \geq 1}$ be a sequence in $\mathbb{R}^k$. Let $\alpha = \limsup_{n \to \infty} \|a_n\|^{1/n}$.*

1. *If $\alpha < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $\alpha > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.*

PROOF:
$\langle 1 \rangle 1.$ If $\alpha < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.
  $\langle 2 \rangle 1.$ ASSUME: $\alpha < 1$
  $\langle 2 \rangle 2.$ PICK β such that $\alpha < \beta < 1$
  $\langle 2 \rangle 3.$ PICK N such that $\forall n \geq N . \|a_n\|^{1/n} < \beta$
    PROOF: Theorem 13.30.
  $\langle 2 \rangle 4.$ $\forall n \geq N . \|a_n\| < \beta^n$
  $\langle 2 \rangle 5.$ $\sum_{n=1}^{\infty} \beta^n$ converges.
    PROOF: Theorem 13.44.
  $\langle 2 \rangle 6.$ $\sum_{n=1}^{\infty} a_n$ converges.
    PROOF: Comparison Test.
$\langle 1 \rangle 2.$ If $\alpha > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.
  $\langle 2 \rangle 1.$ ASSUME: $\alpha > 1$
  $\langle 2 \rangle 2.$ There exists a sequence of positive integers (n_k) such that $\|a_{n_k}\|^{1/n_k} \to \alpha$ as $k \to \infty$.
    PROOF: Theorem 13.28.
  $\langle 2 \rangle 3.$ There are infinitely many n such that $\|a_n\| > 1$.
  $\langle 2 \rangle 4.$ $a_n \nrightarrow 0$ as $n \to \infty$.
  $\langle 2 \rangle 5.$ $\sum_{n=1}^{\infty} a_n$ diverges.

PROOF: Corollary 13.41.1.
∎

Example 13.51. If $a_n = 1/n$ then $|a_n|^{1/n} \to 1$ and $\sum_n a_n$ diverges.
If $a_n = 1/n^2$ then $|a_n|^{1/n} \to 1$ and $\sum_n a_n$ converges.

Theorem 13.52 (Ratio Test)**.** *Let $(a_n)_{n \geq 0}$ be a sequence in $\mathbb{R}^k$.*

1. *If*
$$\limsup_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$$
then $\sum_{n=0}^{\infty} a_n$ converges.

2. *If there exists N such that $\forall n \geq N. \frac{\|a_{n+1}\|}{\|a_n\|} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.*

PROOF:
⟨1⟩1. If $\limsup_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$ then $\sum_{n=0}^{\infty} a_n$ converges.
⟨2⟩1. ASSUME: $\limsup_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$
⟨2⟩2. $\limsup_{n\to\infty} \|a_n\|^{1/n} < 1$
PROOF: Theorem 13.37.
⟨2⟩3. $\sum_{n=0}^{\infty} a_n$ converges.
PROOF: Root Test
⟨1⟩2. If there exists N such that $\forall n \geq N. \frac{\|a_{n+1}\|}{\|a_n\|} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.
PROOF: Since $a_n \not\to 0$ as $n \to \infty$.
∎

Example 13.53. If $a_n = 1/n$ then $a_{n+1}/a_n \to 1$ and $\sum_n a_n$ diverges.
If $a_n = 1/n^2$ then $a_{n+1}/a_n \to 1$ and $\sum_n a_n$ converges.

13.5 The Number e

Lemma 13.54. *The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.*

PROOF:
$$\begin{aligned}\sum_{n=0}^{N} \frac{1}{n!} &\leq 1 + \sum_{n=1}^{N} \frac{1}{2^{n-1}} \\ &< 3\end{aligned}$$
∎

Definition 13.55. The number e is
$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \; .$$

Theorem 13.56.
$$\left(1 + \frac{1}{n}\right)^n \to e \text{ as } n \to \infty$$

PROOF:
⟨1⟩1. For $n \in \mathbb{N}$,
LET: $s_n = \sum_{k=0}^{n} \frac{1}{k!}$
⟨1⟩2. For $n \in \mathbb{Z}^+$,
LET: $t_n = \left(1 + \frac{1}{n}\right)^n$
⟨1⟩3. For $n \in \mathbb{Z}^+$ we have

$$t_n = \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \quad .$$

PROOF:

$$\begin{aligned} t_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} && \text{(Binomial Theorem)} \\ &= \sum_{k=0}^{n} \frac{\prod_{i=0}^{k-1}(n-i)}{k!} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=0}^{k-1} \frac{n-i}{n} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \end{aligned}$$

⟨1⟩4. For $n \in \mathbb{Z}^+$ we have $t_n \leq s_n$.
⟨1⟩5. $\limsup_{n\to\infty} t_n \leq e$
⟨1⟩6. For $m, n \in \mathbb{Z}^+$ with $n \geq m$ we have

$$t_n \geq \sum_{k=0}^{m} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \quad .$$

⟨1⟩7. For $m \in \mathbb{Z}^+$ we have

$$\liminf_{n\to\infty} t_n \geq \sum_{k=0}^{m} \frac{1}{k!} \quad .$$

⟨1⟩8. For $m \in \mathbb{Z}^+$ we have

$$s_m \leq \liminf_{n\to\infty} t_n \quad .$$

⟨1⟩9.

$$e \leq \liminf_{n\to\infty} t_n$$

⟨1⟩10. $t_n \to e$ as $n \to \infty$.
PROOF: From ⟨1⟩5 and ⟨1⟩9.
□

Theorem 13.57. *e is irrational.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $e = p/q$ where p and q are positive integers.
⟨1⟩2. For $n \in \mathbb{N}$,

LET: $s_n = \sum_{k=0}^n \frac{1}{k!}$.

⟨1⟩3. For $n \in \mathbb{Z}^+$ we have

$$0 < e - s_n < \frac{1}{n!n} \quad .$$

PROOF:

$$\begin{aligned} e - s_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &< \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \frac{1}{(n+1)^k} \\ &= \frac{1}{n!n} \end{aligned}$$

⟨1⟩4.

$$0 < q!(e - s_q) < \frac{1}{q}$$

⟨1⟩5. $q!e$ is an integer.

