Summary of Halmos' Naive Set Theory

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August 25, 2023

Contents

Chapter 1

Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*, $\in$. If $x \in A$ we say that x *belongs* to A, x is an *element* of A, or x is *contained* in A. If this does not hold we write $x \notin A$.

Axiom 1.1 (Axiom of Extensionality)**.** *Two sets are equal if and only if they have the same elements.*

Axiom 1.2 (Axiom of Comprehension, Aussonderungsaxiom)**.** *To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

Definition 1.3. Given a set A and a condition $S(x)$, we write $\{x \in A : S(x)\}$ for the set whose elements are exactly those elements x of A for which $S(x)$ holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality. ☐

Axiom 1.4 (Axiom of Pairing)**.** *For any two sets, there exists a set that they both belong to.*

Definition 1.5 ((Unordered) Pair)**.** For any sets a and b, the *(unordered) pair* $\{a, b\}$ is the set whose elements are just a and b.

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality. ☐

Axiom 1.6 (Union Axiom)**.** *For every set A, there exists a set that contains all the elements that belong to at least one element of A.*

Definition 1.7 (Subset)**.** Let A and B be sets. We say that A is a *subset* of B, or B *includes* A, and write $A \subseteq B$ or $B \supseteq A$, iff every element of A is an element of B.

Axiom 1.8 (Power Set Axiom)**.** *For any set A, there exists a set that contains all the subsets of A.*

Definition 1.9 (Empty)**.** A set is *empty* iff it has no elements; otherwise it is *non-empty.*

Axiom 1.10 (Axiom of Infinity)**.** *There exists a set I such that:*

- *I has an element that is empty*
- *for all $x \in I$, there exists $y \in I$ such that the elements of y are exactly x and the elements of x.*

Definition 1.11 (Ordered Pair)**.** For any sets a and b, the *ordered pair* (a, b) is defined by

$$(a, b) := \{\{a\}, \{a, b\}\} \ .$$

Definition 1.12 (Power Set)**.** For any set A, the *power set* of A, $\mathcal{P}A$, is the set whose elements are exactly the subsets of A.

Proof: This exists by the Power Set Axiom and Axiom of Comprehension, and is unique by the Axiom of Extensionality. □

Definition 1.13 (Cartesian Product)**.** For any sets A and B, the *Cartesian product* $A \times B$ is

$$A \times B := \{p \in \mathcal{P}\mathcal{P}(A \cup B) : \exists a \in A. \exists b \in B. p = (a, b)\} \ .$$

Definition 1.14 (Relation)**.** A *relation* is a set of ordered pairs.

If R is a relation, we write xRy for $(x, y) \in R$.

Given sets X and Y, a relation *between* X *and* Y is a subset of $X \times Y$.

Given a set X, a relation *on* X is a relation between X and X.

Definition 1.15 (Function)**.** Let X and Y be sets. A *function, map, mapping, transformation* or *operator* f *from* X *to* Y, $f : X \to Y$, is a relation f between X and Y such that, for all $x \in X$, there exists a unique $f(x) \in Y$, called the *value* of f at the *argument* x, such that $(x, f(x)) \in f$.

Definition 1.16 (Family)**.** Let I and X be sets. A *family* of elements of X *indexed* by I is a function $a : I \to X$. We write a_i for $a(i)$, and $\{a_i\}_{i \in I}$ for a.

Definition 1.17 (Cartesian Product of a Family of Sets)**.** Let $\{A_i\}_{i \in I}$ be a family of sets. The *Cartesian product* $\bigtimes_{i \in I} A_i$ is the set of all families $\{a_i\}_{i \in I}$ such that $\forall i \in I. a_i \in A_i$.

We write A^I for $\bigtimes_{i \in I} A$.

Axiom 1.18 (Axiom of Choice)**.** *The Cartesian product of a non-empty family of non-empty sets is non-empty.*

Chapter 2

Basic Properties and Operations on Sets

2.1 The Subset Relation

Theorem 2.1. *For any set A, we have $A \subseteq A$.*

PROOF: Every element of A is an element of A. $\square$

Theorem 2.2. *For any sets A, B and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

PROOF: If every element of A is an element of B, and every element of B is an element of C, then every element of A is an element of C. $\square$

Theorem 2.3. *For any sets A and B, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

PROOF: If every element of A is an element of B, and every element of B is an element of A, then A and B have the same elements, and therefore are equal by the Axiom of Extensionality. $\square$

Definition 2.4 (Proper Subset)**.** Let A and B be sets. We say that A is a *proper* subset of B, or B *properly* includes A, and write $A \subsetneq B$ or $B \supsetneq A$, iff $A \subseteq B$ and $A \neq B$.

2.2 Comprehension Notation

Theorem 2.5. *There is no set that contains every set.*

PROOF:
⟨1⟩1. LET: A be a set.
PROVE: There exists a set B such that $B \notin A$.
⟨1⟩2. LET: $B = \{x \in A : x \notin x\}$
⟨1⟩3. If $B \in A$ then we have $B \in B$ if and only if $B \notin B$.
⟨1⟩4. $B \notin A$
$\square$

2.3 The Empty Set

Theorem 2.6. *There exists a set with no elements.*

PROOF: Immediate from the Axiom of Infinity. □

Definition 2.7 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

Theorem 2.8. *For any set A we have $\emptyset \subset A$.*

PROOF: Vacuous. □

2.4 Unordered Pairs

Definition 2.9 (Singleton)**.** For any set a, the *singleton* $\{a\}$ is defined to be $\{a, a\}$.

2.5 Unions

Definition 2.10 (Union)**.** For any set $\mathcal{C}$, the *union* of $\mathcal{C}$, $\bigcup \mathcal{C}$, is the set whose elements are the elements of the elements of $\mathcal{C}$.

We write $\bigcup_{X \in \mathcal{A}} t[X]$ for $\bigcup \{t[X] \mid X \in \mathcal{A}\}$.

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality. □

Proposition 2.11.

