

Summary of Halmos' Naive Set Theory

Robin Adams

August 28, 2023

Contents

1	Primitive Terms and Axioms	3
2	Basic Properties and Operations on Sets	5
2.1	The Subset Relation	6
2.2	Comprehension Notation	7
2.3	The Empty Set	7
2.4	Unordered Pairs	7
2.5	Unions	7
2.6	Intersections	8
2.7	Unordered Triples	9
2.8	Relative Complements	9
2.9	Symmetric Difference	12
2.10	Power Sets	13
3	Relations and Functions	15
3.1	Ordered Pairs	15
3.2	Relations	16
3.3	Composition	16
3.4	Inverses	17
3.5	Equivalence Relations	17
3.6	Functions	18
3.7	Families	21
3.8	Inverses and Composites of Functions	23
3.8.1	Inverse Image	23
3.8.2	Inverse of a Function	25
3.9	Choice Functions	28
4	Equivalence	29
5	Order	31
5.1	Partial Orders	31
5.2	Linear Orders	34
5.3	Well Orderings	40

6	Natural Numbers	44
6.1	Natural Numbers	44
7	Ordinal Numbers	49
7.1	Order on the Natural Numbers	52
7.2	Finite Sets	54
7.3	Ordinal Arithmetic	58
7.4	Arithmetic on the Natural Numbers	59
8	Countable Sets	62
9	Cardinal Numbers	64
9.1	Cardinal Arithmetic	64
9.2	Alephs	68
10	Real Numbers	69
10.1	Axioms for Real Numbers	69
10.2	Consequences of the Axioms	70

Chapter 1

Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*, \in . If $x \in A$ we say that x *belongs to* A , x is an *element* of A , or x is *contained in* A . If this does not hold we write $x \notin A$.

Definition 1.1 (Empty). A set is *empty* iff it has no elements; otherwise it is *nonempty*.

Definition 1.2 (Disjoint). Two sets A and B are *disjoint* iff there is no x such that $x \in A$ and $x \in B$.

Definition 1.3 (Pairwise Disjoint). We say the elements of a set x are *pairwise disjoint* iff, for all $y, z \in x$, either y and z are disjoint or $y = z$.

Definition 1.4 (Subset). Let A and B be sets. We say that A is a *subset* of B , or B *includes* A , and write $A \subseteq B$ or $B \supseteq A$, iff every element of A is an element of B .

Axiom 1.5 (Axiom of Extensionality). *If two sets have the same elements then they are equal.*

Axiom 1.6 (Axiom of Regularity). *For any nonempty set A , there exists $m \in A$ such that m and A are disjoint.*

Axiom 1.7 (Axiom of Union). *For every set A , there exists a set that contains all the elements that belong to at least one element of A .*

Axiom 1.8 (Axiom of Pairing). *For any two sets, there exists a set that they both belong to.*

Axiom Schema 1.9 (Axiom Schema of Replacement). *For any property $P(x, y)$, the following is an axiom:*

Let A be a set such that, for all $x \in A$, there exists at most one y such that $P(x, y)$. Then there exists a set B whose elements are exactly those sets y such that $\exists x \in A. P(x, y)$.

Axiom 1.10 (Axiom of Infinity). *There exists a set I that has an empty element and such that, for all $x \in I$, there exists $y \in I$ such that the elements of y are exactly x and the elements of x .*

Axiom 1.11 (Power Set Axiom). *For any set A , there exists a set B such that every subset of A belongs to B .*

Axiom 1.12 (Axiom of Choice). *Let A be a set of nonempty, pairwise disjoint sets. Then there exists a set C such that, for all $x \in A$, there is exactly one y such that $y \in C$ and $y \in x$.*

Chapter 2

Basic Properties and Operations on Sets

Axiom 2.1 (Axiom of Comprehension, Aussonderungsaxiom). *To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

Definition 2.2. Given a set A and a condition $S(x)$, we write $\{x \in A : S(x)\}$ for the set whose elements are exactly those elements x of A for which $S(x)$ holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality. \square

Definition 2.3 ((Unordered) Pair). For any sets a and b , the *(unordered) pair* $\{a, b\}$ is the set whose elements are just a and b .

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality. \square

Axiom 2.4 (Power Set Axiom). *For any set A , there exists a set that contains all the subsets of A .*

Definition 2.5 (Empty). A set is *empty* iff it has no elements; otherwise it is *non-empty*.

Axiom 2.6 (Axiom of Infinity). *There exists a set I such that:*

- *I has an element that is empty*
- *for all $x \in I$, there exists $y \in I$ such that the elements of y are exactly x and the elements of x .*

Definition 2.7 (Ordered Pair). For any sets a and b , the *ordered pair* (a, b) is defined by

$$(a, b) := \{\{a\}, \{a, b\}\} .$$

Definition 2.8 (Power Set). For any set A , the *power set* of A , $\mathcal{P}A$, is the set whose elements are exactly the subsets of A .

PROOF: This exists by the Power Set Axiom and Axiom of Comprehension, and is unique by the Axiom of Extensionality. \square

Definition 2.9 (Cartesian Product). For any sets A and B , the *Cartesian product* $A \times B$ is

$$A \times B := \{p \in \mathcal{P}\mathcal{P}(A \cup B) : \exists a \in A. \exists b \in B. p = (a, b)\} .$$

Definition 2.10 (Relation). A *relation* is a set of ordered pairs.

If R is a relation, we write xRy for $(x, y) \in R$.

Given sets X and Y , a relation *between X and Y* is a subset of $X \times Y$.

Given a set X , a relation *on X* is a relation between X and X .

Definition 2.11 (Function). Let X and Y be sets. A *function*, *map*, *mapping*, *transformation* or *operator f from X to Y* , $f : X \rightarrow Y$, is a relation f between X and Y such that, for all $x \in X$, there exists a unique $f(x) \in Y$, called the *value* of f at the *argument* x , such that $(x, f(x)) \in f$.

Definition 2.12 (Family). Let I and X be sets. A *family* of elements of X indexed by I is a function $a : I \rightarrow X$. We write a_i for $a(i)$, and $\{a_i\}_{i \in I}$ for a .

Definition 2.13 (Cartesian Product of a Family of Sets). Let $\{A_i\}_{i \in I}$ be a family of sets. The *Cartesian product* $\times_{i \in I} A_i$ is the set of all families $\{a_i\}_{i \in I}$ such that $\forall i \in I. a_i \in A_i$.

We write A^I for $\times_{i \in I} A$.

Axiom 2.14 (Axiom of Choice). *The Cartesian product of a non-empty family of non-empty sets is non-empty.*

Axiom 2.15 (Axiom of substitution). *If $S(a, b)$ is a sentence such that for each a in A the set $\{b : S(a, b)\}$ can be formed, then there exists a function F with domain A such that $F(a) = \{b : S(a, b)\}$ for each a in A .*

2.1 The Subset Relation

Theorem 2.16. *For any set A , we have $A \subseteq A$.*

PROOF: Every element of A is an element of A . \square

Theorem 2.17. *For any sets A , B and C , if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

PROOF: If every element of A is an element of B , and every element of B is an element of C , then every element of A is an element of C . \square

Theorem 2.18. *For any sets A and B , if $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

PROOF: If every element of A is an element of B , and every element of B is an element of A , then A and B have the same elements, and therefore are equal by the Axiom of Extensionality. \square

Definition 2.19 (Proper Subset). Let A and B be sets. We say that A is a *proper* subset of B , or B *properly* includes A , and write $A \subsetneq B$ or $B \supsetneq A$, iff $A \subseteq B$ and $A \neq B$.

2.2 Comprehension Notation

Theorem 2.20. *There is no set that contains every set.*

PROOF:

$\langle 1 \rangle 1$. LET: A be a set.

PROVE: There exists a set B such that $B \notin A$.

$\langle 1 \rangle 2$. LET: $B = \{x \in A : x \notin x\}$

$\langle 1 \rangle 3$. If $B \in A$ then we have $B \in B$ if and only if $B \notin B$.

$\langle 1 \rangle 4$. $B \notin A$

\square

2.3 The Empty Set

Theorem 2.21. *There exists a set with no elements.*

PROOF: Immediate from the Axiom of Infinity. \square

Definition 2.22 (Empty Set). The *empty set* \emptyset is the set with no elements.

Theorem 2.23. *For any set A we have $\emptyset \subset A$.*

PROOF: Vacuous. \square

2.4 Unordered Pairs

Definition 2.24 (Singleton). For any set a , the *singleton* $\{a\}$ is defined to be $\{a, a\}$.

2.5 Unions

Definition 2.25 (Union). For any set \mathcal{C} , the *union* of \mathcal{C} , $\bigcup \mathcal{C}$, is the set whose elements are the elements of the elements of \mathcal{C} .

We write $\bigcup_{X \in \mathcal{A}} t[X]$ for $\bigcup \{t[X] \mid X \in \mathcal{A}\}$.

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality. \square

Proposition 2.26.

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of \emptyset . \square

Proposition 2.27. *For any set A , we have $\bigcup\{A\} = A$.*

PROOF: For any x , we have x is an element of an element of $\{A\}$ if and only if x is an element of A . \square

Definition 2.28. We write $A \cup B$ for $\bigcup\{A, B\}$.

Proposition 2.29. *For any set A , we have $A \cup \emptyset = A$.*

PROOF: $x \in A \cup \emptyset$ iff $x \in A$ or $x \in \emptyset$, iff $x \in A$. \square

Proposition 2.30 (Idempotence). *For any set A , we have $A \cup A = A$.*

PROOF: $x \in A$ or $x \in A$ is equivalent to $x \in A$. \square

Proposition 2.31. *For any sets A and B , we have $A \subseteq B$ if and only if $A \cup B = B$.*

PROOF: For any x , the statement "if $x \in A$ then $x \in B$ " is equivalent to " $x \in A$ or $x \in B$ if and only if $x \in B$ ". \square

Proposition 2.32. *For any sets a and b , we have $\{a\} \cup \{b\} = \{a, b\}$.*

PROOF: Immediate from definitions. \square

2.6 Intersections

Definition 2.33 (Intersection). For any sets A and B , the *intersection* $A \cap B$ is defined to be $\{x \in A : x \in B\}$.

Proposition 2.34. *For any set A , we have $A \cap \emptyset = \emptyset$.*

PROOF: There is no x such that $x \in A$ and $x \in \emptyset$. \square

Proposition 2.35. *For any set A , we have*

$$A \cap A = A .$$

PROOF: We have $x \in A$ and $x \in A$ if and only if $x \in A$. \square

Proposition 2.36. *For any sets A and B , we have $A \subseteq B$ if and only if $A \cap B = A$.*

PROOF: For any x , the statement "if $x \in A$ then $x \in B$ " is equivalent to " $x \in A$ and $x \in B$ if and only if $x \in A$ ". \square

Proposition 2.37. *For any sets A , B and C , we have $C \subseteq A$ if and only if $(A \cap B) \cup C = A \cap (B \cup C)$.*

PROOF: The statement "if $x \in C$ then $x \in A$ " is equivalent to the statement " $((x \in A \wedge x \in B) \vee x \in C) \Leftrightarrow (x \in A \wedge (x \in B \vee x \in C))$ ". \square

Definition 2.38 (Disjoint). Two sets A and B are *disjoint* if and only if $A \cap B = \emptyset$.

Definition 2.39 (Pairwise Disjoint). Let A be a set. We say the elements of A are *pairwise disjoint* if and only if, for all $x, y \in A$, if $x \cap y \neq \emptyset$ then $x = y$.

Definition 2.40 (Intersection). For any nonempty set \mathcal{C} , the *intersection* of \mathcal{C} , $\bigcap \mathcal{C}$, is the set that contains exactly those sets that belong to every element of \mathcal{C} .

We write $\bigcap_{X \in \mathcal{A}} t[X]$ for $\bigcap \{t[X] \mid X \in \mathcal{A}\}$.

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{C} be a nonempty set.

$\langle 1 \rangle 2$. There exists a set I whose elements are exactly the sets that belong to every element of \mathcal{C} .

PROOF: Pick $A \in \mathcal{C}$, and take $I = \{x \in A : \forall X \in \mathcal{C}. x \in X\}$.

$\langle 1 \rangle 3$. For any sets I, J , if the elements of I and J are exactly the sets that belong to every element of \mathcal{C} then $I = J$.

PROOF: Axiom of Extensionality.

\square

2.7 Unordered Triples

Definition 2.41 ((Unordered) Triple). Given sets a_1, \dots, a_n , define the (*unordered*) *n-tuple* $\{a_1, \dots, a_n\}$ to be

$$\{a_1, \dots, a_n\} := \{a_1\} \cup \dots \cup \{a_n\} .$$

2.8 Relative Complements

Definition 2.42 (Relative Complement). For any sets A and B , the *difference* or *relative complement* $A - B$ is defined to be

$$A - B := \{x \in A : x \notin B\} .$$

Proposition 2.43. For any sets A and E , we have $A \subseteq E$ if and only if

$$E - (E - A) = A$$

PROOF:

$\langle 1 \rangle 1$. LET: A and E be sets.

$\langle 1 \rangle 2$. If $A \subseteq E$ then $E - (E - A) = A$

$\langle 2 \rangle 1$. ASSUME: $A \subseteq E$

$\langle 2 \rangle 2$. $E - (E - A) \subseteq A$

PROOF: If $x \in E$ and $x \notin E - A$ then $x \in A$.

$\langle 2 \rangle 3$. $A \subseteq E - (E - A)$

PROOF: If $x \in A$ then $x \in E$ and $x \notin E - A$.

$\langle 1 \rangle 3$. If $E - (E - A) = A$ then $A \subseteq E$.

PROOF: Since $E - (E - A) \subseteq E$.

□

Proposition 2.44. *For any set E we have*

$$E - \emptyset = E$$

PROOF: $x \in E$ if and only if $x \in E$ and $x \notin \emptyset$. □

Proposition 2.45. *For any set E we have*

$$E - E = \emptyset .$$

PROOF: There is no x such that $x \in E$ and $x \notin E$. □

Proposition 2.46. *For any sets A and E , we have*

$$A \cap (E - A) = \emptyset .$$

PROOF: There is no x such that $x \in A$ and $x \in E - A$. □

Proposition 2.47. *Let A and E be sets. Then $A \subseteq E$ if and only if*

$$A \cup (E - A) = E .$$

PROOF:

$\langle 1 \rangle 1$. LET: A and E be sets.

$\langle 1 \rangle 2$. If $A \subseteq E$ then $A \cup (E - A) = E$.

$\langle 2 \rangle 1$. ASSUME: $A \subseteq E$

$\langle 2 \rangle 2$. $A \cup (E - A) \subseteq E$

PROOF: If $x \in A$ or $x \in E - A$ then $x \in E$.

$\langle 2 \rangle 3$. $E \subseteq A \cup (E - A)$

PROOF: If $x \in E$ then either $x \in A$ or $x \notin A$. In the latter case, $x \in E - A$.

$\langle 1 \rangle 3$. If $A \cup (E - A) = E$ then $A \subseteq E$

PROOF: Since $A \subseteq A \cup (E - A)$.

□

Proposition 2.48. *Let A , B and E be sets. Then:*

1. *If $A \subseteq B$ then $E - B \subseteq E - A$.*

2. *If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A , B and E be sets.

$\langle 1 \rangle 2$. If $A \subseteq B$ then $E - B \subseteq E - A$.

PROOF: If $A \subseteq B$, $x \in E$ and $x \notin B$, then we have $x \in E$ and $x \notin A$.

⟨1⟩3. If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.

⟨2⟩1. ASSUME: $A \subseteq E$

⟨2⟩2. ASSUME: $E - B \subseteq E - A$

⟨2⟩3. LET: $x \in A$

⟨2⟩4. $x \in E$

⟨2⟩5. $x \notin E - A$

⟨2⟩6. $x \notin E - B$

⟨2⟩7. $x \in B$

□

Example 2.49. We cannot remove the hypothesis $A \subseteq E$ in item 2 above. Let $E = \emptyset$, $A = \{\emptyset\}$ and $B = \emptyset$. Then $E - B = E - A = \emptyset$ but $A \not\subseteq B$.

Proposition 2.50 (De Morgan's Law). *For any sets A , B and E , we have $E - (A \cup B) = (E - A) \cap (E - B)$.*

PROOF: $(x \in E \wedge \neg(x \in A \vee x \in B)) \Leftrightarrow (x \in E \wedge x \notin A \wedge x \in E \wedge x \notin B)$. □

Proposition 2.51 (De Morgan's Law). *For any sets A , B and E , we have $E - (A \cap B) = (E - A) \cup (E - B)$.*

PROOF: $(x \in E \vee \neg(x \in A \wedge x \in B)) \Leftrightarrow (x \in E \wedge x \notin A) \vee (x \in E \wedge x \notin B)$. □

Proposition 2.52. *For any sets A , B and E , if $A \subseteq E$ then*

$$A - B = A \cap (E - B) .$$

PROOF: If $A \subseteq E$ then we have $(x \in A \wedge x \notin B) \Leftrightarrow (x \in A \wedge x \in E \wedge x \notin B)$. □

Proposition 2.53. *For any sets A and B , we have $A \subseteq B$ if and only if $A - B = \emptyset$.*

PROOF: Both are equivalent to the statement that there is no x such that $x \in A$ and $x \notin B$. □

Proposition 2.54. *For any sets A and B , we have*

$$A - (A - B) = A \cap B .$$

PROOF: $(x \in A \wedge \neg(x \in A \wedge x \notin B)) \Leftrightarrow x \in A \wedge x \in B$. □

Proposition 2.55. *For any sets A , B and C , we have*

$$A \cap (B - C) = (A \cap B) - (A \cap C) .$$

PROOF: $(x \in A \wedge x \in B \wedge x \notin C) \Leftrightarrow (x \in A \wedge x \in B \wedge \neg(x \in A \wedge x \in C))$. □

Proposition 2.56. *For any sets A , B , C and E , if $(A \cap B) - C \subseteq E$ then we have*

$$A \cap B \subseteq (A \cap C) \cup (B \cap (E - C)) .$$

PROOF:

$\langle 1 \rangle 1.$ LET: $x \in A \cap B$
 PROVE: $x \in (A \cap C) \cup (B \cap (E - C))$
 $\langle 1 \rangle 2.$ CASE: $x \in C$
 PROOF: Then $x \in A \cap C$.
 $\langle 1 \rangle 3.$ CASE: $x \notin C$
 PROOF: Then $x \in E$ and so $x \in B \cap (E - C)$.
 \square

Proposition 2.57. *For any sets A, B, C and E , we have*

$$(A \cup C) \cap (B \cup (E - C)) \subseteq A \cup B .$$

PROOF: The statement $(x \in A \vee x \in C) \wedge (x \in B \vee (x \in E \wedge x \notin C))$ implies $x \in A \vee x \in B$. \square

Proposition 2.58 (De Morgan's Law). *Let E be a set and \mathcal{C} a nonempty set. Then*

$$E - \bigcup_{X \in \mathcal{C}} X = \bigcap_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy. \square

Proposition 2.59 (De Morgan's Law). *Let E be a set and \mathcal{C} a nonempty set. Then*

$$E - \bigcap_{X \in \mathcal{C}} X = \bigcup_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy. \square

2.9 Symmetric Difference

Definition 2.60 (Symmetric Difference). For any sets A and B , the *symmetric difference* $A + B$ is defined to be

$$A + B := (A - B) \cup (B - A) .$$

Proposition 2.61. *For any sets A and B , we have*

$$A + B = B + A$$

PROOF: From the commutativity of union. \square

Proposition 2.62. *For any sets A, B and C , we have*

$$A + (B + C) = (A + B) + C .$$

PROOF: Each is the set of all x that belong to either exactly one or all three of A, B and C . \square

Proposition 2.63. *For any set A , we have*

$$A + \emptyset = A .$$

PROOF:

$$\begin{aligned} A + \emptyset &= (A - \emptyset) \cup (\emptyset - A) \\ &= A \cup \emptyset \\ &= A \end{aligned}$$

□

Proposition 2.64. *For any set A we have*

$$A + A = \emptyset .$$

PROOF:

$$\begin{aligned} A + A &= (A - A) \cup (A - A) \\ &= \emptyset \cup \emptyset \\ &= \emptyset \end{aligned}$$

□

2.10 Power Sets

Proposition 2.65.

$$\mathcal{P}\emptyset = \{\emptyset\}$$

PROOF: The only subset of \emptyset is \emptyset . □

Proposition 2.66. *For any set a , we have*

$$\mathcal{P}\{a\} = \{\emptyset, \{a\}\} .$$

PROOF: The only subsets of $\{a\}$ are \emptyset and $\{a\}$. □

Proposition 2.67. *For any sets a and b , we have*

$$\mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} .$$

PROOF: The only subsets of $\{a, b\}$ are \emptyset , $\{a\}$, $\{b\}$ and $\{a, b\}$. □

Proposition 2.68. *For any nonempty set \mathcal{C} we have*

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}X = \mathcal{P}\left(\bigcap \mathcal{C}\right) .$$

PROOF:

$$\begin{aligned} x \in \bigcup_{X \in \mathcal{C}} \mathcal{P}X &\Leftrightarrow \forall X \in \mathcal{C}. x \subseteq X \\ &\Leftrightarrow \forall X \in \mathcal{C}. \forall y \in x. y \in X \\ &\Leftrightarrow \forall y \in x. \forall X \in \mathcal{C}. y \in X \\ &\Leftrightarrow x \subseteq \bigcap \mathcal{C} \end{aligned}$$

□

Proposition 2.69. *For any set \mathcal{C} we have*

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}X \subseteq \mathcal{P} \bigcup \mathcal{C} .$$

PROOF: If there exists $X \in \mathcal{C}$ such that $x \subseteq X$ then $x \subseteq \bigcup \mathcal{C}$. \square

Proposition 2.70. *For any set E , we have*

$$\bigcap \mathcal{P}E = \emptyset .$$

PROOF: Since $\emptyset \in \mathcal{P}E$. \square

Proposition 2.71. *For any sets E and F , if $E \subseteq F$ then $\mathcal{P}E \subseteq \mathcal{P}F$.*

PROOF: If $E \subseteq F$ and $X \subseteq E$ then $X \subseteq F$. \square

Chapter 3

Relations and Functions

3.1 Ordered Pairs

Proposition 3.1. *For any sets a, b, x and y , if $(a, b) = (x, y)$ then $a = x$ and $b = y$.*

PROOF:

$\langle 1 \rangle 1$. LET: a, b, x and y be sets.

