

Mathematics

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September 20, 2023

Contents

1	Primitive Terms and Axioms	5
1.1	Primitive Terms	5
1.2	Axioms	5
1.3	Consequences of the Axioms	6
1.3.1	Definitions	6
1.3.2	The Empty Set	6
1.3.3	The Singleton	7
1.3.4	Subsets	7
1.4	Composition	8
1.5	Axioms Part Two	8
1.6	Cartesian Product	9
1.7	Quotient Sets	9
1.8	Partitions	9
2	Category Theory	11
2.1	Categories	11
2.2	Functors	12
3	Group Theory	13
4	Linear Algebra	15
5	Topology	17
5.1	Topological Spaces	17
5.1.1	Subspaces	19
5.1.2	Topological Disjoint Union	19
5.1.3	Product Topology	19
5.1.4	Bases	19
5.1.5	Subbases	19
5.1.6	Countability Axioms	19
5.2	Continuous Functions	20
5.3	Convergence	21
5.4	Connected Spaces	21
5.5	Hausdorff Spaces	22

5.6	Separable Spaces	22
5.7	Sequential Compactness	22
5.8	Compactness	22
5.9	Quotient Spaces	23
5.10	Gluing	24
5.11	Metric Spaces	25
5.12	Complete Metric Spaces	25
5.13	Manifolds	26
6	Homotopy Theory	27
6.1	Homotopies	27
6.2	Homotopy Equivalence	27
7	Simplicial Complexes	29
7.1	Cell Decompositions	29
7.2	CW-complexes	29
8	Topological Groups	31
8.1	Continuous Actions	31
9	Topological Vector Spaces	33
9.1	Cauchy Sequences	33
9.2	Seminorms	34
9.3	Fréchet Spaces	34
9.4	Normed Spaces	34
9.5	Inner Product Spaces	35
9.6	Banach Spaces	35
9.7	Hilbert Spaces	35
9.8	Locally Convex Spaces	36

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*. We write $A : \text{Set}$ for: A is a set.

For any set A , let there be *elements* of A . We write $a : \text{El}(A)$ for: a is an element of A .

For any sets A and B , let there be *functions* from A to B . We write $f : A \rightarrow B$ iff f is a function from A to B .

For any function $f : A \rightarrow B$ and element $a : \text{El}(A)$, let there be an element $f(a) : \text{El}(B)$, the *value* of the function f at the *argument* a .

1.2 Axioms

Axiom Schema 1.1 (Choice). *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x : \text{El}(X)$ and $y : \text{El}(Y)$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a : \text{El}(A)$, there exists $b : \text{El}(B)$ such that $P[A, B, a, b]$. Then there exists a function $f : A \rightarrow B$ such that $\forall a : \text{El}(A). P[A, B, a, f(a)]$.

Axiom 1.2 (Pairing). *For any sets A and B , there exists a set $A \times B$, the Cartesian product of A and B , and functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ such that, for all $a : \text{El}(A)$ and $b : \text{El}(B)$, there exists a unique $(a, b) : \text{El}(A \times B)$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Definition 1.3 (Injective). A function $f : A \rightarrow B$ is *injective* or an *injection* iff, for all $x, y : \text{El}(A)$, if $f(x) = f(y)$ then $x = y$.

Axiom Schema 1.4 (Separation). *For every property $P[X, x]$ where X is a set variable and $x : \text{El}(X)$, the following is an axiom:*

For every set A , there exists a set $S = \{x : \text{El}(A) \mid P[A, x]\}$ and an injection $i : S \rightarrow A$ such that, for all $x : \text{El}(A)$, we have

$$(\exists y : S. i(y) = x) \Leftrightarrow P[A, x] .$$

Axiom 1.5 (Infinity). *There exists a set \mathbb{N} , an element $0 : \text{El}(\mathbb{N})$, and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that:*

- $\forall n : \text{El}(\mathbb{N}). s(n) \neq 0$
- $\forall m, n : \text{El}(\mathbb{N}). s(m) = s(n) \Rightarrow m = n.$

1.3 Consequences of the Axioms

1.3.1 Definitions

Definition 1.6. Let $f, g : A \rightarrow B$. We say f and g are *equal*, $f = g$, iff $\forall x : \text{El}(A). f(x) = g(x)$.

Definition 1.7 (Surjective). A function $f : A \rightarrow B$ is *surjective* iff, for all $y : \text{El}(B)$, there exists $x : \text{El}(A)$ such that $f(x) = y$.

Definition 1.8 (Bijective). A function $f : A \rightarrow B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$ ".