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of $\emptyset$. □

Proposition 2.12. *For any set A, we have $\bigcup \{A\} = A$.*

PROOF: For any x, we have x is an element of an element of $\{A\}$ if and only if x is an element of A. □

Definition 2.13. We write $A \cup B$ for $\bigcup \{A, B\}$.

Proposition 2.14. *For any set A, we have $A \cup \emptyset = A$.*

PROOF: $x \in A \cup \emptyset$ iff $x \in A$ or $x \in \emptyset$, iff $x \in A$. □

Proposition 2.15 (Idempotence)**.** *For any set A, we have $A \cup A = A$.*

PROOF: $x \in A$ or $x \in A$ is equivalent to $x \in A$. □

Proposition 2.16. *For any sets A and B, we have $A \subseteq B$ if and only if $A \cup B = B$.*

PROOF: For any x, the statement "if $x \in A$ then $x \in B$" is equivalent to "$x \in A$ or $x \in B$ if and only if $x \in B$". □

Proposition 2.17. *For any sets a and b, we have $\{a\} \cup \{b\} = \{a, b\}$.*

PROOF: Immediate from definitions. □

2.6 Intersections

Definition 2.18 (Intersection)**.** For any sets A and B, the *intersection* $A \cap B$ is defined to be $\{x \in A : x \in B\}$.

Proposition 2.19. *For any set A, we have $A \cap \emptyset = \emptyset$.*

PROOF: There is no x such that $x \in A$ and $x \in \emptyset$. ∎

Proposition 2.20. *For any set A, we have*

$$A \cap A = A \ .$$

PROOF: We have $x \in A$ and $x \in A$ if and only if $x \in A$. ∎

Proposition 2.21. *For any sets A and B, we have $A \subseteq B$ if and only if $A \cap B = A$.*

PROOF: For any x, the statement "if $x \in A$ then $x \in B$" is equivalent to "$x \in A$ and $x \in B$ if and only if $x \in A$". ∎

Proposition 2.22. *For any sets A, B and C, we have $C \subseteq A$ if and only if $(A \cap B) \cup C = A \cap (B \cup C)$.*

PROOF: The statement "if $x \in C$ then $x \in A$" is equivalent to the statement "$((x \in A \wedge x \in B) \vee x \in C) \Leftrightarrow (x \in A \wedge (x \in B \vee x \in C))$". ∎

Definition 2.23 (Disjoint)**.** Two sets A and B are *disjoint* if and only if $A \cap B = \emptyset$.

Definition 2.24 (Pairwise Disjoint)**.** Let A be a set. We say the elements of A are *pairwise disjoint* if and only if, for all $x, y \in A$, if $x \cap y \neq \emptyset$ then $x = y$.

Definition 2.25 (Intersection)**.** For any nonempty set $\mathcal{C}$, the *intersection* of $\mathcal{C}$, $\bigcap \mathcal{C}$, is the set that contains exactly those sets that belong to every element of $\mathcal{C}$.

We write $\bigcap_{X \in \mathcal{A}} t[X]$ for $\bigcap \{t[X] \mid X \in \mathcal{A}\}$.

PROOF:
⟨1⟩1. LET: $\mathcal{C}$ be a nonempty set.
⟨1⟩2. There exists a set I whose elements are exactly the sets that belong to every element of $\mathcal{C}$.
PROOF: Pick $A \in \mathcal{C}$, and take $I = \{x \in A : \forall X \in \mathcal{C}. x \in X\}$.
⟨1⟩3. For any sets I, J, if the elements of I and J are exactly the sets that belong to every element of $\mathcal{C}$ then $I = J$.
PROOF: Axiom of Extensionality.
∎

2.7 Unordered Triples

Definition 2.26 ((Unordered) Triple)**.** Given sets $a_1, \ldots, a_n$, define the *(unordered) n-tuple* $\{a_1, \ldots, a_n\}$ to be

$$\{a_1, \ldots, a_n\} := \{a_1\} \cup \cdots \cup \{a_n\} \ .$$

2.8 Relative Complements

Definition 2.27 (Relative Complement)**.** For any sets A and B, the *difference* or *relative complement* $A - B$ is defined to be

$$A - B := \{x \in A : x \notin B\} \ .$$

Proposition 2.28. *For any sets A and E, we have $A \subseteq E$ if and only if*

$$E - (E - A) = A$$

PROOF:
$\langle 1 \rangle 1$. LET: A and E be sets.
$\langle 1 \rangle 2$. If $A \subseteq E$ then $E - (E - A) = A$
  $\langle 2 \rangle 1$. ASSUME: $A \subseteq E$
  $\langle 2 \rangle 2$. $E - (E - A) \subseteq A$
    PROOF: If $x \in E$ and $x \notin E - A$ then $x \in A$.
  $\langle 2 \rangle 3$. $A \subseteq E - (E - A)$
    PROOF: If $x \in A$ then $x \in E$ and $x \notin E - A$.
$\langle 1 \rangle 3$. If $E - (E - A) = A$ then $A \subseteq E$.
  PROOF: Since $E - (E - A) \subseteq E$.
$\square$

Proposition 2.29. *For any set E we have*

$$E - \emptyset = E$$

PROOF: $x \in E$ if and only if $x \in E$ and $x \notin \emptyset$. $\square$

Proposition 2.30. *For any set E we have*

$$E - E = \emptyset \ .$$

PROOF: There is no x such that $x \in E$ and $x \notin E$. $\square$

Proposition 2.31. *For any sets A and E, we have*

$$A \cap (E - A) = \emptyset \ .$$

PROOF: There is no x such that $x \in A$ and $x \in E - A$. $\square$

Proposition 2.32. *Let A and E be sets. Then $A \subseteq E$ if and only if*

$$A \cup (E - A) = E \ .$$

PROOF:
$\langle 1 \rangle 1$. LET: A and E be sets.
$\langle 1 \rangle 2$. If $A \subseteq E$ then $A \cup (E - A) = E$.
  $\langle 2 \rangle 1$. ASSUME: $A \subseteq E$
  $\langle 2 \rangle 2$. $A \cup (E - A) \subseteq E$

