Summary of Halmos' Naive Set Theory

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Contents

Chapter 1

Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*, $\in$. If $x \in A$ we say that x *belongs* to A, x is an *element* of A, or x is *contained* in A. If this does not hold we write $x \notin A$.

Axiom 1.1 (Axiom of Extensionality)**.** *Two sets are equal if and only if they have the same elements.*

Axiom 1.2 (Axiom of Comprehension, Aussonderungsaxiom)**.** *To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

Axiom 1.3. *A set exists.*

Axiom 1.4 (Axiom of Pairing)**.** *For any two sets, there exists a set that they both belong to.*

Axiom 1.5 (Union Axiom)**.** *For every set A, there exists a set that contains all the elements that belong to at least one element of A.*

Definition 1.6 (Subset)**.** Let A and B be sets. We say that A is a *subset* of B, or B *includes* A, and write $A \subseteq B$ or $B \supseteq A$, iff every element of A is an element of B.

Axiom 1.7 (Power Set Axiom)**.** *For any set A, there exists a set that contains all the subsets of A.*

Chapter 2

The Subset Relation

Theorem 2.1. *For any set A, we have $A \subseteq A$.*

PROOF: Every element of A is an element of A. $\square$

Theorem 2.2. *For any sets A, B and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

PROOF: If every element of A is an element of B, and every element of B is an element of C, then every element of A is an element of C. $\square$

Theorem 2.3. *For any sets A and B, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

PROOF: If every element of A is an element of B, and every element of B is an element of A, then A and B have the same elements, and therefore are equal by the Axiom of Extensionality. $\square$

Definition 2.4 (Proper Subset)**.** Let A and B be sets. We say that A is a *proper* subset of B, or B *properly* includes A, and write $A \subsetneq B$ or $B \supsetneq A$, iff $A \subseteq B$ and $A \neq B$.

Chapter 3

Comprehension Notation

Definition 3.1. Given a set A and a condition $S(x)$, we write $\{x \in A : S(x)\}$ for the set whose elements are exactly those elements x of A for which $S(x)$ holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality. $\square$

Theorem 3.2. *There is no set that contains every set.*

PROOF:
$\langle 1 \rangle 1$. LET: A be a set.
PROVE: There exists a set B such that $B \notin A$.
$\langle 1 \rangle 2$. LET: $B = \{x \in A : x \notin x\}$
$\langle 1 \rangle 3$. If $B \in A$ then we have $B \in B$ if and only if $B \notin B$.
$\langle 1 \rangle 4$. $B \notin A$
$\square$

Chapter 4

Unordered Pairs

Theorem 4.1. *There exists a set with no elements.*

PROOF: Pick a set A by Axiom 1.3. Then the set $\{x \in A : x \neq x\}$ has no elements. $\square$

Definition 4.2 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

Theorem 4.3. *For any set A we have $\emptyset \subset A$.*

PROOF: Vacuous. $\square$

Definition 4.4 ((Unordered) Pair)**.** For any sets a and b, the *(unordered) pair* $\{a, b\}$ is the set whose elements are just a and b.

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality. $\square$

Definition 4.5 (Singleton)**.** For any set a, the *singleton* $\{a\}$ is defined to be $\{a, a\}$.

Chapter 5

Unions and Intersections

Definition 5.1 (Union)**.** For any set $\mathcal{C}$, the *union* of $\mathcal{C}$, $\bigcup \mathcal{C}$, is the set whose elements are the elements of the elements of $\mathcal{C}$.

We write $\bigcup_{X \in \mathcal{A}} t[X]$ for $\bigcup \{t[X] \mid X \in \mathcal{A}\}$.

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality. □

Proposition 5.2.

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of $\emptyset$. □

Proposition 5.3. *For any set A, we have $\bigcup \{A\} = A$.*

PROOF: For any x, we have x is an element of an element of $\{A\}$ if and only if x is an element of A. □

Definition 5.4. We write $A \cup B$ for $\bigcup \{A, B\}$.

