

Encyclopaedia of Mathematics and Physics

Robin Adams

Contents

1	Set Theory	5
2	Relations	7
3	Order Theory	9
4	Field Theory	11
4.1	Ordered Fields	13
5	Real Analysis	15
5.1	Construction of the Real Numbers	15
5.2	Properties of the Real Numbers	21
5.2.1	Logarithms	27
5.2.2	Intervals	28
5.3	The Extended Real Number System	28
6	Complex Analysis	31
I	Linear Algebra	37
7	Vector Spaces	39
7.1	Convex Sets	39
8	Real Inner Product Spaces	41
8.1	Balls	42
9	Complex Inner Product Spaces	43
9.1	Hilbert Spaces	44
10	Lie Algebras	45
10.1	Lie Algebar Homomorphisms	46

II	Topology	47
11	Metric Spaces	49
11.1	Balls	49
11.2	Limit Points	50
11.3	Closed Sets	50
11.4	Interior Points	50
11.5	Open Sets	50
11.6	Perfect Sets	53
11.7	Bounded Sets	53
11.8	Dense Sets	53
11.9	Closure	53
11.10	Compact Sets	54
III	More Algebra	57
12	Lie Groups	59

Chapter 1

Set Theory

Proposition 1.1. *Every infinite subset of a countably infinite set is countable.*

PROOF:

- $\langle 1 \rangle 1$. LET: $i : A \hookrightarrow \mathbb{N}$ be an infinite subset of \mathbb{N} .
- $\langle 1 \rangle 2$. Define $j : \mathbb{N} \rightarrow A$ by: $j(k)$ is the element such that $i(j(k))$ is least such that $i(j(k)) \notin \{i(j(0)), \dots, i(j(k-1))\}$.
- $\langle 1 \rangle 3$. j is a bijection.

□

Proposition 1.2. *A countable union of countable sets is countable.*

PROOF:

- $\langle 1 \rangle 1$. LET: (A_n) be a sequence of countable sets.
- $\langle 1 \rangle 2$. For $n \in \mathbb{N}$, PICK an enumeration $(e_{nm})_m$ of A_n .
- $\langle 1 \rangle 3$. LET: (p_k) be the following enumeration of $\mathbb{N} \times \mathbb{N}$:
 $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \dots$
- $\langle 1 \rangle 4$. $(e_{\pi_1(p_k)\pi_2(p_k)})_k$ is an enumeration of $\bigcup_n A_n$.

□

Theorem 1.3. $2^{\mathbb{N}}$ is uncountable.

PROOF:

- $\langle 1 \rangle 1$. ASSUME: for a contradiction $f : \mathbb{N} \approx 2^{\mathbb{N}}$
- $\langle 1 \rangle 2$. LET: $S = \{n \in \mathbb{N} : n \notin f(n)\}$
- $\langle 1 \rangle 3$. For all n , we have $n \in S \Leftrightarrow n \notin f(n)$
- $\langle 1 \rangle 4$. For all n we have $S \neq f(n)$.
- $\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 1$.

□

Chapter 2

Relations

Definition 2.1 (Antisymmetric). A relation R on a set A is *antisymmetric* iff, whenever xRy and yRx , then $x = y$.

Definition 2.2 (Transitive). A relation R on a type A is *transitive* iff, whenever xRy and yRz , then xRz .

Chapter 3

Order Theory

Definition 3.1 (Linear Order). A *linear order* on a set A is a binary relation \leq on A that is transitive, antisymmetric and:

$$\forall x, y \in A. x \leq y \vee y \leq x .$$

A *linearly ordered set* is a pair (A, \leq) where A is a set and \leq is a binary relation on A .

We write $x < y$ for $x \leq y$ and $x \neq y$.

Definition 3.2 (Upper Bound). Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is an *upper bound* in E iff $\forall x \in E. x \leq u$. We say E is *bounded above* iff it has an upper bound.

The *up-set* of E , denoted $E \uparrow$, is the set of upper bounds of E .

Definition 3.3 (Lower Bound). Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then l is a *lower bound* in E iff $\forall x \in E. l \leq x$. We say E is *bounded below* iff it has a lower bound.

The *down-set* of E , denoted $E \downarrow$, is the set of lower bounds of E .

Definition 3.4 (Supremum). Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is the *least upper bound* or *supremum* of E iff u is an upper bound for E and, for any upper bound u' for E , we have $u \leq u'$.

Definition 3.5 (Infimum). Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then l is the *greatest lower bound* or *infimum* of E iff l is a lower bound for E and, for any lower bound l' for E , we have $l' \leq l$.

Definition 3.6 (Least Upper Bound Property). A linearly ordered set S has the *least upper bound property* iff every nonempty subset of S that is bounded above has a least upper bound.

Proposition 3.7. Let S be a linearly ordered set and $E \subseteq S$.

1. If $E \downarrow$ has a supremum l , then l is the infimum of E .

2. If $E \uparrow$ has an infimum u , then U is the supremum of E .

PROOF:

- (1)1. If $E \downarrow$ has a supremum l , then l is the infimum of E .
 (2)1. l is a lower bound for E .
 (3)1. LET: $x \in E$
 (3)2. x is an upper bound for $E \downarrow$.
 PROOF: For all $y \in E \downarrow$ we have $y \leq x$.
 (3)3. $l \leq x$
 (2)2. For any lower bound l' for E , we have $l' \leq l$.
 PROOF: Since l is an upper bound for $E \downarrow$.
 (1)2. If $E \uparrow$ has an infimum u , then u is the supremum of E .
 PROOF: Dual.

□

Corollary 3.7.1. *A linearly ordered set has the least upper bound property if and only if every nonempty set bounded below has an infimum.*

Definition 3.8 (Closed Downwards). Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed downwards* iff, whenever $x \in E$ and $y < x$, then $y \in E$.

Definition 3.9 (Closed Upwards). Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed upwards* iff, whenever $x \in E$ and $x < y$, then $y \in E$.

Definition 3.10 (Greatest). Let S be a linearly ordered set and $u \in S$. Then u is *greatest* in S iff $\forall x \in S. x \leq u$.

Definition 3.11 (Least). Let S be a linearly ordered set and $l \in S$. Then l is *least* in S iff $\forall x \in S. l \leq x$.

Proposition 3.12. *Let \leq be a linear order on a set S and $E \subseteq S$. Then $\leq \cap E^2$ is a linear order on E .*

PROOF: Easy. □

Given a linearly ordered set (S, \leq) and $E \subseteq S$, we write just E for the linearly ordered set $(E, \leq \cap E^2)$.

Definition 3.13 (Lexicographic Order). Let A and B be linearly ordered sets. The *lexicographic order* or *dictionary order* on $A \times B$ is the order defined by

$$(a, b) \leq (a', b') \Leftrightarrow a = a' \vee (a < a' \wedge b \leq b') .$$

Proposition 3.14. *The lexicographic order is a linear order.*

Chapter 4

Field Theory

Definition 4.1 (Field). A *field* F consists of a set F , two operations $+, \cdot : F^2 \rightarrow F$ and an element $0 \in F$ such that:

- $+$ is commutative.
- $+$ is associative.
- $\forall x \in F. x + 0 = x$
- $\forall x \in F. \exists y \in F. x + y = 0$
- \cdot is commutative.
- \cdot is associative.
- There exists $1 \in F$ such that $1 \neq 0$ and $\forall x \in F. x1 = x$ and $\forall x \in F. x \neq 0 \Rightarrow \exists y \in F. xy = 1$
- *Distributive Law* $\forall x, y, z \in F. x(y + z) = xy + xz$

Proposition 4.2. *In any field F , the element 0 is the unique element such that $\forall x \in F. x + 0 = x$.*

PROOF: If 0 and $0'$ both have this property then $0 = 0 + 0' = 0'$. \square

Proposition 4.3. *In any field F , given $x \in F$, there is a unique $y \in F$ such that $x + y = 0$.*

PROOF: If $x + y = x + y' = 0$ then

$$\begin{aligned} y &= y + 0 \\ &= y + x + y' \\ &= 0 + y' \\ &= y' \end{aligned}$$

\square

Definition 4.4. Let F be a field. Let $x \in F$. We denote by $-x$ the unique element of F such that $x + (-x) = 0$.

Given $x, y \in F$, we write $x - y$ for $x + (-y)$.

Proposition 4.5. In any field F , if $x + y = x + z$ then $y = z$.

PROOF: If $x + y = x + z$ we have

$$-x + x + y = -x + x + z$$

$$\therefore 0 + y = 0 + z$$

$$\therefore y = z$$

□

Proposition 4.6. In any field F , we have $-(-x) = x$.

PROOF: Since $x + (-x) = 0$. □

Proposition 4.7. In any field F , the element 1 such that $\forall x \in F. x1 = x$ is unique.

PROOF: If 1 and $1'$ both have this property then $1 = 1 \cdot 1' = 1'$. □

Proposition 4.8. In any field F , given $x \in F$ with $x \neq 0$, the element y such that $xy = 1$ is unique.

PROOF: If y and y' both have this property then we have

$$y = y1$$

$$= yxy'$$

$$= 1y'$$

$$= y'$$

□

Definition 4.9. In any field F , if $x \neq 0$, we write x^{-1} for the unique element such that $xx^{-1} = 1$.

We write x/y for xy^{-1} .

Proposition 4.10. In any field F , if $xy = xz$ and $x \neq 0$ then $y = z$.

PROOF:

$$y = 1y$$

$$= x^{-1}xy$$

$$= x^{-1}xz$$

$$= 1z$$

$$= z$$

□

Proposition 4.11. In any field F , if $x \neq 0$ then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.

PROOF: Since $xx^{-1} = 1$. □

Proposition 4.12. In any field F , we have $x0 = 0$.

