Mathematics

Robin Adams

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Contents

Part I

Category Theory

Chapter 1

Foundations

This is a placeholder — I am not sure what foundation I want to use for this project yet. I will try to work in a way which is foundation-independent. What I do could be formalized in ZFC, ETCS, or some other system. I will assume the usual set theoretic constructions as needed.

1.1 Relations

Definition 1.1 (Reflexive)**.** A relation R on a class A is *reflexive* iff, for all $x \in A$, we have xRx.

Definition 1.2 (Transitive)**.** A relation R on a class A is *transitive* iff, whenever xRy and yRz, then xRz.

Chapter 2

Categories

2.1 Definition

Definition 2.1 (Category)**.** A *category* $\mathcal{C}$ consists of:

- A class $|\mathcal{C}|$ of *objects*. We write $A \in \mathcal{C}$ for $A \in |\mathcal{C}|$.
- For any objects A, B, a set $\mathcal{C}[A, B]$ of *morphisms* from A to B. We write $f : A \to B$ for $f \in \mathcal{C}[A, B]$.
- For any object A, a morphism $\mathrm{id}_A : A \to A$, the *identity* morphism on A.
- For any morphisms $f : A \to B$ and $g : B \to C$, a morphism $g \circ f : A \to C$, the *composite* of f and g.

such that:

Associativity Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have $h \circ (g \circ f) = (h \circ g) \circ f$

Left Unit Law For any morphism $f : A \to B$, we have $\mathrm{id}_B \circ f = f$.

Right Unit Law For any morphism $f : A \to B$, we have $f \circ \mathrm{id}_A = f$.

2.2 Examples

Example 2.2 (Category of Sets)**.** The *category of sets* **Set** has objects all sets and morphisms all functions.

Example 2.3 (Category of Finite Sets)**.** The *category of finite sets* $\mathbf{Set_{fin}}$ has objects all finite sets and morphisms all functions.

Example 2.4 (Category of Sets and Relations)**.** The *category of sets and relations* **Rel** has:

- objects all sets
- morphism $A \to B$ all relations between A and B
- the identity on A is $\{(a,a) : a \in A\}$
- given $R \subseteq A \times B$ and $S \subseteq B \times C$, we define

$$S \circ R = \{(a,c) \in A \times C : \exists b \in B. aRb \wedge bSc\} \ .$$

2.3 Subcategories

Definition 2.5 (Subcategory)**.** A category $\mathcal{C}$ is a *subcategory* of a category $\mathcal{D}$ iff:

- $|\mathcal{C}| \subseteq |\mathcal{D}|$
- for all $A, B \in \mathcal{C}$, we have $\mathcal{C}[A,B] \subseteq \mathcal{D}[A,B]$
- for all $A \in \mathcal{C}$, the identity on A is the same in $\mathcal{C}$ and $\mathcal{D}$
- composition in $\mathcal{C}$ and composition in $\mathcal{D}$ agree on composable pairs of morphisms from $\mathcal{C}$.

It is a *full* subcategory iff, for all $A, B \in \mathcal{C}$, we have $\mathcal{C}[A,B] = \mathcal{D}[A,B]$.

Chapter 3

Morphisms

Definition 3.1 (Endomorphism)**.** In a category $\mathcal{C}$, an *endomorphism* on an object A is a morphism $A \to A$. We write $\mathrm{End}_{\mathcal{C}}(A)$ for $\mathcal{C}[A, A]$.

3.1 Monomorphisms and Epimorphisms

Definition 3.2 (Monomorphism)**.** In a category, let $f : A \to B$. Then f is a *monomorphism* or *monic* iff, for every object X and morphism $x, y : X \to A$, if $fx = fy$ then $x = y$.

Definition 3.3 (Epimorphism)**.** In a category, let $f : A \to B$. Then f is a *epimorphism* or *epi* iff, for every object X and morphism $x, y : B \to X$, if $xf = yf$ then $x = y$.

Proposition 3.4. *The composite of two monomorphism is monic.*

PROOF:
$\langle 1 \rangle 1.$ LET: $f : A \rightarrowtail B$ and $g : B \rightarrowtail C$ be monic.
$\langle 1 \rangle 2.$ LET: $x, y : X \to A$
$\langle 1 \rangle 3.$ ASSUME: $g \circ f \circ x = g \circ f \circ y$
$\langle 1 \rangle 4.$ $f \circ x = f \circ y$
$\langle 1 \rangle 5.$ $x = y$
$\square$

Proposition 3.5. *The composite of two epimorphisms is epi.*

PROOF: Dual. $\square$

Proposition 3.6. *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is monic then f is monic.*

PROOF: If $f \circ x = f \circ y$ then $gfx = gfy$ and so $x = y$. $\square$

Proposition 3.7. *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is epi then g is epi.*

PROOF: Dual. □

Proposition 3.8. *A function is a monomorphism in* **Set** *iff it is injective.*

PROOF:
⟨1⟩1. LET: $f : A \to B$
⟨1⟩2. If f is monic then f is injective.
  ⟨2⟩1. ASSUME: f is monic.
  ⟨2⟩2. LET: $x, y \in A$
  ⟨2⟩3. ASSUME: $f(x) = f(y)$
  ⟨2⟩4. LET: $\overline{x}, \overline{y} : 1 \to A$ be the functions such that $\overline{x}(*) = x$ and $\overline{y}(*) = y$
  ⟨2⟩5. $f \circ \overline{x} = f \circ \overline{y}$
  ⟨2⟩6. $\overline{x} = \overline{y}$
    PROOF: By ⟨2⟩1.
  ⟨2⟩7. $x = y$
⟨1⟩3. If f is injective then f is monic.
  ⟨2⟩1. ASSUME: f is injective.
  ⟨2⟩2. LET: X be a set and $x, y : X \to A$.
  ⟨2⟩3. ASSUME: $f \circ x = f \circ y$
    PROVE: $x = y$
  ⟨2⟩4. LET: $t \in X$
    PROVE: $x(t) = y(t)$
  ⟨2⟩5. $f(x(t)) = f(y(t))$
  ⟨2⟩6. $x(t) = y(t)$
    PROOF: By ⟨2⟩1.
□

Proposition 3.9. *A function is an epimorphism in* **Set** *iff it is surjective.*

PROOF:
⟨1⟩1. LET: $f : A \to B$
⟨1⟩2. If f is an epimorphism then f is surjective.
  ⟨2⟩1. ASSUME: f is an epimorphism.
  ⟨2⟩2. LET: $b \in B$
  ⟨2⟩3. LET: $x, y : B \to 2$ be defined by $x(b) = 1$ and $x(t) = 0$ for all other $t \in B$, $y(t) = 0$ for all $t \in B$.
  ⟨2⟩4. $x \neq y$
  ⟨2⟩5. $x \circ f \neq y \circ f$
  ⟨2⟩6. There exists $a \in A$ such that $f(a) = b$.
⟨1⟩3. If f is surjective then f is an epimorphism.
  ⟨2⟩1. ASSUME: f is surjective.
  ⟨2⟩2. LET: $x, y : B \to X$
  ⟨2⟩3. ASSUME: $x \circ f = y \circ f$
    PROVE: $x = y$
  ⟨2⟩4. LET: $b \in B$
    PROVE: $x(b) = y(b)$
  ⟨2⟩5. PICK $a \in A$ such that $f(a) = b$

⟨2⟩6. $x(f(a)) = y(f(a))$
⟨2⟩7. $x(b) = y(b)$
□

Proposition 3.10. *In a preorder, every morphism is monic and epi.*

Proof: Immediate from definitions. □

3.2 Sections and Retractions

Definition 3.11 (Section, Retraction)**.** In a category, let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $r \circ s = \mathrm{id}_B$.

Proposition 3.12. *Every identity morphism is a section and retraction of itself.*

Proof: Immediate from definitions. □

Proposition 3.13. *Let $r, r' : A \to B$ and $s : B \to A$. If r is a retraction of s and r' is a section of s then $r = r'$.*

Proof:

$$\begin{aligned} r &= r \circ \mathrm{id}_A \\ &= r \circ s \circ r' \\ &= \mathrm{id}_B \circ r' \\ &= r' \end{aligned}$$ □

Proposition 3.14. *Let $r_1 : A \to B$, $r_2 : B \to C$, $s_1 : B \to A$ and $s_2 : C \to B$. If r_1 is a retraction of s_1 and r_2 is a retraction of s_2 then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

Proof:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \mathrm{id}_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= \mathrm{id}_C \end{aligned}$$ □

Proposition 3.15. *Every section is monic.*

Proof:
⟨1⟩1. Let: $s : A \to B$ be a section of $r : B \to A$.
⟨1⟩2. Let: $x, y : X \to A$ satisfy $sx = sy$.
⟨1⟩3. $rsx = rsy$
⟨1⟩4. $x = y$
□

Proposition 3.16. *Every retraction is epi.*

Proof: Dual. □

Proposition 3.17. *In **Set**, every epimorphism has a retraction.*

PROOF: By the Axiom of Choice. □

Example 3.18. It is not true in general that every monomorphism in any category has a section. nor that every epimorphism in any category has a retraction.

In the category **2**, the morphism $0 \leq 1$ is monic and epi but has no retraction or section.

3.3 Isomorphisms

Definition 3.19 (Isomorphism)**.** In a category $\mathcal{C}$, a morphism $f : A \to B$ is an *isomorphism*, denoted $f : A \cong B$, iff there exists a morphism $f^{-1} : B \to A$, the *inverse* of f, such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

An *automorphism* on an object A is an isomorphism between A and itself. We write $\mathrm{Aut}_{\mathcal{C}}(A)$ for the set of all automorphisms on A.

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 3.20. *The inverse of an isomorphism is unique.*

PROOF: Proposition 3.13. □

Proposition 3.21. *For any object A we have $\mathrm{id}_A : A \cong A$ and $\mathrm{id}_A^{-1} = \mathrm{id}_A$.*

PROOF: Since $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$ by the Unit Laws. □

Proposition 3.22. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Immediate from definitions. □

Proposition 3.23. *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF: From Proposition 3.14. □

Definition 3.24 (Groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

3.4 Initial and Terminal Objects

Definition 3.25 (Initial Object)**.** An object I in a category is *initial* iff, for any object X, there is exactly one morphism $I \to X$.

Example 3.26. The empty set is the initial object in **Set**.

Definition 3.27 (Terminal Object)**.** An object T in a category is *terminal* iff, for any object X, there is exactly one morphism $X \to T$.

Example 3.28. Every singleton is terminal in **Set**.

Proposition 3.29. *If I and J are initial in a category, then there exists a unique isomorphism $I \cong J$.*

PROOF:
$\langle 1 \rangle 1$. LET: i be the unique morphism $I \to J$.
$\langle 1 \rangle 2$. LET: i^{-1} be the unique morphism $J \to I$.
$\langle 1 \rangle 3$. $i \circ i^{-1} = \mathrm{id}_J$
PROOF: Since there is only one morphism $J \to J$.
$\langle 1 \rangle 4$. $i^{-1} \circ i = \mathrm{id}_I$
PROOF: Since there is only one morphism $I \to I$.
□

Proposition 3.30. *If S and T are terminal in a category, then there exists a unique isomorphism $S \cong T$.*

PROOF: Dual. □

3.5 Comma Categories

Definition 3.31 (Comma Category)**.** Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. The *comma category* $F \downarrow G$ is the category with:

- objects all pairs (C, D, f) where $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f : FC \to GD : \mathcal{E}$
- morphisms $(u, v) : (C, D, f) \to (C', D', g)$ all pairs $u : C \to C' : \mathcal{C}$ and $v : D \to D' : \mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ \downarrow{\scriptstyle Fu} & & \downarrow{\scriptstyle Gv} \\ FC' & \xrightarrow{g} & GD' \end{array}$$

Definition 3.32 (Slice Category)**.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category* over A, denoted $\mathcal{C}/A$, is the comma category $1_{\mathcal{C}} \downarrow K^{\mathbf{1}} A$.

Definition 3.33 (Coslice Category)**.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *coslice category* over A, denoted $\mathcal{C} \backslash A$, is the comma category $K^{\mathbf{1}} A \downarrow 1_{\mathcal{C}}$.

Definition 3.34 (Pointed Sets)**.** The *category of pointed sets* $\mathbf{Set}_*$ is the coslice category $\mathbf{Set} \backslash 1$.

Chapter 4

Functors

Definition 4.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for any morphism $f : A \to B : \mathcal{C}$, a morphism $Ff : FA \to FB : \mathcal{D}$

such that:

- $F\mathrm{id}_A = \mathrm{id}_{FA}$
- $F(g \circ f) = Fg \circ Ff$

Definition 4.2 (Identity Functor)**.** For any category $\mathcal{C}$, the *identity functor* $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined by

$$1_{\mathcal{C}} A = A$$
$$1_{\mathcal{C}} f = f$$

Definition 4.3 (Constant Functor)**.** Given categories $\mathcal{C}$, $\mathcal{D}$ and an object $D \in \mathcal{D}$, the *constant functor* $K^{\mathcal{C}} D : \mathcal{C} \to \mathcal{D}$ is the functor defined by

$$K^{\mathcal{C}} DC = D$$
$$K^{\mathcal{C}} Df = \mathrm{id}_D$$

Definition 4.4 (Composition of Functors)**.** Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, define the *composite* functor $G \circ F : \mathcal{C} \to \mathcal{E}$ by

$$(G \circ F)A = G(FA)$$
$$(G \circ F)f = G(Ff)$$

Definition 4.5 (Category of Categories)**.** For any universe $\mathcal{U}$, let $\mathbf{Cat}_{\mathcal{U}}$ be the category of categories whose sets of objects and morphisms are in $\mathcal{U}$, and functors.

Definition 4.6 (Faithful)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* iff, for any objects $A, B \in \mathcal{C}$ and morphisms $f, g : A \to B$, if $Ff = Fg$ then $f = g$.

Definition 4.7 (Full)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* iff, for any objects $A, B \in \mathcal{C}$ and morphism $g : FA \to FB$, there exists $f : A \to B$ such that $Ff = g$.

Definition 4.8. Given any category $\mathcal{C}$, the *hom functor* $\mathcal{C}[-,+] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is defined by

$$\begin{aligned} \mathcal{C}[A, B] &\text{ is the set of all morphisms } A \to B \\ \mathcal{C}[f, g](h) &= g \circ h \circ f \end{aligned}$$

Chapter 5

Natural Transformations

Definition 5.1 (Natural Transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\tau : F \Rightarrow G$ consists of, for every object $X \in \mathcal{C}$, a morphism $\tau_X : FX \to GX : \mathcal{D}$ such that, for every morphism $f : X \to Y : \mathcal{C}$, the following diagram commutes.

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \downarrow{\scriptstyle \tau_X} & & \downarrow{\scriptstyle \tau_Y} \\ GX & \xrightarrow{Gf} & GY \end{array}$$

Definition 5.2 (Identity Natural Transformation)**.** For any functor $F : \mathcal{C} \to \mathcal{D}$, define the *identity* natural transformation $\mathrm{id}_F : F \Rightarrow F$ by

$$(\mathrm{id}_F)_X = \mathrm{id}_{FX} \qquad (X \in \mathcal{C}) \ .$$

Definition 5.3 (Composition of Natural Transformations)**.** Given natural transformations $\sigma : F \Rightarrow G$ and $\tau : G \Rightarrow H$, define the *composite* $\tau \bullet \sigma : F \Rightarrow H$ by

$$(\tau \bullet \sigma)_X = \tau_X \circ \sigma_X$$

Definition 5.4. Given categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathbf{Cat}[\mathcal{C}, \mathcal{D}]$ be the category of functors $\mathcal{C} \to \mathcal{D}$ and natural transformations.

Definition 5.5 (Natural Isomorphism)**.** A *natural isomorphism* is an isomorphism in a functor category.

Proposition 5.6. *A natural transformation $\theta : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ is a natural isomorphism if and only if, for all $X \in \mathcal{C}$, the morphism $\theta_X : FX \to GX$ is an isomorphism.*

Proof:
$\langle 1 \rangle 1.$ If θ is a natural isomorphism then every θ_X is iso.
  $\langle 2 \rangle 1.$ Assume: θ is a natural isomorphism with inverse $\theta^{-1} : G \Rightarrow F$.

$\langle 2 \rangle 2$. For all X we have θ_X is iso with inverse $(\theta^{-1})_X$.
$\langle 1 \rangle 2$. If every θ_X is iso then θ is a natural isomorphism.
$\langle 2 \rangle 1$. Assume: Every θ_X is iso.
$\langle 2 \rangle 2$. Let: $\theta^{-1} : G \Rightarrow F$ be the natural transformation defined by $(\theta^{-1})_X = (\theta_X)^{-1}$.
$\langle 3 \rangle 1$. Let: $f : X \to Y : \mathcal{C}$
$\langle 3 \rangle 2$. $\theta_Y \circ Ff = Gf \circ \theta_X$
$\langle 3 \rangle 3$. $Ff \circ {\theta_X}^{-1} = {\theta_Y}^{-1} \circ Gf$
$\langle 2 \rangle 3$. $\theta \bullet \theta^{-1} = \mathrm{id}_G$
$\langle 2 \rangle 4$. $\theta^{-1} \bullet \theta = \mathrm{id}_F$
$\square$

Chapter 6

Adjunctions

Definition 6.1 (Adjunction)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *adjunction* $F \dashv U : \mathcal{C} \rightharpoonup \mathcal{D}$ consists of:

- a functor $F : \mathcal{C} \to \mathcal{D}$, the *left adjoint*;
- a functor $U : \mathcal{D} \to \mathcal{C}$, the *right adjoint*;
- a natural isomorphism $\phi_{XY} : \mathcal{D}[FX, Y] \cong \mathcal{C}[X, UY]$

We write $\check{f}$ for $\phi(f)$ and $\hat{f}$ for $\phi^{-1}(f)$. We call f and $\check{f}$ *adjuncts*.

Definition 6.2 (Unit)**.** The *unit* of an adjunction is the natural transformation

$$\eta_X : X \to UFX$$
$$\eta_X = \phi(\mathrm{id}_{FX})$$

Definition 6.3 (Counit)**.** The *counit* of an adjunction is the natural transformation

$$\epsilon_Y : FUY \to Y$$
$$\epsilon_Y = \phi^{-1}(\mathrm{id}_{UY})$$

Proposition 6.4. *Given $g : FX \to Y$ we have*

$$\phi(g) = Ug \circ \eta_X$$

Proposition 6.5. *Given $f : X \to UY$ we have*

$$\phi^{-1}(f) = \epsilon_Y \circ Ff$$

Proposition 6.6. *Let $U : \mathcal{D} \to \mathcal{C}$. Assume that, for all $X \in \mathcal{C}$, there exists an initial object $(FX, \eta_X : X \to UFX)$ in $\mathrm{id}_{\mathcal{C}} \downarrow U$. Then F extends to a functor $F : \mathcal{C} \to \mathcal{D}$ that is left adjoint to U with unit η.*

Proposition 6.7. *Let $F : \mathcal{C} \to \mathcal{D}$. Assume that, for all $Y \in \mathcal{D}$, there exists a terminal object $(UY, \epsilon_Y : FUY \to Y)$ in $F \downarrow \mathrm{id}_{\mathcal{D}}$. Then U extends to a functor $U : \mathcal{D} \to \mathcal{C}$ that is right adjoint to F with counit ϵ.*

Proposition 6.8 (Uniqueness of Right Adjoints)**.** *If $F \dashv U$ and $F \dashv V$, then there exists a unique natural isomorphism $\alpha : U \cong V$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{D}[FX, Y] \arrow[r] \arrow[dr] & \mathcal{C}[X, UY] \arrow[d] \\ & \mathcal{C}[X, VY] \end{tikzcd}$$

Proposition 6.9 (Uniqueness of Left Adjoints)**.** *If $F \dashv U$ and $G \dashv U$, then there exists a unique natural isomorphism $\alpha : F \cong G$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{D}[FX, Y] \arrow[r] \arrow[d] & \mathcal{C}[X, UY] \\ \mathcal{D}[GX, Y] \arrow[ur] & \end{tikzcd}$$

Chapter 7

Limits and Colimits

Definition 7.1 (Limit)**.** Let $D : \mathcal{J} \to \mathcal{C}$ be a functor. The category of *cones* over D is

$$\mathbf{Cone}(D) = K^{\mathcal{C},\mathcal{J}} \downarrow K^{\mathbf{1},\mathbf{Cat}[\mathcal{J},\mathcal{C}]}D$$

That is, a cone over D consists of an object $C \in \mathcal{C}$ and a natural transformation $KC \Rightarrow D : \mathcal{J} \to \mathcal{C}$.

A *limit* of D is a terminal object in $\mathbf{Cone}(D)$.

Definition 7.2 (Colimit)**.** Let $D : \mathcal{J} \to \mathcal{C}$ be a functor. The category of *cocones* under D is

$$\mathbf{Cocone}(D) = K^{\mathbf{1},\mathbf{Cat}[\mathcal{J},\mathcal{C}]}D \downarrow K^{\mathcal{C},\mathcal{J}}$$

That is, a cocone under D consists of an object $C \in \mathcal{C}$ and a natural transformation $D \Rightarrow KC : \mathcal{J} \to \mathcal{C}$.

A *colimit* of D is an initial object in $\mathbf{Cocone}(D)$.

Theorem 7.3 (Uniqueness of Limits)**.** *If $\{\tau_J : L \to DJ\}$ and $\{\sigma_J : M \to DJ\}$ are limits of D, then there exists a unique isomorphism $\phi : L \cong M$ such that the following diagram commutes*

$$\begin{tikzcd}
L \arrow[r, "\tau_J"] \arrow[d, "\phi"] & DJ \\
M \arrow[ur, "\sigma_J"] &
\end{tikzcd}$$

Theorem 7.4 (Uniqueness of Colimits)**.** *If $\{\tau_J : DJ \to L\}$ and $\{\sigma_J : DJ \to M\}$ are limits of D, then there exists a unique isomorphism $\phi : L \cong M$ such that the following diagram commutes*

$$\begin{tikzcd}
DJ \arrow[r, "\tau_J"] \arrow[dr, "\sigma_J"'] & L \arrow[d, "\phi"] \\
& M
\end{tikzcd}$$

Proposition 7.5. *An initial object in $\mathcal{C}$ is the same as a colimit of the unique functor $\mathbf{0} \to \mathcal{C}$.*

Proposition 7.6. *A terminal object in $\mathcal{C}$ is the same as a limit of the unique functor $\mathbf{0} \to \mathcal{C}$.*

Theorem 7.7. *Let $\mathcal{J}$ and $\mathcal{C}$ be categories. Suppose that $\{\tau_J^D : \lim D \to DJ\}$ is a limit of D for every functor $D : \mathcal{J} \to \mathcal{C}$. Then* lim *extends to a functor $\mathbf{Cat}[\mathcal{J}, \mathcal{C}] \to \mathcal{C}$ that is right adjoint to $K^{\mathcal{C},\mathcal{J}}$.*

Proof: Since by defintion $(KD, \tau^D : K(\lim D) \Rightarrow D)$ is terminal in $K \downarrow \mathrm{id}_{\mathbf{Cat}[\mathcal{J},\mathcal{C}]}$ for all D. □

Theorem 7.8. *Let $\mathcal{J}$ and $\mathcal{C}$ be categories. Suppose that $\{\tau_J^D : DJ \to \operatorname{colim} D\}$ is a colimit of D for every functor $D : \mathcal{J} \to \mathcal{C}$. Then* colim *extends to a functor $\mathbf{Cat}[\mathcal{J}, \mathcal{C}] \to \mathcal{C}$ that is left adjoint to $K^{\mathcal{C},\mathcal{J}}$.*

7.1 Products and Coproducts

Definition 7.9 (Product)**.** Let $\{A_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$.

A *product* $\{\pi_i : P \to A_i\}_{i\in I}$ is a limit of A considered as a functor from the discrete category I to $\mathcal{C}$. The components π_i are called the *projections*.

A *coproduct* $\{\kappa_i : A_i \to C\}_{i\in I}$ is a colimit of A considered as a functor from the discrete category I to $\mathcal{C}$. The components κ_i are called the *coprojections* or *injections*.

Chapter 8

Constructions of Categories

8.1 Opposite Category

Definition 8.1 (Opposite Category)**.** For any category $\mathcal{C}$, the *opposite* category $\mathcal{C}^{\mathrm{op}}$ is the category with the same objects as $\mathcal{C}$ and

$$\mathcal{C}^{\mathrm{op}}[A, B] = \mathcal{C}[B, A]$$

8.2 Product Categories

Theorem 8.2. *Let $\{\mathcal{C}_i\}_{i \in I}$ be a family of categories in $\mathbf{Cat}_{\mathcal{U}}$, where $I \in \mathcal{U}$. Then there is a product $\prod_i \mathcal{C}_i$ in $\mathbf{Cat}_{\mathcal{U}}$ with:*

- *objects the families of objects $\{X_i \in \mathcal{C}_i\}_{i \in I}$*
- *morphisms the families of morphisms $\{f_i : X_i \to Y_i\}_{i \in I}$.*

8.3 Arrow Category

Definition 8.3 (Arrow Category)**.** For any category $\mathcal{C}$, the *arrow category* $\mathcal{C}^{\to}$ has:

- objects all triples (A, B, f) where $A, B \in \mathcal{C}$ and $f : A \to B : \mathcal{C}$
- morphisms $(u, v) : (A, B, f) \to (C, D, g)$ all pairs $(u : A \to C, v : B \to D)$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle v} \\ C & \xrightarrow{g} & D \end{array}$$

We have the domain and codomain functors $\mathrm{dom} : \mathcal{C}^{\to} \to \mathcal{C}$ and $\mathrm{cod} : \mathcal{C}^{\to} \to \mathcal{C}$ given by

$$\begin{aligned} \mathrm{dom}(A, B, f) &= A & \qquad \mathrm{cod}(A, B, f) &= B \\ \mathrm{dom}(u, v) &= u & \qquad \mathrm{cod}(u, v) &= v \end{aligned}$$

8.4 Slice Category

Definition 8.4 (Slice Category)**.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *slice category* $\mathcal{C}/A$ is the category with

- objects all pairs (B, f) such that $f : B \to A$
- morphisms $u : (B, f) \to (C, g)$ all morphisms $u : B \to C$ such that $g \circ u = f$.

Example 8.5. id_A is terminal in $\mathcal{C}/A$.

Definition 8.6 (Coslice Category)**.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. The *coslice category* $A\backslash\mathcal{C}$ is the category with

- objects all pairs (B, f) such that $f : A \to B$
- morphisms $u : (B, f) \to (C, g)$ all morphisms $u : B \to C$ such that $u \circ f = g$.

Example 8.7. id_A is initial in $A\backslash\mathcal{C}$.

Example 8.8. The *category of pointed sets* $\mathbf{Set}_*$ is $\mathbf{1}\backslash\mathbf{Set}$.

Chapter 9

Preorders

9.1 Definition

Definition 9.1 (Thin Category)**.** A category $\mathcal{C}$ is *thin* or a *preorder* iff, for any objects A and B, there is at most one morphism $A \to B$. We write $A \leq B$ iff there exists a morphism $A \to B$; this is called the *ordering relation* on $\mathcal{C}$.

Proposition 9.2. *For any preorder $\mathcal{C}$, the relation $\leq$ is reflexive and transitive. Conversely, given any class A and relation $\leq$ on A that is reflexive and transitive, there exists a preorder A with class of objects A, unique up to unique isomorphism that is the identity on objects, such that $\leq$ is the ordering relation on A.*

PROOF: All parts are immediate from definitions. □

Proposition 9.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be preorders and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is* monotone*: for all $x, y \in \mathcal{C}$, if $x \leq y$ then $F(x) \leq F(y)$.*

Conversely, given any monotone function f from the objects of $\mathcal{C}$ to the objects of $\mathcal{D}$, there exists a unique functor whose action on objects is f.

PROOF: Immediate from definitions. □

Example 9.4 (Discrete Category)**.** For any set A, the *discrete* category A is the preorder with objects the elements of A and order relation $=$.

Example 9.5. For any ordinal α, let α be the preorder $\{\beta : \beta < \alpha\}$ under $\leq$.

Definition 9.6. A *meet*, *infimum* or *greatest lower bound* is a product in a preorder. That is, a meet of $\{a_i\}_{i \in I}$ is an element $\bigwedge_i a_i$ such that:

- $\bigwedge_i a_i \leq a_i$ for all i
- if $x \leq a_i$ for all i then $x \leq \bigwedge_i a_i$

Definition 9.7. A *join*, *supremum* or *least upper bound* is a coproduct in a preorder. That is, a join of $\{a_i\}_{i \in I}$ is an element $\bigvee_i a_i$ such that:

- $a_i \leq \bigvee_i a_i$ for all i
- if $a_i \leq x$ for all i then $\bigvee_i a_i \leq x$

9.2 Partial Orders

9.2.1 Definition

Definition 9.8 (Partial Order)**.** A *partial order*, *partially ordered set* or *poset* is a preorder such that, for any x and y, if $x \leq y$ and $y \leq x$ then $x = y$.

Example 9.9. Every discrete category is a poset.

Example 9.10. For any ordinal α, the preorder α is a poset.

Definition 9.11 (Category of Posets)**.** Let **Pos** be full subcategory of **Cat** whose objects are the posets.

Chapter 10

Objects

10.1 Terminal and Initial Objects

Definition 10.1 (Terminal Object)**.** Let $\mathcal{C}$ be a category. An object $T \in \mathcal{C}$ is *terminal* iff, for every object X, there is exactly one morphism $X \to T$.

Example 10.2.

- The terminal objects in **Set** are the singletons.
- **1** is terminal in **Cat**.

Definition 10.3 (Initial Object)**.** Let $\mathcal{C}$ be a category. An object $I \in \mathcal{C}$ is *initial* iff, for every object X, there is exactly one morphism $I \to X$.

Example 10.4.

- The empty set is the initial object in **Set**.
- **0** is initial in **Cat**.

Part II

Number Theory

Definition 10.5 (Partition)**.** A *partition* of a natural number n is a nonincreasing sequence of positive integers whose sum is n.

Part III

Order Theory

Chapter 11

Boolean Algebras

Definition 11.1 (Boolean Algebra)**.** A *Boolean algebra* B is a lattice with a function $\neg : B \to B$ such that, for all $a, b \in B$, we have

$$a \leq \neg b \text{ iff } a \wedge b = \bot$$
$$\neg\neg a = a$$

Example 11.2. For any set A, the power set $\mathcal{P}A$ is a Boolean algebra under inclusion, with $\neg X = A - X$.

Definition 11.3 (Boolean Algebra Homomorphism)**.** Given Boolean algebras B and B', a *Boolean algebra homomorphism* $h : B \to B'$ is a lattice homomorphism such that

$$\forall x \in B. h(\neg x) = \neg h(x) \enspace .$$

Let **BA** be the category of Boolean algebras and Boolean algebra homomorphisms.

Example 11.4. $\mathbf{2} = \mathcal{P}\mathbf{1}$ is initial in **BA**.

Definition 11.5 (Filter)**.** Let B be a Boolean algebra. A *filter* in B is a subset $F \subseteq B$ that is closed upwards and closed under binary meets.

Definition 11.6 (Maximal Filter)**.** A filter F in B is *maximal* or an *ultrafilter* iff $F \neq B$ and, for any filter F', if $F \subseteq F'$ then either $F = F'$ or $F' = B$.

Proposition 11.7. *Let F be a filter in B. Then F is an ultrafilter iff, for all $x \in B$, exactly one of $x \in F$ and $\neg x \in F$ holds.*

Proof:
$\langle 1 \rangle 1.$ If F is an ultrafilter then, for all $x \in B$, we have either $x \in F$ or $\neg x \in F$.
  $\langle 2 \rangle 1.$ Assume: F is an ultrafilter.
  $\langle 2 \rangle 2.$ Let: $x \in B$
  $\langle 2 \rangle 3.$ Let: $F' = \{y \in B \mid \neg x \vee y \in F\}$
  $\langle 2 \rangle 4.$ $F \subseteq F'$

⟨2⟩5. $F' = F$ or $F' = B$

⟨2⟩6. CASE: $F' = F$

PROOF: We have $x \in F'$ hence $x \in B$.

⟨2⟩7. CASE: $F' = B$

PROOF: We have $\bot \in F'$ hence $\neg x \vee \bot \in F$ and so $\neg x \in F$.

⟨1⟩2. If F is an ultrafilter then we do not have $x \in F$ and $\neg x \in F$.

PROOF: If $x \in F$ and $\neg x \in F$ then $\bot \in F$ hence $F = B$.

⟨1⟩3. If, for all $x \in B$, we have exactly one of $x \in F$ and $\neg x \in F$ holds, then F is an ultrafilter.

⟨2⟩1. ASSUME: For all $x \in B$, we have exactly one of $x \in F$ and $\neg x \in F$ holds.

⟨2⟩2. LET: F' be a filter with $F \subset F'$.

PROVE: $F' = B$

⟨2⟩3. PICK $x \in F' - F$

⟨2⟩4. $\neg x \in F$

⟨2⟩5. $x, \neg x \in F'$

⟨2⟩6. $F' = B$

□

Part IV

Group Theory

Chapter 12

Semigroups

Definition 12.1 (Semigroup)**.** A *semigroup* consists of a set S and an associative binary operation $\cdot$ on S.

Definition 12.2 (Unit)**.** Let S be a semigroup. An element $e \in S$ is a *unit* iff $\forall x \in S. xe = ex = x$.

Chapter 13

Monoids

Definition 13.1 (Monoid)**.** A *monoid* is a category with one object.

Proposition 13.2. *Let M be a monoid with object $*$. Then the set of morphisms $M[*,*]$ is a semigroup with a unit. Conversely, given any semigroup with a unit M, there exists a monoid, unique up to isomorphism that is the identity on morphisms, such that the morphisms are the elements of M with composition given by the semigroup operation.*

PROOF: Immediate from definitions. $\square$

Definition 13.3 (Monoid Homomorphism)**.** A *monoid homomorphism* is a functor between monoids.

Proposition 13.4. *The monoid homomorphisms are exactly the semigroup homomorphisms that preserve the unit.*

PROOF: Immediate from definitions. $\square$

Example 13.5. $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ are monoids under addition.

Example 13.6. For any category $\mathcal{C}$ and object $A \in \mathcal{C}$, the full subcategory of $\mathcal{C}$ with only one object A is a monoid. We write $\mathcal{C}[A, A]$ for this monoid.

Definition 13.7. Let **Mon** be the full subcategory of **Cat** whose objects are the monoids.

Let U : **Mon** $\rightarrow$ **Cat** be the functor that maps a monoid to its *underlying* set of morphisms.

13.1 Free Monoids

Theorem 13.8. *Let A be a set. Then there exists a monoid $F(A)$ and function $i : A \rightarrow UF(A)$ such that, for any monoid M and function $f : A \rightarrow UM$, there exists a unique monoid homomorphism $\overline{f} : FA \rightarrow M$ such that*

$$f = U\overline{f} \circ i$$

Proof: Take FA to be the set of all finite sequences in A under concatenation. $\square$

Definition 13.9. We call FA the *free* monoid on A.

Chapter 14

Groups

Definition 14.1 (Group)**.** Let $\mathcal{C}$ be a category with finite products. A *group (object)* in $\mathcal{C}$ consists of an object $G \in \mathcal{C}$ and morphisms

$$m : G^2 \to G, e : 1 \to G, i : G \to G$$

such that the following diagrams commute.

$$\begin{tikzcd}
G^3 \arrow[r, "m\times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times m"] & G^2 \arrow[d, "m"] \\
G^2 \arrow[r, "m"] & G
\end{tikzcd}$$

$$\begin{tikzcd}
1 \times G \arrow[r, "e\times \mathrm{id}_G"] \arrow[dr, "\cong"] & G^2 \arrow[d, "m"] \\
& G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times 1 \arrow[r, "\mathrm{id}_G \times e"] \arrow[dr, "\cong"] & G^2 \arrow[d, "m"] \\
& G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "\Delta"] \arrow[d] & G^2 \arrow[r, "\mathrm{id}_G \times i"] & G^2 \arrow[d, "m"] \\
1 \arrow[rr, "e"] & & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\Delta"] \arrow[d] & G^2 \arrow[r, "i \times \mathrm{id}_G"] & G^2 \arrow[d, "m"] \\
1 \arrow[rr, "e"] & & G
\end{tikzcd}$$

Proposition 14.2. *A group in* **Set** *is exactly (the set of morphisms of) a monoid that is also a groupoid.*

Proof: Immediate from definitions. □

Proposition 14.3. *The inverse of an element is unique.*

Proof: If i and j are inverses of x then $i = ixj = j$. □

Example 14.4.

- The *trivial* group is $\{e\}$ under $ee = e$.
- $\mathbb{Z}$ is a group under addition

- $\mathbb{Q}$ is a group under addition
- $\mathbb{Q} - \{0\}$ is a group under multiplication
- $\mathbb{R}$ is a group under addition
- $\mathbb{R} - \{0\}$ is a group under multiplication
- $\mathbb{C}$ is a group under addition
- $\mathbb{C} - \{0\}$ is a group under multiplication
- $\{-1, 1\}$ is a group under multiplication
- For any category $\mathcal{C}$ and object $A \in \mathcal{C}$, we have $\mathrm{Aut}_{\mathcal{C}}(A)$ is a group under $gf = f \circ g$.

  For A a set, we call $S_A = \mathrm{Aut}_{\mathbf{Set}}(A)$ the *symmetric group* or *group of permutations* of A.
- For $n \geq 3$, the *dihedral group* D_{2n} consists of the set of rigid motions that map the regular n-gon onto itself under composition.
- Let $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ under matrix multiplication.
- The quaternionic group Q_8 is the group

$$\{1, -1, i, -i, j, -j, k, -k\}$$

  with multiplication table

$$\begin{array}{cccccccc}
1 & -1 & i & -i & j & -j & k & -k \\
-1 & 1 & -i & i & -j & j & -k & k \\
i & -i & -1 & 1 & k & -k & -j & j \\
-i & i & 1 & -1 & -k & k & j & -j \\
j & -j & -k & k & -1 & 1 & i & -i \\
-j & j & k & -k & 1 & -1 & -i & i \\
k & -k & j & -j & -i & i & -1 & 1 \\
-k & k & -j & j & i & -i & 1 & -1
\end{array}$$

Example 14.5.

- The only group of order 1 is the trivial group.
- The only group of order 2 is $\mathbb{Z}_2$.
- The only group of order 3 is $\mathbb{Z}_3$.
- There are exactly two groups of order 4: $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ under $(a, b)(c, d) = (ac, bd)$.

Proposition 14.6 (Cancellation)**.** *Let G be a group. Let $a, g, h \in G$. If $ag = ah$ or $ga = ha$ then $g = h$.*

Proof: If $ag = ah$ then $g = a^{-1}ag = a^{-1}ah = h$. Similarly if $ga = ha$. □

Proposition 14.7. *Let G be a group and $g, h \in G$. Then $(gh)^{-1} = h^{-1}g^{-1}$.*

Proof: Since $ghh^{-1}g^{-1} = e$. □

Definition 14.8. Let G be a group. Let $g \in G$. We define $g^n \in G$ for all $n \in \mathbb{Z}$ as follows:

$$\begin{aligned} g^0 &= e \\ g^{n+1} &= g^n g && (n \geq 0) \\ g^{-n} &= (g^{-1})^n && (n > 0) \end{aligned}$$

Proposition 14.9. *Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then*

$$g^{m+n} = g^m g^n \quad .$$

Proof:
⟨1⟩1. For all $k \in \mathbb{Z}$ we have $g^{k+1} = g^k g$
⟨2⟩1. For all $k \geq 0$ we have $g^{k+1} = g^k g$
Proof: Immediate from definition.
⟨2⟩2. $g^{-1+1} = g^{-1}g$
Proof: Both are equal to e.
⟨2⟩3. For all $k > 1$ we have $g^{-k+1} = g^{-k}g$
Proof:

$$\begin{aligned} g^{-k+1} &= (g^{-1})^{k-1} \\ &= (g^{-1})^{k-1}g^{-1}g \\ &= (g^{-1})^k g \\ &= g^{-k}g \end{aligned}$$

⟨1⟩2. For all $k \in \mathbb{Z}$ we have $g^{k-1} = g^k g^{-1}$
Proof: Substitute $k = k - 1$ above and multiply by g^{-1}.
⟨1⟩3. $g^{m+0} = g^m g^0$
Proof: Since $g^m g^0 = g^m e = g^m$.
⟨1⟩4. If $g^{m+n} = g^m g^n$ then $g^{m+n+1} = g^m g^{n+1}$
Proof:

$$\begin{aligned} g^{m+n+1} &= g^{m+n}g && (\langle 1 \rangle 1) \\ &= g^m g^n g \\ &= g^m g^{n+1} && (\langle 1 \rangle 1) \end{aligned}$$

⟨1⟩5. If $g^{m+n} = g^m g^n$ then $g^{m+n-1} = g^m g^{n-1}$
Proof:

$$\begin{aligned} g^{m+n-1}g &= g^{m+n} && (\langle 1 \rangle 1) \\ &= g^m g^n \\ \therefore g^{m+n-1} &= g^m g^n g^{-1} \\ &= g^m g^{n-1} && (\langle 1 \rangle 2) \end{aligned}$$

∎

Proposition 14.10. *Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then*

$$(g^m)^n = g^{mn} \quad .$$

PROOF:
⟨1⟩1. $(g^m)^0 = g^0$
PROOF: Both sides are equal to e.
⟨1⟩2. If $(g^m)^n = g^{mn}$ then $(g^m)^{n+1} = g^{m(n+1)}$.
PROOF:

$$\begin{aligned}(g^m)^{n+1} &= (g^m)^n g^m && \text{(Proposition 14.9)}\\ &= g^{mn} g^m \\ &= g^{mn+m} && \text{(Proposition 14.9)}\end{aligned}$$

⟨1⟩3. If $(g^m)^n = g^{mn}$ then $(g^m)^{n-1} = g^{m(n-1)}$.
PROOF:

$$\begin{aligned}(g^m)^n &= g^{mn} \\ \therefore (g^m)^{n-1} g^m &= g^{mn-m} g^m && \text{(Proposition 14.9)}\\ \therefore (g^m)^{n-1} &= g^{mn-m} && \text{(Cancellation)}\end{aligned}$$

∎

Definition 14.11 (Commute)**.** Let G be a group and $g, h \in G$. We say g and h *commute* iff $gh = hg$.

Definition 14.12. Let G be a group. Given $g \in G$ and $A \subseteq G$, we define

$$gA = \{ga : a \in A\}, \qquad Ag = \{ag : a \in A\} \quad .$$

Given sets $A, B \subseteq G$, we define

$$AB = \{ab : a \in A, b \in B\} \quad .$$

14.1 Symmetric Groups

Definition 14.13. Let n be a natural number and $a_1, \ldots, a_r \in \{1, \ldots, n\}$ be distinct. The *cycle* or *r-cycle*

$$(a_1 \ a_2 \ \cdots \ a_r) \in S_n$$

is the permutation that sends a_i to a_{i+1} $(1 \leq i < r)$ and a_r to a_1.

We call r the *length* of the cycle.

A *transposition* is a 2-cycle.

Proposition 14.14. *Disjoint cycles commute.*

PROOF: Easy. ∎

Proposition 14.15. *For any cycle $(a_1 \ a_2 \ \cdots \ a_r)$ in S_n and $\tau \in S_n$ we have*

$$\tau (a_1 \ a_2 \ \cdots \ a_n) \tau^{-1} = (\tau(a_1) \ \tau(a_2) \ \cdots \ \tau(a_n)) \quad .$$

PROOF: Easy. ∎

14.2 Order of an Element

Definition 14.16 (Order)**.** Let G be a group. Let $g \in G$. Then g has *finite order* iff there exists a positive integer n such that $g^n = e$. In this case, the *order* of g, denoted $|g|$, is the least positive integer n such that $g^n = e$.

If g does not have finite order, we write $|g| = \infty$.

Proposition 14.17. *Let G be a group. Let $g \in G$ and n be a positive integer. If $g^n = e$ then $|g| \mid n$.*

PROOF:
$\langle 1 \rangle 1$. LET: $n = q|g| + d$ where $0 \leq d < |g|$
PROOF: Division Algorithm.
$\langle 1 \rangle 2$. $g^d = e$
PROOF:

$$\begin{aligned} e &= g^n \\ &= g^{q|g|+d} \\ &= (g^{|g|})^q g^d && \text{(Propositions 14.9, 14.10)} \\ &= e^q g^d \\ &= g^d \end{aligned}$$

$\langle 1 \rangle 3$. $d = 0$
PROOF: By minimality of $|g|$.
$\langle 1 \rangle 4$. $n = q|g|$
□

Corollary 14.17.1. *Let G be a group. Let $g \in G$ have finite order and $n \in \mathbb{Z}$. Then $g^n = e$ if and only if $|g| \mid n$.*

Proposition 14.18. *Let G be a group and $g \in G$. Then $|g| \leq |G|$.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: w.l.o.g. G is finite.
$\langle 1 \rangle 2$. PICK i, j with $0 \leq i < j \leq |G|$ such that $g^i = g^j$.
PROOF: Otherwise $g^0, g^1, \ldots, g^{|G|}$ would be $|G| + 1$ distinct elements of G.
$\langle 1 \rangle 3$. $g^{j-i} = e$
$\langle 1 \rangle 4$. g has finite order and $|g| \leq |G|$
PROOF: Since $|g| \leq j - i \leq j \leq |G|$.
□

Proposition 14.19. *Let G be a group. Let $g \in G$ have finite order. Let $m \in \mathbb{N}$. Then*

$$|g^m| = \frac{\operatorname{lcm}(m, |g|)}{m} = \frac{|g|}{\gcd(m, |g|)}$$

Proof: Since for any integer d we have

$$\begin{aligned} g^{md} = e &\Leftrightarrow |g| \mid md && \text{(Corollary 14.17.1)} \\ &\Leftrightarrow \operatorname{lcm}(m, |g|) \mid md \\ &\Leftrightarrow \frac{\operatorname{lcm}(m, |g|)}{m} \mid d \end{aligned}$$ □

and so $|g^m| = \frac{\operatorname{lcm}(m,|g|)}{m}$ by Corollary 14.17.1. □

Corollary 14.19.1. *If g has odd order then $|g^2| = |g|$.*

Proposition 14.20. *Let G be a group. Let $g, h \in G$ have finite order. Assume $gh = hg$. Then $|gh|$ has finite order and*

$$|gh| \mid \operatorname{lcm}(|g|, |h|)$$

Proof: Since $(gh)^{\operatorname{lcm}(|g|,|h|)} = g^{\operatorname{lcm}(|g|,|h|)} h^{\operatorname{lcm}(|g|,|h|)} = e$. □

Example 14.21. This example shows that we cannot remove the hypothesis that $gh = hg$.

In $\mathrm{GL}_2(\mathbb{R})$, take

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} .$$

Then $|g| = 4$, $|h| = 3$ and $|gh| = \infty$.

Proposition 14.22. *Let G be a group and $g, h \in G$ have finite order. If $gh = hg$ and* $\gcd(|g|, |h|) = 1$ *then $|gh| = |g||h|$.*

Proof:
$\langle 1 \rangle 1$. Let: $N = |gh|$
$\langle 1 \rangle 2$. $g^N = (h^{-1})^N$
$\langle 1 \rangle 3$. $g^{N|g|} = e$
$\langle 1 \rangle 4$. $|g^N| \mid |g|$
$\langle 1 \rangle 5$. $h^{-N|h|} = e$
$\langle 1 \rangle 6$. $|g^N| \mid |h|$
$\langle 1 \rangle 7$. $|g^N| = 1$
  Proof: Since $\gcd(|g|, |h|) = 1$.
$\langle 1 \rangle 8$. $g^N = e$
$\langle 1 \rangle 9$. $|g| \mid N$
$\langle 1 \rangle 10$. $h^{-N} = e$
$\langle 1 \rangle 11$. $|h| \mid N$
$\langle 1 \rangle 12$. $N = |g||h|$
  Proof: Using Proposition 14.20.
□

Proposition 14.23. *Let G be a finite group. Assume there is exactly one element $f \in G$ of order 2. Then the product of all the elements of G is f.*

PROOF: Let the elements of G be $g_1, g_2, \ldots, g_n$. Apart from e and f, every element and its inverse are distinct elements of the list. Hence the product of the list is $ef = f$. □

Proposition 14.24. *Let G be a finite group of order n. Let m be the number of elements of G of order 2. Then $n - m$ is odd.*

PROOF: In the list of all elements that are not of order 2, every element and its inverse are distinct except for e. Hence the list has odd length. □

Corollary 14.24.1. *If a finite group has even order, then it contains an element of order 2.*

Proposition 14.25. *Let G be a group and $a, g \in G$. Then $|aga^{-1}| = |g|$.*

PROOF: Since

$$\begin{aligned}(aga^{-1})^n = e &\Leftrightarrow ag^n a^{-1} = e \\ &\Leftrightarrow g^n = e\end{aligned}$$

□

Proposition 14.26. *Let G be a group and $g, h \in G$. Then $|gh| = |hg|$.*

PROOF: Since $|gh| = |ghgg^{-1}| = |hg|$. □

Proposition 14.27. *Let G be a group of order n. Let k be relatively prime to n. Then every element in G has the form x^k for some x.*

⟨1⟩1. PICK integers a and b such that $an + bk = 1$.
⟨1⟩2. LET: $g \in G$
⟨1⟩3. $g = (g^b)^k$
PROOF:

$$\begin{aligned}g &= g.(g^n)^{-a} && (g^n = e) \\ &= g^{1-an} \\ &= g^{bk}\end{aligned}$$

□

14.3 Generators

Definition 14.28 (Generator)**.** Let G be a group and $a \in G$. We say *a generates* the group iff, for all $x \in G$, there exists an integer n such that $x^n = a$.

Example 14.29. $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

PROOF:
⟨1⟩1. LET: $H = \langle s, t \rangle$
⟨1⟩2. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in H$.

PROOF: It is t^q.

⟨1⟩3. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \in H$.

PROOF:

$$\begin{aligned} st^{-q}s^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \end{aligned}$$

⟨1⟩4.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & qa+b \\ c & qc+d \end{pmatrix}$$

⟨1⟩5.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} a+qb & b \\ c+qd & d \end{pmatrix}$$

⟨1⟩6. For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, if c and d are both nonzero, then there exists $N \in H$ such that the bottom row of MN has one entry the same as M and one entry with smaller absolute value.

PROOF: From ⟨1⟩4 and ⟨1⟩5 taking $q = -1$.

⟨1⟩7. For any $M \in \mathrm{SL}_2(\mathbb{Z})$, there exists $N \in H$ such that MN has a zero on the bottom row.

PROOF: Apply ⟨1⟩6 repeatedly.

⟨1⟩8. Any matrix in $\mathrm{SL}_2(\mathbb{Z})$ with a zero on the bottom row is in H.

⟨2⟩1. $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H$

PROOF: ⟨1⟩2

⟨2⟩2. $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \in H$

PROOF: It is $s^2 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ since $s^2 = -I$.

⟨2⟩3. $\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} \in H$

PROOF: It is $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} s$.

⟨2⟩4. $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \in H$

PROOF: It is $s^2 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s$.

⟨1⟩9. Every matrix in $\mathrm{SL}_2(\mathbb{Z})$ is in H.

□

14.4 p-groups

Definition 14.30 (p-group)**.** Let p be a prime. A *p-group* is a finite group whose order is a power of p.

Chapter 15

Group Homomorphisms

Definition 15.1 (Homomorphism)**.** Let G and H be groups. A *(group) homomorphism* $\phi : G \to H$ is a function such that, for all $x, y \in G$,

$$\phi(xy) = \phi(x)\phi(y) \ .$$

Proposition 15.2. *Let G and H be groups with identities e_G and e_H. Let $\phi : G \to H$ be a group homomorphism. Then $\phi(e_G) = e_H$.*

Proof: Since $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$ and so $\phi(e_G) = e_H$ by Cancellation. □

Proposition 15.3. *Let $\phi : G \to H$ be a group homomorphism. For all $x \in G$ we have $\phi(x^{-1}) = \phi(x)^{-1}$.*

Proof: Since $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_H$. □

Proposition 15.4. *Let G, H and K be groups. If $\phi : G \to H$ and $\psi : H \to K$ are homomorphisms then $\psi \circ \phi : G \to K$ is a homomorphism.*

Proof: For $x, y \in G$ we have

$$\psi(\phi(xy)) = \psi(\phi(x)\phi(y)) = \psi(\phi(x))\psi(\phi(y)) \ .$$

Proposition 15.5. *Let G be a group. Then $\mathrm{id}_G : G \to G$ is a group homomorphism.*

Proof: For $x, y \in G$ we have $\mathrm{id}_G(xy) = xy = \mathrm{id}_G(x)\mathrm{id}_G(y)$. □

Proposition 15.6. *Let $\phi : G \to H$ be a group homomorphism. Let $g \in G$ have finite order. Then $|\phi(g)|$ divides $|g|$.*

Proof: Since $\phi(g)^{|g|} = \phi(g^{|g|}) = e$. □

Definition 15.7 (Category of Groups)**.** Let **Grp** be the category of groups and group homomorphisms.