⟨1⟩6. $q!(e - s_q)$ is an integer.

⟨1⟩7. There exists an integer between 0 and 1.

⟨1⟩8. Q.E.D.

PROOF: This is a contradiction.

□

Theorem 13.58. *e is transcendental.*

PROOF: See I. M. Niven. *Irrational Numbers* p. 25. □

13.6 Power Series

Definition 13.59 (Power Series)**.** Let (c_n) be a sequence of complex numbers. The *power series* with *coefficients* (c_n) is the function that maps a complex number z to the series

$$\sum_{n=0}^{\infty} c_n z^n \quad .$$

Definition 13.60 (Radius of Convergence)**.** Let (c_n) be a sequence of complex numbers. Let

$$\begin{aligned} \alpha &:= \limsup_{n \to \infty} |c_n|^{1/n} \\ R &:= \frac{1}{\alpha} \end{aligned}$$

where $R = +\infty$ if $\alpha = 0$ and $R = 0$ if $\alpha = +\infty$. Then R is called the *radius of convergence* of the power series $\sum_n c_n z^n$.

Theorem 13.61. *Let R be the radius of convergence of $\sum_n c_n z^n$.*

1. *If $|z| < R$ then $\sum_{n=0}^{\infty} c_n z^n$ converges.*

2. *If $|z| > R$ then $\sum_{n=0}^{\infty} c_n z^n$ diverges.*

PROOF:
⟨1⟩1. For $z \in \mathbb{C}$ and $n \in \mathbb{N}$,
LET: $a_n(z) = c_n z^n$
⟨1⟩2.
$$\limsup_{n\to\infty} |a_n(z)|^{1/n} = |z|/R$$
⟨1⟩3. If $|z| < R$ then $\sum_{n=0}^{\infty} a_n(z)$ converges.
PROOF: Root Test.
⟨1⟩4. If $|z| > R$ then $\sum_{n=0}^{\infty} a_n(z)$ diverges.
PROOF: Root Test.
□

13.7 Summation by Parts

Theorem 13.62. *Let (a_n), (b_n) be two sequences in $\mathbb{R}^k$. Let*

$$A_n = \sum_{k=0}^{n} a_k \qquad (n \geq -1) \ .$$

Let p and q be integers with $0 \leq p \leq q$. Then

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \ .$$

PROOF:
$$\begin{aligned} \sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \end{aligned}$$
□

Theorem 13.63. *Let (a_n) be a sequence in $\mathbb{R}^k$ and (b_n) be a sequence of real numbers. Assume that:*

1. *The partial sums $\sum_{n=0}^{N} a_n$ form a bounded sequence.*
2. *(b_n) is monotone decreasing.*
3. *$b_n \to 0$ as $n \to \infty$.*

Then $\sum_{n=0}^{\infty} a_n b_n$ converges.

PROOF:

⟨1⟩1. Pick M such that, for all N, we have $\| \sum_{n=0}^{N} a_n \| \leq M$.
⟨1⟩2. Let: $\epsilon > 0$
⟨1⟩3. Pick N such that $b_N \leq \epsilon / 2M$.
⟨1⟩4. Let: $N \leq p \leq q$
⟨1⟩5. For any integer k,
Let: $A_k = \sum_{n=0}^{k} a_n$.
⟨1⟩6. $\| \sum_{n=p}^{q} a_n b_n \| \leq \epsilon$
Proof:

$$\begin{aligned} \left\| \sum_{n=p}^{q} a_n b_n \right\| &= \left\| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right\| \quad \text{(Summation by Parts)} \\ &\leq M \left\| \sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q - b_p \right\| \\ &= 2M b_p \\ &\leq 2M b_N \\ &\leq \epsilon \end{aligned}$$

⟨1⟩7. Q.E.D.
Proof: Cauchy criterion.
□

Corollary 13.63.1 (Alternating Series)**.** *Let (c_n) be a sequence of real numbers. Assume that*

1. *$(|c_n|)$ is monotone decreasing.*
2. *$c_n \geq 0$ for all odd n, and $c_n \leq 0$ for all even n.*
3. *$c_n \to 0$ as $n \to \infty$*

Then $\sum_{n=0}^{\infty} c_n$ converges.

Proof: Take $a_n = (-1)^{n+1}$ and $b_n = |c_n|$. □

Theorem 13.64. *Let $\sum_n c_n z^n$ be a power series with radius of convergence 1. Suppose (c_n) is monotone decreasing with limit 0. Then $\sum_n c_n z^n$ converges at every point on the circle $|z| = 1$ except possibly $z = 1$.*

Proof:
⟨1⟩1. Let: z be a complex number with $|z| = 1$ and $z \neq 1$.
⟨1⟩2. For $n \in \mathbb{N}$,
Let: $a_n = z^n$.
⟨1⟩3. For $n \in \mathbb{N}$,
Let: $b_n = c_n$.
⟨1⟩4. The partial sums $\sum_{n=0}^{N} a_n$ form a bounded sequence.

PROOF:

$$\begin{aligned}\left|\sum_{n=0}^{N} a_n\right| &= \left|\sum_{n=0}^{N} z^n\right| \\ &= \left|\frac{1-z^{N+1}}{1-z}\right| \\ &\leq \frac{2}{|1-z|}\end{aligned}$$

⟨1⟩5. (b_n) is monotone decreasing with limit 0.
⟨1⟩6. Q.E.D.
PROOF: Theorem 13.63.
□

13.8 Absolute Convergence

Definition 13.65 (Absolute Convergence)**.** Let (a_n) be a sequence in $\mathbb{R}^k$. Then the series $\sum_{n=0}^{\infty} a_n$ *converges absolutely* iff $\sum_{n=0}^{\infty} \|a_n\|$ converges.