Definition 1.9 (Composition). Given $f : A \rightarrow B$ and $g : B \rightarrow C$, let $g \circ f$ be the function such that $\forall a : \text{El}(A). (g \circ f)(a) = g(f(a))$.

1.3.2 The Empty Set

Theorem 1.10. *There exists a set which has no elements.*

PROOF:

<1>1. PICK a set A

PROOF: By the Axiom of Infinity, a set exists.

<1>2. LET: $S = \{x : \text{El}(A) \mid \perp\}$ with injection $i : S \rightarrow A$

PROOF: Axiom of Separation.

<1>3. S has no elements.

□

Theorem 1.11. *If E and E' have no elements then $E \approx E'$.*

PROOF:

$\langle 1 \rangle 1$. LET: E and E' have no elements.

$\langle 1 \rangle 2$. PICK a function $F : E \rightarrow E'$.

PROOF: Axiom of Choice since vacuously $\forall x : \text{El}(E) . \exists y : \text{El}(E') . \top$.

$\langle 1 \rangle 3$. F is injective.

PROOF: Vacuously, for all $x, y : \text{El}(E)$, if $F(x) = F(y)$ then $x = y$.

$\langle 1 \rangle 4$. F is surjective.

PROOF: Vacuously, for all $y : \text{El}(E')$, there exists $x : \text{El}(E)$ such that $F(x) = y$.

□

Definition 1.12 (Empty Set). The *empty set* \emptyset is the set with no elements.

1.3.3 The Singleton

Theorem 1.13. *There exists a set that has exactly one element.*

PROOF:

$\langle 1 \rangle 1$. PICK a set A that has an element.

PROOF: By the Axiom of Infinity, there exists a set that has an element.

$\langle 1 \rangle 2$. PICK $a : \text{El}(A)$

$\langle 1 \rangle 3$. PICK a set S and injection $i : S \rightarrow A$ such that, for all $x : \text{El}(A)$, there exists $s : \text{El}(S)$ such that $s = x$ if and only if $x = a$

$\langle 1 \rangle 4$. S has exactly one element.

□

Theorem 1.14. *If A and B both have exactly one element then $A \approx B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B both have exactly one element a and b respectively.

$\langle 1 \rangle 2$. LET: $F : A \rightarrow B$ be the function such that, for all $x : \text{El}(A)$, we have
 $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$. F is a bijection.

□

Definition 1.15 (Singleton). Let 1 be the set that has exactly one element. Let $*$ be its element.

1.3.4 Subsets

Definition 1.16 (Subset). A *subset* of a set A consists of a set S and an injection $i : S \rightarrow A$. We write $(S, i) : \text{Sub}(A)$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Definition 1.17 (Membership). Given $(S, i) : \text{Sub}(A)$ and $a : \text{El}(A)$, we write $a \in S$ for $\exists s : \text{El}(S) . i(s) = a$.

1.4 Composition

Definition 1.18 (Composite). Let $\phi : A \rightarrowtail B$ and $\psi : B \rightarrowtail C$. The *composite* $\psi \circ \phi : A \rightarrowtail C$ is the relation such that $a(\psi \circ \phi)c$ iff there exists b such that $a\phi b$ and $b\psi c$.

Definition 1.19 (Identity). For any set A , the *identity* function $\text{id}_A : A \rightarrow A$ is the function defined by $\text{id}_A(a) = a$.

Theorem 1.20. *Composition of relations is associative, and the identity function is an identity for composition. The composite of functions is a function. The composite of injective functions is injective. The composite of surjective functions is surjective. The composite of bijections is a bijection. A function $f : A \rightarrow B$ is a bijection iff there exists a function $f^{-1} : B \rightarrow A$ such that $f^{-1}f = \text{id}_A$ and $ff^{-1} = \text{id}_B$, in which case f^{-1} is unique.*

1.5 Axioms Part Two

Axiom 1.21 (Power Set). *For any set A , there exists a set $\mathcal{P}A$, the power set of A , and a relation $\in : A \rightarrowtail \mathcal{P}A$, called membership, such that, for any subset S of A , there exists a unique $\bar{S} \in \mathcal{P}A$ such that, for all $x \in A$, we have $x \in \bar{S}$ if and only if $x \in S$.*

We usually write just S for \bar{S} .

Axiom Schema 1.22 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A , there exists a set B , a function $p : B \rightarrow A$, a set Y and a relation $M : B \rightarrowtail Y$ such that:

- $\forall b \in B. P[A, \{y \in Y : bMy\}, p(b)]$
- *For all $a \in A$, if $\exists Y. P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.*

Definition 1.23 (Universe). Let $E : U \rightarrowtail X$ be a relation. Let us say that a set A is *small* iff there exists $u \in U$ such that $A \approx \{x \in X : uEx\}$.