Example 15.8. The trivial group 1 is both initial and terminal in **Grp**.

Example 15.9. There are 49487365402 groups of order 1024 up to isomorphism.

Proposition 15.10. *A group homomorphism $\phi : G \to H$ is an isomorphism in* **Grp** *if and only if it is bijective.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: ϕ is bijective.
PROVE: ϕ^{-1} is a group homomorphism.
$\langle 1 \rangle 2$. LET: $h, h' \in H$
$\langle 1 \rangle 3$. $\phi(\phi^{-1}(hh')) = \phi(\phi^{-1}(h)\phi^{-1}(h'))$
PROOF: Both are equal to hh'.
$\langle 1 \rangle 4$. $\phi^{-1}(hh') = \phi^{-1}(h)\phi^{-1}(h')$
□

Corollary 15.10.1.

$$D_6 \cong C_3$$

PROOF: The canonical homomorphism $D_6 \to C_3$ is bijective. □

Corollary 15.10.2.

$$(\mathbb{R}, +) \cong (\{x \in \mathbb{R} : x > 0\}, \cdot)$$

PROOF: The function that maps x to e^x is a bijective homomorphism. □

Proposition 15.11. *The trivial group is the zero object in* **Grp**.

PROOF: For any group G, the unique function $G \to \{e\}$ is a group homomorphism, and the only group homomorphism $\{e\} \to G$ maps e to e_G. □

Proposition 15.12. *For any groups G and H, the set $G \times H$ under $(g,h)(g',h') = (gg',hh')$ is the product of G and H in* **Grp**.

PROOF:
$\langle 1 \rangle 1$. $G \times H$ is a group.
$\langle 2 \rangle 1$. The multiplication is associative.
PROOF: Since $(g_1,h_1)((g_2,h_2)(g_3,h_3)) = ((g_1,h_1)(g_2,h_2))(g_3,h_3) = (g_1g_2g_3, h_1h_2h_3)$.
$\langle 2 \rangle 2$. (e_G, e_H) is the identity.
PROOF: Since $(g,h)(e_G,e_H) = (e_G,e_H)(g,h) = (g,h)$.
$\langle 2 \rangle 3$. The inverse of (g,h) is (g^{-1},h^{-1}).
PROOF: Since $(g,h)(g^{-1},h^{-1}) = (g^{-1},h^{-1})(g,h) = (e_G,e_H)$.
$\langle 1 \rangle 2$. $\pi_1 : G \times H \to G$ is a group homomorphism.
PROOF: Immediate from definitions.
$\langle 1 \rangle 3$. $\pi_2 : G \times H \to H$ is a group homomorphism.
PROOF: Immediate from definitions.
$\langle 1 \rangle 4$. For any group homomorphism $\phi : K \to G$ and $\psi : K \to H$, the function $\langle \phi, \psi \rangle : K \to G \times H$ where $\langle \phi, \psi \rangle(k) = (\phi(k), \psi(k))$ is a group homomorphism.

PROOF:

$$\begin{aligned}\langle\phi,\psi\rangle(kk') &= (\phi(kk'),\psi(kk')) \\ &= (\phi(k)\phi(k'),\psi(k)\psi(k')) \\ &= (\phi(k),\psi(k))(\phi(k'),\psi(k')) \\ &= \langle\phi,\psi\rangle(k)\langle\phi,\psi\rangle(k')\end{aligned}$$

□

15.1 Subgroups

Definition 15.13 (Subgroup)**.** Let $(G,\cdot)$ and $(H,*)$ be groups such that H is a subset of G. Then H is a *subgroup* of G iff the inclusion $i : H \hookrightarrow G$ is a group homomorphism.

Proposition 15.14. *If $(H,*)$ is a subgroup of $(G,\cdot)$ then $*$ is the restriction of $\cdot$ to H.*

PROOF: Given $x, y \in H$ we have

$$x * y = i(x * y) = i(x) \cdot i(y) = x \cdot y \ . \qquad \square$$

Example 15.15. For any group G we have $\{e\}$ is a subgroup of G.

Proposition 15.16. *Let G be a group. Let H be a subset of G. Then H is a subgroup of G iff H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$.*

PROOF:
$\langle 1\rangle 1$. If H is a subgroup of G then H is nonempty.
PROOF: Since every group has an identity element and so is nonempty.
$\langle 1\rangle 2$. If H is a subgroup of G then, for all $x, y \in H$, we have $xy^{-1} \in H$.
PROOF: Easy.
$\langle 1\rangle 3$. If H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$, then H is a subgroup of G.
$\langle 2\rangle 1$. ASSUME: H is nonempty.
$\langle 2\rangle 2$. ASSUME: $\forall x, y \in H. xy^{-1} \in H$
$\langle 2\rangle 3$. $e \in H$
PROOF: Pick $x \in H$. We have $e = xx^{-1} \in H$.
$\langle 2\rangle 4$. $\forall x \in H. x^{-1} \in H$
PROOF: Given $x \in H$ we have $x^{-1} = ex^{-1} \in H$.
$\langle 2\rangle 5$. H is closed under the restriction of $\cdot$
PROOF: Given $x, y \in H$ we have $xy = x(y^{-1})^{-1} \in H$.
$\langle 2\rangle 6$. H is a group under the restriction of $\cdot$
PROOF: Associativity is inherited from G and the existence of an identity element and inverses follows from $\langle 2\rangle 3$ and $\langle 2\rangle 4$.
$\langle 2\rangle 7$. The inclusion $H \hookrightarrow G$ is a group homomorphism.
PROOF: For $x, y \in H$ we have $i(xy) = i(x)i(y) = xy$.

□

Corollary 15.16.1. *The intersection of a set of subgroups of G is a subgroup of G.*

Corollary 15.16.2. *Let $\phi : G \to H$ be a group homomorphism. Let K be a subgroup of H. Then $\phi^{-1}(K)$ is a subgroup of G.*

PROOF:
$\langle 1\rangle 1.$ $\phi^{-1}(K)$ is nonempty.
PROOF: Since $e \in \phi^{-1}(K)$.
$\langle 1\rangle 2.$ LET: $x, y \in \phi^{-1}(K)$
$\langle 1\rangle 3.$ $\phi(x), \phi(y) \in K$
$\langle 1\rangle 4.$ $\phi(x)\phi(y)^{-1} \in K$
$\langle 1\rangle 5.$ $\phi(xy^{-1}) \in K$
$\langle 1\rangle 6.$ $xy^{-1} \in \phi^{-1}(K)$
$\square$

Corollary 15.16.3. *Let $\phi : G \to H$ be a group homomorphism. Let K be a subgroup of G. Then $\phi(K)$ is a subgroup of H.*

PROOF:
$\langle 1\rangle 1.$ LET: $x, y \in \phi(K)$
$\langle 1\rangle 2.$ PICK $a, b \in K$ such that $x = \phi(a)$ and $y = \phi(b)$
$\langle 1\rangle 3.$ $xy^{-1} = \phi(ab^{-1})$
$\langle 1\rangle 4.$ $xy^{-1} \in \phi(K)$
$\square$

Proposition 15.17. *Let G be a subgroup of $\mathbb{Z}$. Then there exists $d \geq 0$ such that $G = d\mathbb{Z}$.*

PROOF:
$\langle 1\rangle 1.$ ASSUME: w.l.o.g. $G \neq \{0\}$
PROOF: Since $\{0\} = 0\mathbb{Z}$.
$\langle 1\rangle 2.$ LET: d be the least positive element of G.
PROVE: $G = d\mathbb{Z}$
PROOF: If $n \in G$ then $-n \in G$ so G must contain a positive element.
$\langle 1\rangle 3.$ $G \subseteq d\mathbb{Z}$
$\langle 2\rangle 1.$ LET: $n \in G$
$\langle 2\rangle 2.$ LET: q and r be the integers such that $n = qd + r$ and $0 \leq r < d$.
$\langle 2\rangle 3.$ $r \in G$
PROOF: Since $r = n - qd$.
$\langle 2\rangle 4.$ $r = 0$
PROOF: By minimality of d.
$\langle 2\rangle 5.$ $n = qd \in d\mathbb{Z}$
$\langle 1\rangle 4.$ $d\mathbb{Z} \subseteq G$
$\square$

15.2 Kernel

Definition 15.18 (Kernel)**.** Let $\phi : G \to H$ be a group homomorphism. The *kernel* of ϕ is
$$\ker\phi = \{g \in G : \phi(g) = e\} \enspace .$$

Proposition 15.19. *Let $\phi : G \to H$ be a group homomorphism. Then $\ker\phi$ is a subgroup of G.*

PROOF: Corollary 15.16.2. ∎

Proposition 15.20. *Let $\phi : G \to H$ be a group homomorphism. Then the inclusion $i : \ker\phi \hookrightarrow G$ is terminal in the category of pairs $(K, \alpha : K \to G)$ such that $\phi \circ \alpha = 0$.*

PROOF:
⟨1⟩1. $\phi \circ i = 0$
⟨1⟩2. For any group K and homomorphism $\alpha : K \to G$ such that $\phi \circ \alpha = 0$, there exists a unique homomorphism $\beta : K \to \ker\phi$ such that $i \circ \beta = \alpha$.
∎

Proposition 15.21. *Let $\phi : G \to H$ be a group homomorphism. Then the following are equivalent:*

1. *ϕ is monic.*
2. $\ker\phi = \{e\}$
3. *ϕ is injective.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
⟨2⟩1. ASSUME: ϕ is monic.
⟨2⟩2. LET: $i : \ker\phi \hookrightarrow G$, $j : \{e\} \hookrightarrow \ker\phi \hookrightarrow G$ be the inclusions.
⟨2⟩3. $\phi \circ i = \phi \circ j$
⟨2⟩4. $i = j$
⟨1⟩2. $2 \Rightarrow 3$
⟨2⟩1. ASSUME: $\ker\phi = \{e\}$
⟨2⟩2. LET: $x, y \in G$
⟨2⟩3. ASSUME: $\phi(x) = \phi(y)$
⟨2⟩4. $\phi(xy^{-1}) = e$
⟨2⟩5. $xy^{-1} \in \ker\phi$
⟨2⟩6. $xy^{-1} = e$
⟨2⟩7. $x = y$
⟨1⟩3. $3 \Rightarrow 1$
PROOF: Easy.
∎

Proposition 15.22. *A group homomorphism is an epimorphism if and only if it is surjective.*

15.3 Inner Automorphisms

Proposition 15.23. *Let G be a group and $g \in G$. The function $\gamma_g : G \to G$ defined by $\gamma_g(a) = gag^{-1}$ is an automorphism on G.*

Proof:
⟨1⟩1. γ_g is a homomorphism.
Proof:

$$\begin{aligned}\gamma_g(ab) &= gabg^{-1} \\ &= gag^{-1}gbg^{-1} \\ &= \gamma_g(a)\gamma_g(b)\end{aligned}$$

⟨1⟩2. γ_g is injective.
Proof: By Cancellation.
⟨1⟩3. γ_g is surjective.
Proof: Given $b \in G$, we have $\gamma_g(g^{-1}bg) = b$.
□

Definition 15.24 (Inner Automorphism)**.** Let G be a group. An *inner automorphism* on G is a function of the form $\gamma_g(a) = gag^{-1}$ for some $g \in G$.

We write $\mathrm{Inn}(G)$ for the set of inner automorphisms of G.

Proposition 15.25. *Let G be a group. The function $\gamma : G \to \mathrm{Aut}_{\mathbf{Grp}}(G)$ that maps g to γ_g is a group homomorphism.*

Proof: Since $\gamma_{gh}(a) = ghah^{-1}g^{-1} = \gamma_g(\gamma_h(a))$. □

Corollary 15.25.1. $\mathrm{Inn}(G)$ *is a subgroup of* $\mathrm{Aut}_{\mathbf{Grp}}(G)$*.*

15.4 Semidirect Products

Definition 15.26 (Semidirect Product)**.** Let N and H be groups. Let $\theta : H \to \mathrm{Aut}_{\mathbf{Grp}}(N)$ be a homomorphism. The *semidirect product* $N \rtimes_\theta H$ is the group $N \times H$ under

$$(n_1, h_1)(n_2, h_2) = (n_1\theta(h_1)(n_2), h_1h_2)$$

If N and H are subgroups of a group G, we write $N \rtimes H$ for $N \rtimes_\theta H$ where $\theta(n)(h) = nhn^{-1}$.

We prove that this is a group.

Proof:
⟨1⟩1. Associativity
Proof:

$$\begin{aligned}(n_1, h_1)((n_2, h_2)(n_3, h_3)) &= (n_1, h_1)(n_2\theta(h_2)(n_3), h_2h_3) \\ &= (n_1\theta(h_1)(n_2\theta(h_2)(n_3)), h_1h_2h_3) \\ &= (n_1\theta(h_1)(n_2)\theta(h_1h_2)(n_3), h_1h_2h_3) \\ &= (n_1\theta(h_1)(n_2), h_1h_2)(n_3, h_3) \\ &= ((n_1, h_1)(n_2, h_2))(n_3, h_3)\end{aligned}$$

$\langle 1\rangle 2.$ $(e_N, e_H)(n, h) = (n, h)$
PROOF:

$$\begin{aligned}(e_N, e_H)(n, h) &= (e_N\theta(e_H)(n), e_H h)\\ &= (n, h)\end{aligned}$$

$\langle 1\rangle 3.$ $(n, h)(e_N, e_H) = (n, h)$
PROOF:

$$\begin{aligned}(n, h)(e_N, e_H) &= (n\theta(h)(e_N), he_H)\\ &= (n, h)\end{aligned}$$

$\langle 1\rangle 4.$ $(n, h)(\theta(h^{-1})(n^{-1}), h^{-1}) = (e_N, e_H)$
PROOF:

$$\begin{aligned}(n, h)(n^{-1}, h^{-1}) &= (n\theta(h)(\theta(h^{-1})(n^{-1})), hh^{-1})\\ &= (nn^{-1}, hh^{-1})\\ &= (e_N, e_H)\end{aligned}$$

$\langle 1\rangle 5.$ $(\theta(h^{-1})(n^{-1}), h^{-1})(n, h) = (e_N, e_H)$
PROOF:

$$\begin{aligned}(\theta(h^{-1})(n^{-1}), h^{-1})(n, h) &= (\theta(h^{-1})(n^{-1})\theta(h^{-1})(n), h^{-1}h)\\ &= (e_N, e_H)\end{aligned}$$

$\square$

Example 15.27. Let $n > 0$. Let D_{2n} be presented by $(a, b \mid a^n, b^2, (ab)^2)$. Define $\theta : C_2 \to \mathrm{Aut}_{\mathbf{Grp}}(C_n)$ by

$$\theta(1)(i) = n - i$$

Then $\phi : C_n \rtimes_\theta C_2 \cong D_{2n}$ with the isomorphism being given by

$$\phi(i, j) = a^i b^j \qquad (0 \leq i < n, 0 \leq i < 2)\ .$$

Proposition 15.28. *The function $i : N \to N \rtimes_\theta H$ that maps n to (n, e_H) is a group monomorphism.*

PROOF:
$\langle 1\rangle 1.$ $i(nn') = i(n)i(n')$
PROOF:

$$\begin{aligned}i(n)i(n') &= (n, e_H)(n', e_H)\\ &= (n\theta(e_H)(n'), e_H e_H)\\ &= (nn', e_H)\\ &= i(nn')\end{aligned}$$

$\langle 1\rangle 2.$ i is injective.
$\square$

Proposition 15.29. *The function $J : h \to N \rtimes_\theta H$ that maps h to (e_N, h) is a group monomorphism.*

PROOF:
$\langle 1\rangle 1.$ $j(hh') = j(h)j(h')$

PROOF:

$$\begin{aligned} j(h)j(h') &= (e_N, h)(e_N, h') \\ &= (e_N\theta(h)(e_N), hh') \\ &= (e_N, hh') \\ &= j(hh') \end{aligned}$$

⟨1⟩2. i is injective.

□

Proposition 15.30. *The natural projection $N \rtimes_\theta H \to H$ is a surjective group homomorphism with kernel N.*

PROOF: Easy. □

Proposition 15.31. *Let N and H be groups and $\theta : H \to \mathrm{Aut}_{\mathbf{Grp}}(N)$ a homomorphism. Let $G = N \rtimes_\theta H$. Let $i : H \hookrightarrow G$ and $j : N \hookrightarrow G$ be the injections. Then θ is realised by conjugation in G. That is, for all $h \in H$ and $n \in N$ we have*

$$j(\theta(h)(n)) = i(h)j(n)i(h)^{-1}$$

I.e.

$$j \circ \theta(h) = \gamma_{i(h)} \enspace .$$

PROOF:

$$\begin{aligned} i(h)j(n)i(h)^{-1} &= (e_N, h)(n, e_H)(e_N, h)^{-1} \\ &= (e_N, h)(n, e_H)(\theta(h^{-1})(e_N), h^{-1}) \\ &= (e_N, h)(n, e_H)(e_N, h^{-1}) \\ &= (\theta(h)(n), h)(e_N, h^{-1}) \\ &= (\theta(h)(n)\theta(h)(e_N), hh^{-1}) \\ &= (\theta(h)(n), e_H) \\ &= j(\theta(h)(n)) \end{aligned}$$

□

Proposition 15.32. *Let G be a group. Let N and H be subgroups of G with N normal. Assume $N \cap H = \{e\}$ and $G = NH$. Let $\gamma : H \to \mathrm{Aut}_{\mathbf{Grp}}(N)$ be conjugation. Then*

$$G \cong N \rtimes_\gamma H$$

PROOF:

⟨1⟩1. LET: $\phi : N \rtimes_\gamma H \to G$ be the homomorphism

$$\phi(n, h) = nh \enspace .$$

PROOF:

$$\begin{aligned} \phi((n_1, h_1)(n_2, h_2)) &= \phi(n_1\theta(h_1)(n_2), h_1h_2) \\ &= n_1\theta(h_1)(n_2)h_1h_2 \\ &= n_1h_1n_2{h_1}^{-1}h_1h_2 \\ &= n_1h_1n_2h_2 \\ &= \phi(n_1, h_1)\phi(n_2, h_2) \end{aligned}$$

⟨1⟩2. $\ker\phi = \{e\}$
⟨1⟩3. ϕ is surjective.
PROOF: Since $G = NH$.
□

Definition 15.33 (Internal Product)**.** Let G be a group. Let N and H be subgroups of G. Then G is the *internal product* of N and H iff N is normal, $N \cap H = \{e\}$ and $G = NH$.

15.5 Direct Products

Definition 15.34 (Direct Product)**.** The *direct product* of groups G and H is their product in **Grp** (which is the same as their product in **Cat**).

Proposition 15.35. *$G \times H$ is the semidirect product $G \rtimes_\theta H$ where $\theta(g) = e$ for all $g \in G$.*

PROOF: Easy. □

15.6 Free Groups

Proposition 15.36. *Let A be a set. Let $\mathcal{F}^A$ be the category whose objects are pairs (G, j) where G is a group and j is a function $A \to G$, with morphisms $f : (G, j) \to (H, k)$ the group homomorphisms $f : G \to H$ such that $f \circ j = k$. Then $\mathcal{F}^A$ has an initial object.*

PROOF:
⟨1⟩1. LET: $W(A)$ be the set of words in the alphabet whose elements are the elements of A together with $\{a^{-1} : a \in A\}$.
⟨1⟩2. LET: $r : W(A) \to W(A)$ be the function that, given a word w, removes the first pair of letters of the form aa^{-1} or $a^{-1}a$; if there is no such pair, then $r(w) = w$.
⟨1⟩3. Let us say that a word w is a *reduced word* iff $r(w) = w$.
⟨1⟩4. For any word w of length n, we have $r^{\ulcorner \frac{n}{2} \urcorner}(w)$ is a reduced word.
PROOF: Since we cannot remove more than $n/2$ pairs of letters from w.
⟨1⟩5. LET: $R : W(A) \to W(A)$ be the function $R(w) = r^{\ulcorner \frac{n}{2} \urcorner}(w)$, where n is the length of w.
⟨1⟩6. LET: $F(A)$ be the set of reduced words.
⟨1⟩7. Define $\cdot : F(A)^2 \to F(A)$ by $w \cdot w' = R(ww')$
⟨1⟩8. $\cdot$ is associative.
PROOF: Both $w_1 \cdot (w_2 \cdot w_3)$ and $(w_1 \cdot w_2) \cdot w_3$ are equal to $R(w_1w_2w_3)$.
⟨1⟩9. The empty word is the identity element in $F(A)$
⟨1⟩10. The inverse of $a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$ is $a_n^{\mp 1} \cdots a_2^{\mp 1} a_1^{\mp 1}$.
⟨1⟩11. LET: $j : A \to F(A)$ be the function that maps a to the word a of length .
⟨1⟩12. LET: G be any group and $k : A \to G$ any function.

⟨1⟩13. The only morphism $f : (F(A), j) \to (G, k)$ in $\mathcal{F}^A$ is $f(a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}) = k(a_1)^{\pm 1} k(a_2)^{\pm 1} \cdots k(a_n)^{\pm 1}$.

$\square$

Definition 15.37 (Free Group)**.** For any set A, the *free group* on A is the initial object $(F(A), i)$ in $\mathcal{F}^A$.

Proposition 15.38. *$i : A \to F(A)$ is injective.*

Proof:
⟨1⟩1. Let: $x, y \in A$
⟨1⟩2. Assume: $x \neq y$
Prove: $i(x) \neq i(y)$
⟨1⟩3. Let: $f : A \to C_2$ be the function that maps x to 0 and all other elements of A to 1.
⟨1⟩4. Let: $\phi : F(A) \to C_2$ be the group homomorphism such that $f = \phi \circ i$.
⟨1⟩5. $f(x) \neq f(y)$
⟨1⟩6. $\phi(i(x)) \neq \phi(i(y))$
⟨1⟩7. $i(x) \neq i(y)$
$\square$

Proposition 15.39.

$$F(0) \cong \{e\}$$

Proof: For any set A, the unique group homomorphism $\{e\} \to A$ makes the following diagram commute.

$$\begin{tikzcd}
\{e\} \arrow[r] & A \\
\emptyset \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Proposition 15.40. *The free group on 1 is $\mathbb{Z}$ with the injection mapping 0 to 1.*

Proof: Given any group G and function $a : 1 \to G$, the required unique homomorphism $\phi : \mathbb{Z} \to G$ is defined by $\phi(n) = a(0)^n$. $\square$

Proposition 15.41. *For any sets A and B, we have that $F(A + B)$ is the coproduct of $F(A)$ and $F(B)$ in* **Grp**.

$$\begin{tikzcd}
 & G & \\
F(A) \arrow[ur, "f"] \arrow[r, "\kappa_1"] & F(A+B) \arrow[u, "k"] & F(B) \arrow[ul, "g"'] \arrow[l, "\kappa_2"'] \\
A \arrow[u, "i_A"] \arrow[r, "k_1"] & A+B \arrow[u, "j"] & B \arrow[u, "i_B"'] \arrow[l, "k_2"']
\end{tikzcd}$$

Proof:

⟨1⟩1. LET: $i_A : A \to F(A)$, $i_B : B \to F(B)$, $j : A + B \to F(A + B)$ be the canonical injections.
⟨1⟩2. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
⟨1⟩3. LET: G be any group and $f : F(A) \to G$, $g : F(B) \to G$ any group homomorphisms.
⟨1⟩4. LET: $h : A + B \to G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.
⟨1⟩5. LET: $k : F(A + B) \to G$ be the unique group homomorphism such that $k \circ j = h$.
⟨1⟩6. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
⟨1⟩7. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.
□

Definition 15.42 (Subgroup Generated by a Group)**.** Let G be a group and A a subset of G. Let $\phi : F(A) \to G$ be the unique group homomorphism such that $\phi(a) = a$ for all $a \in A$. The subgroup *generated* by A is

$$\langle A \rangle := \operatorname{im} \phi$$

$$\begin{tikzcd}
F(A) \arrow[r, "\phi"] & G \\
A \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Proposition 15.43. *Let G be a group and A a subset of G. Then $\langle A \rangle$ is the set of all elements of the form $a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$ (where $n \geq 0$) such that $a_1, \ldots, a_n \in A$.*

PROOF: Immediate from definitions. □

Corollary 15.43.1. *Let G be a group and $g \in G$. Then*

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\} \ .$$

Proposition 15.44. *Let G be a group and A a subset of G. Then $\langle A \rangle$ is the intersection of all the subgroups of G that include A.*

PROOF: Easy. □

Definition 15.45 (Finitely Generated)**.** Let G be a group. Then G is *finitely generated* iff there exists a finite subset A of G such that $G = \langle A \rangle$.

Proposition 15.46. *Every subgroup of a finitely generated free group is free.*

PROOF: TODO.

Proposition 15.47. $F(2)$ *includes subgroups isomorphic to the free group on arbitrarily many generators.*

PROOF: TODO

Proposition 15.48.

$$[F(2), F(2)] \cong F(\mathbb{Z})$$

PROOF: TODO

15.7 Normal Subgroups

Definition 15.49 (Normal Subgroup)**.** A subgroup N of G is *normal* iff, for all $g \in G$ and $n \in N$, we have $gng^{-1} \in N$.

Example 15.50. Every subgroup of Q_8 is normal.

Proposition 15.51. *Let G be a group and N a subgroup of G. Then the following are equivalent.*

1. *N is normal.*
2. *$\forall g \in G. gNg^{-1} \subseteq N$*
3. *$\forall g \in G. gNg^{-1} = N$*
4. *$\forall g \in G. gN \subseteq Ng$*
5. *$\forall g \in G. gN = Ng$*

PROOF:
$\langle 1 \rangle 1.$ $1 \Leftrightarrow 2$
PROOF: Immediate from definitions.
$\langle 1 \rangle 2.$ $2 \Rightarrow 3$
PROOF: If 2 holds then we have $gNg^{-1} \subseteq N$ and $g^{-1}Ng \subseteq N$ hence $N = gNg^{-1}$.
$\langle 1 \rangle 3.$ $3 \Rightarrow 2$
PROOF: Trivial.
$\langle 1 \rangle 4.$ $2 \Leftrightarrow 4$
PROOF: Easy.
$\langle 1 \rangle 5.$ $3 \Leftrightarrow 5$
PROOF: Easy.
□

Proposition 15.52. *Let $\phi : G \to H$ be a group homomorphism. Then $\ker \phi$ is a normal subgroup of G.*

PROOF: Given $g \in G$ and $n \in \ker \phi$ we have

$$\begin{aligned}\phi(gng^{-1}) &= \phi(g)\phi(n)\phi(g)^{-1} \\ &= \phi(g)\phi(g)^{-1} \\ &= e\end{aligned}$$

and so $gng^{-1} \in \ker \phi$. □

Proposition 15.53. *If H and K are normal subgroups of a group G then HK is normal in G.*

PROOF: For $g \in G$, $h \in H$ and $k \in K$ we have $ghkg^{-1} = (ghg^{-1})(gkg^{-1}) \in HK$. $\square$

15.8 Quotient Groups

Definition 15.54. Let G be a group. Let $\sim$ be an equivalence relation on G. Then we say that $\sim$ is *compatible* with the group operation on G iff, for all $a, a', g \in G$, if $a \sim a'$ then $ga \sim ga'$ and $ag \sim a'g$.

Proposition 15.55. *Let G be a group. Let $\sim$ be an equivalence relation on G. Then there exists an operation $\cdot : (G/\sim)^2 \to G/sim$ such that*

$$\forall a, b \in G.[a][b] = [ab]$$

iff $\sim$ is compatible with the group operation on G. In this case, $G/\sim$ is a group under $\cdot$ and the canonical function $\pi : G \to G/\sim$ is a group homomorphism, and is universal with respect to group homomorphisms $\phi : G \to G'$ such that if $a \sim a'$ then $\phi(a) = \phi(a')$.

PROOF: Easy. $\square$

Definition 15.56 (Quotient Group)**.** Let G be a group. Let $\sim$ be an equivalence relation on G that is compatible with the group operation on G. Then $G/\sim$ is the *quotient group* of G by $\sim$ under $[a][b] = [ab]$.

Proposition 15.57. *Let G be a group and H a subgroup of G. Then H is normal if and only if there exists a group K and homomorphism $\phi : G \to K$ such that $H = \ker \phi$.*

PROOF: One direction is given by Proposition 15.52. For the other direction, take $K = G/H$ and ϕ to be the canonical map $G \to G/H$. $\square$

Definition 15.58 (Modular Group)**.** The *modular group* $\mathrm{PSL}_2(\mathbb{Z})$ is $\mathrm{SL}_2(\mathbb{Z})/\{I, -I\}$.

Proposition 15.59. $\mathrm{PSL}_2(\mathbb{Z})$ *is generated by* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

PROOF: By Example 14.29.

Proposition 15.60 (Roger Alperin)**.** $\mathrm{PSL}_2(\mathbb{Z})$ *is presented by* $(x, y | x^2, y^3)$.

PROOF:
$\langle 1 \rangle 1.$ LET: $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$\langle 1 \rangle 2.$ LET: $y = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

⟨1⟩3. Define an action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{R} - \mathbb{Q}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} r = \frac{ar+b}{cr+d} .$$

⟨2⟩1. Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ and r irrational we have $\frac{ar+b}{cr+d}$ is irrational.

⟨3⟩1. Assume: for a contradiction $\frac{ar+b}{cr+d} = \frac{p}{q}$ where p and q are integers with $q > 0$.

⟨3⟩2. $aqr + bq = cpr + dp$

⟨3⟩3. $(aq - cp)r = dp - bq$

⟨3⟩4. $aq = cp = dp - bq = 0$

⟨3⟩5. $adq - cdp = 0$

⟨3⟩6. $cdp - cbq = 0$

⟨3⟩7. $(ad - cb)q = 0$

Proof: Since $ad - cb = 1$.

⟨3⟩8. $q = 0$

⟨3⟩9. Q.E.D.

Proof: This contradicts ⟨3⟩1.

⟨2⟩2. $-Ir = r$

Proof: Since $-Ir = \frac{-r}{-1} = r$.

⟨2⟩3. Given $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we have $A(Br) = (AB)r$.

Proof:

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} r \right] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{er+f}{gr+h} \\
&= \frac{a\frac{er+f}{gr+h} + b}{c\frac{er+f}{gr+h} + d} \\
&= \frac{a(er+f) + b(gr+h)}{c(er+f) + d(gr+h)} \\
&= \frac{(ae+bg)r + (af+bh)}{(ce+dg)r + (cf+dh)} \\
&= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} r \\
&= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right] r
\end{aligned}$$

⟨1⟩4.

$$yr = 1 - \frac{1}{r}$$

⟨1⟩5.

$$y^{-1} r = \frac{1}{1-r}$$

Proof: Since $y^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

⟨1⟩6.

$$yxr = 1 + r$$

PROOF: Since $yx = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$.

⟨1⟩7.

$$y^{-1}xr = \frac{r}{1+r}$$

PROOF: Since $y^{-1}x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

⟨1⟩8. If $r > -1$ is positive then yxr is positive.

⟨1⟩9. If r is positive then $y^{-1}xr$ is positive.

⟨1⟩10. If $r < -1$ then $y^{-1}xr$ is positive.

⟨1⟩11. If r is negative then yr is positive.

⟨1⟩12. If r is negative then $y^{-1}r$ is positive.

⟨1⟩13. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x)\cdots(y^{\pm 1}x)$$

with one or more factors can equal the identity.

PROOF: If the last factor is (yx), then the product maps numbers in $(-1, 0)$ to positive numbers. If the last factor is $(y^{-1}x)$, then the product maps numbers < -1 to positive numbers.

⟨1⟩14. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x)\cdots(y^{\pm 1}x)y^{\pm 1}$$

with one or more factors can equal the identity.

PROOF: The product maps negative numbers to positive numbers.

⟨1⟩15. $\mathrm{PSL}_2(\mathbb{Z})$ is presented by $(x, y | x^2, y^3)$.

□

Corollary 15.60.1. $\mathrm{PSL}_2(\mathbb{Z})$ *is the coproduct of* C_2 *and* C_3 *in* **Grp**.

Theorem 15.61. *Every group homomorphism* $\phi : G \to H$ *may be decomposed as*

$$G \longrightarrow G/\ker\phi \xrightarrow{\cong} \mathrm{im}\,\phi \longrightarrow H$$

PROOF: Easy. □

Corollary 15.61.1 (First Isomorphism Theorem)**.** *Let* $\phi : G \to H$ *be a surjective group homomorphism. Then* $H \cong G/\ker\phi$.

Proposition 15.62. *Let* H_1 *be a normal subgroup of* G_1 *and* H_2 *a normal subgroup of* G_2*. Then* $H_1 \times H_2$ *is a normal subgroup of* $G_1 \times G_2$*, and*

$$\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2} \ .$$

PROOF: $\pi \times \pi : G_1 \times G_2 \twoheadrightarrow G_1/H_1 \times G_2/H_2$ is a surjective homomorphism with kernel $H_1 \times H_2$. □

Example 15.63.

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

PROOF: Map a real number r to $(\cos r, \sin r)$. The result is a surjective group homomorphism with kernel $\mathbb{Z}$. □

Proposition 15.64. *Let H be a normal subgroup of a group G. For every subgroup K of G that includes H, we have H is a normal subgroup of K, and K/H is a subgroup of G/H. The mapping*

$$u : \{\text{subgroups of } G \text{ including } H\} \to \{\text{subgroups of } G/H\}$$

with $u(K) = K/H$ is a poset isomorphism.

PROOF:
$\langle 1 \rangle 1$. If K is a subgroup of G that includes H then H is normal in K.
$\langle 1 \rangle 2$. If K is a subgroup of G that includes H then K/H is a subgroup of G/H.
$\langle 1 \rangle 3$. If $H \subseteq K_1 \subseteq K_2$ then $K_1/H \subseteq K_2/H$.
$\langle 1 \rangle 4$. If $K_1/H = K_2/H$ then $K_1 = K_2$
 $\langle 2 \rangle 1$. ASSUME: $K_1/H = K_2/H$
 $\langle 2 \rangle 2$. $K_1 \subseteq K_2$
  $\langle 3 \rangle 1$. LET: $k \in K_1$
  $\langle 3 \rangle 2$. $kH \in K_2/H$
  $\langle 3 \rangle 3$. PICK $k' \in K_2$ such that $kH = k'H$
  $\langle 3 \rangle 4$. $kk'^{-1} \in H$
  $\langle 3 \rangle 5$. $kk'^{-1} \in K_2$
  $\langle 3 \rangle 6$. $k \in K_2$
 $\langle 2 \rangle 3$. $K_2 \subseteq K_1$
  PROOF: Similar.
$\langle 1 \rangle 5$. For any subgroup L of G/H, there exists a subgroup K of G that includes H such that $L = K/H$.
 $\langle 2 \rangle 1$. LET: L be a subgroup of G/H.
 $\langle 2 \rangle 2$. LET: $K = \{k \in G : kH \in L\}$
 $\langle 2 \rangle 3$. K is a subgroup of G.
  PROOF: Given $k, k' \in K$ we have $kH, k'H \in L$ hence $kk'^{-1}H \in L$ and so $kk'^{-1} \in K$.
 $\langle 2 \rangle 4$. $H \subseteq K$
  PROOF: For all $h \in H$ we have $hH = H \in L$.
 $\langle 2 \rangle 5$. $L = K/H$
  PROOF: By definition.

□

Proposition 15.65 (Third Isomorphism Theorem)**.** *Let H be a normal subgroup of a group G. Let N be a subgroup of G that includes H. Then N/H is normal in G/H if and only if N is normal in G, in which case*

$$\frac{G/H}{N/H} \cong \frac{G}{N}$$

PROOF:
$\langle 1 \rangle 1$. If N/H is normal in G/H then N is normal in G.
 $\langle 2 \rangle 1$. ASSUME: N/H is normal in G/H.
 $\langle 2 \rangle 2$. LET: $g \in G$ and $n \in N$.

⟨2⟩3. $gng^{-1}H \in N/H$
⟨2⟩4. Pick $n' \in N$ such that $gng^{-1}H = n'H$
⟨2⟩5. $gng^{-1}n'^{-1} \in H$
⟨2⟩6. $gng^{-1}n'^{-1} \in N$
⟨2⟩7. $gng^{-1} \in N$
⟨1⟩2. If N is normal in G then N/H is normal in G/H and $(G/H)/(N/H) \cong G/N$.
⟨2⟩1. Assume: N is normal in G.
⟨2⟩2. Let: $\phi : G/H \to G/N$ be the homomorphism $\phi(gH) = gN$
⟨3⟩1. If $gH = g'H$ then $gN = g'N$
Proof: If $gg'^{-1} \in H$ then $gg'^{-1} \in N$.
⟨3⟩2. $\phi((gH)(g'H)) = \phi(gH)\phi(g'H)$
Proof: Both are $gg'N$.
⟨2⟩3. ϕ is surjective.
⟨2⟩4. $\ker \phi = N/H$
⟨2⟩5. $(G/H)/(N/H) \cong G/N$
Proof: By the First Isomorphism Theorem.
□

Proposition 15.66 (Second Isomorphism Theorem)**.** *Let H and K be subgroups of a group G. Assume that H is normal in G. Then:*

1. *HK is a subgroup of G, and H is normal in HK.*

2. *$H \cap K$ is normal in K, and*

$$\frac{HK}{H} \cong \frac{K}{H \cap K} \quad .$$

Proof:
⟨1⟩1. HK is a subgroup of G.
Proof: Since $hkh'k' = hh'(h'^{-1}kh')k' \in HK$.
⟨1⟩2. H is normal in HK.
⟨1⟩3. $H \cap K$ is normal in K and $HK/H \cong K/(H \cap K)$
Proof: The function that maps k to kH is a surjective homomorphism $K \twoheadrightarrow HK/H$ with kernel $H \cap K$. Surjectivity follows because $hkH = hkh^{-1}H$.
□

See also Proposition 15.81 for a result that holds even if H is not normal.

15.9 Cosets

Proposition 15.67. *Let G be a group. Let $\sim$ be an equivalence relation on G such that, for all $a, b, g \in G$, if $a \sim b$ then $ga \sim gb$. Let $H = \{h \in G : h \sim e\}$. Then H is a subgroup of G and, for all $a, b \in G$, we have*

$$a \sim b \Leftrightarrow a^{-1}b \in H \Leftrightarrow aH = bH \quad .$$

PROOF:
$\langle 1\rangle 1.$ $e \in H$
$\langle 1\rangle 2.$ For all $x, y \in H$ we have $xy^{-1} \in H$.
$\langle 2\rangle 1.$ ASSUME: $x \sim e$ and $y \sim e$.
$\langle 2\rangle 2.$ $e \sim y^{-1}$
PROOF: Since $yy^{-1} \sim ey^{-1}$.
$\langle 2\rangle 3.$ $xy^{-1} \sim e$
PROOF: Since $xy^{-1} \sim ey^{-1} \sim e$.
$\langle 1\rangle 3.$ If $a \sim b$ then $a^{-1}b \in H$.
PROOF: If $a \sim b$ then $a^{-1}b \sim a^{-1}a = e$.
$\langle 1\rangle 4.$ If $a^{-1}b \in H$ then $aH = bH$.
$\langle 2\rangle 1.$ ASSUME: $a^{-1}b \in H$
$\langle 2\rangle 2.$ $bH \subseteq aH$
PROOF: For any $h \in H$ we have $bh = aa^{-1}bh \in aH$.
$\langle 2\rangle 3.$ $aH \subseteq bH$
PROOF: Similar since $b^{-1}a \in H$.
$\langle 1\rangle 5.$ If $aH = bH$ then $a \sim b$.
$\langle 2\rangle 1.$ ASSUME: $aH = bH$
$\langle 2\rangle 2.$ PICK $h \in H$ such that $a = bh$.
$\langle 2\rangle 3.$ $b^{-1}a = h$
$\langle 2\rangle 4.$ $b^{-1}a \in H$
$\langle 2\rangle 5.$ $b^{-1}a \sim e$
$\langle 2\rangle 6.$ $a \sim b$
PROOF: $a = bb^{-1}a \sim be = b$.
$\square$

Definition 15.68 (Coset)**.** Let G be a group and H a subgroup of G. A *left coset* of H is a set of the form aH for $a \in G$. A *right coset* of H is a set of the form Ha for some $a \in G$.

We write G/H for the set of all left cosets of H, and $G\backslash H$ for the set of all right cosets of H.

Proposition 15.69.

$$G/H \cong G\backslash H$$

PROOF: The function that maps aH to Ha^{-1} is a bijection. $\square$

Proposition 15.70. *Let G be a group and H a subgroup of G. Define $\sim_H$ on G by: $a \sim b$ iff $a^{-1}b \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations $\sim$ on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ga \sim gb$. The equivalence class of a is aH.*

PROOF:
$\langle 1\rangle 1.$ For any subgroup H, we have $\sim_H$ is an equivalence relation on G.
$\langle 2\rangle 1.$ $\sim$ is reflexive.
PROOF: For any $a \in G$ we have $a^{-1}a = e \in H$.
$\langle 2\rangle 2.$ $\sim$ is symmetric.
PROOF: If $a^{-1}b \in H$ then $b^{-1}a \in H$.

$\langle 2\rangle 3$. $\sim$ is transitive.
PROOF: If $a^{-1}b \in H$ and $b^{-1}c \in H$ then $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$.
$\langle 1\rangle 2$. If $a \sim_H b$ then $ga \sim_H gb$.
PROOF: If $a^{-1}b \in H$ then $(ga)^{-1}(gb) = a^{-1}g^{-1}gb = a^{-1}b \in H$.
$\langle 1\rangle 3$. For any equivalence relation $\sim$ on G such that, whenever $a \sim b$, then $ga \sim gb$, there exists a subgroup H such that $\sim = \sim_H$.
PROOF: Proposition 15.67.
$\langle 1\rangle 4$. The $\sim_H$-equivalence class of a is aH.
PROOF:

$$\begin{aligned} a \sim b &\Leftrightarrow a^{-1}b \in H \\ &\Leftrightarrow \exists h \in H.a^{-1}b = h \\ &\Leftrightarrow \exists h \in H.b = aH \\ &\Leftrightarrow b \in aH \end{aligned}$$

□

Proposition 15.71. *Let G be a group and H a subgroup of G. Define $\sim_H$ on G by: $a \sim b$ iff $ab^{-1} \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations $\sim$ on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ag \sim bg$. The equivalence class of a is Ha.*

PROOF: Similar. □

Proposition 15.72. *Let G be a group and H be a subgroup of G. Define $\sim_L$ and $\sim_R$ on G by:*

$$a \sim_L b \Leftrightarrow a^{-1}b \in H, \qquad a \sim_R b \Leftrightarrow ab^{-1} \in H \enspace .$$

Then $\sim_L = \sim_R$ if and only if H is normal.

PROOF:
$\langle 1\rangle 1$. If $\sim_L = \sim_R$ then H is normal.
$\langle 2\rangle 1$. ASSUME: $\sim_L = \sim_R$
$\langle 2\rangle 2$. LET: $h \in H$ and $g \in G$
$\langle 2\rangle 3$. $g \sim_L gh^{-1}$
$\langle 2\rangle 4$. $g \sim_R gh^{-1}h$
$\langle 2\rangle 5$. $ghg^{-1} \in H$
$\langle 1\rangle 2$. If H is normal then $\sim_L = \sim_R$.
$\langle 2\rangle 1$. ASSUME: H is normal.
$\langle 2\rangle 2$. If $a \sim_L b$ then $a \sim_R b$.
$\langle 3\rangle 1$. ASSUME: $a \sim_L b$
$\langle 3\rangle 2$. $a^{-1}b \in H$
$\langle 3\rangle 3$. $aa^{-1}ba^{-1} \in H$
$\langle 3\rangle 4$. $ba^{-1} \in H$
$\langle 3\rangle 5$. $a \sim_R b$
$\langle 2\rangle 3$. If $a \sim_R b$ then $a \sim_L b$.
PROOF: Similar.
□

Corollary 15.72.1. *Let G be a group and H be a normal subgroup of G. Define $\sim$ on G by $a \sim b$ iff $a^{-1}b \in H$. Then $G/\sim$ is a group under $[a][b] = [ab]$.*

Definition 15.73 (Quotient Group)**.** Let G be a group and H be a normal subgroup of G. The *quotient group G/H* is $G/\sim$ where $a \sim b$ iff $a^{-1}b \in H$, under $[a][b] = [ab]$ or $(aH)(bH) = abH$.

Corollary 15.73.1. *Let H be a normal subgroup of a group G. For every group homomorphism $\phi : G \to G'$ such that $H \subseteq \ker\phi$, there exists a unique group homomorphism $\overline{\phi} : G/H \to G'$ such that the following diagram commutes.*

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & G' \\ & G/H \arrow[ur, "\overline{\phi}"'] & \end{tikzcd}$$

Proposition 15.74. *$\mathbb{Z}/n\mathbb{Z}$ has exactly n elements.*

Proof: Every integer is congruent to one of 0, 1, …, $n-1$ by the division algorithm, and no two of them are conguent to one another, since if $0 \leq i < j < n$ then $0 < j - i < n$. □

Proposition 15.75. *Let m and n be integers with $n > 0$. The order of m in $\mathbb{Z}/n\mathbb{Z}$ is $\frac{n}{\gcd(m,n)}$.*

Proof: By Proposition 14.19 since the order of 1 is n. □

Proposition 15.76. *The integer m generates $\mathbb{Z}/n\mathbb{Z}$ if and only if* $\gcd(m,n) = 1$.

Proof: By Proposition 15.75. □

Corollary 15.76.1. *If p is prime then every non-zero element in $\mathbb{Z}/p\mathbb{Z}$ is a generator.*

Proposition 15.77.

$$\mathrm{Aut}_{\mathbf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$$

Proof: Every permutation of $\{(1,0),(0,1),(1,1)\}$ gives an automorphism of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

Example 15.78. Not all monomorphisms split in **Grp**.

Define $\phi : \mathbb{Z}/3\mathbb{Z} \to S_3$ by

$$\phi(0) = \mathrm{id}_3, \qquad \phi(1) = (1\ 3\ 2), \qquad \phi(2) = (1\ 2\ 3)\ .$$

Then ϕ is monic but has no retraction.

For if $r : S_3 \to \mathbb{Z}/3\mathbb{Z}$ is a retraction, then we would have

$$r(1\ 2) + r(2\ 3) = 1, \qquad r(2\ 3) + r(1\ 2) = 2$$

which is impossible.

Proposition 15.79. *Let G be a group, H a subgroup of G, and $g \in G$. The function that maps h to gh is a bijection $H \cong gH$.*

PROOF: By Cancellation. □

Proposition 15.80. *Let G be a group, H a subgroup of G, and $g \in G$. The function that maps h to hg is a bijection $H \cong Hg$.*

PROOF: By Cancellation. □

Proposition 15.81. *Let H and K be finite subgroups of a group G. Then*

$$|HK| = \frac{|H||K|}{|H \cap K|} \ .$$

PROOF:
⟨1⟩1. LET: $f : \{hK : h \in H\} \to H/(H \cap K)$ be the function $f(hK) = h(H \cap K)$
  PROOF: This is well-defined because if $hK = h'K$ then $h^{-1}h' \in H \cap K$ so $h(H \cap K) = h'(H \cap K)$.
⟨1⟩2. f is injective.
  PROOF: If $h(H \cap K) = h'(H \cap K)$ then $hK = h'K$.
⟨1⟩3. f is surjective.
  PROOF: Clear.
⟨1⟩4.

$$\frac{|HK|}{|K|} = \frac{|H|}{|H \cap K|}$$

□

15.10 Congruence

Definition 15.82 (Congruence)**.** Given integers a, b, n with n positive, we say a is *congruent* to b *modulo* n, and write $a \equiv b(\operatorname{mod} n)$, iff $a + n\mathbb{Z} = b + n\mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$.

Proposition 15.83. *Given integers a, b, n with n positive, we have $a \equiv b(\operatorname{mod} n)$ iff $n \mid a - b$.*

PROOF: By Proposition 15.67. □

Proposition 15.84. *If $a \equiv a' \bmod n$ and $b \equiv b' \bmod n$ then $a+b \equiv a'+b' \bmod n$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid (a' + b') - (a + b)$. □

Proposition 15.85. *If $a \equiv a' \bmod n$ and $b \equiv b' \bmod n$ then $ab \equiv a'b' \bmod n$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid a'b' - ab = a'(b' - b) + (a' - a)b$. □

15.11 Cyclic Groups

Definition 15.86 (Cyclic Group)**.** The *cyclic* groups are $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ for positive integers n.

Proposition 15.87. *If m and n are positive integers with $\gcd(m, n) = 1$ then $C_{mn} \cong C_m \times C_n$.*

PROOF: The function that maps x to $(x \bmod m, x \bmod n)$ is an isomorphism. ∎

Proposition 15.88. *Let G be a group and $g \in G$. Then $\langle g \rangle$ is cyclic.*

PROOF: If g has finite order then $\langle g \rangle \cong C_{|g|}$, otherwise $\langle g \rangle \cong \mathbb{Z}$. ∎

Proposition 15.89. *Every finitely generated subgroup of $\mathbb{Q}$ is cyclic.*

PROOF:
$\langle 1 \rangle$1. LET: $G = \langle a_1/b, \ldots, a_n/b \rangle$ where a_1, …, a_n, b are integers with $b > 0$
$\langle 1 \rangle$2. LET: $a = \gcd(a_1, \ldots, a_n)$
$\langle 1 \rangle$3. $G = \langle a/b \rangle$
∎

Corollary 15.89.1. *$\mathbb{Q}$ is not finitely generated.*

Proposition 15.90. *Let $n > 0$. Let G be a subgroup of $\mathbb{Z}/n\mathbb{Z}$. Then there exists d such that $d \mid n$ and $G = \langle d \rangle$.*

PROOF:
$\langle 1 \rangle$1. LET: $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the canonical projection.
$\langle 1 \rangle$2. LET: $G' = \pi^{-1}(G)$
$\langle 1 \rangle$3. G' is a cyclic subgroup of $\mathbb{Z}$.
$\langle 1 \rangle$4. PICK $d \in \mathbb{Z}$ such that $d > 0$ and $G' = \langle d \rangle$.
$\langle 1 \rangle$5. $G = \langle d \rangle$
$\langle 1 \rangle$6. $n \in G'$
$\langle 1 \rangle$7. $d \mid n$
∎

15.12 Commutator Subgroup

Definition 15.91 (Commutator)**.** Let G be a group and $g, h \in G$. The *commutator* of g and h is

$$[g, h] = ghg^{-1}h^{-1} \ .$$

Definition 15.92 (Commutator Subgroup)**.** Let G be a group. The *commutator subgroup*, denoted $[G, G]$ or G', is the subgroup generated by the elements of the form $aba^{-1}b^{-1}$.

We write $G^{(i)}$ for the result of taking the commutator subgroup i times starting with G.

Lemma 15.93. *Let $\phi : G_1 \to G_2$ be a group homomorphism. Then, for all $g, h \in G_1$, we have*

$$\phi([g,h]) = [\phi(g), \phi(h)]$$

and so $\phi(G_1') \subseteq G_2'$.