PROOF: If $x \in A$ or $x \in E - A$ then $x \in E$.
⟨2⟩3. $E \subseteq A \cup (E - A)$
PROOF: If $x \in E$ then either $x \in A$ or $x \notin A$. In the latter case, $x \in E - A$.
⟨1⟩3. If $A \cup (E - A) = E$ then $A \subseteq E$
PROOF: Since $A \subseteq A \cup (E - A)$.
□

Proposition 2.33. *Let A, B and E be sets. Then:*

1. *If $A \subseteq B$ then $E - B \subseteq E - A$.*
2. *If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.*

PROOF:
⟨1⟩1. LET: A, B and E be sets.
⟨1⟩2. If $A \subseteq B$ then $E - B \subseteq E - A$.
PROOF: If $A \subseteq B$, $x \in E$ and $x \notin B$, then we have $x \in E$ and $x \notin A$.
⟨1⟩3. If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.
⟨2⟩1. ASSUME: $A \subseteq E$
⟨2⟩2. ASSUME: $E - B \subseteq E - A$
⟨2⟩3. LET: $x \in A$
⟨2⟩4. $x \in E$
⟨2⟩5. $x \notin E - A$
⟨2⟩6. $x \notin E - B$
⟨2⟩7. $x \in B$
□

Example 2.34. We cannot remove the hypothesis $A \subseteq E$ in item 2 above. Let $E = \emptyset$, $A = \{\emptyset\}$ and $B = \emptyset$. Then $E - B = E - A = \emptyset$ but $A \nsubseteq B$.

Proposition 2.35 (De Morgan's Law)**.** *For any sets A, B and E, we have $E - (A \cup B) = (E - A) \cap (E - B)$.*

PROOF: $(x \in E \wedge \neg(x \in A \vee x \in B)) \Leftrightarrow (x \in E \wedge x \notin A \wedge x \in E \wedge x \notin B)$. □

Proposition 2.36 (De Morgan's Law)**.** *For any sets A, B and E, we have $E - (A \cap B) = (E - A) \cup (E - B)$.*

PROOF: $(x \in E \vee \neg(x \in A \wedge x \in B)) \Leftrightarrow (x \in E \wedge x \notin A) \vee (x \in E \wedge x \notin B)$. □

Proposition 2.37. *For any sets A, B and E, if $A \subseteq E$ then*

$$A - B = A \cap (E - B) \ .$$

PROOF: If $A \subseteq E$ then we have $(x \in A \wedge x \notin B) \Leftrightarrow (x \in A \wedge x \in E \wedge x \notin B)$. □

Proposition 2.38. *For any sets A and B, we have $A \subseteq B$ if and only if $A - B = \emptyset$.*

PROOF: Both are equivalent to the statement that there is no x such that $x \in A$ and $x \notin B$. □

Proposition 2.39. *For any sets A and B, we have*

$$A - (A - B) = A \cap B \ .$$

Proof: $(x \in A \wedge \neg(x \in A \wedge x \notin B)) \Leftrightarrow x \in A \wedge x \in B$. ∎

Proposition 2.40. *For any sets A, B and C, we have*

$$A \cap (B - C) = (A \cap B) - (A \cap C) \ .$$

Proof: $(x \in A \wedge x \in B \wedge x \notin C) \Leftrightarrow (x \in A \wedge x \in B \wedge \neg(x \in A \wedge x \in C))$. ∎

Proposition 2.41. *For any sets A, B, C and E, if $(A \cap B) - C \subseteq E$ then we have*

$$A \cap B \subseteq (A \cap C) \cup (B \cap (E - C)) \ .$$

Proof:
⟨1⟩1. Let: $x \in A \cap B$
Prove: $x \in (A \cap C) \cup (B \cap (E - C))$
⟨1⟩2. Case: $x \in C$
Proof: Then $x \in A \cap C$.
⟨1⟩3. Case: $x \notin C$
Proof: Then $x \in E$ and so $x \in B \cap (E - C)$.
∎

Proposition 2.42. *For any sets A, B, C and E, we have*

$$(A \cup C) \cap (B \cup (E - C)) \subseteq A \cup B \ .$$

Proof: The statement $(x \in A \vee x \in C) \wedge (x \in B \vee (x \in E \wedge x \notin C))$ implies $x \in A \vee x \in B$. ∎

Proposition 2.43 (De Morgan's Law)**.** *Let E be a set and $\mathcal{C}$ a nonempty set. Then*

$$E - \bigcup \mathcal{C} = \bigcap_{X \in \mathcal{C}} (E - X) \ .$$

Proof: Easy. ∎

Proposition 2.44 (De Morgan's Law)**.** *Let E be a set and $\mathcal{C}$ a nonempty set. Then*

$$E - \bigcap \mathcal{C} = \bigcup_{X \in \mathcal{C}} (E - X) \ .$$

Proof: Easy. ∎

2.9 Symmetric Difference

Definition 2.45 (Symmetric Difference)**.** For any sets A and B, the *symmetric difference* $A + B$ is defined to be

$$A + B := (A - B) \cup (B - A) \ .$$

Proposition 2.46. *For any sets A and B, we have*

$$A + B = B + A$$

Proof: From the commutativity of union. □

Proposition 2.47. *For any sets A, B and C, we have*

$$A + (B + C) = (A + B) + C \ .$$

Proof: Each is the set of all x that belong to either exactly one or all three of A, B and C. □

Proposition 2.48. *For any set A, we have*

$$A + \emptyset = A \ .$$

Proof:

$$\begin{aligned} A + \emptyset &= (A - \emptyset) \cup (\emptyset - A) \\ &= A \cup \emptyset \\ &= A \end{aligned}$$ □

Proposition 2.49. *For any set A we have*

$$A + A = \emptyset \ .$$

Proof:

$$\begin{aligned} A + A &= (A - A) \cup (A - A) \\ &= \emptyset \cup \emptyset \\ &= \emptyset \end{aligned}$$ □

2.10 Power Sets

Proposition 2.50.