Then (U, X, E) form a *universe* if and only if:

- \mathbb{N} is U -small.
- For any U -small sets A and B and relation $R : A \rightarrowtail B$, the tabulation of R is U -small.
- If A is U -small then so is $\mathcal{P}A$
- Let $f : A \rightarrow B$ be a function. If B is U -small and $f^{-1}(b)$ is U -small for all $b \in B$, then A is U -small.
- If $p : B \twoheadrightarrow A$ is a surjective function such that A is U -small, then there exists a U -small set C , a surjection $q : C \twoheadrightarrow A$, and a function $f : C \rightarrow B$ such that $q = pf$.

Axiom 1.24 (Universe). *There exists a universe.*

Let $E : U \multimap X$ be a universe. We shall say a set is *small* iff it is U -small, and *large* otherwise.

1.6 Cartesian Product

Definition 1.25 (Cartesian Product). Let A and B be sets. The *Cartesian product* of A and B , $A \times B$, is the tabulation of the relation $A \multimap B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

1.7 Quotient Sets

Proposition 1.26. *Let \sim be an equivalence relation on X . Then there exists a set X/\sim , the quotient set of X with respect to \sim , and a surjective function $\pi : X \rightarrow X/\sim$, the canonical projection, such that, for all $x, y : \text{El}(X)$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \rightarrow Q$ is another quotient with respect to \sim , then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

1.8 Partitions

Definition 1.27 (Partition). A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X .

Chapter 2

Category Theory

2.1 Categories

Definition 2.1. A *category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*
- for any objects X and Y , a set $\text{Mor}(X, Y)$ of *morphisms* from X to Y . We write $f : X \rightarrow Y$ for $f : \text{El}(\text{Mor}(X, Y))$.
- for any objects X, Y and Z , a function $\circ : \text{Mor}(Y, Z) \times \text{Mor}(X, Y) \rightarrow \text{Mor}(X, Z)$, called *composition*.

such that:

- Given $f : X \rightarrow Y, g : Y \rightarrow Z$ and $h : Z \rightarrow W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X , there exists a morphism $\text{id}_X : X \rightarrow X$, the *identity morphism* on X , such that:
 - for any object Y and morphism $f : Y \rightarrow X$ we have $\text{id}_X \circ f = f$
 - for any object Y and morphism $f : X \rightarrow Y$ we have $f \circ \text{id}_X = f$

The identity morphism on an object is unique.

Definition 2.2. Let **Set** be the category of small sets and functions.

Section.

Retraction.

Isomorphism.

Define the opposite category.

Initial object.

Terminal object.

Zero object.

Proposition 2.3. *The empty set is initial in Set.*

PROOF: For any set A , the nowhere-defined function is the unique function $\emptyset \rightarrow A$. \square

Proposition 2.4. *1 is terminal in Set.*

PROOF: For any set A , the constant function to $*$ is the only function $A \rightarrow 1$. \square

2.2 Functors

Definition 2.5 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- a function $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism $f : A \rightarrow B$ in \mathcal{C} , a morphism $Ff : FA \rightarrow FB$ in \mathcal{D}

such that:

- for all $A : \text{El}(\text{Ob}(\mathcal{C}))$ we have $F\text{id}_A = \text{id}_{FA}$
- for any morphism $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathcal{C} , we have $F(g \circ f) = Fg \circ Ff$

Define the identity functor, constant functors.

Functors preserve isomorphisms.

Chapter 3

Group Theory

Definition 3.1. Let \mathbf{Grp} be the category of small groups and group homomorphisms.

Definition 3.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G .

Chapter 4

Linear Algebra

Definition 4.1. For any field K , let \mathbf{Vect}_K be the category of small vector spaces over K and linear transformations.

Dual space functor $\mathbf{Vect}_K^{\mathrm{op}} \rightarrow \mathbf{Vect}_K$.

Chapter 5

Topology

5.1 Topological Spaces

Definition 5.1 (Topological Space). Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say (X, \mathcal{O}) is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call \mathcal{O} the *topology* of the topological space, and call its elements *open sets*. We shall often write X for the topological space (X, \mathcal{O}) .

Definition 5.2 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.

Proposition 5.3. *A set B is open if and only if $X - B$ is closed.*

Proposition 5.4. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that \mathcal{C} is the set of closed sets if and only if:*

- For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$
- For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.
- $\emptyset \in \mathcal{C}$

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Definition 5.5 (Neighbourhood). Let X be a topological space, $x \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x , and x is an *interior* point of U , iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 5.6. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 5.7. *Let X be a set and $\mathcal{N} : X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that, for all $x \in X$, we have \mathcal{N}_x is the set of neighbourhoods of x , if and only if:*

- *For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$*
- *For all $x \in X$ we have $X \in \mathcal{N}_x$*
- *For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$*
- *For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$*
- *For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M. M \in \mathcal{N}_y$.*

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$.