PROOF:

$$\begin{aligned}
 x0 + 0 &= x0 \\
 &= x(0 + 0) \\
 &= x0 + x0 \\
 \therefore 0 &= x0 \quad \square
 \end{aligned}$$

Proposition 4.13. *In any field F , if $xy = 0$ then $x = 0$ or $y = 0$.*

PROOF: If $xy = 0$ and $x \neq 0$ then we have $y = x^{-1}xy = x^{-1}0 = 0$. \square

Proposition 4.14. *In any field F , we have $(-x)y = -(xy)$.*

PROOF:

$$\begin{aligned}
 xy + (-x)y &= (x + (-x))y \\
 &= 0y \\
 &= 0 \quad (\text{Proposition 4.12}) \square
 \end{aligned}$$

Corollary 4.14.1. *In any field F , we have $(-x)(-y) = xy$.*

PROOF:

$$\begin{aligned}
 (-x)(-y) &= -(x(-y)) \\
 &= -(-(xy)) \\
 &= xy \quad (\text{Proposition 4.6}) \square
 \end{aligned}$$

Proposition 4.15. *Let K be a field. Let $a, b \in K$. If $a^2 = b^2$ then $a = b$ or $a = -b$.*

PROOF:

$$\begin{aligned}
 a^2 - b^2 &= 0 \\
 \therefore (a - b)(a + b) &= 0
 \end{aligned}$$

Hence either $a - b = 0$ or $a + b = 0$, and the conclusion follows. \square

4.1 Ordered Fields

Definition 4.16 (Ordered Field). An *ordered field* F consists of a field F and a linear order \leq on F such that:

- For all $x, y, z \in F$, if $y < z$ then $x + y < x + z$
- For all $x, y \in F$, if $x > 0$ and $y > 0$ then $xy > 0$.

We call x *positive* iff $x > 0$ and *negative* iff $x < 0$.

Example 4.17. \mathbb{Q} is an ordered field.

Proposition 4.18. *In any ordered field, if x is positive then $-x$ is negative.*

PROOF: If $x > 0$ then $0 = x + (-x) > 0 = (-x) = -x$. \square

Proposition 4.19. *In any ordered field, if $y < z$ and x is positive then $xy < xz$.*

PROOF: If $y < z$ then we have

$$\begin{aligned} 0 &< z - y \\ \therefore 0 &< x(z - y) \\ &= xz - xy \\ \therefore xy &< xz \end{aligned}$$

□

Proposition 4.20. *In any ordered field, if $y < z$ and x is negative then $xy > xz$.*

PROOF:

- (1)1. $-x$ is positive.
- (1)2. $(-x)y < (-x)z$
- (1)3. $-(xy) < -(xz)$
- (1)4. $xz < xy$

□

Proposition 4.21. *In any ordered field, if $x \neq 0$ then $x^2 > 0$.*

PROOF:

- (1)1. If $x > 0$ then $x^2 > 0$.

PROOF: Proposition 4.19.

- (1)2. If $x < 0$ then $x^2 > 0$.

PROOF: Proposition 4.20.

□

Corollary 4.21.1. *In any ordered field, we have $1 > 0$.*

Proposition 4.22. *In any ordered field, if x is positive then x^{-1} is positive.*

PROOF: If $x^{-1} < 0$ then we would have $1 = xx^{-1} < x0 = 0$ contradicting Corollary 4.21.1. □

Proposition 4.23. *In any ordered field, if $0 < x < y$ then $y^{-1} < x^{-1}$.*

PROOF:

- (1)1. ASSUME: $0 < x < y$
- (1)2. x^{-1} and y^{-1} are positive.

PROOF: Proposition 4.22.

- (1)3. $xy^{-1} < yy^{-1} = 1$
- (1)4. $y^{-1} = x^{-1}xy^{-1} < x^{-1}1 = x^{-1}$

□

Lemma 4.24. *Let K be an ordered field. Let $b \in K$ with $b > 1$. Let n be a positive integer. Then*

$$b^n - 1 \geq n(b - 1)$$

PROOF:

$$\begin{aligned} b^n - 1 &= (b - 1)(b^{n-1} + b^{n-2} + \cdots + 1) \\ &\geq (b - 1)(1 + 1 + \cdots + 1) \\ &= n(b - 1) \end{aligned}$$

□

Chapter 5

Real Analysis

5.1 Construction of the Real Numbers

Definition 5.1 (Cut). A *cut* is a subset α of \mathbb{Q} such that:

- $\emptyset \neq \alpha \neq \mathbb{Q}$
- α is closed downwards.
- α has no greatest element.

In this section, we write R for the set of all cuts.

Proposition 5.2. *R is linearly ordered by \subseteq .*

PROOF: The only difficult part is to prove that, for any cuts α and β , either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

$\langle 1 \rangle 1$. ASSUME: $\alpha \not\subseteq \beta$

PROVE: $\beta \subseteq \alpha$

$\langle 1 \rangle 2$. PICK $q \in \alpha$ such that $q \notin \beta$

$\langle 1 \rangle 3$. LET: $r \in \beta$

$\langle 1 \rangle 4$. $q \not\leq r$

$\langle 1 \rangle 5$. $r < q$

$\langle 1 \rangle 6$. $r \in \alpha$

□

Proposition 5.3. *R has the least upper bound property.*

PROOF:

$\langle 1 \rangle 1$. LET: $E \subseteq R$ be nonempty and bounded above.

$\langle 1 \rangle 2$. LET: $s = \bigcup E$

PROVE: s is a cut.

$\langle 1 \rangle 3$. $\emptyset \neq s$

PROOF: Since E is nonempty and every element of E is nonempty.

$\langle 1 \rangle 4$. $s \neq \mathbb{Q}$

- ⟨2⟩1. PICK an upper bound u for E .
- ⟨2⟩2. PICK $q \notin u$
 PROVE: $q \notin s$
- ⟨2⟩3. $\forall \alpha \in E. \alpha \subseteq u$
- ⟨2⟩4. $s \subseteq u$
- ⟨2⟩5. $q \notin s$
- ⟨1⟩5. s is closed downwards.
- ⟨2⟩1. LET: $q \in s$ and $r < q$.
- ⟨2⟩2. PICK $\alpha \in E$ such that $q \in \alpha$.
- ⟨2⟩3. $r \in \alpha$
- ⟨2⟩4. $r \in s$
- ⟨1⟩6. s has no greatest element.
- ⟨2⟩1. LET: $q \in s$
- ⟨2⟩2. PICK $\alpha \in E$ such that $q \in \alpha$.
- ⟨2⟩3. PICK $r \in \alpha$ such that $q < r$.
- ⟨2⟩4. $r \in s$

□

Definition 5.4 (Addition). Given cuts α and β , we define

$$\alpha + \beta = \{q + r : q \in \alpha, r \in \beta\} .$$

Proposition 5.5. *Given cuts α and β , we have $\alpha + \beta$ is a cut.*

PROOF:

- ⟨1⟩1. $\alpha + \beta$ is nonempty.
 PROOF: Since α and β are nonempty.
- ⟨1⟩2. $\alpha + \beta \neq \mathbb{Q}$
 - ⟨2⟩1. PICK $q \in \mathbb{Q} - \alpha$ and $r \in \mathbb{Q} - \beta$.
 PROVE: $q + r \notin \alpha + \beta$
 - ⟨2⟩2. ASSUME: for a contradiction $q + r \in \alpha + \beta$.
 - ⟨2⟩3. PICK $x \in \alpha$ and $y \in \beta$ such that $q + r = x + y$
 - ⟨2⟩4. $x < q$
 - ⟨2⟩5. $y < r$
 - ⟨2⟩6. $x + y < q + r$
 - ⟨2⟩7. Q.E.D.
- PROOF: This is a contradiction.
- ⟨1⟩3. $\alpha + \beta$ is closed downwards.
 - ⟨2⟩1. LET: $q \in \alpha, r \in \beta$ and $x < q + r$
 - ⟨2⟩2. $x - q < r$
 - ⟨2⟩3. $x - q \in \beta$
 - ⟨2⟩4. $x \in \alpha + \beta$
- ⟨1⟩4. $\alpha + \beta$ has no greatest element.
 - ⟨2⟩1. LET: $q \in \alpha$ and $r \in \beta$.
 PROVE: $q + r$ is not greatest in $\alpha + \beta$.
 - ⟨2⟩2. PICK $q' \in \alpha$ with $q < q'$ and $r' \in \beta$ with $r < r'$.
 - ⟨2⟩3. $q + r < q' + r' \in \alpha + \beta$

□

Proposition 5.6. *Addition is commutative and associative on R .*

PROOF: Immediate from definitions and the fact that addition is commutative and associative on \mathbb{Q} . □

Definition 5.7. For any $q \in \mathbb{Q}$, let $q^* = \{r \in \mathbb{Q} : r < q\}$.

Proposition 5.8. *For any $q \in \mathbb{Q}$, we have q^* is a cut.*

PROOF:

⟨1⟩1. $q^* \neq \emptyset$

PROOF: Since $q - 1 \in q^*$.

⟨1⟩2. $q^* \neq \mathbb{Q}$

PROOF: Since $q \notin q^*$.

⟨1⟩3. q^* is closed downwards.

PROOF: Immediate from definition.

⟨1⟩4. q^* has no greatest element.

PROOF: For all $r \in q^*$ we have $r < (q + r)/2 \in q^*$.

□

Proposition 5.9. *For any cut α we have $\alpha + 0^* = \alpha$.*

PROOF:

⟨1⟩1. $\alpha + 0^* \subseteq \alpha$

⟨2⟩1. LET: $q \in \alpha$ and $r \in 0^*$

PROVE: $q + r \in \alpha$

⟨2⟩2. $r < 0$

⟨2⟩3. $q + r < q$

⟨2⟩4. $q + r \in \alpha$

⟨1⟩2. $\alpha \subseteq \alpha + 0^*$

⟨2⟩1. LET: $q \in \alpha$

⟨2⟩2. PICK $r \in \alpha$ such that $q < r$

⟨2⟩3. $q = r + (q - r) \in \alpha + 0^*$

□

Proposition 5.10. *For any cut α , there exists a cut β such that $\alpha + \beta = 0$.*

PROOF:

⟨1⟩1. LET: $\beta = \{p \in \mathbb{Q} : \exists r > 0. -p - r \notin \alpha\}$

⟨1⟩2. β is a cut.