$$\mathcal{P}\emptyset = \{\emptyset\}$$

Proof: The only subset of $\emptyset$ is $\emptyset$. □

Proposition 2.51. *For any set a, we have*

$$\mathcal{P}\{a\} = \{\emptyset, \{a\}\} \ .$$

Proof: The only subsets of $\{a\}$ are $\emptyset$ and $\{a\}$. □

Proposition 2.52. *For any sets a and b, we have*

$$\mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \ .$$

Proof: The only subsets of $\{a, b\}$ are $\emptyset$, $\{a\}$, $\{b\}$ and $\{a, b\}$. □

Proposition 2.53. *For any nonempty set $\mathcal{C}$ we have*

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}X = \mathcal{P}\left(\bigcap \mathcal{C}\right) \enspace .$$

Proof:

$$\begin{aligned} x \in \bigcup_{X \in \mathcal{C}} \mathcal{P}X &\Leftrightarrow \forall X \in \mathcal{C}. x \subseteq X \\ &\Leftrightarrow \forall X \in \mathcal{C}. \forall y \in x. y \in X \\ &\Leftrightarrow \forall y \in x. \forall X \in mathcalC. y \in X \\ &\Leftrightarrow x \subseteq \bigcap \mathcal{C} \end{aligned}$$

∎

Proposition 2.54. *For any set $\mathcal{C}$ we have*

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}X \subseteq \mathcal{P}\bigcup \mathcal{C} \enspace .$$

Proof: If there exists $X \in \mathcal{C}$ such that $x \subseteq X$ then $x \subseteq \bigcup \mathcal{C}$. ∎

Proposition 2.55. *For any set E, we have*

$$\bigcap \mathcal{P}E = \emptyset \enspace .$$

Proof: Since $\emptyset \in \mathcal{P}E$. ∎

Proposition 2.56. *For any sets E and F, if $E \subseteq F$ then $\mathcal{P}E \subseteq \mathcal{P}F$.*

Proof: If $E \subseteq F$ and $X \subseteq E$ then $X \subseteq F$. ∎

Chapter 3

Relations and Functions

3.1 Ordered Pairs

Proposition 3.1. *For any sets a, b, x and y, if $(a, b) = (x, y)$ then $a = x$ and $b = y$.*

PROOF:
$\langle 1 \rangle 1$. LET: a, b, x and y be sets.
$\langle 1 \rangle 2$. ASSUME: $(a, b) = (x, y)$
$\langle 1 \rangle 3$. $a = x$
  PROOF: $\{a\} = \bigcap(a, b) = \bigcap(x, y) = \{x\}$.
$\langle 1 \rangle 4$. $\{a, b\} = \{x, y\}$
$\langle 1 \rangle 5$. CASE: $a = b$
  $\langle 2 \rangle 1$. $x = y$
    PROOF: Since $\{x, y\} = \{a, b\}$ is a singleton.
  $\langle 2 \rangle 2$. $b = y$
    PROOF: $b = a = x = y$
$\langle 1 \rangle 6$. CASE: $a \neq b$
  $\langle 2 \rangle 1$. $x \neq y$
    PROOF: Since $\{x, y\} = \{a, b\}$ is not a singleton.
  $\langle 2 \rangle 2$. $b = y$
    PROOF: $\{b\} = \{a, b\} - \{a\} = \{x, y\} - \{x\} = \{y\}$.
□

Proposition 3.2. *For any sets A, B and X, we have*

$$(A - B) \times X = (A \times X) - (B \times X) \ .$$

PROOF: Easy. □

Proposition 3.3. *For any sets A and B, we have $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.*

PROOF: Easy. □

Proposition 3.4. *For any sets A, B, X and Y, if $A \subseteq X$ and $B \subseteq Y$ then $A \times B \subseteq X \times Y$. The converse holds assuming $A \neq \emptyset$ and $B \neq \emptyset$.*

PROOF: Easy. $\square$

3.2 Relations

Definition 3.5 (Domain)**.** The *domain* of a relation R is the set

$$\operatorname{dom} R := \left\{ x \in \bigcup\bigcup R : \exists y.(x, y) \in R \right\} \ .$$

Definition 3.6 (Range)**.** The *range* of a relation R is the set

$$\operatorname{ran} R := \left\{ y \in \bigcup\bigcup R : \exists x.(x, y) \in R \right\} \ .$$

Definition 3.7 (Reflexive)**.** Let R be a relation on X. Then R is *reflexive* iff, for all $x \in X$, we have xRx.

Definition 3.8 (Symmetric)**.** Let R be a relation on X. Then R is *symmetric* iff, whenever xRy, then yRx.

Definition 3.9 (Antisymmetric)**.** A relation R is *antisymmetric* iff, whenever xRy and yRx, then $x = y$.

Definition 3.10 (Transitive)**.** Let R be a relation on X. Then R is *transitive* iff, whenever xRy and yRz, then xRz.

Definition 3.11 (Identity Relation)**.** For any set X, the *identity relation I_X* on X is

$$I_X = \{(x, x) : x \in X\} \ .$$

3.3 Composition

Definition 3.12 (Composition)**.** Let R be a relation between X and Y, and S a relation between Y and Z. The *composite* or *relative product* $S \circ R = SR$ is the relation between X and Z defined by

$$x(S \circ R)z \Leftrightarrow \exists y \in Y(xRy \wedge ySz) \ .$$

Proposition 3.13. *Let R be a relation between X and Y, S a relation between Y and Z, and T a relation between Z and W. Then*

$$T(SR) = (TS)R \ .$$

PROOF: Easy. $\square$

Example 3.14. Composition of relations is not commutative in general. Let $X = \{a, b\}$ where $a \neq b$. Let $R = \{(a, a), (b, a)\}$ and $S = \{(a, b), (b, b)\}$. Then $SR = S$ but $RS = R \neq S$.