Definition 5.8 (Exterior Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x .

Definition 5.9 (Boundary Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B .

Definition 5.10 (Interior). Let X be a topological space and $B \subseteq X$. The *interior* of B , B° , is the set of all interior points of B .

Proposition 5.11. *The interior of B is the union of all the open sets included in B .*

Definition 5.12 (Closure). Let X be a topological space and $B \subseteq X$. The *closure* of B , \overline{B} , is the set of all points that are not exterior points of B .

Proposition 5.13. *The closure of B is the intersection of all the closed sets that include B .*

Proposition 5.14. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 5.15 (Kuratowski Closure Axioms). *Let X be a set and $- : \mathcal{P}X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} such that, for all $B \subseteq X$, \overline{B} is the closure of B , if and only if:*

- $\overline{\emptyset} = \emptyset$
- *For all $A \subseteq X$ we have $A \subseteq \overline{A}$*
- *For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$*
- *For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

In this case, \mathcal{O} is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

5.1.1 Subspaces

Definition 5.16 (Subspace). Let X be a topological space and $X_0 \subseteq X$. The *subspace topology* on X_0 is $\{U \cap X_0 : U \text{ is open in } X\}$.

Example 5.17. The *unit sphere* S^2 is $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a subspace of \mathbb{R}^3 .

5.1.2 Topological Disjoint Union

Definition 5.18. Let X and Y be topological spaces. The *disjoint union* is $X + Y$ where $U \subseteq X + Y$ is open if and only if $\kappa_1^{-1}(U)$ is open in X and $\kappa_2^{-1}(U)$ is open in Y .

5.1.3 Product Topology

Definition 5.19 (Product Topology). Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

5.1.4 Bases

Definition 5.20 (Basis). Let X be a topological space. A *basis* for the topology on X is a set of open sets \mathcal{B} such that every open set is the union of a subset of \mathcal{B} .

5.1.5 Subbases

Definition 5.21 (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a subset $\mathcal{S} \subseteq \mathcal{P}X$ such that every open set is a union of finite intersections of \mathcal{S} .

Definition 5.22 (Space with Basepoint). A *space with basepoint* is a pair (X, x) where X is a topological space and $x \in \text{El}(X)$.

5.1.6 Countability Axioms

Definition 5.23 (Neighbourhood Basis). Let X be a topological space and $x_0 \in \text{El}(X)$. A *neighbourhood basis* of x_0 is a set \mathcal{U} of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of \mathcal{U} .

Definition 5.24 (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

Definition 5.25 (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

\mathbb{R}^n is second countable.

An uncountable discrete space is first countable but not second countable.

Proposition 5.26. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.*

PROOF:

$\langle 1 \rangle 1$. For all $\lambda : \text{El}(\Lambda)$, PICK U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.

$\langle 1 \rangle 2$. For all $\lambda : \text{El}(\Lambda)$, PICK $x_\lambda \in U_\lambda$.

$\langle 1 \rangle 3$. ASSUME: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda \in \Lambda}$.

$\langle 1 \rangle 4$. PICK $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$

$\langle 1 \rangle 5$. There is no $U \in B$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

□

5.2 Continuous Functions

Definition 5.27 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* iff, for every open set V in Y , the inverse image $f^{-1}(V)$ is open in X .

Proposition 5.28. 1. id_X is continuous

2. The composite of two continuous functions is continuous.

3. If $f : X \rightarrow Y$ is continuous and $X_0 \subseteq X$ then $f|_{X_0} : X_0 \rightarrow Y$ is continuous.

4. If $f : X + Y \rightarrow Z$, then f is continuous iff $f \circ \kappa_1 : X \rightarrow Z$ and $f \circ \kappa_2 : Y \rightarrow Z$ are continuous.

5. If $f : Z \rightarrow X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Definition 5.29 (Homeomorphism). Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Definition 5.30 (Retraction). Let X be a topological space and A a subspace of X . A continuous function $\rho : X \rightarrow A$ is a *retraction* iff $\rho|_A = \text{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 5.31. Let **Top** be the category of small topological spaces and continuous functions.

Forgetful functor **Top** \rightarrow **Set**.

Basepoint preserving continuous functor.