Theorem 13.66. *If $\sum_{n=0}^{\infty} a_n$ converges absolutely then $\sum_{n=0}^{\infty} a_n$ converges.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N such that, for all $p, q \geq N$, we have

$$\sum_{n=p}^{q} \|a_n\| \leq \epsilon \quad .$$

⟨1⟩3. For $p, q \geq N$, we have

$$\left\|\sum_{n=p}^{q} a_n\right\| \leq \epsilon \quad .$$

S
⟨1⟩4. Q.E.D.
PROOF: Cauchy criterion.
□

13.9 Addition and Multiplication of Series

Theorem 13.67. *If $\sum_n a_n = A$ and $\sum_n b_n = B$ then $\sum_n (a_n + b_n) = A + B$.*

PROOF:

$$\begin{aligned}\sum_{n=0}^{N}(a_n + b_n) &= \sum_{n=0}^{N} a_n + \sum_{n=0}^{N} b_n \\ &\to A + B \qquad \text{as } N \to \infty\end{aligned}$$ □

Theorem 13.68. *If $\sum_n a_n = A$ then $\sum_n (ca_n) = cA$.*

PROOF:

$$\sum_{n=0}^{N} ca_n = c\sum_{n=0}^{N} a_n$$
$$\to cA \qquad \text{as } N \to \infty \square$$

Definition 13.69 (Cauchy Product)**.** The *(Cauchy) product* of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} \quad .$$

Theorem 13.70. *Let (a_n) and (b_n) be sequences of complex numbers. Assume:*

1. *$\sum_{n=0}^{\infty} a_n$ converges absolutely.*
2. *$\sum_{n=0}^{\infty} b_n$ converges.*

For $n \in \mathbb{N}$, let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) \quad .$$

PROOF:
⟨1⟩1. LET:

$$A = \sum_{n=0}^{\infty} a_n$$

⟨1⟩2. LET:

$$B = \sum_{n=0}^{\infty} b_n$$

⟨1⟩3. For $n \in \mathbb{N}$,
LET:

$$A_n = \sum_{k=0}^{n} a_k \quad .$$

⟨1⟩4. For $n \in \mathbb{N}$,
LET:

$$B_n = \sum_{k=0}^{n} b_k \quad .$$

⟨1⟩5. For $n \in \mathbb{N}$,
LET:

$$C_n = \sum_{k=0}^{n} a_k b_{n-k} \quad .$$

⟨1⟩6. For $n \in \mathbb{N}$,
LET:

$$\beta_n = B_n - B$$

⟨1⟩7. For $n \in \mathbb{N}$,

$$C_n = A_n B + \sum_{k=0}^{n} a_k \beta_{n-k} \enspace .$$

⟨1⟩8. For $n \in \mathbb{N}$,
LET:

$$\gamma_n = \sum_{k=0}^{n} a_k \beta_{n-k}$$

⟨1⟩9. $A_n B \to AB$ as $n \to \infty$.
⟨1⟩10. $\gamma_n \to 0$ as $n \to \infty$.
⟨2⟩1. LET: $\alpha = \sum_{n=0}^{\infty} |a_n|$
⟨2⟩2. For all $\epsilon > 0$ we have $\limsup_n |\gamma_n| \leq \epsilon\alpha$.
⟨3⟩1. LET: $\epsilon > 0$
⟨3⟩2. PICK N such that $\forall n \geq N . |\beta_n| \leq \epsilon$.
⟨3⟩3. For all $n \geq N$ we have $|\gamma_n| \leq \left| \sum_{k=0}^{N} \beta_k \alpha_{n-k} \right| + \epsilon\alpha$.
PROOF:

$$\begin{aligned} |\gamma_n| &\leq \left| \sum_{k=0}^{N} \beta_k \alpha_{n-k} \right| + \left| \sum_{k=N+1}^{n} \beta_k a_{n-k} \right| \\ &\leq \left| \sum_{k=0}^{N} \beta_k \alpha_{n-k} \right| + \epsilon\alpha \end{aligned}$$

⟨3⟩4.

$$\limsup_{n \to \infty} |\gamma_n| \leq \epsilon\alpha$$

⟨2⟩3. $\limsup_n \gamma_n = 0$
⟨1⟩11. $C_n \to AB$ as $n \to \infty$.
□

Theorem 13.71 (Abel). *Let (a_n) and (b_n) be sequences of complex numbers. Let*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

for all n. If the series $\sum_n a_n$, $\sum_n b_n$ and $\sum_n c_n$ all converge, then

$$\sum_n c_n = \left(\sum_n a_n \right) \left(\sum_n b_n \right) \enspace .$$

Proposition 13.72. *The Cauchy product of two absolutely convergent series is absolutely convergent.*

PROOF:
⟨1⟩1. LET: $\sum_n a_n$ and $\sum_n b_n$ be two absolutely convergent series.
⟨1⟩2. LET: $c_n = \sum_{k=0}^{n} a_k b_{n-k}$
⟨1⟩3. $\sum_n |c_n|$ converges.

PROOF:
$$\sum_{n=0}^{\infty} |c_n| = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} a_k b_{n-k} \right|$$
$$\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k||b_{n-k}|$$
which converges by Theorem 13.70.

13.10 Rearrangements

Definition 13.73 (Rearrangement)**.** A *rearrangement* of a sequence (a_n) is a sequence $(a_{\phi(n)})$ for some bijection $\phi : \mathbb{N} \approx \mathbb{N}$.