⟨2⟩1. $\beta \neq \emptyset$

⟨3⟩1. PICK $q \notin \alpha$

⟨3⟩2. $-q - 1 \in \beta$

⟨2⟩2. $\beta \neq \mathbb{Q}$

⟨3⟩1. PICK $q \in \alpha$

PROVE: $-q \notin \beta$

⟨3⟩2. ASSUME: for a contradiction $-q \in \beta$

- $\langle 3 \rangle 3$. PICK $r > 0$ such that $q - r \notin \alpha$
- $\langle 3 \rangle 4$. $q - r < q$
- $\langle 3 \rangle 5$. Q.E.D.

PROOF: This contradicts the fact that α is closed downwards.

- $\langle 2 \rangle 3$. β is closed downwards.
 - $\langle 3 \rangle 1$. LET: $p \in \beta$ and $q < p$.
 - $\langle 3 \rangle 2$. PICK $r > 0$ such that $-p - r \notin \alpha$
 - $\langle 3 \rangle 3$. $-p - r < -q - r$
 - $\langle 3 \rangle 4$. $-q - r \notin \alpha$
 - $\langle 3 \rangle 5$. $q \in \beta$
- $\langle 2 \rangle 4$. β has no greatest element.
 - $\langle 3 \rangle 1$. LET: $p \in \beta$
 - $\langle 3 \rangle 2$. PICK $r > 0$ such that $-p - r \notin \alpha$
 - $\langle 3 \rangle 3$. $-(p + r/2) - r/2 \notin \alpha$
 - $\langle 3 \rangle 4$. $p + r/2 \in \beta$
- $\langle 1 \rangle 3$. $\alpha + \beta \subseteq 0^*$
 - $\langle 2 \rangle 1$. LET: $p \in \alpha$ and $q \in \beta$.
 - $\langle 2 \rangle 2$. PICK $r > 0$ such that $-q - r \notin \alpha$.
 - $\langle 2 \rangle 3$. $p < -q - r$
 - $\langle 2 \rangle 4$. $p + q < -r$
 - $\langle 2 \rangle 5$. $p + q < 0$
 - $\langle 2 \rangle 6$. $p + q \in 0^*$
- $\langle 1 \rangle 4$. $0^* \subseteq \alpha + \beta$
 - $\langle 2 \rangle 1$. LET: $v \in 0^*$
 - $\langle 2 \rangle 2$. LET: $w = -v/2$
 - $\langle 2 \rangle 3$. $w > 0$
 - $\langle 2 \rangle 4$. PICK an integer n such that $nw \in \alpha$ and $(n + 1)w \notin \alpha$.
 - $\langle 2 \rangle 5$. LET: $p = -(n + 2)w$
 - $\langle 2 \rangle 6$. $p \in \beta$
 - $\langle 2 \rangle 7$. $v = nw + p$
 - $\langle 2 \rangle 8$. $v \in \alpha + \beta$

□

Proposition 5.11. *Given $\alpha, \beta, \gamma \in R$, if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

PROOF:

- $\langle 1 \rangle 1$. $\alpha + \beta \subseteq \alpha + \gamma$
 PROOF: Immediate from definitions.
- $\langle 1 \rangle 2$. $\alpha + \beta \neq \alpha + \gamma$
 PROOF: If $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$ by cancellation.

□

Definition 5.12. Given cuts α and β , define $\alpha\beta$ by:

$$\alpha\beta = \begin{cases} \{p \in \mathbb{Q} : \exists r \in \alpha. \exists s \in \beta (p \leq rs \wedge r > 0 \wedge s > 0)\} & \text{if } \alpha > 0^* \text{ and } \beta > 0^* \\ (-\alpha)(-\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^* \\ -((-\alpha)\beta) & \text{if } \alpha < 0^* \text{ and } \beta > 0^* \\ -(\alpha(-\beta)) & \text{if } \alpha > 0^* \text{ and } \beta < 0^* \\ 0^* & \text{if } \alpha = 0^* \text{ or } \beta = 0^* \end{cases}$$

Proposition 5.13. For any cuts α and β , we have $\alpha\beta$ is a cut.

PROOF:

(1)1. If $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta$ is a cut.

(2)1. $\alpha\beta \neq \emptyset$

(3)1. PICK $q \in \alpha$ and $r \in \beta$ such that $q, r \notin 0^*$

(3)2. ASSUME: w.l.o.g. $0 < q$ and $0 < r$.

PROOF: Since α and β have no greatest element.

(3)3. $qr \in \alpha\beta$

(2)2. $\alpha\beta \neq \mathbb{Q}$

(3)1. PICK $r \notin \alpha$ and $s \notin \beta$

PROVE: $rs \notin \alpha\beta$

(3)2. ASSUME: for a contradiction $rs \in \alpha\beta$.

(3)3. PICK $r' \in \alpha$ and $s' \in \beta$ such that $rs \leq r's'$ and $r' > 0$ and $s' > 0$.

(3)4. $r' < r$ and $s' < s$

(3)5. $r's' < rs$

(3)6. Q.E.D.

PROOF: This is a contradiction.

(2)3. $\alpha\beta$ is closed downwards.

(3)1. LET: $p \in \alpha\beta$ and $p' < p$

(3)2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$

(3)3. $p' \leq rs$

(3)4. $p' \in \alpha\beta$

(2)4. $\alpha\beta$ has no greatest element.

(3)1. LET: $p \in \alpha\beta$

(3)2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$.

(3)3. PICK $r' \in \alpha$ and $s' \in \beta$ with $r < r'$ and $s < s'$.

(3)4. $p < r's' \in \alpha\beta$

(1)2. For any cuts α and β , we have $\alpha\beta$ is a cut.

PROOF: Since if α is a cut then $-\alpha$ is a cut.

□

Proposition 5.14. For any cuts α and β we have $\alpha\beta = \beta\alpha$.

PROOF: Easy from the definitions. □

Proposition 5.15. For any cuts α , β and γ we have

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma .$$

PROOF:

$\langle 1 \rangle 1$. CASE: α , β and γ are all positive.

PROOF: In this case $\alpha(\beta\gamma) = (\alpha\beta)\gamma = \{p \in \mathbb{Q} : \exists r \in \alpha. \exists s \in \beta. \exists t \in \gamma. (p \leq rst \wedge r > 0 \wedge s > 0 \wedge t > 0)\}$.

$\langle 1 \rangle 2$. CASE: One of α , β or γ is 0^* .

PROOF: Then $\alpha(\beta\gamma) = (\alpha\beta)\gamma = 0^*$.

$\langle 1 \rangle 3$. CASE: α and β are positive, γ is negative.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\ &= -(\alpha(\beta(-\gamma))) \\ &= -((\alpha\beta)(-\gamma)) && (\langle 1 \rangle 1) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 4$. CASE: α is positive, β is negative, γ is positive.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-((- \beta)\gamma)) \\ &= -(\alpha((- \beta)\gamma)) \\ &= -((\alpha(-\beta))\gamma) && (\langle 1 \rangle 1) \\ &= -(\alpha(-\beta))\gamma \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 5$. CASE: α is positive, β and γ are negative.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha((- \beta)(- \gamma)) \\ &= (\alpha(-\beta))(-\gamma) && (\langle 1 \rangle 1) \\ &= -(\alpha\beta)(-\gamma) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 6$. CASE: α is negative, β and γ are positive.

PROOF: Similar to $\langle 1 \rangle 3$.

$\langle 1 \rangle 7$. CASE: α is negative, β is positive, γ is negative.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\ &= (-\alpha)(\beta(-\gamma)) \\ &= ((-\alpha)\beta)(-\gamma) && (\langle 1 \rangle 1) \\ &= -(\alpha\beta)(-\gamma) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 8$. CASE: α and β are negative, γ is positive.

PROOF: Similar to $\langle 1 \rangle 5$.

$\langle 1 \rangle 9$. CASE: α , β and γ are all negative.

PROOF:

$$\begin{aligned}
 \alpha(\beta\gamma) &= \alpha(-(-\beta)(-\gamma)) \\
 &= -((- \alpha)((-\beta)(-\gamma))) \\
 &= -(((- \alpha)(-\beta))(-\gamma)) & ((1)1) \\
 &= -((\alpha\beta)(-\gamma)) \\
 &= (\alpha\beta)\gamma
 \end{aligned}$$

□

Proposition 5.16. *For any cut α we have $\alpha 1^* = \alpha$.*

PROOF:

$\langle 1 \rangle 1$. CASE: α is positive.

$\langle 2 \rangle 1$. $\alpha 1^* \subseteq \alpha$

$\langle 2 \rangle 2$. $\alpha \subseteq \alpha 1^*$

$\langle 1 \rangle 2$. CASE: $\alpha = 0^*$

$\langle 1 \rangle 3$. CASE: α is negative.

□

Theorem 5.17. *There exists an ordered field with the least upper bound property.*

Proposition 5.18. *There is no rational p such that $p^2 = 2$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $p^2 = 2$.

$\langle 1 \rangle 2$. PICK integers m, n not both even such that $p = m/n$.

$\langle 1 \rangle 3$. $m^2 = 2n^2$

$\langle 1 \rangle 4$. m is even.

$\langle 1 \rangle 5$. PICK an integer k such that $m = 2k$.

$\langle 1 \rangle 6$. $4k^2 = 2n^2$

$\langle 1 \rangle 7$. $2k^2 = n^2$

$\langle 1 \rangle 8$. n is even.

$\langle 1 \rangle 9$. Q.E.D.

PROOF: $\langle 1 \rangle 2$, $\langle 1 \rangle 4$ and $\langle 1 \rangle 8$ form a contradiction.

□

Theorem 5.19. *Any two complete ordered fields are isomorphic.*

Definition 5.20. Let \mathbb{R} be the complete ordered field. We call its elements *real numbers*.

5.2 Properties of the Real Numbers

Theorem 5.21. \mathbb{Q} is a subfield of \mathbb{R} .

Theorem 5.22 (Archimedean Property). *Let $x, y \in \mathbb{R}$ with $x > 0$. There exists a positive integer n such that $nx > y$.*

PROOF:

- (1)1. LET: $A = \{nx : n \in \mathbb{Z}^+\}$
- (1)2. ASSUME: for a contradiction there is no positive integer n such that $nx > y$.
- (1)3. y is an upper bound for A .
- (1)4. LET: $\alpha = \sup A$
- (1)5. $\alpha - x$ is not an upper bound for A .
- (1)6. PICK a positive integer m such that $\alpha - x < mx$
- (1)7. $\alpha < (m+1)x \in A$
- (1)8. Q.E.D.