5.3 Convergence

Definition 5.32 (Convergence). Let X be a topological space. Let (x_n) be a sequence in X . A point $a \in \text{El}(X)$ is a *limit* of the sequence iff, for every neighbourhood U of a , there exists n_0 such that $\forall n \geq n_0. x_n \in U$.

Convergence in a product space is pointwise convergence.

If $f : X \rightarrow Y$ is continuous and $x_n \rightarrow l$ in X then $f(x_n) \rightarrow f(l)$ in Y .

Example 5.33. The converse does not hold.

Let X be the set of all continuous functions $[0, 1] \rightarrow [-1, 1]$ under the product topology. Let $i : X \rightarrow L^2([0, 1])$ be the inclusion.

If $f_n \rightarrow f$ then $i(f_n) \rightarrow i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K_\epsilon. \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0, 1]} U_\lambda$ where $U_\lambda = [-1, 1]$ except for $\lambda = \lambda_1, \dots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \dots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 5.34. *The converse does hold for first countable spaces. If $f : X \rightarrow Y$ where X is first countable, and Y is a topological space, and whenever $x_n \rightarrow x$ then $f(x_n) \rightarrow f(x)$, then f is continuous.*

5.4 Connected Spaces

Definition 5.35 (Connected). A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

Proposition 5.36. *The continuous image of a connected space is connected.*

Proposition 5.37. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are connected, then X is connected.*

Proposition 5.38. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 5.39 (Path-connected). A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \rightarrow X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 5.40. *The continuous image of a path connected space is path connected.*

Proposition 5.41. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are path connected, then X is path connected.*

Proposition 5.42. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

5.5 Hausdorff Spaces

Definition 5.43 (Hausdorff). A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 5.44. *In a Hausdorff space, a sequence has at most one limit.*

Proposition 5.45. 1. *Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

Proposition 5.46. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \rightarrow B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X , then $f = g$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.

$\langle 1 \rangle 2$. PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.

$\langle 1 \rangle 3$. PICK $x \in f^{-1}(U) \cap g^{-1}(V)$

$\langle 1 \rangle 4$. $f(x) = g(x) \in U \cap V$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 5.47. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$ be uniformly continuous. Let \hat{X} and \hat{Y} be the completions of X and Y . Then f extends uniquely to a continuous map $\hat{X} \rightarrow \hat{Y}$.*

PROOF: The extension maps $\lim_{n \rightarrow \infty} x_n$ to $\lim_{n \rightarrow \infty} f(x_n)$. □

5.6 Separable Spaces

Definition 5.48 (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

5.7 Sequential Compactness

Definition 5.49 (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

5.8 Compactness

Definition 5.50 (Compact). A topological space is *compact* iff every open cover has a finite subcover.

Proposition 5.51. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P . Then $X \in P$.*

PROOF:

- $\langle 1 \rangle 1.$ P is an open cover of X
 $\langle 1 \rangle 2.$ PICK a finite subcover $U_1, \dots, U_n \in P$
 $\langle 1 \rangle 3.$ $X = U_1 \cup \dots \cup U_n \in P$
 \square

Corollary 5.51.1. *Let f be a compact space and $f : X \rightarrow \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. \square

Proposition 5.52. *The continuous image of a compact space is compact.*

Proposition 5.53. *A closed subspace of a compact space is compact.*

Proposition 5.54. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. X and Y are compact.
2. $X + Y$ is compact.
3. $X \times Y$ is compact.

Proposition 5.55. *A compact subspace of a Hausdorff space is closed.*

Proposition 5.56. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 5.57. *A first countable compact space is sequentially compact.*

5.9 Quotient Spaces

Definition 5.58 (Quotient Space). Let X be a topological space and \sim an equivalence relation on X . The *quotient topology* on X/\sim is defined by: $U : \text{El}(\mathcal{P}X)$ is open in X/\sim if and only if $\pi^{-1}(U)$ is open in X .

Proposition 5.59. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $f : X/\sim \rightarrow Y$. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proposition 5.60. *Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $\phi : Y \rightarrow X/\sim$.*

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \rightarrow X$ such that $\pi \circ \Phi = \phi|U$. Then ϕ is continuous.

Proposition 5.61. *A quotient of a connected space is connected.*

Proposition 5.62. *A quotient of a path connected space is path connected.*

Proposition 5.63. *Let X be a topological space and \sim an equivalence relation on X . If X/\sim is Hausdorff then every equivalence class of \sim is closed in X .*

Definition 5.64. Let X be a topological space and $A_1, \dots, A_r \subseteq X$. Then $X/A_1, \dots, A_r$ is the quotient space of X with respect to \sim where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 5.65 (Cone). Let X be a topological space. The *cone over X* is the space $(X \times [0, 1])/(X \times \{1\})$.