Proposition 3.15. *A relation R is transitive if and only if $RR \subseteq R$.*

PROOF: Easy. $\square$

3.4 Inverses

Definition 3.16 (Inverse)**.** Let R be a relation between X and Y. The *inverse* or *converse* R^{-1} is the relation between Y and X defined by

$$yR^{-1}x \Leftrightarrow xRy \ .$$

Proposition 3.17. *For any relation R, we have*

$$\operatorname{dom} R^{-1} = \operatorname{ran} R \ .$$

PROOF: Easy. □

Proposition 3.18. *For any relation R, we have*

$$\operatorname{ran} R^{-1} = \operatorname{dom} R \ .$$

PROOF: Easy. □

Proposition 3.19. *Let R be a relation between X and Y, and S a relation between Y and Z. Then*

$$(SR)^{-1} = R^{-1}S^{-1} \ .$$

PROOF: Easy. □

Proposition 3.20. *A relation R is symmetric if and only if $R \subseteq R^{-1}$.*

PROOF: Easy. □

Proposition 3.21. *Let R be a relation between X and Y. Then*

$$I_Y R = R I_X = R \ .$$

PROOF: Easy. □

Proposition 3.22. *A relation R on a set X is reflexive if and only if $I_X \subseteq R$.*

PROOF: Easy. □

Proposition 3.23. *Let R be a relation on a set X. Then R is antisymmetric iff $R \cap R^{-1} \subseteq I_X$.*

PROOF: Easy. □

3.5 Equivalence Relations

Definition 3.24 (Equivalence Relation)**.** Let R be a relation on X. Then R is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Definition 3.25 (Partition)**.** Let X be a set. A *partition* of X is a pairwise disjoint set of nonempty subsets of X whose union is X.

Definition 3.26 (Equivalence Class)**.** Let R be an equivalence relation on X. Let $x \in X$. The *equivalence class* of x with respect to R is

$$x/R := \{y \in X : xRy\} \ .$$

We write X/R for the set of all equivalence classes with respect to R.

Definition 3.27 (Induced)**.** Let P be a partition of X. The relation *induced* by P is X/P where $x(X/P)y$ iff there exists $X \in P$ such that $x \in X$ and $y \in X$.

Theorem 3.28. *Let R be an equivalence relation on X. Then X/R is a partition of X that induces the relation R.*

PROOF: Easy. □

Theorem 3.29. *Let P be a partition of X. Then X/P is an equivalence relation on X, and $P = X/(X/P)$.*

PROOF: Easy. □

3.6 Functions

Definition 3.30 (Onto)**.** Let $f : X \to Y$. We say f maps X *onto* Y iff $\mathrm{ran}\, f = Y$.

Definition 3.31 (Image)**.** Let $f : X \to Y$ and $A \subseteq X$. The *image* of A under f is

$$f(A) := \{f(x) : x \in A\} \ .$$

Definition 3.32 (Inclusion Map)**.** Let Y be a set and $X \subseteq Y$. Then the *inclusion map* $i : X \hookrightarrow Y$ is the function defined by $i(x) = x$ for all $x \in X$.

Proposition 3.33. *For any set X, the identity relation I_X is a function $X \to X$.*

PROOF: Easy. □

Definition 3.34 (Restriction)**.** Let $f : Y \to Z$ and $X \subseteq Y$. The *restriction* of f to X is the function $f \upharpoonright X : X \to Z$ defined by

$$(f \upharpoonright X)(x) = f(x) \qquad (x \in X) \ .$$

Given sets X, Y and Z with $X \subseteq Y$, if $f : X \to Z$ and $g : Y \to Z$, we say g is an *extension* of f to Y iff $f = g \upharpoonright X$.

Definition 3.35 (Projection)**.** Given sets X and Y, the *projection* maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are defined by

$$\pi_1(x, y) = x, \qquad \pi_2(x, y) = y \qquad (x \in X, y \in Y) \ .$$

Definition 3.36 (Canonical Map)**.** Let X be a set and R an equivalence relation on X. The *canonical map* $\pi : X \to X/R$ is the map defined by $\pi(x) = x/R$.

Definition 3.37 (One-to-One)**.** A function $f : X \to Y$ is *one-to-one*, or a *one-to-one correspondence*, iff, for all $x, y \in X$, if $f(x) = f(y)$ then $x = y$.

Proposition 3.38. *Let $f : X \to Y$. Then the following are equivalent:*

1. *f is one-to-one.*
2. *For all $A, B \subseteq X$, we have $f(A \cap B) = f(A) \cap f(B)$.*
3. *For all $A \subseteq X$, we have $f(X - A) \subseteq Y - f(A)$.*

PROOF: Easy. □

Proposition 3.39. *Let $f : X \to Y$. Then f maps X onto Y if and only if, for all $A \subseteq X$, we have $Y - f(A) \subseteq f(X - A)$.*

PROOF: Easy. □

3.7 Families

Proposition 3.40 (Generalized Associative Law for Unions)**.** *Let $\{I_j\}_{j \in J}$ be a family of sets. Let $K = \bigcup_{j \in J} I_j$. Let $\{A_k\}_{k \in K}$ be a family of sets indexed by K. Then*

$$\bigcup_{k \in K} A_k = \bigcup_{j \in J} \bigcup_{i \in I_j} A_i \ .$$

PROOF: Easy. □

Proposition 3.41 (Generalized Commutative Law for Unions)**.** *Let $\{I_j\}_{j \in J}$ be a family of sets. Let $f : J \to J$ be a one-to-one correspondence from J onto J. Then*

$$\bigcup_{j \in J} I_j = \bigcup_{j \in J} I_{f(j)} \ .$$

PROOF: Easy. □

Proposition 3.42 (Generalized Associative Law for Intersections)**.** *Let $\{I_j\}_{j \in J}$ be a nonempty family of nonempty sets. Let $K = \bigcup_{j \in J} I_j$. Let $\{A_k\}_{k \in K}$ be a family of sets indexed by K. Then*

$$\bigcap_{k \in K} A_k = \bigcap_{j \in J} \bigcap_{i \in I_j} A_i \ .$$

PROOF: Easy. □

Proposition 3.43 (Generalized Commutative Law for Intersections)**.** *Let $\{I_j\}_{j \in J}$ be a nonempty family of sets. Let $f : J \to J$ be a one-to-one correspondence from J onto J. Then*

$$\bigcap_{j \in J} I_j = \bigcap_{j \in J} I_{f(j)} \ .$$

PROOF: Easy. □

Proposition 3.44. *Let B be a set and $\{A_i\}_{i\in I}$ a family of sets. Then*

$$B \cap \bigcup_{i\in I} A_i = \bigcup_{i\in I}(B \cap A_i)$$

PROOF: Easy. □

Proposition 3.45. *Let B be a set and $\{A_i\}_{i\in I}$ a nonempty family of sets. Then*

$$B \cup \bigcap_{i\in I} A_i = \bigcap_{i\in I}(B \cup A_i)$$

PROOF: Easy. □

Definition 3.46 (Projection)**.** Let $\{A_i\}_{i\in I}$ be a family of sets and $i \in I$. The *projection* function $\pi_i : \bigtimes_{i\in I} A_i \to A_i$ is defined by $\pi_i(a) = a_i$.