PROOF: This contradicts (1)4.

□

Theorem 5.23. \mathbb{Q} is dense in \mathbb{R} .

PROOF:

- (1)1. LET: $x, y \in \mathbb{R}$ with $x < y$
- (1)2. PICK a positive integer n such that $n(y-x) > 1$.
- PROOF: Archimedean property.
- (1)3. PICK a positive integer m_1 such that $m_1 > nx$
- PROOF: Archimedean property.
- (1)4. PICK a positive integer m_2 such that $m_2 > -nx$
- PROOF: Archimedean property.
- (1)5. $-m_2 < nx < m_1$
- (1)6. LET: m be the integer such that $m-1 \leq nx < m$.
- (1)7. $nx < m \leq 1 + nx < ny$
- (1)8. $x < m/n < y$

□

Theorem 5.24. For every real number $x > 0$ and positive integer n , there exists a unique positive real number y such that $y^n = x$.

PROOF:

- (1)1. There exists a real $y > 0$ such that $y^n = x$.
- (2)1. LET: $E = \{t \in \mathbb{R}^+ : t^n < x\}$
- (2)2. LET: $y = \sup E$
- (3)1. $E \neq \emptyset$
- (4)1. LET: $t = x/(x+1)$
- (4)2. $0 < t < 1$
- (4)3. $t^n < t < x$
- (4)4. $t \in E$
- (3)2. $x+1$ is an upper bound for E .
- (4)1. LET: $t > x+1$
- (4)2. $t^n > t > x$
- (4)3. $t \notin E$

⟨2⟩3. $y^n = x$

⟨3⟩1. $y^n \not\leq x$

⟨4⟩1. ASSUME: for a contradiction $y^n < x$.

⟨4⟩2. PICK h such that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y + 1)^{n-1}} .$$

⟨4⟩3. $(y + h)^n - y^n < x - y^n$

PROOF:

$$\begin{aligned} (y + h)^n - y^n &= ((y + h) - y) \sum_{i=0}^{n-1} (y + h)^{n-1-i} y^i \\ &= h \sum_{i=0}^{n-1} (y + h)^{n-1-i} y^i \\ &\leq hn(y + h)^{n-1} \\ &\leq hn(y + 1)^{n-1} \\ &< x - y^n \end{aligned}$$

⟨4⟩4. $(y + h)^n < x$

⟨4⟩5. $y + h \in E$

⟨4⟩6. Q.E.D.

PROOF: This contradicts the fact that y is an upper bound for E .

⟨3⟩2. $y^n \not\geq x$

⟨4⟩1. ASSUME: for a contradiction $y^n > x$

⟨4⟩2. LET:

$$k = \frac{y^n - x}{ny^{n-1}}$$

⟨4⟩3. $0 < k < y$

⟨4⟩4. $y - k$ is an upper bound for E .

⟨5⟩1. LET: $t \geq y - k$

⟨5⟩2. $y^n - t^n \leq y^n - x$

PROOF:

$$\begin{aligned} y^n - t^n &\leq y^n - (y - k)^n \\ &= (y - (y - k)) \sum_{i=0}^{n-1} y^{n-i} (y - k)^i \\ &= k \sum_{i=0}^{n-1} y^{n-i} (y - k)^i \\ &\leq kny^{n-1} \\ &= y^n - x \end{aligned}$$

⟨5⟩3. $t^n \geq x$

⟨5⟩4. $t \notin E$

⟨4⟩5. Q.E.D.

PROOF: This contradicts the fact that y is the least upper bound of E .

⟨1⟩2. If y and y' are positive reals with $y^n = y'^n$ then $y = y'$.

PROOF: Since the function that sends y to y^n is strictly monotone.
 \square

Definition 5.25 (*n*th Root). Given any real number $x > 0$ and positive integer n , the *n*th root of x , denoted $x^{1/n}$, is the unique positive real such that

$$(x^{1/n})^n = x .$$

We write \sqrt{x} for $x^{1/2}$.

Proposition 5.26. Let a and b be positive real numbers and n a positive integer. Then

$$(ab)^{1/n} = a^{1/n}b^{1/n} .$$

PROOF: Since $(a^{1/n}b^{1/n})^n = ab$. \square

Lemma 5.27. Let b be a real number with $b > 1$. Let n be a positive integer. Then

$$b - 1 \geq n(b^{1/n} - 1) .$$

PROOF: From Lemma 4.24. \square

Lemma 5.28. Let b and t be real numbers with $b > 1$ and $t > 1$. For any positive integer n , if $n > \frac{b-1}{t-1}$ then $b^{1/n} < t$.

PROOF:

$$\begin{aligned} b - 1 &\geq n(b^{1/n} - 1) \\ \therefore \frac{b - 1}{n} &\geq b^{1/n} - 1 \\ \therefore t - 1 &> b^{1/n} - 1 \\ \therefore t &> b^{1/n} \end{aligned} \quad \square$$

Lemma 5.29. Let b be a real number with $b > 0$. Let m, n, p, q be integers with $n > 0$ and $q > 0$. Assume $m/n = p/q$. Then

$$(b^m)^{1/n} = (b^p)^{1/q} .$$

PROOF:

$$\langle 1 \rangle 1. (b^m)^{1/n} = (b^{1/n})^m$$

PROOF:

$$\begin{aligned} ((b^{1/n})^m)^n &= ((b^{1/n})^n)^m \\ &= b^m \end{aligned}$$

$$\langle 1 \rangle 2. ((b^m)^{1/n})^q = b^p$$

PROOF:

$$\begin{aligned} ((b^m)^{1/n})^q &= (b^{1/n})^{mq} \\ &= (b^{1/n})^{np} \\ &= b^p \end{aligned}$$

\square

Definition 5.30. For a a positive real and q a rational number, we may therefore define a^q by

$$a^{m/n} = (a^m)^{1/n}$$

for m and n integers with $n > 0$.

Proposition 5.31. Let a be a positive real and r, s rational numbers. Then

$$a^{r+s} = a^r a^s .$$

PROOF:

$$\begin{aligned} a^{m/n+p/q} &= a^{(mq+np)/nq} \\ &= (a^{mq+np})^{1/nq} \\ &= (a^{mq})^{1/nq} (a^{np})^{1/nq} \\ &= a^{m/n} a^{p/q} \end{aligned} \quad \square$$

Proposition 5.32. Let $b > 1$ be a real number and q a rational number. Then

$$b^q = \sup\{b^t : t \in \mathbb{Q}, t \leq q\}$$

PROOF: It is the greatest element of this set. \square

Definition 5.33. Let $b > 1$ be a real number and x a real number. Then

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t \leq x\} .$$

Lemma 5.34. Let b, w and y be real numbers with $b > 1$. Assume $b^w < y$. Then there exists a positive integer n such that $b^{w+1/n} < y$.

PROOF:

- $\langle 1 \rangle 1$. LET: $t = yb^{-w}$
- $\langle 1 \rangle 2$. PICK a positive integer n such that $n > \frac{b-1}{t-1}$.
- $\langle 1 \rangle 3$. $b^{1/n} < t$

PROOF: Lemma 5.28.

- $\langle 1 \rangle 4$. $b^{w+1/n} < y$

\square

Lemma 5.35. Let b, w and y be real numbers with $b > 1$. Assume $b^w > y$. Then there exists a positive integer n such that $b^{w-1/n} < y$.

PROOF:

- $\langle 1 \rangle 1$. LET: $t = b^w/y$
- $\langle 1 \rangle 2$. PICK a positive integer n such that $n > \frac{b-1}{t-1}$
- $\langle 1 \rangle 3$. $b^{1/n} < t$

PROOF: Lemma 5.28.

- $\langle 1 \rangle 4$. $y < b^{w-1/n}$

\square

Proposition 5.36. *For b and x real numbers with $b > 1$ we have*

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t < x\} .$$

PROOF:

- $\langle 1 \rangle 1.$ b^x is an upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
- $\langle 1 \rangle 2.$ LET: u be any upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
PROVE: $b^x \leq u$
- $\langle 1 \rangle 3.$ LET: q be a rational number with $q \leq x$.
PROVE: $b^q \leq u$
- $\langle 1 \rangle 4.$ ASSUME: for a contradiction $b^q > u$.
- $\langle 1 \rangle 5.$ PICK a positive integer n such that $b^{q-1/n} > u$.
PROOF: Lemma 5.35.
- $\langle 1 \rangle 6.$ $b^{q-1/n} \leq u$
PROOF: $\langle 1 \rangle 2$
- $\langle 1 \rangle 7.$ Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 4$.

□

Lemma 5.37. *Let A be a set of positive real numbers with supremum $a > 0$ and B a set of positive real numbers with supremum $b > 0$. Then ab is the supremum of $\{xy : x \in A, y \in B\}$.*

PROOF:

- $\langle 1 \rangle 1.$ For all $x \in A$ and $y \in B$ we have $xy \leq ab$.
- $\langle 1 \rangle 2.$ If u is any upper bound for $\{xy : x \in A, y \in B\}$ then $ab \leq u$.
 - $\langle 2 \rangle 1.$ LET: u be an upper bound for $\{xy : x \in A, y \in B\}$.
 - $\langle 2 \rangle 2.$ For all $x \in A$ we have u/x is an upper bound for B .
 - $\langle 2 \rangle 3.$ For all $x \in A$ we have $b \leq u/x$
 - $\langle 2 \rangle 4.$ For all $x \in A$ we have $x \leq u/b$
 - $\langle 2 \rangle 5.$ $a \leq u/b$
 - $\langle 2 \rangle 6.$ $ab \leq u$

□

Proposition 5.38. *Let $b, x, y \in \mathbb{R}$ with $b > 1$. Then*

$$b^{x+y} = b^x b^y .$$

PROOF:

- $\langle 1 \rangle 1.$ For any rational number $q < x + y$, there exist rational numbers $r < x$ and $s < y$ such that $q = r + s$.
 - $\langle 2 \rangle 1.$ $q - x < y$
 - $\langle 2 \rangle 2.$ PICK a rational t such that $q - x < t < y$
 - $\langle 2 \rangle 3.$ $q = t + (q - t)$ and $t < y, q - t < x$
- $\langle 1 \rangle 2.$ $b^x b^y = b^{x+y}$

PROOF:

$$\begin{aligned}
 b^x b^y &= \sup\{b^q b^r : q, r \in \mathbb{Q}, q < x, r < y\} \\
 &= \sup\{b^{q+r} : q, r \in \mathbb{Q}, q < x, r < y\} \\
 &= \sup\{b^q : q \in \mathbb{Q}, q < x + y\} \\
 &= b^{x+y}
 \end{aligned}$$

□

5.2.1 Logarithms

Proposition 5.39. *Let b and y be real numbers with $b > 1$ and $y > 0$. There exists a unique real x such that $b^x = y$.*

PROOF:

⟨1⟩1. LET: $x = \sup\{w : b^w < y\}$

PROVE: $b^x = y$

⟨2⟩1. $\{w : b^w < y\} \neq \emptyset$

PROOF: It contains 0.