Definition 5.66 (Suspension). Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 5.67 (Wedge Product). Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 5.68 (Smash Product). Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 5.69. $D^n/S^{n-1} \cong S^n$

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : D^n/S^{n-1} \rightarrow S^n$ be the function induced by the map $D^n \rightarrow S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.

$\langle 1 \rangle 2$. ϕ is a bijection.

$\langle 1 \rangle 3$. ϕ is a homeomorphism.

PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.

□

5.10 Gluing

Definition 5.70 (Gluing). Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \rightarrow Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where \sim is the equivalence relation generated by $x \sim \phi(x)$ for all $x : \text{El}(X)$.

Proposition 5.71. *Y is a subspace of $Y \cup_\phi X$.*

Definition 5.72. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0, 1])/\alpha$ is the quotient space of $X \times [0, 1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x : \text{El}(X)$.

Definition 5.73 (Möbius Strip). The *Möbius strip* is $([-1, 1] \times [0, 1])/\alpha$ where $\alpha(x) = -x$.

Definition 5.74 (Klein Bottle). The *Klein bottle* is $(S^1 \times [0, 1])/\alpha$ where $\alpha(z) = \bar{z}$.

Proposition 5.75. Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\text{id}_M} M \cong K$.

PROOF:

$\langle 1 \rangle 1$. LET: $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{i\theta/2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-i\theta/2}, t)$.

$\langle 1 \rangle 2$. f induces a bijection $M \cup_{\text{id}_M} M \approx K$

$\langle 1 \rangle 3$. f is a homeomorphism.

□

5.11 Metric Spaces

Definition 5.76 (Metric Space). Let X be a set and $d : X^2 \rightarrow \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geq 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d) . We often write X for the metric space (X, d) .

Definition 5.77 (Topology of a Metric Space). Let (X, d) be a metric space. The topology *induced* by the metric d is defined by: for $V \subseteq X$, we have V is open if and only if, for all $x \in V$, there exists $\epsilon > 0$ such that $\{y \in X : d(x, y) < \epsilon\} \subseteq V$.

Definition 5.78 (Metrizable). A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 5.79. Every metrizable space is Hausdorff.

Every metrizable space is first countable.

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

5.12 Complete Metric Spaces

Definition 5.80 (Complete). A metric space is *complete* iff every Cauchy sequence converges.

Example 5.81. \mathbb{R} is complete.

Proposition 5.82. *The product of two complete metric spaces is complete.*

Proposition 5.83. *Every compact metric space is complete.*

Proposition 5.84. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 5.85 (Completion). Let X be a metric space. A *completion* of X is a complete metric space \hat{X} and injection $i : X \rightarrow \hat{X}$ such that:

- The metric on X is the restriction of the metric on \hat{X}
- X is dense in \hat{X} .

Proposition 5.86. *Let $i_1 : X \rightarrow Y_1$ and $i_2 : X \rightarrow Y_2$ be completions of X . Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

PROOF: Define $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$. \square

Theorem 5.87. *Every metric space has a completion.*

PROOF: Let \hat{X} be the set of Cauchy sequences in X quotiented by \sim where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \rightarrow 0$. \square

5.13 Manifolds

Definition 5.88 (Manifold). An n -dimensional manifold is a second countable Hausdorff space locally homeomorphic to \mathbb{R}^n .

Chapter 6

Homotopy Theory

6.1 Homotopies

Definition 6.1 (Homotopy). Let X and Y be topological spaces. Let $f, g : X \rightarrow Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0, 1] \rightarrow Y$ such that

- $\forall x : \text{El}(X) . h(x, 0) = f(x)$
- $\forall x : \text{El}(X) . h(x, 1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \rightarrow Y$.

Proposition 6.2. Let $f, f' : X \rightarrow Y$ and $g, g' : Y \rightarrow Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.

Definition 6.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

Definition 6.4. A functor $F : \mathbf{Top} \rightarrow \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \rightarrow Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

6.2 Homotopy Equivalence

Definition 6.5 (Homotopy Equivalence). Let X and Y be topological spaces. A *homotopy equivalence* between X and Y , $f : X \simeq Y$, is a continuous function $f : X \rightarrow Y$ such that there exists a continuous function $g : Y \rightarrow X$, the *homotopy inverse* to f , such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.

Definition 6.6 (Contractible). A topological space X is *contractible* iff $X \simeq 1$.

Example 6.7. \mathbb{R}^n is contractible.

Example 6.8. D^n is contractible.