$\langle 1 \rangle 2$. ASSUME: $(a, b) = (x, y)$

$\langle 1 \rangle 3$. $a = x$

PROOF: $\{a\} = \bigcap(a, b) = \bigcap(x, y) = \{x\}$.

$\langle 1 \rangle 4$. $\{a, b\} = \{x, y\}$

$\langle 1 \rangle 5$. CASE: $a = b$

$\langle 2 \rangle 1$. $x = y$

PROOF: Since $\{x, y\} = \{a, b\}$ is a singleton.

$\langle 2 \rangle 2$. $b = y$

PROOF: $b = a = x = y$

$\langle 1 \rangle 6$. CASE: $a \neq b$

$\langle 2 \rangle 1$. $x \neq y$

PROOF: Since $\{x, y\} = \{a, b\}$ is not a singleton.

$\langle 2 \rangle 2$. $b = y$

PROOF: $\{b\} = \{a, b\} - \{a\} = \{x, y\} - \{x\} = \{y\}$.

□

Proposition 3.2. *For any sets A, B and X , we have*

$$(A - B) \times X = (A \times X) - (B \times X) .$$

PROOF: Easy. □

Proposition 3.3. *For any sets A and B , we have $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.*

PROOF: Easy. □

Proposition 3.4. For any sets A, B, X and Y , if $A \subseteq X$ and $B \subseteq Y$ then $A \times B \subseteq X \times Y$. The converse holds assuming $A \neq \emptyset$ and $B \neq \emptyset$.

PROOF: Easy. \square

3.2 Relations

Definition 3.5 (Domain). The *domain* of a relation R is the set

$$\text{dom } R := \left\{ x \in \bigcup \bigcup R : \exists y. (x, y) \in R \right\} .$$

Definition 3.6 (Range). The *range* of a relation R is the set

$$\text{ran } R := \left\{ y \in \bigcup \bigcup R : \exists x. (x, y) \in R \right\} .$$

Definition 3.7 (Reflexive). Let R be a relation on X . Then R is *reflexive* iff, for all $x \in X$, we have xRx .

Definition 3.8 (Symmetric). Let R be a relation on X . Then R is *symmetric* iff, whenever xRy , then yRx .

Definition 3.9 (Antisymmetric). A relation R is *antisymmetric* iff, whenever xRy and yRx , then $x = y$.

Definition 3.10 (Transitive). Let R be a relation on X . Then R is *transitive* iff, whenever xRy and yRz , then xRz .

Definition 3.11 (Identity Relation). For any set X , the *identity relation* I_X on X is

$$I_X = \{(x, x) : x \in X\} .$$

3.3 Composition

Definition 3.12 (Composition). Let R be a relation between X and Y , and S a relation between Y and Z . The *composite* or *relative product* $S \circ R = SR$ is the relation between X and Z defined by

$$x(S \circ R)z \Leftrightarrow \exists y \in Y (xRy \wedge ySz) .$$

Proposition 3.13. Let R be a relation between X and Y , S a relation between Y and Z , and T a relation between Z and W . Then

$$T(SR) = (TS)R .$$

PROOF: Easy. \square

Example 3.14. Composition of relations is not commutative in general. Let $X = \{a, b\}$ where $a \neq b$. Let $R = \{(a, a), (b, a)\}$ and $S = \{(a, b), (b, b)\}$. Then $SR = S$ but $RS = R \neq S$.

Proposition 3.15. A relation R is transitive if and only if $RR \subseteq R$.

PROOF: Easy. \square

3.4 Inverses

Definition 3.16 (Inverse). Let R be a relation between X and Y . The *inverse* or *converse* R^{-1} is the relation between Y and X defined by

$$yR^{-1}x \Leftrightarrow xRy .$$

Proposition 3.17. *For any relation R , we have*

$$\text{dom } R^{-1} = \text{ran } R .$$

PROOF: Easy. \square

Proposition 3.18. *For any relation R , we have*

$$\text{ran } R^{-1} = \text{dom } R .$$

PROOF: Easy. \square

Proposition 3.19. *Let R be a relation between X and Y , and S a relation between Y and Z . Then*

$$(SR)^{-1} = R^{-1}S^{-1} .$$

PROOF: Easy. \square

Proposition 3.20. *A relation R is symmetric if and only if $R \subseteq R^{-1}$.*

PROOF: Easy. \square

Proposition 3.21. *Let R be a relation between X and Y . Then*

$$I_Y R = R I_X = R .$$

PROOF: Easy. \square

Proposition 3.22. *A relation R on a set X is reflexive if and only if $I_X \subseteq R$.*

PROOF: Easy. \square

Proposition 3.23. *Let R be a relation on a set X . Then R is antisymmetric iff $R \cap R^{-1} \subseteq I_X$.*

PROOF: Easy. \square

3.5 Equivalence Relations

Definition 3.24 (Equivalence Relation). Let R be a relation on X . Then R is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Definition 3.25 (Partition). Let X be a set. A *partition* of X is a pairwise disjoint set of nonempty subsets of X whose union is X .

Definition 3.26 (Equivalence Class). Let R be an equivalence relation on X . Let $x \in X$. The *equivalence class* of x with respect to R is

$$x/R := \{y \in X : xRy\} .$$

We write X/R for the set of all equivalence classes with respect to R .

Definition 3.27 (Induced). Let P be a partition of X . The relation *induced* by P is X/P where $x(X/P)y$ iff there exists $X \in P$ such that $x \in X$ and $y \in X$.

Theorem 3.28. Let R be an equivalence relation on X . Then X/R is a partition of X that induces the relation R .

PROOF: Easy. \square

Theorem 3.29. Let P be a partition of X . Then X/P is an equivalence relation on X , and $P = X/(X/P)$.

PROOF: Easy. \square

3.6 Functions

Definition 3.30 (Injective). A function $f : X \rightarrow Y$ is *one-to-one* or *injective* or an *injection* iff, for all $x, y \in X$, if $f(x) = f(y)$ then $x = y$. In this case, we write $f : X \rightarrowtail Y$.

Definition 3.31 (Surjective). Let $f : X \rightarrow Y$. We say f is *surjective*, or a *surjection*, or f maps X *onto* Y iff $\text{ran } f = Y$. In this case, we write $f : X \twoheadrightarrow Y$.

Definition 3.32 (Bijective). Let $f : X \rightarrow Y$. Then f is *bijective*, or a *bijection*, iff it is injective and surjective.

Definition 3.33 (Image). Let $f : X \rightarrow Y$ and $A \subseteq X$. The *image* of A under f is

$$f(A) := \{f(x) : x \in A\} .$$

Proposition 3.34. Let $f : X \rightarrow Y$ and $A \subseteq B \subseteq X$. Then $f(A) \subseteq f(B)$.

PROOF:

$\langle 1 \rangle 1$. LET: X and Y be sets.

$\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$

$\langle 1 \rangle 3$. LET: $A, B \in \mathcal{P}X$ with $A \subseteq B$.

$\langle 1 \rangle 4$. LET: $y \in f(A)$

$\langle 1 \rangle 5$. PICK $x \in A$ such that $f(x) = y$

PROOF: $\langle 1 \rangle 4$

$\langle 1 \rangle 6$. $x \in B$

PROOF: $\langle 1 \rangle 3, \langle 1 \rangle 5$

$\langle 1 \rangle 7$. $y \in f(B)$

PROOF: $\langle 1 \rangle 5, \langle 1 \rangle 6$

□

Proposition 3.35. *Let $f : X \rightarrow Y$. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then $f(\bigcup \mathcal{A}) = \bigcup_{A \in \mathcal{A}} f(A)$.*

PROOF:

$$\begin{aligned}
 y \in f\left(\bigcup \mathcal{A}\right) &\Leftrightarrow \exists x \in \bigcup \mathcal{A}. y = f(x) \\
 &\Leftrightarrow \exists x. \exists A \in \mathcal{A} (x \in A \wedge y = f(x)) \\
 &\Leftrightarrow \exists A \in \mathcal{A}. \exists x \in A. y = f(x) \\
 &\Leftrightarrow \exists A \in \mathcal{A}. y \in f(A) \\
 &\Leftrightarrow y \in \bigcup_{A \in \mathcal{A}} f(A) \quad \square
 \end{aligned}$$

Proposition 3.36. *Let $f : X \rightarrow Y$. Let \mathcal{A} be a nonempty subset of $\mathcal{P}X$. Then $f(\bigcap \mathcal{A}) \subseteq \bigcap_{A \in \mathcal{A}} f(A)$. Equality holds if f is injective.*

PROOF:

- ⟨1⟩1. LET: X and Y be sets.
- ⟨1⟩2. LET: $f : X \rightarrow Y$
- ⟨1⟩3. LET: \mathcal{A} be a nonempty subset of $\mathcal{P}X$.
- ⟨1⟩4. $f(\bigcap \mathcal{A}) \subseteq \bigcap_{A \in \mathcal{A}} f(A)$
 - ⟨2⟩1. LET: $y \in f(\bigcap \mathcal{A})$
 - ⟨2⟩2. PICK $x \in \bigcap \mathcal{A}$ such that $y = f(x)$
 - PROOF: ⟨2⟩1
 - ⟨2⟩3. LET: $A \in \mathcal{A}$
 - ⟨2⟩4. $x \in A$
 - PROOF: ⟨2⟩2, ⟨2⟩3
 - ⟨2⟩5. $y \in f(A)$
 - PROOF: ⟨2⟩2, ⟨2⟩4
- ⟨1⟩5. If f is injective then $f(\bigcap \mathcal{A}) = \bigcap_{A \in \mathcal{A}} f(A)$
 - ⟨2⟩1. ASSUME: f is injective.
 - ⟨2⟩2. LET: $y \in \bigcap_{A \in \mathcal{A}} f(A)$
 - ⟨2⟩3. PICK $A \in \mathcal{A}$
 - PROOF: \mathcal{A} is nonempty by ⟨1⟩3.
 - ⟨2⟩4. $y \in f(A)$
 - PROOF: ⟨2⟩2, ⟨2⟩3
 - ⟨2⟩5. PICK $x \in A$ such that $y = f(x)$
 - PROOF: ⟨2⟩4
 - ⟨2⟩6. $x \in \bigcap \mathcal{A}$
 - ⟨3⟩1. LET: $A' \in \mathcal{A}$
 - ⟨3⟩2. $y \in f(A')$
 - PROOF: ⟨2⟩2, ⟨3⟩1
 - ⟨3⟩3. PICK $x' \in A'$ such that $y = f(x')$
 - PROOF: ⟨3⟩2
 - ⟨3⟩4. $x = x'$
 - PROOF: ⟨2⟩1, ⟨2⟩5, ⟨3⟩3
 - ⟨3⟩5. $x \in A'$

PROOF: $\langle 3 \rangle 3, \langle 3 \rangle 4$
 $\langle 2 \rangle 7. y \in f(\bigcap \mathcal{A})$
 PROOF: $\langle 2 \rangle 5, \langle 2 \rangle 6$

□

Proposition 3.37. *Let X and Y be sets. Let $f : X \rightarrow Y$. Let $A, B \in \mathcal{P}X$. Then $f(A) - f(B) \subseteq f(A - B)$. Equality holds if f is injective.*

PROOF:
 $\langle 1 \rangle 1$. LET: X and Y be sets.
 $\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$
 $\langle 1 \rangle 3$. LET: $A, B \in \mathcal{P}X$
 $\langle 1 \rangle 4$. $f(A) - f(B) \subseteq f(A - B)$
 $\langle 2 \rangle 1$. LET: $y \in f(A) - f(B)$
 $\langle 2 \rangle 2$. $y \in f(A)$
 PROOF: $\langle 2 \rangle 1$
 $\langle 2 \rangle 3$. PICK $x \in A$ such that $y = f(x)$.
 PROOF: $\langle 2 \rangle 2$
 $\langle 2 \rangle 4$. $x \notin B$
 $\langle 3 \rangle 1$. ASSUME: for a contradiction $x \in B$
 $\langle 3 \rangle 2$. $y \in f(B)$
 PROOF: $\langle 2 \rangle 3, \langle 3 \rangle 1$
 $\langle 3 \rangle 3$. Q.E.D.
 PROOF: $\langle 2 \rangle 1$ and $\langle 3 \rangle 2$ form a contradiction.
 $\langle 2 \rangle 5$. $x \in A - B$
 PROOF: $\langle 2 \rangle 3, \langle 2 \rangle 4$
 $\langle 2 \rangle 6$. $y \in f(A - B)$
 PROOF: $\langle 2 \rangle 3, \langle 2 \rangle 5$
 $\langle 1 \rangle 5$. If f is injective then $f(A - B) = f(A) - f(B)$.
 $\langle 2 \rangle 1$. ASSUME: f is injective.
 $\langle 2 \rangle 2$. LET: $y \in f(A - B)$
 $\langle 2 \rangle 3$. PICK $x \in A - B$ such that $y = f(x)$
 PROOF: $\langle 2 \rangle 2$
 $\langle 2 \rangle 4$. $x \in A$
 PROOF: $\langle 2 \rangle 3$
 $\langle 2 \rangle 5$. $y \in f(A)$
 PROOF: $\langle 2 \rangle 3, \langle 2 \rangle 4$
 $\langle 2 \rangle 6$. $y \notin f(B)$
 $\langle 3 \rangle 1$. ASSUME: $y \in f(B)$
 $\langle 3 \rangle 2$. PICK $x' \in B$ such that $y = f(x')$
 PROOF: $\langle 3 \rangle 1$
 $\langle 3 \rangle 3$. $x = x'$
 PROOF: $\langle 2 \rangle 1, \langle 2 \rangle 3, \langle 3 \rangle 2$
 $\langle 3 \rangle 4$. $x \in B$
 PROOF: $\langle 3 \rangle 2, \langle 3 \rangle 3$
 $\langle 3 \rangle 5$. Q.E.D.
 PROOF: $\langle 2 \rangle 3$ and $\langle 3 \rangle 4$ form a contradiction.

$\langle 2 \rangle 7. y \in f(A) - f(B)$

PROOF: $\langle 2 \rangle 5, \langle 2 \rangle 6$

□

Definition 3.38 (Inclusion Map). Let Y be a set and $X \subseteq Y$. Then the *inclusion map* $i : X \hookrightarrow Y$ is the function defined by $i(x) = x$ for all $x \in X$.

Proposition 3.39. For any set X , the identity relation I_X is a function $X \rightarrow X$.

PROOF: Easy. □

Definition 3.40 (Restriction). Let $f : Y \rightarrow Z$ and $X \subseteq Y$. The *restriction* of f to X is the function $f \upharpoonright X : X \rightarrow Z$ defined by

$$(f \upharpoonright X)(x) = f(x) \quad (x \in X) .$$

Given sets X, Y and Z with $X \subseteq Y$, if $f : X \rightarrow Z$ and $g : Y \rightarrow Z$, we say g is an *extension* of f to Y iff $f = g \upharpoonright X$.

Definition 3.41 (Projection). Given sets X and Y , the *projection* maps $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are defined by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y \quad (x \in X, y \in Y) .$$

Definition 3.42 (Canonical Map). Let X be a set and R an equivalence relation on X . The *canonical map* $\pi : X \rightarrow X/R$ is the map defined by $\pi(x) = x/R$.

Proposition 3.43. Let $f : X \rightarrow Y$. Then the following are equivalent:

1. f is one-to-one.
2. For all $A, B \subseteq X$, we have $f(A \cap B) = f(A) \cap f(B)$.
3. For all $A \subseteq X$, we have $f(X - A) \subseteq Y - f(A)$.

PROOF: Easy. □

Proposition 3.44. Let $f : X \rightarrow Y$. Then f maps X onto Y if and only if, for all $A \subseteq X$, we have $Y - f(A) \subseteq f(X - A)$.

PROOF: Easy. □

3.7 Families

Proposition 3.45 (Generalized Associative Law for Unions). Let $\{I_j\}_{j \in J}$ be a family of sets. Let $K = \bigcup_{j \in J} I_j$. Let $\{A_k\}_{k \in K}$ be a family of sets indexed by K . Then

$$\bigcup_{k \in K} A_k = \bigcup_{j \in J} \bigcup_{i \in I_j} A_i .$$

PROOF: Easy. \square

Proposition 3.46 (Generalized Commutative Law for Unions). *Let $\{I_j\}_{j \in J}$ be a family of sets. Let $f : J \rightarrow J$ be a one-to-one correspondence from J onto J . Then*

$$\bigcup_{j \in J} I_j = \bigcup_{j \in J} I_{f(j)} .$$

PROOF: Easy. \square

Proposition 3.47 (Generalized Associative Law for Intersections). *Let $\{I_j\}_{j \in J}$ be a nonempty family of nonempty sets. Let $K = \bigcup_{j \in J} I_j$. Let $\{A_k\}_{k \in K}$ be a family of sets indexed by K . Then*

$$\bigcap_{k \in K} A_k = \bigcap_{j \in J} \bigcap_{i \in I_j} A_i .$$

PROOF: Easy. \square

Proposition 3.48 (Generalized Commutative Law for Intersections). *Let $\{I_j\}_{j \in J}$ be a nonempty family of sets. Let $f : J \rightarrow J$ be a one-to-one correspondence from J onto J . Then*

$$\bigcap_{j \in J} I_j = \bigcap_{j \in J} I_{f(j)} .$$

PROOF: Easy. \square

Proposition 3.49. *Let B be a set and $\{A_i\}_{i \in I}$ a family of sets. Then*

$$B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i)$$

PROOF: Easy. \square

Proposition 3.50. *Let B be a set and $\{A_i\}_{i \in I}$ a nonempty family of sets. Then*

$$B \cup \bigcap_{i \in I} A_i = \bigcap_{i \in I} (B \cup A_i)$$

PROOF: Easy. \square

Definition 3.51 (Projection). *Let $\{A_i\}_{i \in I}$ be a family of sets and $i \in I$. The projection function $\pi_i : \times_{i \in I} A_i \rightarrow A_i$ is defined by $\pi_i(a) = a_i$.*

Proposition 3.52. *Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be families of sets. Then*

$$\left(\bigcup_{i \in I} A_i \right) \times \left(\bigcup_{j \in J} B_j \right) = \bigcup_{i \in I} \bigcup_{j \in J} (A_i \times B_j) .$$

PROOF: Easy. \square

Proposition 3.53. Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be nonempty families of sets. Then

$$\left(\bigcap_{i \in I} A_i \right) \times \left(\bigcap_{j \in J} B_j \right) = \bigcap_{i \in I} \bigcap_{j \in J} (A_i \times B_j) .$$

PROOF: Easy. \square

Proposition 3.54. Let $f : X \rightarrow Y$. Let $\{A_i\}_{i \in I}$ be a family of subsets of X . Then

$$f \left(\bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} f(A_i) .$$

PROOF: Easy. \square

Example 3.55. It is not true in general that, if $f : X \rightarrow Y$ and $\{A_i\}_{i \in I}$ is a nonempty family of subsets of X , then $f(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f(A_i)$.