⟨2⟩2. $\{w : b^w < y\}$ is bounded above.

⟨3⟩1. LET: n be the least integer such that

$$n \geq \frac{y-1}{b-1}$$

PROOF: Archimedean property.

⟨3⟩2. LET: w be a real number with $b^w < y$

PROVE: $w < n$

⟨3⟩3. $b^w < n(b-1) + 1$

⟨3⟩4. $b^w < b^n$

⟨3⟩5. $w < n$

⟨1⟩2. $b^x \leq y$

⟨2⟩1. ASSUME: for a contradiction $b^x > y$

⟨2⟩2. PICK a positive integer n such that $b^{x-1/n} > y$

PROOF: Lemma 5.35.

⟨2⟩3. PICK w such that $x - 1/n < w$ and $b^w < y$

PROOF: Since $x - 1/n$ is not an upper bound for $\{w : b^w < y\}$.

⟨2⟩4. $b^{x-1/n} < y$

⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩2.

⟨1⟩3. $b^x \geq y$

⟨2⟩1. ASSUME: for a contradiction $b^x < y$.

⟨2⟩2. PICK a positive integer n such that $b^{x+1/n} < y$.

⟨2⟩3. $x + 1/n \leq x$

⟨2⟩4. Q.E.D.

PROOF: This is a contradiction.

□

Definition 5.40 (Logarithm). Let b and y be real numbers with $b > 1$ and $y > 0$. The *logarithm* of y to base b , denoted $\log_b y$, is the unique real number

such that

$$b^{\log_b y} = y \ .$$

5.2.2 Intervals

Definition 5.41 (Intervals). Let $a, b \in \mathbb{R}$.

The *open interval* (a, b) is $\{x \in \mathbb{R} : a < x < b\}$.

The *closed interval* $[a, b]$ is $\{x \in \mathbb{R} : a \leq x \leq b\}$.

The *half-open intervals* $[a, b)$ and $(a, b]$ are defined by

$$\begin{aligned} [a, b) &:= \{x \in \mathbb{R} : a \leq x < b\} \\ (a, b] &:= \{x \in \mathbb{R} : a < x \leq b\} \end{aligned}$$

Definition 5.42 (k -cell). Let k be a positive integer. A k -cell is a subset of \mathbb{R}^k of the form

$$\{\vec{x} \in \mathbb{R}^k : \forall i = 1, \dots, k. a_i \leq x_i \leq b_i\}$$

for some real numbers $a_1, \dots, a_k, b_1, \dots, b_k$ with $a_i \leq b_i$ for each i .

5.3 The Extended Real Number System

Definition 5.43 (Extended Real Number System). The *extended real number system* is the set $\mathbb{R} \cup \{+\infty, -\infty\}$.

We extend the ordering \leq to the extended reals by defining

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

We extend $+$, \cdot and $/$ to partial operations on the extended real by defining:

$$\begin{array}{ll}
x + (+\infty) = +\infty & (x \in \mathbb{R}) \\
x + (-\infty) = -\infty & (x \in \mathbb{R}) \\
(+\infty) + x = +\infty & (x \in \mathbb{R}) \\
(+\infty) + (+\infty) \text{ is undefined} & \\
(+\infty) + (-\infty) \text{ is undefined} & \\
(-\infty) + x = -\infty & (x \in \mathbb{R}) \\
(-\infty) + (+\infty) \text{ is undefined} & \\
(-\infty) + (-\infty) \text{ is undefined} & \\
x \cdot (+\infty) = +\infty & (x \in \mathbb{R}) \\
x \cdot (-\infty) = -\infty & (x \in \mathbb{R}) \\
(+\infty) \cdot x = +\infty & (x \in \mathbb{R}) \\
(+\infty) \cdot (+\infty) \text{ is undefined} & \\
(+\infty) \cdot (-\infty) \text{ is undefined} & \\
(-\infty) \cdot x = -\infty & (x \in \mathbb{R}) \\
(-\infty) \cdot (+\infty) \text{ is undefined} & \\
(-\infty) \cdot (-\infty) \text{ is undefined} & \\
x / (+\infty) = 0 & (x \in \mathbb{R}) \\
x / (-\infty) = 0 & (x \in \mathbb{R}) \\
(+\infty) / x \text{ is undefined} & (x \in \mathbb{R}) \\
(+\infty) / (+\infty) \text{ is undefined} & \\
(+\infty) / (-\infty) \text{ is undefined} & \\
(-\infty) / x \text{ is undefined} & (x \in \mathbb{R}) \\
(-\infty) / (+\infty) \text{ is undefined} & \\
(-\infty) / (-\infty) \text{ is undefined} &
\end{array}$$

Chapter 6

Complex Analysis

Definition 6.1 (Complex Numbers). A *complex number* is a pair of real numbers. We write \mathbb{C} for the set of complex numbers.

Define $+$ and \cdot on \mathbb{C} by:

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b)(c, d) &= (ac - bd, ad + bc)\end{aligned}$$

Theorem 6.2. *The complex numbers form a field.*

Theorem 6.3. *The function that maps a to $(a, 0)$ is an embedding of \mathbb{R} in \mathbb{C} .*

Definition 6.4.

$$i = (0, 1)$$

Lemma 6.5.

$$(a, b) = a + ib$$

PROOF: Since $(a, 0) + (0, 1)(b, 0) = (a, b)$. \square

Lemma 6.6.

$$i^2 = -1$$

PROOF: Immediate from definitions. \square

Corollary 6.6.1. *There is no linear order on \mathbb{C} that makes \mathbb{C} into an ordered field.*

Definition 6.7 (Complex Conjugate). For any complex number z , the *complex conjugate* \bar{z} is defined by

$$\overline{a + ib} = a - ib \quad (a, b \in \mathbb{R}) .$$

Definition 6.8 (Real Part). For any complex number z , the *real part* of z , denoted $\operatorname{Re}(z)$, is defined by

$$\operatorname{Re}(a + ib) = a \quad (a, b \in \mathbb{R}) .$$

Definition 6.9 (Imaginary Part). For any complex number z , the *imaginary part* of z , denoted $\text{Im}(z)$, is defined by

$$\text{Im}(a + ib) = b \quad (a, b \in \mathbb{R}) .$$

Theorem 6.10. For all $z, w \in \mathbb{C}$ we have

$$\overline{z + w} = \bar{z} + \bar{w} .$$

PROOF:

$$\begin{aligned} \overline{(a + ib) + (c + id)} &= \overline{(a + c) + i(b + d)} \\ &= (a + c) - i(b + d) \\ &= (a - ib) + (c - id) \\ &= \overline{a + ib} + \overline{c + id} \end{aligned} \quad \square$$

Theorem 6.11. For all $z, w \in \mathbb{C}$ we have

$$\overline{zw} = \bar{z} \cdot \bar{w} .$$

PROOF:

$$\begin{aligned} \overline{(a + ib)(c + id)} &= \overline{(ac - bd) + i(ad + bc)} \\ &= (ac - bd) - i(ad + bc) \\ &= (a - ib)(c - id) \\ &= \overline{a + ib} \cdot \overline{c + id} \end{aligned} \quad \square$$

Theorem 6.12. For all $z \in \mathbb{C}$ we have

$$\text{Re}(z) = \frac{1}{2}(z + \bar{z}) .$$

PROOF:

$$\begin{aligned} (a + ib) + \overline{a + ib} &= (a + ib) + (a - ib) \\ &= 2a \\ &= 2 \text{Re}(a + ib) \end{aligned} \quad \square$$

Theorem 6.13. For all $z \in \mathbb{C}$ we have

$$\text{Im}(z) = \frac{1}{2i}(z - \bar{z}) .$$

PROOF:

$$\begin{aligned} (a + ib) - \overline{a + ib} &= (a + ib) - (a - ib) \\ &= 2ib \\ &= 2i \text{Im}(a + ib) \end{aligned} \quad \square$$

Theorem 6.14. For all $z \in \mathbb{C}$ we have $z\bar{z}$ is a non-negative real.