Proposition 3.47. *Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be families of sets. Then*

$$\left(\bigcup_{i\in I} A_i\right) \times \left(\bigcup_{j\in J} B_j\right) = \bigcup_{i\in I}\bigcup_{j\in J}(A_i \times B_j) \ .$$

PROOF: Easy. □

Proposition 3.48. *Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be nonempty families of sets. Then*

$$\left(\bigcap_{i\in I} A_i\right) \times \left(\bigcap_{j\in J} B_j\right) = \bigcap_{i\in I}\bigcap_{j\in J}(A_i \times B_j) \ .$$

PROOF: Easy. □

Proposition 3.49. *Let $f : X \to Y$. Let $\{A_i\}_{i\in I}$ be a family of subsets of X. Then*

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i) \ .$$

PROOF: Easy. □

Example 3.50. It is not true in general that, if $f : X \to Y$ and $\{A_i\}_{i\in I}$ is a nonempty family of subsets of X, then $f\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f(A_i)$.

Take $X = \{a, b\}$ and $Y = \{c\}$ where $a \neq b$. Take $I = \{i, j\}$ with $i \neq j$. Let $A_i = \{a\}$ and $A_j = \{b\}$. Let f be the unique function $X \to Y$. Then $f\left(\bigcap_{i\in I} A_i\right) = f(\emptyset) = \emptyset$ but $\bigcap_{i\in I} f(A_i) = \{c\}$.

3.8 Inverses and Composites of Functions

Definition 3.51 (Inverse)**.** Given a function $f : X \to Y$, the *inverse* of f is the function $f^{-1} : \mathcal{P}Y \to \mathcal{P}X$ defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \ .$$

We call $f^{-1}(B)$ the *inverse image* of B under f.

Proposition 3.52. *Let $f : X \to Y$. Then f maps X onto Y if and only if the inverse image of any nonempty subset of Y is nonempty.*

Proof: Easy. □

Proposition 3.53. *Let $f : X \to Y$. Then f is one-to-one if and only if the inverse image of any singleton subset of Y is a singleton.*

Proof: Easy. □

Proposition 3.54. *Let $f : X \to Y$. Let $B \subseteq Y$. Then*

$$f(f^{-1}(B)) \subseteq B \ .$$

Proof: Easy. □

Proposition 3.55. *Let $f : X \to Y$. Let $A \subseteq X$. Then*

$$A \subseteq f^{-1}(f(A)) \ .$$

Equality holds if f is one-to-one.

Proof: Easy. □

Proposition 3.56. *Let $f : X \to Y$. Let $\{B_i\}_{i \in I}$ be a family of subsets of Y. Then*

$$f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i) \ .$$

Proof: Easy. □

Proposition 3.57. *Let $f : X \to Y$. Let $\{B_i\}_{i \in I}$ be a nonempty family of subsets of Y. Then*

$$f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i) \ .$$

Proof: Easy. □

Proposition 3.58. *Let $f : X \to Y$ and $B \subseteq Y$. Then $f^{-1}(Y - B) = X - f^{-1}(B)$.*

Proof: Easy. □

Proposition 3.59. *Let $f : X \to Y$ be one-to-one. Then the inverse of f as a relation, f^{-1}, is a function $f^{-1} : \operatorname{ran} f \to X$, and for all $y \in \operatorname{ran} f$, we have $f^{-1}(y)$ is the unique x such that $f(x) = y$.*

Proof: Easy. □

Proposition 3.60. *Let $f : X \to Y$ and $g : Y \to Z$. Then $gf : X \to Z$ and, for all $x \in X$, we have*

$$(g \circ f)(x) = g(f(x)) \ .$$

Proof: Easy. □

Example 3.61. Example 3.14 shows that function composition is not commutative in general.

Proposition 3.62. *Let $f : X \to Y$ and $g : Y \to Z$. Then*

$$(gf)^{-1} = f^{-1}g^{-1} : \mathcal{P}Z \to \mathcal{P}X \ .$$

Proof: Easy. □

Proposition 3.63. *Let $f : X \to Y$ and $g : Y \to X$. If $gf = I_X$ then f is one-to-one and g maps Y onto X.*

Proof: Easy. □

3.9 Choice Functions

Definition 3.64 (Choice Function)**.** A *choice function* for a set X is a function $f : \mathcal{P}X - \{\varnothing\} \to X$ such that $f(S) \in S$ for all S.

Proposition 3.65. *Every set has a choice function.*

Proof: Given a nonempty set X, apply the Axiom of Choice to the family $\{S\}_{S \in \mathcal{P}X - \{\varnothing\}}$. □

Proposition 3.66. *For any relation R, there exists a function $f \subseteq R$ such that* $\operatorname{dom} f = \operatorname{dom} R$.

Proof:
⟨1⟩1. Let: R be a relation.
⟨1⟩2. Pick a choice function g for $\operatorname{ran} R$.
⟨1⟩3. Let: $f : \operatorname{dom} R \to \operatorname{ran} R$ be the function $f(x) = g(\{y \in \operatorname{ran} R : xRy\})$
⟨1⟩4. $f \subseteq R$ and $\operatorname{dom} f = \operatorname{dom} R$.
□

Proposition 3.67. *If $\mathcal{C}$ is a set of pairwise disjoint nonempty sets , then there exists a set A such that, for all $C \in \mathcal{C}$, we have $A \cap C$ is a singleton.*

Proof:
⟨1⟩1. Let: f be a choice function for $\bigcup \mathcal{C}$
⟨1⟩2. Let: $A = \{f(C) : C \in \mathcal{C}\}$
⟨1⟩3. For all $C \in \mathcal{C}$ we have $A \cap C = \{f(C)\}$
□

Chapter 4

Equivalence

Definition 4.1 (Equivalent)**.** Sets E and F are *equivalent*, $E \sim F$, iff there exists a one-to-one correspondence between them.