PROOF: Easy. □

Lemma 15.94. *Let N and H be normal subgroups of a group G. Then $[N,H] \subseteq N \cap H$.*

PROOF:
⟨1⟩1. LET: $n \in N$ and $h \in H$
PROVE: $nhn^{-1}h^{-1} \in N \cap H$
⟨1⟩2. $nhn^{-1} \in H$
PROOF: Since H is normal.
⟨1⟩3. $nhn^{-1}h^{-1} \in H$
⟨1⟩4. $hn^{-1}h^{-1} \in N$
PROOF: Since N is normal.
⟨1⟩5. $nhn^{-1}h^{-1} \in N$
⟨1⟩6. $nhn^{-1}h^{-1} \in N \cap H$
□

Corollary 15.94.1. *Let N and H be normal subgroups of G. If $N \cap H = \{e\}$, then every element in N commutes with every element in H.*

Proposition 15.95. *Let N and H be normal subgroups of G. If $N \cap H = \{e\}$ then $NH \cong N \times H$.*

PROOF: From Proposition 15.32. □

15.13 Presentations

Definition 15.96 (Presentation)**.** A *presentation* of a group G is a pair (A, R) where A is a set and $R \subseteq F(A)$ is a set of words such that

$$G \cong F(A)/N(R)$$

where $N(R)$ is the smallest normal subgroup of $F(A)$ that includes R.

Example 15.97.

- The free group on a set A is presented by $(A, \emptyset)$.
- S_3 is presented by $(x, y | x^2, y^3, xyxy)$.
- $(a, b \mid a^2, b^2, (ab)^n)$ is a presentation of D_{2n}.
- $(x, y \mid x^2y^{-2}, y^4, xyx^{-1}y)$ is a presentation of Q_8.

Proposition 15.98 (Word Problem)**.** *Let (A, R) be a presentation of the group G. Let $w_1, w_2 \in F(A)$ be two words. Then it is undecidable in general if $w_1N(R) = w_2N(R)$ in G.*

Definition 15.99 (Finitely Presented)**.** A group is *finitely presented* iff it has a presentation (A, R) where both A and R are finite.

Proposition 15.100. *Let $(A|R)$ be a presentation of G and $(A'|R')$ a presentation of H. Assume w.l.o.g. A and A' are disjoint. Then the group $G * G'$ presented by $(A \cup A'|R \cup R')$ is the coproduct of G and G' in* **Grp**.

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & A \cup A' \arrow[d] & A' \arrow[l] \arrow[d] \\
F(A) \arrow[r] \arrow[d] & F(A \cup A') \arrow[d] & F(A') \arrow[l] \arrow[d] \\
G \arrow[r, "\kappa_1"] & G * G' & G' \arrow[l, "\kappa_2"']
\end{tikzcd}
$$

PROOF:
⟨1⟩1. LET: $\kappa_1 : G \to G*G'$ and $\kappa_2 : G' \to G*G'$ be the unique homomorphisms that make the diagram above commute.
⟨1⟩2. LET: $\phi : G \to H$ and $\psi : G' \to H$ be any homomorphisms.
⟨1⟩3. LET: $[\phi, \psi] : F(A \cup A') \to H$ be the unique homomorphism such that ...
⟨1⟩4. $R \cup R' \subseteq \ker[\phi, \psi]$
⟨1⟩5. $[\phi, \psi]$ factors uniquely through the morphism $F(A \cup A') \to G * G'$
□

15.14 Index of a Subgroup

Definition 15.101 (Index)**.** Let G be a group and H a subgroup of G. The *index* of H in G, denoted $|G : H|$, is the number of left cosets of H in G if this is finite, otherwise ∞.

Theorem 15.102 (Lagrange's Theorem)**.** *Let G be a finite group and H a subgroup of G. Then*

$$|G| = |G : H||H| \enspace .$$

PROOF: G/H is a partition of G into $|G : H|$ subsets, each of size $|H|$. □

Corollary 15.102.1. *For p a prime number, the only group of order p is C_p.*

PROOF: Let G be a group of order p and $g \in G$ with $g \neq e$. Then $|\langle g \rangle|$ divides p and is not 1, hence is p, that is, $G = \langle g \rangle$. □

Theorem 15.103 (Cauchy's Theorem)**.** *Let G be a finite group. If p is prime and $p \mid |G|$ then the number of cyclic subgroups of order p is congruent to 1 modulo p. In particular, there exists an element of order p.*

PROOF:
⟨1⟩1. LET: $S = \{(a_1, a_2, \ldots, a_p) \in G^p : a_1 a_2 \cdots a_p = e\}$
⟨1⟩2. $|S| = |G|^{p-1}$

PROOF: Given any $a_1, \ldots, a_{p-1} \in G$, there exists a unique a_p such that $(a_1, \ldots, a_p) \in S$, namely $a_p = (a_1 \cdots a_{p-1})^{-1}$.

$\langle 1 \rangle 3.$ $p \mid |S|$

$\langle 1 \rangle 4.$ Define an action of $\mathbb{Z}/p\mathbb{Z}$ on S by

$$m \cdot (a_1, \ldots, a_p) = (a_m, a_{m+1}, \ldots, a_p, a_1, a_2, \ldots, a_{m-1}) \ .$$

PROOF: If $(a_1, \ldots, a_p) \in S$ then $(a_2, a_3, \ldots, a_p, a_1) \in S$ since $a_1 = (a_2 \cdots a_p)^{-1}$.

$\langle 1 \rangle 5.$ LET: Z be the set of fixed points of this action.

$\langle 1 \rangle 6.$ $|Z| \equiv 0 (\mathrm{mod}\, p)$

PROOF: Corollary 17.18.1, $\langle 1 \rangle 3$.

$\langle 1 \rangle 7.$ $Z = \{(a, a, \ldots, a) : a^p = e\}$

$\langle 1 \rangle 8.$ $Z \neq \emptyset$

PROOF: Since $(e, e, \ldots, e) \in Z$.

$\langle 1 \rangle 9.$ An element a has order p iff $(a, a, \ldots, a) \in Z$ and $a \neq e$.

$\langle 1 \rangle 10.$ LET: N be the number of cyclic subgroups of order p.

$\langle 1 \rangle 11.$ The number of elements of order p is $N(p-1)$

$\langle 1 \rangle 12.$ $|Z| = N(p-1) + 1$

$\langle 1 \rangle 13.$ $-N + 1 \equiv 0 (\mathrm{mod}\, p)$

PROOF: From $\langle 1 \rangle 6$.

$\langle 1 \rangle 14.$ $N \equiv 1 (\mathrm{mod}\, p)$

$\square$

Proposition 15.104. *Let G be a group. Let K be a subgroup of G and H a subgroup of K. If $|G : H|$, $|G : K|$ and $|K : H|$ are all finite then*

$$|G : H| = |G : K||K : H| \ .$$

PROOF:

$\langle 1 \rangle 1.$ LET: $G/K = \{g_1 K, g_2 K, \ldots, g_m K\}$

$\langle 1 \rangle 2.$ LET: $K/H = \{k_1 H, k_2 H, \ldots, k_n H\}$

$\langle 1 \rangle 3.$ $G/H = \{g_i k_j H : 1 \leq i \leq m, 1 \leq j \leq n\}$

$\langle 2 \rangle 1.$ LET: $g \in G$

$\langle 2 \rangle 2.$ PICK i such that $gK = g_i K$

$\langle 2 \rangle 3.$ $g^{-1} g_i \in K$

$\langle 2 \rangle 4.$ PICK j such that $g^{-1} g_i H = k_j H$

$\langle 2 \rangle 5.$ $g^{-1} g_i k_j \in H$

$\langle 2 \rangle 6.$ $gH = g_i k_j H$

$\langle 1 \rangle 4.$ If $g_i k_j H = g_{i'} k_{j'} H$ then $i = i'$ and $j = j'$.

$\langle 2 \rangle 1.$ ASSUME: $g_i k_j H = g_{i'} k_{j'} H$

$\langle 2 \rangle 2.$ $g_i K = g_{i'} K$

$\langle 2 \rangle 3.$ $i = i'$

$\langle 2 \rangle 4.$ $k_j H = k_{j'} H$

$\langle 2 \rangle 5.$ $j = j'$

$\square$

15.15 Cokernels

Proposition 15.105. *Let $\phi : G \to H$ be a homomorphism between groups. Then there exists a group K and homomorphism $\pi : H \to K$ that is initial with respect to all homomorphism $\alpha : H \to L$ such that $\alpha \circ \phi = 0$.*

PROOF:
$\langle 1 \rangle 1$. LET: N be the intersection of all the normal subgroups of H that include $\operatorname{im} \phi$.
$\langle 1 \rangle 2$. LET: $K = H/N$ and π be the canonical homomorphism.
$\langle 1 \rangle 3$. LET: $\pi \circ \phi = 0$
$\langle 1 \rangle 4$. LET: $\alpha : H \to L$ satisfy $\alpha \circ \phi = 0$
$\langle 1 \rangle 5$. $\operatorname{im} \phi \subseteq \ker \alpha$
$\langle 1 \rangle 6$. $N \subseteq \ker \alpha$
$\langle 1 \rangle 7$. There exists a unique $\overline{\alpha} : H/ \operatorname{im} \phi \to L$ such that $\overline{\alpha} \circ \pi = \alpha$
□

Definition 15.106 (Cokernel)**.** For any homomorphism $\phi : G \to H$ in **Grp**, the *cokernel* of ϕ is the group $\operatorname{coker} \phi$ and homomorphism $\pi : H \to \operatorname{coker} \phi$ that is initial among homomorphisms $\alpha : H \to L$ such that $\alpha \circ \phi = 0$.

Example 15.107. It is not true that a homomorphism with trivial cokernel is epi. The inclusion $\langle (1\ 2) \rangle \hookrightarrow S_3$ has trivial cokernel but is not epi.

15.16 Cayley Graphs

Definition 15.108 (Cayley Graph)**.** Let G be a finitely generated group. Let A be a finite set of generators for G. The *Cayley graph* of G with respect to A is the directed graph whose vertices are the elements of G, with an edge $g_1 \to g_2$ labelled by $a \in A$ iff $g_2 = g_1 a$.

Proposition 15.109. *G is the free group on A iff the Cayley graph with respect to A is a tree.*

PROOF: Both are equivalent to saying that the product of two different strings of elements of A and/or their inverses are not equal. □

15.17 Characteristic Subgroups

Definition 15.110 (Characteristic Subgroup)**.** Let G be a group. Let H be a subgroup of G. Then H is a *characteristic* subgroup of G iff, for every automorphism ϕ of G, we have $\phi(H) \subseteq H$.

Proposition 15.111. *Characteristic subgroups are normal.*

PROOF: Take ϕ to be conjugation with respect to an arbitrary element. □

Proposition 15.112. *Let G be a group. Let K be a normal subgroup of G and H a characteristic subgroup of K. Then H is normal in G.*

Proof: For any $a \in G$ we have conjugation by a is an automorphism on K, hence H is closed under it. $\square$

Proposition 15.113. *Let G be a group. Let H be a subgroup of G. Suppose there is no other subgroup of G isomorphic to H. Then H is characteristic, hence normal.*

Proof: For any automorphism ϕ on G, we have $\phi(H)$ is isomorphic to H, hence $\phi(H) = H$. $\square$

Proposition 15.114. *Let G be a finite group. Let K be a normal subgroup of G. Assume $|K|$ and $|G/K|$ are relatively prime. Then K is characteristic.*

Proof:
$\langle 1 \rangle 1$. Let: K' be a subgroup of G isomorphic to K.
Prove: $K' = K$
$\langle 1 \rangle 2$. $|K'/(K \cap K')|$ divides both $|K'| = |K|$ and $|G/K|$
$\langle 1 \rangle 3$. $|K'/(K \cap K')| = 1$
$\langle 1 \rangle 4$. $K' = K \cap K'$
$\langle 1 \rangle 5$. $K' = K$
$\square$

Proposition 15.115. *The commutator subgroup of a group is characteristic.*

Proof: Lemma 15.93. $\square$

15.18 Simple Groups

Definition 15.116 (Simple Group)**.** A group G is *simple* iff its only normal subgroups are $\{e\}$ and G.

Proposition 15.117. *Let G be a group. Then G is simple if and only if the only homomorphic images of G are 1 and G.*

Proof: Both are equivalent to saying that, for any surjective homomorphism $\phi : G \to G'$, either ϕ has kernel $\{e\}$ (in which case it is an isomorphism) or ϕ has kernel G (in which case $G' = 1$.) $\square$

15.19 Sylow Subgroups

Definition 15.118 (Sylow Subgroup)**.** Let p be a prime number. Let G be a finite group. A *p-Sylow subgroup* of G is a subgroup of order p^r, where r is the largest integer such that p^r divides $|G|$.

Proposition 15.119. *Let p be prime. Let G be a finite group. Let P be a p-Sylow subgroup of G. If P is normal then P is characteristic.*

Proof: Proposition 15.114. $\square$

Corollary 15.119.1. *Let p be prime. Let G be a finite group. Let P be a p-Sylow subgroup of G. Let H be a subgroup of G that includes P. If P is normal in H and H is normal in G then P is normal in G.*

Proposition 15.120. *Let G be a finite group. Let $P_1, \ldots, P_r$ be its nontrivial Sylow subgroups. Assume all P_i are normal in G. Then*

$$G \cong P_1 \times \cdots \times P_r \ .$$

PROOF:
$\langle 1 \rangle 1.$ $P_1 P_2 \cdots P_r \cong P_1 \times P_2 \times \cdots \times P_r$
$\langle 2 \rangle 1.$ $P_1 \cong P_1$
$\langle 2 \rangle 2.$ For $1 \leq i < r$, if $P_1 P_2 \cdots P_i \cong P_1 \times P_2 \times \cdots \times P_i$ then $P_1 P_2 \cdots P_i P_{i+1} \cong P_1 \times P_2 \times \cdots P_i \times P_{i+1}$
$\langle 3 \rangle 1.$ LET: $1 \leq i < r$
$\langle 3 \rangle 2.$ ASSUME: $P_1 P_2 \cdots P_i \cong P_1 \times P_2 \times \cdots \times P_i$
$\langle 3 \rangle 3.$ $P_1 P_2 \cdots P_i$ is normal in G.
$\langle 3 \rangle 4.$ $P_1 P_2 \cdots P_i \cap P_{i+1} = \{e\}$
$\langle 4 \rangle 1.$ LET: $|P_j| = p_j^{k_j}$ for all j.
$\langle 4 \rangle 2.$ The order of any element of $P_1 P_2 \cdots P_i$ divides $p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i}$
$\langle 4 \rangle 3.$ The order of any element of P_{i+1} divides $p_{i+1}^{k_{i+1}}$
$\langle 4 \rangle 4.$ The p_j are all distinct.
PROOF: Any p_j-Sylow subgroup is congruent to P_j hence equal to P_j since P_j is normal.
$\langle 4 \rangle 5.$ The only element in $P_1 P_2 \cdots P_i$ and P_{i+1} is e.
$\langle 3 \rangle 5.$ $P_1 P_2 \cdots P_i P_{i+1} \cong P_1 P_2 \cdots P_i \times P_{i+1}$
PROOF: Proposition 15.95.
$\langle 3 \rangle 6.$ $P_1 P_2 \cdots P_i P_{i+1} \cong P_1 \times P_2 \times \cdots \times P_i \times P_{i+1}$
$\langle 1 \rangle 2.$ $G = P_1 P_2 \cdots P_r$
PROOF: Since $|G| = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$.
□

15.20 Series of Subgroups

Definition 15.121 (Series of Subgroups)**.** Let G be a group. A *series* of subgroups of G is a sequence (G_n) of subgroups of G such that

$$G = G_0 \supsetneq G_1 \supsetneq G_2 \supsetneq \cdots$$

It is a *normal* series iff G_{n+1} is normal in G_n for all n.

Proposition 15.122. *The maximal length of a normal series in G is 0 iff G is trivial.*

PROOF: Since 1 is normal in G for every G. □

Proposition 15.123. *The maximal length of a normal series in G is 1 iff G is non-trivial and simple.*

PROOF: Immediate from definitions. □

Example 15.124. $\mathbb{Z}$ has normal series of arbitrary length.

PROOF: We have $\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supsetneq \cdots$. □

Example 15.125. The maximal length of a normal series in $\mathbb{Z}/n\mathbb{Z}$ is the number of primes in the prime factorization of n.

PROOF: Let $n = p_1 p_2 \cdots p_k$. A normal series of maximal length is
$\mathbb{Z}/p_1 p_2 \cdots p_k \mathbb{Z} \supsetneq \mathbb{Z}/p_1 p_2 \cdots p_{k-1}\mathbb{Z} \supsetneq \cdots \supsetneq \mathbb{Z}/p_1\mathbb{Z} \supsetneq \{e\}$.□

Definition 15.126 (Equivalent Normal Series)**.** Let

$$\begin{aligned} G = G_0 \supsetneq G_1 \supsetneq G_2 \supsetneq \cdots \supsetneq G_n = \{e\} \\ G = G'_0 \supsetneq G'_1 \supsetneq G'_2 \supsetneq \cdots \supsetneq G'_m = \{e\} \end{aligned}$$

be two normal series in a group G. Then the two series are *equivalent* iff $m = n$ and there exists a permutation $\sigma \in S_n$ such that, for all i, we have $G_i/G_{i+1} \cong G'_{\sigma(i)}/G'_{\sigma(i)+1}$.

Definition 15.127 (Composition Series)**.** Let G be a group. A *composition series* for G is a series of subgroups in G

$$G = G_0 \supsetneq G_1 \supsetneq G_2 \supsetneq \cdots \supsetneq G_n = \{e\}$$

such that, for all i, we have G_i/G_{i+1} is simple.

Proposition 15.128. *A normal series of maximal length in a group is a composition series.*

PROOF: Easy. □

Corollary 15.128.1. *Every finite group has a composition series.*

Corollary 15.128.2. *If a group has a composition series then every normal subgroup has a composition series.*

Definition 15.129 (Refinement)**.** A series of subgroups S_1 is a *refinement* of the series S_2 iff every subgroup in S_2 appears in S_1.

Lemma 15.130. *Let G be a group. Let Q, N and L be subgroups of G. Assume L is a normal subgroup of Q and $qN = Nq$ for all $q \in Q$. Then*

$$\frac{QN}{LN} \cong \frac{Q}{L(Q \cap N)} \ .$$

PROOF:
⟨1⟩1. QN is a subgroup of G.
  PROOF: Since $QN = NQ$.
⟨1⟩2. LN is a subgroup of G.

PROOF: Since $LN = NL$.
⟨1⟩3. LN is normal in QN.
⟨2⟩1. LET: $l \in L$, $q \in Q$, and $n, n' \in N$.
PROVE: $qnln'n^{-1}q^{-1} \in LN$
⟨2⟩2. PICK $n_1 \in N$ such that $nl = ln_1$
⟨2⟩3. PICK $n_2 \in N$ such that $n_1 n' n^{-1} q^{-1} = q^{-1} n_2$
⟨2⟩4. $qnln'n^{-1}q^{-1} = qlq^{-1}n_2 \in LN$
PROOF: Since L is normal in Q.
⟨1⟩4. The function $f : Q \to QN/LN$ that maps q to qLN is a surjective homomorphism.
⟨1⟩5. $\ker f = L(Q \cap N)$
⟨2⟩1. $\ker f \subseteq L(Q \cap N)$
⟨3⟩1. LET: $x \in \ker f$
⟨3⟩2. $x \in LN$
⟨3⟩3. PICK $l \in L$ and $n \in N$ such that $x = ln$
⟨3⟩4. $n = l^{-1}x \in Q \cap N$
⟨3⟩5. $x \in L(Q \cap N)$
⟨2⟩2. $L(Q \cap N) \subseteq \ker f$
PROOF: Since $L(Q \cap N) \subseteq Q$ and $L(Q \cap N) \subseteq LN$.
⟨1⟩6. Q.E.D.
PROOF: First Isomorphism Theorem.
□

Theorem 15.131 (Schreier)**.** *Any two normal series in a group have equivalent refinements.*

PROOF:
⟨1⟩1. LET: G be a group.
⟨1⟩2. LET: $S_1 : G = G_0 \supsetneq G_1 \supsetneq G_2 \supsetneq \cdots \supsetneq G_m = \{e\}$ and $S_2 : G = H_0 \supsetneq H_1 \supsetneq H_2 \supsetneq \cdots \supsetneq H_n = \{e\}$ be two normal series in G.
⟨1⟩3. For each i, we have
$$G_i = G_i \cap H_0 \supseteq G_i \cap H_1 \supseteq \cdots \supseteq G_i \cap H_n = \{e\}$$
is a series of subgroups in G_i.
⟨1⟩4. For each i, we have
$$G_i = (G_i \cap H_0)G_{i+1} \supseteq (G_i \cap H_1)G_{i+1} \supseteq \cdots \supseteq (G_i \cap H_n)G_{i+1} = G_{i+1}$$
is a normal series in G_i.
⟨2⟩1. LET: $0 \leq i < m$ and $0 \leq j < n$
PROVE: $(G_i \cap H_{j+1})G_{i+1}$ is normal in $(G_i \cap H_j)G_{i+1}$
⟨2⟩2. LET: $x \in G_i \cap H_{j+1}$, $y \in G_{i+1}$, $a \in G_i \cap H_j$ and $b \in G_{i+1}$
PROVE: $abxyb^{-1}a^{-1} \in (G_i \cap H_{j+1})G_{i+1}$
⟨2⟩3. $axa^{-1} \in G_i \cap H_{j+1}$
PROOF: Since $a, x \in G_i$ and H_{j+1} is normal in H_j.
⟨2⟩4. $ax^{-1}bxa^{-1} \in G_{i+1}$
PROOF: Since G_{i+1} is normal in G_i.
⟨2⟩5. $yb^{-1} \in G_{i+1}$
⟨2⟩6. $ayb^{-1}a^{-1} \in G_{i+1}$
PROOF: Since G_{i+1} is normal in G_i.

⟨2⟩7. $abxyb^{-1}a^{-1} = (axa^{-1})(ax^{-1}bxa^{-1}ayb^{-1}a^{-1}) \in (G_i \cap H_{j+1})G_{i+1}$

⟨1⟩5. Let S be the series obtained by concatenating the series ⟨1⟩4 for G_0 to G_1, G_1 to G_2, ..., G_{m-1} to G_m

⟨1⟩6. S is a refinement of S_1.

⟨1⟩7. S is normal.

⟨1⟩8. LET: T be the similarly constructed normal refinement of S_2.

⟨1⟩9. For all i, j we have
$$\frac{(G_i \cap H_j)G_{i+1}}{(G_i \cap H_{j+1})G_{i+1}} \cong \frac{G_i \cap H_j}{(G_i \cap H_{j+1})(G_{i+1} \cap H_j)}$$

⟨2⟩1. $G_i \cap H_{j+1}$ is normal in $G_i \cap H_j$

⟨2⟩2. For all $q \in G_i \cap H_j$ we have $qG_{i+1} = G_{i+1}q$

PROOF: Since for all $q \in G_i$ we have $qG_{i+1} = G_{i+1}q$.

⟨2⟩3. Q.E.D.

PROOF: Lemma 15.130

⟨1⟩10. For all i, j we have
$$\frac{(G_i \cap H_j)H_{j+1}}{(G_{i+1} \cap H_j)H_{j+1}} \cong \frac{G_i \cap H_j}{(G_{i+1} \cap H_j)(G_i \cap H_{j+1})}$$

PROOF: Lemma 15.130

⟨1⟩11. For all i, j we have
$$\frac{(G_i \cap H_j)G_{i+1}}{(G_i \cap H_{j+1})G_{i+1}} \cong \frac{(G_i \cap H_j)H_{j+1}}{(G_{i+1} \cap H_j)H_{j+1}}$$

⟨1⟩12. S and T are equivalent.

□

Corollary 15.131.1 (Jordan-Hölder)**.** *Any two composition series for a group are equivalent.*

Definition 15.132 (Composition Factors)**.** Let G be a group that has a composition series. The multiset of *composition factors* of G is the multiset of quotients of any composition series.

Example 15.133. Non-isomorphic groups can have the same composition factors. For example, $C_2 \times C_2$ and C_4 both have composition factors $\{|C_2, C_2|\}$.

Proposition 15.134. *Let G be a group. Let N be a normal subgroup of G. Then G has a composition series if and only if N and G/N both have composition series, in which case the composition factors of G are the union of the composition factors of N and the composition factors of G/N.*

PROOF:

⟨1⟩1. If G has a composition series then N and G/N have composition series.

⟨2⟩1. LET: $G = G_0 \supsetneq G_1 \supsetneq G_2 \supsetneq \cdots \supsetneq G_n = \{e\}$ be a composition series for G.

⟨2⟩2. N has a composition series.

⟨3⟩1. For all i, we have $\frac{G_i \cap N}{G_{i+1} \cap N}$ is either trivial or isomorphic to G_i/G_{i+1}.

⟨4⟩1. The homomorphism $G_i \cap N \hookrightarrow G_i \twoheadrightarrow G_i/G_{i+1}$ has kernel $G_{i+1} \cap N$.

⟨4⟩2. There is an injective homomorphism $(G_i \cap N)/(G_{i+1} \cap N) \to G_i/G_{i+1}$.

PROOF: First Isomorphism Theorem.

⟨4⟩3. $(G_i \cap N)/(G_{i+1} \cap N)$ is either trivial or isomorphic to G_i/G_{i+1}.

PROOF: Since G_i/G_{i+1} is simple.

⟨3⟩2. Eliminating all duplicates from the series $N = G_0 \cap N \supseteq G_1 \cap N \supseteq G_2 \cap N \supseteq \cdots \supseteq G_n \cap N = \{e\}$ gives a composition series for N.

⟨2⟩3. G/N has a composition series.

⟨3⟩1. For all i we have $\frac{(G_i N)/N}{(G_{i+1} N)/N}$ is either trivial or isomorphic to G_i/G_{i+1}.

⟨4⟩1. LET: $0 \leq i < n$

⟨4⟩2. $\frac{(G_i N)/N}{(G_{i+1} N)N} \cong G_i N/G_{i+1} N$

PROOF: Third Isomorphism Theorem.

⟨4⟩3. There exists a surjective homomorphism

$$\frac{G_i}{G_{i+1}} \twoheadrightarrow \frac{G_i N}{G_{i+1} N} \ .$$

⟨5⟩1. LET: f be the homomorphism $G_i \hookrightarrow G_i N \twoheadrightarrow G_i N/G_{i+1} N$

⟨5⟩2. f is surjective.

⟨5⟩3. $f(G_{i+1}) = \{e\}$

⟨5⟩4. Q.E.D.

PROOF: By the universal property of quotient groups.

⟨4⟩4. $G_i N/G_{i+1} N$ is either trivial or isomorphic to G_i/G_{i+1}.

PROOF: Proposition 15.117.

⟨3⟩2. Eliminating all duplicates from the series $G/N = G_0 N/N \supseteq G_1 N/N \supseteq G_2 N/N \supseteq \cdots \supseteq G_n N/N = \{e\}$ gives a composition series for G/N.

⟨1⟩2. If N and G/N have composition series, then G has a composition series, and the composition factors of G are the union of the composition factors of N and the composition factors of G/N.

⟨2⟩1. LET: $N = N_0 \supsetneq N_1 \supsetneq N_2 \supsetneq \cdots \supsetneq N_n = \{e\}$ be a composition series for N.

⟨2⟩2. LET: $G/N = H_0 \supsetneq H_1 \supsetneq H_2 \supsetneq \cdots \supsetneq H_m = \{e\}$ be a composition series for G/N.

⟨2⟩3. $G = \pi^{-1}(H_0) \supsetneq \pi^{-1}(H_1) \supsetneq \cdots \pi^{-1}(H_m) = N_0 \supsetneq N_1 \supsetneq N_2 \supsetneq \cdots \supsetneq N_n$ is a composition series for G.

□

Proposition 15.135. *Let G_1 and G_2 be groups. Then $G_1 \times G_2$ has a composition series if and only if G_1 and G_2 both have composition series.*

PROOF:

⟨1⟩1. If $G_1 \times G_2$ has a composition series then G_1 has a composition series.

⟨2⟩1. LET: $G_1 \times G_2 = A_0 \supsetneq A_1 \supsetneq \cdots \supsetneq A_n = \{e\}$ be a composition series.

⟨2⟩2. For each i, we have $\pi_1(A_i)/\pi_1(A_{i+1})$ is either isomorphic to A_i/A_{i+1} or trivial.

⟨2⟩3. Eliminating duplicates from $G_1 = \pi_1(A_0) \supseteq \pi_1(A_1) \supseteq \cdots \supseteq \pi_1(A_n) = \{e\}$ gives a composition series for G_1.

⟨1⟩2. If $G_1 \times G_2$ has a composotion series then G_2 has a composition series.

PROOF: Similar.

⟨1⟩3. If G_1 and G_2 have composition series then $G_1 \times G_2$ has a composition series.

$\langle 2 \rangle 1.$ LET: $G_1 = H_0 \supsetneq H_1 \supsetneq \cdots \supsetneq H_m = \{e\}$ be a composition series for G_1.

$\langle 2 \rangle 2.$ LET: $G_2 = K_0 \supsetneq K_1 \supsetneq \cdots \supsetneq K_n = \{e\}$ be a composition series for G_2.

$\langle 2 \rangle 3.$ $G_1 \times G_2 = H_0 \times K_0 \supsetneq H_1 \times K_0 \supsetneq \cdots \supsetneq H_m \times K_0 \supsetneq H_m \times K_1 \supsetneq \cdots \supsetneq H_m \times K_n = \{e\}$ is a composition series for $G_1 \times G_2$.

$\square$

Definition 15.136 (Cyclic Series)**.** A normal series of subgroups is *cyclic* iff every quotient is cyclic.

Chapter 16

Abelian Groups

Definition 16.1 (Abelian Group)**.** A group is *Abelian* iff any two elements commute.

In an Abelian group G, we often denote the group operation by $+$, the identity element by 0 and the inverse of an element g by $-g$. We write ng for g^n ($g \in G$, $n \in \mathbb{Z}$).

Example 16.2. Every group of order ≤ 4 is Abelian.

Example 16.3. For any positive integer n, we have $\mathbb{Z}/n\mathbb{Z}$ is an Abelian group under addition.

Example 16.4. S_n is not Abelian for $n \geq 3$. If $x = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}$ then $xy = \begin{pmatrix} 2 & 3 \end{pmatrix}$ and $yx = \begin{pmatrix} 1 & 3 \end{pmatrix}$.

Example 16.5. There are 42 Abelian groups of order 1024 up to isomorphism.

Proposition 16.6. *Let G be a group. If $g^2 = e$ for all $g \in G$ then G is Abelian.*

Proof: For any $g, h \in G$ we have

$$\begin{aligned} ghgh &= e \\ \therefore hgh &= g && \text{(multiplying on the left by } g\text{)} \\ \therefore hg &= gh && \text{(multiplying on the right by } h\text{)} \end{aligned}$$ □

Proposition 16.7. *Let G be a group. Then G is Abelian if and only if the function that maps g to g^{-1} is a group homomorphism.*

Proof:

⟨1⟩1. If G is Abelian then the function that maps g to g^{-1} is a group homomorphism.

Proof: Since $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$.

⟨1⟩2. If the function that maps g to g^{-1} is a group homomorphism then G is Abelian.

PROOF: Since $gh = (g^{-1})^{-1}(h^{-1})^{-1} = (g^{-1}h^{-1})^{-1} = hg$.
□

Proposition 16.8. *Let G be a group. Then G is Abelian if and only if the function that maps g to g^2 is a group homomorphism.*

PROOF:
⟨1⟩1. If G is Abelian then the function that maps g to g^2 is a group homomorphism.
PROOF: Since $(gh)^2 = g^2h^2$.
⟨1⟩2. If the function that maps g to g^2 is a group homomorphism then G is Abelian.
PROOF: Since we have $(gh)^2 = ghgh = g^2h^2$ and so $hg = gh$.
□

Proposition 16.9. *Let G be a group. Then G is Abelian if and only if the homomorphism $\gamma : G \to \mathrm{Aut}_{\mathbf{Grp}}(G)$ is the trivial homomorphism.*

PROOF:
⟨1⟩1. If G is Abelian then γ is trivial.
PROOF: Since $\gamma_g(a) = gag^{-1} = a$.
⟨1⟩2. If γ is trivial then G is Abelian.
PROOF: If $\gamma_g(a) = gag^{-1} = a$ for all g and a then $ga = ag$ for all g, a.
□

Proposition 16.10. *Let G be an Abelian group. Let $g, h \in G$. If g has maximal finite order in G, and h has finite order, then $|h| \mid |g|$.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $|h| \nmid |g|$.
⟨1⟩2. PICK a prime p such that $|g| = p^m r$, $|h| = p^n s$ where $p \nmid r$, $p \nmid s$ and $m < n$.
⟨1⟩3. $|g^{p^m} h^s| = p^n r$
PROOF: Proposition 14.22.
⟨1⟩4. $|g| < |g^{p^m} h^s|$
⟨1⟩5. Q.E.D.
PROOF: This contradicts the maximality of $|g|$.
□

Proposition 16.11. *Given a set A and an Abelian group H, the set H^A is an Abelian group under*

$$(\phi + \psi)(a) = \phi(a) + \psi(a) \qquad (\phi, \psi \in H^A, a \in A) \ .$$

PROOF:
⟨1⟩1. $\phi + (\psi + \chi) = (\phi + \psi) + \chi$
⟨1⟩2. $\phi + \psi = \psi + \phi$
⟨1⟩3. LET: $0 : A \to H$ be the function $0(a) = 0$.
⟨1⟩4. $\phi + 0 = 0 + \phi = \phi$

$\langle 1\rangle 5$. Given $\phi : A \to H$, define $-\phi : A \to H$ by $(-\phi)(a) = -(\phi(a))$.
$\langle 1\rangle 6$. $\phi + (-\phi) = (-\phi) + \phi = 0$
□

Proposition 16.12. *Given a group G and an Abelian group H, the set $\mathbf{Grp}[G, H]$ is a subgroup of H^G.*

Proof:
$\langle 1\rangle 1$. Given $\phi, \psi : G \to H$ group homomorphisms, we have $\phi - \psi$ is a group homomorphism.
Proof:

$$\begin{aligned}(\phi - \psi)(g + g') &= \phi(g + g') - \psi(g + g') \\ &= \phi(g) + \phi(g') - \psi(g) - \psi(g') \\ &= \phi(g) - \psi(g) + \phi(g') - \psi(g') \\ &= (\phi - \psi)(g) + (\phi - \psi)(g')\end{aligned}$$

□

Proposition 16.13. *Let G be a group. The following are equivalent.*

1. $\mathrm{Inn}(G)$ *is cyclic.*
2. $\mathrm{Inn}(G)$ *is trivial.*
3. G *is Abelian.*

Proof:
$\langle 1\rangle 1$. $1 \Rightarrow 2$
$\langle 2\rangle 1$. Assume: $\mathrm{Inn}(G) = \langle \gamma_g \rangle$
$\langle 2\rangle 2$. g commutes with every element of G
$\langle 3\rangle 1$. Let: $x \in G$
$\langle 3\rangle 2$. Pick $n \in \mathbb{Z}$ such that $\gamma_x = \gamma_g^n$
$\langle 3\rangle 3$. $\forall y \in G. xyx^{-1} = g^n y g^{-n}$
$\langle 3\rangle 4$. $xgx^{-1} = g$
$\langle 2\rangle 3$. $\gamma_g = \mathrm{id}_G$
$\langle 1\rangle 2$. $2 \Rightarrow 3$
$\langle 2\rangle 1$. Assume: $\forall g \in G. \gamma_g = \mathrm{id}_G$
$\langle 2\rangle 2$. Let: $x, y \in G$
$\langle 2\rangle 3$. $\gamma_x(y) = y$
$\langle 2\rangle 4$. $xyx^{-1} = y$
$\langle 2\rangle 5$. $xy = yx$
$\langle 1\rangle 3$. $3 \Rightarrow 2$
Proof: If $xy = yx$ for all x, y then $\gamma_x(y) = y$ for all x, y.
$\langle 1\rangle 4$. $2 \Rightarrow 1$
Proof: Easy.
□

Corollary 16.13.1. *If $\mathrm{Aut}_{\mathbf{Grp}}(G)$ is cyclic then G is Abelian.*

Proposition 16.14. *Every subgroup of an Abelian group is normal.*

PROOF: Let G be an Abelian group and N a subgroup of G. Given $g \in G$ and $n \in N$ we have $gng^{-1} = n \in N$. ∎

Proposition 16.15. *For any group G, the group $G/[G,G]$ is Abelian.*

PROOF: For any $g, h \in G$ we have

$$gh(hg)^{-1} \in [G,G]$$
$$\therefore gh[G,G] = hg[G,G]$$

∎

Proposition 16.16. *Let G be a finite Abelian group. Let p be a prime divisor of $|G|$. Then G has an element of order p.*

PROOF:
⟨1⟩1. ASSUME: as induction hypothesis the result holds for all groups smaller than G.
⟨1⟩2. PICK $g \in G - \{0\}$.
⟨1⟩3. PICK an element $h \in \langle g \rangle$ with prime order q.
⟨1⟩4. CASE: $q = p$
  PROOF: h is the required element.
⟨1⟩5. CASE: $q \neq p$
  ⟨2⟩1. PICK $r \in G$ such that $r + \langle h \rangle$ has order p in $G/\langle h \rangle$.
    PROOF: By induction hypothesis since $|G/\langle h \rangle| = |G|/q$.
  ⟨2⟩2. $pr \in \langle h \rangle$
  ⟨2⟩3. PICK k such that $pr = kh$
  ⟨2⟩4. $pqr = e$
  ⟨2⟩5. qr has order p.
∎

Corollary 16.16.1. *For n an odd integer, any Abelian group of order $2n$ has exactly one element of order 2.*

PROOF: If x and y are distinct elements of order 2 then $\langle x, y \rangle = \{e, x, y, xy\}$ has size 4 and so $4 \mid 2n$ which is a contradiction. ∎

Example 16.17. It is not true that, if G is a finite group and $d \mid |G|$, then G has an element of order d. The quaternionic group has no element of order 4.

Proposition 16.18. *If G is a finite Abelian group and $d \mid |G|$ then G has a subgroup of size d.*

PROOF:
⟨1⟩1. ASSUME: as induction hypothesis the result is true for all $d' < d$.
⟨1⟩2. ASSUME: w.l.o.g. $d \neq 1$.
⟨1⟩3. PICK a prime p such that $p \mid d$.
⟨1⟩4. PICK an element $g \in G$ of order p.
⟨1⟩5. $d/p \mid |G/\langle g \rangle|$
⟨1⟩6. PICK a subgrop H of $G/\langle g \rangle$ of size d/p.
⟨1⟩7. $\pi^{-1}(H)$ is a subgroup of G of size d.
∎

Proposition 16.19. *Let $(G, \cdot)$ be a group. Let $\circ : G^2 \to G$ be a group homomorphism such that $(G, \circ)$ is a group. Then $\circ$ and $\cdot$ coincide, and G is Abelian.*

PROOF:
⟨1⟩1. For all $g_1, g_2, h_1, h_2 \in G$ we have
$$(g_1 g_2) \circ (h_1 h_2) = (g_1 \circ h_1)(g_2 \circ h_2)$$
⟨1⟩2. $e \circ e = e$
PROOF:
$$\begin{aligned} e \circ e &= (ee) \circ (ee) \\ &= (e \circ e)(e \circ e) \end{aligned}$$
Hence $e \circ e = e$ by Cancellation.
⟨1⟩3. e is the identity of $(G, \circ)$
⟨1⟩4. For all $g, h \in G$ we have
$$g \circ h = gh$$
PROOF:
$$\begin{aligned} g \circ h &= (ge) \circ (eh) \\ &= (g \circ e)(e \circ h) \\ &= gh \end{aligned}$$
⟨1⟩5. For all $g, h \in G$ we have $gh = hg$.
PROOF:
$$\begin{aligned} gh &= (e \circ g)(h \circ e) \\ &= (eh) \circ (ge) \\ &= h \circ g \\ &= hg \end{aligned}$$
□

Corollary 16.19.1. *If $(G, m : G^2 \to G, e : 1 \to G, i : G \to G)$ is a group object in **Grp** then m is the multiplication of G, $e(*)$ is the identity of G, $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Grp** where $e(*) = e$ and $i(g) = g^{-1}$.*

Proposition 16.20. *Let G be a group. If every element of G has order ≤ 2 then G is Abelian.*

PROOF:
⟨1⟩1. LET: $x, y \in G$
PROVE: $xy = yx$
⟨1⟩2. ASSUME: w.l.o.g. $x \neq e \neq y$.
⟨1⟩3. $x^2 = e = y^2$
⟨1⟩4. $x^{-1} = x$ and $y^{-1} = y$.
⟨1⟩5. CASE: $xy = e$
PROOF: Then $y = x^{-1}$ and so $xy = yx = e$.
⟨1⟩6. CASE: $xy \neq e$
⟨2⟩1. $(xy)^2 = e$
⟨2⟩2. $xyxy = e$

$\langle 2\rangle 3.$ $xy = y^{-1}x^{-1}$
$\langle 2\rangle 4.$ $xy = yx$
□

Proposition 16.21. *Every Abelian group is solvable.*

Proof: If G is Abelian then $G' = \{e\}$. □

Proposition 16.22. *The only non-trivial simple finite Abelian groups are $\mathbb{Z}/p\mathbb{Z}$ for p a prime.*

Proof:
$\langle 1\rangle 1.$ Let: G be a non-trivial simple finite Abelian group.
$\langle 1\rangle 2.$ Pick a prime p that divides $|G|$.
$\langle 1\rangle 3.$ Pick an element $a \in G$ of order p.
Proof: Cauchy's Theorem.
$\langle 1\rangle 4.$ $\langle a\rangle = G$
□

Proposition 16.23. *If $N \rtimes_\theta H$ is Abelian then $N \rtimes_\theta H \cong N \times H$.*

Proof: By Proposition 15.35 since $\theta(h)(n) = hnh^{-1} = n$. □

Lemma 16.24. *Let p be a prime integer and $r \geq 1$. Let G be a noncyclic Abelian group of order p^{r+1}, and let $g \in G$ be an element of order p^r. Then there exists an element $h \in G$ such that $h \notin \langle g\rangle$ and $|h| = p$.*

Proof:
$\langle 1\rangle 1.$ Let: $K = \langle G\rangle$
$\langle 1\rangle 2.$ Pick $h' \in G$ such that $h' \notin K$.
$\langle 1\rangle 3.$ $|G/K| = p$
$\langle 1\rangle 4.$ $ph' \in K$
$\langle 1\rangle 5.$ Let: $k = ph'$
$\langle 1\rangle 6.$ $|k|$ is a power of p.
$\langle 1\rangle 7.$ $|k| \neq p^r$
Proof: If $|k| = p^r$ then $|h'| = p^{r+1}$ contradicting the hypothesis that G is not cyclic.
$\langle 1\rangle 8.$ Pick $s < r$ such that $|\langle k\rangle| = p^s$.
$\langle 1\rangle 9.$ $\langle k\rangle = \langle p^{r-s}g\rangle$
Proof: Proposition 15.90.
$\langle 1\rangle 10.$ Pick $m \in \mathbb{Z}$ such that $k = mpg$.
$\langle 1\rangle 11.$ Let: $h = h' - mg$
$\langle 1\rangle 12.$ $|h| = p$
Proof:

$$\begin{aligned} ph &= ph' - pmg \\ &= k - k \\ &= 0 \end{aligned}$$

□

16.1 The Category of Abelian Groups

Definition 16.25 (Category of Abelian Groups)**.** Let **Ab** be the full subcategory of **Grp** whose objects are the Abelian groups.

Proposition 16.26. *If $(G, m : G^2 \to G, e : 1 \to G, i : G \to G)$ is a group object in* **Ab** *then m is the multiplication of G, $e(*)$ is the identity of G, $i(g) = g^{-1}$, and G is Abelian.*

Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Ab** *where $e(*) = e$ and $i(g) = g^{-1}$.*

PROOF: Immediate from Corollary 16.19.1. □

Definition 16.27 (Direct Sum)**.** Given Abelian groups G and H, we also call the direct product of G and H the *direct sum* and denote it $G \oplus H$.

Proposition 16.28. *Given Abelian groups G and H, the direct sum $G \oplus H$ is the coproduct of G and H in* **Ab***.*

PROOF:
⟨1⟩1. LET: $\kappa_1 : G \to G \oplus H$ be the group homomorphism $\kappa_1(g) = (g, e_H)$.
⟨1⟩2. LET: $\kappa_2 : H \to G \oplus H$ be the group homomorphism $\kappa_2(h) = (e_G, h)$.
⟨1⟩3. Given group homomorphism $\phi : G \to K$ and $\psi : H \to K$, define $[\phi, \psi] : G \oplus H \to K$ by $[\phi, \psi](g, h) = \phi(g) + \psi(h)$.
⟨1⟩4. $[\phi, \psi]$ is a group homomorphism.
PROOF:

$$\begin{aligned} [\phi,\psi]((g,h)+(g',h')) &= [\phi,\psi](g+g',h+h') \\ &= \phi(g+g')+\psi(h+h') \\ &= \phi(g)+\phi(g')+\psi(h)+\psi(h') \\ &= \phi(g)+\psi(h)+\phi(g')+\psi(h') \\ &= [\phi,\psi](g,h)+[\phi,\psi](g',h') \end{aligned}$$

⟨1⟩5. $[\phi, \psi] \circ \kappa_1 = \phi$
PROOF:

$$\begin{aligned} [\phi,\psi](\kappa_1(g)) &= [\phi,\psi](g,e_h) \\ &= \phi(g)+\psi(e_H) \\ &= \phi(g)+e_K \\ &= \phi(g) \end{aligned}$$

⟨1⟩6. $[\phi, \psi] \circ \kappa_2 = \psi$
PROOF: Similar.
⟨1⟩7. If $f : G \oplus H \to K$ is a group homomorphism with $f \circ \kappa_1 = \phi$ and $f \circ \kappa_2 = \psi$ then $f = [\phi, \psi]$.
PROOF:

$$\begin{aligned} f(g,h) &= f((g,e_H)+(e_G,h)) \\ &= f(\kappa_1(g))+f(\kappa_2(h)) \\ &= \phi(g)+\psi(h) \end{aligned}$$

$\square$

Theorem 16.29. *Every finitely generated Abelian group is a direct sum of cyclic groups.*

PROOF: TODO $\square$

Proposition 16.30. *Let G be an Abelian group. Let H and K be subgroups of G such that $|H|$ and $|K|$ are relatively prime. Then $H + K \cong H \oplus K$.*

PROOF: Proposition 15.95. $\square$

Corollary 16.30.1. *Every finite Abelian group is the direct sum of its Sylow subgroups.*

16.2 Free Abelian Groups

Proposition 16.31. *Let A be a set. Let $\mathcal{F}^A$ be the category whose objects are pairs (G, j) where G is an Abelian group and j is a function $A \to G$, with morphisms $f : (G, j) \to (H, k)$ the group homomorphisms $f : G \to H$ such that $f \circ j = k$. Then $\mathcal{F}^A$ has an initial object.*

PROOF:
⟨1⟩1. LET: $\mathbb{Z}^{\oplus A}$ be the subgroup of $\mathbb{Z}^A$ consisting of all functions $\alpha : A \to \mathbb{Z}$ such that $\alpha(a) = 0$ for only finitely many $a \in A$.
⟨1⟩2. LET: $i : A \to \mathbb{Z}^{\oplus A}$ be the function such that $i(a)(b) = 1$ if $a = b$ and 0 if $a \neq b$.
⟨1⟩3. LET: G be any Abelian group and $j : A \to G$ any function.
⟨1⟩4. The unique homomorphism $\phi : \mathbb{Z}^{\oplus A} \to G$ required is defined by $\phi(\alpha) = \sum_{a \in A} \alpha(a) j(a)$

$\square$

Definition 16.32 (Free Abelian Group)**.** For any set A, the *free Abelian group* on A is the initial object $(F^{ab}(A), i)$ in $\mathcal{F}^A$.

Proposition 16.33. *For any sets A and B, we have that $F^{ab}(A + B)$ is the coproduct of $F^{ab}(A)$ and $F^{ab}(B)$ in* **Grp**.

$$\begin{tikzcd}
 & G & \\
F^{ab}(A) \arrow[ur, "f"] \arrow[r, "\kappa_1"] & F^{ab}(A+B) \arrow[u, "k"] & F^{ab}(B) \arrow[ul, "g"'] \arrow[l, "\kappa_2"] \\
A \arrow[u, "i_A"] \arrow[r, "k_1"] & A+B \arrow[u, "j"] & B \arrow[u, "i_B"'] \arrow[l, "k_2"]
\end{tikzcd}$$

PROOF:

⟨1⟩1. LET: $i_A : A \to F^{ab}(A)$, $i_B : B \to F^{ab}(B)$, $j : A + B \to F^{ab}(A + B)$ be the canonical injections.
⟨1⟩2. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
⟨1⟩3. LET: G be any group and $f : F^{ab}(A) \to G$, $g : F^{ab}(B) \to G$ any group homomorphisms.
⟨1⟩4. LET: $h : A + B \to G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.
⟨1⟩5. LET: $k : F^{ab}(A + B) \to G$ be the unique group homomorphism such that $k \circ j = h$.
⟨1⟩6. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
⟨1⟩7. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.
□

Proposition 16.34. *For A and B finite sets, if $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$ then $A \cong B$.*

PROOF:
⟨1⟩1. For any set C, define $\sim$ on $F^{\text{ab}}(C)$ by: $f \sim f'$ iff there exists $g \in F^{\text{ab}}(C)$ such that $f - f' = 2g$.
⟨1⟩2. For any set C, $\sim$ is an equivalence relation on $F^{\text{ab}}(C)$.
⟨1⟩3. For any set C, we have $F^{\text{ab}}(C) / \sim$ is finite if and only if C is finite, in which case $|F^{\text{ab}}(C) / \sim| = 2^{|C|}$.
PROOF: There is a bijection between $F^{\text{ab}}(C) / \sim$ and the finite subsets of C, which maps f to $\{c \in C : f(c) \text{ is odd}\}$.
⟨1⟩4. If $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$ then $A \cong B$.
PROOF: If $|F^{\text{ab}}(A) / \sim| = |F^{\text{ab}}(B) / \sim|$ then $2^{|A|} = 2^{|B|}$ and so $|A| = |B|$.
□

Proposition 16.35. *Let G be an Abelian group. Then G is finitely generated if and only if there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n.*

PROOF:
⟨1⟩1. If G is finitely generated then there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n.
PROOF: Let $G = \langle a_1, \ldots, a_n \rangle$. Define $\phi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$ by $\phi(i_1, \ldots, i_n) = i_1 \cdot a_1 + \cdots + i_n \cdot a_n$.
⟨1⟩2. If there exists a surjective homomorphism $\phi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n then G is finitely generated.
PROOF: G is generated by $\phi(1, 0, \ldots, 0)$, $\phi(0, 1, 0, \ldots, 0)$, $\ldots$, $\phi(0, \ldots, 0, 1)$.
□

Proposition 16.36. *Let A be a set. Let $i : A \hookrightarrow F(A)$ be the free group on A. Then $\pi \circ i : A \to F(A)/[F(A), F(A)]$ is the free Abelian group on A.*

$$\begin{tikzcd}
F(A)/[F(A),F(A)] \arrow[drr, "h"] & & \\
F(A) \arrow[u, "\pi"] \arrow[rr, "g"] & & G \\
A \arrow[u, "i"] \arrow[urr, "f"'] & &
\end{tikzcd}$$

PROOF:
⟨1⟩1. LET: G be an Abelian group and $f : A \to G$ a function.
⟨1⟩2. LET: $g : F(A) \to G$ be the unique group homomorphism such that $g \circ i = f$.
⟨1⟩3. $[F(A), F(A)] \subseteq \ker g$
PROOF: For all $x, y \in F(A)$ we have $g(xyx^{-1}y^{-1}) = g(x)+g(y)-g(x)-g(y) = 0$.
⟨1⟩4. LET: $h : F(A)/[F(A), F(A)]$ be the unique group homomorphism such that $h \circ \pi = g$.
⟨1⟩5. h is the unique group homomorphism such that $h \circ \pi \circ i = f$.
□

Corollary 16.36.1. *Let A and B be sets. Let $F(A)$ and $F(B)$ be the free groups on A and B respectively. If $F(A) \cong F(B)$ then $A \cong B$.*

PROOF: Proposition 16.34. □

16.3 Cokernels

Proposition 16.37. *Let $\phi : G \to H$ be a homomorphism between Abelian groups. Then there exists an Abelian group K and homomorphism $\pi : H \to K$ that is initial with respect to all homomorphism $\alpha : H \to L$ such that $\alpha \circ \phi = 0$.*

PROOF:
⟨1⟩1. LET: $K = H/\operatorname{im}\phi$ and π be the canonical homomorphism.
⟨1⟩2. LET: $\pi \circ \phi = 0$
⟨1⟩3. LET: $\alpha : H \to L$ satisfy $\alpha \circ \phi = 0$
⟨1⟩4. $\operatorname{im}\phi \subseteq \ker \alpha$
⟨1⟩5. There exists a unique $\overline{\alpha} : H/\operatorname{im}\phi \to L$ such that $\overline{\alpha} \circ \pi = \alpha$
□

Definition 16.38 (Cokernel)**.** For any homomorphism $\phi : G \to H$ in **Ab**, the *cokernel* of ϕ is the Abelian group $\operatorname{coker}\phi$ and homomorphism $\pi : H \to \operatorname{coker}\phi$ that is initial among homomorphisms $\alpha : H \to L$ such that $\alpha \circ \phi = 0$.