Theorem 13.74 (Riemann)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a series that converges but not absolutely. Let α and β be extended reals with $\alpha \leq \beta$. Then there exists a rearrangement of $\sum_n a_n$ with partial sums s'_n such that*
$$\limsup_{n\to\infty} s'_n = \alpha, \qquad \liminf_{n\to\infty} s'_n = \beta \ .$$

PROOF:
⟨1⟩1. For $n \in \mathbb{Z}^+$,
LET:
$$p_n = \frac{|a_n| + a_n}{2} \ .$$
⟨1⟩2. For $n \in \mathbb{Z}^+$,
LET:
$$q_n = \frac{|a_n| - a_n}{2} \ .$$
⟨1⟩3. $\forall n \in \mathbb{Z}^+ . p_n - q_n = a_n$
⟨1⟩4. $\forall n \in \mathbb{Z}^+ . p_n + q_n = |a_n|$
⟨1⟩5. $\forall n \in \mathbb{Z}^+ . p_n \geq 0$
⟨1⟩6. $\forall n \in \mathbb{Z}^+ . q_n \geq 0$
⟨1⟩7. $\sum_n p_n$ and $\sum_n q_n$ both diverge.
⟨2⟩1. It is not the case than $\sum_n p_n$ and $\sum_n q_n$ both converge.
PROOF: This would imply that $\sum_n |a_n|$ converges by ⟨1⟩4.
⟨2⟩2. It is not the case that $\sum_n p_n$ converges and $\sum_n q_n$ diverges.
PROOF: This would imply that $\sum_n a_n$ diverges by ⟨1⟩3.
⟨2⟩3. It is not the case that $\sum_n p_n$ diverges and $\sum_n q_n$ converges.
PROOF: This would imply that $\sum_n a_n$ diverges by ⟨1⟩3.
⟨1⟩8. LET: (P_n) be the subsequence of (a_n) consisting of the non-negative terms.
⟨1⟩9. LET: (Q_n) be the subsequence of $(|a_n|)$ consisting only of the terms such that a_n is negative.
⟨1⟩10. $\sum_n P_n$ diverges.
PROOF: It is the series $\sum_n p_n$ with the zero terms removed.
⟨1⟩11. $\sum_n Q_n$ diverges.
PROOF: It is the series $\sum_n q_n$ with the zero terms removed.

⟨1⟩12. Pick sequences of real numbers (α_n), (β_n) such that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha_n < \beta_n$ for all n, and $\beta_1 > 0$.

⟨1⟩13. Pick strictly increasing sequences of natural numbers $(m_n)_{n \geq 1}$, $(k_n)_{n \geq 1}$ such that, for all n,

$$\sum_{i=1}^{n-1} \left(\sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) + \sum_{j=m_{n-1}+1}^{m_n} P_j > \beta_n$$

$$\sum_{i=1}^{n} \left(\sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) < \alpha_n$$

and m_n and k_n are the smallest integers that make these inequalities true.

Proof: Given the choice of $m_1, \ldots, m_n$ and $k_1, \ldots, k_n$, there must exist such an m_{n+1} by ⟨1⟩10, and then there must exist such a k_{n+1} by ⟨1⟩11.

⟨1⟩14. For $n \in \mathbb{Z}^+$,

Let: $x_n = \sum_{i=1}^{n-1} \left(\sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right) + \sum_{j=m_{n-1}+1}^{m_n} P_j$

⟨1⟩15. For $n \in \mathbb{Z}^+$,

Let: $y_n = \sum_{i=1}^{n} \left(\sum_{j=m_{i-1}+1}^{m_i} P_j - \sum_{j=k_{i-1}+1}^{k_i} Q_j \right)$

⟨1⟩16. For $n \in \mathbb{Z}^+$ we have

$$|x_n - \beta_n| \leq P_{m_n} \quad .$$

Proof: By minimality of m_n.

⟨1⟩17. For $n \in \mathbb{Z}^+$ we have

$$|y_n - \alpha_n| \leq Q_{k_n} \quad .$$

Proof: By minimality of k_n.

⟨1⟩18. $P_n \to 0$ as $n \to \infty$.

Proof: Since $a_n \to 0$ as $n \to \infty$.

⟨1⟩19. $Q_n \to 0$ as $n \to \infty$.

Proof: Since $a_n \to 0$ as $n \to \infty$.

⟨1⟩20. $x_n \to \beta$ as $n \to \infty$.

Proof: ⟨1⟩16, ⟨1⟩18

⟨1⟩21. $y_n \to \alpha$ as $n \to \infty$.

Proof: ⟨1⟩17, ⟨1⟩19

⟨1⟩22. No number less than α or greater than β is a subsequential limit of the partial sums of the series $P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots$

Proof: Since every partial sum after the $m_n + k_n$ term is between $\alpha_n - Q_{k_n}$ and $\beta_n + P_{m_n}$.

□

Theorem 13.75. *If $\sum_n a_n$ is a series in $\mathbb{R}^k$ that converges absolutely to s, then every rearrangement of $\sum_n a_n$ converges to s.*

Proof:

⟨1⟩1. Let: $\sum_n a'_n = \sum_n a_{k_n}$ be a rearrangement with partial sums s'_n.

⟨1⟩2. Let: $\epsilon > 0$

$\langle 1\rangle 3$. Pick N such that, for all $m \geq n \geq N$, we have

$$\sum_{i=n}^{m} \|a_i\| \leq \epsilon/3 \ .$$

$\langle 1\rangle 4$. Pick p such that $\{1, \ldots, N\} \subseteq \{k_1, k_2, \ldots, k_p\}$.

$\langle 1\rangle 5$. For all $n > p$ we have $\|s_n - s'_n\| \leq \epsilon$.

Proof:

$$\begin{aligned} \|s_n - s'_n\| &= \left\| \sum_{i=1}^{N} a_i + \sum_{i=N+1}^{n} a_i - \sum_{i=1}^{p} a_{k_i} - \sum_{i=p+1}^{n} a_{k_i} \right\| \\ &\leq \left\| \sum_{i=N+1}^{n} a_i \right\| + \left\| \sum_{\substack{1 \leq i \leq p \\ k_i > N}} a_{k_i} \right\| + \left\| \sum_{i=p+1}^{n} a_{k_i} \right\| \\ &\leq \epsilon & (\langle 1\rangle 3) \end{aligned}$$

$\langle 1\rangle 6$. $s'_n \to s$ as $n \to \infty$.