Proposition 5.5. *For any set A, we have $A \cup \emptyset = A$.*

PROOF: $x \in A \cup \emptyset$ iff $x \in A$ or $x \in \emptyset$, iff $x \in A$. □

Proposition 5.6 (Commutativity)**.** *For any sets A and B, we have $A \cup B = B \cup A$.*

PROOF: $x \in A \cup B$ iff $x \in A$ or $x \in B$, iff $x \in B$ or $x \in A$, iff $x \in B \cup A$. □

Proposition 5.7 (Associativity)**.** *For any sets A, B and C, we have $A \cup (B \cup C) = (A \cup B) \cup C$.*

PROOF: Each is the set of all x such that $x \in A$ or $x \in B$ or $x \in C$. □

Proposition 5.8 (Idempotence)**.** *For any set A, we have $A \cup A = A$.*

PROOF: $x \in A$ or $x \in A$ is equivalent to $x \in A$. □

Proposition 5.9. *For any sets A and B, we have $A \subseteq B$ if and only if $A \cup B = B$.*

PROOF: For any x, the statement "if $x \in A$ then $x \in B$" is equivalent to "$x \in A$ or $x \in B$ if and only if $x \in B$". □

Proposition 5.10. *For any sets a and b, we have $\{a\} \cup \{b\} = \{a, b\}$.*

PROOF: Immediate from definitions. □

Definition 5.11 ((Unordered) Triple)**.** Given sets a_1, ..., a_n, define the *(unordered) n-tuple* $\{a_1, \ldots, a_n\}$ to be

$$\{a_1, \ldots, a_n\} := \{a_1\} \cup \cdots \cup \{a_n\} \ .$$

Definition 5.12 (Intersection)**.** For any sets A and B, the *intersection* $A \cap B$ is defined to be $\{x \in A : x \in B\}$.

Proposition 5.13. *For any set A, we have $A \cap \emptyset = \emptyset$.*

PROOF: There is no x such that $x \in A$ and $x \in \emptyset$. □

Proposition 5.14. *For any sets A and B, we have*

$$A \cap B = B \cap A \ .$$

PROOF: $x \in A$ and $x \in B$ if and only if $x \in B$ and $x \in A$. □

Proposition 5.15. *For any sets A, B and C, we have*

$$A \cap (B \cap C) = (A \cap B) \cap C \ .$$

PROOF: Each is the set of all x such that $x \in A$ and $x \in B$ and $x \in C$. □

Proposition 5.16. *For any set A, we have*

$$A \cap A = A \ .$$

PROOF: We have $x \in A$ and $x \in A$ if and only if $x \in A$. □

Proposition 5.17. *For any sets A and B, we have $A \subseteq B$ if and only if $A \cap B = A$.*

PROOF: For any x, the statement "if $x \in A$ then $x \in B$" is equivalent to "$x \in A$ and $x \in B$ if and only if $x \in A$". □

Definition 5.18 (Disjoint)**.** Two sets A and B are *disjoint* if and only if $A \cap B = \emptyset$.

Definition 5.19 (Pairwise Disjoint)**.** Let A be a set. We say the elements of A are *pairwise disjoint* if and only if, for all $x, y \in A$, if $x \cap y \neq \emptyset$ then $x = y$.

Proposition 5.20 (Distributive Law)**.** *For any sets A, B and C, we have*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

PROOF:

$x \in A \wedge (x \in B \vee x \in C) \Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$.□

Proposition 5.21 (Distributive Law)**.** *For any sets A, B and C, we have*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

PROOF:

$x \in A \vee (x \in B \wedge x \in C) \Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$.□

Proposition 5.22. *For any sets A, B and C, we have $C \subseteq A$ if and only if $(A \cap B) \cup C = A \cap (B \cup C)$.*

PROOF: The statement "if $x \in C$ then $x \in A$" is equivalent to the statement "$((x \in A \wedge x \in B) \vee x \in C) \Leftrightarrow (x \in A \wedge (x \in B \vee x \in C))$". □

Definition 5.23 (Intersection)**.** For any nonempty set $\mathcal{C}$, the *intersection* of $\mathcal{C}$, $\bigcap \mathcal{C}$, is the set that contains exactly those sets that belong to every element of $\mathcal{C}$.

We write $\bigcap_{X \in \mathcal{A}} t[X]$ for $\bigcap\{t[X] \mid X \in \mathcal{A}\}$.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{C}$ be a nonempty set.

$\langle 1 \rangle 2$. There exists a set I whose elements are exactly the sets that belong to every element of $\mathcal{C}$.

PROOF: Pick $A \in \mathcal{C}$, and take $I = \{x \in A : \forall X \in \mathcal{C}. x \in X\}$.

$\langle 1 \rangle 3$. For any sets I, J, if the elements of I and J are exactly the sets that belong to every element of $\mathcal{C}$ then $I = J$.

PROOF: Axiom of Extensionality.

□

Chapter 6

Complements and Powers

Definition 6.1 (Relative Complement)**.** For any sets A and B, the *difference* or *relative complement* $A - B$ is defined to be

$$A - B := \{x \in A : x \notin B\} \ .$$

Proposition 6.2. *For any sets A and E, we have $A \subseteq E$ if and only if*

$$E - (E - A) = A$$

PROOF:
⟨1⟩1. LET: A and E be sets.
⟨1⟩2. If $A \subseteq E$ then $E - (E - A) = A$
  ⟨2⟩1. ASSUME: $A \subseteq E$
  ⟨2⟩2. $E - (E - A) \subseteq A$
    PROOF: If $x \in E$ and $x \notin E - A$ then $x \in A$.
  ⟨2⟩3. $A \subseteq E - (E - A)$
    PROOF: If $x \in A$ then $x \in E$ and $x \notin E - A$.
⟨1⟩3. If $E - (E - A) = A$ then $A \subseteq E$.
  PROOF: Since $E - (E - A) \subseteq E$.
□

Proposition 6.3. *For any set E we have*

$$E - \emptyset = E$$

PROOF: $x \in E$ if and only if $x \in E$ and $x \notin \emptyset$. □

Proposition 6.4. *For any set E we have*

$$E - E = \emptyset \ .$$

PROOF: There is no x such that $x \in E$ and $x \notin E$. □

Proposition 6.5. *For any sets A and E, we have*

$$A \cap (E - A) = \emptyset \ .$$

PROOF: There is no x such that $x \in A$ and $x \in E - A$. $\square$

Proposition 6.6. *Let A and E be sets. Then $A \subseteq E$ if and only if*

$$A \cup (E - A) = E \ .$$

PROOF:
$\langle 1 \rangle 1$. LET: A and E be sets.
$\langle 1 \rangle 2$. If $A \subseteq E$ then $A \cup (E - A) = E$.
  $\langle 2 \rangle 1$. ASSUME: $A \subseteq E$
  $\langle 2 \rangle 2$. $A \cup (E - A) \subseteq E$
    PROOF: If $x \in A$ or $x \in E - A$ then $x \in E$.
  $\langle 2 \rangle 3$. $E \subseteq A \cup (E - A)$
    PROOF: If $x \in E$ then either $x \in A$ or $x \notin A$. In the latter case, $x \in E - A$.
$\langle 1 \rangle 3$. If $A \cup (E - A) = E$ then $A \subseteq E$
  PROOF: Since $A \subseteq A \cup (E - A)$.
$\square$

Proposition 6.7. *Let A, B and E be sets. Then:*

1. *If $A \subseteq B$ then $E - B \subseteq E - A$.*
2. *If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.*

PROOF:
$\langle 1 \rangle 1$. LET: A, B and E be sets.
$\langle 1 \rangle 2$. If $A \subseteq B$ then $E - B \subseteq E - A$.
  PROOF: If $A \subseteq B$, $x \in E$ and $x \notin B$, then we have $x \in E$ and $x \notin A$.
$\langle 1 \rangle 3$. If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.
  $\langle 2 \rangle 1$. ASSUME: $A \subseteq E$
  $\langle 2 \rangle 2$. ASSUME: $E - B \subseteq E - A$
  $\langle 2 \rangle 3$. LET: $x \in A$
  $\langle 2 \rangle 4$. $x \in E$
  $\langle 2 \rangle 5$. $x \notin E - A$
  $\langle 2 \rangle 6$. $x \notin E - B$
  $\langle 2 \rangle 7$. $x \in B$
$\square$

Example 6.8. We cannot remove the hypothesis $A \subseteq E$ in item 2 above. Let $E = \emptyset$, $A = \{\emptyset\}$ and $B = \emptyset$. Then $E - B = E - A = \emptyset$ but $A \nsubseteq B$.

Proposition 6.9 (De Morgan's Law)**.** *For any sets A, B and E, we have $E - (A \cup B) = (E - A) \cap (E - B)$.*

PROOF: $(x \in E \wedge \neg(x \in A \vee x \in B)) \Leftrightarrow (x \in E \wedge x \notin A \wedge x \in E \wedge x \notin B)$. $\square$

Proposition 6.10 (De Morgan's Law)**.** *For any sets A, B and E, we have $E - (A \cap B) = (E - A) \cup (E - B)$.*

PROOF: $(x \in E \vee \neg(x \in A \wedge x \in B)) \Leftrightarrow (x \in E \wedge x \notin A) \vee (x \in E \wedge x \notin B)$. $\square$

Proposition 6.11. *For any sets A, B and E, if $A \subseteq E$ then*

$$A - B = A \cap (E - B) \ .$$

PROOF: If $A \subseteq E$ then we have $(x \in A \wedge x \notin B) \Leftrightarrow (x \in A \wedge x \in E \wedge x \notin B)$. □

Proposition 6.12. *For any sets A and B, we have $A \subseteq B$ if and only if $A - B = \emptyset$.*

PROOF: Both are equivalent to the statement that there is no x such that $x \in A$ and $x \notin B$. □

Proposition 6.13. *For any sets A and B, we have*

$$A - (A - B) = A \cap B \ .$$

PROOF: $(x \in A \wedge \neg(x \in A \wedge x \notin B)) \Leftrightarrow x \in A \wedge x \in B$. □

Proposition 6.14. *For any sets A, B and C, we have*

$$A \cap (B - C) = (A \cap B) - (A \cap C) \ .$$

PROOF: $(x \in A \wedge x \in B \wedge x \notin C) \Leftrightarrow (x \in A \wedge x \in B \wedge \neg(x \in A \wedge x \in C))$. □

Proposition 6.15. *For any sets A, B, C and E, if $(A \cap B) - C \subseteq E$ then we have*

$$A \cap B \subseteq (A \cap C) \cup (B \cap (E - C)) \ .$$

PROOF:
$\langle 1 \rangle 1.$ LET: $x \in A \cap B$
PROVE: $x \in (A \cap C) \cup (B \cap (E - C))$
$\langle 1 \rangle 2.$ CASE: $x \in C$
PROOF: Then $x \in A \cap C$.
$\langle 1 \rangle 3.$ CASE: $x \notin C$
PROOF: Then $x \in E$ and so $x \in B \cap (E - C)$.
□

Proposition 6.16. *For any sets A, B, C and E, we have*

$$(A \cup C) \cap (B \cup (E - C)) \subseteq A \cup B \ .$$

PROOF: The statement $(x \in A \vee x \in C) \wedge (x \in B \vee (x \in E \wedge x \notin C))$ implies $x \in A \vee x \in B$. □

Definition 6.17 (Symmetric Difference)**.** For any sets A and B, the *symmetric difference* $A + B$ is defined to be

$$A + B := (A - B) \cup (B - A) \ .$$

Proposition 6.18. *For any sets A and B, we have*

$$A + B = B + A$$

PROOF: From the commutativity of union. □

Proposition 6.19. *For any sets A, B and C, we have*

$$A + (B + C) = (A + B) + C \ .$$

PROOF: Each is the set of all x that belong to either exactly one or all three of A, B and C. □

Proposition 6.20. *For any set A, we have*

$$A + \emptyset = A \ .$$

PROOF:

$$\begin{aligned} A + \emptyset &= (A - \emptyset) \cup (\emptyset - A) \\ &= A \cup \emptyset \\ &= A \end{aligned}$$

□

Proposition 6.21. *For any set A we have*

$$A + A = \emptyset \ .$$

PROOF:

$$\begin{aligned} A + A &= (A - A) \cup (A - A) \\ &= \emptyset \cup \emptyset \\ &= \emptyset \end{aligned}$$

□

Definition 6.22 (Power Set)**.** For any set A, the *power set* of A, $\mathcal{P}A$, is the set whose elements are exactly the subsets of A.

PROOF: This exists by the Power Set Axiom and Axiom of Comprehension, and is unique by the Axiom of Extensionality. □

Proposition 6.23.

$$\mathcal{P}\emptyset = \{\emptyset\}$$

PROOF: The only subset of $\emptyset$ is $\emptyset$. □

Proposition 6.24. *For any set a, we have*

$$\mathcal{P}\{a\} = \{\emptyset, \{a\}\} \ .$$

PROOF: The only subsets of $\{a\}$ are $\emptyset$ and $\{a\}$. □

Proposition 6.25. *For any sets a and b, we have*

$$\mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \ .$$

PROOF: The only subsets of $\{a, b\}$ are $\emptyset$, $\{a\}$, $\{b\}$ and $\{a, b\}$. □

Proposition 6.26 (De Morgan's Law)**.** *Let E be a set and $\mathcal{C}$ a nonempty set. Then*

$$E - \bigcup \mathcal{C} = \bigcap_{X \in \mathcal{C}} (E - X) \enspace .$$

PROOF: Easy. □

Proposition 6.27 (De Morgan's Law)**.** *Let E be a set and $\mathcal{C}$ a nonempty set. Then*

$$E - \bigcap \mathcal{C} = \bigcup_{X \in \mathcal{C}} (E - X) \enspace .$$

PROOF: Easy. □

Proposition 6.28. *For any nonempty set $\mathcal{C}$ we have*

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}X = \mathcal{P}\left(\bigcap \mathcal{C}\right) \enspace .$$

PROOF:

$$\begin{aligned} x \in \bigcup_{X \in \mathcal{C}} \mathcal{P}X &\Leftrightarrow \forall X \in \mathcal{C}. x \subseteq X \\ &\Leftrightarrow \forall X \in \mathcal{C}. \forall y \in x. y \in X \\ &\Leftrightarrow \forall y \in x. \forall X \in mathcalC. y \in X \\ &\Leftrightarrow x \subseteq \bigcap \mathcal{C} \end{aligned}$$

□

Proposition 6.29. *For any set $\mathcal{C}$ we have*

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}X \subseteq \mathcal{P} \bigcup \mathcal{C} \enspace .$$

PROOF: If there exists $X \in \mathcal{C}$ such that $x \subseteq X$ then $x \subseteq \bigcup \mathcal{C}$. □

Proposition 6.30. *For any set E, we have*

$$\bigcap \mathcal{P}E = \emptyset \enspace .$$

PROOF: Since $\emptyset \in \mathcal{P}E$. □

Proposition 6.31. *For any sets E and F, if $E \subseteq F$ then $\mathcal{P}E \subseteq \mathcal{P}F$.*

PROOF: If $E \subseteq F$ and $X \subseteq E$ then $X \subseteq F$. □