PROOF:

$$\begin{aligned}(a + ib)(\overline{a + ib}) &= (a + ib)(a - ib) \\ &= a^2 + b^2\end{aligned}\quad \square$$

Theorem 6.15. *For any $z \in \mathbb{C}$, if $z\bar{z} = 0$ then $z = 0$.*

PROOF: Let $z = a + ib$. Then $z\bar{z} = a^2 + b^2 = 0$ iff $a = b = 0$. \square

Definition 6.16 (Absolute Value). For $z \in \mathbb{C}$, the *absolute value* of z is

$$|z| = (z\bar{z})^{1/2}.$$

Proposition 6.17. *For x a non-negative real we have $|x| = x$.*

PROOF: Since $|x| = \sqrt{x^2} = x$. \square

Proposition 6.18. *For x a negative real we have $|x| = -x$.*

PROOF: Since $|x| = \sqrt{x^2} = -x$. \square

Theorem 6.19. *For any complex number z we have $|z| \geq 0$.*

PROOF: Immediate from definition. \square

Theorem 6.20. *For any complex number z , if $|z| = 0$ then $z = 0$.*

PROOF: From Theorem 6.15. \square

Theorem 6.21. *For any complex number z we have*

$$|\bar{z}| = |z|.$$

PROOF: Immediate from definitions. \square

Theorem 6.22. *For any complex numbers z and w we have*

$$|zw| = |z||w|.$$

PROOF:

$$\begin{aligned}|zw| &= \sqrt{zw\bar{z}\bar{w}} \\ &= \sqrt{z\bar{z}}\sqrt{w\bar{w}} && \text{(Proposition 5.26)} \\ &= |z||w|\end{aligned}\quad \square$$

Theorem 6.23. *For any complex number z we have*

$$|\operatorname{Re} z| \leq |z|$$

PROOF: Let $z = a + ib$. Then

$$|\operatorname{Re} z| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}. \square$$

Theorem 6.24. *For any complex numbers z and w we have*

$$|z + w| \leq |z| + |w|.$$

PROOF:

$$\begin{aligned}
 |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\
 &= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\
 &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 && \text{(Theorem 6.12)} \\
 &\leq |z|^2 + 2|z\bar{w}| + |w|^2 && \text{(Theorem 6.23)} \\
 &= |z|^2 + 2|z||w| + |w|^2 && \text{(Theorem 6.22)} \\
 &= (|z| + |w|)^2 && \square
 \end{aligned}$$

Theorem 6.25 (Schwarz Inequality). *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be complex numbers. Then*

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2 .$$

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } A = \sum_{j=1}^n |a_j|^2$$

$$\langle 1 \rangle 2. \text{ LET: } B = \sum_{j=1}^n |b_j|^2$$

$$\langle 1 \rangle 3. \text{ LET: } C = \sum_{j=1}^n a_j \bar{b}_j$$

$$\langle 1 \rangle 4. \text{ ASSUME: w.l.o.g. } B > 0$$

PROOF: If $B = 0$ then $b_1 = \dots = b_n = 0$ and both sides of the inequality are 0.

$$\langle 1 \rangle 5. \sum_{j=1}^n |Ba_j - Cb_j|^2 = B(AB - |C|^2)$$

PROOF:

$$\begin{aligned}
 \sum_{j=1}^n |Ba_j - Cb_j|^2 &= \sum_{j=1}^n (Ba_j - Cb_j)(B\bar{a}_j - \bar{C}\bar{b}_j) \\
 &= B^2 \sum_{j=1}^n |a_j|^2 - B\bar{C} \sum_{j=1}^n a_j \bar{b}_j - BC \sum_{j=1}^n \bar{a}_j b_j + |C|^2 \sum_{j=1}^n |b_j|^2 \\
 &= B^2 A - 2B|C|^2 + B|C|^2 \\
 &= B(AB - |C|^2)
 \end{aligned}$$

$$\langle 1 \rangle 6. B(AB - |C|^2) \geq 0$$

$$\langle 1 \rangle 7. AB \geq |C|^2$$

\square

Proposition 6.26. *For any non-zero complex number w , there are exactly two complex numbers z such that $z^2 = w$.*

PROOF:

$$\langle 1 \rangle 1. \text{ There are at most two complex numbers } z \text{ such that } z^2 = w.$$

PROOF: Proposition 4.15.

$$\langle 1 \rangle 2. \text{ There are at least two complex numbers } z \text{ such that } z^2 = w.$$

$$\langle 2 \rangle 1. \text{ LET: } w = u + iv$$

$$\langle 2 \rangle 2. \text{ LET: } a = \sqrt{\frac{|w|+u}{2}}$$

$$\langle 2 \rangle 3. \text{ LET: } b = \sqrt{\frac{|w|-u}{2}}$$

$\langle 2 \rangle 4.$ CASE: $v \geq 0$

$\langle 3 \rangle 1.$ LET: $z = a + ib$

$\langle 3 \rangle 2.$ $z^2 = w$

PROOF:

$$\begin{aligned} z^2 &= (a + ib)^2 \\ &= a^2 - b^2 + 2iab \\ &= u + i\sqrt{|w|^2 - u^2} \\ &= u + iv \\ &= w \end{aligned}$$

$\langle 3 \rangle 3.$ $(-z)^2 = w$

$\langle 2 \rangle 5.$ CASE: $v \leq 0$

$\langle 3 \rangle 1.$ LET: $z = a - ib$

$\langle 3 \rangle 2.$ $z^2 = w$

PROOF:

$$\begin{aligned} z^2 &= (a - ib)^2 \\ &= a^2 - b^2 - 2iab \\ &= u - i\sqrt{|w|^2 - u^2} \\ &= u - i|v| \\ &= w \end{aligned}$$

$\langle 3 \rangle 3.$ $(-z)^2 = w$

□

Part I

Linear Algebra

Chapter 7

Vector Spaces

7.1 Convex Sets

Definition 7.1 (Convex). Let $E \subseteq \mathbb{R}^k$. Then E is *convex* iff, for all $\vec{x}, \vec{y} \in E$ and $\lambda \in (0, 1)$,

$$\lambda \vec{x} + (1 - \lambda) \vec{y} \in E \text{ .}$$

Proposition 7.2. *Every k -cell is convex.*

PROOF:

$\langle 1 \rangle 1$. LET: $C = \{\vec{x} \in \mathbb{R}^k : \forall i. a_i \leq x_i \leq b_i\}$ be a k -cell.

$\langle 1 \rangle 2$. LET: $\vec{x}, \vec{y} \in C$ and $\lambda \in (0, 1)$.

PROVE: $\lambda \vec{x} + (1 - \lambda) \vec{y} \in C$

$\langle 1 \rangle 3$. For each i we have $a_i \leq \lambda x_i + (1 - \lambda) y_i \leq b_i$

PROOF: Since $\lambda a_1 + (1 - \lambda) a_i \leq \lambda x_i + (1 - \lambda) y_i \leq \lambda b_i + (1 - \lambda) b_i$.

□

Chapter 8

Real Inner Product Spaces

Definition 8.1 (Inner Product). Given $\vec{x}, \vec{y} \in \mathbb{R}^k$, define the *inner product* $\vec{x} \cdot \vec{y}$ by

$$(x_1, \dots, x_k) \cdot (y_1, \dots, y_k) = x_1 y_1 + \dots + x_k y_k \ .$$

Definition 8.2 (Norm). Define the *norm* of a vector $\vec{x} \in \mathbb{R}^k$ by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} \ .$$

Proposition 8.3.

$$\|\vec{x}\| \geq 0$$

PROOF: Immediate from the definition. \square

Proposition 8.4. If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$.

PROOF: If $\|\vec{x}\| = 0$ then $x_1^2 + \dots + x_n^2 = 0$ so $x_1 = \dots = x_n = 0$. \square

Proposition 8.5. For $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^k$,

$$\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| \ .$$

PROOF: Easy. \square

Proposition 8.6. For $\vec{x}, \vec{y} \in \mathbb{R}^k$, we have

$$\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\| \|\vec{y}\| \ .$$

PROOF: By the Schwarz inequality. \square

Proposition 8.7. For $\vec{x}, \vec{y} \in \mathbb{R}^k$ we have

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \ .$$

PROOF:

$$\begin{aligned}
 \|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
 &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\
 &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Proposition 8.6)} \\
 &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square
 \end{aligned}$$

Corollary 8.7.1. For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ we have

$$\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| .$$

8.1 Balls

Definition 8.8 (Closed Ball). Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *closed ball* with centre \vec{x} and radius r is

$$\{y \in \mathbb{R}^k : \|y - x\| \leq r\} .$$

Proposition 8.9. Every closed ball is convex.

PROOF:

(1)1. LET: B be the closed ball with center \vec{a} and radius r .

(1)2. LET: $\vec{x}, \vec{y} \in B$

(1)3. LET: $\lambda \in (0, 1)$

(1)4. $\lambda\vec{x} + (1 - \lambda)\vec{y} \in B$

PROOF:

$$\begin{aligned}
 \|\lambda\vec{x} + (1 - \lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1 - \lambda)(\vec{y} - \vec{a})\| \\
 &= \lambda\|\vec{x} - \vec{a}\| + (1 - \lambda)\|\vec{y} - \vec{a}\| \\
 &\leq \lambda r + (1 - \lambda)r \\
 &= r && \square
 \end{aligned}$$

\square

Chapter 9

Complex Inner Product Spaces

Definition 9.1 (Inner Product). Let V be a complex vector space. An *inner product* on V is a function $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{C}$ such that, for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

- $\langle y, x \rangle = \overline{\langle x, y \rangle}$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- $\langle x, x \rangle \geq 0$
- If $\langle x, x \rangle = 0$ then $x = 0$.

An *inner product space* consists of a complex vector space V and an inner product on V .

Definition 9.2 (Norm). Let V be an inner product space and $x \in V$. The *norm* of x is

$$\|x\| = \sqrt{\langle x, x \rangle} .$$

Proposition 9.3. *An inner product space is a metric space under*

$$d(x, y) = \|x - y\| .$$

Definition 9.4 (Bounded). Let V_1 and V_2 be inner product spaces and $T : V_1 \rightarrow V_2$ a linear transformation. Then T is *bounded* iff $\{\|T(x)\| : \|x\| = 1\}$ is bounded above.

Proposition 9.5. *Every linear transformation between finite dimensional inner product spaces is bounded.*

Definition 9.6 (Outer Product). Let V be an inner product space and $|\psi\rangle, |\phi\rangle \in V$. The *outer product* of $|\psi\rangle$ and $|\phi\rangle$ is

$$|\psi\rangle \langle \phi| : V \rightarrow V .$$

9.1 Hilbert Spaces

Definition 9.7 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Theorem 9.8 (Completeness Relation). Let \mathcal{H} be a Hilbert space. Let $\{|e_n\rangle\}_{n \in \mathbb{N}}$ be a countable orthonormal basis for \mathcal{H} . Then

$$\sum_{n=0}^{\infty} |e_n\rangle \langle e_n| = I \quad .$$

PROOF:

(1)1. LET: $|\psi\rangle \in \mathcal{H}$

(1)2. LET: $|\psi\rangle = \sum_{n=0}^{\infty} \alpha_n |e_n\rangle$

(1)3. $\sum_{n=0}^{\infty} \langle e_n | \phi \rangle |e_n\rangle = |\psi\rangle$

PROOF:

$$\begin{aligned} \sum_{n=0}^{\infty} \langle e_n | \phi \rangle |e_n\rangle &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_m \langle e_n | e_m \rangle |e_n\rangle \\ &= \sum_{n=0}^{\infty} \alpha_n |e_n\rangle \\ &= |\psi\rangle \end{aligned}$$

□

□

Definition 9.9 (Separable). A Hilbert space is *separable* iff it has a countable dense orthonormal basis.