Take $X = \{a, b\}$ and $Y = \{c\}$ where $a \neq b$. Take $I = \{i, j\}$ with $i \neq j$. Let $A_i = \{a\}$ and $A_j = \{b\}$. Let f be the unique function $X \rightarrow Y$. Then $f(\bigcap_{i \in I} A_i) = f(\emptyset) = \emptyset$ but $\bigcap_{i \in I} f(A_i) = \{c\}$.

3.8 Inverses and Composites of Functions

3.8.1 Inverse Image

Definition 3.56 (Inverse Image). Let $f : X \rightarrow Y$. Let B be a subset of Y . Then the *inverse image* of B under f is

$$f^{-1}(B) = \{x \in X : f(x) \in B\} .$$

Proposition 3.57. Let $f : X \rightarrow Y$. Let $B \subseteq Y$. Then

$$f(f^{-1}(B)) \subseteq B .$$

Equality holds if f is surjective.

PROOF:

- $\langle 1 \rangle 1$. LET: X and Y be sets.
- $\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$
- $\langle 1 \rangle 3$. LET: $B \subseteq Y$
- $\langle 1 \rangle 4$. $f(f^{-1}(B)) \subseteq B$
 - $\langle 2 \rangle 1$. LET: $y \in f(f^{-1}(B))$
 - $\langle 2 \rangle 2$. PICK $x \in f^{-1}(B)$ such that $f(x) = y$
 - $\langle 2 \rangle 3$. $f(x) \in B$
 - $\langle 2 \rangle 4$. $y \in B$
- $\langle 1 \rangle 5$. If f is surjective then $f(f^{-1}(B)) = B$
 - $\langle 2 \rangle 1$. ASSUME: f is surjective.
 - $\langle 2 \rangle 2$. LET: $y \in B$

$\langle 2 \rangle 3$. PICK $x \in X$ such that $f(x) = y$

PROOF: $\langle 2 \rangle 1, \langle 2 \rangle 2$

$\langle 2 \rangle 4$. $x \in f^{-1}(B)$

PROOF: $\langle 2 \rangle 2, \langle 2 \rangle 3$

$\langle 2 \rangle 5$. $y \in f(f^{-1}(B))$

PROOF: $\langle 2 \rangle 3, \langle 2 \rangle 4$

□

Proposition 3.58. *Let $f : X \rightarrow Y$. Let $A \subseteq X$. Then*

$$A \subseteq f^{-1}(f(A)) .$$

Equality holds if f is one-to-one.

PROOF:

$\langle 1 \rangle 1$. LET: X and Y be sets.

$\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$

$\langle 1 \rangle 3$. LET: $A \subseteq X$

$\langle 1 \rangle 4$. $A \subseteq f^{-1}(f(A))$

$\langle 2 \rangle 1$. LET: $x \in A$

$\langle 2 \rangle 2$. $f(x) \in f(A)$

$\langle 2 \rangle 3$. $x \in f^{-1}(f(x))$

$\langle 1 \rangle 5$. If f is injective then $f^{-1}(f(A)) = A$

$\langle 2 \rangle 1$. ASSUME: f is injective.

$\langle 2 \rangle 2$. LET: $x \in f^{-1}(f(A))$

$\langle 2 \rangle 3$. $f(x) \in f(A)$

PROOF: $\langle 2 \rangle 2$

$\langle 2 \rangle 4$. PICK $x' \in A$ such that $f(x') = f(x)$

PROOF: $\langle 2 \rangle 3$

$\langle 2 \rangle 5$. $x' = x$

PROOF: $\langle 2 \rangle 1, \langle 2 \rangle 4$

$\langle 2 \rangle 6$. $x \in A$

PROOF: $\langle 2 \rangle 4, \langle 2 \rangle 5$

□

Proposition 3.59. *Let $f : X \rightarrow Y$. Let $A \subseteq B \subseteq Y$. Then $f^{-1}(A) \subseteq f^{-1}(B)$.*

PROOF:

$\langle 1 \rangle 1$. LET: X and Y be sets.

$\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$

$\langle 1 \rangle 3$. LET: $A \subseteq B \subseteq Y$

$\langle 1 \rangle 4$. LET: $x \in f^{-1}(A)$

$\langle 1 \rangle 5$. $f(x) \in A$

$\langle 1 \rangle 6$. $f(x) \in B$

$\langle 1 \rangle 7$. $x \in f^{-1}(B)$

□

Proposition 3.60. *Let $f : X \rightarrow Y$. Let $\mathcal{B} \subseteq Y$. Then*

$$f^{-1}\left(\bigcup \mathcal{B}\right) = \bigcup_{B \in \mathcal{B}} f^{-1}(B) .$$

PROOF:

$$\begin{aligned}
 x \in f^{-1}\left(\bigcup \mathcal{B}\right) &\Leftrightarrow f(x) \in \bigcup \mathcal{B} \\
 &\Leftrightarrow \exists B \in \mathcal{B}. f(x) \in B \\
 &\Leftrightarrow \exists B \in \mathcal{B}. x \in f^{-1}(B) \\
 &\Leftrightarrow x \in \bigcup_{B \in \mathcal{B}} f^{-1}(B) \quad \square
 \end{aligned}$$

Proposition 3.61. *Let $f : X \rightarrow Y$. Let $\{B_i\}_{i \in I}$ be a nonempty family of subsets of Y . Then*

$$f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i) .$$

PROOF: Easy. \square

Proposition 3.62. *Let $f : X \rightarrow Y$ and $B \subseteq Y$. Then $f^{-1}(Y - B) = X - f^{-1}(B)$.*

PROOF: Easy. \square

3.8.2 Inverse of a Function

Proposition 3.63. *Let $f : X \approx Y$. Then f^{-1} is a function, and is a bijection $f^{-1} : Y \approx X$.*

PROOF:

$\langle 1 \rangle 1$. LET: X and Y be sets.

$\langle 1 \rangle 2$. LET: $f : X \approx Y$

$\langle 1 \rangle 3$. f^{-1} is a function.

$\langle 2 \rangle 1$. LET: $(x, y), (x, z) \in f^{-1}$

$\langle 2 \rangle 2$. $(y, x), (z, x) \in f$

$\langle 2 \rangle 3$. $y = z$

PROOF: f is injective.

$\langle 1 \rangle 4$. $\text{dom } f^{-1} = Y$

PROOF: Proposition 3.17, $\langle 1 \rangle 2$

$\langle 1 \rangle 5$. $\text{ran } f^{-1} = X$

PROOF:

$$\begin{aligned}
 x \in \text{ran } f^{-1} &\Leftrightarrow \exists y. (y, x) \in f^{-1} \\
 &\Leftrightarrow \exists y. (x, y) \in f \\
 &\Leftrightarrow x \in \text{dom } f \\
 &\Leftrightarrow x \in X
 \end{aligned}$$

$\langle 1 \rangle 6$. f^{-1} is injective.

$\langle 2 \rangle 1$. LET: $y, y' \in Y$

$\langle 2 \rangle 2$. ASSUME: $f^{-1}(y) = f^{-1}(y')$

$\langle 2 \rangle 3$. $y = y'$

PROOF: $y = f(f^{-1}(y)) = f(f^{-1}(y')) = y'$.

□

Proposition 3.64. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Then $gf : X \rightarrow Z$ and, for all $x \in X$, we have*

$$(g \circ f)(x) = g(f(x)) .$$

PROOF: Easy. □

Example 3.65. Example 3.14 shows that function composition is not commutative in general.

Proposition 3.66. *The composite of two injective functions is injective.*

PROOF:

⟨1⟩1. LET: $f : X \rightarrow Y$ and $g : Y \rightarrow Z$

⟨1⟩2. LET: $x, y \in X$

⟨1⟩3. ASSUME: $(g \circ f)(x) = (g \circ f)(y)$

⟨1⟩4. $g(f(x)) = g(f(y))$

⟨1⟩5. $f(x) = f(y)$

PROOF: g is injective.

⟨1⟩6. $x = y$

PROOF: f is injective.

□

Proposition 3.67. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If $g \circ f$ is injective then f is injective.*

PROOF: If $f(x) = f(y)$ then $g(f(x)) = g(f(y))$ and so $x = y$. □

Proposition 3.68. *The composite of two surjective functions is surjective.*

PROOF:

⟨1⟩1. LET: $f : X \rightarrow Y$ and $g : Y \rightarrow Z$

⟨1⟩2. LET: $z \in Z$

⟨1⟩3. PICK $y \in Y$ such that $g(y) = z$

PROOF: Since g is surjective.

⟨1⟩4. PICK $x \in X$ such that $f(x) = y$

PROOF: Since f is surjective.

⟨1⟩5. $(g \circ f)(x) = z$

□

Proposition 3.69. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If $g \circ f$ is surjective then g is surjective.*

PROOF: Let $z \in Z$. Pick $x \in X$ such that $g(f(x)) = z$. Then there exists y such that $g(y) = z$, namely $y = f(x)$. □

Proposition 3.70. *The composite of two bijective functions is bijective.*

PROOF: Propositions 3.66 and 3.68. □

Proposition 3.71. *Let X, Y and Z be sets. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Let $A \subseteq Z$. Then*

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) .$$

PROOF:

$\langle 1 \rangle 1$. LET: X, Y and Z be sets.

$\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$ and $g : Y \rightarrow Z$

$\langle 1 \rangle 3$. LET: $A \subseteq Z$

$\langle 1 \rangle 4$. $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$

PROOF:

$\langle 2 \rangle 1$. LET: $x \in X$

$\langle 2 \rangle 2$. $x \in (g \circ f)^{-1}(A) \Leftrightarrow x \in f^{-1}(g^{-1}(A))$

PROOF:

$$\begin{aligned} x \in (g \circ f)^{-1}(A) &\Leftrightarrow (g \circ f)(x) \in A \\ &\Leftrightarrow g(f(x)) \in A && \text{(Proposition 3.64, } \langle 1 \rangle 2, \langle 2 \rangle 1) \\ &\Leftrightarrow f(x) \in g^{-1}(A) \\ &\Leftrightarrow x \in f^{-1}(g^{-1}(A)) \end{aligned}$$

□

Proposition 3.72. *Let $f : X \approx Y$ and $g : Y \approx Z$. Then*

$$(gf)^{-1} = f^{-1}g^{-1} : Z \rightarrow X .$$

PROOF: Easy. □

Definition 3.73 (Left Inverse, Right Inverse). Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$. Then g is a *left inverse* of f , and f is a *right inverse* of g , iff $g \circ f = I_X$.

Proposition 3.74. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$. If $gf = I_X$ then f is one-to-one and g maps Y onto X .*

PROOF: Easy. □

Lemma 3.75. *Let $f : A \rightarrow B$. If there are functions $g : B \rightarrow A$ and $h : B \rightarrow A$ such that $\forall a \in A. g(f(a)) = a$ and $\forall b \in B. f(h(b)) = b$, then f is bijective and $g = h = f^{-1}$.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B be sets.

$\langle 1 \rangle 2$. LET: $f : A \rightarrow B$ and $g, h : B \rightarrow A$

$\langle 1 \rangle 3$. ASSUME: $\forall a \in A. g(f(a)) = a$

$\langle 1 \rangle 4$. ASSUME: $\forall b \in B. f(h(b)) = b$

$\langle 1 \rangle 5$. f is injective.

PROOF: Proposition 3.74, $\langle 1 \rangle 2, \langle 1 \rangle 3$.

$\langle 1 \rangle 6$. f is surjective.

PROOF: Proposition 3.74, $\langle 1 \rangle 2, \langle 1 \rangle 4$.

$\langle 1 \rangle 7$. $g = h$

$\langle 2 \rangle 1$. LET: $b \in B$
 $\langle 2 \rangle 2$. $g(b) = h(b)$
 PROOF:

$$\begin{aligned} g(b) &= g(f(h(b))) & (\langle 1 \rangle 4, \langle 2 \rangle 1) \\ &= h(b) & (\langle 1 \rangle 3, \langle 1 \rangle 2, \langle 2 \rangle 1) \end{aligned}$$
 $\langle 1 \rangle 8$. $h = f^{-1}$
 $\langle 2 \rangle 1$. LET: $b \in B$
 $\langle 2 \rangle 2$. $f(h(b)) = b$
 PROOF: $\langle 1 \rangle 4, \langle 2 \rangle 1$
 $\langle 2 \rangle 3$. $h(b) = f^{-1}(b)$
 \square

3.9 Choice Functions

Definition 3.76 (Choice Function). A *choice function* for a set X is a function $f : \mathcal{P}X - \{\emptyset\} \rightarrow X$ such that $f(S) \in S$ for all S .

Proposition 3.77. *Every set has a choice function.*

PROOF: Given a nonempty set X , apply the Axiom of Choice to the family $\{S\}_{S \in \mathcal{P}X - \{\emptyset\}}$. \square

Proposition 3.78. *For any relation R , there exists a function $f \subseteq R$ such that $\text{dom } f = \text{dom } R$.*

PROOF:
 $\langle 1 \rangle 1$. LET: R be a relation.
 $\langle 1 \rangle 2$. PICK a choice function g for $\text{ran } R$.
 $\langle 1 \rangle 3$. LET: $f : \text{dom } R \rightarrow \text{ran } R$ be the function $f(x) = g(\{y \in \text{ran } R : xRy\})$
 $\langle 1 \rangle 4$. $f \subseteq R$ and $\text{dom } f = \text{dom } R$.
 \square

Proposition 3.79. *If \mathcal{C} is a set of pairwise disjoint nonempty sets, then there exists a set A such that, for all $C \in \mathcal{C}$, we have $A \cap C$ is a singleton.*

PROOF:
 $\langle 1 \rangle 1$. LET: f be a choice function for $\bigcup \mathcal{C}$
 $\langle 1 \rangle 2$. LET: $A = \{f(C) : C \in \mathcal{C}\}$
 $\langle 1 \rangle 3$. For all $C \in \mathcal{C}$ we have $A \cap C = \{f(C)\}$
 \square

Chapter 4

Equivalence

Definition 4.1 (Equivalent). Sets E and F are *equivalent*, $E \sim F$, iff there exists a one-to-one correspondence between them.

Proposition 4.2. *For any set X , equivalence is an equivalence relation on $\mathcal{P}X$.*

PROOF: Easy.

Theorem 4.3 (Schröder-Bernstein). *Let X and Y be sets. If there exist injective functions $X \rightarrow Y$ and $Y \rightarrow X$, then $X \sim Y$.*

PROOF:

- $\langle 1 \rangle 1$. LET: $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be one-to-one.
- $\langle 1 \rangle 2$. ASSUME: w.l.o.g. $X \cap Y = \emptyset$
- $\langle 1 \rangle 3$. For $x \in X$, let us say that x is the *parent* of $f(x)$; and for $y \in Y$, let us say that y is the *parent* of $g(y)$.
- $\langle 1 \rangle 4$. For $z \in X \cup Y$, let the set of *descendants* of z be the intersection of all the subsets S of $X \cup Y$ such that $z \in S$ and, if $t \in S$ and t is the parent of u then $u \in S$.
- $\langle 1 \rangle 5$. LET: X_X be the set of all elements of X that are descendants of the elements of X that have no parent.
- $\langle 1 \rangle 6$. LET: X_Y be the set of all elements of X that are descendants of the elements of Y that have no parent.
- $\langle 1 \rangle 7$. LET: $X_\infty = X - X_X - X_Y$
- $\langle 1 \rangle 8$. LET: Y_X be the set of all elements of Y that are descendants of the elements of X that have no parent.
- $\langle 1 \rangle 9$. LET: Y_Y be the set of all elements of Y that are descendants of the elements of Y that have no parent.
- $\langle 1 \rangle 10$. LET: $Y_\infty = Y - Y_X - Y_Y$
- $\langle 1 \rangle 11$. $f|X_X : X_X \sim Y_X$
- $\langle 1 \rangle 12$. $g|Y_Y : Y_Y \sim X_Y$
- $\langle 1 \rangle 13$. $f|X_\infty : X_\infty \sim Y_\infty$
- $\langle 1 \rangle 14$. Define $h : X \rightarrow Y$ by $h(x) = g^{-1}(x)$ if $x \in X_Y$, and $f(x)$ if not.

15. $h : X \sim Y$
 \square

Theorem 4.4 (Cantor). *For any set X we have $X \not\sim \mathcal{P}X$.*

PROOF: If $f : X \rightarrow \mathcal{P}X$ then $\{x \in X : x \notin f(x)\}$ is a subset of X not in $\text{ran } f$. \square

Chapter 5

Order

5.1 Partial Orders

Definition 5.1 (Partial Order). A *partial order* on a set X is a relation on X that is reflexive, antisymmetric and transitive.

A *partially ordered set* or *poset* is a pair (X, \leq) such that \leq is a partial order on X . We write X for the poset (X, \leq) .

Given a partial order \leq , we write \geq for the inverse of \leq .

We write $x < y$ or $y > x$ for $x \leq y \wedge x \neq y$. When this holds, we say x is *less than* y or *smaller than* y ; and y is *greater than* x , *larger than* x .

Proposition 5.2. *For any set X , the relation \subseteq is a partial order on $\mathcal{P}X$.*

PROOF: From Theorems 2.16, 2.17 and 2.18. \square

Proposition 5.3. *Let X be a set. Let $<$ be a relation on X . Then there exists a partial order \leq on X such that*

$$\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$$

if and only if $<$ is irreflexive and transitive. In this case, \leq is unique and is defined by

$$\forall x, y \in X (x \leq y \Leftrightarrow x < y \vee x = y) .$$

PROOF:

$\langle 1 \rangle 1$. LET: X be a set.

$\langle 1 \rangle 2$. LET: $<$ be a relation on X .

$\langle 1 \rangle 3$. If there exists a partial order \leq on X such that $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$, then $<$ is irreflexive.

PROOF: Trivial.

$\langle 1 \rangle 4$. If there exists a partial order \leq on X such that $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$, then $<$ is transitive.

$\langle 2 \rangle 1$. LET: \leq be a partial order on X .