□

13.11 Completion of a Metric Space

Definition 13.76 (Completion)**.** Let X be a metric space. Let X^* be the set of all Cauchy sequences in X, quotiented by: $(p_n) \sim (q_n)$ iff $d(p_n, q_n) \to 0$. Define the distance function on X^* by:

$$\Delta((p_n), (q_n)) = \lim_{n \to \infty} d(p_n, q_n) \ .$$

Then the metric space X^* is called the *completion* of X.

Theorem 13.77. *The completion of X^* is a complete metric space, and X is a dense subspace under the embedding that maps $p \in X$ to the constant sequence (p).*

Example 13.78. $\mathbb{R}$ is the completion of $\mathbb{Q}$.

Chapter 14

Continuity

14.1 Limit of a Function

Definition 14.1 (Limit)**.** Let X and Y be metric spaces. Let $E \subseteq X$ and $f : E \to Y$. Let p be a limit point of E and $q \in Y$. Then we say q is the *limit* of f at p, and write

$$f(x) \to q \text{ as } x \to p, \text{ or } \lim_{x \to p} f(x) = q \enspace ,$$

iff for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in E$, if $0 < d(x, p) < \delta$ then $d(f(x), q) < \epsilon$.

Theorem 14.2. *Let X and Y be metric spaces. Let $E \subseteq X$ and $f : E \to Y$. Let p be a limit point of E and $q \in Y$. Then $f(x) \to q$ as $x \to p$ if and only if, for every sequence (p_n) in $E - \{p\}$ with limit p, we have $f(p_n) \to q$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. If $f(x) \to q$ as $x \to p$ then, for every sequence (p_n) in $E - \{p\}$ with limit p, we have $f(p_n) \to q$ as $n \to \infty$.
  $\langle 2 \rangle 1$. ASSUME: $f(x) \to q$ as $x \to p$.
  $\langle 2 \rangle 2$. LET: (p_n) be a sequence in $E - \{p\}$ with limit p.
  $\langle 2 \rangle 3$. LET: $\epsilon > 0$
  $\langle 2 \rangle 4$. PICK $\delta > 0$ such that, for all $x \in E$, if $0 < d(x, p) < \delta$ then $d(f(x), q) < \epsilon$.
  $\langle 2 \rangle 5$. PICK N such that, for all $n \geq N$, we have $d(p_n, p) < \delta$
  $\langle 2 \rangle 6$. $\forall n \geq N . d(f(p_n), q) < \epsilon$
$\langle 1 \rangle 2$. If, for every sequence (p_n) in $E - \{p\}$ with limit p, we have $f(p_n) \to q$ as $n \to \infty$, then $f(x) \to q$ as $x \to p$.
  $\langle 2 \rangle 1$. ASSUME: $f(x) \nrightarrow q$ as $x \to p$.
  $\langle 2 \rangle 2$. PICK $\epsilon > 0$ such that, for all $\delta > 0$, there exists a $x \in E$ such that $0 < d(x, p) < \delta$ and $d(f(x), q) \geq \epsilon$.
  $\langle 2 \rangle 3$. For all $n \in \mathbb{Z}^+$, PICK $p_n \in E$ such that $0 < d(p_n, p) < 1/n$ and $d(f(p_n), q) \geq \epsilon$.

⟨2⟩4. $p_n \to p$ as $n \to \infty$.
⟨2⟩5. $f(p_n) \not\to q$ as $n \to \infty$.
□

Corollary 14.2.1. *A function has at most one limit at any point.*

Theorem 14.3. *Let X be a metric space, $E \subseteq X$, and p a limit point of E. Let $f, g : E \to \mathbb{R}^k$. Assume $f(x) \to a$ as $x \to p$ and $g(x) \to b$ as $x \to p$. Then*

$$f(x) + g(x) \to a + b \text{ as } x \to p \ .$$

PROOF:
⟨1⟩1. LET: (p_n) be a sequence in E that converges to p.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $g(p_n) \to b$ as $n \to \infty$.
⟨1⟩4. $f(p_n) + g(p_n) \to a + b$ as $n \to \infty$.
PROOF: Proposition 13.5.
⟨1⟩5. Q.E.D.
PROOF: Theorem 14.2.
□

Theorem 14.4. *Let X be a metric space, $E \subseteq X$, and p a limit point of E. Let $f, g : E \to \mathbb{C}$. Assume $f(x) \to a$ as $x \to p$ and $g(x) \to b$ as $x \to p$. Then*

$$f(x)g(x) \to ab \text{ as } x \to p \ .$$

PROOF:
⟨1⟩1. LET: (p_n) be a sequence in E that converges to p.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $g(p_n) \to b$ as $n \to \infty$.
⟨1⟩4. $f(p_n)g(p_n) \to ab$ as $n \to \infty$.
PROOF: Proposition 13.7.
⟨1⟩5. Q.E.D.
PROOF: Theorem 14.2.
□

Theorem 14.5. *Let X be a metric space, $E \subseteq X$, and p a limit point of E. Let $f : E \to \mathbb{C} - \{0\}$. Assume $f(x) \to a \neq 0$ as $x \to p$. Then*

$$f(x)^{-1} \to a^{-1} \text{ as } x \to p \ .$$

PROOF:
⟨1⟩1. LET: (p_n) be a sequence in E that converges to p.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $f(p_n)^{-1} \to a^{-1}$ as $n \to \infty$.
PROOF: Proposition 13.8.
⟨1⟩4. Q.E.D.
PROOF: Theorem 14.2.

∎

Theorem 14.6. *Let X be a metric space, $E \subseteq X$, and p a limit point of E. Let $f, g : E \to \mathbb{R}^k$. Assume $f(x) \to a$ as $x \to p$ and $g(x) \to b$ as $x \to p$. Then*

$$f(x) \cdot g(x) \to a \cdot b \text{ as } x \to p \ .$$

Proof:
⟨1⟩1. Let: (p_n) be a sequence in E that converges to p.
⟨1⟩2. $f(p_n) \to a$ as $n \to \infty$.
⟨1⟩3. $g(p_n) \to b$ as $n \to \infty$.
⟨1⟩4. $f(p_n) \cdot g(p_n) \to a \cdot b$ as $n \to \infty$.
Proof: Proposition 13.10.
⟨1⟩5. Q.E.D.
Proof: Theorem 14.2.
∎

14.2 Continuous Functions

Definition 14.7 (Continuous)**.** Let X be a metric space, $E \subseteq X$ and $p \in E$. Then f is *continuous* at p iff, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in E$, if $d(x, p) < \delta$ then

$$d(f(x), f(p)) < \epsilon \ .$$

f is *continuous* or *continuous on E* iff f is continuous at every point.

