Mathematics

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Contents

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B, and call A the *domain* of f and B the *codomain*.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $g \circ f : A \to C$, the *composite* of f and g.

Axiom 0.1 (Associativity). *Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h(gf) = (hg)f \quad .$$

Axiom 0.2 (Identity). *For any set A, there exists a function $i : A \to A$ such that:*

- *for any set B and function $f : A \to B$, we have $fi = f$*
- *for any set B and function $f : B \to A$, we have $if = f$.*

Proposition 0.3. *For any set A, there exists a unique function $i : A \to A$ such that:*

- *for any set B and function $f : A \to B$, we have $fi = f$*
- *for any set B and function $f : B \to A$, we have $if = f$.*

PROOF: If i and j both satisfy these conditions then $i = ij = j$. □

Definition 0.4 (Identity Function). For any set A, the *identity function* on A, id_A, is the unique function $A \to A$ such that:

- for any set B and function $f : A \to B$, we have $f\mathrm{id}_A = f$
- for any set B and function $f : B \to A$, we have $\mathrm{id}_B f = f$.

Definition 0.5 (Isomorphism). A function $f : A \to B$ is an *isomorphism*, $f : A \cong B$, iff there exists a function $g : B \to A$ such that $fg = \mathrm{id}_B$ and $gf = \mathrm{id}_A$.

Axiom 0.6 (Terminal Set). *There exists a* terminal *set* 1 *such that, for any set A, there exists exists exactly one function $A \to 1$.*

Proposition 0.7. *If S and T are terminal sets then there exists a unique isomorphism $S \cong T$.*

PROOF:
$\langle 1 \rangle$1. LET: f be the unique function $S \to T$
$\langle 1 \rangle$2. LET: f^{-1} be the unique function $T \to S$
$\langle 1 \rangle$3. $ff^{-1} = \mathrm{id}_T$
PROOF: Each is the unique function $T \to T$.
$\langle 1 \rangle$4. $f^{-1}f = \mathrm{id}_S$
PROOF: Each is the unique function $S \to S$.
$\square$

Definition 0.8 (Terminal Set)**.** Let 1 be the set such that, for any set A, there exists exactly one function $A \to 1$.

Definition 0.9 (Element)**.** An *element* of a set A is a function $1 \to A$. We write $a \in A$ for $a : 1 \to A$.

Given $f : A \to B$ and $a \in A$, we write $f(a)$ for fa.

Definition 0.10 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Definition 0.11 (Injective)**.** A function $f : A \to B$ is *injective* iff, for all $x, x' \in A$, if $f(x) = f(x')$ then $x = x'$.

Definition 0.12 (Bijective)**.** A function is *bijective* iff it is injective and surjective.