Mathematics

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Chapter 1

Sets and Classes

1.1 Classes

Our language is the language of first-order logic with equality over one primitive binary predicate $\in$. We call all the objects we reason about *sets*. When $a \in b$, we say a is a *member* or *element* of b, or b *contains* a. We write $b \ni a$ for $a \in b$, and $a \notin b$ for $\neg(a \in b)$. We write $\forall x \in a.\phi$ as an abbreviation for $\forall x(x \in a \to \phi)$, and $\exists x \in a.\phi$ as an abbreviation for $\exists x(x \in a \wedge \phi)$.

We shall speak informally of *classes* as an abbreviation for talking about predicates. A *class* is determined by a unary predicate $\phi[x]$ (possibly with parameters). We write $\{x \mid \phi[x]\}$ or $\{x : \phi[x]\}$ for the class determined by $\phi[x]$. We write 'a is an element of $\{x \mid \phi[x]\}$' or '$a \in \{x \mid \phi[x]\}$' for $\phi[a]$.

We say two classes $\mathbf{A}$ and $\mathbf{B}$ are *equal*, and write $\mathbf{A} = \mathbf{B}$, iff $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$.

The following are all valid formulas of first-order logic:

Proposition Schema 1.1.1. *For any classes* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$*, the following are theorems:*

1. $\mathbf{A} = \mathbf{A}$
2. *If* $\mathbf{A} = \mathbf{B}$ *then* $\mathbf{B} = \mathbf{A}$.
3. *If* $\mathbf{A} = \mathbf{B}$ *and* $\mathbf{B} = \mathbf{C}$ *then* $\mathbf{A} = \mathbf{C}$.

Definition 1.1.2 (Subclass)**.** We say a class $\mathbf{A}$ is a *subclass* of $\mathbf{B}$, or $\mathbf{B}$ is a *superclass* of $\mathbf{A}$, or $\mathbf{B}$ *includes* $\mathbf{A}$, and write $\mathbf{A} \subseteq \mathbf{B}$ or $\mathbf{B} \supseteq \mathbf{A}$, iff every element of $\mathbf{A}$ is an element of $\mathbf{B}$. Otherwise we write $\mathbf{A} \nsubseteq \mathbf{B}$ or $\mathbf{B} \nsupseteq \mathbf{A}$.

We say $\mathbf{A}$ is a *proper* subclass of $\mathbf{B}$, $\mathbf{B}$ is a *proper* superclass of $\mathbf{A}$, or $\mathbf{B}$ *properly* includes $\mathbf{A}$, and write $\mathbf{A} \subsetneq \mathbf{B}$ or $\mathbf{B} \supsetneq \mathbf{A}$, iff in addition $\mathbf{A} \neq \mathbf{B}$.

The following are all valid formulas of first-order logic:

Proposition Schema 1.1.3. *For any classes* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$*, the following are theorems:*

1. $\mathbf{A} \subseteq \mathbf{A}$
2. *If* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{A}$ *then* $\mathbf{A} = \mathbf{B}$.
3. *If* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{C}$ *then* $\mathbf{A} \subseteq \mathbf{C}$.

Definition 1.1.4 (Empty Class)**.** The *empty class* $\emptyset$ is $\{x \mid \bot\}$.

Proposition 1.1.5. *For any class* $\mathbf{A}$*, we have* $\emptyset \subseteq \mathbf{A}$.

Proof: Vacuously, every element of $\emptyset$ is an element of $\mathbf{A}$. □

Definition 1.1.6 (Universal Class)**.** The *universal class* $\mathbf{V}$ is $\{x \mid \top\}$.

Proposition 1.1.7. *For any class* $\mathbf{A}$*, we have* $\mathbf{A} \subseteq \mathbf{V}$.

Proof: Trivially, every element of $\mathbf{A}$ is an element of $\mathbf{V}$. □

Definition 1.1.8 (Union)**.** The *union* of two classes $\mathbf{A}$ and $\mathbf{B}$ is the class $\mathbf{A} \cup \mathbf{B} = \{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$.

Proposition 1.1.9. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$*, we have*

$$\mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A}$$
$$\mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C}$$
$$\mathbf{A} \cup \emptyset = \mathbf{A}$$

Proof: These are valid formulas of first-order logic. □

Definition 1.1.10 (Intersection)**.** The *intersection* of two classes $\mathbf{A}$ and $\mathbf{B}$ is the class $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$.

Proposition 1.1.11. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$*, we have*

$$\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A}$$
$$\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$$
$$\mathbf{A} \cap \emptyset = \emptyset$$

Proof: These are valid formulas of first-order logic. □

Proposition 1.1.12 (Distributive Laws)**.** *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$*, we have*

$$\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$$
$$\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$$

Proof: These are valid formulas of first-order logic. □

Definition 1.1.13 (Union)**.** The *union* of a class $\mathbf{A}$ is $\{x \mid \exists X \in \mathbf{A}. x \in X\}$. We write $\bigcup_{P(x)} t(x)$ for $\bigcup\{t(x) \mid P(x)\}$.

Proposition 1.1.14. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, if* $\mathbf{A} \subseteq \mathbf{B}$ *then* $\bigcup \mathbf{A} \subseteq \bigcup \mathbf{B}$.

PROOF: First-order logic. □

Definition 1.1.15 (Intersection)**.** The *intersection* of a class $\mathbf{A}$ is $\{x \mid \forall X \in \mathbf{A}.x \in X\}$. We write $\bigcap_{P(x)} t(x)$ for $\bigcap\{t(x) \mid P(x)\}$.

Definition 1.1.16 (Relative Complement)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be classes. The *relative complement* of $\mathbf{B}$ in $\mathbf{A}$ is the class $\mathbf{A} - \mathbf{B} = \{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

Proposition 1.1.17 (De Morgan's Laws)**.** *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, *we have*

$$\mathbf{A} - (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} - \mathbf{B}) \cap (\mathbf{A} - \mathbf{C})$$
$$\mathbf{A} - (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} - \mathbf{B}) \cup (\mathbf{A} - \mathbf{C})$$

PROOF: First-order logic. □

Proposition 1.1.18. *If* $\mathbf{A} \subseteq \mathbf{B}$ *then* $\mathbf{C} - \mathbf{B} \subseteq \mathbf{C} - \mathbf{A}$.

PROOF: First-order logic. □

Definition 1.1.19 (Symmetric Difference)**.** The *symmetric difference* of classes $\mathbf{A}$ and $\mathbf{B}$ is the class $\mathbf{A} + \mathbf{B} := (\mathbf{A} - \mathbf{B}) \cup (\mathbf{B} - \mathbf{A})$.

Proposition 1.1.20. *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, *we have*

$$\mathbf{A} \cap (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) + (\mathbf{A} \cap \mathbf{C})$$
$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

PROOF: First-order logic. □

1.2 Axioms

Axiom 1.2.1 (Extensionality)**.** *If two sets have exactly the same members, they are equal.*

Thanks to this axiom, we may identify a set a with the class $\{x \mid x \in a\}$. Our use of the symbols $\in$ and $=$ is consistent. We say a class $\mathbf{A}$ *is a set* iff there exists a set a such that $a = \mathbf{A}$; that is, $\{x \mid \phi[x]\}$ is a set iff $\exists a \forall x (x \in a \leftrightarrow \phi[x])$. Otherwise, $\mathbf{A}$ is a *proper class.*

Axiom 1.2.2 (Union)**.** *The union of a set is a set.*

Axiom 1.2.3 (Power Set)**.** *For any set A, the class $\mathcal{P}A = \{x \mid x \subseteq A\}$ is a set, called the* power set *of A.*

Axiom 1.2.4 (Infinity)**.** *There exists a set I such that:*

- *There exists an element of I that has no members*
- *For every $x \in I$, there exists a set $y \in I$ such that the elements of y are exactly x and the members of x.*

Axiom 1.2.5 (Choice)**.** *For any set A of pairwise disjoint, nonempty sets, there exists a set C such that, for all $x \in A$, $x \cap C$ has exactly one element.*

Axiom Schema 1.2.6 (Replacement)**.** *For any predicate $P(x, y)$, the following is an axiom:*

Let A be a set. Assume that, for all $x \in A$, there exists at most one y such that $P(x, y)$. Then $\{y \mid \exists x \in A.P(x, y)\}$ is a set.

Axiom 1.2.7 (Regularity)**.** *For any nonempty set A, there exists $m \in A$ such that $m \cap A = \emptyset$.*

1.3 Basic Constructions on Sets

1.3.1 Consequences of the Axioms

Proposition 1.3.1. *The class $\emptyset = \{x \mid \bot\}$ is a set.*

PROOF: Immediate from the Axiom of Infinity. □

Proposition 1.3.2 (Pairing)**.** *For any sets a and b, the class $\{a, b\} = \{x \mid x = a \vee x = b\}$ is a set.*

PROOF:
$\langle 1 \rangle$1. LET: $P(x, y)$ be the predicate $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$.
$\langle 1 \rangle$2. For all $x \in \mathcal{P}\mathcal{P}\emptyset$, there exists at most one y such that $P(x, y)$.
  $\langle 2 \rangle$1. LET: $x \in \mathcal{P}\mathcal{P}\emptyset$
  $\langle 2 \rangle$2. LET: y and y' be sets.
  $\langle 2 \rangle$3. ASSUME: $P(x, y)$ and $P(x, y')$
  $\langle 2 \rangle$4. $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$
    PROOF: From $\langle 2 \rangle$3.
  $\langle 2 \rangle$5. $(x = \emptyset \wedge y' = a) \vee (x = \mathcal{P}\emptyset \wedge y' = b)$
    PROOF: From $\langle 2 \rangle$3.
  $\langle 2 \rangle$6. $\emptyset \neq \mathcal{P}\emptyset$
    PROOF: Since $\emptyset \in \mathcal{P}\emptyset$ and $\emptyset \notin \emptyset$.
  $\langle 2 \rangle$7. $y = y'$
$\langle 1 \rangle$3. LET: A be the set $\{y \mid \exists x \in \mathcal{P}\mathcal{P}\emptyset.P(x, y)\}$.
$\langle 1 \rangle$4. $A = \{a, b\}$
□

Proposition 1.3.3. *The union of two sets is a set.*

PROOF: The union of two sets A and B is $\bigcup\{A, B\}$. □

Proposition Schema 1.3.4. *For any sets a_1, ..., a_n, the class $\{a_1, \ldots, a_n\} = \{x \mid x = a_1 \vee \cdots \vee x = a_n\}$ is a set.*

PROOF: The case $n = 1$ follows from Pairing since $\{a\} = \{a, a\}$.
If we have proved the theorem for n we have $\{a_1, \ldots, a_n, a_{n+1}\} = \{a_1, \ldots, a_n\} \cup \{a_{n+1}\}$. □

Proposition 1.3.5. *For any sets A and B, if $A \subseteq B$ then $\mathcal{P}A \subseteq \mathcal{P}B$.*

PROOF: From Proposition 1.1.3. □

Proposition 1.3.6. *For any set A we have $\bigcup \mathcal{P}A = A$.*

PROOF:
⟨1⟩1. $\bigcup \mathcal{P}A \subseteq A$
⟨2⟩1. LET: $x \in \bigcup \mathcal{P}A$
⟨2⟩2. PICK $X \in \mathcal{P}A$ such that $x \in X$
PROOF: ⟨2⟩1
⟨2⟩3. $X \subseteq A$
PROOF: ⟨2⟩2
⟨2⟩4. $x \in A$
PROOF: ⟨2⟩2, ⟨2⟩3
⟨1⟩2. $A \subseteq \bigcup \mathcal{P}A$
PROOF: For all $x \in A$ we have $x \in \{x\} \in \mathcal{P}A$.
⟨1⟩3. Q.E.D.
PROOF: By Proposition 1.1.3.
□

1.3.2 Comprehension

Proposition Schema 1.3.7 (Comprehension)**.** *For any predicate $P(x)$, the following is a theorem:*
For any set A, the class $\{x \in A \mid P(x)\}$ is a set.

PROOF:
⟨1⟩1. LET: A be a set.
⟨1⟩2. LET: $Q(x, y)$ be the predicate $P(x) \wedge y = x$.
⟨1⟩3. For all $x \in A$, there exists at most one y such that $Q(x, y)$.
⟨2⟩1. LET: $x \in A$
⟨2⟩2. LET: y and y' be sets.
⟨2⟩3. ASSUME: $Q(x, y)$ and $Q(x, y')$
⟨2⟩4. $x \in A \wedge P(x) \wedge y = x \wedge y' = x$
PROOF: From ⟨2⟩3.
⟨2⟩5. $y = y'$
PROOF: From ⟨2⟩4.
⟨1⟩4. LET: B be the set $\{y \mid \exists x \in A.Q(x, y)\}$
PROOF: This is a set by an Axiom of Replacement and ⟨1⟩3.
⟨1⟩5. $B = \{y \in A \mid P(y)\}$
PROOF:

$$\begin{aligned} y \in B &\Leftrightarrow \exists x \in A.Q(x, y) && (\langle 1\rangle 4) \\ &\Leftrightarrow \exists x \in A(P(x) \wedge y = x) && (\langle 1\rangle 2) \\ &\Leftrightarrow P(y) \end{aligned}$$

□

Corollary 1.3.7.1. *The intersection of a set and a class is a set.*

Corollary 1.3.7.2. *The intersection of a nonempty class is a set.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathbf{A}$ be a nonempty class.
$\langle 1 \rangle 2$. PICK$A \in \mathbf{A}$
$\langle 1 \rangle 3$. $\bigcap \mathbf{A} = \{x \in A \mid \forall X \in \mathbf{A}. x \in X\}$ which is a set.
□

Corollary 1.3.7.3. *The relative complement of a class in a set is a set.*

Corollary 1.3.7.4 (Russell's Paradox)**.** $\mathbf{V}$ *is a proper class.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathbf{R} = \{x \mid x \notin x\}$
$\langle 1 \rangle 2$. $\mathbf{R}$ is a proper class.
  $\langle 2 \rangle 1$. ASSUME: for a contradiction $\mathbf{R}$ is a set
  $\langle 2 \rangle 2$. $\mathbf{R} \in \mathbf{R}$ iff $\mathbf{R} \notin \mathbf{R}$
  $\langle 2 \rangle 3$. This is a contradiction.
$\langle 1 \rangle 3$. $\mathbf{V}$ is a proper class.
  PROOF: From Comprehension and $\langle 1 \rangle 2$.
□

Definition 1.3.8. For any sets A and B, the *relative complement* $A - B$ is the set $\{x \in A \mid x \notin B\}$.

Proposition 1.3.9 (Distributive Laws)**.** *For any set A and class $\mathbf{B}$, we have*

$$A \cup \bigcap \mathbf{B} = \bigcap \{A \cup X \mid X \in \mathbf{B}\}$$
$$A \cap \bigcup \mathbf{B} = \bigcup \{A \cap X \mid X \in \mathbf{B}\}$$

PROOF: First-order logic. □

Proposition 1.3.10 (De Morgan's Laws)**.** *For any set C and class $\mathbf{A}$, we have*

$$C - \bigcap \mathbf{A} = \bigcup \{C - X \mid X \in \mathbf{A}\}$$
$$C - \bigcup \mathbf{A} = \bigcap \{C - X \mid X \in \mathbf{A}\}$$

PROOF: First-order logic. □

Definition 1.3.11 (Transitive Class)**.** A class $\mathbf{A}$ is a *transitive class* iff whenever $x \in y \in \mathbf{A}$ then $x \in \mathbf{A}$.

Proposition 1.3.12. *Let A be a set. Then the following are equivalent.*

1. *A is a transitive class.*
2. $\bigcup A \subseteq A$
3. *Every element of A is a subset of A.*

4. $A \subseteq \mathcal{P}A$

Proof: Immediate from definitions. □

Proposition 1.3.13. *For any set a, we have a is a transitive set if and only if $\mathcal{P}a$ is a transitive set.*

Proof:
⟨1⟩1. If a is a transitive set then $\mathcal{P}a$ is a transitive set.
⟨2⟩1. Assume: a is a transitive set.
⟨2⟩2. $a \subseteq \mathcal{P}a$
Proof: Proposition 1.3.12, ⟨2⟩1.
⟨2⟩3. $\mathcal{P}a \subseteq \mathcal{P}\mathcal{P}a$
Proof: Proposition 1.3.5, ⟨2⟩2.
⟨2⟩4. $\mathcal{P}a$ is a transitive set.
Proof: Proposition 1.3.12, ⟨2⟩3.
⟨1⟩2. If $\mathcal{P}a$ is a transitive set then a is a transitive set.
⟨2⟩1. Assume: $\mathcal{P}a$ is a transitive set.
⟨2⟩2. $\bigcup \mathcal{P}a \subseteq \mathcal{P}a$
Proof: Proposition 1.3.12, ⟨2⟩1.
⟨2⟩3. $a \subseteq \mathcal{P}a$
Proof: Proposition 1.3.6, ⟨2⟩2
⟨2⟩4. a is a transitive set.
Proof: Proposition 1.3.12, ⟨2⟩3.
□

Proposition 1.3.14. *If $\mathbf{A}$ is a transitive class then $\bigcup \mathbf{A}$ is a transitive class.*

Proof:
⟨1⟩1. Assume: $\mathbf{A}$ is a transitive class.
⟨1⟩2. Let: $x \in y \in \bigcup \mathbf{A}$
⟨1⟩3. $y \in \mathbf{A}$
Proof: ⟨1⟩1, ⟨1⟩2
⟨1⟩4. $x \in \mathbf{A}$
Proof: ⟨1⟩1, ⟨1⟩2, ⟨1⟩3
□

Proposition 1.3.15. *If every member of $\mathbf{A}$ is a transitive set then $\bigcup \mathbf{A}$ is a transitive class.*

Proof:
⟨1⟩1. Assume: Every member of $\mathbf{A}$ is a transitive set.
⟨1⟩2. Let: $x \in y \in \bigcup \mathbf{A}$
⟨1⟩3. Pick $A \in \mathbf{A}$ such that $y \in A$.
⟨1⟩4. $x \in A$
⟨1⟩5. $x \in \bigcup \mathbf{A}$
□

Chapter 2

Relations

2.1 Ordered Pairs

Definition 2.1.1 (Ordered Pair)**.** For any sets a and b, the *ordered pair* (a, b) is defined to be $\{\{a\}, \{a, b\}\}$.

Theorem 2.1.2. *For any sets a, b, c, d, we have $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.*

PROOF:
$\langle 1 \rangle 1$. If $(a, b) = (c, d)$ then $a = c$ and $b = d$.
  $\langle 2 \rangle 1$. ASSUME: $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$
  $\langle 2 \rangle 2$. $\bigcap\{\{a\}, \{a, b\}\} = \bigcap\{\{c\}, \{c, d\}\}$
  $\langle 2 \rangle 3$. $\{a\} = \{c\}$
  $\langle 2 \rangle 4$. $a = c$
  $\langle 2 \rangle 5$. $\bigcup\{\{a\}, \{a, b\}\} = \bigcup\{\{c\}, \{c, d\}\}$
  $\langle 2 \rangle 6$. $\{a, b\} = \{c, d\}$
  $\langle 2 \rangle 7$. $b = c$ or $b = d$
  $\langle 2 \rangle 8$. $a = d$ or $b = d$
  $\langle 2 \rangle 9$. If $b = c$ and $a = d$ then $b = d$
    PROOF: By $\langle 2 \rangle 4$.
  $\langle 2 \rangle 10$. $b = d$
    PROOF: From $\langle 2 \rangle 7$, $\langle 2 \rangle 8$, $\langle 2 \rangle 9$.
$\langle 1 \rangle 2$. If $a = c$ and $b = d$ then $(a, b) = (c, d)$.
  PROOF: First-order logic.
$\square$

Definition 2.1.3 (Cartesian Product)**.** The *Cartesian product* of classes $\mathbf{A}$ and $\mathbf{B}$ is the class $\mathbf{A} \times \mathbf{B} := \{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$.

Proposition 2.1.4. *If A and B are sets then $A \times B$ is a set.*

PROOF: It is a subset of $\mathcal{PP}(A \cup B)$. $\square$

Proposition 2.1.5. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, we have* $\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$*.*

Proof:

$$\begin{aligned}(x, y) \in \mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) &\Leftrightarrow x \in \mathbf{A} \wedge (y \in \mathbf{B} \vee y \in \mathbf{C}) \\ &\Leftrightarrow (x \in \mathbf{A} \wedge y \in \mathbf{B}) \vee (x \in \mathbf{A} \wedge y \in \mathbf{C}) \\ &\Leftrightarrow (x, y) \in (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})\end{aligned}$$

□

Proposition 2.1.6. *If* $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ *and* $\mathbf{A}$ *is nonempty then* $\mathbf{B} = \mathbf{C}$*.*

Proof:

⟨1⟩1. Pick $a \in \mathbf{A}$

⟨1⟩2. For all x we have $x \in \mathbf{B}$ iff $x \in \mathbf{C}$.

Proof:

$$\begin{aligned}x \in \mathbf{B} &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{B} \\ &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{C} \\ &\Leftrightarrow x \in \mathbf{C}\end{aligned}$$

□

Proposition 2.1.7. *For any set* A *and class* $\mathbf{B}$*, we have* $A \times \bigcup \mathbf{B} = \bigcup \{A \times X \mid X \in \mathbf{B}\}$*.*

Proof:

$$\begin{aligned}(x, y) \in A \times \bigcup \mathbf{B} &\Leftrightarrow x \in A \wedge \exists Y \in \mathbf{B}. y \in Y \\ &\Leftrightarrow \exists Y \in \mathbf{B}(x \in A \wedge y \in Y) \\ &\Leftrightarrow (x, y) \in \bigcup \{A \times X \mid X \in \mathbf{B}\}\end{aligned}$$

□

2.2 Relations

Definition 2.2.1 (Relation)**.** A *relation* is a class of ordered pairs.

Definition 2.2.2 (Domain)**.** The *domain* of a class $\mathbf{R}$ is the class

$$\operatorname{dom} \mathbf{R} := \{x \mid \exists y.(x, y) \in \mathbf{R}\} \ .$$

Definition 2.2.3 (Range)**.** The *range* of a class $\mathbf{R}$ is the class

$$\operatorname{ran} \mathbf{R} := \{x \mid \exists y.(y, x) \in \mathbf{R}\} \ .$$

Definition 2.2.4 (Field)**.** The *field* of a class $\mathbf{R}$ is the class

$$\operatorname{fld} \mathbf{R} := \operatorname{dom} \mathbf{R} \cup \operatorname{ran} \mathbf{R} \ .$$

Proposition 2.2.5. *For any set* R*, the classes* $\operatorname{dom} R$*,* $\operatorname{ran} R$*,* $\operatorname{fld} R$ *are sets.*

Proof: They are all subsets of $\bigcup \bigcup R$. □

Definition 2.2.6 (Single-Rooted)**.** A class $\mathbf{R}$ is *single-rooted* iff, for all $y \in \operatorname{ran}\mathbf{R}$, there is exactly one x such that $(x, y) \in \mathbf{R}$.

Definition 2.2.7 (Inverse)**.** The *inverse* of a class $\mathbf{F}$ is the class

$$\mathbf{F}^{-1} := \{(x, y) \mid (y, x) \in \mathbf{F}\} \ .$$

Proposition 2.2.8. *For any class* $\mathbf{F}$*, we have* $\operatorname{dom}\mathbf{F}^{-1} = \operatorname{ran}\mathbf{F}$

Proof:

$$\begin{aligned} y \in \operatorname{dom}\mathbf{F}^{-1} &\Leftrightarrow \exists x.(y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists x.(x, y) \in \mathbf{F} \\ &\Leftrightarrow y \in \operatorname{ran}\mathbf{F} \end{aligned} \qquad \square$$

Proposition 2.2.9. *For any class* $\mathbf{F}$*, we have* $\operatorname{ran}\mathbf{F}^{-1} = \operatorname{dom}\mathbf{F}$.

Proof:

$$\begin{aligned} y \in \operatorname{ran}\mathbf{F}^{-1} &\Leftrightarrow \exists x.(x, y) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists x.(y, x) \in \mathbf{F} \\ &\Leftrightarrow y \in \operatorname{dom}\mathbf{F} \end{aligned} \qquad \square$$

Proposition 2.2.10. *For any relation* $\mathbf{F}$*, we have* $(\mathbf{F}^{-1})^{-1} = \mathbf{F}$.

Proof:

$$\begin{aligned} (x, y) \in (\mathbf{F}^{-1})^{-1} &\Leftrightarrow (y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow (x, y) \in \mathbf{F} \end{aligned} \qquad \square$$

Definition 2.2.11 (Composition)**.** The *composition* of classes $\mathbf{F}$ and $\mathbf{G}$ is the class

$$\mathbf{F} \circ \mathbf{G} := \{(x, z) \mid \exists y.(x, y) \in \mathbf{G} \wedge (y, z) \in \mathbf{F}\} \ .$$

Proposition 2.2.12. *For any classes* $\mathbf{F}$ *and* $\mathbf{G}$*,*

$$(\mathbf{F} \circ \mathbf{G})^{-1} = \mathbf{G}^{-1} \circ \mathbf{F}^{-1} \ .$$

Proof:

$$\begin{aligned} (z, x) \in (\mathbf{F} \circ \mathbf{G})^{-1} &\Leftrightarrow (x, z) \in \mathbf{F} \circ \mathbf{G} \\ &\Leftrightarrow \exists y.(x, y) \in \mathbf{G} \wedge (y, z) \in \mathbf{F} \\ &\Leftrightarrow \exists y.(y, x) \in \mathbf{G}^{-1} \wedge (z, y) \in \mathbf{F}^{-1} \\ &\Leftrightarrow (z, x) \in \mathbf{G}^{-1} \circ \mathbf{F}^{-1} \end{aligned} \qquad \square$$

Definition 2.2.13 (Restriction)**.** The *restriction* of the class $\mathbf{F}$ to the class $\mathbf{A}$ is the class $\mathbf{F} \restriction \mathbf{A} := \{(x, y) \mid x \in \mathbf{A}, (x, y) \in \mathbf{F}\}$.

Definition 2.2.14 (Image)**.** The *image* of the class $\mathbf{A}$ under the class $\mathbf{F}$ is the set $F(A) := \operatorname{ran}(F \restriction A) = \{y \mid \exists x \in \mathbf{A}.(x, y) \in \mathbf{F}\}$.

Proposition 2.2.15. *For any classes* $\mathbf{F}$*,* $\mathbf{A}$ *and* $\mathbf{B}$*, we have*

$$\mathbf{F}(\mathbf{A} \cup \mathbf{B}) = \mathbf{F}(\mathbf{A}) \cup \mathbf{F}(\mathbf{B}) \enspace .$$

PROOF:

$$\begin{aligned} y \in \mathbf{F}(\mathbf{A} \cup \mathbf{B}) &\Leftrightarrow \exists x \in \mathbf{A} \cup \mathbf{B}.(x,y) \in \mathbf{F} \\ &\Leftrightarrow \exists x \in \mathbf{A}.(x,y) \in \mathbf{F} \vee \exists x \in \mathbf{B}.(x,y) \in \mathbf{F} \\ &\Leftrightarrow y \in \mathbf{F}(\mathbf{A}) \cup \mathbf{F}(\mathbf{B}) \end{aligned}$$

∎

Proposition 2.2.16. *For any classes* $\mathbf{F}$ *and* $\mathbf{A}$ *we have* $\mathbf{F}(\bigcup \mathbf{A}) = \bigcup\{\mathbf{F}(X) \mid X \in \mathbf{A}\}$.

PROOF:

$$\begin{aligned} y \in \mathbf{F}(\bigcup \mathbf{A}) &\Leftrightarrow \exists x \in \bigcup \mathbf{A}.(x,y) \in \mathbf{F} \\ &\Leftrightarrow \exists x.\exists X.X \in \mathbf{A} \wedge x \in X \wedge (x,y) \in \mathbf{F} \\ &\Leftrightarrow \exists X \in \mathbf{F}.y \in \mathbf{F}(X) \end{aligned}$$

∎

Proposition 2.2.17. *For any classes* $\mathbf{F}$*,* $\mathbf{A}$ *and* $\mathbf{B}$*, we have* $\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$*. Equality holds if* $\mathbf{F}$ *is single-rooted.*

PROOF:
$\langle 1 \rangle 1.$ $\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
  $\langle 2 \rangle 1.$ LET: $y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$
  $\langle 2 \rangle 2.$ PICK $x \in \mathbf{A} \cap \mathbf{B}$ such that $(x,y) \in \mathbf{F}$
  $\langle 2 \rangle 3.$ $y \in \mathbf{F}(\mathbf{A})$
    PROOF: Since $x \in \mathbf{A}$.
  $\langle 2 \rangle 4.$ $y \in \mathbf{F}(\mathbf{B})$
    PROOF: Since $x \in \mathbf{B}$.
$\langle 1 \rangle 2.$ If $\mathbf{F}$ is single-rooted then $\mathbf{F}(\mathbf{A} \cap \mathbf{B}) = \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$.
  $\langle 2 \rangle 1.$ ASSUME: $\mathbf{F}$ is single-rooted.
  $\langle 2 \rangle 2.$ LET: $y \in \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
  $\langle 2 \rangle 3.$ PICK $x \in \mathbf{A}$ such that $(x,y) \in \mathbf{F}$
  $\langle 2 \rangle 4.$ PICK $x' \in \mathbf{B}$ such that $(x',y) \in \mathbf{F}$
  $\langle 2 \rangle 5.$ $x = x'$
    PROOF: $\langle 2 \rangle 1$
  $\langle 2 \rangle 6.$ $x \in \mathbf{A} \cap \mathbf{B}$
  $\langle 2 \rangle 7.$ $y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$
∎

Proposition 2.2.18. *For any classes* $\mathbf{F}$ *and* $\mathbf{A}$ *we have*

$$\mathbf{F}\left(\bigcap \mathbf{A}\right) \subseteq \bigcap\{\mathbf{F}(X) \mid X \in \mathbf{A}\} \enspace .$$

Equality holds if $\mathbf{F}$ *is single-rooted and* $\mathbf{A}$ *is nonempty.*

PROOF:
$\langle 1 \rangle 1.$ $\mathbf{F}\left(\bigcap \mathbf{A}\right) \subseteq \bigcap\{\mathbf{F}(X) \mid X \in \mathbf{A}\}$

⟨2⟩1. Let: $y \in \mathbf{F}(\bigcap \mathbf{A})$
⟨2⟩2. Pick $x \in \bigcap \mathbf{A}$ such that $(x, y) \in \mathbf{F}$
⟨2⟩3. Let: $X \in \mathbf{A}$
Prove: $y \in \mathbf{F}(X)$
⟨2⟩4. $x \in X$
⟨2⟩5. $y \in \mathbf{F}(X)$
⟨1⟩2. If $\mathbf{F}$ is single-rooted then $\mathbf{F}(\bigcap \mathbf{A}) = \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
⟨2⟩1. Assume: $\mathbf{F}$ is single-rooted.
⟨2⟩2. Assume: $\mathbf{A}$ is nonempty.
⟨2⟩3. Let: $y \in \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
⟨2⟩4. Pick $X_0 \in \mathbf{A}$
⟨2⟩5. Pick $x \in X_0$ such that $(x, y) \in \mathbf{F}$
⟨2⟩6. $x \in \bigcap \mathbf{A}$
⟨3⟩1. Let: $X \in \mathbf{A}$
⟨3⟩2. Pick $x' \in X$ such that $(x', y) \in \mathbf{F}$.
⟨3⟩3. $x = x'$
Proof: ⟨2⟩1
⟨3⟩4. $x \in X$
⟨2⟩7. $y \in \mathbf{F}(\bigcap \mathbf{A})$
□

Proposition 2.2.19. *For any classes* $\mathbf{F}$, $\mathbf{A}$ *and* $\mathbf{B}$, *we have*

$$\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B}) \ .$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

Proof:
⟨1⟩1. $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
⟨2⟩1. Let: $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
⟨2⟩2. Pick $x \in \mathbf{A}$ such that $(x, y) \in \mathbf{F}$
⟨2⟩3. $x \notin \mathbf{B}$
⟨2⟩4. $x \in \mathbf{A} - \mathbf{B}$
⟨2⟩5. $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
⟨1⟩2. If $\mathbf{F}$ is single-rooted then $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) = \mathbf{F}(\mathbf{A} - \mathbf{B})$
⟨2⟩1. Assume: $\mathbf{F}$ is single-rooted.
⟨2⟩2. Let: $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
⟨2⟩3. Pick $x \in \mathbf{A} - \mathbf{B}$ such that $(x, y) \in \mathbf{F}$
⟨2⟩4. $y \in \mathbf{F}(\mathbf{A})$
⟨2⟩5. $y \notin \mathbf{F}(\mathbf{B})$
⟨3⟩1. Assume: for a contradiction $y \in \mathbf{F}(\mathbf{B})$
⟨3⟩2. Pick $x' \in \mathbf{B}$ such that $(x', y) \in \mathbf{F}$
⟨3⟩3. $x = x'$
Proof: ⟨2⟩1
⟨3⟩4. $x \in \mathbf{B}$
⟨3⟩5. Q.E.D.
Proof: This contradicts ⟨2⟩3.

□

Definition 2.2.20 (Reflexive)**.** Let $\mathbf{R}$ be a binary relation on $\mathbf{A}$. Then $\mathbf{R}$ is *reflexive* on $\mathbf{A}$ iff $\forall x \in \mathbf{A}.(x, x) \in \mathbf{R}$.

Definition 2.2.21 (Irreflexive)**.** A relation $\mathbf{R}$ is *irreflexive* iff there is no x such that $(x, x) \in \mathbf{R}$.

Definition 2.2.22 (Symmetric)**.** A relation $\mathbf{R}$ is *symmetric* iff, whenever $(x, y) \in \mathbf{R}$, then $(y, x) \in \mathbf{R}$.

Definition 2.2.23 (Transitive)**.** A relation $\mathbf{R}$ is *transitive* iff, whenever $(x, y), (y, z) \in \mathbf{R}$, then $(x, z) \in \mathbf{R}$.

Proposition 2.2.24. *If* $\mathbf{R}$ *is transitive then* $\mathbf{R}^{-1}$ *is transitive.*

PROOF:
⟨1⟩1. ASSUME: $(x, y), (y, z) \in \mathbf{R}^{-1}$
⟨1⟩2. $(y, x), (z, y) \in \mathbf{R}$
⟨1⟩3. $(z, x) \in \mathbf{R}$
⟨1⟩4. $(x, z) \in \mathbf{R}^{-1}$
□

2.3 n-ary Relations

Definition Schema 2.3.1. For any sets $a_1, \ldots, a_n$, define the *ordered n-tuple* $(a_1, \ldots, a_n)$ by

$$(a_1) := a_1$$
$$(a_1, \ldots, a_n, a_{n+1}) = ((a_1, \ldots, a_n), a_{n+1})$$

Definition Schema 2.3.2. An *n-ary relation on* $\mathbf{A}$ is a class of ordered n-tuples all of whose components are in $\mathbf{A}$.

2.4 Equivalence Relations

Definition 2.4.1 (Equivalence Relation)**.** An *equivalence relation* on a class $\mathbf{A}$ is a relation on $\mathbf{A}$ that is reflexive on $\mathbf{A}$, symmetric and transitive.

Proposition 2.4.2. *If* $\mathbf{R}$ *is a symmetric and transitive relation, then* $\mathbf{R}$ *is an equivalence relation on* $\operatorname{fld}\mathbf{R}$*.*

PROOF:
⟨1⟩1. LET: $x \in \operatorname{fld}\mathbf{R}$
PROVE: $(x, x) \in \mathbf{R}$
⟨1⟩2. PICK y such that either $(x, y) \in \mathbf{R}$ or $(y, x) \in \mathbf{R}$
⟨1⟩3. $(x, y) \in \mathbf{R}$ and $(y, x) \in \mathbf{R}$
PROOF: Symmetry.

$\langle 1\rangle 4.$ $(x, x) \in \mathbf{R}$
  PROOF: Transitivity.
□

Definition 2.4.3 (Equivalence Class)**.** Let $\mathbf{R}$ be an equivalence relation on $\mathbf{A}$ and $a \in \mathbf{A}$. The *equivalence class* of a *modulo* $\mathbf{R}$ is

$$[a]_{\mathbf{R}} := \{x \mid (a, x) \in \mathbf{R}\} \ .$$

Proposition 2.4.4. *Let* $\mathbf{R}$ *be an equivalence relation on* $\mathbf{A}$ *and* $a, b \in \mathbf{A}$*. Then* $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ *if and only if* $(a, b) \in \mathbf{R}$*.*

PROOF:
$\langle 1\rangle 1.$ If $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ then $(a, b) \in \mathbf{R}$.
  $\langle 2\rangle 1.$ ASSUME: $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$
  $\langle 2\rangle 2.$ $(b, b) \in \mathbf{R}$
    PROOF: Reflexivity
  $\langle 2\rangle 3.$ $b \in [b]_{\mathbf{R}}$
  $\langle 2\rangle 4.$ $b \in [a]_{\mathbf{R}}$
  $\langle 2\rangle 5.$ $(a, b) \in \mathbf{R}$
$\langle 1\rangle 2.$ If $(a, b) \in \mathbf{R}$ then $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$.
  $\langle 2\rangle 1.$ For all $x, y \in \mathbf{A}$, if $(x, y) \in \mathbf{R}$ then $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$
    $\langle 3\rangle 1.$ LET: $x, y \in \mathbf{A}$
    $\langle 3\rangle 2.$ ASSUME: $(x, y) \in \mathbf{R}$
    $\langle 3\rangle 3.$ LET: $t \in [y]_{\mathbf{R}}$
    $\langle 3\rangle 4.$ $(y, t) \in \mathbf{R}$
      PROOF: $\langle 3\rangle 3$
    $\langle 3\rangle 5.$ $(x, t) \in \mathbf{R}$
      PROOF: Transitivity, $\langle 3\rangle 2$, $\langle 3\rangle 4$.
    $\langle 3\rangle 6.$ $t \in [x]_{\mathbf{R}}$
      PROOF: $\langle 3\rangle 5$
  $\langle 2\rangle 2.$ ASSUME: $(a, b) \in \mathbf{R}$
  $\langle 2\rangle 3.$ $[b]_{\mathbf{R}} \subseteq [a]_{\mathbf{R}}$
    PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 2$.
  $\langle 2\rangle 4.$ $(b, a) \in \mathbf{R}$
    PROOF: Symmetry, $\langle 2\rangle 2$.
  $\langle 2\rangle 5.$ $[a]_{\mathbf{R}} \subseteq [b]_{\mathbf{R}}$
    PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 4$.
  $\langle 2\rangle 6.$ $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$
    PROOF: $\langle 2\rangle 3$, $\langle 2\rangle 5$.
□

Definition 2.4.5 (Partition)**.** A *partition* Π of a set A is a set of nonempty subsets of A that is disjoint and exhaustive, i.e.

1. no two different sets in Π have any common elements, and
2. each element of A is in some set in Π.

Definition 2.4.6. Let R be an equivalence relation on a set A. The *quotient set* A/R is the set of all equivalence classes.

Proposition 2.4.7. *Let R be an equivalence relation on a set A. Then A/R is a partition of A.*

PROOF:
⟨1⟩1. Every member of A/R is nonempty.
  PROOF: Since $a \in [a]_R$ by reflexivity.
⟨1⟩2. No two different sets in A/R have any common elements.
  ⟨2⟩1. LET: $[a]_R, [b]_R \in A/R$
  ⟨2⟩2. LET: $c \in [a]_R \cap [b]_R$
    PROVE: $[a]_R = [b]_R$
  ⟨2⟩3. $(a, c) \in R$
    PROOF: ⟨2⟩2
  ⟨2⟩4. $(b, c) \in R$
    PROOF: ⟨2⟩2
  ⟨2⟩5. $(c, b) \in R$
    PROOF: Symmetry, ⟨2⟩4
  ⟨2⟩6. $(a, b) \in R$
    PROOF: Transitivity, ⟨2⟩3, ⟨2⟩5
  ⟨2⟩7. $[a]_R = [b]_R$
    PROOF: Proposition 2.4.4, ⟨2⟩6
⟨1⟩3. Each element of A is in some set in A/R.
  PROOF: Since $a \in [a]_R$ by reflexivity.
□

2.5 Ordering Relations

Definition 2.5.1 (Linear Ordering)**.** Let $\mathbf{A}$ be a class. A *linear ordering* or *total ordering* on $\mathbf{A}$ is a relation $\mathbf{R}$ on $\mathbf{A}$ such that:

1. $\mathbf{R}$ is transitive.
2. *Trichotomy.* For all $x, y \in \mathbf{A}$, exactly one of the following holds:

$$(x, y) \in \mathbf{R}, \qquad (y, x) \in \mathbf{R}, \qquad x = y \ .$$

We often use the symbol $<$ for a linear ordering, and then write $x < y$ for $(x, y) \in <$.

Theorem 2.5.2. *Any linear ordering on a class is irreflexive.*

PROOF: Immediate from trichotomy. □

Proposition 2.5.3. *If $\mathbf{R}$ is a linear ordering on $\mathbf{A}$ then $\mathbf{R}^{-1}$ is also a linear ordering on $\mathbf{A}$.*

PROOF:

$\langle 1\rangle 1.$ $\mathbf{R}^{-1}$ is transitive.
PROOF: Proposition 2.2.24.
$\langle 1\rangle 2.$ $\mathbf{R}^{-1}$ satisfies trichotomy.
$\langle 2\rangle 1.$ LET: $x, y \in \mathbf{A}$
$\langle 2\rangle 2.$ Exactly one of $(x, y) \in \mathbf{R}$, $(y, x) \in \mathbf{R}$, $x = y$ holds.
$\langle 2\rangle 3.$ Exactly one of $(y, x) \in \mathbf{R}^{-1}$, $(x, y) \in \mathbf{R}^{-1}$, $x = y$ holds.
$\square$

Definition 2.5.4 (Lexicographic Ordering)**.** Let A and B be linearly ordered sets. The *lexicographic ordering* $<$ on $A \times B$ is defined by:

$$(a, b) < (a', b') \Leftrightarrow a < a' \vee (a = a' \wedge b < b') \ .$$

Proposition 2.5.5. *Let A and B be linearly ordered sets. Then the lexicographic ordering on $A \times B$ is a linear ordering.*

PROOF:
$\langle 1\rangle 1.$ $<$ is transitive.
$\langle 2\rangle 1.$ LET: $(a_1, b_1) < (a_2, b_2) < (a_3, b_3)$
PROVE: $(a_1, b_1) < (a_3, b_3)$
$\langle 2\rangle 2.$ CASE: $a_1 < a_2$
$\langle 3\rangle 1.$ $a_2 < a_3$ or $a_2 = a_3$
PROOF: $\langle 2\rangle 1$
$\langle 3\rangle 2.$ $a_1 < a_3$
PROOF: Transitivity
$\langle 3\rangle 3.$ $(a_1, b_1) < (a_3, b_3)$
$\langle 2\rangle 3.$ CASE: $a_1 = a_2$ and $b_1 < b_2$ and $a_2 < a_3$
PROOF: We have $a_1 < a_3$ so $(a_1, b_1) < (a_3, b_3)$.
$\langle 2\rangle 4.$ CASE: $a_1 = a_2$ and $b_1 < b_2$ and $a_2 = a_3$ and $b_2 < b_3$
PROOF: We have $a_1 = a_3$ and $b_1 < b_3$ so $(a_1, b_1) < (a_3, b_3)$.
$\langle 1\rangle 2.$ $<$ satisfies trichotomy.
$\langle 2\rangle 1.$ LET: $(a_1, b_1), (a_2, b_2) \in A \times B$
$\langle 2\rangle 2.$ Exactly one of $a_1 < a_2$, $a_1 = a_2$, $a_2 < a_1$ holds.
$\langle 2\rangle 3.$ CASE: $a_1 < a_2$
PROOF: We have $(a_1, b_1) < (a_2, b_2)$, $(a_1, b_1) \neq (a_2, b_2)$, and $(a_2, b_2) \not< (a_1, b_1)$.
$\langle 2\rangle 4.$ CASE: $a_1 = a_2$
$\langle 3\rangle 1.$ Exactly one of $b_1 < b_2$, $b_1 = b_2$, $b_2 < b_1$ holds.
$\langle 3\rangle 2.$ Exactly one of $(a_1, b_1) < (a_2, b_2)$, $(a_1, b_1) = (a_2, b_2)$, $(a_2, b_2) < (a_1, b_1)$ holds.
$\langle 2\rangle 5.$ CASE: $a_2 < a_1$
PROOF: We have $(a_2, b_2) < (a_1, b_1)$, $(a_2, b_2) \neq (a_1, b_1)$, and $(a_1, b_1) \not< (a_2, b_2)$.

Chapter 3

Functions

3.1 Functions

Definition 3.1.1 (Function)**.** A *function* is a relation $\mathbf{F}$ such that, for all $x \in \operatorname{dom} \mathbf{F}$, there is only one y such that $(x, y) \in \mathbf{F}$. We denote this y by $\mathbf{F}(x)$.

We say that $\mathbf{F}$ is a function *from* $\mathbf{A}$ *into* $\mathbf{B}$, or that $\mathbf{F}$ *maps* $\mathbf{A}$ *into* $\mathbf{B}$, and write $\mathbf{F} : \mathbf{A} \to \mathbf{B}$, iff $\mathbf{F}$ is a function, $\operatorname{dom} \mathbf{F} = \mathbf{A}$ and $\operatorname{ran} \mathbf{F} \subseteq \mathbf{B}$.

Proposition 3.1.2. *For any class* $\mathbf{F}$*,* $\mathbf{F}^{-1}$ *is a function if and only if* $\mathbf{F}$ *is single-rooted.*

Proof: Immediate from definitions. □

Proposition 3.1.3. *For any relation* $\mathbf{F}$*,* $\mathbf{F}$ *is a function if and only if* $\mathbf{F}^{-1}$ *is single-rooted.*

Proof: Immediate from definitions. □

Proposition 3.1.4. *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be functions. Then* $\mathbf{F} \circ \mathbf{G}$ *is a function, its domain is*

$$\{x \in \operatorname{dom} \mathbf{G} \mid \mathbf{G}(x) \in \operatorname{dom} \mathbf{F}\} \enspace ,$$

and for x *in this domain,* $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$*.*

Proof:
$\langle 1 \rangle 1.$ $\mathbf{F} \circ \mathbf{G}$ is a function.
  $\langle 2 \rangle 1.$ Let: $(x, z), (x, z') \in \mathbf{F} \circ \mathbf{G}$
  $\langle 2 \rangle 2.$ Pick y, y' such that $(x, y) \in \mathbf{G}$, $(y, z) \in \mathbf{F}$, $(x, y') \in \mathbf{G}$, $(y', z') \in \mathbf{F}$
  $\langle 2 \rangle 3.$ $y = y'$
    Proof: $\mathbf{G}$ is a function.
  $\langle 2 \rangle 4.$ $z = z'$
    Proof: $\mathbf{F}$ is a function.
$\langle 1 \rangle 2.$ $\operatorname{dom}(\mathbf{F} \circ \mathbf{G}) = \{x \in \operatorname{dom} \mathbf{G} \mid \mathbf{G}(x) \in \operatorname{dom} \mathbf{F}\}$

PROOF:

$$\begin{aligned} x \in \mathrm{dom}(\mathbf{F} \circ \mathbf{G}) &\Leftrightarrow \exists z.(x,z) \in \mathbf{F} \circ \mathbf{G} \\ &\Leftrightarrow \exists y,z((x,y) \in \mathbf{G} \wedge (y,z) \in \mathbf{F}) \\ &\Leftrightarrow \exists y((x,y) \in \mathbf{G} \wedge y \in \mathrm{dom}\,\mathbf{F}) \\ &\Leftrightarrow x \in \mathrm{dom}\,\mathbf{G} \wedge \mathbf{G}(y) \in \mathrm{dom}\,\mathbf{F} \end{aligned}$$

$\langle 1\rangle 3.$ $\forall x \in \mathrm{dom}(\mathbf{F} \circ \mathbf{G}).(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$

PROOF:

$\langle 2\rangle 1.$ LET: $x \in \mathrm{dom}(\mathbf{F} \circ \mathbf{G})$

$\langle 2\rangle 2.$ $(x, (\mathbf{F} \circ \mathbf{G})(x)) \in \mathbf{F} \circ \mathbf{G}$

$\langle 2\rangle 3.$ PICK y such that $(x,y) \in \mathbf{G}$ and $(y, (\mathbf{F} \circ \mathbf{G})(x)) \in \mathbf{F}$

$\langle 2\rangle 4.$ $y = \mathbf{G}(x)$

$\langle 2\rangle 5.$ $\mathbf{F}(\mathbf{G}(x)) = (\mathbf{F} \circ \mathbf{G})(x)$

$\square$

Proposition 3.1.5. *For any set A there exists a function $F : \mathcal{P}A - \{\emptyset\} \to A$ (a* choice function *for A) such that, for every nonempty $B \subseteq A$, we have $F(B) \in B$.*

PROOF:

$\langle 1\rangle 1.$ LET: A be a set.

$\langle 1\rangle 2.$ LET: $\mathcal{A} = \{\{B\} \times B \mid B \in \mathcal{P}A - \{\emptyset\}\}$

$\langle 1\rangle 3.$ Every member of $\mathcal{A}$ is nonempty.

$\langle 1\rangle 4.$ Any two distinct members of $\mathcal{A}$ are disjoint.

$\langle 1\rangle 5.$ PICK a set C such that, for all $X \in \mathcal{A}$, we have $C \cap X$ is a singleton.

PROOF: Axiom of Choice.

$\langle 1\rangle 6.$ LET: $F = C \cap \bigcup \mathcal{A}$

$\langle 1\rangle 7.$ $F : \mathcal{P}A - \{\emptyset\} \to A$

$\langle 2\rangle 1.$ F is a function.

$\langle 3\rangle 1.$ LET: $(B,b), (B,b') \in F$

$\langle 3\rangle 2.$ $(B,b), (B,b') \in \{B\} \times B$

PROOF: Since $(B,b), (B,b') \in \bigcup \mathcal{A}$.

$\langle 3\rangle 3.$ $(B,b), (B,b') \in C \cap (\{B\} \times B)$

$\langle 3\rangle 4.$ $(B,b) = (B,b')$

PROOF: From $\langle 1\rangle 5$.

$\langle 3\rangle 5.$ $b = b'$

$\langle 2\rangle 2.$ $\mathrm{dom}\,F = \mathcal{P}A - \{\emptyset\}$

PROOF:

$$\begin{aligned} B \in \mathrm{dom}\,F &\Leftrightarrow \exists b.(B,b) \in F \\ &\Leftrightarrow \exists b.((B,b) \in \bigcup \mathcal{A} \wedge (B,b) \in C) \\ &\Leftrightarrow \exists b.\exists B' \in \mathcal{P}A - \{\emptyset\}.((B,b) \in \{B'\} \times B' \wedge (B,b) \in C) \\ &\Leftrightarrow B \in \mathcal{P}A - \{\emptyset\} \wedge \exists b \in B.(B,b) \in C \\ &\Leftrightarrow B \in \mathcal{P}A - \{\emptyset\} && (\langle 1\rangle 5) \end{aligned}$$

$\langle 2\rangle 3.$ $\mathrm{ran}\,F \subseteq A$

$\langle 1\rangle 8.$ For every nonempty $B \subseteq A$ we have $F(B) \in B$

$\square$

Proposition 3.1.6. *For any relation R there exists a function $H \subseteq R$ with* $\operatorname{dom} H = \operatorname{dom} R$.

PROOF:
$\langle 1 \rangle 1$. LET: R be a relation.
$\langle 1 \rangle 2$. PICK a choice function G for $\operatorname{ran} R$.
$\langle 1 \rangle 3$. Define $H : domR \to \operatorname{ran} R$ by $H(x) = G(\{y \mid xRy\})$
$\langle 1 \rangle 4$. $H \subseteq R$
□

Proposition 3.1.7. *For any function $\mathbf{G}$ and nonempty class $\mathbf{A}$, we have*

$$\mathbf{G}^{-1}\left(\bigcap \mathbf{A}\right) = \bigcap \{\mathbf{G}^{-1}(X) \mid X \in \mathbf{A}\} \enspace .$$

PROOF: Propositions 2.2.18 and 3.1.3. □

Proposition 3.1.8. *For any function $\mathbf{G}$ and classes $\mathbf{A}$ and $\mathbf{B}$, we have*

$$\mathbf{G}^{-1}(\mathbf{A} - \mathbf{B}) = \mathbf{G}^{-1}(\mathbf{A}) - \mathbf{G}^{-1}(\mathbf{B}) \enspace .$$

PROOF: Proposition 2.2.19 and 3.1.3. □

Definition 3.1.9 (Identity Function)**.** For any class $\mathbf{A}$, the *identity function* on $\mathbf{A}$ is $\mathbf{I_A} = \{(x, x) \mid x \in \mathbf{A}\}$.

Definition 3.1.10 (Injective)**.** A function is *one-to-one*, *injective* or an *injection* iff it is single-rooted.

Proposition 3.1.11. *Let $\mathbf{F}$ be a one-to-one function. Let $x \in \operatorname{dom} \mathbf{F}$. Then* $\mathbf{F}^{-1}(\mathbf{F}(x)) = x$.

PROOF:
$\langle 1 \rangle 1$. $\mathbf{F}^{-1}$ is a function.
PROOF: Proposition 3.1.2.
$\langle 1 \rangle 2$. $(x, \mathbf{F}(x)) \in \mathbf{F}$
$\langle 1 \rangle 3$. $(\mathbf{F}(x), x) \in \mathbf{F}^{-1}$
□

Proposition 3.1.12. *Let $\mathbf{F}$ be a one-to-one function. Let $y \in \operatorname{ran} \mathbf{F}$. Then* $\mathbf{F}(\mathbf{F}^{-1}(y)) = y$.

PROOF:
$\langle 1 \rangle 1$. $\mathbf{F}^{-1}$ is a function.
PROOF: Proposition 3.1.2.
$\langle 1 \rangle 2$. $y \in \operatorname{dom} \mathbf{F}^{-1}$
PROOF: Proposition 2.2.8.
$\langle 1 \rangle 3$. $(y, \mathbf{F}^{-1}(y)) \in \mathbf{F}^{-1}$
$\langle 1 \rangle 4$. $(\mathbf{F}^{-1}(y), y) \in \mathbf{F}$
□

Proposition 3.1.13. *Let $F : A \to B$ where A is nonempty. There exists $G : B \to A$ (a* left inverse*) such that $G \circ F = I_A$ if and only if F is one-to-one.*

Proof:
⟨1⟩1. If there exists $G : B \to A$ such that $G \circ F = I_A$ then F is one-to-one.
⟨2⟩1. Assume: $G : B \to A$ and $G \circ F = I_A$
⟨2⟩2. Let: $x, y \in A$
⟨2⟩3. Assume: $F(x) = F(y)$
⟨2⟩4. $x = y$
Proof: $x = G(F(x)) = G(F(y)) = y$
⟨1⟩2. If F is one-to-one then there exists $G : B \to A$ such that $G \circ F = I_A$.
⟨2⟩1. Assume: F is one-to-one.
⟨2⟩2. Pick $a \in A$
⟨2⟩3. Let: $G : B \to A$ be the function defined by: $G(b) = F^{-1}(b)$ if $b \in \operatorname{ran} F$, $G(b) = a$ otherwise.
Prove: $G \circ F = I_A$
⟨2⟩4. Let: $x \in A$
⟨2⟩5. $G(F(x)) = x$
∎

Definition 3.1.14 (Surjective)**.** Let $F : A \to B$. We say that F is *surjective*, or maps A *onto* B, and write $F : A \twoheadrightarrow B$, iff for all $y \in B$ there exists $x \in A$ such that $F(x) = y$.

Proposition 3.1.15. *Let $F : A \to B$. There exists $H : B \to A$ (a* right inverse*) such that $F \circ H = I_B$ if and only if F maps A onto B.*

Proof:
⟨1⟩1. If F has a right inverse then F is surjective.
⟨2⟩1. Assume: F has a right inverse $H : B \to A$.
⟨2⟩2. Let: $y \in B$
⟨2⟩3. $F(H(y)) = y$
⟨2⟩4. There exists $x \in A$ such that $F(x) = y$
⟨1⟩2. If F is surjective then F has a right inverse.
⟨2⟩1. Assume: F is surjective.
⟨2⟩2. Pick a function H such that $H \subseteq F^{-1}$ and $\operatorname{dom} H = \operatorname{dom} F^{-1} = B$
⟨2⟩3. $H : B \to A$
⟨2⟩4. $F \circ H = I_B$
⟨3⟩1. Let: $y \in B$
⟨3⟩2. $(y, H(y)) \in F^{-1}$
⟨3⟩3. $F(H(y)) = y$
∎

Definition 3.1.16 (Function Set)**.** Given a set A and a class $\mathbf{B}$, we write $\mathbf{B}^A$ for the class of all functions $A \to \mathbf{B}$.

Proposition 3.1.17. *If A and B are sets then A^B is a set.*

Proof: It is a subset of $\mathcal{P}(A \times B)$. ∎

Definition 3.1.18 (Natural Map)**.** Let A be a set and R an equivalence relation on A. The *natural map* $A \to A/R$ is the function that maps $a \in A$ to $[a]_R$.

Definition 3.1.19 (Respects)**.** Let $\mathbf{R}$ be an equivalence relation on $\mathbf{A}$ and $\mathbf{F} : \mathbf{A} \to \mathbf{B}$. Then $\mathbf{F}$ *respects* $\mathbf{A}$ iff, whenever $(x, y) \in \mathbf{R}$, then $\mathbf{F}(x) = \mathbf{F}(y)$.

Theorem 3.1.20. *Let A be a set and $\mathbf{B}$ a class. Let R be an equivalence relation on A and $F : A \to \mathbf{B}$. Then F respects R if and only if there exists $\hat{F} : A/R \to \mathbf{B}$ such that*

$$\forall a \in A.\hat{F}([a]_R) = F(a) \ .$$

In this case, $\hat{F}$ is unique.

Proof:
⟨1⟩1. If F respects R then there exists $\hat{F} : A/R \to \mathbf{B}$ such that $\forall a \in A.\hat{F}([a]_R) = F(a)$.
⟨2⟩1. Assume: F respects R.
⟨2⟩2. Let: $\hat{F} = \{([a]_R, F(a)) \mid a \in A\}$
⟨2⟩3. $\hat{F}$ is a function.
⟨3⟩1. Assume: $a, a' \in A$ and $[a]_R = [a']_R$
Prove: $F(a) = F(a')$
⟨3⟩2. $(a, a') \in R$
Proof: Proposition 2.4.4.
⟨3⟩3. $F(a) = F(a')$
Proof: ⟨2⟩1
⟨2⟩4. $\mathrm{dom}\, \hat{F} = A/R$
⟨2⟩5. $\mathrm{ran}\, \hat{F} \subseteq \mathbf{B}$
⟨2⟩6. $\forall a \in A.\hat{F}([a]_R) = F(a)$
⟨1⟩2. If there exists $\hat{F} : A/R \to \mathbf{B}$ such that $\forall a \in A.\hat{F}([a]_R) = F(a)$ then F respects R.
⟨2⟩1. Assume: $\hat{F} : A/R \to \mathbf{B}$ and $\forall a \in A.\hat{F}([a]_R) = F(a)$
⟨2⟩2. Let: $a, a' \in A$
⟨2⟩3. Assume: $(a, a') \in R$
⟨2⟩4. $[a]_R = [a']_R$
Proof: Proposition 2.4.4.
⟨2⟩5. $F(a) = F(a')$
Proof: ⟨2⟩1
⟨1⟩3. If $G, H : A/R \to \mathbf{B}$ and $\forall a \in A.G([a]_R) = H([a]_R)$ then $G = H$.
□

Definition 3.1.21 (Strictly Monotone)**.** Let $(A, <_A)$ and $(B, <_B)$ be linearly ordered sets. A function $f : A \to B$ is *strictly monotone* iff, whenever $x <_A y$, then $f(x) <_B f(y)$.

Proposition 3.1.22. *A strictly monotone function is injective.*

Proof:
⟨1⟩1. Let: $(A, <_A)$ and $(B, <_B)$ be linearly ordered sets.

⟨1⟩2. LET: $f : A \to B$ be strictly monotone.
⟨1⟩3. LET: $x, y \in A$
⟨1⟩4. ASSUME: $f(x) = f(y)$
⟨1⟩5. $f(x) \not< f(y)$ and $f(y) \not< f(x)$
PROOF: Trichotomy.
⟨1⟩6. $x \not< y$ and $y \not< x$
⟨1⟩7. $x = y$
PROOF: Trichotomy.
□

Proposition 3.1.23. *Let A and B be linearly ordered sets. Let $f : A \to B$. Let $x, y \in A$. If f is strictly monotone and $f(x) < f(y)$ then $x < y$.*

PROOF:
⟨1⟩1. $f(x) \neq f(y)$ and $f(y) \not< f(x)$
PROOF: Trichotomy.
⟨1⟩2. $x \neq y$ and $y \not< x$
⟨1⟩3. $x < y$
PROOF: Trichotomy.
□

Definition 3.1.24 (Closed)**.** Let $\mathbf{F}$ be a function and $\mathbf{A} \subseteq \operatorname{dom} \mathbf{F}$. Then $\mathbf{A}$ is *closed* under $\mathbf{F}$ iff $\forall x \in \mathbf{A}.\mathbf{F}(x) \in \mathbf{A}$.

3.2 Dependent Product Sets

Definition 3.2.1. Let I be a set and let $\mathbf{H}(i)$ be a class for all $i \in I$. We write $\prod_{i \in I} \mathbf{H}(i)$ for the class of all functions f with $\operatorname{dom} f = I$ and $\forall i \in I.f(i) \in \mathbf{H}(i)$.

Proposition 3.2.2. *If I is a set and $H(i)$ is a set for all $i \in I$, then $\prod_{i \in I} H(i)$ is a set.*

PROOF:
⟨1⟩1. $\{H(i) \mid i \in I\}$ is a set.
PROOF: Axiom of Replacement.
⟨1⟩2. $\prod_{i \in I} H(i) \subseteq \bigcup \{H(i) \mid i \in I\}^I$
□

Proposition 3.2.3. *Let I be a set. Let $H(i)$ be a set for all $i \in I$. If $\forall i \in I.H(i) \neq \emptyset$ then $\prod_{i \in I} H(i) \neq \emptyset$.*

PROOF:
⟨1⟩1. ASSUME: $\forall i \in I.H(i) \neq \emptyset$
⟨1⟩2. LET: $R = \{(i, x) \mid i \in I, x \in H(i)\}$
⟨1⟩3. PICK a function $f \subseteq R$ such that $\operatorname{dom} f = \operatorname{dom} R$
⟨1⟩4. $f \in \prod_{i \in I} H(i)$
□

Chapter 4

Natural Numbers

4.1 Inductive Sets

Definition 4.1.1 (Successor)**.** The *successor* of a set a is the set $a^+ := a \cup \{a\}$.

Proposition 4.1.2. *For a transitive set a,*

$$\bigcup(a^+) = a \quad .$$

PROOF:
$\langle 1 \rangle 1.$ $\bigcup(a^+) \subseteq a$
  $\langle 2 \rangle 1.$ LET: $x \in \bigcup(a^+)$
    PROVE: $x \in a$
  $\langle 2 \rangle 2.$ PICK $y \in a^+$ such that $x \in y$.
  $\langle 2 \rangle 3.$ $y \in a$ or $y = a$.
  $\langle 2 \rangle 4.$ CASE: $y \in a$
    PROOF: Then $x \in a$ because a is a transitive set.
  $\langle 2 \rangle 5.$ CASE: $y = a$
    PROOF: Then $x \in a$ immediately.
$\langle 1 \rangle 2.$ $a \subseteq \bigcup(a^+)$
  PROOF: Since $a \in a^+$.
□

Proposition 4.1.3. *For any set a, we have a is a transitive set if and only if a^+ is a transitive set.*

PROOF:
$\langle 1 \rangle 1.$ If a is a transitive set then a^+ is a transitiveset.
  PROOF: If a is a transitive set then $\bigcup(a^+) = a \subseteq a^+$ by Proposition 4.1.2 and so a^+ is a transitive set.
$\langle 1 \rangle 2.$ If a^+ is a transitive set then a is a transitive set.
  $\langle 2 \rangle 1.$ ASSUME: a^+ is a transitive set.
  $\langle 2 \rangle 2.$ LET: $x \in y \in a$

$\langle 2\rangle 3.$ $x \in y \in a^+$
$\langle 2\rangle 4.$ $x \in a^+$
Proof: $\langle 2\rangle 1$
$\langle 2\rangle 5.$ $x \neq a$
Proof: From $\langle 2\rangle 2$ and the Axiom of Regularity.
$\langle 2\rangle 6.$ $x \in a$
□

Definition 4.1.4. We write 0 for $\emptyset$, 1 for $\emptyset^+$, 2 for $\emptyset^{++}$, etc.

Definition 4.1.5 (Inductive)**.** A set I is *inductive* iff $\emptyset \in I$ and $\forall x \in I.x^+ \in I$.

Definition 4.1.6 (Natural Number)**.** A *natural number* is a set that belongs to every inductive set.

Theorem 4.1.7. *The class* $\mathbb{N}$ *of natural numbers is a set.*

Proof:
$\langle 1\rangle 1.$ Pick an inductive set I.
Proof: Axiom of Infinity.
$\langle 1\rangle 2.$ $\mathbb{N} \subseteq I$
□

Theorem 4.1.8. $\mathbb{N}$ *is inductive, and is a subset of every other inductive set.*

Proof:
$\langle 1\rangle 1.$ $\mathbb{N}$ is inductive.
$\langle 2\rangle 1.$ $0 \in \mathbb{N}$
Proof: Since 0 is a member of every inductive set.
$\langle 2\rangle 2.$ $\forall n \in \mathbb{N}.n^+ \in \mathbb{N}$
$\langle 3\rangle 1.$ Let: $n \in \mathbb{N}$
$\langle 3\rangle 2.$ Let: I be any inductive set.
Prove: $n^+ \in I$
$\langle 3\rangle 3.$ $n \in I$
Proof: $\langle 3\rangle 1$, $\langle 3\rangle 2$
$\langle 3\rangle 4.$ $n^+ \in I$
Proof: $\langle 3\rangle 2$, $\langle 3\rangle 3$
$\langle 1\rangle 2.$ $\mathbb{N}$ is a subset of every inductive set.
Proof: Immediate from definitions.
□

Corollary 4.1.8.1 (Induction Principle for $\mathbb{N}$)**.** *Any inductive subset of* $\mathbb{N}$ *coincides with* $\mathbb{N}$.

Theorem 4.1.9. *Every natural number except 0 is the successor of some natural number.*

Proof: Trivially by induction. □

Proposition 4.1.10. *Every natural number is a transitive set.*

Proof:
$\langle 1 \rangle 1$. 0 is a transitive set.
Proof: Vacuously, if $x \in y \in 0$ then $x \in 0$.
$\langle 1 \rangle 2$. For every natural number n, if n is a transitive set then n^+ is a transitive set.
Proof: Proposition 4.1.3.
$\square$

Proposition 4.1.11. $\mathbb{N}$ *is a transitive set.*

Proof:
$\langle 1 \rangle 1$. $0 \subseteq \mathbb{N}$
$\langle 1 \rangle 2$. $\forall n \in \mathbb{N}. n \subseteq \mathbb{N} \Rightarrow n^+ \subseteq \mathbb{N}$
$\langle 1 \rangle 3$. $\forall n \in \mathbb{N}. n \subseteq \mathbb{N}$
Proof: From $\langle 1 \rangle 1$ and $\langle 1 \rangle 2$ by induction.
$\square$

Chapter 5

Complex Analysis

Definition 5.0.1. For $p \geq 1$, let l^p be the set of all sequences of complex numbers (x_n) such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

Proposition 5.0.2. *If $(x_n), (y_n) \in l^p$ then $(x_n + y_n) \in l^p$.*

PROOF:
⟨1⟩1. LET: $(x_n), (y_n) \in l^p$
⟨1⟩2. $\sum_{n=1}^{\infty} |x_n + y_n|^p \leq 2^p(\sum_{n=1}^{\infty} |x_n|^p + \sum_{n=1}^{\infty} |y_n|^p)$
PROOF:
⟨2⟩1. For all $n \in \mathbb{N}$ we have $|x_n + y_n|^p \leq 2^p(|x_n|^p + |y_n|^p)$.
PROOF:

$$\begin{aligned} |x_n + y_n|^p &\leq (|x_n| + |y_n|)^p && \text{(Triangle Inequality)} \\ &\leq (2\max(|x_n|, |y_n|))^p \\ &\leq 2^p(|x_n|^p + |y_n|^p) \end{aligned}$$

□

Theorem 5.0.3 (Hölder's Inequality)**.** *Let p and q be reals such that $p > 1$, $q > 1$ and $1/p + 1/q = 1$. Let $(x_n) \in l^p$ and $(y_n) \in l^q$. Then*

$$\sum_n |x_n y_n| \leq \left(\sum_n |x_n|^p\right)^{1/p} \left(\sum_n |y_n|^q\right)^{1/q}$$

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. neither (x_n) nor (y_n) are all zero.
⟨1⟩2. For $0 \leq x \leq 1$ we have

$$x^{1/p} \leq \frac{1}{p}x + \frac{1}{q} \ .$$

⟨2⟩1. LET: $f(x) = x/p + 1/q - x^{1/p}$
⟨2⟩2. $f'(x) = 1/p(1 - x^{(1-p)/p})$
⟨2⟩3. $f'(x) \geq 0$ for all $x \in [0, 1]$
⟨2⟩4. f is a monotonically decreasing function on $[0, 1]$

⟨2⟩5. $f(0) = 1/q$
⟨2⟩6. $f(1) = 0$
⟨2⟩7. $f(x) \geq 0$ for all $x \in [0, 1]$
⟨1⟩3. For any $a, b \geq 0$ we have
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad .$$
⟨2⟩1. CASE: $a^p \leq b^q$
⟨3⟩1. $ab^{-q/p} \leq \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q}$
PROOF: Substituting $x = a^p/b^q$ in ⟨1⟩2.
⟨3⟩2. $ab^{1-q} \leq \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q}$
PROOF: From ⟨3⟩1 since $1 - q = -q/p$.
⟨3⟩3. $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$
PROOF: Multiplying ⟨3⟩2 by b^q.
⟨2⟩2. CASE: $b^q \leq a^p$
PROOF: Similar.
⟨1⟩4. For any integers $1 \leq j \leq n$, we have
$$\frac{|x_j|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}} \frac{|y_j|}{\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}} \leq \frac{1}{p}\frac{|x_j|^p}{\sum_{k=1}^n |x_k|^p} + \frac{1}{q}\frac{|y_j|^q}{\sum_{k=1}^n |y_k|^q}$$
PROOF: From ⟨1⟩3 substituting
$$a = \frac{|x_j|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}} \text{ and } b = \frac{|y_j|}{\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}}$$
⟨1⟩5. For any positive integer n we have
$$\frac{\sum_{k=1}^n |x_k||y_k|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}} \leq 1$$
PROOF:
$$\begin{aligned}\frac{\sum_{j=1}^n |x_j||y_j|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}} &\leq \frac{1}{p} + \frac{1}{q} \quad \text{(Summing ⟨1⟩4 from } j = 1 \text{ to } n) \\ &= 1\end{aligned}$$
⟨1⟩6.
$$\sum_n |x_n y_n| \leq \left(\sum_n |x_n|^p\right)^{1/p} \left(\sum_n |y_n|^q\right)^{1/q}$$
PROOF: Taking the limit $n \to \infty$ in ⟨1⟩5.
□

Theorem 5.0.4 (Minkowski's Inequality)**.** *Let $p \geq 1$. Let $(x_n), (y_n) \in l^p$. Then*
$$\left(\sum_{n=1}^\infty |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^\infty |y_n|^p\right)^{1/p}$$

PROOF:
⟨1⟩1. CASE: $p = 1$
PROOF: This is just the Triangle Inequality.
⟨1⟩2. CASE: $p > 1$
⟨2⟩1. LET: $q = p/(p - 1)$

⟨2⟩2.
$$\sum_{n=1}^{\infty} |x_n + y_n|^p \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q}$$

PROOF:

⟨3⟩1. $(|x_n + y_n|^{p-1}) \in l^q$

PROOF:
$$\begin{aligned} \sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q} &= \sum_{n=1}^{\infty} |x_n + y_n|^p && (\langle 2 \rangle 2) \\ &< \infty && \text{(Proposition 5.0.2)} \end{aligned}$$

⟨3⟩2. Q.E.D.

PROOF:
$$\begin{aligned} \sum_{n=1}^{\infty} |x_n + y_n|^p &= \sum_{n=1}^{\infty} |x_n + y_n| |x_n + y_n|^{p-1} \\ &\leq \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n| |x_n + y_n|^{p-1} \\ &\leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \\ &\quad + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} && \text{(Hölder's Inequality, } \langle 2 \rangle 2) \end{aligned}$$

⟨2⟩3.
$$\sum_{n=1}^{\infty} |x_n + y_n|^p \leq \left\{ \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \right\} \left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{1/q}$$

⟨3⟩1. $q(p-1) = p$

PROOF: ⟨2⟩2

⟨3⟩2. Q.E.D.

PROOF: From ⟨2⟩2, ⟨3⟩1.

□

Part I

Linear Algebra

Chapter 6

Vector Spaces

6.1 Vector Spaces

Definition 6.1.1 (Vector Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *vector space* over K is a triple $(V, +, \cdot)$ such that:

- V is a nonempty set, whose elemnts are called *vectors*;
- $+ : V^2 \to V$
- $\cdot : K \times V \to V$

such that the following hold for all $u, v, w \in V$ and $\alpha, \beta \in K$:

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. For every $u, v \in V$ there exists $w \in V$ such that $u + w = v$
4. $\alpha(\beta v) = (\alpha\beta)v$
5. $(\alpha + \beta)v = \alpha v + \beta v$
6. $\alpha(u + v) = \alpha u + \alpha v$
7. $1v = v$

Elements of K are called *scalars*.

We write *real vector space* for 'vector space over $\mathbb{R}$', and *complex vector space* for 'vector space over $\mathbb{C}$'.

Proposition 6.1.2. *Let K be either $\mathbb{R}$ and $\mathbb{C}$. The set $\{0\}$ is a vector space over K under the unique functions $+ : \{0\}^2 \to \{0\}$, $\cdot : K \times \{0\} \to \{0\}$.*

Proof: Each axiom holds trivially because $x = y$ holds for all $x, y \in \{0\}$. □

Proposition 6.1.3. *The set $\mathbb{R}$ is a real vector space under real addition and real multiplication.*

PROOF: TODO — after we have proved these facts about $\mathbb{R}$. ∎

Proposition 6.1.4. *The set $\mathbb{C}$ is a real vector space under complex addition and complex multiplication.*

PROOF: TODO

Proposition 6.1.5. *The set $\mathbb{C}$ is a complex vector space under complex addition and complex multiplication.*

PROOF: TODO

Proposition 6.1.6. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let $\{V_i\}_{i \in I}$ be a family of vector spaces over K. Then $\prod_{i \in I} V_i$ is a vector space over K under the operations given by*

$$\begin{aligned} \{x_i\}_{i \in I} + \{y_i\}_{i \in I} &= \{x_i + y_i\}_{i \in I} \\ \alpha \{x_i\}_{i \in I} &= \{\alpha x_i\}_{i \in I} \end{aligned}$$

PROOF: Each axiom follows from the corresponding axiom in V_i. ∎

Corollary 6.1.6.1. *Let V be a vector space over K. For any set I, we have V^I is a vector space over K.*

Corollary 6.1.6.2. *Let $n \in \mathbb{Z}_+$. Then $\mathbb{R}^n$ is a real vector space, and $\mathbb{C}^n$ is both a real and a complex vector space, under*

$$\begin{aligned} (x_1, \ldots, x_n) + (y_1, \ldots, y_n) &= (x_1 + y_1, \ldots, x_n + y_n) \\ \lambda(x_1, \ldots, x_n) &= (\lambda x_1, \ldots, \lambda x_n) \end{aligned}$$

Proposition 6.1.7. *Let V be a vector space over K. Then there exists a unique $0 \in V$ such that, for all $v \in V$, we have $v + 0 = v$.*

PROOF:
$\langle 1 \rangle 1.$ There exists $0 \in V$ such that $\forall v \in V. v + 0 = v$
  $\langle 2 \rangle 1.$ Pick $v \in V$
  $\langle 2 \rangle 2.$ Pick $0 \in V$ such that $v + 0 = v$
    PROOF: Axiom 3.
  $\langle 2 \rangle 3.$ For all $u \in V$, we have $u + 0 = u$
    $\langle 3 \rangle 1.$ LET: $u \in V$
    $\langle 3 \rangle 2.$ PICK $u' \in V$ such that $v + u' = u$
      PROOF: Axiom 3.
    $\langle 3 \rangle 3.$ $u + 0 = u$

$$\begin{aligned} u + 0 &= v + u' + 0 && (\langle 3 \rangle 2) \\ &= v + u' && (\langle 2 \rangle 2) \\ &= u && (\langle 3 \rangle 2) \end{aligned}$$

⟨1⟩2. If $0, 0' \in V$ are such that $\forall v \in V. v + 0 = v$ and $\forall v \in V. v + 0' = v$, then $0 = 0'$.
⟨2⟩1. LET: $0, 0' \in V$
⟨2⟩2. ASSUME: $\forall v \in V. v + 0 = v$
⟨2⟩3. ASSUME: $\forall v \in V. v + 0' = v$
⟨2⟩4. $0 = 0'$

$$\begin{aligned} 0 &= 0 + 0' && (\langle 2\rangle 2) \\ &= 0' && (\langle 2\rangle 3) \end{aligned}$$

□

Proposition 6.1.8. *Let V be a vector space. For any $v \in V$, there exists a unique $-v \in V$ such that $v + (-v) = 0$.*

PROOF:
⟨1⟩1. LET: $v \in V$
⟨1⟩2. There exists $-v \in V$ such that $v + (-v) = u$
PROOF: Axiom 3.
⟨1⟩3. If $v + x = 0$ and $v + y = 0$ then $x = y$
⟨2⟩1. ASSUME: $v + x = 0$
⟨2⟩2. ASSUME: $v + y = 0$
⟨2⟩3. $x = y$
PROOF:

$$\begin{aligned} x &= x + 0 && \text{(Proposition 6.1.7)} \\ &= x + v + y && (\langle 2\rangle 2) \\ &= 0 + y && (\langle 2\rangle 1) \\ &= y && \text{(Proposition 6.1.7)} \end{aligned}$$

□

Proposition 6.1.9. *Let V be a vector space. For any $u, v \in V$, there exists a unique $u - v \in V$ such that $v + (u - v) = u$, namely $u - v = u + (-v)$.*

PROOF:
⟨1⟩1. LET: $u, v \in V$
⟨1⟩2. $v + (u + (-v)) = u$
PROOF:

$$\begin{aligned} v + u + (-v) &= u + 0 && \text{(Proposition 6.1.8)} \\ &= u && \text{(Proposition 6.1.7)} \end{aligned}$$

⟨1⟩3. For all $x \in V$, if $v + x = u$ then $x = u + (-v)$.
⟨2⟩1. LET: $x \in V$
⟨2⟩2. ASSUME: $v + x = u$
⟨2⟩3. $x = u + (-v)$
PROOF:

$$\begin{aligned} u + (-v) &= v + x + (-v) && (\langle 2\rangle 2) \\ &= x + 0 && \text{(Proposition 6.1.8)} \\ &= x && \text{(Proposition 6.1.7)} \end{aligned}$$

□

Proposition 6.1.10. *Let V be a vector space over K. Let $u, v, w \in V$. If $u + v = u + w$ then $v = w$.*

Proof:
$\langle 1 \rangle 1$. Assume: $u + v = u + w$
$\langle 1 \rangle 2$. $v = w$
Proof:

$$\begin{aligned} v &= v + 0 && \text{(Proposition 6.1.7)} \\ &= v + u + (-u) && \text{(Proposition 6.1.8)} \\ &= w + u + (-u) && (\langle 1 \rangle 1) \\ &= w + 0 && \text{(Proposition 6.1.8)} \\ &= w && \text{(Proposition 6.1.7)} \end{aligned}$$

Proposition 6.1.11. *Let V be a vector space over K. Let $\lambda \in K$. Then $\lambda 0 = 0$.*

Proof:
$\langle 1 \rangle 1$. $\lambda 0 + \lambda 0 = \lambda 0 + 0$
Proof:

$$\begin{aligned} \lambda 0 + \lambda 0 &= \lambda(0 + 0) && \text{(Axiom 6)} \\ &= \lambda 0 && \text{(Proposition 6.1.7)} \end{aligned}$$

$\langle 1 \rangle 2$. $\lambda 0 = 0$
Proof: Proposition 6.1.10.

□

Proposition 6.1.12. *Let V be a vector space over K. Let $\lambda \in K$ and $v \in V$. If $\lambda v = 0$ then $\lambda = 0$ or $v = 0$.*

Proof:
$\langle 1 \rangle 1$. Assume: $\lambda \neq 0$
$\langle 1 \rangle 2$. Assume: $\lambda v = 0$
$\langle 1 \rangle 3$. $v = 0$
Proof:

$$\begin{aligned} v &= 1v && \text{(Axiom 7)} \\ &= \lambda^{-1} \lambda v \\ &= \lambda^{-1} 0 && (\langle 1 \rangle 2) \\ &= 0 \end{aligned}$$

□

Proposition 6.1.13. *Let V be a vector space over K. For all $v \in V$ we have $0v = 0$.*

Proof:
$\langle 1 \rangle 1$. $0v + 0 = 0v + 0v$

$$
\begin{aligned}
0v + 0 &= 0v && \text{(Proposition 6.1.7)} \\
&= (0+0)v \\
&= 0v + 0v && \text{(Axiom 5)}
\end{aligned}
$$

$\langle 1 \rangle 2.$ $0v = 0$

PROOF: Proposition 6.1.10, $\langle 1 \rangle 1$.

□

Proposition 6.1.14. *Let V be a vector space over K. Let $v \in V$. Then $(-1)v = -v$.*

PROOF:

$\langle 1 \rangle 1.$ $v + (-1)v = 0$

PROOF:

$$
\begin{aligned}
v + (-1)v &= 1v + (-1)v && \text{(Axiom 7)} \\
&= (1 + (-1))v && \text{(Axiom 5)} \\
&= 0v \\
&= 0 && \text{(Proposition 6.1.13)}
\end{aligned}
$$

$\langle 1 \rangle 2.$ Q.E.D.

PROOF: Proposition 6.1.8.

□

6.2 Subspaces

Definition 6.2.1 (Subspace)**.** Let V be a vector space over K and $U \subseteq V$. Then U is a *subspace* of V iff $\forall \alpha, \beta \in K. \forall u, v \in U. \alpha u + \beta v \in U$. It is a *proper* subspace iff in addition $U \neq V$.

Proposition 6.2.2. *Let V be a vector space over K and U a subspace of V. Then U is a vector space over K under the restrictions of the operations of V.*

PROOF: Each of the axioms follows from the corresponding axiom in V. For axiom 3, we have if $u, v \in U$ then $v - u = 1v + (-1)u \in U$. □

Proposition 6.2.3. *Every vector space is a subspace of itself.*

PROOF: Trivial. □

Proposition 6.2.4. *Let Ω be a subset of $\mathbb{R}^N$. Let $\mathcal{C}(\Omega)$ be the set of all continuous functions $\Omega \to \mathbb{C}$. Then $\mathcal{C}(\Omega)$ is a subspace of $\mathbb{C}^\Omega$.*

PROOF: If $f, g : \Omega \to \mathbb{C}$ are continuous then so is $\alpha f + \beta g$. □

Proposition 6.2.5. *Let Ω be an open set in $\mathbb{R}^N$. Let $\mathcal{C}^k(\Omega)$ be the set of all continuous functions $\Omega \to \mathbb{C}$ with continuous partial derivatives of order k. Then $\mathcal{C}^k(\Omega)$ is a subspace of $\mathbb{C}^\Omega$.*

PROOF: If $f, g : \Omega \to \mathbb{C}$ have continuous partial derivatives of order k then so does $\alpha f + \beta g$. $\square$

Proposition 6.2.6. *Let Ω be an open set in $\mathbb{R}^N$. Let $\mathcal{C}^\infty(\Omega)$ be the set of all infinitely differentiable functions $\Omega \to \mathbb{C}$. Then $\mathcal{C}^\infty(\Omega)$ is a subspace of $\mathbb{C}^\Omega$.*

PROOF: If $f, g : \Omega \to \mathbb{C}$ are infinitely differentiable then so is $\alpha f + \beta g$. $\square$

Proposition 6.2.7. *Let Ω be an open set in $\mathbb{R}^N$. Let $\mathcal{P}(\Omega)$ be the set of all polynomials in N variables considered as functions $\Omega \to \mathbb{C}$. Then $\mathcal{P}(\Omega)$ is a subspace of $\mathbb{C}^\Omega$.*

PROOF: If $f, g : \Omega \to \mathbb{C}$ are polynomials in N variables then so is $\alpha f + \beta g$. $\square$

Proposition 6.2.8. *Let V be a vector space and U_1, U_2 subspaces of V. If $U_1 \subseteq U_2$ then U_1 is a subspace of U_2.*

PROOF: Trivial. $\square$

Proposition 6.2.9. *Let V be a vector space over K . The intersection of a set of subspaces of V is a subspace of V.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{U}$ be a set of subspaces of V.
$\langle 1 \rangle 2$. LET: $u, v \in \bigcap \mathcal{U}$ and $\lambda, \mu \in K$
$\langle 1 \rangle 3$. $\lambda u + \mu v \in \bigcap \mathcal{U}$
  $\langle 2 \rangle 1$. LET: $U \in \mathcal{U}$
  $\langle 2 \rangle 2$. $u, v \in U$
    PROOF: $\langle 1 \rangle 2$, $\langle 2 \rangle 1$.
  $\langle 2 \rangle 3$. $\lambda u + \beta v \in U$
    PROOF: $\langle 1 \rangle 1$, $\langle 1 \rangle 2$, $\langle 2 \rangle 1$, $\langle 2 \rangle 2$.
$\square$

Proposition 6.2.10. *The set of all bounded complex sequences is a proper subspace of $\mathbb{C}^\mathbb{N}$.*

PROOF: If (x_n) and (y_n) are bounded then so is $(\lambda x_n + \mu y_n)$. $\square$

Proposition 6.2.11. *The set of all convergent complex sequences is a proper subspace of the space of all bounded complex sequences.*

PROOF: If (x_n) and (y_n) converge then so does $(\lambda x_n + \mu y_n)$. $\square$

Proposition 6.2.12. *The set l^p of all sequences (x_n) in $\mathbb{C}$ such that $\sum_n |x_n|^p < \infty$ is a subspace of $\mathbb{C}^\mathbb{N}$.*

PROOF: It is closed under addition by Proposition 5.0.2, and it is easy to see that it is closed under scalar multiplication. $\square$

6.3 Linear Independence and Bases

Definition 6.3.1 (Linear Combination)**.** Let V be a vector space over K. Let $v, v_1, \ldots, v_n \in V$. Then v is a *linear combination* of $v_1, \ldots, v_n$ iff there exist scalars $\lambda_1, \ldots, \lambda_n \in K$ such that

$$v = \lambda_1 v_1 + \cdots + \lambda_n v_n \ .$$

Definition 6.3.2 (Linearly Independent)**.** Let V be a vector space over K. Let $A \subseteq V$. Then A is *linearly independent* iff, for all $\lambda_1, \ldots, \lambda_n \in K$ and $v_1, \ldots, v_n \in A$, if $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ then $\lambda_1 = \cdots = \lambda_n = 0$.

Definition 6.3.3 (Span)**.** Let V be a vector space over K and $A \subseteq V$. The *span* of A, or the subspace of V *spanned* by A, is the set of all linear combinations of vectors in A.

Proposition 6.3.4. *Let V be a vector space over K and $A \subseteq V$. Then* $\operatorname{span} A$ *is a subspace of V.*

PROOF: Given $\alpha, \beta \in K$ and $\lambda_1 u_1 + \cdots + \lambda_m u_m, \mu_1 v_1 + \cdots + \mu_n v_n \in \operatorname{span} A$, we have

$$\begin{aligned}
&\alpha(\lambda_1 u_1 + \cdots + \lambda_m u_m) + \beta(\mu_1 v_1 + \cdots + \mu_n v_n) \\
&= \alpha\lambda_1 u_1 + \cdots + \alpha\lambda_m u_m + \beta\mu_1 v_1 + \cdots + \beta\mu_n v_n \\
&\in \operatorname{span} A
\end{aligned}$$ ∎

Definition 6.3.5 (Basis)**.** Let V be a vector space over K and $B \subseteq V$. Then B is a *basis* for V iff B is linearly independent and $\operatorname{span} B = V$.

Definition 6.3.6 (Finite Dimensional)**.** A vector space is *finite dimensional* iff there exists a finite basis; otherwise it is *infinite dimensional*.

Proposition 6.3.7. *In a finite dimensional space, any two bases have the same size.*

TODO

Definition 6.3.8 (Dimension)**.** The *dimension* of a finite dimensional vector space V, $\dim V$, is the number of vectors in any basis.

Proposition 6.3.9. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Then K^n as a vector space over K has dimension n.*

PROOF: The vectors with one component 1 and all other components 0 form a basis. ∎

Proposition 6.3.10. *As a real vector space, $\mathbb{C}^n$ has dimension $2n$.*

PROOF: The vectors with one component either 1 or i and all other components 0 form a basis. ∎

Proposition 6.3.11. *Let Ω be a nonempty open set in $\mathbb{R}^n$. The space $\mathcal{C}(\Omega)$ is infinite dimensional.*

PROOF: Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}$ be the first projection. The functions 1, $\pi_1(x)$, $\pi_1(x)^2$, $\pi_1(x)^3$, ... form an infinite linearly independent set in $\mathcal{C}(\Omega)$. □

Proposition 6.3.12. *The spaces $\mathcal{C}^k(\mathbb{R}^n)$ and $\mathcal{C}^\infty(\mathbb{R}^n)$ are infinite dimensional.*

PROOF: The monomials 1, x, x^2, ... form an infinite linearly independent set. □

6.4 Linear Mappings

Definition 6.4.1 (Kernel)**.** Let U and V be vector spaces and $T : U \to V$. The *kernel* of T is

$$\ker T := \{u \in U \mid T(u) = 0\} .$$

Definition 6.4.2 (Linear Mapping)**.** Let U and V be vector spaces over K. A function $L : U \to V$ is a *linear mapping* iff $\forall x, y \in U . \forall \alpha, \beta \in K . L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$.

Proposition 6.4.3. *Let U and V be vector spaces over K. The set of linear mappings from U to V is a subspace of V^U.*

6.5 Eigenvalues and Eigenvectors

Definition 6.5.1 (Eigenvalue and Eigenvector)**.** Let V be a vector space over K. Let $A : V \to V$ be a linear transformation. Let $v \in V$ and $\lambda \in K$. Then v is an *eigenvector* of A with *eigenvalue* λ iff $A(v) = \lambda v$.

Chapter 7

Normed Spaces

Definition 7.0.1 (Norm)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. A *norm* on V is a function $\| \; \| : V \to \mathbb{R}$ such that, for all $u, v \in V$ and $\lambda \in K$:

1. If $\|v\| = 0$ then $v = 0$.

2. $\|\lambda v\| = |\lambda| \|v\|$

3. (*Triangle Inequality*) $\|u + v\| \leq \|u\| + \|v\|$

A *normed space* over K is a pair $(V, \| \; \|)$ where V is a vector space over K and $\| \; \|$ is a norm on V.

Proposition 7.0.2. *In a normed space,* $\|0\| = 0$.

Proof: $\|0\| = |0| \|0\| = 0$ by Axiom 2. □

Proposition 7.0.3. *Let V be a normed vector space over K. For all $v \in V$ we have $\|v\| \geq 0$.*

Proof:

$$\begin{aligned} 0 &= \|0\| && \text{(Proposition 7.0.2)} \\ &= \|v - v\| \\ &\leq \|v\| + \|-v\| && \text{(Triangle Inequality)} \\ &= 2\|v\| && \text{(Axiom 2)} \end{aligned}$$

Proposition 7.0.4. *Let V be a normed space. Let $u, v \in V$. Then*

$$|\|u\| - \|v\|| \leq \|u - v\| \ .$$

Proof:

$$\begin{aligned} \|u\| &\leq \|u-v\| + \|v\| && \text{(Triangle Inequality)} \\ \therefore \|u\| - \|v\| &\leq \|u-v\| \\ \|v\| &\leq \|v-u\| + \|u\| && \text{(Triangle Inequality)} \\ &= \|u-v\| + \|u\| && \text{(Axiom 2)} \\ \therefore \|v\| - \|u\| &\leq \|u-v\| \end{aligned}$$

Definition 7.0.5 (Euclidean Norm)**.** The *Euclidean norm* on K^n is defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2} \ .$$

Proposition 7.0.6. *The Euclidean norm on K^n is a norm.*

Proof:
⟨1⟩1. If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$
Proof: If $\sqrt{|x_1|^2 + \cdots + |x_n|^2} = 0$ then $x_1 = \cdots = x_n = 0$.
⟨1⟩2. $\|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$
Proof:

$$\begin{aligned} \|\lambda\vec{x}\| & \sqrt{|\lambda x_1|^2 + \cdots + |\lambda x_n|^2} \\ &= \sqrt{|\lambda|^2|x_1|^2 + \cdots + |\lambda|^2|x_n|^2} \\ &= |\lambda|\sqrt{|x_1|^2 + \cdots + |x_n|^2} \\ &= |\lambda|\|\vec{x}\| \end{aligned}$$

⟨1⟩3. $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
Proof:

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= |u_1 + v_1|^2 + \cdots + |u_n + v_n|^2 \\ &= |u_1|^2 + \cdots + |u_n|^2 + |v_1|^2 + \cdots + |v_n|^2 \\ &+ 2|u_1||v_1| + \cdots + 2|u_n||v_n| \\ &\leq \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2|u_1 v_1 + \cdots + u_n v_n| \\ &\leq \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\| && \text{(Cauchy-Schwarz)} \\ &= (\|\vec{u}\| + \|\vec{v}\|)^2 \end{aligned}$$

□

Corollary 7.0.6.1. *The absolute value function $|\ |$ is a norm on K.*

Proposition 7.0.7. *The function $\|\vec{x}\| = |x_1| + \cdots + |x_n|$ is a norm on $\mathbb{C}^n$.*

Proof:
⟨1⟩1. If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$
Proof: If $|x_1| + \cdots + |x_n| = 0$ then $x_1 = \cdots = x_n = 0$.
⟨1⟩2. $\|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$
Proof:

$$\begin{aligned} \|\lambda\vec{x}\| & |\lambda x_1| + \cdots + |\lambda x_n| \\ \\ &= |\lambda|(|x_1| + \cdots + |x_n|) \\ &= |\lambda|\|\vec{x}\| \end{aligned}$$

$\langle 1 \rangle 3.$ $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
PROOF:

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= |u_1+v_1| + \cdots + |u_n+v_n| \\
&\leq |u_1| + |v_1| + \cdots + |u_n| + |v_n| \\
&= \|\vec{u}\| + \|\vec{v}\|
\end{aligned}$$

□

Proposition 7.0.8. *The function* $\|\vec{x}\| = \max(|x_1|, \ldots, |x_n|)$ *is a norm on* $\mathbb{C}^n$*.*

PROOF:
$\langle 1 \rangle 1.$ If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$
PROOF: If $\max(|x_1|, \ldots, |x|n|) = 0$ then $x_1 = \cdots = x_n = 0$.
$\langle 1 \rangle 2.$ $\|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$
PROOF:

$$\begin{aligned}
\|\lambda\vec{x}\| &= \max(|\lambda x_1|, \ldots, |\lambda x_n|) \\
&= |\lambda| \max(|x_1|, \ldots, |x_n|) \\
&= |\lambda|\|\vec{x}\|
\end{aligned}$$

$\langle 1 \rangle 3.$ $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
PROOF:

$$\begin{aligned}
\|\vec{u}+\vec{v}\| &= \max(|u_1+v_1|, \ldots, |u_n+v_n|) \\
&\leq \max(|u_1|+|v_1|, \ldots, |u_n|+|v_n|) \\
&\leq \max(|u_1|, \ldots, |u_n|) + \max(|v_1|, \ldots, |v_n|)
\end{aligned}$$

□

Definition 7.0.9 (Uniform Convergence Norm)**.** Let Ω be a closed bounded subset of $\mathbb{R}^n$. The *uniform convergence norm* on $\mathcal{C}(\Omega)$ is the function defined by $\|f\| = \max_{x \in \Omega} |f(x)|$.

Proposition 7.0.10. *Let* Ω *be a closed bounded subset of* $\mathbb{R}^n$*. The uniform convergence norm is a norm on* $\mathcal{C}(\Omega)$*.*

PROOF:
$\langle 1 \rangle 1.$ If $\|f\| = 0$ then $f = 0$
PROOF: If $\max_x |f(x)| = 0$ then $f(x) = 0$ for all x.
$\langle 1 \rangle 2.$ $\|\lambda f\| = |\lambda|\|f\|$
PROOF:

$$\begin{aligned}
\|\lambda f\| &= \max_x |\lambda f(x)| \\
&= |\lambda| \max_x |f(x)| \\
&= |\lambda|\|f\|
\end{aligned}$$

$\langle 1 \rangle 3.$ $\|f+g\| \leq \|f\| + \|g\|$

PROOF:

$$\begin{aligned}\|f+g\| &= \max_x |f(x)+g(x)| \\ &\leq \max_x (|f(x)|+|g(x)|) \\ &\leq \max_x |f(x)| + \max_x |g(x)| \\ &= \|f\|+\|g\|\end{aligned}$$

□

Proposition 7.0.11. *Let $p \geq 1$. The function $\|(z_n)\| = (\sum_{n=1}^{\infty} |z_n|^p)^{1/p}$ is a norm on l^p.*

PROOF:
⟨1⟩1. If $\|(z_n)\| = 0$ then $(z_n) = (0)$
PROOF: If $(\sum_n |z_n|^p)^{1/p} = 0$ then $\sum_n |z_n|^p = 0$ so $|z_n|^p = 0$ for all n, and so $z_n = 0$ for all n.
⟨1⟩2. $\|(\lambda z_n)\| = |\lambda| \|(z_n)\|$
PROOF:

$$\begin{aligned}\|(\lambda z_n)\| &= \left(\sum_n |\lambda z_n|^p\right)^{1/p} \\ &= |\lambda| \left(\sum_n |z_n|^p\right)^{1/p} \\ &= |\lambda|\,|(z_n)|\end{aligned}$$

⟨1⟩3. The triangle inequality holds.
PROOF: This is Minkowski's Inequality. □

Proposition 7.0.12. *Let V be a normed space and U a vector subspace of V. Then U is a normed space under the restriction of the norm to U.*

PROOF: Each axiom follows from the fact it holds in V. □

Proposition 7.0.13. *Let V be a normed space over K. Let $x_1, \ldots, x_n$ be linearly independent elements of V. Then there exists a real number $c > 0$ such that, for all $\alpha_1, \ldots, \alpha_n \in K$, we have*

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|) \ .$$

PROOF:
⟨1⟩1. Define $f : K^n \to \mathbb{R}$ by

$$f(\alpha_1, \ldots, \alpha_n) = \|\alpha_1 x_1 + \cdots + \alpha_n x_n\|$$

⟨1⟩2. f is continuous.
⟨2⟩1. LET: $(\alpha_1, \ldots, \alpha_n) \in K^n$ and $\epsilon > 0$
⟨2⟩2. LET: $\delta = \epsilon / (\|x_1\| + \cdots + \|x_n\|)$
PROOF: $x_1, \ldots, x_n$ are not all zero because they are linearly independent.
⟨2⟩3. LET: $(\beta_1, \ldots, \beta_n)$ with $|\alpha_i - \beta_i| < \delta$ for all i
⟨2⟩4. $\|(\alpha_1 x_1 + \cdots + \alpha_n x_n) - (\beta_1 x_1 + \beta_n x_n)\| < \epsilon$

PROOF:

$$\begin{aligned}
&\|(\alpha_1 x_1 + \cdots + \alpha_n x_n) - (\beta_1 x_1 + \beta_n x_n)\| \\
\leq & |\alpha_1 - \beta_1|\|x_1\| + \cdots + |\alpha_n - \beta_n|\|x_n\| && \text{(Axioms 2 and 3)} \\
< & \delta(\|x_1\| + \cdots + \|x_n\|) && (\langle 2\rangle 3) \\
= & \epsilon && (\langle 2\rangle 2)
\end{aligned}$$

$\langle 1\rangle 3$. PICK $(\beta_1, \ldots, \beta_n) \in \{(\beta_1, \ldots, \beta_n) \in K^n \mid |\beta_1| + \cdots + |\beta_n| = 1\}$ at which f attains its minimum.

PROOF: Extreme Value Theorem.

$\langle 1\rangle 4$. Let $c = f(\beta_1, \ldots, \beta_n)$

$\langle 1\rangle 5$. $c > 0$

PROOF: Linear independence.

$\langle 1\rangle 6$. LET: $\alpha_1, \ldots, \alpha_n \in K$

$\langle 1\rangle 7$. $\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|)$

$\langle 2\rangle 1$. ASSUME: w.l.o.g. $\alpha_1 \ldots, \alpha_n$ are not all zero.

$\langle 2\rangle 2$. LET: $\beta_i = \alpha_i/(|\alpha_1| + \cdots + |\alpha_n|)$ for $i = 1, \ldots, n$

$\langle 2\rangle 3$. $|\beta_1| + \cdots + |\beta_n| = 1$

$\langle 2\rangle 4$. $f(\beta_1, \ldots, \beta_n) \geq c$

$\langle 2\rangle 5$. Q.E.D.

PROOF: Multiply both sides by $|\alpha_1| + \cdots + |\alpha_n|$.

□

Proposition 7.0.14. *Let V be a normed space over K. Define $d : V^2 \to \mathbb{R}$ by $d(x, y) = \|x - y\|$. Then d is a metric on V.*

PROOF:

$\langle 1\rangle 1$. For all $x, y \in V$ we have $d(x, y) \geq 0$

PROOF: Proposition 7.0.3.

$\langle 1\rangle 2$. For all $x, y \in V$ we have $d(x, y) = 0$ iff $x = y$

$\langle 2\rangle 1$. If $d(x, y) = 0$ then $x = y$

PROOF: Axiom 1.

$\langle 2\rangle 2$. If $x = y$ then $d(x, y) = 0$

PROOF: Proposition 7.0.2.

$\langle 1\rangle 3$. $\forall x, y \in V. d(x, y) = d(y, x)$

PROOF: By Axiom 2.

$\langle 1\rangle 4$. $\forall x, y, z \in V. d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Axiom 3.

□

Henceforth we identify any normed space with this metric space.

7.1 Convergence

Proposition 7.1.1. *Let V be a normed space over K. Let (x_n) be a sequence in V and $l \in V$. Then $x_n \to l$ as $n \to \infty$ in V if and only if $\|x_n - l\| \to 0$ as $n \to \infty$ in $\mathbb{R}$.*

PROOF: Immediate from definitions. □

Proposition 7.1.2. *In a normed space, a sequence has at most one limit.*

PROOF:
$\langle 1\rangle 1$. LET: V be a vector space over K.
$\langle 1\rangle 2$. ASSUME: $x_n \to l$ and $x_n \to m$ as $n \to \infty$.
$\langle 1\rangle 3$. ASSUME: for a contradiction $l \neq m$
$\langle 1\rangle 4$. LET: $\epsilon = \|l - m\|/2$
$\langle 1\rangle 5$. PICK N such that $\forall n \geq N.\|x_n - l\| < \epsilon$ and $\forall n \geq N.\|x_n - m\| < \epsilon$
PROOF: $\langle 1\rangle 2$, $\langle 1\rangle 4$
$\langle 1\rangle 6$. $\|l - m\| < \|l - m\|$
PROOF:
$$\begin{aligned} \|l - m\| &\leq \|x_N - l\| + \|x_N - m\| && \text{(Triangle Inequality)} \\ &< 2\epsilon && (\langle 1\rangle 5) \\ &= \|l - m\| && (\langle 1\rangle 4 \end{aligned}$$
$\langle 1\rangle 7$. Q.E.D.
PROOF: This is a contradiction.
□

Definition 7.1.3 (Bounded)**.** Let V be a normed space over K. A sequence (x_n) in V is *bounded* iff there exists B such that $\forall n \leq N.\|x_n\| < B$.

Proposition 7.1.4. *Every convergent sequence is bounded.*

PROOF:
$\langle 1\rangle 1$. LET: $x_n \to l$ as $n \to \infty$
$\langle 1\rangle 2$. PICK N such that $\forall n \geq N.\|x_n - l\| < 1$
$\langle 1\rangle 3$. LET: $B = \max(\|x_1\|, \|x_2\|, \ldots, \|x_{N-1}\|, \|l\| + 1)$
$\langle 1\rangle 4$. LET: $n \in \mathbb{N}$
$\langle 1\rangle 5$. $\|x_n\| \leq B$
$\langle 2\rangle 1$. CASE: $n < N$
PROOF: $\|x_n\| \leq B$ from $\langle 1\rangle 3$.
$\langle 2\rangle 2$. CASE: $n \geq N$
PROOF:
$$\begin{aligned} \|x_n\| &\leq \|l\| + \|x_n - l\| && \text{(Triangle Inequality)} \\ &< \|l\| + 1 && (\langle 1\rangle 2) \\ &\leq B && (\langle 1\rangle 3) \end{aligned}$$
□

Proposition 7.1.5. *Let V be a normed space over K. If $x_n \to l$ as $n \to \infty$ in V, and $\lambda_n \to \lambda$ as $n \to \infty$ in K, then $\lambda_n x_n \to \lambda l$ as $n \to \infty$.*

PROOF:
$\langle 1\rangle 1$. LET: V be a normed space over K.
$\langle 1\rangle 2$. LET: $x_n \to l$ as $n \to \infty$
$\langle 1\rangle 3$. LET: $\lambda_n \to \lambda$ as $n \to \infty$
$\langle 1\rangle 4$. LET: $\epsilon > 0$

⟨1⟩5. PICK N such that, for all $n \geq N$, we have $\|x_n - l\| < \epsilon/2|\lambda|$ and $|\lambda_n - \lambda| < \sqrt{\epsilon/2}$ and $\|x_n\| < \sqrt{\epsilon/2}$
⟨1⟩6. LET: $n \geq N$
⟨1⟩7. $\|\lambda_n x_n - \lambda l\| < \epsilon$
PROOF:

$$\begin{aligned} \|\lambda_n x_n - \lambda l\| &\leq \|\lambda_n x_n - \lambda x_n\| + \|\lambda x_n - \lambda l\| && \text{(Triangle Inequality)} \\ &= |\lambda_n - \lambda| \|x_n\| + |\lambda| \|x_n - l\| && \text{(Axiom 2)} \\ &< \sqrt{\epsilon/2}\sqrt{\epsilon/2} + |\lambda| \epsilon/2|\lambda| && (\langle 1 \rangle 5) \\ &= \epsilon \end{aligned}$$

□

Proposition 7.1.6. *Let V be a normed space over K. If $x_n \to l$ and $y_n \to m$ as $n \to \infty$, then $x_n + y_n \to l + m$ as $n \to \infty$.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N such that, for all $n \geq N$, we have $\|x_n - l\| < \epsilon/2$ and $\|y_n - m\| < \epsilon/2$
⟨1⟩3. LET: $n \geq N$
⟨1⟩4. $\|(x_n + y_n) - (l + m)\| < \epsilon$
PROOF:

$$\begin{aligned} \|(x_n + y_n) - (l + m)\| &\leq \|x_n - l\| + \|y_n - m\| && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1 \rangle 2) \\ &= \epsilon \end{aligned}$$

□

Definition 7.1.7 (Uniform Convergence)**.** Let Ω be a closed bounded subset of $\mathbb{R}^n$. Let (f_n) be a sequence in $\mathcal{C}(\Omega)$ and $f \in \mathcal{C}(\Omega)$. Then (f_n) *converges uniformly* to f iff, for every $\epsilon > 0$, there exists N such that $\forall x \in \Omega . \forall n \geq N . |f_n(x) - f(x)| < \epsilon$.

Proposition 7.1.8. *Let Ω be a closed bounded subset of $\mathbb{R}^n$. Let (f_n) be a sequence in $\mathcal{C}(\Omega)$ and $f \in \mathcal{C}(\Omega)$. Then (f_n)* converges uniformly *to f iff f_n converges to f under the uniform convergence norm.*

PROOF:

$$\begin{aligned} & (f_n) \text{ converges to } f \text{ under the uniform convergence norm} \\ \Leftrightarrow & \forall \epsilon > 0 . \exists N . \forall n \geq N . \|f_n - f\| < \epsilon \\ \Leftrightarrow & \forall \epsilon > 0 . \exists N . \forall n \geq N . \forall x \in X . |f_n(x) - f(x)| < \epsilon \end{aligned}$$

□

Definition 7.1.9 (Pointwise Convergence)**.** Let (f_n) be a sequence in $\mathcal{C}([0,1])$ and $f \in \mathcal{C}([0,1])$. Then (f_n) *converges pointwise* to f iff, for all $t \in [0,1]$, we have $|f_n(t) - f(t)| \to 0$ as $n \to \infty$.

Proposition 7.1.10. *There is no norm n on $\mathcal{C}([0,1])$ such that, for every sequence (f_n) and function f in $\mathcal{C}([0,1])$, we have (f_n) converges pointwise to f if and only if (f_n) converges to f under n.*

PROOF:

$\langle 1\rangle 1$. ASSUME: for a contradiction $\| \ \|$ is a norm on $\mathcal{C}([0,1])$ such that, for every sequence (f_n) and function f in $\mathcal{C}([0,1])$, we have (f_n) converges pointwise to f if and only if (f_n) converges to f under $\| \ \|$.

$\langle 1\rangle 2$. For $n \in \mathbb{Z}_+$, define $g_n \in \mathcal{C}([0,1])$ by

$$g_n(t) = \begin{cases} 2^n t & \text{if } 0 \leq t \leq 2^{-n} \\ 2 - 2^n t & \text{if } 2^{-n} \leq t \leq 2^{1-n} \\ 0 & \text{if } 2^{1-n} \leq t \leq 1 \end{cases}$$

$\langle 1\rangle 3$. For all n, $\|g_n\| \neq 0$

PROOF: Axiom 1.

$\langle 1\rangle 4$. For $n \in \mathbb{Z}_+$, define $f_n \in \mathcal{C}([0,1])$ by $f_n = g_n / \|g_n\|$

$\langle 1\rangle 5$. For all n, $\|f_n\| = 1$

PROOF: Axiom 2.

$\langle 1\rangle 6$. (f_n) does not converge under $\| \ \|$

$\langle 1\rangle 7$. (f_n) converges pointwise to 0.

$\langle 1\rangle 8$. This is a contradiction.

□

Definition 7.1.11 (Equivalence of Norms)**.** Let $\| \ \|_1$ and $\| \ \|_2$ be two norms on the same vector space V. Then the norms are *equivalent* if and only if, for any sequence (x_n) in V and $l \in V$, we have that (x_n) converges to l under $\| \ \|_1$ if and only if (x_n) converges to l under $\| \ \|_2$.

Theorem 7.1.12. *Let $\| \ \|_1$ and $\| \ \|_2$ be two norms on the same vector space E over K. Then $\| \ \|_1$ and $\| \ \|_2$ are equivalent if and only if there exist positive real numbers α and β such that, for all $x \in E$,*

$$\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1 \ .$$

PROOF:

$\langle 1\rangle 1$. If $\| \ \|_1$ and $\| \ \|_2$ are equivalent then there exist positive real numbers α and β such that, for all $x \in E$, $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$.

$\langle 2\rangle 1$. ASSUME: $\| \ \|_1$ and $\| \ \|_2$ are equivalent.

$\langle 2\rangle 2$. There exists $\alpha > 0$ such that, for all $x \in E$, we have $\alpha\|x\|_1 \leq \|x\|_2$

$\langle 3\rangle 1$. ASSUME: for a contradiction there is no $\alpha > 0$ such that, for all $x \in E$, we have $\alpha\|x\|_1 \leq \|x\|_2$.

$\langle 3\rangle 2$. For all $n \in \mathbb{Z}_+$, PICK $x_n \in E$ such that $1/n\|x_n\|_1 > \|x\|_2$

$\langle 3\rangle 3$. For all $n \in \mathbb{Z}_+$,

LET:

$$y_n = \frac{1}{\sqrt{n}} \frac{x_n}{\|x_n\|_2}$$

$\langle 3\rangle 4$. (y_n) converges to 0 under $\| \ \|_2$

$\langle 3\rangle 5$. (y_n) converges to 0 under $\| \ \|_1$

$\langle 3\rangle 6$. For all $n \in \mathbb{Z}_+$, we have $\|y_n\| > \sqrt{n}$

$\langle 3\rangle 7$. This is a contradiction.

$\langle 2\rangle 3$. There exists $\beta > 0$ such that, for all $x \in E$, we have $\|x\|_2 \leq \beta\|x\|_1$

PROOF: Similar.

⟨1⟩2. If there exist positive real numbers α and β such that, for all $x \in E$, $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$. then $\| \|_1$ and $\| \|_2$ are equivalent.

⟨2⟩1. ASSUME: α and β are positive reals with $\forall x \in E.\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$.

⟨2⟩2. Let (x_n) be a sequence in E and $l \in E$

⟨2⟩3. If (x_n) converges to l under $\| \|_1$ then (x_n) converges to l under $\| \|_2$.

⟨3⟩1. ASSUME: (x_n) converges to l under $\| \|_1$

⟨3⟩2. LET: $\epsilon > 0$

⟨3⟩3. PICK N such that $\forall n \geq N.\|x_n - l\|_1 < \epsilon/\beta$

⟨3⟩4. $\forall n \geq N.\|x_n - l\|_2 < \epsilon$

⟨2⟩4. If (x_n) converges to l under $\| \|_2$ then (x_n) converges to l under $\| \|_1$.

PROOF: Similar.

□

Theorem 7.1.13. *Any two norms on a finite dimensional vector space are equivalent.*

PROOF:

⟨1⟩1. LET: V be a finite dimensional vector space over K.

⟨1⟩2. ASSUME: w.l.o.g. $\dim V > 0$

⟨1⟩3. PICK a basis $\{e_1, \ldots, e_n\}$ for V.

⟨1⟩4. LET: $\| \|_0 : V \to \mathbb{R}$ be the function: $\|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 = |\alpha_1| + \cdots + |\alpha_n|$.

⟨1⟩5. $\| \|_0$ is a norm.

⟨2⟩1. If $\|v\|_0 = 0$ then $v = 0$

PROOF: If $|\alpha_1| + \cdots + |\alpha_n| = 0$ then $\alpha_1 = \cdots = \alpha_n = 0$ so $\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$.

⟨2⟩2. $\|\lambda v\|_0 = |\lambda|\|v\|_0$

PROOF:

$$\begin{aligned}
\|\lambda(\alpha_1 e_1 + \cdots + \alpha_n e_n)\|_0 &= \|\lambda\alpha_1 e_1 + \cdots + \lambda\alpha_n e_n\|_0 \\
&= |\lambda\alpha_1| + \cdots + |\lambda\alpha_n| && (\langle 1\rangle 4) \\
&= |\lambda|(|\alpha_1| + \cdots + |\alpha_n|) \\
&= |\lambda|\|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 && (\langle 1\rangle 4)
\end{aligned}$$

⟨2⟩3. $\|u + v\|_0 \leq \|u\|_0 + \|v\|_0$

PROOF:

$$\begin{aligned}
\|(\alpha_1 e_1 + \cdots + \alpha_n e_n) + (\beta_1 e_1 + \cdots + \beta_n e_n)\| &= |\alpha_1 + \beta_1| + \cdots + |\alpha_n + \beta_n| \\
&\leq |\alpha_1| + \cdots + |\alpha_n| + |\beta_1| + \cdots + |\beta_n| \\
&= \|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 + \|\beta_1 e_1 + \cdots + \beta_n e_n\|_0
\end{aligned}$$

⟨1⟩6. Any norm on V is equivalent to $\| \|_0$.

⟨2⟩1. LET: $\| \|$ be any norm on V.

⟨2⟩2. PICK $\alpha > 0$ such that, for all $\alpha_1, \ldots, \alpha_n \in K$, we have $\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \geq \alpha(|\alpha_1| + \cdots + |\alpha_n|)$

PROOF: Proposition 7.0.13, ⟨2⟩1, ⟨1⟩3.

⟨2⟩3. LET: $\beta = \max(\|e_1\|, \ldots, \|e_n\|)$

⟨2⟩4. $\beta > 0$

PROOF: $e_1, \ldots, e_n$ cannot all be zero by ⟨1⟩3.

⟨2⟩5. For all $x \in V$ we have $\alpha\|x\|_0 \leq \|x\| \leq \beta\|x\|_0$
⟨3⟩1. LET: $x \in V$
⟨3⟩2. $\alpha\|x\|_0 \leq \|x\|$
PROOF: ⟨1⟩3, ⟨1⟩4, ⟨2⟩2.
⟨3⟩3. $\|x\| \leq \beta\|x\|_0$
⟨4⟩1. LET: $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$
⟨4⟩2. Q.E.D.
PROOF:

$$\begin{aligned} \|x\| &= \|\alpha_1 e_1 + \cdots + \alpha_n e_n\| && (\langle 4\rangle 1) \\ &\leq |\alpha_1|\|e_1\| + \cdots + |\alpha_n|\|e_n\| && (\langle 2\rangle 1) \\ &\leq \beta(|\alpha_1| + \cdots + |\alpha_n|) && (\langle 2\rangle 3) \\ &= \beta\|x\|_0 && (\langle 1\rangle 4) \end{aligned}$$

⟨2⟩6. Q.E.D.
PROOF: Theorem 7.1.12, ⟨1⟩5, ⟨2⟩1, ⟨2⟩2, ⟨2⟩4, ⟨2⟩5.
□

Definition 7.1.14 (Open Ball)**.** Let V be a normed space over K. Let $x \in V$. Let $r > 0$. The *open ball* with *centre* x and *radius* r is

$$B(x, r) := \{y \in V \mid \|y - x\| < r\} \ .$$

Definition 7.1.15 (Closed Ball)**.** Let V be a normed space over K. Let $x \in V$. Let $r > 0$. The *closed ball* with *centre* x and *radius* r is

$$\overline{B}(x, r) := \{y \in V \mid \|y - x\| \leq r\} \ .$$

Definition 7.1.16 (Sphere)**.** Let V be a normed space over K. Let $x \in V$. Let $r > 0$. The *sphere* with *centre* x and *radius* r is

$$S(x, r) := \{y \in V \mid \|y - x\| = r\} \ .$$

Definition 7.1.17 (Open Set)**.** Let V be a normed space over K. A set $S \subseteq V$ is *open* iff, for all $x \in S$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S$.

Proposition 7.1.18. *Equivalent norms define the same set of open sets.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. LET: $\| \ \|_1$ and $\| \ \|_2$ be equivalent norms on V.
⟨1⟩3. PICK reals $\alpha, \beta > 0$ such that, for all $x \in V$, we have $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$
⟨1⟩4. LET: $S \subseteq V$
⟨1⟩5. If S is open under $\| \ \|_1$ then S is open under $\| \ \|_2$.
⟨2⟩1. ASSUME: S is open under $\| \ \|_1$.
⟨2⟩2. LET: $x \in S$
⟨2⟩3. PICK $\epsilon > 0$ such that $\{y \in V \mid \|x - y\|_1 < \epsilon\} \subseteq S$.
⟨2⟩4. LET: $\delta = \alpha\epsilon$

$\langle 2\rangle 5.$ $\{y \in V \mid \|x - y\|_2 < \delta\} \subseteq S$
$\langle 1\rangle 6.$ If S is open under $\| \; \|_2$ then S is open under $\| \; \|_1$.
PROOF: Similar.
□

Proposition 7.1.19. *Every open ball is open.*

PROOF:
$\langle 1\rangle 1.$ LET: V be a normed space over K.
$\langle 1\rangle 2.$ LET: $c \in V$ and $r > 0$
PROVE: $B(c, r)$ is open.
$\langle 1\rangle 3.$ LET: $x \in B(c, r)$
$\langle 1\rangle 4.$ LET: $\epsilon = r - \|x - c\|$
PROVE: $B(x, \epsilon) \subseteq B(c, r)$
$\langle 1\rangle 5.$ LET: $y \in B(x, \epsilon)$
PROVE: $y \in B(c, r)$
$\langle 1\rangle 6.$ $\|y - c\| < r$
PROOF:

$$\begin{aligned}
\|y - c\| &\leq \|y - x\| + \|x - c\| && \text{(Triangle Inequality)} \\
&< \epsilon + \|x - c\| && (\langle 1\rangle 5) \\
&= r && (\langle 1\rangle 4)
\end{aligned}$$

□

Proposition 7.1.20. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega . g(x) < f(x)\}$ is open.*

PROOF:
$\langle 1\rangle 1.$ LET: $g \in U$
$\langle 1\rangle 2.$ LET: $\epsilon = \max_{x \in \Omega}(f(x) - g(x))$
PROVE: $B(g, \epsilon) \subseteq S$
$\langle 1\rangle 3.$ $\epsilon > 0$
$\langle 1\rangle 4.$ LET: $h \in B(g, \epsilon/2)$
PROVE: $h \in S$
$\langle 1\rangle 5.$ LET: $x \in \Omega$
$\langle 1\rangle 6.$ $h(x) < f(x)$
PROOF:

$$\begin{aligned}
h(x) &\leq g(x) + \epsilon/2 && (\langle 1\rangle 4) \\
&< g(x) + \epsilon && (\langle 1\rangle 3) \\
&\leq f(x) && (\langle 1\rangle 2)
\end{aligned}$$

□

Proposition 7.1.21. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega . g(x) > f(x)\}$ is open.*

PROOF: Given $g \in U$, let $\epsilon = \max_x(g(x) - f(x))/2$. Then $B(g, \epsilon) \subseteq U$. □

Proposition 7.1.22. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$ be such that $f(x) > 0$ for all $x \in \Omega$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega . |g(x)| < f(x)\}$ is open.*

PROOF: Given $g \in U$, let $\epsilon = \max_x(f(x) - |g(x)|)/2$. Then $B(g, \epsilon) \subseteq U$. □

Proposition 7.1.23. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$ be such that $f(x) > 0$ for all $x \in \Omega$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega.|g(x)| > f(x)\}$ is open.*

PROOF: Given $g \in U$, let $\epsilon = \max_x(|g(x)| - f(x))/2$. Then $B(g, \epsilon) \subseteq U$. □

Proposition 7.1.24. *The union of a set of open sets is open.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. LET: $\mathcal{U}$ be a set of open sets in V.
⟨1⟩3. LET: $x \in \bigcup \mathcal{U}$
⟨1⟩4. PICK $U \in \mathcal{U}$ such that $x \in U$.
⟨1⟩5. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
⟨1⟩6. $B(x, \epsilon) \subseteq \bigcup \mathcal{U}$
□

Proposition 7.1.25. *The intersection of two open sets is open.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. LET: U_1 and U_2 be open sets in V.
⟨1⟩3. LET: $x \in U_1 \cap U_2$
⟨1⟩4. PICK $\epsilon_1 > 0$ such that $B(x, \epsilon_1) \subseteq U_1$
⟨1⟩5. PICK $\epsilon_2 > 0$ such that $B(x, \epsilon_2) \subseteq U_2$
⟨1⟩6. LET: $\epsilon = \min(\epsilon_1, \epsilon_2)$
⟨1⟩7. $B(x, \epsilon) \subseteq U_1 \cap U_2$
□

Proposition 7.1.26. *In any normed space, $\emptyset$ is open.*

PROOF: Vacuous. □

Proposition 7.1.27. *In any normed space V, the whole space V is open.*

PROOF: For any $x \in V$ we have $B(x, 1) \subseteq V$. □

Definition 7.1.28 (Closed Set)**.** Let V be a normed space over K. A set $S \subseteq V$ is *closed* iff $V - S$ is open.

Proposition 7.1.29. *Every closed ball is closed.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. LET: $c \in V$ and $r > 0$
  PROVE: $\overline{B}(c, r)$ is closed.
⟨1⟩3. LET: $x \in V - \overline{B}(c, r)$
⟨1⟩4. LET: $\epsilon = \|x - c\| - r$
  PROVE: $B(x, \epsilon) \subseteq V - \overline{B}(c, r)$

$\langle 1 \rangle 5.$ $\epsilon > 0$
PROOF: Since $\|x - c\| > r$ by $\langle 1 \rangle 3$.
$\langle 1 \rangle 6.$ LET: $y \in B(x, \epsilon)$
$\langle 1 \rangle 7.$ $\|y - c\| > r$
PROOF:

$$\begin{aligned} \|y - c\| &\geq \|x - c\| - \|x - y\| && \text{(Triangle Inequality)} \\ &> \|x - c\| - \epsilon && (\langle 1 \rangle 6) \\ &= r && (\langle 1 \rangle 4) \end{aligned}$$

□

Proposition 7.1.30. *The intersection of a set of closed sets is closed.*

PROOF: From Proposition 7.1.24. □

Proposition 7.1.31. *The union of two closed sets is closed.*

PROOF: From Proposition 7.1.25. □

Proposition 7.1.32. *Every sphere is closed.*

PROOF: $S(c, r) = \overline{B}(c, r) - B(c, r)$. □

Proposition 7.1.33. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega. g(x) \leq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega. g(x) > f(x)\}$. □

Proposition 7.1.34. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega. g(x) \geq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega. g(x) < f(x)\}$. □

Proposition 7.1.35. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega. |g(x)| \leq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega. |g(x)| > f(x)\}$. □

Proposition 7.1.36. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega. |g(x)| \geq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega. |g(x)| < f(x)\}$. □

Proposition 7.1.37. *Let Ω be a closed bounded set in $\mathbb{R}^n$. Let $x_0 \in \Omega$ and $\lambda \in \mathbb{C}$. Then $C = \{g \in \mathcal{C}(\Omega) \mid g(x_0) = \lambda\}$ is closed.*

PROOF: Given $g \in \mathcal{C}(\Omega) - C$, let $\epsilon = |g(x_0) - \lambda|/2$. Then $B(g, \epsilon) \subseteq \mathcal{C}(\Omega) - C$. □

Proposition 7.1.38. *In any normed space V, we have $\emptyset$ is closed.*

PROOF: Since $V - \emptyset = V$ is open. □

Proposition 7.1.39. *In any normed space V, the whole space V is closed.*

PROOF: Since $V - V = \emptyset$ is open. □

Theorem 7.1.40. *Let V be a normed space over K. Let S be a subset of V. Then S is closed if and only if, for any sequence (x_n) in S, if $x_n \to l$ as $n \to \infty$ then $l \in S$.*

PROOF:
$\langle 1\rangle 1$. If S is closed then, for any sequence (x_n) in S, if $x_n \to l$ as $n \to \infty$ then $l \in S$.
  $\langle 2\rangle 1$. ASSUME: S is closed.
  $\langle 2\rangle 2$. LET: (x_n) be a sequence in S.
  $\langle 2\rangle 3$. ASSUME: $x_n \to l$ as $n \to \infty$.
  $\langle 2\rangle 4$. ASSUME: for a contradiction $l \notin S$.
  $\langle 2\rangle 5$. PICK $\epsilon > 0$ such that $B(l, \epsilon) \subseteq V - S$
  $\langle 2\rangle 6$. PICK N such that $\forall n \geq N. x_n \in B(l, \epsilon)$
  $\langle 2\rangle 7$. $x_N \in V - S$
  $\langle 2\rangle 8$. This contradicts $\langle 2\rangle 2$.
$\langle 1\rangle 2$. If, for any sequence (x_n) in S, if $x_n \to l$ as $n \to \infty$ then $l \in S$, then S is closed.
  $\langle 2\rangle 1$. ASSUME: for any sequence (x_n) in S, if $x_n \to l$ as $n \to \infty$ then $l \in S$.
  $\langle 2\rangle 2$. LET: $x \in V - S$
  $\langle 2\rangle 3$. ASSUME: for a contradiction there is no $\epsilon > 0$ such that $B(x, \epsilon) \subseteq V - S$.
  $\langle 2\rangle 4$. For $n \in \mathbb{Z}_+$, PICK $x_n \in B(x, 1/n) \cap S$
  $\langle 2\rangle 5$. $x_n \to x$ as $n \to \infty$
  $\langle 2\rangle 6$. $x \in S$
  $\langle 2\rangle 7$. This contradicts $\langle 2\rangle 2$.
□

Definition 7.1.41 (Closure)**.** Let V be a normed space over K. Let S be a subset of V. The *closure* of S, $\mathrm{cl}\, S$, is the intersection of the set of closed sets that include S.

Proposition 7.1.42. *Let V be a normed space over K. Let S be a subset of V. Then the closure of S is the smallest closed set that includes S.*

PROOF: Proposition 7.1.30. □

Theorem 7.1.43. *Let V be a normed space over K. Let S be a subset of V. Then*

$$\mathrm{cl}\, S = \{l \in V \mid \exists \text{ a sequence } (x_n) \text{ in } S. x_n \to l \text{ as } n \to \infty\} \ .$$

PROOF:
$\langle 1\rangle 1$. For all $l \in \mathrm{cl}\, S$, there exists a sequence (x_n) in S such that $x_n \to l$ as $n \to \infty$.
  $\langle 2\rangle 1$. LET: $l \in \mathrm{cl}\, S$
  $\langle 2\rangle 2$. For $n \in \mathbb{Z}_+$, pick $x_n \in B(l, 1/n) \cap S$
  PROOF: There must be such an x_n otherwise $S - B(l, 1/n)$ would be a smaller closed set that includes S.

⟨2⟩3. $x_n \to l$ as $n \to \infty$

⟨1⟩2. For any sequence (x_n) in S, if $x_n \to l$ as $n \to \infty$ then $l \in \operatorname{cl} S$.

PROOF: Theorem 7.1.40.

□

Definition 7.1.44 (Dense)**.** Let V be a normed space over K. Let $S \subseteq V$. Then S is *dense* if and only if $\operatorname{cl} S = V$.

Theorem 7.1.45 (Weierstrass Approximation Theorem)**.** *Let a and b be real numbers with $a < b$. In $\mathcal{C}([a,b])$, the set of polynomials is dense.*

PROOF:TODO

Proposition 7.1.46. *Let $p \geq 1$. The set of all sequences that have only finitely many non-zero terms is dense in l^p.*

PROOF:

⟨1⟩1. LET: $(z_n) \in l^p$

⟨1⟩2. LET: $\epsilon > 0$

PROVE: There exists a sequence (x_n) with only finitely many non-zero terms such that $\left(\sum_{n=1}^{\infty} |z_n - x_n|^p\right)^{1/p} < \epsilon$

⟨1⟩3. PICK N such that $|\sum_{n=1}^{\infty} |z_n|^p - \sum_{n=1}^{N} |z_n|^p| < \epsilon^p$

⟨1⟩4. LET: (x_n) be the sequence that agrees with (z_n) up to term N, and then zeros after that.

⟨1⟩5. $\left(\sum_{n=1}^{\infty} |z_n - x_n|^p\right)^{1/p} < \epsilon$

PROOF:

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} |z_n - x_n|^p\right)^{1/p} &= \left(\sum_{n=N+1}^{\infty} |z_n|^p\right)^{1/p} && (\langle 1\rangle 4) \\
&< \epsilon && (\langle 1\rangle 2)
\end{aligned}$$

□

Theorem 7.1.47. *Let V be a normed space over K. Let $S \subseteq V$. Then the following are equivalent.*

1. *S is dense.*
2. *For all $l \in V$, there exists a sequence (x_n) in S such that $x_n \to l$ as $n \to \infty$.*
3. *Every nonempty open subset of V intersects S.*

PROOF:

⟨1⟩1. $1 \Leftrightarrow 2$

PROOF: Theorem 7.1.43.

⟨1⟩2. $1 \Rightarrow 3$

⟨2⟩1. ASSUME: S is dense.

⟨2⟩2. LET: U be a nonempty open subset of V.

⟨2⟩3. $X - U$ does not include S.

PROOF: Lest we have $\mathrm{cl}\, S \subseteq X - U$.
⟨2⟩4. U intersects S.
⟨1⟩3. $3 \Rightarrow 1$
⟨2⟩1. ASSUME: Every nonempty subset of V intersects S.
⟨2⟩2. Every closed proper subset of V does not include S.
⟨2⟩3. $\mathrm{cl}\, S = V$
□

Definition 7.1.48 (Compact)**.** Let V be a normed space over K and $S \subseteq V$. Then S is *compact* if and only if every sequence in S has a convergent subsequence whose limit is in S.

Proposition 7.1.49. *In K^n, a set is compact if and only if it is bounded and closed.*

PROOF: TODO

Definition 7.1.50 (Bounded)**.** Let V be a normed space over K and $S \subseteq V$. Then S is *bounded* iff there exists $r > 0$ such that $V \subseteq B(0, r)$.

Theorem 7.1.51. *Every compact set is closed and bounded.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. LET: $S \subseteq V$ be compact.
⟨1⟩3. S is closed.
⟨2⟩1. LET: (x_n) be a sequence in S that converges to l
⟨2⟩2. PICK a sequence (x_{n_r}) that converges to $x \in S$
PROOF: ⟨1⟩2, ⟨2⟩1
⟨2⟩3. $x_{n_r} \to l$ as $n \to \infty$
PROOF: ⟨2⟩1, ⟨2⟩2
⟨2⟩4. $l = x$
PROOF: Proposition 7.1.2.
⟨2⟩5. $l \in S$
PROOF: ⟨2⟩2, ⟨2⟩4
⟨2⟩6. Q.E.D.
PROOF: Theorem 7.1.40.
⟨1⟩4. S is bounded.
⟨2⟩1. ASSUME: for a contradiction S is unbounded.
⟨2⟩2. For $n \in \mathbb{Z}_+$, PICK $x_n \in S - B(0, n)$
⟨2⟩3. PICK a convergent subsequence (x_{n_r}) that converges to l, say.
⟨2⟩4. PICK $N \in \mathbb{Z}_+$ such that $\|l\| < N$
⟨2⟩5. PICK r such that $n_r > N$ and $\|x_{n_r} - l\| < N - \|l\|$
⟨2⟩6. $\|x_{n_r}\| < N < n_r$
⟨2⟩7. This contradicts ⟨2⟩2.
□

Proposition 7.1.52. *In $\mathcal{C}([0, 1])$, the closed ball $\overline{B}(0, 1)$ is closed and bounded but not compact.*

PROOF: The sequence of functions (x^n) is in $\overline{B}(0,1)$ but has no convergent subsequence. □

Theorem 7.1.53 (Riesz's Lemma)**.** *Let V be a normed vector space over K. Let X be a closed proper subspace of V. Let $0 < \epsilon < 1$. Then there exists $x \in V$ such that $\|x\| = 1$ and $\forall y \in X . \|x - y\| \geq \epsilon$.*

PROOF:
⟨1⟩1. PICK $z \in V - X$
⟨1⟩2. LET: $d = \inf_{x \in X} \|z - x\|$
⟨1⟩3. $d > 0$
PROOF: Since X is closed, there exists $e > 0$ such that $B(z, d) \subseteq V - X$ and hence $\|z - x\| \geq d$ for all $x \in X$.
⟨1⟩4. PICK $x_0 \in X$ such that $d \leq \|z - x_0\| \leq d/\epsilon$
PROOF: One exists since d/ϵ is not a lower bound for $\{\|z - x\| \mid x \in X\}$.
⟨1⟩5. LET: $x = (z - x_0)/\|z - x_0\|$
⟨1⟩6. LET: $y \in X$
⟨1⟩7. $\|x - y\| \geq \epsilon$
PROOF:

$$\begin{aligned} \|x - y\| &= \left\| \frac{z - x_0}{\|z - x_0\|} - y \right\| && (\langle 1 \rangle 5) \\ &= \frac{1}{\|z - x_0\|} \|z - (x_0 + \|z - x_0\| y)\| \\ &\geq \frac{1}{\|z - x_0\|} d && (\langle 1 \rangle 2) \\ &\geq \epsilon && (\langle 1 \rangle 4) \end{aligned}$$

□

Theorem 7.1.54. *Let V be a normed space over K. Then V is finite dimensional if and only if $\overline{B}(0,1)$ is compact.*

PROOF:
⟨1⟩1. If V is finite dimensional then $\overline{B}(0,1)$ is compact.
⟨2⟩1. ASSUME: V is finite dimensional.
⟨2⟩2. PICK a basis $\{e_1, \ldots, e_n\}$
⟨2⟩3. ASSUME: w.l.o.g. $\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| = |\alpha_1| + \cdots + |\alpha_n|$
⟨2⟩4. LET: $(\alpha_{k1} e_1 + \cdots + \alpha_{kn} e_n)$ be a sequence in $\overline{B}(0,1)$
⟨2⟩5. PICK a convergent subsequence $(\alpha_{k_r 1})$ of (α_{k1}), a convergent subsequence $(\alpha_{k'_r} 2)$ of $(\alpha_{k_r 2})$, …, a convergent subsequence $(\alpha_{k''_r} n)$.
⟨2⟩6. $(\alpha_{k''_r 1} e_1 + \cdots + \alpha_{k''_r} n)$ converges.
⟨1⟩2. If V is infinite dimensional then $\overline{B}(0,1)$ is not compact.
⟨2⟩1. ASSUME: V is infinite dimensional.
⟨2⟩2. Choose a sequence (x_n) with $\|x_n\| = 1$ and $\|x_m - x_n\| \geq 1/2$ for $m \neq n$
⟨3⟩1. ASSUME: $x_1, \ldots, x_n$ satisfy $\|x_i\| = 1$ and $\|x_i - x_j\| \geq 1/2$ for $i \neq j$
⟨3⟩2. PICK $x_{n+1} \in V$ such that $\|x_{n+1}\| = 1$ and for all $y \in \operatorname{span}\{x_1, \ldots, x_n\}$ we have $\|x_{n+1} - y\| \geq 1/2$

⟨4⟩1. $\mathrm{span}\{x_1, \ldots, x_n\}$ is closed.
⟨5⟩1. LET: $S = \mathrm{span}\{x_1, \ldots, x_n\}$
⟨5⟩2. LET: (a_n) be a sequence in S that converges to $a \in V$
PROVE: $a \in S$
⟨5⟩3. (a_n) is a Cauchy sequence in V.
⟨5⟩4. (a_n) is a Cauchy sequence in S.
⟨5⟩5. A finite dimensional normed space is a Banach space.
⟨5⟩6. S is complete.
⟨5⟩7. $a \in S$
⟨4⟩2. $\mathrm{span}\{x_1, \ldots, x_n\}$ is a proper subspace of V.
PROOF: ⟨2⟩1
⟨4⟩3. Q.E.D.
PROOF: Riesz's Lemma.
⟨2⟩3. ASSUME: for a contradiction (x_{n_r}) is a subsequence that converges to l
⟨2⟩4. For all $r \in \mathbb{N}$, we have $\|x_{n_r} - l\| + \|x_{n_{r+1}} - l\| \geq 1/2$
⟨2⟩5. This is a contradiction.
□

Proposition 7.1.55. *Let V be a normed space. The closure of a subspace of V is a subspace.*

PROOF:
⟨1⟩1. LET: U be a subspace of V
⟨1⟩2. LET: $x, y \in \mathrm{cl}\, U$ and $\alpha, \beta \in K$
⟨1⟩3. PICK sequences (x_n), (y_n) in U that converge to x and y respectively.
⟨1⟩4. $\alpha x_n + \beta y_n \to \alpha x + \beta y$ as $n \to \infty$
⟨1⟩5. $\alpha x + \beta y \in \mathrm{cl}\, U$
□

7.2 Continuous Linear Mappings

Definition 7.2.1 (Continuous)**.** Let U and V be normed spaces. Let $f : U \to V$ and $x \in U$. Then f is *continuous at* x iff, for any sequence (x_n) in U, if $x_n \to x$ as $n \to \infty$ then $f(x_n) \to f(x)$ as $n \to \infty$. The function f is *continuous* iff f is continuous at every point.

Proposition 7.2.2. *Let V be a normed space. Then the norm is a continuous function $V \to \mathbb{R}$.*

PROOF: From Proposition 7.0.4. □

Proposition 7.2.3. *Let U and V be normed space. Let $f : U \to V$. Then the following are equivalent.*

1. *f is continuous.*
2. *For any open set S in V, we have $f^{-1}(S)$ is open in U.*

3. *For any closed set C in V, we have $f^{-1}(C)$ is closed in U.*

Proposition 7.2.4. *Let U and V be normed spaces over K. Let $T : U \to V$ be a linear transformation. If T is continuous at some point, then it is continuous.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: T is continuous at u_0
$\langle 1 \rangle 2$. LET: $x_n \to l$ as $n \to \infty$ in U
$\langle 1 \rangle 3$. $x_n - l + u_0 \to u_0$ as $n \to \infty$.
$\langle 1 \rangle 4$. $T(x_n - l + u_0) \to T(u_0)$ as $n \to \infty$.
$\langle 1 \rangle 5$. $T(x_n) - T(l) + T(u_0) \to T(u_0)$ as $n \to \infty$.
$\langle 1 \rangle 6$. $T(x_n) \to T(l)$ as $n \to \infty$.
$\square$

Definition 7.2.5 (Bounded)**.** Let U and V be normed spaces over K. Let $T : U \to V$ be a linear transformation. Then T is *bounded* iff there exists $\alpha > 0$ such that, for all $x \in U$, we have $\|T(x)\| \leq \alpha \|x\|$.

Theorem 7.2.6. *Let U and V be normed spaces over K. Let $T : U \to V$ be a linear transformation. Then T is continuous if and only if it is bounded.*

PROOF:
$\langle 1 \rangle 1$. If T is continuous then T is bounded.
  $\langle 2 \rangle 1$. ASSUME: T is not bounded.
  $\langle 2 \rangle 2$. For $n \in \mathbb{Z}_+$, PICK $x_n \in U$ such that $\|T(x_n)\| > n\|x_n\|$.
  $\langle 2 \rangle 3$. For $n \in \mathbb{Z}_+$,
    LET:
$$y_n = \frac{x_n}{n\|x_n\|}$$
  $\langle 2 \rangle 4$. $y_n \to 0$ as $n \to \infty$
  $\langle 2 \rangle 5$. $T(y_n) \not\to 0$ as $n \to \infty$
  $\langle 2 \rangle 6$. T is not continuous.
$\langle 1 \rangle 2$. If T is bounded then T is continuous.
  $\langle 2 \rangle 1$. ASSUME: T is bounded.
  $\langle 2 \rangle 2$. PICK $\alpha > 0$ such that $\forall x \in U . \|T(x)\| \leq \alpha \|x\|$.
  $\langle 2 \rangle 3$. T is continuous at 0.
    $\langle 3 \rangle 1$. LET: (x_n) be a sequence that converges to 0 in U
    $\langle 3 \rangle 2$. $T(x_n) \to 0$ as $n \to \infty$
      PROOF:
$$\begin{aligned} \|T(x_n)\| &\leq \alpha \|x_n\| && (\langle 2 \rangle 2) \\ &\to 0 && \text{as } n \to \infty \end{aligned}$$
  $\langle 2 \rangle 4$. T is continuous.
    PROOF: Proposition 7.2.4.
$\square$

Proposition 7.2.7. *Let U and V be normed spaces over K where U is finite dimensional. Let $T : U \to V$ be a linear transformation. Then T is bounded.*

PROOF:

⟨1⟩1. Pick a basis $\{e_1, \ldots, e_n\}$ of unit vectors for U.
⟨1⟩2. Let: $M = \max(\|T(e_1)\|, \ldots, \|T(e_n)\|)$
⟨1⟩3. Pick $C > 0$ such that, for all $\alpha_1, \ldots, \alpha_n \in K$, we have $|\alpha_1| + \cdots + |\alpha_n| \leq C\|\alpha_1 e_1 + \cdots + \alpha_n e_n\|$
Proof: Theorem 7.1.13.
⟨1⟩4. Let: $x \in U$
Prove: $\|T(x)\| \leq CM\|x\|$
⟨1⟩5. Let: $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$
⟨1⟩6. $\|T(x)\| \leq CM\|x\|$
Proof:

$$\begin{aligned} \|T(x)\| &= \|\alpha_1 T(e_1) + \cdots + \alpha_n T(e_n)\| && (T \text{ linear}) \\ &\leq |\alpha_1|\|T(e_1)\| + \cdots + |\alpha_n|\|T(e_n)\| && (\text{Triangle inequality}) \\ &\leq M(|\alpha_1| + \cdots + |\alpha_n|) && (\langle 1\rangle 2) \\ &\leq CM\|x\| && (\langle 1\rangle 3) \end{aligned}$$

□

Corollary 7.2.7.1. *Let U and V be normed spaces over K where U is finite dimensional. Let $T : U \to V$ be a linear transformation. Then T is continuous.*

Proposition 7.2.8. *Let U and V be normed spaces over K. Let $T : U \to V$ be a linear transformation. If T is continuous, then T is uniformly continuous.*

Proof:
⟨1⟩1. Assume: T is continuous
⟨1⟩2. Pick $B > 0$ such that $\forall x \in U. \|T(x)\| \leq B\|x\|$
⟨1⟩3. Let: $\epsilon > 0$
⟨1⟩4. Let: $\delta = \epsilon / B$
⟨1⟩5. Let: $x, y \in U$
⟨1⟩6. Assume: $\|x - y\| < \delta$
⟨1⟩7. $\|T(x) - T(y)\| < \epsilon$
Proof:

$$\begin{aligned} \|T(x) - T(y)\| &= \|T(x - y)\| && (T \text{ linear}) \\ &\leq B\|x - y\| && (\langle 1\rangle 2) \\ &< B\delta && (\langle 1\rangle 6) \\ &= \epsilon && (\langle 1\rangle 4) \end{aligned}$$

□

Proposition 7.2.9. *Let U and V be normed spaces over K. The set $\mathcal{B}(U, V)$ of all bounded linear maps from U to V forms a subspace of the space of all linear maps from U to V.*

Proof:
⟨1⟩1. Let: $S, T : U \to V$ be bounded linear maps and $\alpha, \beta \in K$.
Prove: $\alpha S + \beta T$ is bounded.
⟨1⟩2. Pick $B, C > 0$ such that $\forall x \in U. \|S(x)\| \leq B\|x\|$ and $\|T(x)\| \leq C\|x\|$
⟨1⟩3. $\forall x \in U. \|(\alpha S + \beta T)(x)\| \leq (|\alpha| B + |\beta| C)\|x\|$

□

Proposition 7.2.10. *Let U and V be normed spaces over K. Define the* operator norm $\| \ \|$ *on $\mathcal{B}(U,V)$ by $\|T\| := \sup\{\|T(x)\| \mid x \in U, \|x\| = 1\}$. Then $\| \ \|$ is a norm on $\mathcal{B}(U,V)$.*

PROOF:
⟨1⟩1. For all $T \in \mathcal{B}(U,V)$, the set $\{\|T(x)\| \mid x \in U, \|x\| = 1\}$ is bounded above.
  ⟨2⟩1. LET: $T \in \mathcal{B}(U,V)$
  ⟨2⟩2. PICK B such that $\forall x \in U.\|T(x)\| \leq B\|x\|$.
  ⟨2⟩3. B is an upper bound for $\{\|T(x)\| \mid x \in U, \|x\| = 1\}$.
⟨1⟩2. If $\|T\| = 0$ then $T = 0$.
  ⟨2⟩1. ASSUME: $\|T\| = 0$
  ⟨2⟩2. LET: $x \in U$
    PROVE: $T(x) = 0$
  ⟨2⟩3. ASSUME: w.l.o.g. $\|x\| \neq 0$
  ⟨2⟩4. LET: $y = x/\|x\|$
  ⟨2⟩5. $\|y\| = 1$
  ⟨2⟩6. $\|T(y)\| = 0$
  ⟨2⟩7. $T(y) = 0$
  ⟨2⟩8. $T(x) = 0$
⟨1⟩3. For all $\lambda \in K$ and $T \in \mathcal{B}(U,V)$, we have $\|\lambda T\| = |\lambda|\|T\|$
  ⟨2⟩1. LET: $\lambda \in K$ and $T \in \mathcal{B}(U,V)$
  ⟨2⟩2. $\|\lambda T\| = |\lambda|\|T\|$
    PROOF:

$$\begin{aligned}\|\lambda T\| &= \sup\{\|\lambda T(x)\| \mid x \in U, \|x\| = 1\} \\ &= \sup\{|\lambda|\|T(x)\| \mid x \in U, \|x\| = 1\} \\ &= |\lambda| \sup\{\|T(x)\| \mid x \in U, \|x\| = 1\} \\ &= |\lambda|\|T\|\end{aligned}$$

⟨1⟩4. For all $S, T \in \mathcal{B}(U,V)$, we have $\|S + T\| \leq \|S\| + \|T\|$.
  ⟨2⟩1. LET: $S, T \in \mathcal{B}(U,V)$
  ⟨2⟩2. $\|S + T\| \leq \|S\| + \|T\|$
    PROOF:

$$\begin{aligned}\|S + T\| &= \sup\{\|S(x) + T(x)\| \mid x \in U, \|x\| = 1\} \\ &\leq \sup\{\|S(x)\| + \|T(x)\| \mid x \in U, \|x\| = 1\} \\ &\leq \sup\{\|S(x)\| \mid x \in U, \|x\| = 1\} + \sup\{\|T(x)\| \mid x \in U, \|x\| = 1\} \\ &= \|S\| + \|T\|\end{aligned}$$

□

Proposition 7.2.11. *Let U and V be normed spaces. Let $T \in \mathcal{B}(U,V)$. Then $\|T\|$ is the least number such that $\forall u \in U.\|T(u)\| \leq \|T\|\|u\|$.*

PROOF:
⟨1⟩1. $\forall u \in U.\|T(u)\| \leq \|T\|\|u\|$
  ⟨2⟩1. LET: $u \in U$
  ⟨2⟩2. LET: $v = u/\|u\|$

$\langle 2\rangle 3.$ $\|T(v)\| \leq \|T\|$
$\langle 2\rangle 4.$ $\|T(u)\| \leq \|T\|\|u\|$
$\langle 1\rangle 2.$ If α satisfies $\forall u \in U.\|T(u)\| \leq \alpha\|u\|$, then $\|T\| \leq \alpha$
$\langle 2\rangle 1.$ ASSUME: $\forall u \in U.\|T(u)\| \leq \alpha\|u\|$
$\langle 2\rangle 2.$ For all $x \in U$, if $\|x\| = 1$ then $\|T(x)\| \leq \alpha$
$\langle 2\rangle 3.$ $\|T\| \leq \alpha$
□

Proposition 7.2.12. *Let V be a normed space. Then id_V is a bounded linear function $V \to V$, and $\|\mathrm{id}_V\| = 1$.*

Proposition 7.2.13. *Let U and V be normed spaces. The constant zero function $U \to V$ is a bounded linear transformation with norm 0.*

Proposition 7.2.14. *Let $N \in \mathbb{N}$. Let $T : \mathbb{C}^N \to \mathbb{C}^N$ be a linear transformation with matrix $A = (a_{ij})$. Then T is bounded and*

$$\|T\| \leq \sqrt{\sum_{i=1}^N \sum_{j=1}^N |a_{ij}|^2} \ .$$

Definition 7.2.15 (Uniform Convergence)**.** Let U and V be normed spaces. Let (T_n) be a sequence in $\mathcal{B}(U,V)$ and $T \in \mathcal{B}(U,V)$. Then (T_n) *converges uniformly* to T iff (T_n) converges to T under the standard norm defined above.

Theorem 7.2.16. *Let U and V be normed spaces. Let $T : U \to V$ be a continuous linear function. Then $\ker T$ is a closed subspace of U.*

PROOF:
$\langle 1\rangle 1.$ $\ker T$ is a subspace of U
PROOF: If $x, y \in \ker T$ and $\alpha, \beta \in K$ then $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) = 0$.
$\langle 1\rangle 2.$ $\ker T$ is closed.
PROOF: Let (x_n) be a sequence in $\ker T$ and $x_n \to l$. Then $T(l) = \lim_{n\to\infty} T(x_n) = 0$.
□

Theorem 7.2.17. *Let U and V be normed spaces. Let W be a closed subspace of U and $T : W \to V$ be a continuous linear mapping. Then the graph $G = \{(x, T(x)) \mid x \in W\}$ is closed in $U \times V$.*

PROOF:
$\langle 1\rangle 1.$ ASSUME: w.l.o.g. $T \neq 0$
$\langle 1\rangle 2.$ LET: $(x,y) \in (U \times V) - G$, i.e. $y \neq T(x)$
$\langle 1\rangle 3.$ LET: $\epsilon = \|y - T(x)\| > 0$
$\langle 1\rangle 4.$ LET: $x' \in W$ with $\|x - x'\| < \epsilon/3\|T\|$
$\langle 1\rangle 5.$ LET: $y' \in V$ with $\|y - y'\| < \epsilon/3$
$\langle 1\rangle 6.$ $y' \neq T(x')$

PROOF:

$$\begin{aligned}\|y' - T(x')\| &\geq \|y - T(x)\| - \|y - y'\| - \|T(x) - T(x')\| \\ &> \epsilon/3 \\ &> 0\end{aligned}$$

□

Theorem 7.2.18 (Diagonal Theorem)**.** *Let E be a normed space over K. Let (x_{ij}) be an infinite matrix of elements of V. If:*

1. *For all $j \in \mathbb{Z}_+$, we have $x_{ij} \to 0$ as $i \to \infty$;*
2. *Every increasing sequence of positive integers (p_j) has a subsequence (p_{j_r}) such that*
$$\sum_{s=1}^{\infty} x_{p_{j_r} p_{j_s}} \to 0 \text{ as } r \to \infty$$

then $x_{ii} \to 0$ as $i \to \infty$.

PROOF:
⟨1⟩1. ASSUME: for a contradiction $x_{ii} \not\to 0$ as $i \to \infty$
⟨1⟩2. PICK $\epsilon > 0$ such that, for all N, there exists $n \geq N$ such that $\|x_{nn}\| \geq \epsilon$
⟨1⟩3. PICK an increasing sequence of integers (p_j) such that $\|x_{p_j p_j}\| \geq \epsilon$ for all j.
⟨1⟩4. PICK a subsequence (q_i) such that $\sum_{j=1}^{\infty} x_{q_i q_j} \to 0$ as $i \to \infty$
⟨1⟩5. For all i, we have $x_{q_i q_j} \to 0$ as $j \to \infty$
⟨1⟩6. For all j, we have $x_{q_i q_j} \to 0$ as $i \to \infty$
⟨1⟩7. Define a subsequence (r_n) of (q_i) by $r_1 = q_1$ and, for all n, r_{n+1} is the first entry such that $r_{n+1} > r_n$, $\|x_{q_i r_n}\| < \epsilon/2^{n+1}$ for all $q_i \geq r_{n+1}$, and $\|x_{r_j r_{n+1}}\| < \epsilon/2^{n+2}$ for $j = 1, \ldots, n$.
⟨1⟩8. $\|x_{r_i r_j}\| < \epsilon/2^{j+1}$ for all i, j such that $i \neq j$
⟨1⟩9. PICK a subsequence (s_j) of (r_j) such that $\sum_{j=1}^{\infty} x_{s_i s_j} \to 0$ as $i \to \infty$
⟨1⟩10. For all i we have $\|\sum_{j=1}^{\infty} x_{s_i s_j}\| \geq \epsilon/2$
PROOF:

$$\begin{aligned}\left\|\sum_{j=1}^{\infty} x_{s_i s_j}\right\| &= \left\|x_{s_i s_i} + \sum_{i \neq j} x_{s_i s_j}\right\| \\ &\geq \left| \|x_{s_i s_i}\| - \left\|\sum_{i \neq j} x_{s_i s_j}\right\| \right| && \text{(Proposition 7.0.4)} \\ &\geq \left| \|x_{s_i s_i}\| - \sum_{i \neq j} \|x_{s_i s_j}\| \right| \\ &\geq \epsilon/2 && (\langle 1\rangle 2,\ \langle 1\rangle 8)\end{aligned}$$

⟨1⟩11. Q.E.D.
PROOF: ⟨1⟩9 and ⟨1⟩10 form a contradiction.
□

7.3 Banach Spaces

Definition 7.3.1 (Cauchy Sequence)**.** Let V be a normed space over K. A *Cauchy sequence* is a sequence of points (x_n) such that, for every $\epsilon > 0$, there exists N such that $\forall m, n \geq N . \|x_m - x_n\| < \epsilon$.

Theorem 7.3.2. *Let V be a normed space over K. Let (x_n) be a sequence in V. The following are equivalent.*

1. *(x_n) is Cauchy.*
2. *For every pair of increasing sequences of positive integers (p_n) and (q_n), we have $\|x_{p_n} - x_{q_n}\| \to 0$ as $n \to \infty$.*
3. *For every increasing sequence of positive integers (p_n), we have $\|x_{p_n} - x_n\| \to 0$ as $n \to \infty$.*

Proof:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. Assume: (x_n) is Cauchy.
  ⟨2⟩2. Let: (p_n) and (q_n) are increasing sequences of positive integers.
  ⟨2⟩3. Let: $\epsilon > 0$
  ⟨2⟩4. Pick N such that $\forall m, n \geq N . \|x_m - x_n\| < \epsilon$
  ⟨2⟩5. $\forall n \geq N . \|x_{p_n} - x_{q_n}\| < \epsilon$
    Proof: Since $p_n, q_n \geq n \geq N$.
⟨1⟩2. $2 \Rightarrow 3$
  Proof: Trivial.
⟨1⟩3. $2 \Rightarrow 1$
  ⟨2⟩1. Assume: (x_n) is not Cauchy
  ⟨2⟩2. Pick $\epsilon > 0$ such that, for every $N \in \mathbb{Z}_+$, there exist $m_N, n_N \geq N$ such that $\|x_{m_N} - x_{n_N}\| \geq \epsilon$
  ⟨2⟩3. Assume: w.l.o.g. (m_N) and (n_N) are increasing sequences.
  ⟨2⟩4. $\|x_{m_N} - x_{n_N}\| \not\to 0$ as $N \to \infty$.
⟨1⟩4. $3 \Rightarrow 2$
  ⟨2⟩1. Assume: 3
  ⟨2⟩2. Let: (p_n) and (q_n) be increasing sequences of positive integers.
  ⟨2⟩3. Let: $\epsilon > 0$
  ⟨2⟩4. Pick N such that $\forall n \geq N . \|x_{p_n} - x_n\| < \epsilon/2$ and $\forall n \geq N . \|x_{q_n} - x_n\| < \epsilon/2$
  ⟨2⟩5. $\forall n \geq N . \|x_{p_n} - x_{q_n}\| < \epsilon$
□

Proposition 7.3.3. *Every convergent sequence is Cauchy.*

Proof:
⟨1⟩1. Let: $x_n \to l$ as $n \to \infty$.
⟨1⟩2. Let: $\epsilon > 0$
⟨1⟩3. Pick N such that $\forall n \geq N . \|x_n - l\| < \epsilon/2$

⟨1⟩4. For all $m, n \geq N$ we have $\|x_m - x_n\| < \epsilon$.
□

Proposition 7.3.4. *In $\mathcal{P}([0,1])$, let*

$$P_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} \quad .$$

Then (P_n) is Cauchy but does not converge.

PROOF: It converges to e^x in $\mathcal{C}([0,1])$, therefore it is Cauchy in $\mathcal{C}([0,1])$, hence Cauchy in $\mathcal{P}([0,1])$. Since $e^x \notin \mathcal{P}([0,1])$, it does not converge in that space. □

Proposition 7.3.5. *Let V be a normed space over K. Let (x_n) be a Cauchy sequence in V. Then $(\|x_n\|)$ converges in $\mathbb{R}$.*

PROOF:
⟨1⟩1. $(\|x_n\|)$ is Cauchy.
  ⟨2⟩1. LET: $\epsilon > 0$
  ⟨2⟩2. PICK N such that $\forall m, n \geq N . \|x_m - x_n\| < \epsilon$
  ⟨2⟩3. $\forall m, n \geq N . |\|x_m\| - \|x_n\|| < \epsilon$
    PROOF: Proposition 7.0.4.
⟨1⟩2. Q.E.D.
  PROOF: Since every Cauchy sequence in $\mathbb{R}$ converges.
□

Proposition 7.3.6. *Every Cauchy sequence is bounded.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. LET: (x_n) be a Cauchy sequence in V.
⟨1⟩3. PICK N such that $\forall m, n \geq N . \|x_m - x_n\| < 1$.
⟨1⟩4. LET: $B = \max(\|x_1\|, \ldots, \|x_{N-1}\|, \|x_N\| + 1$
⟨1⟩5. $\forall n . \|x_n\| \leq B$
□

Definition 7.3.7 (Banach Space)**.** A normed space V over K is *complete* or a *Banach space* iff every Cauchy sequence converges.

Proposition 7.3.8. *l^2 is complete.*

PROOF:
⟨1⟩1. LET: (a_n) be a Cauchy sequence in l^2 where $a_n = (\alpha_{n1}, \alpha_{n2}, \ldots)$.
⟨1⟩2. For all $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}_+$ such that $\forall m, n \geq n_0 . \sum_{k=1}^{\infty} |\alpha_{mk} - \alpha_{nk}|^2 < \epsilon^2$.
⟨1⟩3. For every $k \in \mathbb{Z}_+$ and $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}_+$ such that $\forall m, n \geq n_0 . |\alpha_{mk} - \alpha_{nk}| < \epsilon$.
⟨1⟩4. For every $k \in \mathbb{Z}_+$, (α_{nk}) is Cauchy in $\mathbb{C}$.
⟨1⟩5. For every $k \in \mathbb{Z}_+$, (α_{nk}) converges in $\mathbb{C}$.
⟨1⟩6. For $k \in \mathbb{Z}_+$,

LET: $\alpha_k = \lim_{n\to\infty} \alpha_{nk}$

$\langle 1\rangle 7$. Let a be the sequence (α_k)

$\langle 1\rangle 8$. For all $\epsilon > 0$, there exists n_0 such that $\forall n \geq n_0. \sum_{k=1}^{\infty} |\alpha_k - \alpha_{nk}|^2 \leq \epsilon^2$.

PROOF: Letting $m \to \infty$ in $\langle 1\rangle 2$.

$\langle 1\rangle 9$. $a \in l^2$

$\langle 2\rangle 1$. PICK n_0 such that $\forall n \geq n_0. \sum_{k=1}^{\infty} |\alpha_k - \alpha_{nk}|^2 \leq 1$

PROOF: $\langle 1\rangle 8$

$\langle 2\rangle 2$. $(\alpha_k - \alpha_{n_0 k}) \in l^2$

$\langle 2\rangle 3$. $(\alpha_{n_0 k}) \in l^2$

PROOF: By $\langle 1\rangle 1$ since the sequence is a_{n_0}.

$\langle 2\rangle 4$. $(\alpha_k) \in l^2$

PROOF: Proposition 5.0.2.

$\langle 1\rangle 10$. $a_n \to a$ as $n \to \infty$

PROOF: By $\langle 1\rangle 8$ since $\|a - a_n\| = \sqrt{\sum_{k=1}^{\infty} |\alpha_k - \alpha_{nk}|^2}$.

□

Proposition 7.3.9. *Let a and b be real numbers with $a < b$. The space $\mathcal{C}([a,b])$ is complete.*

PROOF:

$\langle 1\rangle 1$. LET: $X = [a,b]$

$\langle 1\rangle 2$. LET: (f_n) be a Cauchy sequence in $\mathcal{C}([a,b])$.

$\langle 1\rangle 3$. For all $\epsilon > 0$, there exists n_0 such that $\forall n, m \geq n_0. \|f_n - f_m\| < \epsilon$.

$\langle 1\rangle 4$. For all $\epsilon > 0$, there exists n_0 such that $\forall n, m \geq n_0. \forall x \in X. |f_n(x) - f_m(x)| < \epsilon$.

$\langle 1\rangle 5$. For all $x \in [a,b]$, $(f_n(x))$ is Cauchy.

$\langle 1\rangle 6$. Define $f : [a,b] \to \mathbb{C}$ by $f(x) = \lim_{n\to\infty} f_n(x)$.

$\langle 1\rangle 7$. For all $\epsilon > 0$, there exists n_0 such that $\forall n \geq n_0. \forall x \in X. |f_n(x) - f(x)| < \epsilon$

PROOF: Letting $m \to \infty$ in $\langle 1\rangle 4$.

$\langle 1\rangle 8$. f is continuous

$\langle 2\rangle 1$. LET: $x_0 \in X$

$\langle 2\rangle 2$. LET: $\epsilon > 0$

$\langle 2\rangle 3$. PICK n_0 such that $\forall n \geq n_0. \forall x \in X. |f_n(x) - f(x)| < \epsilon/3$

PROOF: By $\langle 1\rangle 7$.

$\langle 2\rangle 4$. PICK $\delta > 0$ such that $\forall x \in X. |x - x_0| < \delta \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \epsilon/3$

PROOF: Since f_{n_0} is continuous.

$\langle 2\rangle 5$. LET: $x \in X$

$\langle 2\rangle 6$. ASSUME: $|x - x_0| < \delta$

$\langle 2\rangle 7$. $|f(x) - f(x_0)| < \epsilon$

PROOF:

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| && \text{(Triangle Inequality)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 2\rangle 3,\ \langle 2\rangle 4) \\ &= \epsilon \end{aligned}$$

$\langle 1\rangle 9$. (f_n) converges to f uniformly.

PROOF: From $\langle 1\rangle 7$

□

Definition 7.3.10 (Series)**.** Let V be a normed space over K. A *convergent series* in V is a sequence (x_n) in V such that $(x_1 + \cdots + x_n)$ is a convergent sequence, in which case we write $\sum_{n=1}^{\infty} x_n$ for its limit.

Definition 7.3.11 (Absolutely Convergent Series)**.** Let V be a normed space over K. An *absolutely convergent series* in V is a sequence (x_n) such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Proposition 7.3.12. *In $\mathcal{P}([0,1])$, the series $\sum_{n=0}^{\infty} x^n/n!$ is absolutely convergent but not convergent.*

PROOF: Proposition 7.3.4. □

Theorem 7.3.13. *A normed space is complete if and only if every absolutely convergent series is convergent.*

PROOF:
⟨1⟩1. LET: V be a normed space over K.
⟨1⟩2. If V is complete then every absolutely convergent series is convergent.
⟨2⟩1. ASSUME: V is complete.
⟨2⟩2. LET: $\sum_{n=1}^{\infty} a_n$ be absolutely convergent in V.
⟨2⟩3. For $n \in \mathbb{Z}_+$,
LET: $s_n = \sum_{k=1}^{n} a_k$
.
⟨2⟩4. (s_n) is Cauchy.
⟨3⟩1. LET: $\epsilon > 0$
⟨3⟩2. PICK k such that $\sum_{n=k+1}^{\infty} \|a_n\| < \epsilon$
⟨3⟩3. LET: $m > n > k$
⟨3⟩4. $\|s_m - s_n\| < \epsilon$
PROOF:

$$\begin{aligned} \|s_m - s_n\| &= \left\| \sum_{i=n+1}^{m} a_i \right\| && (\langle 2\rangle 3,\ \langle 3\rangle 3) \\ &\leq \sum_{i=s+1}^{m} \|a_i\| && \text{(Triangle inequality)} \\ &\leq \sum_{i=k+1}^{\infty} \|a_i\| \\ &< \epsilon && (\langle 3\rangle 2,\ \langle 3\rangle 3) \end{aligned}$$

⟨2⟩5. (s_n) converges.
⟨1⟩3. If every absolutely convergent series is convergent then V is complete.
⟨2⟩1. ASSUME: Every absolutely convergent series in V is convergent.
⟨2⟩2. LET: (a_n) be a Cauchy sequence in V.
⟨2⟩3. PICK a strictly increasing sequence of positive integers (p_n) such that $\forall k. \forall m, n \geq p_k . \|x_m - x_n\| < 2^{-k}$.
⟨2⟩4. $\sum_{k=1}^{\infty} (x_{p_{k+1}} - x_{p_k})$ is absolutely convergent.

PROOF:
$$\sum_{k=1}^{\infty} \|x_{p_{k+1}} - x_{p_k}\| < \sum_{k=1}^{\infty} 2^{-k} \qquad (\langle 2\rangle 3)$$
$$< \infty$$

$\langle 2\rangle 5$. $\sum_{k=1}^{\infty}(x_{p_{k+1}} - x_{p_k})$ is convergent.
PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 5$
$\langle 2\rangle 6$. LET: $s = \sum_{k=1}^{\infty}(x_{p_{k+1}} - x_{p_k})$
$\langle 2\rangle 7$. $x_{p_k} \to s + x_{p_1}$ as $k \to \infty$.
$\langle 3\rangle 1$. $\sum_{i=1}^{k}(x_{p_{i+1}} - x_{p_i}) \to s$ as $k \to \infty$
PROOF: $\langle 2\rangle 6$
$\langle 3\rangle 2$. $x_{p_{k+1}} - x_{p_1} \to s$ as $k \to \infty$
$\langle 2\rangle 8$. $x_n \to s + x_{p_1}$ as $k \to \infty$.
PROOF:
$\langle 3\rangle 1$. LET: $\epsilon > 0$
$\langle 3\rangle 2$. PICK N such that $\forall k \geq N. \|x_{p_k} - (s + x_{p_1})\| < \epsilon/2$ and $\forall m, n \geq N. \|x_m - x_n\| < \epsilon/2$
PROOF: $\langle 2\rangle 2$, $\langle 2\rangle 7$
$\langle 3\rangle 3$. $\forall n \geq N. \|x_n - (s + x_{p_1})\| < \epsilon$
□

Theorem 7.3.14. *A closed vector subspace of a Banach space is a Banach space.*

PROOF:
$\langle 1\rangle 1$. LET: V be a Banach space over K.
$\langle 1\rangle 2$. LET: U be a closed vector subspace of V.
$\langle 1\rangle 3$. LET: (a_n) be a Cauchy sequence in U.
$\langle 1\rangle 4$. (a_n) is a Cauchy sequence in V.
$\langle 1\rangle 5$. LET: $l = \lim_{n\to\infty} a_n$
$\langle 1\rangle 6$. $l \in U$
PROOF: Theorem 7.1.40.
$\langle 1\rangle 7$. $a_n \to l$ as $n \to \infty$ in U.
□

Definition 7.3.15 (Completion)**.** Let V be a normed space over K. A *completion* of V consists of a normed space W over K and an injection $\phi : V \to W$ such that:

1. $\forall x, y \in V. \forall \alpha, \beta \in K. \phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$
2. $\forall x \in V. \|\phi(x)\| = \|x\|$
3. $\phi(V)$ is dense in W.
4. W is complete.

Proposition 7.3.16. *Every normed space has a completion.*

PROOF:

$\langle 1\rangle 1.$ LET: V be a normed space over K.
$\langle 1\rangle 2.$ Let us say two Cauchy sequences (x_n), (y_n) ore *equivalent* iff $x_n - y_n \to 0$ as $n \to \infty$.
$\langle 1\rangle 3.$ Equivalence is an equivalence relation on the set of Cauchy sequences.
$\langle 1\rangle 4.$ LET: W be the set of Cauchy sequences in V quotiented by equivalence.
$\langle 1\rangle 5.$ Define addition and multiplication on W by

$$[(x_n)] + [(y_n)] = [(x_n + y_n)]$$
$$\lambda[(x_n)] = [(\lambda x_n)]$$

$\langle 1\rangle 6.$ Define a norm on W by $\|[(x_n)]\| = \lim_{n\to\infty} \|x_n\|$
$\langle 1\rangle 7.$ Define $\phi : V \to W$ by $\phi(v) = [(v)]$.
$\langle 1\rangle 8.$ ϕ is injective.
$\langle 1\rangle 9.$ $\forall x, y \in V.\forall \alpha, \beta \in K.\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$
$\langle 1\rangle 10.$ $\forall x \in V.\|\phi(x)\| = \|x\|$
$\langle 1\rangle 11.$ $\phi(V)$ is dense in W.
$\langle 2\rangle 1.$ LET: $[(a_n)] \in W$ and $\epsilon > 0$.
PROVE: $B([(a_n)], \epsilon)$ intersects $\phi(V)$.
$\langle 2\rangle 2.$ PICK N such that $\forall m, n \geq N.\|a_m - a_n\| < \epsilon/2$
$\langle 2\rangle 3.$ $\phi(a_N) \in B([(a_n)], \epsilon)$
PROOF:

$$\begin{aligned} \|[(a_n)] - \phi(a_N)\| &= \lim_{n\to\infty} \|a_n - a_N\| \\ &\leq \epsilon/2 && (\langle 2\rangle 2) \\ &< \epsilon \end{aligned}$$

$\langle 1\rangle 12.$ W is complete.
$\langle 2\rangle 1.$ LET: (X_n) be a Cauchy sequence in W.
$\langle 2\rangle 2.$ For $n \in \mathbb{Z}_+$, PICK $x_n \in V$ such that

$$\|\phi(x_n) - X_n\| < 1/n \ .$$

$\langle 2\rangle 3.$ (x_n) is Cauchy in V.
$\langle 3\rangle 1.$ LET: $\epsilon > 0$
$\langle 3\rangle 2.$ PICK N such that $\forall m, n \geq N.\|X_n - X_m\| < \epsilon/3$ and $1/N < \epsilon/3$
$\langle 3\rangle 3.$ LET: $m, n \geq N$
$\langle 3\rangle 4.$ $\|x_m - x_n\| < \epsilon$
PROOF:

$$\begin{aligned} \|x_m - x_n\| &= \|\phi(x_m) - \phi(x_n)\| \\ &\leq \|\phi(x_m) - X_m\| + \|X_m - X_n\| + \|X_n - \phi(x_n)\| \\ &< \|X_m - X_n\| + 1/m + 1/n \\ &< \epsilon \end{aligned}$$

$\langle 2\rangle 4.$ LET: $X = [(x_n)]$
$\langle 2\rangle 5.$ $X_n \to X$ as $n \to \infty$
PROOF:

$$\begin{aligned} \|X_n - X\| &\leq \|X_n - \phi(x_n)\| + \|\phi(x_n) - X\| \\ &< \|\phi(x_n) - X\| + 1/n \\ &\to 0 && \text{as } n \to \infty \end{aligned}$$

□

Proposition 7.3.17. *Let U be a normed space and V a Banach space. Then $\mathcal{B}(U, V)$ is a Banach space.*

PROOF:
⟨1⟩1. LET: (T_n) be a Cauchy sequence in $\mathcal{B}(U, V)$
⟨1⟩2. For all $u \in U$, $(T_n(u))$ is a Cauchy sequence in V.
⟨2⟩1. LET: $u \in U$
⟨2⟩2. LET: $\epsilon > 0$
PROVE: $\exists N. \forall m, n \geq N. \|T_m(u) - T_n(u)\| < \epsilon$
⟨2⟩3. ASSUME: w.l.o.g. $u \neq 0$
⟨2⟩4. PICK N such that $\forall m, n \geq N. \|T_m - T_n\| < \epsilon / \|u\|$
⟨2⟩5. LET: $m, n \geq N$
⟨2⟩6. $\|T_m(u) - T_n(u)\| < \epsilon$
PROOF:

$$\begin{aligned} \|T_m(u) - T_n(u)\| &\leq \|T_m - T_n\| \|u\| && \text{(Proposition 7.2.11)} \\ &< \epsilon \end{aligned}$$

⟨1⟩3. Define $T : U \to V$ by $T(u) = \lim_{n \to \infty} T_n(u)$
⟨1⟩4. $T \in \mathcal{B}(U, V)$
⟨2⟩1. For all $x, y \in U$ and $\alpha, \beta \in K$ we have $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$
⟨3⟩1. LET: $x, y \in U$
⟨3⟩2. LET: $\alpha, \beta \in K$
⟨3⟩3. $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$
PROOF:

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_{n \to \infty} T_n(\alpha x + \beta y) \\ &= \lim_{n \to \infty} (\alpha T_n(x) + \beta T_n(y)) \\ &= \alpha T(x) + \beta T(y) \end{aligned}$$

⟨2⟩2. T is bounded.
⟨3⟩1. PICK N such that $\forall n \geq N. \|T_n - T\| < 1$
⟨3⟩2. PICK $B > 0$ such that $\forall x \in U. \|T_N(x)\| \leq B\|x\|$
⟨3⟩3. LET: $x \in U$
⟨3⟩4. $\|T(x)\| \leq (B + 1)\|x\|$
PROOF:

$$\begin{aligned} \|T(x)\| &\leq \|T_N(x) - T(x)\| + \|T(x)\| && \text{(Triangle inequality)} \\ &\leq \|T_N - T\| \|x\| + \|T\| \|x\| && \text{(Proposition 7.2.11)} \\ &< \|x\| + B\|x\| && (\langle 3 \rangle 1,\ \langle 3 \rangle 2) \\ &= (B + 1)\|x\| \end{aligned}$$

⟨1⟩5. $T_n \to T$ as $n \to \infty$
⟨2⟩1. LET: $\epsilon > 0$
⟨2⟩2. PICK N such that $\forall m, n \geq N. \|T_m - T_n\| < \epsilon / 2$
⟨2⟩3. LET: $n \geq N$
PROVE: $\|T_n - T\| < \epsilon$
⟨2⟩4. LET: $x \in U$ with $\|x\| = 1$
⟨2⟩5. $\|T_n(x) - T(x)\| \leq \epsilon / 2$
PROOF: Let $n \to \infty$ in ⟨2⟩2.

□

Corollary 7.3.17.1. *For any normed space V over K, the space $\mathcal{B}(V, K)$ is a Banach space.*

Theorem 7.3.18. *Let U be a normed space and V a Banach space. Let W be a subspace of U. Let $T : W \to V$ be a continuous linear transformation. Then T has a unique extension to a continuous linear transformation $\mathrm{cl}\, W \to V$.*

PROOF:
⟨1⟩1. Define $S : \mathrm{cl}\, W \to V$ by: $S(x) = \lim_{n\to\infty} T(x_n)$, where (x_n) is any sequence in W that converges to x.
⟨2⟩1. For all $x \in \mathrm{cl}\, W$, there exists a sequence (x_n) in W that converges to x.
PROOF: Theorem 7.1.43.
⟨2⟩2. If (x_n) is a Cauchy sequence in W then $(T(x_n))$ is Cauchy in V.
⟨3⟩1. ASSUME: w.l.o.g. $T \neq 0$
⟨3⟩2. LET: (x_n) be a Cauchy sequence in W.
⟨3⟩3. PICK $B > 0$ such that $\forall x \in W. \|T(x)\| \leq B\|x\|$
⟨3⟩4. LET: $\epsilon > 0$
⟨3⟩5. PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon / \|T\|$
⟨3⟩6. LET: $m, n \geq N$
⟨3⟩7. $\|T(x_m) - T(x_n)\| < \epsilon$
⟨2⟩3. If (x_n) and (y_n) are sequences in W that converge to the same element in $\mathrm{cl}\, W$ then $(T(x_n))$ and $(T(y_n))$ have the same limit in V.
⟨3⟩1. ASSUME: w.l.o.g. $T \neq 0$
⟨3⟩2. ASSUME: $x_n \to l$ and $y_n \to l$ as $n \to \infty$
⟨3⟩3. LET: $T(x_n) \to a$ and $T(y_n) \to b$ as $n \to \infty$
⟨3⟩4. ASSUME: for a contradiction $a \neq b$
⟨3⟩5. LET: $\epsilon = \|a - b\|$
⟨3⟩6. PICK N such that $\forall n \geq N. \|x_n - l\| < \epsilon/3\|T\|$ and $\forall n \geq N. \|y_n - l\| < \epsilon/3\|T\|$
⟨3⟩7. $\forall n \geq N. \|T(x_n) - T(y_n)\| < 2\epsilon/3$
⟨3⟩8. $\|a - b\| \leq 2\epsilon/3$
⟨3⟩9. This contradicts ⟨3⟩5.
⟨1⟩2. S extends T
⟨2⟩1. LET: $w \in W$
⟨2⟩2. $w \to w$ as $n \to \infty$
⟨2⟩3. $T(w) \to T(w)$ as $n \to \infty$
⟨2⟩4. $S(w) = T(w)$
⟨1⟩3. S is bounded.
⟨2⟩1. LET: $x \in \mathrm{cl}\, W$
PROVE: $\|S(x)\| \leq \|T\|\|x\|$
⟨2⟩2. PICK a sequence (x_n) in W that converges to x.
⟨2⟩3. $\|T(x_n)\| \leq \|T\|\|x_n\|$ for all n.
⟨2⟩4. $\|S(x)\| \leq \|T\|\|x\|$
PROOF: Taking the limit as $n \to \infty$.
⟨1⟩4. S is linear.

⟨2⟩1. LET: $x, y \in \operatorname{cl} W$ and $\alpha, \beta \in K$
⟨2⟩2. PICK sequences (x_n) and (y_n) in W that converge to x and y.
⟨2⟩3. $T(\alpha x_n + \beta y_n) = \alpha T(x_n) + \beta T(y_n)$ for all n.
⟨2⟩4. $S(\alpha x + \beta y) = \alpha S(x) + \beta S(y)$
PROOF: Taking the limit as $n \to \infty$.
⟨1⟩5. S is unique.
⟨2⟩1. LET: S' be a continuous linear extension of S defined on $\operatorname{cl} W$.
⟨2⟩2. LET: $x \in W$
PROVE: $S(x) = S'(x)$
⟨2⟩3. PICK a sequence (x_n) in W that converges to x.
⟨2⟩4. $T(x_n) = S'(x_n) \to S'(x)$ as $n \to \infty$
⟨2⟩5. $S'(x) = S(x)$
□

Corollary 7.3.18.1. *Let U be a normed space and V a Banach space. Let W be a dense subspace of U. Let $T : W \to V$ be a continuous linear transformation. Then T has a unique extension to a continuous linear transformation $U \to V$.*

Definition 7.3.19 (Functional)**.** Let V be a normed space over K. A *functional* on V is a bounded linear mapping $V \to K$. The *dual space* of V is the space $\mathcal{B}(V, K)$ of all functionals.

Theorem 7.3.20 (Banach-Steinhaus Theorem)**.** *Let $\mathcal{T}$ be a family of bounded linear mappings from a Banach space X into a normed space Y. If, for every $x \in X$, there exists a constant M_x such that $\forall T \in \mathcal{T}. \|T(x)\| \leq M_x$, then there exists a constant $M > 0$ such that $\forall T \in \mathcal{T}. \|T\| \leq M$.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction no such M exists.
⟨1⟩2. For $n \in \mathbb{Z}_+$, PICK $T_n \in \mathcal{T}$ such that $\|T_n\| > n2^n$.
⟨1⟩3. For $n \in \mathbb{Z}_+$, PICK $x_n \in X$ such that $\|x_n\| = 1$ and $\|T_n(x_n)\| > n2^n$.
⟨1⟩4. For $n \in \mathbb{Z}_+$,

$$\left\| \frac{1}{n} T_n \left(\frac{x_n}{2^n} \right) \right\| > 1 \ .$$

⟨1⟩5. For $i, j \in \mathbb{Z}_+$,
LET: $y_{ij} = \frac{1}{i} T_i(\frac{x_j}{2^j})$.
⟨1⟩6. For all $j \in \mathbb{Z}_+$, $y_{ij} \to 0$ as $i \to \infty$
⟨2⟩1. LET: $j \in \mathbb{Z}_+$
⟨2⟩2. PICK M such that $\forall T \in \mathcal{T}. \|T(x_j/2^j)\| \leq M$
⟨2⟩3. For all i, $\|y_{ij}\| \leq M/i$
⟨1⟩7. For any increasing sequence of positive integers (p_i), we have $\sum_{j=1}^\infty y_{p_i p_j} \to 0$ as $i \to \infty$
⟨2⟩1. LET: (p_i) be an increasing sequence of positive integers.
⟨2⟩2. LET: $z = \sum_{j=1}^\infty x_{p_j}/2^{p_j}$
PROOF: This converges by Theorem 7.3.13.
⟨2⟩3. PICK C such that $\forall T \in \mathcal{T}. \|T(z)\| \leq C$
⟨2⟩4. For all i, $\| \sum_{j=1}^\infty y_{p_i p_j} \| \leq C/p_i$.

PROOF:
$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} y_{p_i p_j} \right\| &= \left\| \sum_{j=1}^{\infty} \frac{1}{p_i} T_{p_i} \left(\frac{x_{p_j}}{2^{p_j}} \right) \right\| && (\langle 1 \rangle 5) \\
&= \frac{1}{p_i} \left\| T_{p_i} \left(\sum_{j=1}^{\infty} \frac{x_{p_j}}{2^{p_j}} \right) \right\| && (T_{p_i} \text{ continuous}) \\
&= \frac{1}{p_i} \| T_{p_i}(z) \| && (\langle 2 \rangle 2) \\
&\leq \frac{C}{p_i} && (\langle 2 \rangle 3)
\end{aligned}$$

$\langle 2 \rangle 5$. $\sum_{j=1}^{\infty} y_{p_i p_j} \to 0$ as $i \to \infty$

$\langle 1 \rangle 8$. $y_{ii} \to 0$ as $i \to \infty$

PROOF: Diagonal Theorem, $\langle 1 \rangle 6$, $\langle 1 \rangle 7$.

$\langle 1 \rangle 9$. Q.E.D.

PROOF: $\langle 1 \rangle 4$ and $\langle 1 \rangle 8$ form a contradiction.

□

7.4 Contraction Mappings

Definition 7.4.1 (Contraction Mapping)**.** Let E be a normed space over K. Let $A \subseteq E$. A function $f : A \to E$ is a *contraction (mapping)* iff there exists a real α such that $0 < \alpha < 1$ and

$$\forall x, y \in A. \| f(x) - f(y) \| \leq \alpha \| x - y \| \ .$$

Proposition 7.4.2. *Contraction mappings are uniformly continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: E be a normed space over K.

$\langle 1 \rangle 2$. LET: $A \subseteq E$

$\langle 1 \rangle 3$. LET: $f : A \to E$ be a contraction mapping.

$\langle 1 \rangle 4$. PICK α such that $0 < \alpha < 1$ and $\forall x, y \in A. \| f(x) - f(y) \| \leq \alpha \| x - y \|$.

$\langle 1 \rangle 5$. LET: $\epsilon > 0$

$\langle 1 \rangle 6$. LET: $\delta = \epsilon / \alpha$

$\langle 1 \rangle 7$. For all $x, y \in A$, if $\| x - y \| < \delta$ then $\| f(x) - f(y) \| < \epsilon$.

□

Theorem 7.4.3 (Banach Fixed Point Theorem)**.** *Let E be a Banach space over K. Let F be a nonempty closed subset of E. Let $f : F \to F$ be a contraction mapping. Then there exists a unique $z \in F$ such that $f(z) = z$.*

PROOF:

$\langle 1 \rangle 1$. PICK α such that $0 < \alpha < 1$ and
$$\forall x, y \in F. \| f(x) - f(y) \| \leq \alpha \| x - y \| \ .$$

$\langle 1 \rangle 2$. PICK $x_0 \in F$

$\langle 1\rangle 3.$ For $n \in \mathbb{Z}_+$,
LET: $x_n = f^n(x_0)$.
$\langle 1\rangle 4.$ (x_n) is a Cauchy sequence.
$\langle 2\rangle 1.$ For all $n \in \mathbb{Z}_+$ we have $\|x_{n+1} - x_n\| \leq \alpha^n \|x_1 - x_0\|$.
$\langle 2\rangle 2.$ For all $m, n \in \mathbb{Z}_+$ with $m < n$ we have $\|x_n - x_m\| < \alpha^m \|x_1 - x_0\|/(1-\alpha)$.
PROOF:

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\| && \text{(Triangle inequality)} \\
&\leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m)\|x_1 - x_0\| && (\langle 2\rangle 1) \\
&< \frac{\|x_1 - x_0\|}{1-\alpha}\alpha^m
\end{aligned}$$

$\langle 2\rangle 3.$ LET: $\epsilon > 0$
$\langle 2\rangle 4.$ PICK N such that $\alpha^N \|x_1 - x_0\|/(1-\alpha) < \epsilon$
$\langle 2\rangle 5.$ For all $m, n \geq N$, we have $\|x_n - x_m\| < \epsilon$
$\langle 1\rangle 5.$ LET: $z = \lim_{n\to\infty} x_n$
$\langle 1\rangle 6.$ $f(z) = z$
PROOF:

$$\begin{aligned}
f(z) &= f\left(\lim_{n\to\infty} x_n\right) \\
&= \lim_{n\to\infty} f(x_n) && \text{(Proposition 7.4.2)} \\
&= \lim_{n\to\infty} x_{n+1} \\
&= z
\end{aligned}$$

$\langle 1\rangle 7.$ For any $w \in F$, if $f(w) = w$ then $w = z$.
$\langle 2\rangle 1.$ LET: $w \in F$
$\langle 2\rangle 2.$ ASSUME: $f(w) = w$
$\langle 2\rangle 3.$ $\|z - w\| \leq \alpha\|z - w\|$
PROOF: $\|z - w\| = \|f(z) - f(w)\| \leq \alpha\|z - w\|$
$\langle 2\rangle 4.$ $\|z - w\| = 0$
$\langle 2\rangle 5.$ $z = w$
$\square$

Chapter 8

Inner Product Spaces

Definition 8.0.1 (Inner Product Space)**.** Let E be a complex vector space. An *inner product* on E is a function $\langle \; , \; \rangle : E^2 \to \mathbb{C}$ such that, for all $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$, we have:

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
3. $\langle x, x \rangle \geq 0$
4. If $\langle x, x \rangle = 0$ then $x = 0$

An *inner product space* consists of a complex vector space V and an inner product on V.

Proposition 8.0.2. *Let E be an inner product space. For any $x \in E$, we have $\langle x, x \rangle$ is real.*

PROOF: Since $\langle x, x \rangle = \overline{\langle x, x \rangle}$. $\square$

Proposition 8.0.3.

$$\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle$$

Proposition 8.0.4.

$$\langle 0, y \rangle = \langle x, 0 \rangle = 0$$

Proposition 8.0.5. *The function $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$ is an inner product on $\mathbb{C}^n$.*

Proposition 8.0.6. *The function $\langle (x_n), (y_n) \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$ is an inner product on l^2.*

Proposition 8.0.7. *The function $\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$ is an inner product on $\mathcal{C}([a, b])$.*

Proposition 8.0.8. *Let $p > 1$ and $\Omega \subseteq \mathbb{R}^N$. Let $L^p(\Omega)$ be the set of all functions $f : \Omega \to \mathbb{C}$ such that $|f|^p$ is Lebesgue integrable.*

The function $\langle f, g\rangle = \int_\Omega f(x)\overline{g(x)}dx$ is an inner product on $L^2(\Omega)$.

Proposition 8.0.9. *Let E_1 and E_2 be inner product spaces. Then the function $\langle(e_1, e_2), (e_1', e_2')\rangle = \langle e_1, e_1'\rangle + \langle e_2, e_2'\rangle$ is an inner product on $E_1 \times E_2$.*

Definition 8.0.10 (Norm)**.** In an inner product space, define $\|x\| = \sqrt{\langle x, x\rangle}$.

Proposition 8.0.11 (Schwarz's Inequality)**.** *In any inner product space,*

$$|\langle x, y\rangle| \leq \|x\|\|y\| \ .$$

Equality holds iff x and y are linearly dependent.

PROOF:

$\langle 1\rangle$1. ASSUME: w.l.o.g. $y \neq 0$

$\langle 1\rangle$2. $|\langle x, y\rangle| \leq \|x\|\|y\|$

  $\langle 2\rangle$1. For all $\alpha \in \mathbb{C}$ we have $\langle x, x\rangle + \overline{\alpha}\langle x, y\rangle + \alpha\langle y, x\rangle + |\alpha|^2\langle y, y\rangle$

    PROOF: The right-hand side is $\langle x + \alpha y, x + \alpha y\rangle$.

  $\langle 2\rangle$2. $\langle x, x\rangle\langle y, y\rangle - |\langle x, y\rangle|^2 \geq 0$

    PROOF: Taking $\alpha = -\langle x, x\rangle/\langle y, y\rangle$ in $\langle 2\rangle$1.

$\langle 1\rangle$3. If $|\langle x, y\rangle| = \|x\|\|y\|$ then x and y are linearly dependent.

  $\langle 2\rangle$1. ASSUME: $|\langle x, y\rangle| = \|x\|\|y\|$

  $\langle 2\rangle$2. $\langle x, y\rangle\langle y, x\rangle = \langle x, x\rangle\langle y, y\rangle$

  $\langle 2\rangle$3. $\langle y, y\rangle x - \langle x, x\rangle y = 0$

    PROOF:

$$\begin{aligned} \langle\langle y, y\rangle x - \langle x, y\rangle y, \langle y, y\rangle x - \langle x, y\rangle y\rangle &= \langle y, y\rangle^2\langle x, x\rangle - \langle y, y\rangle\langle y, x\rangle\langle x, y\rangle - \langle x, y\rangle\langle y, y\rangle\langle y, x\rangle + \langle x, y\rangle\langle y, x \\ &= 0 \end{aligned}$$

$\langle 1\rangle$4. If x and y are linearly dependent then $|\langle x, y\rangle| = \|x\|\|y\|$

  $\langle 2\rangle$1. ASSUME: x and y are linearly dependent.

  $\langle 2\rangle$2. LET: $y = \alpha x$

  $\langle 2\rangle$3. $|\langle x, y\rangle| = \|x\|\|y\|$

    PROOF:

$$\begin{aligned} |\langle x, y\rangle| &= |\langle x, \alpha x\rangle| \\ &= |\alpha||\langle x, x\rangle| \\ &= |\alpha|\|x\|^2 \\ &= \|x\|\|\alpha x\| \\ &= \|x\|\|y\| \end{aligned}$$

□

Corollary 8.0.11.1 (Triangle Inequality)**.** *In any inner product space,*

$$\|x + y\| \leq \|x\| + \|y\|$$

PROOF:

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &\leq \langle x,x\rangle + 2|\langle x,y\rangle| + \langle y,y\rangle \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 && \text{(Schwarz's Inequality)} \\ &= (\|x\| + \|y\|)^2\end{aligned}$$

□

Corollary 8.0.11.2. *The norm in an inner product space is a norm.*

Theorem 8.0.12 (Parallelogram Law)**.** *In any inner product space,*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

PROOF:
⟨1⟩1. $\|x+y\|^2 = \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2$
⟨1⟩2. $\|x-y\|^2 = \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2$
⟨1⟩3. Q.E.D.
PROOF: Add ⟨1⟩1 and ⟨1⟩2.
□

Proposition 8.0.13. *Let E be a normed space over $\mathbb{C}$. Then there exists an inner product on E that induces the norm of E iff E satisfies the Parallelogram Law.*

PROOF: If E satisfies the parallelogram law, define

$$\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2) \ .$$

Definition 8.0.14 (Orthogonal)**.** Vectors x and y in an inner product space are *orthogonal*, $x \perp y$, iff $\langle x,y\rangle = 0$.

Theorem 8.0.15 (Pythagorean Formula)**.** *If x and y are orthogonal then*

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \ .$$

Definition 8.0.16 (Weak Convergence)**.** Let E be an inner product space. Let (x_n) be a sequence in E and $l \in E$. Then (x_n) *weakly converges* to l, $x_n \xrightarrow{w} l$ as $n \to \infty$, iff $\forall y \in E. \langle x_n, y\rangle \to \langle l, y\rangle$ as $n \to \infty$.

Proposition 8.0.17. *In any inner product space E, the inner product $\langle\ ,\ \rangle : E^2 \to \mathbb{C}$ is continuous.*

PROOF:
⟨1⟩1. LET: $x_n \to x$ and $y_n \to y$ in E.
⟨1⟩2. $\langle x_n, y_n\rangle \to \langle x,y\rangle$
PROOF:

$$\begin{aligned}|\langle x_n,y_n\rangle - \langle x,y\rangle| &\leq |\langle x_n,y_n\rangle - \langle x_n,y\rangle| + |\langle x_n,y\rangle - \langle x,y\rangle| \\ &= |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\ &\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| && \text{(Schwarz's Inequality)} \\ &\to 0\end{aligned}$$

using the fact that (x_n) is bounded.

□

Theorem 8.0.18. *$x_n \to l$ if and only if $x_n \overset{w}{\to} l$ and $\|x_n\| \to \|x\|$.*

PROOF:
⟨1⟩1. If $x_n \to l$ then $x_n \overset{w}{\to} l$ and $\|x_n\| \to \|l\|$.
PROOF: Easy using the fact that the inner product is continuous.
⟨1⟩2. If $x_n \overset{w}{\to} l$ and $\|x_n\| \to \|l\|$ then $x_n \to l$.
⟨2⟩1. ASSUME: $x_n \overset{w}{\to} l$ and $\|x_n\| \to \|l\|$
⟨2⟩2. $\langle x_n, l\rangle \to \|l\|^2$
⟨2⟩3. $\|x_n - l\| \to 0$
PROOF:

$$\begin{aligned}\|x_n - l\|^2 &= \langle x_n - l, x_n - l\rangle \\ &= \langle x_n, x_n\rangle - \langle x_n, l\rangle - \langle l, x_n\rangle + \langle l, l\rangle \\ &= \|x_n\|^2 - \langle x_n, l\rangle - \overline{\langle x_n, l\rangle} + \|l\|^2 \\ &\to \|l\|^2 - 2\|l\|^2 + \|l\|^2 \\ &= 0\end{aligned}$$

□

Theorem 8.0.19. *Let S be a subset of an inner product space E such that $\operatorname{span} S$ is dense in E. If (x_n) is a bounded sequence in E and, for all $y \in S$, we have $\langle x_n, y\rangle \to \langle x, y\rangle$ then $x_n \overset{w}{\to} x$.*

PROOF:
⟨1⟩1. For all $y \in \operatorname{span} S$, we have $\langle x_n, y\rangle \to \langle x, y\rangle$
⟨1⟩2. LET: $z \in E$
PROVE: $\langle x_n, z\rangle \to \langle x, z\rangle$
⟨1⟩3. LET: $\epsilon > 0$
PROVE: There exists n_0 such that $\forall n \geq n_0 . |\langle x_n, z\rangle - \langle x, z\rangle| < \epsilon$
⟨1⟩4. PICK $M > 0$ such that $\|x\| \leq M$ and $\forall n \in \mathbb{Z}_+ . \|x_n\| \leq M$.
⟨1⟩5. PICK $y_0 \in \operatorname{span} S$ such that $\|z - y_0\| < \epsilon/3M$
⟨1⟩6. PICK $n_0 \in \mathbb{Z}_+$ such that, for all $n \geq n_0$, we have $|\langle x_n, y_0\rangle - \langle x, y_0\rangle| < \epsilon/3$
⟨1⟩7. LET: $n \geq n_0$
⟨1⟩8. $|\langle x_n, z\rangle - \langle x, z\rangle| < \epsilon$
PROOF:

$$\begin{aligned}|\langle x_n, z\rangle - \langle x, z\rangle| &\leq |\langle x_n, z\rangle - \langle x_n, y_0\rangle| + |\langle x_n, y_0\rangle - \langle x, y_0\rangle| + |\langle x, y_0\rangle - \langle x, z\rangle| \\ &< \|x_n\|\|z - y_0\| + \epsilon/3 + \|x\|\|y_0 - z\| \\ &< M(\epsilon/3M) + \epsilon/3 + M(\epsilon/3M) \\ &= \epsilon\end{aligned}$$

□

8.1 Orthonormal Bases

Definition 8.1.1 (Orthogonal)**.** Let V be an inner product space and $S \subseteq V$. Then S is *orthogonal* iff any two distinct elements of S are orthogonal.

Definition 8.1.2 (Orthonormal)**.** Let V be an inner product space and $S \subseteq V$. Then S is *orthonormal* iff it is orthogonal and $\forall x \in S. \|x\| = 1$.

Proposition 8.1.3. *Orthonormal sets are linearly independent.*

PROOF:
$\langle 1 \rangle$1. LET: S be orthonormal
$\langle 1 \rangle$2. ASSUME: $\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$ where $e_1, \ldots, e_n \in S$
$\langle 1 \rangle$3. $|\alpha_1|^2 + \cdots |\alpha_n|^2 = 0$
PROOF:

$$\begin{aligned} 0 &= \sum_{m=1}^n \langle 0, \alpha_m e_m \rangle \\ &= \sum_{m=1}^n \langle \sum_{k=1}^n \alpha_k e_k, \alpha_m e_m \rangle \\ &= \sum_{m=1}^n \sum_{k=1}^n \alpha_k \overline{\alpha_m} \langle e_k, e_m \rangle \\ &= \sum_{k=1}^n |\alpha_k|^2 \end{aligned}$$

$\langle 1 \rangle$4. $\alpha_1 = \cdots = \alpha_n = 0$
$\square$

Proposition 8.1.4. *In l^2, let e_n be the sequence whose nth element is 1 and whose other elements are 0. Then $\{e_n \mid n \in \mathbb{Z}_+\}$ is orthonormal.*

Proposition 8.1.5. *In $L^2([-\pi, \pi])$, let $\phi_n(x) = e^{inx}/\sqrt{2\pi}$ for $n \in \mathbb{Z}$. Then $\{\phi_n \mid n \in \mathbb{Z}\}$ is orthonormal.*

Definition 8.1.6 (Legendre Polynomials)**.** The *Legendre polynomials* $P_n \in \mathbb{Q}[x]$ for $n \in \mathbb{N}$ are defined by

$$\begin{aligned} P_0 &= 1 \\ P_n &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \end{aligned}$$

Proposition 8.1.7. *Let P_n be the nth Legendre polynomial. Then $\{P_n \mid n \in \mathbb{N}\}$ is orthogonal in $L^2([-1, 1])$.*

Definition 8.1.8 (Hermite Polynomial)**.** The *Hermite polynomials* $H_n \in \mathbb{R}[x]$ for $n \in \mathbb{N}$ are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad .$$

Proposition 8.1.9. *Let H_n be the nth Hermite polynomial. Then $\{e^{-x^2/2} H_n(x) \mid n \in \mathbb{N}\}$ is orthogonal in $L^2(\mathbb{R})$.*

Theorem 8.1.10. *Let V be an inner product space. If $x_1, \ldots, x_n \in V$ are orthogonal then*

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2 \ .$$

Theorem 8.1.11 (Bessel's Equality)**.** *Let V be an inner product space. Let $x_1, \ldots, x_n \in V$ be orthonormal. Let $x \in V$. Then*

$$\left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \ .$$

Proof:

$$\begin{aligned}
\left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 &= \left\langle x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k, x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\rangle \\
&= \langle x, x \rangle - \left\langle x, \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\rangle - \left\langle \sum_{k=1}^{n} \langle x, x_k \rangle x_k, x \right\rangle \\
&+ \left\langle \sum_{k=1}^{n} \langle x, x_k \rangle x_k, \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\rangle \\
&= \langle x, x \rangle - 2 \sum_{k=1}^{n} \langle x, x_k \rangle \langle x_k, x \rangle + \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, x_i \rangle \langle x_j, x \rangle \langle x_i, x_j \rangle \\
&= \|x\|^2 - 2 \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 + \sum_{i=1}^{n} \langle x, x_i \rangle \langle x_i, x \rangle \\
&= \|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2
\end{aligned}$$

□

Corollary 8.1.11.1 (Bessel's Inequality)**.** *Let V be an inner product space. Let $x_1, \ldots, x_n \in V$ be orthonormal. Let $x \in E$. Then*

$$\sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \leq \|x\|^2 \ .$$

Corollary 8.1.11.2. *Orthonormal sequences are weakly convergent to 0.*

Proof: Let (x_n) be an orthonormal sequence. Taking the limit in Bessel's inequality we have $\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \leq \|x\|^2 < \infty$ and so $\langle x, x_k \rangle \to 0$ as $k \to \infty$. □

Corollary 8.1.11.3 (Generalized Fourier Series)**.** *Let V be an inner product space. Let (e_n) be an orthonormal sequence in V. For any $x \in V$, the* generalized Fourier series *of x is*

$$\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \ ,$$

and $\langle x, e_n \rangle$ *is called the* n*th* generalized Fourier coefficient *of* x *with respect to* (e_n)*. We have* $(\langle x, e_n \rangle e_n)_n \in l^2$*.*

Definition 8.1.12 (Complete Orthonormal Sequence)**.** Let E be an inner product space. Let (x_n) be an orthonormal sequence in E. Then (x_n) is *complete* iff, for all $x \in E$, we have

$$\sum_{n=1}^{\infty} \langle x, x_n \rangle x_n = x \quad .$$

Chapter 9

Hilbert Spaces

Definition 9.0.1 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

Proposition 9.0.2. *For $n \in \mathbb{N}$, $\mathbb{C}^n$ is a Hilbert space.*

Proposition 9.0.3. *l^2 is a Hilbert space.*

Proposition 9.0.4. *$L^2(\mathbb{R})$ is a Hilbert space.*

Proposition 9.0.5. *$L^2([a,b])$ is a Hilbert space.*

Proposition 9.0.6. *Let ρ be a measurable function on $[a,b]$ such that $\rho(x) > 0$ almost everywhere. Let $L^{2\rho}([a,b])$ be the set of all measurable functions $f : [a,b] \to \mathbb{C}$ such that*

$$\int_a^b |f(x)|^2 \rho(x) dx < \infty \ .$$

Define an inner product on $L^{2\rho}([a,b])$ by

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\rho(x) dx \ .$$

Then $L^{2\rho}([a,b])$ is a Hilbert space.

Proposition 9.0.7. *Let m and N be positive integers. Let Ω be an open set in $\mathbb{R}^N$. Let $\tilde{H}^m(\Omega)$ be the set of all $f \in \mathcal{C}^m(\Omega)$ such that, for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$ with $|\alpha| := \alpha_1 + \cdots + \alpha_N \leq m$, we have*

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} \in L^2(\Omega) \ .$$

Define an inner product on $\tilde{H}^m(\Omega)$ by

$$\langle f, g \rangle := \int_\Omega \sum_\alpha D^\alpha f \overline{D^\alpha g} \ .$$

Let $H^m(\Omega)$ be the completion of $\tilde{H}^m(\Omega)$. Then $H^m(\Omega)$ is a Hilbert space.

Theorem 9.0.8. *Weakly convergent sequences in a Hilbert space are bounded.*

PROOF:
$\langle 1\rangle 1$. LET: H be a Hilbert space.
$\langle 1\rangle 2$. LET: (x_n) be a weakly convergent sequence in H.
$\langle 1\rangle 3$. For $n \in \mathbb{Z}_+$,
LET: $f_n : H \to \mathbb{C}$, $f_n(x) = \langle x, x_n\rangle$
$\langle 1\rangle 4$. For $n \in \mathbb{Z}_+$, f_n is a bounded linear functional.
$\langle 1\rangle 5$. For every $x \in H$, the sequence $(f_n(x))$ is bounded.
PROOF: Since it converges.
$\langle 1\rangle 6$. PICK $M > 0$ such that, for all $n \in \mathbb{Z}_+$, we have $\|f_n\| \leq M$.
PROOF: Banach-Steinhaus Theorem, $\langle 1\rangle 4$, $\langle 1\rangle 5$.
$\langle 1\rangle 7$. $\forall n \in \mathbb{Z}_+ . \|f_n\| = \|x_n\|$
$\langle 2\rangle 1$. LET: $n \in \mathbb{Z}_+$
$\langle 2\rangle 2$. $\|f_n\| \leq \|x_n\|$
PROOF: Since for all $x \in H$ we have $|f_n(x)| = |\langle x, x_n\rangle| \leq \|x\|\|x_n\|$ by Schwarz's Inequality.
$\langle 2\rangle 3$. $\|x_n\| \leq \|f_n\|$
PROOF: Since $\|x_n\|^2 = |\langle x_n, x_n\rangle| = |f_n(x_n)| \leq \|f_n\|\|x_n\|$.
$\langle 1\rangle 8$. $\forall n \in \mathbb{Z}_+ . \|x_n\| \leq M$
PROOF: $\langle 1\rangle 6$, $\langle 1\rangle 7$
□

Theorem 9.0.9. *Let H be a Hilbert space. Let (x_n) be an orthonormal sequence in H and let (α_n) be a sequence of complex numbers. Then the series $\sum_{n=1}^\infty \alpha_n x_n$ converges in H if and only if $\sum_{n=1}^\infty |\alpha_n|$ converges in $\mathbb{R}$, in which case*

$$\left\| \sum_{n=1}^\infty \alpha_n x_n \right\|^2 = \sum_{n=1}^\infty |\alpha_n|^2 \enspace .$$

PROOF:
$\langle 1\rangle 1$. For $m > k > 0$ we have

$$\left\| \sum_{n=k}^m \alpha_n x_n \right\|^2 = \sum_{n=k}^m |\alpha_n|^2 \enspace .$$

PROOF: Theorem 8.1.10.
$\langle 1\rangle 2$. If $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$ then $\sum_{n=1}^\infty \alpha_n x_n$ converges.
$\langle 2\rangle 1$. ASSUME: $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$
$\langle 2\rangle 2$. $(\sum_{n=1}^m \alpha_n x_n)_m$ is Cauchy.
PROOF: From $\langle 1\rangle 1$.
$\langle 2\rangle 3$. $\sum_{n=1}^\infty \alpha_n x_n$ converges.
$\langle 1\rangle 3$. If $\sum_{n=1}^\infty \alpha_n x_n$ converges then $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$.
PROOF: From $\langle 1\rangle 1$.
$\langle 1\rangle 4$. If $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$ then

$$\left\| \sum_{n=1}^\infty \alpha_n x_n \right\|^2 = \sum_{n=1}^\infty |\alpha_n|^2 \enspace .$$

PROOF: From ⟨1⟩1.

∎

Proposition 9.0.10. *Every complete orthonormal sequence in a Hilbert space is a basis.*

PROOF:
⟨1⟩1. LET: E be an inner product space.
⟨1⟩2. LET: (e_n) be a complete orthonormal sequence in E.
⟨1⟩3. For all $x \in E$, there exists a sequence (α_n) in $\mathbb{C}$ such that $x = \sum_n \alpha_n e_n$.
PROOF: Immediate from ⟨1⟩2.
⟨1⟩4. If $\sum_n \alpha_n e_n = \sum_n \beta_n e_n$ then $\alpha_n = \beta_n$ for all n.
⟨2⟩1. LET: $x = \sum_n \alpha_n e_n = \sum_n \beta_n e_n$
⟨2⟩2. $\sum_n |\alpha_n - \beta_n|^2 = 0$
PROOF:

$$\begin{aligned} 0 &= \|x - x\|^2 \\ &= \left\| \sum_{n=1}^{\infty} \alpha_n e_n - \sum_{n=1}^{\infty} \beta_n e_n \right\|^2 && (\langle 2\rangle 1) \\ &= \left\| \sum_{n=1}^{\infty} (\alpha_n - \beta_n) e_n \right\|^2 \\ &= \sum_{n=1}^{\infty} |\alpha_n - \beta_n|^2 && (\text{Theorem 9.0.9}) \end{aligned}$$

⟨2⟩3. $\alpha_n = \beta_n$ for all n.

∎

Theorem 9.0.11. *An orthonormal sequence (x_n) in a Hilbert space H is complete if and only if, for all $x \in H$, if $\forall n. \langle x, x_n \rangle = 0$ then $x = 0$.*

PROOF:
⟨1⟩1. If (x_n) is complete then, for all $x \in H$, if $\forall n. \langle x, x_n \rangle = 0$ then $x = 0$.
⟨2⟩1. ASSUME: (x_n) is complete.
⟨2⟩2. LET: $x \in H$
⟨2⟩3. ASSUME: $\forall n. \langle x, x_n \rangle = 0$
⟨2⟩4. $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n = 0$
⟨1⟩2. If, for all $x \in H$, if $\forall n. \langle x, x_n \rangle = 0$ then $x = 0$, then (x_n) is complete.
⟨2⟩1. ASSUME: For all $x \in H$, if $\forall n. \langle x, x_n \rangle = 0$, then $x = 0$.
⟨2⟩2. LET: $y = x - \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$
⟨2⟩3. For all n, $\langle y, x_n \rangle = 0$
⟨3⟩1. LET: $n \in \mathbb{Z}_+$
⟨3⟩2. $\langle y, x_n \rangle = 0$

PROOF:

$$\begin{aligned}\langle y, x_n\rangle &= \left\langle x - \sum_{m=1}^{\infty} \langle x, x_m\rangle x_m, x_n \right\rangle \\ &= \langle x, x_n\rangle - \sum_{m=1}^{\infty} \langle x, x_m\rangle\langle x_m, x_n\rangle \\ &= \langle x, x_n\rangle - \langle x, x_n\rangle \\ &= 0\end{aligned}$$

⟨2⟩4. $y = 0$
⟨2⟩5. $x = \sum_{n=1}^{\infty} \langle x, x_n\rangle x_n$
□

Theorem 9.0.12 (Parseval's Formula)**.** *Let H be a Hilbert space. Let (x_n) be an orthonormal sequence in H. Then (x_n) is complete if and only if, for all $x \in H$,*

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \quad .$$

PROOF:
⟨1⟩1. If (x_n) is complete then for all $x \in H$ we have $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2$.
⟨2⟩1. ASSUME: (x_n) is complete.
⟨2⟩2. LET: $x \in H$
⟨2⟩3. $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2$
PROOF:

$$\begin{aligned}\|x\|^2 &= \left\| \sum_{n=1}^{\infty} \langle x, x_n\rangle x_n \right\|^2 && (\langle 2\rangle 1) \\ &= \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 && (\text{Theorem 9.0.9})\end{aligned}$$

⟨1⟩2. If, for all $x \in H$, we have $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2$, then (x_n) is complete.
⟨2⟩1. ASSUME: For all $x \in H$, we have $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2$
⟨2⟩2. LET: $x \in H$
⟨2⟩3. $x = \sum_{n=1}^{\infty} \langle x, x_n\rangle x_n$
□

Proposition 9.0.13. *For $n \in \mathbb{Z}$, let $\pi_n(x) = e^{inx}/\sqrt{2\pi}$. Then $\{\pi_n \mid n \in \mathbb{Z}\}$ is a complete orthonormal set in $L^2([-\pi, \pi])$.*

TODO

Proposition 9.0.14. $B = \{1/\sqrt{2\pi}\} \cup \{\cos nx/\sqrt{\pi} \mid n \in \mathbb{Z}_+\} \cup \{\sin nx/\sqrt{\pi} \mid n \in \mathbb{Z}_+\}$ *is a complete orthonormal set in $L^2([-\pi, \pi])$.*

PROOF:
⟨1⟩1. For all $f \in B$ we have $\|f\| = 1$
⟨2⟩1. $\|1/\sqrt{2\pi}\| = 1$

PROOF:

$$\begin{aligned}\|1/\sqrt{2\pi}\| &= \int_{-\pi}^{\pi} dx/2\pi \\ &= 1\end{aligned}$$

$\langle 2\rangle 2$. For all $n \in \mathbb{Z}_+$ we have $\|\cos nx/\sqrt{\pi}\| = 1$

PROOF:

$$\begin{aligned}\|\cos nx/\sqrt{\pi}\| &= 1/\pi \int_{-\pi}^{\pi} \cos^2 nx\, dx \\ &= 1/2\pi \int_{-\pi}^{\pi} (\cos 2nx + 1) dx \\ &= 1/2\pi\, [1/2n \sin 2nx + x]_{-\pi}^{\pi} \\ &= (1/2\pi)(2\pi) \\ &= 1\end{aligned}$$

$\langle 2\rangle 3$. For all $n \in \mathbb{Z}_+$ we have $\|\sin nx/\sqrt{\pi}\| = 1$

PROOF:

$$\begin{aligned}\|\sin nx/\sqrt{\pi}\| &= 1/\pi \int_{-\pi}^{\pi} \sin^2 nx\, dx \\ &= -1/2\pi \int_{-\pi}^{\pi} (\cos 2nx - 1) dx \\ &= -1/2\pi\, [1/2n \sin 2nx - x]_{-\pi}^{\pi} \\ &= (-1/2\pi)(-2\pi) \\ &= 1\end{aligned}$$

$\langle 1\rangle 2$. For all $f, g \in B$ with $f \neq g$ we have $\langle f, g\rangle = 0$

$\langle 2\rangle 1$. $\langle 1, \cos nx\rangle = 0$

PROOF:

$$\begin{aligned}\int_{-\pi}^{\pi} \cos nx\, dx &= [1/n \sin nx]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

$\langle 2\rangle 2$. $\langle 1, \sin nx\rangle = 0$

PROOF:

$$\begin{aligned}\int_{-\pi}^{\pi} \sin nx\, dx &= [-1/n \cos nx]_{-\pi}^{\pi} \\ &= -1/n \cos n\pi + 1/n \cos n\pi \\ &= 0\end{aligned}$$

$\langle 2\rangle 3$. If $m \neq n$ then $\langle \cos mx, \cos nx\rangle = 0$

PROOF:

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx \cos nx\, dx &= 1/2 \int_{-\pi}^{\pi} (\cos(n+m)x - \cos(n-m)x) dx \\ &= 1/2 \left[\frac{1}{n+m} \sin(n+m)x - \frac{1}{n-m} \sin(n-m)x\right]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

$\langle 2\rangle 4$. $\langle \cos mx, \sin nx\rangle = 0$

PROOF:
$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 1/2 \int_{-\pi}^{\pi} (\sin(n+m)x - \sin(n-m)x) dx$$
$$= 1/2 \left[-\frac{1}{n+m} \cos(n+m)x + \frac{1}{n-m} \cos(n-m)x \right]_{-\pi}^{\pi}$$
$$= 0 \qquad (\cos \text{ is odd})$$

⟨2⟩5. If $m \neq n$ then $\langle \sin mx, \sin nx \rangle = 0$

PROOF:
$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 1/2 \int_{-\pi}^{\pi} (\cos(n-m)x - \cos(n+m)x) dx$$
$$= 1/2 \left[\frac{1}{n-m} \sin(n-m)x - \frac{1}{n+m} \sin(n+m)x \right]_{-\pi}^{\pi}$$
$$= 0$$

⟨1⟩3. For all $f \in L^2([-\pi, \pi])$, if $\forall g \in B. \langle f, g \rangle = 0$ then $f = 0$

⟨2⟩1. LET: $f \in L^2([-\pi, \pi])$

⟨2⟩2. ASSUME: $\forall g \in B. \langle f, g \rangle = 0$

⟨2⟩3. For all $n \in \mathbb{Z}$, $\langle f, e^{inx} \rangle = 0$

PROOF: Since $e^{inx} = \cos nx + i \sin nx$.

⟨2⟩4. $f = 0$

PROOF: From Proposition 9.0.13.

□

Proposition 9.0.15. $\{\frac{1}{\sqrt{\pi}}\} \cup \{\sqrt{\frac{2}{\pi}} \cos nx \mid n \in \mathbb{Z}_+\}$ *is a complete orthonormal set in* $L^2([0, \pi])$.

Proposition 9.0.16. $\{\sqrt{\frac{2}{\pi}} \sin nx \mid n \in \mathbb{Z}_+\}$ *is a complete orthonormal set in* $L^2([0, \pi])$.

Definition 9.0.17 (Signum)**.** The *signum* function $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathrm{sgn}\, x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Definition 9.0.18 (Rademacher Functions)**.** The *Rademarcher functions* $R : \mathbb{N} \times [0, 1] \to \{-1, 0, 1\}$ are defined by

$$R(m, x) = \mathrm{sgn}(\sin(2^m \pi x)) \ .$$

Proposition 9.0.19. *The Rademacher functios* $\{R(m, -) \mid m \in \mathbb{N}\}$ *are orthonormal in* $L^2([0, 1])$.

PROOF:

⟨1⟩1. $\forall m \in \mathbb{N}. \|R(m, -)\| = 1$

PROOF: $\int_0^1 \mathrm{sgn}(\sin(2^m \pi x))^2 dx = 1$ since the integrand is 1 except for finitely many points in $[0, 1]$.

⟨1⟩2. Given natural numbers $m \neq n$, we have $\langle R(m,-), R(n,-)\rangle = 0$

⟨2⟩1. Given reals a, b and a natural number m, we have $\int_a^b R(m,x)dx = 0$ whenever $2^m(b-a)$ is an even integer.

Proof: If $m > 0$, or if $m = 0$ and $b - a$ is an even integer, then the regions where $R(m,x) = 1$ are isometric with the regions where $R(m,x) = -1$.

⟨2⟩2. Let: m and n be natural numbers with $n < m$.

⟨2⟩3. $\langle R(m,-), R(n,-)\rangle = 0$

Proof:

$$\begin{aligned}\int_0^1 R(m,x)R(n,x)dx &= \sum_{k=1}^{2^n} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} R(m,x)R(n,x)dx \\ &= \sum_{k=1}^{2^n} (-i)^{k+1} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} R(m,x)dx \\ &= 0 && \left(\langle 2\rangle 1, 2^m\left(\frac{k}{2^n} - \frac{k-1}{2^n}\right) = 2^{m-n} \text{ is an even integer}\right)\end{aligned}$$

□

Proposition 9.0.20. *The set of Rademacher functions is not complete.*

Proof:

⟨1⟩1. Define $f : [0,1] \to \mathbb{C}$ by $f(x) = 0$ if $0 \leq x < 1/4$, $f(x) = 1$ if $1/4 \leq x \leq 3/4$, $f(x) = 0$ if $3/4 < x \leq 1$.

⟨1⟩2. $f \in L^2([0,1])$

⟨1⟩3. $\langle R(0,-), f\rangle = 1/2$

⟨1⟩4. $\langle R(m,-), f\rangle = 0$ for $m \geq 1$

⟨1⟩5. $f \neq 1/2R(0,-)$

□

Definition 9.0.21 (Walsh Functions)**.** Define the *Walsh functions* $W : \mathbb{N} \times [0,1] \to \{-1,0,1\}$ as follows. Given $m \in \mathbb{N}$, let $m = \sum_{k=1}^n 2^{k-1}a_k$ where each a_k is either 0 or 1. Then

$$W(m,x) = \prod_{k=1}^n R(k,x)^{a_k} \quad .$$

Proposition 9.0.22. *The set of Walsh functions* $\{W(m,-) \mid m \in \mathbb{N}\}$ *is a compete orthonormal set.*

TODO