$\langle 2 \rangle 2$. ASSUME: $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$

$\langle 2 \rangle 3$. LET: $x < y$ and $y < z$
 $\langle 2 \rangle 4$. $x \leq y$
PROOF: $\langle 2 \rangle 3$
 $\langle 2 \rangle 5$. $x \neq y$
PROOF: $\langle 2 \rangle 3$
 $\langle 2 \rangle 6$. $y \leq z$
PROOF: $\langle 2 \rangle 3$
 $\langle 2 \rangle 7$. $y \neq z$
PROOF: $\langle 2 \rangle 3$
 $\langle 2 \rangle 8$. $x \leq z$
PROOF: From $\langle 2 \rangle 4$ and $\langle 2 \rangle 6$ since \leq is transitive by $\langle 2 \rangle 1$.
 $\langle 2 \rangle 9$. $x \neq z$
 $\langle 3 \rangle 1$. ASSUME: for a contradiction $x = z$
 $\langle 3 \rangle 2$. $y \leq x$
PROOF: $\langle 2 \rangle 6$, $\langle 3 \rangle 1$
 $\langle 3 \rangle 3$. $x = y$
PROOF: From $\langle 2 \rangle 4$ and $\langle 3 \rangle 2$ since \leq is antisymmetric by $\langle 2 \rangle 1$.
 $\langle 3 \rangle 4$. Q.E.D.
PROOF: $\langle 2 \rangle 5$ and $\langle 3 \rangle 3$ form a contradiction.
 $\langle 2 \rangle 10$. $x < z$
PROOF: $\langle 2 \rangle 8$, $\langle 2 \rangle 9$
 $\langle 1 \rangle 5$. If there exists a partial order \leq on X such that $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$, then $\forall x, y \in X (x \leq y \Leftrightarrow x < y \vee x = y)$.
PROOF: Trivial.
 $\langle 1 \rangle 6$. If $<$ is irreflexive and transitive, then the relation \leq defined by $\forall x, y \in X (x \leq y \Leftrightarrow x < y \vee x = y)$ is a partial order on X .
 $\langle 2 \rangle 1$. ASSUME: $<$ is irreflexive.
 $\langle 2 \rangle 2$. ASSUME: $<$ is transitive.
 $\langle 2 \rangle 3$. LET: \leq be the relation defined by $\forall x, y \in X (x \leq y \Leftrightarrow x < y \vee x = y)$
 $\langle 2 \rangle 4$. \leq is reflexive.
PROOF: Immediate from $\langle 2 \rangle 3$.
 $\langle 2 \rangle 5$. \leq is transitive.
 $\langle 3 \rangle 1$. LET: $x, y, z \in X$
 $\langle 3 \rangle 2$. ASSUME: $x \leq y$ and $y \leq z$
 $\langle 3 \rangle 3$. $x < y$ or $x = y$
PROOF: $\langle 2 \rangle 4$, $\langle 3 \rangle 2$
 $\langle 3 \rangle 4$. $y < z$ or $y = z$
PROOF: $\langle 2 \rangle 4$, $\langle 3 \rangle 2$
 $\langle 3 \rangle 5$. CASE: $x < y$ and $y < z$
 $\langle 4 \rangle 1$. $x < z$
PROOF: $\langle 2 \rangle 2$
 $\langle 4 \rangle 2$. $x \leq z$
PROOF: $\langle 2 \rangle 4$, $\langle 4 \rangle 1$
 $\langle 3 \rangle 6$. CASE: $x = y$
PROOF: Then $x \leq z$ from $\langle 3 \rangle 2$
 $\langle 3 \rangle 7$. CASE: $y = z$

PROOF: Then $x \leq z$ from $\langle 3 \rangle 2$
 $\langle 2 \rangle 6$. \leq is antisymmetric.
 $\langle 3 \rangle 1$. LET: $x, y \in X$
 $\langle 3 \rangle 2$. ASSUME: $x \leq y$
 $\langle 3 \rangle 3$. ASSUME: $y \leq x$
 $\langle 3 \rangle 4$. $x < y$ or $x = y$
PROOF: $\langle 2 \rangle 4, \langle 3 \rangle 2$
 $\langle 3 \rangle 5$. $y < x$ or $y = x$
PROOF: $\langle 2 \rangle 4, \langle 3 \rangle 2$
 $\langle 3 \rangle 6$. $\neg(x < y \wedge y < x)$
 $\langle 4 \rangle 1$. ASSUME: for a contradiction $x < y$ and $y < x$
 $\langle 4 \rangle 2$. $x < x$
PROOF: $\langle 2 \rangle 2, \langle 4 \rangle 1$
 $\langle 4 \rangle 3$. Q.E.D.
PROOF: $\langle 2 \rangle 1$ and $\langle 4 \rangle 2$ form a contradiction.
 $\langle 3 \rangle 7$. $x = y$
PROOF: $\langle 3 \rangle 4, \langle 3 \rangle 5, \langle 3 \rangle 6$
 $\langle 1 \rangle 7$. If $<$ is irreflexive, then the relation \leq defined by $\forall x, y \in X (x \leq y \Leftrightarrow x < y \vee x = y)$ satisfies $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$.
PROOF: Trivial.
 \square

Proposition 5.4. *In a poset, we never have $x < y$ and $y < x$.*

PROOF: We would then have $x < x$ by transitivity, contradicting the irreflexivity of $<$. \square

Definition 5.5 ((Strict) Initial Segment). Let X be a poset and $a \in X$. The (strict) initial segment determined by a is

$$s(a) := \{x \in X : x < a\} .$$

Definition 5.6 (Weak Initial Segment). Let X be a poset and $a \in X$. The weak initial segment determined by a is

$$\bar{s}(a) := \{x \in X : x \leq a\} .$$

Definition 5.7 (Immediate Successor). Let X be a poset and $x, y \in X$. Then y is the (immediate) successor of x , and x is the (immediate) predecessor of y , iff $x < y$ and there is no z such that $x < z < y$.

Definition 5.8 (Least). Let X be a partial order and $a \in X$. Then a is least in X iff $\forall x \in X. a \leq x$.

Proposition 5.9. *A poset has at most one least element.*

PROOF: If a and b are least then $a \leq b$ and $b \leq a$, hence $a = b$. \square

Definition 5.10 (Greatest). Let X be a partial order and $a \in X$. Then a is greatest in X iff $\forall x \in X. x \leq a$.

Proposition 5.11. *A poset has at most one greatest element.*

PROOF: If a and b are greatest then $a \leq b$ and $b \leq a$, hence $a = b$. \square

Definition 5.12 (Minimal). Let X be a poset and $a \in X$. Then a is *minimal* iff there is no $x \in X$ such that $x < a$.

Definition 5.13 (Maximal). Let X be a poset and $a \in X$. Then a is *maximal* iff there is no $x \in X$ such that $a < x$.

Definition 5.14 (Lower Bound). Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is a *lower bound* for E iff $\forall x \in E. a \leq x$.

Definition 5.15 (Upper Bound). Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is an *upper bound* for E iff $\forall x \in E. x \leq a$.

Definition 5.16 (Greatest Lower Bound, Infimum). Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is the *greatest lower bound* or *infimum* for E iff a is the greatest element in the set of lower bounds for E .

Definition 5.17 (Least Upper Bound, Supremum). Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is the *least upper bound* or *supremum* for E iff a is the least element in the set of upper bounds for E .

5.2 Linear Orders

Definition 5.18 (Linear Order). A partial order \leq on a set X is a *linear order*, *total order* or *simple order* iff, for all $x, y \in X$, either $x \leq y$ or $y \leq x$. We then call the poset (X, \leq) a *linearly ordered set* or a *chain*.

Proposition 5.19. *Let R be a partial order on X . Then R is total if and only if $X^2 \subseteq R \cup R^{-1}$.*

PROOF: Easy. \square

Proposition 5.20. *Let X be a set and $<$ a relation on X . Then there exists a linear order \leq on X such that*

$$\forall x, y \in X. (x < y \Leftrightarrow x \leq y \wedge x \neq y)$$

if and only if $<$ is irreflexive, transitive, and:

$$\forall x, y \in X. (x < y \vee x = y \vee y < x) .$$

In this case, \leq is unique and is defined by

$$\forall x, y \in X. (x \leq y \Leftrightarrow x < y \vee x = y)$$

PROOF:

$\langle 1 \rangle$ 1. LET: X be a set.

- $\langle 1 \rangle 2$. LET: $<$ be a relation on X
 $\langle 1 \rangle 3$. If there exists a linear order \leq on X such that $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$, then $<$ is irreflexive.
 PROOF: Trivial.
 $\langle 1 \rangle 4$. If there exists a linear order \leq on X such that $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$, then $<$ is transitive.
 $\langle 2 \rangle 1$. LET: \leq be a linear order on X .
 $\langle 2 \rangle 2$. ASSUME: $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$
 $\langle 2 \rangle 3$. LET: $x < y$ and $y < z$
 $\langle 2 \rangle 4$. $x \leq y$
 PROOF: $\langle 2 \rangle 3$
 $\langle 2 \rangle 5$. $x \neq y$
 PROOF: $\langle 2 \rangle 3$
 $\langle 2 \rangle 6$. $y \leq z$
 PROOF: $\langle 2 \rangle 3$
 $\langle 2 \rangle 7$. $y \neq z$
 PROOF: $\langle 2 \rangle 3$
 $\langle 2 \rangle 8$. $x \leq z$
 PROOF: From $\langle 2 \rangle 4$ and $\langle 2 \rangle 6$ since \leq is transitive.
 $\langle 2 \rangle 9$. $x \neq z$
 $\langle 3 \rangle 1$. ASSUME: for a contradiction $x = z$
 $\langle 3 \rangle 2$. $y \leq x$
 PROOF: $\langle 2 \rangle 6$, $\langle 3 \rangle 1$
 $\langle 3 \rangle 3$. $x = y$
 PROOF: From $\langle 2 \rangle 4$ and $\langle 3 \rangle 2$ since \leq is antisymmetric.
 $\langle 3 \rangle 4$. Q.E.D.
 PROOF: $\langle 2 \rangle 5$ and $\langle 3 \rangle 3$ form a contradiction.
 $\langle 2 \rangle 10$. $x < z$
 PROOF: $\langle 2 \rangle 8$, $\langle 2 \rangle 9$
 $\langle 1 \rangle 5$. If there exists a linear order \leq on X such that $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$, then $\forall x, y \in X (x < y \vee x = y \vee y < x)$.
 $\langle 2 \rangle 1$. LET: \leq be a linear order on X .
 $\langle 2 \rangle 2$. ASSUME: $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$
 $\langle 2 \rangle 3$. LET: $x, y \in X$
 $\langle 2 \rangle 4$. $x \leq y$ or $y \leq x$
 $\langle 2 \rangle 5$. CASE: $x \leq y$
 PROOF: Then $x < y$ or $x = y$ using $\langle 2 \rangle 2$.
 $\langle 2 \rangle 6$. CASE: $y \leq x$
 PROOF: Then $y < x$ or $x = y$ using $\langle 2 \rangle 2$.
 $\langle 1 \rangle 6$. If there exists a linear order \leq on X such that $\forall x, y \in X (x < y \Leftrightarrow x \leq y \wedge x \neq y)$, then $\forall x, y \in X (x \leq y \Leftrightarrow x < y \vee x = y)$.
 PROOF: Trivial.
 $\langle 1 \rangle 7$. If $<$ is irreflexive, transitive and satisfies $\forall x, y \in X (x < y \vee x = y \vee y < x)$, then the relation \leq defined by $\forall x, y \in X (x \leq y \Leftrightarrow x < y \vee x = y)$ is a linear order on X .
 $\langle 2 \rangle 1$. ASSUME: $<$ is irreflexive.

$\langle 2 \rangle 2$. ASSUME: $<$ is transitive.
 $\langle 2 \rangle 3$. ASSUME: $\forall x, y \in X (x < y \vee x = y \vee y < x)$
 $\langle 2 \rangle 4$. LET: \leq be the relation defined by $\forall x, y (x \leq y \Leftrightarrow x < y \vee x = y)$
 $\langle 2 \rangle 5$. \leq is reflexive.
 PROOF: Immediate from $\langle 2 \rangle 4$.
 $\langle 2 \rangle 6$. \leq is transitive.
 $\langle 3 \rangle 1$. LET: $x, y, z \in X$
 $\langle 3 \rangle 2$. ASSUME: $x \leq y$ and $y \leq z$
 $\langle 3 \rangle 3$. $x < y$ or $x = y$
 PROOF: $\langle 2 \rangle 4, \langle 3 \rangle 2$
 $\langle 3 \rangle 4$. $y < z$ or $y = z$
 PROOF: $\langle 2 \rangle 4, \langle 3 \rangle 2$
 $\langle 3 \rangle 5$. CASE: $x < y$ and $y < z$
 $\langle 4 \rangle 1$. $x < z$
 PROOF: $\langle 2 \rangle 2$
 $\langle 4 \rangle 2$. $x \leq z$
 PROOF: $\langle 2 \rangle 4, \langle 4 \rangle 1$
 $\langle 3 \rangle 6$. CASE: $x = y$
 PROOF: Then $x \leq z$ from $\langle 3 \rangle 2$
 $\langle 3 \rangle 7$. CASE: $y = z$
 PROOF: Then $x \leq z$ from $\langle 3 \rangle 2$
 $\langle 2 \rangle 7$. \leq is antisymmetric.
 $\langle 3 \rangle 1$. LET: $x, y \in X$
 $\langle 3 \rangle 2$. ASSUME: $x \leq y$
 $\langle 3 \rangle 3$. ASSUME: $y \leq x$
 $\langle 3 \rangle 4$. $x < y$ or $x = y$
 PROOF: $\langle 2 \rangle 4, \langle 3 \rangle 2$
 $\langle 3 \rangle 5$. $y < x$ or $y = x$
 PROOF: $\langle 2 \rangle 4, \langle 3 \rangle 2$
 $\langle 3 \rangle 6$. $\neg(x < y \wedge y < x)$
 $\langle 4 \rangle 1$. ASSUME: for a contradiction $x < y$ and $y < x$
 $\langle 4 \rangle 2$. $x < x$
 PROOF: $\langle 2 \rangle 2, \langle 4 \rangle 1$
 $\langle 4 \rangle 3$. Q.E.D.
 PROOF: $\langle 2 \rangle 1$ and $\langle 4 \rangle 2$ form a contradiction.
 $\langle 3 \rangle 7$. $x = y$
 PROOF: $\langle 3 \rangle 4, \langle 3 \rangle 5, \langle 3 \rangle 6$
 $\langle 2 \rangle 8$. $\forall x, y \in X (x \leq y \vee y \leq x)$
 $\langle 3 \rangle 1$. LET: $x, y \in X$
 $\langle 3 \rangle 2$. $x < y$ or $x = y$ or $y < x$
 PROOF: $\langle 2 \rangle 3, \langle 3 \rangle 1$
 $\langle 3 \rangle 3$. $x \leq y$ or $y \leq x$
 PROOF: $\langle 2 \rangle 4, \langle 3 \rangle 2$

□

Theorem 5.21 (Zorn's Lemma). *Let X be a poset such that every chain in X*

has an upper bound. Then X has a maximal element.

PROOF:

⟨1⟩1. PICK a choice function f for X .

⟨1⟩2. LET: \mathcal{X} be the set of chains in X .

⟨1⟩3. For all $A \in \mathcal{X}$,

LET: $\hat{A} = \{x \in X : A \cup \{x\} \in \mathcal{X}\}$

⟨1⟩4. LET: $g : \mathcal{X} \rightarrow \mathcal{X}$ be the function

$$g(A) = \begin{cases} A \cup \{f(\hat{A} - A)\} & \text{if } \hat{A} - A \neq \emptyset \\ A & \text{if } \hat{A} - A = \emptyset \end{cases}$$

⟨1⟩5. For $\mathcal{T} \subseteq \mathcal{X}$, let us say \mathcal{T} is a *tower* iff:

- $\emptyset \in \mathcal{T}$
- $\forall A \in \mathcal{T}. g(A) \in \mathcal{T}$
- For every chain \mathcal{C} in \mathcal{T} , we have $\bigcup \mathcal{C} \in \mathcal{T}$

⟨1⟩6. LET: \mathcal{T}_0 be the intersection of the set of all towers.

PROOF: The set of all towers is nonempty since \mathcal{X} is a tower.

⟨1⟩7. LET: $A = \bigcup \mathcal{T}_0$

⟨1⟩8. A is a chain in X .

⟨2⟩1. \mathcal{T}_0 is a chain under \subseteq

⟨3⟩1. Given $C \in \mathcal{T}_0$, let us say that C is *comparable* iff, for all $A \in \mathcal{T}_0$, either $A \subseteq C$ or $C \subseteq A$.

⟨3⟩2. For all $A, C \in \mathcal{T}_0$, if C is comparable and $A \subsetneq C$ then $g(A) \subseteq C$.

PROOF: Since $g(A) - A$ has at most one element, so if $A \subsetneq C \subseteq g(A)$ then $C = g(A)$.

⟨3⟩3. For $C \in \mathcal{T}_0$ comparable,

LET: $\mathcal{U}_C = \{A \in \mathcal{T}_0 : A \subseteq C \vee g(C) \subseteq A\}$

⟨3⟩4. For $C \in \mathcal{T}_0$ comparable, \mathcal{U}_C is a tower.

⟨4⟩1. LET: $C \in \mathcal{T}_0$ be comparable

⟨4⟩2. $\emptyset \in \mathcal{U}_C$

PROOF: Since $\emptyset \subseteq C$.

⟨4⟩3. $\forall A \in \mathcal{U}_C. g(A) \in \mathcal{U}_C$

PROOF: By ⟨1⟩8.

⟨4⟩4. For every chain $\mathcal{C} \subseteq \mathcal{U}_C$ we have $\bigcup \mathcal{C} \in \mathcal{U}_C$

⟨5⟩1. LET: $\mathcal{C} \subseteq \mathcal{U}_C$ be a chain.

⟨5⟩2. CASE: $\exists A \in \mathcal{C}. g(C) \subseteq A$

PROOF: Then $g(C) \subseteq \bigcup \mathcal{C}$

⟨5⟩3. CASE: $\forall A \in \mathcal{C}. A \subseteq C$

PROOF: Then $\bigcup \mathcal{C} \subseteq C$.

⟨3⟩5. For $C \in \mathcal{T}_0$ comparable, $\mathcal{U}_C = \mathcal{T}_0$.

⟨3⟩6. For $C \in \mathcal{T}_0$ comparable we have $g(C)$ is comparable.

PROOF: Since for all $A \in \mathcal{T}_0$ either $A \subseteq C \subseteq g(C)$ or $g(C) \subseteq A$.

⟨3⟩7. The set of comparable sets in \mathcal{T}_0 is a tower.

⟨4⟩1. \emptyset is comparable.

PROOF: $\forall A \in \mathcal{T}_0. \emptyset \subseteq A$

$\langle 4 \rangle 2$. For all $C \in \mathcal{T}_0$, if A is comparable then $g(C)$ is comparable.
 PROOF: $\langle 3 \rangle 6$
 $\langle 4 \rangle 3$. For every chain $\mathcal{C} \subseteq \mathcal{T}_0$ of comparable sets, we have $\bigcup \mathcal{C}$ is comparable.
 $\langle 5 \rangle 1$. LET: $\mathcal{C} \subseteq \mathcal{T}_0$ be a chain of comparable sets.
 $\langle 5 \rangle 2$. LET: $A \in \mathcal{T}_0$
 $\langle 5 \rangle 3$. CASE: there exists $C \in \mathcal{C}$ such that $A \subseteq C$
 PROOF: Then $A \subseteq \bigcup \mathcal{C}$.
 $\langle 5 \rangle 4$. CASE: for all $C \in \mathcal{C}$ we have $C \subseteq A$
 PROOF: Then $\bigcup \mathcal{C} \subseteq A$.
 $\langle 3 \rangle 8$. Every set in \mathcal{T}_0 is comparable.
 $\langle 2 \rangle 2$. LET: $x, y \in A$
 $\langle 2 \rangle 3$. PICK $A, C \in \mathcal{T}_0$ such that $x \in A$ and $y \in C$
 $\langle 2 \rangle 4$. ASSUME: w.l.o.g. $A \subseteq C$
 $\langle 2 \rangle 5$. $x, y \in C$
 $\langle 2 \rangle 6$. $x \leq y$ or $y \leq x$
 PROOF: Since $C \in \mathcal{X}$ so C is a chain.
 $\langle 1 \rangle 9$. PICK an upper bound u for A .
 $\langle 1 \rangle 10$. $A \in \mathcal{T}_0$
 PROOF: Since \mathcal{T}_0 is a chain in \mathcal{T}_0 so $\bigcup \mathcal{T}_0 \in \mathcal{T}_0$.
 $\langle 1 \rangle 11$. $g(A) \in \mathcal{T}_0$
 $\langle 1 \rangle 12$. $g(A) \subseteq A$
 $\langle 1 \rangle 13$. $g(A) = A$
 $\langle 1 \rangle 14$. $\hat{A} - A = \emptyset$
 $\langle 1 \rangle 15$. $u \in A$
 PROOF: Since $A \cup \{u\}$ is a chain so $u \in \hat{A}$ and therefore $u \in A$.
 $\langle 1 \rangle 16$. u is maximal in X .
 $\langle 2 \rangle 1$. LET: $x \in X$
 $\langle 2 \rangle 2$. ASSUME: $u \leq x$
 $\langle 2 \rangle 3$. $A \cup \{x\}$ is a chain.
 $\langle 2 \rangle 4$. $x \in A$
 $\langle 2 \rangle 5$. $x \leq u$
 $\langle 2 \rangle 6$. $x = u$

□

Definition 5.22 (Cofinal). Let X be a poset and $A \subseteq X$. Then A is *cofinal* iff, for all $x \in X$, there exists $a \in A$ such that $x \leq a$.

Definition 5.23 (Order Isomorphism). Two posets X and Y are (*order*) *isomorphic*, $X \cong Y$ iff there exists an order preserving one-to-one correspondence f between them. We write $f : X \cong Y$ and call f a (*order*) *isomorphism*.

Proposition 5.24. Let X and Y be posets. Let f be a one-to-one correspondence between X and Y . Then f is a similarity if and only if, for all $x, y \in X$, we have $x < y$ iff $f(x) < f(y)$.

PROOF: Easy. □

Proposition 5.25. *For any poset X we have $I_X : X \cong X$.*

PROOF: Easy. \square

Proposition 5.26. *If $f : X \cong Y$ then $f^{-1} : Y \cong X$.*

PROOF: Easy. \square

Proposition 5.27. *If $f : X \cong Y$ and $g : Y \cong Z$ then $g \circ f : X \cong Z$.*

PROOF: Easy. \square

Corollary 5.27.1. *For any set E , similarity is an equivalence relation on the set of all posets that are subsets of E .*

Definition 5.28 (Open Interval). Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *open interval* (a, b) is

$$(a, b) := \{x \in X : a < x < b\} .$$

Definition 5.29 (Lexicographical Order). Let A and B be linearly ordered sets. The *lexicographical order* on $A \times B$ is the relation \leq defined by $(a, b) \leq (x, y)$ iff $a < x$ or $(a = x \text{ and } b \leq y)$.

Definition 5.30 (Least Upper Bound Property). A linearly ordered set A has the *least upper bound property* iff every nonempty subset of A bounded above has a least upper bound.

Proposition 5.31. *Let A be a linearly ordered set. Then A has the least upper bound property if and only if every nonempty subset of A bounded below has a greatest lower bound.*

PROOF:

$\langle 1 \rangle 1$. For any linearly ordered set A , if A has the least upper bound property then every nonempty subset of A bounded below has a greatest lower bound.

$\langle 2 \rangle 1$. LET: A be a linearly ordered set.

$\langle 2 \rangle 2$. ASSUME: A has the least upper bound property.

$\langle 2 \rangle 3$. LET: S be a nonempty subset of A bounded below.

$\langle 2 \rangle 4$. LET: $S \downarrow$ be the set of all lower bounds for S .

$\langle 2 \rangle 5$. $S \downarrow \neq \emptyset$

PROOF: Since S is bounded below by $\langle 2 \rangle 3$.

$\langle 2 \rangle 6$. $S \downarrow$ is bounded above.

$\langle 3 \rangle 1$. PICK $s \in S$

PROOF: S is nonempty by $\langle 2 \rangle 3$.

$\langle 3 \rangle 2$. s is an upper bound for $S \downarrow$.

$\langle 2 \rangle 7$. LET: l be the supremum of $S \downarrow$.

PROOF: $\langle 2 \rangle 2, \langle 2 \rangle 5, \langle 2 \rangle 6$

$\langle 2 \rangle 8$. l is a lower bound for S .

$\langle 3 \rangle 1$. LET: $s \in S$

$\langle 3 \rangle 2$. s is an upper bound for $S \downarrow$

$\langle 3 \rangle 3$. $l \leq s$

PROOF: $\langle 2 \rangle 7$, $\langle 3 \rangle 2$

$\langle 2 \rangle 9$. For any lower bound l' for S we have $l' \leq l$.

PROOF: Since l is an upper bound for $S \downarrow$ by $\langle 2 \rangle 7$.

$\langle 1 \rangle 2$. For any linearly ordered set A , if every nonempty subset of A bounded below has a greatest lower bound then A has the least upper bound property.

$\langle 2 \rangle 1$. LET: (A, \leq) be a linearly ordered set.

$\langle 2 \rangle 2$. ASSUME: Every nonempty subset of A bounded below has a greatest lower bound.

$\langle 2 \rangle 3$. (A, \geq) has the least upper bound property.

$\langle 2 \rangle 4$. Every nonempty subset of A bounded below with respect to \geq has a greatest lower bound with respect to \geq .

$\langle 2 \rangle 5$. Every nonempty subset of A bounded above with respect to \leq has a least upper bound with respect to \leq .

□

5.3 Well Orderings

Definition 5.32 (Well Ordered Set). A poset X is *well ordered*, and its ordering is a *well ordering*, iff every nonempty subset of X has a least element.

Proposition 5.33. *Every well ordered set is totally ordered.*

PROOF: For all x and y we have $\{x, y\}$ has a least element, so $x \leq y$ or $y \leq x$. □

Theorem 5.34 (Transfinite Induction). *Let X be a well ordered set. Let $S \subseteq X$ satisfy:*

$$\forall x \in X (\forall y < x. y \in S) \Rightarrow x \in S .$$

Then $S = X$.

PROOF: We have $X - S$ has no least element, so $X - S = \emptyset$. □

Definition 5.35 (Continuation). Let A and B be well ordered sets. Then B is a *continuation* of A iff there exists $b \in B$ such that $A = s(b)$ and the order on A is the restriction of the order on B to A .

Proposition 5.36. *Let \mathcal{C} be a set of well ordered sets that is totally ordered under continuation. Then there exists a unique well ordering on $\bigcup \mathcal{C}$ such that $\bigcup \mathcal{C}$ is a continuation of every element of \mathcal{C} .*

PROOF: Define \leq on $\bigcup \mathcal{C}$ by: $x \leq y$ iff there exists $C \in \mathcal{C}$ such that $x, y \in C$ and $x \leq y$ in C . □

Proposition 5.37. *Every totally ordered set has a cofinal well ordered subset.*

PROOF:

- ⟨1⟩1. LET: X be a totally ordered set.
- ⟨1⟩2. LET: \mathcal{C} be the poset of all well ordered subsets of X under continuation.
- ⟨1⟩3. Every chain in \mathcal{C} has an upper bound.
PROOF: Proposition 5.36.
- ⟨1⟩4. PICK a maximal element C of \mathcal{C}
PROVE: C is cofinal
PROOF: Zorn's Lemma
- ⟨1⟩5. LET: $x \in X$
- ⟨1⟩6. We cannot have $\forall c \in C. c < x$
PROOF: Then $C \cup \{x\}$ would be a larger chain.
- ⟨1⟩7. $\exists c \in C. x \leq c$
□

Theorem 5.38 (Well Ordering Theorem). *Every set can be well ordered.*

PROOF:

- ⟨1⟩1. LET: X be a set.
- ⟨1⟩2. LET: \mathcal{W} be the poset of all well ordered subsets of X under continuation.
- ⟨1⟩3. Every chain in \mathcal{W} has an upper bound.
PROOF: Proposition 5.36.
- ⟨1⟩4. PICK a maximal $M \in \mathcal{W}$
PROOF: Zorn's Lemma
- ⟨1⟩5. $M = X$
PROOF: If $x \in X - M$ then $M \cup \{x\}$ with x as the greatest element is a continuation of M .
□

Theorem 5.39 (Transfinite Recursion). *Let W be a well ordered set and X a set. Let S be the set of all functions f such that $\text{ran } f \subseteq X$, and there exists $a \in W$ such that $\text{dom } f = s(a)$. Then there exists a unique function $U : W \rightarrow X$ such that*

$$\forall a \in W. U(a) = f(U \upharpoonright s(a)) .$$

PROOF:

- ⟨1⟩1. Let us say that a subset $A \subseteq W \times X$ is *f-closed* iff, whenever $a \in W$ and $t : s(a) \rightarrow X$ satisfies $\forall c < a. (c, t(c)) \in A$, then $(a, f(t)) \in A$.
- ⟨1⟩2. LET: U be the intersection of the set of *f-closed* subsets of $W \times X$
PROOF: This set is nonempty since $W \times X$ is *f-closed*.
- ⟨1⟩3. U is *f-closed*.
- ⟨1⟩4. U is a function.
 - ⟨2⟩1. LET: $P(a)$ be the property: there is at most one $x \in X$ such that $(a, x) \in U$
 - ⟨2⟩2. LET: $a \in W$
 - ⟨2⟩3. ASSUME: as transfinite induction hypothesis $\forall c < a. P(c)$
 - ⟨2⟩4. LET: $(a, x), (a, y) \in U$
 - ⟨2⟩5. $x = f(U \upharpoonright c)$
PROOF: If not then $U - \{(a, x)\}$ would be *f-closed*.

$\langle 2 \rangle 6. y = f(U \upharpoonright c)$
 $\langle 2 \rangle 7. x = y$
 $\langle 1 \rangle 5. \text{dom } U = W$
 $\langle 2 \rangle 1. \text{LET: } a \in W$
 $\langle 2 \rangle 2. \text{ASSUME: as transfinite induction hypothesis } \forall c < a. c \in \text{dom } U$
 $\langle 2 \rangle 3. (a, f(U \upharpoonright s(a))) \in U$
 $\langle 1 \rangle 6. \text{If } U' : W \rightarrow X \text{ and } \forall a \in W. U'(a) = f(U' \upharpoonright s(a)), \text{ then } U' = U.$
 PROOF: Prove $U'(a) = U(a)$ by transfinite induction on a .
 \square

Proposition 5.40. *Let X be a well ordered set and f a similarity between X and a subset of X . Then, for all $a \in X$, we have $a \leq f(a)$.*

PROOF:
 $\langle 1 \rangle 1. \text{LET: } a \in X$
 $\langle 1 \rangle 2. \text{ASSUME: as transfinite induction hypothesis } \forall c < a. c \leq f(c)$
 $\langle 1 \rangle 3. \text{ASSUME: for a contradiction } f(a) < a$
 $\langle 1 \rangle 4. f(a) \leq f(f(a))$
 PROOF: $\langle 1 \rangle 2$
 $\langle 1 \rangle 5. f(f(a)) < f(a)$
 PROOF: From $\langle 1 \rangle 3$ since f is a similarity.
 $\langle 1 \rangle 6. \text{Q.E.D.}$
 PROOF: This is a contradiction.
 \square

Proposition 5.41. *Let X and Y be well ordered sets. Then there is at most one similarity between them.*

PROOF:
 $\langle 1 \rangle 1. \text{LET: } f, g : X \cong Y$
 PROVE: $\forall a \in X. f(a) = g(a)$
 $\langle 1 \rangle 2. \text{LET: } a \in X$
 $\langle 1 \rangle 3. \text{ASSUME: as transfinite induction hypothesis } \forall c < a. f(c) = g(c)$
 $\langle 1 \rangle 4. f(a) \text{ is the least element of } Y - \{f(c) : c < a\}$
 $\langle 1 \rangle 5. g(a) \text{ is the least element of } Y - \{g(c) : c < a\}$
 $\langle 1 \rangle 6. f(a) = g(a)$
 \square

Proposition 5.42. *A well ordered set is not similar to any of its initial segments.*

PROOF:
 $\langle 1 \rangle 1. \text{LET: } X \text{ be a well ordered set.}$
 $\langle 1 \rangle 2. \text{ASSUME: for a contradiction } f : X \cong s(a) \text{ for some } a \in X$
 $\langle 1 \rangle 3. f(a) < a$
 $\langle 1 \rangle 4. \text{Q.E.D.}$
 PROOF: This contradicts Proposition 5.40.
 \square

Theorem 5.43 (Comparability Theorem). *Given well ordered sets X and Y , either $X \cong Y$, or X is similar to an initial segment of Y , or Y is similar to an initial segment of X .*

PROOF:

- $\langle 1 \rangle 1$. LET: $X_0 = \{a \in X : \exists b \in Y. s(a) \cong s(b)\}$
- $\langle 1 \rangle 2$. LET: $U : X_0 \rightarrow Y$ be the function: for $a \in X_0$, we have $U(a)$ is the unique element in Y such that $s(a) \cong s(U(a))$
- $\langle 1 \rangle 3$. LET: $Y_0 = \text{ran } U$
- $\langle 1 \rangle 4$. Either $X_0 = X$ or there exists $a \in X$ such that $X_0 = s(a)$
 - $\langle 2 \rangle 1$. ASSUME: $X_0 \neq X$
 - $\langle 2 \rangle 2$. LET: a be the least element of $X - X_0$
 - $\langle 2 \rangle 3$. LET: $x \in X_0$
 - PROVE: $x < a$
 - $\langle 2 \rangle 4$. PICK $f : s(x) \cong s(U(x))$
 - $\langle 2 \rangle 5$. ASSUME: for a contradiction $a < x$
 - $\langle 2 \rangle 6$. $f \upharpoonright s(a) : s(a) \cong s(f(a))$
 - $\langle 2 \rangle 7$. $a \in X_0$
 - $\langle 2 \rangle 8$. Q.E.D.
- PROOF: This is a contradiction.
- $\langle 1 \rangle 5$. Either $Y_0 = Y$ or there exists $b \in Y$ such that $Y_0 = s(b)$
 - PROOF: Similar.
- $\langle 1 \rangle 6$. CASE: $X_0 = X$ and $Y_0 = Y$
 - PROOF: Then $U : X \cong Y$.
- $\langle 1 \rangle 7$. CASE: $X_0 = X$ and $Y_0 \neq Y$
 - PROOF: Then $U : X \cong s(b)$ where $Y_0 = s(b)$.
- $\langle 1 \rangle 8$. CASE: $X_0 \neq X$ and $Y_0 = Y$
 - PROOF: Then $U : s(a) \cong Y$ where $X_0 = s(a)$.
- $\langle 1 \rangle 9$. CASE: $X_0 \neq X$ and $Y_0 \neq Y$
 - $\langle 2 \rangle 1$. LET: $X_0 = s(a)$ and $Y_0 = s(b)$
 - $\langle 2 \rangle 2$. $U : s(a) \cong s(b)$
 - $\langle 2 \rangle 3$. $a \in X_0$
 - $\langle 2 \rangle 4$. Q.E.D.
- PROOF: This is a contradiction.

□

Corollary 5.43.1. *Let X be a well ordered set. Then any subset A of X is either similar to X or to an initial segment of X .*

PROOF: We cannot have X is similar to an initial segment of A , say $f : X \cong \{x \in A : x < a\}$, because then we would have $f(a) < a$ contradicting Proposition 5.40. □

Corollary 5.43.2. *For any sets X and Y , either there exists an injective function $X \rightarrow Y$, or there exists an injective function $Y \rightarrow X$.*

PROOF: Using the Well Ordering Theorem. □

Chapter 6

Natural Numbers

6.1 Natural Numbers

Definition 6.1 (Successor). The *successor* of a set x , x^+ , is defined by

$$x^+ := x \cup \{x\} .$$

Definition 6.2. We define

$$\begin{aligned} 0 &= \emptyset \\ 1 &= 0^+ \\ 2 &= 1^+ \end{aligned}$$

etc.

Definition 6.3 (Characteristic Function). Let X be a set and $A \subseteq X$. The *characteristic function* of A is the function $\chi_A : X \rightarrow 2$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Theorem 6.4. Let X be a set. The function $\chi : \mathcal{P}X \rightarrow 2^X$ that maps a subset A of X to χ_A is a one-to-one correspondence.

PROOF: Easy. \square

Definition 6.5. The set ω of *natural numbers* is the set such that:

- $0 \in \omega$
- For all $n \in \omega$ we have $n^+ \in \omega$
- For any set X , if $0 \in X$ and $\forall n \in X. n^+ \in X$ then $\omega \subseteq X$

PROOF: To show this exists, pick a set A such that $0 \in A$ and $\forall n \in A. n^+ \in A$ (by the Axiom of Infinity), and let $\omega = \bigcap \{X \in \mathcal{P}A : 0 \in X \wedge \forall n \in X. n^+ \in X\}$.
 \square

Definition 6.6 (Sequence). A *finite sequence* is a family whose index set is a natural number. An *infinite sequence* is a family whose index set is ω .

Given a finite sequence of sets $\{A_i\}_{i \in n^+}$, we write $\bigcup_{i=0}^n A_i$ for $\bigcup_{i \in n^+} A_i$. Given an infinite sequence of sets $\{A_i\}_{i \in \omega}$, we write $\bigcup_{i=0}^{\infty} A_i$ for $\bigcup_{i \in \omega} A_i$.

We make similar definitions for \bigcap and \times .

Proposition 6.7. For any natural numbers m and n , if $m \in n$ then $m^+ \in n^+$.

PROOF:

$\langle 1 \rangle 1$. LET: $P(n)$ be the property $\forall m \in n. m^+ \in n^+$

$\langle 1 \rangle 2$. $P(0)$

PROOF: Vacuous.

$\langle 1 \rangle 3$. For any natural number n , if $P(n)$ then $P(n^+)$.

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. ASSUME: $P(n)$

$\langle 2 \rangle 3$. LET: $m \in n^+$

$\langle 2 \rangle 4$. $m \in n$ or $m = n$

$\langle 2 \rangle 5$. $m^+ \in n^+$ or $m^+ = n^+$

PROOF: $\langle 2 \rangle 2$

$\langle 2 \rangle 6$. CASE: $m^+ \in n^{++}$

\square

Theorem 6.8 (Principle of Mathematical Induction). For any subset S of ω , if $0 \in S$ and $\forall n \in S. n^+ \in S$, then $S = \omega$.

PROOF: From the definition of ω . \square

Proposition 6.9.

$$\forall n \in \omega. \forall x \in n. n \not\subseteq x$$

PROOF:

$\langle 1 \rangle 1$. $\forall x \in 0. 0 \not\subseteq x$

PROOF: Vacuous.

$\langle 1 \rangle 2$. For any natural number n , if $\forall x \in n. n \not\subseteq x$ then $\forall x \in n^+. n^+ \not\subseteq x$.

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. ASSUME: $\forall x \in n. n \not\subseteq x$

$\langle 2 \rangle 3$. LET: $x \in n^+$

$\langle 2 \rangle 4$. ASSUME: for a contradiction $n^+ \subseteq x$

$\langle 2 \rangle 5$. $x \in n$ or $x = n$

$\langle 2 \rangle 6$. CASE: $x \in n$

PROOF: Then we have $n \subseteq n^+ \subseteq x$ contradicting $\langle 2 \rangle 2$.

$\langle 2 \rangle 7$. CASE: $x = n$

PROOF: Then we have $n \in n^+ \subseteq x = n$ and $n \subseteq n$ contradicting $\langle 2 \rangle 2$.

\square

Corollary 6.9.1. *For any natural number n we have $n \notin n$.*

Corollary 6.9.2. *For any natural number n we have $n \neq n^+$.*

Definition 6.10 (Transitive Set). A set E is a *transitive set* iff, whenever $x \in y \in E$, then $x \in E$.

Proposition 6.11. *Every natural number is a transitive set.*

PROOF:

$\langle 1 \rangle 1$. 0 is a transitive set.

PROOF: Vacuously, if $x \in y \in 0$ then $x \in 0$.

$\langle 1 \rangle 2$. For any natural number n , if n is a transitive set, then n^+ is a transitive set.

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. ASSUME: n is a transitive set.

$\langle 2 \rangle 3$. LET: $x \in y \in n^+$

$\langle 2 \rangle 4$. $y \in n$ or $y = n$

$\langle 2 \rangle 5$. CASE: $y \in n$

$\langle 3 \rangle 1$. $x \in n$

PROOF: $\langle 2 \rangle 2$, $\langle 2 \rangle 3$, $\langle 2 \rangle 5$.

$\langle 3 \rangle 2$. $x \in n^+$

$\langle 2 \rangle 6$. CASE: $y = n$

$\langle 3 \rangle 1$. $x \in n$

PROOF: $\langle 2 \rangle 3$, $\langle 2 \rangle 6$

$\langle 3 \rangle 2$. $x \in n^+$

□

Proposition 6.12. *For any natural numbers m and n , if $m^+ = n^+$ then $m = n$.*

PROOF:

$\langle 1 \rangle 1$. LET: m and n be natural numbers.

$\langle 1 \rangle 2$. ASSUME: $m^+ = n^+$

$\langle 1 \rangle 3$. $m \in m^+ = n^+$

$\langle 1 \rangle 4$. $m \in n$ or $m = n$

$\langle 1 \rangle 5$. $n \in n^+ = m^+$

$\langle 1 \rangle 6$. $n \in m$ or $n = m$

$\langle 1 \rangle 7$. We cannot have $m \in n$ and $n \in m$

$\langle 2 \rangle 1$. ASSUME: for a contradiction $m \in n$ and $n \in m$

$\langle 2 \rangle 2$. $m \in m$

PROOF: Since m is a transitive set (Proposition 6.11).

$\langle 2 \rangle 3$. Q.E.D.

PROOF: This contradicts Proposition 6.9.

$\langle 1 \rangle 8$. $m = n$

□

Theorem 6.13 (Recursion Theorem). *Let X be a set. Let $a \in X$. Let $f : X \rightarrow X$. There exists a function $u : \omega \rightarrow X$ such that $u(0) = a$ and, for all $n \in \omega$, we have $u(n^+) = f(u(n))$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{C} = \{A \in \mathcal{P}(\omega \times X) : (0, a) \in A \wedge \forall n \in \omega. \forall x \in X. (n, x) \in A \Rightarrow (n^+, f(x)) \in A\}$

$\langle 1 \rangle 2$. $\mathcal{C} \neq \emptyset$

PROOF: $\omega \times X \in \mathcal{C}$

$\langle 1 \rangle 3$. LET: $u = \bigcap \mathcal{C}$

$\langle 1 \rangle 4$. $u \in \mathcal{C}$

$\langle 1 \rangle 5$. u is a function.

$\langle 2 \rangle 1$. LET: $P(n)$ be the property: $\forall x, y \in X. (n, x) \in u \wedge (n, y) \in u \Rightarrow x = y$

$\langle 2 \rangle 2$. $P(0)$

$\langle 3 \rangle 1$. $\forall x \in X. (0, x) \in u \Rightarrow x = a$

PROOF: If $(0, x) \in u$ and $x \neq a$ then $u - \{(0, x)\} \in \mathcal{C}$ and so $u - \{(0, x)\} \subseteq u$, which is impossible.

$\langle 2 \rangle 3$. For every natural number n , if $P(n)$ then $P(n^+)$.

$\langle 3 \rangle 1$. LET: n be a natural number.

$\langle 3 \rangle 2$. ASSUME: $P(n)$

$\langle 3 \rangle 3$. LET: $x, y \in X$

$\langle 3 \rangle 4$. ASSUME: $(n^+, x), (n^+, y) \in u$

$\langle 3 \rangle 5$. PICK $x', y' \in X$ such that $(n, x') \in u$, $(n, y') \in u$ and $f(x') = x$ and $f(y') = y$

PROOF: If no such x' exists then $u - \{(n^+, x)\} \in \mathcal{C}$ and so $u - \{(n^+, x)\} \subseteq u$ which is impossible. Similarly for y' .

$\langle 3 \rangle 6$. $x' = y'$

PROOF: $\langle 3 \rangle 2$

$\langle 3 \rangle 7$. $x = y$

□

Proposition 6.14. *For any natural number n , either $n = 0$ or there exists a natural number m such that $n = m^+$.*

PROOF: Easy induction on n . □

Proposition 6.15. *ω is a transitive set.*

PROOF:

$\langle 1 \rangle 1$. LET: $P(n)$ be the property $\forall x \in n. x \in \omega$

$\langle 1 \rangle 2$. $P(0)$

PROOF: Vacuous.

$\langle 1 \rangle 3$. For any natural number n , if $P(n)$ then $P(n^+)$.

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. ASSUME: $P(n)$

$\langle 2 \rangle 3$. LET: $x \in n^+$

$\langle 2 \rangle 4$. $x \in n$ or $x = n$

$\langle 2 \rangle 5$. CASE: $x \in n$

PROOF: Then $x \in \omega$ by $\langle 2 \rangle 2$.

$\langle 2 \rangle 6$. CASE: $x = n$

PROOF: Then $x \in \omega$ by $\langle 2 \rangle 1$.

□

Proposition 6.16. *For any natural number n and any nonempty subset $E \subseteq n$, there exists $k \in E$ such that $\forall m \in E. k = m \vee k \in m$.*

PROOF:

⟨1⟩1. LET: $P(n)$ be the property: for any nonempty subset $E \subseteq n$, there exists $k \in E$ such that $\forall m \in E. k = m \vee k \in m$

⟨1⟩2. $P(0)$

PROOF: Vacuous as there is no nonempty subset of 0.

⟨1⟩3. For any natural number n , if $P(n)$ then $P(n^+)$.

⟨2⟩1. LET: n be a natural number.

⟨2⟩2. ASSUME: $P(n)$

⟨2⟩3. LET: E be a nonempty subset of n^+

⟨2⟩4. CASE: $E - \{n\} = \emptyset$

PROOF: Then $E = \{n\}$ so take $k = n$.

⟨2⟩5. CASE: $E - \{n\} \neq \emptyset$

⟨3⟩1. PICK $k \in E - \{n\}$ such that $\forall m \in E - \{n\}. k = m \vee k \in m$

PROOF: By ⟨2⟩2.

⟨3⟩2. $\forall m \in E. k = m \vee k \in m$

PROOF: Since $k \in n$.

□

Chapter 7

Ordinal Numbers

Definition 7.1 (Ordinal (Number)). An *ordinal (number)* is a well ordered set α such that $\forall \xi \in \alpha. s(\xi) = \xi$.

Given ordinals α, β , we write $\alpha < \beta$ iff $\alpha \in \beta$.

Proposition 7.2. *Every natural number is an ordinal.*

PROOF: Easy. \square

Proposition 7.3. ω is an ordinal.

PROOF: Easy. \square

Proposition 7.4. If α is an ordinal number then so is α^+ .

PROOF: Easy. \square

Proposition 7.5. Let α be an ordinal and $\eta, \xi \in \alpha$. Then $\eta < \xi$ if and only if $\eta \in \xi$.

PROOF: Easy. \square

Proposition 7.6. Every ordinal is a transitive set.

PROOF: Easy. \square

Proposition 7.7. Every element of an ordinal is an ordinal.

PROOF: Easy. \square

Proposition 7.8. Similar ordinals are equal.

PROOF:

$\langle 1 \rangle 1$. LET: α, β be ordinals.

$\langle 1 \rangle 2$. LET: $f : \alpha \cong \beta$ be a similarity.

PROVE: $\forall \xi \in \alpha. f(\xi) = \xi$

$\langle 1 \rangle 3$. LET: $\xi \in \alpha$

$\langle 1 \rangle 4$. ASSUME: as transfinite induction hypothesis $\forall \eta < \xi. f(\eta) = \eta$
 $\langle 1 \rangle 5$. $f(\xi) \subseteq \xi$
 $\langle 2 \rangle 1$. LET: $\eta \in f(\xi)$
 $\langle 2 \rangle 2$. PICK $\zeta \in \alpha$ such that $f(\zeta) = \eta$
 $\langle 2 \rangle 3$. $\zeta \in \xi$
PROOF: Since $f(\zeta) \in f(\xi)$ and f is a similarity.
 $\langle 2 \rangle 4$. $f(\zeta) = \zeta$
PROOF: $\langle 1 \rangle 4$
 $\langle 2 \rangle 5$. $\eta = \zeta$
PROOF: $\langle 2 \rangle 2, \langle 2 \rangle 4$
 $\langle 2 \rangle 6$. $\eta \in \xi$
PROOF: $\langle 2 \rangle 3, \langle 2 \rangle 5$
 $\langle 1 \rangle 6$. $\xi \subseteq f(\xi)$
 $\langle 2 \rangle 1$. LET: $\eta \in \xi$
 $\langle 2 \rangle 2$. $\eta = f(\eta) \in f(\xi)$
 $\langle 1 \rangle 7$. $f(\xi) = \xi$
 \square

Proposition 7.9. *Let α and β be ordinals. Then the following are equivalent.*

1. $\alpha \in \beta$
2. $\alpha \subsetneq \beta$
3. β is a continuation of α .

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 3$
PROOF: If $\alpha \in \beta$ then $\alpha = s(\alpha)$.
 $\langle 1 \rangle 2$. $3 \Rightarrow 2$
PROOF: Immediate from definitions.
 $\langle 1 \rangle 3$. $2 \Rightarrow 1$
 $\langle 2 \rangle 1$. LET: γ be the least element of β such that $\gamma \notin \alpha$
 $\langle 2 \rangle 2$. $\alpha \subseteq \gamma$
 $\langle 3 \rangle 1$. LET: $\eta \in \alpha$
 $\langle 3 \rangle 2$. $\eta \subseteq \alpha$
 $\langle 3 \rangle 3$. $\gamma \notin \eta$
 $\langle 3 \rangle 4$. $\eta \in \gamma$ or $\eta = \gamma$
 $\langle 3 \rangle 5$. $\eta \neq \gamma$
PROOF: Since $\eta \in \alpha$ and $\gamma \notin \alpha$.
 $\langle 3 \rangle 6$. $\eta \in \gamma$
 $\langle 2 \rangle 3$. $\gamma \subseteq \alpha$
PROOF: For all $\eta \in \gamma$ we have $\eta \in \alpha$ by leastness of γ .
 $\langle 2 \rangle 4$. $\gamma = \alpha$
 $\langle 2 \rangle 5$. $\alpha \in \beta$
 \square

Proposition 7.10. *For any ordinal numbers α and β , either $\alpha = \beta$, or $\alpha < \beta$, or $\beta < \alpha$.*

PROOF:

- ⟨1⟩1. Either $\alpha = \beta$, or α is similar to an initial segment of β , or β is similar to an initial segment of α .
- ⟨1⟩2. CASE: α is similar to an initial segment of β .
 - ⟨2⟩1. PICK $\eta \in \beta$ such that $\alpha \sim s(\eta)$
 - ⟨2⟩2. $\alpha \sim \eta$
 - ⟨2⟩3. $\alpha = \eta$
 - PROOF: Proposition 7.8.
 - ⟨2⟩4. $\alpha \in \beta$
- ⟨1⟩3. CASE: β is similar to an initial segment of α .
 PROOF: Then $\beta \in \alpha$ similarly.

□

Proposition 7.11. *Every set of ordinals is well ordered by $<$.*

PROOF:

- ⟨1⟩1. LET: E be a set of ordinals.
- ⟨1⟩2. LET: A be a nonempty subset of E .
- ⟨1⟩3. PICK $\alpha \in A$
- ⟨1⟩4. CASE: $\alpha \cap A = \emptyset$
 PROOF: Then α is least in A .
- ⟨1⟩5. CASE: $\alpha \cap A \neq \emptyset$
 PROOF: Then $\alpha \cap A$ has a least element, which is least in A .

□

Definition 7.12 (Limit Ordinal). A *limit ordinal* is an ordinal number that is not 0 and not α^+ for any ordinal α .

Proposition 7.13. *For any set E of ordinal numbers, $\bigcup E$ is an ordinal and is the supremum of E .*

PROOF: Proposition 5.36. □

Theorem 7.14 (Burali-Forti Paradox). *There is no set whose members are exactly the ordinal numbers.*

PROOF: For any set of ordinals E , we have $(\bigcup E)^+$ is an ordinal that is not in E . □

Theorem 7.15 (Counting Theorem). *Every well ordered set is similar to a unique ordinal.*

PROOF:

- ⟨1⟩1. LET: X be a well ordered set.
- ⟨1⟩2. There exists an ordinal α such that $X \cong \alpha$.
 - ⟨2⟩1. For all $a \in X$, there exists a unique ordinal α such that $s(a) \cong \alpha$
 - ⟨3⟩1. LET: $a \in X$
 - ⟨3⟩2. ASSUME: as transfinite induction hypothesis that, for all $b < a$, there exists a unique ordinal β such that $s(b) \cong \beta$

$\langle 3 \rangle 3$. LET: $\alpha = \{\beta : \beta \text{ is an ordinal} \wedge \exists b < a. s(b) \cong \beta\}$
 PROOF: This is a set by the Axiom of Substitution.
 $\langle 3 \rangle 4$. α is an ordinal
 $\langle 4 \rangle 1$. LET: $\gamma \in \beta \in \alpha$
 $\langle 4 \rangle 2$. PICK $b < a$ and $f : s(b) \cong \beta$
 $\langle 4 \rangle 3$. PICK $c < b$ such that $f(c) = \gamma$
 $\langle 4 \rangle 4$. $f \upharpoonright s(c) : s(c) \cong \gamma$
 $\langle 3 \rangle 5$. $s(a) \cong \alpha$
 PROOF: The function $f : s(a) \rightarrow \alpha$ defined by $f(b)$ is the ordinal such that $s(b) \cong f(b)$ is a similarity.
 $\langle 3 \rangle 6$. α is unique.
 PROOF: Proposition 7.8.
 $\langle 2 \rangle 2$. LET: $\alpha = \{\beta : \beta \text{ is an ordinal} \wedge \exists a \in X. s(a) \cong \beta\}$
 PROOF: This is a set by the Axiom of Substitution.
 $\langle 2 \rangle 3$. α is an ordinal.
 PROOF: Similar.
 $\langle 2 \rangle 4$. $X \cong \alpha$
 PROOF: Similar.
 $\langle 1 \rangle 3$. For any ordinals α and β , if $X \cong \alpha$ and $X \cong \beta$ then $\alpha = \beta$.
 PROOF: Proposition 7.8.
 \square

7.1 Order on the Natural Numbers

Proposition 7.16. *For natural numbers m, n and k , if $m < n$ then $m + k < n + k$.*

PROOF:
 $\langle 1 \rangle 1$. LET: $m, n \in \omega$
 $\langle 1 \rangle 2$. ASSUME: $m < n$
 $\langle 1 \rangle 3$. $m + 0 < n + 0$
 $\langle 1 \rangle 4$. $\forall k \in \omega. m + k < n + k \Rightarrow m + k^+ < n + k^+$
 PROOF: By Proposition 6.7.
 \square

Proposition 7.17. *For natural numbers m, n and k , if $m < n$ and $k \neq 0$ then $mk < nk$.*

PROOF:
 $\langle 1 \rangle 1$. LET: $m, n \in \omega$
 $\langle 1 \rangle 2$. ASSUME: $m < n$
 $\langle 1 \rangle 3$. $m1 < n1$
 $\langle 1 \rangle 4$. For all $k \in \omega$, if $k \neq 0$ and $mk < nk$ then $m(k + 1) < n(k + 1)$

PROOF:

$$\begin{aligned}
m(k+1) &= mk + m \\
&< mk + n && \text{(Proposition 7.16)} \\
&< nk + n && \text{(Proposition 7.16)} \\
&= n(k+1)
\end{aligned}$$

□

Proposition 7.18. *Let n be a natural number. Let X be a proper subset of n . Then there exists $m < n$ such that $X \sim m$.*

PROOF:

⟨1⟩1. LET: $P(n)$ be the property: for every proper subset $X \subsetneq n$, there exists $m < n$ such that $X \sim m$.

⟨1⟩2. $P(0)$

PROOF: Vacuous.

⟨1⟩3. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

⟨2⟩1. LET: $n \in \omega$

⟨2⟩2. ASSUME: $P(n)$

⟨2⟩3. LET: X be a proper subset of $n+1$

⟨2⟩4. CASE: $X - \{n\} = n$

PROOF: Then $X = n$ so $X \sim n < n+1$.

⟨2⟩5. CASE: $X - \{n\} \subsetneq n$

⟨3⟩1. PICK $m < n$ such that $X - \{n\} \sim m$

⟨3⟩2. $X \sim m$ or $X \sim m+1$

PROOF: If $n \in X$ then $X \sim m+1$. If $n \notin X$ then $X \sim m$.

□

Proposition 7.19. *For every natural number n , we have n is not equivalent to a proper subset of n .*

PROOF:

⟨1⟩1. LET: $P(n)$ be the property: every one-to-one function $n \rightarrow n$ is onto.

⟨1⟩2. $P(0)$

PROOF: The only function $0 \rightarrow 0$ is \emptyset .

⟨1⟩3. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

⟨2⟩1. LET: $n \in \omega$

⟨2⟩2. ASSUME: $P(n)$

⟨2⟩3. ASSUME: $f : n+1 \rightarrow n+1$ is one-to-one.

⟨2⟩4. LET: $g : n \rightarrow n$ be the function

$$g(k) = \begin{cases} f(k) & \text{if } f(k) < n \\ f(n) & \text{if } f(k) = n \end{cases}$$

PROOF: If $k < n$ and $f(k) = n$ then $f(n) < n$ since f is one-to-one.

⟨2⟩5. g is one-to-one.

⟨3⟩1. LET: $k, l < n$

⟨3⟩2. ASSUME: $g(k) = g(l)$

⟨3⟩3. CASE: $f(k) < n$ and $f(l) < n$

PROOF: Then $f(k) = g(k) = g(l) = f(l)$ so $k = l$ since f is one-to-one.

⟨3⟩4. CASE: $f(k) < n$ and $f(l) = n$
PROOF: Then $f(k) = g(k) = g(l) = f(n)$ contradicting the fact that f is one-to-one.

⟨3⟩5. CASE: $f(k) = n$ and $f(l) < n$
PROOF: Similar.

⟨3⟩6. CASE: $f(k) = n$ and $f(l) = n$
PROOF: Then $k = l$ since f is one-to-one.

⟨2⟩6. g maps n onto n .
PROOF: ⟨2⟩2

⟨2⟩7. f maps $n + 1$ onto $n + 1$.
⟨3⟩1. LET: $l < n + 1$
⟨3⟩2. CASE: $l < n$
⟨4⟩1. PICK $k < n$ such that $g(k) = l$
⟨4⟩2. $f(k) = l$ or $f(n) = l$
⟨3⟩3. CASE: $l = n$
⟨4⟩1. CASE: $f(n) = n$
PROOF: Then $l \in \text{ran } f$ as required.
⟨4⟩2. CASE: $f(n) < n$
⟨5⟩1. PICK $k < n$ such that $g(k) = f(n)$
⟨5⟩2. $f(k) = n$

□

Corollary 7.19.1. *Equivalent natural numbers are equal.*

Proposition 7.20. *The lexicographical order is a well ordering on $\omega \times \omega$.*

PROOF: Easy. □

7.2 Finite Sets

Definition 7.21 (Finite). A set is *finite* iff it is equivalent to a natural number; otherwise, it is *infinite*.

Proposition 7.22. *No finite set is equivalent to one of its proper subsets.*

PROOF: From Proposition 7.19. □

Proposition 7.23. *ω is infinite.*

PROOF: Since the function that maps n to $n + 1$ is a one-to-one correspondence between ω and $\omega - \{0\}$. □

Proposition 7.24. *Every subset of a finite set is finite.*

PROOF: Proposition 7.18. □

Definition 7.25 (Number of Elements). For any finite set E , the *number of elements* in E , $\sharp(E)$, is the unique natural number such that $E \sim \sharp(E)$.

Proposition 7.26. *Let E and F be finite sets. If $E \subseteq F$ then $\sharp(E) \leq \sharp(F)$.*

PROOF: Proposition 7.18. \square

Proposition 7.27. *Let E and F be disjoint finite sets. Then $E \cup F$ is finite and $\sharp(E \cup F) = \sharp(E) + \sharp(F)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $P(n)$ be the statement: $n \in \omega$ and for any $m \in \omega$, if $E \sim m$, $F \sim n$ and $E \cap F = \emptyset$, then $E \cup F \sim m + n$

$\langle 1 \rangle 2$. $P(0)$

$\langle 2 \rangle 1$. LET: $m \in \omega$

$\langle 2 \rangle 2$. LET: $E \sim m$ and $F \sim 0$

$\langle 2 \rangle 3$. $F = \emptyset$

$\langle 2 \rangle 4$. $E \cup F = E \sim m = m + 0$

$\langle 1 \rangle 3$. $\forall n \in \omega. P(n) \Rightarrow P(n + 1)$

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: $P(n)$

$\langle 2 \rangle 3$. LET: $m \in \omega$

$\langle 2 \rangle 4$. LET: $E \sim m$ and $F \sim n + 1$

$\langle 2 \rangle 5$. ASSUME: $E \cap F = \emptyset$

$\langle 2 \rangle 6$. PICK $f \in F$

$\langle 2 \rangle 7$. $F - \{f\} \sim n$

$\langle 2 \rangle 8$. $E \cap (F - \{f\}) = \emptyset$

$\langle 2 \rangle 9$. $E \cup (F - \{f\}) \sim m + n$

PROOF: $\langle 2 \rangle 2$

$\langle 2 \rangle 10$. $E \cup F \sim m + n + 1$

\square

Corollary 7.27.1. *The union of two finite sets is finite.*

PROOF: Since, if E and F are finite, then $E \cup F = (E - F) \cup (E \cap F) \cup (F - E)$ and these are finite and disjoint. \square

Proposition 7.28. *If E and F are finite sets then $E \times F$ is finite and $\sharp(E \times F) = \sharp(E)\sharp(F)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $P(n)$ be the statement: $n \in \omega$ and for all $m \in \omega$, if $E \sim m$ and $F \sim n$ then $E \times F \sim mn$

$\langle 1 \rangle 2$. $P(0)$

PROOF: If $F \sim 0$ then $F = \emptyset$ so $E \times F = \emptyset \sim 0$.

$\langle 1 \rangle 3$. $\forall n \in \omega. P(n) \Rightarrow P(n + 1)$

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: $P(n)$

$\langle 2 \rangle 3$. LET: $m \in \omega$

$\langle 2 \rangle 4$. ASSUME: $E \sim m$ and $F \sim n + 1$

$\langle 2 \rangle 5$. PICK $f \in F$

- ⟨2⟩6. $F - \{f\} \sim n$
- ⟨2⟩7. $E \times (F - \{f\}) \sim mn$
- ⟨2⟩8. $E \times F = (E \times (F - \{f\})) \cup (E \times \{f\})$
- ⟨2⟩9. $E \times \{f\} \sim m$
- ⟨2⟩10. $E \times F \sim mn + m$

PROOF: Proposition 7.27.

□

Proposition 7.29. *For any finite sets E and F , we have E^F is finite and $\sharp(E^F) = \sharp(E)^{\sharp(F)}$.*

PROOF:

- ⟨1⟩1. LET: $P(n)$ be the property: $n \in \omega$ and for all $m \in \omega$, if $E \sim m$ and $F \sim n$ then $E^F \sim m^n$
- ⟨1⟩2. $P(0)$
PROOF: Since $E^\emptyset = \{\emptyset\} \sim 1$
- ⟨1⟩3. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$
 - ⟨2⟩1. LET: $n \in \omega$
 - ⟨2⟩2. ASSUME: $P(n)$
 - ⟨2⟩3. LET: $m \in \omega$
 - ⟨2⟩4. LET: $E \sim m$ and $F \sim n+1$
 - ⟨2⟩5. PICK $f \in F$
 - ⟨2⟩6. $F - \{f\} \sim n$
 - ⟨2⟩7. LET: $\phi : E^F \rightarrow E^{F-\{f\}} \times E$ be the function $\phi(g) = (g \upharpoonright (F - \{f\}), g(f))$
 - ⟨2⟩8. ϕ is a one-to-one correspondence
 - ⟨2⟩9. $\sharp(E^F) = m^{n+1}$

PROOF:

$$\begin{aligned}
 \sharp(E^F) &= \sharp(E^{F-\{f\}} \times E) \\
 &= \sharp(E^{F-\{f\}}) \sharp(E) && \text{(Proposition 7.28)} \\
 &= m^n m && (\langle 2 \rangle 2, \langle 2 \rangle 4) \\
 &= m^{n+1}
 \end{aligned}$$

□

Corollary 7.29.1. *If E is finite then $\mathcal{P}E$ is finite and $\sharp(\mathcal{P}E) = 2^{\sharp(E)}$.*

Proposition 7.30. *The union of a finite set of finite sets is finite.*

PROOF:

- ⟨1⟩1. LET: $P(n)$ be the property: for any set E , if $E \sim n$ and every element of E is finite, then $\bigcup E$ is finite.
- ⟨1⟩2. $P(0)$
PROOF: Since $\bigcup \emptyset = \emptyset$ is finite.
- ⟨1⟩3. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$
 - ⟨2⟩1. LET: n be a natural number.
 - ⟨2⟩2. ASSUME: $P(n)$
 - ⟨2⟩3. LET: $E \sim n+1$

- ⟨2⟩4. PICK $X \in E$
- ⟨2⟩5. $E - \{X\} \sim n$
- ⟨2⟩6. $\bigcup(E - \{X\})$ is finite.
- PROOF: ⟨2⟩2
- ⟨2⟩7. $\bigcup E = \bigcup(E - \{X\}) \cup X$
- ⟨2⟩8. $\bigcup E$ is finite.
- PROOF: Corollary 7.27.1.

□

Proposition 7.31. *Every nonempty finite set of natural numbers has a greatest element.*

PROOF:

- ⟨1⟩1. LET: $P(n)$ be the property: for every $E \subseteq \mathbb{N}$, if $E \sim n$ then E has a greatest element.
- ⟨1⟩2. $P(1)$
- PROOF: Since k is the greatest element of $\{k\}$.
- ⟨1⟩3. $\forall n \geq 1. P(n) \Rightarrow P(n+1)$
- ⟨2⟩1. LET: $n \geq 1$
- ⟨2⟩2. ASSUME: $P(n)$
- ⟨2⟩3. ASSUME: $E \subseteq \omega$ and $E \sim n+1$
- ⟨2⟩4. PICK $k \in E$
- ⟨2⟩5. LET: l be the greatest element of $E - \{k\}$
- ⟨2⟩6. Either k or l is greatest in E .

□

Proposition 7.32. *Every infinite set has a subset equivalent to ω .*

PROOF:

- ⟨1⟩1. LET: X be an infinite set.
- ⟨1⟩2. PICK a choice function f for X .
- ⟨1⟩3. LET: \mathcal{C} be the set of all finite subsets of X .
- ⟨1⟩4. For all $A \in \mathcal{C}$ we have $X - A \in \text{dom } f$.
- PROOF: For all $A \in \mathcal{C}$ we have $X - A \neq \emptyset$.
- ⟨1⟩5. LET: $U : \omega \rightarrow \mathcal{C}$ be the function defined recursively by $U(0) = \emptyset$ and $U(n+1) = U(n) \cup \{f(X - U(n))\}$ for all $n \in \omega$.
- ⟨1⟩6. LET: $v : \omega \rightarrow X$ be the function $v(n) = f(X - U(n))$
- PROVE: v is one-to-one.
- ⟨1⟩7. $\forall n \in \omega. v(n) \notin U(n)$
- PROOF: Since $v(n) = f(X - U(n)) \in X - U(n)$.
- ⟨1⟩8. $\forall n \in \omega. v(n) \in U(n+1)$
- ⟨1⟩9. $\forall m, n \in \omega. n \leq m \Rightarrow U(n) \subseteq U(m)$
- PROOF: Since $U(n) \subseteq U(n+1)$ for all n .
- ⟨1⟩10. $\forall m, n \in \omega. n < m \Rightarrow v(n) \neq v(m)$
- PROOF: Since $v(n) \in U(m)$ and $v(m) \notin U(m)$.

□

Corollary 7.32.1. *A set is infinite if and only if it is equivalent to a proper subset.*

7.3 Ordinal Arithmetic

Definition 7.33 (Addition). Let I be a well ordered set and $(\alpha_i)_{i \in I}$ be a sequence of ordinals. Choose a well ordered set A_i such that $A_i \cong \alpha_i$ for each $i \in I$, and assume the sets A_i are pairwise disjoint. The *sum* $\sum_{i \in I} \alpha_i$ is the ordinal of the well ordered set $\bigcup_{i \in I} A_i$, where:

- for $x, y \in A_i$, we have $x <_{\bigcup_{i \in I} A_i} y$ if and only if $x <_{A_i} y$
- for $x \in A_i$ and $y \in A_j$ with $i \neq j$, we have $x <_{\bigcup_{i \in I} A_i} y$ iff $i <_I j$

We write $\alpha + \beta$ for $\sum_{i \in 2} \gamma_i$ where $\gamma_0 = \alpha$ and $\gamma_1 = \beta$.

Proposition 7.34.

$$\begin{aligned}\alpha + 0 &= \alpha \\ 0 + \alpha &= \alpha \\ \alpha + 1 &= \alpha^+ \\ \alpha + (\beta + \gamma) &= (\alpha + \beta) + \gamma\end{aligned}$$

PROOF: Easy. \square

Proposition 7.35. For any ordinals α and β , we have $\alpha < \beta$ if and only if there exists $\gamma \neq 0$ such that $\beta = \alpha + \gamma$.

PROOF: Easy. \square

Proposition 7.36.

$$1 + \omega = \omega$$

PROOF: Easy. \square

Definition 7.37 (Multiplication). Given ordinals α and β , the *product* $\alpha\beta$ is the ordinal of $\alpha \times \beta$ under the *reverse lexicographic order*: $(a, b) < (c, d)$ iff $b < d$ or $(b = d \text{ and } a < c)$.

Proposition 7.38.

$$\begin{aligned}\alpha 0 &= 0 \\ 0 \alpha &= 0 \\ \alpha 1 &= \alpha \\ 1 \alpha &= \alpha \\ \alpha(\beta \gamma) &= (\alpha \beta) \gamma \\ \alpha(\beta + \gamma) &= \alpha \beta + \alpha \gamma\end{aligned}$$

PROOF: Easy. \square

Proposition 7.39. For ordinals α and β , if $\alpha\beta = 0$ then $\alpha = 0$ or $\beta = 0$.

PROOF: Easy. \square

Example 7.40. The commutative law fails:

$$2\omega = \omega \neq \omega 2$$

PROOF: Easy. \square

Example 7.41. The right distributive law fails:

$$(1 + 1)\omega = \omega \neq 1\omega + 1\omega = \omega 2$$

Definition 7.42 (Exponentiation). Given ordinals α and β , define the ordinal α^β by

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta \alpha \\ \alpha^\lambda &= \bigcup_{\beta < \lambda} \alpha^\beta \quad (\lambda \text{ a limit ordinal}) \end{aligned}$$

Proposition 7.43.

$$\begin{aligned} 0^\alpha &= 0 & (\alpha \geq 1) \\ 1^\gamma &= 1 \\ \alpha^{\beta+\gamma} &= \alpha^\beta \alpha^\gamma \\ \alpha^{\beta\gamma} &= (\alpha^\beta)^\gamma \end{aligned}$$

PROOF: Easy. \square

Example 7.44. $(\alpha\beta)^\gamma$ is different from $\alpha^\gamma\beta^\gamma$ in general:

$$(2 \cdot 2)^\omega = \omega \neq 2^\omega 2^\omega = \omega^2 .$$

7.4 Arithmetic on the Natural Numbers

Proposition 7.45. For all $m, n \in \omega$, we have

$$m + n = n + m .$$

PROOF:

$\langle 1 \rangle 1$. LET: $P(m)$ be the property $\forall n \in \omega. m + n = n + m$

$\langle 1 \rangle 2$. $P(0)$

$\langle 2 \rangle 1$. LET: $Q(n)$ be the property $0 + n = n + 0$

$\langle 2 \rangle 2$. $Q(0)$

PROOF: Trivial.

$\langle 2 \rangle 3$. $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$. LET: $n \in \omega$

$\langle 3 \rangle 2$. ASSUME: $Q(n)$
 $\langle 3 \rangle 3$. $0 + n^+ = n^+ + 0$

PROOF:

$$\begin{aligned}
 0 + n^+ &= (0 + n)^+ \\
 &= (n + 0)^+ & (\langle 3 \rangle 2) \\
 &= n^+ \\
 &= n^+ + 0
 \end{aligned}$$

$\langle 1 \rangle 3$. $\forall m \in \omega. P(m) \Rightarrow P(m^+)$

$\langle 2 \rangle 1$. LET: $m \in \omega$

$\langle 2 \rangle 2$. ASSUME: $P(m)$

$\langle 2 \rangle 3$. LET: $Q(n)$ be the property $m^+ + n = n + m^+$

$\langle 2 \rangle 4$. $Q(0)$

PROOF: $\langle 1 \rangle 2$

$\langle 2 \rangle 5$. $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$. LET: $n \in \omega$

$\langle 3 \rangle 2$. ASSUME: $Q(n)$

$\langle 3 \rangle 3$. $Q(n^+)$

PROOF:

$$\begin{aligned}
 m^+ + n^+ &= (m^+ + n)^+ \\
 &= (n + m^+)^+ & (\langle 3 \rangle 2) \\
 &= (n + m)^{++} \\
 &= (m + n)^{++} & (\langle 2 \rangle 2) \\
 &= (m + n^+)^+ \\
 &= (n^+ + m)^+ & (\langle 2 \rangle 2) \\
 &= n^+ + m^+
 \end{aligned}$$

□

Proposition 7.46. *For all $m, n \in \omega$, we have*

$$mn = nm .$$

PROOF:

$\langle 1 \rangle 1$. LET: $P(m)$ be the statement $\forall n \in \omega. mn = nm$

$\langle 1 \rangle 2$. $P(0)$

$\langle 2 \rangle 1$. LET: $Q(n)$ be the statement $0n = n0$

$\langle 2 \rangle 2$. $Q(0)$

PROOF: Trivial.

$\langle 2 \rangle 3$. $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$. LET: $n \in \omega$

$\langle 3 \rangle 2$. ASSUME: $Q(n)$

$\langle 3 \rangle 3$. $Q(n^+)$

PROOF:

$$\begin{aligned}
0n^+ &= 0n + 0 \\
&= 0n \\
&= n0 & (\langle 3 \rangle 2) \\
&= 0 \\
&= n^+0
\end{aligned}$$

$\langle 1 \rangle 3. \forall m \in \omega. P(m) \Rightarrow P(m^+)$

$\langle 2 \rangle 1. \text{ LET: } m \in \omega$

$\langle 2 \rangle 2. \text{ ASSUME: } P(m)$

$\langle 2 \rangle 3. \text{ LET: } Q(n) \text{ be the statement } m^+n = nm^+$

$\langle 2 \rangle 4. Q(0)$

PROOF: $\langle 1 \rangle 2$

$\langle 2 \rangle 5. \forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1. \text{ LET: } n \in \omega$

$\langle 3 \rangle 2. \text{ ASSUME: } Q(n)$

$\langle 3 \rangle 3. Q(n^+)$

PROOF:

$$\begin{aligned}
m^+n^+ &= m^+n + m^+ \\
&= (m^+n + m)^+ \\
&= (nm^+ + m)^+ & (\langle 3 \rangle 2) \\
&= (nm + n + m)^+ \\
&= (mn + m + n)^+ & (\langle 2 \rangle 2, \text{ Proposition 7.45}) \\
&= (mn^+ + n)^+ \\
&= (n^+m + n)^+ & (\langle 2 \rangle 2) \\
&= n^+m + n^+ \\
&= n^+m^+
\end{aligned}$$

□

Chapter 8

Countable Sets

Definition 8.1 (Countable). A set A is *countable* or *denumerable* iff there exists an injective function $A \rightarrow \omega$.

Definition 8.2 (Countably Infinite). A set is *countably infinite* iff it is similar to ω .

Proposition 8.3. *Every subset of a countable set is countable.*

PROOF: Easy. \square

Proposition 8.4. *Let X be a set. If there exists a function from ω onto X , then X is countable.*

PROOF:

$\langle 1 \rangle 1$. LET: f be a function from ω onto X .

$\langle 1 \rangle 2$. Choose a function $g : X \rightarrow \omega$ such that, for all $x \in X$, we have $f(g(x)) = x$.

$\langle 1 \rangle 3$. g is one-to-one.

\square

Proposition 8.5. $\omega \times \omega$ is countable.

PROOF: The sequence

$(0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), \dots$

is an enumeration of $\omega \times \omega$. \square

Corollary 8.5.1. *A countable union of countable sets is countable.*

PROOF:

$\langle 1 \rangle 1$. LET: A be a countable set of countable sets.

$\langle 1 \rangle 2$. PICK a surjection $f : \omega \rightarrow A$

$\langle 1 \rangle 3$. For $n \in \omega$, PICK a surjection $g_n : \omega \rightarrow f(n)$

$\langle 1 \rangle 4$. PICK a surjection $h : \omega \rightarrow \omega \times \omega$

$\langle 1 \rangle 5$. $\lambda n \in \omega. g_{\pi_1(h(n))}(\pi_2(h(n)))$ is a surjection $\omega \rightarrow \bigcup A$

\square

Corollary 8.5.2. *The Cartesian product of two countable sets is countable.*

Corollary 8.5.3. *For any countable set A , the set of all finite subsets of A is countable.*

PROOF: Prove by induction on n that the set of all subsets of size n is countable. The set of all finite subsets is then the union of these. \square

Proposition 8.6. *$\mathcal{P}\omega$ is uncountable.*

PROOF: Cantor's Theorem. \square

Chapter 9

Cardinal Numbers

Definition 9.1 (Cardinal Number). A *cardinal number* or *initial ordinal* is an ordinal α such that, for all $\beta < \alpha$, we have $\beta \not\sim \alpha$.

Definition 9.2 (Cardinality). For any set X , the *cardinality* of X , $\text{card } X$, is the least ordinal that is equivalent to X .

Proposition 9.3. *Given sets X and Y , we have $X \sim Y$ if and only if $\text{card } X = \text{card } Y$.*

PROOF: Easy. \square

Proposition 9.4. *For sets X and Y , we have $\text{card } X \leq \text{card } Y$ if and only if there exists an injective function $X \rightarrow Y$.*

PROOF: Easy. \square

Proposition 9.5. *Every natural number is a cardinal. ω is a cardinal.*

PROOF: Easy. \square

Proposition 9.6. *Every infinite cardinal is a limit ordinal.*

PROOF: For α infinite we have $f : \alpha^+ \sim \alpha$ where $f(\alpha) = 0$ and $f(\beta) = \beta^+$ for all other β . \square

9.1 Cardinal Arithmetic

Definition 9.7 (Addition). Given a family of cardinal numbers $\{\kappa_i\}_{i \in I}$, let $\sum_{i \in I} \kappa_i$ be $\text{card} \bigcup_{i \in I} A_i$, where $\{A_i\}_{i \in I}$ is a pairwise disjoint family of sets with $\text{card } A_i = \kappa_i$ for all i .

We write $\kappa + \lambda$ for $\sum_{i \in 2} \kappa_i$ where $\kappa_0 = \kappa$ and $\kappa_1 = \lambda$.

Proposition 9.8.

$$\begin{aligned}\kappa + \lambda &= \lambda + \kappa \\ \kappa + (\lambda + \mu) &= (\kappa + \lambda) + \mu\end{aligned}$$

PROOF: Easy. \square

Proposition 9.9. *Cardinal addition agrees with ordinal addition on the natural numbers.*

PROOF: Easy induction. \square

Proposition 9.10. *If $\kappa \leq \kappa'$ then $\kappa + \lambda \leq \kappa' + \lambda$.*

PROOF: Easy. \square

Proposition 9.11. *If κ is an infinite cardinal number then $\kappa + \kappa = \kappa$.*

PROOF:

$\langle 1 \rangle 1$. LET: A be an infinite set.

PROVE: $A \times 2 \sim A$

$\langle 1 \rangle 2$. LET: \mathcal{F} be the set of all functions f such that there exists $X \subseteq A$ such that $f : X \times 2 \sim X$.

$\langle 1 \rangle 3$. \mathcal{F} is non-empty.

PROOF: Pick a subset $X \subseteq A$ such that $X \sim \omega$, and a bijection $X \times 2 \sim X$.

$\langle 1 \rangle 4$. \mathcal{F} is partially ordered by extension.

$\langle 1 \rangle 5$. Every chain in \mathcal{F} has an upper bound.

PROOF: If $\mathcal{C} \subseteq \mathcal{F}$ is a chain then $\bigcup \mathcal{C} \in \mathcal{F}$.

$\langle 1 \rangle 6$. PICK $f \in \mathcal{F}$ maximal.

$\langle 1 \rangle 7$. PICK $X \subseteq A$ such that $f : X \times 2 \sim X$

$\langle 1 \rangle 8$. $X - A$ is finite.

$\langle 2 \rangle 1$. ASSUME: for a contradiction $X - A$ is infinite.

$\langle 2 \rangle 2$. PICK $Y \subseteq X - A$ such that $Y \sim \omega$.

$\langle 2 \rangle 3$. PICK $g : Y \times 2 \sim Y$

$\langle 2 \rangle 4$. $f \cup g : (X \cup Y) \times 2 \sim X \cup Y$

$\langle 2 \rangle 5$. Q.E.D.

PROOF: This contradicts the maximality of f .

$\langle 1 \rangle 9$. $\text{card } A + \text{card } A = \text{card } A$

PROOF:

$$\begin{aligned}
 2 \text{ card } A &= 2(\text{card } X + \text{card}(A - X)) \\
 &= 2 \text{ card } X + 2 \text{ card}(A - X) \\
 &= \text{card } X + 2 \text{ card}(A - X) && (\langle 1 \rangle 7) \\
 &= \text{card } X && (\langle 1 \rangle 8) \\
 &= \text{card } X + \text{card}(A - X) && (\langle 1 \rangle 8) \\
 &= \text{card } A
 \end{aligned}$$

\square

Corollary 9.11.1. *For any cardinals κ and λ that are not both finite, we have*

$$\kappa + \lambda = \max(\kappa, \lambda) .$$

Definition 9.12 (Multiplication). Given a family of cardinal numbers $\{\kappa_i\}_{i \in I}$, let $\prod_{i \in I} \kappa_i = \text{card} \times_{i \in I} \kappa_i$.