Definition 6.9 (Deformation Retract). Let X be a topological space and A a subspace of X . A retraction $\rho : X \rightarrow A$ is a *deformation retraction* iff $i \circ \rho \simeq \text{id}_X$, where i is the inclusion $A \hookrightarrow X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 6.10 (Strong Deformation Retract). Let X be a topological space and A a subspace of X . A *strong deformation retraction* $\rho : X \rightarrow A$ is a continuous function such that there exists a homotopy $h : X \times [0, 1] \rightarrow X$ between $i \circ \rho$ and id_X such that, for all $a : \text{El}(X)$ and $t : \text{El}([0, 1])$, we have $h(a, t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 6.11. $\{0\}$ is a strong deformation retract of \mathbb{R}^n and of D^n .

Example 6.12. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 6.13. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 6.14. For any topological space X , the singleton consisting of the vertex is a strong deformation retract of the cone over X .

Chapter 7

Simplicial Complexes

Definition 7.1 (Simplex). A k -dimensional simplex or k -simplex in \mathbb{R}^n is the convex hull $s(x_0, \dots, x_k)$ of $k + 1$ points in general position.

Definition 7.2 (Face). A sub-simplex or face of $s(x_0, \dots, x_k)$ is the convex hull of a subset of $\{x_0, \dots, x_k\}$.

Definition 7.3 (Simplicial Complex). A simplicial complex in \mathbb{R}^n is a set K of simplices such that:

- for every simplex s in K , every face of s is in K .
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2 .
- K is locally finite, i.e. every point of \mathbb{R}^n has a neighbourhood that only intersects finitely many elements of K .

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of \mathbb{R}^n .

7.1 Cell Decompositions

Definition 7.4 (n -cell). An n -cell is a topological space homeomorphic to \mathbb{R}^n .

Definition 7.5 (Cell Decomposition). Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n -cells.

Definition 7.6 (n -skeleton). Given a cell decomposition of X , the n -skeleton X^n is the union of all the cells of dimension $\leq n$.

7.2 CW-complexes

Definition 7.7 (CW-Complex). A *CW-complex* consists of a topological space X and a cell decomposition \mathcal{E} of X such that:

1. *Characteristic Maps* For every n -cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \rightarrow X$ such that $\Phi_e((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in \mathcal{E} of dimension $< n$.
2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have \bar{e} intersects only finitely many other cells in \mathcal{E} .
3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \bar{e}$ is closed.

Proposition 7.8. *If a cell decomposition \mathcal{E} satisfies the Characteristic Maps axiom, then for every n -cell $e \in \mathcal{E}$ we have $\bar{e} = \Phi_e(D^n)$. Therefore \bar{e} is compact and $\bar{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

PROOF:

$\langle 1 \rangle 1.$ $e \subseteq \Phi_e(D^n) \subseteq \bar{e}$

PROOF:

$$\begin{aligned}
 e &= \Phi_e((D^n)^\circ) \\
 &\subseteq \Phi_e(D^n) \\
 &= \Phi_e(\overline{(D^n)^\circ}) \\
 &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
 &= \bar{e}
 \end{aligned}$$

$\langle 1 \rangle 2.$ $\Phi_e(D^n)$ is compact.

PROOF: Because D^n is compact.

$\langle 1 \rangle 3.$ $\Phi_e(D^n)$ is closed.

$\langle 1 \rangle 4.$ $\Phi_e(D^n) = \bar{e}$

□

Chapter 8

Topological Groups

Definition 8.1 (Topological Group). A *topological group* is a group G with a topology such that the function $G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.

Example 8.2. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 8.3. Any subgroup of a topological group is a topological group under the subspace topology.

Definition 8.4 (Homogeneous Space). A *homogeneous space* is a topological space of the form G/H , where G is a topological group and H is a normal subgroup of G , under the quotient topology.

Proposition 8.5. Let G be a topological group and H a normal subgroup of G . Then G/H is Hausdorff if and only if H is closed.

PROOF: See Bourbaki, N., General Topology. III.12 \square

8.1 Continuous Actions

Definition 8.6 (Continuous Action). Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \rightarrow X$ such that:

- $\forall x : \text{El}(X) . ex = x$
- $\forall g, h : \text{El}(G) . \forall x : \text{El}(X) . g(hx) = (gh)x$

A G -space consists of a topological space X and a continuous action of G on X .

Definition 8.7 (Orbit). Let X be a G -space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 8.8. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:

$\langle 1 \rangle 1$. LET: $f_3 : S^2/SO(2) \rightarrow [-1, 1]$ be the function induced by $\pi_3 : S^2 \rightarrow [-1, 1]$

$\langle 1 \rangle 2$. f_3 is bijective.

$\langle 1 \rangle 3$. $S^2/SO(2)$ is compact.

PROOF: It is the continuous image of S^2 which is compact.