Proposition 16.39. *$\pi : H \to \operatorname{coker}\phi$ is initial among functions $f : H \to X$ such that, for all $x, y \in H$, if $x + \operatorname{im}\phi = y + \operatorname{im}\phi$ then $f(x) = f(y)$.*

PROOF: Easy. □

Proposition 16.40. *Let $\phi : G \to H$ be a homomorphism of Abelian groups. Then the following are equivalent.*

- *ϕ is an epimorphism.*
- $\operatorname{coker}\phi$ *is trivial.*
- *ϕ is surjective.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. ASSUME: ϕ is epi.
  ⟨2⟩2. LET: $\pi : H \to \operatorname{coker}\phi$ be the canonical homomorphism.
  ⟨2⟩3. $\pi \circ \phi = 0 \circ \phi$
  ⟨2⟩4. $\pi = 0$
  ⟨2⟩5. $\operatorname{coker}\phi = \operatorname{im}\pi$ is trivial.
⟨1⟩2. $2 \Rightarrow 3$
  PROOF: If $\operatorname{coker}\phi = H/\operatorname{im}\phi$ is trivial then $\operatorname{im}\phi = H$.
⟨1⟩3. $3 \Rightarrow 1$
  PROOF: If it is surjective then it is epi in **Set**.
□

16.4 Commutator Subgroups

Proposition 16.41. *Let G be a group. Let G' be the commutator subgroup of G. Then G/G' is Abelian.*

PROOF: Since $ghg^{-1}h^{-1}G' = G'$ so $ghG' = hgG'$. □

Proposition 16.42. *Let G be a group and A an Abelian group. Let $\alpha : G \to A$ be a homomorphism. Then $G' \subseteq \ker\alpha$.*

PROOF: Since $\phi([g,h]) = \phi(g)\phi(h)\phi(g)^{-1}\phi(h)^{-1} = e$. □

Corollary 16.42.1. *Let G be a group. The canonical projection $G \twoheadrightarrow G/G'$ is initial in the category of homomorphisms from G to an Abelian group.*

Definition 16.43 (Abelian Series)**.** A normal series of subgroups is *Abelian* iff every quotient is Abelian.

Lemma 16.44. *Let G be a group. Let H be a normal subgroup of G. If G/H is Abelian then $G' \subseteq G/H$.*

PROOF: Given $g, h \in G$ we have

$$ghH = hgH$$
$$\therefore ghg^{-1}h^{-1} \in H$$

□

16.5 Derived Series

Definition 16.45 (Derived Series)**.** Let G be a group. The *derived series* of G is the series of subgroups

$$G \supseteq G' \supseteq G'' \supseteq G''' \supseteq \cdots$$

where G' is the commutator subgroup of G.

We write $G^{(i)}$ for the $i+1$st entry in the derived series

Proposition 16.46. *Each $G^{(i)}$ is characteristic.*

PROOF:
⟨1⟩1. G is characteristic in G.
 PROOF: Trivial.
⟨1⟩2. If $G^{(i)}$ is characteristic in G then $G^{(i+1)}$ is characteristic in G.
 ⟨2⟩1. ASSUME: $G^{(i)}$ is characteristic.
 ⟨2⟩2. LET: $\phi : G \cong G$ be an automorphism of G.
 ⟨2⟩3. For all $g, h \in G^{(i)}$ we have $\phi([g, h]) \in G^{(i+1)}$.
  PROOF: Since $\phi([g, h]) = [\phi(g), \phi(h)]$ and $\phi(g), \phi(h) \in G^{(i)}$.
 ⟨2⟩4. $\phi(G^{(i+1)}) \subseteq G^{(i+1)}$
□

16.6 Solvable Groups

Definition 16.47 (Solvable)**.** A group is *solvable* iff its derived series terminates in $\{e\}$.

Theorem 16.48 (Feit-Thompson)**.** *Every finite group of odd order is solvable.*

Corollary 16.48.1. *Every non-Abelian finite simple group has even order.*

PROOF: A non-Abelian finite simple group of odd order is solvable, hence its composition factors are all Abelian. But a simple group is its own only composition factor. □

Proposition 16.49. *Let H be a nontrivial normal subgroup of a solvable group G. Then H contains a nontrivial Abelian subgroup that is normal in G.*

PROOF:
⟨1⟩1. LET: r be the largest number such that $H \cap G^{(r)}$ is non-trivial.
⟨1⟩2. LET: $K = H \cap G^{(r)}$
⟨1⟩3. K is Abelian.
 PROOF: Since $[K, K] \subseteq G^{(r+1)} = \{e\}$.
⟨1⟩4. K is normal.
 PROOF: Proposition 16.46.
□

Theorem 16.50 (Burnside)**.** *Let p and q be primes. Every group of order $p^a q^b$ is solvable.*

Proposition 16.51. *The semidirect product of two solvable groups is solvable.*

PROOF:
⟨1⟩1. LET: N and H be solvable groups.
⟨1⟩2. LET: $\theta : H \to \mathrm{Aut}_{\mathbf{Grp}}(N)$

$$\begin{aligned}[(n_1,h_1),(n_2,h_2)] &= (n_1,h_1)(n_2,h_2)(n_1,h_1)^{-1}(n_2,h_2)^{-1} \\ &= (n_1,h_1)(n_2,h_2)(\theta(h_1{}^{-1})(n_1{}^{-1}),h_1{}^{-1})(\theta(h_2{}^{-1})(n_2{}^{-1}),h_2{}^{-1}) \\ &= (n_1\theta(h_1)(n_2),h_1h_2)(\theta(h_1{}^{-1})(n_1{}^{-1})\theta(h_1{}^{-1})(\theta(h_2{}^{-1})(n_2{}^{-1})),h_1{}^{-1}h_2{}^{-1}) \\ &= (n_1\theta(h_1)(n_2),h_1h_2)(\theta(h_1{}^{-1})(n_1{}^{-1}\theta(h_2{}^{-1})(n_2{}^{-1})),h_1{}^{-1}h_2{}^{-1}) \\ &= (n_1\theta(h_1)(n_2)\theta(h_1h_2)(\theta(h_1{}^{-1})(n_1{}^{-1}\theta(h_2{}^{-1})(n_2{}^{-1}))),[h_1,h_2]) \\ &= (n_1\theta_{h_1}(n_2)\theta_{h_1h_2h_1{}^{-1}}(n_1{}^{-1})\theta_{[h_1,h_2]}(n_2{}^{-1}),[h_1,h_2])\end{aligned}$$

Proposition 16.52. *Let G be a finite group. The following are equivalent.*

1. *All composition factors of G are cyclic.*
2. *G has a cyclic series of subgroups ending in $\{e\}$.*
3. *G has an Abelian series of subgroups ending in $\{e\}$.*
4. *G is solvable.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
PROOF: Trivial.
⟨1⟩2. $2 \Rightarrow 3$
PROOF: Trivial.
⟨1⟩3. $3 \Rightarrow 4$
⟨2⟩1. LET: $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\}$ be an Abelian series of subgroups.
⟨2⟩2. For all i we have $G^{(i)} \subseteq G_i$.
PROOF: Lemma 16.44.
⟨2⟩3. $G^{(n)} = \{e\}$
⟨1⟩4. $4 \Rightarrow 1$
PROOF: Extend the derived series of G to a composition series, using the fact that every simple Abelian group is cyclic.
□

Corollary 16.52.1. *All p-groups are solvable.*

PROOF: Their composition factors are simple p-groups, hence cyclic. □

Corollary 16.52.2. *Let G be a group and N a normal subgroup. Then G is solvable if and only if both N and G/N are solvable.*

PROOF: By Proposition 15.134. □

Corollary 16.52.3. *The semidirect product of two solvable groups is solvable.*

Corollary 16.52.4. *Let G be a finite solvable group. Then the composition factors of G are exactly C_p for p a prime factor of G (with the same multiplicities).*

PROOF: Since each composition factor is simple and cyclic hence removes one prime factor in $|G|$. $\square$

Chapter 17

Group Actions

17.1 Group Actions

Definition 17.1 (Action)**.** Let G be a group. Let A be an object of a category $\mathcal{C}$. A *(left) action* of G on A is a group homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(A)$.

It is *faithful* or *effective* iff it is injective.

Proposition 17.2. *Let A be a set. An action of the group G on the set A is given by a function $\cdot : G \times A \to A$ such that*

- $\forall a \in A . ea = a$
- $\forall g, h \in G . \forall a \in A . (gh)a = g(ha)$

PROOF: Just unfolding definitions. □

Example 17.3. Left multiplication defines a faithful action of any group on its own underlying set.

In fact, for any subgroup H of a group G, left multiplication defines an action of G on G/H.

Corollary 17.3.1 (Cayley's Theorem)**.** *Every group G is a subgroup of a symmetric group, namely* $\mathrm{Aut}_{\mathbf{Set}}(G)$.

Example 17.4. Conjugation $g * h = ghg^{-1}$ is an action of any group on its own underlying set.

Definition 17.5 (Transitive)**.** An action of a group G on a set A is *transitive* iff, for all $a, b \in A$, there exists $g \in G$ such that $ga = b$.

Example 17.6. Left multiplication of a group G is a transitive action of G on G.

Definition 17.7 (Orbit)**.** Given an action of a group G on a set A and $a \in A$, the *orbit* of a is

$$\mathrm{O}_G(a) := \{ga : g \in G\} \ .$$

Proposition 17.8. *Given an action of a group G on a set A, the orbits form a partition of A.*

PROOF:
⟨1⟩1. Every element of A is in some orbit.
 PROOF: Since $a \in \mathrm{O}_G(a)$.
⟨1⟩2. Distinct orbits are disjoint.
 ⟨2⟩1. LET: $a \in \mathrm{O}_G(b) \cap \mathrm{O}_G(c)$
 ⟨2⟩2. PICK $g, h \in G$ such that $a = gb = hc$.
 ⟨2⟩3. $\mathrm{O}_G(b) \subseteq \mathrm{O}_G(c)$
  PROOF: For all $k \in G$ we have $kb = kg^{-1}hc$.
 ⟨2⟩4. $\mathrm{O}_G(c) \subseteq \mathrm{O}_G(b)$
  PROOF: Similar.
□

Proposition 17.9. *Given an action of a group G on a set A and $a \in A$, the action is transitive on* $\mathrm{O}_G(a)$.

PROOF:
⟨1⟩1. The restriction of the action is an action on $\mathrm{O}_G(a)$.
 PROOF: Since $g(ha) = (gh)a$, the action maps $\mathrm{O}_G(a)$ to itself.
⟨1⟩2. The restricted action is transitive.
 PROOF: Given $ga, ha \in \mathrm{O}_G(a)$, we have $ha = (hg^{-1})(ga)$.
□

Definition 17.10 (Stabilizer Subgroup)**.** Given an action of a group G on a set A and $a \in A$, the *stabilizer subgroup* of a is

$$\mathrm{Stab}_G\left(a\right) := \{g \in G : ga = a\} \enspace .$$

Proposition 17.11. *Stabilizer subgroups are subgroups.*

PROOF: If $g, h \in \mathrm{Stab}_G\left(a\right)$ then $gh^{-1}a = a$ so $gh^{-1} \in \mathrm{Stab}_G\left(a\right)$. □

Proposition 17.12. *Let G act on a set A. Let $a \in A$ and $g \in G$. Then*

$$\mathrm{Stab}_G\left(ga\right) = g\mathrm{Stab}_G\left(a\right)g^{-1} \enspace .$$

PROOF:

$$\begin{aligned} h \in \mathrm{Stab}_G\left(ga\right) &\Leftrightarrow hga = ga \\ &\Leftrightarrow g^{-1}hga = a \\ &\Leftrightarrow g^{-1}hg \in \mathrm{Stab}_G\left(a\right) \\ &\Leftrightarrow h \in g\mathrm{Stab}_G\left(a\right)g^{-1} \end{aligned}$$ □

Corollary 17.12.1. *Let G be an action on a set A and $a \in A$. If* $\mathrm{Stab}_G\left(a\right)$ *is normal in G, then for any $b \in \mathrm{O}_G(a)$ we have* $\mathrm{Stab}_G\left(a\right) = \mathrm{Stab}_G\left(b\right)$.

Definition 17.13 (Free)**.** An action of a group G on a set A is *free* iff, whenever $ga = a$, then $g = e$.

Example 17.14. The action of left multiplication is free.

Proposition 17.15. *Let G be a group. Let H be a subgroup of G of finite index n. Then H includes a subgroup K that is normal in G and such that $|G : K|$ divides* $\gcd(|G|, n!)$.

PROOF:
$\langle 1 \rangle 1$. LET: $\sigma : G \to \mathrm{Aut}_{\mathbf{Set}}(G/H)$ be the action of left multiplication.
$\langle 1 \rangle 2$. LET: $K = \ker \sigma$
$\langle 1 \rangle 3$. $K \subseteq H$
  $\langle 2 \rangle 1$. LET: $g \in K$
  $\langle 2 \rangle 2$. $\sigma(g)(H) = H$
  $\langle 2 \rangle 3$. $gH = H$
  $\langle 2 \rangle 4$. $g \in H$
$\langle 1 \rangle 4$. K is normal in G.
  PROOF: Proposition 15.52.
$\langle 1 \rangle 5$. $|G : K| \mid |G|$
  PROOF: Lagrange's Theorem.
$\langle 1 \rangle 6$. $|G : K| \mid n!$
  PROOF: Since G/K is a subgroup of $\mathrm{Aut}_{\mathbf{Set}}(G/H)$.
$\square$

Corollary 17.15.1. *Let G be a finite group. Let H be a subgroup of G of index p where p is the smallest prime that divides $|G|$. Then H is normal in G.*

PROOF:
$\langle 1 \rangle 1$. PICK a subgroup K of H normal in G such that $|G : K|$ divides $\gcd(|G|, p!)$.
$\langle 1 \rangle 2$. $|G : K|$ divides p.
$\langle 1 \rangle 3$. $|G : H||H : K|$ divides p.
$\langle 1 \rangle 4$. $|H : K| = 1$
$\langle 1 \rangle 5$. $H = K$
$\langle 1 \rangle 6$. H is normal.
$\square$

Corollary 17.15.2. *Any subgroup of index 2 is normal.*

Proposition 17.16. *Let G be a group with finite set of generators A. Then left multiplication defines a free action of G on its Cayley graph.*

PROOF: Easy since if $g_2 = g_1 a$ then $hg_2 = hg_1 a$. $\square$

Corollary 17.16.1. *A free group acts freely on a tree.*

Theorem 17.17. *If a group G acts freely on a tree then G is free.*

Corollary 17.17.1. *Every subgroup of the free group on a finite set is free.*

PROOF: If H is a subgroup of $F(A)$ then left multiplication defines a free action of H on the Cayley graph of $F(A)$, which is a tree. $\square$

Proposition 17.18. *Let S be a finite set. Let G be a group acting on S. Let Z be the set of fixed points of the action:*

$$Z = \{a \in S : \forall g \in G.ga = a\} \enspace .$$

Let A be a set of representatives for the nontrivial orbits of the action. Then

$$|S| = |Z| + \sum_{a \in A} [G : \mathrm{Stab}_G(a)] \enspace .$$

PROOF: Immediate from the fact that the orbits partition S. ∎

Corollary 17.18.1. *Let p be a prime. Let S be a finite set. Let G be a p-group acting on S. Let Z be the set of fixed points of the action. Then $|Z| \cong |S| (\mathrm{mod}\, p)$.*

Corollary 17.18.2. *Let p be a prime. Let S be a finite set. Let G be a p-group acting on S. If p does not divide $|S|$ then the action has a fixed point.*

17.2 Category of G-Sets

Definition 17.19. Given a group G, let $G - \mathbf{Set}$ be the category with:

- objects all pairs (A, ρ) such that A is a set and $\rho : G \times A \to A$ is an action of G on A;
- morphisms $f : (A, \rho) \to (B, \sigma)$ are functions $f : A \to B$ that are *(G-)equivariant*, i.e.

$$\forall g \in G. \forall a \in A. f(\rho(g, a)) = \sigma(g, f(a)) \enspace .$$

Proposition 17.20. *A G-equivariant function $f : A \to B$ is an isomorphism in $G - \mathbf{Set}$ if and only if it is bijective.*

PROOF:
⟨1⟩1. LET: $f : A \to B$ be G-equivariant and bijective.
PROVE: f^{-1} is G-equivariant.
⟨1⟩2. LET: $g \in G$ and $b \in B$
⟨1⟩3. $f^{-1}(gb) = gf^{-1}(b)$
PROOF:

$$\begin{aligned} f(f^{-1}(gb)) &= gb \\ &= gf(f^{-1}(b)) \\ &= f(gf^{-1}(b)) \end{aligned}$$

∎

Proposition 17.21. *Let G be a group and A a transitive G-set. Let $a \in A$. Then A is isomorphic to $G / \mathrm{Stab}_G(a)$ under left multiplication.*

PROOF:
⟨1⟩1. LET: $f : G/\mathrm{Stab}_G(a) \to A$ be the function $f(g\mathrm{Stab}_G(a)) = ga$.
⟨2⟩1. ASSUME: $g\mathrm{Stab}_G(a) = h\mathrm{Stab}_G(a)$
PROVE: $ga = ha$
⟨2⟩2. $g^{-1}h \in \mathrm{Stab}_G(a)$
⟨2⟩3. $g^{-1}ha = a$
⟨2⟩4. $ha = ga$
⟨1⟩2. f is G-equivariant.
PROOF: Since $f(gh\mathrm{Stab}_G(a)) = gha = gf(h\mathrm{Stab}_G(a))$.
⟨1⟩3. f is injective.
PROOF: If $ga = ha$ then $g^{-1}h \in \mathrm{Stab}_G(a)$ so $g\mathrm{Stab}_G(a) = h\mathrm{Stab}_G(a)$.
⟨1⟩4. f is surjective.
PROOF: Since for all $b \in A$ there exists $g \in G$ such that $ga = b$.
□

Corollary 17.21.1. *If O is an orbit of the action of a finite group G on a set A, then O is finite and $|O|$ divides $|G|$.*

Corollary 17.21.2. *Let H be a subgroup of G and $g \in G$. Then*

$$G/H \cong G/(gHg^{-1})$$

in $G - \mathbf{Set}$.

PROOF: Taking $A = G/H$ and $a = gH$. □

Proposition 17.22. *Given a family of G-sets $\{A_i\}_{i \in I}$, we have $\prod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g\{a_i\}_{i \in I} = \{ga_i\}_{i \in I} \ .$$

PROOF: Easy. □

Proposition 17.23. *Given a family of G-sets $\{A_i\}_{i \in I}$, we have $\coprod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g(i, a_i) = (i, ga_i) \ .$$

PROOF: Easy. □

Proposition 17.24. *Every finite G-set is a coproduct of G-sets of the form G/H.*

PROOF: If $O(a_1)$, …, $O(a_n)$ are the orbits of the G-set A, then G is the coproduct of $G/\mathrm{Stab}_G(a_1)$, …, $G/\mathrm{Stab}_G(a_n)$. □

Proposition 17.25. *For any group G we have $G \cong \mathrm{Aut}_{G-\mathbf{Set}}(G)$ (considering G as a G-set under left multiplication).*

PROOF:

$\langle 1 \rangle 1$. Define $\phi : G \to \mathrm{Aut}_{G-\mathbf{Set}}(G)$ by $\phi(g)(g') = g'g^{-1}$.
$\langle 2 \rangle 1$. LET: $g \in G$
PROVE: $\lambda g' \in G.g'g^{-1}$ is an automorphism of G in $G - \mathbf{Set}$.
$\langle 2 \rangle 2$. $\phi(g)$ is G-equivariant.
PROOF: Since $\phi(g)(h_1 h_2) = h_1 h_2 g^{-1} = h_1 \phi(g)(h_2)$.
$\langle 2 \rangle 3$. $\phi(g)$ is injective.
PROOF: By Cancellation.
$\langle 2 \rangle 4$. $\phi(g)$ is surjective.
PROOF: For any $h \in G$ we ahev $h = \phi(g)(hg)$.
$\langle 1 \rangle 2$. ϕ is a group homomorphism.
PROOF: $\phi(g_1 g_2)(h) = h {g_2}^{-1} {g_1}^{-1} = \phi(g_1)(\phi(g_2)(h))$.
$\langle 1 \rangle 3$. ϕ is injective.
PROOF: If $\phi(g) = \phi(g')$ then $g = \phi(g)(e) = \phi(g')(e) = g'$.
$\langle 1 \rangle 4$. ϕ is surjective.
$\langle 2 \rangle 1$. LET: $\sigma \in \mathrm{Aut}_{G-\mathbf{Set}}(G)$
$\langle 2 \rangle 2$. LET: $g = \sigma(e)$
PROVE: $\sigma = \phi(g^{-1})$
$\langle 2 \rangle 3$. $\sigma(h) = hg$
PROOF: $\sigma(h) = \sigma(he) = h\sigma(e) = hg$.
$\square$

17.3 Center

Definition 17.26 (Center)**.** The *center* of a group G, $Z(G)$, is the kernel of the conjugation action $\sigma : G \to S_G$.

Proposition 17.27. *The center of a group G is*

$$Z(G) = \{g \in G : \forall a \in G.ag = ga\} \ .$$

PROOF: Immediate from definitions. $\square$

Lemma 17.28. *Let G be a finite group. Assume $G/Z(G)$ is cyclic. Then G is Abelian and so $G/Z(G)$ is trivial.*

PROOF:
$\langle 1 \rangle 1$. PICK $g \in G$ such that $gZ(G)$ generates $G/Z(G)$.
$\langle 1 \rangle 2$. LET: $a, b \in G$
$\langle 1 \rangle 3$. PICK $r, s \in \mathbb{Z}$ such that $aZ(G) = g^r Z(G)$ and $bZ(G) = g^s Z(G)$
$\langle 1 \rangle 4$. LET: $z = g^{-r} a \in Z(G)$ and $w = g^{-s} b \in Z(G)$
$\langle 1 \rangle 5$. $a = g^r z$ and $b = g^s w$
$\langle 1 \rangle 6$. $ab = ba$
PROOF:

$$\begin{aligned} ab &= g^r z g^s w \\ &= g^{r+s} z w \\ &= g^s w g^r z \\ &= ba \end{aligned}$$

□

Proposition 17.29. *Let G be a group. Let N be a subgroup of $Z(G)$. Then N is normal in G.*

PROOF: For all $n \in N$ and $g \in G$ we have $gng^{-1} = ngg^{-1} = n \in N$ since $n \in Z(G)$. □

Proposition 17.30. *For any group G we have $G/Z(G) \cong \mathrm{Inn}\,(G)$.*

PROOF: The homomorphism $g \mapsto \gamma_g$ is a surjective homomorphism with kernel $Z(G)$. □

Proposition 17.31. *Let p and q be prime integers. Let G be a group of order pq. Then either G is Abelian or the center of G is trivial.*

PROOF: Otherwise we would have $|Z(G)| = p$ say and so $|\mathrm{Inn}\,(G)\,| = q$, meaning $\mathrm{Inn}\,(G)$ is cyclic, hence trivial, which is a contradiction. □

Theorem 17.32 (First Sylow Theorem)**.** *Let p be a prime and $k \in \mathbb{N}$. Let G be a finite group. If p^k divides $|G|$ then G has a subgroup of order p^k.*

PROOF:
$\langle 1\rangle$1. ASSUME: as induction hypothesis the statement is true for all groups smaller than G.
$\langle 1\rangle$2. ASSUME: w.l.o.g. $k \neq 0$ and $|G| \neq p$
$\langle 1\rangle$3. CASE: There exists a proper subgroup H of G such that p does not divide $[G : H]$.
PROOF: Then H has a subgroup of order p^k by induction hypothesis $\langle 1\rangle$1.
$\langle 1\rangle$4. CASE: For every proper subgroup H of G we have p divides $[G : H]$.
$\langle 2\rangle$1. p divides $|Z(G)|$.
PROOF: By the Class Formula.
$\langle 2\rangle$2. PICK $a \in Z(G)$ that has order p.
PROOF: Cauchy's Theorem.
$\langle 2\rangle$3. LET: $N = \langle a\rangle$
$\langle 2\rangle$4. N is normal.
PROOF: Proposition 17.29.
$\langle 2\rangle$5. p^{k-1} divides $|G/N|$.
$\langle 2\rangle$6. PICK a subgroup Q of G/N of order p^{k-1}.
PROOF: Induction hypothesis $\langle 1\rangle$1.
$\langle 2\rangle$7. LET: $P = \pi^{-1}(Q)$
$\langle 2\rangle$8. $|P| = p^k$

□

Theorem 17.33 (Second Sylow Theorem)**.** *Let G be a finite group. Let p be a prime. Let P be a p-Sylow subgroup of G. Let H be a subgroup of G that is a p-group. Then H is a subgroup of a conjugate of P.*

PROOF:

⟨1⟩1. PICK a fixed point gP for the action of H on the set of left cosets of P by left multiplication.

PROOF: Corollary 17.18.2.

⟨1⟩2. For all $h \in H$ we have $hgP = gP$

⟨1⟩3. $H \subseteq gPg^{-1}$

□

Proposition 17.34.

$$Z(G \times H) = Z(G) \times Z(H)$$

PROOF:

$$\begin{aligned}(g,h) \in Z(G \times H) &\Leftrightarrow \forall g' \in G.\forall h' \in H.(g,h)(g',h') = (g',h')(g,h) \\ &\Leftrightarrow \forall g' \in G.\forall h' \in H.(gg',hh') = (g'g,h'h) \\ &\Leftrightarrow \forall g' \in G.\forall h' \in H(gg' = g'g \wedge hh' = h'h) \\ &\Leftrightarrow g \in Z(G) \wedge h \in Z(H)\end{aligned}$$

□

17.4 Centralizer

Definition 17.35 (Centralizer)**.** Let G be a group. Let $a \in G$. The *centralizer* or *normalizer* of a, denoted $Z_G(a)$, is the stabilizer of a under the action of conjugation.

Proposition 17.36.

$$Z_G(a) = \{g \in G : ga = ag\}$$

PROOF: Immediate from definitions. □

17.5 Conjugacy Class

Definition 17.37 (Conjugacy Class)**.** Let G be a group. Let $a \in G$. The *conjugacy class* of a, denoted $[a]$, is the orbit of a under the action of conjugation.

Proposition 17.38 (Class Formula)**.** *Let G be a finite group. Let A be a set of representatives of the non-trivial conjugacy classes. Then*

$$|G| = |Z(G)| + \sum_{a \in A} [G : Z(a)] \ .$$

PROOF: Proposition 17.18. □

Corollary 17.38.1. *Let p be a prime. Let G be a p-group and H a nontrivial normal subgroup of G. Then $H \cap Z(G) \neq \{e\}$.*

PROOF: Let A be a set of representatives of the non-trivial conjugacy classes. Let $A \cap H = \{a_1, \ldots, a_n\}$. Then

$$|H| = |H \cap Z(G)| + \sum_{i=1}^{n} [G : Z(a_i)] \ .$$

Since $p \mid |H|$ and $p \mid [G : Z(a_i)]$ for all i, we have $p \mid |H \cap Z(G)|$. □

Corollary 17.38.2. *Let p be a prime. Every p-group has a non-trivial center.*

Corollary 17.38.3. *Let p be a prime. Every group G of order p^2 is Abelian.*

Proof: By Proposition 17.31. □

Proposition 17.39. *Let p be a prime and r a non-negative integer. Let G be a group of order p^r. Then, for $k = 0, 1, \ldots, r$, we have G has a normal subgroup of order p^k.*

Proof:
$\langle 1 \rangle 1$. Assume: as induction hypothesis the result holds for $r' < r$.
$\langle 1 \rangle 2$. Assume: w.l.o.g. $k > 0$
  Proof: Since $\{e\}$ is a normal subgroup of order p^0.
$\langle 1 \rangle 3$. Pick a subgroup N of $Z(G)$ of order p.
  $\langle 2 \rangle 1$. $p \mid |Z(G)|$
    Proof: From Corollary 17.38.2.
  $\langle 2 \rangle 2$. $Z(G)$ has a subgroup of order p.
    Proof: Cauchy's Theorem.
$\langle 1 \rangle 4$. N is normal.
  Proof: Proposition 17.29.
$\langle 1 \rangle 5$. Pick a normal subgroup M of G/N of order p^{k-1}.
  Proof: From the induction hypothesis $\langle 1 \rangle 1$.
$\langle 1 \rangle 6$. $\pi^{-1}(M)$ is a normal subgroup of G of order p^k.
□

Example 17.40. The only non-Abelian group of order 6 is S_3.

Proof:
$\langle 1 \rangle 1$. Let: G be a non-Adelian group of order 6.
$\langle 1 \rangle 2$. $Z(G) = \{e\}$
  Proof: Otherwise $Z(G)$ has order 2 or 3 and is cyclic, contradicting Lemma 17.28.
$\langle 1 \rangle 3$. G has three conjugacy classes: $Z(G)$, a class of size 2 and a class of size 3.
  Proof: By the Class Formula since the only way to make 5 using non-trivial factors of 6 is $2 + 3$.
$\langle 1 \rangle 4$. Pick an element $y \in G$ of order 3.
  Proof: It cannot be that every element is of order ≤ 2 by Proposition 16.20.
$\langle 1 \rangle 5$. $\langle y \rangle$ is normal in G.
  Proof: Since it has index 2.
$\langle 1 \rangle 6$. The conjugacy class y is $\{y, y^2\}$.
  Proof: Since $\langle y \rangle$ must be a union of conjugacy classes.
$\langle 1 \rangle 7$. The conjugacy class of size 2 is $\{y, y^2\}$.
  Proof: Since y^2 has order 3 and so its conjugacy class is of size 2 similarly, and there is only one conjugacy class of size 2.
$\langle 1 \rangle 8$. Pick $x \in G$ such that $yx = xy^2$.
  Proof: y^2 is conjugate to y so there exists x such that $x^{-1}yx = y^2$.

⟨1⟩9. x has order 2.
PROOF: x is not in the conjugacy class of size 2 so its order cannot be 3.
⟨1⟩10. x and y generate G.
PROOF: Since e, y, y^2, x, xy, xy^2 are all distinct.
⟨1⟩11. $G \cong S_3$
PROOF: We now know the entire multiplication table of G.
□

Proposition 17.41. *Let G be a finite group. Let H be a subgroup of G of order 2. Let $a \in H$. Let $[a]_H$ be the conjugacy class of a in H, and $[a]_G$ the conjugacy class of a in G. If $Z_G(a) \subseteq H$ then $[a]_H$ is half the size of $[a]_G$; otherwise, $[a]_H = [a]_G$.*

PROOF:
⟨1⟩1. H is normal in G.
PROOF: Corollary 17.15.2.
⟨1⟩2. $HZ_G(a)$ is a subgroup of G.
⟨1⟩3. H is normal in $HZ_G(a)$.
⟨1⟩4. $H \cap Z_G(a)$ is normal in $Z_G(a)$.
⟨1⟩5.
$$\frac{HZ_G(a)}{H} \cong \frac{Z_G(a)}{H \cap Z_G(a)}$$
⟨1⟩6. If $Z_G(a) \subseteq H$ then $|[a]_H| = |[a]_G|/2$.
PROOF: In this case we have $Z_H(a) = Z_G(a)$ and so $|[a]_H| = |H|/|Z_H(a)| = (|G|/2)/|Z_G(a)| = |[a]_G|/2$.
⟨1⟩7. If $Z_G(a) \nsubseteq H$ then $[a]_H = [a]_G$.
PROOF:
⟨2⟩1. PICK $b \in Z_G(a) - H$
⟨2⟩2. $Hb^{-1} = G - H$
⟨2⟩3. $G = HZ_G(a)$
PROOF: For $x \in H$ we have $x = xe$ and for $x \notin H$ we have $x \in Hb^{-1}$ hence $xb \in H$ and $x = (xb)b$.
⟨2⟩4. $|[a]_H| = |[a]_G|$
PROOF:
$$\begin{aligned} |[a]_H| &= \frac{|H|}{|Z_H(a)|} \\ &= \frac{|H|}{|H \cap Z_G(a)|} \\ &= \frac{|Z_G(a)||H|}{|Z_G(a)||H \cap Z_G(a)|} \\ &= \frac{|HZ_G(a)|}{|Z_G(a)|} \\ &= \frac{|G|}{|Z_G(a)|} \\ &= |[a]_G| \end{aligned}$$

∎

17.6 Conjugation on Sets

Definition 17.42 (Conjugation)**.** Let G be a group. Define an action of G on $\mathcal{P}G$ called *conjugation* that takes g and A to

$$gAg^{-1} = \{gag^{-1} : a \in A\} \ .$$

Proposition 17.43. *The conjugate of a subgroup is a subgroup.*

Proof: Let H be a subgroup of G. Given $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$, we have

$$(gh_1g^{-1})(gh_2g^{-1})^{-1} = gh_1{h_2}^{-1}g^{-1} \in gHg^{-1} \ . \quad ∎$$

Definition 17.44 (Normalizer)**.** Let G be a group and $A \subseteq G$. The *normalizer* of A, denoted $N_G(A)$, is its stabilizer under conjugation.

Proposition 17.45. *Let G be a group, $g \in G$ and A a finite subset of G. If $gAg^{-1} \subseteq A$ then $gAg^{-1} = A$ and so $g \in N_G(A)$.*

Proof: Conjugation by g is an injection from A into A, hence a bijection. ∎

Proposition 17.46. *Let G be a group and H a subgroup of G. Then $N_G(H)$ is the largest subgroup of G that includes H such that H is normal in $N_G(H)$.*

Proof:
⟨1⟩1. $N_G(H)$ is a subgroup of G.
  Proof: If $a, b \in N_G(H)$ then $ab^{-1}Hba^{-1} = aHa^{-1} = H$ so $ab^{-1} \in N_G(H)$.
⟨1⟩2. $H \subseteq N_G(H)$
  Proof: Easy.
⟨1⟩3. H is normal in $N_G(H)$.
  Proof: If $a \in N_G(H)$ then $aHa^{-1} = H$ by definition.
⟨1⟩4. For any subgroup K of G, if $H \subseteq K$ and H is normal in K then $K \subseteq N_G(H)$.
  Proof: H is normal in K means that, for all $a \in K$, we have $aHa^{-1} = H$ and so $a \in N_G(H)$.
∎

Corollary 17.46.1. *Let G be a group and H a subgroup of G. Then H is normal if and only if $G = N_G(H)$.*

Proposition 17.47. *Let G be a group and H a subgroup of G. If $[G : N_G(H)]$ is finite, then it is the number of subgroups conjugate to H.*

Proof: By the Orbit-Stabilizer Theorem. ∎

Corollary 17.47.1. *Let G be a group and H a subgroup of G. If $[G : H]$ is finite, the the number of subgroups conjugate to H is finite and divides $[G : H]$.*

Lemma 17.48. *Let H be a p-group that is a subgroup of a finite group G. Then*

$$[N_G(H) : H] \equiv [G : H] (\bmod p) \ .$$

PROOF:
⟨1⟩1. ASSUME: w.l.o.g. H is non-trivial.
⟨1⟩2. gH is a fixed point of the action of H on the set of left cosets of H by left multiplication if and only if $g \in N_G(H)$.
PROOF:

$$\begin{aligned} gH \text{ is a fixed point} &\Leftrightarrow \forall h \in H . hgH = gH \\ &\Leftrightarrow H \subseteq gHg^{-1} \\ &\Leftrightarrow H = gHg^{-1} && (|gHg^{-1}| = |H|) \\ &\Leftrightarrow g \in N_G(H) \end{aligned}$$

⟨1⟩3. The number of fixed points in $[N_G(H) : H]$.
⟨1⟩4. Q.E.D.
PROOF: Corollary 17.18.1.
□

Proposition 17.49. *Let H be a p-subgroup of a finite group G that is not a p-Sylow subgroup. Then there exists a p-subgroup H' of G such that H is a normal subgroup of H' and $[H' : H] = p$.*

PROOF:
⟨1⟩1. p divides $[N_G(H) : H]$.
PROOF: Lemma 17.48.
⟨1⟩2. PICK $gH \in N_G(H)/H$ of order p.
PROOF: Cauchy's Theorem.
⟨1⟩3. LET: $H' = \pi^{-1}(\langle gH \rangle)$
⟨1⟩4. H is a normal subgroup of H'.
⟨1⟩5. $[H' : H] = p$
□

Corollary 17.49.1. *No p-group of order $\geq p^2$ is simple.*

Lemma 17.50. *Let p be a prime. Let G be a finite group. Let P be a p-Sylow subgroup of G. Every p-subgroup of $N_G(P)$ is a subgroup of P.*

PROOF:
⟨1⟩1. LET: H be a p-subgroup of $N_G(P)$.
⟨1⟩2. P is normal in $N_G(P)$.
PROOF: Proposition 17.46.
⟨1⟩3. PH is a subgroup of $N_G(P)$.
PROOF: Second Isomorphism Theorem.
⟨1⟩4. $|PH/P| = |H/(P \cap H)|$
PROOF: Second Isomorphism Theorem.
⟨1⟩5. PH is a p-group.
⟨2⟩1. ASSUME: for a contradiction q is prime, $q \mid |PH|$ and $q \neq p$

$\langle 2\rangle 2.$ $q \mid |PH/P|$
$\langle 2\rangle 3.$ $q \mid |H/(P \cap H)|$
$\langle 2\rangle 4.$ $q \mid |H|$
$\langle 2\rangle 5.$ Q.E.D.
PROOF: This contradicts the fact that H is a p-group, $\langle 1\rangle 1$.
$\langle 1\rangle 6.$ $PH = P$
PROOF: By maximality of P.
$\langle 1\rangle 7.$ $H \subseteq P$
□

Lemma 17.51. *Let p be a prime. Let G be a finite group. Let P be a p-Sylow subgroup of G. Let P act by conjugation on the set of p-Sylow subgroups of G. Then P is the unique fixed point of this action.*

PROOF:
$\langle 1\rangle 1.$ P is a fixed point of this action.
PROOF: For any $x \in P$ we have $xPx^{-1} = P$.
$\langle 1\rangle 2.$ If Q is any fixed point of the action then $Q = P$.
$\langle 2\rangle 1.$ LET: Q be a fixed point of the action.
$\langle 2\rangle 2.$ For all $x \in P$ we have $xQx^{-1} = Q$.
$\langle 2\rangle 3.$ $P \subseteq N_G(Q)$
$\langle 2\rangle 4.$ $P \subseteq Q$
PROOF: Lemma 17.50.
$\langle 2\rangle 5.$ $P = Q$
PROOF: Since $|P| = |Q|$.
□

Theorem 17.52 (Third Sylow Theorem)**.** *Let p be a prime. Let G be a finite group of order $p^r m$ where p does not divide m. Then the number of p-Sylow subgroups of G divides m and is congruent to 1 modulo p.*

PROOF:
$\langle 1\rangle 1.$ LET: N_p be the number of p-Sylow subgroups of G.
$\langle 1\rangle 2.$ PICK a p-Sylow subgroup P.
PROOF: One exists by the First Sylow Theorem.
$\langle 1\rangle 3.$ The p-Sylow subgroups of G are exactly the conjugates of P.
PROOF: Second Sylow Theorem
$\langle 1\rangle 4.$ $m = N_p[N_G(P) : P]$
PROOF: Since $N_p = [G : N_G(P)]$ by Proposition 17.47.
$\langle 1\rangle 5.$ N_p divides m.
$\langle 1\rangle 6.$ $mN_p \equiv m (\mathrm{mod}\, p)$
$\langle 2\rangle 1.$ $m \equiv [N_G(P) : P] (\mathrm{mod}\, p)$
PROOF: Lemma 17.48.
$\langle 2\rangle 2.$ $mN_p \equiv m (\mathrm{mod}\, p)$
PROOF: By $\langle 1\rangle 4$.
$\langle 1\rangle 7.$ $N_p \equiv 1 (\mathrm{mod}\, p)$
□

PROOF:

$\langle 1 \rangle 1$. LET: N_p be the number of p-Sylow subgroups of G.
$\langle 1 \rangle 2$. PICK a p-Sylow subgroup P of G.
  PROOF: First Sylow Theorem.
$\langle 1 \rangle 3$. N_p is the number of conjugates of P.
  PROOF: Second Sylow Theorem.
$\langle 1 \rangle 4$. $N_p \mid m$
  PROOF: Corollary 17.47.1.
$\langle 1 \rangle 5$. P acts on the set of conjugates of P with one fixed point.
  PROOF: Lemma 17.51.
$\langle 1 \rangle 6$. $N_p \equiv 1 (\bmod p)$
  PROOF: Corollary 17.18.1.
□

Corollary 17.52.1. *Let G be a finite group. Let p be a prime number. If $|G| = mp^r$ and the only divisor d of m such that $d \equiv 1 (\bmod p)$ is $d = 1$, then G is not simple.*

PROOF: There must be 1 p-Sylow subgroup, which has order p^r and is normal.
□

Corollary 17.52.2. *Let G be a finite group. Let p be a prime number. If $|G| = mp^r$ where $1 < m < p$ then G is not simple.*

Proposition 17.53. *Let p and q be prime numbers with $p < q$. Let G be a group of order pq with a normal subgroup H of order p. Then G is cyclic.*

PROOF:
$\langle 1 \rangle 1$. LET: $\gamma : G \to \mathrm{Aut}_{\mathbf{Grp}}(H)$ be the action of conjugation.
$\langle 1 \rangle 2$. H is cyclic of order p.
$\langle 1 \rangle 3$. $|\mathrm{Aut}_{\mathbf{Grp}}(H)| = p - 1$
$\langle 1 \rangle 4$. $|\operatorname{im} \gamma| \mid pq$
  PROOF: Since $\operatorname{im} \gamma$ is a quotient group of G.
$\langle 1 \rangle 5$. $|\operatorname{im} \gamma| \mid p - 1$
$\langle 1 \rangle 6$. $|\operatorname{im} \gamma| = 1$
$\langle 1 \rangle 7$. $\gamma = 0$
$\langle 1 \rangle 8$. $H \subseteq Z(G)$
$\langle 1 \rangle 9$. G is Abelian.
  PROOF: Lemma 17.28.
$\langle 1 \rangle 10$. PICK an element g of order p.
  PROOF: Cauchy's Theorem.
$\langle 1 \rangle 11$. PICK an element h of order q.
  PROOF: Cauchy's Theorem.
$\langle 1 \rangle 12$. $|gh| = pq$
  PROOF: Proposition 14.22.
□

PROOF:
$\langle 1 \rangle 1$. ASSUME: w.l.o.g. $q \not\equiv 1 (\bmod p)$

PROOF: Since the only non-cyclic group of order 6 is S_3 which does not have a normal subgroup of order 2.

⟨1⟩2. PICK a subgroup K of order q.

⟨1⟩3. K is normal.

PROOF: Since K is the unique q-Sylow subgroup by the Third Sylow Theorem.

⟨1⟩4. $H \cap K = \{e\}$

⟨1⟩5. $HK \cong H \times K$

PROOF: Proposition **??**.

⟨1⟩6. $|HK| = pq$

⟨1⟩7. $HK = G$

⟨1⟩8. $G \cong \mathbb{Z}/pq\mathbb{Z}$

PROOF:

$$\begin{aligned} G &\cong H \times K \\ &\cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \\ &\cong \mathbb{Z}/pq\mathbb{Z} \end{aligned}$$

□

Corollary 17.53.1. *Let p and q be prime numbers with $p < q$ and $q \not\equiv 1 (\mathrm{mod}\, p)$. Then the only group of order pq is the cyclic group.*

PROOF: By the Third Sylow Theorem, such a group must have exactly one p-Sylow subgroup, which is therefore normal. □

Proposition 17.54. *Let p be prime. Let G be a finite group. Let P be a p-Sylow subgroup of G. Then*

$$N_G(N_G(P)) = N_G(P) \ .$$

PROOF:

⟨1⟩1. P is normal in $N_G(P)$.

PROOF: Proposition 17.46.

⟨1⟩2. $N_G(P)$ is normal in $N_G(N_G(P))$.

PROOF: Proposition 17.46.

⟨1⟩3. P is normal in $N_G(N_G(P))$.

PROOF: Corollary 15.119.1.

⟨1⟩4. $N_G(N_G(P)) \subseteq N_G(P)$

PROOF: Proposition 17.46.

⟨1⟩5. $N_G(N_G(P)) = N_G(P)$

□

Proposition 17.55. *Let p, q and r be three distinct prime numbers. Then there is no simple group of order pqr.*

PROOF:

⟨1⟩1. LET: G be a group of order pqr.

⟨1⟩2. ASSUME: w.l.o.g. $p < q < r$

⟨1⟩3. ASSUME: for a contradiction G is simple.

⟨1⟩4. The number of subgroups of order p is at least $p + 1$.

PROOF: Third Sylow Theorem

$\langle 1\rangle 5$. The number of subgroups of order q is at least $q+1$.

PROOF: Third Sylow Theorem

$\langle 1\rangle 6$. The number of subgroups of order r is pq.

PROOF: By the Third Sylow Theorem, the number divides pq, and it cannot be 1 (lest that subgroup be normal) or p or q (as these are less than r hence not congruent to 1 modulo r).

$\langle 1\rangle 7$. There are at least p^2-1 elements of order p.

$\langle 1\rangle 8$. There are at least q^2-1 elements of order q.

$\langle 1\rangle 9$. There are at least $pqr-pq$ elements of order r.

$\langle 1\rangle 10$. Q.E.D.

PROOF: This is a contradiction as the total number of elements of order 1, p, q and r is

$$\begin{aligned} 1+(p^2-1)+(q^2-1)+(pqr-pq) &= p^2+q^2+pqr-pq-1 \\ &> pqr+p^2-1 \\ &> pqr \end{aligned}$$

□

Proposition 17.56. *Let G be a finite simple group. Let H be a subgroup of G of index $N>1$. Then $|G|$ divides $N!$.*

PROOF:

$\langle 1\rangle 1$. PICK a subgroup K of H that is normal in G such that $[G:K]$ divides $\gcd(|G|,N!)$.

$\langle 1\rangle 2$. $K=\{e\}$

$\langle 1\rangle 3$. $[G:K]=|G|$

$\langle 1\rangle 4$. $|G|$ divides $N!$

□

Corollary 17.56.1. *Let G be a finite simple group. Let p be a prime factor of $|G|$. Let N_p be the number of p-Sylow subgroups of G. Then $|G|$ divides $N_p!$.*

PROOF: Since $N_p=[G:N_G(P)]$ and $N_p>1$ since G is simple. □

Definition 17.57 (Centralizer)**.** Let G be a group and $A\subseteq G$. The *centralizer* of A is

$$Z_G(A):=\{g\in G:\forall a\in A.gag^{-1}=a\}\ .$$

Proposition 17.58. *Let H and K be subgroups of G with $H\subseteq N_G(K)$. Then the function $\gamma:H\to \mathrm{Aut}_{\mathbf{Grp}}(K)$ defined by conjugation*

$$\gamma_h(k)=hkh^{-1}$$

is a homomorphism of groups with $\ker\gamma=H\cap Z_G(K)$.

PROOF:

$\langle 1\rangle 1$. For all $g,h\in H$ we have $\gamma_{gh}=\gamma_g\circ\gamma_h$.

PROOF: Since $\gamma_{gh}(k)=\gamma_g(\gamma_h(k))=ghkh^{-1}g^{-1}$.

$\langle 1\rangle 2$. For all $h\in H$ we have $\gamma_h=\mathrm{id}_K$ iff $h\in Z_G(K)$.

PROOF: Both are equivalent to $\forall k\in K.hkh^{-1}=k$, i.e. $\forall k\in K.hk=kh$.

□

17.7 Nilpotent Groups

Definition 17.59 (Nilpotent)**.** Let G be a group. Define inductively a sequence (Z_n) of subgroups of G by $Z_0 = \{e\}$, and Z_{i+1} is the inverse image under π of the center of G/Z_i.

Then G is *nilpotent* iff $Z_n = G$ for some n.

We prove this is well-defined by proving that, for all i, we have Z_i is normal in G.

PROOF:
⟨1⟩1. ASSUME: as induction hypothesis Z_i is normal in G.
PROVE: Z_{i+1} is normal in G.
⟨1⟩2. LET: $x \in Z_{i+1}$ and $g \in G$
PROVE: $gxg^{-1} \in Z_{i+1}$
PROVE: For all $h \in G$ we have $gxg^{-1}hZ_i = hgxg^{-1}Z_i$
⟨1⟩3. LET: $h \in G$
⟨1⟩4. $gxg^{-1}hZ_i = hgxg^{-1}Z_i$
PROOF:

$$\begin{aligned} gxg^{-1}hZ_i &= gg^{-1}hxZ_i \\ &= hxZ_i \\ &= hgg^{-1}xZ_i \\ &= hgxg^{-1}Z_i \end{aligned}$$

□

Proposition 17.60. *Every Abelian group is nilpotent.*

PROOF: Let G be an Abelian group. The center of G/Z_0 is G/Z_0, hence $Z_1 = G$.
□

Example 17.61. The semidirect product of two nilpotent groups is not necessarily nilpotent. S_3 is the semidirect product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ but is not nilpotent.

Proposition 17.62. *Let G be a group. Then G is nilpotent if and only if $G/Z(G)$ is nilpotent.*

PROOF:
⟨1⟩1. LET: (Z_n) be the sequence of subgroups of G where $Z_0 = \{e\}$ and Z_{n+1} is the inverse image of the center of G/Z_n.
⟨1⟩2. $G/Z_0 \cong G$
⟨1⟩3. $Z_1 = Z(G)$
⟨1⟩4. The corresponding sequence of subgroups for $G/Z(G)$ is $G/Z(G)$, $Z_2/Z(G)$, $Z_3/Z(G)$, ...
⟨1⟩5. G is nilpotent iff $G/Z(G)$ is nilpotent.
PROOF: Both are equivalent to $\exists n. Z_n = g$ and to $\exists n. Z_n/Z(G) = G/Z(G)$.
□

Proposition 17.63. *Every p-group is nilpotent.*

Proof: Each Z_n is a p-group and so has non-trivial center, hence each Z_{n+1} is larger than Z_n and so the sequence must terminate. □

Proposition 17.64. *Every nilpotent group is solvable.*

Proof: Let (Z_n) be the defining sequence of subgroups. Then $Z_{n+1}/Z_n = Z(G/Z_n)$ is Abelian for all n, hence the group is solvable by Proposition 16.52. □

Example 17.65. The converse is not true — S_3 is solvable but not nilpotent.

Proposition 17.66. *Let G be a nilpotent group. Then every nontrivial normal subgroup of G intersects $Z(G)$ non-trivially.*

Proof:
⟨1⟩1. Let: H be a nontrivial normal subgroup of G.
⟨1⟩2. Let: (Z_n) be the sequence of subgroups with $Z_0 = \{e\}$ and Z_{n+1} the inverse image of $Z(G/Z_n)$.
⟨1⟩3. Let: r be least such that $H \cap Z_r \neq \{e\}$.
⟨1⟩4. Pick $h \in H \cap Z_r$ with $h \neq e$.
⟨1⟩5. $hZ_{r-1} \in Z(G/Z_{r-1})$
⟨1⟩6. For all $g \in G$ we have $ghZ_{r-1} = hgZ_{r-1}$
⟨1⟩7. For all $g \in G$ we have $ghg^{-1}h^{-1} \in Z_{r-1}$
⟨1⟩8. For all $g \in G$ we have $ghg^{-1}h^{-1} = e$
  Proof: Since $ghg^{-1}h^{-1} \in H$ and $H \cap Z_{r-1} = \{e\}$.
⟨1⟩9. For all $g \in G$ we have $gh = hg$
⟨1⟩10. $h \in H \cap Z(G)$
□

Example 17.67. We cannot weaken the hypothesis to G being solvable. S_3 is solvable and $\mathbb{Z}/2\mathbb{Z}$ is a nontrivial normal subgroup but its intersection with $Z(S_3)$ is just $\{e\}$.

Proposition 17.68. *Let G be a finite nilpotent group. Let H be a proper subgroup of G. Then $H \subsetneq N_G(H)$.*

Proof:
⟨1⟩1. Assume: as induction hypothesis the theorem holds for all groups smaller than G.
⟨1⟩2. $Z(G)$ is non-trivial.
⟨1⟩3. Case: $Z(G) \nsubseteq H$
  ⟨2⟩1. Pick $g \in Z(G) - H$
  ⟨2⟩2. $g \in N_G(H) - H$
⟨1⟩4. Case: $Z(G) \subseteq H$
  ⟨2⟩1. $H/Z(G) \subsetneq N_{G/Z(G)}(H/Z(G))$
    Proof: By induction hypothesis ⟨1⟩1.
  ⟨2⟩2. Pick g such that $gZ(G) \in N_{G/Z(G)}(H/Z(G)) - H/Z(G)$
  ⟨2⟩3. $g \in N_G(H)$
    ⟨3⟩1. Let: $h \in H$

PROVE: $ghg^{-1} \in H$
⟨3⟩2. $ghg^{-1}Z(G) \in H/Z(G)$
⟨3⟩3. PICK $h_1 \in H$ such that $ghg^{-1}Z(G) = h_1 Z(G)$
⟨3⟩4. $ghg^{-1}{h_1}^{-1} \in Z(G)$
⟨3⟩5. $ghg^{-1}{h_1}^{-1} \in H$
PROOF: ⟨1⟩4
⟨3⟩6. $ghg^{-1} \in H$
⟨2⟩4. $g \notin H$
□

Corollary 17.68.1. *Let G be a finite group. Then G is nilpotent if and only if every Sylow subgroup of G is normal.*

PROOF:
⟨1⟩1. If G is nilpotent then every Sylow subgroup of G is normal.
⟨2⟩1. ASSUME: G is nilpotent.
⟨2⟩2. LET: P be Sylow subgroup of G
⟨2⟩3. $N_G(P) = N_G(N_G(P))$
PROOF: Proposition 17.54.
⟨2⟩4. $N_G(P) = G$
PROOF: Proposition 17.68.
⟨2⟩5. P is normal.
⟨1⟩2. If every Sylow subgroup of G is normal then G is nilpotent.
⟨2⟩1. ASSUME: As induction hypothesis the result holds for all groups smaller than G.
⟨2⟩2. ASSUME: Every Sylow subgroup of G is normal.
⟨2⟩3. LET: $P_1, \ldots, P_r$ be the nontrivial Sylow subgroups of G.
⟨2⟩4. $G \cong P_1 \times \cdots \times P_r$
PROOF: Proposition 15.120.
⟨2⟩5. ASSUME: w.l.o.g. $r > 1$
PROOF: The case $r = 1$ holds by Proposition 17.63.
⟨2⟩6. $Z(G) \cong Z(P_1) \times \cdots \times Z(P_r)$
PROOF: Proposition 17.34.
⟨2⟩7. $G/Z(G) \cong P_1/Z(P_1) \times \cdots \times P_r/Z(P_r)$
PROOF: Proposition 15.62.
⟨2⟩8. The nontrivial Sylow subgroups of $G/Z(G)$ are $P_1/Z(P_1), \ldots, P_r/Z(P_r)$.
⟨2⟩9. Every Sylow subgroup of $G/Z(G)$ is normal.
⟨2⟩10. $|G/Z(G)| < |G|$
PROOF: Because of Corollary 17.38.2.
⟨2⟩11. $G/Z(G)$ is nilpotent.
PROOF: By the induction hypothesis **??**.
⟨2⟩12. G is nilpotent.
PROOF: Proposition 17.62.
□

17.8 Symmetric Groups

Proposition 17.69. *Every permutation in S_n is the product of a unique set of disjoint cycles.*

Proof: Since any permutation acts as a cycle on any of its orbits. □

Corollary 17.69.1. *The transpositions generate S_n.*

Proof: Since any cycle is a product of transpositions:

$$(a_1\ a_2\ \cdots\ a_n) = (a_1\ a_n) \circ \cdots \circ (a_1\ a_3) \circ (a_1\ a_2)\ .$$ □

Corollary 17.69.2. *S_n is generated by $(1\ 2)$ and $(1\ 2\ 3\ \cdots\ n)$.*

Proof:
⟨1⟩1. Any transposition of the form $(i\ i+1)$ is in the subgroup generated by these two permutations.
  Proof: It is $(1\ 2\ \cdots\ n)^i(1\ 2)(1\ 2\ \cdots\ n)^{-i}$.
⟨1⟩2. Any transposition of the form $(1\ i)$ is in the subgroup generated by these two permutations.
  Proof: It is $(i-1\ i)\cdots(3\ 4)(2\ 3)(1\ 2)(2\ 3)\cdots(i-1\ i)$.
⟨1⟩3. Any transposition is in the subgroup generated by these two permutations.
  Proof: Since $(i\ j) = (1\ i)(1\ j)(1\ i)$
⟨1⟩4. These two permutations generate S_n.
  Proof: By the previous Corollary.
□

Definition 17.70 (Type)**.** For any $\sigma \in S_n$, the *type* of σ is the partition of n consisting of the sizes of the orbits of σ.

Proposition 17.71. *Two permutations in S_n are conjugate if and only if they have the same type.*

Proof:
⟨1⟩1. Two permutations that are conjugate have the same type.
  Proof: Since
  $\tau(a_1\,a_2\,\cdots\,a_r)(b_1\,b_2\,\cdots\,b_s)\cdots(c_1\,c_2\,\cdots\,c_t)tau^{-1} = (\tau(a_1)\,\tau(a_2)\,\cdots\,\tau(a_r))(\tau(b_1)\,\tau(b_2)\,\cdots\,\tau(b_s))\cdots($
⟨1⟩2. Two permutaitons with the same type are conjugate.
  ⟨2⟩1. Let: $\rho = (a_1\ a_2\ \cdots\ a_r)(b_1\ b_2\ \cdots\ b_s)\cdots(c_1\ c_2\ \cdots\ c_t)$ and $\sigma = (a'_1\ a'_2\ \cdots\ a'_r)(b'_1\ b'_2\ \cdots\ b'_s)\cdots(c'_1\ c'_2\ \cdots\ c'_t)$
  ⟨2⟩2. Let: τ be the permutation $\tau(a_i) = a'_i$, $\tau(b_i) = b'_i$, ..., $\tau(c_i) = c'_i$
  ⟨2⟩3. $\sigma = \tau\rho\tau^{-1}$
□

Corollary 17.71.1. *The number of conjugacy classes in S_n equals the number of permutations of n.*

Definition 17.72 (Sign)**.** Define $\Delta_n \in \mathbb{Z}[x_1, \ldots, x_n]$ by

$$\Delta_n = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

Define an action of S_n on $\mathbb{Z}[x_1, \ldots, x_n]$ by

$$\sigma p(x_1, \ldots, x_n) = p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \ .$$

The *sign* of a permutation $\sigma \in S_n$ is the number $\epsilon(\sigma) \in \{1, -1\}$ such that

$$\sigma \Delta_n = \epsilon(\sigma) \Delta_n \ .$$

We say σ is *even* if $\epsilon(\sigma) = 1$ and *odd* if $\epsilon(\sigma) = -1$.

Proposition 17.73. *ϵ is a group homomorphism $S_n \to \mathbb{Z}^*$.*

PROOF:
$\langle 1 \rangle 1$. LET: $\rho, \sigma \in S_n$
$\langle 1 \rangle 2$. $(\rho \circ \sigma)\Delta_n = \rho(\sigma \Delta_n)$
$\langle 1 \rangle 3$. $\epsilon(\rho \circ \sigma)\Delta_n = \epsilon(\rho)\epsilon(\sigma)\Delta_n$
$\langle 1 \rangle 4$. $\epsilon(\rho \circ \sigma) = \epsilon(\rho)\epsilon(\sigma)$
$\square$

Proposition 17.74. *Let $\sigma = \tau_1 \cdots \tau_r$ where each τ_i is a transposition. Then σ is even if and only if r is even.*

PROOF: Since every transposition is odd and ϵ is a homomorphism, we have $\epsilon(\tau_1 \cdots \tau_r) = (-1)^r$. $\square$

Corollary 17.74.1. *A cycle is even if and only if its length is odd.*

17.8.1 Transitive Subgroups

Definition 17.75 (Transitive)**.** A subgroup of S_n is *transitive* iff its action on $\{1, \ldots, n\}$ is transitive.

Proposition 17.76. *If G is a transitive subgroup of S_n then $n \mid |G|$.*

PROOF: By Proposition 17.18 we have
$$n = [G : \mathrm{Stab}_G(1)]$$
and so $n \mid |G|$. $\square$

17.9 Alternating Groups

Definition 17.77. Let $n \in \mathbb{N}$. The *alternating group A_n* is the subgroup of S_n consisting of the even permutations.

We call A_5 the *icosahedral (rotating) group.*

Proposition 17.78. *For $n \geq 2$ we have A_n is normal in S_n and*

$$[S_n : A_n] = 2 \ .$$

PROOF: Since $\epsilon : S_n \to \{1, -1\}$ is a homomorphism with kernel A_n. $\square$

Proposition 17.79. *Let $n \geq 2$ and $\sigma \in A_n$. Let $[\sigma]_{A_n}$ be the conjugacy class of σ in A_n, and $[\sigma]_{S_n}$ the conjugacy class of σ is S_n. Then:*

1. *If $Z_{S_n}(\sigma) \subseteq A_n$ then $|[\sigma]_{S_n}| = 2|[\sigma]_{A_n}|$.*

2. *If not then $[\sigma]_{S_n} = [\sigma]_{A_n}$.*

PROOF:
⟨1⟩1. $Z_{A_n}(\sigma) = A_n \cap Z_{S_n}(\sigma)$
⟨1⟩2. $|[\sigma]_{S_n}| = [S_n : Z_{S_n}(\sigma)]$
PROOF: Orbit-Stabilizer Theorem.
⟨1⟩3. $|[\sigma]_{A_n}| = [A_n : Z_{A_n}(\sigma)]$
PROOF: Orbit-Stabilizer Theorem.
⟨1⟩4. If $Z_{S_n}(\sigma) \subseteq A_n$ then $|[\sigma]_{S_n}| = 2|[\sigma]_{A_n}|$.
PROOF:

$$\begin{aligned} |[\sigma]_{S_n}| &= [S_n : Z_{S_n}(\sigma)] \\ &= [S_n : A_n][A_n : Z_{S_n}(\sigma)] \\ &= 2|[\sigma]_{A_n}| \end{aligned}$$

⟨1⟩5. If $Z_{S_n}(\sigma) \nsubseteq A_n$ then $[\sigma]_{S_n} = [\sigma]_{A_n}$.
⟨2⟩1. ASSUME: $Z_{S_n}(\sigma) \nsubseteq A_n$
⟨2⟩2. $A_n Z_{S_n}(\sigma) = S_n$
PROOF: Since $A_n \subseteq A_n Z_{S_n}(\sigma)$ and $[S_n : A_n] = 2$.
⟨2⟩3. $|[\sigma]_{S_n}| = |[\sigma]_{A_n}|$
PROOF:

$$\begin{aligned} |[\sigma]_{S_n}| &= [S_n : Z_{S_n}(\sigma)] \\ &= [A_n Z_{S_n}(\sigma) : Z_{S_n}(\sigma)] \\ &= [A_n : A_n \cap Z_{S_n}(\sigma)] && \text{(Second Isomorphism Theorem)} \\ &= [A_n : Z_{A_n}(\sigma)] \\ &= |[\sigma]_{A_n}| \end{aligned}$$

□

Proposition 17.80. *Let $n \geq 2$. Let $\sigma \in A_n$. Then $|[\sigma]_{S_n}| = 2|[\sigma]_{A_n}|$ if and only if the type of σ consists of distinct odd numbers.*

PROOF:
⟨1⟩1. If $|[\sigma]_{S_n}| = 2|[\sigma]_{A_n}|$ then the type of σ consists of distinct odd numbers.
⟨2⟩1. If the type of σ has an even number then $Z_{S_n}(\sigma) \nsubseteq A_n$.
PROOF: If $(a_1\ a_2\ \cdots\ a_n)$ is an even cycle that is a factor of σ then $(1\ 2\ \cdots\ n)$ is an odd permutation in $Z_{S_n}(\sigma)$.
⟨2⟩2. If the type of σ has an odd number repeated then $Z_{S_n}(\sigma) \nsubseteq A_n$.
PROOF: If $(a_1\ a_2\ \cdots\ a_n)$ and $(b_1\ b_2\ \cdots\ b_n)$ are two distinct odd factors of σ then $(a_1\ b_1)(a_2\ b_2)\cdots(a_n\ b_n)$ is an odd permutation in $Z_{S_n}(\sigma)$.
⟨2⟩3. Q.E.D.
PROOF: Proposition 17.79
⟨1⟩2. If the type of σ consists of distinct odd numbers then $|[\sigma]_{S_n}| = 2|[\sigma]_{A_n}|$.

⟨2⟩1. LET: $\sigma = (a_{11} \ \cdots \ a_{1\lambda_1})(b_{21} \ \cdots \ b_{2\lambda_2}) \cdots (c_{n1} \ \cdots \ c_{n\lambda_n})$ where the λ_i are all odd and distinct.
⟨2⟩2. LET: $\tau \in Z_{S_n}(\sigma)$
PROVE: τ is even.
⟨2⟩3. $(\tau(a_{i1}) \ \cdots \ \tau(a_{i\lambda_i})) = (\tau_{i1} \ \cdots \ \tau_{i\lambda_i})$
⟨2⟩4. The action of τ on $\{a_{i1}, \ldots, a_{i\lambda_i}\}$ is $(a_{i1} \ \cdots \ a_{i\lambda_i})^{r_i}$ for some r_i
⟨2⟩5. $\tau = \prod_{i=1}^{n} (a_{i1} \ \cdots \ a_{i\lambda_i})^{r_i}$
⟨2⟩6. τ is even.
□

Corollary 17.80.1. A_5 *is simple.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction G is a non-trivial proper normal subgroup of A_5.
⟨1⟩2. $|G|$ is one of 2, 3, 4, 5, 6, 10, 12, 15, 20 or 30.
⟨1⟩3. There are conjugacy classes in A_5 whose sizes total to 1, 2, 3, 4, 5, 9, 11, 14, 19 or 29.
⟨1⟩4. The types of the even permutations in S_5 are $[1,1,1,1,1]$, $[2,2,1]$, $[3,1,1]$ and $[5]$.
⟨1⟩5. The size of the conjugacy class of type $[2,2,1]$ in S_5 is 15.
PROOF: There are 5 ways to choose the element not in the 2-cycles, and then 3 ways to arrange the other 4 elements into two 2-cycles.
⟨1⟩6. The size of the conjugacy class of type $[2,2,1]$ in A_5 is 15.
PROOF: Proposition 17.80.
⟨1⟩7. The size of the conjugacy class of type $[3,1,1]$ in S_5 is 20.
PROOF: There are 10 ways to choose the three elements in the 3-cycle, and then two 3-cycles that they can form.
⟨1⟩8. The size of the conjugacy class of type $[3,1,1]$ in A_5 is 20.
PROOF: Proposition 17.80.
⟨1⟩9. The size of the conjugacy class of type $[5]$ in S_5 is 24.
PROOF: There are four choices for the value at 1, then three choices for its value, then two choices for its value, then one choice for its value.
⟨1⟩10. The size of the conjugacy class of type $[5]$ in S_5 is 12.
PROOF: Proposition 17.80.
⟨1⟩11. Q.E.D.
PROOF: This contradicts ⟨1⟩3.
□

Proposition 17.81. A_6 *is simple.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction G is a non-trivial proper normal subgroup of A_6.
⟨1⟩2. $|G|$ is one of 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180.
⟨1⟩3. There are conjugacy classes in A_6 whose sizes total to 1, 2, 3, 4, 5, 7, 8, 9, 11, 14, 17, 19, 23, 29, 35, 39, 44, 59, 71, 89, 119 or 179.

$\langle 1 \rangle 4$. The types of the even permutations in S_6 are $[1,1,1,1,1,1]$, $[2,2,1,1]$, $[3,1,1,1]$, $[3,3]$, $[4,2]$, $[5,1]$.
$\langle 1 \rangle 5$. The size of the conjugacy class of type $[2,2,1,1]$ in S_6 is 45.
$\langle 1 \rangle 6$. The size of the conjugacy class of type $[2,2,1,1]$ in A_6 is 45.
$\langle 1 \rangle 7$. The size of the conjugacy class of type $[3,1,1,1]$ in S_6 is 40.
$\langle 1 \rangle 8$. The size of the conjugacy class of type $[3,1,1,1]$ in A_6 is 40.
$\langle 1 \rangle 9$. The size of the conjugacy class of type $[3,3]$ in S_6 is 80.
$\langle 1 \rangle 10$. The size of the conjugacy class of type $[3,3]$ in A_6 is 80.
$\langle 1 \rangle 11$. The size of the conjugacy class of type $[4,2]$ in S_6 is 90.
$\langle 1 \rangle 12$. The size of the conjugacy class of type $[4,2]$ in A_6 is 90.
$\langle 1 \rangle 13$. The size of the conjugacy class of type $[5,1]$ in S_6 is 144.
$\langle 1 \rangle 14$. The size of the conjugacy class of type $[5,1]$ in A_6 is 72.
$\langle 1 \rangle 15$. The size of the conjugacy class of type $[6]$ in S_6 is 120.
$\langle 1 \rangle 16$. The size of the conjugacy class of type $[6]$ in A_6 is 120.
$\langle 1 \rangle 17$. Q.E.D.
Proof: This is a contradiction.
□

Proposition 17.82. *The icosahedral group A_5 is the group of symmetries of an icosahedron obtained through rigid motions.*

Proof: Routine. □

Proposition 17.83. *The alternating group A_n is generated by 3-cycles.*

Proof:
$\langle 1 \rangle 1$. The product of two transpositions is generated by 3-cycles.
$\langle 2 \rangle 1$. $(ab)(ab) = e$
$\langle 2 \rangle 2$. $(ab)(ac) = (acb)$ for $b \neq c$
$\langle 2 \rangle 3$. $(ab)(cd) = (adc)(abc)$ for $c \neq d$ and $c, d \notin \{a, b\}$
□

Proposition 17.84. *Let $n \geq 5$. If a normal subgroup of A_n contains a 3-cycle, then it contains all 3-cycles.*

Proof:
$\langle 1 \rangle 1$. Let: N be a normal subgroup of A_n.
$\langle 1 \rangle 2$. Let: $(abc) \in N$
$\langle 1 \rangle 3$. N contains the conjugacy class of (abc).
$\langle 1 \rangle 4$. The conjugacy class of (abc) in N is the same as its conjugacy class in S_n.
Proof: Proposition 17.80 since the type of (abc) is $[3,1,1,\ldots]$.
$\langle 1 \rangle 5$. N contains all 3-cycles.
□

Proposition 17.85. *For $n \geq 4$, the center of A_n is trivial.*

Theorem 17.86. *For $n \geq 5$, the alternating group A_n is simple.*

Proof:

$\langle 1 \rangle 1.$ A_5 is simple.
  PROOF: Corollary 17.80.1.
$\langle 1 \rangle 2.$ For $n \geq 6$ we have A_n is simple.
  $\langle 2 \rangle 1.$ LET: $n \geq 6$
  $\langle 2 \rangle 2.$ LET: N be a nontrivial normal subgroup of A_n.
  $\langle 2 \rangle 3.$ N contains a 3-cycle.
    $\langle 3 \rangle 1.$ PICK $\tau \in N$ such that $\tau \neq \mathrm{id}$ and τ acts on at most 6 elements.
    $\langle 3 \rangle 2.$ PICK $T \subseteq \{1, \ldots, n\}$ with $|T| = 6$ such that τ acts on T.
    $\langle 3 \rangle 3.$ Consider A_6 as a subgroup of A_n by letting it act on T.
    $\langle 3 \rangle 4.$ $N \cap A_6$ is normal.
    $\langle 3 \rangle 5.$ $N \cap A_6$ is nontrivial.
    $\langle 3 \rangle 6.$ $N \cap A_6 = A_6$
      PROOF: Proposition 17.81.
    $\langle 3 \rangle 7.$ N contains a 3-cycle.
  $\langle 2 \rangle 4.$ N contains all 3-cycles.
    PROOF: Proposition 17.84.
  $\langle 2 \rangle 5.$ $N = A_n$
    PROOF: Proposition 17.83.
□

Corollary 17.86.1. *For $n \geq 5$, we have S_n is unsolvable.*

PROOF: Since the composition factors of S_n are C_2 and A_n. □

Chapter 18

Extensions

Definition 18.1 (Extension)**.** Let G, N and H be groups. Then G is an *extension* of H by N iff there exist homomorphisms $phi : N \to G$ and $\psi : G \to H$ such that

$$1 \to N \to G \to H \to 1$$

is an exact sequence; i.e. ϕ is injective, ψ is surjective, and $\operatorname{im} \phi = \ker \psi$.

Proposition 18.2. *Let G be an extension of H by N. Then the composition factors of G are the union of the composition factors of H and the composition factors of N.*

Proof: From Proposition 15.134 since $H \cong G/N$. □

Definition 18.3 (Split Extension)**.** An exact sequence of groups

$$1 \to N \to G \to H \to 1$$

splits iff H is a subgroup of G and $N \cap H = \{e\}$.

Example 18.4. The sequence

$$0 \to \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is exact but does not split as there is no subgroup of $\mathbb{Z}$ isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Proposition 18.5. *Let N be a normal subgroup of G and let H be a subgroup such that $G = NH$ and $N \cap H = \{e\}$. Then G is a split extension of H by N.*

Proof:
⟨1⟩1. $G/N \cong H$
  ⟨2⟩1. Let: α be the homomorphism $H \hookrightarrow G \twoheadrightarrow G/N$
  ⟨2⟩2. α is injective.
    Proof: Since $\ker \alpha = \{e\}$
  ⟨2⟩3. α is surjective.

Proof: For all $g \in G$, pick $n \in N$ and $h \in H$ such that $g^{-1} = nh$. Then $gN = \alpha(h^{-1})$.
⟨2⟩4. $\alpha : H \cong G/N$ is an isomorphism.
⟨1⟩2. The exact sequnce $1 \to N \to G \twoheadrightarrow G/N \cong H \to 1$ splits.
□

Proposition 18.6. *Let N and H be groups. Let $\theta : H \to \mathrm{Aut}_{\mathbf{Grp}}(N)$ be a homomorphism. The sequence*

$$1 \to N \to N \rtimes_\theta H \to H \to 1$$

is split exact.

Proof: Easy. □

Proposition 18.7. *Let G be an Abelian p-group. Let $g \in G$ be an element of maximal order. Then the exact sequence*

$$0 \to \langle g \rangle \to G \to G/\langle g \rangle \to 0$$

splits.

Proof:
⟨1⟩1. Assume: as induction hypothesis the proposition is true for all Abelian p-groups smaller than G.
⟨1⟩2. Assume: w.l.o.g. G is non-trivial.
⟨1⟩3. Let: $|g| = p^r$
⟨1⟩4. Let: $K = \langle g \rangle$
⟨1⟩5. K is normal.
⟨1⟩6. Assume: w.l.o.g. $G \neq K$
⟨1⟩7. Pick an element $h + K \in G/K$ of order p.
Proof: Cauchy's Theorem
⟨1⟩8. Let: $G' = \pi^{-1}(\langle h + K \rangle)$
⟨1⟩9. $|G'| = p^{r+1}$
⟨1⟩10. $K \subseteq G'$
⟨1⟩11. G' is not cyclic.
Proof: By maximality of the order of g.
⟨1⟩12. Pick $h \in G'$ with $h \notin K$ and $|h| = p$.
Proof: Lemma 16.24.
⟨1⟩13. Let: $H = \langle h \rangle$
⟨1⟩14. $K \cap H = \{0\}$
⟨2⟩1. Let: $x \in K \cap H$
⟨2⟩2. Let: $x = ih$ where $0 \leq i < p$
⟨2⟩3. $x + K = K$
⟨2⟩4. $ih + K = K$
⟨2⟩5. $i = 0$
Proof: Since the order of $h + K$ is p.
⟨1⟩15. $|G/H| < |G|$

⟨1⟩16. LET: $K' = \langle g + H \rangle$
⟨1⟩17. K' is a cyclic subgroup of maximal order in G/H.
PROOF:

$$\begin{aligned} K' &= \frac{K+H}{H} \\ &\cong \frac{K}{K \cap H} && \text{(Second Isomorphism Theorem)} \\ &\cong K \end{aligned}$$

⟨1⟩18. PICK a subgroup L' of G/H such that $K'+L' = G/H$ and $K' \cap L' = \{0\}$.
PROOF: By the induction hypothesis ⟨1⟩1.
⟨1⟩19. LET: $L = \pi^{-1}(L')$
⟨1⟩20. $H \subseteq L$
⟨1⟩21. $K + L = G$
⟨1⟩22. $K \cap L = \{0\}$
□

Proposition 18.8. *Let p be a prime. If*

$$G = \frac{\mathbb{Z}}{p^{r_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p^{r_m}\mathbb{Z}} \cong \frac{\mathbb{Z}}{p^{s_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p^{s_n}\mathbb{Z}}$$

with $r_1 \geq \cdots \geq r_m$ and $s_1 \geq \cdots \geq s_n$ then $m = n$ and $r_i = s_i$ for all i.

PROOF:
⟨1⟩1. ASSUME: as induction hypothesis the result is true for all groups smaller than G.
⟨1⟩2. LET: pG be the image of the homomorphsim $g \mapsto pg$
⟨1⟩3.

$$pG \cong \frac{\mathbb{Z}}{p^{r_1-1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p^{r_m-1}\mathbb{Z}} \cong \frac{\mathbb{Z}}{p^{s_1-1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p^{s_n-1}\mathbb{Z}}$$

⟨1⟩4. Q.E.D.
PROOF: The result follows by induction.
□

Corollary 18.8.1. *Every finite Abelian group is the direct sum of a unique multiset of cyclic p-groups.*

Definition 18.9. Let G be a finite Abelian group. The multiset of *elementary divisors* of G are the numbers $e_1, \ldots, e_n$ such that each is a power of a prime and

$$G \cong \frac{\mathbb{Z}}{e_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{e_n\mathbb{Z}} \ .$$

Proposition 18.10. *For any finite Abelian group G, there exist positive integers $d_1, \ldots, d_s$ such that*

$$1 < d_1 \mid \cdots \mid d_s$$

and

$$G \cong \frac{\mathbb{Z}}{d_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{d_s\mathbb{Z}} \ .$$

PROOF:
⟨1⟩1. LET: $p_1, \ldots, p_s$ be the prime factors of $|G|$.
⟨1⟩2. For $i = 1, \ldots, s$, let n_{ij} be the integers such that $n_{i1} \geq n_{i2} \geq \cdots$ such that either $p_i^{n_{ij}}$ is an elementary divisor of G, or $n_{ij} = 0$
⟨1⟩3. LET: r be the greatest integer such that some n_{ir} is non-zero.
⟨1⟩4. LET: $d_{r-j+1} = \prod_i p_i^{n_{ij}}$
⟨1⟩5.
$$\frac{\mathbb{Z}}{d_{r-j+1}\mathbb{Z}} \cong \frac{\mathbb{Z}}{p_1^{n_{1j}}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_s^{n_{sj}}\mathbb{Z}}$$
⟨1⟩6.
$$G \cong \frac{\mathbb{Z}}{d_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{d_r\mathbb{Z}}$$
□

Definition 18.11 (Invariant Factors)**.** For any finite Abelian group G, the *invariant factors* of G are the positive integers $d_1, \ldots, d_s$ such that

$$1 < d_1 \mid \cdots \mid d_s$$

and
$$G \cong \frac{\mathbb{Z}}{d_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{d_s\mathbb{Z}} \ .$$

Lemma 18.12. *Let G be a finite Abelian group. Assume that, for every n, the number of elements g such that $ng = 0$ is at most n. Then G is cyclic.*

PROOF:
⟨1⟩1. LET: $1 < d_1 \mid \cdots \mid d_r$ be the invariant factors of G.
PROVE: $r = 1$ hence $G \cong \mathbb{Z}/d_1\mathbb{Z}$
⟨1⟩2. ASSUME: for a contradiction $s > 1$
⟨1⟩3. $|G| > d_s$
⟨1⟩4. For all $g \in G$ we have $d_s g = 0$.
⟨1⟩5. Q.E.D.
PROOF: This contradicts the assumption that the number of elements g such that $d_s g = 0$ is at most d_s.
□

Chapter 19

Classification of Groups

Example 19.1.

- The only group of order 1 is the trivial group.
- The only group of order 2 is C_2.
- The only group of order 3 is C_3.
- There are two groups of order 4: C_4 and $C_2 \times C_2$.
- The only group of order 5 is C_5.
- There are two groups of order 6: C_6 and S_3.
- The only group of order 7 is C_7.
- There are two groups of order 9: C_9 and $C_3 \times C_3$.
- There are two groups of order 10: C_{10} and D_{10}.
- The only group of order 11 is C_{11}.
- The only group of order 13 is C_{13}.
- There are two groups of order 14: C_{14} and D_{14}.
- The only group of order 15 is C_{15}.

Proposition 19.2. *The only non-Abelian groups of order 8 are D_8 and Q_8.*

PROOF:
$\langle 1 \rangle 1$. LET: G be a non-Abelian group of order 8.
$\langle 1 \rangle 2$. G has no element of order 8.
 PROOF: If it does then it is C_8 and hence Abelian.
$\langle 1 \rangle 3$. PICK an element y of order 4.
 $\langle 2 \rangle 1$. PICK an element a of order 2.
 $\langle 2 \rangle 2$. $G / \langle a \rangle$ is isomorphic to C_4 or $C_2 \times C_2$.
 $\langle 2 \rangle 3$. PICK an element $y \langle a \rangle$ of order 2 in $G / \langle a \rangle$

⟨2⟩4. $y^2 \in \langle a \rangle$

⟨2⟩5. CASE:
$y^2 = a$

PROOF: In this case y is of order 4.

⟨2⟩6. CASE:
$y^2 = e$

PROOF: In this case $G \cong C_2^3$ which is Abelian.

⟨1⟩4. PICK $x \notin \langle y \rangle$ such that $x^2 = e$ or $x^2 = y^2$

⟨2⟩1. $G/\langle y \rangle \cong C_2$

⟨2⟩2. PICK $x\langle y \rangle \in G/\langle y \rangle$ of order 2.

⟨2⟩3. $x^2 \in \langle y \rangle$

⟨2⟩4. $x^2 \neq y$ and $x^2 \neq y^3$

⟨2⟩5. $x^2 = e$ or $x^2 = y^2$

⟨1⟩5. $xy = y^3 x$

⟨2⟩1. $xy \neq e$

PROOF: Since $y^{-1} = y^3 \neq x$.

⟨2⟩2. $xy \neq y$

PROOF: $xy = y$ implies $x = e$.

⟨2⟩3. $xy \neq y^2$

PROOF: $xy = y^2$ implies $x = y$.

⟨2⟩4. $xy \neq y^3$

PROOF: $xy = y^3$ implies $x = y^2$.

⟨2⟩5. $xy \neq x$

PROOF: $xy = x$ implies $y = e$.

⟨2⟩6. $xy \neq yx$

PROOF: $xy = yx$ implies G is Abelian.

⟨2⟩7. $xy \neq y^2 x$

⟨3⟩1. ASSUME: for a contradiction $xy = y^2 x$

⟨3⟩2. $xy^2 = x$

PROOF:

$$\begin{aligned} xy^2 &= y^2 xy \\ &= y^4 x \\ &= x \end{aligned}$$

⟨3⟩3. $y^2 = e$

⟨1⟩6. The multiplication table of G is one of the following.

e	y	y^2	y^3	x	yx	y^2x	y^3x
y	y^2	y^3	e	yx	y^2x	y^3x	x
y^2	y^3	e	y	y^2x	y^3x	x	yx
y^3	e	y	y^2	y^3x	x	yx	y^2x
x	y^3x	y^2x	yx	e	y^3	y^2	y
yx	x	y^3x	y^2x	y	e	y^3	y^2
y^2x	yx	x	y^3x	y^2	y	e	y^3
y^3x	y^2x	yx	x	y^3	y^2	y	e

e	y	y^2	y^3	x	yx	y^2x	y^3x
y	y^2	y^3	e	yx	y^2x	y^3x	x
y^2	y^3	e	y	y^2x	y^3x	x	yx
y^3	e	y	y^2	y^3x	x	yx	y^2x
x	y^3x	y^2x	yx	y^2	y	e	y^3
yx	x	y^3x	y^2x	y^3	y^2	y	e
y^2x	yx	x	y^3x	e	y^3	y^2	y
y^3x	y^2x	yx	x	y	e	y^3	y^2

$\langle 1 \rangle 7$. $G \cong D_8$ or $G \cong Q_8$.
$\square$

Corollary 19.2.1. *The groups of order 8 are D_8, Q_8, $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

Proposition 19.3. *Let q be an odd prime. Then D_{2q} is the only non-Abelian group of order $2q$.*

Proof:
$\langle 1 \rangle 1$. Let: G be a non-Abelian group of order $2q$.
$\langle 1 \rangle 2$. Pick $y \in G$ of order q.
  Proof: Cauchy's Theorem
$\langle 1 \rangle 3$. $\langle y \rangle$ is the only subgroup of order q.
  Proof: Third Sylow Theorem
$\langle 1 \rangle 4$. $\langle y \rangle$ is normal.
$\langle 1 \rangle 5$. Pick $x \in G - \langle y \rangle - \{e\}$
$\langle 1 \rangle 6$. $|x| = 2$
  Proof: We cannot have $|x| = 2q$ since G is not cyclic, and $|x| \neq q$ since $\langle x \rangle$ is not the subgroup of order q.
$\langle 1 \rangle 7$. $xyx^{-1} \in \langle y \rangle$
  Proof: Since $x\langle y \rangle x^{-1} = \langle y \rangle$ by $\langle 1 \rangle 3$.
$\langle 1 \rangle 8$. Pick r such that $0 \leq r < q$ and $xyx^{-1} = y^r$.
$\langle 1 \rangle 9$. $y^{r^2} = y$
  Proof:

$$\begin{aligned} y^{r^2} &= (xyx^{-1})^r && (\langle 1 \rangle 8) \\ &= xy^r x^{-1} \\ &= x^2 y x^{-2} && (\langle 1 \rangle 8) \\ &= y && (\langle 1 \rangle 6) \end{aligned}$$

$\langle 1 \rangle 10$. $q \mid (r-1)(r+1)$
  Proof: Since $y^{(r-1)(r+1)} = e$ and $|y| = q$ by $\langle 1 \rangle 2$.
$\langle 1 \rangle 11$. $r = 1$ or $r = q - 1$
  Proof: Since $0 \leq r < q$ by $\langle 1 \rangle 8$.
$\langle 1 \rangle 12$. $r \neq 1$
  $\langle 2 \rangle 1$. Assume: for a contradiction $r = 1$.
  $\langle 2 \rangle 2$. $xy = yx$
    Proof: $\langle 1 \rangle 8$
  $\langle 2 \rangle 3$. $|xy| = 2q$

PROOF: Proposition 14.22

⟨2⟩4. G is cyclic.

⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1.

⟨1⟩13. $x^2 = e$ and $y^q = e$ and $yx = xy^{q-1}$

⟨1⟩14. $G \cong D_{2q}$

□

Corollary 19.3.1. *For q an odd prime, the only groups of order $2q$ are C_{2q} and D_{2q}.*

Proposition 19.4. *There is no non-Abelian simple group of order less than 60.*

PROOF:We rule out the other sizes as follows:

- 1 — Only group is the trivial group.
- 2 — Prime therefore cyclic
- 3 — Prime therefore cyclic
- 4 — Corollary 17.49.1
- 5 — Prime therefore cyclic
- 6 — Corollary 17.52.2
- 7 — Prime therefore cyclic
- 8 — Corollary 17.49.1
- 9 — Corollary 17.49.1
- 10 — Corollary 17.52.2
- 11 — Prime therefore cyclic
- 12 —

  ⟨1⟩1. There is no simple non-Abelian group of order 12.

  ⟨2⟩1. ASSUME: for a contradiction G is a simple non-Abelian group of order 12.

  ⟨2⟩2. G has 4 3-Sylow subgroups.

  ⟨2⟩3. G has 8 elements of order 3.

  ⟨2⟩4. G has 3 elements of order 2 or 4.

  ⟨2⟩5. G has one 2-Sylow subgroup.

  ⟨2⟩6. The 2-Sylow subgroup of G is normal.

  ⟨2⟩7. Q.E.D.

  PROOF: This contradicts ⟨2⟩1.

- 13 — Prime therefore cyclic

- 14 — Corollary 17.52.2
- 15 — Corollary 17.52.2
- 16 — Corollary 17.49.1
- 17 — Prime therefore cyclic
- 18 — Corollary 17.52.2
- 19 — Prime therefore cyclic
- 20 — Corollary 17.52.2
- 21 — Corollary 17.52.2
- 22 — Corollary 17.52.2
- 23 — Prime therefore cyclic
- 24 —

  ⟨1⟩2. There is no simple non-Abelian group of order 24.

  ⟨2⟩1. ASSUME: for a contradiction G is a simple non-Abelian group of order 24.
  ⟨2⟩2. G has 3 2-Sylow subgroups.
  ⟨2⟩3. LET: $\gamma : G \to S_3$ be the action of conjugation of G on the set of 2-Sylow subgroups.
  ⟨2⟩4. $\ker \gamma \neq \{e\}$
  PROOF: γ cannot be injective since $|G| > |S_3|$.
  ⟨2⟩5. $\ker \gamma \neq G$
  ⟨2⟩6. $\ker \gamma$ is a proper non-trivial normal subgroup of G.
  ⟨2⟩7. Q.E.D.
  PROOF: This contradicts ⟨2⟩1.

- 25 — Corollary 17.49.1
- 26 — Corollary 17.52.2
- 27 — Corollary 17.49.1
- 28 — Corollary 17.52.2
- 29 — Prime therefore cyclic
- 30 — Proposition 17.55
- 31 — Prime therefore cyclic
- 32 — Corollary 17.49.1
- 33 — Corollary 17.52.2

- 34 — Corollary 17.52.2
- 35 — Corollary 17.52.2
- 36 —

  $\langle 1\rangle 3$. There is no simple non-Abelian group of order 36.

  $\langle 2\rangle 1$. ASSUME: for a contradiction G is a simple non-Abelian group of order 36.
  $\langle 2\rangle 2$. G has 4 3-Sylow subgroups.
  $\langle 2\rangle 3$. LET: $\gamma : G \to S_4$ be the action of conjugation of G on the set of 2-Sylow subgroups.
  $\langle 2\rangle 4$. $\ker \gamma \neq \{e\}$
  PROOF: γ cannot be injective since $|G| > |S_4|$.
  $\langle 2\rangle 5$. $\ker \gamma \neq G$
  $\langle 2\rangle 6$. $\ker \gamma$ is a proper non-trivial normal subgroup of G.
  $\langle 2\rangle 7$. Q.E.D.
  PROOF: This contradicts $\langle 2\rangle 1$.

- 37 — Prime therefore cyclic
- 38 — Corollary 17.52.2
- 39 — Corollary 17.52.2
- 40 — There can be only 1 5-Sylow subgroup.
- 41 — Prime therefore cyclic
- 42 — Proposition 17.55
- 43 — Prime therefore cyclic
- 44 — Corollary 17.52.2
- 45 — There can be only 1 5-Sylow subgroup.
- 46 — Corollary 17.52.2
- 47 — Prime therefore cyclic
- 48 —

  $\langle 1\rangle 4$. There is no simple non-Abelian group of order 48.

  $\langle 2\rangle 1$. ASSUME: for a contradiction G is a simple non-Abelian group of order 48.
  $\langle 2\rangle 2$. G has 3 2-Sylow subgroups.
  $\langle 2\rangle 3$. LET: $\gamma : G \to S_3$ be the action of conjugation of G on the set of 2-Sylow subgroups.
  $\langle 2\rangle 4$. $\ker \gamma \neq \{e\}$
  PROOF: γ cannot be injective since $|G| > |S_3|$.

⟨2⟩5. $\ker \gamma \neq G$
⟨2⟩6. $\ker \gamma$ is a proper non-trivial normal subgroup of G.
⟨2⟩7. Q.E.D.
PROOF: This contradicts ⟨2⟩1.

- 49 — Corollary 17.49.1
- 50 — Corollary 17.52.2
- 51 — Corollary 17.52.2
- 52 — Corollary 17.52.2
- 53 — Prime therefore cyclic
- 54 — Corollary 17.52.2
- 55 — Corollary 17.52.2
- 56 — Corollary 17.52.2
- 57 — Corollary 17.52.2
- 58 — Corollary 17.52.2
- 59 — Prime therefore cyclic

Proposition 19.5. *Every simple group of order 60 has a subgroup of index 5.*

PROOF:
⟨1⟩1. LET: G be a simple group of order 60.
⟨1⟩2. The number of 2-Sylow subgroups of G is either 5 or 15.
⟨2⟩1. LET: n be the number of 2-Sylow subgroups.
⟨2⟩2. $60|n!$
PROOF: Corollary 17.56.1.
⟨2⟩3. $n \geq 5$
⟨2⟩4. $n \mid 15$
PROOF: Third Sylow Theorem
⟨2⟩5. $n = 5$ or $n = 15$
⟨1⟩3. ASSUME: w.l.o.g. G has 15 2-Sylow subgroups.
⟨1⟩4. G has 4 or 10 3-Sylow subgroups.
⟨1⟩5. G has 10 3-Sylow subgroups.
PROOF: Corollary 17.56.1.
⟨1⟩6. G has exactly 6 5-Sylow subgroups.
⟨1⟩7. The number of elements of order 3 is 20.
⟨1⟩8. The number of elements of order 5 is 24.
⟨1⟩9. The number of elements of order 2 or 4 is 15.
⟨1⟩10. PICK two 2-Sylow subgroups H_1 and H_2 with non-trivial intersection.
⟨1⟩11. LET: $g \in G$ be such that $H_1 \cap H_2 = \{e, g\}$.
⟨1⟩12. LET: $K = Z_G(H_1 \cap H_2)$

$\langle 1 \rangle 13.$ $|K| = 12$ or $|K| = 20$
PROOF: We have $4 \mid |K|$ since $H_1 \leq K$, and $|K| \geq 6$ since $H_1 \cup H_2 \subseteq K$. We also have $|K| \mid 60$.
$\langle 1 \rangle 14.$ $[G : K] \neq 3$
PROOF: There cannot be an embedding of G in S_3.
$\langle 1 \rangle 15.$ $[G : K] = 5$
$\square$

Theorem 19.6. *A_5 is the only simple group of order 60.*

PROOF:
$\langle 1 \rangle 1.$ LET: G be a simple group of order 60.
$\langle 1 \rangle 2.$ PICK a subgroup K of G of index 5.
$\langle 1 \rangle 3.$ LET: $\phi : G \rightarrow S_5$ be the action of G on G/K of left multiplication.
$\langle 1 \rangle 4.$ ϕ is injective.
PROOF: Since $\ker \phi$ is a proper normal subgroup of G hence $\ker \phi = \{e\}$.
$\langle 1 \rangle 5.$ $\phi(G)$ is a subgroup of S_5 of index 2.
$\langle 1 \rangle 6.$ $\phi(G)$ is normal in S_5.
$\langle 1 \rangle 7.$ $\phi(G) \cap A_5$ is a normal subgroup of A_5
$\langle 1 \rangle 8.$ $\phi(G) \cap A_5 = \{e\}$ or $\phi(G) \cap A_5 = A_5$
PROOF: Corollary 17.80.1.
$\langle 1 \rangle 9.$ $\phi(G) \cap A_5 = A_5$
PROOF: We cannot have $\phi(G) \cap A_5 = \{e\}$ lest

$$|\phi(G)A_5| = |\phi(G)||A_5|/|\phi(G) \cap A_5| = 3600$$

by the Second Isomorphism Theorem.
$\langle 1 \rangle 10.$ $\phi(G) = A_5$
$\langle 1 \rangle 11.$ $\phi : G \cong A_5$
$\square$

Proposition 19.7. *There is no non-Abelian simple group of order between 60 and 168.*

PROOF: We rule out the other sizes as follows:

- 61 — prime therefore cyclic
- 62 — Corollary 17.52.2
- 63 — Corollary 17.52.1
- 64 — Corollary 17.49.1
- 65 — Corollary 17.52.2
- 66 — Corollary 17.52.2
- 67 — prime therefore cyclic
- 68 — Corollary 17.52.2
- 69 — Corollary 17.52.2

- 70 — Proposition 17.55
- 71 — prime therefore cyclic
- 72

  ⟨1⟩1. There is no simple non-Abelian group of order 72

  PROOF:
  ⟨2⟩1. ASSUME: for a contradiction G is a simple non-Abelian group of order 72.
  ⟨2⟩2. G has 4 3-Sylow subgroups.
  ⟨2⟩3. LET: $\gamma : G \to S_4$ be the action of conjugation on the set of 3-Sylow subgroups.
  ⟨2⟩4. $\ker \gamma \neq 1$
  PROOF: Since $|G| > |S_4|$.
  ⟨2⟩5. $\ker \gamma$ is a non-trivial proper subgroup of G.
  ⟨2⟩6. Q.E.D.
  PROOF: This is a contradiction.

- 73 — prime therefore cyclic
- 74 — Corollary 17.52.2
- 75 — Corollary 17.52.2
- 76 — Corollary 17.52.2
- 77 — Corollary 17.52.2
- 78 — Corollary 17.52.2
- 79 — prime therefore cyclic
- 80

  ⟨1⟩2. There is no simple non-Abelian group of order 80.

  PROOF:
  ⟨2⟩1. ASSUME: for a contradiction G is a simple non-Abelian group of order 80.
  ⟨2⟩2. G has 5 2-Sylow subgroups.
  ⟨2⟩3. LET: $\gamma : G \to S_5$ be the action of conjugation on the set of 2-Sylow subgroups.
  ⟨2⟩4. $\ker \gamma \neq 1$
  PROOF: Otherwise $\operatorname{im} \gamma$ would be a subgroup of S_5 of order 80, contradicting Lagrange's Theorem.
  ⟨2⟩5. $\ker \gamma$ is a non-trivial normal subgroup of G.
  ⟨2⟩6. Q.E.D.
  PROOF: This is a contradiction.

- 81 — Corollary 17.49.1

- 82 — Corollary 17.52.2
- 83 — prime therefore cyclic
- 84 — Corollary 17.52.1
- 85 — Corollary 17.52.2
- 86 — Corollary 17.52.2
- 87 — Corollary 17.52.2
- 88 — Corollary 17.52.2
- 89 — prime therefore cyclic
- 90 — Corollary 17.52.1
- 91 — Corollary 17.52.2
- 92 — Corollary 17.52.2
- 93 — Corollary 17.52.2
- 94 — Corollary 17.52.2
- 95 — Corollary 17.52.2
- 96 — There are 3 2-Sylow subgroups. The kernel of the action of conjugation $G \to S_3$ is a non-trivial normal subgroup of G.
- 97 — prime therefore cyclic
- 98 — Corollary 17.52.2
- 99 — Corollary 17.52.2
- 100 — Corollary 17.52.2
- 101 — prime therefore cyclic
- 102 — Proposition 17.55
- 103 — prime therefore cyclic
- 104 — Corollary 17.52.2
- 105 — Proposition 17.55
- 106 — Corollary 17.52.2
- 107 — prime therefore cyclic
- 108 — There are 4 3-Sylow subgroups. The kernel of the action of conjugation $G \to S_4$ is a non-trivial normal subgroup of G.

- 109 — prime therefore cyclic
- 110 — Proposition 17.55
- 111 — Corollary 17.52.2
- 112

⟨1⟩3. There is no simple non-Abelian group of order 112.

⟨2⟩1. Assume: for a contradiction G is a simple non-Abelian group of order 112.
⟨2⟩2. G has exactly 7 2-Sylow subgroups.
⟨2⟩3. Let: $\gamma : G \to A_7$ be the action of conjugation of G on the set of 2-Sylow subgroups.
Proof: $\gamma(g)$ is always an even permutation since G has no subgroup of index 2.
⟨2⟩4. $\ker \gamma \neq 1$
Proof: Since $|G|$ does not divide $|A_7| = 7!/2$.
⟨2⟩5. $\ker \gamma$ is a non-trivial normal subgroup of G.
⟨2⟩6. Q.E.D.

- 113 — prime therefore cyclic
- 114 — Proposition 17.55
- 115 — Corollary 17.52.2
- 116 — Corollary 17.52.2
- 117 — Corollary 17.52.2
- 118 — Corollary 17.52.2
- 119 — Corollary 17.52.2
- 120

⟨1⟩4. There is no simple non-Abelian group of order 120.

Proof:
⟨2⟩1. Assume: for a contradiction G is a simple non-Abelian group of order 120.
⟨2⟩2. There are exactly 6 5-Sylow subgroups.
⟨2⟩3. Let: $\gamma : G \to A_6$ be the action of conjugation on the set of 5-Sylow subgroups.
⟨2⟩4. $\operatorname{im} \gamma$ is a subgroup of A_6 of order 120.
⟨2⟩5. Q.E.D.
Proof: This is a contradiction by inspection of the list of subgroups of A_6.

- 121 — Corollary 17.49.1

- 122 — Corollary 17.52.2
- 123 — Corollary 17.52.2
- 124 — Corollary 17.52.2
- 125 — Corollary 17.49.1
- 126 — Corollary 17.52.1
- 127 — prime therefore cyclic
- 128 — Corollary 17.49.1
- 129 — Corollary 17.52.2
- 130 — Proposition 17.55
- 131 — prime therefore cyclic
- 132

  ⟨1⟩5. There is no simple non-Abelian group of order 132.

  ⟨2⟩1. ASSUME: for a contradiction G is a simple non-Abelian group of order 132.
  ⟨2⟩2. There are at least 4 3-Sylow subgroups.
  ⟨2⟩3. There are at least 8 elements of order 3.
  ⟨2⟩4. There are exactly 12 11-Sylow subgroups.
  ⟨2⟩5. There are exactly 120 elements of order 11.
  ⟨2⟩6. There are exactly 3 elements of order 2.
  ⟨2⟩7. There is a unique 2-Sylow subgroups.
  ⟨2⟩8. Q.E.D.
  PROOF: This is a contradiction.