Proposition 4.2. *For any set X, equivalence is an equivalence relation on $\mathcal{P}X$.*

PROOF: Easy.

Chapter 5

Order

Definition 5.1 (Partial Order)**.** A *partial order* on a set X is a relation on X that is reflexive, antisymmetric and transitive.

A *partially ordered set* or *poset* is a pair $(X, \leqslant)$ such that $\leqslant$ is a partial order on X. We write X for the poset $(X, \leqslant)$.

Given a partial order $\leqslant$, we write $\geqslant$ for the inverse of $\leqslant$.

We write $x < y$ or $y > x$ for $x \leqslant y \wedge x \neq y$. When this holds, we say x is *less than* y, *smaller* than y, or a *predecessor* of y; and y is *greater than* x, *larger* than x, or a *successor* of x.

Proposition 5.2. *For any set X, the relation $\subseteq$ is a partial order on $\mathcal{P}X$.*

PROOF: Easy. ∎

Proposition 5.3. *In a poset, we never have $x < y$ and $y < x$.*

PROOF: We would then have $x \leqslant y$ and $y \leqslant x$ hence $x = y$ by antisymmetry. But if $x < y$ or $y < x$ then $x \neq y$. ∎

Proposition 5.4. *The relation $<$ is transitive.*

PROOF:
$\langle 1 \rangle$1. ASSUME: $x < y$ and $y < z$
$\langle 1 \rangle$2. $x \leqslant y$ and $y \leqslant z$
$\langle 1 \rangle$3. $x \leqslant z$
  PROOF: Since $\leqslant$ is transitive.
$\langle 1 \rangle$4. $x \neq z$
  PROOF: By Proposition 5.3.
∎

Proposition 5.5. *Let $<$ be a transitive relation on X such that we never have $x < y$ and $y < x$. Define $\leqslant$ by: $x \leqslant y$ iff $x < y$ or $x = y$. Then $\leqslant$ is a partial order on X.*

PROOF:

⟨1⟩1. $\leqslant$ is reflexive.
PROOF: By definition.
⟨1⟩2. $\leqslant$ is asymmetric.
PROOF: If $x \leqslant y$ and $y \leqslant x$, we must have $x = y$, because otherwise we would have $x < y$ and $y < x$.
⟨1⟩3. $\leqslant$ is transitive.
⟨2⟩1. LET: $x \leqslant y$ and $y \leqslant z$
⟨2⟩2. CASE: $x = y$
PROOF: We have $y \leqslant z$ so $x \leqslant z$.
⟨2⟩3. CASE: $y = z$
PROOF: We have $x \leqslant y$ so $x \leqslant z$.
⟨2⟩4. CASE: $x < y$ and $y < z$
PROOF: We have $x < z$ by transitivity, so $x \leqslant z$.
□

Definition 5.6 ((Strict) Initial Segment)**.** Let X be a poset and $a \in X$. The *(strict) initial segment* determined by a is

$$s(a) := \{x \in X : x < a\} \ .$$

Definition 5.7 (Weak Initial Segment)**.** Let X be a poset and $a \in X$. The *weak initial segment* determined by a is

$$\bar{s}(a) := \{x \in X : x \leqslant a\} \ .$$

Definition 5.8 (Immediate Successor)**.** Let X be a poset and $x, y \in X$. Then y is the *immediate successor* of x, and x is the *immediate predecessor* of y, iff $x < y$ and there is no z such that $x < z < y$.

Definition 5.9 (Least)**.** Let X be a partial order and $a \in X$. Then a is *least* in X iff $\forall x \in X.a \leqslant x$.

Proposition 5.10. *A poset has at most one least element.*

PROOF: If a and b are least then $a \leqslant b$ and $b \leqslant a$, hence $a = b$. □

Definition 5.11 (Greatest)**.** Let X be a partial order and $a \in X$. Then a is *greatest* in X iff $\forall x \in X.x \leqslant a$.

Proposition 5.12. *A poset has at most one greatest element.*

PROOF: If a and b are greatest then $a \leqslant b$ and $b \leqslant a$, hence $a = b$. □

Definition 5.13 (Minimal)**.** Let X be a poset and $a \in X$. Then a is *minimal* iff there is no $x \in X$ such that $x < a$.

Definition 5.14 (Maximal)**.** Let X be a poset and $a \in X$. Then a is *maximal* iff there is no $x \in X$ such that $a < x$.

Definition 5.15 (Lower Bound)**.** Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is a *lower bound* for E iff $\forall x \in E.a \leqslant x$.

Definition 5.16 (Upper Bound)**.** Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is an *upper bound* for E iff $\forall x \in E.x \leqslant a$.

Definition 5.17 (Greatest Lower Bound, Infimum)**.** Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is the *greatest lower bound* or *infimum* for E iff a is the greatest element in the set of lower bounds for E.

Definition 5.18 (Least Upper Bound, Supremum)**.** Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is the *least upper bound* or *supremum* for E iff a is the least element in the set of upper bounds for E.

Definition 5.19 (Total Order)**.** A partial order $\leqslant$ on a set X is a *total order*, *simple order* or *linear order* iff, for all $x, y \in X$, either $x \leqslant y$ or $y \leqslant x$. We then call the poset $(X, \leqslant)$ a *linearly ordered set* or a *chain*.