$\langle 1 \rangle 4$. $[-1, 1]$ is Hausdorff.

$\langle 1 \rangle 5$. f_3 is a homeomorphism.

□

Definition 8.9 (Stabilizer). Let X be a G -space and $x \in X$. The *stabilizer* of x is $G_x := \{g : \text{El}(G) \mid gx = x\}$.

Proposition 8.10. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx .*

PROOF:

$\langle 1 \rangle 1$. If $gG_x = hG_x$ then $gx = hx$.

$\langle 2 \rangle 1$. ASSUME: $gG_x = hG_x$

$\langle 2 \rangle 2$. $g^{-1}h \in G_x$

$\langle 2 \rangle 3$. $g^{-1}hx = x$

$\langle 2 \rangle 4$. $gx = hx$

$\langle 1 \rangle 2$. If $gx = hx$ then $gG_x = hG_x$.

PROOF: Similar.

$\langle 1 \rangle 3$. The function is continuous.

PROOF: Proposition 5.59.

□

Chapter 9

Topological Vector Spaces

Definition 9.1 (Topological Vector Space). Let K be either \mathbb{R} or \mathbb{C} . A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Subtraction is a continuous function $E^2 \rightarrow E$
- Multiplication is a continuous function $K \times E \rightarrow E$

Proposition 9.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. \square

Theorem 9.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 \square

Proposition 9.4. *Let E be a topological vector space and E_0 a subspace of E . Then $\overline{E_0}$ is a subspace of E .*

Definition 9.5. Let E be a topological vector space. The topological space associated with E is $E/\overline{\{0\}}$.

9.1 Cauchy Sequences

Definition 9.6 (Cauchy Sequence). Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geq n_0, x_n - x_m \in U$.

Definition 9.7 (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

9.2 Seminorms

Definition 9.8 (Seminorm). Let E be a vector space over K . A *seminorm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ such that:

1. $\forall x : \text{El}(E) . \|x\| \geq 0$
2. $\forall \alpha : \text{El}(K) . \forall x : \text{El}(E) . \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y : \text{El}(E) . \|x + y\| \leq \|x\| + \|y\|$

Example 9.9. The function that maps (x_1, \dots, x_n) to $|x_i|$ is a seminorm on \mathbb{R}^n .

Definition 9.10. Let E be a vector space over K . Let Λ be a set of seminorms on E . The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x : \text{El}(E)$.

Proposition 9.11. E is a topological vector space under this topology. It is Hausdorff iff, for all $x : \text{El}(E)$, if $\forall \lambda \in \Lambda . \lambda(x) = 0$ then $x = 0$.

9.3 Fréchet Spaces

Definition 9.12 (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 9.13. Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$. Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 9.14 (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

9.4 Normed Spaces

Definition 9.15 (Normed Space). Let E be a vector space over K . A *norm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ is a seminorm such that, $\forall x \in E . \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 9.16. If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E .

Proposition 9.17. *Let $\| \cdot \|$ be a seminorm on the vector space E . Then $\| \cdot \|$ defines a norm on $E/\{0\}$.*

Proposition 9.18. *Let E and F be normed spaces. Any continuous linear map $E \rightarrow F$ is uniformly continuous.*

Definition 9.19. For $p \geq 1$, let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\{0\}.$$

9.5 Inner Product Spaces

Proposition 9.20. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E .*

9.6 Banach Spaces

Definition 9.21 (Banach Space). A *Banach space* is a complete normed space.

Example 9.22. For any topological space X , the set $C(X)$ of bounded continuous functions $X \rightarrow \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 9.23. *The completion of a normed space is a Banach space.*

Proposition 9.24. *Let E and F be normed spaces. Let $f : E \rightarrow F$ be a continuous linear map. Then the extension to the completions $\hat{E} \rightarrow \hat{F}$ is linear.*

Proposition 9.25. $L^p(\mathbb{R}^n)$ is a Banach space.

Proposition 9.26. $C(\mathbb{R})$ is first countable but not second countable.

PROOF: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n . Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.

It is first countable because it is metrizable. \square

9.7 Hilbert Spaces

Definition 9.27 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Example 9.28. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \rightarrow \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

Proposition 9.29. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

9.8 Locally Convex Spaces

Definition 9.30 (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 9.31. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Proposition 9.32. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Example 9.33. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \rightarrow \mathbb{R}$ is continuous as a map $E' \rightarrow \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 9.34 (Thom Space). Let E be a vector bundle with a Riemannian metric, $DE = \{x : \text{El}(E) \mid \|x\| \leq 1\}$ its disc bundle and $SE := \{v : \text{El}(E) \mid \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE .