

Mathematics

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Chapter 1

Sets and Classes

1.1 Classes

Our language is the language of first-order logic with equality over one primitive binary predicate \in . We call all the objects we reason about *sets*. When $a \in b$, we say a is a *member* or *element* of b , or b *contains* a . We write $b \ni a$ for $a \in b$, and $a \notin b$ for $\neg(a \in b)$. We write $\forall x \in a. \phi$ as an abbreviation for $\forall x(x \in a \rightarrow \phi)$, and $\exists x \in a. \phi$ as an abbreviation for $\exists x(x \in a \wedge \phi)$.

We shall speak informally of *classes* as an abbreviation for talking about predicates. A *class* is determined by a unary predicate $\phi[x]$ (possibly with parameters). We write $\{x \mid \phi[x]\}$ or $\{x : \phi[x]\}$ for the class determined by $\phi[x]$. We write ' a is an element of $\{x \mid \phi[x]\}$ ' or ' $a \in \{x \mid \phi[x]\}$ ' for $\phi[a]$.

We write $\{t[x_1, \dots, x_n] \mid P[x_1, \dots, x_n]\}$ for

$$\{y \mid \exists x_1, \dots, x_n (y = t[x_1, \dots, x_n] \wedge P[x_1, \dots, x_n])\} .$$

We say two classes \mathbf{A} and \mathbf{B} are *equal*, and write $\mathbf{A} = \mathbf{B}$, iff $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$.

Proposition Schema 1.1.1. *For any class \mathbf{A} , the following is a theorem.*

$$\mathbf{A} = \mathbf{A}$$

PROOF: We have $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{A})$. \square

Proposition Schema 1.1.2. *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem.*

If $\mathbf{A} = \mathbf{B}$ then $\mathbf{B} = \mathbf{A}$.

PROOF: If $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$ then $\forall x(x \in \mathbf{B} \leftrightarrow x \in \mathbf{A})$. \square

Proposition Schema 1.1.3. *For any classes \mathbf{A} , \mathbf{B} and \mathbf{C} , the following is a theorem.*

If $\mathbf{A} = \mathbf{B}$ and $\mathbf{B} = \mathbf{C}$ then $\mathbf{A} = \mathbf{C}$.

PROOF: If $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$ and $\forall x(x \in \mathbf{B} \leftrightarrow x \in \mathbf{C})$ then $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{C})$. \square

1.1.1 Subclasses

Definition 1.1.4 (Subclass). We say a class \mathbf{A} is a *subclass* of \mathbf{B} , or \mathbf{B} is a *superclass* of \mathbf{A} , or \mathbf{B} *includes* \mathbf{A} , and write $\mathbf{A} \subseteq \mathbf{B}$ or $\mathbf{B} \supseteq \mathbf{A}$, iff every element of \mathbf{A} is an element of \mathbf{B} . Otherwise we write $\mathbf{A} \not\subseteq \mathbf{B}$ or $\mathbf{B} \not\supseteq \mathbf{A}$.

We say \mathbf{A} is a *proper* subclass of \mathbf{B} , \mathbf{B} is a *proper* superclass of \mathbf{A} , or \mathbf{B} *properly* includes \mathbf{A} , and write $\mathbf{A} \subsetneq \mathbf{B}$ or $\mathbf{B} \supsetneq \mathbf{A}$, iff $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$.

Proposition Schema 1.1.5. *For any class \mathbf{A} , the following is a theorem.*

$$\mathbf{A} \subseteq \mathbf{A}$$

PROOF: Every element of \mathbf{A} is an element of \mathbf{A} . \square

Proposition Schema 1.1.6. *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem.*

If $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$ then $\mathbf{A} = \mathbf{B}$.

PROOF: If every element of \mathbf{A} is an element of \mathbf{B} , and every element of \mathbf{B} is an element of \mathbf{A} , then \mathbf{A} and \mathbf{B} have exactly the same elements. \square

Proposition Schema 1.1.7. *For any classes \mathbf{A} , \mathbf{B} and \mathbf{C} , the following is a theorem.*

If $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{C}$ then $\mathbf{A} \subseteq \mathbf{C}$.

PROOF: If every element of \mathbf{A} is an element of \mathbf{B} and every element of \mathbf{B} is an element of \mathbf{C} then every element of \mathbf{A} is an element of \mathbf{C} . \square

1.1.2 Constructions of Classes

Definition 1.1.8 (Empty Class). The *empty class* \emptyset is $\{x \mid \perp\}$. Every other class is *nonempty*.

Definition 1.1.9 (Universal Class). The *universal class* \mathbf{V} is $\{x \mid \top\}$.

Definition 1.1.10 (Enumeration). Given objects a_1, \dots, a_n , we define the class $\{a_1, \dots, a_n\}$ to be the class $\{x \mid x = a_1 \vee \dots \vee x = a_n\}$.

Definition 1.1.11 (Intersection). For any classes \mathbf{A} and \mathbf{B} , the *intersection* $\mathbf{A} \cap \mathbf{B}$ is $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$.

Definition 1.1.12 (Union). For any classes \mathbf{A} and \mathbf{B} , the *union* $\mathbf{A} \cup \mathbf{B}$ is $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$.

Definition 1.1.13 (Relative Complement). Let \mathbf{A} and \mathbf{B} be classes. The *relative complement* of \mathbf{B} in \mathbf{A} is the class $\mathbf{A} - \mathbf{B} := \{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

Definition 1.1.14 (Symmetric Difference). For any classes \mathbf{A} and \mathbf{B} , the *symmetric difference* is the class $\mathbf{A} + \mathbf{B} := (\mathbf{A} - \mathbf{B}) \cup (\mathbf{B} - \mathbf{A})$.

Definition 1.1.15 (Pairwise disjoint). Let \mathbf{A} be a class. We say the elements of \mathbf{A} are *pairwise disjoint* iff, for all $x, y \in \mathbf{A}$, if $x \cap y \neq \emptyset$ then $x = y$.

1.2 Sets and the Axiom of Extensionality

Axiom 1.2.1 (Extensionality). *If two sets have exactly the same members, they are equal.*

Thanks to this axiom, we may identify a set a with the class $\{x \mid x \in a\}$. Our use of the symbols \in and $=$ is consistent. We say a class \mathbf{A} *is a set* iff there exists a set a such that $a = \mathbf{A}$; that is, $\{x \mid \phi[x]\}$ is a set iff $\exists a \forall x (x \in a \leftrightarrow \phi[x])$. Otherwise, \mathbf{A} is a *proper class*.

Definition 1.2.2 (Subset). If A is a set and $A \subseteq \mathbf{B}$, we say A is a *subset* of \mathbf{B} .

Definition 1.2.3 (Union). The *union* of a class \mathbf{A} is $\{x \mid \exists X \in \mathbf{A}. x \in X\}$. We write $\bigcup_{P(x)} t(x)$ for $\bigcup \{t(x) \mid P(x)\}$.

Definition 1.2.4 (Intersection). The *intersection* of a class \mathbf{A} is $\{x \mid \forall X \in \mathbf{A}. x \in X\}$. We write $\bigcap_{P(x)} t(x)$ for $\bigcap \{t(x) \mid P(x)\}$.

Definition 1.2.5 (Power Class). For any class \mathbf{A} , the *power class* $\mathcal{P}\mathbf{A}$ is $\{X \mid X \subseteq \mathbf{A}\}$.

1.3 The Other Axioms

Axiom 1.3.1 (Union). *The union of a set is a set.*

Axiom 1.3.2 (Power Set). *The power class of a set is a set, which we call a power set.*

Axiom 1.3.3 (Infinity). *There exists a set I such that:*

- *There exists an element of I that has no members*
- *For every $x \in I$, there exists a set $y \in I$ such that the elements of y are exactly x and the members of x .*

Axiom 1.3.4 (Choice). *For any set A of pairwise disjoint, nonempty sets, there exists a set C such that, for all $x \in A$, we have $x \cap C$ has exactly one element.*

Axiom Schema 1.3.5 (Replacement). *For any predicate $P(x, y)$, the following is an axiom:*

Let A be a set. Assume that, for all $x \in A$, there exists at most one y such that $P(x, y)$. Then $\{y \mid \exists x \in A. P(x, y)\}$ is a set.

Axiom 1.3.6 (Regularity). *For any A , if A has a member, then there exists $m \in A$ such that $m \cap A = \emptyset$.*

1.4 Consequences of the Axioms

Proposition 1.4.1. *The empty class \emptyset is a set.*

PROOF: Immediate from the Axiom of Infinity. \square

Proposition 1.4.2 (Russell's Paradox). *The class $\{x \mid x \notin x\}$ is a proper class.*

PROOF: Assume for a contradiction $R = \{x \mid x \notin x\}$ is a set. Then $R \in R$ if and only if $R \notin R$, which is impossible. \square

Proposition 1.4.3 (Pairing). *For any sets a and b , the class $\{a, b\}$ is a set.*

PROOF:

$\langle 1 \rangle 1$. LET: $P(x, y)$ be the predicate $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$.

$\langle 1 \rangle 2$. For all $x \in \mathcal{P}\mathcal{P}\emptyset$, there exists at most one y such that $P(x, y)$.

$\langle 2 \rangle 1$. LET: $x \in \mathcal{P}\mathcal{P}\emptyset$

$\langle 2 \rangle 2$. LET: y and y' be sets.

$\langle 2 \rangle 3$. ASSUME: $P(x, y)$ and $P(x, y')$

$\langle 2 \rangle 4$. $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$

PROOF: From $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. $(x = \emptyset \wedge y' = a) \vee (x = \mathcal{P}\emptyset \wedge y' = b)$

PROOF: From $\langle 2 \rangle 3$.

$\langle 2 \rangle 6$. $\emptyset \neq \mathcal{P}\emptyset$

PROOF: Since $\emptyset \in \mathcal{P}\emptyset$ and $\emptyset \notin \emptyset$.

$\langle 2 \rangle 7$. $y = y'$

$\langle 1 \rangle 3$. LET: A be the set $\{y \mid \exists x \in \mathcal{P}\mathcal{P}\emptyset. P(x, y)\}$.

$\langle 1 \rangle 4$. $A = \{a, b\}$

\square

Proposition 1.4.4. *The union of two sets is a set.*

PROOF: Because $A \cup B = \bigcup \{A, B\}$. \square

Proposition Schema 1.4.5. *For any number n , the following is a theorem:*

For any sets a_1, \dots, a_n , the class $\{a_1, \dots, a_n\} = \{x \mid x = a_1 \vee \dots \vee x = a_n\}$ is a set.

PROOF: The case $n = 1$ follows from Pairing since $\{a\} = \{a, a\}$.

If we have proved the theorem for n we have $\{a_1, \dots, a_n, a_{n+1}\} = \{a_1, \dots, a_n\} \cup \{a_{n+1}\}$. \square

Proposition Schema 1.4.6 (Comprehension). *For any predicate $P(x)$, the following is a theorem:*

For any set A , the class $\{x \in A \mid P(x)\}$ is a set.

PROOF:

$\langle 1 \rangle 1$. LET: A be a set.

$\langle 1 \rangle 2$. LET: $Q(x, y)$ be the predicate $P(x) \wedge y = x$.

$\langle 1 \rangle 3$. For all $x \in A$, there exists at most one y such that $Q(x, y)$.

⟨2⟩1. LET: $x \in A$
 ⟨2⟩2. LET: y and y' be sets.
 ⟨2⟩3. ASSUME: $Q(x, y)$ and $Q(x, y')$
 ⟨2⟩4. $x \in A \wedge P(x) \wedge y = x \wedge y' = x$
 PROOF: From ⟨2⟩3.
 ⟨2⟩5. $y = y'$
 PROOF: From ⟨2⟩4.
 ⟨1⟩4. LET: B be the set $\{y \mid \exists x \in A. Q(x, y)\}$
 PROOF: This is a set by an Axiom of Replacement and ⟨1⟩3.
 ⟨1⟩5. $B = \{y \in A \mid P(y)\}$
 PROOF:

$$y \in B \Leftrightarrow \exists x \in A. Q(x, y) \quad (\langle 1 \rangle 4)$$

$$\Leftrightarrow \exists x \in A (P(x) \wedge y = x) \quad (\langle 1 \rangle 2)$$

$$\Leftrightarrow P(y)$$

□

Corollary Schema 1.4.6.1. *For any class \mathbf{A} , the following is a theorem:
For any set B , if $\mathbf{A} \subseteq B$ then \mathbf{A} is a set.*

Corollary 1.4.6.1. *The universal class \mathbf{V} is a proper class.*

Corollary 1.4.6.2. *The intersection of a set and a class is a set.*

Corollary 1.4.6.3. *The relative complement of a class in a set is a set.*

Corollary 1.4.6.4. *The symmetric difference of two sets is a set.*

Corollary 1.4.6.5. *The intersection of a nonempty class is a set.*

Proposition Schema 1.4.7. *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem:*

If $\mathbf{A} \subseteq \mathbf{B}$ then $\mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathbf{B}$.

PROOF: Every subset of \mathbf{A} is a subset of \mathbf{B} . □

Proposition Schema 1.4.8. *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem:*

If $\mathbf{A} \subseteq \mathbf{B}$ then $\bigcup \mathbf{A} \subseteq \bigcup \mathbf{B}$.

PROOF: If $x \in X \in \mathbf{A}$ then $x \in X \in \mathbf{B}$. □

Proposition Schema 1.4.9. *For any class \mathbf{A} , the following is a theorem:*

$$\mathbf{A} = \bigcup \mathcal{P}\mathbf{A}$$

PROOF:

⟨1⟩1. $\mathbf{A} \subseteq \bigcup \mathcal{P}\mathbf{A}$

PROOF: For all $x \in \mathbf{A}$ we have $x \in \{x\} \in \mathcal{P}\mathbf{A}$.

⟨1⟩2. $\bigcup \mathcal{P}\mathbf{A} \subseteq \mathbf{A}$

- $\langle 2 \rangle 1.$ LET: $x \in \bigcup \mathcal{P}\mathbf{A}$
- $\langle 2 \rangle 2.$ PICK $X \in \mathcal{P}\mathbf{A}$ such that $x \in X$
- $\langle 2 \rangle 3.$ $X \subseteq \mathbf{A}$
- $\langle 2 \rangle 4.$ $x \in \mathbf{A}$

□

Proposition 1.4.10. *No set is a member of itself.*

PROOF:

- $\langle 1 \rangle 1.$ LET: x be any set.
- $\langle 1 \rangle 2.$ PICK $m \in \{x\}$ such that $m \cap \{x\} = \emptyset$.
- $\langle 1 \rangle 3.$ $m = x$
- $\langle 1 \rangle 4.$ $x \cap \{x\} = \emptyset$
- $\langle 1 \rangle 5.$ $x \notin x$

□

Proposition 1.4.11. *There are no sets a and b such that $a \in b$ and $b \in a$.*

PROOF:

- $\langle 1 \rangle 1.$ LET: a and b be any sets.
- $\langle 1 \rangle 2.$ PICK $m \in \{a, b\}$ such that $m \cap \{a, b\} = \emptyset$
- $\langle 1 \rangle 3.$ CASE: $m = a$
PROOF: Then $b \notin a$.
- $\langle 1 \rangle 4.$ CASE: $m = b$
PROOF: Then $a \notin b$.

□

1.5 Transitive Classes

Definition 1.5.1 (Transitive Class). A class \mathbf{A} is a *transitive class* iff whenever $x \in y \in \mathbf{A}$ then $x \in \mathbf{A}$.

Proposition Schema 1.5.2. *For any class \mathbf{A} , the following is a theorem:
The following are equivalent.*

- 1. \mathbf{A} is a transitive class.
- 2. $\bigcup \mathbf{A} \subseteq \mathbf{A}$
- 3. Every element of \mathbf{A} is a subset of \mathbf{A} .
- 4. $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$

PROOF: Immediate from definitions. □

Proposition Schema 1.5.3. *For any class \mathbf{A} , the following is a theorem:
If \mathbf{A} is a transitive class then $\bigcup \mathbf{A}$ is a transitive class.*

PROOF:

- $\langle 1 \rangle 1.$ ASSUME: \mathbf{A} is a transitive class.

$\langle 1 \rangle 2$. LET: $x \in y \in \bigcup \mathbf{A}$

$\langle 1 \rangle 3$. $y \in \mathbf{A}$

PROOF: Since $\bigcup \mathbf{A} \subseteq \mathbf{A}$ by Proposition 1.5.2.

$\langle 1 \rangle 4$. $x \in \bigcup \mathbf{A}$

□

Proposition Schema 1.5.4. *For any class \mathbf{A} , the following is a theorem:
We have \mathbf{A} is a transitive class if and only if $\mathcal{P}\mathbf{A}$ is a transitive class.*

PROOF:

$\langle 1 \rangle 1$. If \mathbf{A} is a transitive class then $\mathcal{P}\mathbf{A}$ is a transitive class.

$\langle 2 \rangle 1$. ASSUME: \mathbf{A} is a transitive class.

$\langle 2 \rangle 2$. $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$

PROOF: Proposition 1.5.2.

$\langle 2 \rangle 3$. $\mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathcal{P}\mathbf{A}$

PROOF: Proposition 1.4.7.

$\langle 2 \rangle 4$. $\mathcal{P}\mathbf{A}$ is a transitive class.

PROOF: Proposition 1.5.2.

$\langle 1 \rangle 2$. If $\mathcal{P}\mathbf{A}$ is a transitive class then \mathbf{A} is a transitive class.

$\langle 2 \rangle 1$. ASSUME: $\mathcal{P}\mathbf{A}$ is a transitive class.

$\langle 2 \rangle 2$. $\bigcup \mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$

PROOF: Proposition 1.5.2.

$\langle 2 \rangle 3$. $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$

PROOF: Proposition 1.4.9.

$\langle 2 \rangle 4$. \mathbf{A} is a transitive class.

PROOF: Proposition 1.5.2.

□

Proposition Schema 1.5.5. *For any class \mathbf{A} , the following is a theorem:
If every member of \mathbf{A} is a transitive set then $\bigcup \mathbf{A}$ is a transitive class.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: Every member of \mathbf{A} is a transitive set.

$\langle 1 \rangle 2$. LET: $x \in y \in \bigcup \mathbf{A}$

$\langle 1 \rangle 3$. PICK $A \in \mathbf{A}$ such that $y \in A$.

$\langle 1 \rangle 4$. $x \in A$

PROOF: Since A is a transitive set.

$\langle 1 \rangle 5$. $x \in \bigcup \mathbf{A}$

□

Proposition Schema 1.5.6. *For any class \mathbf{A} , the following is a theorem:
If every member of \mathbf{A} is a transitive set then $\bigcap \mathbf{A}$ is a transitive class.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: Every member of \mathbf{A} is a transitive set.

$\langle 1 \rangle 2$. LET: $x \in y \in \bigcap \mathbf{A}$

PROVE: $x \in \bigcap \mathbf{A}$

$\langle 1 \rangle 3$. LET: $A \in \mathbf{A}$

$\langle 1 \rangle 4. y \in A$

$\langle 1 \rangle 5. x \in A$

PROOF: Since A is a transitive set.

□

Chapter 2

Relations

2.1 Ordered Pairs

Definition 2.1.1 (Ordered Pair). For any sets a and b , the *ordered pair* (a, b) is defined to be $\{\{a\}, \{a, b\}\}$.

Theorem 2.1.2. For any sets a, b, c, d , we have $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

PROOF:

$\langle 1 \rangle 1$. If $(a, b) = (c, d)$ then $a = c$ and $b = d$.

$\langle 2 \rangle 1$. ASSUME: $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$

$\langle 2 \rangle 2$. $\bigcap \{\{a\}, \{a, b\}\} = \bigcap \{\{c\}, \{c, d\}\}$

$\langle 2 \rangle 3$. $\{a\} = \{c\}$

$\langle 2 \rangle 4$. $a = c$

$\langle 2 \rangle 5$. $\bigcup \{\{a\}, \{a, b\}\} = \bigcup \{\{c\}, \{c, d\}\}$

$\langle 2 \rangle 6$. $\{a, b\} = \{c, d\}$

$\langle 2 \rangle 7$. $b = c$ or $b = d$

$\langle 2 \rangle 8$. $a = d$ or $b = d$

$\langle 2 \rangle 9$. If $b = c$ and $a = d$ then $b = d$

PROOF: By $\langle 2 \rangle 4$.

$\langle 2 \rangle 10$. $b = d$

PROOF: From $\langle 2 \rangle 7$, $\langle 2 \rangle 8$, $\langle 2 \rangle 9$.

$\langle 1 \rangle 2$. If $a = c$ and $b = d$ then $(a, b) = (c, d)$.

PROOF: First-order logic.

□

Definition 2.1.3 (Cartesian Product). The *Cartesian product* of classes \mathbf{A} and \mathbf{B} is the class $\mathbf{A} \times \mathbf{B} := \{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$.

Proposition 2.1.4. For any sets A and B , the class $A \times B$ is a set.

PROOF: It is a subset of $\mathcal{PP}(A \cup B)$. □

Proposition Schema 2.1.5. *For any classes \mathbf{A} , \mathbf{B} and \mathbf{C} , the following is a theorem:*

$$\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$$

PROOF:

$$\begin{aligned} (x, y) \in \mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) &\Leftrightarrow x \in \mathbf{A} \wedge (y \in \mathbf{B} \vee y \in \mathbf{C}) \\ &\Leftrightarrow (x \in \mathbf{A} \wedge y \in \mathbf{B}) \vee (x \in \mathbf{A} \wedge y \in \mathbf{C}) \\ &\Leftrightarrow (x, y) \in (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C}) \quad \square \end{aligned}$$

Proposition Schema 2.1.6. *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem:*

If $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ and \mathbf{A} is nonempty then $\mathbf{B} = \mathbf{C}$.

PROOF:

- $\langle 1 \rangle 1.$ PICK $a \in \mathbf{A}$
 $\langle 1 \rangle 2.$ For all x we have $x \in \mathbf{B}$ iff $x \in \mathbf{C}$.

PROOF:

$$\begin{aligned} x \in \mathbf{B} &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{B} \\ &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{C} \\ &\Leftrightarrow x \in \mathbf{C} \end{aligned}$$

\square

Proposition Schema 2.1.7. *For any classes \mathbf{A} and \mathbf{B} , the following is a theorem:*

$$\mathbf{A} \times \bigcup \mathbf{B} = \{(a, b) \mid \exists Y \in \mathbf{B}. (a \in \mathbf{A} \wedge b \in Y)\}$$

PROOF:

$$\begin{aligned} (x, y) \in \mathbf{A} \times \bigcup \mathbf{B} &\Leftrightarrow x \in \mathbf{A} \wedge \exists Y \in \mathbf{B}. y \in Y \\ &\Leftrightarrow \exists Y \in \mathbf{B} (x \in \mathbf{A} \wedge y \in Y) \quad \square \end{aligned}$$

2.2 Relations

Definition 2.2.1 (Relation). A *relation* \mathbf{R} between classes \mathbf{A} and \mathbf{B} is a subclass of $\mathbf{A} \times \mathbf{B}$.

A *(binary) relation on \mathbf{A}* is a relation between \mathbf{A} and \mathbf{A} .

We write $x\mathbf{R}y$ for $(x, y) \in \mathbf{R}$.

2.2.1 Identity Functions

Definition 2.2.2 (Identity Function). For any class \mathbf{A} , the *identity function* or *diagonal relation* $\text{id}_{\mathbf{A}}$ on \mathbf{A} is

$$\text{id}_{\mathbf{A}} := \{(x, x) \mid x \in \mathbf{A}\} .$$

2.2.2 Inverses

Definition 2.2.3 (Inverse). The *inverse* of a relation \mathbf{R} between \mathbf{A} and \mathbf{B} is the relation \mathbf{R}^{-1} between \mathbf{B} and \mathbf{A} defined by

$$b\mathbf{R}^{-1}a \Leftrightarrow a\mathbf{R}b .$$

Proposition Schema 2.2.4. *For any classes \mathbf{A} , \mathbf{B} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is a relation between \mathbf{A} and \mathbf{B} , we have $(\mathbf{R}^{-1})^{-1} = \mathbf{R}$.

PROOF:

$$\begin{aligned} x(\mathbf{R}^{-1})^{-1}y &\Leftrightarrow y\mathbf{R}^{-1}x \\ &\Leftrightarrow x\mathbf{R}y \end{aligned}$$

□

2.2.3 Composition

Definition 2.2.5 (Composition). Let \mathbf{R} be a relation between \mathbf{A} and \mathbf{B} , and \mathbf{S} be a relation between \mathbf{B} and \mathbf{C} . The *composition* $\mathbf{S} \circ \mathbf{R}$ is the relation between \mathbf{A} and \mathbf{C} defined by

$$a(\mathbf{S} \circ \mathbf{R})c \Leftrightarrow \exists b(a\mathbf{R}b \wedge b\mathbf{S}c) .$$

Proposition Schema 2.2.6. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{R} and \mathbf{S} , the following is a theorem:*

If \mathbf{R} is a relation between \mathbf{A} and \mathbf{B} , and \mathbf{S} is a relation between \mathbf{B} and \mathbf{C} , then

$$(\mathbf{S} \circ \mathbf{R})^{-1} = \mathbf{R}^{-1} \circ \mathbf{S}^{-1} .$$

PROOF:

$$\begin{aligned} z(\mathbf{S} \circ \mathbf{R})^{-1}x &\Leftrightarrow x(\mathbf{S} \circ \mathbf{R})z \\ &\Leftrightarrow \exists y.(x\mathbf{R}y \wedge y\mathbf{S}z) \\ &\Leftrightarrow \exists y.(y\mathbf{R}^{-1}x \wedge z\mathbf{S}^{-1}y) \\ &\Leftrightarrow z(\mathbf{R}^{-1} \circ \mathbf{S}^{-1})x \end{aligned}$$

□

2.2.4 Properties of Relations

Definition 2.2.7 (Reflexive). Let \mathbf{R} be a binary relation on \mathbf{A} . Then \mathbf{R} is *reflexive* on \mathbf{A} iff $\forall x \in \mathbf{A} . (x, x) \in \mathbf{R}$.

Proposition Schema 2.2.8. *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is a reflexive relation on \mathbf{A} then so is \mathbf{R}^{-1} .

PROOF:

⟨1⟩1. LET: $x \in \mathbf{A}$

⟨1⟩2. $x\mathbf{R}x$

PROOF: Since \mathbf{R} is reflexive.

$\langle 1 \rangle 3. x\mathbf{R}^{-1}x$

□

Definition 2.2.9 (Irreflexive). A relation \mathbf{R} is *irreflexive* iff there is no x such that $(x, x) \in \mathbf{R}$.

Definition 2.2.10 (Symmetric). A relation \mathbf{R} is *symmetric* iff, whenever $x\mathbf{R}y$, then $y\mathbf{R}x$.

Definition 2.2.11 (Antisymmetric). A relation \mathbf{R} is *antisymmetric* iff, whenever $x\mathbf{R}y$ and $y\mathbf{R}x$, then $x = y$.

Proposition Schema 2.2.12. *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is an antisymmetric relation on \mathbf{A} then so is \mathbf{R}^{-1} .

PROOF:

$\langle 1 \rangle 1.$ ASSUME: $x\mathbf{R}^{-1}y$ and $y\mathbf{R}^{-1}x$

$\langle 1 \rangle 2.$ $y\mathbf{R}x$ and $x\mathbf{R}y$

$\langle 1 \rangle 3.$ $x = y$

PROOF: Since \mathbf{R} is antisymmetric.

□

Definition 2.2.13 (Transitive). A relation \mathbf{R} is *transitive* iff, whenever $x\mathbf{R}y$ and $y\mathbf{R}z$, then $x\mathbf{R}z$.

Proposition Schema 2.2.14. *For any classes \mathbf{A} , \mathbf{B} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is a transitive relation between \mathbf{A} and \mathbf{B} then \mathbf{R}^{-1} is transitive.

PROOF:

$\langle 1 \rangle 1.$ ASSUME: $(x, y), (y, z) \in \mathbf{R}^{-1}$

$\langle 1 \rangle 2.$ $(y, x), (z, y) \in \mathbf{R}$

$\langle 1 \rangle 3.$ $(z, x) \in \mathbf{R}$

$\langle 1 \rangle 4.$ $(x, z) \in \mathbf{R}^{-1}$

□

2.3 n-ary Relations

Definition Schema 2.3.1. For any sets a_1, \dots, a_n , define the *ordered n -tuple* (a_1, \dots, a_n) by

$$(a_1) := a_1$$

$$(a_1, \dots, a_n, a_{n+1}) = ((a_1, \dots, a_n), a_{n+1})$$

Definition Schema 2.3.2. An *n -ary relation on \mathbf{A}* is a class of ordered n -tuples all of whose components are in \mathbf{A} .

Chapter 3

Functions

3.1 Functions

Definition 3.1.1 (Function). A *function* from \mathbf{A} to \mathbf{B} is a relation \mathbf{F} between \mathbf{A} and \mathbf{B} such that, for all $x \in \mathbf{A}$, there is only one y such that $x\mathbf{F}y$. We denote this y by $\mathbf{F}(x)$.

A *binary operation* on a class \mathbf{A} is a function $\mathbf{A}^2 \rightarrow \mathbf{A}$.

Definition 3.1.2 (Closed). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{A}$ be a function and $\mathbf{B} \subseteq \mathbf{A}$. Then \mathbf{B} is *closed* under \mathbf{F} iff $\forall x \in \mathbf{B}. \mathbf{F}(x) \in \mathbf{B}$.

Proposition 3.1.3. *For any class \mathbf{A} , the following is a theorem:*

$$\text{id}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}$$

PROOF: For all $x \in \mathbf{A}$, the only y such that $(x, y) \in \text{id}_{\mathbf{A}}$ is $y = x$. \square

Proposition Schema 3.1.4. *For any classes $\mathbf{A}, \mathbf{B}, \mathbf{C}$, \mathbf{F} and \mathbf{G} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{G} : \mathbf{B} \rightarrow \mathbf{C}$. Then $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \rightarrow \mathbf{C}$ and, for all $x \in \mathbf{A}$, we have

$$(\mathbf{G} \circ \mathbf{F})(x) = \mathbf{G}(\mathbf{F}(x)) .$$

PROOF:

$\langle 1 \rangle 1. \forall x \in \mathbf{A}. (x, \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G} \circ \mathbf{F}$

PROOF: Because $(x, \mathbf{F}(x)) \in \mathbf{F}$ and $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$.

$\langle 1 \rangle 2. \text{ If } (x, z) \in \mathbf{F} \circ \mathbf{G} \text{ then } z = \mathbf{G}(\mathbf{F}(x))$

$\langle 2 \rangle 1. \text{ PICK } y \in \mathbf{B} \text{ such that } x\mathbf{F}y \text{ and } y\mathbf{G}z$

$\langle 2 \rangle 2. y = \mathbf{F}(x)$

$\langle 2 \rangle 3. z = \mathbf{G}(y)$

$\langle 2 \rangle 4. z = \mathbf{G}(\mathbf{F}(x))$

\square

Proposition 3.1.5. *For any set A there exists a function $F : \mathcal{P}A - \{\emptyset\} \rightarrow A$ (a choice function for A) such that, for every nonempty $B \subseteq A$, we have $F(B) \in B$.*

PROOF:

- ⟨1⟩1. LET: A be a set.
- ⟨1⟩2. LET: $\mathcal{A} = \{\{B\} \times B \mid B \in \mathcal{P}A - \{\emptyset\}\}$
- ⟨1⟩3. Every member of \mathcal{A} is nonempty.
- ⟨1⟩4. Any two distinct members of \mathcal{A} are disjoint.
- ⟨1⟩5. PICK a set C such that, for all $X \in \mathcal{A}$, we have $C \cap X$ is a singleton.

PROOF: Axiom of Choice.

- ⟨1⟩6. LET: $F = C \cap \bigcup \mathcal{A}$
- ⟨1⟩7. $F : \mathcal{P}A - \{\emptyset\} \rightarrow A$
 - ⟨2⟩1. F is a function.
 - ⟨3⟩1. LET: $(B, b), (B, b') \in F$
 - ⟨3⟩2. $(B, b), (B, b') \in \{B\} \times B$
 - PROOF: Since $(B, b), (B, b') \in \bigcup \mathcal{A}$.
 - ⟨3⟩3. $(B, b), (B, b') \in C \cap (\{B\} \times B)$
 - ⟨3⟩4. $(B, b) = (B, b')$
 - PROOF: From ⟨1⟩5.
 - ⟨3⟩5. $b = b'$
 - ⟨2⟩2. $\text{dom } F = \mathcal{P}A - \{\emptyset\}$

PROOF:

$$\begin{aligned}
 & B \in \text{dom } F \\
 & \Leftrightarrow \exists b. (B, b) \in F \\
 & \Leftrightarrow \exists b. ((B, b) \in \bigcup \mathcal{A} \wedge (B, b) \in C) \\
 & \Leftrightarrow \exists b. \exists B' \in \mathcal{P}A - \{\emptyset\}. ((B, b) \in \{B'\} \times B' \wedge (B, b) \in C) \\
 & \Leftrightarrow B \in \mathcal{P}A - \{\emptyset\} \wedge \exists b \in B. (B, b) \in C \\
 & \Leftrightarrow B \in \mathcal{P}A - \{\emptyset\} \tag{⟨1⟩5}
 \end{aligned}$$

- ⟨2⟩3. $\text{ran } F \subseteq A$

- ⟨1⟩8. For every nonempty $B \subseteq A$ we have $F(B) \in B$

□

Proposition 3.1.6. *For any relation R between A and B , there exists a function $H : A \rightarrow B$ such that $H \subseteq R$ (i.e. $\forall x \in A. xRH(x)$).*

PROOF:

- ⟨1⟩1. LET: R be a relation between A and B .
- ⟨1⟩2. PICK a choice function G for B .
- ⟨1⟩3. Define $H : A \rightarrow B$ by $H(x) = G(\{y \mid xRy\})$
- ⟨1⟩4. $H \subseteq R$

□

3.1.1 Injective Functions

Definition 3.1.7 (Injective). A function $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ is *one-to-one*, *injective* or an *injection*, $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$, iff, for all $x, y \in \mathbf{A}$, if $\mathbf{F}(x) = \mathbf{F}(y)$, then $x = y$.

Proposition 3.1.8. *For any class \mathbf{A} , the following is a theorem:*

$\text{id}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}$ is injective.

PROOF: If $\text{id}_A(x) = \text{id}_A(y)$ then immediately $x = y$. \square

Proposition Schema 3.1.9. *For any classes A, B, C, F, G , the following is a theorem:*

Assume $F : A \rightarrow B$ and $G : B \rightarrow C$. Then $G \circ F : A \rightarrow C$.

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in A$

$\langle 1 \rangle 2$. ASSUME: $(G \circ F)(x) = (G \circ F)(y)$

$\langle 1 \rangle 3$. $G(F(x)) = G(F(y))$

$\langle 1 \rangle 4$. $F(x) = F(y)$

PROOF: Since G is injective.

$\langle 1 \rangle 5$. $x = y$

PROOF: Since F is injective.

\square

Proposition 3.1.10. *Let $F : A \rightarrow B$ where A is nonempty. There exists $G : B \rightarrow A$ (a left inverse) such that $G \circ F = \text{id}_A$ if and only if F is one-to-one.*

PROOF:

$\langle 1 \rangle 1$. If there exists $G : B \rightarrow A$ such that $G \circ F = \text{id}_A$ then F is one-to-one.

$\langle 2 \rangle 1$. ASSUME: $G : B \rightarrow A$ and $G \circ F = I_A$

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: $F(x) = F(y)$

$\langle 2 \rangle 4$. $x = y$

PROOF: $x = G(F(x)) = G(F(y)) = y$

$\langle 1 \rangle 2$. If F is one-to-one then there exists $G : B \rightarrow A$ such that $G \circ F = I_A$.

$\langle 2 \rangle 1$. ASSUME: F is one-to-one.

$\langle 2 \rangle 2$. PICK $a \in A$

$\langle 2 \rangle 3$. LET: $G : B \rightarrow A$ be the function defined by: $G(b)$ is the (unique) $x \in A$ such that $F(x) = b$ if there exists such an x , $G(b) = a$ otherwise.

$\langle 2 \rangle 4$. For all $x \in A$ we have $G(F(x)) = x$.

\square

3.1.2 Surjective Functions

Definition 3.1.11 (Surjective). Let $F : A \rightarrow B$. We say that F is *surjective*, or maps A *onto* B , and write $F : A \twoheadrightarrow B$, iff for all $y \in B$ there exists $x \in A$ such that $F(x) = y$.

Proposition Schema 3.1.12. *For any class A , the following is a theorem:*

$\text{id}_A : A \rightarrow A$ is surjective.

PROOF: For any $y \in A$ we have $\text{id}_A(y) = y$. \square

Proposition Schema 3.1.13. *For any classes A, B, C, F and G , the following is a theorem:*

If $F : A \twoheadrightarrow B$ and $G : B \twoheadrightarrow C$, then $G \circ F : A \twoheadrightarrow C$.

PROOF:

- ⟨1⟩1. LET: $c \in \mathbf{C}$
- ⟨1⟩2. PICK $b \in \mathbf{B}$ such that $\mathbf{G}(b) = c$.
- ⟨1⟩3. PICK $a \in \mathbf{A}$ such that $\mathbf{F}(a) = b$.
- ⟨1⟩4. $(\mathbf{G} \circ \mathbf{F})(a) = c$

□

Proposition 3.1.14. *Let $F : A \rightarrow B$. There exists $H : B \rightarrow A$ (a right inverse) such that $F \circ H = \text{id}_B$ if and only if F maps A onto B .*

PROOF:

- ⟨1⟩1. If F has a right inverse then F is surjective.
 - ⟨2⟩1. ASSUME: F has a right inverse $H : B \rightarrow A$.
 - ⟨2⟩2. LET: $y \in B$
 - ⟨2⟩3. $F(H(y)) = y$
 - ⟨2⟩4. There exists $x \in A$ such that $F(x) = y$
- ⟨1⟩2. If F is surjective then F has a right inverse.
 - ⟨2⟩1. ASSUME: F is surjective.
 - ⟨2⟩2. PICK a function $H : B \rightarrow A$ such that $H \subseteq F^{-1}$

PROOF: Proposition 3.1.6.

- ⟨2⟩3. $F \circ H = \text{id}_B$
 - ⟨3⟩1. LET: $y \in B$
 - ⟨3⟩2. $(y, H(y)) \in F^{-1}$
 - ⟨3⟩3. $F(H(y)) = y$

□

3.1.3 Bijections

Definition 3.1.15 (Bijection). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Then \mathbf{F} is *bijective* or a *bijection*, $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$, iff it is injective and surjective.

Proposition Schema 3.1.16. *For any class \mathbf{A} , the following is a theorem:*
The identity function $\text{id}_{\mathbf{A}} : \mathbf{A} \approx \mathbf{A}$ is a bijection.

PROOF: Proposition 3.1.8 and 3.1.12. □

Proposition Schema 3.1.17. *For any classes \mathbf{A} , \mathbf{B} and \mathbf{F} , the following is a theorem:*

If $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$ then $\mathbf{F}^{-1} : \mathbf{B} \approx \mathbf{A}$.

PROOF:

- ⟨1⟩1. $\mathbf{F}^{-1} : \mathbf{B} \rightarrow \mathbf{A}$
 - ⟨2⟩1. LET: $b \in \mathbf{B}$
 - ⟨2⟩2. PICK $a \in \mathbf{A}$ such that $\mathbf{F}(a) = b$.

PROOF: Since \mathbf{F} is surjective.
 - ⟨2⟩3. $(b, a) \in \mathbf{F}^{-1}$
 - ⟨2⟩4. If $(b, a') \in \mathbf{F}^{-1}$ then $a' = a$.
 - ⟨3⟩1. LET: $a' \in \mathbf{A}$ such that $(b, a') \in \mathbf{F}^{-1}$

$$\langle 3 \rangle 2. \mathbf{F}(a') = \mathbf{F}(a)$$

$$\langle 3 \rangle 3. a' = a$$

PROOF: Since \mathbf{F} is injective.

$$\langle 1 \rangle 2. \mathbf{F}^{-1} \text{ is injective.}$$

$$\langle 2 \rangle 1. \text{ LET: } x, y \in \mathbf{B}$$

$$\langle 2 \rangle 2. \text{ ASSUME: } \mathbf{F}^{-1}(x) = \mathbf{F}^{-1}(y)$$

$$\langle 2 \rangle 3. x = y$$

$$\text{PROOF: } x = \mathbf{F}(\mathbf{F}^{-1}(x)) = \mathbf{F}(\mathbf{F}^{-1}(y)) = y.$$

$$\langle 1 \rangle 3. \mathbf{F}^{-1} \text{ is surjective.}$$

$$\text{PROOF: For all } a \in \mathbf{A} \text{ we have } \mathbf{F}^{-1}(\mathbf{F}(a)) = a.$$

□

Proposition Schema 3.1.18. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{F} and \mathbf{G} , the following is a theorem:*

$$\text{If } \mathbf{F} : \mathbf{A} \approx \mathbf{B} \text{ and } \mathbf{G} : \mathbf{B} \approx \mathbf{C} \text{ then } \mathbf{G} \circ \mathbf{F} : \mathbf{A} \approx \mathbf{C}.$$

PROOF: Propositions 3.1.9 and 3.1.13. □

3.1.4 Restrictions

Definition 3.1.19 (Restriction). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Let $\mathbf{C} \subseteq \mathbf{A}$. The *restriction* of \mathbf{F} to \mathbf{C} , denoted $\mathbf{F} \upharpoonright \mathbf{C}$, is the function

$$\begin{aligned} \mathbf{F} \upharpoonright \mathbf{C} : \mathbf{C} &\rightarrow \mathbf{B} \\ (\mathbf{F} \upharpoonright \mathbf{C})(x) &= \mathbf{F}(x) \quad (x \in \mathbf{C}) \end{aligned}$$

3.1.5 Images

Definition 3.1.20 (Image). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C} \subseteq \mathbf{A}$. The *image* of \mathbf{C} under \mathbf{F} is the class

$$\mathbf{F}(\mathbf{C}) := \{\mathbf{F}(x) \mid x \in \mathbf{C}\}.$$

Proposition Schema 3.1.21. *For any classes \mathbf{F} , \mathbf{A} and \mathbf{B} , the following is a theorem.*

$$\text{If } \mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}, \text{ then for any subset } S \subseteq \mathbf{A}, \text{ the class } \mathbf{F}(S) \text{ is a set.}$$

PROOF: By an Axiom of Replacement. □

Proposition Schema 3.1.22. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{F} , the following is a theorem:*

$$\text{Assume } \mathbf{F} : \mathbf{A} \rightarrow \mathbf{B} \text{ and } \mathbf{C} \subseteq \mathcal{P}\mathbf{A}. \text{ Then}$$

$$\mathbf{F}\left(\bigcup \mathbf{C}\right) = \{y \mid \exists X \in \mathbf{C}. y \in \mathbf{F}(X)\}$$

PROOF:

$$\begin{aligned} y \in \mathbf{F}\left(\bigcup \mathbf{C}\right) &\Leftrightarrow \exists x \in \bigcup \mathbf{C}. y = \mathbf{F}(x) \\ &\Leftrightarrow \exists x. \exists X. X \in \mathbf{C} \wedge x \in X \wedge y = \mathbf{F}(x) \\ &\Leftrightarrow \exists X \in \mathbf{C}. y \in \mathbf{F}(X) \end{aligned}$$

□

Proposition Schema 3.1.23. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} and \mathbf{F} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$. Then

$$\mathbf{F}(\mathbf{C} \cup \mathbf{D}) = \mathbf{F}(\mathbf{C}) \cup \mathbf{F}(\mathbf{D}) .$$

PROOF:

$$\begin{aligned} y \in \mathbf{F}(\mathbf{C} \cup \mathbf{D}) &\Leftrightarrow \exists x \in \mathbf{C} \cup \mathbf{D}. y = \mathbf{F}(x) \\ &\Leftrightarrow \exists x \in \mathbf{C}. y = \mathbf{F}(x) \vee \exists x \in \mathbf{D}. y = \mathbf{F}(x) \\ &\Leftrightarrow y \in \mathbf{F}(\mathbf{C}) \cup \mathbf{F}(\mathbf{D}) \quad \square \end{aligned}$$

Proposition 3.1.24. *For any classes \mathbf{F} , \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$. Then

$$\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B}) .$$

Equality holds if \mathbf{F} is injective.

PROOF:

- $\langle 1 \rangle 1. \mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
- $\langle 2 \rangle 1. \text{ LET: } y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$
- $\langle 2 \rangle 2. \text{ PICK } x \in \mathbf{A} \cap \mathbf{B} \text{ such that } y = \mathbf{F}(x)$
- $\langle 2 \rangle 3. y \in \mathbf{F}(\mathbf{A})$
- PROOF: Since $x \in \mathbf{A}$.
- $\langle 2 \rangle 4. y \in \mathbf{F}(\mathbf{B})$
- PROOF: Since $x \in \mathbf{B}$.
- $\langle 1 \rangle 2. \text{ If } \mathbf{F} \text{ is injective then } \mathbf{F}(\mathbf{A} \cap \mathbf{B}) = \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B}).$
- $\langle 2 \rangle 1. \text{ ASSUME: } \mathbf{F} \text{ is injective.}$
- $\langle 2 \rangle 2. \text{ LET: } y \in \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
- $\langle 2 \rangle 3. \text{ PICK } x \in \mathbf{A} \text{ such that } y = \mathbf{F}(x)$
- $\langle 2 \rangle 4. \text{ PICK } x' \in \mathbf{B} \text{ such that } y = \mathbf{F}(x')$
- $\langle 2 \rangle 5. x = x'$
- PROOF: $\langle 2 \rangle 1$
- $\langle 2 \rangle 6. x \in \mathbf{A} \cap \mathbf{B}$
- $\langle 2 \rangle 7. y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$

\square

Proposition Schema 3.1.25. *For any classes \mathbf{F} , \mathbf{A} , \mathbf{B} , and \mathbf{C} , the following is a theorem:*

Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C} \subseteq \mathcal{P}\mathbf{A}$. Then

$$\mathbf{F}\left(\bigcap \mathbf{C}\right) \subseteq \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\} .$$

Equality holds if \mathbf{F} is injective and \mathbf{A} is nonempty.

PROOF:

- $\langle 1 \rangle 1. \mathbf{F}(\bigcap \mathbf{A}) \subseteq \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$

- ⟨2⟩1. LET: $y \in \mathbf{F}(\bigcap \mathbf{A})$
- ⟨2⟩2. PICK $x \in \bigcap \mathbf{A}$ such that $y = \mathbf{F}(x)$
- ⟨2⟩3. LET: $X \in \mathbf{A}$
- ⟨2⟩4. $x \in X$
- ⟨2⟩5. $y \in \mathbf{F}(X)$
- ⟨1⟩2. If \mathbf{F} is injective then $\mathbf{F}(\bigcap \mathbf{A}) = \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
 - ⟨2⟩1. ASSUME: \mathbf{F} is injective.
 - ⟨2⟩2. ASSUME: \mathbf{A} is nonempty.
 - ⟨2⟩3. LET: $y \in \bigcap \{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
 - ⟨2⟩4. PICK $X_0 \in \mathbf{A}$
 - ⟨2⟩5. PICK $x \in X_0$ such that $(x, y) \in \mathbf{F}$
 - ⟨2⟩6. $x \in \bigcap \mathbf{A}$
 - ⟨3⟩1. LET: $X \in \mathbf{A}$
 - ⟨3⟩2. PICK $x' \in X$ such that $(x', y) \in \mathbf{F}$.
 - ⟨3⟩3. $x = x'$
 - PROOF: ⟨2⟩1
 - ⟨3⟩4. $x \in X$
 - ⟨2⟩7. $y \in \mathbf{F}(\bigcap \mathbf{A})$

□

Proposition 3.1.26. *For any classes $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ and \mathbf{F} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$. Then

$$\mathbf{F}(\mathbf{C}) - \mathbf{F}(\mathbf{D}) \subseteq \mathbf{F}(\mathbf{C} - \mathbf{D}) .$$

Equality holds if \mathbf{F} is injective.

PROOF:

- ⟨1⟩1. $\mathbf{F}(\mathbf{C}) - \mathbf{F}(\mathbf{D}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
 - ⟨2⟩1. LET: $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
 - ⟨2⟩2. PICK $x \in \mathbf{A}$ such that $y = \mathbf{F}(x)$
 - ⟨2⟩3. $x \notin \mathbf{B}$
 - ⟨2⟩4. $x \in \mathbf{A} - \mathbf{B}$
 - ⟨2⟩5. $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
- ⟨1⟩2. If \mathbf{F} is injective then $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) = \mathbf{F}(\mathbf{A} - \mathbf{B})$
 - ⟨2⟩1. ASSUME: \mathbf{F} is injective.
 - ⟨2⟩2. LET: $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
 - ⟨2⟩3. PICK $x \in \mathbf{A} - \mathbf{B}$ such that $y = \mathbf{F}(x)$
 - ⟨2⟩4. $y \in \mathbf{F}(\mathbf{A})$
 - ⟨2⟩5. $y \notin \mathbf{F}(\mathbf{B})$
 - ⟨3⟩1. ASSUME: for a contradiction $y \in \mathbf{F}(\mathbf{B})$
 - ⟨3⟩2. PICK $x' \in \mathbf{B}$ such that $y = \mathbf{F}(x')$
 - ⟨3⟩3. $x = x'$
 - PROOF: ⟨2⟩1
 - ⟨3⟩4. $x \in \mathbf{B}$
 - ⟨3⟩5. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 3$.

□

3.1.6 Inverse Images

Definition 3.1.27 (Inverse Image). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C} \subseteq \mathbf{B}$. Then the *inverse image* of \mathbf{C} under \mathbf{F} is

$$\mathbf{F}^{-1}(\mathbf{C}) = \{x \in \mathbf{A} \mid \mathbf{F}(x) \in \mathbf{C}\} .$$

Proposition Schema 3.1.28. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{F} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C} \subseteq \mathcal{PB}$. Then

$$\mathbf{F}^{-1}\left(\bigcap \mathbf{C}\right) = \bigcap \{\mathbf{F}^{-1}(X) \mid X \in \mathbf{C}\} .$$

PROOF:

$$\begin{aligned} x \in \mathbf{F}^{-1}\left(\bigcap \mathbf{C}\right) &\Leftrightarrow \mathbf{F}(x) \in \bigcap \mathbf{C} \\ &\Leftrightarrow \forall X \in \mathbf{C}. \mathbf{F}(x) \in X \\ &\Leftrightarrow \forall X \in \mathbf{C}. x \in \mathbf{F}^{-1}(X) \end{aligned} \quad \square$$

Proposition Schema 3.1.29. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} and \mathbf{F} , the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{C}, \mathbf{D} \subseteq \mathbf{B}$. Then

$$\mathbf{F}^{-1}(\mathbf{C} - \mathbf{D}) = \mathbf{F}^{-1}(\mathbf{C}) - \mathbf{F}^{-1}(\mathbf{D}) .$$

PROOF:

$$\begin{aligned} x \in \mathbf{F}^{-1}(\mathbf{C} - \mathbf{D}) &\Leftrightarrow \mathbf{F}(x) \in \mathbf{C} - \mathbf{D} \\ &\Leftrightarrow \mathbf{F}(x) \in \mathbf{C} \wedge \mathbf{F}(x) \notin \mathbf{D} \\ &\Leftrightarrow x \in \mathbf{F}^{-1}(\mathbf{C}) \wedge x \notin \mathbf{F}^{-1}(\mathbf{D}) \\ &\Leftrightarrow x \in \mathbf{F}^{-1}(\mathbf{C}) - \mathbf{F}^{-1}(\mathbf{D}) \end{aligned} \quad \square$$

3.1.7 Function Sets

Proposition 3.1.30. *For any classes \mathbf{B} and \mathbf{F} , the following is a theorem:*

Let A be a set. If $\mathbf{F} : A \rightarrow \mathbf{B}$ then \mathbf{F} is a set.

PROOF: By an Axiom of Replacement, we have $R = \{\mathbf{F}(x) \mid x \in A\}$ is a set. Hence \mathbf{F} is a set since $\mathbf{F} \subseteq A \times R$. □

Definition 3.1.31 (Dependent Product Class). Let I be a set and let $\mathbf{H}(i)$ be a class for all $i \in I$. We write $\prod_{i \in I} \mathbf{H}(i)$ for the class of all functions $f : I \rightarrow \bigcup_{i \in I} \mathbf{H}(i)$ such that $\forall i \in I. f(i) \in \mathbf{H}(i)$.

We write \mathbf{B}^I for $\prod_{i \in I} \mathbf{B}$ where \mathbf{B} does not depend on I .

Proposition Schema 3.1.32. *Let I be a set. Let $H(i)$ be a set for every $i \in I$. Then $\prod_{i \in I} \mathbf{H}(i)$ is a set.*

PROOF:

$\langle 1 \rangle 1.$ $\{\mathbf{H}(i) \mid i \in I\}$ is a set.

PROOF: By an Axiom of Replacement.

$\langle 1 \rangle 2.$ $\bigcup_{i \in I} \mathbf{H}(i)$ is a set.

$\langle 1 \rangle 3.$ $\prod_{i \in I} \mathbf{H}(i)$ is a set.

PROOF: It is a subset of $\mathcal{P}(I \times \bigcup_{i \in I} \mathbf{H}(i))$.

□

Proposition 3.1.33. *Let I be a set. Let $H(i)$ be a set for all $i \in I$. If $\forall i \in I. H(i) \neq \emptyset$ then $\prod_{i \in I} H(i) \neq \emptyset$.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: $\forall i \in I. H(i) \neq \emptyset$

$\langle 1 \rangle 2.$ LET: $R = \{(i, x) \mid i \in I, x \in H(i)\}$

$\langle 1 \rangle 3.$ PICK a function $f : I \rightarrow \bigcup_{i \in I} H(i)$ such that $f \subseteq R$

PROOF: Proposition 3.1.6.

$\langle 1 \rangle 4.$ $f \in \prod_{i \in I} H(i)$

□

3.2 Equinumerosity

Definition 3.2.1 (Equinumerous). Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between A and B .

Theorem 3.2.2 (Cantor 1873). *No set is equinumerous to its power set.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: for a contradiction $f : A \approx \mathcal{P}A$

$\langle 1 \rangle 2.$ LET: $S = \{x \in A \mid x \notin f(x)\}$

$\langle 1 \rangle 3.$ PICK $a \in A$ such that $f(a) = S$

$\langle 1 \rangle 4.$ $a \in S$ if and only if $a \notin S$

$\langle 1 \rangle 5.$ Q.E.D.

PROOF: This is a contradiction.

□

3.3 Domination

Definition 3.3.1 (Dominate). A set A is *dominated* by a set B , $A \preceq B$, iff there exists an injection $A \rightarrow B$.

Proposition 3.3.2. *Given sets A and B , if $A \neq \emptyset$ or $B = \emptyset$, then we have $A \preceq B$ iff there exists a surjective function $B \rightarrow A$.*

PROOF:

- ⟨1⟩1. If $A \preceq B$ and $A \neq \emptyset$ then there exists a surjective function $B \rightarrow A$.
 - ⟨2⟩1. ASSUME: $f : A \rightarrow B$ be injective.
 - ⟨2⟩2. PICK $a \in A$
 - ⟨2⟩3. LET: $g : B \rightarrow A$ be the function defined by $g(b) = f^{-1}(b)$ if $b \in \text{ran } f$,
and $g(b) = a$ otherwise.
 - ⟨2⟩4. g is surjective.
- ⟨1⟩2. If there exists a surjective function $B \rightarrow A$ then $A \preceq B$.
 - ⟨2⟩1. ASSUME: there exists a surjective function $g : B \rightarrow A$
 - ⟨2⟩2. $\forall a \in A. \exists b \in B. g(b) = a$
 - ⟨2⟩3. Choose a function $f : A \rightarrow B$ such that $\forall a \in A. g(f(a)) = a$
 - ⟨2⟩4. f is injective.

□

Chapter 4

Equivalence Relations

Definition 4.0.1 (Equivalence Relation). An *equivalence relation* on a class \mathbf{A} is a binary relation on \mathbf{A} that is reflexive, symmetric and transitive.

Proposition 4.0.2. *Equinumerosity is an equivalence relation on the class of all sets.*

PROOF: Propositions 3.1.16, 3.1.17, 3.1.18. \square

Definition 4.0.3 (Respects). Let \mathbf{R} be an equivalence relation on \mathbf{A} and $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Then \mathbf{F} *respects* \mathbf{A} iff, whenever $(x, y) \in \mathbf{R}$, then $\mathbf{F}(x) = \mathbf{F}(y)$.

Definition 4.0.4 (Equivalence Class). Let \mathbf{R} be an equivalence relation on \mathbf{A} and $a \in \mathbf{A}$. The *equivalence class* of a modulo \mathbf{R} is

$$[a]_{\mathbf{R}} := \{x \mid a\mathbf{R}x\} .$$

Proposition Schema 4.0.5. *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem.*

Assume \mathbf{R} be an equivalence relation on \mathbf{A} . Let $a, b \in \mathbf{A}$. Then $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ if and only if $a\mathbf{R}b$.

PROOF:

$\langle 1 \rangle 1$. If $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ then $a\mathbf{R}b$.

$\langle 2 \rangle 1$. ASSUME: $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$

$\langle 2 \rangle 2$. $b\mathbf{R}b$

PROOF: Reflexivity

$\langle 2 \rangle 3$. $b \in [b]_{\mathbf{R}}$

$\langle 2 \rangle 4$. $b \in [a]_{\mathbf{R}}$

$\langle 2 \rangle 5$. $a\mathbf{R}b$

$\langle 1 \rangle 2$. If $a\mathbf{R}b$ then $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$.

$\langle 2 \rangle 1$. For all $x, y \in \mathbf{A}$, if $x\mathbf{R}y$ then $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$

$\langle 3 \rangle 1$. LET: $x, y \in \mathbf{A}$

$\langle 3 \rangle 2$. ASSUME: $x\mathbf{R}y$

$\langle 3 \rangle 3$. LET: $t \in [y]_{\mathbf{R}}$
 $\langle 3 \rangle 4$. $y \mathbf{R} t$
 $\langle 3 \rangle 5$. $x \mathbf{R} t$
 PROOF: Transitivity, $\langle 3 \rangle 2$, $\langle 3 \rangle 4$.
 $\langle 3 \rangle 6$. $t \in [x]_{\mathbf{R}}$
 $\langle 2 \rangle 2$. ASSUME: $a \mathbf{R} b$
 $\langle 2 \rangle 3$. $[b]_{\mathbf{R}} \subseteq [a]_{\mathbf{R}}$
 PROOF: $\langle 2 \rangle 1$, $\langle 2 \rangle 2$.
 $\langle 2 \rangle 4$. $b \mathbf{R} a$
 PROOF: Symmetry, $\langle 2 \rangle 2$.
 $\langle 2 \rangle 5$. $[a]_{\mathbf{R}} \subseteq [b]_{\mathbf{R}}$
 PROOF: $\langle 2 \rangle 1$, $\langle 2 \rangle 4$.
 $\langle 2 \rangle 6$. $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$
 PROOF: $\langle 2 \rangle 3$, $\langle 2 \rangle 5$.

□

Definition 4.0.6 (Partition). A *partition* Π of a set A is a set of nonempty subsets of A that is disjoint and exhaustive, i.e.

1. no two different sets in Π have any common elements, and
2. each element of A is in some set in Π .

Definition 4.0.7. Let R be an equivalence relation on a set A . The *quotient set* A/R is the set of all equivalence classes.

Theorem 4.0.8. Let A be a set and \mathbf{B} a class. Let R be an equivalence relation on A and $F : A \rightarrow \mathbf{B}$. Then F respects R if and only if there exists $\hat{F} : A/R \rightarrow \mathbf{B}$ such that

$$\forall a \in A. \hat{F}([a]_R) = F(a) \text{ .}$$

In this case, \hat{F} is unique.

PROOF:

$\langle 1 \rangle 1$. If F respects R then there exists $\hat{F} : A/R \rightarrow \mathbf{B}$ such that $\forall a \in A. \hat{F}([a]_R) = F(a)$.
 $\langle 2 \rangle 1$. ASSUME: F respects R .
 $\langle 2 \rangle 2$. LET: $\hat{F} = \{([a]_R, F(a)) \mid a \in A\}$
 $\langle 2 \rangle 3$. \hat{F} is a function.
 $\langle 3 \rangle 1$. ASSUME: $a, a' \in A$ and $[a]_R = [a']_R$
 PROVE: $F(a) = F(a')$
 $\langle 3 \rangle 2$. $(a, a') \in R$
 PROOF: Proposition 4.0.5.
 $\langle 3 \rangle 3$. $F(a) = F(a')$
 PROOF: $\langle 2 \rangle 1$
 $\langle 2 \rangle 4$. $\text{dom } \hat{F} = A/R$
 $\langle 2 \rangle 5$. $\text{ran } \hat{F} \subseteq \mathbf{B}$
 $\langle 2 \rangle 6$. $\forall a \in A. \hat{F}([a]_R) = F(a)$

- $\langle 1 \rangle 2$. If there exists $\hat{F} : A/R \rightarrow \mathbf{B}$ such that $\forall a \in A. \hat{F}([a]_R) = F(a)$ then F respects R .
 $\langle 2 \rangle 1$. ASSUME: $\hat{F} : A/R \rightarrow \mathbf{B}$ and $\forall a \in A. \hat{F}([a]_R) = F(a)$
 $\langle 2 \rangle 2$. LET: $a, a' \in A$
 $\langle 2 \rangle 3$. ASSUME: $(a, a') \in R$
 $\langle 2 \rangle 4$. $[a]_R = [a']_R$
 PROOF: Proposition 4.0.5.
 $\langle 2 \rangle 5$. $F(a) = F(a')$
 PROOF: $\langle 2 \rangle 1$
 $\langle 1 \rangle 3$. If $G, H : A/R \rightarrow \mathbf{B}$ and $\forall a \in A. G([a]_R) = H([a]_R)$ then $G = H$.
 \square

Proposition 4.0.9. *Let R be an equivalence relation on a set A . Then A/R is a partition of A .*

PROOF:

- $\langle 1 \rangle 1$. Every member of A/R is nonempty.
 PROOF: Since $a \in [a]_R$ by reflexivity.
 $\langle 1 \rangle 2$. No two different sets in A/R have any common elements.
 $\langle 2 \rangle 1$. LET: $[a]_R, [b]_R \in A/R$
 $\langle 2 \rangle 2$. LET: $c \in [a]_R \cap [b]_R$
 PROVE: $[a]_R = [b]_R$
 $\langle 2 \rangle 3$. aRc
 PROOF: $\langle 2 \rangle 2$
 $\langle 2 \rangle 4$. bRc
 PROOF: $\langle 2 \rangle 2$
 $\langle 2 \rangle 5$. cRb
 PROOF: Symmetry, $\langle 2 \rangle 4$
 $\langle 2 \rangle 6$. aRb
 PROOF: Transitivity, $\langle 2 \rangle 3$, $\langle 2 \rangle 5$
 $\langle 2 \rangle 7$. $[a]_R = [b]_R$
 PROOF: Proposition 4.0.5, $\langle 2 \rangle 6$
 $\langle 1 \rangle 3$. Each element of A is in some set in A/R .
 PROOF: Since $a \in [a]_R$ by reflexivity.
 \square

Proposition 4.0.10. *For any partition P of a set A , there exists a unique equivalence relation R on A such that $A/R = P$, namely xRy iff $\exists X \in P (x \in X \wedge y \in X)$.*

PROOF: Easy. \square

Definition 4.0.11 (Natural Map). Let A be a set and R an equivalence relation on A . The *natural map* $A \rightarrow A/R$ is the function that maps $a \in A$ to $[a]_R$.

Chapter 5

Ordering Relations

5.1 Partial Orders

Definition 5.1.1 (Partial Ordering). Let \mathbf{A} be a class. A *partial ordering* on \mathbf{A} is a relation \mathbf{R} on \mathbf{A} that is reflexive, antisymmetric and transitive.

We often write \leq for a partial ordering, and then write $x < y$ for $x \leq y \wedge x \neq y$.

Proposition Schema 5.1.2. *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is a partial order on \mathbf{A} then so is \mathbf{R}^{-1} .

PROOF:

$\langle 1 \rangle 1.$ \mathbf{R}^{-1} is reflexive.

PROOF: Proposition 2.2.8.

$\langle 1 \rangle 2.$ \mathbf{R}^{-1} is antisymmetric.

PROOF: Proposition 2.2.12.

$\langle 1 \rangle 3.$ \mathbf{R}^{-1} is transitive.

$\langle 2 \rangle 1.$ ASSUME: $x\mathbf{R}^{-1}y$ and $y\mathbf{R}^{-1}z$

$\langle 2 \rangle 2.$ $y\mathbf{R}x$ and $z\mathbf{R}y$

$\langle 2 \rangle 3.$ $z\mathbf{R}x$

PROOF: Since \mathbf{R} is transitive.

$\langle 2 \rangle 4.$ $x\mathbf{R}^{-1}z$

□

Proposition Schema 5.1.3. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{F} and \mathbf{R} , the following is a theorem:*

Assume \mathbf{R} is a partial order on \mathbf{B} and $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ is injective. Define \mathbf{S} on \mathbf{A} by $x\mathbf{S}y$ iff $\mathbf{F}(x)\mathbf{R}\mathbf{F}(y)$. Then \mathbf{S} is a partial order on \mathbf{A} .

PROOF:

$\langle 1 \rangle 1.$ \mathbf{S} is reflexive.

PROOF: For any $x \in \mathbf{A}$ we have $\mathbf{F}(x)\mathbf{R}\mathbf{F}(x)$.

- ⟨1⟩2. **S** is antisymmetric.
 ⟨2⟩1. LET: $x, y \in \mathbf{A}$
 ⟨2⟩2. ASSUME: $x\mathbf{S}y$ and $y\mathbf{S}x$
 ⟨2⟩3. $\mathbf{F}(x)\mathbf{R}\mathbf{F}(y)$ and $\mathbf{F}(y)\mathbf{R}\mathbf{F}(x)$
 ⟨2⟩4. $\mathbf{F}(x) = \mathbf{F}(y)$
 PROOF: **R** is antisymmetric.
 ⟨2⟩5. $x = y$
 ⟨1⟩3. **S** is transitive.
 □

Corollary Schema 5.1.3.1. *For any classes **A**, **B** and **R**, the following is a theorem:*

*Assume **R** be a partial order on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Then $\mathbf{R} \cap \mathbf{B}^2$ is a partial order on **B**.*

Definition 5.1.4 (Partially Ordered Set). A *partially ordered set* or *poset* is a pair (A, \leq) where A is a set and \leq is a partial ordering on A . We often write just A for (A, \leq) .

If (A, \leq) is a poset and $B \subseteq A$ we write just B for the poset $(B, \leq \cap B^2)$.

Definition 5.1.5 (Strictly Monotone). Let $(A, <_A)$ and $(B, <_B)$ be posets. A function $f : A \rightarrow B$ is *strictly monotone* iff, whenever $x <_A y$, then $f(x) <_B f(y)$.

Definition 5.1.6 (Minimal). Let \leq be a partial order on **A**. An element $m \in \mathbf{A}$ is *minimal* iff there is no $x \in \mathbf{A}$ such that $x < m$.

Definition 5.1.7 (Maximal). Let \leq be a partial order on **A**. An element $m \in \mathbf{A}$ is *maximal* iff there is no $x \in \mathbf{A}$ such that $m < x$.

Definition 5.1.8 (Least). Let \leq be a partial order on **A**. An element $m \in \mathbf{A}$ is *least* iff for all $x \in \mathbf{A}$ we have $m \leq x$.

Proposition 5.1.9. *A partial order has at most one least element.*

PROOF: If m and m' are least then $m \leq m'$ and $m' \leq m$, so $m = m'$. □

Definition 5.1.10 (Greatest). Let \leq be a partial order on **A**. An element $m \in \mathbf{A}$ is *greatest* iff for all $x \in \mathbf{A}$ we have $x \leq m$.

Proposition 5.1.11. *A poset has at most one greatest element.*

PROOF: If m and m' are greatest then $m \leq m'$ and $m' \leq m$, so $m = m'$. □

Definition 5.1.12 (Upper Bound). Let \leq be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Let $u \in \mathbf{A}$. Then u is an *upper bound* for **B** iff $\forall x \in \mathbf{B}. x \leq u$.

Definition 5.1.13 (Lower Bound). Let \leq be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Let $l \in \mathbf{A}$. Then l is a *lower bound* for **B** iff $\forall x \in \mathbf{B}. l \leq x$.

Definition 5.1.14 (Bounded Above). Let \leq be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Then **B** is *bounded above* iff it has an upper bound.

Definition 5.1.15 (Bounded Below). Let \leq be a partial ordering on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Then \mathbf{B} is *bounded below* iff it has a lower bound.

Definition 5.1.16 (Least Upper Bound). Let \leq be a partial ordering on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Let $s \in \mathbf{A}$. Then s is the *least upper bound* or *supremum* of \mathbf{B} iff s is an upper bound for \mathbf{B} and, for every upper bound u for \mathbf{B} , we have $s \leq u$.

Definition 5.1.17 (Greatest Lower Bound). Let \leq be a partial ordering on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Let $i \in \mathbf{A}$. Then i is the *greatest lower bound* or *infimum* of \mathbf{B} iff i is a lower bound for \mathbf{B} and, for every lower bound l for \mathbf{B} , we have $i \leq l$.

Definition 5.1.18 (Complete). A partial order is *complete* iff every nonempty subset bounded above has a supremum, and every nonempty subset bounded below has an infimum.

Definition 5.1.19 (Order Isomorphism). Let A and B be posets. An *order isomorphism* between A and B , $f : A \cong B$, is a bijection $f : A \approx B$ such that f and f^{-1} are monotone.

Theorem 5.1.20 (Knaster Fixed-Point Theorem). *Let A be a complete poset with a greatest and least element. Let $\phi : A \rightarrow A$ be monotone. Then there exists $a \in A$ such that $\phi(a) = a$.*

PROOF:

$\langle 1 \rangle 1$. LET: $B = \{x \in A \mid x \leq \phi(x)\}$

$\langle 1 \rangle 2$. LET: $a = \sup B$

PROOF: B is nonempty because the least element of A is in B , and it is bounded above by the greatest element of A .

$\langle 1 \rangle 3$. For all $b \in B$ we have $b \leq \phi(a)$

$\langle 2 \rangle 1$. LET: $b \in B$

$\langle 2 \rangle 2$. $b \leq \phi(b)$

$\langle 2 \rangle 3$. $b \leq a$

$\langle 2 \rangle 4$. $\phi(b) \leq \phi(a)$

$\langle 2 \rangle 5$. $b \leq \phi(a)$

$\langle 1 \rangle 4$. $a \leq \phi(a)$

$\langle 1 \rangle 5$. $\phi(a) \leq \phi(\phi(a))$

$\langle 1 \rangle 6$. $\phi(a) \in B$

$\langle 1 \rangle 7$. $\phi(a) \leq a$

$\langle 1 \rangle 8$. $\phi(a) = a$

□

Definition 5.1.21 (Dense). Let \leq be a partial order on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Then \mathbf{B} is *dense* iff, for all $x, y \in \mathbf{A}$, if $x < y$ then there exists $z \in \mathbf{B}$ such that $x < z < y$.

Proposition 5.1.22. *Let A be a complete poset with no least element. Let $B \subseteq A$ be dense. Let $\theta : A \rightarrow A$ be a monotone map that is the identity on B . Then $\theta = \text{id}_A$.*

PROOF:

- ⟨1⟩1. LET: $a \in A$
 PROVE: $\theta(a) = a$
- ⟨1⟩2. LET: $S(a) = \{b \in B \mid b < a\}$
- ⟨1⟩3. $S(a)$ is nonempty and bounded above.
 - ⟨2⟩1. $S(a)$ is nonempty.
 - ⟨3⟩1. PICK $a_1 < a$
 PROOF: Since a is not least.
 - ⟨3⟩2. There exists $b \in B$ such that $a_1 < b < a$.
 - ⟨2⟩2. $S(a)$ is bounded above by a .
- ⟨1⟩4. $\sup S(a) \leq a$
- ⟨1⟩5. $\sup S(a) = a$
 - ⟨2⟩1. ASSUME: for a contradiction $\sup S(a) < a$
 - ⟨2⟩2. PICK $b \in B$ such that $\sup S(a) < b < a$
 - ⟨2⟩3. $b \in S(a)$
 - ⟨2⟩4. Q.E.D.
- PROOF: This contradicts the fact that $\sup S(a) < b$.
- ⟨1⟩6. For all $b \in S(a)$ we have $b \leq \theta(a)$
 - ⟨2⟩1. LET: $b \in S(a)$
 - ⟨2⟩2. $b < a$
 - ⟨2⟩3. $\theta(b) \leq \theta(a)$
 - ⟨2⟩4. $b \leq \theta(a)$
 - PROOF: $\theta(b) = b$
- ⟨1⟩7. $a \leq \theta(a)$
 PROOF: Since $a = \sup S(a)$ and $\theta(a)$ is an upper bound for $S(a)$.
- ⟨1⟩8. $a \not\leq \theta(a)$
 - ⟨2⟩1. ASSUME: for a contradiction $a < \theta(a)$.
 - ⟨2⟩2. PICK $b \in B$ such that $a < b < \theta(a)$
 - ⟨2⟩3. $\theta(a) \leq \theta(b) = b$
 - ⟨2⟩4. Q.E.D.
 - PROOF: This contradicts the fact that $b < \theta(a)$.
- ⟨1⟩9. $\theta(a) = a$

□

Theorem 5.1.23. *Let A and P be complete posets with no least or greatest element. Let B be dense in A and Q be dense in P . Every order isomorphism $\phi : B \cong Q$ extends uniquely to an order isomorphism $A \cong P$.*

PROOF:

- ⟨1⟩1. For $a \in A$, let $S(a) = \{b \in B \mid b < a\}$.
- ⟨1⟩2. Define $\bar{\phi} : A \rightarrow P$ by $\bar{\phi}(a) = \sup \phi(S(a))$.
 - ⟨2⟩1. $\phi(S(a))$ is nonempty.
 - ⟨3⟩1. PICK $a_1 < a$
 PROOF: Since a is not least.
 - ⟨3⟩2. PICK $b \in B$ such that $a_1 < b < a$.
 - ⟨3⟩3. $\phi(b) \in \phi(S(a))$
 - ⟨2⟩2. $\phi(S(a))$ is bounded above.
 - ⟨3⟩1. PICK $a_2 > a$

PROOF: Since a is not greatest.

$\langle 3 \rangle 2$. PICK $b \in B$ such that $a < b < a_2$

$\langle 3 \rangle 3$. $\phi(b)$ is an upper bound for $\phi(S(a))$.

$\langle 1 \rangle 3$. $\bar{\phi}$ is monotone.

PROOF: If $a \leq a'$ then $S(a) \subseteq S(a')$ and so $\bar{\phi}(a) \leq \bar{\phi}(a')$.

$\langle 1 \rangle 4$. $\bar{\phi}$ extends ϕ .

$\langle 2 \rangle 1$. LET: $b \in B$

PROVE: $\phi(b) = \sup \phi(S(b))$

$\langle 2 \rangle 2$. $\phi(b)$ is an upper bound for $\phi(S(b))$

$\langle 2 \rangle 3$. LET: u be any upper bound for $\phi(S(b))$

PROVE: $\phi(b) \leq u$

$\langle 2 \rangle 4$. ASSUME: for a contradiction $u < \phi(b)$

$\langle 2 \rangle 5$. PICK $q \in Q$ such that $u < q < \phi(b)$

$\langle 2 \rangle 6$. PICK $b' \in B$ such that $\phi(b') = q$

$\langle 2 \rangle 7$. $b' < b$

$\langle 2 \rangle 8$. $b' \in S(b)$

$\langle 2 \rangle 9$. $q = \phi(b') \leq u$

$\langle 2 \rangle 10$. Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 5$. LET: $\bar{\psi} = \phi^{-1}$

$\langle 1 \rangle 6$. LET: $\bar{\psi} : P \rightarrow A$ be the function $\bar{\psi}(p) = \sup\{\psi(q) \mid q \in Q, q < p\}$

$\langle 1 \rangle 7$. $\bar{\psi}$ is monotone and extends ψ

PROOF: Similar.

$\langle 1 \rangle 8$. $\bar{\psi} \circ \bar{\phi} : A \rightarrow A$ is monotone and the identity on B .

$\langle 1 \rangle 9$. $\bar{\psi} \circ \bar{\phi} = \text{id}_A$

PROOF: Proposition 5.1.22.

$\langle 1 \rangle 10$. $\bar{\phi} \circ \bar{\psi} = \text{id}_B$

PROOF: Proposition 5.1.22.

$\langle 1 \rangle 11$. If $\phi^* : A \cong P$ is any order isomorphism that extends ϕ then $\phi^* = \bar{\phi}$.

$\langle 2 \rangle 1$. LET: $a \in A$

PROVE: $\phi^*(a) = \sup \phi(S(a))$

$\langle 2 \rangle 2$. $\phi^*(a)$ is an upper bound for $\phi(S(a))$

$\langle 2 \rangle 3$. LET: u be any upper bound for $\phi(S(a))$

PROVE: $\phi^*(a) \leq u$

$\langle 2 \rangle 4$. ASSUME: for a contradiction $u < \phi^*(a)$

$\langle 2 \rangle 5$. PICK $q \in Q$ such that $u < q < \phi^*(a)$

$\langle 2 \rangle 6$. PICK $b \in B$ such that $q = \phi(b)$

$\langle 2 \rangle 7$. $b < a$

$\langle 2 \rangle 8$. $b \in S(a)$

$\langle 2 \rangle 9$. $q = \phi(b) \leq u$

$\langle 2 \rangle 10$. Q.E.D.

PROOF: This is a contradiction.

□

Definition 5.1.24 (Initial Segment). Let \leq be a partial order on \mathbf{A} and $t \in A$.

The *initial segment* up to t is the class

$$\text{seg } t := \{x \in \mathbf{A} \mid x < t\} .$$

Definition 5.1.25 (Lexicographic Ordering). Let \mathbf{R} be a partial order on \mathbf{A} and \mathbf{S} a partial order on \mathbf{B} . The *lexicographic ordering* \leq on $\mathbf{A} \times \mathbf{B}$ is defined by:

$$(a, b) \leq (a', b') \Leftrightarrow (a\mathbf{R}a' \wedge a \neq a') \vee (a = a' \wedge b\mathbf{S}b') .$$

Proposition Schema 5.1.26. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{R} and \mathbf{S} , the following is a theorem:*

If \mathbf{R} is a partial order on \mathbf{A} and \mathbf{S} is a partial order on \mathbf{B} then the lexicographic ordering on $\mathbf{A} \times \mathbf{B}$ is a partial order.

PROOF:

$\langle 1 \rangle 1$. LET: \leq be the lexicographic ordering on $\mathbf{A} \times \mathbf{B}$

$\langle 1 \rangle 2$. \leq is reflexive.

PROOF: For any $a \in \mathbf{A}$ and $b \in \mathbf{B}$ we have $a = a$ and $b\mathbf{S}b$, so $(a, b) \leq (a, b)$.

$\langle 1 \rangle 3$. \leq is antisymmetric.

$\langle 2 \rangle 1$. ASSUME: $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$

$\langle 2 \rangle 2$. $(a\mathbf{R}a' \wedge a \neq a') \vee (a = a' \wedge b\mathbf{S}b')$

$\langle 2 \rangle 3$. $(a'\mathbf{R}a \wedge a' \neq a) \vee (a' = a \wedge b\mathbf{S}b')$

$\langle 2 \rangle 4$. CASE: $a = a'$

PROOF: Then $b\mathbf{S}b'$ and $b'\mathbf{S}b$ hence $b = b'$ and $(a, b) = (a', b')$.

$\langle 2 \rangle 5$. CASE: $a \neq a'$

PROOF: Then $a\mathbf{R}a'$ and $a'\mathbf{R}a$ hence $a = a'$ which is a contradiction.

$\langle 1 \rangle 4$. \leq is transitive.

$\langle 2 \rangle 1$. ASSUME: $(a_1, b_1) \leq (a_2, b_2) \leq (a_3, b_3)$

$\langle 2 \rangle 2$. $(a_1\mathbf{R}a_2 \wedge a_1 \neq a_2) \vee (a_1 = a_2 \wedge b_1\mathbf{S}b_2)$

$\langle 2 \rangle 3$. $(a_2\mathbf{R}a_3 \wedge a_2 \neq a_3) \vee (a_2 = a_3 \wedge b_2\mathbf{S}b_3)$

$\langle 2 \rangle 4$. CASE: $a_1\mathbf{R}a_2, a_1 \neq a_2, a_2\mathbf{R}a_3, a_2 \neq a_3$

$\langle 3 \rangle 1$. $a_1\mathbf{R}a_3$

PROOF: Since \mathbf{R} is transitive.

$\langle 3 \rangle 2$. $a_1 \neq a_3$

PROOF: If $a_1 = a_3$ then $a_1\mathbf{R}a_2$ and $a_2\mathbf{R}a_1$ so $a_1 = a_2$ which is a contradiction.

$\langle 2 \rangle 5$. CASE: $a_1\mathbf{R}a_2, a_1 \neq a_2, a_2 = a_3, b_2\mathbf{S}b_3$

PROOF: Then $a_1\mathbf{R}a_3$ and $a_1 \neq a_3$.

$\langle 2 \rangle 6$. CASE: $a_1 = a_2, b_1\mathbf{S}b_2, a_2\mathbf{R}a_3, a_2 \neq a_3$

PROOF: Then $a_1\mathbf{R}a_3$ and $a_1 \neq a_3$.

$\langle 2 \rangle 7$. CASE: $a_1 = a_2, b_1\mathbf{S}b_2, a_2 = a_3, b_2\mathbf{S}b_3$

PROOF: Then $a_1 = a_3$ and $b_1\mathbf{S}b_3$.

□

5.2 Linear Orders

Definition 5.2.1 (Linear Ordering). Let \mathbf{A} be a class. A *linear ordering* or *total ordering* on \mathbf{A} is a partial ordering \leq on \mathbf{A} that is *total*, i.e.

$$\forall x, y \in \mathbf{A}. x \leq y \vee y \leq x$$

We often use the symbol $<$ for a linear ordering, and then write $x < y$ for $(x, y) \in <$.

Proposition Schema 5.2.2 (Trichotomy). *For any classes \mathbf{A} and \leq , the following is a theorem:*

Assume \leq be a linear ordering on \mathbf{A} . For any $x, y \in \mathbf{A}$, exactly one of $x < y$, $x = y$, $y < x$ holds.

PROOF: Immediate from definitions. \square

Proposition Schema 5.2.3. *For any classes \mathbf{A} and $<$, the following is a theorem:*

Let $<$ be a transitive relation on \mathbf{A} that satisfies trichotomy. Define \leq on \mathbf{A} by $x \leq y$ iff $x < y$ or $x = y$. Then \leq is a linear ordering on \mathbf{A} and $x < y$ iff $x \leq y$ and $x \neq y$.

PROOF:

$\langle 1 \rangle 1.$ \leq is reflexive.

PROOF: By definition we have $\forall x \in \mathbf{A}. x \leq x$.

$\langle 1 \rangle 2.$ \leq is antisymmetric.

$\langle 2 \rangle 1.$ ASSUME: $x \leq y$ and $y \leq x$

$\langle 2 \rangle 2.$ $x < y$ or $x = y$

$\langle 2 \rangle 3.$ $y < x$ or $y = x$

$\langle 2 \rangle 4.$ We cannot have $x < y$ and $y < x$

PROOF: Trichotomy.

$\langle 2 \rangle 5.$ $x = y$

$\langle 1 \rangle 3.$ \leq is transitive.

$\langle 2 \rangle 1.$ ASSUME: $x \leq y$ and $y \leq z$

$\langle 2 \rangle 2.$ $x < y$ or $x = y$

$\langle 2 \rangle 3.$ $y < z$ or $y = z$

$\langle 2 \rangle 4.$ CASE: $x < y$ and $y < z$

PROOF: Then $x < z$ by transitivity, so $x \leq z$.

$\langle 2 \rangle 5.$ CASE: $x = y$

PROOF: Then we have $y \leq z$ and so $x \leq z$.

$\langle 2 \rangle 6.$ CASE: $y = z$

PROOF: Then we have $x \leq y$ and so $x \leq z$.

$\langle 1 \rangle 4.$ \leq is total.

PROOF: Immediate from trichotomy.

\square

Proposition Schema 5.2.4. *For any classes \mathbf{A} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} is a linear ordering on \mathbf{A} then \mathbf{R}^{-1} is also a linear ordering on \mathbf{A} .

PROOF:

$\langle 1 \rangle 1.$ \mathbf{R}^{-1} is a partial order on \mathbf{A} .

PROOF: Proposition 5.1.2.

$\langle 1 \rangle 2.$ \mathbf{R}^{-1} is total.

$\langle 2 \rangle 1.$ LET: $x, y \in \mathbf{A}$

$\langle 2 \rangle 2.$ $x\mathbf{R}y$ or $y\mathbf{R}x$.

$\langle 2 \rangle 3.$ $y\mathbf{R}^{-1}x$ or $x\mathbf{R}^{-1}y$.

□

Proposition Schema 5.2.5. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{F} , \mathbf{R} , \mathbf{S} , the following is a theorem:*

Assume \mathbf{R} is a linear order on \mathbf{A} , \mathbf{S} is a partial order on \mathbf{B} , and $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. If \mathbf{F} is strictly monotone then it is injective.

PROOF:

$\langle 1 \rangle 1.$ LET: $x, y \in \mathbf{A}$

$\langle 1 \rangle 2.$ ASSUME: $x \neq y$

PROVE: $\mathbf{F}(x) \neq \mathbf{F}(y)$

$\langle 1 \rangle 3.$ ASSUME: w.l.o.g. $x\mathbf{R}y$

PROOF: \mathbf{R} is total.

$\langle 1 \rangle 4.$ $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$ and $\mathbf{F}(x) \neq \mathbf{F}(y)$

PROOF: \mathbf{F} is strictly monotone.

□

Proposition Schema 5.2.6. *For any classes \mathbf{A} , \mathbf{B} , \leq , \preccurlyeq and \mathbf{F} , the following is a theorem:*

Assume \leq is a linear order on \mathbf{A} and \preccurlyeq is a linear order on \mathbf{B} . Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and \mathbf{F} is strictly monotone. For all $x, y \in \mathbf{A}$, if $\mathbf{F}(x) \prec \mathbf{F}(y)$ then $x < y$.

PROOF:

$\langle 1 \rangle 1.$ $\mathbf{F}(x) \neq \mathbf{F}(y)$ and $\mathbf{F}(y) \not\prec \mathbf{F}(x)$

PROOF: Trichotomy.

$\langle 1 \rangle 2.$ $x \neq y$ and $y \not< x$

PROOF: \mathbf{F} is strictly monotone.

$\langle 1 \rangle 3.$ $x < y$

PROOF: Trichotomy.

□

Corollary Schema 5.2.6.1. *For any classes \mathbf{A} , \mathbf{B} , \leq , \preccurlyeq and \mathbf{F} , the following is a theorem:*

Assume \leq is a linear order on \mathbf{A} and \preccurlyeq is a linear order on \mathbf{B} . Assume $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and \mathbf{F} is strictly monotone. Then \mathbf{F} is an order isomorphism.

Proposition Schema 5.2.7. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{F} and \mathbf{S} , the following is a theorem:*

Assume \mathbf{S} is a linear order on \mathbf{B} and $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Define \mathbf{R} on \mathbf{A} by $x\mathbf{R}y$ if and only if $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$. Then \mathbf{R} is a linear order on \mathbf{A} .

PROOF:

$\langle 1 \rangle 1.$ \mathbf{R} is a partial order on \mathbf{A} .

PROOF: Proposition 5.1.3.

$\langle 1 \rangle 2.$ \mathbf{R} is total.

PROOF: For all $x, y \in \mathbf{A}$ we have $\mathbf{F}(x)\mathbf{SF}(y)$ or $\mathbf{F}(y)\mathbf{SF}(x)$.

□

Corollary Schema 5.2.7.1. *For any classes \mathbf{A} , \mathbf{B} and \mathbf{R} , the following is a theorem:*

Assume \mathbf{R} be a linear order on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Then $\mathbf{R} \cap \mathbf{B}^2$ is a linear order on \mathbf{B} .

Proposition Schema 5.2.8. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{R} and \mathbf{S} , the following is a theorem:*

Assume \mathbf{R} is a linear order on \mathbf{A} and \mathbf{S} is a linear order on \mathbf{B} . Then the lexicographic ordering is a linear order on $\mathbf{A} \times \mathbf{B}$.

PROOF:

$\langle 1 \rangle 1.$ LET: \leq be the lexicographic order on $\mathbf{A} \times \mathbf{B}$

$\langle 1 \rangle 2.$ \leq is a partial order.

PROOF: Proposition 5.1.26.

$\langle 1 \rangle 3.$ \leq is total.

$\langle 2 \rangle 1.$ LET: $a, a' \in \mathbf{A}$ and $b, b' \in \mathbf{B}$

$\langle 2 \rangle 2.$ CASE: $a\mathbf{R}a'$ and $a \neq a'$

PROOF: Then $(a, b) \leq (a', b')$.

$\langle 2 \rangle 3.$ CASE: $a = a'$

PROOF: We have $b\mathbf{S}b'$ or $b'\mathbf{S}b$, so $(a, b) \leq (a', b')$ or $(a', b') \leq (a, b)$.

$\langle 2 \rangle 4.$ CASE: $a'\mathbf{R}a$ and $a \neq a'$

PROOF: Then $(a', b') \leq (a, b)$.

□

5.3 Well Orderings

Definition 5.3.1 (Well Ordering). A *well ordering* on a class \mathbf{A} is a linear ordering on \mathbf{A} such that:

- For all $t \in \mathbf{A}$ we have $\text{seg } t$ is a set.
- Every nonempty subset of \mathbf{A} has a least element.

Proposition Schema 5.3.2. *For any classes \mathbf{A} , \mathbf{B} and \leq , the following is a theorem:*

Assume \leq well orders \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$ is nonempty. Then \mathbf{B} has a least element.

PROOF:

$\langle 1 \rangle 1.$ PICK $b \in \mathbf{B}$

$\langle 1 \rangle 2.$ LET: $S = \text{seg } b \cap \mathbf{B}$

⟨1⟩3. CASE: $S = \emptyset$

PROOF: In this case b is the least element of \mathbf{B} .

⟨1⟩4. CASE: $S \neq \emptyset$

PROOF: In this cases S has a least element, which is the least element of \mathbf{B} .

□

Proposition Schema 5.3.3. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{F} and \mathbf{S} , the following is a theorem:*

Assume \mathbf{S} well orders \mathbf{B} and $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Define \mathbf{R} on \mathbf{A} by $x\mathbf{R}y$ if and only if $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$. Then \mathbf{R} well orders \mathbf{A} .

PROOF:

⟨1⟩1. \mathbf{R} linearly orders \mathbf{A} .

PROOF: Proposition 5.2.7.

⟨1⟩2. For all $t \in \mathbf{A}$ we have $\{x \in \mathbf{A} \mid x\mathbf{R}t \wedge x \neq t\}$ is a set.

⟨2⟩1. LET: $t \in \mathbf{A}$

⟨2⟩2. LET: $S = \{y \in \mathbf{B} \mid y\mathbf{S}\mathbf{F}(t) \wedge y \neq \mathbf{F}(t)\}$

⟨2⟩3. LET: $P(x, y)$ be the property $\mathbf{F}(y) = x$

⟨2⟩4. For all $x \in S$ there exists at most one y such that $P(x, y)$

PROOF: \mathbf{F} is injective.

⟨2⟩5. LET: $T = \{y \mid \exists x \in S.P(x, y)\}$

PROOF: Axiom of Replacement.

⟨2⟩6. $T = \{x \in \mathbf{A} \mid x\mathbf{R}t \wedge x \neq t\}$

⟨1⟩3. Every nonempty subset of \mathbf{A} has a least element.

⟨2⟩1. LET: S be a nonempty subset of \mathbf{A} .

⟨2⟩2. LET: y be the least element of $\mathbf{F}(S)$.

⟨2⟩3. PICK $x \in S$ such that $\mathbf{F}(x) = y$.

⟨2⟩4. x is least in S .

□

Corollary Schema 5.3.3.1. *For any classes \mathbf{A} , \mathbf{B} and \mathbf{R} , the following is a theorem:*

If \mathbf{R} well orders \mathbf{B} and $\mathbf{A} \subseteq \mathbf{B}$ then $\mathbf{R} \cap \mathbf{A}^2$ well orders \mathbf{A} .

Proposition 5.3.4. *For any well ordered sets A and B , the lexicographic order well orders $A \times B$.*

PROOF:

⟨1⟩1. $A \times B$ is linearly ordered.

PROOF: Proposition 5.2.8.

⟨1⟩2. Every nonempty subset of $A \times B$ has a least element.

⟨2⟩1. LET: S be a nonempty subset of $A \times B$.

⟨2⟩2. LET: a be the least element of $\{x \in A \mid \exists y \in B.(x, y) \in S\}$.

⟨2⟩3. LET: b be the least element of $\{y \in B \mid (a, y) \in S\}$.

⟨2⟩4. (a, b) is least in S .

□

Definition 5.3.5 (End Extension). Let A and B be well ordered sets. Then B is an *end extension* of A iff $A \subseteq B$ and:

- Whenever $x, y \in A$ then $x \leq_A y$ iff $x \leq_B y$.
- Whenever $x \in A$ and $y \in B - A$ then $x < y$.

5.3.1 Transfinite Induction

Theorem Schema 5.3.6 (Transfinite Induction Principle). *For any classes \mathbf{A} , \mathbf{B} and \leq , the following is a theorem:*

Assume \leq is a well ordering on \mathbf{A} and $\mathbf{B} \subseteq \mathbf{A}$. Assume that, for all $t \in \mathbf{A}$,

$$\text{seg } t \subseteq \mathbf{B} \Rightarrow t \in \mathbf{B} .$$

Then $\mathbf{B} = \mathbf{A}$.

PROOF:

- $\langle 1 \rangle 1$. ASSUME: for a contradiction $\mathbf{B} \neq \mathbf{A}$
- $\langle 1 \rangle 2$. LET: m be the least element of $\mathbf{A} - \mathbf{B}$
- PROOF: Proposition 5.3.2.
- $\langle 1 \rangle 3$. $\text{seg } m \subseteq \mathbf{B}$
- PROOF: By leastness of m .
- $\langle 1 \rangle 4$. $m \in \mathbf{B}$
- $\langle 1 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

□

Theorem 5.3.7. *Let \leq be a linear ordering on A . Assume that, for any $B \subseteq A$ such that $\forall t \in A. \text{seg } t \subseteq B \Rightarrow t \in B$, we have $B = A$. Then \leq is a well ordering on A .*

PROOF:

- $\langle 1 \rangle 1$. LET: $C \subseteq A$ be nonempty.
- $\langle 1 \rangle 2$. LET: $B = \{t \in A \mid \forall x \in C. t < x\}$
- $\langle 1 \rangle 3$. $B \cap C = \emptyset$
- $\langle 1 \rangle 4$. $B \neq A$
- $\langle 1 \rangle 5$. PICK $t \in A$ such that $\text{seg } t \subseteq B$ and $t \notin B$
- $\langle 1 \rangle 6$. t is least in C .

□

Proposition Schema 5.3.8. *For any classes \mathbf{A} , \mathbf{B} , \mathbf{F} , \mathbf{G} , \leq and \preceq , the following is a theorem:*

Assume \leq well orders \mathbf{A} and \preceq well orders \mathbf{B} . Assume \mathbf{F} and \mathbf{G} are order isomorphisms between \mathbf{A} and \mathbf{B} . Then $\mathbf{F} = \mathbf{G}$.

PROOF:

- $\langle 1 \rangle 1$. For all $x \in \mathbf{A}$, if $\forall t < x. \mathbf{F}(t) = \mathbf{G}(t)$, then $\mathbf{F}(x) = \mathbf{G}(x)$
- $\langle 2 \rangle 1$. LET: $x \in \mathbf{A}$
- $\langle 2 \rangle 2$. ASSUME: $\forall t < x. \mathbf{F}(t) = \mathbf{G}(t)$
- $\langle 2 \rangle 3$. $\mathbf{F}(\text{seg } x) = \mathbf{G}(\text{seg } x)$

- ⟨2⟩4. $\mathbf{F}(x)$ is the least element of $\mathbf{B} - \mathbf{F}(\text{seg } x)$
- ⟨2⟩5. $\mathbf{G}(x)$ is the least element of $\mathbf{B} - \mathbf{G}(\text{seg } x)$
- ⟨2⟩6. $\mathbf{F}(x) = \mathbf{G}(x)$
- ⟨1⟩2. $\forall x \in \mathbf{A}. \mathbf{F}(x) = \mathbf{G}(x)$

PROOF: Transfinite induction.

□

5.3.2 Transfinite Recursion

Theorem Schema 5.3.9 (Transfinite Recursion Theorem Schema). *Let \mathbf{A} and $<$ be classes. For any property $G(x, y)$, there exists a class \mathbf{F} such that the following is a theorem:*

Assume that $<$ is a well ordering on the class \mathbf{A} . Assume that, for any f , there exists a unique y such that $G(f, y)$. Then $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{V}$ is the unique function such that, for all $t \in \mathbf{A}$, we have $\mathbf{F} \upharpoonright \text{seg } t$ is a set and

$$G(\mathbf{F} \upharpoonright \text{seg } t, \mathbf{F}(t)) .$$

PROOF:

- ⟨1⟩1. For $t \in \mathbf{A}$, let us say that a function $v : \text{seg } t \rightarrow \mathbf{V}$ is *G-constructed up to t* iff

$$\forall x < t. G(v \upharpoonright \text{seg } x, v(x)) .$$

- ⟨1⟩2. Let \mathbf{K} be the class of all functions v such that there exists $t \in \mathbf{A}$ such that v is *G-constructed up to t* .
- ⟨1⟩3. LET: $\mathbf{F} = \bigcup \mathbf{K}$
- ⟨1⟩4. For all $t_1, t_2 \in \mathbf{A}$ with $t_1 \leq t_2$, if v_1 is *G-constructed up to t_1* and v_2 is *G-constructed up to t_2* , then $\forall x \leq t_1. v_1(x) = v_2(x)$.

- ⟨2⟩1. LET: $x \in \mathbf{V}$
- ⟨2⟩2. ASSUME: $\forall y < x. (y \leq t_1 \Rightarrow v_1(y) = v_2(y))$
- ⟨2⟩3. ASSUME: $x \leq t_1$
- ⟨2⟩4. $G(v_1 \upharpoonright \text{seg } x, v_1(x))$
- ⟨2⟩5. $G(v_2 \upharpoonright \text{seg } x, v_2(x))$
- ⟨2⟩6. $v_1 \upharpoonright \text{seg } x = v_2 \upharpoonright \text{seg } x$
- ⟨2⟩7. $v_1(x) = v_2(x)$
- ⟨2⟩8. Q.E.D.

PROOF: By transfinite induction.

- ⟨1⟩5. \mathbf{F} is a function.
 - ⟨2⟩1. LET: $(x, y_1), (x, y_2) \in \mathbf{F}$
 - ⟨2⟩2. PICK $v_1, v_2 \in \mathbf{K}$ such that $v_1(x) = y_1$ and $v_2(x) = y_2$.
 - ⟨2⟩3. PICK $t_1, t_2 \in \mathbf{A}$ such that v_1 is *G-constructed up to t_1* and v_2 is *G-constructed up to t_2* .
 - ⟨2⟩4. ASSUME: w.l.o.g. $t_1 \leq t_2$
 - ⟨2⟩5. $v_1(x) = v_2(x)$
- PROOF: ⟨1⟩4.
- ⟨2⟩6. $y_1 = y_2$
- ⟨1⟩6. $\forall x \in \text{dom } \mathbf{F}. G(\mathbf{F} \upharpoonright \text{seg } x, \mathbf{F}(x))$

- ⟨2⟩1. LET: $x \in \text{dom } F$
- ⟨2⟩2. PICK $v \in \mathbf{K}$ such that $v(x) = \mathbf{F}(x)$
- ⟨2⟩3. PICK t such that v is G -constructed up to t .
- ⟨2⟩4. $G(v \upharpoonright \text{seg } x, v(x))$
- ⟨2⟩5. $v \upharpoonright \text{seg } x = \mathbf{F} \upharpoonright \text{seg } x$
 - ⟨3⟩1. LET: $y < x$
 - PROVE: $v(y) = \mathbf{F}(y)$
 - ⟨3⟩2. $y \in \text{dom } \mathbf{F}$
 - PROOF: Since $y \in \text{dom } v$ and $v \in \mathbf{K}$.
 - ⟨3⟩3. PICK $u \in \mathbf{K}$ such that $u(y) = F(y)$
 - ⟨3⟩4. $u(y) = v(y)$
 - PROOF: ⟨1⟩4.
 - ⟨3⟩5. $v(y) = \mathbf{F}(y)$
- ⟨2⟩6. $G(\mathbf{F} \upharpoonright \text{seg } x, \mathbf{F}(x))$
- ⟨1⟩7. $\text{dom } \mathbf{F} = \mathbf{A}$
 - ⟨2⟩1. ASSUME: $\text{dom } \mathbf{F} \neq \mathbf{A}$
 - ⟨2⟩2. LET: t be least in $\mathbf{A} - \text{dom } \mathbf{F}$
 - ⟨2⟩3. $\text{seg } t = \text{dom } \mathbf{F}$
 - ⟨2⟩4. \mathbf{F} is a set.
 - PROOF: Axiom of Replacement.
 - ⟨2⟩5. LET: y be the unique object such that $G(\mathbf{F}, y)$
 - ⟨2⟩6. LET: $v = \mathbf{F} \cup \{(t, y)\}$
 - ⟨2⟩7. v is G -constructed up to t .
 - ⟨2⟩8. $t \in \text{dom } \mathbf{F}$
 - ⟨2⟩9. Q.E.D.
 - PROOF: This is a contradiction.
- ⟨1⟩8. \mathbf{F} is unique.
- PROOF: If \mathbf{F}' also satisfies the theorem, prove $\mathbf{F}(x) = \mathbf{F}'(x)$ by transfinite induction on x .

□

Theorem 5.3.10. *Let A and B be well ordered sets. Then one of the following holds: $A \cong B$; there exists $b \in B$ such that $A \cong \text{seg } b$; there exists $a \in A$ such that $\text{seg } a \cong B$.*

PROOF:

- ⟨1⟩1. PICK e that is not in A or B .
- ⟨1⟩2. LET: $F : A \rightarrow B \cup \{e\}$ be the function defined by transfinite recursion thus:

$$F(t) = \begin{cases} \text{the least element of } B - F(\text{seg } t) & \text{if } B - F(\text{seg } t) \neq \emptyset \\ e & \text{if } B - F(\text{seg } t) = \emptyset \end{cases}$$
- ⟨1⟩3. CASE: $e \in \text{ran } F$
 - ⟨2⟩1. LET: t be least such that $F(t) = e$
 - ⟨2⟩2. $F \upharpoonright \text{seg } t : \text{seg } t \cong B$
- ⟨1⟩4. CASE: $\text{ran } F = B$
- PROOF: We have $F : A \cong B$

$\langle 1 \rangle 5$. CASE: $\text{ran } F \subsetneq B$

$\langle 2 \rangle 1$. LET: b be the least element of $B - \text{ran } F$

$\langle 2 \rangle 2$. $F : A \cong \text{seg } b$

□

Chapter 6

Ordinal Numbers

6.1 Ordinals

Definition 6.1.1 (Ordinal Number). An *ordinal (number)* is a transitive set α that is *well-ordered by* \in ; that is, such that $\{(x, y) \in \alpha^2 \mid x \in y \vee x = y\}$ well orders α .

Given $x, y \in \alpha$, we write $x < y$ iff $x \in y$, and $x \leq y$ iff $x \in y$ or $x = y$.

Let \mathbf{On} be the class of ordinal numbers. For $\alpha, \beta \in \mathbf{On}$, we write $\alpha < \beta$ iff $\alpha \in \beta$, and $\alpha \leq \beta$ iff $\alpha < \beta$ or $\alpha = \beta$.

Proposition 6.1.2. For any ordinal numbers α and β , if $\alpha \cong \beta$ then $\alpha = \beta$.

PROOF:

$\langle 1 \rangle 1$. LET: $f : \alpha \cong \beta$

$\langle 1 \rangle 2$. For all $x \in \alpha$, if $\forall t < x. f(t) = t$ then $f(x) = x$

$\langle 2 \rangle 1$. $f(x) \subseteq x$

$\langle 3 \rangle 1$. LET: $y \in f(x)$

$\langle 3 \rangle 2$. $y \in \beta$

$\langle 3 \rangle 3$. PICK $t \in \alpha$ such that $f(t) = y$

PROOF: f is surjective.

$\langle 3 \rangle 4$. $f(t) \in f(x)$

$\langle 3 \rangle 5$. $t \in x$

PROOF: Since f is an order isomorphism.

$\langle 3 \rangle 6$. $f(t) = t$

PROOF: Induction hypothesis.

$\langle 3 \rangle 7$. $y = t$

$\langle 3 \rangle 8$. $y \in x$

$\langle 2 \rangle 2$. $x \subseteq f(x)$

$\langle 3 \rangle 1$. LET: $t \in x$

$\langle 3 \rangle 2$. $f(t) \in f(x)$

$\langle 3 \rangle 3$. $f(t) = t$

$\langle 3 \rangle 4$. $t \in f(x)$

$\langle 1 \rangle 3. \forall x \in \alpha. f(x) = x$

PROOF: Transfinite induction.

$\langle 1 \rangle 4. \alpha = \beta$

PROOF: Since $\beta = \{f(t) \mid t \in \alpha\} = \{t \mid t \in \alpha\} = \alpha$.

□

Theorem 6.1.3. *Every well-ordered set is isomorphic to a unique ordinal.*

PROOF:

$\langle 1 \rangle 1.$ For any well-ordered set A , there exists an ordinal α such that $A \cong \alpha$.

$\langle 2 \rangle 1.$ LET: A be a well-ordered set.

$\langle 2 \rangle 2.$ Define the function E on A by transfinite recursion thus:

$$E(t) = \{E(x) \mid x < t\} \quad (t \in A) .$$

$\langle 2 \rangle 3.$ LET: $\alpha = \{E(x) \mid x \in A\}$

$\langle 2 \rangle 4.$ α is an ordinal.

$\langle 3 \rangle 1.$ α is a transitive set.

$\langle 4 \rangle 1.$ LET: $x \in y \in \alpha$

$\langle 4 \rangle 2.$ PICK $t \in A$ such that $y = E(t)$

$\langle 4 \rangle 3.$ $x \in E(t) = \{E(s) \mid s < t\}$

$\langle 4 \rangle 4.$ PICK $s < t$ such that $x = E(s)$

$\langle 4 \rangle 5.$ $x \in \alpha$

$\langle 3 \rangle 2.$ α is well-ordered by \in .

$\langle 4 \rangle 1.$ LET: $< = \{(x, y) \in \alpha \mid x \in y\}$

$\langle 4 \rangle 2.$ $<$ is transitive.

$\langle 5 \rangle 1.$ LET: $x, y, z \in \alpha$ with $x \in y \in z$

$\langle 5 \rangle 2.$ PICK $t \in A$ such that $z = E(t)$

$\langle 5 \rangle 3.$ PICK $s \in A$ such that $s < t$ and $y = E(s)$

$\langle 5 \rangle 4.$ PICK $r \in A$ such that $r < s$ and $x = E(r)$

$\langle 5 \rangle 5.$ $r < t$

$\langle 5 \rangle 6.$ $x \in z$

$\langle 4 \rangle 3.$ $<$ satisfies trichotomy.

$\langle 5 \rangle 1.$ LET: $x, y \in \alpha$

$\langle 5 \rangle 2.$ PICK $s, t \in A$ such that $E(s) = x$ and $E(t) = y$

$\langle 5 \rangle 3.$ Exactly one of $s < t$, $s = t$, $t < s$ holds.

$\langle 5 \rangle 4.$ CASE: $s < t$

$\langle 6 \rangle 1.$ $x \in y$

$\langle 6 \rangle 2.$ $x \neq y$ and $y \notin x$

PROOF: Axiom of Regularity.

$\langle 5 \rangle 5.$ CASE: $s = t$

$\langle 6 \rangle 1.$ $x = y$

$\langle 6 \rangle 2.$ $x \notin y$ and $y \notin x$

PROOF: Axiom of Regularity.

$\langle 5 \rangle 6.$ CASE: $t < s$

PROOF: Similar to $\langle 5 \rangle 4.$

$\langle 4 \rangle 4.$ \leq is a linear order on α .

PROOF: Proposition 5.2.3.

$\langle 4 \rangle 5.$ Every nonempty subset of α has a least element.

- ⟨5⟩1. LET: S be a nonempty subset of α
 - ⟨5⟩2. LET: $T = \{x \in A \mid E(x) \in S\}$
 - ⟨5⟩3. LET: t be the least element of T .
PROVE: $E(t)$ is least in S
 - ⟨5⟩4. LET: $y \in S$
 - ⟨5⟩5. PICK $s \in T$ such that $E(s) = y$
 - ⟨5⟩6. $t \leq s$
 - ⟨5⟩7. $x \leq y$
 - ⟨2⟩5. E is surjective.
PROOF: By definition of α .
 - ⟨2⟩6. E is strictly monotone.
PROOF: If $s < t$ then $E(s) \in E(t)$ by definition of $E(t)$.
 - ⟨2⟩7. Q.E.D.
PROOF: Corollary 5.2.6.1.
 - ⟨1⟩2. For any ordinals α and β , if $\alpha \cong \beta$ then $\alpha = \beta$.
PROOF: Proposition 6.1.2.
-

Proposition 6.1.4. *The class **On** is a transitive class. That is, every element of an ordinal is an ordinal.*

PROOF:

- ⟨1⟩1. LET: α be an ordinal.
 - ⟨1⟩2. LET: $\beta \in \alpha$
 - ⟨1⟩3. β is a transitive set.
⟨2⟩1. LET: $x \in y \in \beta$
⟨2⟩2. $y \in \alpha$
PROOF: α is transitive.
 - ⟨2⟩3. $x \in \alpha$
PROOF: α is transitive.
 - ⟨2⟩4. $x \in \beta$
PROOF: Since $\{(x, y) \in \alpha^2 \mid x \in y\}$ is transitive.
 - ⟨1⟩4. β is well ordered by \in .
PROOF: By Corollary 5.3.3.1.
-

Proposition 6.1.5. *Given two ordinal numbers α, β , exactly one of $\alpha \in \beta$, $\alpha = \beta$, $\beta \in \alpha$ holds.*

PROOF:

- ⟨1⟩1. At most one holds.
PROOF: Since every ordinal is a transitive set and we never have $\alpha \in \alpha$.
- ⟨1⟩2. At least one holds.
⟨2⟩1. Either $\alpha \cong \beta$ or $\exists t \in \beta. \alpha \cong \text{seg } t$ or $\exists t \in \alpha. \text{seg } t \cong \beta$.
- ⟨2⟩2. CASE: $\alpha \cong \beta$
PROOF: Then $\alpha = \beta$ by Proposition 6.1.2.
- ⟨2⟩3. CASE: There exists $t \in \beta$ such that $\alpha \cong \text{seg } t$

- ⟨3⟩1. t is an ordinal number.
PROOF: Proposition 6.1.4.
- ⟨3⟩2. $t = \text{seg } t$
 - ⟨4⟩1. $t \subseteq \text{seg } t$
 - ⟨5⟩1. LET: $s \in t$
 - ⟨5⟩2. $s \in \beta$
PROOF: β is a transitive set.
 - ⟨5⟩3. $s \in \text{seg } t$
 - ⟨4⟩2. $\text{seg } t \subseteq t$
PROOF: Immediate from definitions.
- ⟨3⟩3. $\alpha = t$
PROOF: Proposition 6.1.2.
- ⟨3⟩4. $\alpha \in \beta$
- ⟨2⟩4. CASE: There exists $t \in \alpha$ such that $\text{seg } t \cong \beta$
PROOF: $\beta \in \alpha$ similarly.

□

Proposition 6.1.6. *Any nonempty set S of ordinal numbers has a least element.*

PROOF:

- ⟨1⟩1. PICK $\beta \in S$
- ⟨1⟩2. CASE: $\beta \cap S = \emptyset$
PROOF: Then β is least in S .
- ⟨1⟩3. CASE: $\beta \cap S \neq \emptyset$
PROOF: The least element of $\beta \cap S$ is least in S .

□

Theorem 6.1.7. *The class \mathbf{On} is well ordered by \in .*

PROOF:

- ⟨1⟩1. LET: $\mathbf{E} = \{(x, y) \in \mathbf{On}^2 \mid x \in y\}$
- ⟨1⟩2. \mathbf{E} is transitive.
PROOF: If $\alpha \in \beta \in \gamma$ then $\alpha \in \gamma$ because every ordinal is a transitive set.
- ⟨1⟩3. \mathbf{E} satisfies trichotomy.
PROOF: Proposition 6.1.5.
- ⟨1⟩4. \mathbf{E} linearly orders \mathbf{On} .
PROOF: Proposition 5.2.3.
- ⟨1⟩5. For any $\alpha \in \mathbf{On}$, the class $\{\beta \in \mathbf{On} \mid \beta \mathbf{E} \alpha\}$ is a set.
PROOF: It is equal to α .
- ⟨1⟩6. Every nonempty subset of \mathbf{On} has an \mathbf{E} -least element.
PROOF: Proposition 6.1.6.

□

Corollary 6.1.7.1 (Burali-Forti Paradox). *The class \mathbf{On} is a proper class.*

PROOF: If it were a set, it would be a transitive set well-ordered by \in , and hence a member of itself, contradicting the Axiom of Regularity.

Proposition 6.1.8. *Any transitive set of ordinal numbers is an ordinal number.*

PROOF: It is well-ordered by \in by Corollary 5.3.3.1 and Theorem 6.1.7. \square

Proposition 6.1.9. \emptyset is an ordinal number.

PROOF: Vacuously, it is a transitive set well-ordered by \in .

Definition 6.1.10. We define $0 = \emptyset$.

Proposition 6.1.11. *If A is a set of ordinal numbers then $\bigcup A$ is an ordinal number.*

PROOF:

$\langle 1 \rangle 1.$ $\bigcup A$ is a transitive set.

PROOF: Proposition 1.5.3.

$\langle 1 \rangle 2.$ $\bigcup A$ is a set of ordinals.

PROOF: Proposition 6.1.4.

\square

Corollary 6.1.11.1. *The poset \mathbf{On} is complete.*

PROOF: For any nonempty set A of ordinals, $\bigcup A$ is its supremum. \square

Proposition 6.1.12. *Let α be an ordinal and $S \subseteq \alpha$. Then S is well-ordered by \in and the ordinal of (S, \in) is $\leq \alpha$.*

PROOF:

$\langle 1 \rangle 1.$ S is well ordered by \in .

$\langle 1 \rangle 2.$ LET: β be the ordinal of (S, \in)

$\langle 1 \rangle 3.$ LET: $E : S \approx \beta$ be the unique isomorphism.

$\langle 1 \rangle 4.$ $\forall \gamma \in S. E(\gamma) \leq \gamma$

$\langle 2 \rangle 1.$ LET: $\gamma \in S$

$\langle 2 \rangle 2.$ ASSUME: as transfinite induction hypothesis $\forall \delta < \gamma. E(\delta) \leq \delta$

$\langle 2 \rangle 3.$ $E(\gamma)$ is the least element of β that is greater than $E(\delta)$ for all $\delta < \gamma$

$\langle 2 \rangle 4.$ γ is greater than $E(\delta)$ for all $\delta < \gamma$

$\langle 2 \rangle 5.$ $E(\gamma) \leq \gamma$

$\langle 1 \rangle 5.$ $\beta \leq \alpha$

$\langle 2 \rangle 1.$ $\forall \gamma < \beta. \gamma < \alpha$

$\langle 3 \rangle 1.$ LET: $\gamma < \beta$

$\langle 3 \rangle 2.$ PICK $\delta \in S$ such that $E(\delta) = \gamma$

$\langle 3 \rangle 3.$ $\gamma = E(\delta) \leq \delta < \alpha$

\square

Proposition 6.1.13. *Let α be a set. Then the following are equivalent.*

1. α is an ordinal.
2. α is a transitive set and, for all $x, y \in \alpha$, either $x = y$ or $x \in y$ or $y \in x$.
3. α is a transitive set of transitive sets.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: α is a transitive set and, for all $x, y \in \alpha$, either $x = y$ or $x \in y$ or $y \in x$

$\langle 2 \rangle 2.$ LET: $z \in \alpha$

PROVE: z is transitive.

$\langle 2 \rangle 3.$ LET: $x \in y \in z$

$\langle 2 \rangle 4.$ $y \in \alpha$

$\langle 2 \rangle 5.$ $x \in \alpha$

$\langle 2 \rangle 6.$ Either $x = z$ or $x \in z$ or $z \in x$

$\langle 2 \rangle 7.$ $x \neq z$

PROOF: We cannot have $x \in y \in x$ by the Axiom of Regularity.

$\langle 2 \rangle 8.$ $z \notin x$

PROOF: We cannot have $x \in y \in z \in x$ by the Axiom of Regularity.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

$\langle 2 \rangle 1.$ LET: x be a transitive set of transitive sets.

$\langle 2 \rangle 2.$ ASSUME: as \in -induction hypothesis that, for all $y \in x$, if y is a transitive set of transitive sets then y is a transitive set of ordinals.

$\langle 2 \rangle 3.$ Every element of x is an ordinal.

$\langle 3 \rangle 1.$ LET: $y \in x$

$\langle 3 \rangle 2.$ y is transitive.

$\langle 3 \rangle 3.$ Every element of y is transitive.

PROOF: Since every element of y is an element of x , because x is transitive.

$\langle 3 \rangle 4.$ y is an ordinal.

PROOF: $\langle 2 \rangle 2$

$\langle 2 \rangle 4.$ Q.E.D.

PROOF: Proposition 6.1.8.

□

Lemma 6.1.14. *Let A and B be well-ordered sets. If B is an end extension of A then the ordinal of A is \leq the ordinal of B .*

PROOF:

$\langle 1 \rangle 1.$ LET: α be the ordinal of A and β the ordinal of B .

$\langle 1 \rangle 2.$ LET: $E_A : A \cong \alpha$ and $E_B : B \cong \beta$ be the canonical isomorphisms.

$\langle 1 \rangle 3.$ $\forall a \in A. E_A(a) = E_B(a)$

$\langle 2 \rangle 1.$ LET: $a \in A$

$\langle 2 \rangle 2.$ ASSUME: as transfinite induction hypothesis $\forall x < a. E_A(x) = E_B(x)$

$\langle 2 \rangle 3.$ $E_A(a)$ is the least ordinal that is greater than $E_A(x)$ for all $x < a$

$\langle 2 \rangle 4.$ $E_B(a)$ is the least ordinal that is greater than $E_B(x)$ for all $x < b$

$\langle 2 \rangle 5.$ $\{x \in A \mid x <_A a\} = \{x \in B \mid x <_B a\}$

$\langle 2 \rangle 6.$ $E_A(a) = E_B(a)$

$\langle 1 \rangle 4.$ $\alpha \subseteq \beta$

$\langle 1 \rangle 5.$ $\alpha \leq \beta$

□

Lemma 6.1.15. *Let \mathcal{C} be a set of well ordered sets such that, for any $A, B \in \mathcal{C}$, we have that one of A and B is an end extension of the other. Let $W = \bigcup \mathcal{C}$ under $x \leq y$ iff there exists $A \in \mathcal{C}$ such that $x, y \in A$ and $x \leq y$. Then W is a well ordered set whose ordinal is the supremum of the ordinals of the members of \mathcal{C} .*

PROOF:

- ⟨1⟩1. \leq is reflexive on W .
 - ⟨2⟩1. LET: $x \in W$
 - ⟨2⟩2. PICK $A \in \mathcal{C}$ such that $x \in A$.
 - ⟨2⟩3. $x \leq x$
- ⟨1⟩2. \leq is antisymmetric on W .
 - ⟨2⟩1. LET: $x, y \in W$
 - ⟨2⟩2. ASSUME: $x \leq y$ and $y \leq x$
 - ⟨2⟩3. PICK $A \in \mathcal{C}$ such that $x, y \in A$ and $x \leq_A y$, and $B \in \mathcal{C}$ such that $x, y \in B$ and $y \leq_B x$
 - ⟨2⟩4. ASSUME: w.l.o.g. B is an end extension of A
 - ⟨2⟩5. $x \leq_B y$ and $y \leq_B x$
 - ⟨2⟩6. $x = y$
- ⟨1⟩3. \leq is transitive on W .
 - ⟨2⟩1. ASSUME: $x \leq y \leq z$
 - ⟨2⟩2. PICK $A, B \in \mathcal{C}$ such that $x \leq_A y$ and $y \leq_B z$
 - ⟨2⟩3. CASE: A is an end extension of B .
 - ⟨3⟩1. $x \leq_A y$ and $y \leq_A z$
 - ⟨3⟩2. $x \leq_A z$
 - ⟨3⟩3. $x \leq z$
 - ⟨2⟩4. CASE: B is an end extension of A .

PROOF: Similar.
- ⟨1⟩4. \leq is total on W .
 - ⟨2⟩1. LET: $x, y \in W$
 - ⟨2⟩2. PICK $A, B \in \mathcal{C}$ such that $x \in A$ and $y \in B$
 - ⟨2⟩3. ASSUME: w.l.o.g. B is an end extension of A
 - ⟨2⟩4. $x \leq_B y$ or $y \leq_B x$
 - ⟨2⟩5. $x \leq_W y$ or $y \leq_W x$
- ⟨1⟩5. Every nonempty subset of W has a least element.
 - ⟨2⟩1. LET: S be a nonempty subset of W
 - ⟨2⟩2. PICK $s \in S$
 - ⟨2⟩3. PICK $A \in \mathcal{C}$ such that $s \in A$
 - ⟨2⟩4. LET: a be the \leq_A -least element of $S \cap A$

PROVE: a is least in S
 - ⟨2⟩5. LET: $x \in S$

PROVE: $a \leq x$
 - ⟨2⟩6. PICK $B \in \mathcal{C}$ such that $x \in B$
 - ⟨2⟩7. CASE: A is an end extension of B
 - ⟨3⟩1. $a \leq_A x$

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6.2 Successors

Definition 6.2.1 (Successor). The *successor* of a set a is the set $a^+ := a \cup \{a\}$.

Proposition 6.2.2. *A set a is a transitive set if and only if*

$$\bigcup(a^+) = a .$$

PROOF:

$\langle 1 \rangle 1$. If a is a transitive set then $\bigcup(a^+) = a$.

$\langle 2 \rangle 1$. ASSUME: a is a transitive set.

$\langle 2 \rangle 2$. $\bigcup(a^+) \subseteq a$

$\langle 3 \rangle 1$. LET: $x \in \bigcup(a^+)$

PROVE: $x \in a$

$\langle 3 \rangle 2$. PICK $y \in a^+$ such that $x \in y$.

$\langle 3 \rangle 3$. $y \in a$ or $y = a$.

$\langle 3 \rangle 4$. CASE: $y \in a$

PROOF: Then $x \in a$ because a is a transitive set.

$\langle 3 \rangle 5$. CASE: $y = a$

PROOF: Then $x \in a$ immediately.

$\langle 2 \rangle 3$. $a \subseteq \bigcup(a^+)$

PROOF: Since $a \in a^+$.

$\langle 1 \rangle 2$. If $\bigcup(a^+) = a$ then a is a transitive set.

$\langle 2 \rangle 1$. ASSUME: $\bigcup(a^+) = a$

$\langle 2 \rangle 2$. $\bigcup a \subseteq a$

PROOF:

$$\begin{aligned} \bigcup a &\subseteq \bigcup(a^+) && \text{(Proposition 1.4.8)} \\ &= a && (\langle 2 \rangle 1) \end{aligned}$$

$\langle 2 \rangle 3$. a is a transitive set.

PROOF: Proposition 1.5.2.

□

Proposition 6.2.3. *For any set a , we have a is a transitive set if and only if a^+ is a transitive set.*

PROOF:

$\langle 1 \rangle 1$. If a is a transitive set then a^+ is a transitive set.

PROOF: If a is a transitive set then $\bigcup(a^+) = a \subseteq a^+$ by Proposition 6.2.2 and so a^+ is a transitive set.

$\langle 1 \rangle 2$. If a^+ is a transitive set then a is a transitive set.

$\langle 2 \rangle 1$. ASSUME: a^+ is a transitive set.

$\langle 2 \rangle 2$. LET: $x \in y \in a$

$\langle 2 \rangle 3$. $x \in y \in a^+$

$\langle 2 \rangle 4$. $x \in a^+$

PROOF: $\langle 2 \rangle 1$

$\langle 2 \rangle 5$. $x \neq a$

PROOF: From $\langle 2 \rangle 2$ and the Axiom of Regularity.

□ $\langle 2 \rangle 6. x \in a$

Definition 6.2.4. We write 0 for \emptyset , 1 for \emptyset^+ , 2 for \emptyset^{++} , etc.

Proposition 6.2.5. For any set A we have $\mathcal{P}A \approx 2^A$.

PROOF: The function $H : \mathcal{P}A \rightarrow 2^A$ defined by $H(S)(a) = \{\emptyset\}$ if $a \in S$ and \emptyset if $a \notin S$ is a bijection. □

Proposition 6.2.6. For any ordinal number α we have α^+ is an ordinal number.

PROOF:

$\langle 1 \rangle 1. \alpha^+$ is a transitive set.

PROOF: Proposition 6.2.3.

$\langle 1 \rangle 2. \alpha^+$ is well-ordered by \in .

$\langle 2 \rangle 1. \text{ For all } x, y, z \in \alpha^+, \text{ if } x \in y \in z \text{ then } x \in z$

$\langle 3 \rangle 1. \text{ CASE: } z = \alpha$

PROOF: Then $x \in \alpha$ since α is a transitive set.

$\langle 3 \rangle 2. \text{ CASE: } z \in \alpha$

PROOF: Then $x \in z$ since α is well-ordered by \in .

$\langle 2 \rangle 2. \text{ For all } x, y \in \alpha^+ \text{ we have } x \in y \text{ or } x = y \text{ or } y \in x$

$\langle 3 \rangle 1. \text{ CASE: } x, y \in \alpha$

PROOF: The result follows because α is well-ordered by \in .

$\langle 3 \rangle 2. \text{ CASE: } x \in \alpha, y = \alpha$

PROOF: Then $x \in y$.

$\langle 3 \rangle 3. \text{ CASE: } x = \alpha, y \in \alpha$

PROOF: Then $y \in x$.

$\langle 3 \rangle 4. \text{ CASE: } x = \alpha, y = \alpha$

PROOF: Then $x = y$.

$\langle 2 \rangle 3. \text{ Every nonempty subset of } \alpha^+ \text{ has an } \in\text{-least element.}$

$\langle 3 \rangle 1. \text{ LET: } S \subseteq \alpha^+ \text{ be nonempty}$

$\langle 3 \rangle 2. \text{ CASE: } S = \{\alpha\}$

PROOF: α is least in S .

$\langle 3 \rangle 3. \text{ CASE: } S \neq \{\alpha\}$

$\langle 4 \rangle 1. S - \{\alpha\}$ is a nonempty subset of α

$\langle 4 \rangle 2. \text{ LET: } \beta \text{ be least in } S - \{\alpha\}$

$\langle 4 \rangle 3. \beta$ is least in S .

□

Proposition 6.2.7. For ordinals α and β , if $\alpha^+ = \beta^+$ then $\alpha = \beta$.

PROOF: If $\alpha^+ = \beta^+$ then

$$\alpha = \bigcup (\alpha^+) \quad (\text{Proposition 6.2.2})$$

$$= \bigcup (\beta^+)$$

$$= \beta \quad (\text{Proposition 6.2.2})$$

Proposition 6.2.8. *For ordinals α and β , we have $\alpha < \beta$ if and only if $\alpha^+ < \beta^+$.*

PROOF:

$$\begin{aligned}\alpha < \beta &\Leftrightarrow \alpha^+ \leq \beta \\ &\Leftrightarrow \alpha^+ < \beta^+ \quad \square\end{aligned}$$

Definition 6.2.9 (Successor Ordinal). An ordinal α is a *successor ordinal* iff $\alpha = \beta^+$ for some β .

Definition 6.2.10 (Limit Ordinal). A *limit ordinal* is an ordinal that is neither 0 nor a successor ordinal.

Proposition 6.2.11. *If λ is a limit ordinal and $\beta < \lambda$ then $\beta^+ < \lambda$.*

PROOF: Since $\beta^+ \leq \lambda$ and $\beta^+ \neq \lambda$. \square

6.3 The Well-Ordering Theorem and Zorn's Lemma

Theorem 6.3.1 (Hartogs). *For any set A , there exists an ordinal not dominated by A .*

PROOF:

- (1)1. LET: α be the class of all ordinals β such that $\beta \prec A$
 PROVE: α is a set.
- (1)2. LET: $W = \{(B, R) \mid B \subseteq A, R \text{ is a well ordering on } B\}$
- (1)3. α is the class of the ordinals of the elements of W .
 - (2)1. For all $(B, R) \in W$, the ordinal of (B, R) is in α .
 - (3)1. LET: $(B, R) \in W$
 - (3)2. LET: β be the ordinal of (B, R)
 - (3)3. LET: $E : B \cong \beta$ be the canonical isomorphism.
 - (3)4. LET: $i : B \hookrightarrow A$ be the inclusion
 - (3)5. $i \circ E^{-1}$ is an injection $\beta \rightarrow A$
 - (3)6. $\beta \in \alpha$
 - (2)2. For all $\beta \in \alpha$, there exists $(B, R) \in W$ such that β is the ordinal number of (B, R) .
 - (3)1. LET: $\beta \in \alpha$
 - (3)2. PICK an injection $f : \beta \rightarrow A$
 - (3)3. Define \leq on $\text{ran } f$ by $f(x) \leq f(y)$ iff $x \leq y$
 - (3)4. $(\text{ran } f, \leq) \in W$
 - (3)5. β is the ordinal number of $(\text{ran } f, \leq)$
- (1)4. α is a set.
 PROOF: By an Axiom of Replacement.
- (1)5. α is an ordinal.
 PROOF: It is a transitive set of ordinals.
- (1)6. $\alpha \not\prec A$
 PROOF: Since $\alpha \notin \alpha$.

□

Theorem 6.3.2 (Numeration Theorem). *Every set is equinumerous with some ordinal.*

PROOF:

- ⟨1⟩1. LET: A be any set.
- ⟨1⟩2. PICK an ordinal α not dominated by A .
- ⟨1⟩3. PICK a choice function G for A .
- ⟨1⟩4. PICK $e \notin A$
- ⟨1⟩5. LET: $F : \alpha \rightarrow A \cup \{e\}$ by transfinite recursion:

$$F(\gamma) = \begin{cases} G(A - F(\{\delta \mid \delta < \gamma\})) & \text{if } A - F(\{\delta \mid \delta < \gamma\}) \neq \emptyset \\ e & \text{if } A - F(\{\delta \mid \delta < \gamma\}) = \emptyset \end{cases}$$

- ⟨1⟩6. $e \in \text{ran } F$
 - ⟨2⟩1. ASSUME: for a contradiction $e \notin \text{ran } F$
 - ⟨2⟩2. F is an injection $\alpha \rightarrow A$.
 - ⟨3⟩1. LET: $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$
PROVE: $F(\beta) \neq F(\gamma)$
 - ⟨3⟩2. ASSUME: w.l.o.g. $\beta < \gamma$
 - ⟨3⟩3. $F(\gamma) \in A - F(\{\delta \mid \delta < \gamma\})$
 - ⟨3⟩4. $F(\gamma) \notin F(\{\delta \mid \delta < \gamma\})$
 - ⟨3⟩5. $F(\gamma) \neq F(\beta)$
 - ⟨2⟩3. Q.E.D.
PROOF: This contradicts ⟨1⟩2.
- ⟨1⟩7. LET: δ be least such that $F(\delta) = e$
- ⟨1⟩8. $F \upharpoonright \delta : \delta \approx A$

Theorem 6.3.3 (Well-Ordering Theorem). *Any set can be well ordered.*

PROOF:

- ⟨1⟩1. PICK an ordinal δ and a bijection $F : A \approx \delta$
- ⟨1⟩2. Define \leq on A by $F(x) \leq F(y)$ iff $x \leq y$ for $x, y \in \delta$
- ⟨1⟩3. \leq is a well ordering on A .

□

Theorem 6.3.4 (Zorn's Lemma). *Let \mathcal{A} be a set such that, for every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$. Then \mathcal{A} has a maximal element.*

PROOF:

- ⟨1⟩1. PICK a well ordering $<$ on \mathcal{A} .
- ⟨1⟩2. LET: $F : \mathcal{A} \rightarrow 2$ be the function defined by transfinite recursion by:

$$F(A) = \begin{cases} 1 & \text{if } A \text{ includes every set } B < A \text{ for which } F(B) = 1 \\ 0 & \text{otherwise} \end{cases}$$
- ⟨1⟩3. LET: $\mathcal{C} = \{A \in \mathcal{A} \mid F(A) = 1\}$
PROVE: $\bigcup \mathcal{C}$ is a maximal element of \mathcal{A}
- ⟨1⟩4. For all $A \in \mathcal{A}$, we have $A \in \mathcal{C}$ iff $\forall B < A. B \in \mathcal{C} \Rightarrow B \subseteq A$
- ⟨1⟩5. \mathcal{C} is a chain.

- ⟨2⟩1. LET: $A, A' \in \mathcal{C}$
 - ⟨2⟩2. ASSUME: w.l.o.g. $A \leq A'$
 - ⟨2⟩3. $A \subseteq A'$
 - PROOF: By ⟨1⟩4
 - ⟨1⟩6. $\bigcup \mathcal{C} \in \mathcal{A}$
 - ⟨1⟩7. $\bigcup \mathcal{C}$ is maximal in \mathcal{A} .
 - ⟨2⟩1. LET: $A \in \mathcal{A}$ and $\bigcup \mathcal{C} \subseteq A$
 - ⟨2⟩2. $A \in \mathcal{C}$
 - PROOF: By ⟨1⟩4 since $\forall B \in \mathcal{C}. B \subseteq A$.
 - ⟨2⟩3. $A \subseteq \bigcup \mathcal{C}$
 - ⟨2⟩4. $A = \bigcup \mathcal{C}$
-

Proposition 6.3.5 (Teichmüller-Tukey Lemma). *Let \mathcal{A} be a nonempty set such that, for every B , we have $B \in \mathcal{A}$ if and only if every finite subset of B is a member of \mathcal{A} . Then \mathcal{A} has a maximal element.*

PROOF:

- ⟨1⟩1. For every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$
 - ⟨2⟩1. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain.
 - ⟨2⟩2. Every finite subset of $\bigcup \mathcal{B}$ is a member of \mathcal{A} .
 - ⟨3⟩1. LET: C be a finite subset of $\bigcup \mathcal{B}$.
 - ⟨3⟩2. PICK $B \in \mathcal{B}$ such that $C \subseteq B$.
 - ⟨3⟩3. $B \in \mathcal{A}$
 - ⟨3⟩4. Every finite subset of B is in \mathcal{A} .
 - ⟨3⟩5. $C \in \mathcal{A}$
 - ⟨2⟩3. $\bigcup \mathcal{B} \in \mathcal{A}$.
 - ⟨1⟩2. Q.E.D.
 - PROOF: Zorn's lemma.
-

Theorem Schema 6.3.6. *For any class \mathbf{A} , there exists a class \mathbf{F} such that the following is a theorem:*

If \mathbf{A} is a proper class of ordinals, then $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{A}$ is an order isomorphism.

PROOF:

- ⟨1⟩1. Define $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{A}$ by transfinite recursion as follows: $\mathbf{F}(\alpha)$ is the least element of \mathbf{A} that is different from $\mathbf{F}(\beta)$ for all $\beta < \alpha$.
- ⟨1⟩2. For all $\alpha, \beta \in \mathbf{On}$, if $\alpha < \beta$ then $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$
- PROOF: We have $\mathbf{F}(\alpha) \neq \mathbf{F}(\beta)$ by the definition of $\mathbf{F}(\beta)$, and $\mathbf{F}(\beta) \not< \mathbf{F}(\alpha)$ by the leastness of $\mathbf{F}(\alpha)$.
- ⟨1⟩3. \mathbf{F} is surjective.
- ⟨2⟩1. LET: $\alpha \in \mathbf{A}$
- ⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall \beta \in \mathbf{A}$, if $\beta < \alpha$ then there exists γ such that $\beta = \mathbf{F}(\gamma)$.
- ⟨2⟩3. LET: $\gamma = \{\delta \in \mathbf{On} \mid \mathbf{F}(\delta) < \alpha\}$
- ⟨2⟩4. γ is a set.

PROOF: Axiom of Replacement applied to α .

$\langle 2 \rangle 5$. γ is a transitive set.

PROOF: If $\mathbf{F}(\delta) < \alpha$ and $\epsilon < \delta$ then $\mathbf{F}(\epsilon) < \alpha$ by $\langle 1 \rangle 2$.

$\langle 2 \rangle 6$. γ is an ordinal.

PROOF: Proposition 6.1.8.

$\langle 2 \rangle 7$. $\mathbf{F}(\gamma) = \alpha$

$\langle 3 \rangle 1$. $\mathbf{F}(\gamma)$ is the least element of \mathbf{A} different from $\mathbf{F}(\delta)$ for all $\delta < \gamma$

$\langle 3 \rangle 2$. $\mathbf{F}(\gamma)$ is the least element of \mathbf{A} different from x for all $x \in \mathbf{A}$ with $x < \alpha$

$\langle 3 \rangle 3$. $\mathbf{F}(\gamma) = \alpha$

□

6.4 Ordinal Operations

Definition 6.4.1 (Ordinal Operation). An *ordinal operation* is a function $\mathbf{On} \rightarrow \mathbf{On}$.

Definition 6.4.2 (Continuous). An ordinal operation $\mathbf{T} : \mathbf{On} \rightarrow \mathbf{On}$ is *continuous* iff, for every limit ordinal λ , we have $\mathbf{T}(\lambda) = \bigcup_{\alpha < \lambda} \mathbf{T}(\alpha)$.

Definition 6.4.3 (Normal). An ordinal operation is *normal* iff it is continuous and strictly monotone.

Proposition Schema 6.4.4. For any class \mathbf{T} , the following is a theorem.

If \mathbf{T} is a continuous ordinal operation and $\forall \gamma. \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$, then \mathbf{T} is normal.

PROOF:

$\langle 1 \rangle 1$. LET: $P[\beta]$ be the property $\forall \gamma < \beta. \mathbf{T}(\gamma) < \mathbf{T}(\beta)$

$\langle 1 \rangle 2$. $P[0]$

PROOF: Vacuous.

$\langle 1 \rangle 3$. For any ordinal γ , if $P[\gamma]$ then $P[\gamma^+]$

$\langle 2 \rangle 1$. ASSUME: $P[\gamma]$

$\langle 2 \rangle 2$. LET: $\delta < \gamma^+$

$\langle 2 \rangle 3$. CASE: $\delta < \gamma$

PROOF: Then $\mathbf{T}(\delta) < \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$.

$\langle 2 \rangle 4$. CASE: $\delta = \gamma$

PROOF: Then $\mathbf{T}(\delta) = \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$.

$\langle 1 \rangle 4$. For any limit ordinal λ , if $\forall \gamma < \lambda. P[\gamma]$ then $P[\lambda]$.

$\langle 2 \rangle 1$. ASSUME: $\forall \gamma < \lambda. P[\gamma]$

$\langle 2 \rangle 2$. LET: $\delta < \lambda$

$\langle 2 \rangle 3$. $\mathbf{T}(\delta) < \mathbf{T}(\lambda)$

PROOF:

$$\begin{aligned} \mathbf{T}(\delta) &< \mathbf{T}(\delta^+) \\ &\leq \bigcup_{\epsilon < \lambda} \mathbf{T}(\epsilon) \\ &= \mathbf{T}(\lambda) \end{aligned}$$

□

Proposition Schema 6.4.5. *For any class \mathbf{T} , the following is a theorem:*

Assume \mathbf{T} is a normal ordinal operation. For every ordinal α , we have $\alpha \leq \mathbf{T}(\alpha)$.

PROOF:

$\langle 1 \rangle 1$. LET: γ be an ordinal.

$\langle 1 \rangle 2$. ASSUME: as induction hypothesis $\forall \delta < \gamma. \mathbf{T}(\delta) \geq \delta$

$\langle 1 \rangle 3$. For all $\delta < \gamma$ we have $\delta < \mathbf{T}(\gamma)$

PROOF: \mathbf{T} is strictly monotone.

$\langle 1 \rangle 4$. $\gamma \leq \mathbf{T}(\gamma)$

□

Proposition Schema 6.4.6. *For any class \mathbf{T} , the following is a theorem:*

Assume \mathbf{T} is a normal ordinal operation. For any ordinal $\beta \geq \mathbf{T}(0)$, there exists a greatest ordinal γ such that $\mathbf{T}(\gamma) \leq \beta$.

PROOF:

$\langle 1 \rangle 1$. There exists γ such that $\mathbf{T}(\gamma) > \beta$

$\langle 2 \rangle 1$. For all γ we have $\mathbf{T}(\gamma) \geq \gamma$

PROOF: Proposition 6.4.5.

$\langle 2 \rangle 2$. $\mathbf{T}(\beta^+) > \beta$

$\langle 1 \rangle 2$. LET: δ be least such that $\mathbf{T}(\delta) > \beta$

$\langle 1 \rangle 3$. δ is a successor ordinal.

$\langle 2 \rangle 1$. $\delta \neq 0$

PROOF: Since $\mathbf{T}(0) \leq \beta$.

$\langle 2 \rangle 2$. δ is not a limit ordinal.

$\langle 3 \rangle 1$. ASSUME: for a contradiction δ is a limit ordinal.

$\langle 3 \rangle 2$. $\beta < \bigcup_{\epsilon < \delta} \mathbf{T}(\epsilon)$

PROOF: \mathbf{T} is continuous.

$\langle 3 \rangle 3$. There exists $\epsilon < \delta$ such that $\beta < \mathbf{T}(\epsilon)$

$\langle 3 \rangle 4$. Q.E.D.

PROOF: This contradicts the minimality of δ .

$\langle 1 \rangle 4$. LET: $\delta = \gamma^+$

$\langle 1 \rangle 5$. γ is greatest such that $\mathbf{T}(\gamma) \leq \beta$

□

Theorem Schema 6.4.7. *For any class \mathbf{T} , the following is a theorem:*

Assume that \mathbf{T} is a normal ordinal operation. For any nonempty set of ordinals S , we have

$$\mathbf{T}(\sup S) = \sup_{\alpha \in S} \mathbf{T}(\alpha) .$$

PROOF:

$\langle 1 \rangle 1$. $\forall \alpha \in S. \mathbf{T}(\alpha) \leq \mathbf{T}(\sup S)$

PROOF: Since \mathbf{T} is monotone.

$\langle 1 \rangle 2$. For any ordinal β , if $\forall \alpha \in S. \mathbf{T}(\alpha) \leq \beta$, then $\mathbf{T}(\sup S) \leq \beta$

- ⟨2⟩1. LET: β be an ordinal.
- ⟨2⟩2. LET: $\gamma = \sup S$
- ⟨2⟩3. ASSUME: $\forall \alpha \in S. \mathbf{T}(\alpha) \leq \beta$
- ⟨2⟩4. CASE: γ is 0 or a successor ordinal
PROOF: Then we must have $\gamma \in S$ so $\mathbf{T}(\gamma) \leq \beta$ from ⟨2⟩3.
- ⟨2⟩5. CASE: γ is a limit ordinal
 - ⟨3⟩1. $\mathbf{T}(\gamma) = \sup_{\alpha < \gamma} \mathbf{T}(\alpha)$
PROOF: \mathbf{T} is continuous.
 - ⟨3⟩2. ASSUME: for a contradiction $\beta < \mathbf{T}(\gamma)$
 - ⟨3⟩3. PICK $\alpha < \gamma$ such that $\beta < \mathbf{T}(\alpha)$
PROOF: ⟨3⟩1, ⟨3⟩2
 - ⟨3⟩4. PICK $\alpha' \in S$ such that $\alpha < \alpha'$
PROOF: ⟨2⟩2, ⟨3⟩3
 - ⟨3⟩5. $\beta < \mathbf{T}(\alpha') \leq \beta$
PROOF: \mathbf{T} is strictly monotone, ⟨3⟩3, ⟨3⟩4, ⟨2⟩3.
 - ⟨3⟩6. Q.E.D.
PROOF: This is a contradiction.

□

Proposition 6.4.8. *For any classes \mathbf{A} and \mathbf{T} , the following is a theorem:*

Assume \mathbf{A} is a proper class of ordinals such that, for every set $S \subseteq \mathbf{A}$, we have $\bigcup S \in \mathbf{A}$. Assume \mathbf{T} is the unique order isomorphism $\mathbf{On} \cong \mathbf{A}$. Then \mathbf{T} is normal.

PROOF:

- ⟨1⟩1. \mathbf{T} is strictly monotone.
PROOF: Since it is an order isomorphism.
- ⟨1⟩2. \mathbf{T} is continuous.
 - ⟨2⟩1. LET: λ be a limit ordinal.
 - ⟨2⟩2. $\mathbf{T}'(\lambda)$ is the least member of \mathbf{A} that is greater than $\mathbf{T}'(\alpha)$ for all $\alpha < \lambda$
 - ⟨2⟩3. $\mathbf{T}'(\lambda) = \sup_{\alpha < \lambda} \mathbf{T}'(\alpha)$

□

Proposition Schema 6.4.9. *For any class \mathbf{T} , the following is a theorem:*

If \mathbf{T} is a normal ordinal operation, then for any limit ordinal λ , we have $\mathbf{T}(\lambda)$ is a limit ordinal.

PROOF:

- ⟨1⟩1. $\mathbf{T}(\lambda) \neq 0$
PROOF: Since $0 \leq \mathbf{T}(0) < \mathbf{T}(\lambda)$.
- ⟨1⟩2. $\mathbf{T}(\lambda)$ is not a successor ordinal.
 - ⟨2⟩1. ASSUME: for a contradiction $\mathbf{T}(\lambda) = \alpha^+$
 - ⟨2⟩2. $\alpha < \mathbf{T}(\lambda) = \sup_{\beta < \lambda} \mathbf{T}(\beta)$
 - ⟨2⟩3. PICK $\beta < \lambda$ such that $\alpha < \mathbf{T}(\beta)$
 - ⟨2⟩4. $\alpha^+ \leq \mathbf{T}(\beta) < \mathbf{T}(\lambda)$
 - ⟨2⟩5. Q.E.D.

PROOF: This is a contradiction.

□

6.5 Ordinal Arithmetic

6.5.1 Addition

Definition 6.5.1. Let A and B be disjoint well-ordered sets. The *concatenation* of A and B is the set $A \cup B$ under the relation:

- if $a, a' \in A$ then $a \leq a'$ iff $a \leq a'$ in A
- if $b, b' \in B$ then $b \leq b'$ iff $b \leq b'$ in B
- if $a \in A$ and $b \in B$ then $a \leq b$ and $b \not\leq a$.

Proposition 6.5.2. *If A and B are disjoint well-ordered sets, then their concatenation is well-ordered.*

PROOF:

$\langle 1 \rangle 1.$ \leq is reflexive.

PROOF: For all $a \in A$ we have $a \leq a$, and for all $b \in B$ we have $b \leq b$.

$\langle 1 \rangle 2.$ \leq is antisymmetric.

$\langle 2 \rangle 1.$ ASSUME: $x \leq y \leq x$

$\langle 2 \rangle 2.$ CASE: $x, y \in A$

PROOF: Then $x = y$ since the order on A is antisymmetric.

$\langle 2 \rangle 3.$ CASE: $x \in A$ and $y \in B$

PROOF: This is impossible as it would imply $y \not\leq x$.

$\langle 2 \rangle 4.$ CASE: $x \in B$ and $y \in A$

PROOF: This is impossible as it would imply $x \not\leq y$.

$\langle 2 \rangle 5.$ CASE: $x, y \in B$

PROOF: Then $x = y$ since the order on B is antisymmetric.

$\langle 1 \rangle 3.$ \leq is transitive.

$\langle 2 \rangle 1.$ ASSUME: $x \leq y \leq z$

$\langle 2 \rangle 2.$ CASE: $x, z \in A$

PROOF: In this case $y \in A$ since $y \leq z$, and so $x \leq z$ since the order on A is transitive.

$\langle 2 \rangle 3.$ CASE: $x \in A$ and $z \in B$

PROOF: Then $x \leq z$ immediately.

$\langle 2 \rangle 4.$ CASE: $x \in B$ and $z \in A$

PROOF: This is impossible because we have $y \notin A$ since $x \leq y$ and $y \notin B$ since $y \leq z$.

$\langle 2 \rangle 5.$ CASE: $x, z \in B$

PROOF: In this case $y \in B$ since $x \leq y$, and so $x \leq z$ since the order on B is transitive.

$\langle 1 \rangle 4.$ \leq is total.

$\langle 2 \rangle 1.$ LET: $x, y \in A \cup B$

$\langle 2 \rangle 2.$ CASE: $x, y \in A$

PROOF: Then $x \leq y$ or $y \leq x$ because the order on A is total.

$\langle 2 \rangle 3.$ CASE: $x \in A$ and $y \in B$

PROOF: Then $x \leq y$.

⟨2⟩4. CASE: $x \in B$ and $y \in A$

PROOF: Then $y \leq x$.

⟨2⟩5. CASE: $x, y \in B$

PROOF: Then $x \leq y$ or $y \leq x$ because the order on B is total.

⟨1⟩5. Every nonempty subset of $A \cup B$ has a least element.

⟨2⟩1. LET: S be a nonempty subset of $A \cup B$

⟨2⟩2. CASE: $S \cap A = \emptyset$

PROOF: Then $S \subseteq B$ and so S has a least element.

⟨2⟩3. CASE: $S \cap A \neq \emptyset$

PROOF: The least element of $S \cap A$ is the least element of S .

□

Definition 6.5.3 (Ordinal Addition). Let α and β be ordinal numbers. Then $\alpha + \beta$ is the ordinal number of the concatenation of A and B , where A is any well ordered set with ordinal α and B is any well ordered set with ordinal β .

Theorem 6.5.4 (Associative Law for Addition). For any ordinals ρ , σ and τ , we have

$$\rho + (\sigma + \tau) = (\rho + \sigma) + \tau .$$

PROOF: Given disjoint well ordered sets A , B and C , the concatenation of A with (the concatenation of B and C) is the same as the concatenation of (the concatenation of A and B) and C . □

Theorem 6.5.5. For any ordinal ρ we have

$$\rho + 0 = 0 + \rho = \rho .$$

PROOF: For any well ordered set A , the concatenation of A with \emptyset is A , and the concatenation of \emptyset with A is A . □

Theorem 6.5.6. For any ordinal α we have $\alpha + 1 = \alpha^+$.

PROOF: Since α^+ is the concatenation of α and $\{\alpha\}$. □

Theorem 6.5.7. For any ordinal α , the operation that maps β to $\alpha + \beta$ is normal.

PROOF:

⟨1⟩1. For any limit ordinal λ , we have $\alpha + \lambda = \sup_{\beta < \lambda} (\alpha + \beta)$.

⟨2⟩1. LET: λ be a limit ordinal.

⟨2⟩2. $(\{0\} \times \alpha) \cup (\{1\} \times \lambda) = \bigcup_{\beta \in \lambda} ((\{0\} \times \alpha) \cup (\{1\} \times \beta))$, where the order on the right hand side is as in Lemma 6.1.15.

PROOF:

$$\begin{aligned} (\{0\} \times \alpha) \cup (\{1\} \times \lambda) &= (\{0\} \times \alpha) \cup (\{1\} \times \bigcup_{\beta < \lambda} \beta) \\ &= (\{0\} \times \alpha) \cup \bigcup_{\beta < \lambda} (\{1\} \times \beta) \\ &= \bigcup_{\beta < \lambda} ((\{0\} \times \alpha) \cup (\{1\} \times \beta)) \end{aligned}$$

(1)2. For any ordinal β we have $\alpha + \beta < \alpha + \beta^+$

PROOF: Since $\alpha + \beta^+ = \alpha + \beta + 1 = (\alpha + \beta)^+$.

□

Corollary 6.5.7.1. *For any ordinals α, β, γ , we have $\beta < \gamma$ if and only if $\alpha + \beta < \alpha + \gamma$.*

Corollary 6.5.7.2 (Left Cancellation for Addition). *For any ordinals α, β and γ , if $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$.*

Theorem 6.5.8. *For any ordinals α, β, γ , if $\beta \leq \gamma$ then $\beta + \alpha \leq \gamma + \alpha$.*

PROOF: Transfinite induction on α . □

Theorem 6.5.9 (Subtraction Theorem). *Let α and β be ordinals with $\alpha \leq \beta$. Then there exists a unique ordinal δ such that $\alpha + \delta = \beta$.*

PROOF:

(1)1. For all ordinals α and β with $\alpha \leq \beta$, there exists δ such that $\alpha + \delta = \beta$

(2)1. LET: α and β be ordinals with $\alpha \leq \beta$

(2)2. LET: δ be the greatest ordinal such that $\alpha + \delta \leq \beta$

PROOF: Proposition 6.4.6.

(2)3. $\alpha + \delta = \beta$

PROOF: If $\alpha + \delta < \beta$ then $\alpha + \delta + 1 \leq \beta$ contradicting the greatestness of δ .

(1)2. Q.E.D.

PROOF: Uniqueness follows from the Left Cancellation Law.

□

6.5.2 Multiplication

Definition 6.5.10 (Ordinal Multiplication). Let α and β be ordinal numbers. Then $\alpha\beta$ is the ordinal number of $A \times B$ under the lexicographic order, where A is any well ordered set with ordinal α and B is any well ordered set with ordinal β .

This is well defined by Proposition 5.3.4.

Theorem 6.5.11 (Associative Law). *For any ordinals ρ, σ and τ , we have*

$$\rho(\sigma\tau) = (\rho\sigma)\tau .$$

PROOF: Let A, B and C be well ordered sets with ordinals ρ, σ and τ . Then both $\rho(\sigma\tau)$ and $(\rho\sigma)\tau$ are the ordinal of $A \times B \times C$ under

$$(a, b, c) \leq (a', b', c') \Leftrightarrow a \leq a' \vee (a = a' \wedge b \leq b') \vee (a = a' \wedge b = b' \wedge c \leq c') . \quad \square$$

Theorem 6.5.12 (Left Distributive Law). *For any ordinals ρ, σ and τ , we have*

$$\rho(\sigma + \tau) = \rho\sigma + \rho\tau$$

PROOF: Let A , B and C be well ordered sets with ordinals ρ , σ and τ and with $B \cap C = \emptyset$. Then both $\rho(\sigma + \tau)$ and $\rho\sigma + \rho\tau$ are the ordinal of $A \times (B \cup C)$ under the lexicographic ordering. \square

Theorem 6.5.13. *For any ordinal ρ we have $\rho 0 = 0\rho = 0$.*

PROOF: For any well ordered set A we have $A \times \emptyset = \emptyset \times A = \emptyset$. \square

Theorem 6.5.14. *For any ordinal ρ we have $\rho 1 = 1\rho = \rho$.*

PROOF: Easy. \square

Theorem 6.5.15. *For any ordinals ρ and σ , if $\rho\sigma = 0$ then $\rho = 0$ or $\sigma = 0$.*

PROOF: If $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$. \square

Theorem 6.5.16. *For any non-zero ordinal α , the operation that maps β to $\alpha\beta$ is normal.*

PROOF:

$\langle 1 \rangle 1$. For any limit ordinal λ , we have $\alpha\lambda = \bigcup_{\beta < \lambda} \alpha\beta$

$\langle 2 \rangle 1$. LET: λ be a limit ordinal

$\langle 2 \rangle 2$. $\alpha \times \lambda = \bigcup_{\beta < \lambda} (\alpha \times \beta)$ as well-ordered sets

$\langle 1 \rangle 2$. For any ordinal β we have $\alpha\beta < \alpha\beta^+$

PROOF: $\alpha\beta^+ = \alpha\beta + \alpha > \alpha\beta$

\square

Corollary 6.5.16.1. *For any ordinals α , β , γ , if $\alpha \neq 0$ then $\beta < \gamma$ if and only if $\alpha\beta < \alpha\gamma$.*

Corollary 6.5.16.2 (Left Cancellation for Multiplication). *For any ordinals α , β , γ , if $\alpha \neq 0$ and $\alpha\beta = \alpha\gamma$ then $\beta = \gamma$.*

Theorem 6.5.17. *For any ordinals α , β and γ , if $\beta \leq \gamma$ then $\beta\alpha \leq \gamma\alpha$.*

PROOF: Transfinite induction on α . \square

Theorem 6.5.18 (Division Theorem). *Let α and δ be ordinal numbers with $\delta \neq 0$. Then there exist unique ordinals β and γ with $\gamma < \delta$ and*

$$\alpha = \delta\beta + \gamma .$$

PROOF:

$\langle 1 \rangle 1$. For any ordinal numbers α and δ with $\delta \neq 0$, there exist ordinals β and γ such that $\gamma < \delta$ and $\alpha = \delta\beta + \gamma$

$\langle 2 \rangle 1$. LET: α and δ be ordinals with $\delta \neq 0$

$\langle 2 \rangle 2$. LET: β be the greatest ordinal such that $\delta\beta \leq \alpha$

PROOF: Proposition 6.4.6.

$\langle 2 \rangle 3$. There exists an ordinal γ such that $\alpha = \delta\beta + \gamma$

PROOF: Subtraction Theorem

- (1)2. For any ordinals $\delta, \beta, \beta', \gamma, \gamma'$, if $\delta\beta + \gamma = \delta\beta' + \gamma'$ and $\delta \neq 0$ and $\gamma, \gamma' < \delta$
 then $\beta = \beta'$ and $\gamma = \gamma'$
 (2)1. LET: $\delta, \beta, \beta', \gamma, \gamma'$ be ordinals.
 (2)2. ASSUME: $\delta \neq 0$ and $\delta\beta + \gamma = \delta\beta' + \gamma'$
 (2)3. $\beta = \beta'$
 (3)1. $\beta \not\leq \beta'$
 PROOF: If $\beta < \beta'$ then

$$\begin{aligned}
 \delta\beta' + \gamma' &\geq \delta\beta' \\
 &\geq \delta(\beta + 1) \\
 &= \delta\beta + \delta \\
 &> \delta\beta + \gamma
 \end{aligned}$$
 (3)2. $\beta' \not\leq \beta$
 PROOF: Similar.
 (2)4. $\gamma = \gamma'$
 PROOF: By Cancellation.
-

6.5.3 Exponentiation

Definition 6.5.19. Given ordinals α and β , define the ordinal α^β as follows:

$$\begin{aligned}
 0^\alpha &:= 0 & (\alpha > 0) \\
 \alpha^0 &:= 1 \\
 \alpha^{\beta^+} &:= \alpha^\beta \alpha & (\alpha > 0) \\
 \alpha^\lambda &:= \sup_{\beta < \lambda} \alpha^\beta & (\alpha > 0, \lambda \text{ a limit ordinal})
 \end{aligned}$$

Theorem 6.5.20. Let α be an ordinal ≥ 2 . The operation that maps β to α^β is normal.

PROOF:

- (1)1. For λ a limit ordinal we have $\alpha^\lambda = \sup_{\beta < \lambda} \alpha^\beta$

PROOF: By definition.

- (1)2. For any ordinal β we have $\alpha^\beta < \alpha^{\beta^+}$

PROOF: We have $\alpha^{\beta^+} = \alpha^\beta \alpha > \alpha^\beta$ by Theorem 6.5.16 since $\alpha > 1$ and $\alpha^\beta \neq 0$.

□

Corollary 6.5.20.1. For any ordinals α, β, γ , if $\alpha \geq 2$ then $\beta < \gamma$ if and only if $\alpha^\beta < \alpha^\gamma$.

Corollary 6.5.20.2 (Cancellation for Exponentiation). For any ordinals α, β, γ , if $\alpha \geq 2$ and $\alpha^\beta = \alpha^\gamma$ then $\beta = \gamma$.

Theorem 6.5.21. For any ordinals α, β and γ , if $\beta \leq \gamma$ then $\beta^\alpha \leq \gamma^\alpha$.

PROOF: Transfinite induction on α .

Theorem 6.5.22 (Logarithm Theorem). *Let α and β be ordinal numbers with $\alpha \neq 0$ and $\beta > 1$. Then there exist unique ordinals γ , δ and ρ such that*

$$\alpha = \beta^\gamma \delta + \rho, \quad 0 \neq \delta < \beta, \quad \rho < \beta^\gamma.$$

PROOF:

$\langle 1 \rangle 1$. For any ordinals α and β with $\alpha \neq 0$ and $\beta > 1$, there exist ordinals γ , δ , ρ such that

$$\alpha = \beta^\gamma \delta + \rho, \quad 0 \neq \delta < \beta, \quad \rho < \beta^\gamma.$$

$\langle 2 \rangle 1$. LET: α and β be ordinals with $\alpha \neq 0$ and $\beta > 1$.

$\langle 2 \rangle 2$. LET: γ be the greatest ordinal such that $\beta^\gamma \leq \alpha$.

PROOF: Proposition 6.4.6.

$\langle 2 \rangle 3$. LET: δ and ρ be the unique ordinals with $\rho < \beta^\gamma$ such that $\alpha = \beta^\gamma \delta + \rho$.

PROOF: By the Division Theorem.

$\langle 2 \rangle 4$. $\delta \neq 0$

PROOF: If $\delta = 0$ then $\alpha = \beta^\gamma 0 + \rho = \rho < \beta^\gamma \leq \alpha$ which is a contradiction.

$\langle 2 \rangle 5$. $\delta < \beta$

PROOF: If $\beta \leq \delta$ then $\alpha \geq \beta^\gamma \delta \geq \beta^\gamma \beta = \beta^{\gamma+1}$, contradicting the greatestness of γ .

$\langle 1 \rangle 2$. If $\beta^\gamma \delta + \rho = \beta^{\gamma'} \delta' + \rho'$ with $\beta > 1$, $0 \neq \delta < \beta$, $0 \neq \delta' < \beta$, $\rho < \beta^\gamma$ and $\rho' < \beta^{\gamma'}$, then $\gamma = \gamma'$, $\delta = \delta'$ and $\rho = \rho'$.