Proposition 5.20. *Let R be a partial order on X. Then R is total if and only if $X^2 \subseteq R \cup R^{-1}$.*

Proof: Easy. □

Proposition 5.21. *For any set X, the relation $\subseteq$ is a total order on X iff X is either $\emptyset$ or a singleton.*

Proof: Easy. □

Theorem 5.22 (Zorn's Lemma)**.** *Let X be a poset such that every chain in X has an upper bound. Then X has a maximal element.*

Proof:
⟨1⟩1. Pick a choice function f for X.
⟨1⟩2. Let: $\mathcal{X}$ be the set of chains in X.
⟨1⟩3. For all $A \in \mathcal{X}$,
Let: $\hat{A} = \{x \in X : A \cup \{x\} \in \mathcal{X}\}$
⟨1⟩4. Let: $g : \mathcal{X} \rightarrow \mathcal{X}$ be the function

$$g(A) = \begin{cases} A \cup \{f(\hat{A} - A)\} & \text{if } \hat{A} - A \neq \emptyset \\ A & \text{if } \hat{A} - A = \emptyset \end{cases}$$

⟨1⟩5. For $\mathcal{T} \subseteq \mathcal{X}$, let us say $\mathcal{T}$ is a *tower* iff:

- $\emptyset \in \mathcal{T}$
- $\forall A \in \mathcal{T}.g(A) \in \mathcal{T}$
- For every chain $\mathcal{C}$ in $\mathcal{T}$, we have $\bigcup \mathcal{C} \in \mathcal{T}$

⟨1⟩6. Let: $\mathcal{T}_0$ be the intersection of the set of all towers.
Proof: The set of all towers is nonempty since $\mathcal{X}$ is a tower.
⟨1⟩7. Let: $A = \bigcup \mathcal{T}_0$
⟨1⟩8. A is a chain in X.
⟨2⟩1. $\mathcal{T}_0$ is a chain under $\subseteq$
⟨3⟩1. Given $C \in \mathcal{T}_0$, let us say that C is *comparable* iff, for all $A \in \mathcal{T}_0$, either $A \subseteq C$ or $C \subseteq A$.

⟨3⟩2. For all $A, C \in \mathcal{T}_0$, if C is comparable and $A \subsetneq C$ then $g(A) \subseteq C$.
PROOF: Since $g(A) - A$ has at most one element, so if $A \subsetneq C \subseteq g(A)$ then $C = g(A)$.
⟨3⟩3. For $C \in \mathcal{T}_0$ comparable,
LET: $\mathcal{U}_C = \{A \in \mathcal{T}_0 : A \subseteq C \vee g(C) \subseteq A\}$
⟨3⟩4. For $C \in \mathcal{T}_0$ comparable, $\mathcal{U}_C$ is a tower.
⟨4⟩1. LET: $C \in \mathcal{T}_0$ be comparable
⟨4⟩2. $\emptyset \in \mathcal{U}_C$
PROOF: Since $\emptyset \subseteq C$.
⟨4⟩3. $\forall A \in \mathcal{U}_C . g(A) \in \mathcal{U}_C$
PROOF: By ⟨1⟩8.
⟨4⟩4. For every chain $\mathcal{C} \subseteq \mathcal{U}_C$ we have $\bigcup \mathcal{C} \in \mathcal{U}_C$
⟨5⟩1. LET: $\mathcal{C} \subseteq \mathcal{U}_C$ be a chain.
⟨5⟩2. CASE: $\exists A \in \mathcal{C} . g(C) \subseteq A$
PROOF: Then $g(C) \subseteq \bigcup \mathcal{C}$
⟨5⟩3. CASE: $\forall A \in \mathcal{C} . A \subseteq C$
PROOF: Then $\bigcup \mathcal{C} \subseteq C$.
⟨3⟩5. For $C \in \mathcal{T}_0$ comparable, $\mathcal{U}_C = \mathcal{T}_0$.
⟨3⟩6. For $C \in \mathcal{T}_0$ comparable we have $g(C)$ is comparable.
PROOF: Since for all $A \in \mathcal{T}_0$ either $A \subseteq C \subseteq g(C)$ or $g(C) \subseteq A$.
⟨3⟩7. The set of comparable sets in $\mathcal{T}_0$ is a tower.
⟨4⟩1. $\emptyset$ is comparable.
PROOF: $\forall A \in \mathcal{T}_0 . \emptyset \subseteq A$
⟨4⟩2. For all $C \in \mathcal{T}_0$, if A is comparable then $g(C)$ is comparable.
PROOF: ⟨3⟩6
⟨4⟩3. For every chain $\mathcal{C} \subseteq \mathcal{T}_0$ of comparable sets, we have $\bigcup \mathcal{C}$ is comparable.
⟨5⟩1. LET: $\mathcal{C} \subseteq \mathcal{T}_0$ be a chain of comparable sets.
⟨5⟩2. LET: $A \in \mathcal{T}_0$
⟨5⟩3. CASE: there exists $C \in \mathcal{C}$ such that $A \subseteq C$
PROOF: Then $A \subseteq \bigcup \mathcal{C}$.
⟨5⟩4. CASE: for all $C \in \mathcal{C}$ we have $C \subseteq A$
PROOF: Then $\bigcup \mathcal{C} \subseteq A$.
⟨3⟩8. Every set in $\mathcal{T}_0$ is comparable.
⟨2⟩2. LET: $x, y \in A$
⟨2⟩3. PICK $A, C \in \mathcal{T}_0$ such that $x \in A$ and $y \in C$
⟨2⟩4. ASSUME: w.l.o.g. $A \subseteq C$
⟨2⟩5. $x, y \in C$
⟨2⟩6. $x \leqslant y$ or $y \leqslant x$
PROOF: Since $C \in \mathcal{X}$ so C is a chain.
⟨1⟩9. PICK an upper bound u for A.
⟨1⟩10. $A \in \mathcal{T}_0$
PROOF: Since $\mathcal{T}_0$ is a chain in $\mathcal{T}_0$ so $\bigcup \mathcal{T}_0 \in \mathcal{T}_0$.
⟨1⟩11. $g(A) \in \mathcal{T}_0$
⟨1⟩12. $g(A) \subseteq A$
⟨1⟩13. $g(A) = A$

$\langle 1 \rangle 14.\ \hat{A} - A = \varnothing$