- 133 — Corollary 17.52.2
- 134 — Corollary 17.52.2
- 135 — Corollary 17.52.1
- 136 — Corollary 17.52.2
- 137 — prime therefore cyclic
- 138 — Proposition 17.55
- 139 — prime therefore cyclic
- 140 — Corollary 17.52.1
- 141 — Corollary 17.52.2
- 142 — Corollary 17.52.2

- 143 — Corollary 17.52.2
- 144 — Burnside's Theorem
- 145 — Burnside's Theorem
- 146 — Burnside's Theorem
- 147 — Burnside's Theorem
- 148 — Burnside's Theorem
- 149 — prime therefore cyclic
- 150 — There are exactly 6 5-Sylow subgroups. The kernel of the action of conjugation $G \to A_5$ is a non-trivial normal subgroup since 150 does not divide $|A_5| = 60$.
- 151 — prime therefore cyclic
- 152 — Burnside's Theorem
- 153 — Burnside's Theorem
- 154 — Proposition 17.55
- 155 — Burnside's Theorem
- 156 — Corollary 17.52.2
- 157 — prime therefore cyclic
- 158 — Burnside's Theorem
- 159 — Burnside's Theorem
- 160 — Burnside's Theorem
- 161 — Burnside's Theorem
- 162 — Burnside's Theorem
- 163 — prime therefore cyclic
- 164 — Burnside's Theorem
- 165 — Proposition 17.55
- 166 — Burnside's Theorem
- 167 — prime therefore cyclic

Proposition 19.8. *Every group of order* < 120 *and* $\neq 60$ *is solvable.*

PROOF:

$\langle 1 \rangle 1$. LET: G be a group of order n where $n < 120$ and $n \neq 60$.
$\langle 1 \rangle 2$. If n is odd then G is solvable.
PROOF: Feit-Thompson Theorem
$\langle 1 \rangle 3$. If n has at most two prime factors then G is solvable.
PROOF: Burnside's Theorem
$\langle 1 \rangle 4$. CASE: $n = pqr$ for some primes p, q, r
PROOF: Its composition factors must be C_p, C_q and C_r.
$\langle 1 \rangle 5$. CASE: $n = 84$
PROOF: By the Third Sylow Theorem, the 7-Sylow subgroup is normal. Since every group of order 12 is solvable, so is every group of order 84.
□

Proposition 19.9. *Let p and q be primes with $p < q$.*

1. *If $q \not\equiv 1 (\mathrm{mod}\, p)$, then the only group of order pq is C_{pq}*
2. *If $q \equiv 1 (\mathrm{mod}\, p)$, then there are exactly two groups of order pq: the cyclic group C_{pq} and a non-Abelian group.*

PROOF:
$\langle 1 \rangle 1$. LET: $|G| = pq$
$\langle 1 \rangle 2$. ASSUME: G is not cyclic.
$\langle 1 \rangle 3$. There is exactly one q-Sylow subgroup $\langle a \rangle$, say.
PROOF: Third Sylow Theorem.
$\langle 1 \rangle 4$. There is more than one p-Sylow subgroup.
$\langle 1 \rangle 5$. The number of p-Sylow subgroups divides q and is congruent to 1 modulo p.
$\langle 1 \rangle 6$. $q \equiv 1 (\mathrm{mod}\, p)$
$\langle 1 \rangle 7$. PICK an element b of order q.
$\langle 1 \rangle 8$. LET: $N = \langle a \rangle$ and $H = \langle b \rangle$
$\langle 1 \rangle 9$. $N \cap H = \{e\}$
$\langle 1 \rangle 10$. $G = NH$
$\langle 1 \rangle 11$. Define $\gamma : H \to \mathrm{Aut}_{\mathbf{Grp}}(N)$ by $\gamma(h)(n) = hnh^{-1}$
$\langle 1 \rangle 12$. $\mathrm{Aut}_{\mathbf{Grp}}(N) \cong C_{q-1}$
$\langle 1 \rangle 13$. $\mathrm{Aut}_{\mathbf{Grp}}(N)$ has a unique subgroup of order p.
□

Part V

Ring Theory

Chapter 20

Rngs

Definition 20.1 (Ring)**.** A *rng* consists of a set R and binary operations $+, \cdot :$ $R^2 \to R$ such that:

- $(R, +)$ is an Abelian group
- $\cdot$ is associative.
- The *distributive properties* hold: for all $r, s, t \in R$ we have

$$(r + s)t = rt + st, \qquad r(s + t) = rs + rt \ .$$

Example 20.2.

- The *zero rng* is $\{0\}$.
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are rngs.
- $2\mathbb{Z}$ is a rng.
- Given a rng R and natural number n, then the set $\mathfrak{gl}_n(R)$ of all $n \times n$ matrices with entries in R is a rng under matrix addition and matrix multiplication.
- For any set S, the power set $\mathcal{P}S$ is a rng under $A + B = (A \cup B) - (A \cap B)$ and $AB = A \cap B$.
- Given a rng R and a set S, then R^S is a rng under $(f + g)(s) = f(s) + g(s)$ and $(fg)(s) = f(s)g(s)$ for all $f, g \in R^S$ and $s \in S$.
- The set $\mathfrak{sl}_n(\mathbb{R}) = \{M \in \mathfrak{gl}_n(\mathbb{R}) : \operatorname{tr} M = 0\}$ is a rng.
- The set $\mathfrak{sl}_n(\mathbb{C}) = \{M \in \mathfrak{gl}_n(\mathbb{C}) : \operatorname{tr} M = 0\}$ is a rng.
- $\mathbb{Z}/n\mathbb{Z}$ is a rng.

- The ring $\mathbb{H}$ of *quaternions* is $\mathbb{R}^4$ under the following operations, where we write (a, b, c, d) as $a + bi + cj + dk$:

$$\begin{aligned}
(a + bi + cj + dk) + (a' + b'i + c'j + d'k) &= (a + a') + (b + b')i \\
&\quad + (c + c')j + (d + d')k \\
(a + bi + cj + dk)(a' + b'i + c'j + d'k) &= (aa' - bb' - cc' - dd') \\
&\quad + (ab' + ba' + cd' - dc')i \\
&\quad + (ac' - bd' + ca' + db')j \\
&\quad + (ad' + bc' - cb' + da')k
\end{aligned}$$

- For any Abelian group G, the set $\mathrm{End}_{\mathbf{Ab}}(G)$ is a ring under pointwise addition and composition.

Proposition 20.3. *In any rng R we have*

$$\forall x \in R. x0 = 0x = 0 \enspace .$$

Proof:

$$\begin{aligned}
x0 &= x(0 + 0) \\
&= x0 + x0
\end{aligned}$$

and so $x0 = 0$ by Cancellation. Similarly $0x = 0$. □

Definition 20.4 (Zero Divisor)**.** Let R be a rng and $a \in R$.

Then a is a *left-zero-divisor* iff there exists $b \in R - \{0\}$ such that $ab = 0$.

The element a is a *right-zero-divisor* iff there exists $b \in R - \{0\}$ such that $ba = 0$.

Example 20.5. 0 is a left- and right-zero-divisor in every non-zero rng.

The zero rng is the only ring with no zero-divisors.

Proposition 20.6. *Let R be a rng and $a \in R$. Then a is not a left-zero-divisor if and only if left multiplication by a is an injective function $R \to R$.*

Proof:

$\langle 1 \rangle 1$. If a is not a left-zero-divisor then left multiplication by a is injective.

  $\langle 2 \rangle 1$. Assume: a is not a left-zero-divisor.

  $\langle 2 \rangle 2$. Let: $ab = ac$

  $\langle 2 \rangle 3$. $a(b - c) = 0$

  $\langle 2 \rangle 4$. $b - c = 0$

  $\langle 2 \rangle 5$. $b = c$

$\langle 1 \rangle 2$. If a is a left-zero-divisor then left multiplication by a is not injective.

  $\langle 2 \rangle 1$. Pick $b \neq 0$ such that $ab = 0$.

  $\langle 2 \rangle 2$. $ab = a0$ but $b \neq 0$

□

20.1 Commutative Rngs

Definition 20.7 (Commutative)**.** A rng R is *commutative* iff $\forall x, y \in R. xy = yx$.

Example 20.8.
- The zero rng is commutative.
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are commutative.
- $2\mathbb{Z}$ is commutative.
- $\mathfrak{gl}_2(\mathbb{R})$ is not commutative.
- For any set S, the rng $\mathcal{P}S$ is commutative.
- If R is commutative then R^S is commutative.

20.2 Rng Homomorphisms

Definition 20.9. Let R and S be rngs. A *rng homomorphism* $\phi : R \to S$ is a function such that, for all $x, y \in R$, we have

$$\phi(x + y) = \phi(x) + \phi(y)$$
$$\phi(xy) = \phi(x)\phi(y)$$

Let **Rng** be the category of rngs and rng homomorphisms.

20.3 Quaternions

Definition 20.10 (Norm)**.** The *norm* of a quaternion is defined by

$$N(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2 \ .$$

Chapter 21

Rings

Definition 21.1 (Ring)**.** A *ring* R is a rng such that there exists $1 \in R$, the *multiplicative identity*, such that

$$\forall x \in R. x1 = 1x = x \ .$$

Example 21.2.

- The zero rng is a ring with $1 = 0$.
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are rngs.
- $2\mathbb{Z}$ is not a ring.
- If R is a ring then $\mathfrak{gl}_n(R)$ is a ring.
- For any set S, the rng $\mathcal{P}S$ is a ring with $1 = S$.
- If R is a ring then R^S is a ring.
- $\mathfrak{sl}_n(\mathbb{R})$ is not a ring for $n > 0$.
- $\mathfrak{sl}_n(\mathbb{C})$ is not a ring for $n > 0$.
- $\mathfrak{so}_n(\mathbb{R}) = \{M \in \mathfrak{sl}_n(\mathbb{R}) : M + M^T = 0\}$ is not a ring.
- $\mathbb{Z}/n\mathbb{Z}$ is a ring.

Proposition 21.3. *In any ring R, if $0 = 1$ then R is the zero ring.*

Proof: For any $x \in R$ we have $x = 1x = 0x = 0$. □

Proposition 21.4. *In any ring we have $(-1)x = -x$.*

Proof: Since

$$\begin{aligned} x + (-1)x &= 1x + (-1)x \\ &= (1 + (-1))x \\ &= 0x \\ &= 0 \end{aligned}$$

□

21.1 Units

Definition 21.5 (Left-Unit, Right-Unit)**.** Let R be a ring and $a \in R$. Then a is a *left-unit* iff there exists $b \in R$ such that $ab = 1$. The element a is a *right-unit* iff there exists $b \in R$ such that $ba = 1$.

An element is a *unit* iff it is a left-unit and a right-unit.

Proposition 21.6. *Let R be a ring and $a \in R$. Then a is a left-unit iff left multiplication by a is a surjective function $R \to R$.*

PROOF:
$\langle 1 \rangle 1.$ If a is a left-unit then left multiplication by a is surjective.
  $\langle 2 \rangle 1.$ PICK $b \in R$ such that $ab = 1$.
  $\langle 2 \rangle 2.$ For all $c \in R$ we have $c = a(bc)$.
$\langle 1 \rangle 2.$ If left multiplication by a is surjective then a is a left-unit.
  PROOF: Immediate.
□

Proposition 21.7. *Let R be a ring and $a \in R$. Then a is a right-unit iff right multiplication by a is a surjective function $R \to R$.*

PROOF: Similar. □

Proposition 21.8. *No left-unit is a right-zero-divisor.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: for a contradiction $ab = 1$ and $ca = 0$ where $c \neq 0$.
$\langle 1 \rangle 2.$ $c = 0$
  PROOF:

$$\begin{aligned} 0 &= 0b \\ &= cab \\ &= c1 \\ &= c \end{aligned}$$

$\langle 1 \rangle 3.$ Q.E.D.
  PROOF: This is a contradiction.
□

Proposition 21.9. *No right-unit is a left-zero-divisor.*

PROOF: Similar. □

Proposition 21.10. *The inverse of a unit is unique.*

PROOF: If $ba = 1$ and $ac = 1$ then $b = bac = c$. □

Proposition 21.11. *The units of a ring form a group under multiplication.*

PROOF:
$\langle 1 \rangle 1.$ If a and b are units then ab is a unit.
  PROOF: We have $b^{-1}a^{-1}ab = 1$ and $abb^{-1}a^{-1} = 1$.

$\langle 1 \rangle 2.$ 1 is a unit.
  PROOF: Since $1 \cdot 1 = 1$.
$\langle 1 \rangle 3.$ If a is a unit then its inverse is a unit.
  PROOF: Immediate from definitions.
□

Definition 21.12 (Group of Units)**.** For any ring R, we write R^* for the group of the units of R under multiplication.

Example 21.13. The quaternionic group is a subgroup of $\mathbb{H}^*$.

Example 21.14. The norm is a group homomorphism $\mathbb{H}^* \to \mathbb{R}^+$ where $\mathbb{R}^+$ is the group of positive real numbers under multiplication with kernel isomorphic to $\mathrm{SU}_2(\mathbb{C})$. The isomorphism maps a quaternion $a + bi + cj + dk$ to $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$.

Theorem 21.15 (Fermat's Little Theorem)**.** *Let p be a prime number and a any integer. Then $a^p \equiv a (\bmod p)$.*

PROOF: If $p \mid a$ then $a^p \equiv a \equiv 0 (\bmod p)$. Otherwise, we have $a^{p-1} \equiv 1 (\bmod p)$ by applying Lagrange's Theorem to $(\mathbb{Z}/p\mathbb{Z})^*$. □

Example 21.16. It is not true that, if $n \mid |G|$, then G has a subgroup of order n. The group A_4 has order 12 but no subgroup of order 6.

Proposition 21.17. *If p is prime then $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.*

PROOF:
$\langle 1 \rangle 1.$ LET: g be an element of maximal order in $(\mathbb{Z}/p\mathbb{Z})^*$.
$\langle 1 \rangle 2.$ For all $h \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $h^{|g|} = 1$.
  PROOF: Proposition 16.10.
$\langle 1 \rangle 3.$ There are at most $|g|$ elements x such that $x^{|g|} = 1$ in $\mathbb{Z}/p\mathbb{Z}$
$\langle 1 \rangle 4.$ $p - 1 \leq |g|$
$\langle 1 \rangle 5.$ $|g| = p - 1$
$\langle 1 \rangle 6.$ g generates $(\mathbb{Z}/p\mathbb{Z})^*$.
□

Example 21.18. $(\mathbb{Z}/12\mathbb{Z})^*$ is not cyclic. Its elements are 1, 5, 7 and 11 with orders 1, 2, 2 and 2.

Theorem 21.19 (Wilson's Theorem)**.** *A positive integer p is prime if and only if $(p-1)! \equiv 1 (\bmod p)$.*

$\langle 1 \rangle 1.$ If p is prime then $(p-1)! \equiv 1 (\bmod p)$.
  $\langle 2 \rangle 1.$ ASSUME: p is prime.
  $\langle 2 \rangle 2.$ $(p-1)!$ is the product of all the elements of $(\mathbb{Z}/p\mathbb{Z})^*$
  $\langle 2 \rangle 3.$ The only element of $(\mathbb{Z}/p\mathbb{Z})^*$ with order 2 is -1.
  $\langle 2 \rangle 4.$ $(p-1)! \equiv -1 (\bmod p)$
    PROOF: Proposition 14.23.

$\langle 1\rangle 2$. If $(p-1)! \equiv -1 (\bmod p)$ then p is prime.
  $\langle 2\rangle 1$. ASSUME: (
    $(p-1)! \equiv -1 (\bmod p)$)
  $\langle 2\rangle 2$. LET: d be a proper divisor of p.
    PROVE: $d = 1$
  $\langle 2\rangle 3$. $d \mid (p-1)!$
  $\langle 2\rangle 4$. $d \mid 1$
   PROOF: Since $d \mid p \mid (p-1)! + 1$.
  $\langle 2\rangle 5$. $d = 1$

$\square$

Proposition 21.20. *If p and q are distinct odd primes then $(\mathbb{Z}/pq\mathbb{Z})^*$ is not cyclic.*

PROOF:
$\langle 1\rangle 1$. $|(\mathbb{Z}/pq\mathbb{Z})^*| = (p-1)(q-1)$
$\langle 1\rangle 2$. LET: $g \in (\mathbb{Z}/pq\mathbb{Z})^*$
  PROVE: g does not have order $(p-1)(q-1)$
$\langle 1\rangle 3$. $g^{(p-1)(q-1)/2} \equiv 1 (\bmod p)$
$\langle 1\rangle 4$. $g^{(p-1)(q-1)/2} \equiv 1 (\bmod q)$
$\langle 1\rangle 5$. $pq \mid g^{(p-1)(q-1)/2} - 1$
$\langle 1\rangle 6$. $g^{(p-1)(q-1)/2} \equiv 1 (\bmod pq)$
$\langle 1\rangle 7$. $|g| \mid (p-1)(q-1)/2$

$\square$

Proposition 21.21. *For any prime p, we have $\mathrm{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$.*

PROOF:
$\langle 1\rangle 1$. LET: $\phi : \mathrm{Aut}_{\mathbf{Grp}}(C_p) \to (\mathbb{Z}/p\mathbb{Z})^*$ be the function $\phi(\alpha) = \alpha(1)$.
  PROOF: $\alpha(1)$ has order p in C_p and so is coprime with p.
$\langle 1\rangle 2$. ϕ is a homomorphism.
  PROOF: $\phi(\alpha \circ \beta) = \alpha(\beta(1)) = \alpha(\beta(1)1) = \beta(1)\alpha(1) = \phi(\alpha)\phi(\beta)$
$\langle 1\rangle 3$. ϕ is injective.
  PROOF: If $\phi(\alpha) = \phi(\beta)$ then for any n we have $\alpha(n) = n\alpha(1) = n\phi(\alpha) = n\phi(\beta) = n\beta(1) = \beta(n)$.
$\langle 1\rangle 4$. ϕ is surjective.
  PROOF: For any $r \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $r = \phi(\alpha)$ where $\alpha(n) = nr \bmod p$.
$\langle 1\rangle 5$. $(\mathbb{Z}/p\mathbb{Z})^* \cong C_{p-1}$

$\square$

21.2 Euler's ϕ-function

Proposition 21.22. *For n a positive integer, we have $(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$.*

PROOF:

$$\begin{aligned} m \in (\mathbb{Z}/n\mathbb{Z})^* &\Leftrightarrow \exists a.am \equiv 1 (\operatorname{mod} n) \\ &\Leftrightarrow \exists a, b.am + bn = 1 \\ &\Leftrightarrow \gcd(m, n) = 1 \end{aligned}$$

□

Definition 21.23 (Euler's Totient Function)**.** For n a positive integer, let $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$.

Proposition 21.24. *If n is an odd positive integer then $\phi(2n) = \phi(n)$.*

PROOF:
$\langle 1 \rangle 1$. LET: n be an odd positive integer.
$\langle 1 \rangle 2$. For any integer m, if $\gcd(m, n) = 1$ then $\gcd(2m + n, 2n) = 1$
PROOF: For p a prime, if $p \mid 2m + n$ and $p \mid 2n$ then $p \neq 2$ (since $2m + n$ is odd) so $p \mid n$ and hence $p \mid m$, which is a contradiction.
$\langle 1 \rangle 3$. For any integer r, if $\gcd(r, 2n) = 1$ then $\gcd(\frac{r+n}{2}, n) = 1$
PROOF: If $p \mid n$ and $p \mid \frac{r+n}{2}$ then $p \mid r + n$ so $p \mid r$ which is a contradiction.
$\langle 1 \rangle 4$. The function that maps m to $2m + n$ is a bijection between $(\mathbb{Z}/n\mathbb{Z})^*$ and $(\mathbb{Z}/2n\mathbb{Z})^*$.
□

Theorem 21.25. *For any positive integer n we have*

$$\sum_{m>0, m|n} \phi(m) = n \ .$$

PROOF:
$\langle 1 \rangle 1$. Define $\chi : \{0, 1, \ldots, n-1\} \to \{(m, d) : m > 0, m \mid n, d \text{ generates } \langle n/m \rangle\}$ by: $\chi(x) = (\gcd(x, n), x)$.
$\langle 1 \rangle 2$. χ is injective.
$\langle 1 \rangle 3$. χ is surjective.
PROOF: Given (m, d) such that d generates $\langle n/m \rangle$ we have $\chi(d) = (m, d)$.
$\langle 1 \rangle 4$. $n = \sum_{m>0, m|n} \phi(m)$
PROOF: Since $\langle n/m \rangle \cong C_m$ and so has $\phi(m)$ generators.
□

Proposition 21.26. *For any positive integers a and n, we have $n \mid \phi(a^n - 1)$.*

PROOF: Since the order of a is n in $(\mathbb{Z}/(a^n - 1)\mathbb{Z})^*$. □

Theorem 21.27 (Euler's Theorem)**.** *For any coprime integers a and n we have $a^{\phi(n)} \equiv a (\operatorname{mod} n)$.*

PROOF: Immediate from Lagrange's Theorem. □

Proposition 21.28.

$$|\mathrm{Aut}_{\mathbf{Grp}}(C_n)| = \phi(n)$$

PROOF: An automorphism α is determined by $\alpha(1)$ which is any element of order n, and g has order n iff $\gcd(g, n) = 1$. □

Example 21.29.

$$\mathrm{Aut}_{\mathbf{Grp}}(\mathbb{Z}) \cong C_2$$

PROOF: The only automorphisms are the identity and multiplication by -1. □

21.3 Nilpotent Elements

Definition 21.30 (Nilpotent)**.** Let R be a ring and $a \in R$. Then a is *nilpotent* iff there exists n such that $a^n = 0$.

Proposition 21.31. *Let R be a ring and $a, b \in R$. If a and b are nilpotent and $ab = ba$ then $a + b$ is nilpotent.*

PROOF:
⟨1⟩1. PICK m and n such that $a^m = b^n = 0$.
⟨1⟩2. $(a + b)^{m+n} = 0$

PROOF: Since $(a+b)^{m+n} = \sum_k \begin{pmatrix} m+n \\ k \end{pmatrix} a^k b^{m+n-k}$ and every term in this sum is 0 since, for every k, either $k \geq m$ or $m + n - k \geq n$.

□

Proposition 21.32. *m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$ if and only if m is divisible by all the prime factors of n.*

PROOF:
⟨1⟩1. If m is nilpotent then m is divisible by all the prime factors of n.
⟨2⟩1. ASSUME: $m^a \equiv 0 (\bmod n)$
⟨2⟩2. For every prime p, if $p \mid n$ then $p \mid m^a$.
⟨2⟩3. For every prime p, if $p \mid n$ then $p \mid m$.
⟨1⟩2. If m is divisible by all the prime factors of n then m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.
⟨2⟩1. ASSUME: m is divisible by all the prime factors of n.
⟨2⟩2. LET: a be the largest number such that $p^a \mid n$ for some prime p.
⟨2⟩3. For every prime p that divides n we have $p^a \mid m^a$
⟨2⟩4. $n \mid m^a$
⟨2⟩5. $m^a \equiv 0 (\bmod n)$
⟨2⟩6. m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

□

Chapter 22

Ring Homomorphisms

Definition 22.1 (Ring Homomorphism)**.** Let R and S be rings. A *ring homomorphism* $\phi : R \to S$ is a rng homomorphism such that $\phi(1) = 1$.

Proposition 22.2. *The zero-ring is terminal in* **Ring**.

Proof: Easy. $\square$

Proposition 22.3. *The ring* $\mathbb{Z}$ *is initial in* **Ring**.

Proof: Easy. $\square$

Proposition 22.4. *Let R and S be rings and $\phi : R \to S$ be a rng homomorphism. If ϕ is surjective, then ϕ is a ring homomorphism.*

Proof:
$\langle 1 \rangle 1$. Pick $a \in R$ such that $\phi(a) = 1$
$\langle 1 \rangle 2$. $\phi(1) = 1$
Proof:

$$\begin{aligned} \phi(1) &= \phi(1)\phi(a) \\ &= \phi(1a) \\ &= \phi(a) \\ &= 1 \end{aligned} \qquad \square$$

Example 22.5. For any set S we have $\mathcal{P}S \cong (\mathbb{Z}/2\mathbb{Z})^S$ in **Ring** with the isomorphism

$$\phi : \mathcal{P}S \cong (\mathbb{Z}/2\mathbb{Z})^S$$
$$\phi(A)(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$

Example 22.6. The function $\mathbb{H} \to \mathfrak{gl}_4(\mathbb{R})$ that maps $a + bi + cj + dk$ to

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

is a monomorphism in **Ring**, as is the function $\mathbb{H} \to \mathfrak{sl}_2(\mathbb{C})$ that maps $a + bi + cj + dk$ to

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} .$$

Proposition 22.7. *Ring homomorphisms preserve units.*

PROOF: If $uv = 1$ then $\phi(u)\phi(v) = 1$. □

Proposition 22.8. *Let $\phi : R \to S$ be a ring homomorphism. Then the following are equivalent.*

1. *ϕ is a monomorphism.*
2. $\ker \phi = \{0\}$
3. *ϕ is injective.*

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
  $\langle 2 \rangle 1.$ ASSUME: ϕ is a monomorphism.
  $\langle 2 \rangle 2.$ LET: $r \in \ker \phi$
  $\langle 2 \rangle 3.$ LET: $\mathrm{ev}_r : \mathbb{Z}[x] \to R$ be the unique ring homomorphism such that $\mathrm{ev}_r(x) = r$.
  $\langle 2 \rangle 4.$ LET: $\mathrm{ev}_0 : \mathbb{Z}[x] \to R$ be the unique ring homomorphism such that $\mathrm{ev}_0(x) = 0$.
  $\langle 2 \rangle 5.$ $\phi \circ \mathrm{ev}_r = \phi \circ \mathrm{ev}_0$
  $\langle 2 \rangle 6.$ $\mathrm{ev}_r = \mathrm{ev}_0$
  $\langle 2 \rangle 7.$ $r = 0$
$\langle 1 \rangle 2.$ $2 \Rightarrow 3$
  PROOF: Proposition 15.21.
$\langle 1 \rangle 3.$ $3 \Rightarrow 1$
  PROOF: Easy.
□

Example 22.9. It is not true that every epimorphism in **Ring** is surjective. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism but not surjective.

The same example shows that a ring homomorphism may be a monomorphism and an epimorphism but not be an isomorphism.

Example 22.10.

$$\mathrm{End}_{\mathbf{Ab}}(\mathbb{Z}) \cong \mathbb{Z}$$

The isomorphism maps any group endomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ to $\phi(1)$.

Example 22.11. The group of units of $\mathrm{End}_{\mathbf{Ab}}(G)$ is $\mathrm{Aut}_{\mathbf{Ab}}(G)$.

Example 22.12. Let R be a ring. Then the function $\lambda : R \to \mathrm{End}_{\mathbf{Ab}}(R)$ defined by

$$\lambda(a)(b) = ab$$

is a ring monomorphism.

PROOF: Easy. □

22.1 Products

Proposition 22.13. *Let R and S be rings. Then $R \times S$ is a ring under componentwise addition and multiplication, and this ring is the product of R and S in* **Ring**.

PROOF: Easy. □

Chapter 23

Subrings

Definition 23.1 (Subring)**.** Let S be a ring. A *subring* of S is a ring R such that R is a subset of S and the inclusion $R \hookrightarrow S$ is a ring homomorphism.

Proposition 23.2. *Let R and S be rings. Then R is a subring of S if and only if R is a subset of S, the unit 1 of S is an element of R, and the operations of R are the restrictions of the operations of S to R.*

PROOF: Easy. $\square$

Corollary 23.2.1. *The zero ring is not a subring of any non-zero ring.*

Proposition 23.3. *Let $\phi : R \to S$ be a ring homomorphism. Then $\phi(R)$ is a subring of S.*

PROOF: Easy. $\square$

23.1 Centralizer

Definition 23.4 (Centralizer)**.** Let R be a ring and $a \in R$. The *centralizer* of a is $\{r \in R : ar = ra\}$.

Proposition 23.5. *The centralizer of a is a subring of R.*

PROOF: Easy. $\square$

23.2 Center

Definition 23.6 (Center)**.** The *center* of a ring R is $\{x \in R : \forall y \in R. xy = yx\}$.

Proposition 23.7. *The center of a ring is a subring.*

PROOF: Easy. $\square$

Proposition 23.8. *Let R be a ring. The center of* $\mathrm{End}_{\mathbf{Ab}}(R)$ *is isomorphic to the center of R.*

PROOF:
⟨1⟩1. LET: $\lambda : R \to \mathrm{End}_{\mathbf{Ab}}(R)$ be left multiplication.
⟨1⟩2. λ maps $Z(R)$ to $Z(\mathrm{End}_{\mathbf{Ab}}(R))$.
  ⟨2⟩1. LET: $a \in Z(R)$
  ⟨2⟩2. LET: $\phi \in \mathrm{End}_{\mathbf{Ab}}(R)$
    PROVE: $\lambda(a) \circ \phi = \phi \circ \lambda(a)$
  ⟨2⟩3. LET: $x \in R$
  ⟨2⟩4. $a + \phi(x) = \phi(a + x)$
⟨1⟩3. $\lambda(Z(R)) = Z(\mathrm{End}_{\mathbf{Ab}}(R))$
  ⟨2⟩1. LET: $\phi \in Z(\mathrm{End}_{\mathbf{Ab}}(R))$
  ⟨2⟩2. For all $r \in R$,
    LET: $\mu_r \in \mathrm{End}_{\mathbf{Ab}}(R)$ be right multiplication by r.
  ⟨2⟩3. For all $r \in R$ we have $\phi \circ \mu_r = \mu_r \circ \phi$.
  ⟨2⟩4. For all $r, x \in R$ we have $\phi(xr) = \phi(x)r$
  ⟨2⟩5. For all $r \in R$ we have $\phi(r) = \phi(1)r$
  ⟨2⟩6. $\phi = \lambda(\phi(1))$
□

Corollary 23.8.1. *If R is a commutative ring then R is isomorphic to the center of* $\mathrm{End}_{\mathbf{Ab}}(R)$.

Example 23.9. For n a positive integer we have $\mathbb{Z}/n\mathbb{Z} \cong \mathrm{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$.

Since, for any $\phi \in \mathrm{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$ we have $\phi(m) = m\phi(1)$ and so the whole of $\mathrm{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$ is the image of λ.

Chapter 24

Monoid Rings

Definition 24.1 (Monoid Ring)**.** Let R be a ring and M a monoid. Define $R[M]$ to be the ring whose elements are the families $\{a_m\}_{m\in M}$ such that $a_m = 0$ for all but finitely many $m \in M$, written

$$\sum_{m\in M} a_m m \ ,$$

under

$$\sum_m a_m m + \sum_m b_m m = \sum_m (a_m + b_m) m$$
$$\left(\sum_m a_m m\right)\left(\sum_m b_m m\right) = \sum_{m\in M} \sum_{m_1 m_2 = m} a_{m_1} b_{m_2} m$$

Example 24.2. Ring homomorphisms do not necessarily preserve zero-divisors. The canonical homomorphism $\pi : \mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ maps the non-zero-divisor 2 to a zero-divisor.

24.1 Polynomials

Definition 24.3 (Polynomial)**.** Let R be a ring. The ring of *polynomials* $R[x]$ is $R[\mathbb{N}]$. We write

$$\sum_n a_n x^n \text{ for } \sum_n a_n n \ .$$

Concretely, a *polynomial* in R is a sequence (a_n) in R such that there exists N such that $\forall n \geq N. a_n = 0$. We write the polynomial as

$$\sum_{n=0}^{N-1} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1} \ .$$

We write $R[x]$ for the set of all polynomials in R.

Define addition and multiplication on $R[x]$ by

$$\sum_n a_n x^n + \sum_n b_n x^n = \sum_n (a_n + b_n) x^n$$
$$\left(\sum_n a_n x^n\right)\left(\sum_n b_n x^n\right) = \sum_n \sum_{i+j=n} a_i b_j x^n$$

A *constant* is a polynomial of the form $a + 0x + 0x^2 + \cdots$ for some $a \in R$. We write $R[x_1, \ldots, x_n]$ for $R[x_1][x_2]\cdots[x_n]$.

Proposition 24.4. *For any ring R, the set of polynomials $R[x]$ is a ring.*

PROOF: Easy. □

Definition 24.5 (Degree)**.** The *degree* of a polynomial $\sum_n a_n x^n$ is the largest integer d such that $a_d \neq 0$. We take the degree of the zero polynomial to be $-\infty$.

Proposition 24.6. *Let R be a ring and $f, g \in R[x]$ be nonzero polynomials. Then*

$$\deg(f+g) \leq \max(\deg f, \deg g) \ .$$

PROOF: If $a_n + b_n \neq 0$ then $a_n \neq 0$ or $b_n \neq 0$. □

Proposition 24.7. *The function $i : n \to \mathbb{Z}[x_1, \ldots, x_n]$ that maps k to x_k is initial in the category with:*

- *objects all pairs $j : n \to R$ where R is a commutative ring and j a function*
- *morphisms $\phi : (j_1, R_1) \to (j_2, R_2)$ are ring homomorphisms $\phi : R_1 \to R_2$ such that $\phi \circ j_1 = j_2$.*

PROOF: The unique morphism $(i, \mathbb{Z}[x_1, \ldots, x_n]) \to (j, R)$ maps a polynomial p to $p(j(0), j(1), \ldots, j(n-1))$. □

Proposition 24.8. *Let $\alpha : R \to S$ be a ring homomorphism. Let $s \in S$ commute with $\alpha(r)$ for all $r \in R$. Then there exists a unique ring homomorphism $\overline{\alpha} : R[x] \to S$ such that $\overline{\alpha}(x) = s$ and the following diagram commutes:*

$$\begin{tikzcd} R[x] \arrow[r, "\overline{\alpha}"] & S \\ R \arrow[u] \arrow[ur, "\alpha"] & \end{tikzcd}$$

PROOF: The map $\overline{\alpha}$ is given by
$\overline{\alpha}(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = \alpha(a_0) + \alpha(a_1)s + \alpha(a_2)s^2 + \cdots + \alpha(a_n)s^n$.
□

Definition 24.9. Let R be a commutative ring. Given a polynomial $p \in R[x]$, the *polynomial function* $p : R \to R$ is the function given by: $p(r) = \alpha_r(p)$, where $\alpha_r : R[x] \to R$ is the unique ring homomorphism such that the following diagram commutes.

$$\begin{tikzcd} R[x] \arrow[r, "\alpha_r"] & R \\ 1 \arrow[u, "x"] \arrow[ur, "r"'] & \end{tikzcd}$$

Proposition 24.10. *$\mathbb{Z}[x, y]$ is the coproduct of $\mathbb{Z}[x]$ and $\mathbb{Z}[y]$ in the category of commutative rings.*

PROOF: Given ring homomorphisms $f : \mathbb{Z}[x] \to R$ and $g : \mathbb{Z}[y] \to R$, the required morphism $\mathbb{Z}[x, y] \to R$ maps $p(x, y)$ to $p(f(x), g(y))$. □

Example 24.11. $\mathbb{Z}[x, y]$ is not the coproduct of $\mathbb{Z}[x]$ and $\mathbb{Z}[y]$ in **Ring**. Given $f : \mathbb{Z}[x] \to R$ and $g : \mathbb{Z}[y] \to R$ with $f(x) \neq g(y)$, the mediating morphism $\mathbb{Z}[x, y] \to R$ cannot exist since it must map xy to both $f(x)g(y)$ and $g(y)f(x)$. □

Definition 24.12. A polynomial is *monic* iff its last non-zero coefficient is 1.

Proposition 24.13. *A monic polynomial is not a left- or right-zero-divisor.*

PROOF: Easy. □

Proposition 24.14. *Let R be a ring. Let $f, g \in R[x]$ with f monic. Then there exist unique polynomials $q, r \in R[x]$ with $\deg r < \deg f$ such that*

$$g = qf + r \ .$$

PROOF:
⟨1⟩1. LET: $d = \deg f$
⟨1⟩2. For all $a \in R$ and $n > d$, there exists $h \in R[x]$ with $\deg h < n$ such that
$$ax^n = ax^{n-d} f + h \ .$$
PROOF: Take $h = ax^n - ax^{n-d} f$.
⟨1⟩3. For all $a \in R$ and $n > d$, there exists $q, h \in R[x]$ with $\deg h \leq d$ such that
$$ax^n = qf + h \ .$$
PROOF: Repeating ⟨1⟩2 by induction.
⟨1⟩4. LET: $g = \sum_{i=0}^{n} a_i x^i$
⟨1⟩5. For $i > d$, PICK $q_i h_i \in R[x]$ with $\deg h < \deg f$ such that $a_i x^i = q_i f + h_i$
⟨1⟩6. $g = \left(\sum_{i=d+1}^{n} q_i\right) f + \sum_{i=d+1}^{n} h_i$
⟨1⟩7. q and r are unique.
PROOF: If $q_1 f + r_1 = q_2 f + r_2$ then $r_1 - r_2 = (q_2 - q_1)f$ and so $r_1 - r_2 = (q_2 - q_1)f = 0$ since $\deg(r_1 - r_2) < \deg f$.
□

24.2 Laurent Polynomials

Definition 24.15 (Laurent Polynomial)**.** Let R be a ring. The ring of *Laurent polynomials* is the group ring $R[\mathbb{Z}]$. We write $\sum_{n \in \mathbb{Z}} a_n x^n$ for $\sum_n a_n n$.

24.3 Power Series

Definition 24.16 (Power Series)**.** Let R be a ring. A *power series* in R is a sequence (a_n) in R. We write the power series as

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \ .$$

We write $R[[x]]$ for the set of all power series in R.

Define addition and multiplication on $R[[x]]$ by

$$\sum_n a_n x^n + \sum_n b_n x^n = \sum_n (a_n + b_n) x^n$$

$$\left(\sum_n a_n x^n \right) \left(\sum_n b_n x^n \right) = \sum_n \sum_{i+j=n} a_i b_j x^n$$

Proposition 24.17. *For any ring R, the set of power series $R[[x]]$ is a ring.*

PROOF: Easy. □

Proposition 24.18. *A power series $\sum_n a_n x^n$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R.*

PROOF:
⟨1⟩1. If $\sum_n a_n x^n$ is a unit then a_0 is a unit.
⟨2⟩1. LET: $\sum_n b_n x^n$ be the inverse of $\sum_n a_n x^n$.
⟨2⟩2. $a_0 b_0 = b_0 a_0 = 1$
⟨1⟩2. If a_0 is a unit then $\sum_n a_n x^n$ is a unit.
PROOF: Define the sequence (b_n) in R by

$$b_n = -{a_0}^{-1} \sum_{i=1}^{n} a_i b_{n-i}$$

Then $\sum_n b_n x^n$ is the inverse of $\sum_n a_n x^n$.
□

Chapter 25

Ideals

Definition 25.1 (Left-Ideal)**.** Let R be a ring.

A subgroup I of R is a *left-ideal* iff, for all $r \in R$, we have $rI \subseteq I$.

A subgroup I of R is a *right-ideal* iff, for all $r \in R$, we have $Ir \subseteq I$.

A subgroup I of R is a *(two-sided) ideal* iff it is a left-ideal and a right-ideal.

Example 25.2. Let R be a ring and $a \in R$. Then Ra is a left-ideal and aR is a right-ideal.

In particular, $\{0\}$ is always a two-sided ideal.

Example 25.3. Let S be a set and $T \subseteq S$. Then $\{X \in \mathcal{P}S : X \subseteq T\}$ is an ideal in $\mathcal{P}S$.

Proposition 25.4. *Let S be a finite set. Then every ideal in $\mathcal{P}S$ is of the form $\{X \in \mathcal{P}S : X \subseteq T\}$ for some $T \subseteq S$.*

Proof:

$\langle 1 \rangle 1$. Let: I be an ideal in $\mathcal{P}S$.

$\langle 1 \rangle 2$. Let: $T = \bigcup I$

$\langle 1 \rangle 3$. For all $i \in T$ we have $\{i\} \in I$.

  $\langle 2 \rangle 1$. Let: $i \in T$

  $\langle 2 \rangle 2$. Pick $X \in I$ such that $i \in X$

  $\langle 2 \rangle 3$. $\{i\} = \{i\} \cap X \in I$

$\langle 1 \rangle 4$. For all $X \subseteq T$ we have $X \in I$.

  Proof: If $X = \{x_1, \ldots, x_n\}$ then $X = \{x_1\} + \cdots + \{x_n\} \in I$.

□

Example 25.5. If S is an infinite set, then there is always an ideal in $\mathcal{P}S$ that is not of the form $\{X \in \mathcal{P}S : X \subseteq T\}$ for some $T \subseteq S$, namely the set of all finite subsets of S.

Proposition 25.6. *Let $\phi : R \twoheadrightarrow S$ be a surjective ring homomorphism. Let J be an ideal in R. Then $\phi(J)$ is an ideal in S.*

Proof:

⟨1⟩1. LET: $j \in J$ and $s \in S$
PROVE: $s\phi(j), \phi(j)s \in \phi(J)$
⟨1⟩2. PICK $r \in R$ such that $\phi(r) = s$
⟨1⟩3. $rj, jr \in J$
⟨1⟩4. $s\phi(j), \phi(j)s \in \phi(J)$
□

Example 25.7. We cannot remove the hypothesis that ϕ is surjective.

Let $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion. Then $i(2\mathbb{Z}) = 2\mathbb{Z}$ is not an ideal in $\mathbb{Q}$.

Proposition 25.8. *Let $\phi : R \to S$ be a ring homomorphism and I a (left-, right-)ideal in S. Then $\phi^{-1}I$ is a (left-, right-)ideal in R.*

PROOF: Easy. □

Corollary 25.8.1. *Let $\phi : R \to S$ be a ring homomorphism. Then* $\ker \phi$ *is an ideal in R.*

Definition 25.9 (Quotient Ring)**.** Let I be an ideal in R. The *quotient ring* R/I is the quotient group R/I under

$$(a + I)(b + I) = ab + I \ .$$

This is well-defined as, if $a + I = a' + I$ and $b + I = b' + I$ then

$$\begin{aligned} a - a' &\in I \\ b - b' &\in I \\ \therefore ab - a'b &\in I \\ a'b - a'b' &\in I \\ \therefore ab - a'b' &\in I \end{aligned}$$

Proposition 25.10. *Let I be an ideal in R. Then the canonical group homomorphism $\pi : R \to R/I$ is a ring homomorphism.*

PROOF: By construction. □

Proposition 25.11. *Let I be an ideal in a ring R. For every ring homomorphism $\phi : R \to S$ such that $I \subseteq \ker\phi$, there exists a unique ring homomorphism $\overline{\phi} : R/I \to S$ such that the following diagram commutes.*

$$\begin{tikzcd} R \arrow[rr, "\phi"] \arrow[dr, two heads, "\pi"'] & & S \\ & R/I \arrow[ur, "\overline{\phi}"'] & \end{tikzcd}$$

PROOF: Easy. □

Corollary 25.11.1. *Every ring homomorphism $\phi : R \to S$ decomposes as follows.*

$$\begin{tikzcd} R \arrow[r, two heads] \arrow[rrr, bend left, "\phi"] & R/\ker\phi \arrow[r, "\cong"] & \operatorname{im}\phi \arrow[r, hook] & S \end{tikzcd}$$

Corollary 25.11.2 (First Isomorphism Theorem)**.** *Let $\phi : R \twoheadrightarrow S$ be a surjective ring homomorphism. Then*

$$S \cong R/\ker\phi \ .$$

Theorem 25.12 (Third Isomorphism Theorem)**.** *Let I and J be ideals in R with $I \subseteq J$. Then J/I is an ideal in R/I, and*

$$\frac{R/I}{J/I} \cong R/J$$

Proof: Since the function $R/I \to R/J$ that maps $r + I$ to $r + J$ is a surjective ring homomorphism with kernel J/I. □

Corollary 25.12.1. *Let $\phi : R \twoheadrightarrow S$ be a surjective ring homomorphism. Let J be an ideal in R. Then*

$$\frac{S}{\phi(J)} \cong \frac{R}{\ker S + J}$$

Proposition 25.13. *Let R be a ring and J an ideal in $\mathfrak{gl}_n(R)$. Let $A \in \mathfrak{gl}_n(R)$. Then $A \in J$ if and only if the matrices obtained by placing any entry of A in any position and zeros elsewhere all belong to J.*

Proof: Each such matrix can be obtained by pre- and post-multiplying A by matrices which have a single 1 and 0s elsewhere. Conversely, A is a sum of such matrices. □

Corollary 25.13.1. *Let R be a ring. Let J be an ideal in $\mathfrak{gl}_n(R)$. Let I be the set of all entries of elements of J. Then I is an ideal in R, and J is the set of all matrices whose entries are in I.*

Proposition 25.14. *Let R be a ring. Let $\{I_\alpha\}_{\alpha\in A}$ be a family of ideals in R. Let*

$$\sum_{\alpha\in A} I_\alpha = \{\sum_{\alpha\in A} r_\alpha : \forall\alpha . r_\alpha \in I_\alpha, r_\alpha = 0 \textit{ for all but finitely many } \alpha \in A\} \ .$$

Then $\sum_{\alpha\in A} I_\alpha$ is an ideal, and is the smallest ideal that includes every I_α.

Proof: Easy. □

Proposition 25.15. *The intersection of a set of ideals is an ideal.*

Proof: Easy. □

25.1 Characteristic

Definition 25.16 (Characteristic)**.** The *characteristic* of a ring R is the non-negative integer n such that $n\mathbb{Z}$ is the kernel of the unique ring homomorphism $\mathbb{Z} \to R$.

Proposition 25.17. *Let R be a ring. If the unit 1 has finite order in R, then its order is the characteristic of R; otherwise, the characteristic of R is 0.*

PROOF: Easy. □

Example 25.18. The zero ring is the only ring with characteristic 1.

25.2 Nilradical

Definition 25.19 (Nilradical)**.** Let R be a commutative ring. The *nilradical* of R is the set of all nilpotent elements.

Proposition 25.20. *Let R be a commutative ring. The nilradical of R is an ideal in R.*

PROOF: If $a^n = 0$ then for any b we have $(ba)^n = 0$. □

Example 25.21. We cannot remove the assumption that R is commutative. In $\mathfrak{gl}_2(\mathbb{R})$ we have that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not.

25.3 Principal Ideals

Definition 25.22 (Principal Ideal)**.** Let R be a commutative ring and $a \in R$. The *principal ideal* generated by a is $(a) = Ra = aR$.

Example 25.23. $\{0\} = (0)$ and $R = \{1\}$ are principal ideals.

Definition 25.24. Let R be a commutative ring and $\{a_\alpha\}_{\alpha \in A}$ be a family of elements of R. The *ideal generated by the elements a_α* is

$$(a_\alpha)_{\alpha \in A} := \sum_{\alpha \in A} (a_\alpha) \ .$$

An ideal is *finitely generated* iff it is generated by a finite family of elements.

Definition 25.25. Let R be a commutative ring and I, J be ideals in R. Then IJ is the ideal generated by $\{ij\}_{i \in I, j \in J}$.

Proposition 25.26.

$$IJ \subseteq I \cap J$$

PROOF: Easy. ∎

Proposition 25.27. *Let R be a commutative ring. Let I and J be ideals in R. If $I + J = R$ then $IJ = I \cap J$.*

PROOF:
$\langle 1 \rangle 1$. LET: $r \in I \cap J$
$\langle 1 \rangle 2$. PICK $i \in I$ and $j \in J$ such that $i + j = 1$.
$\langle 1 \rangle 3$. $ri, rj \in IJ$
$\langle 1 \rangle 4$. $r = ri + rj \in IJ$
∎

Proposition 25.28. *Let R be a commutative ring. Let $f \in R[x]$ be a monic polynomial of degree d. Then the function*

$$\phi : R[x] \to R^{\oplus d}$$

that sends a polynomial g to the remainder of the division of g by f induces an isomorphism of Abelian groups

$$\frac{R[x]}{(f(x))} \cong R^{\oplus d} \ .$$

PROOF: It is clearly a group homomorphism; it is surjective since it maps any polynomial of degree $< d$ to itself, and its kernel is $(f(x))$ since these are the polynomials with remainder 0. ∎

Corollary 25.28.1. *Let R be a commutative ring and $a \in R$. Then we have*

$$\frac{R[x]}{(x - a)} \cong R$$

PROOF:
$\langle 1 \rangle 1$. LET: $\phi : R[x] \to R$ be evaluation at a.
$\langle 1 \rangle 2$. $\phi(g)$ is the remainder when dividing g by $x - a$.
PROOF: If $g = (x - a)q + r$ then $g(a) = (a - a)q(a) + r = r$.
$\langle 1 \rangle 3$. ϕ induces a group isomorphism $R[x]/(x - a) \cong R$
PROOF: By the theorem.
$\langle 1 \rangle 4$. This isomorphism is a ring isomorphism.
PROOF: Since evaluation at a is a ring homomorphism.
∎

Example 25.29. We have

$$\frac{\mathbb{R}[x]}{(x^2 + 1)} \cong \mathbb{C}$$

as rings.

25.4 Maximal Ideals

Definition 25.30 (Maximal Ideal)**.** Let R be a ring and I an ideal in R. Then I is a *maximal ideal* iff $I \neq R$ and, whenever J is an ideal with $I \subseteq J$, then either $I = J$ or $J = R$.

Chapter 26

Integral Domains

Definition 26.1 (Integral Domain)**.** An *integral domain* is a non-trivial commutative ring with no nonzero zero-divisors.

Example 26.2. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are integral domains.

Proposition 26.3. *$\mathbb{Z}/n\mathbb{Z}$ is an integral domain if and only if n is prime.*

PROOF:

$$\begin{aligned} n \text{ is prime} &\Leftrightarrow \forall a, b \in \mathbb{Z}(n \mid ab \Rightarrow n \mid a \vee n \mid b) \\ &\Leftrightarrow \forall a, b \in \mathbb{Z}/n\mathbb{Z}(ab \cong 0 (\bmod n) \Rightarrow a \cong 0 (\bmod n) \vee b \cong 0 (\bmod n)) \\ &\Leftrightarrow \mathbb{Z}/n\mathbb{Z} \text{ is an integral domain} \end{aligned}$$ ∎

Proposition 26.4. *In an integral domain, if $x^2 = 1$ then $x = \pm 1$.*

PROOF: We have $x^2 - 1 = (x-1)(x+1) = 0$ so $x - 1 = 0$ or $x + 1 = 0$. ∎

Proposition 26.5. *Let R be an integral domain and $f, g \in R[x]$. Then*

$$\deg(fg) = \deg f + \deg g$$

PROOF:

⟨1⟩1. LET: $f = \sum_n a_n x^n$ and $g = \sum_n b_n x^n$.

⟨1⟩2. LET: $d = \deg f$ and $e = \deg g$.

⟨1⟩3. The $d + e$th term of fg is

$$a_d b_e x^{d+e}$$

which is non-zero.

⟨1⟩4. For $n > d + e$ the nth term of fg is 0.

∎

Corollary 26.5.1. *Let R be a ring. Then $R[x]$ is an integral domain if and only if R is an integral domain.*

Proposition 26.6. *Let R be a ring. Then $R[[x]]$ is an integral domain if and only if R is an integral domain.*

PROOF:
$\langle 1 \rangle 1.$ If $R[[x]]$ is an integral domain then R is an integral domain.
  PROOF: Easy.
$\langle 1 \rangle 2.$ If R is an integral domain then $R[[x]]$ is an integral domain.
  $\langle 2 \rangle 1.$ ASSUME: R is an integral domain.
  $\langle 2 \rangle 2.$ LET: $(\sum_n a_n x^n)(\sum_n b_n x^n) = 0$
  $\langle 2 \rangle 3.$ $a_0 b_0 = 0$
  $\langle 2 \rangle 4.$ $a_0 = 0$ or $b_0 = 0$
  $\langle 2 \rangle 5.$ ASSUME: w.l.o.g. $b_0 \neq 0$
  PROVE: For all n we have $a_n = 0$
  $\langle 2 \rangle 6.$ ASSUME: as induction hypothesis $a_0 = a_1 = \cdots = a_{n-1} = 0$
  $\langle 2 \rangle 7.$ $\sum_{i=0}^{n} a_i b_{n-i} = 0$
  $\langle 2 \rangle 8.$ $a_n b_0 = 0$
  $\langle 2 \rangle 9.$ $a_n = 0$
□

Proposition 26.7. *Let R be a ring and S an integral domain. Every rng homomorphism $\phi : R \to S$ is a ring homomorphism.*

PROOF:
$$\begin{aligned} \phi(1) &= \phi(1 \cdot 1) \\ &= \phi(1)\phi(1) \end{aligned}$$
and so $\phi(1) = 1$ by Cancellation. □

Proposition 26.8. *The characteristic of an integral domain is either 0 or a prime number.*

PROOF:
$\langle 1 \rangle 1.$ LET: D be an integral domain.
$\langle 1 \rangle 2.$ LET: n be the characteristic of D
$\langle 1 \rangle 3.$ ASSUME: $n \neq 0$
$\langle 1 \rangle 4.$ ASSUME: $n = ab$
$\langle 1 \rangle 5.$ $ab = 0$ in D
$\langle 1 \rangle 6.$ $a = 0$ or $b = 0$ in D
$\langle 1 \rangle 7.$ $n \mid a$ or $n \mid b$
$\langle 1 \rangle 8.$ One of a, b is 1 and the other is n.
□

26.1 Prime Ideals

Definition 26.9 (Prime Ideal)**.** Let I be an ideal in a commutative ring R. Then I is a *prime ideal* iff R/I is an integral domain.

Example 26.10. Let R be a commutative ring and $a \in R$. Then $(x - a)$ is a prime ideal in R iff R is an integral domain.

Proposition 26.11. *Let R be a commutative ring and I a proper ideal in R. Then I is prime iff, whenever $ab \in I$, then $a \in I$ or $b \in I$.*

PROOF: The condition is the same as saying that, if $(a+I)(b+I)=I$, then $a+I=I$ or $b+I=I$. $\square$

Definition 26.12 (Spectrum)**.** The *spectrum* of a commutative ring R, $\operatorname{Spec} R$, is the set of prime ideals.