$\langle 2 \rangle 1$. LET: $\alpha = \beta^\gamma \delta + \rho = \beta^{\gamma'} \delta' + \rho'$

$\langle 2 \rangle 2$. $\beta^\gamma \leq \alpha < \beta^{\gamma+1}$

$\langle 2 \rangle 3$. $\beta^{\gamma'} \leq \alpha < \beta^{\gamma'+1}$

$\langle 2 \rangle 4$. $\beta^\gamma < \beta^{\gamma'+1}$ and $\beta^{\gamma'} < \beta^{\gamma+1}$

$\langle 2 \rangle 5$. $\gamma < \gamma' + 1$ and $\gamma' < \gamma + 1$

$\langle 2 \rangle 6$. $\gamma = \gamma'$

$\langle 2 \rangle 7$. $\delta = \delta'$ and $\rho = \rho'$

PROOF: By the Division Theorem.

□

Chapter 7

Cardinal Numbers

7.1 Cardinal Numbers

Definition 7.1.1 (Cardinality). For any set A , the *cardinality* or *cardinal number* $|A|$ of A is the least ordinal equinumerous with A .

Let **Card** be the class of all cardinal numbers.

Proposition 7.1.2. For any sets A and B , we have $A \approx B$ iff $|A| = |B|$.

PROOF: Easy. \square

Definition 7.1.3 (Addition). Given cardinal numbers κ and λ , we define $\kappa + \lambda$ to be $|A \cup B|$ where A and B are disjoint sets of cardinality κ and λ respectively.

We prove this is well-defined.

PROOF:

$\langle 1 \rangle 1$. ASSUME: $A \approx A'$, $B \approx B'$, and $A \cap B = A' \cap B' = \emptyset$

$\langle 1 \rangle 2$. PICK bijections $f : A \approx A'$ and $g : B \approx B'$

$\langle 1 \rangle 3$. The function $A \cup B \rightarrow A' \cup B'$ that maps $a \in A$ to $f(a)$ and $b \in B$ to $g(b)$ is a bijection.

\square

Proposition 7.1.4. For any cardinal number κ , we have $\kappa + 0 = \kappa$.

PROOF: Let A and B be disjoint sets of cardinality κ and 0 . Then $B = \emptyset$ so $A \cup B = A$ and so $|A \cup B| = \kappa$. \square

Theorem 7.1.5 (Associative Law for Addition). For any cardinal numbers κ , λ , μ we have $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$.

PROOF: Since $A \cup (B \cup C) = (A \cup B) \cup C$. \square

Proposition 7.1.6. For any cardinal numbers κ and λ we have $\kappa + \lambda = \lambda + \kappa$.

PROOF: Since $A \cup B = B \cup A$. \square

Definition 7.1.7 (Multiplication). For κ and λ cardinal numbers, we define $\kappa\lambda$ to be the cardinal number of $A \times B$, where $|A| = \kappa$ and $|B| = \lambda$.

We prove this is well-defined.

PROOF: If $f : A \approx A'$ and $g : B \approx B'$ then the function that maps (a, b) to $(f(a), g(b))$ is a bijection $A \times B \approx A' \times B'$. \square

Proposition 7.1.8. For any cardinal number κ we have $\kappa \cdot 0 = 0$.

PROOF: Since $A \times \emptyset = \emptyset$. \square

Proposition 7.1.9. For any cardinal number κ we have $\kappa \cdot 1 = \kappa$.

PROOF: The function that maps (a, e) to a is a bijection $A \times \{e\} \approx A$. \square

Theorem 7.1.10 (Distributive Law). For any cardinal numbers κ , λ and μ , we have $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$.

PROOF: Since $A \times (B \cup C) = (A \times B) \cup (A \times C)$. \square

Theorem 7.1.11 (Associative Law for Multiplication). For any cardinal numbers κ , λ and μ , we have $\kappa(\lambda\mu) = (\kappa\lambda)\mu$.

PROOF: Since $A \times (B \times C) \approx (A \times B) \times C$. \square

Theorem 7.1.12 (Commutative Law for Multiplication). For any cardinal numbers κ and λ , we have $\kappa\lambda = \lambda\kappa$.

PROOF: Since $A \times B \approx B \times A$. \square

Theorem 7.1.13. For any cardinal numbers κ and λ , if $\kappa\lambda = 0$ then $\kappa = 0$ or $\lambda = 0$.

PROOF: if $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$. \square

Definition 7.1.14 (Exponentiation). Given cardinal numbers κ and λ , we define κ^λ to be $|A^B|$, where $|A| = \kappa$ and $|B| = \lambda$.

We prove this is well-defined.

PROOF: If $f : A \approx A'$ and $g : B \approx B'$, then the function that maps $h : B \rightarrow A$ to $f \circ h \circ g^{-1}$ is a bijection $A^B \approx A'^{B'}$. \square

Proposition 7.1.15. For any cardinal numbers κ , λ and μ ,

$$\kappa^{\lambda+\mu} = (\kappa^\lambda)^\mu$$

PROOF: The function that maps $f : A \times B \rightarrow C$ to $\lambda a \in A. \lambda b \in B. f(a, b)$ is a bijection $A^{B \times C} \approx (A^B)^C$. \square

Proposition 7.1.16. For any cardinal numbers κ , λ and μ ,

$$(\kappa\lambda)^\mu = \kappa^\mu \lambda^\mu.$$

PROOF: The function $f : A^C \times B^C \rightarrow (A \times B)^C$ with $f(g, h)(c) = (g(c), h(c))$ is a bijection. \square

Proposition 7.1.17. *For any cardinal numbers κ , λ and μ , we have*

$$\kappa^{\lambda+\mu} = \kappa^\lambda \kappa^\mu .$$

PROOF: If $B \cap C = \emptyset$, then $f : A^B \times A^C \rightarrow A^{B \cup C}$ given by $f(g, h)(b) = g(b)$ and $f(g, h)(c) = h(c)$ is a bijection. \square

Proposition 7.1.18. *For any cardinal number κ , we have $\kappa^0 = 1$.*

PROOF: For any set A , we have $A^\emptyset = \{\emptyset\}$. \square

Proposition 7.1.19. *For any cardinal number κ , we have $\kappa^1 = \kappa$.*

PROOF: For any sets A and B , if $B = \{b\}$ then the function $f : A \rightarrow A^B$ with $f(a)(b) = a$ is a bijection. \square

Proposition 7.1.20. *For any non-zero cardinal number κ we have $0^\kappa = 0$.*

PROOF: If A is nonempty then there is no function $A \rightarrow \emptyset$. \square

Proposition 7.1.21. *For any set A we have $|\mathcal{P}A| = 2^{|A|}$.*

PROOF: The function $f : \mathcal{P}A \rightarrow 2^A$ where $f(X)(a) = 0$ if $a \notin X$ and $f(X)(a) = 1$ if $a \in X$. \square

Corollary 7.1.21.1. *For any cardinal number κ we have $\kappa < 2^\kappa$.*

PROOF: By Cantor's Theorem. \square

7.2 Ordering on Cardinal Numbers

Definition 7.2.1. Given cardinal numbers κ and λ , we have $\kappa \leq \lambda$ iff $A \preceq B$, where $|A| = \kappa$ and $|B| = \lambda$.

PROOF:

$\langle 1 \rangle 1$. LET: $|A| = \kappa$ and $|B| = \lambda$

$\langle 1 \rangle 2$. PICK bijections $f : A \approx \kappa$ and $g : B \approx \lambda$

$\langle 1 \rangle 3$. If $\kappa \leq \lambda$ then $A \preceq B$

PROOF: Let $i : \kappa \hookrightarrow \lambda$ be the inclusion. Then $g^{-1} \circ i \circ f$ is an injection $A \rightarrow B$.

$\langle 1 \rangle 4$. If $A \preceq B$ then $\kappa \leq \lambda$

$\langle 2 \rangle 1$. ASSUME: $A \preceq B$

$\langle 2 \rangle 2$. PICK an injection $h : A \rightarrow B$

$\langle 2 \rangle 3$. $g(h(A)) \subseteq B$ is well-ordered by \in

$\langle 2 \rangle 4$. LET: γ be the ordinal number of $(g(h(A)), \in)$

$\langle 2 \rangle 5$. $\gamma \leq \lambda$

PROOF: Proposition 6.1.12.

$\langle 2 \rangle 6$. $\kappa \leq \gamma$

PROOF: By the leastness of κ , since A is equinumerous with γ .

$\langle 2 \rangle 7. \kappa \leq \lambda$

□

Corollary 7.2.1.1. *There is no largest cardinal number.*

Proposition 7.2.2. *For any cardinal numbers κ, λ, μ , if $\kappa \leq \lambda$ then $\kappa + \mu \leq \lambda + \mu$.*

PROOF: If $f : A \rightarrow B$ is injective, and $A \cap C = B \cap C = \emptyset$, then the function $A \cup C \rightarrow B \cup C$ that maps a to $f(a)$ and maps c to c is an injection. □

Proposition 7.2.3. *For any cardinal numbers κ, λ, μ , if $\kappa \leq \lambda$ then $\kappa\mu \leq \lambda\mu$.*

PROOF: If $f : A \rightarrow B$ is injective, then the function $A \times C \rightarrow B \times C$ that maps (a, c) to $(f(a), c)$ is injective. □

Proposition 7.2.4. *For any cardinal numbers κ, λ, μ , if $\kappa \leq \lambda$ then $\kappa^\mu \leq \lambda^\mu$.*

PROOF: Given an injection $f : A \rightarrow B$, the function that maps $A^C \rightarrow B^C$ that maps g to $f \circ g$ is an injection. □

Proposition 7.2.5. *For any cardinal numbers κ, λ, μ , if $\kappa \leq \lambda$ and μ and κ are not both 0, then $\mu^\kappa \leq \mu^\lambda$.*

PROOF:

$\langle 1 \rangle 1.$ LET: A, B and C be sets with A and C not both empty.

$\langle 1 \rangle 2.$ LET: $f : A \rightarrow B$ be an injection.

PROVE: $C^A \preceq C^B$

$\langle 1 \rangle 3.$ CASE: $C = \emptyset$

PROOF: Then $A \neq \emptyset$ so $C^A = \emptyset \preceq C^B$.

$\langle 1 \rangle 4.$ CASE: $C \neq \emptyset$

$\langle 2 \rangle 1.$ PICK $c \in C$

$\langle 2 \rangle 2.$ LET: $g : C^A \rightarrow C^B$ be the function $g(h)(f(a)) = h(a)$, $g(h)(b) = c$ if $b \notin f(A)$

$\langle 2 \rangle 3.$ g is an injection.

□

Proposition 7.2.6. *Let \mathcal{A} be a set such that $\forall X \in \mathcal{A}. |X| \leq \kappa$. Then*

$$\left| \bigcup \mathcal{A} \right| \leq |\mathcal{A}| \kappa .$$

PROOF:

$\langle 1 \rangle 1.$ For $X \in \mathcal{A}$, choose a surjection $f_X : \kappa \rightarrow X$.

$\langle 1 \rangle 2.$ Define $g : \mathcal{A} \times \kappa \rightarrow \bigcup \mathcal{A}$ by $g(X, \alpha) = f_X(\alpha)$

$\langle 1 \rangle 3.$ g is surjective.

□

Lemma 7.2.7. *The union of a set of cardinal numbers is a cardinal number.*

PROOF:

$\langle 1 \rangle 1$. LET: A be a set of cardinal numbers.

PROVE: $\bigcup A$ is the smallest ordinal equinumerous with $\bigcup A$

$\langle 1 \rangle 2$. LET: $\alpha < \bigcup A$

PROVE: $\alpha \not\approx \bigcup A$

$\langle 1 \rangle 3$. PICK $\kappa \in A$ such that $\alpha < \kappa$

$\langle 1 \rangle 4$. $\alpha \prec \kappa$

$\langle 1 \rangle 5$. $\alpha \prec \bigcup A$

\square

Chapter 8

Natural Numbers

8.1 Inductive Sets

Definition 8.1.1 (Inductive). A set I is *inductive* iff $0 \in I$ and $\forall x \in I. x^+ \in I$.

Definition 8.1.2 (Natural Number). A *natural number* is a set that belongs to every inductive set.

Theorem 8.1.3. *The class \mathbb{N} of natural numbers is a set.*

PROOF:

$\langle 1 \rangle 1$. PICK an inductive set I .

PROOF: Axiom of Infinity.

$\langle 1 \rangle 2$. $\mathbb{N} \subseteq I$

□

Theorem 8.1.4. *\mathbb{N} is inductive, and is a subset of every other inductive set.*

PROOF:

$\langle 1 \rangle 1$. \mathbb{N} is inductive.

$\langle 2 \rangle 1$. $0 \in \mathbb{N}$

PROOF: Since 0 is a member of every inductive set.

$\langle 2 \rangle 2$. $\forall n \in \mathbb{N}. n^+ \in \mathbb{N}$

$\langle 3 \rangle 1$. LET: $n \in \mathbb{N}$

$\langle 3 \rangle 2$. LET: I be any inductive set.

PROVE: $n^+ \in I$

$\langle 3 \rangle 3$. $n \in I$

PROOF: $\langle 3 \rangle 1$, $\langle 3 \rangle 2$

$\langle 3 \rangle 4$. $n^+ \in I$

PROOF: $\langle 3 \rangle 2$, $\langle 3 \rangle 3$

$\langle 1 \rangle 2$. \mathbb{N} is a subset of every inductive set.

PROOF: Immediate from definitions.

□

Corollary 8.1.4.1 (Induction Principle for \mathbb{N}). *Any inductive subset of \mathbb{N} coincides with \mathbb{N} .*

Theorem 8.1.5. *Every natural number except 0 is the successor of some natural number.*

PROOF: Trivially by induction. \square

Proposition 8.1.6. *Every natural number is an ordinal.*

PROOF: By induction. \square

Proposition 8.1.7. *\mathbb{N} is a transitive set.*

PROOF:

$\langle 1 \rangle 1. 0 \subseteq \mathbb{N}$

$\langle 1 \rangle 2. \forall n \in \mathbb{N}. n \subseteq \mathbb{N} \Rightarrow n^+ \subseteq \mathbb{N}$

$\langle 1 \rangle 3. \forall n \in \mathbb{N}. n \subseteq \mathbb{N}$

PROOF: From $\langle 1 \rangle 1$ and $\langle 1 \rangle 2$ by induction.

\square

Corollary 8.1.7.1. *\mathbb{N} is an ordinal.*

Definition 8.1.8. We define $\omega = \mathbb{N}$.

Proposition 8.1.9 (Dependent Choice). *Let A be a nonempty set and R a relation on A such that $\forall x \in A. \exists y \in A. (y, x) \in R$. Then there exists a function $f : \mathbb{N} \rightarrow A$ such that $\forall n \in \mathbb{N}. (f(n+1), f(n)) \in R$.*

PROOF:

$\langle 1 \rangle 1.$ PICK a choice function F for A .

$\langle 1 \rangle 2.$ PICK $a \in A$

$\langle 1 \rangle 3.$ Define $f : \mathbb{N} \rightarrow A$ by $f(0) = a$ and $f(n+1) = F(\{y \in A \mid (y, f(n)) \in R\})$.

\square

Theorem 8.1.10. *Let $<$ be a linear ordering on A . Then $<$ is a well ordering on A if and only if there does not exist a function $f : \mathbb{N} \rightarrow A$ such that $\forall n \in \mathbb{N}. f(n+1) < f(n)$.*

PROOF:

$\langle 1 \rangle 1.$ If there exists a function $f : \mathbb{N} \rightarrow A$ such that $\forall n \in \mathbb{N}. f(n+1) < f(n)$ then $<$ is not a well ordering on A .

PROOF: $\text{ran } f$ is a nonempty subset of A with no least element.

$\langle 1 \rangle 2.$ If $<$ is not a well ordering on A then there exists a function $f : \mathbb{N} \rightarrow A$ such that $\forall n \in \mathbb{N}. f(n+1) < f(n)$.

$\langle 2 \rangle 1.$ ASSUME: $<$ is not a well ordering on A .

$\langle 2 \rangle 2.$ PICK a nonempty subset $B \subseteq A$ that has no least element.

$\langle 2 \rangle 3.$ $\forall x \in B. \exists y \in B. y < x$

$\langle 2 \rangle 4.$ Choose a function $g : B \rightarrow B$ such that $\forall x \in B. g(x) < x$

$\langle 2 \rangle 5.$ PICK $b \in B$

- ⟨2⟩6. Define $f : \mathbb{N} \rightarrow A$ recursively by $f(0) = b$ and $\forall n \in \mathbb{N}. f(n+1) = g(f(n))$
 ⟨2⟩7. $\forall n \in \mathbb{N}. f(n+1) < f(n)$

□

Proposition 8.1.11. *There is no function $f : \mathbb{N} \rightarrow \mathbf{V}$ such that $\forall n. f(n^+) \in f(n)$.*

PROOF:

- ⟨1⟩1. LET: $f : \mathbb{N} \rightarrow \mathbf{V}$
 ⟨1⟩2. PICK $m \in f(\mathbb{N})$ such that $m \cap f(\mathbb{N}) = \emptyset$
 PROOF: Axiom of Regularity.
 ⟨1⟩3. PICK $n \in \mathbb{N}$ such that $m = f(n)$.
 ⟨1⟩4. $f(n) \cap f(\mathbb{N}) = \emptyset$
 ⟨1⟩5. $f(n^+) \notin f(n)$

□

8.2 Cardinality

Definition 8.2.1 (Finite). A set is *finite* iff it is equinumerous to some natural number; otherwise it is *infinite*.

Theorem 8.2.2 (Pigeonhole Principle). *No natural number is equinumerous to a proper subset of itself.*

PROOF:

- ⟨1⟩1. LET: $P(n)$ be the property: any one-to-one function $n \rightarrow n$ is surjective.
 ⟨1⟩2. $P(0)$
 PROOF: The only function $0 \rightarrow 0$ is injective.
 ⟨1⟩3. For every natural number n , if $P(n)$ then $P(n+1)$.
 ⟨2⟩1. ASSUME: $P(n)$
 ⟨2⟩2. LET: f be a one-to-one function $n+1 \rightarrow n+1$
 ⟨2⟩3. $f \upharpoonright n$ is a one-to-one function $n \rightarrow n+1$
 ⟨2⟩4. CASE: $n \notin \text{ran } f$
 ⟨3⟩1. $f \upharpoonright n : n \rightarrow n$
 ⟨3⟩2. $\text{ran}(f \upharpoonright n) = n$
 ⟨3⟩3. $f(n) = n$
 PROOF: ⟨2⟩1.
 ⟨3⟩4. $\text{ran } f = n+1$
 ⟨2⟩5. CASE: $n \in \text{ran } f$
 ⟨3⟩1. PICK $p \in n$ such that $f(p) = n$
 ⟨3⟩2. LET: $\hat{f} : n \rightarrow n$ be the function

$$\hat{f}(p) = f(n)$$

$$\hat{f}(x) = f(x) \quad (x \neq p)$$

 ⟨3⟩3. \hat{f} is one-to-one
 ⟨3⟩4. $\text{ran } \hat{f} = n$
 PROOF: ⟨2⟩1

$\langle 3 \rangle 5. \text{ ran } f = n + 1$
 $\langle 1 \rangle 4. \text{ For every natural number } n, P(n).$
 \square

Corollary 8.2.2.1. *No finite set is equinumerous to a proper subset of itself.*

Corollary 8.2.2.2. *Every natural number is a cardinal number.*

PROOF: For any natural number n , we have that n is the least ordinal such that $n \approx n$. \square

Corollary 8.2.2.3. \mathbb{N} is a cardinal number.

Corollary 8.2.2.4. \mathbb{N} is infinite.

PROOF: The function that maps n to $n + 1$ is a bijection between \mathbb{N} and $\mathbb{N} - \{0\}$. \square

Corollary 8.2.2.5. *If C is a proper subset of a natural number n , then there exists $m < n$ such that $C \approx m$.*

PROOF: By Proposition 6.1.12. \square

Corollary 8.2.2.6. *Any subset of a finite set is finite.*

Proposition 8.2.3. *For any natural numbers m and n we have $m + n$ (cardinal addition) is a natural number.*

PROOF: Induction on n . \square

Corollary 8.2.3.1. *The union of two finite sets is finite.*

Corollary 8.2.3.2. *The union of a finite set of finite sets is finite.*

PROOF: By induction on the number of elements. \square

Proposition 8.2.4. *For natural numbers m and n , the cardinal sum $m + n$ is equal to the ordinal sum $m + n$.*

PROOF: Induction on n . \square

Proposition 8.2.5. *For any natural numbers m and n , we have mn (cardinal multiplication) is a natural number.*

Corollary 8.2.5.1. *If A and B are finite sets then $A \times B$ is finite.*

Proposition 8.2.6. *For natural numbers m and n , the cardinal product mn is equal to the ordinal product mn .*

PROOF: Induction on n . \square

Proposition 8.2.7. *For any natural numbers m and n we have m^n (cardinal exponentiation) is a natural number.*

PROOF: Induction on n .

Corollary 8.2.7.1. *If A and B are finite sets then A^B are finite.*

Proposition 8.2.8. *For natural numbers m and n , the cardinal exponentiation m^n and the ordinal exponentiation m^n agree.*

PROOF: Induction on n . \square

Proposition 8.2.9. $\mathbb{N}^2 \approx \mathbb{N}$

PROOF: The function $J : \mathbb{N}^2 \rightarrow \mathbb{N}$ defined by $J(m, n) = ((m + n)^2 + 3m + n)/2$ is a bijection. \square

Proposition 8.2.10. *For any infinite cardinal κ we have $\aleph_0 \leq \kappa$.*

PROOF:

$\langle 1 \rangle 1$. LET: A be an infinite set.

PROVE: $\mathbb{N} \preceq A$

$\langle 1 \rangle 2$. PICK a choice function F for A .

$\langle 1 \rangle 3$. Define $h : \mathbb{N} \rightarrow \{X \in \mathcal{P}A \mid X \text{ is finite}\}$ by

$$h(0) = \emptyset$$

$$h(n+1) = h(n) \cup \{F(A - \{h(m) \mid m < n\})\}$$

$\langle 1 \rangle 4$. Define $g : \mathbb{N} \rightarrow A$ by $g(n) = F(A - \{h(m) \mid m < n\})$

$\langle 1 \rangle 5$. g is injective.

PROOF: If $m < n$ then $g(m) \neq g(n)$.

\square

8.3 Countable Sets

Definition 8.3.1 (Countable). A set A is *countable* iff $|A| \leq \aleph_0$.

Theorem 8.3.2. *The union of a countable set of countable sets is countable.*

PROOF: Proposition 7.2.6. \square

8.4 Arithmetic

Definition 8.4.1 (Even). A natural number n is *even* iff there exists $m \in \mathbb{N}$ such that $n = 2m$.

Definition 8.4.2 (Odd). A natural number n is *odd* iff there exists $p \in \mathbb{N}$ such that $n = 2p + 1$.

Proposition 8.4.3. *Every natural number is either even or odd.*

PROOF:

$\langle 1 \rangle 1$. 0 is even.

PROOF: $0 = 2 \times 0$.

⟨1⟩2. For any natural number n , if n is either even or odd then n^+ is either even or odd.

PROOF:

⟨2⟩1. LET: $n \in \mathbb{N}$

⟨2⟩2. If n is even then n^+ is odd.

PROOF: If $n = 2p$ then $n^+ = 2p + 1$.

⟨2⟩3. If n is odd then n^+ is even.

PROOF: If $n = 2p + 1$ then $n^+ = 2(p + 1)$.

□

Proposition 8.4.4. *No natural number is both even and odd.*

PROOF:

⟨1⟩1. 0 is not odd.

PROOF: For any p we have $2p + 1 = (2p)^+ \neq 0$.

⟨1⟩2. For any natural number n , if n is not both even and odd, then n^+ is not both even and odd.

⟨2⟩1. LET: n be a natural number.

⟨2⟩2. If n^+ is even then n is odd.

⟨3⟩1. ASSUME: n^+ is even.

⟨3⟩2. PICK p such that $n^+ = 2p$

⟨3⟩3. $p \neq 0$

PROOF: Since $n^+ \neq 0$.

⟨3⟩4. PICK q such that $p = q^+$

PROOF: Theorem 8.1.5.

⟨3⟩5. $n^+ = 2q + 2$

PROOF: ⟨3⟩2, ⟨3⟩4.

⟨3⟩6. $n = 2q + 1$

PROOF: Proposition 6.2.7, ⟨3⟩5

⟨3⟩7. n is odd.

⟨2⟩3. If n^+ is odd then n is even.

⟨3⟩1. ASSUME: n^+ is odd.

⟨3⟩2. PICK p such that $n^+ = 2p + 1$

⟨3⟩3. $n = 2p$

PROOF: Proposition 6.2.7, ⟨3⟩2

⟨3⟩4. n is even.

□

Proposition 8.4.5. *Let m, n, p, q be natural numbers. Assume $m + n = p + q$. Then $m < p$ if and only if $q < n$.*

PROOF:

⟨1⟩1. If $m < p$ then $q < n$.

PROOF: If $m < p$ and $n \leq q$ then $m + n < p + q$.

⟨1⟩2. If $q < n$ then $m < p$.

PROOF: Similar.

□

Proposition 8.4.6. *Let m, n, p and q be natural numbers. Assume $n < m$ and $q < p$. Then*

$$mq + np < mp + nq .$$

PROOF:

$\langle 1 \rangle 1$. PICK positive natural numbers a and b such that $m = n + a$ and $p = q + b$.

$\langle 1 \rangle 2$. $mp + nq > mq + np$

PROOF:

$$\begin{aligned} mp + nq &= (n + a)(q + b) + nq \\ &= 2nq + nb + aq + ab \\ mq + np &= (n + a)q + n(q + b) \\ &= 2nq + aq + nb \\ \therefore mp + nq &= mq + np + ab \\ &> mq + np \end{aligned}$$

□

8.5 Sequences

Definition 8.5.1 (Sequence). Let A be a set. A *finite sequence* in A is a function $a : n \rightarrow A$ for some natural number n ; we write it as $(a(0), a(1), \dots, a(n-1))$. An (*infinite*) *sequence* in A is a function $\mathbb{N} \rightarrow A$.

We write A^* for the set of all finite sequences in A .

Proposition 8.5.2. *If A is countable then A^* is countable.*

PROOF: For any n , the set A^n is countable, and A^* is equinumerous with $\bigcup_n A^n$.

□

8.6 Transitive Closure

Proposition 8.6.1. *For any set A , there exists a unique transitive set C such that:*

- $A \subseteq C$
- For any transitive set X , if $A \subseteq X$ then $C \subseteq X$

PROOF:

$\langle 1 \rangle 1$. Define a function $F : \mathbb{N} \rightarrow \mathbf{V}$ by

$$F(0) = A$$

$$F(n+1) = A \cup \bigcup (F(0) \cup \dots \cup F(n))$$

$\langle 1 \rangle 2$. For all $n \in \mathbb{N}$ and $a \in F(n)$ we have $a \subseteq F(n+1)$

PROOF: $a \in F(0) \cup \dots \cup F(n)$ so $a \subseteq \bigcup (F(0) \cup \dots \cup F(n)) \subseteq F(n+1)$.

$\langle 1 \rangle 3$. LET: $C = \bigcup_{n \in \mathbb{N}} F(n)$

$\langle 1 \rangle 4$. C is transitive.

- ⟨2⟩1. LET: $x \in y \in C$
- ⟨2⟩2. PICK $n \in \mathbb{N}$ such that $y \in F(n)$
- ⟨2⟩3. $y \subseteq F(n+1)$
- PROOF: ⟨1⟩2
- ⟨2⟩4. $x \in F(n+1)$
- ⟨2⟩5. $x \in C$
- ⟨1⟩5. $A \subseteq C$
- PROOF: Since $F(0) = A$.
- ⟨1⟩6. For any transitive set X , if $A \subseteq X$ then $C \subseteq X$
- ⟨2⟩1. LET: X be a transitive set
- ⟨2⟩2. ASSUME: $A \subseteq X$
- ⟨2⟩3. For all $n \in \mathbb{N}$ we have $F(n) \subseteq X$.
- ⟨3⟩1. $F(0) \subseteq X$
- PROOF: ⟨2⟩2
- ⟨3⟩2. For all $n \in \mathbb{N}$, if $F(n) \subseteq X$, then $F(n+1) \subseteq X$.
- ⟨4⟩1. LET: $n \in \mathbb{N}$
- ⟨4⟩2. ASSUME: $\forall m < n. F(m) \subseteq X$
- ⟨4⟩3. $F(0) \cup \dots \cup F(n) \subseteq X$
- ⟨4⟩4. $\bigcup(F(0) \cup \dots \cup F(n)) \subseteq X$
- PROOF: Since X is transitive.
- ⟨4⟩5. $F(n+1) \subseteq X$
- ⟨2⟩4. $C \subseteq X$
- ⟨1⟩7. Let D be a transitive set such that $A \subseteq D$ and, for any transitive set X , if $A \subseteq X$ then $D \subseteq X$. Then $D = C$.
- PROOF: We have $C \subseteq D$ and $D \subseteq C$.

□

8.7 The Veblen Fixed Point Theorem

Theorem Schema 8.7.1 (Veblen Fixed Point Theorem). *For any class \mathbf{T} , the following is a theorem:*

Assume \mathbf{T} is a normal ordinal operation. For every ordinal β , there exists $\gamma \geq \beta$ such that $\mathbf{T}(\gamma) = \gamma$.

PROOF:

- ⟨1⟩1. LET: β be an ordinal.
- ⟨1⟩2. ASSUME: w.l.o.g. $\beta < \mathbf{T}(\beta)$
- PROOF: We have $\beta \leq \mathbf{T}(\beta)$ by Proposition 6.4.5, and if $\beta = \mathbf{T}(\beta)$ we take $\gamma := \beta$.
- ⟨1⟩3. Define $f : \mathbb{N} \rightarrow \mathbf{On}$ by recursion thus:

$$f(0) = \beta$$

$$f(n^+) = \mathbf{T}(f(n))$$
- ⟨1⟩4. LET: $\gamma = \sup_{n \in \mathbb{N}} f(n)$
- ⟨1⟩5. $\beta \leq \gamma$
- PROOF: Since $\beta = f(0)$.

$\langle 1 \rangle 6. \mathbf{T}(\gamma) = \gamma$

$\langle 2 \rangle 1. \mathbf{T}(\gamma) \leq \gamma$

PROOF:

$$\mathbf{T}(\gamma) = \sup_{n \in \mathbb{N}} \mathbf{T}(f(n)) \quad (\text{Theorem 6.4.7})$$

$$= \sup_{n \in \mathbb{N}} f(n^+) \quad (\langle 1 \rangle 3)$$

$$\leq \sup_{n \in \mathbb{N}} f(n)$$

$$= \gamma$$

$\langle 2 \rangle 2. \gamma \leq \mathbf{T}(\gamma)$

PROOF: Proposition 6.4.5.

□

Definition 8.7.2 (Derived Operation). Let \mathbf{T} be a normal ordinal operation. The *derived* operation $\mathbf{T}' : \mathbf{On} \rightarrow \mathbf{V}$ is the unique order isomorphism between \mathbf{On} and the fixed points of \mathbf{T} .

Proposition Schema 8.7.3. *For any class \mathbf{T} , the following is a theorem:*

If \mathbf{T} is a normal ordinal operation, then the derived operation is normal.

PROOF:

$\langle 1 \rangle 1.$ For any set S of fixed points of \mathbf{T} , we have $\bigcup S$ is a fixed point of \mathbf{T}

$\langle 2 \rangle 1.$ LET: S be a set of fixed points of \mathbf{T} .

$\langle 2 \rangle 2. \mathbf{T}(\sup S) = \sup S$

PROOF:

$$\mathbf{T}(\sup S) = \sup_{\alpha \in S} \mathbf{T}(\alpha) \quad (\text{Theorem 6.4.7})$$

$$= \sup_{\alpha \in S} \alpha \quad (\langle 2 \rangle 1)$$

$$= \sup S$$

$\langle 1 \rangle 2.$ Q.E.D.

PROOF: Proposition 6.4.8.

□

Chapter 9

Infinite Cardinals

9.1 Alephs

Definition 9.1.1 (Aleph). Let \aleph be the unique order isomorphism between **On** and the class of infinite cardinals.

Proposition 9.1.2. *The operation \aleph is normal.*

PROOF: Proposition 6.4.8 and Lemma 7.2.7. \square

Definition 9.1.3 (Continuum Hypothesis). The *continuum hypothesis* is the statement that $\aleph_1 = 2^{\aleph_0}$.

Definition 9.1.4 (Generalised Continuum Hypothesis). The *generalised continuum hypothesis* is the statement that, for all α , $\aleph_{\alpha+} = 2^{\aleph_\alpha}$.

9.2 Beths

Definition 9.2.1 (Beth). Define the operation $\beth : \mathbf{On} \rightarrow \mathbf{Card}$ by transfinite recursion as follows:

$$\begin{aligned}\beth_0 &:= \aleph_0 \\ \beth_{\alpha+} &:= 2^{\beth_\alpha} \\ \beth_\lambda &:= \bigcup_{\alpha < \lambda} \beth_\alpha \quad (\lambda \text{ a limit ordinal})\end{aligned}$$

Proposition 9.2.2. \beth is a normal operation.

PROOF: It is continuous by definition, and $\beth_\alpha < \beth_{\alpha+}$ by Cantor's Theorem. \square

Proposition 9.2.3. *The continuum hypothesis is equivalent to the statement $\beth_1 = \aleph_1$.*

The generalised continuum hypothesis is equivalent to the statement $\beth = \aleph$.

PROOF: Immediate from definitions. \square

Chapter 10

The Cumulative Hierarchy

Definition 10.0.1 (Cumulative Hierarchy). Define the function $V : \mathbf{On} \rightarrow \mathbf{V}$ by transfinite recursion thus:

$$V_\alpha = \bigcup_{\beta < \alpha} \mathcal{P}V_\beta$$

Proposition 10.0.2. *For all $\alpha \in \mathbf{On}$, V_α is a transitive set.*

PROOF:

$\langle 1 \rangle 1.$ LET: $\alpha \in \mathbf{On}$

$\langle 1 \rangle 2.$ ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha. V_\beta$ is a transitive set.

$\langle 1 \rangle 3.$ For all $\beta < \alpha$, $\mathcal{P}V_\beta$ is a transitive set.

PROOF: Proposition 1.5.4.

$\langle 1 \rangle 4.$ V_α is a transitive set.

PROOF: Proposition 1.5.3.

□

Proposition 10.0.3. *For any ordinals α and β , if $\beta < \alpha$ then $V_\beta \subseteq V_\alpha$.*

PROOF: Since $V_\beta \in \mathcal{P}V_\beta \subseteq V_\alpha$ and V_α is a transitive set. □

Theorem 10.0.4.

1. $V_0 = \emptyset$

2. $\forall \alpha \in \mathbf{On}. V_{\alpha+} = \mathcal{P}V_\alpha$

3. For any limit ordinal λ , $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$.

PROOF:

$\langle 1 \rangle 1.$ $V_0 = \emptyset$

PROOF: Immediate from definition.

$\langle 1 \rangle 2.$ $\forall \alpha \in \mathbf{On}. V_{\alpha+} = \mathcal{P}V_\alpha$

PROOF:

- ⟨2⟩1. LET: $\alpha \in \mathbf{On}$
 ⟨2⟩2. For all $\beta < \alpha$ we have $\mathcal{P}V_\beta \subseteq \mathcal{P}V_\alpha$
 PROOF: Propositions 1.4.7 and 10.0.3.
 ⟨2⟩3. $V_{\alpha^+} = \mathcal{P}V_\alpha$

$$\begin{aligned}
 V_{\alpha^+} &= \bigcup_{\beta < \alpha^+} \mathcal{P}V_\beta \\
 &= \bigcup_{\beta < \alpha} \mathcal{P}V_\beta \cup \mathcal{P}V_\alpha \\
 &\quad \mathcal{P}V_\alpha
 \end{aligned}$$

□

- ⟨1⟩3. For any limit ordinal λ , $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$

PROOF:

- ⟨2⟩1. $V_\lambda \subseteq \bigcup_{\alpha < \lambda} V_\alpha$

PROOF:

$$\begin{aligned}
 V_\lambda &= \bigcup_{\alpha < \lambda} \mathcal{P}V_\alpha \\
 &= \bigcup_{\alpha < \lambda} V_{\alpha^+} & (\langle 1 \rangle 2) \\
 &\subseteq \bigcup_{\alpha < \lambda} V_\alpha
 \end{aligned}$$

- ⟨2⟩2. $\bigcup_{\alpha < \lambda} V_\alpha \subseteq V_\lambda$

PROOF: Proposition 10.0.3.

□

Proposition 10.0.5. *For every set A , there exists an ordinal α such that $A \in V_\alpha$.*

PROOF:

- ⟨1⟩1. Let us say a set A is *grounded* iff there exists an ordinal α such that $A \in V_\alpha$.
 ⟨1⟩2. For any set A , if every element of A is grounded, then A is grounded.
 ⟨2⟩1. LET: A be a set.
 ⟨2⟩2. $S = \{\alpha \mid \exists a \in A. \alpha \text{ is the least ordinal such that } a \in V_\alpha\}$
 PROOF: S is a set by an Axiom of Replacement.
 ⟨2⟩3. LET: $\beta = \sup S$
 ⟨2⟩4. $A \subseteq V_\beta$
 ⟨3⟩1. LET: $a \in A$
 ⟨3⟩2. LET: α be the least ordinal such that $a \in V_\alpha$
 ⟨3⟩3. $\alpha \in S$
 ⟨3⟩4. $\alpha \leq \beta$
 ⟨3⟩5. $a \in V_\beta$
 ⟨2⟩5. $A \in V_{\beta^+}$
 ⟨1⟩3. ASSUME: for a contradiction there exists an ungrounded set.
 ⟨1⟩4. PICK a transitive set B that has an ungrounded member.
 PROOF: Pick a transitive set c , and take B to be the transitive closure of $\{c\}$.

- ⟨1⟩5. LET: $A = \{x \in B \mid x \text{ is ungrounded}\}$
 ⟨1⟩6. PICK $m \in A$ such that $m \cap A = \emptyset$
 PROOF: Axiom of Regularity.
 ⟨1⟩7. Every member of m is grounded.
 ⟨2⟩1. ASSUME: for a contradiction $x \in m$ is ungrounded.
 ⟨2⟩2. $x \in B$
 PROOF: Since B is transitive (⟨1⟩4).
 ⟨2⟩3. $x \in A$
 PROOF: ⟨1⟩5
 ⟨2⟩4. Q.E.D.
 PROOF: This contradicts ⟨1⟩6.
 ⟨1⟩8. m is grounded.
 PROOF: ⟨1⟩2
 ⟨1⟩9. Q.E.D.
 PROOF: This contradicts ⟨1⟩6.
 □

Definition 10.0.6 (Rank). The *rank* of a set A is the least ordinal α such that $A \in V_{\alpha^+}$.

Proposition 10.0.7. For any set A we have

$$\text{rank } A = \bigcup_{a \in A} (\text{rank } a)^+$$

PROOF:

- ⟨1⟩1. LET: $\alpha = \bigcup_{a \in A} (\text{rank } a)^+$
 ⟨1⟩2. $A \subseteq V_\alpha$
 ⟨2⟩1. LET: $a \in A$
 ⟨2⟩2. $a \in V_{(\text{rank } a)^+}$
 ⟨2⟩3. $a \in V_\alpha$
 ⟨1⟩3. $A \in V_{\alpha^+}$
 ⟨1⟩4. If $A \subseteq V_\beta$ then $\alpha \leq \beta$
 ⟨2⟩1. ASSUME: $A \subseteq V_\beta$
 ⟨2⟩2. For all $a \in A$ we have $(\text{rank } a)^+ \leq \beta$
 PROOF: Since $a \in V_\beta$.
 ⟨2⟩3. $\alpha \leq \beta$
 □

Corollary 10.0.7.1. For any sets a and b , if $a \in b$ then $\text{rank } a < \text{rank } b$.

Proposition 10.0.8. For any ordinal number α we have $\text{rank } \alpha = \alpha$.

PROOF:

- ⟨1⟩1. LET: α be an ordinal.
 ⟨1⟩2. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha. \text{rank } \beta = \beta$
 ⟨1⟩3. $\text{rank } \alpha = \bigcup_{\beta < \alpha} \beta^+$

PROOF:

$$\begin{aligned}\text{rank } \alpha &= \bigcup_{\beta < \alpha} (\text{rank } \beta)^+ \\ &= \bigcup_{\beta < \alpha} \beta^+\end{aligned}$$

$$\langle 1 \rangle 4. \bigcup_{\beta < \alpha} \beta^+ \leq \alpha$$

PROOF: Since for all $\beta < \alpha$ we have $\beta^+ \leq \alpha$.

$$\langle 1 \rangle 5. \alpha \leq \bigcup_{\beta < \alpha} \beta^+$$

$$\langle 2 \rangle 1. \text{ LET: } \gamma = \bigcup_{\beta < \alpha} \beta^+$$

$$\langle 2 \rangle 2. \text{ ASSUME: for a contradiction } \gamma < \alpha$$

$$\langle 2 \rangle 3. \gamma^+ \leq \bigcup_{\beta < \alpha} \beta^+ = \gamma$$

$$\langle 2 \rangle 4. \text{ Q.E.D.}$$

PROOF: This is a contradiction.

□

Chapter 11

Group Theory

11.1 Groups

Definition 11.1.1 (Group). A *group* G consists of a set G and a function $\cdot : G^2 \rightarrow G$ such that:

1. \cdot is associative
2. There exists $e \in G$ such that $\forall x \in G. xe = x$ and $\forall x \in G. \exists y \in G. xy = e$.

Proposition 11.1.2. *The inverse of an element in a group is unique.*

PROOF:

$\langle 1 \rangle$ 1. ASSUME: b and b' are inverses of a .

$\langle 1 \rangle$ 2. $b = b'$

PROOF:

$$\begin{aligned} b &= be \\ &= bab' \\ &= eb' \\ &= b' \end{aligned}$$

□

Definition 11.1.3. We write x^{-1} for the inverse of x .

Proposition 11.1.4. *In any group, if $ab = ac$ then $b = c$.*

PROOF:

$$\begin{aligned} b &= eb \\ &= a^{-1}ab \\ &= a^{-1}ac \\ &= ec \\ &= c \end{aligned}$$

□

11.2 Abelian Groups

Definition 11.2.1 (Abelian group). An *Abelian group* is a group whose multiplication is commutative.

We may say we are writing an Abelian group *additively*, meaning we write $a + b$ for ab , 0 for e and $-a$ for a^{-1} . In this case we write $a - b$ for ab^{-1} .

Chapter 12

Ring Theory

12.1 Rings

Definition 12.1.1 (Commutative Ring). A *commutative ring* consists of a set R and two binary operations $+$, \cdot on R such that:

- D is an Abelian group under $+$. Let us write 0 for its identity element.
- \cdot is commutative and associative, and distributes over $+$.
- \cdot has an identity element 1 that is different from 0 .

Proposition 12.1.2. *In any commutative ring, $0x = 0$.*

PROOF:

$$\begin{aligned}(0 + 0)x &= 0x \\ \therefore 0x + 0x &= 0x + 0 \\ \therefore 0x &= 0 && \text{(Proposition 11.1.4)} \square\end{aligned}$$

Proposition 12.1.3. *In any commutative ring, $(-a)b = -(ab)$.*

PROOF:

$$\begin{aligned}ab + (-a)b &= (a + (-a))b \\ &= 0b \\ &= 0 && \text{(Proposition 12.1.2)} \square\end{aligned}$$

12.2 Ordered Rings

Definition 12.2.1 (Ordered Commutative Ring). An *ordered commutative ring* consists of a commutative ring R with a linear order $<$ on R such that:

- for all $x, y, z \in R$, we have $x < y$ if and only if $x + z < y + z$.

- for all $x, y, z \in R$, if $0 < z$ then we have $x < y$ if and only if $xz < yz$.

Proposition 12.2.2. *In any ordered commutative ring, $0 < 1$.*

PROOF: If $1 < 0$ then we have $0 < -1$ and so $0 < (-1)(-1) = 1$, which is a contradiction. \square

Proposition 12.2.3. *The ordering on an ordered commutative ring is dense; that is, if $x < y$ then there exists z such that $x < z < y$.*

PROOF: Take $z = (x + y)/2$. \square

12.3 Integral Domains

Definition 12.3.1 (Integral Domain). An *integral domain* is a commutative ring such that, for all $a, b \in D$, if $ab = 0$ then $a = 0$ or $b = 0$.

Proposition 12.3.2. *In any integral domain, if $ab = ac$ and $a \neq 0$ then $b = c$.*

PROOF: We have $a(b - c) = 0$ and $a \neq 0$ so $b - c = 0$ hence $b = c$. \square

Definition 12.3.3 (Ordered Integral Domain). An *ordered integral domain* is an ordered commutative ring that is an integral domain.

Chapter 13

Field Theory

13.1 Fields

Definition 13.1.1 (Field). A *field* F is a commutative ring such that $0 \neq 1$ and, for all $x \in F$, if $x \neq 0$ then there exists $y \in F$ such that $xy = 1$.

Proposition 13.1.2. *Every field is an integral domain.*

PROOF: If $ab = 0$ and $a \neq 0$ then $b = a^{-1}ab = 0$. \square

Proposition 13.1.3. *In any field F , we have $F - \{0\}$ is an Abelian group under multiplication.*

PROOF: Immediate from the definition. \square

Definition 13.1.4 (Field of Fractions). Let D be an integral domain. The *field of fractions* of D is the quotient set $F = (D \times (D - \{0\})) / \sim$ where

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

under

$$\begin{aligned} [(a, b)] + [(c, d)] &= [(ad + bc, bd)] \\ [(a, b)][(c, d)] &= [(ac, bd)] \end{aligned}$$

We prove this is a field.

PROOF:

$\langle 1 \rangle 1.$ \sim is an equivalence relation on $D \times (D - \{0\})$.

PROOF:

$\langle 2 \rangle 1.$ \sim is reflexive.

PROOF: We always have $ab = ba$.

$\langle 2 \rangle 2.$ \sim is symmetric.

PROOF: If $ad = bc$ then $cb = da$.

$\langle 2 \rangle 3$. \sim is transitive.

$\langle 3 \rangle 1$. ASSUME: $(a, b) \sim (c, d) \sim (e, f)$

$\langle 3 \rangle 2$. $ad = bc$ and $cf = de$

$\langle 3 \rangle 3$. $adf = bde$

PROOF: $adf = bcf = bde$

$\langle 3 \rangle 4$. $af = be$

PROOF: Proposition 12.3.2.

□

$\langle 1 \rangle 2$. Addition is well-defined.

PROOF:

$\langle 2 \rangle 1$. If $b \neq 0$ and $d \neq 0$ then $bd \neq 0$.

PROOF: Since D is an integral domain.

$\langle 2 \rangle 2$. If $ab' = a'b$ and $cd' = c'd$ then $(ad + bc)b'd' = (a'd' + b'c')bd$.

PROOF:

$$\begin{aligned} (ad + bc)b'd' &= ab'dd' + bb'cd' \\ &= a'bdd' + bb'c'd \\ &= (a'd' + b'c')bd \end{aligned}$$

□

$\langle 1 \rangle 3$. Multiplication is well-defined.

PROOF:

$\langle 2 \rangle 1$. If $b \neq 0$ and $d \neq 0$ then $bd \neq 0$.

PROOF: Since D is an integral domain.

$\langle 2 \rangle 2$. If $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$ then $[(ac, bd)] = [(a'c', b'd')]$.

PROOF: If $ab' = a'b$ and $cd' = c'd$ then $acb'd' = a'c'bd$.

□

$\langle 1 \rangle 4$. Addition is commutative.

PROOF: $[(a, b)] + [(c, d)] = [(ad + bc, bd)] = [(cb + da, db)] = [(c, d)] + [(a, b)]$ □

$\langle 1 \rangle 5$. Addition is associative.

PROOF:

$$\begin{aligned} [(a, b)] + ([[(c, d)] + [(e, f)]] &= [(a, b)] + [(cf + de, df)] \\ &= [(adf + bcf + bde, bdf)] \\ &= [(ad + bc, bd)] + [(e, f)] \\ &= ([[(a, b)] + [(c, d)]] + [(e, f)]) \quad \square \end{aligned}$$

$\langle 1 \rangle 6$. For any $x \in F$ we have $x + [(0, 1)] = x$

PROOF: $[(a, b)] + [(0, 1)] = [(a \cdot 1 + b \cdot 0, b \cdot 1)] = [(a, b)]$ □

$\langle 1 \rangle 7$. For any $x \in F$, there exists $y \in F$ such that $x + y = [(0, 1)]$.

PROOF: $[(a, b)] + [(-a, b)] = [(ab - ab, b^2)] = [(0, b^2)] = [(0, 1)]$ □

$\langle 1 \rangle 8$. Multiplication is commutative.

PROOF: $[(a, b)][(c, d)] = [(c, d)][(a, b)] = [(ac, bd)]$. □

$\langle 1 \rangle 9$. Multiplication is associative.

PROOF: $[(a, b)]([[(c, d)][(e, f)]] = ([[(a, b)][(c, d)]][(e, f)] = [(ace, bdf)]$. □

$\langle 1 \rangle 10$. For any $x \in F$ we have $x[(1, 1)] = x$

PROOF: $[(a, b)][(1, 1)] = [(a, b)]$ □

$\langle 1 \rangle 11$. For any non-zero $x \in F$, there exists $y \in F$ such that $xy = [(1, 1)]$.

PROOF:

- $\langle 2 \rangle 1$. LET: $[(a, b)] \in \mathbb{Q}$
- $\langle 2 \rangle 2$. ASSUME: $[(a, b)] \neq [(0, 1)]$
- $\langle 2 \rangle 3$. $a \neq 0$
- $\langle 2 \rangle 4$. $[(a, b)][(b, a)] = [(1, 1)]$

□

□

Definition 13.1.5. For any field F , let $N(F)$ be the intersection of all the subsets $S \subseteq F$ such that $1 \in S$ and $\forall x \in S. x + 1 \in S$.

Definition 13.1.6 (Characteristic Zero). A field F has *characteristic 0* iff $0 \notin N(F)$.

Proposition 13.1.7. In a field F with characteristic 0, the function $n : \mathbb{N} \rightarrow N(F)$ defined by

$$\begin{aligned} n(0) &= 1 \\ n(x + 1) &= n(x) + 1 \end{aligned}$$

is a bijection.

PROOF:

- $\langle 1 \rangle 1$. n is injective.
- $\langle 2 \rangle 1$. ASSUME: for a contradiction $n(i) = n(j)$ with $i \neq j$
- $\langle 2 \rangle 2$. ASSUME: w.l.o.g. $i < j$
- $\langle 2 \rangle 3$. $n(j - i) = 0$
- $\langle 2 \rangle 4$. Q.E.D.

PROOF: This contradicts the fact that F has characteristic 0.

- $\langle 1 \rangle 2$. n is surjective.

PROOF: Since $\text{ran } n$ is a subset of F that includes 1 and is closed under $+1$.

□

Definition 13.1.8. In any field F , let

$$I(F) = N(F) \cup \{0\} \cup \{-x \mid x \in N(F)\}$$

Definition 13.1.9. In any field F , let

$$Q(F) = \{x/y \mid x, y \in I(F), y \neq 0\}$$

Proposition 13.1.10. $Q(F)$ is the smallest subfield of F .

PROOF: $Q(F)$ is closed under $+$ and \cdot , and any subset of F closed under $+$ and \cdot that contains 0 and 1 must include $Q(F)$. □

Theorem 13.1.11. Let F and G be fields of characteristic 0. Then there exists a unique field isomorphism between $Q(F)$ and $Q(G)$.

PROOF:

- (1)1. LET: $\phi : N(F) \rightarrow N(G)$ be the unique function such that $\phi(1) = 1$ and $\forall x \in N(F). \phi(x+1) = \phi(x) + 1$.
- (1)2. ϕ is a bijection.
 PROOF: Similar to Proposition 13.1.7.
- (1)3. $\forall x, y \in N(F). \phi(x+y) = \phi(x) + \phi(y)$
 PROOF: Induction on y .
- (1)4. $\forall x, y \in N(F). \phi(xy) = \phi(x)\phi(y)$
 PROOF: Induction on y .
- (1)5. Extend ϕ to a bijection $I(F) \cong I(G)$ such that $\forall x, y \in I(F). \phi(x+y) = \phi(x) + \phi(y)$ and $\forall x, y \in I(F). \phi(xy) = \phi(x)\phi(y)$
- (2)1. Define $\phi(0) = 0$ and $\phi(-x) = -\phi(x)$ for $x \in N(F)$
- (3)1. $0 \notin N(F)$
- (3)2. For all $x \in N(F)$ we have $-x \notin N(F)$
 PROOF: Then we would have $x + -x = 0 \in N(F)$.
- (3)3. For all $x \in N(F)$ we have $-x \neq 0$
- (2)2. For all $x, y \in I(F)$ we have $\phi(x+y) = \phi(x) + \phi(y)$
 PROOF: Case analysis on x and y .
- (2)3. For all $x, y \in I(F)$ we have $\phi(xy) = \phi(x)\phi(y)$
 PROOF: Case analysis on x and y .
- (1)6. Extend ϕ to a bijection $Q(F) \cong Q(G)$ such that $\forall x, y \in Q(F). \phi(x+y) = \phi(x) + \phi(y)$ and $\forall x, y \in Q(F). \phi(xy) = \phi(x)\phi(y)$
- (2)1. Define $\phi(x/y) = \phi(x)/\phi(y)$
- (1)7. ϕ is unique.
- (2)1. LET: θ satisfy the theorem.
- (2)2. For all $x \in N(F)$ we have $\theta(x) = \phi(x)$
- (2)3. For all $x \in I(F)$ we have $\theta(x) = \phi(x)$
- (2)4. For all $x \in Q(F)$ we have $\theta(x) = \phi(x)$

□

13.2 Ordered Fields

Definition 13.2.1 (Ordered Field). An *ordered field* is an ordered commutative ring that is a field.

Proposition 13.2.2. Every ordered field F has characteristic 0.

PROOF: We have $0 < n$ for all $n \in N(F)$. □

Proposition 13.2.3. Let F be a field of characteristic 0. Then there exists a unique relation $<$ on $Q(F)$ that makes $Q(F)$ into an ordered field.

PROOF: Easy. □

Corollary 13.2.3.1. Let F and G be ordered fields. Let ϕ be the unique field isomorphism between $Q(F)$ and $Q(G)$. Then ϕ is an ordered field isomorphism.