Theorem 14.8. *Let X be a metric space, $E \subseteq X$ and $p \in E$ be a limit point of E. Then f is continuous at p iff $f(x) \to f(p)$ as $x \to p$.*

Proof: Easy. ∎

Theorem 14.9. *Let X, Y and Z be metric spaces. Let $E \subseteq X$. Let $f : E \to Y$ and $g : f(E) \to Z$. Let $p \in E$. If f is continuous at p and g is continuous at $f(p)$ then $g \circ f$ is continuous at p.*

Proof:
⟨1⟩1. Let: $\epsilon > 0$
⟨1⟩2. Pick $\delta_1 > 0$ such that, for all $y \in f(E)$, if $d(y, f(p)) < \delta_1$ then $d(g(y), g(f(p))) < \epsilon$.
⟨1⟩3. Pick $\delta_2 > 0$ such that, for all $x \in E$, if $d(x, p) < \delta_2$ then $d(f(x), f(p)) < \delta_1$.
⟨1⟩4. For all $x \in E$, if $d(x, p) < \delta_2$ then $d(g(f(x)), g(f(p))) < \epsilon$.
∎

Theorem 14.10. *Let X and Y be metric spaces. Let $f : X \to Y$. Then f is continuous if and only if, for every open set $V \subseteq Y$, we have $f^{-1}(V)$ is open in X.*

PROOF:

$\langle 1\rangle 1$. If f is continuous then, for every open set V in Y, we have $f^{-1}(V)$ is open in X.

$\langle 2\rangle 1$. ASSUME: f is continuous.

$\langle 2\rangle 2$. LET: V be an open set in Y.
PROVE: $f^{-1}(V)$ is open.

$\langle 2\rangle 3$. LET: $x \in f^{-1}(V)$

$\langle 2\rangle 4$. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$.

$\langle 2\rangle 5$. PICK $\delta > 0$ such that, for all $x' \in X$, if $d(x', x) < \delta$ then $d(f(x'), f(x)) < \epsilon$.

$\langle 2\rangle 6$. $B(x, \delta) \subseteq f^{-1}(V)$

$\langle 1\rangle 2$. If, for every open set V in Y, we have $f^{-1}(V)$ is open in X, then f is continuous.

$\langle 2\rangle 1$. ASSUME: For every open set V in Y, we have $f^{-1}(V)$ is open in X.

$\langle 2\rangle 2$. LET: $p \in X$

$\langle 2\rangle 3$. LET: $\epsilon > 0$

$\langle 2\rangle 4$. $f^{-1}(B(f(p), \epsilon))$ is open in X.

$\langle 2\rangle 5$. PICK $\delta > 0$ such that $B(p, \delta) \subseteq f^{-1}(B(f(p), \epsilon))$.

$\langle 2\rangle 6$. For all $x \in X$, if $d(x, p) < \delta$ then $d(f(x), f(p)) < \epsilon$.

$\square$

Corollary 14.10.1. *Let X and Y be metric spaces. Let $f : X \to Y$. Then f is continuous if and only if, for every closed set C in Y, we have $f^{-1}(C)$ is closed in X.*

Theorem 14.11. *Let X be a metric space. Let $f : X \to \mathbb{R}^k$. Then f is continuous if and only if, for $i = 1, \ldots, k$, we have $\pi_i \circ f$ is continuous.*

PROOF:

$\langle 1\rangle 1$. Each π_i is continuous.

$\langle 2\rangle 1$. LET: $\vec{p} \in \mathbb{R}^k$

$\langle 2\rangle 2$. LET: $\epsilon > 0$

$\langle 2\rangle 3$. LET: $\vec{q} \in \mathbb{R}^k$

$\langle 2\rangle 4$. ASSUME: $\|\vec{p} - \vec{q}\| < \epsilon$

$\langle 2\rangle 5$. $|p_i - q_i| < \epsilon$

$\langle 1\rangle 2$. If, for all i, we have $\pi_i \circ f$ is continuous, then f is continuous.

$\langle 2\rangle 1$. ASSUME: For all i, we have $\pi_i \circ f$ is continuous.

$\langle 2\rangle 2$. LET: $p \in X$

$\langle 2\rangle 3$. LET: $\epsilon > 0$

$\langle 2\rangle 4$. For $i = 1, \ldots, k$, PICK $\delta_i > 0$ such that, for all $x \in X$, we have if $d(x, p) < \delta_i$ then $|\pi_i(f(p)) - \pi_i(f(x))| < \epsilon/\sqrt{k}$

$\langle 2\rangle 5$. LET: $\delta = \min(\delta_1, \ldots, \delta_k)$

$\langle 2\rangle 6$. LET: $q \in X$ with $d(p, q) < \delta$.

$\langle 2\rangle 7$. $\|f(p) - f(q)\| < \epsilon$

PROOF:

$$\begin{aligned}\|f(p) - f(q)\| &= \sqrt{\sum_{i=1}^{k} |\pi_i(f(p)) - \pi_i(f(q))|^2} \\ &< \sqrt{\sum_{i=1}^{k} \epsilon^2/k} \\ &= \epsilon\end{aligned}$$

□

Theorem 14.12. *Let X be a compact metric space and Y a metric space. Let $f : X \to Y$ be continuous. Then $f(X)$ is compact.*

PROOF:
⟨1⟩1. LET: $\mathcal{V}$ be an open cover of $f(X)$.
⟨1⟩2. $\{f^{-1}(V) : V \in \mathcal{V}\}$ is an open cover of X.
⟨1⟩3. PICK a finite subcover $\{f^{-1}(V_1), \ldots, f^{-1}(V_n)\}$.
⟨1⟩4. $\{V_1, \ldots, V_n\}$ covers Y.
□

Corollary 14.12.1. *Every continuous function from a compact metric space to $\mathbb{R}^k$ is bounded.*

Example 14.13. If $E \subseteq \mathbb{R}$ is not compact, then there exists a continuous function $E \to \mathbb{R}$ that is not bounded.