Chapter 10

Lie Algebras

Definition 10.1 (Lie Algebra). Let K be a field. A *Lie algebra* \mathcal{L} over K consists of a vector space \mathcal{L} over K and an operation

$$[\cdot, \cdot] : \mathcal{L}^2 \rightarrow \mathcal{L} ,$$

the *Lie bracket* or *commutator*, such that, for all $x, y, z \in \mathcal{L}$ and $\alpha \in K$:

$$\begin{aligned} [x + y, z] &= [x, z] + [y, z] \\ [x, y + z] &= [x, y] + [x, z] \\ [\alpha x, y] &= \alpha[x, y] \\ [x, x] &= 0 \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \end{aligned} \quad (\text{Jacobi identity})$$

Lemma 10.2. If K has characteristic 0 then the condition $[x, x] = 0$ can be replaced with $[x, y] = -[y, x]$.

Proposition 10.3. The commutator is determined by its values on any basis for \mathcal{L} .

Example 10.4. \mathbb{R}^3 with the cross product is a real Lie algebra.

Example 10.5. For any $n \geq 0$, we have $GL(n, K)$ is a Lie algebra over K under

$$[A, B] = AB - BA .$$

Definition 10.6 (Linear Lie Algebra). A *linear Lie algebra* over K is a Lie algebra over K that is a subalgebra of $GL(n, K)$ for some n .

Example 10.7 (Special Linear Algebra). The *special Linear algebra* $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \text{tr} = 0\}$ is a real linear Lie algebra.

Example 10.8 (Orthogonal Lie Algebra). The *orthogonal Lie algebra* $SO(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A \text{ is skew-symmetric}\}$ is a real linear Lie algebra.

Example 10.9. Let $u(n)$ be the set of all skew-Hermitian $n \times n$ -matrices as a real Lie algebra.

Let $su(n) = u(n) \cap SL(n, \mathbb{R})$.

Proposition 10.10. $SU(2)$ is spanned by the Pauli matrices

$$\sigma_x = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_y = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which satisfy

$$\begin{aligned} [\sigma_x, \sigma_y] &= \sigma_z \\ [\sigma_y, \sigma_z] &= \sigma_x \\ [\sigma_z, \sigma_x] &= \sigma_y \end{aligned}$$

10.1 Lie Algebar Homomorphisms

Definition 10.11 (Homomorphism). Let L_1 and L_2 be Lie algebras over the same field. A *Lie algebra homomorphism* $\phi : L_1 \rightarrow L_2$ is a linear transformation such that

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in L_1$.

Lemma 10.12. *Every bijective Lie algebra homomorphism is an isomorphism.*

Definition 10.13 (Representation). Let L be a real (complex) Lie algebra. A *representation* of L is a Lie algebra homomorphism $L \rightarrow GL(n, \mathbb{R})$ ($GL(n, \mathbb{C})$) for some n .

Example 10.14. The linear transformation $\mathbb{R}^3 \rightarrow su(2)$ defined by

$$i \mapsto \sigma_x, j \mapsto \sigma_y, k \mapsto \sigma_z$$

is a representation of \mathbb{R}^3 .

Part II

Topology

Chapter 11

Metric Spaces

Definition 11.1 (Metric). A *metric* on a set X is a function $d : X^2 \rightarrow \mathbb{R}$ such that, for all $x, y, z \in X$:

- $d(x, y) \geq 0$
- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- **Triangle Inequality** $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* X consists of a set X and a metric on X .

Example 11.2. \mathbb{R}^k is a metric space under $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$. The triangle inequality is Corollary 8.7.1.

Proposition 11.3. Let (X, d) be a metric space and Y a subset of X . Then $d \upharpoonright Y^2$ is a metric on Y .

PROOF: Easy. \square

11.1 Balls

Definition 11.4 (Open Ball). Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *open ball* with *centre* \vec{x} and *radius* r is

$$\{y \in \mathbb{R}^k : \|y - x\| < r\} .$$

Proposition 11.5. Every open ball in \mathbb{R}^k is convex.

PROOF:

(1)1. LET: B be the open ball with center \vec{a} and radius r .

(1)2. LET: $\vec{x}, \vec{y} \in B$

(1)3. LET: $\lambda \in (0, 1)$

(1)4. $\lambda\vec{x} + (1 - \lambda)\vec{y} \in B$

PROOF:

$$\begin{aligned}
 \|\lambda\vec{x} + (1 - \lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1 - \lambda)(\vec{y} - \vec{a})\| \\
 &= \lambda\|\vec{x} - \vec{a}\| + (1 - \lambda)\|\vec{y} - \vec{a}\| \\
 &< \lambda r + (1 - \lambda)r \\
 &= r
 \end{aligned}$$

□

□

11.2 Limit Points

Definition 11.6 (Limit Point). Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is a *limit point* of E iff every open ball with centre p contains a point of E other than p .

Proposition 11.7. Let X be a metric space. Let $E \subseteq X$. Let p be a *limit point* of E . Then every neighbourhood of p contains infinitely many points of E .

PROOF:

(1)1. ASSUME: for a contradiction N is a neighbourhood of p that contains only finitely many points q_1, \dots, q_n of $E - \{p\}$.

(1)2. LET: $r = \min(q_1, \dots, q_n)$

(1)3. LET: B be the open ball with centre p and radius r .

(1)4. B is a neighbourhood of p that contains no points of E other than p .

□

Corollary 11.7.1. A finite set has no limit points.

Definition 11.8 (Isolated Point). Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *isolated point* of E iff $p \in E$ and p is not a limit point of E .

11.3 Closed Sets

Definition 11.9 (Closed Set). Let X be a metric space. Let $E \subseteq X$. Then E is *closed* iff every limit point of E is a member of E .

11.4 Interior Points

Definition 11.10 (Interior Point). Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *interior point* of E iff there exists an open ball B with centre p such that $B \subseteq E$.

11.5 Open Sets

Definition 11.11 (Open Sets). Let X be a metric space. Let $E \subseteq X$. Then E is *open* iff every point in E is an interior point of E .

Proposition 11.12. *Every open ball is open.*

PROOF:

- $\langle 1 \rangle 1$. LET: B be an open ball with centre c and radius r .
- $\langle 1 \rangle 2$. LET: $x \in B$
- $\langle 1 \rangle 3$. LET: $\epsilon = r - d(x, c)$
- $\langle 1 \rangle 4$. LET: B' be the open ball with centre x and radius ϵ .
- PROVE: $B' \subseteq B$
- $\langle 1 \rangle 5$. LET: $y \in B'$
- $\langle 1 \rangle 6$. $d(y, c) < r$

PROOF:

$$\begin{aligned}
 d(y, c) &\leq d(y, x) + d(x, c) && \text{(Triangle Inequality)} \\
 &< \epsilon + d(x, c) && (\langle 1 \rangle 5) \\
 &= r && (\langle 1 \rangle 3)
 \end{aligned}$$

□

Proposition 11.13. *A set is open if and only if its complement is closed.*

PROOF:

- $\langle 1 \rangle 1$. LET: $E \subseteq X$
- $\langle 1 \rangle 2$. If E is open then $X - E$ is closed.
 - $\langle 2 \rangle 1$. ASSUME: E is open.
 - $\langle 2 \rangle 2$. LET: p be a limit point of $X - E$.
 - PROVE: $p \in X - E$
 - $\langle 2 \rangle 3$. ASSUME: for a contradiction $p \in E$.
 - $\langle 2 \rangle 4$. PICK an open ball B with centre p such that $B \subseteq E$.
 - $\langle 2 \rangle 5$. B contains a point of $X - E$.
 - PROOF: $\langle 2 \rangle 2$
 - $\langle 2 \rangle 6$. Q.E.D.
 - PROOF: This contradicts $\langle 2 \rangle 4$.
- $\langle 1 \rangle 3$. If $X - E$ is closed then E is open.
 - $\langle 2 \rangle 1$. ASSUME: $X - E$ is closed.
 - $\langle 2 \rangle 2$. LET: $p \in E$
 - $\langle 2 \rangle 3$. ASSUME: for a contradiction no open ball with centre p is a subset of E .
 - $\langle 2 \rangle 4$. Every open ball with centre p intersects $X - E$.
 - $\langle 2 \rangle 5$. p is a limit point of $X - E$.
 - $\langle 2 \rangle 6$. $p \in X - E$
 - PROOF: $\langle 2 \rangle 1$
 - $\langle 2 \rangle 7$. Q.E.D.
 - PROOF: This contradicts $\langle 2 \rangle 2$.

□

Corollary 11.13.1. *A set is closed if and only if its complement is open.*

Proposition 11.14. *The union of a set of open sets is open.*

PROOF:

- $\langle 1 \rangle 1$. LET: \mathcal{U} be a set of open sets.
- $\langle 1 \rangle 2$. LET: $p \in \bigcup \mathcal{U}$
- $\langle 1 \rangle 3$. PICK $U \in \mathcal{U}$ such that $p \in U$.
- $\langle 1 \rangle 4$. PICK an open ball B with centre p such that $B \subseteq U$.
- $\langle 1 \rangle 5$. $B \subseteq \bigcup \mathcal{U}$

□

Corollary 11.14.1. *The intersection of a set of closed sets is closed.*

Proposition 11.15. *The intersection of two open sets is open.*

PROOF:

- $\langle 1 \rangle 1$. LET: U and V be open.
- $\langle 1 \rangle 2$. LET: $p \in U \cap V$
- $\langle 1 \rangle 3$. PICK open balls B_1 and B_2 with centre p such that $B_1 \subseteq U$ and $B_2 \subseteq V$.
- $\langle 1 \rangle 4$. ASSUME: w.l.o.g. the radius of B_1 is \leq the radius of B_2 .
- $\langle 1 \rangle 5$. $B_1 \subseteq U \cap V$

□

Corollary 11.15.1. *The union of two closed sets is closed.*

Example 11.16. The intersection of a set of open sets is not necessarily open.

For every positive integer n , we have $(-1/n, 1/n)$ is open in \mathbb{R} , but $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$ is not open.

Theorem 11.17. *Let X be a metric space. Let $Y \subseteq X$ and $E \subseteq Y$. Then E is open in Y if and only if there exists an open subset G of X such that $E = G \cap Y$.*

PROOF:

- $\langle 1 \rangle 1$. If E is open in Y then there exists an open subset G of X such that $E = G \cap Y$.
- $\langle 2 \rangle 1$. ASSUME: E is open in Y .
- $\langle 2 \rangle 2$. For $p \in E$, PICK $r_p > 0$ such that the open ball in Y with centre p and radius r_p is included in E .
- $\langle 2 \rangle 3$. For $p \in E$,
LET: V_p be the open ball in X with centre p and radius r_p .
- $\langle 2 \rangle 4$. LET: $G = \bigcup_{p \in E} V_p$
- $\langle 2 \rangle 5$. G is open in Y .
- PROOF: Proposition 11.14.
- $\langle 2 \rangle 6$. $E = G \cap Y$
- $\langle 3 \rangle 1$. $E \subseteq G \cap Y$
 - $\langle 4 \rangle 1$. LET: $p \in E$
 - $\langle 4 \rangle 2$. $p \in V_p$
 - $\langle 4 \rangle 3$. $p \in G$
- $\langle 3 \rangle 2$. $G \cap Y \subseteq E$
 - $\langle 4 \rangle 1$. LET: $x \in G \cap Y$
 - $\langle 4 \rangle 2$. PICK $p \in E$ such that $x \in V_p$
 - $\langle 4 \rangle 3$. $d(x, p) < r_p$

- $\langle 4 \rangle 4. x \in E$
 $\langle 1 \rangle 2.$ For any open subset G of X , we have $G \cap Y$ is open in Y .
 $\langle 2 \rangle 1.$ LET: G be an open subset of X .
 $\langle 2 \rangle 2.$ LET: $p \in G \cap Y$
 $\langle 2 \rangle 3.$ PICK $r > 0$ such that the open ball in X with centre p and radius r is included in G .
 $\langle 2 \rangle 4.$ The open ball in Y with centre p and radius r is included in $G \cap Y$.
 \square

11.6 Perfect Sets

Definition 11.18 (Perfect Set). Let X be a metric space. Let $E \subseteq X$. Then E is *perfect* iff E is closed and every point in E is a limit point of E .

11.7 Bounded Sets

Definition 11.19 (Bounded Set). Let X be a metric space. Let $E \subseteq X$. Then E is *bounded* iff there exists a real number M and $q \in X$ such that, for all $p \in E$, we have $d(p, q) < M$.

11.8 Dense Sets

Definition 11.20 (Dense Set). Let X be a metric space. Let $E \subseteq X$. Then E is *dense* iff every point of X is either a limit point of E or a point of E , or both.

11.9 Closure

Definition 11.21 (Closure). Let X be a metric space. Let $E \subseteq X$. Then the *closure* of E , denoted \overline{E} , is the union of E and the set of limit points of E .

Proposition 11.22. \overline{E} is the smallest closed set that includes E .

PROOF:

- $\langle 1 \rangle 1.$ \overline{E} is closed.
 $\langle 2 \rangle 1.$ LET: p be a limit point of \overline{E} .
 $\langle 2 \rangle 2.$ ASSUME: $p \notin E$
 PROVE: p is a limit point of E .
 $\langle 2 \rangle 3.$ LET: B be the open ball with centre p and radius r .
 PROVE: B intersects E .
 $\langle 2 \rangle 4.$ PICK a point $q \in B \cap \overline{E}$.
 $\langle 2 \rangle 5.$ PICK an open ball B' with centre q such that $B' \subseteq B$.
 $\langle 2 \rangle 6.$ PICK a point $r \in E \cap B'$
 $\langle 2 \rangle 7.$ $r \in E \cap B$
 $\langle 1 \rangle 2.$ If C is closed and $E \subseteq C$ then $\overline{E} \subseteq C$.

- ⟨2⟩1. ASSUME: C is closed.
- ⟨2⟩2. ASSUME: $E \subseteq C$
- ⟨2⟩3. LET: $p \in \overline{E}$
- ⟨2⟩4. ASSUME: for a contradiction $p \notin C$
- ⟨2⟩5. p is a limit point of C .
 - ⟨3⟩1. LET: B be an open ball with centre p .
 - ⟨3⟩2. B intersects E .
 - ⟨3⟩3. B intersects C .
 - ⟨3⟩4. B intersects C in a point other than p .
- PROOF: ⟨2⟩3
- ⟨2⟩6. Q.E.D.
- PROOF: This contradicts ⟨2⟩1.

□

Corollary 11.22.1. E is closed if and only if $E = \overline{E}$.

Theorem 11.23. Let E be a nonempty set of real numbers bounded above. Then $\sup E \in \overline{E}$.

PROOF:

- ⟨1⟩1. ASSUME: $\sup E \notin E$
 - PROVE: $\sup E$ is a limit point of E .
- ⟨1⟩2. LET: B be an open ball with centre $\sup E$ and radius r .
- ⟨1⟩3. There exists $x \in E$ such that $x > \sup E - r$.
- ⟨1⟩4. E intersects B in a point other than p .

□

11.10 Compact Sets

Definition 11.24 (Open Cover). Let X be a metric space. Let $E \subseteq X$. An *open cover* of E is a set \mathcal{U} of open sets such that $E \subseteq \bigcup \mathcal{U}$.

Definition 11.25 (Compact Set). Let X be a metric space. Let $K \subseteq X$. Then K is *compact* iff every open cover of K includes a finite subcover.

Proposition 11.26. Every finite set is compact.

PROOF: Easy. □

Theorem 11.27. Let X be a metric space. Let $Y \subseteq X$ and $K \subseteq Y$. Then K is compact in Y if and only if K is compact in X .

PROOF:

- ⟨1⟩1. If K is compact in Y then K is compact in X .
- ⟨2⟩1. ASSUME: K is compact in Y .
- ⟨2⟩2. LET: \mathcal{U} be an open cover of K in X .
- ⟨2⟩3. $\{U \cap Y : U \in \mathcal{U}\}$ is an open cover of K in Y .
- ⟨2⟩4. PICK a finite subcover $\{U_1 \cap Y, \dots, U_n \cap Y\}$

- $\langle 2 \rangle 5$. $\{U_1, \dots, U_n\}$ is a finite subset of \mathcal{U} that is an open cover of K in X .
- $\langle 1 \rangle 2$. If K is compact in X then K is compact in Y .
- $\langle 2 \rangle 1$. ASSUME: K is compact in X .
- $\langle 2 \rangle 2$. LET: \mathcal{U} be an open cover of K in Y .
- $\langle 2 \rangle 3$. $\{U \text{ open in } X : U \cap Y \in \mathcal{U}\}$ is an open cover of K in X .
- $\langle 2 \rangle 4$. PICK a finite subcover $\{U_1, \dots, U_n\}$.
- $\langle 2 \rangle 5$. $\{U_1 \cap Y, \dots, U_n \cap Y\}$ is a subset of \mathcal{U} that is an open cover of E in Y .

□

Proposition 11.28. *Every compact set is closed.*

PROOF:

- $\langle 1 \rangle 1$. LET: E be compact.
- $\langle 1 \rangle 2$. LET: $p \in X - E$
 PROVE: There exists an open ball with centre p that is a subset of $X - E$.
- $\langle 1 \rangle 3$. For all $q \in E$, there exist disjoint open balls B with centre q and B' with centre p .
- $\langle 1 \rangle 4$. The set of open balls B such that there exists a disjoint open ball B' with centre p is an open cover of E .
- $\langle 1 \rangle 5$. PICK a finite subcover $\{B_1, \dots, B_n\}$.
- $\langle 1 \rangle 6$. For $i = 1, \dots, n$, PICK an open ball B'_i with centre p such that $B_i \cap B'_i = \emptyset$.
- $\langle 1 \rangle 7$. $B'_1 \cap \dots \cap B'_n$ is an open ball with centre p that is a subset of $X - E$.

□

Proposition 11.29. *Every closed subset of a compact set is compact.*

PROOF:

- $\langle 1 \rangle 1$. LET: E be compact and $C \subseteq E$ be closed.
- $\langle 1 \rangle 2$. LET: \mathcal{U} be an open cover of C .
- $\langle 1 \rangle 3$. $\mathcal{U} \cup \{X - C\}$ is an open cover of E .
- $\langle 1 \rangle 4$. PICK a finite subcover $\{U_1, \dots, U_n\}$ or $\{U_1, \dots, U_n, X - C\}$.
- $\langle 1 \rangle 5$. $\{U_1, \dots, U_n\}$ covers C .

□

Part III

More Algebra

Chapter 12

Lie Groups

Definition 12.1 (Lie Group). A *Lie group* G is a group G that is also an analytic differentiable manifold such that the group operation and inverse operation are analytic.

A *homomorphism of Lie groups* is a group homomorphism that is an analytic function.

Lemma 12.2. *Every bijective Lie group homomorphism is an isomorphism.*

Definition 12.3 (Unitary Group). The *unitary group* $U(n)$ is the Lie group of all $n \times n$ unitary matrices.

Definition 12.4 (Special Unitary Group). The *special unitary group* $SU(n)$ is the Lie group of all $n \times n$ unitary matrices with determinant 1.

Definition 12.5 (Lie Subgroup). Let G be a Lie group. A *Lie subgroup* of G is a subgroup that is also an analytic submanifold of G .

Example 12.6. $U(n)$ and $SU(n)$ are Lie subgroups of $GL(n, \mathbb{C})$.