Definition 13.2.4 (Archimedean). An ordered field F is *Archimedean* iff

$$\forall x \in F. \exists n \in N(F). n > x .$$

Proposition 13.2.5. *Let F be an Archimedean ordered field. Let $x, y \in F$ with $x > 0$. Then there exists $n \in N(F)$ such that $nx > y$.*

PROOF: Pick $n > y/x$. \square

Proposition 13.2.6. *Let F be an Archimedean ordered field. For all $x, y \in F$, if $x < y$, then there exists $r \in Q(F)$ such that $x < r < y$.*

PROOF:

$\langle 1 \rangle 1$. CASE: $x > 0$

$\langle 2 \rangle 1$. PICK $n \in N(F)$ such that $n(y - x) > 1$

PROOF: Proposition 13.2.5.

$\langle 2 \rangle 2$. $ny > 1 + nx$

$\langle 2 \rangle 3$. LET: m be the least element of $N(F)$ such that $m > nx$.

$\langle 2 \rangle 4$. $m - 1 \leq nx$

$\langle 2 \rangle 5$. $nx < m < ny$

$\langle 2 \rangle 6$. $x < m/n < y$

$\langle 1 \rangle 2$. CASE: $x \leq 0$

$\langle 2 \rangle 1$. PICK $k \in N(F)$ such that $k > -x$

$\langle 2 \rangle 2$. $0 < x + k < y + k$

$\langle 2 \rangle 3$. PICK $r \in Q(F)$ such that $x + k < r < y + k$

PROOF: $\langle 1 \rangle 1$

$\langle 2 \rangle 4$. $x < r - k < y$

Definition 13.2.7 (Complete). An ordered field F is *complete* iff every nonempty subset of F bounded above has a least upper bound.

Proposition 13.2.8. *Every complete ordered field is Archimedean.*

PROOF:

$\langle 1 \rangle 1$. LET: F be a complete ordered field.

$\langle 1 \rangle 2$. LET: $x \in F$

$\langle 1 \rangle 3$. ASSUME: for a contradiction there is no member of $N(F)$ greater than x .

$\langle 1 \rangle 4$. x is an upper bound for $N(F)$.

$\langle 1 \rangle 5$. LET: $y = \sup N(F)$

$\langle 1 \rangle 6$. PICK $n \in N(F)$ such that $y - 1 < n$

$\langle 1 \rangle 7$. $y < n + 1$

$\langle 1 \rangle 8$. Q.E.D.

PROOF: This is a contradiction.

\square

Proposition 13.2.9. *Let F be a complete ordered field and $a \in F$ be nonnegative. Then there exists $b \in F$ such that $b^2 = a$.*

PROOF:

$\langle 1 \rangle 1$. LET: $B = \{x \in F \mid 0 \leq x \leq 1 + a\}$

$\langle 1 \rangle 2$. LET: $\phi : B \rightarrow B$ be the function

$$\phi(x) = x + \frac{1}{2(1+a)}(a - x^2) .$$

- ⟨1⟩3. ϕ is strictly monotone.
 ⟨2⟩1. LET: $0 \leq x < y \leq 1 + a$
 ⟨2⟩2. $1 - \frac{x+y}{2(1+a)} > 0$
 ⟨2⟩3. $\phi(y) - \phi(x) = (y - x)(1 - \frac{x+y}{2(1+a)}) > 0$
 ⟨2⟩4. $\phi(x) < \phi(y)$
 ⟨1⟩4. PICK $b \in B$ such that $\phi(b) = b$.
 PROOF: Knaster Fixed-Point Theorem.
 ⟨1⟩5. $b^2 = a$
 \square

Theorem 13.2.10 (Uniqueness of the Complete Ordered Field). *If F and G are complete ordered fields, then there exists a unique bijection $\phi : F \cong G$ such that, for all $x, y \in F$,*

$$\begin{aligned}\phi(x + y) &= \phi(x) + \phi(y) \\ \phi(xy) &= \phi(x)\phi(y)\end{aligned}$$

This bijection also satisfies: for all $x, y \in F$,

$$x < y \Leftrightarrow \phi(x) < \phi(y) .$$

PROOF:

- ⟨1⟩1. PICK a bijection $\phi : Q(F) \cong Q(G)$ such that, for all $x, y \in Q(F)$,
 $\phi(x + y) = \phi(x) + \phi(y)$
 $\phi(xy) = \phi(x)\phi(y)$
 $x < y \Leftrightarrow \phi(x) < \phi(y)$

PROOF: Corollary 13.2.3.1.

- ⟨1⟩2. $Q(F)$ intersects every interval in F .

PROOF: Proposition 13.2.6.

- ⟨1⟩3. $Q(G)$ intersects every interval in G .

PROOF: Proposition 13.2.6.

- ⟨1⟩4. PICK an order isomorphism $\psi : F \cong G$ that extends ϕ .

PROOF: Theorem 5.1.23.

- ⟨1⟩5. $\forall x, y \in F. \psi(x + y) = \psi(x) + \psi(y)$
 ⟨2⟩1. LET: $x, y \in F$
 ⟨2⟩2. $\psi(x) + \psi(y) \not\leq \psi(x + y)$
 ⟨3⟩1. ASSUME: for a contradiction $\psi(x) + \psi(y) < \psi(x + y)$
 ⟨3⟩2. PICK $r' \in Q(G)$ such that $\psi(x) < r' < \psi(x + y) - \psi(y)$
 ⟨3⟩3. PICK $s' \in Q(G)$ such that $\psi(y) < s' < \psi(x + y) - r'$
 ⟨3⟩4. $r' + s' < \psi(x + y)$
 ⟨3⟩5. PICK $r, s \in Q(F)$ such that $\phi(r) = r'$ and $\phi(s) = s'$
 ⟨3⟩6. $\phi(r + s) = r' + s'$
 ⟨3⟩7. $\psi(x) < \psi(r)$
 ⟨3⟩8. $\psi(y) < \psi(s)$
 ⟨3⟩9. $\psi(x + y) > \psi(r + s)$
 ⟨3⟩10. $x < r$

- $\langle 3 \rangle 11. y < s$
- $\langle 3 \rangle 12. x + y > r + s$
- $\langle 3 \rangle 13. \text{Q.E.D.}$

PROOF: This is a contradiction.

- $\langle 2 \rangle 3. \psi(x + y) \not\leq \psi(x) + \psi(y)$

PROOF: Similar.

- $\langle 1 \rangle 6. \forall x, y \in F. \psi(xy) = \psi(x)\psi(y)$

- $\langle 2 \rangle 1. \text{LET: } x, y \in F$

- $\langle 2 \rangle 2. \text{CASE: } x \text{ and } y \text{ are positive.}$

- $\langle 3 \rangle 1. \psi(x)\psi(y) \not\leq \psi(xy)$

- $\langle 4 \rangle 1. \text{ASSUME: for a contradiction } \psi(x)\psi(y) < \psi(xy)$

- $\langle 4 \rangle 2. \text{PICK } r' \in Q(G) \text{ such that } \psi(x) < r' < \psi(xy)/\psi(y)$

- $\langle 4 \rangle 3. \text{PICK } s' \in Q(G) \text{ such that } \psi(y) < s' < \psi(xy)/r'$

- $\langle 4 \rangle 4. r's' < \psi(xy)$

- $\langle 4 \rangle 5. \text{PICK } r, s \in Q(F) \text{ such that } \phi(r) = r' \text{ and } \phi(s) = s'$

- $\langle 4 \rangle 6. \phi(rs) = r's'$

- $\langle 4 \rangle 7. x < r, y < s \text{ and } rs < xy$

- $\langle 4 \rangle 8. \text{Q.E.D.}$

PROOF: This is a contradiction.

- $\langle 3 \rangle 2. \psi(xy) \not\leq \psi(x)\psi(y)$

PROOF: Similar.

- $\langle 2 \rangle 3. \text{CASE: } x \text{ and } y \text{ are not both positive.}$

PROOF: Follows from $\langle 2 \rangle 2$ since $\psi(-x) = -\psi(x)$ by $\langle 1 \rangle 5$.

- $\langle 1 \rangle 7. \text{For any field isomorphism } \theta : F \cong G, \text{ we have } \theta = \psi.$

- $\langle 2 \rangle 1. \theta \upharpoonright Q(F) = \phi$

PROOF: Theorem 13.1.11.

- $\langle 2 \rangle 2. \theta \text{ is strictly monotone.}$

- $\langle 3 \rangle 1. \text{LET: } x, y \in F \text{ with } x < y$

- $\langle 3 \rangle 2. y - x > 0$

- $\langle 3 \rangle 3. \text{PICK } z \in F \text{ such that } z^2 = y - x$

- $\langle 3 \rangle 4. \theta(z)^2 = \theta(y) - \theta(x)$

- $\langle 3 \rangle 5. \theta(y) - \theta(x) > 0$

- $\langle 3 \rangle 6. \theta(x) < \theta(y)$

- $\langle 2 \rangle 3. \theta = \psi$

PROOF: By the uniqueness of ψ .

□

Chapter 14

Number Systems

14.1 The Integers

Definition 14.1.1. The set of *integers* \mathbb{Z} is the quotient set \mathbb{N}^2 / \sim , where $(m, n) \sim (p, q)$ iff $m + q = n + p$.

We prove \sim is an equivalence relation on \mathbb{N}^2 .

PROOF:

$\langle 1 \rangle 1.$ \sim is reflexive.

PROOF: For all $m, n \in \mathbb{N}$ we have $m + n = n + m$.

$\langle 1 \rangle 2.$ \sim is symmetric.

PROOF: If $m + q = n + p$ then $p + n = q + m$.

$\langle 1 \rangle 3.$ \sim is transitive.

$\langle 2 \rangle 1.$ ASSUME: $(m, n) \sim (p, q) \sim (r, s)$

$\langle 2 \rangle 2.$ $m + q = n + p$ and $p + s = q + r$

$\langle 2 \rangle 3.$ $m + q + s = n + q + r$

$\langle 2 \rangle 4.$ $m + s = n + r$

PROOF: By cancellation.

□

Definition 14.1.2 (Addition). Define *addition* $+$ on \mathbb{Z} by $[(m, n)] + [(p, q)] = [(m + p, n + q)]$.

We prove this is well-defined.

PROOF: If $m + n' = n + m'$ and $p + q' = q + p'$ then $m + p + n' + q' = n + q + m' + p'$.

□

Proposition 14.1.3. *Addition on \mathbb{Z} is commutative.*

PROOF: $[(m, n)] + [(p, q)] = [(m + p, n + q)] = [(p + m, q + n)] = [(p, q)] + [(m, n)]$.

□

Proposition 14.1.4. *Addition on \mathbb{Z} is associative.*

PROOF: $[(m, n)] + [(p, q)] + [(r, s)] = [(m + p + r, n + q + s)] = [(m, n)] + [(p, q)] + [(r, s)]$. \square

Proposition 14.1.5. *Given natural numbers m and n , we have $[(m, 0)] = [(n, 0)]$ iff $m = n$.*

PROOF: Immediate from definitions. \square

Definition 14.1.6. We identify any natural number n with the integer $[(n, 0)]$.

Proposition 14.1.7. *Addition on integers agrees with addition on natural numbers.*

PROOF: Since $[(m, 0)] + [(n, 0)] = [(m + n, 0)]$. \square

Proposition 14.1.8. *For all $a \in \mathbb{Z}$ we have $a + 0 = a$.*

PROOF: $[(m, n)] + [(0, 0)] = [(m + 0, n + 0)] = [(m, n)]$. \square

Proposition 14.1.9. *For all $a \in \mathbb{Z}$, there exists $b \in \mathbb{Z}$ such that $a + b = 0$.*

PROOF: $[(m, n)] + [(n, m)] = [(m + n, m + n)] = [(0, 0)]$ \square

Proposition 14.1.10. *The integers form an Abelian group under addition.*

PROOF: Proposition 14.1.3, 14.1.4, 14.1.8, 14.1.9. \square

Definition 14.1.11. Define multiplication \cdot on \mathbb{Z} by: $[(m, n)][(p, q)] = [(mp + nq, mq + np)]$.

We prove this is well defined.

PROOF:

$\langle 1 \rangle 1$. ASSUME: $m + n' = n + m'$ and $p + q' = q + p'$

PROVE: $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$

$\langle 1 \rangle 2$. $mp + n'p = np + m'p$

$\langle 1 \rangle 3$. $nq + m'q = mq + n'q$

$\langle 1 \rangle 4$. $m'p + m'q' = m'q + m'p'$

$\langle 1 \rangle 5$. $n'q + n'p' = n'p + n'q'$

$\langle 1 \rangle 6$. $mp + n'p + nq + m'q + m'p + m'q' + n'q + n'p' = np + m'p + mq + n'q + m'q + m'p' + n'p + n'q'$

$\langle 1 \rangle 7$. $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$

PROOF: By cancellation.

\square

Proposition 14.1.12. *Multiplication on integers agrees with multiplication on natural numbers.*

PROOF: Since $[(m, 0)][(n, 0)] = [(mn + 0, m0 + n0)] = [(mn, 0)]$. \square

Proposition 14.1.13. *Multiplication on \mathbb{Z} is commutative.*

PROOF: $[(m, n)][(p, q)] = [(mp + nq, mq + np)] = [(pm + qn, pn + qm)] = [(p, q)][(m, n)]$. \square

Proposition 14.1.14. *Multiplication on \mathbb{Z} is associative.*

PROOF:

$$\begin{aligned}
 [(m, n)][(p, q)][(r, s)] &= [(m, n)][(pr + qs, ps + qr)] \\
 &= [(mpr + mqs + nps + nqr, mps + mqr + npr + nqs)] \\
 &= [(mp + nq, mq + np)][(r, s)] \\
 &= [(m, n)][(p, q)][(r, s)] \quad \square
 \end{aligned}$$

Proposition 14.1.15. *Multiplication distributes over addition.*

PROOF:

$$\begin{aligned}
 [(m, n)][(p, q)] + [(m, n)][(r, s)] &= [(m, n)][(p + r, q + s)] \\
 &= [(mp + mr + nq + ns, mp + nr + mq + ms)] \\
 [(m, n)][(p, q)] + [(m, n)][(r, s)] &= [(mp + nq, mq + np)] + [(mr + ns, ms + nr)] \\
 &= [(mp + nq + mr + ns, mq + np + ms + nr)] \quad \square
 \end{aligned}$$

Proposition 14.1.16. *For any integer a we have $a1 = a$.*

PROOF: Since $[(m, n)][(1, 0)] = [(m1 + n0, m0 + n1)] = [(m, n)]$. \square

Proposition 14.1.17. *For any integers a and b , if $ab = 0$ then $a = 0$ or $b = 0$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: $[(m, n)][(p, q)] = [(0, 0)]$

$\langle 1 \rangle 2$. $mp + nq = mq + np$

$\langle 1 \rangle 3$. ASSUME: $[(m, n)] \neq [(0, 0)]$

$\langle 1 \rangle 4$. $m \neq n$

PROVE: $p = q$

$\langle 1 \rangle 5$. CASE: $m < n$

$\langle 2 \rangle 1$. $p \not\leq q$

PROOF: If $p < q$ then $mq + np < mp + nq$ by Proposition 8.4.6.

$\langle 2 \rangle 2$. $q \not\leq p$

PROOF: If $q < p$ then $mp + nq < mq + np$ by Proposition 8.4.6.

$\langle 2 \rangle 3$. $p = q$

PROOF: By trichotomy.

$\langle 1 \rangle 6$. CASE: $n < m$

PROOF: Similar.

\square

Proposition 14.1.18. *The integers \mathbb{Z} form an integral domain.*

PROOF: Propositions 14.1.13, 14.1.14, 14.1.15, 14.1.16, 14.1.17, 14.1.10. \square

Definition 14.1.19. Define $<$ on \mathbb{Z} by $[(m, n)] < [(p, q)]$ if and only if $m + q < n + p$.

We prove this is well-defined.

PROOF:

$\langle 1 \rangle 1$. ASSUME: $m + n' = n + m'$ and $p + q' = q + p'$.

PROVE: $m + q < n + p$ if and only if $m' + q' < n' + p'$

$\langle 1 \rangle 2$. $m + q < n + p$ if and only if $m' + q' < n' + p'$

PROOF:

$$m + q < n + p \Leftrightarrow m + n' + q < n + n' + p \quad (\text{Corollary 6.5.7.1})$$

$$\Leftrightarrow m' + n + q < n + n' + p$$

$$\Leftrightarrow m' + q < n' + p \quad (\text{Corollary 6.5.7.1})$$

$$\Leftrightarrow m' + q + p' < n' + p + p' \quad (\text{Corollary 6.5.7.1})$$

$$\Leftrightarrow m' + q' + p < n' + p + p'$$

$$\Leftrightarrow m' + q' < n' + p' \quad (\text{Corollary 6.5.7.1}) \square$$

Proposition 14.1.20. *The ordering on the integers agrees with the ordering on the natural numbers.*

PROOF: We have $[(m, 0)] < [(n, 0)]$ iff $m < n$. \square

Proposition 14.1.21. *$<$ is a linear order on \mathbb{Z} .*

PROOF:

$\langle 1 \rangle 1$. $<$ is irreflexive.

PROOF: We never have $m + n < m + n$.

$\langle 1 \rangle 2$. $<$ is transitive.

$\langle 2 \rangle 1$. ASSUME: $[(m, n)] < [(p, q)] < [(r, s)]$

$\langle 2 \rangle 2$. $m + q < n + p$ and $p + s < q + r$

$\langle 2 \rangle 3$. $m + q + s < n + q + r$

PROOF: $m + q + s < n + p + s < n + q + r$

$\langle 2 \rangle 4$. $m + s < n + r$

PROOF: Corollary 6.5.7.1.

$\langle 1 \rangle 3$. $<$ is total.

PROOF: Given natural numbers m, n, p and q , either $m + q < n + p$, or $m + q = n + p$, or $n + p < m + q$.

\square

Definition 14.1.22 (Positive). An integer a is *positive* iff $a > 0$.

Theorem 14.1.23. *For any integers a, b and c , we have $a < b$ if and only if $a + c < b + c$.*

PROOF:

$\langle 1 \rangle 1$. If $a < b$ then $a + c < b + c$.

$\langle 2 \rangle 1$. LET: $a = [(m, n)]$, $b = [(p, q)]$ and $c = [(r, s)]$.

$\langle 2 \rangle 2$. ASSUME: $a < b$

$\langle 2 \rangle 3$. $m + q < n + p$

$\langle 2 \rangle 4$. $m + r + q + s < n + r + p + s$

$\langle 2 \rangle 5$. $[(m + r, n + s)] < [(p + r, q + s)]$

$\langle 2 \rangle 6$. $a + c < b + c$

$\langle 1 \rangle 2$. If $a + c < b + c$ then $a < b$.

PROOF: From $\langle 1 \rangle 1$ and Proposition 5.2.6.

□

Proposition 14.1.24. *Let a , b and c be integers. If $0 < c$, then $a < b$ if and only if $ac < bc$.*

PROOF:

$\langle 1 \rangle 1$. LET: $c = [(r, s)]$

$\langle 1 \rangle 2$. ASSUME: $0 < c$

$\langle 1 \rangle 3$. $s < r$

$\langle 1 \rangle 4$. For all integers a and b , if $a < b$ then $ac < bc$

$\langle 2 \rangle 1$. LET: $a = [(m, n)]$, $b = [(p, q)]$.

$\langle 2 \rangle 2$. ASSUME: $a < b$

$\langle 2 \rangle 3$. $m + q < n + p$

$\langle 2 \rangle 4$. $(m + q)r + (p + n)s < (m + q)s + (p + n)r$

PROOF: Proposition 8.4.6, $\langle 1 \rangle 3$, $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. $mr + ns + ps + qr < ms + nr + pr + qs$

$\langle 2 \rangle 6$. $[(mr + ns, ms + nr)] < [(pr + qs, ps + qr)]$

$\langle 2 \rangle 7$. $ac < bc$

$\langle 1 \rangle 5$. For all integers a and b , if $ac < bc$ then $a < b$

PROOF: From $\langle 1 \rangle 4$ and Proposition 5.2.6.

□

Proposition 14.1.25. *Let a be a positive integer. For any integer b , there exists $k \in \mathbb{N}$ such that $b < ak$.*

PROOF:

$\langle 1 \rangle 1$. CASE: $b \leq 0$

PROOF: Take $k = 1$.

$\langle 1 \rangle 2$. CASE: $b > 0$

PROOF: Take $k = b + 1$.

□

14.2 The Rationals

Definition 14.2.1 (Rational Numbers). The set \mathbb{Q} of *rational numbers* is the field of fractions over the integers.

Proposition 14.2.2. *For any integers a and b , we have $[(a, 1)] = [(b, 1)]$ iff $a = b$.*

PROOF: Immediate from definitions. □

Henceforth we identify any integer a with the rational number $[(a, 1)]$.

Proposition 14.2.3. *Addition on the rationals agrees with addition on the integers.*

PROOF: $[(a, 1)] + [(b, 1)] = [(a \cdot 1 + b \cdot 1, 1 \cdot 1)] = [(a + b, 1)]$. \square

Proposition 14.2.4. *Multiplication on the rationals agrees with multiplication on the integers.*

PROOF: $[(a, 1)][(b, 1)] = [(ab, 1)]$ \square

Definition 14.2.5. Define the ordering $<$ on the rationals by: if b and d are positive, then $[(a, b)] < [(c, d)]$ iff $ad < bc$.

We prove this is well-defined.

PROOF:

$\langle 1 \rangle 1$. For any rational q , there exist integers a, b with b positive such that $q = [(a, b)]$.

PROOF: Since $[(a, b)] = [(-a, -b)]$, and if $b \neq 0$ then one of b and $-b$ is positive.

$\langle 1 \rangle 2$. If b, b', d and d' are positive, $[(a, b)] = [(a', b')]$, and $[(c, d)] = [(c', d')]$, then $ad < bc$ iff $a'd' < b'c'$.

PROOF:

$\langle 2 \rangle 1$. If $ad < bc$ then $a'd' < b'c'$.

$\langle 3 \rangle 1$. ASSUME: $ad < bc$

$\langle 3 \rangle 2$. $ab'd < bb'c$

$\langle 3 \rangle 3$. $a'bd < bb'c$

$\langle 3 \rangle 4$. $a'd < b'c$

$\langle 3 \rangle 5$. $a'dd' < b'cd'$

$\langle 3 \rangle 6$. $a'dd' < b'c'd$

$\langle 3 \rangle 7$. $a'd' < b'c'$

$\langle 2 \rangle 2$. If $a'd' < b'c'$ then $ad < bc$.

PROOF: Similar.

\square

Proposition 14.2.6. *The ordering on the rationals agrees with the ordering on the integers.*

PROOF: We have $[(a, 1)] < [(b, 1)]$ if and only if $a < b$. \square

Proposition 14.2.7. *The relation $<$ is a linear ordering on \mathbb{Q} .*

PROOF:

$\langle 1 \rangle 1$. $<$ is irreflexive.

PROOF: We never have $ab < ab$.

$\langle 1 \rangle 2$. $<$ is transitive.

$\langle 2 \rangle 1$. ASSUME: $[(a, b)] < [(c, d)] < [(e, f)]$ where b, d and f are positive.

$\langle 2 \rangle 2$. $ad < bc$ and $cf < de$

$\langle 2 \rangle 3$. $adf < bde$

PROOF: $adf < bcf < bde$

$\langle 2 \rangle 4$. $af < be$

$\langle 1 \rangle 3$. $<$ is total.

PROOF: For any integers a, b, c, d , we have $ad < bc$ or $ad = bc$ or $bc < ad$.

□

Proposition 14.2.8. *For any rationals r , s and t , we have $r < s$ if and only if $r + t < s + t$.*

PROOF:

⟨1⟩1. LET: a, b, c, d, e, f be integers with b, d and f positive.

⟨1⟩2. $[(a, b)] + [(e, f)] < [(c, d)] + [(e, f)]$ if and only if $[(a, b)] < [(c, d)]$.

PROOF:

$$\begin{aligned}
 [(a, b)] + [(e, f)] < [(c, d)] + [(e, f)] &\Leftrightarrow [(af + be, bf)] < [(cf + de, df)] \\
 &\Leftrightarrow (af + be)df < (cf + de)bf \\
 &\Leftrightarrow afd f + bedf < cfbf + debf \\
 &\Leftrightarrow afd f < cfbf \\
 &\Leftrightarrow ad < bc \\
 &\Leftrightarrow [(a, b)] < [(c, d)]
 \end{aligned}$$

□

Corollary 14.2.8.1. *For any rational r , we have $r < 0$ if and only if $0 < -r$.*

Definition 14.2.9 (Absolute Value). For any rational r , the *absolute value* of r is defined by

$$|r| := \begin{cases} -r & \text{if } 0 < -r \\ r & \text{otherwise} \end{cases}$$

Proposition 14.2.10. *For any rationals r , s and t , if t is positive then $r < s$ iff $rt < st$.*

PROOF:

⟨1⟩1. LET: $r = [(a, b)]$, $s = [(c, d)]$ and $t = [(e, f)]$ where b, d and f are positive.

⟨1⟩2. ASSUME: $0 < t$

⟨1⟩3. $e > 0$

⟨1⟩4. $rt < st$ iff $r < s$

PROOF:

$$\begin{aligned}
 rt < st &\Leftrightarrow [(ae, bf)] < [(ce, df)] \\
 &\Leftrightarrow aedf < cebf \\
 &\Leftrightarrow ad < bc \\
 &\Leftrightarrow r < s
 \end{aligned}$$

□

Corollary 14.2.10.1. *The rationals form an ordered field.*

Proposition 14.2.11. *Let p be a positive rational. For any rational number r , there exists $k \in \mathbb{N}$ such that $r < pk$.*

PROOF:

⟨1⟩1. LET: $p = a/b$ and $r = c/d$ where a, b and d are positive.

⟨1⟩2. PICK $k \in \mathbb{N}$ such that $bc < adk$

PROOF: Proposition 14.1.25.

⟨1⟩3. $r < pk$

□

Proposition 14.2.12. $\mathbb{Q} \approx \mathbb{N}$

PROOF: Arrange the rationals in order $0/1, 1/1, 1/2, 0/2, -1/2, -1/1, -2/1, -2/2, -2/3, -1/3, 0/3, 1/3, 2/3$, etc. then remove all duplicates. □

14.3 The Real Numbers

Definition 14.3.1 (Cauchy Sequence). A *Cauchy sequence* is a sequence (q_n) of rationals such that, for every positive rational ϵ , there exists $k \in \mathbb{N}$ such that $\forall m, n > k. |q_m - q_n| < \epsilon$.

Definition 14.3.2 (Dedekind Cut). A *Dedekind cut* is a set $x \subseteq \mathbb{Q}$ such that:

1. $\emptyset \neq x \neq \mathbb{Q}$
2. x is closed downwards.
3. x has no greatest member.

The set \mathbb{R} of *real numbers* is the set of Dedekind cuts.

Proposition 14.3.3. For any rational q , we have $\{r \in \mathbb{Q} \mid r < q\} \in \mathbb{R}$.

PROOF:

⟨1⟩1. LET: $q \in \mathbb{Q}$

⟨1⟩2. LET: $q \downarrow = \{r \mid r < q\}$

⟨1⟩3. $q \notin q \downarrow$

PROOF: We have $q - 1 \in q \downarrow$.

⟨1⟩4. $q \notin q \downarrow$

PROOF: Since $q \notin q \downarrow$.

⟨1⟩5. $q \downarrow$ is closed downwards.

PROOF: Trivial.

⟨1⟩6. $q \downarrow$ has no greatest element.

PROOF: For all $r \in q \downarrow$ we have $r < (q + r)/2 \in q \downarrow$.

□

Proposition 14.3.4. For rationals q and r , we have $q = r$ if and only if $\{s \in \mathbb{Q} \mid s < q\} = \{s \in \mathbb{Q} \mid s < r\}$.

PROOF:

⟨1⟩1. LET: $q \downarrow = \{s \in \mathbb{Q} \mid s < q\}$

⟨1⟩2. LET: $r \downarrow = \{s \in \mathbb{Q} \mid s < r\}$

⟨1⟩3. If $q = r$ then $q \downarrow = r \downarrow$

PROOF: Trivial.

(1)4. If $q < r$ then $q \downarrow \neq r \downarrow$

PROOF: We have $q \in r \downarrow$ and $q \notin q \downarrow$.

(1)5. If $r < q$ then $q \downarrow \neq r \downarrow$

PROOF: We have $r \in q \downarrow$ and $q \notin q \downarrow$.

□

Henceforth we identify a rational q with the real number $\{r \in \mathbb{Q} \mid r < q\}$.

Definition 14.3.5. Define the ordering $<$ on \mathbb{R} by: $x < y$ iff $x \subsetneq y$.

Proposition 14.3.6. *The ordering on the reals agrees with the ordering on the rationals.*

PROOF:

(1)1. LET: $q, r \in \mathbb{Q}$

(1)2. LET: $q \downarrow = \{s \in \mathbb{Q} \mid s < q\}$.

(1)3. LET: $r \downarrow = \{s \in \mathbb{Q} \mid s < r\}$.

PROVE: $q < r$ iff $q \downarrow \subsetneq r \downarrow$

(1)4. If $q < r$ then $q \downarrow \subsetneq r \downarrow$

(2)1. ASSUME: $q < r$

(2)2. $q \downarrow \subseteq r \downarrow$

PROOF: If $s < q$ then $s < r$.

(2)3. $q \downarrow \neq r \downarrow$

PROOF: Proposition 14.3.4.

(1)5. If $q \downarrow \subsetneq r \downarrow$ then $q < r$

(2)1. ASSUME: $q \downarrow \subsetneq r \downarrow$

(2)2. PICK $s \in r \downarrow$ such that $s \notin q \downarrow$

(2)3. $q \leq s < r$

□

Proposition 14.3.7. *The ordering $<$ is a linear ordering on \mathbb{R} .*

PROOF:

(1)1. $<$ is irreflexive.

PROOF: No set is a proper subset of itself.

(1)2. $<$ is transitive.

PROOF: Since the relationship \subsetneq is transitive on the class of all sets.

(1)3. $<$ is total.

(2)1. LET: x, y be Dedekind cuts.

(2)2. ASSUME: $x \not\subseteq y$

PROVE: $y \subsetneq x$

(2)3. PICK $q \in x$ such that $q \notin y$

(2)4. LET: $r \in y$

PROVE: $r \in x$

(2)5. $q \not\leq r$

PROOF: Since y is closed downwards.

(2)6. $r < q$

(2)7. $r \in x$

PROOF: Since x is closed downwards.

□

Proposition 14.3.8. *Any bounded nonempty subset of \mathbb{R} has a least upper bound.*

PROOF:

⟨1⟩1. LET: A be a bounded nonempty subset of \mathbb{R} .

⟨1⟩2. $\bigcup A$ is a Dedekind cut.

⟨2⟩1. $\bigcup A \neq \emptyset$

⟨3⟩1. PICK $x \in A$

⟨3⟩2. PICK $q \in x$

⟨3⟩3. $q \in \bigcup A$

⟨2⟩2. $\bigcup A \neq \mathbb{Q}$

⟨3⟩1. PICK an upper bound u for A

⟨3⟩2. PICK $q \notin u$

PROVE: $q \notin \bigcup A$

⟨3⟩3. ASSUME: for a contradiction $q \in \bigcup A$

⟨3⟩4. PICK $x \in A$ such that $q \in x$

⟨3⟩5. $x \leq u$

⟨3⟩6. $q \in u$

⟨3⟩7. Q.E.D.

PROOF: This is a contradiction.

⟨2⟩3. $\bigcup A$ is closed downwards.

⟨3⟩1. LET: $q \in \bigcup A$ and $r < q$

⟨3⟩2. PICK $x \in A$ such that $q \in x$

⟨3⟩3. $r \in x$

⟨3⟩4. $r \in \bigcup A$

⟨2⟩4. $\bigcup A$ has no greatest element.

⟨3⟩1. LET: $q \in \bigcup A$

⟨3⟩2. PICK $x \in A$ such that $q \in x$

⟨3⟩3. PICK $r \in x$ such that $q < r$

⟨3⟩4. $r \in \bigcup A$

⟨1⟩3. $\bigcup A$ is an upper bound for A .

PROOF: For all $x \in A$ we have $x \subseteq \bigcup A$.

⟨1⟩4. For any upper bound u for $\bigcup A$ we have $\bigcup A \leq u$.

PROOF: If $\forall x \in A. x \subseteq u$ we have $\bigcup A \subseteq u$.

□

Definition 14.3.9 (Addition). Define *addition* $+$ on the reals by

$$x + y := \{q + r \mid q \in x, r \in y\} .$$

We prove this is well-defined.

PROOF:

⟨1⟩1. LET: $x, y \in \mathbb{R}$

PROVE: $X + y$ is a Dedekind cut.

$\langle 1 \rangle 2. x + y \neq \emptyset$

PROOF: Pick $q \in x$ and $r \in y$; then $q + r \in x + y$.

$\langle 1 \rangle 3. x + y \neq \mathbb{Q}$

$\langle 2 \rangle 1. \text{ PICK } q \notin x \text{ and } r \notin y$

PROVE: $q + r \notin x + y$

$\langle 2 \rangle 2. \text{ ASSUME: for a contradiction } q + r \in x + y$

$\langle 2 \rangle 3. \text{ PICK } q' \in x \text{ and } r' \in y \text{ such that } q + r = q' + r'$

$\langle 2 \rangle 4. q' < q \text{ and } r' < r$

$\langle 2 \rangle 5. q' + r' < q + r$

$\langle 2 \rangle 6. \text{ Q.E.D.}$

PROOF: This is a contradiction.

$\langle 1 \rangle 4. x + y$ is closed downwards.

$\langle 2 \rangle 1. \text{ LET: } q \in x \text{ and } r \in y$

$\langle 2 \rangle 2. \text{ LET: } s < q + r$

PROVE: $s \in x + y$

$\langle 2 \rangle 3. s - r < q$

$\langle 2 \rangle 4. s - r \in x$

$\langle 2 \rangle 5. s = (s - r) + r \in x + y$

$\langle 1 \rangle 5. x + y$ has no greatest element.

$\langle 2 \rangle 1. \text{ LET: } q \in x \text{ and } r \in y$

PROVE: There exists $s \in x + y$ such that $q + r < s$

$\langle 2 \rangle 2. \text{ PICK } q' \in x \text{ and } r' \in y \text{ such that } q < q' \text{ and } r < r'$

$\langle 2 \rangle 3. q + r < q' + r' \in x + y$

□

Proposition 14.3.10. *Addition on the reals agrees with addition on the rationals.*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } q, r \in \mathbb{Q}$

$\langle 1 \rangle 2. q \downarrow + r \downarrow \subseteq (q + r) \downarrow$

PROOF: If $s_1 < q$ and $s_2 < r$ then $s_1 + s_2 < q + r$.

$\langle 1 \rangle 3. (q + r) \downarrow \subseteq q \downarrow + r \downarrow$

$\langle 2 \rangle 1. \text{ LET: } s < q + r$

$\langle 2 \rangle 2. s - r < q$

$\langle 2 \rangle 3. \text{ PICK } t \text{ such that } s - r < t < q$

$\langle 2 \rangle 4. s - t < r$

$\langle 2 \rangle 5. s = t + (s - t) \in q \downarrow + r \downarrow$

□

Proposition 14.3.11. *Addition is associative.*

PROOF:

$$\begin{aligned} x + (y + z) &= \{q + r \mid q \in x, r \in y + z\} \\ &= \{q + s_1 + s_2 \mid q \in x, s_1 \in y, s_2 \in z\} \\ &= \{r + s_2 \mid r \in x + y, s_2 \in z\} \\ &= (x + y) + z \end{aligned}$$

□

Proposition 14.3.12. *Addition is commutative.*

PROOF:

$$\begin{aligned} x + y &= \{q + r \mid q \in x, r \in y\} \\ &= \{r + q \mid r \in y, q \in x\} \\ &= y + x \end{aligned}$$

□

Proposition 14.3.13. *For any $x \in \mathbb{R}$ we have $x + 0 = x$.*

PROOF:

$\langle 1 \rangle 1. x + 0 \subseteq x$

PROOF: If $q \in x$ and $r < 0$ then $q + r < q$ so $q + r \in x$.

$\langle 1 \rangle 2. x \subseteq x + 0$

$\langle 2 \rangle 1. \text{ LET: } q \in x$

$\langle 2 \rangle 2. \text{ PICK } r \in x \text{ such that } q < r.$

PROOF: x has no greatest element.

$\langle 2 \rangle 3. q - r < 0$

$\langle 2 \rangle 4. q = r + (q - r) \in x + 0$

□

Definition 14.3.14. For $x \in \mathbb{R}$, define $-x := \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$.

Proposition 14.3.15. *For all $x \in \mathbb{R}$ we have $-x \in \mathbb{R}$.*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } x \in \mathbb{R}$

$\langle 1 \rangle 2. -x \neq \emptyset$

$\langle 2 \rangle 1. \text{ PICK } s \notin x$

$\langle 2 \rangle 2. -s - 1 \in -x$

$\langle 1 \rangle 3. -x \neq \mathbb{Q}$

$\langle 2 \rangle 1. \text{ PICK } s \in x$

PROVE: $-s \notin -x$

$\langle 2 \rangle 2. \text{ ASSUME: for a contradiction } -s \in -x$

$\langle 2 \rangle 3. \text{ PICK } r > -s \text{ such that } -r \notin x$

$\langle 2 \rangle 4. -r < s$

$\langle 2 \rangle 5. \text{ Q.E.D.}$

PROOF: This contradicts the fact that x is closed downwards.

$\langle 1 \rangle 4. -x$ is closed downwards.

PROOF: Immediate from definition.

$\langle 1 \rangle 5. -x$ has no greatest element.

$\langle 2 \rangle 1. \text{ LET: } q \in -x$

$\langle 2 \rangle 2. \text{ PICK } r > q \text{ such that } -r \notin x$

$\langle 2 \rangle 3. \text{ PICK } s \text{ such that } q < s < r$

$\langle 2 \rangle 4. s \in -x$

□

Lemma 14.3.16. *Let p be a positive rational number. For any real number x , there exists a rational $q \in x$ such that $p + q \notin x$.*

PROOF:

- $\langle 1 \rangle 1$. PICK $q_0 \in x$
- $\langle 1 \rangle 2$. There exists $k \in \mathbb{N}$ such that $q_0 + kp \notin x$
 - $\langle 2 \rangle 1$. PICK $q_1 \notin x$
 - $\langle 2 \rangle 2$. PICK $k \in \mathbb{N}$ such that $q_1 - q_0 < pk$
 - PROOF: Proposition 14.2.11.
 - $\langle 2 \rangle 3$. $q_1 < q_0 + kp$
 - $\langle 2 \rangle 4$. $q_0 + kp \notin x$
- $\langle 1 \rangle 3$. LET: k be the least natural number such that $q_0 + kp \notin x$
- $\langle 1 \rangle 4$. $k \neq 0$
- PROOF: $\langle 1 \rangle 1$
- $\langle 1 \rangle 5$. LET: $q = q_0 + (k-1)p$
- $\langle 1 \rangle 6$. $q \in x$ and $q + p \notin x$.

□

Proposition 14.3.17. *For every real x we have $x + (-x) = 0$.*

PROOF:

- $\langle 1 \rangle 1$. LET: x be a real number.
- $\langle 1 \rangle 2$. $x + (-x) \subseteq 0$
 - $\langle 2 \rangle 1$. LET: $q_1 \in x$ and $q_2 \in -x$
 - $\langle 2 \rangle 2$. PICK $r > q_2$ such that $-r \notin x$
 - $\langle 2 \rangle 3$. $q_1 < -r$
 - $\langle 2 \rangle 4$. $r < -q_1$
 - $\langle 2 \rangle 5$. $q_2 < -q_1$
 - $\langle 2 \rangle 6$. $q_1 + q_2 < 0$
- $\langle 1 \rangle 3$. $0 \subseteq x + (-x)$
 - $\langle 2 \rangle 1$. LET: $p < 0$
 - $\langle 2 \rangle 2$. $0 < -p$
 - $\langle 2 \rangle 3$. PICK $q \in x$ such that $q - p/2 \notin x$
 - PROOF: Lemma 14.3.16.
 - $\langle 2 \rangle 4$. LET: $s = p/2 - q$
 - $\langle 2 \rangle 5$. $-s \notin x$
 - $\langle 2 \rangle 6$. $p - q < s$
 - $\langle 2 \rangle 7$. $p - q \in -x$
 - $\langle 2 \rangle 8$. $p \in x + (-x)$

□

Corollary 14.3.17.1. *The reals form an Abelian group under addition.*

Proposition 14.3.18. *For any reals x, y and z , we have $x < y$ if and only if $x + z < y + z$.*

PROOF:

- $\langle 1 \rangle 1$. $\forall x, y, z \in \mathbb{R}. x \leq y \Rightarrow x + z \leq y + z$
 - $\langle 2 \rangle 1$. LET: $x, y, z \in \mathbb{R}$
 - $\langle 2 \rangle 2$. ASSUME: $x \leq y$
 - $\langle 2 \rangle 3$. For all $q \in x$ and $r \in z$ we have $q + r \in y + z$

⟨1⟩2. $\forall x, y, z \in \mathbb{R}. x + z = y + z \Leftrightarrow x = y$

PROOF: Proposition 11.1.4.

⟨1⟩3. $\forall x, y, z \in \mathbb{R}. x < y \Rightarrow x + z < y + z$

⟨1⟩4. Q.E.D.

PROOF: Proposition 5.2.6.

□

Definition 14.3.19 (Absolute Value). The *absolute value* of a real number x is defined to be

$$|x| = \begin{cases} x & \text{if } 0 \leq x \\ -x & \text{if } x < 0 \end{cases}$$

Definition 14.3.20 (Multiplication). Define *multiplication* \cdot on \mathbb{R} as follows:

- If x and y are non-negative then

$$xy = 0 \cup \{rs \mid 0 \leq r \in x \wedge 0 \leq s \in y\} .$$

- If x and y are both negative then

$$xy = (-x)(-y) .$$

- If one of x and y is negative and one is non-negative then

$$xy = -(|x||y|) .$$

We prove this is well-defined.

PROOF:

⟨1⟩1. LET: x and y be non-negative reals.

PROVE: xy is real.

⟨1⟩2. $xy \neq \emptyset$

PROOF: Since $-1 \in xy$.

⟨1⟩3. $xy \neq \mathbb{Q}$

⟨2⟩1. PICK $r \notin x$ and $s \notin y$

PROVE: $rs \notin xy$

⟨2⟩2. $0 \leq r$ and $0 \leq s$

PROOF: Since $0 \subseteq x$ and $0 \subseteq y$.

⟨2⟩3. ASSUME: for a contradiction $rs \in xy$

⟨2⟩4. PICK r' and s' such that $0 \leq r' \in x$, $0 \leq s' \in y$ and $rs = r's'$

⟨2⟩5. $r' < r$

⟨2⟩6. $s' < s$

⟨2⟩7. $r's' < rs$

⟨2⟩8. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩4. xy is closed downwards.

⟨2⟩1. LET: $q \in xy$ and $r < q$

- ⟨2⟩2. CASE: $q \in 0$
 PROOF: Then $r < q < 0$ so $r \in xy$
- ⟨2⟩3. CASE: $q = s_1 s_2$ where $0 \leq s_1 \in x$ and $0 \leq s_2 \in y$
 - ⟨3⟩1. ASSUME: w.l.o.g. $0 \leq r$
 - ⟨3⟩2. $0 < s_1$ and $0 < s_2$
 - ⟨3⟩3. $r/s_2 < s_1$
 - ⟨3⟩4. $r/s_2 \in x$
 - ⟨3⟩5. $r = (r/s_2)s_2 \in xy$
- ⟨1⟩5. xy has no greatest element.
 - ⟨2⟩1. LET: $q \in xy$
 - ⟨2⟩2. CASE: $q \in 0$
 PROOF: $q < q/2 \in 0$
 - ⟨2⟩3. CASE: $q = rs$ where $0 \leq r \in x$ and $0 \leq s \in y$
 - ⟨3⟩1. PICK r' and s' with $r < r' \in x$ and $s < s' \in y$
 - ⟨3⟩2. $q < r's' \in xy$

□

Proposition 14.3.21. *Multiplication is commutative.*

PROOF: Immediate from definition. □

Proposition 14.3.22. *Multiplication is associative.*

PROOF:

- ⟨1⟩1. For non-negative reals x, y and z , we have $x(yz) = (xy)z$
 PROOF: It computes to $0 \cup \{qrs \mid 0 \leq q \in x, 0 \leq r \in y, 0 \leq s \in z\}$.
- ⟨1⟩2. For all reals x, y and z , we have $x(yz) = (xy)z$
 PROOF: It is equal to $|x||y||z|$ if an even number of them are negative, and $-(|x||y||z|)$ otherwise.

□

Proposition 14.3.23. *Multiplication distributes over addition.*

PROOF:

- ⟨1⟩1. For all non-negative reals x, y and z , we have $x(y + z) = xy + xz$
 - ⟨2⟩1. LET: x, y and z be non-negative reals.
 - ⟨2⟩2. $x(y + z) \subseteq xy + xz$
 - ⟨3⟩1. LET: $q \in x(y + z)$
 - ⟨3⟩2. CASE: $q < 0$
 PROOF: Then we have $q/2 \in xy$ and $q/2 \in xz$ so $q \in xy + xz$.
 - ⟨3⟩3. CASE: $q = rs$ where $0 \leq r \in x$ and $0 \leq s \in y + z$
 - ⟨4⟩1. PICK $s_1 \in y$ and $s_2 \in z$ such that $s = s_1 + s_2$
 - ⟨4⟩2. $rs_1 \in xy$
 PROOF: If $s_1 < 0$ then $rs_1 < 0$ so $rs_1 \in xy$. If $0 \leq s_1$ then we also have $rs_1 \in xy$.
 - ⟨4⟩3. $rs_2 \in xz$
 PROOF: Similar.
 - ⟨4⟩4. $q \in xy + xz$

PROOF: Since $q = rs_1 + rs_2$.

$\langle 2 \rangle 3$. $xy + xz \subseteq x(y + z)$

$\langle 3 \rangle 1$. LET: $q \in xy$ and $r \in xz$.

PROVE: $q + r \in x(y + z)$

$\langle 3 \rangle 2$. CASE: $q < 0$ and $r < 0$

PROOF: Then $q + r < 0$ so $q + r \in x(y + z)$.

$\langle 3 \rangle 3$. CASE: $q < 0$ and $r = r_1r_2$ where $0 \leq r_1 \in x$ and $0 \leq r_2 \in z$

$\langle 4 \rangle 1$. $q + r < r$

$\langle 4 \rangle 2$. $q + r \in xz$

$\langle 4 \rangle 3$. ASSUME: w.l.o.g. $0 \leq q + r$

PROOF: Otherwise $q + r \in x(y + z)$ immediately.

$\langle 4 \rangle 4$. PICK s_1, s_2 with $0 \leq s_1 \in x$, $0 \leq s_2 \in y$ and $q + r = s_1s_2$

$\langle 4 \rangle 5$. $s_2 \in y + z$

PROOF: Since $0 \in z$ so $s_2 = s_2 + 0 \in y + z$.

$\langle 4 \rangle 6$. $q + r \in x(y + z)$

$\langle 3 \rangle 4$. CASE: $q = q_1q_2$ where $0 \leq q_1 \in x$ and $0 \leq q_2 \in y$ and $r < 0$

PROOF: Similar.

$\langle 3 \rangle 5$. CASE: $q = q_1q_2$ where $0 \leq q_1 \in x$ and $0 \leq q_2 \in y$ and $r = r_1r_2$ where $0 \leq r_1 \in x$ and $0 \leq r_2 \in z$

$\langle 4 \rangle 1$. ASSUME: w.l.o.g. $q_1 \leq r_1$

$\langle 4 \rangle 2$. $q + r \leq r_1(q_2 + r_2) \in x(y + z)$

$\langle 1 \rangle 2$. For any negative real x and non-negative reals y and z , we have $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned} x(y + z) &= -(-x)(y + z) = -((-x)y + (-x)z) & (\langle 1 \rangle 1) \\ &= -((-x)y) - ((-x)z) \\ &= xy + xz \end{aligned}$$

$\langle 1 \rangle 3$. For any non-negative real x and reals y and z with one negative and one non-negative, we have $x(y + z) = xy + xz$

$\langle 2 \rangle 1$. ASSUME: w.l.o.g. y is negative and z is non-negative.

$\langle 2 \rangle 2$. CASE: $0 \leq y + z$

PROOF:

$$\begin{aligned} xy + xz &= xy + x(-y + y + z) \\ &= -(x(-y)) + x(-y + y + z) \\ &= -(x(-y)) + x(-y) + x(y + z) & (\langle 1 \rangle 1) \\ &= x(y + z) \end{aligned}$$

$\langle 2 \rangle 3$. CASE: $y + z < 0$

$\langle 3 \rangle 1$. $-y - z > 0$

$\langle 3 \rangle 2$. $-y = z - y - z$

$\langle 3 \rangle 3$. $xy + xz = x(y + z)$

PROOF:

$$\begin{aligned}
 xy + xz &= -(x(-y)) + xz \\
 &= -(x(z - y - z)) + xz \\
 &= -(xz + x(-y - z)) + xz & ((1)1) \\
 &= -xy - x(-y - z) + xz \\
 &= -x(-y - z) \\
 &= x(y + z)
 \end{aligned}$$

(1)4. For any non-negative real x and negative reals y and z , we have $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned}
 x(y + z) &= -x(-y - z) \\
 &= -(x(-y) + x(-z)) & ((1)1) \\
 &= -x(-y) - x(-z) \\
 &= xy + xz
 \end{aligned}$$

(1)5. For any negative real x and reals y and z with one negative and one non-negative, we have $x(y + z) = xy + xz$

(2)1. ASSUME: w.l.o.g. y is negative and z is non-negative.

(2)2. CASE: $0 \leq y + z$

PROOF:

$$\begin{aligned}
 x(y + z) &= -((-x)(y + z)) \\
 &= -((-x)y + (-x)z) & ((1)3) \\
 &= -((-x)y) - ((-x)z) \\
 &= (-x)(-y) - ((-x)z) \\
 &= xy + xz
 \end{aligned}$$

(2)3. CASE: $y + z < 0$

PROOF:

$$\begin{aligned}
 x(y + z) &= (-x)(-y - z) \\
 &= (-x)(-y) + (-x)(-z) & ((1)3) \\
 &= xy + xz
 \end{aligned}$$

(1)6. For any negative reals x , y and z , we have $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned}
 x(y + z) &= (-x)(-y - z) \\
 &= (-x)(-y) + (-x)(-z) & ((1)1) \\
 &= xy + xz
 \end{aligned}$$

□

Proposition 14.3.24. *For any real x we have $x1 = x$.*

PROOF:

(1)1. CASE: $0 \leq x$

(2)1. $x1 \subseteq x$

(3)1. LET: $q \in x1$

- ⟨3⟩2. CASE: $q < 0$
 PROOF: Then $q \in x$ because $0 \leq x$.
 ⟨3⟩3. $q = rs$ where $0 \leq r \in x$ and $0 \leq s < 1$
 PROOF: Then $q < r$ so $q \in x$.
 ⟨2⟩2. $x \subseteq x1$
 ⟨3⟩1. LET: $q \in x$
 ⟨3⟩2. ASSUME: w.l.o.g. $0 \leq q$
 ⟨3⟩3. PICK r such that $q < r \in x$
 ⟨3⟩4. $0 \leq q/r < 1$
 ⟨3⟩5. $q = r(q/r) \in x1$
 ⟨1⟩2. CASE: $x < 0$
 PROOF:

$$\begin{aligned}
 x1 &= -((-x)1) \\
 &= -(-x) && (\langle 1 \rangle 1) \\
 &= x
 \end{aligned}$$

□

Lemma 14.3.25. *Let $x \in \mathbb{R}$ and c be a positive rational. Then there exists $a \in x$ and a non-least rational upper bound b for x such that $b - a = c$.*

PROOF:

- ⟨1⟩1. PICK $a_1 \in x$ such that if x has a rational supremum s then $a_1 > s - c$
 ⟨1⟩2. There exists a natural number n such that $a_1 + nc$ is an upper bound for x .
 ⟨2⟩1. PICK a non-least upper bound b_1 for x .
 ⟨2⟩2. PICK a natural number n such that $nc > b_1 - a_1$
 PROOF: Proposition 14.2.11.
 ⟨2⟩3. $a_1 + nc > b_1$
 ⟨2⟩4. $a_1 + nc$ is an upper bound for x .
 ⟨1⟩3. LET: k be the least natural number such that $a_1 + kc$ is an upper bound for x .
 ⟨1⟩4. $a_1 + (k-1)c \in x$
 ⟨1⟩5. $a_1 + kc$ is not the supremum of x .
 ⟨2⟩1. ASSUME: for a contradiction $a_1 + kc$ is the supremum of x .
 ⟨2⟩2. $a_1 > a_1 + (k-1)c$
 PROOF: ⟨1⟩1
 ⟨2⟩3. Q.E.D.
 PROOF: This is a contradiction.
 ⟨1⟩6. LET: $a = a_1 + (k-1)c$
 ⟨1⟩7. LET: $b = a_1 + kc$
 ⟨1⟩8. $b - a = c$

□

Proposition 14.3.26. *For any non-zero real x , there exists a real y such that $xy = 1$.*

PROOF:

- ⟨1⟩1. CASE: $0 < x$
- ⟨2⟩1. LET: $y = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{u^{-1} \mid u \text{ is an upper bound for } x \text{ but not the supremum of } x\}$
- ⟨2⟩2. y is a real number.
 - ⟨3⟩1. $y \neq \emptyset$
PROOF: Since $0 \in y$.
 - ⟨3⟩2. $y \neq \mathbb{Q}$
 - ⟨4⟩1. PICK $q \in x$ such that $0 < q$
 - ⟨4⟩2. $q^{-1} \notin y$
 - ⟨3⟩3. y is closed downwards.
 - ⟨4⟩1. LET: $q \in y$ and $r < q$
PROVE: $r \in y$
 - ⟨4⟩2. ASSUME: w.l.o.g. $0 < r$
 - ⟨4⟩3. q^{-1} is a non-least upper bound for x .
 - ⟨4⟩4. $q^{-1} < r^{-1}$
 - ⟨4⟩5. r^{-1} is a non-least upper bound for x .
 - ⟨4⟩6. $r \in y$
 - ⟨3⟩4. y has no greatest element.
 - ⟨4⟩1. LET: $q \in y$
PROVE: There exists $r \in y$ such that $q < r$
 - ⟨4⟩2. CASE: $q \leq 0$
 - ⟨5⟩1. PICK a non-least upper bound u for x .
 - ⟨5⟩2. $q < u^{-1} \in x$
 - ⟨4⟩3. CASE: $q = u^{-1}$ where u is a non-least upper bound for x .
 - ⟨5⟩1. PICK a non-least upper bound v with $v < u$
 - ⟨5⟩2. $u^{-1} < v^{-1} \in y$
- ⟨2⟩3. $0 < y$
- ⟨2⟩4. $xy \subseteq 1$
 - ⟨3⟩1. LET: $q \in xy$
 - ⟨3⟩2. ASSUME: w.l.o.g. $0 < q$
 - ⟨3⟩3. PICK $0 < r \in x$ and $0 < s \in y$ such that $q = rs$
 - ⟨3⟩4. s^{-1} is a non-least upper bound for x
 - ⟨3⟩5. $r < s^{-1}$
 - ⟨3⟩6. $rs < 1$
- ⟨2⟩5. $1 \subseteq xy$
 - ⟨3⟩1. LET: $q < 1$
PROVE: $q \in xy$
 - ⟨3⟩2. ASSUME: w.l.o.g. $0 < q$
 - ⟨3⟩3. PICK a_1 with $0 < a_1 \in x$
 - ⟨3⟩4. $(1 - q)a_1 > 0$
 - ⟨3⟩5. PICK $a \in x$ and a non-least upper bound w of x such that $w - a = (1 - q)a_1$
PROOF: Lemma 14.3.25.
 - ⟨3⟩6. $w - a < (1 - q)w$
 - ⟨3⟩7. $qw < a$
 - ⟨3⟩8. $w < a/q$
 - ⟨3⟩9. a/q is a non-least upper bound for x

PROOF: There are countably many integer polynomials, each with finitely many roots. \square

Corollary 14.3.30.1. *There are uncountably many transcendental numbers.*

Proposition 14.3.31. *Let A be a set of disks in the plane, no two of which intersect. Then A is countable.*

PROOF: Every circle includes a point with rational coordinates. Define $f : \{q \in \mathbb{Q}^2 \mid \exists C \in A. q \in C\} \rightarrow A$ by $f(q) = C$ iff $q \in C$. Then f is surjective. \square

Proposition 14.3.32. *There exists an uncountable set of circles in the plane that do not intersect.*

PROOF: The set of all circles with origin O is uncountable. \square

Chapter 15

Complex Analysis

Definition 15.0.1. For $p \geq 1$, let l^p be the set of all sequences of complex numbers (x_n) such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

Proposition 15.0.2. If $(x_n), (y_n) \in l^p$ then $(x_n + y_n) \in l^p$.

PROOF:

$\langle 1 \rangle 1$. LET: $(x_n), (y_n) \in l^p$

$\langle 1 \rangle 2$. $\sum_{n=1}^{\infty} |x_n + y_n|^p \leq 2^p (\sum_{n=1}^{\infty} |x_n|^p + \sum_{n=1}^{\infty} |y_n|^p)$

PROOF:

$\langle 2 \rangle 1$. For all $n \in \mathbb{N}$ we have $|x_n + y_n|^p \leq 2^p (|x_n|^p + |y_n|^p)$.

PROOF:

$$\begin{aligned} |x_n + y_n|^p &\leq (|x_n| + |y_n|)^p && \text{(Triangle Inequality)} \\ &\leq (2 \max(|x_n|, |y_n|))^p \\ &\leq 2^p (|x_n|^p + |y_n|^p) \end{aligned}$$

□

Theorem 15.0.3 (Hölder's Inequality). Let p and q be reals such that $p > 1$, $q > 1$ and $1/p + 1/q = 1$. Let $(x_n) \in l^p$ and $(y_n) \in l^q$. Then

$$\sum_n |x_n y_n| \leq \left(\sum_n |x_n|^p \right)^{1/p} \left(\sum_n |y_n|^q \right)^{1/q}$$

PROOF:

$\langle 1 \rangle 1$. ASSUME: w.l.o.g. neither (x_n) nor (y_n) are all zero.

$\langle 1 \rangle 2$. For $0 \leq x \leq 1$ we have

$$x^{1/p} \leq \frac{1}{p}x + \frac{1}{q}.$$

$\langle 2 \rangle 1$. LET: $f(x) = x/p + 1/q - x^{1/p}$

$\langle 2 \rangle 2$. $f'(x) = 1/p(1 - x^{(1-p)/p})$

$\langle 2 \rangle 3$. $f'(x) \geq 0$ for all $x \in [0, 1]$

$\langle 2 \rangle 4$. f is a monotonically decreasing function on $[0, 1]$

- ⟨2⟩5. $f(0) = 1/q$
 ⟨2⟩6. $f(1) = 0$
 ⟨2⟩7. $f(x) \geq 0$ for all $x \in [0, 1]$
 ⟨1⟩3. For any $a, b \geq 0$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

- ⟨2⟩1. CASE: $a^p \leq b^q$
 ⟨3⟩1. $ab^{-q/p} \leq \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q}$
 PROOF: Substituting $x = a^p/b^q$ in ⟨1⟩2.
 ⟨3⟩2. $ab^{1-q} \leq \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q}$
 PROOF: From ⟨3⟩1 since $1 - q = -q/p$.
 ⟨3⟩3. $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$
 PROOF: Multiplying ⟨3⟩2 by b^q .
 ⟨2⟩2. CASE: $b^q \leq a^p$
 PROOF: Similar.

- ⟨1⟩4. For any integers $1 \leq j \leq n$, we have

$$\frac{|x_j|}{(\sum_{k=1}^n |x_k|^p)^{1/p}} \frac{|y_j|}{(\sum_{k=1}^n |y_k|^q)^{1/q}} \leq \frac{1}{p} \frac{|x_j|^p}{\sum_{k=1}^n |x_k|^p} + \frac{1}{q} \frac{|y_j|^q}{\sum_{k=1}^n |y_k|^q}$$

PROOF: From ⟨1⟩3 substituting

$$a = \frac{|x_j|}{(\sum_{k=1}^n |x_k|^p)^{1/p}} \text{ and } b = \frac{|y_j|}{(\sum_{k=1}^n |y_k|^q)^{1/q}}$$

- ⟨1⟩5. For any positive integer n we have

$$\frac{\sum_{k=1}^n |x_k| |y_k|}{(\sum_{k=1}^n |x_k|^p)^{1/p} (\sum_{k=1}^n |y_k|^q)^{1/q}} \leq 1$$

PROOF:

$$\frac{\sum_{j=1}^n |x_j| |y_j|}{(\sum_{k=1}^n |x_k|^p)^{1/p} (\sum_{k=1}^n |y_k|^q)^{1/q}} \leq \frac{1}{p} + \frac{1}{q} \quad (\text{Summing } \langle 1 \rangle 4 \text{ from } j = 1 \text{ to } n)$$

$$= 1$$

- ⟨1⟩6.

$$\sum_n |x_n y_n| \leq \left(\sum_n |x_n|^p \right)^{1/p} \left(\sum_n |y_n|^q \right)^{1/q}$$

PROOF: Taking the limit $n \rightarrow \infty$ in ⟨1⟩5.

□

Theorem 15.0.4 (Minkowski's Inequality). *Let $p \geq 1$. Let $(x_n), (y_n) \in l^p$. Then*

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p}$$

PROOF:

- ⟨1⟩1. CASE: $p = 1$

PROOF: This is just the Triangle Inequality.

- ⟨1⟩2. CASE: $p > 1$

- ⟨2⟩1. LET: $q = p/(p-1)$

⟨2⟩2.

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n + y_n|^p &\leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \\ &\quad + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \end{aligned}$$

PROOF:

⟨3⟩1. $(|x_n + y_n|^{p-1}) \in l^q$

PROOF:

$$\sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q} = \sum_{n=1}^{\infty} |x_n + y_n|^p \quad (\langle 2 \rangle 2)$$

$$< \infty$$

(Proposition 15.0.2)

⟨3⟩2. Q.E.D.

PROOF:

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n + y_n|^p &= \sum_{n=1}^{\infty} |x_n + y_n| |x_n + y_n|^{p-1} \\ &\leq \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n| |x_n + y_n|^{p-1} \\ &\leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \\ &\quad + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} \quad (\text{Hölder's Inequality, } \langle 2 \rangle 2) \end{aligned}$$

⟨2⟩3.

$$\sum_{n=1}^{\infty} |x_n + y_n|^p \leq \left\{ \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \right\} \left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{1/q}$$

⟨3⟩1. $q(p-1) = p$

PROOF: ⟨2⟩2

⟨3⟩2. Q.E.D.

PROOF: From ⟨2⟩2, ⟨3⟩1.

□

Part I

Linear Algebra

Chapter 16

Vector Spaces

16.1 Vector Spaces

Definition 16.1.1 (Vector Space). Let K be either \mathbb{R} or \mathbb{C} . A *vector space* over K is a triple $(V, +, \cdot)$ such that:

- V is a nonempty set, whose elements are called *vectors*;
- $+: V^2 \rightarrow V$
- $\cdot: K \times V \rightarrow V$

such that the following hold for all $u, v, w \in V$ and $\alpha, \beta \in K$:

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. For every $u, v \in V$ there exists $w \in V$ such that $u + w = v$
4. $\alpha(\beta v) = (\alpha\beta)v$
5. $(\alpha + \beta)v = \alpha v + \beta v$
6. $\alpha(u + v) = \alpha u + \alpha v$
7. $1v = v$

Elements of K are called *scalars*.

We write *real vector space* for 'vector space over \mathbb{R} ', and *complex vector space* for 'vector space over \mathbb{C} '.

Proposition 16.1.2. *Let K be either \mathbb{R} and \mathbb{C} . The set $\{0\}$ is a vector space over K under the unique functions $+: \{0\}^2 \rightarrow \{0\}$, $\cdot: K \times \{0\} \rightarrow \{0\}$.*

PROOF: Each axiom holds trivially because $x = y$ holds for all $x, y \in \{0\}$. \square

Proposition 16.1.3. *The set \mathbb{R} is a real vector space under real addition and real multiplication.*

PROOF: TODO — after we have proved these facts about \mathbb{R} . \square

Proposition 16.1.4. *The set \mathbb{C} is a real vector space under complex addition and complex multiplication.*

PROOF: TODO

Proposition 16.1.5. *The set \mathbb{C} is a complex vector space under complex addition and complex multiplication.*

PROOF: TODO

Proposition 16.1.6. *Let K be either \mathbb{R} or \mathbb{C} . Let $\{V_i\}_{i \in I}$ be a family of vector spaces over K . Then $\prod_{i \in I} V_i$ is a vector space over K under the operations given by*

$$\begin{aligned}\{x_i\}_{i \in I} + \{y_i\}_{i \in I} &= \{x_i + y_i\}_{i \in I} \\ \alpha \{x_i\}_{i \in I} &= \{\alpha x_i\}_{i \in I}\end{aligned}$$

PROOF: Each axiom follows from the corresponding axiom in V_i . \square

Corollary 16.1.6.1. *Let V be a vector space over K . For any set I , we have V^I is a vector space over K .*

Corollary 16.1.6.2. *Let $n \in \mathbb{Z}_+$. Then \mathbb{R}^n is a real vector space, and \mathbb{C}^n is both a real and a complex vector space, under*

$$\begin{aligned}(x_1, \dots, x_n) + (y_1, \dots, y_n) &= (x_1 + y_1, \dots, x_n + y_n) \\ \lambda(x_1, \dots, x_n) &= (\lambda x_1, \dots, \lambda x_n)\end{aligned}$$

Proposition 16.1.7. *Let V be a vector space over K . Then there exists a unique $0 \in V$ such that, for all $v \in V$, we have $v + 0 = v$.*

PROOF:

$\langle 1 \rangle$ 1. There exists $0 \in V$ such that $\forall v \in V. v + 0 = v$

$\langle 2 \rangle$ 1. Pick $v \in V$

$\langle 2 \rangle$ 2. Pick $0 \in V$ such that $v + 0 = v$

PROOF: Axiom 3.

$\langle 2 \rangle$ 3. For all $u \in V$, we have $u + 0 = u$

$\langle 3 \rangle$ 1. LET: $u \in V$

$\langle 3 \rangle$ 2. PICK $u' \in V$ such that $v + u' = u$

PROOF: Axiom 3.

$\langle 3 \rangle$ 3. $u + 0 = u$

$$\begin{aligned}u + 0 &= v + u' + 0 && (\langle 3 \rangle 2) \\ &= v + u' && (\langle 2 \rangle 2) \\ &= u && (\langle 3 \rangle 2)\end{aligned}$$

$\langle 1 \rangle 2$. If $0, 0' \in V$ are such that $\forall v \in V. v + 0 = v$ and $\forall v \in V. v + 0' = v$, then $0 = 0'$.

$\langle 2 \rangle 1$. LET: $0, 0' \in V$

$\langle 2 \rangle 2$. ASSUME: $\forall v \in V. v + 0 = v$

$\langle 2 \rangle 3$. ASSUME: $\forall v \in V. v + 0' = v$

$\langle 2 \rangle 4$. $0 = 0'$

$$0 = 0 + 0' \quad (\langle 2 \rangle 2)$$

$$= 0' \quad (\langle 2 \rangle 3)$$

□

Proposition 16.1.8. *Let V be a vector space. For any $v \in V$, there exists a unique $-v \in V$ such that $v + (-v) = 0$.*

PROOF:

$\langle 1 \rangle 1$. LET: $v \in V$

$\langle 1 \rangle 2$. There exists $-v \in V$ such that $v + (-v) = u$

PROOF: Axiom 3.

$\langle 1 \rangle 3$. If $v + x = 0$ and $v + y = 0$ then $x = y$

$\langle 2 \rangle 1$. ASSUME: $v + x = 0$

$\langle 2 \rangle 2$. ASSUME: $v + y = 0$

$\langle 2 \rangle 3$. $x = y$

PROOF:

$$x = x + 0 \quad (\text{Proposition 16.1.7})$$

$$= x + v + y \quad (\langle 2 \rangle 2)$$

$$= 0 + y \quad (\langle 2 \rangle 1)$$

$$= y \quad (\text{Proposition 16.1.7})$$

□

Proposition 16.1.9. *Let V be a vector space. For any $u, v \in V$, there exists a unique $u - v \in V$ such that $v + (u - v) = u$, namely $u - v = u + (-v)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $u, v \in V$

$\langle 1 \rangle 2$. $v + (u + (-v)) = u$

PROOF:

$$v + u + (-v) = u + 0 \quad (\text{Proposition 16.1.8})$$

$$= u \quad (\text{Proposition 16.1.7})$$

$\langle 1 \rangle 3$. For all $x \in V$, if $v + x = u$ then $x = u + (-v)$.

$\langle 2 \rangle 1$. LET: $x \in V$

$\langle 2 \rangle 2$. ASSUME: $v + x = u$

$\langle 2 \rangle 3$. $x = u + (-v)$

PROOF:

$$u + (-v) = v + x + (-v) \quad (\langle 2 \rangle 2)$$

$$= x + 0 \quad (\text{Proposition 16.1.8})$$

$$= x \quad (\text{Proposition 16.1.7})$$

□

Proposition 16.1.10. *Let V be a vector space over K . Let $u, v, w \in V$. If $u + v = u + w$ then $v = w$.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: $u + v = u + w$

$\langle 1 \rangle 2.$ $v = w$

PROOF:

$$\begin{aligned}
 v &= v + 0 && \text{(Proposition 16.1.7)} \\
 &= v + u + (-u) && \text{(Proposition 16.1.8)} \\
 &= w + u + (-u) && ((1)1) \\
 &= w + 0 && \text{(Proposition 16.1.8)} \\
 &= w && \text{(Proposition 16.1.7)}
 \end{aligned}$$

Proposition 16.1.11. *Let V be a vector space over K . Let $\lambda \in K$. Then $\lambda 0 = 0$.*

PROOF:

$\langle 1 \rangle 1.$ $\lambda 0 + \lambda 0 = \lambda 0 + 0$

PROOF:

$$\begin{aligned}
 \lambda 0 + \lambda 0 &= \lambda(0 + 0) && \text{(Axiom 6)} \\
 &= \lambda 0 && \text{(Proposition 16.1.7)}
 \end{aligned}$$

$\langle 1 \rangle 2.$ $\lambda 0 = 0$

PROOF: Proposition 16.1.10.

□

Proposition 16.1.12. *Let V be a vector space over K . Let $\lambda \in K$ and $v \in V$. If $\lambda v = 0$ then $\lambda = 0$ or $v = 0$.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: $\lambda \neq 0$

$\langle 1 \rangle 2.$ ASSUME: $\lambda v = 0$

$\langle 1 \rangle 3.$ $v = 0$

PROOF:

$$\begin{aligned}
 v &= 1v && \text{(Axiom 7)} \\
 &= \lambda^{-1}\lambda v \\
 &= \lambda^{-1}0 && ((1)2) \\
 &= 0
 \end{aligned}$$

□

Proposition 16.1.13. *Let V be a vector space over K . For all $v \in V$ we have $0v = 0$.*

PROOF:

$\langle 1 \rangle 1.$ $0v + 0 = 0v + 0v$

$$\begin{aligned}
0v + 0 &= 0v && \text{(Proposition 16.1.7)} \\
&= (0 + 0)v \\
&= 0v + 0v && \text{(Axiom 5)}
\end{aligned}$$

$\langle 1 \rangle 2.$ $0v = 0$

PROOF: Proposition 16.1.10, $\langle 1 \rangle 1.$

□

Proposition 16.1.14. *Let V be a vector space over K . Let $v \in V$. Then $(-1)v = -v$.*

PROOF:

$\langle 1 \rangle 1.$ $v + (-1)v = 0$

PROOF:

$$\begin{aligned}
v + (-1)v &= 1v + (-1)v && \text{(Axiom 7)} \\
&= (1 + (-1))v && \text{(Axiom 5)} \\
&= 0v \\
&= 0 && \text{(Proposition 16.1.13)}
\end{aligned}$$

$\langle 1 \rangle 2.$ Q.E.D.

PROOF: Proposition 16.1.8.

□

16.2 Subspaces

Definition 16.2.1 (Subspace). Let V be a vector space over K and $U \subseteq V$. Then U is a *subspace* of V iff $\forall \alpha, \beta \in K. \forall u, v \in U. \alpha u + \beta v \in U$. It is a *proper* subspace iff in addition $U \neq V$.

Proposition 16.2.2. *Let V be a vector space over K and U a subspace of V . Then U is a vector space over K under the restrictions of the operations of V .*

PROOF: Each of the axioms follows from the corresponding axiom in V . For axiom 3, we have if $u, v \in U$ then $v - u = 1v + (-1)u \in U$. □

Proposition 16.2.3. *Every vector space is a subspace of itself.*

PROOF: Trivial. □

Proposition 16.2.4. *Let Ω be a subset of \mathbb{R}^N . Let $\mathcal{C}(\Omega)$ be the set of all continuous functions $\Omega \rightarrow \mathbb{C}$. Then $\mathcal{C}(\Omega)$ is a subspace of \mathbb{C}^Ω .*

PROOF: If $f, g : \Omega \rightarrow \mathbb{C}$ are continuous then so is $\alpha f + \beta g$. □

Proposition 16.2.5. *Let Ω be an open set in \mathbb{R}^N . Let $\mathcal{C}^k(\Omega)$ be the set of all continuous functions $\Omega \rightarrow \mathbb{C}$ with continuous partial derivatives of order k . Then $\mathcal{C}^k(\Omega)$ is a subspace of \mathbb{C}^Ω .*

PROOF: If $f, g : \Omega \rightarrow \mathbb{C}$ have continuous partial derivatives of order k then so does $\alpha f + \beta g$. \square

Proposition 16.2.6. *Let Ω be an open set in \mathbb{R}^N . Let $\mathcal{C}^\infty(\Omega)$ be the set of all infinitely differentiable functions $\Omega \rightarrow \mathbb{C}$. Then $\mathcal{C}^\infty(\Omega)$ is a subspace of \mathbb{C}^Ω .*

PROOF: If $f, g : \Omega \rightarrow \mathbb{C}$ are infinitely differentiable then so is $\alpha f + \beta g$. \square

Proposition 16.2.7. *Let Ω be an open set in \mathbb{R}^N . Let $\mathcal{P}(\Omega)$ be the set of all polynomials in N variables considered as functions $\Omega \rightarrow \mathbb{C}$. Then $\mathcal{P}(\Omega)$ is a subspace of \mathbb{C}^Ω .*

PROOF: If $f, g : \Omega \rightarrow \mathbb{C}$ are polynomials in N variables then so is $\alpha f + \beta g$. \square

Proposition 16.2.8. *Let V be a vector space and U_1, U_2 subspaces of V . If $U_1 \subseteq U_2$ then U_1 is a subspace of U_2 .*

PROOF: Trivial. \square

Proposition 16.2.9. *Let V be a vector space over K . The intersection of a set of subspaces of V is a subspace of V .*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{U} be a set of subspaces of V .

$\langle 1 \rangle 2$. LET: $u, v \in \bigcap \mathcal{U}$ and $\lambda, \mu \in K$

$\langle 1 \rangle 3$. $\lambda u + \mu v \in \bigcap \mathcal{U}$

$\langle 2 \rangle 1$. LET: $U \in \mathcal{U}$

$\langle 2 \rangle 2$. $u, v \in U$

PROOF: $\langle 1 \rangle 2, \langle 2 \rangle 1$.

$\langle 2 \rangle 3$. $\lambda u + \beta v \in U$

PROOF: $\langle 1 \rangle 1, \langle 1 \rangle 2, \langle 2 \rangle 1, \langle 2 \rangle 2$.

\square

Proposition 16.2.10. *The set of all bounded complex sequences is a proper subspace of $\mathbb{C}^\mathbb{N}$.*

PROOF: If (x_n) and (y_n) are bounded then so is $(\lambda x_n + \mu y_n)$. \square

Proposition 16.2.11. *The set of all convergent complex sequences is a proper subspace of the space of all bounded complex sequences.*

PROOF: If (x_n) and (y_n) converge then so does $(\lambda x_n + \mu y_n)$. \square

Proposition 16.2.12. *The set l^p of all sequences (x_n) in \mathbb{C} such that $\sum_n |x_n|^p < \infty$ is a subspace of $\mathbb{C}^\mathbb{N}$.*

PROOF: It is closed under addition by Proposition 15.0.2, and it is easy to see that it is closed under scalar multiplication. \square

16.3 Linear Independence and Bases

Definition 16.3.1 (Linear Combination). Let V be a vector space over K . Let $v, v_1, \dots, v_n \in V$. Then v is a *linear combination* of v_1, \dots, v_n iff there exist scalars $\lambda_1, \dots, \lambda_n \in K$ such that

$$v = \lambda_1 v_1 + \dots + \lambda_n v_n .$$

Definition 16.3.2 (Linearly Independent). Let V be a vector space over K . Let $A \subseteq V$. Then A is *linearly independent* iff, for all $\lambda_1, \dots, \lambda_n \in K$ and $v_1, \dots, v_n \in A$, if $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ then $\lambda_1 = \dots = \lambda_n = 0$.

Definition 16.3.3 (Span). Let V be a vector space over K and $A \subseteq V$. The *span* of A , or the subspace of V *spanned* by A , is the set of all linear combinations of vectors in A .

Proposition 16.3.4. Let V be a vector space over K and $A \subseteq V$. Then $\text{span } A$ is a subspace of V .

PROOF: Given $\alpha, \beta \in K$ and $\lambda_1 u_1 + \dots + \lambda_m u_m, \mu_1 v_1 + \dots + \mu_n v_n \in \text{span } A$, we have

$$\begin{aligned} & \alpha(\lambda_1 u_1 + \dots + \lambda_m u_m) + \beta(\mu_1 v_1 + \dots + \mu_n v_n) \\ &= \alpha\lambda_1 u_1 + \dots + \alpha\lambda_m u_m + \beta\mu_1 v_1 + \dots + \beta\mu_n v_n \\ &\in \text{span } A \end{aligned} \quad \square$$

Definition 16.3.5 (Basis). Let V be a vector space over K and $B \subseteq V$. Then B is a *basis* for V iff B is linearly independent and $\text{span } B = V$.

Definition 16.3.6 (Finite Dimensional). A vector space is *finite dimensional* iff there exists a finite basis; otherwise it is *infinite dimensional*.

Proposition 16.3.7. In a finite dimensional space, any two bases have the same size.

TODO

Definition 16.3.8 (Dimension). The *dimension* of a finite dimensional vector space V , $\dim V$, is the number of vectors in any basis.

Proposition 16.3.9. Let K be either \mathbb{R} or \mathbb{C} . Then K^n as a vector space over K has dimension n .

PROOF: The vectors with one component 1 and all other components 0 form a basis. \square

Proposition 16.3.10. As a real vector space, \mathbb{C}^n has dimension $2n$.

PROOF: The vectors with one component either 1 or i and all other components 0 form a basis. \square

Proposition 16.3.11. *Let Ω be a nonempty open set in \mathbb{R}^n . The space $\mathcal{C}(\Omega)$ is infinite dimensional.*

PROOF: Let $\pi_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ be the first projection. The functions $1, \pi_1(x), \pi_1(x)^2, \pi_1(x)^3, \dots$ form an infinite linearly independent set in $\mathcal{C}(\Omega)$. \square

Proposition 16.3.12. *The spaces $\mathcal{C}^k(\mathbb{R}^n)$ and $\mathcal{C}^\infty(\mathbb{R}^n)$ are infinite dimensional.*

PROOF: The monomials $1, x, x^2, \dots$ form an infinite linearly independent set. \square

16.4 Linear Mappings

Definition 16.4.1 (Kernel). Let U and V be vector spaces and $T : U \rightarrow V$. The *kernel* of T is

$$\ker T := \{u \in U \mid T(u) = 0\} .$$

Definition 16.4.2 (Linear Mapping). Let U and V be vector spaces over K . A function $L : U \rightarrow V$ is a *linear mapping* iff $\forall x, y \in U. \forall \alpha, \beta \in K. L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$.

Proposition 16.4.3. *Let U and V be vector spaces over K . The set of linear mappings from U to V is a subspace of V^U .*

16.5 Eigenvalues and Eigenvectors

Definition 16.5.1 (Eigenvalue and Eigenvector). Let V be a vector space over K . Let $A : V \rightarrow V$ be a linear transformation. Let $v \in V$ and $\lambda \in K$. Then v is an *eigenvector* of A with *eigenvalue* λ iff $A(v) = \lambda v$.

Chapter 17

Normed Spaces

Definition 17.0.1 (Norm). Let K be either \mathbb{R} or \mathbb{C} . Let V be a vector space over K . A *norm* on V is a function $\| \cdot \| : V \rightarrow \mathbb{R}$ such that, for all $u, v \in V$ and $\lambda \in K$:

1. If $\|v\| = 0$ then $v = 0$.
2. $\|\lambda v\| = |\lambda| \|v\|$
3. (*Triangle Inequality*) $\|u + v\| \leq \|u\| + \|v\|$

A *normed space* over K is a pair $(V, \| \cdot \|)$ where V is a vector space over K and $\| \cdot \|$ is a norm on V .

Proposition 17.0.2. *In a normed space, $\|0\| = 0$.*

PROOF: $\|0\| = |0| \|0\| = 0$ by Axiom 2. \square

Proposition 17.0.3. *Let V be a normed vector space over K . For all $v \in V$ we have $\|v\| \geq 0$.*

PROOF:

$$\begin{aligned}
 0 &= \|0\| && \text{(Proposition 17.0.2)} \\
 &= \|v - v\| \\
 &\leq \|v\| + \|-v\| && \text{(Triangle Inequality)} \\
 &= 2\|v\| && \text{(Axiom 2)}
 \end{aligned}$$

Proposition 17.0.4. *Let V be a normed space. Let $u, v \in V$. Then*

$$|\|u\| - \|v\|| \leq \|u - v\| .$$

PROOF:

$$\begin{aligned}
 \|u\| &\leq \|u - v\| + \|v\| && \text{(Triangle Inequality)} \\
 \therefore \|u\| - \|v\| &\leq \|u - v\| \\
 \|v\| &\leq \|v - u\| + \|u\| && \text{(Triangle Inequality)} \\
 &= \|u - v\| + \|u\| && \text{(Axiom 2)} \\
 \therefore \|v\| - \|u\| &\leq \|u - v\|
 \end{aligned}$$

Definition 17.0.5 (Euclidean Norm). The *Euclidean norm* on K^n is defined by

$$\|(x_1, \dots, x_n)\| = \sqrt{|x_1|^2 + \dots + |x_n|^2}.$$

Proposition 17.0.6. The Euclidean norm on K^n is a norm.

PROOF:

$\langle 1 \rangle 1$. If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$

PROOF: If $\sqrt{|x_1|^2 + \dots + |x_n|^2} = 0$ then $x_1 = \dots = x_n = 0$.

$\langle 1 \rangle 2$. $\|\lambda \vec{x}\| = |\lambda| \|\vec{x}\|$

PROOF:

$$\begin{aligned}
 \|\lambda \vec{x}\| &= \sqrt{|\lambda x_1|^2 + \dots + |\lambda x_n|^2} \\
 &= \sqrt{|\lambda|^2 |x_1|^2 + \dots + |\lambda|^2 |x_n|^2} \\
 &= |\lambda| \sqrt{|x_1|^2 + \dots + |x_n|^2} \\
 &= |\lambda| \|\vec{x}\|
 \end{aligned}$$

$\langle 1 \rangle 3$. $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

PROOF:

$$\begin{aligned}
 \|\vec{u} + \vec{v}\|^2 &= |u_1 + v_1|^2 + \dots + |u_n + v_n|^2 \\
 &= |u_1|^2 + \dots + |u_n|^2 + |v_1|^2 + \dots + |v_n|^2 \\
 &\quad + 2|u_1||v_1| + \dots + 2|u_n||v_n| \\
 &\leq \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2|u_1 v_1 + \dots + u_n v_n| \\
 &\leq \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\| && \text{(Cauchy-Schwarz)} \\
 &= (\|\vec{u}\| + \|\vec{v}\|)^2
 \end{aligned}$$

□

Corollary 17.0.6.1. The absolute value function $|\cdot|$ is a norm on K .

Proposition 17.0.7. The function $\|\vec{x}\| = |x_1| + \dots + |x_n|$ is a norm on \mathbb{C}^n .

PROOF:

$\langle 1 \rangle 1$. If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$

PROOF: If $|x_1| + \dots + |x_n| = 0$ then $x_1 = \dots = x_n = 0$.

$\langle 1 \rangle 2$. $\|\lambda \vec{x}\| = |\lambda| \|\vec{x}\|$

PROOF:

$$\begin{aligned}
 \|\lambda \vec{x}\| &= |\lambda x_1| + \cdots + |\lambda x_n| \\
 &= |\lambda|(|x_1| + \cdots + |x_n|) \\
 &= |\lambda| \|\vec{x}\|
 \end{aligned}$$

$\langle 1 \rangle 3. \|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

PROOF:

$$\begin{aligned}
 \|\vec{u} + \vec{v}\|^2 &= |u_1 + v_1|^2 + \cdots + |u_n + v_n|^2 \\
 &\leq |u_1|^2 + |v_1|^2 + \cdots + |u_n|^2 + |v_n|^2 \\
 &= \|\vec{u}\|^2 + \|\vec{v}\|^2
 \end{aligned}$$

□

Proposition 17.0.8. *The function $\|\vec{x}\| = \max(|x_1|, \dots, |x_n|)$ is a norm on \mathbb{C}^n .*

PROOF:

$\langle 1 \rangle 1.$ If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$

PROOF: If $\max(|x_1|, \dots, |x_n|) = 0$ then $x_1 = \cdots = x_n = 0$.

$\langle 1 \rangle 2. \|\lambda \vec{x}\| = |\lambda| \|\vec{x}\|$

PROOF:

$$\begin{aligned}
 \|\lambda \vec{x}\| &= \max(|\lambda x_1|, \dots, |\lambda x_n|) \\
 &= |\lambda| \max(|x_1|, \dots, |x_n|) \\
 &= |\lambda| \|\vec{x}\|
 \end{aligned}$$

$\langle 1 \rangle 3. \|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

PROOF:

$$\begin{aligned}
 \|\vec{u} + \vec{v}\| &= \max(|u_1 + v_1|, \dots, |u_n + v_n|) \\
 &\leq \max(|u_1| + |v_1|, \dots, |u_n| + |v_n|) \\
 &\leq \max(|u_1|, \dots, |u_n|) + \max(|v_1|, \dots, |v_n|)
 \end{aligned}$$

□

Definition 17.0.9 (Uniform Convergence Norm). Let Ω be a closed bounded subset of \mathbb{R}^n . The *uniform convergence norm* on $\mathcal{C}(\Omega)$ is the function defined by $\|f\| = \max_{x \in \Omega} |f(x)|$.

Proposition 17.0.10. *Let Ω be a closed bounded subset of \mathbb{R}^n . The uniform convergence norm is a norm on $\mathcal{C}(\Omega)$.*

PROOF:

$\langle 1 \rangle 1.$ If $\|f\| = 0$ then $f = 0$

PROOF: If $\max_x |f(x)| = 0$ then $f(x) = 0$ for all x .

$\langle 1 \rangle 2. \|\lambda f\| = |\lambda| \|f\|$

PROOF:

$$\begin{aligned}
 \|\lambda f\| &= \max_x |\lambda f(x)| \\
 &= |\lambda| \max_x |f(x)| \\
 &= |\lambda| \|f\|
 \end{aligned}$$

⟨1⟩3. $\|f + g\| \leq \|f\| + \|g\|$

PROOF:

$$\begin{aligned}\|f + g\| &= \max_x |f(x) + g(x)| \\ &\leq \max_x (|f(x)| + |g(x)|) \\ &\leq \max_x |f(x)| + \max_x |g(x)| \\ &= \|f\| + \|g\|\end{aligned}$$

□

Proposition 17.0.11. *Let $p \geq 1$. The function $\|(z_n)\| = (\sum_{n=1}^{\infty} |z_n|^p)^{1/p}$ is a norm on l^p .*

PROOF:

⟨1⟩1. If $\|(z_n)\| = 0$ then $(z_n) = (0)$

PROOF: If $(\sum_n |z_n|^p)^{1/p} = 0$ then $\sum_n |z_n|^p = 0$ so $|z_n|^p = 0$ for all n , and so $z_n = 0$ for all n .

⟨1⟩2. $\|(\lambda z_n)\| = |\lambda| \|(z_n)\|$

PROOF:

$$\begin{aligned}\|(\lambda z_n)\| &= \left(\sum_n |\lambda z_n|^p \right)^{1/p} \\ &= |\lambda| \left(\sum_n |z_n|^p \right)^{1/p} \\ &= |\lambda| \|(z_n)\|\end{aligned}$$

⟨1⟩3. The triangle inequality holds.

PROOF: This is Minkowski's Inequality. □

Proposition 17.0.12. *Let V be a normed space and U a vector subspace of V . Then U is a normed space under the restriction of the norm to U .*

PROOF: Each axiom follows from the fact it holds in V . □

Proposition 17.0.13. *Let V be a normed space over K . Let x_1, \dots, x_n be linearly independent elements of V . Then there exists a real number $c > 0$ such that, for all $\alpha_1, \dots, \alpha_n \in K$, we have*

$$\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \geq c(|\alpha_1| + \dots + |\alpha_n|) .$$

PROOF:

⟨1⟩1. Define $f : K^n \rightarrow \mathbb{R}$ by

$$f(\alpha_1, \dots, \alpha_n) = \|\alpha_1 x_1 + \dots + \alpha_n x_n\|$$

⟨1⟩2. f is continuous.

⟨2⟩1. LET: $(\alpha_1, \dots, \alpha_n) \in K^n$ and $\epsilon > 0$

⟨2⟩2. LET: $\delta = \epsilon / (\|x_1\| + \dots + \|x_n\|)$

PROOF: x_1, \dots, x_n are not all zero because they are linearly independent.

⟨2⟩3. LET: $(\beta_1, \dots, \beta_n)$ with $|\alpha_i - \beta_i| < \delta$ for all i

$$\langle 2 \rangle 4. \|(\alpha_1 x_1 + \cdots + \alpha_n x_n) - (\beta_1 x_1 + \beta_n x_n)\| < \epsilon$$

PROOF:

$$\|(\alpha_1 x_1 + \cdots + \alpha_n x_n) - (\beta_1 x_1 + \beta_n x_n)\|$$

$$\leq |\alpha_1 - \beta_1| \|x_1\| + \cdots + |\alpha_n - \beta_n| \|x_n\| \quad (\text{Axioms 2 and 3})$$

$$< \delta(\|x_1\| + \cdots + \|x_n\|) \quad (\langle 2 \rangle 3)$$

$$= \epsilon \quad (\langle 2 \rangle 2)$$

$\langle 1 \rangle 3.$ PICK $(\beta_1, \dots, \beta_n) \in \{(\beta_1, \dots, \beta_n) \in K^n \mid |\beta_1| + \cdots + |\beta_n| = 1\}$ at which f attains its minimum.

PROOF: Extreme Value Theorem.

$\langle 1 \rangle 4.$ Let $c = f(\beta_1, \dots, \beta_n)$

$\langle 1 \rangle 5.$ $c > 0$

PROOF: Linear independence.

$\langle 1 \rangle 6.$ LET: $\alpha_1, \dots, \alpha_n \in K$

$\langle 1 \rangle 7.$ $\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|)$

$\langle 2 \rangle 1.$ ASSUME: w.l.o.g. $\alpha_1, \dots, \alpha_n$ are not all zero.

$\langle 2 \rangle 2.$ LET: $\beta_i = \alpha_i / (|\alpha_1| + \cdots + |\alpha_n|)$ for $i = 1, \dots, n$

$\langle 2 \rangle 3.$ $|\beta_1| + \cdots + |\beta_n| = 1$

$\langle 2 \rangle 4.$ $f(\beta_1, \dots, \beta_n) \geq c$

$\langle 2 \rangle 5.$ Q.E.D.

PROOF: Multiply both sides by $|\alpha_1| + \cdots + |\alpha_n|$.

□

Proposition 17.0.14. Let V be a normed space over K . Define $d : V^2 \rightarrow \mathbb{R}$ by $d(x, y) = \|x - y\|$. Then d is a metric on V .

PROOF:

$\langle 1 \rangle 1.$ For all $x, y \in V$ we have $d(x, y) \geq 0$

PROOF: Proposition 17.0.3.

$\langle 1 \rangle 2.$ For all $x, y \in V$ we have $d(x, y) = 0$ iff $x = y$

$\langle 2 \rangle 1.$ If $d(x, y) = 0$ then $x = y$

PROOF: Axiom 1.

$\langle 2 \rangle 2.$ If $x = y$ then $d(x, y) = 0$

PROOF: Proposition 17.0.2.

$\langle 1 \rangle 3.$ $\forall x, y \in V. d(x, y) = d(y, x)$

PROOF: By Axiom 2.

$\langle 1 \rangle 4.$ $\forall x, y, z \in V. d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Axiom 3.

□

Henceforth we identify any normed space with this metric space.

17.1 Convergence

Proposition 17.1.1. Let V be a normed space over K . Let (x_n) be a sequence in V and $l \in V$. Then $x_n \rightarrow l$ as $n \rightarrow \infty$ in V if and only if $\|x_n - l\| \rightarrow 0$ as $n \rightarrow \infty$ in \mathbb{R} .

PROOF: Immediate from definitions. \square

Proposition 17.1.2. *In a normed space, a sequence has at most one limit.*

PROOF:

$\langle 1 \rangle 1$. LET: V be a vector space over K .

$\langle 1 \rangle 2$. ASSUME: $x_n \rightarrow l$ and $x_n \rightarrow m$ as $n \rightarrow \infty$.

$\langle 1 \rangle 3$. ASSUME: for a contradiction $l \neq m$

$\langle 1 \rangle 4$. LET: $\epsilon = \|l - m\|/2$

$\langle 1 \rangle 5$. PICK N such that $\forall n \geq N, \|x_n - l\| < \epsilon$ and $\forall n \geq N, \|x_n - m\| < \epsilon$

PROOF: $\langle 1 \rangle 2, \langle 1 \rangle 4$

$\langle 1 \rangle 6$. $\|l - m\| < \|l - m\|$

PROOF:

$$\|l - m\| \leq \|x_N - l\| + \|x_N - m\| \quad (\text{Triangle Inequality})$$

$$< 2\epsilon \quad (\langle 1 \rangle 5)$$

$$= \|l - m\| \quad (\langle 1 \rangle 4)$$

$\langle 1 \rangle 7$. Q.E.D.

PROOF: This is a contradiction.

\square

Definition 17.1.3 (Bounded). Let V be a normed space over K . A sequence (x_n) in V is *bounded* iff there exists B such that $\forall n, \|x_n\| < B$.

Proposition 17.1.4. *Every convergent sequence is bounded.*

PROOF:

$\langle 1 \rangle 1$. LET: $x_n \rightarrow l$ as $n \rightarrow \infty$

$\langle 1 \rangle 2$. PICK N such that $\forall n \geq N, \|x_n - l\| < 1$

$\langle 1 \rangle 3$. LET: $B = \max(\|x_1\|, \|x_2\|, \dots, \|x_{N-1}\|, \|l\| + 1)$

$\langle 1 \rangle 4$. LET: $n \in \mathbb{N}$

$\langle 1 \rangle 5$. $\|x_n\| \leq B$

$\langle 2 \rangle 1$. CASE: $n < N$

PROOF: $\|x_n\| \leq B$ from $\langle 1 \rangle 3$.

$\langle 2 \rangle 2$. CASE: $n \geq N$

PROOF:

$$\|x_n\| \leq \|l\| + \|x_n - l\| \quad (\text{Triangle Inequality})$$

$$< \|l\| + 1 \quad (\langle 1 \rangle 2)$$

$$\leq B \quad (\langle 1 \rangle 3)$$

\square

Proposition 17.1.5. *Let V be a normed space over K . If $x_n \rightarrow l$ as $n \rightarrow \infty$ in V , and $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ in K , then $\lambda_n x_n \rightarrow \lambda l$ as $n \rightarrow \infty$.*

PROOF:

$\langle 1 \rangle 1$. LET: V be a normed space over K .

$\langle 1 \rangle 2$. LET: $x_n \rightarrow l$ as $n \rightarrow \infty$

$\langle 1 \rangle 3$. LET: $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$

- (1)4. LET: $\epsilon > 0$
 (1)5. PICK N such that, for all $n \geq N$, we have $\|x_n - l\| < \epsilon/2|\lambda|$ and $|\lambda_n - \lambda| < \sqrt{\epsilon/2}$ and $\|x_n\| < \sqrt{\epsilon/2}$
 (1)6. LET: $n \geq N$
 (1)7. $\|\lambda_n x_n - \lambda l\| < \epsilon$

PROOF:

$$\begin{aligned}
 \|\lambda_n x_n - \lambda l\| &\leq \|\lambda_n x_n - \lambda x_n\| + \|\lambda x_n - \lambda l\| && \text{(Triangle Inequality)} \\
 &= |\lambda_n - \lambda| \|x_n\| + |\lambda| \|x_n - l\| && \text{(Axiom 2)} \\
 &< \sqrt{\epsilon/2} \sqrt{\epsilon/2} + |\lambda| \epsilon/2|\lambda| && ((1)5) \\
 &= \epsilon
 \end{aligned}$$

□

Proposition 17.1.6. *Let V be a normed space over K . If $x_n \rightarrow l$ and $y_n \rightarrow m$ as $n \rightarrow \infty$, then $x_n + y_n \rightarrow l + m$ as $n \rightarrow \infty$.*

PROOF:

- (1)1. LET: $\epsilon > 0$
 (1)2. PICK N such that, for all $n \geq N$, we have $\|x_n - l\| < \epsilon/2$ and $\|y_n - m\| < \epsilon/2$
 (1)3. LET: $n \geq N$
 (1)4. $\|(x_n + y_n) - (l + m)\| < \epsilon$

PROOF:

$$\begin{aligned}
 \|(x_n + y_n) - (l + m)\| &\leq \|x_n - l\| + \|y_n - m\| && \text{(Triangle Inequality)} \\
 &< \epsilon/2 + \epsilon/2 && ((1)2) \\
 &= \epsilon
 \end{aligned}$$

□

Definition 17.1.7 (Uniform Convergence). Let Ω be a closed bounded subset of \mathbb{R}^n . Let (f_n) be a sequence in $\mathcal{C}(\Omega)$ and $f \in \mathcal{C}(\Omega)$. Then (f_n) *converges uniformly* to f iff, for every $\epsilon > 0$, there exists N such that $\forall x \in \Omega. \forall n \geq N. |f_n(x) - f(x)| < \epsilon$.

Proposition 17.1.8. *Let Ω be a closed bounded subset of \mathbb{R}^n . Let (f_n) be a sequence in $\mathcal{C}(\Omega)$ and $f \in \mathcal{C}(\Omega)$. Then (f_n) converges uniformly to f iff f_n converges to f under the uniform convergence norm.*

PROOF:

$$\begin{aligned}
 &(f_n) \text{ converges to } f \text{ under the uniform convergence norm} \\
 &\Leftrightarrow \forall \epsilon > 0. \exists N. \forall n \geq N. \|f_n - f\| < \epsilon \\
 &\Leftrightarrow \forall \epsilon > 0. \exists N. \forall n \geq N. \forall x \in X. |f_n(x) - f(x)| < \epsilon
 \end{aligned}$$

□

Definition 17.1.9 (Pointwise Convergence). Let (f_n) be a sequence in $\mathcal{C}([0, 1])$ and $f \in \mathcal{C}([0, 1])$. Then (f_n) *converges pointwise* to f iff, for all $t \in [0, 1]$, we have $|f_n(t) - f(t)| \rightarrow 0$ as $n \rightarrow \infty$.

Proposition 17.1.10. *There is no norm n on $\mathcal{C}([0, 1])$ such that, for every sequence (f_n) and function f in $\mathcal{C}([0, 1])$, we have (f_n) converges pointwise to f if and only if (f_n) converges to f under n .*

PROOF:

(1)1. ASSUME: for a contradiction $\| \cdot \|$ is a norm on $\mathcal{C}([0, 1])$ such that, for every sequence (f_n) and function f in $\mathcal{C}([0, 1])$, we have (f_n) converges pointwise to f if and only if (f_n) converges to f under $\| \cdot \|$.

(1)2. For $n \in \mathbb{Z}_+$, define $g_n \in \mathcal{C}([0, 1])$ by

$$g_n(t) = \begin{cases} 2^n t & \text{if } 0 \leq t \leq 2^{-n} \\ 2 - 2^n t & \text{if } 2^{-n} \leq t \leq 2^{1-n} \\ 0 & \text{if } 2^{1-n} \leq t \leq 1 \end{cases}$$

(1)3. For all n , $\|g_n\| \neq 0$

PROOF: Axiom 1.

(1)4. For $n \in \mathbb{Z}_+$, define $f_n \in \mathcal{C}([0, 1])$ by $f_n = g_n / \|g_n\|$

(1)5. For all n , $\|f_n\| = 1$

PROOF: Axiom 2.

(1)6. (f_n) does not converge under $\| \cdot \|$

(1)7. (f_n) converges pointwise to 0.

(1)8. This is a contradiction.

□

Definition 17.1.11 (Equivalence of Norms). Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be two norms on the same vector space V . Then the norms are *equivalent* if and only if, for any sequence (x_n) in V and $l \in V$, we have that (x_n) converges to l under $\| \cdot \|_1$ if and only if (x_n) converges to l under $\| \cdot \|_2$.

Theorem 17.1.12. Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be two norms on the same vector space E over K . Then $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent if and only if there exist positive real numbers α and β such that, for all $x \in E$,

$$\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1 .$$

PROOF:

(1)1. If $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent then there exist positive real numbers α and β such that, for all $x \in E$, $\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1$.

(2)1. ASSUME: $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent.

(2)2. There exists $\alpha > 0$ such that, for all $x \in E$, we have $\alpha \|x\|_1 \leq \|x\|_2$

(3)1. ASSUME: for a contradiction there is no $\alpha > 0$ such that, for all $x \in E$, we have $\alpha \|x\|_1 \leq \|x\|_2$.

(3)2. For all $n \in \mathbb{Z}_+$, PICK $x_n \in E$ such that $1/n \|x_n\|_1 > \|x_n\|_2$

(3)3. For all $n \in \mathbb{Z}_+$,

LET:

$$y_n = \frac{1}{\sqrt{n}} \frac{x_n}{\|x_n\|_2}$$

(3)4. (y_n) converges to 0 under $\| \cdot \|_2$

(3)5. (y_n) converges to 0 under $\| \cdot \|_1$

(3)6. For all $n \in \mathbb{Z}_+$, we have $\|y_n\| > \sqrt{n}$

(3)7. This is a contradiction.

(2)3. There exists $\beta > 0$ such that, for all $x \in E$, we have $\|x\|_2 \leq \beta \|x\|_1$

PROOF: Similar.

- (1)2. If there exist positive real numbers α and β such that, for all $x \in E$,
 $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.
 (2)1. ASSUME: α and β are positive reals with $\forall x \in E. \alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$.
 (2)2. Let (x_n) be a sequence in E and $l \in E$
 (2)3. If (x_n) converges to l under $\|\cdot\|_1$ then (x_n) converges to l under $\|\cdot\|_2$.
 (3)1. ASSUME: (x_n) converges to l under $\|\cdot\|_1$
 (3)2. LET: $\epsilon > 0$
 (3)3. PICK N such that $\forall n \geq N. \|x_n - l\|_1 < \epsilon/\beta$
 (3)4. $\forall n \geq N. \|x_n - l\|_2 < \epsilon$
 (2)4. If (x_n) converges to l under $\|\cdot\|_2$ then (x_n) converges to l under $\|\cdot\|_1$.
 PROOF: Similar.

□

Theorem 17.1.13. *Any two norms on a finite dimensional vector space are equivalent.*

PROOF:

- (1)1. LET: V be a finite dimensional vector space over K .
 (1)2. ASSUME: w.l.o.g. $\dim V > 0$
 (1)3. PICK a basis $\{e_1, \dots, e_n\}$ for V .
 (1)4. LET: $\|\cdot\|_0 : V \rightarrow \mathbb{R}$ be the function: $\|\alpha_1 e_1 + \dots + \alpha_n e_n\|_0 = |\alpha_1| + \dots + |\alpha_n|$.
 (1)5. $\|\cdot\|_0$ is a norm.
 (2)1. If $\|v\|_0 = 0$ then $v = 0$
 PROOF: If $|\alpha_1| + \dots + |\alpha_n| = 0$ then $\alpha_1 = \dots = \alpha_n = 0$ so $\alpha_1 e_1 + \dots + \alpha_n e_n = 0$.
 (2)2. $\|\lambda v\|_0 = |\lambda| \|v\|_0$
 PROOF:

$$\begin{aligned} \|\lambda(\alpha_1 e_1 + \dots + \alpha_n e_n)\|_0 &= \|\lambda \alpha_1 e_1 + \dots + \lambda \alpha_n e_n\|_0 \\ &= |\lambda \alpha_1| + \dots + |\lambda \alpha_n| & (\langle 1 \rangle 4) \\ &= |\lambda|(|\alpha_1| + \dots + |\alpha_n|) \\ &= |\lambda| \|\alpha_1 e_1 + \dots + \alpha_n e_n\|_0 & (\langle 1 \rangle 4) \end{aligned}$$

 (2)3. $\|u + v\|_0 \leq \|u\|_0 + \|v\|_0$
 PROOF:

$$\begin{aligned} \|(\alpha_1 e_1 + \dots + \alpha_n e_n) + (\beta_1 e_1 + \dots + \beta_n e_n)\| &= |\alpha_1 + \beta_1| + \dots + |\alpha_n + \beta_n| \\ &\leq |\alpha_1| + \dots + |\alpha_n| + |\beta_1| + \dots + |\beta_n| \\ &= \|\alpha_1 e_1 + \dots + \alpha_n e_n\|_0 + \|\beta_1 e_1 + \dots + \beta_n e_n\|_0 \end{aligned}$$

 (1)6. Any norm on V is equivalent to $\|\cdot\|_0$.
 (2)1. LET: $\|\cdot\|$ be any norm on V .
 (2)2. PICK $\alpha > 0$ such that, for all $\alpha_1, \dots, \alpha_n \in K$, we have $\|\alpha_1 e_1 + \dots + \alpha_n e_n\| \geq \alpha(|\alpha_1| + \dots + |\alpha_n|)$
 PROOF: Proposition 17.0.13, (2)1, (1)3.
 (2)3. LET: $\beta = \max(\|e_1\|, \dots, \|e_n\|)$
 (2)4. $\beta > 0$
 PROOF: e_1, \dots, e_n cannot all be zero by (1)3.

⟨2⟩5. For all $x \in V$ we have $\alpha\|x\|_0 \leq \|x\| \leq \beta\|x\|_0$

⟨3⟩1. LET: $x \in V$

⟨3⟩2. $\alpha\|x\|_0 \leq \|x\|$

PROOF: ⟨1⟩3, ⟨1⟩4, ⟨2⟩2.

⟨3⟩3. $\|x\| \leq \beta\|x\|_0$

⟨4⟩1. LET: $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$

⟨4⟩2. Q.E.D.

PROOF:

$$\|x\| = \|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \quad (\langle 4 \rangle 1)$$

$$\leq |\alpha_1| \|e_1\| + \cdots + |\alpha_n| \|e_n\| \quad (\langle 2 \rangle 1)$$

$$\leq \beta(|\alpha_1| + \cdots + |\alpha_n|) \quad (\langle 2 \rangle 3)$$

$$= \beta\|x\|_0 \quad (\langle 1 \rangle 4)$$

⟨2⟩6. Q.E.D.

PROOF: Theorem 17.1.12, ⟨1⟩5, ⟨2⟩1, ⟨2⟩2, ⟨2⟩4, ⟨2⟩5.

□

Definition 17.1.14 (Open Ball). Let V be a normed space over K . Let $x \in V$. Let $r > 0$. The *open ball* with *centre* x and *radius* r is

$$B(x, r) := \{y \in V \mid \|y - x\| < r\} .$$

Definition 17.1.15 (Closed Ball). Let V be a normed space over K . Let $x \in V$. Let $r > 0$. The *closed ball* with *centre* x and *radius* r is

$$\overline{B}(x, r) := \{y \in V \mid \|y - x\| \leq r\} .$$

Definition 17.1.16 (Sphere). Let V be a normed space over K . Let $x \in V$. Let $r > 0$. The *sphere* with *centre* x and *radius* r is

$$S(x, r) := \{y \in V \mid \|y - x\| = r\} .$$

Definition 17.1.17 (Open Set). Let V be a normed space over K . A set $S \subseteq V$ is *open* iff, for all $x \in S$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S$.

Proposition 17.1.18. *Equivalent norms define the same set of open sets.*

PROOF:

⟨1⟩1. LET: V be a normed space over K .

⟨1⟩2. LET: $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent norms on V .

⟨1⟩3. PICK reals $\alpha, \beta > 0$ such that, for all $x \in V$, we have $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$

⟨1⟩4. LET: $S \subseteq V$

⟨1⟩5. If S is open under $\|\cdot\|_1$ then S is open under $\|\cdot\|_2$.

⟨2⟩1. ASSUME: S is open under $\|\cdot\|_1$.

⟨2⟩2. LET: $x \in S$

⟨2⟩3. PICK $\epsilon > 0$ such that $\{y \in V \mid \|x - y\|_1 < \epsilon\} \subseteq S$.

⟨2⟩4. LET: $\delta = \alpha\epsilon$

- $\langle 2 \rangle 5. \{y \in V \mid \|x - y\|_2 < \delta\} \subseteq S$
 $\langle 1 \rangle 6. \text{ If } S \text{ is open under } \|\cdot\|_2 \text{ then } S \text{ is open under } \|\cdot\|_1.$

PROOF: Similar.

□

Proposition 17.1.19. *Every open ball is open.*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } V \text{ be a normed space over } K.$

$\langle 1 \rangle 2. \text{ LET: } c \in V \text{ and } r > 0$

PROVE: $B(c, r)$ is open.

$\langle 1 \rangle 3. \text{ LET: } x \in B(c, r)$

$\langle 1 \rangle 4. \text{ LET: } \epsilon = r - \|x - c\|$

PROVE: $B(x, \epsilon) \subseteq B(c, r)$

$\langle 1 \rangle 5. \text{ LET: } y \in B(x, \epsilon)$

PROVE: $y \in B(c, r)$

$\langle 1 \rangle 6. \|y - c\| < r$

PROOF:

$$\begin{aligned} \|y - c\| &\leq \|y - x\| + \|x - c\| && \text{(Triangle Inequality)} \\ &< \epsilon + \|x - c\| && (\langle 1 \rangle 5) \\ &= r && (\langle 1 \rangle 4) \end{aligned}$$

□

Proposition 17.1.20. *Let Ω be a closed bounded set in \mathbb{R}^n . Let $f \in \mathcal{C}(\Omega)$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega. g(x) < f(x)\}$ is open.*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } g \in U$

$\langle 1 \rangle 2. \text{ LET: } \epsilon = \max_{x \in \Omega} (f(x) - g(x))$

PROVE: $B(g, \epsilon) \subseteq U$

$\langle 1 \rangle 3. \epsilon > 0$

$\langle 1 \rangle 4. \text{ LET: } h \in B(g, \epsilon/2)$

PROVE: $h \in U$

$\langle 1 \rangle 5. \text{ LET: } x \in \Omega$

$\langle 1 \rangle 6. h(x) < f(x)$

PROOF:

$$\begin{aligned} h(x) &\leq g(x) + \epsilon/2 && (\langle 1 \rangle 4) \\ &< g(x) + \epsilon && (\langle 1 \rangle 3) \\ &\leq f(x) && (\langle 1 \rangle 2) \end{aligned}$$

□

Proposition 17.1.21. *Let Ω be a closed bounded set in \mathbb{R}^n . Let $f \in \mathcal{C}(\Omega)$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega. g(x) > f(x)\}$ is open.*

PROOF: Given $g \in U$, let $\epsilon = \max_x (g(x) - f(x))/2$. Then $B(g, \epsilon) \subseteq U$. □

Proposition 17.1.22. *Let Ω be a closed bounded set in \mathbb{R}^n . Let $f \in \mathcal{C}(\Omega)$ be such that $f(x) > 0$ for all $x \in \Omega$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega. |g(x)| < f(x)\}$ is open.*

PROOF: Given $g \in U$, let $\epsilon = \max_x (f(x) - |g(x)|)/2$. Then $B(g, \epsilon) \subseteq U$. \square

Proposition 17.1.23. *Let Ω be a closed bounded set in \mathbb{R}^n . Let $f \in \mathcal{C}(\Omega)$ be such that $f(x) > 0$ for all $x \in \Omega$. Then $U = \{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega, |g(x)| > f(x)\}$ is open.*

PROOF: Given $g \in U$, let $\epsilon = \max_x (|g(x)| - f(x))/2$. Then $B(g, \epsilon) \subseteq U$. \square

Proposition 17.1.24. *The union of a set of open sets is open.*

PROOF:

- $\langle 1 \rangle 1$. LET: V be a normed space over K .
- $\langle 1 \rangle 2$. LET: \mathcal{U} be a set of open sets in V .
- $\langle 1 \rangle 3$. LET: $x \in \bigcup \mathcal{U}$
- $\langle 1 \rangle 4$. PICK $U \in \mathcal{U}$ such that $x \in U$.
- $\langle 1 \rangle 5$. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
- $\langle 1 \rangle 6$. $B(x, \epsilon) \subseteq \bigcup \mathcal{U}$

\square

Proposition 17.1.25. *The intersection of two open sets is open.*

PROOF:

- $\langle 1 \rangle 1$. LET: V be a normed space over K .
- $\langle 1 \rangle 2$. LET: U_1 and U_2 be open sets in V .
- $\langle 1 \rangle 3$. LET: $x \in U_1 \cap U_2$
- $\langle 1 \rangle 4$. PICK $\epsilon_1 > 0$ such that $B(x, \epsilon_1) \subseteq U_1$
- $\langle 1 \rangle 5$. PICK $\epsilon_2 > 0$ such that $B(x, \epsilon_2) \subseteq U_2$
- $\langle 1 \rangle 6$. LET: $\epsilon = \min(\epsilon_1, \epsilon_2)$
- $\langle 1 \rangle 7$. $B(x, \epsilon) \subseteq U_1 \cap U_2$

\square

Proposition 17.1.26. *In any normed space, \emptyset is open.*

PROOF: Vacuous. \square

Proposition 17.1.27. *In any normed space V , the whole space V is open.*

PROOF: For any $x \in V$ we have $B(x, 1) \subseteq V$. \square

Definition 17.1.28 (Closed Set). Let V be a normed space over K . A set $S \subseteq V$ is *closed* iff $V - S$ is open.

Proposition 17.1.29. *Every closed ball is closed.*

PROOF:

- $\langle 1 \rangle 1$. LET: V be a normed space over K .
- $\langle 1 \rangle 2$. LET: $c \in V$ and $r > 0$
 PROVE: $\overline{B}(c, r)$ is closed.
- $\langle 1 \rangle 3$. LET: $x \in V - \overline{B}(c, r)$
- $\langle 1 \rangle 4$. LET: $\epsilon = \|x - c\| - r$
 PROVE: $B(x, \epsilon) \subseteq V - \overline{B}(c, r)$

$\langle 1 \rangle 5. \epsilon > 0$

PROOF: Since $\|x - c\| > r$ by $\langle 1 \rangle 3$.

$\langle 1 \rangle 6. \text{ LET: } y \in B(x, \epsilon)$

$\langle 1 \rangle 7. \|y - c\| > r$

PROOF:

$$\begin{aligned} \|y - c\| &\geq \|x - c\| - \|x - y\| && \text{(Triangle Inequality)} \\ &> \|x - c\| - \epsilon && (\langle 1 \rangle 6) \\ &= r && (\langle 1 \rangle 4) \end{aligned}$$

□

Proposition 17.1.30. *The intersection of a set of closed sets is closed.*

PROOF: From Proposition 17.1.24. □

Proposition 17.1.31. *The union of two closed sets is closed.*

PROOF: From Proposition 17.1.25. □

Proposition 17.1.32. *Every sphere is closed.*

PROOF: $S(c, r) = \overline{B}(c, r) - B(c, r)$. □

Proposition 17.1.33. *Let Ω be a closed bounded set in \mathbb{R}^n . Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega, g(x) \leq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega, g(x) > f(x)\}$. □

Proposition 17.1.34. *Let Ω be a closed bounded set in \mathbb{R}^n . Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega, g(x) \geq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega, g(x) < f(x)\}$. □

Proposition 17.1.35. *Let Ω be a closed bounded set in \mathbb{R}^n . Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega, |g(x)| \leq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega, |g(x)| > f(x)\}$. □

Proposition 17.1.36. *Let Ω be a closed bounded set in \mathbb{R}^n . Let $f \in \mathcal{C}(\Omega)$. Then $\{g \in \mathcal{C}(\Omega) \mid \forall x \in \Omega, |g(x)| \geq f(x)\}$ is closed.*

PROOF: It is $\mathcal{C}(\Omega) - \{g \mid \forall x \in \Omega, |g(x)| < f(x)\}$. □

Proposition 17.1.37. *Let Ω be a closed bounded set in \mathbb{R}^n . Let $x_0 \in \Omega$ and $\lambda \in \mathbb{C}$. Then $C = \{g \in \mathcal{C}(\Omega) \mid g(x_0) = \lambda\}$ is closed.*

PROOF: Given $g \in \mathcal{C}(\Omega) - C$, let $\epsilon = |g(x_0) - \lambda|/2$. Then $B(g, \epsilon) \subseteq \mathcal{C}(\Omega) - C$. □

Proposition 17.1.38. *In any normed space V , we have \emptyset is closed.*

PROOF: Since $V - \emptyset = V$ is open. □

Proposition 17.1.39. *In any normed space V , the whole space V is closed.*

PROOF: Since $V - V = \emptyset$ is open. \square

Theorem 17.1.40. *Let V be a normed space over K . Let S be a subset of V . Then S is closed if and only if, for any sequence (x_n) in S , if $x_n \rightarrow l$ as $n \rightarrow \infty$ then $l \in S$.*

PROOF:

- (1)1. If S is closed then, for any sequence (x_n) in S , if $x_n \rightarrow l$ as $n \rightarrow \infty$ then $l \in S$.
 - (2)1. ASSUME: S is closed.
 - (2)2. LET: (x_n) be a sequence in S .
 - (2)3. ASSUME: $x_n \rightarrow l$ as $n \rightarrow \infty$.
 - (2)4. ASSUME: for a contradiction $l \notin S$.
 - (2)5. PICK $\epsilon > 0$ such that $B(l, \epsilon) \subseteq V - S$
 - (2)6. PICK N such that $\forall n \geq N. x_n \in B(l, \epsilon)$
 - (2)7. $x_N \in V - S$
 - (2)8. This contradicts (2)2.
- (1)2. If, for any sequence (x_n) in S , if $x_n \rightarrow l$ as $n \rightarrow \infty$ then $l \in S$, then S is closed.
 - (2)1. ASSUME: for any sequence (x_n) in S , if $x_n \rightarrow l$ as $n \rightarrow \infty$ then $l \in S$.
 - (2)2. LET: $x \in V - S$
 - (2)3. ASSUME: for a contradiction there is no $\epsilon > 0$ such that $B(x, \epsilon) \subseteq V - S$.
 - (2)4. For $n \in \mathbb{Z}_+$, PICK $x_n \in B(x, 1/n) \cap S$
 - (2)5. $x_n \rightarrow x$ as $n \rightarrow \infty$
 - (2)6. $x \in S$
 - (2)7. This contradicts (2)2.

\square

Definition 17.1.41 (Closure). Let V be a normed space over K . Let S be a subset of V . The *closure* of S , $\text{cl } S$, is the intersection of the set of closed sets that include S .

Proposition 17.1.42. *Let V be a normed space over K . Let S be a subset of V . Then the closure of S is the smallest closed set that includes S .*

PROOF: Proposition 17.1.30. \square

Theorem 17.1.43. *Let V be a normed space over K . Let S be a subset of V . Then*

$$\text{cl } S = \{l \in V \mid \exists \text{ a sequence } (x_n) \text{ in } S. x_n \rightarrow l \text{ as } n \rightarrow \infty\} .$$

PROOF:

- (1)1. For all $l \in \text{cl } S$, there exists a sequence (x_n) in S such that $x_n \rightarrow l$ as $n \rightarrow \infty$.
 - (2)1. LET: $l \in \text{cl } S$
 - (2)2. For $n \in \mathbb{Z}_+$, pick $x_n \in B(l, 1/n) \cap S$
- PROOF: There must be such an x_n otherwise $S - B(l, 1/n)$ would be a smaller closed set that includes S .

- ⟨2⟩3. $x_n \rightarrow l$ as $n \rightarrow \infty$
 ⟨1⟩2. For any sequence (x_n) in S , if $x_n \rightarrow l$ as $n \rightarrow \infty$ then $l \in \text{cl } S$.
 PROOF: Theorem 17.1.40.

□

Definition 17.1.44 (Dense). Let V be a normed space over K . Let $S \subseteq V$. Then S is *dense* if and only if $\text{cl } S = V$.

Theorem 17.1.45 (Weierstrass Approximation Theorem). Let a and b be real numbers with $a < b$. In $\mathcal{C}([a, b])$, the set of polynomials is dense.

PROOF: TODO

Proposition 17.1.46. Let $p \geq 1$. The set of all sequences that have only finitely many non-zero terms is dense in l^p .

PROOF:

- ⟨1⟩1. LET: $(z_n) \in l^p$
 ⟨1⟩2. LET: $\epsilon > 0$
 PROVE: There exists a sequence (x_n) with only finitely many non-zero terms such that $(\sum_{n=1}^{\infty} |z_n - x_n|^p)^{1/p} < \epsilon$
 ⟨1⟩3. PICK N such that $|\sum_{n=1}^{\infty} |z_n|^p - \sum_{n=1}^N |z_n|^p| < \epsilon^p$
 ⟨1⟩4. LET: (x_n) be the sequence that agrees with (z_n) up to term N , and then zeros after that.
 ⟨1⟩5. $(\sum_{n=1}^{\infty} |z_n - x_n|^p)^{1/p} < \epsilon$

PROOF:

$$\left(\sum_{n=1}^{\infty} |z_n - x_n|^p \right)^{1/p} = \left(\sum_{n=N+1}^{\infty} |z_n|^p \right)^{1/p} < \epsilon \quad (\langle 1 \rangle 4)$$

$$< \epsilon \quad (\langle 1 \rangle 2)$$

□

Theorem 17.1.47. Let V be a normed space over K . Let $S \subseteq V$. Then the following are equivalent.

1. S is dense.
2. For all $l \in V$, there exists a sequence (x_n) in S such that $x_n \rightarrow l$ as $n \rightarrow \infty$.
3. Every nonempty open subset of V intersects S .

PROOF:

- ⟨1⟩1. $1 \Leftrightarrow 2$
 PROOF: Theorem 17.1.43.
 ⟨1⟩2. $1 \Rightarrow 3$
 ⟨2⟩1. ASSUME: S is dense.
 ⟨2⟩2. LET: U be a nonempty open subset of V .
 ⟨2⟩3. $X - U$ does not include S .

PROOF: Lest we have $\text{cl } S \subseteq X - U$.

$\langle 2 \rangle 4$. U intersects S .

$\langle 1 \rangle 3$. $3 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: Every nonempty subset of V intersects S .

$\langle 2 \rangle 2$. Every closed proper subset of V does not include S .

$\langle 2 \rangle 3$. $\text{cl } S = V$

□

Definition 17.1.48 (Compact). Let V be a normed space over K and $S \subseteq V$. Then S is *compact* if and only if every sequence in S has a convergent subsequence whose limit is in S .

Proposition 17.1.49. *In K^n , a set is compact if and only if it is bounded and closed.*

PROOF: TODO

Definition 17.1.50 (Bounded). Let V be a normed space over K and $S \subseteq V$. Then S is *bounded* iff there exists $r > 0$ such that $S \subseteq B(0, r)$.

Theorem 17.1.51. *Every compact set is closed and bounded.*

PROOF:

$\langle 1 \rangle 1$. LET: V be a normed space over K .

$\langle 1 \rangle 2$. LET: $S \subseteq V$ be compact.

$\langle 1 \rangle 3$. S is closed.

$\langle 2 \rangle 1$. LET: (x_n) be a sequence in S that converges to l

$\langle 2 \rangle 2$. PICK a sequence (x_{n_r}) that converges to $x \in S$

PROOF: $\langle 1 \rangle 2$, $\langle 2 \rangle 1$

$\langle 2 \rangle 3$. $x_{n_r} \rightarrow l$ as $n \rightarrow \infty$

PROOF: $\langle 2 \rangle 1$, $\langle 2 \rangle 2$

$\langle 2 \rangle 4$. $l = x$

PROOF: Proposition 17.1.2.

$\langle 2 \rangle 5$. $l \in S$

PROOF: $\langle 2 \rangle 2$, $\langle 2 \rangle 4$

$\langle 2 \rangle 6$. Q.E.D.

PROOF: Theorem 17.1.40.

$\langle 1 \rangle 4$. S is bounded.

$\langle 2 \rangle 1$. ASSUME: for a contradiction S is unbounded.

$\langle 2 \rangle 2$. For $n \in \mathbb{Z}_+$, PICK $x_n \in S - B(0, n)$

$\langle 2 \rangle 3$. PICK a convergent subsequence (x_{n_r}) that converges to l , say.

$\langle 2 \rangle 4$. PICK $N \in \mathbb{Z}_+$ such that $\|l\| < N$

$\langle 2 \rangle 5$. PICK r such that $n_r > N$ and $\|x_{n_r} - l\| < N - \|l\|$

$\langle 2 \rangle 6$. $\|x_{n_r}\| < N < n_r$

$\langle 2 \rangle 7$. This contradicts $\langle 2 \rangle 2$.

□

Proposition 17.1.52. *In $\mathcal{C}([0, 1])$, the closed ball $\overline{B}(0, 1)$ is closed and bounded but not compact.*

PROOF: The sequence of functions (x^n) is in $\overline{B}(0, 1)$ but has no convergent subsequence. \square

Theorem 17.1.53 (Riesz's Lemma). *Let V be a normed vector space over K . Let X be a closed proper subspace of V . Let $0 < \epsilon < 1$. Then there exists $x \in V$ such that $\|x\| = 1$ and $\forall y \in X, \|x - y\| \geq \epsilon$.*

PROOF:

$\langle 1 \rangle 1$. PICK $z \in V - X$

$\langle 1 \rangle 2$. LET: $d = \inf_{x \in X} \|z - x\|$

$\langle 1 \rangle 3$. $d > 0$

PROOF: Since X is closed, there exists $e > 0$ such that $B(z, d) \subseteq V - X$ and hence $\|z - x\| \geq d$ for all $x \in X$.

$\langle 1 \rangle 4$. PICK $x_0 \in X$ such that $d \leq \|z - x_0\| \leq d/\epsilon$

PROOF: One exists since d/ϵ is not a lower bound for $\{\|z - x\| \mid x \in X\}$.

$\langle 1 \rangle 5$. LET: $x = (z - x_0)/\|z - x_0\|$

$\langle 1 \rangle 6$. LET: $y \in X$

$\langle 1 \rangle 7$. $\|x - y\| \geq \epsilon$

PROOF:

$$\|x - y\| = \left\| \frac{z - x_0}{\|z - x_0\|} - y \right\| \quad (\langle 1 \rangle 5)$$

$$= \frac{1}{\|z - x_0\|} \|z - (x_0 + \|z - x_0\|y)\|$$

$$\geq \frac{1}{\|z - x_0\|} d \quad (\langle 1 \rangle 2)$$

$$\geq \epsilon \quad (\langle 1 \rangle 4)$$

\square

Theorem 17.1.54. *Let V be a normed space over K . Then V is finite dimensional if and only if $\overline{B}(0, 1)$ is compact.*

PROOF:

$\langle 1 \rangle 1$. If V is finite dimensional then $\overline{B}(0, 1)$ is compact.

$\langle 2 \rangle 1$. ASSUME: V is finite dimensional.

$\langle 2 \rangle 2$. PICK a basis $\{e_1, \dots, e_n\}$

$\langle 2 \rangle 3$. ASSUME: w.l.o.g. $\|\alpha_1 e_1 + \dots + \alpha_n e_n\| = |\alpha_1| + \dots + |\alpha_n|$

$\langle 2 \rangle 4$. LET: $(\alpha_{k1} e_1 + \dots + \alpha_{kn} e_n)$ be a sequence in $\overline{B}(0, 1)$

$\langle 2 \rangle 5$. PICK a convergent subsequence $(\alpha_{k_r, 1})$ of (α_{k1}) , a convergent subsequence $(\alpha_{k_r', 2})$ of $(\alpha_{k_r, 2})$, \dots , a convergent subsequence $(\alpha_{k_r'', n})$.

$\langle 2 \rangle 6$. $(\alpha_{k_r'', 1} e_1 + \dots + \alpha_{k_r'', n} e_n)$ converges.

$\langle 1 \rangle 2$. If V is infinite dimensional then $\overline{B}(0, 1)$ is not compact.

$\langle 2 \rangle 1$. ASSUME: V is infinite dimensional.

$\langle 2 \rangle 2$. Choose a sequence (x_n) with $\|x_n\| = 1$ and $\|x_m - x_n\| \geq 1/2$ for $m \neq n$

$\langle 3 \rangle 1$. ASSUME: x_1, \dots, x_n satisfy $\|x_i\| = 1$ and $\|x_i - x_j\| \geq 1/2$ for $i \neq j$

$\langle 3 \rangle 2$. PICK $x_{n+1} \in V$ such that $\|x_{n+1}\| = 1$ and for all $y \in \text{span}\{x_1, \dots, x_n\}$ we have $\|x_{n+1} - y\| \geq 1/2$

- ⟨4⟩1. $\text{span}\{x_1, \dots, x_n\}$ is closed.
 - ⟨5⟩1. LET: $S = \text{span}\{x_1, \dots, x_n\}$
 - ⟨5⟩2. LET: (a_n) be a sequence in S that converges to $a \in V$
PROVE: $a \in S$
 - ⟨5⟩3. (a_n) is a Cauchy sequence in V .
 - ⟨5⟩4. (a_n) is a Cauchy sequence in S .
 - ⟨5⟩5. A finite dimensional normed space is a Banach space.
 - ⟨5⟩6. S is complete.
 - ⟨5⟩7. $a \in S$
 - ⟨4⟩2. $\text{span}\{x_1, \dots, x_n\}$ is a proper subspace of V .
PROOF: ⟨2⟩1
 - ⟨4⟩3. Q.E.D.
PROOF: Riesz's Lemma.
 - ⟨2⟩3. ASSUME: for a contradiction (x_{n_r}) is a subsequence that converges to l
 - ⟨2⟩4. For all $r \in \mathbb{N}$, we have $\|x_{n_r} - l\| + \|x_{n_{r+1}} - l\| \geq 1/2$
 - ⟨2⟩5. This is a contradiction.
-

Proposition 17.1.55. *Let V be a normed space. The closure of a subspace of V is a subspace.*

PROOF:

- ⟨1⟩1. LET: U be a subspace of V
 - ⟨1⟩2. LET: $x, y \in \text{cl}U$ and $\alpha, \beta \in K$
 - ⟨1⟩3. PICK sequences $(x_n), (y_n)$ in U that converge to x and y respectively.
 - ⟨1⟩4. $\alpha x_n + \beta y_n \rightarrow \alpha x + \beta y$ as $n \rightarrow \infty$
 - ⟨1⟩5. $\alpha x + \beta y \in \text{cl}U$
-

17.2 Continuous Linear Mappings

Definition 17.2.1 (Continuous). Let U and V be normed spaces. Let $f : U \rightarrow V$ and $x \in U$. Then f is *continuous at x* iff, for any sequence (x_n) in U , if $x_n \rightarrow x$ as $n \rightarrow \infty$ then $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$. The function f is *continuous* iff f is continuous at every point.

Proposition 17.2.2. *Let V be a normed space. Then the norm is a continuous function $V \rightarrow \mathbb{R}$.*

PROOF: From Proposition 17.0.4. □

Proposition 17.2.3. *Let U and V be normed space. Let $f : U \rightarrow V$. Then the following are equivalent.*

1. f is continuous.
2. For any open set S in V , we have $f^{-1}(S)$ is open in U .

3. For any closed set C in V , we have $f^{-1}(C)$ is closed in U .

Proposition 17.2.4. *Let U and V be normed spaces over K . Let $T : U \rightarrow V$ be a linear transformation. If T is continuous at some point, then it is continuous.*

PROOF:

- $\langle 1 \rangle 1$. ASSUME: T is continuous at u_0
- $\langle 1 \rangle 2$. LET: $x_n \rightarrow l$ as $n \rightarrow \infty$ in U
- $\langle 1 \rangle 3$. $x_n - l + u_0 \rightarrow u_0$ as $n \rightarrow \infty$.
- $\langle 1 \rangle 4$. $T(x_n - l + u_0) \rightarrow T(u_0)$ as $n \rightarrow \infty$.
- $\langle 1 \rangle 5$. $T(x_n) - T(l) + T(u_0) \rightarrow T(u_0)$ as $n \rightarrow \infty$.
- $\langle 1 \rangle 6$. $T(x_n) \rightarrow T(l)$ as $n \rightarrow \infty$.

□

Definition 17.2.5 (Bounded). Let U and V be normed spaces over K . Let $T : U \rightarrow V$ be a linear transformation. Then T is *bounded* iff there exists $\alpha > 0$ such that, for all $x \in U$, we have $\|T(x)\| \leq \alpha\|x\|$.

Theorem 17.2.6. *Let U and V be normed spaces over K . Let $T : U \rightarrow V$ be a linear transformation. Then T is continuous if and only if it is bounded.*

PROOF:

- $\langle 1 \rangle 1$. If T is continuous then T is bounded.
- $\langle 2 \rangle 1$. ASSUME: T is not bounded.
- $\langle 2 \rangle 2$. For $n \in \mathbb{Z}_+$, PICK $x_n \in U$ such that $\|T(x_n)\| > n\|x_n\|$.
- $\langle 2 \rangle 3$. For $n \in \mathbb{Z}_+$,
LET:

$$y_n = \frac{x_n}{n\|x_n\|}$$

- $\langle 2 \rangle 4$. $y_n \rightarrow 0$ as $n \rightarrow \infty$
- $\langle 2 \rangle 5$. $T(y_n) \not\rightarrow 0$ as $n \rightarrow \infty$
- $\langle 2 \rangle 6$. T is not continuous.
- $\langle 1 \rangle 2$. If T is bounded then T is continuous.
- $\langle 2 \rangle 1$. ASSUME: T is bounded.
- $\langle 2 \rangle 2$. PICK $\alpha > 0$ such that $\forall x \in U, \|T(x)\| \leq \alpha\|x\|$.
- $\langle 2 \rangle 3$. T is continuous at 0.
- $\langle 3 \rangle 1$. LET: (x_n) be a sequence that converges to 0 in U
- $\langle 3 \rangle 2$. $T(x_n) \rightarrow 0$ as $n \rightarrow \infty$

PROOF:

$$\begin{array}{ll} \|T(x_n)\| \leq \alpha\|x_n\| & (\langle 2 \rangle 2) \\ \rightarrow 0 & \text{as } n \rightarrow \infty \end{array}$$

- $\langle 2 \rangle 4$. T is continuous.

PROOF: Proposition 17.2.4.

□

Proposition 17.2.7. *Let U and V be normed spaces over K where U is finite dimensional. Let $T : U \rightarrow V$ be a linear transformation. Then T is bounded.*

PROOF:

- ⟨1⟩1. PICK a basis $\{e_1, \dots, e_n\}$ of unit vectors for U .
 ⟨1⟩2. LET: $M = \max(\|T(e_1)\|, \dots, \|T(e_n)\|)$
 ⟨1⟩3. PICK $C > 0$ such that, for all $\alpha_1, \dots, \alpha_n \in K$, we have $|\alpha_1| + \dots + |\alpha_n| \leq C\|\alpha_1 e_1 + \dots + \alpha_n e_n\|$

PROOF: Theorem 17.1.13.

- ⟨1⟩4. LET: $x \in U$
 PROVE: $\|T(x)\| \leq CM\|x\|$
 ⟨1⟩5. LET: $x = \alpha_1 e_1 + \dots + \alpha_n e_n$
 ⟨1⟩6. $\|T(x)\| \leq CM\|x\|$

PROOF:

$$\begin{aligned}
 \|T(x)\| &= \|\alpha_1 T(e_1) + \dots + \alpha_n T(e_n)\| && (T \text{ linear}) \\
 &\leq |\alpha_1| \|T(e_1)\| + \dots + |\alpha_n| \|T(e_n)\| && (\text{Triangle inequality}) \\
 &\leq M(|\alpha_1| + \dots + |\alpha_n|) && (\langle 1 \rangle 2) \\
 &\leq CM\|x\| && (\langle 1 \rangle 3)
 \end{aligned}$$

□

Corollary 17.2.7.1. *Let U and V be normed spaces over K where U is finite dimensional. Let $T : U \rightarrow V$ be a linear transformation. Then T is continuous.*

Proposition 17.2.8. *Let U and V be normed spaces over K . Let $T : U \rightarrow V$ be a linear transformation. If T is continuous, then T is uniformly continuous.*

PROOF:

- ⟨1⟩1. ASSUME: T is continuous
 ⟨1⟩2. PICK $B > 0$ such that $\forall x \in U. \|T(x)\| \leq B\|x\|$
 ⟨1⟩3. LET: $\epsilon > 0$
 ⟨1⟩4. LET: $\delta = \epsilon/B$
 ⟨1⟩5. LET: $x, y \in U$
 ⟨1⟩6. ASSUME: $\|x - y\| < \delta$
 ⟨1⟩7. $\|T(x) - T(y)\| < \epsilon$

PROOF:

$$\begin{aligned}
 \|T(x) - T(y)\| &= \|T(x - y)\| && (T \text{ linear}) \\
 &\leq B\|x - y\| && (\langle 1 \rangle 2) \\
 &< B\delta && (\langle 1 \rangle 6) \\
 &= \epsilon && (\langle 1 \rangle 4)
 \end{aligned}$$

□

Proposition 17.2.9. *Let U and V be normed spaces over K . The set $\mathcal{B}(U, V)$ of all bounded linear maps from U to V forms a subspace of the space of all linear maps from U to V .*

PROOF:

- ⟨1⟩1. LET: $S, T : U \rightarrow V$ be bounded linear maps and $\alpha, \beta \in K$.
 PROVE: $\alpha S + \beta T$ is bounded.
 ⟨1⟩2. PICK $B, C > 0$ such that $\forall x \in U. \|S(x)\| \leq B\|x\|$ and $\|T(x)\| \leq C\|x\|$
 ⟨1⟩3. $\forall x \in U. \|(\alpha S + \beta T)(x)\| \leq (|\alpha|B + |\beta|C)\|x\|$

□

Proposition 17.2.10. *Let U and V be normed spaces over K . Define the operator norm $\| \cdot \|$ on $\mathcal{B}(U, V)$ by $\|T\| := \sup\{\|T(x)\| \mid x \in U, \|x\| = 1\}$. Then $\| \cdot \|$ is a norm on $\mathcal{B}(U, V)$.*

PROOF:

⟨1⟩1. For all $T \in \mathcal{B}(U, V)$, the set $\{\|T(x)\| \mid x \in U, \|x\| = 1\}$ is bounded above.

⟨2⟩1. LET: $T \in \mathcal{B}(U, V)$

⟨2⟩2. PICK B such that $\forall x \in U. \|T(x)\| \leq B\|x\|$.

⟨2⟩3. B is an upper bound for $\{\|T(x)\| \mid x \in U, \|x\| = 1\}$.

⟨1⟩2. If $\|T\| = 0$ then $T = 0$.

⟨2⟩1. ASSUME: $\|T\| = 0$

⟨2⟩2. LET: $x \in U$

PROVE: $T(x) = 0$

⟨2⟩3. ASSUME: w.l.o.g. $\|x\| \neq 0$

⟨2⟩4. LET: $y = x/\|x\|$

⟨2⟩5. $\|y\| = 1$

⟨2⟩6. $\|T(y)\| = 0$

⟨2⟩7. $T(y) = 0$

⟨2⟩8. $T(x) = 0$

⟨1⟩3. For all $\lambda \in K$ and $T \in \mathcal{B}(U, V)$, we have $\|\lambda T\| = |\lambda|\|T\|$

⟨2⟩1. LET: $\lambda \in K$ and $T \in \mathcal{B}(U, V)$

⟨2⟩2. $\|\lambda T\| = |\lambda|\|T\|$

PROOF:

$$\begin{aligned} \|\lambda T\| &= \sup\{\|\lambda T(x)\| \mid x \in U, \|x\| = 1\} \\ &= \sup\{|\lambda|\|T(x)\| \mid x \in U, \|x\| = 1\} \\ &= |\lambda| \sup\{\|T(x)\| \mid x \in U, \|x\| = 1\} \\ &= |\lambda|\|T\| \end{aligned}$$

⟨1⟩4. For all $S, T \in \mathcal{B}(U, V)$, we have $\|S + T\| \leq \|S\| + \|T\|$.

⟨2⟩1. LET: $S, T \in \mathcal{B}(U, V)$

⟨2⟩2. $\|S + T\| \leq \|S\| + \|T\|$

PROOF:

$$\begin{aligned} \|S + T\| &= \sup\{\|S(x) + T(x)\| \mid x \in U, \|x\| = 1\} \\ &\leq \sup\{\|S(x)\| + \|T(x)\| \mid x \in U, \|x\| = 1\} \\ &\leq \sup\{\|S(x)\| \mid x \in U, \|x\| = 1\} + \sup\{\|T(x)\| \mid x \in U, \|x\| = 1\} \\ &= \|S\| + \|T\| \end{aligned}$$

□

Proposition 17.2.11. *Let U and V be normed spaces. Let $T \in \mathcal{B}(U, V)$. Then $\|T\|$ is the least number such that $\forall u \in U. \|T(u)\| \leq \|T\|\|u\|$.*

PROOF:

⟨1⟩1. $\forall u \in U. \|T(u)\| \leq \|T\|\|u\|$

⟨2⟩1. LET: $u \in U$

⟨2⟩2. LET: $v = u/\|u\|$

- ⟨2⟩3. $\|T(v)\| \leq \|T\|$
- ⟨2⟩4. $\|T(u)\| \leq \|T\|\|u\|$
- ⟨1⟩2. If α satisfies $\forall u \in U. \|T(u)\| \leq \alpha\|u\|$, then $\|T\| \leq \alpha$
- ⟨2⟩1. ASSUME: $\forall u \in U. \|T(u)\| \leq \alpha\|u\|$
- ⟨2⟩2. For all $x \in U$, if $\|x\| = 1$ then $\|T(x)\| \leq \alpha$
- ⟨2⟩3. $\|T\| \leq \alpha$

□

Proposition 17.2.12. *Let V be a normed space. Then id_V is a bounded linear function $V \rightarrow V$, and $\|\text{id}_V\| = 1$.*

Proposition 17.2.13. *Let U and V be normed spaces. The constant zero function $U \rightarrow V$ is a bounded linear transformation with norm 0.*

Proposition 17.2.14. *Let $N \in \mathbb{N}$. Let $T : \mathbb{C}^N \rightarrow \mathbb{C}^N$ be a linear transformation with matrix $A = (a_{ij})$. Then T is bounded and*

$$\|T\| \leq \sqrt{\sum_{i=1}^N \sum_{j=1}^N |a_{ij}|^2}.$$

Definition 17.2.15 (Uniform Convergence). *Let U and V be normed spaces. Let (T_n) be a sequence in $\mathcal{B}(U, V)$ and $T \in \mathcal{B}(U, V)$. Then (T_n) converges uniformly to T iff (T_n) converges to T under the standard norm defined above.*

Theorem 17.2.16. *Let U and V be normed spaces. Let $T : U \rightarrow V$ be a continuous linear function. Then $\ker T$ is a closed subspace of U .*

PROOF:

⟨1⟩1. $\ker T$ is a subspace of U

PROOF: If $x, y \in \ker T$ and $\alpha, \beta \in K$ then $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) = 0$.

⟨1⟩2. $\ker T$ is closed.

PROOF: Let (x_n) be a sequence in $\ker T$ and $x_n \rightarrow l$. Then $T(l) = \lim_{n \rightarrow \infty} T(x_n) = 0$.

□

Theorem 17.2.17. *Let U and V be normed spaces. Let W be a closed subspace of U and $T : W \rightarrow V$ be a continuous linear mapping. Then the graph $G = \{(x, T(x)) \mid x \in W\}$ is closed in $U \times V$.*

PROOF:

⟨1⟩1. ASSUME: w.l.o.g. $T \neq 0$

⟨1⟩2. LET: $(x, y) \in (U \times V) - G$, i.e. $y \neq T(x)$

⟨1⟩3. LET: $\epsilon = \|y - T(x)\| > 0$

⟨1⟩4. LET: $x' \in W$ with $\|x - x'\| < \epsilon/3\|T\|$

⟨1⟩5. LET: $y' \in V$ with $\|y - y'\| < \epsilon/3$

⟨1⟩6. $y' \neq T(x')$

PROOF:

$$\begin{aligned}\|y' - T(x')\| &\geq \|y - T(x)\| - \|y - y'\| - \|T(x) - T(x')\| \\ &> \epsilon/3 \\ &> 0\end{aligned}$$

□

Theorem 17.2.18 (Diagonal Theorem). *Let E be a normed space over K . Let (x_{ij}) be an infinite matrix of elements of V . If:*

1. *For all $j \in \mathbb{Z}_+$, we have $x_{ij} \rightarrow 0$ as $i \rightarrow \infty$;*
2. *Every increasing sequence of positive integers (p_j) has a subsequence (p_{j_r}) such that*

$$\sum_{s=1}^{\infty} x_{p_{j_r} p_{j_s}} \rightarrow 0 \text{ as } r \rightarrow \infty$$

then $x_{ii} \rightarrow 0$ as $i \rightarrow \infty$.

PROOF:

- (1)1. ASSUME: for a contradiction $x_{ii} \not\rightarrow 0$ as $i \rightarrow \infty$
- (1)2. PICK $\epsilon > 0$ such that, for all N , there exists $n \geq N$ such that $\|x_{nn}\| \geq \epsilon$
- (1)3. PICK an increasing sequence of integers (p_j) such that $\|x_{p_j p_j}\| \geq \epsilon$ for all j .
- (1)4. PICK a subsequence (q_i) such that $\sum_{j=1}^{\infty} x_{q_i q_j} \rightarrow 0$ as $i \rightarrow \infty$
- (1)5. For all i , we have $x_{q_i q_j} \rightarrow 0$ as $j \rightarrow \infty$
- (1)6. For all j , we have $x_{q_i q_j} \rightarrow 0$ as $i \rightarrow \infty$
- (1)7. Define a subsequence (r_n) of (q_i) by $r_1 = q_1$ and, for all n , r_{n+1} is the first entry such that $r_{n+1} > r_n$, $\|x_{q_i r_n}\| < \epsilon/2^{n+1}$ for all $q_i \geq r_{n+1}$, and $\|x_{r_j r_{n+1}}\| < \epsilon/2^{n+2}$ for $j = 1, \dots, n$.
- (1)8. $\|x_{r_i r_j}\| < \epsilon/2^{j+1}$ for all i, j such that $i \neq j$
- (1)9. PICK a subsequence (s_j) of (r_j) such that $\sum_{j=1}^{\infty} x_{s_i s_j} \rightarrow 0$ as $i \rightarrow \infty$
- (1)10. For all i we have $\|\sum_{j=1}^{\infty} x_{s_i s_j}\| \geq \epsilon/2$

PROOF:

$$\begin{aligned}\left\| \sum_{j=1}^{\infty} x_{s_i s_j} \right\| &= \left\| x_{s_i s_i} + \sum_{i \neq j} x_{s_i s_j} \right\| \\ &\geq \left| \|x_{s_i s_i}\| - \left\| \sum_{i \neq j} x_{s_i s_j} \right\| \right| \quad (\text{Proposition 17.0.4}) \\ &\geq \left| \|x_{s_i s_i}\| - \sum_{i \neq j} \|x_{s_i s_j}\| \right| \\ &\geq \epsilon/2 \quad ((1)2, (1)8)\end{aligned}$$

(1)11. Q.E.D.

PROOF: (1)9 and (1)10 form a contradiction.

□

17.3 Banach Spaces

Definition 17.3.1 (Cauchy Sequence). Let V be a normed space over K . A *Cauchy sequence* is a sequence of points (x_n) such that, for every $\epsilon > 0$, there exists N such that $\forall m, n \geq N, \|x_m - x_n\| < \epsilon$.

Theorem 17.3.2. Let V be a normed space over K . Let (x_n) be a sequence in V . The following are equivalent.

1. (x_n) is Cauchy.
2. For every pair of increasing sequences of positive integers (p_n) and (q_n) , we have $\|x_{p_n} - x_{q_n}\| \rightarrow 0$ as $n \rightarrow \infty$.
3. For every increasing sequence of positive integers (p_n) , we have $\|x_{p_n} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

- $\langle 2 \rangle 1$. ASSUME: (x_n) is Cauchy.
 - $\langle 2 \rangle 2$. LET: (p_n) and (q_n) are increasing sequences of positive integers.
 - $\langle 2 \rangle 3$. LET: $\epsilon > 0$
 - $\langle 2 \rangle 4$. PICK N such that $\forall m, n \geq N, \|x_m - x_n\| < \epsilon$
 - $\langle 2 \rangle 5$. $\forall n \geq N, \|x_{p_n} - x_{q_n}\| < \epsilon$
- PROOF: Since $p_n, q_n \geq n \geq N$.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: Trivial.

$\langle 1 \rangle 3. 2 \Rightarrow 1$

- $\langle 2 \rangle 1$. ASSUME: (x_n) is not Cauchy
- $\langle 2 \rangle 2$. Pick $\epsilon > 0$ such that, for every $N \in \mathbb{Z}_+$, there exist $m_N, n_N \geq N$ such that $\|x_{m_N} - x_{n_N}\| \geq \epsilon$
- $\langle 2 \rangle 3$. ASSUME: w.l.o.g. (m_N) and (n_N) are increasing sequences.
- $\langle 2 \rangle 4$. $\|x_{m_N} - x_{n_N}\| \not\rightarrow 0$ as $N \rightarrow \infty$.

$\langle 1 \rangle 4. 3 \Rightarrow 2$

- $\langle 2 \rangle 1$. ASSUME: 3
- $\langle 2 \rangle 2$. LET: (p_n) and (q_n) be increasing sequences of positive integers.
- $\langle 2 \rangle 3$. LET: $\epsilon > 0$
- $\langle 2 \rangle 4$. PICK N such that $\forall n \geq N, \|x_{p_n} - x_n\| < \epsilon/2$ and $\forall n \geq N, \|x_{q_n} - x_n\| < \epsilon/2$
- $\langle 2 \rangle 5$. $\forall n \geq N, \|x_{p_n} - x_{q_n}\| < \epsilon$

□

Proposition 17.3.3. Every convergent sequence is Cauchy.

PROOF:

- $\langle 1 \rangle 1$. LET: $x_n \rightarrow l$ as $n \rightarrow \infty$.
- $\langle 1 \rangle 2$. LET: $\epsilon > 0$
- $\langle 1 \rangle 3$. PICK N such that $\forall n \geq N, \|x_n - l\| < \epsilon/2$

⟨1⟩4. For all $m, n \geq N$ we have $\|x_m - x_n\| < \epsilon$.
 \square

Proposition 17.3.4. *In $\mathcal{P}([0, 1])$, let*

$$P_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} .$$

Then (P_n) is Cauchy but does not converge.

PROOF: It converges to e^x in $\mathcal{C}([0, 1])$, therefore it is Cauchy in $\mathcal{C}([0, 1])$, hence Cauchy in $\mathcal{P}([0, 1])$. Since $e^x \notin \mathcal{P}([0, 1])$, it does not converge in that space. \square

Proposition 17.3.5. *Let V be a normed space over K . Let (x_n) be a Cauchy sequence in V . Then $(\|x_n\|)$ converges in \mathbb{R} .*

PROOF:

⟨1⟩1. $(\|x_n\|)$ is Cauchy.

⟨2⟩1. LET: $\epsilon > 0$

⟨2⟩2. PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$

⟨2⟩3. $\forall m, n \geq N. \|\|x_m\| - \|x_n\|\| < \epsilon$

PROOF: Proposition 17.0.4.

⟨1⟩2. Q.E.D.

PROOF: Since every Cauchy sequence in \mathbb{R} converges.

\square

Proposition 17.3.6. *Every Cauchy sequence is bounded.*

PROOF:

⟨1⟩1. LET: V be a normed space over K .

⟨1⟩2. LET: (x_n) be a Cauchy sequence in V .

⟨1⟩3. PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < 1$.

⟨1⟩4. LET: $B = \max(\|x_1\|, \dots, \|x_{N-1}\|, \|x_N\|) + 1$

⟨1⟩5. $\forall n. \|x_n\| \leq B$

\square

Definition 17.3.7 (Banach Space). A normed space V over K is *complete* or a *Banach space* iff every Cauchy sequence converges.

Proposition 17.3.8. *l^2 is complete.*

PROOF:

⟨1⟩1. LET: (a_n) be a Cauchy sequence in l^2 where $a_n = (\alpha_{n1}, \alpha_{n2}, \dots)$.

⟨1⟩2. For all $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}_+$ such that $\forall m, n \geq n_0. \sum_{k=1}^{\infty} |\alpha_{mk} - \alpha_{nk}|^2 < \epsilon^2$.

⟨1⟩3. For every $k \in \mathbb{Z}_+$ and $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}_+$ such that $\forall m, n \geq n_0. |\alpha_{mk} - \alpha_{nk}| < \epsilon$.

⟨1⟩4. For every $k \in \mathbb{Z}_+$, (α_{nk}) is Cauchy in \mathbb{C} .

⟨1⟩5. For every $k \in \mathbb{Z}_+$, (α_{nk}) converges in \mathbb{C} .

⟨1⟩6. For $k \in \mathbb{Z}_+$,

- LET: $\alpha_k = \lim_{n \rightarrow \infty} \alpha_{nk}$
- $\langle 1 \rangle 7$. Let a be the sequence (α_k)
- $\langle 1 \rangle 8$. For all $\epsilon > 0$, there exists n_0 such that $\forall n \geq n_0. \sum_{k=1}^{\infty} |\alpha_k - \alpha_{nk}|^2 \leq \epsilon^2$.
- PROOF: Letting $m \rightarrow \infty$ in $\langle 1 \rangle 2$.
- $\langle 1 \rangle 9$. $a \in l^2$
- $\langle 2 \rangle 1$. PICK n_0 such that $\forall n \geq n_0. \sum_{k=1}^{\infty} |\alpha_k - \alpha_{nk}|^2 \leq 1$
- PROOF: $\langle 1 \rangle 8$
- $\langle 2 \rangle 2$. $(\alpha_k - \alpha_{n_0 k}) \in l^2$
- $\langle 2 \rangle 3$. $(\alpha_{n_0 k}) \in l^2$
- PROOF: By $\langle 1 \rangle 1$ since the sequence is a_{n_0} .
- $\langle 2 \rangle 4$. $(\alpha_k) \in l^2$
- PROOF: Proposition 15.0.2.
- $\langle 1 \rangle 10$. $a_n \rightarrow a$ as $n \rightarrow \infty$
- PROOF: By $\langle 1 \rangle 8$ since $\|a - a_n\| = \sqrt{\sum_{k=1}^{\infty} |\alpha_k - \alpha_{nk}|^2}$.
-

Proposition 17.3.9. *Let a and b be real numbers with $a < b$. The space $\mathcal{C}([a, b])$ is complete.*

PROOF:

- $\langle 1 \rangle 1$. LET: $X = [a, b]$
- $\langle 1 \rangle 2$. LET: (f_n) be a Cauchy sequence in $\mathcal{C}([a, b])$.
- $\langle 1 \rangle 3$. For all $\epsilon > 0$, there exists n_0 such that $\forall n, m \geq n_0. \|f_n - f_m\| < \epsilon$.
- $\langle 1 \rangle 4$. For all $\epsilon > 0$, there exists n_0 such that $\forall n, m \geq n_0. \forall x \in X. |f_n(x) - f_m(x)| < \epsilon$.
- $\langle 1 \rangle 5$. For all $x \in [a, b]$, $(f_n(x))$ is Cauchy.
- $\langle 1 \rangle 6$. Define $f : [a, b] \rightarrow \mathbb{C}$ by $f(x) = \lim_{n \rightarrow \infty} f_n(x)$.
- $\langle 1 \rangle 7$. For all $\epsilon > 0$, there exists n_0 such that $\forall n \geq n_0. \forall x \in X. |f_n(x) - f(x)| < \epsilon$
- PROOF: Letting $m \rightarrow \infty$ in $\langle 1 \rangle 4$.
- $\langle 1 \rangle 8$. f is continuous
- $\langle 2 \rangle 1$. LET: $x_0 \in X$
- $\langle 2 \rangle 2$. LET: $\epsilon > 0$
- $\langle 2 \rangle 3$. PICK n_0 such that $\forall n \geq n_0. \forall x \in X. |f_n(x) - f(x)| < \epsilon/3$
- PROOF: By $\langle 1 \rangle 7$.
- $\langle 2 \rangle 4$. PICK $\delta > 0$ such that $\forall x \in X. |x - x_0| < \delta \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \epsilon/3$
- PROOF: Since f_{n_0} is continuous.
- $\langle 2 \rangle 5$. LET: $x \in X$
- $\langle 2 \rangle 6$. ASSUME: $|x - x_0| < \delta$
- $\langle 2 \rangle 7$. $|f(x) - f(x_0)| < \epsilon$
- PROOF:
- $$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| \quad (\text{Triangle Inequality})$$
- $$< \epsilon/3 + \epsilon/3 + \epsilon/3 \quad (\langle 2 \rangle 3, \langle 2 \rangle 4)$$
- $$= \epsilon$$
- $\langle 1 \rangle 9$. (f_n) converges to f uniformly.
- PROOF: From $\langle 1 \rangle 7$
-

Definition 17.3.10 (Series). Let V be a normed space over K . A *convergent series* in V is a sequence (x_n) in V such that $(x_1 + \cdots + x_n)$ is a convergent sequence, in which case we write $\sum_{n=1}^{\infty} x_n$ for its limit.

Definition 17.3.11 (Absolutely Convergent Series). Let V be a normed space over K . An *absolutely convergent series* in V is a sequence (x_n) such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Proposition 17.3.12. In $\mathcal{P}([0, 1])$, the series $\sum_{n=0}^{\infty} x^n/n!$ is absolutely convergent but not convergent.

PROOF: Proposition 17.3.4. \square

Theorem 17.3.13. A normed space is complete if and only if every absolutely convergent series is convergent.

PROOF:

$\langle 1 \rangle 1$. LET: V be a normed space over K .

$\langle 1 \rangle 2$. If V is complete then every absolutely convergent series is convergent.

$\langle 2 \rangle 1$. ASSUME: V is complete.

$\langle 2 \rangle 2$. LET: $\sum_{n=1}^{\infty} a_n$ be absolutely convergent in V .

$\langle 2 \rangle 3$. For $n \in \mathbb{Z}_+$,
LET: $s_n = \sum_{k=1}^n a_k$

$\langle 2 \rangle 4$. (s_n) is Cauchy.

$\langle 3 \rangle 1$. LET: $\epsilon > 0$

$\langle 3 \rangle 2$. PICK k such that $\sum_{n=k+1}^{\infty} \|a_n\| < \epsilon$

$\langle 3 \rangle 3$. LET: $m > n > k$

$\langle 3 \rangle 4$. $\|s_m - s_n\| < \epsilon$

PROOF:

$$\begin{aligned} \|s_m - s_n\| &= \left\| \sum_{i=n+1}^m a_i \right\| && (\langle 2 \rangle 3, \langle 3 \rangle 3) \\ &\leq \sum_{i=n+1}^m \|a_i\| && (\text{Triangle inequality}) \\ &\leq \sum_{i=k+1}^{\infty} \|a_i\| \\ &< \epsilon && (\langle 3 \rangle 2, \langle 3 \rangle 3) \end{aligned}$$

$\langle 2 \rangle 5$. (s_n) converges.

$\langle 1 \rangle 3$. If every absolutely convergent series is convergent then V is complete.

$\langle 2 \rangle 1$. ASSUME: Every absolutely convergent series in V is convergent.

$\langle 2 \rangle 2$. LET: (a_n) be a Cauchy sequence in V .

$\langle 2 \rangle 3$. PICK a strictly increasing sequence of positive integers (p_n) such that
 $\forall k. \forall m, n \geq p_k. \|x_m - x_n\| < 2^{-k}$.

$\langle 2 \rangle 4$. $\sum_{k=1}^{\infty} (x_{p_{k+1}} - x_{p_k})$ is absolutely convergent.

PROOF:

$$\sum_{k=1}^{\infty} \|x_{p_{k+1}} - x_{p_k}\| < \sum_{k=1}^{\infty} 2^{-k} \quad (\langle 2 \rangle 3)$$

$$< \infty$$

$\langle 2 \rangle 5$. $\sum_{k=1}^{\infty} (x_{p_{k+1}} - x_{p_k})$ is convergent.

PROOF: $\langle 2 \rangle 1$, $\langle 2 \rangle 5$

$\langle 2 \rangle 6$. LET: $s = \sum_{k=1}^{\infty} (x_{p_{k+1}} - x_{p_k})$

$\langle 2 \rangle 7$. $x_{p_k} \rightarrow s + x_{p_1}$ as $k \rightarrow \infty$.

$\langle 3 \rangle 1$. $\sum_{i=1}^k (x_{p_{i+1}} - x_{p_i}) \rightarrow s$ as $k \rightarrow \infty$

PROOF: $\langle 2 \rangle 6$

$\langle 3 \rangle 2$. $x_{p_{k+1}} - x_{p_1} \rightarrow s$ as $k \rightarrow \infty$

$\langle 2 \rangle 8$. $x_n \rightarrow s + x_{p_1}$ as $k \rightarrow \infty$.

PROOF:

$\langle 3 \rangle 1$. LET: $\epsilon > 0$

$\langle 3 \rangle 2$. PICK N such that $\forall k \geq N, \|x_{p_k} - (s + x_{p_1})\| < \epsilon/2$ and $\forall m, n \geq N, \|x_m - x_n\| < \epsilon/2$

PROOF: $\langle 2 \rangle 2$, $\langle 2 \rangle 7$

$\langle 3 \rangle 3$. $\forall n \geq N, \|x_n - (s + x_{p_1})\| < \epsilon$

□

Theorem 17.3.14. *A closed vector subspace of a Banach space is a Banach space.*

PROOF:

$\langle 1 \rangle 1$. LET: V be a Banach space over K .

$\langle 1 \rangle 2$. LET: U be a closed vector subspace of V .

$\langle 1 \rangle 3$. LET: (a_n) be a Cauchy sequence in U .

$\langle 1 \rangle 4$. (a_n) is a Cauchy sequence in V .

$\langle 1 \rangle 5$. LET: $l = \lim_{n \rightarrow \infty} a_n$

$\langle 1 \rangle 6$. $l \in U$

PROOF: Theorem 17.1.40.

$\langle 1 \rangle 7$. $a_n \rightarrow l$ as $n \rightarrow \infty$ in U .

□

Definition 17.3.15 (Completion). Let V be a normed space over K . A *completion* of V consists of a normed space W over K and an injection $\phi : V \rightarrow W$ such that:

$$1. \forall x, y \in V, \forall \alpha, \beta \in K, \phi(\alpha x + \beta y) = \alpha \phi(x) + \beta \phi(y)$$

$$2. \forall x \in V, \|\phi(x)\| = \|x\|$$

3. $\phi(V)$ is dense in W .

4. W is complete.

Proposition 17.3.16. *Every normed space has a completion.*

PROOF:

- (1)1. LET: V be a normed space over K .
 (1)2. Let us say two Cauchy sequences $(x_n), (y_n)$ are *equivalent* iff $x_n - y_n \rightarrow 0$ as $n \rightarrow \infty$.
 (1)3. Equivalence is an equivalence relation on the set of Cauchy sequences.
 (1)4. LET: W be the set of Cauchy sequences in V quotiented by equivalence.
 (1)5. Define addition and multiplication on W by

$$\begin{aligned}
 [(x_n)] + [(y_n)] &= [(x_n + y_n)] \\
 \lambda[(x_n)] &= [(\lambda x_n)]
 \end{aligned}$$

- (1)6. Define a norm on W by $\|[(x_n)]\| = \lim_{n \rightarrow \infty} \|x_n\|$
 (1)7. Define $\phi : V \rightarrow W$ by $\phi(v) = [(v)]$.
 (1)8. ϕ is injective.
 (1)9. $\forall x, y \in V. \forall \alpha, \beta \in K. \phi(\alpha x + \beta y) = \alpha \phi(x) + \beta \phi(y)$
 (1)10. $\forall x \in V. \|\phi(x)\| = \|x\|$
 (1)11. $\phi(V)$ is dense in W .

- (2)1. LET: $[(a_n)] \in W$ and $\epsilon > 0$.

PROVE: $B([(a_n)], \epsilon)$ intersects $\phi(V)$.

- (2)2. PICK N such that $\forall m, n \geq N. \|a_m - a_n\| < \epsilon/2$

- (2)3. $\phi(a_N) \in B([(a_n)], \epsilon)$

PROOF:

$$\begin{aligned}
 \|[(a_n)] - \phi(a_N)\| &= \lim_{n \rightarrow \infty} \|a_n - a_N\| \\
 &\leq \epsilon/2 & (\langle 2 \rangle 2) \\
 &< \epsilon
 \end{aligned}$$

- (1)12. W is complete.

- (2)1. LET: (X_n) be a Cauchy sequence in W .

- (2)2. For $n \in \mathbb{Z}_+$, PICK $x_n \in V$ such that

$$\|\phi(x_n) - X_n\| < 1/n.$$

- (2)3. (x_n) is Cauchy in V .

- (3)1. LET: $\epsilon > 0$

- (3)2. PICK N such that $\forall m, n \geq N. \|X_n - X_m\| < \epsilon/3$ and $1/N < \epsilon/3$

- (3)3. LET: $m, n \geq N$

- (3)4. $\|x_m - x_n\| < \epsilon$

PROOF:

$$\begin{aligned}
 \|x_m - x_n\| &= \|\phi(x_m) - \phi(x_n)\| \\
 &\leq \|\phi(x_m) - X_m\| + \|X_m - X_n\| + \|X_n - \phi(x_n)\| \\
 &< \|X_m - X_n\| + 1/m + 1/n \\
 &< \epsilon
 \end{aligned}$$

- (2)4. LET: $X = [(x_n)]$

- (2)5. $X_n \rightarrow X$ as $n \rightarrow \infty$

PROOF:

$$\begin{aligned}
 \|X_n - X\| &\leq \|X_n - \phi(x_n)\| + \|\phi(x_n) - X\| \\
 &< \|\phi(x_n) - X\| + 1/n \\
 &\rightarrow 0
 \end{aligned}$$

as $n \rightarrow \infty$

□

Proposition 17.3.17. *Let U be a normed space and V a Banach space. Then $\mathcal{B}(U, V)$ is a Banach space.*

PROOF:

$\langle 1 \rangle 1$. LET: (T_n) be a Cauchy sequence in $\mathcal{B}(U, V)$

$\langle 1 \rangle 2$. For all $u \in U$, $(T_n(u))$ is a Cauchy sequence in V .

$\langle 2 \rangle 1$. LET: $u \in U$

$\langle 2 \rangle 2$. LET: $\epsilon > 0$

PROVE: $\exists N. \forall m, n \geq N. \|T_m(u) - T_n(u)\| < \epsilon$

$\langle 2 \rangle 3$. ASSUME: w.l.o.g. $u \neq 0$

$\langle 2 \rangle 4$. PICK N such that $\forall m, n \geq N. \|T_m - T_n\| < \epsilon/\|u\|$

$\langle 2 \rangle 5$. LET: $m, n \geq N$

$\langle 2 \rangle 6$. $\|T_m(u) - T_n(u)\| < \epsilon$

PROOF:

$$\|T_m(u) - T_n(u)\| \leq \|T_m - T_n\| \|u\| \quad (\text{Proposition 17.2.11})$$

$$< \epsilon$$

$\langle 1 \rangle 3$. Define $T : U \rightarrow V$ by $T(u) = \lim_{n \rightarrow \infty} T_n(u)$

$\langle 1 \rangle 4$. $T \in \mathcal{B}(U, V)$

$\langle 2 \rangle 1$. For all $x, y \in U$ and $\alpha, \beta \in K$ we have $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

$\langle 3 \rangle 1$. LET: $x, y \in U$

$\langle 3 \rangle 2$. LET: $\alpha, \beta \in K$

$\langle 3 \rangle 3$. $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

PROOF:

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_{n \rightarrow \infty} T_n(\alpha x + \beta y) \\ &= \lim_{n \rightarrow \infty} (\alpha T_n(x) + \beta T_n(y)) \\ &= \alpha T(x) + \beta T(y) \end{aligned}$$

$\langle 2 \rangle 2$. T is bounded.

$\langle 3 \rangle 1$. PICK N such that $\forall n \geq N. \|T_n - T\| < 1$

$\langle 3 \rangle 2$. PICK $B > 0$ such that $\forall x \in U. \|T_N(x)\| \leq B\|x\|$

$\langle 3 \rangle 3$. LET: $x \in U$

$\langle 3 \rangle 4$. $\|T(x)\| \leq (B + 1)\|x\|$

PROOF:

$$\begin{aligned} \|T(x)\| &\leq \|T_N(x) - T(x)\| + \|T_N(x)\| && (\text{Triangle inequality}) \\ &\leq \|T_N - T\| \|x\| + \|T_N\| \|x\| && (\text{Proposition 17.2.11}) \\ &< \|x\| + B\|x\| && (\langle 3 \rangle 1, \langle 3 \rangle 2) \\ &= (B + 1)\|x\| \end{aligned}$$

$\langle 1 \rangle 5$. $T_n \rightarrow T$ as $n \rightarrow \infty$

$\langle 2 \rangle 1$. LET: $\epsilon > 0$

$\langle 2 \rangle 2$. PICK N such that $\forall m, n \geq N. \|T_m - T_n\| < \epsilon/2$

$\langle 2 \rangle 3$. LET: $n \geq N$

PROVE: $\|T_n - T\| < \epsilon$

$\langle 2 \rangle 4$. LET: $x \in U$ with $\|x\| = 1$

$\langle 2 \rangle 5$. $\|T_n(x) - T(x)\| \leq \epsilon/2$

PROOF: Let $n \rightarrow \infty$ in $\langle 2 \rangle 2$.

□

Corollary 17.3.17.1. *For any normed space V over K , the space $\mathcal{B}(V, K)$ is a Banach space.*

Theorem 17.3.18. *Let U be a normed space and V a Banach space. Let W be a subspace of U . Let $T : W \rightarrow V$ be a continuous linear transformation. Then T has a unique extension to a continuous linear transformation $\text{cl } W \rightarrow V$.*

PROOF:

- (1)1. Define $S : \text{cl } W \rightarrow V$ by: $S(x) = \lim_{n \rightarrow \infty} T(x_n)$, where (x_n) is any sequence in W that converges to x .
- (2)1. For all $x \in \text{cl } W$, there exists a sequence (x_n) in W that converges to x .
PROOF: Theorem 17.1.43.
- (2)2. If (x_n) is a Cauchy sequence in W then $(T(x_n))$ is Cauchy in V .
 - (3)1. ASSUME: w.l.o.g. $T \neq 0$
 - (3)2. LET: (x_n) be a Cauchy sequence in W .
 - (3)3. PICK $B > 0$ such that $\forall x \in W. \|T(x)\| \leq B\|x\|$
 - (3)4. LET: $\epsilon > 0$
 - (3)5. PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon/\|T\|$
 - (3)6. LET: $m, n \geq N$
 - (3)7. $\|T(x_m) - T(x_n)\| < \epsilon$
- (2)3. If (x_n) and (y_n) are sequences in W that converge to the same element in $\text{cl } W$ then $(T(x_n))$ and $(T(y_n))$ have the same limit in V .
 - (3)1. ASSUME: w.l.o.g. $T \neq 0$
 - (3)2. ASSUME: $x_n \rightarrow l$ and $y_n \rightarrow l$ as $n \rightarrow \infty$
 - (3)3. LET: $T(x_n) \rightarrow a$ and $T(y_n) \rightarrow b$ as $n \rightarrow \infty$
 - (3)4. ASSUME: for a contradiction $a \neq b$
 - (3)5. LET: $\epsilon = \|a - b\|$
 - (3)6. PICK N such that $\forall n \geq N. \|x_n - l\| < \epsilon/3\|T\|$ and $\forall n \geq N. \|y_n - l\| < \epsilon/3\|T\|$
 - (3)7. $\forall n \geq N. \|T(x_n) - T(y_n)\| < 2\epsilon/3$
 - (3)8. $\|a - b\| \leq 2\epsilon/3$
 - (3)9. This contradicts (3)5.
- (1)2. S extends T
 - (2)1. LET: $w \in W$
 - (2)2. $w \rightarrow w$ as $n \rightarrow \infty$
 - (2)3. $T(w) \rightarrow T(w)$ as $n \rightarrow \infty$
 - (2)4. $S(w) = T(w)$
- (1)3. S is bounded.
 - (2)1. LET: $x \in \text{cl } W$
PROVE: $\|S(x)\| \leq \|T\|\|x\|$
 - (2)2. PICK a sequence (x_n) in W that converges to x .
 - (2)3. $\|T(x_n)\| \leq \|T\|\|x_n\|$ for all n .
 - (2)4. $\|S(x)\| \leq \|T\|\|x\|$
 - PROOF: Taking the limit as $n \rightarrow \infty$.
- (1)4. S is linear.

- ⟨2⟩1. LET: $x, y \in \text{cl } W$ and $\alpha, \beta \in K$
- ⟨2⟩2. PICK sequences (x_n) and (y_n) in W that converge to x and y .
- ⟨2⟩3. $T(\alpha x_n + \beta y_n) = \alpha T(x_n) + \beta T(y_n)$ for all n .
- ⟨2⟩4. $S(\alpha x + \beta y) = \alpha S(x) + \beta S(y)$

PROOF: Taking the limit as $n \rightarrow \infty$.

- ⟨1⟩5. S is unique.
- ⟨2⟩1. LET: S' be a continuous linear extension of S defined on $\text{cl } W$.
- ⟨2⟩2. LET: $x \in W$
- PROVE: $S(x) = S'(x)$
- ⟨2⟩3. PICK a sequence (x_n) in W that converges to x .
- ⟨2⟩4. $T(x_n) = S'(x_n) \rightarrow S'(x)$ as $n \rightarrow \infty$
- ⟨2⟩5. $S'(x) = S(x)$

□

Corollary 17.3.18.1. *Let U be a normed space and V a Banach space. Let W be a dense subspace of U . Let $T : W \rightarrow V$ be a continuous linear transformation. Then T has a unique extension to a continuous linear transformation $U \rightarrow V$.*

Definition 17.3.19 (Functional). Let V be a normed space over K . A *functional* on V is a bounded linear mapping $V \rightarrow K$. The *dual space* of V is the space $\mathcal{B}(V, K)$ of all functionals.

Theorem 17.3.20 (Banach-Steinhaus Theorem). *Let \mathcal{T} be a family of bounded linear mappings from a Banach space X into a normed space Y . If, for every $x \in X$, there exists a constant M_x such that $\forall T \in \mathcal{T}. \|T(x)\| \leq M_x$, then there exists a constant $M > 0$ such that $\forall T \in \mathcal{T}. \|T\| \leq M$.*

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction no such M exists.
- ⟨1⟩2. For $n \in \mathbb{Z}_+$, PICK $T_n \in \mathcal{T}$ such that $\|T_n\| > n2^n$.
- ⟨1⟩3. For $n \in \mathbb{Z}_+$, PICK $x_n \in X$ such that $\|x_n\| = 1$ and $\|T_n(x_n)\| > n2^n$.
- ⟨1⟩4. For $n \in \mathbb{Z}_+$,

$$\left\| \frac{1}{n} T_n \left(\frac{x_n}{2^n} \right) \right\| > 1 .$$

- ⟨1⟩5. For $i, j \in \mathbb{Z}_+$,
LET: $y_{ij} = \frac{1}{i} T_i \left(\frac{x_j}{2^j} \right)$.
 - ⟨1⟩6. For all $j \in \mathbb{Z}_+$, $y_{ij} \rightarrow 0$ as $i \rightarrow \infty$
 - ⟨2⟩1. LET: $j \in \mathbb{Z}_+$
 - ⟨2⟩2. PICK M such that $\forall T \in \mathcal{T}. \|T(x_j/2^j)\| \leq M$
 - ⟨2⟩3. For all i , $\|y_{ij}\| \leq M/i$
 - ⟨1⟩7. For any increasing sequence of positive integers (p_i) , we have $\sum_{j=1}^{\infty} y_{p_i p_j} \rightarrow 0$ as $i \rightarrow \infty$
 - ⟨2⟩1. LET: (p_i) be an increasing sequence of positive integers.
 - ⟨2⟩2. LET: $z = \sum_{j=1}^{\infty} x_{p_j} / 2^{p_j}$
- PROOF: This converges by Theorem 17.3.13.
- ⟨2⟩3. PICK C such that $\forall T \in \mathcal{T}. \|T(z)\| \leq C$
 - ⟨2⟩4. For all i , $\|\sum_{j=1}^{\infty} y_{p_i p_j}\| \leq C/p_i$.

PROOF:

$$\left\| \sum_{j=1}^{\infty} y_{p_i p_j} \right\| = \left\| \sum_{j=1}^{\infty} \frac{1}{p_i} T_{p_i} \left(\frac{x_{p_j}}{2^{p_j}} \right) \right\| \quad (\langle 1 \rangle 5)$$

$$= \frac{1}{p_i} \left\| T_{p_i} \left(\sum_{j=1}^{\infty} \frac{x_{p_j}}{2^{p_j}} \right) \right\| \quad (T_{p_i} \text{ continuous})$$

$$= \frac{1}{p_i} \|T_{p_i}(z)\| \quad (\langle 2 \rangle 2)$$

$$\leq \frac{C}{p_i} \quad (\langle 2 \rangle 3)$$

$$\langle 2 \rangle 5. \sum_{j=1}^{\infty} y_{p_i p_j} \rightarrow 0 \text{ as } i \rightarrow \infty$$

$$\langle 1 \rangle 8. y_{ii} \rightarrow 0 \text{ as } i \rightarrow \infty$$

PROOF: Diagonal Theorem, $\langle 1 \rangle 6$, $\langle 1 \rangle 7$. $\langle 1 \rangle 9$. Q.E.D.PROOF: $\langle 1 \rangle 4$ and $\langle 1 \rangle 8$ form a contradiction.

□

17.4 Contraction Mappings

Definition 17.4.1 (Contraction Mapping). Let E be a normed space over K . Let $A \subseteq E$. A function $f : A \rightarrow E$ is a *contraction (mapping)* iff there exists a real α such that $0 < \alpha < 1$ and

$$\forall x, y \in A. \|f(x) - f(y)\| \leq \alpha \|x - y\| .$$

Proposition 17.4.2. *Contraction mappings are uniformly continuous.*

PROOF:

 $\langle 1 \rangle 1$. LET: E be a normed space over K . $\langle 1 \rangle 2$. LET: $A \subseteq E$ $\langle 1 \rangle 3$. LET: $f : A \rightarrow E$ be a contraction mapping. $\langle 1 \rangle 4$. PICK α such that $0 < \alpha < 1$ and $\forall x, y \in A. \|f(x) - f(y)\| \leq \alpha \|x - y\|$. $\langle 1 \rangle 5$. LET: $\epsilon > 0$ $\langle 1 \rangle 6$. LET: $\delta = \epsilon/\alpha$ $\langle 1 \rangle 7$. For all $x, y \in A$, if $\|x - y\| < \delta$ then $\|f(x) - f(y)\| < \epsilon$.

□

Theorem 17.4.3 (Banach Fixed Point Theorem). Let E be a Banach space over K . Let F be a nonempty closed subset of E . Let $f : F \rightarrow F$ be a contraction mapping. Then there exists a unique $z \in F$ such that $f(z) = z$.

PROOF:

 $\langle 1 \rangle 1$. PICK α such that $0 < \alpha < 1$ and

$$\forall x, y \in F. \|f(x) - f(y)\| \leq \alpha \|x - y\| .$$

 $\langle 1 \rangle 2$. PICK $x_0 \in F$

⟨1⟩3. For $n \in \mathbb{Z}_+$,

LET: $x_n = f^n(x_0)$.

⟨1⟩4. (x_n) is a Cauchy sequence.

⟨2⟩1. For all $n \in \mathbb{Z}_+$ we have $\|x_{n+1} - x_n\| \leq \alpha^n \|x_1 - x_0\|$.

⟨2⟩2. For all $m, n \in \mathbb{Z}_+$ with $m < n$ we have $\|x_n - x_m\| < \alpha^m \|x_1 - x_0\| / (1 - \alpha)$.

PROOF:

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\| \quad (\text{Triangle inequality}) \\ &\leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m) \|x_1 - x_0\| \quad (\langle 2 \rangle 1) \\ &< \frac{\|x_1 - x_0\|}{1 - \alpha} \alpha^m \end{aligned}$$

⟨2⟩3. LET: $\epsilon > 0$

⟨2⟩4. PICK N such that $\alpha^N \|x_1 - x_0\| / (1 - \alpha) < \epsilon$

⟨2⟩5. For all $m, n \geq N$, we have $\|x_n - x_m\| < \epsilon$

⟨1⟩5. LET: $z = \lim_{n \rightarrow \infty} x_n$

⟨1⟩6. $f(z) = z$

PROOF:

$$\begin{aligned} f(z) &= f\left(\lim_{n \rightarrow \infty} x_n\right) \\ &= \lim_{n \rightarrow \infty} f(x_n) \quad (\text{Proposition 17.4.2}) \\ &= \lim_{n \rightarrow \infty} x_{n+1} \\ &= z \end{aligned}$$

⟨1⟩7. For any $w \in F$, if $f(w) = w$ then $w = z$.

⟨2⟩1. LET: $w \in F$

⟨2⟩2. ASSUME: $f(w) = w$

⟨2⟩3. $\|z - w\| \leq \alpha \|z - w\|$

PROOF: $\|z - w\| = \|f(z) - f(w)\| \leq \alpha \|z - w\|$

⟨2⟩4. $\|z - w\| = 0$

⟨2⟩5. $z = w$

□

Chapter 18

Inner Product Spaces

Definition 18.0.1 (Inner Product Space). Let E be a complex vector space. An *inner product* on E is a function $\langle \cdot, \cdot \rangle : E^2 \rightarrow \mathbb{C}$ such that, for all $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$, we have:

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
3. $\langle x, x \rangle \geq 0$
4. If $\langle x, x \rangle = 0$ then $x = 0$

An *inner product space* consists of a complex vector space V and an inner product on V .

Proposition 18.0.2. Let E be an inner product space. For any $x \in E$, we have $\langle x, x \rangle$ is real.

PROOF: Since $\langle x, x \rangle = \overline{\langle x, x \rangle}$. \square

Proposition 18.0.3.

$$\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle$$

Proposition 18.0.4.

$$\langle 0, y \rangle = \langle x, 0 \rangle = 0$$

Proposition 18.0.5. The function $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^n x_i \overline{y_i}$ is an inner product on \mathbb{C}^n .

Proposition 18.0.6. The function $\langle (x_n), (y_n) \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$ is an inner product on l^2 .

Proposition 18.0.7. The function $\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$ is an inner product on $\mathcal{C}([a, b])$.

Proposition 18.0.8. *Let $p > 1$ and $\Omega \subseteq \mathbb{R}^N$. Let $L^p(\Omega)$ be the set of all functions $f : \Omega \rightarrow \mathbb{C}$ such that $|f|^p$ is Lebesgue integrable.*

The function $\langle f, g \rangle = \int_{\Omega} f(x)g(x)dx$ is an inner product on $L^2(\Omega)$.

Proposition 18.0.9. *Let E_1 and E_2 be inner product spaces. Then the function $\langle (e_1, e_2), (e'_1, e'_2) \rangle = \langle e_1, e'_1 \rangle + \langle e_2, e'_2 \rangle$ is an inner product on $E_1 \times E_2$.*

Definition 18.0.10 (Norm). In an inner product space, define $\|x\| = \sqrt{\langle x, x \rangle}$.

Proposition 18.0.11 (Schwarz's Inequality). *In any inner product space,*

$$|\langle x, y \rangle| \leq \|x\| \|y\| .$$

Equality holds iff x and y are linearly dependent.

PROOF:

$\langle 1 \rangle 1$. ASSUME: w.l.o.g. $y \neq 0$

$\langle 1 \rangle 2$. $|\langle x, y \rangle| \leq \|x\| \|y\|$

$\langle 2 \rangle 1$. For all $\alpha \in \mathbb{C}$ we have $\langle x, x \rangle + \bar{\alpha} \langle x, y \rangle + \alpha \langle y, x \rangle + |\alpha|^2 \langle y, y \rangle$

PROOF: The right-hand side is $\langle x + \alpha y, x + \alpha y \rangle$.

$\langle 2 \rangle 2$. $\langle x, x \rangle \langle y, y \rangle - |\langle x, y \rangle|^2 \geq 0$

PROOF: Taking $\alpha = -\langle x, x \rangle / \langle y, y \rangle$ in $\langle 2 \rangle 1$.

$\langle 1 \rangle 3$. If $|\langle x, y \rangle| = \|x\| \|y\|$ then x and y are linearly dependent.

$\langle 2 \rangle 1$. ASSUME: $|\langle x, y \rangle| = \|x\| \|y\|$

$\langle 2 \rangle 2$. $\langle x, y \rangle \langle y, x \rangle = \langle x, x \rangle \langle y, y \rangle$

$\langle 2 \rangle 3$. $\langle y, y \rangle x - \langle x, x \rangle y = 0$

PROOF:

$$\begin{aligned} \langle \langle y, y \rangle x - \langle x, x \rangle y, \langle y, y \rangle x - \langle x, x \rangle y \rangle &= \langle y, y \rangle^2 \langle x, x \rangle - \langle y, y \rangle \langle y, x \rangle \langle x, y \rangle - \langle x, y \rangle \langle y, y \rangle \langle y, x \rangle + \langle x, y \rangle \langle y, x \rangle \langle x, x \rangle \\ &= 0 \end{aligned}$$

$\langle 1 \rangle 4$. If x and y are linearly dependent then $|\langle x, y \rangle| = \|x\| \|y\|$

$\langle 2 \rangle 1$. ASSUME: x and y are linearly dependent.

$\langle 2 \rangle 2$. LET: $y = \alpha x$

$\langle 2 \rangle 3$. $|\langle x, y \rangle| = \|x\| \|y\|$

PROOF:

$$\begin{aligned} |\langle x, y \rangle| &= |\langle x, \alpha x \rangle| \\ &= |\alpha| |\langle x, x \rangle| \\ &= |\alpha| \|x\|^2 \\ &= \|x\| \|\alpha x\| \\ &= \|x\| \|y\| \end{aligned}$$

□

Corollary 18.0.11.1 (Triangle Inequality). *In any inner product space,*

$$\|x + y\| \leq \|x\| + \|y\|$$

PROOF:

$$\begin{aligned}
 \|x + y\|^2 &= \langle x + y, x + y \rangle \\
 &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
 &\leq \langle x, x \rangle + 2|\langle x, y \rangle| + \langle y, y \rangle \\
 &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad (\text{Schwarz's Inequality}) \\
 &= (\|x\| + \|y\|)^2 \quad \square
 \end{aligned}$$

Corollary 18.0.11.2. *The norm in an inner product space is a norm.*

Theorem 18.0.12 (Parallelogram Law). *In any inner product space,*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

PROOF:

$$\begin{aligned}
 \langle 1 \rangle 1. \quad &\|x + y\|^2 = \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 \\
 \langle 1 \rangle 2. \quad &\|x - y\|^2 = \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2 \\
 \langle 1 \rangle 3. \quad &\text{Q.E.D.}
 \end{aligned}$$

PROOF: Add $\langle 1 \rangle 1$ and $\langle 1 \rangle 2$.

□

Proposition 18.0.13. *Let E be a normed space over \mathbb{C} . Then there exists an inner product on E that induces the norm of E iff E satisfies the Parallelogram Law.*

PROOF: If E satisfies the parallelogram law, define

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2) .$$

Definition 18.0.14 (Orthogonal). Vectors x and y in an inner product space are *orthogonal*, $x \perp y$, iff $\langle x, y \rangle = 0$.

Theorem 18.0.15 (Pythagorean Formula). *If x and y are orthogonal then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 .$$

Definition 18.0.16 (Weak Convergence). Let E be an inner product space. Let (x_n) be a sequence in E and $l \in E$. Then (x_n) *weakly converges* to l , $x_n \xrightarrow{w} l$ as $n \rightarrow \infty$, iff $\forall y \in E. \langle x_n, y \rangle \rightarrow \langle l, y \rangle$ as $n \rightarrow \infty$.

Proposition 18.0.17. *In any inner product space E , the inner product $\langle \cdot, \cdot \rangle : E^2 \rightarrow \mathbb{C}$ is continuous.*

PROOF:

$\langle 1 \rangle 1.$ LET: $x_n \rightarrow x$ and $y_n \rightarrow y$ in E .

$\langle 1 \rangle 2.$ $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$

PROOF:

$$\begin{aligned}
 |\langle x_n, y_n \rangle - \langle x, y \rangle| &\leq |\langle x_n, y_n \rangle - \langle x_n, y \rangle| + |\langle x_n, y \rangle - \langle x, y \rangle| \\
 &= |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle| \\
 &\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \quad (\text{Schwarz's Inequality}) \\
 &\rightarrow 0
 \end{aligned}$$

using the fact that (x_n) is bounded.

□

Theorem 18.0.18. $x_n \rightarrow l$ if and only if $x_n \xrightarrow{w} l$ and $\|x_n\| \rightarrow \|l\|$.

PROOF:

⟨1⟩1. If $x_n \rightarrow l$ then $x_n \xrightarrow{w} l$ and $\|x_n\| \rightarrow \|l\|$.

PROOF: Easy using the fact that the inner product is continuous.

⟨1⟩2. If $x_n \xrightarrow{w} l$ and $\|x_n\| \rightarrow \|l\|$ then $x_n \rightarrow l$.

⟨2⟩1. ASSUME: $x_n \xrightarrow{w} l$ and $\|x_n\| \rightarrow \|l\|$

⟨2⟩2. $\langle x_n, l \rangle \rightarrow \|l\|^2$

⟨2⟩3. $\|x_n - l\| \rightarrow 0$

PROOF:

$$\begin{aligned} \|x_n - l\|^2 &= \langle x_n - l, x_n - l \rangle \\ &= \langle x_n, x_n \rangle - \langle x_n, l \rangle - \langle l, x_n \rangle + \langle l, l \rangle \\ &= \|x_n\|^2 - \langle x_n, l \rangle - \overline{\langle x_n, l \rangle} + \|l\|^2 \\ &\rightarrow \|l\|^2 - 2\|l\|^2 + \|l\|^2 \\ &= 0 \end{aligned}$$

□

Theorem 18.0.19. Let S be a subset of an inner product space E such that $\text{span } S$ is dense in E . If (x_n) is a bounded sequence in E and, for all $y \in S$, we have $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$ then $x_n \xrightarrow{w} x$.

PROOF:

⟨1⟩1. For all $y \in \text{span } S$, we have $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$

⟨1⟩2. LET: $z \in E$

PROVE: $\langle x_n, z \rangle \rightarrow \langle x, z \rangle$

⟨1⟩3. LET: $\epsilon > 0$

PROVE: There exists n_0 such that $\forall n \geq n_0, |\langle x_n, z \rangle - \langle x, z \rangle| < \epsilon$

⟨1⟩4. PICK $M > 0$ such that $\|x\| \leq M$ and $\forall n \in \mathbb{Z}_+, \|x_n\| \leq M$.

⟨1⟩5. PICK $y_0 \in \text{span } S$ such that $\|z - y_0\| < \epsilon/3M$

⟨1⟩6. PICK $n_0 \in \mathbb{Z}_+$ such that, for all $n \geq n_0$, we have $|\langle x_n, y_0 \rangle - \langle x, y_0 \rangle| < \epsilon/3$

⟨1⟩7. LET: $n \geq n_0$

⟨1⟩8. $|\langle x_n, z \rangle - \langle x, z \rangle| < \epsilon$

PROOF:

$$\begin{aligned} |\langle x_n, z \rangle - \langle x, z \rangle| &\leq |\langle x_n, z \rangle - \langle x_n, y_0 \rangle| + |\langle x_n, y_0 \rangle - \langle x, y_0 \rangle| + |\langle x, y_0 \rangle - \langle x, z \rangle| \\ &< \|x_n\| \|z - y_0\| + \epsilon/3 + \|x\| \|y_0 - z\| \\ &< M(\epsilon/3M) + \epsilon/3 + M(\epsilon/3M) \\ &= \epsilon \end{aligned}$$

□

18.1 Orthonormal Bases

Definition 18.1.1 (Orthogonal). Let V be an inner product space and $S \subseteq V$. Then S is *orthogonal* iff any two distinct elements of S are orthogonal.

Definition 18.1.2 (Orthonormal). Let V be an inner product space and $S \subseteq V$. Then S is *orthonormal* iff it is orthogonal and $\forall x \in S, \|x\| = 1$.

Proposition 18.1.3. *Orthonormal sets are linearly independent.*

PROOF:

$\langle 1 \rangle 1$. LET: S be orthonormal

$\langle 1 \rangle 2$. ASSUME: $\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$ where $e_1, \dots, e_n \in S$

$\langle 1 \rangle 3$. $|\alpha_1|^2 + \cdots + |\alpha_n|^2 = 0$

PROOF:

$$\begin{aligned} 0 &= \sum_{m=1}^n \langle 0, \alpha_m e_m \rangle \\ &= \sum_{m=1}^n \left\langle \sum_{k=1}^n \alpha_k e_k, \alpha_m e_m \right\rangle \\ &= \sum_{m=1}^n \sum_{k=1}^n \alpha_k \overline{\alpha_m} \langle e_k, e_m \rangle \\ &= \sum_{k=1}^n |\alpha_k|^2 \end{aligned}$$

$\langle 1 \rangle 4$. $\alpha_1 = \cdots = \alpha_n = 0$

□

Proposition 18.1.4. *In l^2 , let e_n be the sequence whose n th element is 1 and whose other elements are 0. Then $\{e_n \mid n \in \mathbb{Z}_+\}$ is orthonormal.*

Proposition 18.1.5. *In $L^2([-\pi, \pi])$, let $\phi_n(x) = e^{inx}/\sqrt{2\pi}$ for $n \in \mathbb{Z}$. Then $\{\phi_n \mid n \in \mathbb{Z}\}$ is orthonormal.*

Definition 18.1.6 (Legendre Polynomials). The *Legendre polynomials* $P_n \in \mathbb{Q}[x]$ for $n \in \mathbb{N}$ are defined by

$$\begin{aligned} P_0 &= 1 \\ P_n &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \end{aligned}$$

Proposition 18.1.7. *Let P_n be the n th Legendre polynomial. Then $\{P_n \mid n \in \mathbb{N}\}$ is orthogonal in $L^2([-1, 1])$.*

Definition 18.1.8 (Hermite Polynomial). The *Hermite polynomials* $H_n \in \mathbb{R}[x]$ for $n \in \mathbb{N}$ are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Proposition 18.1.9. *Let H_n be the n th Hermite polynomial. Then $\{e^{-x^2/2} H_n(x) \mid n \in \mathbb{N}\}$ is orthogonal in $L^2(\mathbb{R})$.*

Theorem 18.1.10. *Let V be an inner product space. If $x_1, \dots, x_n \in V$ are orthogonal then*

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2 .$$

Theorem 18.1.11 (Bessel's Equality). *Let V be an inner product space. Let $x_1, \dots, x_n \in V$ be orthonormal. Let $x \in V$. Then*

$$\left\| x - \sum_{k=1}^n \langle x, x_k \rangle x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^n |\langle x, x_k \rangle|^2 .$$

PROOF:

$$\begin{aligned} \left\| x - \sum_{k=1}^n \langle x, x_k \rangle x_k \right\|^2 &= \left\langle x - \sum_{k=1}^n \langle x, x_k \rangle x_k, x - \sum_{k=1}^n \langle x, x_k \rangle x_k \right\rangle \\ &= \langle x, x \rangle - \left\langle x, \sum_{k=1}^n \langle x, x_k \rangle x_k \right\rangle - \left\langle \sum_{k=1}^n \langle x, x_k \rangle x_k, x \right\rangle \\ &\quad + \left\langle \sum_{k=1}^n \langle x, x_k \rangle x_k, \sum_{k=1}^n \langle x, x_k \rangle x_k \right\rangle \\ &= \langle x, x \rangle - 2 \sum_{k=1}^n \langle x, x_k \rangle \langle x_k, x \rangle + \sum_{i=1}^n \sum_{j=1}^n \langle x, x_i \rangle \langle x_j, x \rangle \langle x_i, x_j \rangle \\ &= \|x\|^2 - 2 \sum_{k=1}^n |\langle x, x_k \rangle|^2 + \sum_{i=1}^n \langle x, x_i \rangle \langle x_i, x \rangle \\ &= \|x\|^2 - \sum_{k=1}^n |\langle x, x_k \rangle|^2 \end{aligned} \quad \square$$

Corollary 18.1.11.1 (Bessel's Inequality). *Let V be an inner product space. Let $x_1, \dots, x_n \in V$ be orthonormal. Let $x \in E$. Then*

$$\sum_{k=1}^n |\langle x, x_k \rangle|^2 \leq \|x\|^2 .$$

Corollary 18.1.11.2. *Orthonormal sequences are weakly convergent to 0.*

PROOF: Let (x_n) be an orthonormal sequence. Taking the limit in Bessel's inequality we have $\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \leq \|x\|^2 < \infty$ and so $\langle x, x_k \rangle \rightarrow 0$ as $k \rightarrow \infty$. \square

Corollary 18.1.11.3 (Generalized Fourier Series). *Let V be an inner product space. Let (e_n) be an orthonormal sequence in V . For any $x \in V$, the generalized Fourier series of x is*

$$\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n ,$$

and $\langle x, e_n \rangle$ is called the n th generalized Fourier coefficient of x with respect to (e_n) . We have $(\langle x, e_n \rangle e_n)_n \in l^2$.

Definition 18.1.12 (Complete Orthonormal Sequence). Let E be an inner product space. Let (x_n) be an orthonormal sequence in E . Then (x_n) is *complete* iff, for all $x \in E$, we have

$$\sum_{n=1}^{\infty} \langle x, x_n \rangle x_n = x \quad .$$

Chapter 19

Hilbert Spaces

Definition 19.0.1 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Proposition 19.0.2. For $n \in \mathbb{N}$, \mathbb{C}^n is a Hilbert space.

Proposition 19.0.3. l^2 is a Hilbert space.

Proposition 19.0.4. $L^2(\mathbb{R})$ is a Hilbert space.

Proposition 19.0.5. $L^2([a, b])$ is a Hilbert space.

Proposition 19.0.6. Let ρ be a measurable function on $[a, b]$ such that $\rho(x) > 0$ almost everywhere. Let $L^{2\rho}([a, b])$ be the set of all measurable functions $f : [a, b] \rightarrow \mathbb{C}$ such that

$$\int_a^b |f(x)|^2 \rho(x) dx < \infty .$$

Define an inner product on $L^{2\rho}([a, b])$ by

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} \rho(x) dx .$$

Then $L^{2\rho}([a, b])$ is a Hilbert space.

Proposition 19.0.7. Let m and N be positive integers. Let Ω be an open set in \mathbb{R}^N . Let $\tilde{H}^m(\Omega)$ be the set of all $f \in C^m(\Omega)$ such that, for every $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{Z}_+^N$ with $|\alpha| := \alpha_1 + \dots + \alpha_N \leq m$, we have

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} \in L^2(\Omega) .$$

Define an inner product on $\tilde{H}^m(\Omega)$ by

$$\langle f, g \rangle := \int_\Omega \sum_\alpha D^\alpha f \overline{D^\alpha g} .$$

Let $H^m(\Omega)$ be the completion of $\tilde{H}^m(\Omega)$. Then $H^m(\Omega)$ is a Hilbert space.

Theorem 19.0.8. *Weakly convergent sequences in a Hilbert space are bounded.*

PROOF:

$\langle 1 \rangle 1$. LET: H be a Hilbert space.

$\langle 1 \rangle 2$. LET: (x_n) be a weakly convergent sequence in H .

$\langle 1 \rangle 3$. For $n \in \mathbb{Z}_+$,

LET: $f_n : H \rightarrow \mathbb{C}$, $f_n(x) = \langle x, x_n \rangle$

$\langle 1 \rangle 4$. For $n \in \mathbb{Z}_+$, f_n is a bounded linear functional.

$\langle 1 \rangle 5$. For every $x \in H$, the sequence $(f_n(x))$ is bounded.

PROOF: Since it converges.

$\langle 1 \rangle 6$. PICK $M > 0$ such that, for all $n \in \mathbb{Z}_+$, we have $\|f_n\| \leq M$.

PROOF: Banach-Steinhaus Theorem, $\langle 1 \rangle 4$, $\langle 1 \rangle 5$.

$\langle 1 \rangle 7$. $\forall n \in \mathbb{Z}_+.$ $\|f_n\| = \|x_n\|$

$\langle 2 \rangle 1$. LET: $n \in \mathbb{Z}_+$

$\langle 2 \rangle 2$. $\|f_n\| \leq \|x_n\|$

PROOF: Since for all $x \in H$ we have $|f_n(x)| = |\langle x, x_n \rangle| \leq \|x\| \|x_n\|$ by Schwarz's Inequality.

$\langle 2 \rangle 3$. $\|x_n\| \leq \|f_n\|$

PROOF: Since $\|x_n\|^2 = |\langle x_n, x_n \rangle| = |f_n(x_n)| \leq \|f_n\| \|x_n\|$.

$\langle 1 \rangle 8$. $\forall n \in \mathbb{Z}_+.$ $\|x_n\| \leq M$

PROOF: $\langle 1 \rangle 6$, $\langle 1 \rangle 7$

□

Theorem 19.0.9. *Let H be a Hilbert space. Let (x_n) be an orthonormal sequence in H and let (α_n) be a sequence of complex numbers. Then the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in H if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2$ converges in \mathbb{R} , in which case*

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2 .$$

PROOF:

$\langle 1 \rangle 1$. For $m > k > 0$ we have

$$\left\| \sum_{n=k}^m \alpha_n x_n \right\|^2 = \sum_{n=k}^m |\alpha_n|^2 .$$

PROOF: Theorem 18.1.10.

$\langle 1 \rangle 2$. If $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ then $\sum_{n=1}^{\infty} \alpha_n x_n$ converges.

$\langle 2 \rangle 1$. ASSUME: $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$

$\langle 2 \rangle 2$. $(\sum_{n=1}^m \alpha_n x_n)_m$ is Cauchy.

PROOF: From $\langle 1 \rangle 1$.

$\langle 2 \rangle 3$. $\sum_{n=1}^{\infty} \alpha_n x_n$ converges.

$\langle 1 \rangle 3$. If $\sum_{n=1}^{\infty} \alpha_n x_n$ converges then $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$.

PROOF: From $\langle 1 \rangle 1$.

$\langle 1 \rangle 4$. If $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ then

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2 .$$

PROOF: From $\langle 1 \rangle 1$.

□

Proposition 19.0.10. *Every complete orthonormal sequence in a Hilbert space is a basis.*

PROOF:

$\langle 1 \rangle 1$. LET: E be an inner product space.

$\langle 1 \rangle 2$. LET: (e_n) be a complete orthonormal sequence in E .

$\langle 1 \rangle 3$. For all $x \in E$, there exists a sequence (α_n) in \mathbb{C} such that $x = \sum_n \alpha_n e_n$.

PROOF: Immediate from $\langle 1 \rangle 2$.

$\langle 1 \rangle 4$. If $\sum_n \alpha_n e_n = \sum_n \beta_n e_n$ then $\alpha_n = \beta_n$ for all n .

$\langle 2 \rangle 1$. LET: $x = \sum_n \alpha_n e_n = \sum_n \beta_n e_n$

$\langle 2 \rangle 2$. $\sum_n |\alpha_n - \beta_n|^2 = 0$

PROOF:

$$\begin{aligned}
 0 &= \|x - x\|^2 \\
 &= \left\| \sum_{n=1}^{\infty} \alpha_n e_n - \sum_{n=1}^{\infty} \beta_n e_n \right\|^2 && (\langle 2 \rangle 1) \\
 &= \left\| \sum_{n=1}^{\infty} (\alpha_n - \beta_n) e_n \right\|^2 \\
 &= \sum_{n=1}^{\infty} |\alpha_n - \beta_n|^2 && (\text{Theorem 19.0.9})
 \end{aligned}$$

$\langle 2 \rangle 3$. $\alpha_n = \beta_n$ for all n .

□

Theorem 19.0.11. *An orthonormal sequence (x_n) in a Hilbert space H is complete if and only if, for all $x \in H$, if $\forall n. \langle x, x_n \rangle = 0$ then $x = 0$.*

PROOF:

$\langle 1 \rangle 1$. If (x_n) is complete then, for all $x \in H$, if $\forall n. \langle x, x_n \rangle = 0$ then $x = 0$.

$\langle 2 \rangle 1$. ASSUME: (x_n) is complete.

$\langle 2 \rangle 2$. LET: $x \in H$

$\langle 2 \rangle 3$. ASSUME: $\forall n. \langle x, x_n \rangle = 0$

$\langle 2 \rangle 4$. $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n = 0$

$\langle 1 \rangle 2$. If, for all $x \in H$, if $\forall n. \langle x, x_n \rangle = 0$ then $x = 0$, then (x_n) is complete.

$\langle 2 \rangle 1$. ASSUME: For all $x \in H$, if $\forall n. \langle x, x_n \rangle = 0$, then $x = 0$.

$\langle 2 \rangle 2$. LET: $y = x - \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$

$\langle 2 \rangle 3$. For all n , $\langle y, x_n \rangle = 0$

$\langle 3 \rangle 1$. LET: $n \in \mathbb{Z}_+$

$\langle 3 \rangle 2$. $\langle y, x_n \rangle = 0$

PROOF:

$$\begin{aligned}
 \langle y, x_n \rangle &= \left\langle x - \sum_{m=1}^{\infty} \langle x, x_m \rangle x_m, x_n \right\rangle \\
 &= \langle x, x_n \rangle - \sum_{m=1}^{\infty} \langle x, x_m \rangle \langle x_m, x_n \rangle \\
 &= \langle x, x_n \rangle - \langle x, x_n \rangle \\
 &= 0
 \end{aligned}$$

$\langle 2 \rangle 4.$ $y = 0$

$\langle 2 \rangle 5.$ $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$

□

Theorem 19.0.12 (Parseval's Formula). *Let H be a Hilbert space. Let (x_n) be an orthonormal sequence in H . Then (x_n) is complete if and only if, for all $x \in H$,*

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2.$$

PROOF:

$\langle 1 \rangle 1.$ If (x_n) is complete then for all $x \in H$ we have $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$.

$\langle 2 \rangle 1.$ ASSUME: (x_n) is complete.

$\langle 2 \rangle 2.$ LET: $x \in H$

$\langle 2 \rangle 3.$ $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$

PROOF:

$$\begin{aligned}
 \|x\|^2 &= \left\| \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n \right\|^2 && (\langle 2 \rangle 1) \\
 &= \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 && (\text{Theorem 19.0.9})
 \end{aligned}$$

$\langle 1 \rangle 2.$ If, for all $x \in H$, we have $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$, then (x_n) is complete.

$\langle 2 \rangle 1.$ ASSUME: For all $x \in H$, we have $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$

$\langle 2 \rangle 2.$ LET: $x \in H$

$\langle 2 \rangle 3.$ $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$

□

Proposition 19.0.13. *For $n \in \mathbb{Z}$, let $\pi_n(x) = e^{inx}/\sqrt{2\pi}$. Then $\{\pi_n \mid n \in \mathbb{Z}\}$ is a complete orthonormal set in $L^2([-\pi, \pi])$.*

TODO

Proposition 19.0.14. $B = \{1/\sqrt{2\pi}\} \cup \{\cos nx/\sqrt{\pi} \mid n \in \mathbb{Z}_+\} \cup \{\sin nx/\sqrt{\pi} \mid n \in \mathbb{Z}_+\}$ is a complete orthonormal set in $L^2([-\pi, \pi])$.

PROOF:

$\langle 1 \rangle 1.$ For all $f \in B$ we have $\|f\| = 1$

$\langle 2 \rangle 1.$ $\|1/\sqrt{2\pi}\| = 1$

PROOF:

$$\begin{aligned}\|1/\sqrt{2\pi}\| &= \int_{-\pi}^{\pi} dx/2\pi \\ &= 1\end{aligned}$$

$\langle 2 \rangle 2$. For all $n \in \mathbb{Z}_+$ we have $\|\cos nx/\sqrt{\pi}\| = 1$

PROOF:

$$\begin{aligned}\|\cos nx/\sqrt{\pi}\| &= 1/\pi \int_{-\pi}^{\pi} \cos^2 nx \, dx \\ &= 1/2\pi \int_{-\pi}^{\pi} (\cos 2nx + 1) \, dx \\ &= 1/2\pi [1/2n \sin 2nx + x]_{-\pi}^{\pi} \\ &= (1/2\pi)(2\pi) \\ &= 1\end{aligned}$$

$\langle 2 \rangle 3$. For all $n \in \mathbb{Z}_+$ we have $\|\sin nx/\sqrt{\pi}\| = 1$

PROOF:

$$\begin{aligned}\|\sin nx/\sqrt{\pi}\| &= 1/\pi \int_{-\pi}^{\pi} \sin^2 nx \, dx \\ &= -1/2\pi \int_{-\pi}^{\pi} (\cos 2nx - 1) \, dx \\ &= -1/2\pi [1/2n \sin 2nx - x]_{-\pi}^{\pi} \\ &= (-1/2\pi)(-2\pi) \\ &= 1\end{aligned}$$

$\langle 1 \rangle 2$. For all $f, g \in B$ with $f \neq g$ we have $\langle f, g \rangle = 0$

$\langle 2 \rangle 1$. $\langle 1, \cos nx \rangle = 0$

PROOF:

$$\begin{aligned}\int_{-\pi}^{\pi} \cos nx \, dx &= [1/n \sin nx]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

$\langle 2 \rangle 2$. $\langle 1, \sin nx \rangle = 0$

PROOF:

$$\begin{aligned}\int_{-\pi}^{\pi} \sin nx \, dx &= [-1/n \cos nx]_{-\pi}^{\pi} \\ &= -1/n \cos n\pi + 1/n \cos n\pi \\ &= 0\end{aligned}$$

$\langle 2 \rangle 3$. If $m \neq n$ then $\langle \cos mx, \cos nx \rangle = 0$

PROOF:

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= 1/2 \int_{-\pi}^{\pi} (\cos(n+m)x - \cos(n-m)x) \, dx \\ &= 1/2 \left[\frac{1}{n+m} \sin(n+m)x - \frac{1}{n-m} \sin(n-m)x \right]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

$\langle 2 \rangle 4$. $\langle \cos mx, \sin nx \rangle = 0$

PROOF:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \sin nx \, dx &= 1/2 \int_{-\pi}^{\pi} (\sin(n+m)x - \sin(n-m)x) \, dx \\ &= 1/2 \left[-\frac{1}{n+m} \cos(n+m)x + \frac{1}{n-m} \cos(n-m)x \right]_{-\pi}^{\pi} \\ &= 0 \end{aligned} \quad (\cos \text{ is odd})$$

$\langle 2 \rangle 5$. If $m \neq n$ then $\langle \sin mx, \sin nx \rangle = 0$

PROOF:

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= 1/2 \int_{-\pi}^{\pi} (\cos(n-m)x - \cos(n+m)x) \, dx \\ &= 1/2 \left[\frac{1}{n-m} \sin(n-m)x - \frac{1}{n+m} \sin(n+m)x \right]_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

$\langle 1 \rangle 3$. For all $f \in L^2([-\pi, \pi])$, if $\forall g \in B. \langle f, g \rangle = 0$ then $f = 0$

$\langle 2 \rangle 1$. LET: $f \in L^2([-\pi, \pi])$

$\langle 2 \rangle 2$. ASSUME: $\forall g \in B. \langle f, g \rangle = 0$

$\langle 2 \rangle 3$. For all $n \in \mathbb{Z}$, $\langle f, e^{inx} \rangle = 0$

PROOF: Since $e^{inx} = \cos nx + i \sin nx$.

$\langle 2 \rangle 4$. $f = 0$

PROOF: From Proposition 19.0.13.

□

Proposition 19.0.15. $\{\frac{1}{\sqrt{\pi}}\} \cup \{\sqrt{\frac{2}{\pi}} \cos nx \mid n \in \mathbb{Z}_+\}$ is a complete orthonormal set in $L^2([0, \pi])$.

Proposition 19.0.16. $\{\sqrt{\frac{2}{\pi}} \sin nx \mid n \in \mathbb{Z}_+\}$ is a complete orthonormal set in $L^2([0, \pi])$.

Definition 19.0.17 (Signum). The *signum* function $\text{sgn} : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\text{sgn } x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Definition 19.0.18 (Rademacher Functions). The *Rademacher functions* $R : \mathbb{N} \times [0, 1] \rightarrow \{-1, 0, 1\}$ are defined by

$$R(m, x) = \text{sgn}(\sin(2^m \pi x)) \quad .$$

Proposition 19.0.19. The Rademacher functions $\{R(m, -) \mid m \in \mathbb{N}\}$ are orthonormal in $L^2([0, 1])$.

PROOF:

$\langle 1 \rangle 1$. $\forall m \in \mathbb{N}. \|R(m, -)\| = 1$

PROOF: $\int_0^1 \text{sgn}(\sin(2^m \pi x))^2 \, dx = 1$ since the integrand is 1 except for finitely many points in $[0, 1]$.

$\langle 1 \rangle 2$. Given natural numbers $m \neq n$, we have $\langle R(m, -), R(n, -) \rangle = 0$

$\langle 2 \rangle 1$. Given reals a, b and a natural number m , we have $\int_a^b R(m, x) dx = 0$ whenever $2^m(b - a)$ is an even integer.

PROOF: If $m > 0$, or if $m = 0$ and $b - a$ is an even integer, then the regions where $R(m, x) = 1$ are isometric with the regions where $R(m, x) = -1$.

$\langle 2 \rangle 2$. LET: m and n be natural numbers with $n < m$.

$\langle 2 \rangle 3$. $\langle R(m, -), R(n, -) \rangle = 0$

PROOF:

$$\begin{aligned} \int_0^1 R(m, x) R(n, x) dx &= \sum_{k=1}^{2^n} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} R(m, x) R(n, x) dx \\ &= \sum_{k=1}^{2^n} (-i)^{k+1} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} R(m, x) dx \end{aligned}$$

$$= 0$$

$$(\langle 2 \rangle 1, 2^m \left(\frac{k}{2^n} - \frac{k-1}{2^n} \right) = 2^{m-n} \text{ is an even integer})$$

□

Proposition 19.0.20. *The set of Rademacher functions is not complete.*

PROOF:

$\langle 1 \rangle 1$. Define $f : [0, 1] \rightarrow \mathbb{C}$ by $f(x) = 0$ if $0 \leq x < 1/4$, $f(x) = 1$ if $1/4 \leq x \leq 3/4$, $f(x) = 0$ if $3/4 < x \leq 1$.

$\langle 1 \rangle 2$. $f \in L^2([0, 1])$

$\langle 1 \rangle 3$. $\langle R(0, -), f \rangle = 1/2$

$\langle 1 \rangle 4$. $\langle R(m, -), f \rangle = 0$ for $m \geq 1$

$\langle 1 \rangle 5$. $f \neq 1/2 R(0, -)$

□

Definition 19.0.21 (Walsh Functions). Define the *Walsh functions* $W : \mathbb{N} \times [0, 1] \rightarrow \{-1, 0, 1\}$ as follows. Given $m \in \mathbb{N}$, let $m = \sum_{k=1}^n 2^{k-1} a_k$ where each a_k is either 0 or 1. Then

$$W(m, x) = \prod_{k=1}^n R(k, x)^{a_k} .$$

Proposition 19.0.22. *The set of Walsh functions $\{W(m, -) \mid m \in \mathbb{N}\}$ is a complete orthonormal set.*

TODO