Summary of Halmos' Naive Set Theory

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Chapter 1

Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*, $\in$. If $x \in A$ we say that x *belongs* to A, x is an *element* of A, or x is *contained* in A. If this does not hold we write $x \notin A$.

Axiom 1.1 (Axiom of Extensionality)**.** *Two sets are equal if and only if they have the same elements.*

Axiom 1.2 (Axiom of Comprehension, Aussonderungsaxiom)**.** *To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

Axiom 1.3. *A set exists.*

Axiom 1.4 (Axiom of Pairing)**.** *For any two sets, there exists a set that they both belong to.*

Axiom 1.5 (Union Axiom)**.** *For every set A, there exists a set that contains all the elements that belong to at least one element of A.*

Chapter 2

The Subset Relation

Definition 2.1 (Subset)**.** Let A and B be sets. We say that A is a *subset* of B, or B *includes* A, and write $A \subseteq B$ or $B \supseteq A$, iff every element of A is an element of B.

Theorem 2.2. *For any set A, we have $A \subseteq A$.*

PROOF: Every element of A is an element of A. $\square$

Theorem 2.3. *For any sets A, B and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

PROOF: If every element of A is an element of B, and every element of B is an element of C, then every element of A is an element of C. $\square$

Theorem 2.4. *For any sets A and B, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

PROOF: If every element of A is an element of B, and every element of B is an element of A, then A and B have the same elements, and therefore are equal by the Axiom of Extensionality. $\square$

Definition 2.5 (Proper Subset)**.** Let A and B be sets. We say that A is a *proper* subset of B, or B *properly* includes A, and write $A \subsetneq B$ or $B \supsetneq A$, iff $A \subseteq B$ and $A \neq B$.

Chapter 3

Comprehension Notation

Definition 3.1. Given a set A and a condition $S(x)$, we write $\{x \in A : S(x)\}$ for the set whose elements are exactly those elements x of A for which $S(x)$ holds.

Proof: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality. □

Theorem 3.2. *There is no set that contains every set.*

Proof:
⟨1⟩1. Let: A be a set.
  Prove: There exists a set B such that $B \notin A$.
⟨1⟩2. Let: $B = \{x \in A : x \notin x\}$
⟨1⟩3. If $B \in A$ then we have $B \in B$ if and only if $B \notin B$.
⟨1⟩4. $B \notin A$
□

Chapter 4

Unordered Pairs

Theorem 4.1. *There exists a set with no elements.*

PROOF: Pick a set A by Axiom 1.3. Then the set $\{x \in A : x \neq x\}$ has no elements. $\square$

Definition 4.2 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

Theorem 4.3. *For any set A we have $\emptyset \subset A$.*

PROOF: Vacuous. $\square$

Definition 4.4 ((Unordered) Pair)**.** For any sets a and b, the *(unordered) pair* $\{a, b\}$ is the set whose elements are just a and b.

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality. $\square$

Definition 4.5 (Singleton)**.** For any set a, the *singleton* $\{a\}$ is defined to be $\{a, a\}$.

Chapter 5

Unions and Intersections

Definition 5.1 (Union)**.** For any set $\mathcal{C}$, the *union* of $\mathcal{C}$, $\bigcup \mathcal{C}$, is the set whose elements are the elements of the elements of $\mathcal{C}$.

Proof: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality. $\square$

Proposition 5.2.

$$\bigcup \emptyset = \emptyset$$

Proof: There is no set that is an element of an element of $\emptyset$. $\square$

Proposition 5.3. *For any set A, we have $\bigcup\{A\} = A$.*

Proof: For any x, we have x is an element of an element of $\{A\}$ if and only if x is an element of A. $\square$

Definition 5.4. We write $A \cup B$ for $\bigcup\{A, B\}$.

Proposition 5.5. *For any set A, we have $A \cup \emptyset = A$.*

Proof: $x \in A \cup \emptyset$ iff $x \in A$ or $x \in \emptyset$, iff $x \in A$. $\square$

Proposition 5.6 (Commutativity)**.** *For any sets A and B, we have $A \cup B = B \cup A$.*

Proof: $x \in A \cup B$ iff $x \in A$ or $x \in B$, iff $x \in B$ or $x \in A$, iff $x \in B \cup A$. $\square$

Proposition 5.7 (Associativity)**.** *For any sets A, B and C, we have $A \cup (B \cup C) = (A \cup B) \cup C$.*

Proof: Each is the set of all x such that $x \in A$ or $x \in B$ or $x \in C$. $\square$

Proposition 5.8 (Idempotence)**.** *For any set A, we have $A \cup A = A$.*

Proof: $x \in A$ or $x \in A$ is equivalent to $x \in A$. $\square$

Proposition 5.9. *For any sets A and B, we have $A \subseteq B$ if and only if $A \cup B = B$.*

Proof: For any x, the statement "if $x \in A$ then $x \in B$" is equivalent to "$x \in A$ or $x \in B$ if and only if $x \in B$". $\square$

Proposition 5.10. *For any sets a and b, we have $\{a\} \cup \{b\} = \{a, b\}$.*

Proof: Immediate from definitions. $\square$

Definition 5.11 ((Unordered) Triple)**.** Given sets a_1, …, a_n, define the *(unordered) n-tuple* $\{a_1, \ldots, a_n\}$ to be

$$\{a_1, \ldots, a_n\} := \{a_1\} \cup \cdots \cup \{a_n\} \ .$$

Definition 5.12 (Intersection)**.** For any sets A and B, the *intersection* $A \cap B$ is defined to be $\{x \in A : x \in B\}$.

Proposition 5.13. *For any set A, we have $A \cap \emptyset = \emptyset$.*

Proof: There is no x such that $x \in A$ and $x \in \emptyset$. $\square$

Proposition 5.14. *For any sets A and B, we have*

$$A \cap B = B \cap A \ .$$

Proof: $x \in A$ and $x \in B$ if and only if $x \in B$ and $x \in A$. $\square$