PROOF:
⟨1⟩1. CASE: E is bounded.
⟨2⟩1. PICK a limit point x_0 of E that is not in E.
⟨2⟩2. Define $f : E \to \mathbb{R}$ by $f(x) = 1/(x - x_0)$.
⟨2⟩3. f is continuous and unbounded.
⟨1⟩2. CASE: E is unbounded.
PROOF: The inclusion $E \hookrightarrow \mathbb{R}$ is continuous and unbounded.
□

Theorem 14.14 (Extreme Values Theorem)**.** *Let X be a compact metric space. Let $f : X \to \mathbb{R}$. Let $M = \sup f(X)$ and $m = \inf f(X)$. Then there exist $p, q \in X$ such that $f(p) = M$ and $f(q) \in m$.*

PROOF: Since $f(X)$ is compact and hence closed. □

Example 14.15. For any $E \subseteq \mathbb{R}$ that is not compact, there exists a continuous and bounded function $E \to \mathbb{R}$ that does not attain its supremum.

PROOF:
⟨1⟩1. CASE: E is bounded.
⟨2⟩1. PICK a limit point x_0 for E such that $x_0 \notin E$.
⟨2⟩2. Define $g : E \to \mathbb{R}$ by $g(x) = 1/(1 + (x - x_0)^2)$.

⟨2⟩3. g is continuous and bounded but does not attain its supremum 1.
⟨1⟩2. Case: E is unbounded.
Proof: Then $h(x) = x^2/(1 + x^2)$ is continuous and bounded but does not attain its supremum 1.
□

Theorem 14.16. *Let X be a compact metric space and Y a metric space. Let $f : X \approx Y$ be a continuous bijection. Then f^{-1} is continuous.*

Proof:
⟨1⟩1. Let: V be open in X.
⟨1⟩2. $X - V$ is compact.
⟨1⟩3. $f(X - V)$ is compact.
⟨1⟩4. $Y - f(V)$ is compact.
⟨1⟩5. $Y - f(V)$ is closed.
⟨1⟩6. $f(V)$ is open.
□

Example 14.17. This example shows we cannot remove the hypothesis of compactness of X, even if Y is compact.

Let $X = [0, 2\pi)$. Let $f : X \to S^1$ be the function $f(t) = (\cos t, \sin t)$. Then f is a continuous bijection $X \approx S^1$, but the inverse f^{-1} is not continuous.

Proposition 14.18. *The continuous image of a connected metric space is connected.*

Proof:
⟨1⟩1. Let: X be a connected metric space and Y a metric space.
⟨1⟩2. Let: $f : X \twoheadrightarrow Y$ be a continuous surjection.
⟨1⟩3. Assume: for a contradiction A and B form a separation of Y.
⟨1⟩4. $f^{-1}(A)$ and $f^{-1}(B)$ form a separation of X.
⟨1⟩5. Q.E.D.
Proof: This contradicts ⟨1⟩1.
□

Corollary 14.18.1 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < c < f(b)$ or $f(a) > c > f(b)$, then there exists a real number $x \in (a, b)$ such that $f(x) = c$.*

Proof: Since $f([a, b])$ is connected. □

Example 14.19. The converse does not hold. Let $f : [-1, 1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For all $a, b \in [-1, 1]$ with $a < b$, and all c with $f(a) < c < f(b)$, there exists $x \in (a, b)$ such that $f(x) = c$. Nevertheless, f is discontinuous at 0.

14.3 Limits from the Left and the Right

Definition 14.20 (Limit from the Left)**.** Let $f : (a, b) \to \mathbb{R}$. Let $c \in (a, b]$ and $q \in \mathbb{R}$. Then we say q is the *limit* as f approaches c *from the left*, and write

$$f(x) \to q \text{ as } x \to c-$$

or

$$\lim_{x \to c-} f(x) = q$$

iff, for every sequence (t_n) in (a, c) such that $t_n \to c$ as $n \to \infty$, we have $f(t_n) \to q$ as $n \to \infty$.

Definition 14.21 (Limit from the Right)**.** Let $f : (a, b) \to \mathbb{R}$. Let $c \in [a, b)$ and $q \in \mathbb{R}$. Then we say q is the *limit* as f approaches c *from the right*, and write

$$f(x) \to q \text{ as } x \to c+$$

or

$$\lim_{x \to c+} f(x) = q$$

iff, for every sequence (t_n) in (c, b) such that $t_n \to c$ as $n \to \infty$, we have $f(t_n) \to q$ as $n \to \infty$.

Proposition 14.22. *Let $f : (a, b) \to \mathbb{R}$. Let $c \in (a, b)$ and $q \in \mathbb{R}$. Then $f(x) \to q$ as $x \to c$ iff $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$.*