Proposition 26.13. *Let $\phi : R \to S$ be a ring homomorphism. If I is a prime ideal in S then $\phi^{-1}(I)$ is a prime ideal in R.*

PROOF:If $ab \in \phi^{-1}(I)$ then $\phi(a)\phi(b) \in I$ so either $\phi(a) \in I$ or $\phi(b) \in I$, i.e. either $a \in \phi^{-1}(I)$ or $b \in \phi^{-1}(I)$. $\square$

Proposition 26.14. *Let R be a commutative ring. Suppose there exists a prime ideal P in R such that the only zero-divisor in P is 0. Then R is an integral domain.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $ab = 0$ in R
$\langle 1 \rangle 2$. $ab \in P$
$\langle 1 \rangle 3$. $a \in P$ or $b \in P$
$\langle 1 \rangle 4$. $a = 0$ or $b = 0$
$\square$

Proposition 26.15. *Let R be a commutative ring. The nilradical of R is included in every prime ideal of R.*

PROOF: Let P be a prime ideal. If $a^n = 0$ then $a^n \in P$ hence $a \in P$. $\square$

Definition 26.16 (Krull Dimension)**.** The *(Krull) dimension* of a commutative ring R is the length of the longest chain of prime ideals in R.

Example 26.17. $\mathbb{Z}[x]$ has Krull dimension 2.

Chapter 27

Unique Factorization Domains

Example 27.1. $\mathbb{Z}$ is a UFD.

Chapter 28

Principal Ideal Domains

Definition 28.1 (Principal Ideal Domain)**.** A commutative ring is a *principal ideal domain* (*PID*) iff every ideal is principal.

Example 28.2. $\mathbb{Z}$ is a PID by Proposition 15.17.

Example 28.3. $\mathbb{Z}[x]$ is not a PID. The ideal $(2, x)$ is not principal.

Proposition 28.4. *Every nonzero prime ideal in a PID is maximal.*

PROOF:
$\langle 1 \rangle 1.$ LET: R be a PID.
$\langle 1 \rangle 2.$ LET: I be a nonzero prime ideal in R.
$\langle 1 \rangle 3.$ PICK $a \in R$ such that $I = (a)$.
$\langle 1 \rangle 4.$ LET: J be an ideal such that $I \subseteq J$
$\langle 1 \rangle 5.$ PICK $b \in R$ such that $J = (b)$.
$\langle 1 \rangle 6.$ PICK $t \in R$ such that $a = bt$.
$\langle 1 \rangle 7.$ $b \in I$ or $t \in I$
$\langle 1 \rangle 8.$ CASE: $b \in I$
  PROOF: Then $J \subseteq I$ so $I = J$.
$\langle 1 \rangle 9.$ CASE: $t \in I$
  $\langle 2 \rangle 1.$ PICK $s \in R$ such that $t = as$.
  $\langle 2 \rangle 2.$ $a = ast$
  $\langle 2 \rangle 3.$ $st = 1$
    PROOF: Since R is an integral domain.
  $\langle 2 \rangle 4.$ $1 \in I$
  $\langle 2 \rangle 5.$ $I = R$
$\square$

Corollary 28.4.1. *Any PID has Krull dimension 1.*

Chapter 29

Euclidean Domains

Example 29.1. $\mathbb{Z}$ is a Euclidean domain.

Chapter 30

Division Rings

Definition 30.1 (Division Ring)**.** A *division ring* is a ring in which every nonzero element is a two-sided unit.

Example 30.2. The quaternions form a division ring, with the inverse of a non-zero element $a + bi + cj + dk$ being

$$\frac{1}{a^2 + b^2 + c^2 + d^2}(a - bi - cj - dk) \ .$$

Example 30.3. For any ring R, the ring of polynomials $R[x]$ is not a division ring, since x has no inverse.

Proposition 30.4. *Every centralizer in a division ring is a division ring.*

PROOF: If $ar = ra$ then $ra^{-1} = a^{-1}r$. □

Proposition 30.5. *A non-trivial ring R is a division ring if and only if its only left-ideals and right-ideals are $\{0\}$ and R.*

PROOF:
$\langle 1\rangle 1$. If R is a division ring then the only left-ideals and right-ideals are $\{0\}$ and R.
  $\langle 2\rangle 1$. ASSUME: R is a division ring.
  $\langle 2\rangle 2$. The only left-ideals are $\{0\}$ and R.
    $\langle 3\rangle 1$. LET: I be a left-ideal that is not $\{0\}$.
      PROVE: $I = R$
    $\langle 3\rangle 2$. PICK $a \in I - \{0\}$
    $\langle 3\rangle 3$. PICK a left inverse b for a
    $\langle 3\rangle 4$. $1 \in I$
      PROOF: Since $1 = ba$.
    $\langle 3\rangle 5$. $I = R$
      PROOF: For any $r \in R$ we have $r = r1 \in I$.
  $\langle 2\rangle 3$. The only right-ideals are $\{0\}$ and R.

PROOF: Similar.

⟨1⟩2. If the only left-ideals and right-ideals are $\{0\}$ and R then R is a division ring.

□

Proposition 30.6. *Let K be a division ring and R a non-trivial ring. Every ring homomorphism $K \to R$ is injective.*

PROOF:

⟨1⟩1. LET: $\phi : K \to R$ be a ring homomorphism.
PROVE: $\ker \phi = \{0\}$

⟨1⟩2. LET: $x \in \ker \phi$

⟨1⟩3. ASSUME: for a contradiction $x \neq 0$.

⟨1⟩4. $\phi(xx^{-1}) = 1$

⟨1⟩5. $0 = 1$

⟨1⟩6. Q.E.D.

PROOF: This contradicts the assumption that R is non-trivial.

□

Chapter 31

Simple Rings

Definition 31.1 (Simple Ring)**.** A non-trivial ring is R *simple* iff its only two-sided ideals are $\{0\}$ and R.

Example 31.2. For any simple ring R we have $\mathfrak{gl}_n(R)$ is simple, by Corollary 25.13.1.

Proposition 31.3. *Let R be a ring and I an ideal in R. Then I is maximal iff R/I is simple.*

PROOF:

$$\begin{aligned} R/I \text{ is simple} &\Leftrightarrow \text{the only ideals in } R/I \text{ are } \{I\} \text{ and } R/I \\ &\Leftrightarrow \text{the only ideals in } R \text{ that include } I \text{ are } I \text{ and } R \\ &\Leftrightarrow I \text{ is maximal} \end{aligned}$$

∎

Chapter 32

Reduced Rings

Definition 32.1 (Reduced Ring)**.** A ring is *reduced* iff it has no non-zero nilpotent elements.

Proposition 32.2. *Let R be a commutative ring. Let N be its nilradical. Then R/N is reduced.*

PROOF:
$\langle 1 \rangle 1$. LET: $r + N$ be nilpotent.
$\langle 1 \rangle 2$. PICK n such that $(r + N)^n = N$
$\langle 1 \rangle 3$. $r^n \in N$
$\langle 1 \rangle 4$. PICK k such that $(r^n)^k = 0$
$\langle 1 \rangle 5$. $r^{nk} = 0$
$\langle 1 \rangle 6$. $r \in N$
$\langle 1 \rangle 7$. $r + N = N$
□

Proposition 32.3. *Let R be a commutative ring. Let I and J be ideals in R. If R/IJ is reduced then $IJ = I \cap J$.*

PROOF:
$\langle 1 \rangle 1$. LET: $r \in I \cap J$
PROVE: $r \in IJ$
$\langle 1 \rangle 2$. $r^2 \in IJ$
$\langle 1 \rangle 3$. $(r + IJ)^2 = IJ$
$\langle 1 \rangle 4$. $r + IJ = IJ$
PROOF: Since R/IJ is reduced.
$\langle 1 \rangle 5$. $r \in IJ$
□

Chapter 33

Boolean Rings

Definition 33.1 (Boolean)**.** A ring is *Boolean* iff $a^2 = a$ for every element a.

Example 33.2. For any set S, the ring $\mathcal{P}S$ is Boolean.

Proposition 33.3. *Every non-trivial Boolean ring has characteristic 2.*

PROOF: We have $4 = 2$ and so $2 = 0$. $\square$

Proposition 33.4. *Every Boolean ring is commutative.*

PROOF:

$$\begin{aligned}
(a+b)^2 &= a + b \\
\therefore a^2 + ab + ba + b^2 &= a + b \\
\therefore a + ab + ba + b &= a + b \\
\therefore ab + ba &= 0 \\
\therefore ab &= -ba \\
&= ba && \text{(Proposition 33.3)}
\end{aligned}$$

Example 33.5. The only Boolean integral domain is $\mathbb{Z}/2\mathbb{Z}$. For, if D is a Boolean integral domain and $x \in D$, we have $x^2 = x$, so $x^2 - x = x(x-1) = 0$ and so $x = 0$ or $x = 1$, i.e. $D = \{0, 1\}$.

Proposition 33.6. *Every Boolean ring has Krull dimension 0.*

PROOF:
$\langle 1 \rangle 1$. LET: R be a Boolean ring.
$\langle 1 \rangle 2$. LET: I be a prime ideal in R.
PROVE: I is maximal.
$\langle 1 \rangle 3$. LET: J be an ideal with $I \subsetneq J$
$\langle 1 \rangle 4$. PICK $a \in J$ with $a \notin I$
$\langle 1 \rangle 5$. $a^2 - a = 0 \in I$
$\langle 1 \rangle 6$. $a(a-1) \in I$

$\langle 1 \rangle 7.$ $a - 1 \in I$
$\langle 1 \rangle 8.$ $a - 1 \in J$
$\langle 1 \rangle 9.$ $1 \in J$
$\langle 1 \rangle 10.$ $J = R$
$\square$

Chapter 34

Modules

Definition 34.1 (Left Module)**.** Let R be a ring and M an Abelian group. A *left-action* of R on M is a ring homomorphism

$$R \to \mathrm{End}_{\mathbf{Ab}}(M) \quad .$$

A *left R-module* consists of an Abelian group M and a left-action of R on M.

Proposition 34.2. *Let R be a ring and M an Abelian group. Let $\cdot : R \times M \to M$. Then $\cdot$ defines a left-action of R on M if and only if, for all $r, s \in R$ and $m, n \in M$:*

- $r(m + n) = rm + rn$
- $(r + s)m = rm + sm$
- $(rs)m = r(sm)$
- $1m = m$

Proof: Immediate from definitions. □

Proposition 34.3. *In any R-module M we have $0m = 0$ for all $m \in M$.*

Proof: Since $0m = (0 + 0)m = 0m + 0m$ and so $0m = 0$ by cancellation in M. □

Proposition 34.4. *In any R-module M we have $(-1)m = -m$ for all $m \in M$.*

Proof: Since $m + (-1)m = 1m + (-1)m = (1 + (-1))m = 0m = 0$. □

Proposition 34.5. *Every Abelian group is a $\mathbb{Z}$-module in exactly one way.*

Proof: Since $\mathbb{Z}$ is initial in **Ring**. □

Definition 34.6 (Right Module)**.** Let R be a ring. A *right R-module* consists of an Abelian group M and a function $\cdot : M \times R \to M$ such that, for all $r, s \in R$ and $m, n \in M$:

- $(m+n)r = mr + nr$
- $m(r+s) = mr + ms$
- $m(rs) = (mr)s$
- $m1 = m$

34.1 Homomorphisms

Definition 34.7 (Homomorphism of Left-Modules)**.** Let R be a ring. Let M and N be left-R-modules. A *homomorphism of left-R-modules* $\phi : M \to N$ is a group homomorphism such that, for all $r \in R$ and $m \in M$, we have $\phi(rm) = r\phi(m)$.

Let $R - \mathbf{Mod}$ be the category of left-R-modules and left-R-module homomorphisms.

Example 34.8.

$$\mathbb{Z} - \mathbf{Mod} \cong \mathbf{Ab}$$

Example 34.9. The trivial group 0 is the zero object in $R - \mathbf{Mod}$.

Proposition 34.10. *Every bijective R-module homomorphism is an isomorphism.*

Proof: Easy. □

Proposition 34.11. *Let R be a ring. Let M be an R-module. Then*

$$M \cong R - \mathbf{Mod}[R, M]$$

as R-modules.

Proof: The isomorphism maps m to the function $\lambda r.rm$. Its inverse maps an R-module homomorphism α to $\alpha(1)$. □

Proposition 34.12. *Let R be a commutative ring. Let M be an R-module. Then there is a bijection between the set of $R[x]$-module structures on M that extend the given R-module structure and* $\mathrm{End}_{R-\mathbf{Mod}}(M)$.

Proof:

$\langle 1 \rangle 1$. Let: $\alpha : R \to \mathrm{End}_{\mathbf{Ab}}(M)$ be the given R-module structure on M.

$\langle 1 \rangle 2$. An $R[x]$-module structure on M that extends α is a ring homomorphism $\beta : R[x] \to \mathrm{End}_{\mathbf{Ab}}(M)$ such that $\beta \circ i = \alpha$, where i is the inclusion $R \to R[x]$.

$\langle 1 \rangle 3$. There is a bijection between the $R[x]$-module structures on M that extend α and the elements $s \in \mathrm{End}_{\mathbf{Ab}}(M)$ that commute with $\alpha(r)$ for all $r \in R$.

Proof: By the universal property for polynomials.

$\langle 1 \rangle 4$. There is a bijection between the $R[x]$-module structures on M that extend α and the R-module homomorphisms $(M, \alpha) \to (M, \alpha)$.

□

Proposition 34.13. *Let R be a commutative ring. Let M and N be R-modules. Then $R - \mathbf{Mod}[M, N]$ is an R-module under*

$$(\phi + \psi)(m) = \phi(m) + \psi(m)$$
$$(r\phi)(m) = r\phi(m)$$

PROOF: Easy. □

Proposition 34.14. *Let R be an integral domain. Let I be a nonzero principal ideal of R. Then $I \cong R$ in $R - \mathbf{Mod}$.*

PROOF:
$\langle 1\rangle 1$. PICK $a \in R$ such that $I = (a)$.
$\langle 1\rangle 2$. LET: $\phi : R \to I$ be the map $\phi(r) = ra$.
$\langle 1\rangle 3$. ϕ is an R-module homomorphism.
PROOF: Since $(r + s)a = ra + sa$ and $(rs)a = r(sa)$.
$\langle 1\rangle 4$. ϕ is surjective.
$\langle 1\rangle 5$. ϕ is injective.
PROOF: If $ra = sa$ then $(r - s)a = 0$ so $r - s = 0$ and $r = s$.
$\langle 1\rangle 6$. $\phi : R \cong I$
□

34.2 Submodules

Definition 34.15 (Submodule)**.** Let M be a left-R-module and $N \subseteq M$. Then N is a *submodule* of M iff N is a subgroup of M and $\forall r \in R.\forall n \in N.rn \in N$.

Proposition 34.16. *Let R be a ring and $I \subseteq R$. Then I is a left-ideal in R iff I is a submodule of R as an R-module.*

PROOF: Immediate from definitions. □

Proposition 34.17. *Let R be a ring. Let M and N be left-R-modules and $\phi : M \to N$ an R-module homomorphism. Then $\ker \phi$ is a submodule of M and $\operatorname{im} \phi$ is a submodule of N.*

PROOF: Easy. □

Proposition 34.18. *Let R be a commutative ring. Let M be a left-R-module. Let $r \in R$. Then $rM = \{rm : m \in M\}$ is a submodule of M.*

PROOF: Easy. □

Proposition 34.19. *Let R be a ring. Let M be a left-R-module. Let I be a left-ideal in R. Then $IM = \{rm : r \in I, m \in M\}$ is a submodule of M.*

PROOF:

⟨1⟩1. IM is a subgroup of M.

⟨2⟩1. LET: $r, s \in I$ and $m, n \in M$.
PROVE: $rm + sn \in IM$

⟨2⟩2. $rm + sn = r(m - n) + (s - r)n$

⟨1⟩2. For all $r \in R$ and $x \in IM$ we have $rx \in IM$.

□

34.3 Quotient Modules

Definition 34.20 (Quotient Module)**.** Let R be a ring. Let M be a left-R-module. Let N be a submodule of M. Then the *quotient module* M/N is the quotient group M/N under

$$r(m + N) = rm + N \ .$$

Proposition 34.21. *Let R be a ring. Let M and P be left-R-modules. Let N be a submodule of M. Let $\phi : M \to P$ be an R-module homomorphism. If $N \subseteq \ker\phi$, then there exists a unique R-module homomorphism $\overline{\phi} : M/N \to P$ such that the following diagram commutes.*

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr] & & P \\ & M/N \arrow[ur, "\overline{\phi}"'] & \end{tikzcd}$$

PROOF: Easy. □

Theorem 34.22. *Every R-module homomorphism $\phi : M \to M'$ may be decomposed as:*

$$M \longrightarrow M/\ker\phi \xrightarrow{\cong} \operatorname{im}\phi \longrightarrow N$$

PROOF: Easy. □

Corollary 34.22.1 (First Isomorphism Theorem)**.** *Let $\phi : M \to M'$ be a surjective R-module homomorphism. Then*

$$M' \cong \frac{M}{\ker\phi} \ .$$

Proposition 34.23 (Second Isomorphism Theorem)**.** *Let R be a ring. Let M be a left-R-module. Let N and P be submodules of M. Then $N + P$ is a submodule of M, $N \cap P$ is a submodule of P, and*

$$\frac{N + P}{N} \cong \frac{P}{N \cap P}$$

PROOF: The function that maps P to $p + N$ is a surjective homomorphism $P \to (N + P)/N$ with kernel $N \cap P$. □

Proposition 34.24 (Third Isomorphism Theorem)**.** *Let R be a ring. Let M be a left-R-module. Let N be a submodule of M and P a submodule of N. Then N/P is a submodule of M/P and*

$$\frac{M/P}{N/P} \cong \frac{M}{N}$$

PROOF: The canonical map $M \to M/N$ induces a surjective homomorphism $M/P \to M/N$ which has kernel N/P. □

Proposition 34.25. *Let R be a ring. Let M be a left-R-module. The sum and intersection of a family of submodules of M are submodules of M.*

PROOF: Easy. □

34.4 Products

Proposition 34.26. *$R-\mathbf{Mod}$ has products.*

PROOF: Given a family $\{M_\alpha\}_{\alpha\in A}$ of left-R-modules, we make $\prod_{\alpha\in A} M_\alpha$ into a left-R-module by

$$\begin{aligned}(f+g)(\alpha) &= f(\alpha)+g(\alpha)\\ (rf)(\alpha) &= rf(\alpha)\end{aligned}$$

□

34.5 Coproducts

Proposition 34.27. *$R-\mathbf{Mod}$ has coproducts.*

PROOF: Given a family $\{M_\alpha\}_{\alpha\in A}$ of left-R-modules, take $\bigoplus_{\alpha\in A} M_\alpha$ to be $\{f \in \prod_{\alpha\in A} M_\alpha : f(\alpha) = 0$ for all but finitely many $\alpha \in A\}$. □

34.6 Direct Sum

Definition 34.28 (Direct Sum)**.** Let R be a ring. Let M and N be left-R-modules. Then the direct sum $M \oplus N$ is an R-module under

$$r(m,n) = (rm, rn) \ .$$

Proposition 34.29. *$M \oplus N$ is the biproduct of M and N in $R-\mathbf{Mod}$.*

PROOF: Easy. □

Example 34.30. Infinite products and coproducts are in general different. We have $\mathbb{Z}^{\mathbb{N}} \ncong \mathbb{Z}^{\oplus\mathbb{N}}$ since $\mathbb{Z}^{\mathbb{N}}$ is uncountable but $\mathbb{Z}^{\oplus\mathbb{N}}$ is countable.

34.7 Kernels and Cokernels

Proposition 34.31. *Let R be a ring. Let $\phi : M \to N$ be a left-R-module homomorphism. Then $\ker\phi \hookrightarrow M$ is terminal in the category of left-R-module homomorphisms $\alpha : P \to M$ such that $\phi \circ \alpha = 0$.*

Proof: Easy. □

Proposition 34.32. *Let R be a ring. Let $\phi : M \to N$ be a left-R-module homomorphism. Then $N \twoheadrightarrow \operatorname{coker}\phi$ is initial in the category of left-R-module homomorphisms $\alpha : N \to P$ such that $\alpha \circ \phi = 0$.*

Proof: Easy. □

Proposition 34.33. *Let R be a ring. Let $\phi : M \to N$ be a left-R-module homomorphism. Then the following are equivalent.*

- *ϕ is a monomorphism.*
- *$\ker\phi$ is trivial.*
- *ϕ is injective.*

Proof: Easy. □

Proposition 34.34. *Let R be a ring. Let $\phi : M \to N$ be a left-R-module homomorphism. Then the following are equivalent.*

- *ϕ is an epimorphism.*
- *$\operatorname{coker}\phi$ is trivial.*
- *ϕ is surjective.*

Proof: Easy. □

Proposition 34.35. *Every monomorphism in $R-\mathbf{Mod}$ is the kernel of some homomorphism.*

Proof: If $\phi : M \to N$ is a monomorphism then it is the kernel of $N \twoheadrightarrow N/\operatorname{im}\phi$.
□

Proposition 34.36. *Every epimorphism in $R-\mathbf{Mod}$ is the cokernel of some homomorphism.*

Proof: If $\phi : M \to N$ is epi then it is the cokernel of $\ker\phi \hookrightarrow M$. □

Example 34.37. Monomorphisms do not split in $R-\mathbf{Mod}$. Multiplication by 2 is a monomorphism $\mathbb{Z} \to \mathbb{Z}$ but has no left inverse.

Example 34.38. Epimorphisms do not split in $R-\mathbf{Mod}$. The canonical map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is an epimorphism without a right inverse.

34.8 Free Modules

Proposition 34.39. *Let R be a ring and A a set. Then there exists a left-R-module $F^R(A)$ and function $j : A \to F^R(A)$ such that, for any left-R-module M and function $f : A \to M$, there exists a unique left-R-module homomorphism $\overline{f} : F^R(A) \to M$ such that the following diagram commutes.*

$$\begin{tikzcd} F^R(A) \arrow[r, "\overline{f}"] & M \\ A \arrow[u, "j"] \arrow[ur, "f"'] & \end{tikzcd}$$

PROOF:
⟨1⟩1. LET: $R^{\oplus A} = \{\alpha : A \to R : \alpha(a) = 0$ for all but finitely many $a \in A\}$ under the operations

$$(\alpha + \beta)(a) = \alpha(a) + \beta(a)$$
$$(r\alpha)(a) = r\alpha(a)$$

⟨1⟩2. $R^{\oplus A}$ is a left-R-module.
⟨1⟩3. LET: $j : A \to R^{\oplus A}$ be the function

$$j(a)(a') = \begin{cases} 1 & \text{if } a = a' \\ 0 & \text{if } a \neq a' \end{cases}$$

⟨1⟩4. LET: M be any left-R-module.
⟨1⟩5. LET: $f : A \to M$ be a function.
⟨1⟩6. LET: $\overline{f} : R^{\oplus A} \to M$ be the function

$$\overline{f}(\alpha) = \sum_{a \in A, \alpha(a) \neq 0} \alpha(a) f(a)$$

⟨1⟩7. $\overline{f}$ is a left-R-module homomorphism.
⟨1⟩8. $\overline{f} \circ j = f$
⟨1⟩9. $\overline{f}$ is unique.

Definition 34.40. We call $j : A \to F^R(A)$ the *free* left-R-module over A.

Proposition 34.41. *j is injective.*

PROOF: By the proof of the previous proposition. □

Proposition 34.42. *Let R be a ring. Let F be a non-zero free left-R-module. Let $\phi : M \to N$ be a left-R-module homomorphism. Then ϕ is onto if and only if, for every left-R-module homomorphism $\alpha : F \to N$, there exists a left-R-module homomorphism $\beta : F \to M$ such that the diagram below commutes.*

$$\begin{tikzcd} M \arrow[r, "\phi"] & N \\ F \arrow[u, "\beta"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

PROOF:

⟨1⟩1. LET: F be the free left-R-module over A with injection $j : A \to F$.
⟨1⟩2. If ϕ is onto then, for every homomorphism $\alpha : F \to N$, there exists a homomorphism $\beta : F \to M$ such that $\phi \circ \beta = \alpha$.
  ⟨2⟩1. ASSUME: ϕ is onto.
  ⟨2⟩2. LET: $\alpha : F \to N$ be a homomorphism.
  ⟨2⟩3. For $a \in A$, PICK $f(a) \in M$ such that $\phi(f(a)) = \alpha(j(a))$
  ⟨2⟩4. LET: $\beta : F \to M$ be the unique homomorphism such that $\beta \circ j = f$
  ⟨2⟩5. $\phi \circ \beta = \alpha$
    PROOF: Each is the unique homomorphism such that $\alpha \circ j = \phi \circ f$.
  □

$$\begin{tikzcd}
& M \arrow[r, "\phi"] & N \\
A \arrow[ur, "f"] \arrow[r, "j"] & F \arrow[u, "\beta"] \arrow[ur, "\alpha"] &
\end{tikzcd}$$

⟨1⟩3. If, for every homomorphism $\alpha : F \to N$, there exists a homomorphism $\beta : F \to M$ such that $\phi \circ \beta = \alpha$, then ϕ is onto.
  ⟨2⟩1. ASSUME: For every homomorphism $\alpha : F \to N$ there exists a homomorphism $\beta : F \to M$ such that $\phi \circ \alpha = \beta$.
  ⟨2⟩2. LET: $n \in N$
  ⟨2⟩3. LET: $\alpha : F \to N$ be the unique homomorphism such that, for all $a \in A$, we have $\alpha(j(a)) = n$
  ⟨2⟩4. PICK a homomorphism $\beta : F \to M$ such that $\phi \circ \beta = \alpha$
  ⟨2⟩5. PICK $a \in A$
  ⟨2⟩6. $\phi(\beta(j(a))) = n$
□

34.9 Generators

Definition 34.43 (Submodule Generated by a Set)**.** Let R be a ring. Let M be a left-R-module. Let A be a subset of M. Let $\phi_A : F^R(A) \to M$ be the unique left-R-module homomorphism such that the following diagram commutes.

$$\begin{tikzcd}
F^R(A) \arrow[r, "\phi_A"] & M \\
A \arrow[u] \arrow[ur, hook] &
\end{tikzcd}$$

The submodule of M *generated* by A, denoted $\langle A \rangle$, is defined to be $\operatorname{im} \phi_A$.

Definition 34.44 (Finitely Generated)**.** Let R be a ring. Let M be a left-R-module. Then M is *finitely generated* iff there exists a finite set $A \subseteq M$ such that $M = \langle A \rangle$.

Example 34.45. A submodule of a finitely generated module is not necessarily finitely generated.

Let $R = \mathbb{Z}[x_1, x_2, \ldots]$. Then R is finitely generated as an R-module, but $(x_1, x_2, \ldots)$ is not.

Proposition 34.46. *The homomorphic image of a finitely generated module is finitely generated.*

PROOF: Easy. □

Proposition 34.47. *Let R be a ring. Let M be a left-R-module. Let N be a submodule of M. If N and M/N are finitely generated then M is finitely generated.*

PROOF:
$\langle 1\rangle 1$. PICK $a_1, \dots, a_n$ that generate N.
$\langle 1\rangle 2$. PICK $b_1, \dots, b_m$ such that $b_1 + N, \dots, b_m + N$ generate M/N.
PROVE: $a_1, \dots, a_n, b_1, \dots, b_m$ generate M.
$\langle 1\rangle 3$. LET: $m \in M$
$\langle 1\rangle 4$. PICK $r_1, \dots, r_m \in R$ such that $m + N = r_1 b_1 + \cdots + r_m b_m + N$
$\langle 1\rangle 5$. $m - r_1 b_1 - \cdots - r_m b_m \in N$
$\langle 1\rangle 6$. PICK $s_1, \dots, s_n \in R$ such that $m - r_1 b_1 - \cdots - r_m b_m = s_1 a_1 + \cdots + s_n a_n$
$\langle 1\rangle 7$. $m = r_1 b_1 + \cdots + r_m b_m + s_1 a_1 + \cdots + s_n a_n$
□

34.10 Projections

Definition 34.48 (Projection)**.** Let R be a ring. Let M be a left-R-module. Let $p : M \to M$ be a left-R-module homomorphism. Then p is a *projection* iff $p^2 = p$.

Proposition 34.49. *Let R be a ring. Let M be a left-R-module. Let $p : M \to M$ be a projection. Then*

$$M \cong \ker p \oplus \operatorname{im} p \ .$$

PROOF:
$\langle 1\rangle 1$. LET: $\phi : M \to \ker p \oplus \operatorname{im} p$ be the map $\phi(m) = (m - p(m), p(m))$
$\langle 1\rangle 2$. ϕ is a left-R-module homomorphism.
$\langle 1\rangle 3$. ϕ is injective.
$\langle 1\rangle 4$. ϕ is surjective.
□

34.11 Pullbacks

Proposition 34.50. $R-\mathbf{Mod}$ *has pullbacks.*

PROOF:
$\langle 1\rangle 1$. LET: $\mu : M \to Z$, $\nu : N \to Z$ be left-R-module homomorphisms.
$\langle 1\rangle 2$. LET: $M \times_Z N = \{(m, n) \in M \times N : \mu(m) = \nu(n)\}$ under

$$(m, n) + (m', n') = (m + m', n + n')$$

$$r(m, n) = (rm, rn)$$

$\langle 1\rangle 3$. $M \times_Z N$ is the pullback of M and N.
□

34.12 Pushouts

Proposition 34.51. $R-\mathbf{Mod}$ *has pushouts.*

PROOF:
⟨1⟩1. LET: $\mu : A \to M$ and $\nu : A \to N$ be left-R-module homomorphisms.

Chapter 35

Cyclic Modules

Definition 35.1 (Cyclic Module)**.** Let R be a ring. Let M be a left-R-module. Then M is *cyclic* iff there exists $m \in M$ such that $M = \langle m \rangle$.

Proposition 35.2. *Let R be a ring. Let M be a left-R-module. Then M is cyclic if and only if there exists a left-ideal I in R such that $M \cong R/I$.*

PROOF:
$\langle 1 \rangle 1.$ If M is cyclic then there exists a left-ideal I in R such that $M \cong R/I$.
  $\langle 2 \rangle 1.$ ASSUME: M is cyclic.
  $\langle 2 \rangle 2.$ PICK $m \in M$ such that $M = \langle m \rangle$
  $\langle 2 \rangle 3.$ LET: $\phi : R \to M$ be the left-R-module homomorphism $\phi(r) = rm$.
  $\langle 2 \rangle 4.$ ϕ is surjective.
  $\langle 2 \rangle 5.$ $M \cong R/\ker \phi$
$\langle 1 \rangle 2.$ For every left-ideal I in R, we have that R/I is cyclic.
  PROOF: R/I is generated by $1 + I$.
$\square$

Proposition 35.3. *A quotient of a cyclic module is cyclic.*

PROOF: If M is generated by m then M/N is generated by $m + N$. $\square$

Proposition 35.4. *Let R be a ring. For any left-ideal I in R and any left-R-module N, we have*

$$R - \mathbf{Mod}[R/I, N] \cong \{n \in N : \forall a \in I . an = 0\} \ .$$

PROOF:
$\langle 1 \rangle 1.$ LET: $\Phi : R - \mathbf{Mod}[R/I, N] \to \{n \in N : \forall a \in I . an = 0\}$ be the function $\Phi(\alpha) = \alpha(1 + I)$
  PROOF: For all $a \in I$ we have $a\alpha(1 + I) = \alpha(a + I) = \alpha(I) = 0$.
$\langle 1 \rangle 2.$ Φ is injective.
  PROOF: If $\alpha(1+I) = \beta(1+I)$ then $\alpha(r+I) = r\alpha(1+I) = r\beta(1+I) = \beta(r+I)$ for all $r \in R$, hence $\alpha = \beta$.

$\langle 1 \rangle 3$. Φ is surjective.
PROOF: Given $n \in N$ such that $\forall a \in I.an = 0$, define $\alpha : R/I \to N$ by $\alpha(r + I) = rn$.
$\langle 1 \rangle 4$. If R is commutative then Φ is an R-module homomorphism.
$\square$

Corollary 35.4.1. *For all $a, b \in \mathbb{Z}$ we have $\mathbf{Ab}[\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}] \cong \mathbb{Z}/\gcd(a, b)\mathbb{Z}$.*

PROOF:

$$\begin{aligned}
\mathbf{Ab}[\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}] &\cong \mathbb{Z} - \mathbf{Mod}[\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}] \\
&\cong \{n \in \mathbb{Z}/b\mathbb{Z} : \forall x \in a\mathbb{Z}.xn \cong 0 (\mathrm{mod}\, b)\} \\
&\cong \{n \in \mathbb{Z}/b\mathbb{Z} : \forall x \in \mathbb{Z}.b \mid xan\} \\
&= \{n \in \mathbb{Z}/b\mathbb{Z} : b \mid an\}
\end{aligned}$$

Chapter 36

Simple Modules

Definition 36.1 (Simple Module)**.** Let R be a ring. An R-module M is *simple* or *irreducible* iff its only submodules are $\{0\}$ and M.

Proposition 36.2 (Schur's Lemma)**.** *Let R be a ring. Let M and N be simple R-modules. Let $\phi : M \to N$ be an R-module homomorphism. Then either $\phi = 0$ or ϕ is an isomorphism.*

PROOF:
⟨1⟩1. ASSUME: $\phi \neq 0$
⟨1⟩2. $\ker \phi = 0$
  PROOF: Since $\ker \phi$ is a submodule of M that is not M.
⟨1⟩3. $\operatorname{im} \phi = N$
  PROOF: Since $\operatorname{im} \phi$ is a submodule of N that is not $\{0\}$.
□

Proposition 36.3. *Every simple module is cyclic.*

PROOF:
⟨1⟩1. LET: M be a simple module.
⟨1⟩2. ASSUME: w.l.o.g. $M \neq \{0\}$
  PROOF: $\{0\} = \langle 0 \rangle$ is cyclic.
⟨1⟩3. PICK $m \in M$ with $m \neq 0$
⟨1⟩4. $\langle m \rangle = M$
  PROOF: Since $\langle m \rangle$ is a submodule of M that is not $\{0\}$.
□

Chapter 37

Noetherian Modules

Definition 37.1 (Noetherian Module)**.** Let R be a ring. A left-R-module is *Noetherian* iff every submodule is finitely generated.

Proposition 37.2. *Let R be a ring. Let M be a left-R-module and N a submodule of M. Then M is Noetherian if and only if N and M/N are Noetherian.*

PROOF:
$\langle 1\rangle 1$. If M is Noetherian then N is Noetherian.
  PROOF: Every submodule of N is a submodule of M, hence finitely generated.
$\langle 1\rangle 2$. If M is Noetherian then M/N is Noetherian.
  $\langle 2\rangle 1$. ASSUME: M is Noetherian.
  $\langle 2\rangle 2$. LET: $\pi : M \twoheadrightarrow M/N$ be the canonical epimorphism.
  $\langle 2\rangle 3$. LET: P be a submodule of M/N.
  $\langle 2\rangle 4$. PICK $a_1, \ldots, a_n \in M$ that generate $\pi^{-1}(P)$.
  $\langle 2\rangle 5$. $a_1 + N, \ldots, a_n + N$ generate P.
$\langle 1\rangle 3$. If N and M/N are Noetherian then M is Noetherian.
  $\langle 2\rangle 1$. ASSUME: N and M/N are Noetherian.
  $\langle 2\rangle 2$. LET: P be a submodule of M.
  $\langle 2\rangle 3$. PICK $a_1, \ldots, a_m \in P$ such that $a_1 + N, \ldots, a_m + N$ generate $\pi(P)$.
  $\langle 2\rangle 4$. PICK $b_1, \ldots, b_n \in M$ that generated $P \cap N$.
    PROVE: $a_1, \ldots, a_m, b_1, \ldots, b_n$ generate P.
  $\langle 2\rangle 5$. LET: $p \in P$
  $\langle 2\rangle 6$. PICK $r_1, \ldots, r_m \in R$ such that $p + N = r_1 a_1 + \cdots + r_m a_m + N$
  $\langle 2\rangle 7$. $p - r_1 a_1 - \cdots - r_m a_m \in P \cap N$
  $\langle 2\rangle 8$. PICK $s_1, \ldots, s_n \in R$ such that $p - r_1 a_1 - \cdots - r_m a_m = s_1 b_1 + \cdots + s_n b_n$
  $\langle 2\rangle 9$. $p = r_1 a_1 + \cdots + r_m a_m + s_1 b_1 + \cdots + s_n b_n$
□

Proposition 37.3. *Let R be a commutative ring. Let M be an R-module. Then the following are equivalent.*

1. *M is Noetherian.*

2. Ascending Chain Condition (a.c.c.) *Every ascending chain of submodules of M stabilizes; that is, if*

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$$

is a chain of submodules of M, then there exists i such that $\forall j \geq i. N_i = N_j$.

3. *Every nonempty set of submodules of M has a maximal element.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. ASSUME: M is Noetherian.
  ⟨2⟩2. LET: $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be an ascending chain of submodules of M.
  ⟨2⟩3. PICK generators $a_1, \ldots, a_k$ that generate $\bigcup_i N_i$
  ⟨2⟩4. PICK j such that $a_1, \ldots, a_k \in N_j$
  ⟨2⟩5. N_j is maximal.
⟨1⟩2. $2 \Rightarrow 3$
  PROOF: If $\mathcal{S}$ is a nonempty set of submodules of M with no maximal element, then we can choose a sequence (N_i) in $\mathcal{S}$ with $N_i \subsetneq N_{i+1}$ for all i.
⟨1⟩3. $3 \Rightarrow 2$
  PROOF: Pick i such that N_i is maximal in $\{N_j : j \geq 1\}$.
⟨1⟩4. $2 \Rightarrow 1$
  ⟨2⟩1. ASSUME: M is not Noetherian.
    PROVE: a.c.c. does not hold.
  ⟨2⟩2. PICK a submodule N of M that is not finitely generated.
  ⟨2⟩3. Choose a sequence of elements (n_i) in N such that $n_{i+1} \notin \langle n_1, \ldots, n_i \rangle$.
  ⟨2⟩4. LET: $N_i = \langle n_1, \ldots, n_i \rangle$ for all i.
  ⟨2⟩5. $N_1 \subsetneq N_2 \subsetneq \cdots$

□

Chapter 38

Noetherian Rings

Definition 38.1 (Noetherian Ring)**.** A commutative ring is *Noetherian* iff it is Noetherian as a module over itself.

Proposition 38.2. *The homomorphic image of a Noetherian ring is Noetherian.*

PROOF:
$\langle 1 \rangle 1.$ LET: R be a Noetherian ring, S be a commutative ring, and $\phi : R \to S$ a surjective ring homomorphism.
$\langle 1 \rangle 2.$ LET: I be an ideal in S.
$\langle 1 \rangle 3.$ LET: $\phi^{-1}(I) = (a_1, \ldots, a_n)$
$\langle 1 \rangle 4.$ $I = (\phi(a_1), \ldots, \phi(a_n))$
$\square$

Proposition 38.3. *Every PID is Noetherian.*

PROOF: Trivial. $\square$

Proposition 38.4. *If R is a Noetherian ring then $R^{\oplus n}$ is a Noetherian left-R-module.*

PROOF: The proof is by induction on n. The case $n = 1$ is immediate.
The induction step holds by Proposition 37.2 since $R^{\oplus(n+1)}/R^{\oplus n} \cong R$. $\square$

Corollary 38.4.1. *If R is a Noetherian ring and M is a finitely generated left-R-module then M is Noetherian.*

PROOF: There is a surjective homomorphism $R^{\oplus n} \twoheadrightarrow M$ for some n, so M is a quotient of $R^{\oplus n}$. $\square$

Proposition 38.5. *A ring is Noetherian iff every ascending chain of ideals stabilizes.*

PROOF: Proposition 37.3. $\square$

Proposition 38.6. *Let R be a commutative Noetherian ring and I an ideal of R. Then R/I is Noetherian.*

PROOF:
⟨1⟩1. LET: J be an ideal in R/I.
⟨1⟩2. $\pi^{-1}(J)$ is an ideal in R.
⟨1⟩3. $\pi^{-1}(J)$ is finitely generated.
⟨1⟩4. J is finitely generated.
□

Lemma 38.7 (Hilbert's Basis Theorem)**.** *If R is a commutative Noetherian ring then $R[x]$ is a Noetherian ring.*

PROOF:
⟨1⟩1. LET: R be a commutative Noetherian ring.
⟨1⟩2. LET: I be an ideal of $R[x]$.
PROVE: I is finitely generated.
⟨1⟩3. LET: $A = \{0\} \cup \{a \in R : a \text{ is the leading coefficient of an element of } I\}$
⟨1⟩4. A is an ideal of R.
⟨2⟩1. $\forall a, b \in A . a - b \in A$
⟨3⟩1. LET: $a, b \in A$
⟨3⟩2. ASSUME: w.l.o.g. $a \neq 0 \neq b$
⟨3⟩3. PICK $f, g \in I$ such that a is the leading coefficient of f and b is the leading coefficient of g.
⟨3⟩4. LET: $d = \deg f$
⟨3⟩5. LET: $e = \deg g$
⟨3⟩6. ASSUME: w.l.o.g. $d \leq e$
⟨3⟩7. $a - b$ is the leading coefficient of $x^{e-d} f - g \in I$
⟨3⟩8. $a - b \in A$
⟨2⟩2. $\forall r \in R . \forall a \in A . ra \in A$
PROOF: If a is the leading coefficient of f then ra is the leading coefficient of rf.
⟨1⟩5. PICK $a_1, \ldots, a_r \in A$ that generate A.
PROOF: Since R is Noetherian.
⟨1⟩6. PICK $f_1, \ldots, f_r \in I$ such that a_i is the leading coefficient of f_i.
⟨1⟩7. For $i = 1, \ldots, r$,
LET: $d_i = \deg f_i$.
⟨1⟩8. LET: $d = \max(d_1, \ldots, d_r)$
⟨1⟩9. LET: M be the following submodule of $R[x]$: $M = \langle 1, x, x^2, \ldots, x^{d-1} \rangle$.
⟨1⟩10. M is a Noetherian R-module.
PROOF: Corollary 38.4.1.
⟨1⟩11. $M \cap I$ is a finitely generated R-module.
⟨1⟩12. PICK $g_1, \ldots, g_s \in M \cap I$ that generate $M \cap I$.
PROVE: $I = (f_1, \ldots, f_r, g_1, \ldots, g_s)$
⟨1⟩13. For all $\alpha \in I$, there exist $\beta_1, \ldots, \beta_r \in R[x]$ such that $\deg(\alpha + \beta_1 f_1 + \cdots + \beta_r f_r) < d$
⟨2⟩1. For all $\alpha \in I$ with $\deg \alpha \geq d$, there exist $\beta_1, \ldots, \beta_r \in R[x]$ such that $\deg(\alpha + \beta_1 f_1 + \cdots + \beta_r f_r) < \deg \alpha$

$\langle 3 \rangle$1. LET: $\alpha \in I$
$\langle 3 \rangle$2. LET: $e = \deg \alpha$
$\langle 3 \rangle$3. LET: a be the leading coefficient of α.
$\langle 3 \rangle$4. PICK $b_1, \ldots, b_r \in R$ such that $a = b_1 a_1 + \cdots + b_r a_r$.
$\langle 3 \rangle$5. $\deg(\alpha - b_1 x^{e-d_1} f_1 - \cdots - b_r x^{e-d_r} f_r) < e$

$\langle 1 \rangle$14. LET: $\alpha \in I$
$\langle 1 \rangle$15. PICK $\beta_1, \ldots, \beta_r \in R[x]$ such that $\deg(\alpha + \beta_1 f_1 + \cdots + \beta_r f_r) < d$
$\langle 1 \rangle$16. $\alpha + \beta_1 f_1 + \cdots + \beta_r f_r \in M \cap I = (g_1, \ldots, g_s)$
$\langle 1 \rangle$17. $\alpha \in (f_1, \ldots, f_r, g_1, \ldots, g_s)$

$\square$

Chapter 39

Algebras

Definition 39.1 (Algebra)**.** Let R be a commutative ring. An *R-algebra* consists of a ring S and a ring homomorphism $\alpha : R \to S$ such that $\alpha(R)$ is included in the center of S. We write rs for $\alpha(r)s$.

Proposition 39.2. *Let R be a commutative ring and S a ring. Let $\cdot : R \times S \to S$. Then there exists $\alpha : R \to S$ that makes S into an R-algebra such that*

$$rs = \alpha(r)s \qquad (r \in R, s \in S)$$

iff S is an R-module under $\cdot$ and, for all $r_1, r_2 \in R$ and $s_1, s_2 \in S$,

$$(r_1 s_1)(r_2 s_2) = (r_1 r_2)(s_1 s_2) \ .$$

Proof: Immediate from definitions. □

Example 39.3. Let R be a commutative ring. Then R is an R-algebra under multiplication.

Example 39.4. Let R be a commutative ring and I an ideal in R. Then R/I is an R-algebra.

Example 39.5. Let R be a commutative ring and M an R-module. Then $\mathrm{End}_{R-\mathbf{Mod}}(M)$ is an R-algebra under composition.

Example 39.6. Let R be a commutative ring. Then $\mathfrak{gl}_n(R)$ is an R-algebra under matrix multiplication.

Definition 39.7 (Algebra Homomorphism)**.** Let R be a commutative ring. Let S and T be R-algebras. An *R-algebra homomorphism* $\phi : S \to T$ is a ring homomorphism such that, for all $r \in R$ and $s \in S$, we have $\phi(rs) = r\phi(s)$.

Let $R - \mathbf{Alg}$ be the category of R-algebras and R-algebra homomorphisms.

Example 39.8.

$$\mathbb{Z} - \mathbf{Alg} \cong \mathbf{Ring}$$

Example 39.9. Let R be a commutative ring. Then $R[x_1, \ldots, x_n]$, and any quotient ring of $R[x_1, \ldots, x_n]$, is a commutative R-algebra.

Example 39.10. R is the initial object in $R - \mathbf{Alg}$.

39.1 Rees Algebra

Definition 39.11 (Rees Algebra)**.** Let R be a commutative ring. Let I be an ideal in R. The *Rees algebra* is the direct sum

$$\mathrm{Rees}_R(I) = \bigoplus_{j \geq 0} I^j$$

under the multiplication

$$(r_0, r_1, r_2, r_3, \ldots)(s_0, s_1, s_2, \ldots) = (r_0 s_0, r_1 s_0 + r_0 s_1, r_0 s_2 + r_1 s_1 + r_2 s_0, \ldots)$$
$$r(r_0, r_1, r_2, \ldots) = (r r_0, r r_1, r r_2, \ldots)$$

Proposition 39.12. *Let R be a commutative ring. Let $a \in R$ be a non-zero-divisor. Then $R[x]$ is the Rees algebra of (a).*

PROOF:
⟨1⟩1. LET: $\phi : R[x] \to \mathrm{Rees}_R((a))$ be the function $\phi(r_0 + r_1 x + r_2 x^2 + \cdots) = (r_0, r_1 a, r_2 a^2, \ldots)$.
⟨1⟩2. ϕ is an R-algebra homomorphism.
⟨1⟩3. ϕ is injective.
⟨2⟩1. LET: $\phi(r_0 + r_1 x + r_2 x^2 + \cdots) = \phi(s_0 + s_1 x + s_2 x^2 + \cdots)$
⟨2⟩2. For all n we have $r_n a^n = s_n a^n$
⟨2⟩3. $(r_n - s_n) a^n = 0$
⟨2⟩4. $r_n - s_n = 0$
PROOF: Since a is not a zero-divisor.
⟨2⟩5. $r_n = s_n$
⟨1⟩4. ϕ is surjective.
□

Proposition 39.13. *Let R be a commutative ring. Let $a \in R$ be a non-zero-divisor. Let I be an ideal of R. Then $\mathrm{Rees}_R(I) \cong \mathrm{Rees}_R(aI)$.*

PROOF:
⟨1⟩1. LET: $\phi : \mathrm{Rees}_R(I) \to \mathrm{Rees}_R(aI)$ be the function $\phi(r_0, r_1, r_2, \ldots) = (r_0, a r_1, a^2 r_2, \ldots)$.
⟨1⟩2. ϕ is an R-algebra homomorphism.
⟨1⟩3. ϕ is injective.
⟨1⟩4. ϕ is surjective.
□

39.2 Free Algebras

Proposition 39.14. *Let R be a ring. Then $R[x_1, \ldots, x_n]$ is the free commutative R-algebra on $\{1, \ldots, n\}$.*

PROOF: Easy. □

Proposition 39.15. *Let R be a ring and A a set. Let A^* be the free monoid on A. Then the monoid ring $R[A^*]$ is the free R-algebra on A.*

PROOF: Easy. □

Proposition 39.16. *Let R be a commutative ring and S a commutative R-algebra. Then S is finitely generated as an R-algebra if and only if S is finitely generated as a commutative R-algebra.*

PROOF: Since a subalgebra of a commutative subalgebra is commutative, so the smallest algebra that contains $\{a_1, \ldots, a_n\}$ is the smallest commutative subalgebra that contains $\{a_1, \ldots, a_n\}$. □

Chapter 40

Algebras of Finite Type

Definition 40.1 (Algebra of Finite Type)**.** Let R be a ring. Let S be an R-algebra. Then R is of *finite type* iff S is a finitely generated R-algebra.

Theorem 40.2. *Let R be a Noetherian ring. Let S be a finite-type R-algebra. Then S is a Noetherian ring.*

PROOF: $S \cong R[x_1, \ldots, x_n]/J$ for some n and some ideal J in $R[x_1, \ldots, x_n]$. We have that $R[x_1, \ldots, x_n]$ is Noetherian by Hilbert's Basis Theorem, hence $R[x_1, \ldots, x_n]/J$ is Noetherian by Proposition 38.6. $\square$

Chapter 41

Finite Algebras

Definition 41.1 (Finite Algebra)**.** Let R be a ring. Let S be an R-algebra. Then S is a *finite* R-algebra iff it is a finitely generated left-R-module.

Proposition 41.2. *Let R be a ring. Every finite R-algebra is of finite type.*

PROOF: If S is generated by $a_1, \ldots, a_n$ as an R-module, then it is generated by $a_1, \ldots, a_n$ as an R-algebra. □

Example 41.3. The converse does not hold. $R[x]$ is of finite type but is not finite.

Chapter 42

Division Algebras

Definition 42.1 (Division Algebra)**.** Let R be a commutative ring. A *division R-algebra* is an R-algebra that is a division ring.

Example 42.2. Let R be a commutative ring. Let M be a simple R-algebra. Then $\mathrm{End}_{R-\mathbf{Mod}}(M)$ is a division algebra. For if $\phi \circ \psi = 0$ then ϕ and ψ cannot both be isomorphisms, hence $\phi = 0$ or $\psi = 0$ by Schur's Lemma.

Chapter 43

Chain Complexes

Definition 43.1 (Chain Complex)**.** Let R be a ring. A *chain complex of left-R-modules* $M_\bullet = (M_\bullet, d_\bullet)$ consists of a family of left-R-modules $\{M_i\}_{i\in\mathbb{Z}}$ and a family of left-R-module homomorphisms $\{d_i : M_i \to M_{i-1}\}_{i\in\mathbb{Z}}$ such that, for all i,

$$d_i \circ d_{i+1} = 0 \ .$$

We call each d_i a *differential* and the family $\{d_i\}_i$ the *boundary* of the chain complex.

Definition 43.2 (Exact)**.** A chain complex $M_\bullet$ is *exact* at M_i iff $\operatorname{im} d_{i+1} = \ker d_i$.

It is *exact* or an *exact sequence* iff it is exact at M_i for all i.

Proposition 43.3. *A complex*

$$\cdots \to 0 \to L \xrightarrow{\alpha} M \to \cdots$$

is exact at L iff α is a monomorphism.

Proof: Since both are equivalent to $\ker\alpha = 0$. □

Proposition 43.4. *A complex*

$$\cdots \to M \xrightarrow{\beta} N \to 0 \to \cdots$$

is exact at N iff β is a epimorphism.

Proof: Since both are equivalent to $\operatorname{im}\beta = N$. □

Definition 43.5 (Short Exact Sequence)**.** A *short exact sequence* is an exact complex of the form

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0 \ .$$

Proposition 43.6 (Four-Lemma). *If*

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & D_1 \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\
A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{h_2} & D_2
\end{array}$$

is a commutative diagram of left-R-modules with exact rows, α is an epimorphism, and β and δ are monomorphisms, then γ is an monomorphism.