We write $\kappa \lambda$ for $\prod_{i \in 2} \kappa_i$ where $\kappa_0 = \kappa$ and $\kappa_1 = \lambda$.

Proposition 9.13.

$$\begin{aligned}\kappa\lambda &= \lambda\kappa \\ \kappa(\lambda\mu) &= (\kappa\lambda)\mu \\ \kappa(\lambda + \mu) &= \kappa\lambda + \kappa\mu\end{aligned}$$

Proposition 9.14. *Cardinal multiplication agrees with ordinal multiplication on the natural numbers.*

PROOF: Easy induction. \square

Proposition 9.15. *If $\kappa \leq \kappa'$ then $\kappa\lambda \leq \kappa'\lambda$.*

PROOF: Easy. \square

Proposition 9.16. *Let $\{\kappa_i\}_{i \in I}$ and $\{\lambda_i\}_{i \in I}$ be families of cardinal numbers with the same index set. If $\kappa_i < \lambda_i$ for all i , then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$.*

PROOF:

$\langle 1 \rangle 1$. Choose a one-to-one function $f_i : \kappa_i \rightarrow \lambda_i$ for each $i \in I$

$\langle 1 \rangle 2$. $\sum_{i \in I} \kappa_i \leq \prod_{i \in I} \lambda_i$

PROOF: Define $g : \sum_{i \in I} \kappa_i \rightarrow \prod_{i \in I} \lambda_i$ by

$$g(i, \eta)(j) = \begin{cases} f_i(\eta) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\langle 1 \rangle 3$. There is no surjective function $\sum_{i \in I} \kappa_i \rightarrow \prod_{i \in I} \lambda_i$

$\langle 2 \rangle 1$. LET: $h : \sum_i \kappa_i \rightarrow \prod_i \lambda_i$

$\langle 2 \rangle 2$. Choose $t(i) < \lambda_i$ for each $i \in I$ such that, for all $\eta < \kappa_i$, we have $t(i) \neq h(i, \eta)(i)$.

PROOF: Since the function that maps η to $h(i, \eta)(i)$ cannot be surjective $\kappa_i \rightarrow \lambda_i$.

$\langle 2 \rangle 3$. For all $i \in I$ and $\eta < \kappa_i$ we have $h \neq t(i, \eta)$.

\square

Proposition 9.17. *If κ is an infinite cardinal then $\kappa\kappa = \kappa$.*

PROOF:

$\langle 1 \rangle 1$. LET: A be an infinite set.

$\langle 1 \rangle 2$. LET: \mathcal{F} be the set of all functions f such that there exists $X \subseteq A$ such that $f : X \times X \sim X$

$\langle 1 \rangle 3$. \mathcal{F} is nonempty.

PROOF: Pick a countably infinite $X \subseteq A$. Then $X \times X \sim X$.

$\langle 1 \rangle 4$. \mathcal{F} is partially ordered by extension.

$\langle 1 \rangle 5$. Every chain in \mathcal{F} has an upper bound.

$\langle 1 \rangle 6$. PICK $f \in \mathcal{F}$ maximal.

$\langle 1 \rangle 7$. PICK $X \subseteq A$ such that $f : X \times X \sim X$.

$\langle 1 \rangle 8$. $\text{card } X = \text{card } A$

$\langle 2 \rangle 1$. ASSUME: for a contradiction $\text{card } X < \text{card } A$

$\langle 2 \rangle 2$. $\text{card } A = \text{card}(A - X)$

PROOF: Corollary 9.11.1.

$\langle 2 \rangle 3$. $\text{card } X < \text{card}(A - X)$

$\langle 2 \rangle 4$. PICK $Y \subseteq A - X$ such that $Y \sim X$

$\langle 2 \rangle 5$. PICK $g : (X \times Y) \cup (Y \times X) \cup (Y \times Y) \sim Y$

PROOF:

$$(X \times Y) \cup (Y \times X) \cup (Y \times Y) \sim 3 \times X \times X \quad (\langle 2 \rangle 4)$$

$$\sim 3 \times X \quad (\langle 1 \rangle 7)$$

$$\sim X \quad (\text{Corollary 9.11.1})$$

$$\sim Y \quad (\langle 2 \rangle 4)$$

$\langle 2 \rangle 6$. $f \cup g : (X \cup Y) \times (X \cup Y) \sim X \cup Y$

$\langle 2 \rangle 7$. Q.E.D.

PROOF: This contradicts the maximality of f .

□

Corollary 9.17.1. *If κ and λ are non-zero cardinals that are not both finite, then*

$$\kappa\lambda = \max(\kappa, \lambda) \text{ .}$$

Definition 9.18 (Exponentiation). Given cardinal numbers κ and λ , let κ^λ be the cardinality of the set of all functions $\lambda \rightarrow \kappa$.

Proposition 9.19.

$$\kappa^{\lambda+\mu} = \kappa^\lambda \kappa^\mu$$

$$(\kappa\lambda)^\mu = \kappa^\mu \lambda^\mu$$

$$\kappa^{\lambda^\mu} = (\kappa^\lambda)^\mu$$

PROOF: Easy. □

Proposition 9.20. *Cardinal exponentiation and ordinal exponentiation agree on the natural numbers.*

PROOF: Easy. □

Proposition 9.21.

$$\text{card } \mathcal{P}X = 2^{\text{card } X}$$

PROOF: Define $\chi : \mathcal{P}X \sim 2^X$ to be the function that maps S to the function $\chi_S : X \rightarrow 2$ where $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \notin S$. □

Proposition 9.22. *For any infinite cardinal κ we have $\kappa < 2^\kappa$.*

PROOF: Proposition 9.16. □

Proposition 9.23. *If $\kappa \leq \lambda$ then $\kappa^\mu \leq \lambda^\mu$.*

PROOF: Easy. □

9.2 Alephs

Definition 9.24 (Aleph). Define the cardinal \aleph_α for every ordinal α as follows: \aleph_α is the least infinite cardinal greater than \aleph_β for all $\beta < \alpha$.

Proposition 9.25.

$$\aleph_0 = \omega$$

PROOF: Easy. \square

Definition 9.26 (Continuum Hypothesis). The *continuum hypothesis* is the statement $\aleph_1 = 2^{\aleph_0}$.

Definition 9.27 (Generalized Continuum Hypothesis). The *generalized continuum hypothesis* is the statement: for every ordinal α we have $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$.

Chapter 10

Real Numbers

10.1 Axioms for Real Numbers

Let there be a set \mathbb{R} , whose elements are called *real numbers*.

Let there be two functions $+, \cdot : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Let there be a relation $< \subseteq \mathbb{R}^2$.

Axiom 10.1 (Associativity of Addition).

$$\forall x, y, z \in \mathbb{R}. x + (y + z) = (x + y) + z$$

Axiom 10.2 (Associativity of Multiplication).

$$\forall x, y, z \in \mathbb{R}. x(yz) = (xy)z$$

Axiom 10.3 (Commutativity of Addition).

$$\forall x, y \in \mathbb{R}. x + y = y + x$$

Axiom 10.4 (Commutativity of Multiplication).

$$\forall x, y \in \mathbb{R}. xy = yx$$

Axiom 10.5 (Identity for Addition). *There exists a unique $z \in \mathbb{R}$ such that $\forall x \in \mathbb{R}. x + z = x$.*

Definition 10.6 (Zero). The real number *zero*, 0, is the unique real number such that $\forall x \in \mathbb{R}. x + 0 = x$.

Axiom 10.7 (Identity for Multiplication). *There exists a unique $i \in \mathbb{R}$ such that $\forall x \in \mathbb{R}. xi = x$. Further, we have $i \neq 0$.*

Definition 10.8 (One). The real number *one*, 1, is the unique real number such that $\forall x \in \mathbb{R}. x1 = x$.

Axiom 10.9 (Additive Inverses). *For all $x \in \mathbb{R}$, there exists a unique $y \in \mathbb{R}$ such that $x + y = 0$.*

Axiom 10.10 (Multiplicative Inverses). *For all $x \in \mathbb{R}$, if $x \neq 0$ then there exists a unique $y \in \mathbb{R}$ such that $xy = 1$.*

Axiom 10.11 (Distributive Law).

$$\forall x, y, z \in \mathbb{R}. x(y + z) = xy + xz$$

Axiom 10.12 (Monotonicity of Addition). *For all $x, y, z \in \mathbb{R}$, if $x < y$ then $x + z < y + z$.*

Axiom 10.13 (Monotonicity of Multiplication). *For all $x, y, z \in \mathbb{R}$, if $x < y$ and $0 < z$ then $xz < yz$.*

Axiom 10.14 (Least Upper Bound Property). *The relation $<$ is a strict linear order on \mathbb{R} with the least upper bound property.*

Axiom 10.15 (Density). *For all $x, y \in \mathbb{R}$, if $x < y$ then there exists $z \in \mathbb{R}$ such that $x < z < y$.*

10.2 Consequences of the Axioms

Definition 10.16 (Negation). For any real number x , the *negation* of x , $-x$, is the unique real number such that $x + (-x) = 0$.

Theorem 10.17. *For any real numbers x and y , if $x + y = x$ then $y = 0$.*

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 2$. ASSUME: $x + y = x$

$\langle 1 \rangle 3$. $y = 0$

PROOF:

$$\begin{aligned} y &= y + 0 && \text{(Definition of zero)} \\ &= y + (x + (-x)) && \text{(Definition of } -x) \\ &= (y + x) + (-x) && \text{(Associativity of Addition)} \\ &= (x + y) + (-x) && \text{(Commutativity of Addition)} \\ &= x + (-x) && (\langle 1 \rangle 2) \\ &= 0 && \text{(Definition of } -x) \end{aligned}$$

□

Theorem 10.18.

$$\forall x \in \mathbb{R}. 0x = 0$$

PROOF:

$\langle 1 \rangle 1$. LET: $x \in \mathbb{R}$

$\langle 1 \rangle 2$. $0x + 0x = 0x$

PROOF:

$$\begin{aligned} xx + 0x &= (x + 0)x && \text{(Distributive Law)} \\ &= xx && \text{(Definition of 0)} \end{aligned}$$

$\langle 1 \rangle 3. 0x = 0$

PROOF: Theorem 10.17, $\langle 1 \rangle 2$.

□

Theorem 10.19.

$$-0 = 0$$

PROOF: Since $0 + 0 = 0$. □

Theorem 10.20.

$$\forall x \in \mathbb{R}. -(-x) = x$$

PROOF: Since $-x + x = 0$. □

Theorem 10.21.

$$\forall x, y \in \mathbb{R}. x(-y) = -(xy)$$

PROOF:

$$\begin{aligned} x(-y) + xy &= x((-y) + y) && \text{(Distributive Law)} \\ &= x0 && \text{(Definition of } -y) \\ &= 0 && \text{(Theorem 10.18)} \end{aligned} \quad \square$$

Theorem 10.22.

$$\forall x \in \mathbb{R}. (-1)x = -x$$

PROOF:

$$\begin{aligned} (-1)x &= -(1 \cdot x) && \text{(Theorem 10.21)} \\ &= -x && \text{(Definition of 1)} \end{aligned} \quad \square$$

Definition 10.23 (Subtraction). Define *subtraction* $- : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $x - y = x + (-y)$.

Theorem 10.24.

$$\forall x, y, z \in \mathbb{R}. x(y - z) = xy - xz$$

PROOF:

$$\begin{aligned} x(y - z) &= x(y + (-z)) && \text{(Definition of subtraction)} \\ &= xy + x(-z) && \text{(Distributive Law)} \\ &= xy + -(xz) && \text{(Theorem 10.21)} \\ &= xy - xz && \text{(Definition of subtraction)} \end{aligned} \quad \square$$

Theorem 10.25.

$$\forall x, y \in \mathbb{R}. -(x + y) = -x - y$$

PROOF:

$$\begin{aligned}
-(x + y) &= (-1)(x + y) && \text{(Theorem 10.22)} \\
&= (-1)x + (-1)y && \text{(Distributive Law)} \\
&= -x + (-y) && \text{(Theorem 10.22)} \\
&= -x - y && \text{(Definition of subtraction)} \quad \square
\end{aligned}$$

Theorem 10.26.

$$\forall x, y \in \mathbb{R}. -(x - y) = -x + y$$

PROOF:

$$\begin{aligned}
-(x - y) &= -(x + (-y)) && \text{(Definition of subtraction)} \\
&= -x - (-y) && \text{(Theorem 10.25)} \\
&= -x + (-(-y)) && \text{(Definition of subtraction)} \\
&= -x + y && \text{(Theorem 10.20)} \quad \square
\end{aligned}$$

Definition 10.27 (Reciprocal). Given $x \in \mathbb{R}$ with $x \neq 0$, the *reciprocal* of x , $1/x$, is the unique real number such that $x \cdot 1/x = 1$.

Theorem 10.28. For any real numbers x and y , if $x \neq 0$ and $xy = x$ then $y = 1$.

PROOF:

- $\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$
- $\langle 1 \rangle 2$. ASSUME: $x \neq 0$
- $\langle 1 \rangle 3$. ASSUME: $xy = x$
- $\langle 1 \rangle 4$. $y = 1$

PROOF:

$$\begin{aligned}
y &= y1 && \text{(Definition of 1)} \\
&= y(x \cdot 1/x) && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\
&= (yx)1/x && \text{(Associativity of Multiplication)} \\
&= (xy)1/x && \text{(Commutativity of Multiplication)} \\
&= x \cdot 1/x && (\langle 1 \rangle 3) \\
&= 1 && \text{(Definition of } 1/x, \langle 1 \rangle 2)
\end{aligned}$$

\square

Definition 10.29 (Quotient). Given real numbers x and y with $y \neq 0$, the *quotient* x/y is defined by

$$x/y = x \cdot 1/y .$$

Theorem 10.30. For any real number x , if $x \neq 0$ then $x/x = 1$.

PROOF: Immediate from definitions. \square

Theorem 10.31.

$$\forall x \in \mathbb{R}. x/1 = x$$

PROOF:

$\langle 1 \rangle 1$. LET: $x \in \mathbb{R}$

$\langle 1 \rangle 2$. $1/1 = 1$

PROOF: Since $1 \cdot 1 = 1$.

$\langle 1 \rangle 3$. $x/1 = x$

PROOF: Since $x/1 = x \cdot 1/1 = x \cdot 1 = x$.

□

Theorem 10.32. For any real numbers x and y , if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 2$. ASSUME: $xy = 0$ and $x \neq 0$

PROVE: $y = 0$

$\langle 1 \rangle 3$. $y = 0$

PROOF:

$$\begin{aligned} y &= 1y && \text{(Definition of 1)} \\ &= (1/x)xy && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\ &= (1/x)0 && (\langle 1 \rangle 2) \\ &= 0 && \text{(Theorem 10.18)} \end{aligned}$$

□

Theorem 10.33. For any real numbers y and z , if $y \neq 0$ and $z \neq 0$ then $(1/y)(1/z) = 1/(yz)$.

PROOF: Since $yz(1/y)(1/z) = 1 \cdot 1 = 1$. □

Corollary 10.33.1. For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have $(x/y)(z/w) = (xz)/(yw)$.

Theorem 10.34. For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + yz}{yw}$$

PROOF:

$$\begin{aligned} yw \left(\frac{x}{y} + \frac{z}{w} \right) &= yw \frac{x}{y} + yw \frac{z}{w} \\ &= wx + yz \end{aligned} \quad \square$$

Theorem 10.35. For any real number x , if $x \neq 0$ then $1/x \neq 0$.

PROOF: Since $x \cdot 1/x = 1 \neq 0$. □

Theorem 10.36. For any real numbers w, z , if $w \neq 0 \neq z$ then $1/(w/z) = z/w$.

PROOF: Since $(z/w)(w/z) = (wz)/(wz) = 1$. □

Theorem 10.37. For any real numbers a, x and y , if $y \neq 0$ then $(ax)/y = a(x/y)$

PROOF: Since $ya(x/y) = ax$. \square

Theorem 10.38. For any real numbers x and y , if $y \neq 0$ then $(-x)/y = x/(-y) = -(x/y)$.

PROOF:

$\langle 1 \rangle 1$. $(-x)/y = -(x/y)$

PROOF: Take $a = -1$ in Theorem 10.37.

$\langle 1 \rangle 2$. $x/(-y) = -(x/y)$

PROOF: Since $(-y)(-(x/y)) = y(x/y) = x$.

\square

Theorem 10.39. For any real numbers x, y, z and w , if $x > y$ and $w > z$ then $x + w > y + z$.

PROOF: We have $y + z < x + z < x + w$ by Monotonicity of Addition twice. \square

Corollary 10.39.1. For any real numbers x and y , if $x > 0$ and $y > 0$ then $x + y > 0$.

Theorem 10.40. For any real numbers x and y , if $x > 0$ and $y > 0$ then $xy > 0$.

PROOF:

$$\begin{aligned} xy &> 0y && \text{(Monotonicity of Multiplication)} \\ &= 0 && \text{(Theorem 10.18)} \end{aligned} \quad \square$$

Theorem 10.41. For any real number x , we have $x > 0$ iff $-x < 0$.

PROOF:

$\langle 1 \rangle 1$. If $0 < x$ then $-x < 0$

PROOF: By Monotonicity of Addition adding $-x$ to both sides.

$\langle 1 \rangle 2$. If $-x < 0$ then $0 < x$

PROOF: By Monotonicity of Addition adding x to both sides.

\square

Theorem 10.42. For any real numbers x and y , we have $x > y$ iff $-x < -y$.

PROOF:

$\langle 1 \rangle 1$. If $y < x$ then $-x < -y$.

PROOF: By Monotonicity of Addition adding $-x - y$ to both sides.

$\langle 1 \rangle 2$. If $-x < -y$ then $y < x$.

PROOF: By Monotonicity of Addition adding $x + y$ to both sides.

\square

Theorem 10.43. For any real numbers x, y and z , if $x > y$ and $z < 0$ then $xz < yz$.

PROOF:

$\langle 1 \rangle 1$. LET: x, y and z be real numbers.

$\langle 1 \rangle 2$. ASSUME: $x > y$

$\langle 1 \rangle 3$. ASSUME: $z < 0$

$\langle 1 \rangle 4$. $-z > 0$

PROOF: Theorem 10.41, $\langle 1 \rangle 3$.

$\langle 1 \rangle 5$. $x(-z) > y(-z)$

PROOF: $\langle 1 \rangle 2$, $\langle 1 \rangle 4$, Monotonicity of Multiplication.

$\langle 1 \rangle 6$. $-(xz) > -(yz)$

PROOF: Theorem 10.21, $\langle 1 \rangle 5$.

$\langle 1 \rangle 7$. $xz < yz$

PROOF: Theorem 10.41, $\langle 1 \rangle 6$.

□

Theorem 10.44. *For any real number x , if $x \neq 0$ then $xx > 0$.*

PROOF:

$\langle 1 \rangle 1$. If $x > 0$ then $xx > 0$

PROOF: By Monotonicity of Multiplication.

$\langle 1 \rangle 2$. If $x < 0$ then $xx > 0$

PROOF: Theorem 10.43.

□

Theorem 10.45.

$$0 < 1$$

PROOF: By Theorem 10.44 since $1 = 1 \cdot 1$. □

Definition 10.46 (Positive). A real number x is *positive* iff $x > 0$.

Theorem 10.47. *For any real numbers x and y , we have xy is positive if and only if x and y are both positive or both negative.*

PROOF: By the Monotonicity of Multiplication and Theorem 10.43. □

Corollary 10.47.1. *For any real number x , if $x > 0$ then $1/x > 0$.*

PROOF: Since $x \cdot 1/x = 1$ is positive. □

Theorem 10.48. *For any real numbers x and y , if $x > y > 0$ then $1/x < 1/y$.*

PROOF: If $1/y \leq 1/x$ then $1 < 1$ by Monotonicity of Multiplication. □

Theorem 10.49. *For any real numbers x and y , if $x < y$ then $x < (x+y)/2 < y$.*

PROOF: We have $2x < x+y$ and $x+y < 2y$ by Monotonicity of Addition, hence $x < (x+y)/2 < y$ by Monotonicity of Multiplication since $1/2 > 0$. □