PROOF:
$\langle 1 \rangle 1$. If $f(x) \to q$ as $x \to c$ then $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$.
  PROOF: Theorem 14.2.
$\langle 1 \rangle 2$. If $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$ then $f(x) \to q$ as $x \to c$.
  $\langle 2 \rangle 1$. ASSUME: $f(x) \to q$ as $x \to c-$ and $f(x) \to q$ as $x \to c+$.
  $\langle 2 \rangle 2$. ASSUME: for a contradiction $f(x) \not\to q$ as $x \to c$.
  $\langle 2 \rangle 3$. PICK a sequence (p_n) such that $p_n \to c$ as $n \to \infty$, $f(p_n) \not\to q$ as $n \to \infty$, and $p_n \neq c$ for all n.
  $\langle 2 \rangle 4$. CASE: There are only finitely many n such that $p_n > c$.
    $\langle 3 \rangle 1$. LET: (q_n) be the subsequence of (p_n) consisting of all the terms such that $p_n < c$.
    $\langle 3 \rangle 2$. $q_n \to c$ as $n \to \infty$.
    $\langle 3 \rangle 3$. $f(q_n) \not\to q$ as $n \to \infty$.
    $\langle 3 \rangle 4$. Q.E.D.
      PROOF: This contradicts $\langle 2 \rangle 1$.
  $\langle 2 \rangle 5$. CASE: There are only finitely many n such that $p_n < c$.
    PROOF: Similar.
  $\langle 2 \rangle 6$. CASE: There are infinitely many n such that $p_n > c$ and infinitely many n such that $p_n < c$.
    $\langle 3 \rangle 1$. LET: (q_n) the subsequence of (p_n) consisting of all the terms such that $p_n > c$, and (r_n) the subsequence consisting of all the terms such that $p_n < c$.

⟨3⟩2. $q_n \to c$ as $n \to \infty$ and $r_n \to c$ as $n \to \infty$.
⟨3⟩3. It is not the case that $f(q_n) \to q$ as $n \to \infty$ and $f(r_n) \to q$ as $n \to \infty$.
PROOF: If $f(q_n) \to q$ as $n \to \infty$ and $f(r_n) \to q$ as $n \to \infty$ then $f(p_n) \to q$ as $n \to \infty$.
⟨3⟩4. Q.E.D.
PROOF: This contradicts ⟨2⟩1.
□

Proposition 14.23. *Let $f : (a,b) \to \mathbb{R}$ be monotonic. Then, for all $c \in (a,b)$ we have $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ both exist, and*

$$\sup_{a<x<c} f(x) = \lim_{x\to c-} f(x) \leq f(c) \leq \lim_{x\to c+} f(x) = \inf_{c<x<b} f(x) \enspace .$$

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. f is monotonically increasing on (a,b).
⟨1⟩2. $f(x) \to \sup_{a<x<c} f(x)$ as $x \to c-$.
⟨2⟩1. LET: (t_n) be a sequence in (a,c) such that $t_n \to c$ as $n \to \infty$.
PROVE: $f(t_n) \to \sup_{a<x<c} f(x)$ as $n \to \infty$.
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK $x \in (a,c)$ such that $f(x)$
⟨1⟩3. $f(x) \to \inf_{c<x<b} f(x)$ as $x \to c+$.
PROOF: Similar.
□

14.4 Discontinuities

Definition 14.24 (Simple Discontinuity)**.** Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$. We say f has a *simple discontinuity* or *discontinuity of the first kind* at c iff f is discontinuous at c but $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ both exist.

Definition 14.25 (Discontinuity of the Second Kind)**.** Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$. We say f has a *discontinuity of the second kind* at c iff $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ do not both exist.

14.5 Uniform Continuity

Definition 14.26 (Uniformly Continuous)**.** Let X and Y be metric spaces. Let $f : X \to Y$. Then f is *uniformly continuous* iff, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $p, q \in X$, if $d(p,q) < \delta$ then $d(f(p), f(q)) < \epsilon$.

Theorem 14.27. *Let X be a compact metric space and Y a metric space. Let $f : X \to Y$. If f is continuous then f is uniformly continuous.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$

$\langle 1\rangle 2$. For all $p \in X$, PICK $\phi(p) > 0$ such that, for all $q \in X$, if $d(p,q) < \phi(x)$ then $d(f(p), f(q)) < \epsilon/2$.
$\langle 1\rangle 3$. For all $p \in X$,
LET: $J(p) = B(p, \phi(x)/2)$.
$\langle 1\rangle 4$. $\{J(p) : p \in X\}$ is an open cover of X.
$\langle 1\rangle 5$. PICK a finite subcover $\{J(p_1), \ldots, J(p_n)\}$.
$\langle 1\rangle 6$. LET: $\delta = \min(\phi(p_1), \ldots, \phi(p_n))/2$
$\langle 1\rangle 7$. LET: $p, q \in X$ with $d(p,q) < \delta$.
$\langle 1\rangle 8$. PICK m such that $p \in J(p_m)$.
$\langle 1\rangle 9$. $d(p, p_m) < \phi(p_m)/2$
$\langle 1\rangle 10$. $d(q, p_m) < \phi(p_m)$
$\langle 1\rangle 11$. $d(f(p), f(q)) < \epsilon$
$\square$

Example 14.28. Let $E \subseteq \mathbb{R}$ be bounded but not compact. Then there exists a continuous function $E \to \mathbb{R}$ that is not uniformly continuous.

PROOF: Pick a limit point x_0 for E that is not in E. Then the function $f(x) = 1/(x - x_0)$ is continuous but not uniformly continuous. $\square$

Proposition 14.29. *Every linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous.*

PROOF: Since $\|A\vec{x} - A\vec{y}\| \leq \|A\|\|\vec{x} - \vec{y}\|$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. $\square$

Part IV

More Algebra

Chapter 15

Lie Groups

Definition 15.1 (Lie Group)**.** A *Lie group* G is a group G that is also an analytic differentiable manifold such that the group operation and inverse operation are analytic.

A *homomorphism of Lie groups* is a group homomorphism that is an analytic function.

Lemma 15.2. *Every bijective Lie group homomorphism is an isomorphism.*

Definition 15.3 (Unitary Group)**.** The *unitary group* $U(n)$ is the Lie group of all $n \times n$ unitary matrices.

Definition 15.4 (Special Unitary Group)**.** The *special unitary group* $SU(n)$ is the Lie group of all $n \times n$ unitary matrices with determinant 1.

Definition 15.5 (Lie Subgroup)**.** Let G be a Lie group. A *Lie subgroup* of G is a subgroup that is also an analytic submanifold of G.

Example 15.6. $U(n)$ and $SU(n)$ are Lie subgroups of $GL(n, \mathbb{C})$.