PROOF:
$\langle 1\rangle$1. LET: $x, y \in C_1$
$\langle 1\rangle$2. ASSUME: $\gamma(x) = \gamma(y)$
$\langle 1\rangle$3. $\delta(h_1(x)) = \delta(h_1(y))$
$\langle 1\rangle$4. $h_1(x) = h_1(y)$
PROOF: δ is injective.
$\langle 1\rangle$5. $x - y \in \ker h_1$
$\langle 1\rangle$6. $x - y \in \operatorname{im} g_1$
$\langle 1\rangle$7. PICK $b \in B_1$ such that $g_1(b) = x - y$.
$\langle 1\rangle$8. $g_2(\beta(b)) = 0$
PROOF: $g_2(\beta(b)) = \gamma(g_1(b)) = \gamma(x - y) = 0$
$\langle 1\rangle$9. $\beta(b) \in \ker g_2$
$\langle 1\rangle$10. $\beta(b) \in \operatorname{im} f_2$
$\langle 1\rangle$11. PICK $a' \in A_2$ such that $f_2(a') = \beta(b)$
$\langle 1\rangle$12. PICK $a \in A_1$ such that $\alpha(a) = a'$
PROOF: α is surjective.
$\langle 1\rangle$13. $\beta(f_1(a)) = \beta(b)$
$\langle 1\rangle$14. $f_1(a) = b$
PROOF: β is injective.
$\langle 1\rangle$15. $0 = g_1(b)$
PROOF: Since $g_1(b) = g_1(f_1(a)) = 0$.
$\langle 1\rangle$16. $x = y$
PROOF: $\langle 1\rangle$7
□

Proposition 43.7 (Four-Lemma). *If*

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & D_1 \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\
A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{h_2} & D_2
\end{array}$$

is a commutative diagram of left-R-modules with exact rows, β and δ are epimorphisms, and ϵ is a monomorphism, then γ is an epimorphism.

PROOF:
$\langle 1\rangle$1. LET: $b_2 \in B_2$

$\langle 1\rangle 2.$ PICK $c_1 \in C_1$ such that $\delta(c_1) = g_2(b_2)$
PROOF: δ is surjective.
$\langle 1\rangle 3.$ $\epsilon(h_1(c_1)) = 0$
$\langle 1\rangle 4.$ $h_1(c_1) = 0$
PROOF: ϵ is injective.
$\langle 1\rangle 5.$ $c_1 \in \ker h_1$
$\langle 1\rangle 6.$ $c_1 \in \operatorname{im} g_1$
$\langle 1\rangle 7.$ PICK $b_1 \in B_1$ such that $g_1(b_1) = c_1$
$\langle 1\rangle 8.$ $g_2(\gamma(b_1)) = g_2(b_2)$
$\langle 1\rangle 9.$ $\gamma(b_1) - b_2 \in \ker g_2$
$\langle 1\rangle 10.$ $\gamma(b_1) - b_2 \in \operatorname{im} f_2$
$\langle 1\rangle 11.$ PICK $a_2 \in A_2$ such that $f_2(a_2) = \gamma(b_1) - b_2$.
$\langle 1\rangle 12.$ PICK $a_1 \in A_1$ such that $\beta(a_1) = a_2$.
PROOF: β is surjective.
$\langle 1\rangle 13.$ $\gamma(f_1(a_1)) = \gamma(b_1) - b_2$
$\langle 1\rangle 14.$ $b_2 = \gamma(b_1 - f_1(a_1))$
$\square$

Theorem 43.8 (Snake Lemma)**.** *Suppose we have R-modules and homomorphisms*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\
0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 & \longrightarrow & 0
\end{array}$$

such that the diagram commutes and the two rows are short exact sequences. Then there exists a homomorphism $\delta : \ker \nu \to \operatorname{coker} \lambda$ such that the following is an exact sequence.

$$0 \to \ker \lambda \xrightarrow{\alpha_1} \ker \mu \xrightarrow{\beta_1} \ker \nu \xrightarrow{\delta} \operatorname{coker} \lambda \xrightarrow{\alpha_0} \operatorname{coker} \mu \xrightarrow{\beta_0} \operatorname{coker} \nu \to 0 \ .$$

PROOF:
$\langle 1\rangle 1.$ Define $\delta : \ker \nu \to \operatorname{coker} \lambda$.
$\langle 2\rangle 1.$ LET: $a \in \ker \nu$
$\langle 2\rangle 2.$ PICK $c \in M_1$ such that $\beta_1(c) = a$.
PROOF: Since β_1 is surjective.
$\langle 2\rangle 3.$ LET: $d = \mu(c)$
$\langle 2\rangle 4.$ $d \in \ker \beta_0 = \operatorname{im} \alpha_0$
PROOF: Since $\beta_0(d) = \beta_0(\mu(c)) = \nu(a) = 0$.
$\langle 2\rangle 5.$ LET: $e \in L_0$ be the element such that $\alpha_0(e) = d$.
$\langle 2\rangle 6.$ LET: $\delta(a) = e + \operatorname{im} \lambda$
$\langle 1\rangle 2.$ δ is a left-R-module homomorphism.
$\langle 2\rangle 1.$ For $a, a' \in \ker \nu$ we have $\delta(a + a') = \delta(a) + \delta(a')$.
$\langle 3\rangle 1.$ LET: $a, a' \in \ker \nu$

⟨3⟩2. LET: $c, c', c'' \in M_1$ and $e, e', e'' \in L_0$ be the elements such that

$$\begin{aligned} \beta_1(c) &= a \\ \beta_1(c') &= a' \\ \beta_1(c'') &= a + a' \\ \alpha_0(e) &= \mu(c) \\ \alpha_0(e') &= \mu(c') \\ \alpha_0(e'') &= \mu(c'') \\ \delta(a) &= e + \operatorname{im}\lambda \\ \delta(a') &= e' + \operatorname{im}\lambda \\ \delta(a+a') &= e'' + \operatorname{im}\lambda \end{aligned}$$

⟨3⟩3. $c'' - c - c' \in \ker\beta_1 = \operatorname{im}\alpha_1$
⟨3⟩4. PICK $g \in L_1$ such that $\alpha_1(g) = c'' - c - c'$.
⟨3⟩5. $\alpha_0(\lambda(g)) = \alpha_0(e'' - e - e')$
⟨3⟩6. $\lambda(g) = e'' - e - e'$
⟨3⟩7. $e'' - e - e' \in \operatorname{im}\lambda$
⟨3⟩8. $e'' + \operatorname{im}\lambda = e + e' + \operatorname{im}\lambda$
⟨3⟩9. $\delta(a + a') = \delta(a) + \delta(a')$
⟨2⟩2. For $r \in R$ and $a \in \ker\nu$ we have $\delta(ra) = r\delta(a)$.
⟨3⟩1. LET: $r \in R$ and $a \in \ker\nu$
⟨3⟩2. LET: $c, c' \in M_1$ and $e, e' \in L_0$ be the elements such that

$$\begin{aligned} \beta_1(c) &= a \\ \beta_1(c') &= ra \\ \alpha_0(e) &= \mu(c) \\ \alpha_0(e') &= \mu(c') \\ \delta(a) &= e + \operatorname{im}\lambda \\ \delta(ra) &= e' + \operatorname{im}\lambda \end{aligned}$$

⟨3⟩3. $rc - c' \in \ker\beta_1 = \operatorname{im}\alpha_1$
⟨3⟩4. PICK $g \in L_1$ such that $\alpha_1(g) = rc - c'$.
⟨3⟩5. $\alpha_0(\lambda(g)) = \alpha_0(re - e')$
⟨3⟩6. $\lambda(g) = re - e'$
⟨3⟩7. $re - e' \in \operatorname{im}\lambda$
⟨3⟩8. $re + \operatorname{im}\lambda = e' + \operatorname{im}\lambda$
⟨3⟩9. $r\delta(a) = \delta(ra)$
⟨1⟩3. The sequence is exact at $\ker\lambda$.
PROOF: Since α_1 is injective.
⟨1⟩4. The sequence is exact at $\ker\mu$.
PROOF: Since $\operatorname{im}\alpha_1 = \ker\beta_1$.
⟨1⟩5. The sequence is exact at $\ker\nu$, i.e.
$beta_1(\ker\mu) = \ker\delta$.
⟨2⟩1. LET: $a \in \ker\nu$
⟨2⟩2. LET: $c \in M_1$ and $e \in L_0$ be the elements such that $\beta_1(c) = a$, $\alpha_0(e) = \mu(c)$, and $\delta(a) = e + \operatorname{im}\lambda$.

⟨2⟩3. If $\delta(a) = \operatorname{im}\lambda$ then $a \in \beta_1(\ker\mu)$
⟨3⟩1. Assume: $\delta(a) = \operatorname{im}\lambda$
⟨3⟩2. $e \in \operatorname{im}\lambda$
⟨3⟩3. Pick $g \in L_1$ such that $\lambda(g) = e$
⟨3⟩4. $\mu(\alpha_1(g)) = \mu(c)$
⟨3⟩5. $c - \alpha_1(g) \in \ker\mu$
⟨3⟩6. $a = \beta_1(c - \alpha_1(g))$
⟨2⟩4. If $a \in \beta_1(\ker\mu)$ then $\delta(a) = \operatorname{im}\lambda$
⟨3⟩1. Assume: $c' \in \ker\mu$ and $a = \beta_1(c')$
⟨3⟩2. $c - c' \in \ker\beta_1 = \operatorname{im}\alpha_1$
⟨3⟩3. Pick $g \in L_1$ such that $\alpha_1(g) = c - c'$
⟨3⟩4. $\alpha_0(\lambda(g)) = \mu(c) - \mu(c') = \alpha_0(e) - 0 = \alpha_0(e)$
⟨3⟩5. $\lambda(g) = e$
⟨3⟩6. $e \in \operatorname{im}\lambda$
⟨3⟩7. $\delta(a) = \operatorname{im}\lambda$
⟨1⟩6. THe sequence is exact at $\operatorname{coker}\lambda$.
⟨2⟩1. Let: $e \in L_0$
Prove: $e + \operatorname{im}\lambda \in \operatorname{im}\delta$ iff $\alpha_0(e) \in \operatorname{im}\mu$.
⟨2⟩2. For all $a \in \ker\nu$, if $\delta(a) = e + \operatorname{im}\lambda$ then $\alpha_0(e) \in \operatorname{im}\mu$
Proof: From ⟨1⟩1 and the fact that α_0 is injective hence e is unique given a.
⟨2⟩3. For all $e \in L_0$, if $\alpha_0(e) \in \operatorname{im}\mu$ then $e + \operatorname{im}\lambda \in \operatorname{im}\delta$.
⟨3⟩1. Let: $e \in L_0$
⟨3⟩2. Assume: $\alpha_0(e) \in \operatorname{im}\mu$
⟨3⟩3. Pick $c \in M_1$ such that $\mu(c) = \alpha_0(e)$.
Prove: $e + \operatorname{im}\lambda = \delta(\beta_1(c))$
⟨3⟩4. Pick $c' \in M_1$ and $e' \in L_0$ such that $\beta_1(c') = \beta_1(c)$, $\alpha_0(e') = \mu(c')$ and $\delta(\beta_1(c)) = e' + \operatorname{im}\lambda$
⟨3⟩5. $c - c' \in \ker\beta_1 = \operatorname{im}\alpha_1$
⟨3⟩6. Pick $g \in L_1$ such that $\alpha_1(g) = c - c'$.
⟨3⟩7. $\alpha_0(\lambda(g)) = \alpha_0(e - e')$
⟨3⟩8. $\lambda(g) = e - e'$
⟨3⟩9. $e + \operatorname{im}\lambda = e' + \operatorname{im}\lambda = \delta(\beta_1(c))$
⟨1⟩7. The sequence is exact at $\operatorname{coker}\mu$.
Proof: Since $\operatorname{im}\alpha_0 = \ker\beta_0$.
⟨1⟩8. The sequence is exact at $\operatorname{coker}\nu$.
Proof: Since β_0 is surjective.
□

Corollary 43.8.1. *Suppose we have R-modules and homomorphisms*

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r, "\alpha_1"] \arrow[d, "\lambda"] & M_1 \arrow[r, "\beta_1"] \arrow[d, "\mu"] & N_1 \arrow[r] \arrow[d, "\nu"] & 0 \\
0 \arrow[r] & L_0 \arrow[r, "\alpha_0"] & M_0 \arrow[r, "\beta_0"] & N_0 \arrow[r] & 0
\end{tikzcd}$$

such that the diagram commutes and the two rows are short exact sequences.

Suppose μ is surjective and ν is injective. Then λ is surjective and ν is an isomorphism.

PROOF: We have $\ker\nu = \operatorname{coker}\mu = 0$ and so $0 \xrightarrow{\delta} \operatorname{coker}\lambda \xrightarrow{\alpha_0} 0$ is an exact sequence, hence $\operatorname{coker}\lambda = 0$ and so λ is surjective.
Since $\operatorname{coker}\mu = 0$ we have $0 \to \operatorname{coker}\nu \to 0$ is an exact sequence and so $\operatorname{coker}\nu = 0$, hence ν is surjective, hence ν is an isomorphism. ☐

Proposition 43.9 (Short Five-Lemma)**.** *Suppose we have R-modules and homomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\ 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 & \longrightarrow & 0 \end{array}$$

such that the diagram commutes and the two rows are short exact sequences. If λ and ν are isomorphisms then μ is an isomorphism.

PROOF:
⟨1⟩1. There exists a homomorphism $\delta : 0 \to L_0$ such that the following is an exact sequence.
$$0 \to 0 \to \ker\mu \to 0 \xrightarrow{\delta} L_0 \xrightarrow{\alpha_0} \operatorname{coker}\mu \xrightarrow{\beta_0} N_0 \to 0 \ .$$
PROOF: Snake Lemma
⟨1⟩2. $\ker\mu = 0$
⟨1⟩3. $\operatorname{coker}\mu = M_0$
☐

Proposition 43.10. *If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is an exact sequence and L and N are Noetherian then M is Noetherian.*

PROOF:
⟨1⟩1. LET: P be a submodule of M.
⟨1⟩2. PICK $a_1, \ldots, a_m$ generate $\alpha^{-1}(P)$.
⟨1⟩3. PICK $c_1, \ldots, c_n$ that generate $\beta(P)$.
⟨1⟩4. For $i = 1, \ldots, n$, PICK b_i such that $\beta(b_i) = c_i$.
PROVE: $\alpha(a_1), \ldots, \alpha(a_m), b_1, \ldots, b_n$ generate P.
⟨1⟩5. LET: $p \in P$
⟨1⟩6. PICK $r_1, \ldots, r_n \in R$ such that $r_1c_1 + \cdots + r_nc_n = \beta(p)$
⟨1⟩7. $r_1b_1 + \cdots + r_nb_n - p \in \ker\beta = \operatorname{im}\alpha$
⟨1⟩8. PICK $s_1, \ldots, s_m \in R$ such that $\alpha(s_1a_1 + \cdots + s_ma_m) = r_1b_1 + \cdots + r_nb_n - p$.
⟨1⟩9. $p = s_1\alpha(a_1) + \cdots + s_m\alpha(a_m) + r_1b_1 + \cdots + r_nb_n$
☐

Proposition 43.11. *Let R be a ring. Let*

$$0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} P \to 0$$

be a short exact sequence of left-R-modules. Let L be an R-module. Then the following is an exact sequence:

$$0 \to R{-}\mathbf{Mod}[P,L] \overset{R-\mathbf{Mod}[\beta,\mathrm{id}_L]}{\longrightarrow} R{-}\mathbf{Mod}[N,L] \overset{R-\mathbf{Mod}[\alpha,\mathrm{id}_L]}{\longrightarrow} R{-}\mathbf{Mod}[M,L] \ .$$

PROOF:
⟨1⟩1. $R-\mathbf{Mod}[\beta,\mathrm{id}_L]$ is injective.
  PROOF: Since β is epi.
⟨1⟩2. $\mathrm{im}\, R-\mathbf{Mod}[\beta,\mathrm{id}_L] = \ker R-\mathbf{Mod}[\alpha,\mathrm{id}_L]$
  ⟨2⟩1. $\mathrm{im}\, R-\mathbf{Mod}[\beta,\mathrm{id}_L] \subseteq \ker R-\mathbf{Mod}[\alpha,\mathrm{id}_L]$
    PROOF: For any $\gamma \in R{-}\mathbf{Mod}[P,L]$ we have $\gamma\circ\beta\circ\alpha = 0$ because $\beta\circ\alpha = 0$.
  ⟨2⟩2. $\ker R-\mathbf{Mod}[\alpha,\mathrm{id}_L] \subseteq \mathrm{im}\, R-\mathbf{Mod}[\beta,\mathrm{id}_L]$
    ⟨3⟩1. LET: $\gamma \in \ker R-\mathbf{Mod}[\alpha,\mathrm{id}_L]$
    ⟨3⟩2. $\gamma\circ\alpha = 0$
    ⟨3⟩3. PICK $\delta : P \to L$ by: for all $p \in P$, we have $\delta(p) = \gamma(n)$ where $n \in N$ is an element such that $\beta(n) = p$.
      PROVE: $\delta\circ\beta = \gamma$
    ⟨3⟩4. LET: $n \in N$
      PROVE: $\delta(\beta(n)) = \gamma(n)$
    ⟨3⟩5. PICK $n' \in N$ such that $\delta(\beta(n)) = \gamma(n')$ and $\beta(n') = \beta(n)$
    ⟨3⟩6. $n - n' \in \ker\beta = \mathrm{im}\,\alpha$
    ⟨3⟩7. PICK $m \in M$ such that $\alpha(m) = n - n'$
    ⟨3⟩8. $0 = \gamma(\alpha(m)) = \gamma(n) - \gamma(n')$
    ⟨3⟩9. $\gamma(n) = \gamma(n') = \delta(\beta(n))$

□

Theorem 43.12 (Nine-Lemma)**.** *Let the following be a commuting diagram of left-R-modules.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L_2 \arrow[r, "f_2"] \arrow[d, "\alpha_1"] & M_2 \arrow[r, "g_2"] \arrow[d, "\beta_1"] & N_2 \arrow[r] \arrow[d, "\gamma_1"] & 0 \\
0 \arrow[r] & L_1 \arrow[r, "f_1"] \arrow[d, "\alpha_0"] & M_1 \arrow[r, "g_1"] \arrow[d, "\beta_0"] & N_1 \arrow[r] \arrow[d, "\gamma_0"] & 0 \\
0 \arrow[r] & L_0 \arrow[r, "f_0"] \arrow[d] & M_0 \arrow[r, "g_0"] \arrow[d] & N_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the rows are exact and the two rightmost columns are exact then the left column is exact.

PROOF:
⟨1⟩1. (L_2, f_2) is the kernel of g_2, (L_1, f_1) is the kernel of g_1 and (L_0, f_0) is the kernel of g_0.

⟨1⟩2. 0 is the cokernel of g_2, g_1 and g_0.
⟨1⟩3. PICK a homomomorphism $\delta : L_0 \to 0$ such that the following is an exact sequence:

$$L_2 \overset{\beta_1 \restriction L_2}{\to} L_1 \overset{\beta_0 \restriction L_1}{\to} L_0 \overset{\delta}{\to} 0 \to 0 \to 0$$

PROOF: Snake Lemma.
⟨1⟩4. $\beta_1 \restriction L_2 = \alpha_1$
⟨1⟩5. $\beta_0 \restriction L_1 = \alpha_0$
⟨1⟩6. The following is an exact sequence:

$$0 \to L_2 \overset{\alpha_1}{\to} L_1 \overset{\alpha_0}{\to} L_0 \to 0$$

□

Theorem 43.13. *Let the following be a commuting diagram of left-R-modules.*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & L_{i+1} \arrow[r] \arrow[d, "\alpha_{i+1}"] & M_{i+1} \arrow[r] \arrow[d] & N_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_i \arrow[r] \arrow[d, "\alpha_i"] & M_i \arrow[r] \arrow[d] & N_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_{i-1} \arrow[r, "f_{i-1}"] \arrow[d] & M_{i-1} \arrow[r] \arrow[d] & N_{i-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Assume the central column is a complex and every row is an exact complex. Then the left and right columns are complexes. Further, if any two of the columns are exact, then so is the third.

PROOF:
⟨1⟩1. The left column is a complex.
⟨2⟩1. LET: $x \in L_{i+1}$
⟨2⟩2. $f_{i-1}(\alpha_i(\alpha_{i+1}(x))) = 0$
⟨2⟩3. $\alpha_i(\alpha_{i+1}(x)) = 0$
PROOF: f_{i-1} is injective.
⟨1⟩2. The right column is a complex.
⟨2⟩1. LET: $x \in N_{i+1}$
⟨2⟩2. PICK $y \in N_{i+1}$ such that $g_{i+1}(y) = x$
⟨2⟩3. $\gamma_i(\gamma_{i+1}(x)) = 0$
PROOF:

$$\begin{aligned}
\gamma_i(\gamma_{i+1}(x)) &= \gamma_i(\gamma_{i+1}(g_{i+1}(y))) \\
&= g_{i-1}(\beta_i(\beta_{i+1}(y))) \\
&= g_{i-1}(0) \\
&= 0
\end{aligned}$$

⟨1⟩3. If the left and center columns are exact then the right column is exact.
⟨2⟩1. LET: $n_i \in \ker \gamma_{i-1}$
PROVE: $n_i \in \operatorname{im} \gamma_i$
⟨2⟩2. PICK $m_i \in M_i$ such that $g_i(m_i) = n_i$
⟨2⟩3. $g_{i-1}(\beta_i(m_i)) = 0$
⟨2⟩4. $\beta_i(m_i) \in \ker g_{i-1} = \operatorname{im} f_{i-1}$
⟨2⟩5. PICK $l_{i-1} \in L_{i-1}$ such that $f_{i-1}(l_{i-1}) = \beta_i(m_i)$
⟨2⟩6. $\beta_{i-1}(f_{i-1}(l_{i-1})) = 0$
⟨2⟩7. $f_{i-2}(\alpha_{i-1}(l_{i-1})) = 0$
⟨2⟩8. $\alpha_{i-1}(l_{i-1}) = 0$
⟨2⟩9. $l_{i-1} \in \ker \alpha_{i-1} = \operatorname{im} \alpha_i$
⟨2⟩10. PICK $l_i \in L_i$ such that $\alpha_i(l_i) = l_{i-1}$
⟨2⟩11. $\beta_i(f_i(l_i)) = \beta_i(m_i)$
⟨2⟩12. $f_i(l_i) - m_i \in \ker \beta_i = \operatorname{im} \beta_{i+1}$
⟨2⟩13. PICK $m_{i+1} \in M_{i+1}$ such that $\beta_{i+1}(m_{i+1}) = f_i(l_i) - m_i$
⟨2⟩14. $\gamma_{i+1}(-g_{i+1}(m_{i+1})) = n_i$
⟨1⟩4. If the left and right columns are exact then the center column is exact.
⟨2⟩1. LET: $x \in \ker \beta_i$
PROVE: $x \in \operatorname{im} \beta_{i+1}$
⟨2⟩2. $g_{i-1}(\beta_i(x)) = 0$
⟨2⟩3. $\gamma_i(g_i(x)) = 0$
⟨2⟩4. $g_i(x) \in \ker \gamma_i = \operatorname{im} \gamma_{i+1}$
⟨2⟩5. PICK $n_{i+1} \in N_{i+1}$ such that $\gamma_{i+1}(n_{i+1}) = g_i(x)$
⟨2⟩6. PICK $m_{i+1} \in M_{i+1}$ such that $g_{i+1}(m_{i+1}) = n_{i+1}$
⟨2⟩7. $g_i(\beta_{i+1}(m_{i+1})) = g_i(x)$
⟨2⟩8. $\beta_{i+1}(m_{i+1}) - x \in \ker g_i = \operatorname{im} f_i$
⟨2⟩9. PICK $l_i \in L_i$ such that $f_i(l_i) = \beta_{i+1}(m_{i+1}) - x$
⟨2⟩10. $\beta_i(f_i(l_i)) = 0$
⟨2⟩11. $f_{i-1}(\alpha_i(l_i)) = 0$
⟨2⟩12. $\alpha_i(l_i) = 0$
⟨2⟩13. $l_i \in \ker \alpha_i = \operatorname{im} \alpha_{i+1}$
⟨2⟩14. PICK $l_{i+1} \in L_{i+1}$ such that $\alpha_{i+1}(l_{i+1}) = l_i$
⟨2⟩15. $\beta_{i+1}(f_{i+1}(l_{i+1})) = \beta_{i+1}(m_{i+1}) - x$
⟨2⟩16. $x = \beta_{i+1}(m_{i+1} - f_{i+1}(l_{i+1}))$
⟨1⟩5. If the center and right columns are exact then the left column is exact.
⟨2⟩1. LET: $l_i \in \ker \alpha_i$
PROVE: $l_i \in \operatorname{im} \alpha_{i+1}$
⟨2⟩2. $\beta_i(f_i(l_i)) = 0$
⟨2⟩3. $f_i(l_i) \in \ker \beta_i = \operatorname{im} \beta_{i+1}$
⟨2⟩4. PICK $m_{i+1} \in M_{i+1}$ such that $\beta_{i+1}(m_{i+1}) = f_i(l_i)$
⟨2⟩5. $\gamma_{i+1}(g_{i+1}(m_{i+1})) = 0$
⟨2⟩6. $g_{i+1}(m_{i+1}) \in \ker \gamma_{i+1} = \operatorname{im} \gamma_{i+2}$
⟨2⟩7. PICK $n_{i+2} \in N_{i+2}$ such that $\gamma_{i+2}(n_{i+2}) = g_{i+1}(m_{i+1})$
⟨2⟩8. PICK $m_{i+2} \in M_{i+2}$ such that $g_{i+2}(m_{i+2}) = n_{i+2}$
⟨2⟩9. $g_{i+1}(\beta_{i+2}(n_{i+2})) = g_{i+1}(m_{i+1})$
⟨2⟩10. $\beta_{i+2}(n_{i+2}) - m_{i+1} \in \ker g_{i+1} = \operatorname{im} f_{i+1}$

⟨2⟩11. PICK $l_{i+1} \in L_{i+1}$ such that $f_{i+1}(l_{i+1}) = \beta_{i+2}(n_{i+2}) - m_{i+1}$
⟨2⟩12. $f_i(\alpha_{i+1}(l_{i+1})) = -f_i(l_i)$
⟨2⟩13. $l_i = \alpha_{i+1}(-l_{i+1})$
□

Corollary 43.13.1 (Nine-Lemma)**.** *Let the following be a commuting diagram of left-R-modules.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L_2 \arrow[r, "f_2"] \arrow[d, "\alpha_1"] & M_2 \arrow[r, "g_2"] \arrow[d, "\beta_1"] & N_2 \arrow[r] \arrow[d, "\gamma_1"] & 0 \\
0 \arrow[r] & L_1 \arrow[r, "f_1"] \arrow[d, "\alpha_0"] & M_1 \arrow[r, "g_1"] \arrow[d, "\beta_0"] & N_1 \arrow[r] \arrow[d, "\gamma_0"] & 0 \\
0 \arrow[r] & L_0 \arrow[r, "f_0"] \arrow[d] & M_0 \arrow[r, "g_0"] \arrow[d] & N_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the rows are exact and the two leftmost columns are exact then the right column is exact.

Proposition 43.14. *Let the following be a commuting diagram of left-R-modules.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L_2 \arrow[r, "f_2"] \arrow[d, "\alpha_1"] & M_2 \arrow[r, "g_2"] \arrow[d, "\beta_1"] & N_2 \arrow[r] \arrow[d, "\gamma_1"] & 0 \\
0 \arrow[r] & L_1 \arrow[r, "f_1"] \arrow[d, "\alpha_0"] & M_1 \arrow[r, "g_1"] \arrow[d, "\beta_0"] & N_1 \arrow[r] \arrow[d, "\gamma_0"] & 0 \\
0 \arrow[r] & L_0 \arrow[r, "f_0"] \arrow[d] & M_0 \arrow[r, "g_0"] \arrow[d] & N_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the rows are exact and the left and right columns are exact then β_1 is monic.

PROOF: By the Snake Lemma, the following is an exact sequence

$$0 \to \ker \alpha_1 \to \ker \beta_1 \to \ker \gamma_1$$

But $\ker \alpha_1 = \ker \gamma_1 = 0$ so $\ker \beta_1 = 0$. □

Proposition 43.15. *Let the following be a commuting diagram of left-R-modules.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L_2 \arrow[r, "f_2"] \arrow[d, "\alpha_1"] & M_2 \arrow[r, "g_2"] \arrow[d, "\beta_1"] & N_2 \arrow[r] \arrow[d, "\gamma_1"] & 0 \\
0 \arrow[r] & L_1 \arrow[r, "f_1"] \arrow[d, "\alpha_0"] & M_1 \arrow[r, "g_1"] \arrow[d, "\beta_0"] & N_1 \arrow[r] \arrow[d, "\gamma_0"] & 0 \\
0 \arrow[r] & L_0 \arrow[r, "f_0"] \arrow[d] & M_0 \arrow[r, "g_0"] \arrow[d] & N_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the rows are exact and the left and right columns are exact then β_0 is epi.

PROOF: Similar. □

Proposition 43.16. *Let the following be a commuting diagram of left-R-modules.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L_2 \arrow[r, "f_2"] \arrow[d, "\alpha_1"] & M_2 \arrow[r, "g_2"] \arrow[d, "\beta_1"] & N_2 \arrow[r] \arrow[d, "\gamma_1"] & 0 \\
0 \arrow[r] & L_1 \arrow[r, "f_1"] \arrow[d, "\alpha_0"] & M_1 \arrow[r, "g_1"] \arrow[d, "\beta_0"] & N_1 \arrow[r] \arrow[d, "\gamma_0"] & 0 \\
0 \arrow[r] & L_0 \arrow[r, "f_0"] \arrow[d] & M_0 \arrow[r, "g_0"] \arrow[d] & N_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the rows are exact, the left and right columns are exact, and the central column is a complex, then the central column is exact.

PROOF:
⟨1⟩1. LET: $x \in \ker \beta_0$
PROVE: $x \in \operatorname{im} \beta_1$
⟨1⟩2. $\gamma_0(g_1(x)) = 0$
⟨1⟩3. $g_1(x) \in \ker \gamma_0 = \operatorname{im} \gamma_1$
⟨1⟩4. PICK $n_2 \in N_2$ such that $\gamma_1(n_2) = g_1(x)$
⟨1⟩5. PICK $m_2 \in M_2$ such that $g_2(m_2) = n_2$
⟨1⟩6. $g_1(\beta_1(m_2)) = g_1(x)$
⟨1⟩7. $\beta_1(m_2) - x \in \ker g_1 = \operatorname{im} f_1$
⟨1⟩8. PICK $l_1 \in L_1$ such that $f_1(l) = \beta_1(m_2) - x$.

$\langle 1 \rangle 9.$ $f_0(\alpha_0(l_1)) = 0$
$\langle 1 \rangle 10.$ $\alpha_0(l_1) = 0$
$\langle 1 \rangle 11.$ $l_1 \in \ker \alpha_0 = \operatorname{im} \alpha_1$
$\langle 1 \rangle 12.$ Pick $l_2 \in L_2$ such that $\alpha_1(l_2) = l_1$.
$\langle 1 \rangle 13.$ $\beta_1(f_2(l_2)) = \beta_1(m_2) - x$
$\langle 1 \rangle 14.$ $x = \beta_1(m_2 - f_2(l_2))$
$\square$

Example 43.17. We cannot remove the hypothesis that the central column is a complex. Consider the situation

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, equal] \arrow[d, "\Delta"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\kappa_1"] \arrow[d, equal] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This diagram commutes, the rows are exact, the left and right columns are exact, but the central column is not a complex and $\operatorname{im} \Delta \neq \ker \pi_1$.

43.1 Split Exact Sequences

Definition 43.18 (Split Sequence)**.** Let $0 \to M_1 \xrightarrow{\alpha} N \xrightarrow{\beta} M_2 \to 0$ be a short exact sequence. Then this sequence *splits* iff there exists an isomorphism

$$\phi : N \cong M_1 \oplus M_2$$

such that $\phi \circ \alpha = \kappa_1 : M_1 \to M_1 \oplus M_2$ and $\beta \circ \phi^{-1} = \pi_2 : M_1 \oplus M_2 \to M_2$.

Proposition 43.19. *Let $\phi : M \to N$ be a left-R-module homomorphism. Then ϕ has a left-inverse if and only if the sequence*

$$0 \to M \xrightarrow{\phi} N \to \operatorname{coker} \phi \to 0$$

splits.

Proof:
$\langle 1 \rangle 1.$ If ϕ has a left-inverse then the sequence splits.
$\langle 2 \rangle 1.$ Assume: ϕ has a left-inverse $\psi : N \to M$.
$\langle 2 \rangle 2.$ Define $i : N \to M \oplus \operatorname{coker} \phi$ by $i(n) = (\psi(n), n + \operatorname{im} \phi)$.

⟨2⟩3. Define $i^{-1} : M \oplus \operatorname{coker}\phi$ by $i^{-1}(m, x + \operatorname{im}\phi) = \phi(m) + x - \phi(\psi(x))$.
⟨2⟩4. $i \circ i^{-1} = \mathrm{id}_{M \oplus \operatorname{coker}\phi}$
PROOF:

$$\begin{aligned}\psi(\phi(m) + x - \phi(\psi(x))) &= m + \psi(x) - \psi(x) \\ &= m\end{aligned}$$

⟨2⟩5. $i^{-1} \circ i = \mathrm{id}_N$
PROOF:

$$\begin{aligned}i^{-1}(\psi(n), n + \operatorname{im}\phi) &= \phi(\psi(n)) + n - \phi(\psi(n)) \\ &= n\end{aligned}$$

⟨2⟩6. $i \circ \phi = \kappa_1 : M \to M \oplus \operatorname{coker}\phi$
PROOF:

$$\begin{aligned}i(\phi(m)) &= (\psi(\phi(m)), \phi(m) + \operatorname{im}\phi) \\ &= (m, \operatorname{im}\phi)\end{aligned}$$

⟨2⟩7. $\pi \circ i^{-1} = \pi_2 : M \oplus \operatorname{coker}\phi \to \operatorname{coker}\phi$
PROOF:

$$\begin{aligned}i^{-1}(\psi(n), n + \operatorname{im}\phi) + \operatorname{im}\phi &= \phi(\psi(n)) + n - \phi(\psi(n)) + \operatorname{im}\phi \\ &= n + \operatorname{im}\phi\end{aligned}$$

⟨1⟩2. If the sequence splits then ϕ has a left-inverse.
PROOF: Since $\kappa_1 : M \to M \oplus \operatorname{coker}\phi$ has left inverse π_1.
□

Proposition 43.20. *Let $\phi : M \to N$ be a left-R-module homomorphism. Then ϕ has a right-inverse if and only if the sequence*

$$0 \to \ker\phi \to M \xrightarrow{\phi} N \to 0$$

splits.

PROOF:
⟨1⟩1. If ϕ has a right-inverse then the sequence splits.
⟨2⟩1. LET: $\psi : N \to M$ be a right inverse to ϕ.
⟨2⟩2. LET: $i : M \to \ker\phi \oplus N$ be the function $i(m) = (m - \psi(\phi(m)), \phi(m))$.
PROOF: $m - \psi(\phi(m)) \in \ker\phi$ since $\phi(m - \psi(\phi(m))) = \phi(m) - \phi(m) = 0$.
⟨2⟩3. LET: $i^{-1} : \ker\phi \oplus N \to M$ be the function $i^{-1}(x, n) = x + \psi(n)$.
⟨2⟩4. $i \circ i^{-1} = \mathrm{id}_{\ker\phi \oplus N}$
PROOF:

$$\begin{aligned}i(i^{-1}(x, n)) &= i(x + \psi(n)) \\ &= (x + \psi(n) - \psi(\phi(x)) - \psi(\phi(\psi(n))), \phi(x) + \phi(\psi(n))) \\ &= (x + \psi(n) - \psi(n), n) \\ &= (x, n)\end{aligned}$$

⟨2⟩5. $i^{-1} \circ i = \mathrm{id}_M$
PROOF:

$$\begin{aligned}i^{-1}(i(m)) &= m - \psi(\phi(m)) + \psi(\phi(m)) \\ &= m\end{aligned}$$

⟨2⟩6. $i \circ \iota = \kappa_1$

PROOF: For $m \in \ker \phi$ we have $i(m) = (m - \psi(\phi(m)), \phi(m)) = (m, 0)$.

$\langle 2 \rangle 7.$ $\phi \circ i^{-1} = \pi_2$

PROOF:

$$\begin{aligned} \phi(i^{-1}(x,n)) &= \phi(x) + \phi(\psi(n)) \\ &= 0 + n \\ &= n \end{aligned}$$

$\langle 1 \rangle 2.$ If the sequence splits then ϕ has a right-inverse.

PROOF: Since $\kappa_2 : N \to M \oplus N$ is a right-inverse to π_2.

□

Proposition 43.21. *Let*

$$0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} F \to 0$$

be a short exact sequence where F is free. Then the sequence splits.

PROOF:

$\langle 1 \rangle 1.$ LET: $F = R^{\oplus A}$

$\langle 1 \rangle 2.$ PICK $\gamma : F \to N$ such that $\mathrm{id}_F = \beta \circ \gamma$

$\langle 1 \rangle 3.$ LET: $i : M \oplus F \to N$ be the homomorphism $i(m, f) = \alpha(m) + \gamma(f)$

$\langle 1 \rangle 4.$ i is injective.

$\langle 2 \rangle 1.$ ASSUME: $i(m, f) = i(m', f')$

$\langle 2 \rangle 2.$ $\alpha(m) + \gamma(f) = \alpha(m') + \gamma(f')$

$\langle 2 \rangle 3.$ $\alpha(m - m') = \gamma(f - f')$

$\langle 2 \rangle 4.$ $f - f' = 0$

PROOF: Applying β to both sides of $\langle 2 \rangle 3$.

$\langle 2 \rangle 5.$ $f = f'$

$\langle 2 \rangle 6.$ $\alpha(m - m') = 0$

$\langle 2 \rangle 7.$ $m = m'$

PROOF: Since α is injective.

$\langle 1 \rangle 5.$ i is surjective.

$\langle 2 \rangle 1.$ LET: $n \in N$

$\langle 2 \rangle 2.$ $n - \gamma(\beta(n)) \in \ker \beta = \operatorname{im} \alpha$

$\langle 2 \rangle 3.$ PICK $m \in M$ such that $\alpha(m) = n - \gamma(\beta(n))$

$\langle 2 \rangle 4.$ $n = i(m, \beta(n))$

$\langle 1 \rangle 6.$ $\alpha = i \circ \kappa_1$

$\langle 1 \rangle 7.$ $\beta \circ i = \pi_2$

□

Chapter 44

Homology

Definition 44.1 (Homology)**.** Let $(M_\bullet, d_\bullet)$ be a chain complex. The *ith homology* of the complex is the R-module

$$H_i(M_\bullet) := \frac{\ker d_i}{\operatorname{im} d_{i+1}} \ .$$

Proposition 44.2. *Consider the complex*

$$0 \to M_1 \xrightarrow{\phi} M_0 \to 0 \ .$$

The 1st homology is $\ker \phi$*, and the 0th homology is* $\operatorname{coker} \phi$*.*

Part VI

Field Theory

Chapter 45

Fields

Definition 45.1 (Field)**.** A *field* is a non-trivial commutative division ring.

Example 45.2. $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields.

Proposition 45.3. *Every field is an integral domain.*

PROOF: By Propositions 21.8 and 21.9. □

Example 45.4. The converse does not hold: $\mathbb{Z}$ is an integral domain but not a field.

Proposition 45.5. *Every finite integral domain is a field.*

PROOF: In a finite integral domain, multiplication by any non-zero element is injective, hence surjective. □

Corollary 45.5.1. *For any positive integer n, the following are equivalent:*

- *n is prime.*
- *$\mathbb{Z}/n\mathbb{Z}$ is an integral domain.*
- *$\mathbb{Z}/n\mathbb{Z}$ is a field.*

Theorem 45.6 (Wedderburn's Little Theorem)**.** *Every finite division ring is a field.*

Proposition 45.7. *Every subring of a field is an integral domain.*

PROOF: Easy. □

Proposition 45.8. *The center of a division ring is a field.*

PROOF:
$\langle 1 \rangle 1$. LET: R be a division ring.
$\langle 1 \rangle 2$. LET: Z be the center of R.
$\langle 1 \rangle 3$. Z is non-trivial.

PROOF: Since $1 \in Z$.
$\langle 1 \rangle 4.$ Z is commutative.
$\langle 1 \rangle 5.$ Z is a division ring.
$\langle 2 \rangle 1.$ LET: $a \in Z$
$\langle 2 \rangle 2.$ $a^{-1} \in Z$
$\langle 3 \rangle 1.$ LET: $x \in R$
$\langle 3 \rangle 2.$ $ax = xa$
$\langle 3 \rangle 3.$ $xa^{-1} = a^{-1}x$
□

Definition 45.9. For any prime p and positive integer r, define a multiplication on $(\mathbb{Z}/p\mathbb{Z})^r$ that makes this group into a field by:

Proposition 45.10. *A commutative ring is a field if and only if it is simple.*

PROOF: Proposition 30.5. □

Corollary 45.10.1. *Every field has Krull dimension 0.*

Proposition 45.11. *Let K be a field. Then $K[x]$ is a PID, and every non-zero ideal in $K[x]$ is generated by a unique monic polynomial.*

PROOF:
$\langle 1 \rangle 1.$ LET: I be a non-zero ideal in $K[x]$
$\langle 1 \rangle 2.$ PICK a monic polynomial $f \in K[x]$ of minimal degree.
PROVE: $I = (f)$
$\langle 1 \rangle 3.$ LET: $g \in I$
$\langle 1 \rangle 4.$ PICK polynomials q, r with $\deg r < \deg f$ such that $g = qf + r$
$\langle 1 \rangle 5.$ $r \in I$
$\langle 1 \rangle 6.$ $r = 0$
$\langle 1 \rangle 7.$ $g \in (f)$
□

Proposition 45.12. *Let R be a commutative ring and I an ideal in R. Then I is maximal iff R/I is a field.*

PROOF: From Proposition 31.3. □

Example 45.13. Let R be a commutative ring and $a \in R$. Then $(x - a)$ is a maximal ideal in $R[x]$ iff R is a field, since $R[x]/(x - a) \cong R$.

Example 45.14. The ideal $(2, x)$ is a maximal ideal in $\mathbb{Z}[x]$, since $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}/2\mathbb{Z}$.

Proposition 45.15. *Every maximal ideal in a commutative ring is a prime ideal.*

PROOF: Since every field is an integral domain. □

Proposition 45.16. *Let R be a commutative ring and I an ideal in R. If I is a prime ideal and R/I is finite then I is a maximal ideal.*

PROOF: Since every finite integral domain is a field. ∎

Proposition 45.17. *Let R be a commutative ring and I a proper ideal in R. Then I is maximal iff, whenever J is an ideal and $I \subseteq J$, then $I = J$ or $J = R$.*

Example 45.18. The inverse image of a maximal ideal under a homomorphism is not necessarily maximal.

Let $i : \mathbb{Z}[x] \to \mathbb{Q}[x]$ be the inclusion. Then (x) is maximal in $\mathbb{Q}[x]$ but its inverse image (x) is not maximal in $\mathbb{Z}[x]$.

Definition 45.19 (Maximal Spectrum)**.** Let R be a commutative ring. The *maximal spectrum* of R is the set of all maximal ideals in R.

Proposition 45.20. *Let K be a field. The Krull dimension of $K[x_1, \ldots, x_n]$ is n.*

Theorem 45.21 (Hilbert's Nullstellensatz)**.** *Let K be a field and L a subfield of K. If K is an L-algebra of finite type, then K is a finite L-algebra.*

Proposition 45.22. *Let K be a subfield of L. Then L is a K-algebra under multiplication.*

PROOF: Easy. ∎

Theorem 45.23. *Let F be a field. Let G be a finite subgroup of F^*. Then G is cyclic.*

PROOF:
$\langle 1 \rangle 1$. For every n, there are at most n elements $a \in G$ such that $a^n = 1$.
  PROOF: Since the polynomial $x^n - 1$ in $F[x]$ can have at most n linear factors $(x - a)$.
$\langle 1 \rangle 2$. Q.E.D.
  PROOF: Lemma 18.12.
∎

Chapter 46

Algebraically Closed Fields

Definition 46.1 (Algebraically Closed)**.** A field K is *algebraically closed* iff, for every $f \in K[x]$ that is not constant, there exists $r \in K$ such that $f(r) = 0$.

Theorem 46.2. *$\mathbb{C}$ is algebraically closed.*

Proposition 46.3. *Let K be an algebraically closed field. Let I be an ideal in $K[x]$. Then I is maximal if and only if $I = (x - c)$ for some $c \in K$.*

PROOF:
$\langle 1 \rangle 1$. If I is maximal then there exists $c \in K$ such that $I = (x - c)$.
  $\langle 2 \rangle 1$. ASSUME: I is maximal.
  $\langle 2 \rangle 2$. PICK f monic of minimal degree such that $f \in I$.
  $\langle 2 \rangle 3$. f is not constant.
    PROOF: Otherwise $f = 1$ and $I = K[x]$.
  $\langle 2 \rangle 4$. PICK $c \in K$ such that $f(c) = 0$
  $\langle 2 \rangle 5$. $x - c \mid f$
  $\langle 2 \rangle 6$. $I \subseteq (x - c)$
  $\langle 2 \rangle 7$. $I = (x - c)$
$\langle 1 \rangle 2$. For all $c \in K$ we have $(x - c)$ is maximal.
  PROOF: Example 45.13.
$\square$

Part VII

Linear Algebra

Chapter 47

Vector Spaces

Definition 47.1 (Vector Space)**.** Let K be a field. A *K-vector space* is a K-module. A *linear map* is a homomorphism of K-modules. We write $K - \mathbf{Vect}$ for $K - \mathbf{Mod}$.

Definition 47.2. Let $\mathrm{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ real matrices. $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication.

Definition 47.3. Let $\mathrm{GL}_n(\mathbb{C})$ be the group of invertible $n \times n$ complex matrices. $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n$ by matrix multiplication.

Definition 47.4. Let $\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : \det M = 1\}$.

Proposition 47.5. $\mathrm{SL}_n(\mathbb{R})$ *is a normal subgroup of* $\mathrm{GL}_n(\mathbb{R})$.

PROOF: If $\det M = 1$ then $\det(AMA^{-1}) = (\det A)(\det M)(\det A)^{-1} = 1$. □

Proposition 47.6.

$$\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$$

Definition 47.7. Let $\mathrm{SL}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : \det M = 1\}$.

Definition 47.8. Let $\mathrm{O}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : MM^{\mathrm{T}} = M^{\mathrm{T}}M = I_n\}$.

Proposition 47.9. *The action of* $\mathrm{O}_n(\mathbb{R})$ *on* $\mathbb{R}^n$ *preserves lengths and angles.*

Definition 47.10. Let $\mathrm{SO}_n(\mathbb{R}) = \{M \in \mathrm{O}_n(\mathbb{R}) : \det M = 1\}$.

Definition 47.11. Let $\mathrm{U}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : MM^\dagger = M^\dagger M = I_n\}$.

Definition 47.12. Let $\mathrm{SU}_n(\mathbb{C}) = \{M \in U_n(\mathbb{C}) : \det M = 1\}$.

Proposition 47.13. *Every matrix in* $\mathrm{SU}_2(\mathbb{C})$ *can be written in the form*

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$ *with* $a^2 + b^2 + c^2 + d^2 = 1$.

PROOF:
⟨1⟩1. LET: $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SU}_2(\mathbb{C})$
⟨1⟩2. $M^{-1} = M^\dagger$
⟨1⟩3. $\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix}$
⟨1⟩4. LET: $\alpha = a + bi$ and $\beta = c + di$.
⟨1⟩5. $\delta = \overline{\alpha} = a - bi$
⟨1⟩6. $\gamma = -\overline{\beta} = -c + di$
⟨1⟩7. $\det M = a^2 + b^2 + c^2 + d^2 = 1$
□

Corollary 47.13.1. $\mathrm{SU}_2(\mathbb{C})$ *is simply connected.*

Corollary 47.13.2.
$$\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{C})/\{I, -I\}$$

PROOF: The function that maps $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$ to $\begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(\\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(\\ 2(bd-ac) & 2(ab+cd) & a^2-b \end{pmatrix}$
is a surjective homomorphism with kernel $\{I, -I\}$. □

Corollary 47.13.3. *The fundamental group of* $\mathrm{SO}_3(\mathbb{R})$ *is* C_2.

Part VIII

Linear Algebra

Chapter 48

Vector Spaces

Definition 48.1 (Vector Space)**.** Let K be a field. A *vector space* over K is a module over K. A *linear transformation* is a K-module homomorphism.

Definition 48.2 (Bilinear Map)**.** Let K be a field. Let U, V and W be vector spaces over K. A function $f : U \times V \to W$ is *bilinear* iff, for all $u_1, u_2 \in U$ and $v_1, v_2 \in V$ and $\alpha \in K$,

$$f(u_1 + \alpha u_2, v_1) = f(u_1, v_1) + \alpha f(u_2, v_1)$$
$$f(u_1, v_1 + \alpha v_2) = f(u_1, v_1) + \alpha f(u_1, v_2)$$

Theorem 48.3. *Let K be a field. Let U and V be vector spaces. There exists a vector space $U \otimes V$ over K and bilinear map $- \otimes - : U \times V \to U \otimes V$, unique up to isomorphism, such that, for every vector space W over K and bilinear map $f : U \times V \to W$, there exists a unique linear map $\overline{f} : U \otimes V \to W$ such that the following diagram commutes.*

$$\begin{tikzcd}
U \otimes V \arrow[r, "\overline{f}"] & W \\
U \times V \arrow[u, "- \otimes -"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Further, $- \otimes -$ is injective and its image spans $U \otimes V$.

Proof: We can construct $U \otimes V$ as follows. Let L be the free vector space generated by $U \times V$. Let R be the subspace generated by all vectors of the form
$$(u_1 + \alpha u_2, v) - (u_1, v) - \alpha(u_2, v)(u, v_1 + \alpha v_2) - (u, v_1) - \alpha(u, v_2)$$
Take $U \otimes V := L/R$. □

Proposition 48.4. *If $\sum_{i=1}^n u_i \otimes v_i = 0$ and $v_1, \ldots, v_n$ are linearly independent in V then $u_1 = \cdots = u_n = 0$.*

Proof:
⟨1⟩1. Let: $f : U \times V \to V^{U^*}$ be the function $f(u, v)(\Phi) = \Phi(u)v$

$\langle 1 \rangle 2.$ f is bilinear.
$\langle 1 \rangle 3.$ LET: $\overline{f} : U \otimes V \to V^{U^*}$ be the induced linear transformation.
$\langle 1 \rangle 4.$ $\overline{f}(\sum_{i=1}^{n} u_i \otimes v_i) = 0$
$\langle 1 \rangle 5.$ $\sum_{i=1}^{n} f(u_i, v_i) = 0$
$\langle 1 \rangle 6.$ For all $\Phi \in U^*$ we have $\sum_{i=1}^{n} \Phi(u_i) v_i = 0$
$\langle 1 \rangle 7.$ For all $\Phi \in U^*$ we have $\Phi(u_1) = \cdots = \Phi(u_n) = 0$
$\langle 1 \rangle 8.$ $u_1 = \cdots = u_n = 0$
□

Proposition 48.5. *Let U and V be vector spaces over K with bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Then $\mathcal{B} = \{b_1 \otimes b_2 : b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2\}$ is a basis for $U \otimes V$.*

PROOF:
$\langle 1 \rangle 1.$ $\mathcal{B}$ is linearly independent.
 $\langle 2 \rangle 1.$ ASSUME: $\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} b_i \otimes b'_j = 0$
 $\langle 2 \rangle 2.$ For all j we have $\sum_{i=1}^{m} \alpha_{ij} b_i = 0$
  PROOF: Proposition 48.4.
 $\langle 2 \rangle 3.$ Each α_{ij} is 0.
$\langle 1 \rangle 2.$ $\mathcal{B}$ spans $U \otimes V$.
 PROOF: If $u = \alpha_1 b_1 + \cdots + \alpha_m b_m$ and $v = \beta_1 b'_1 + \cdots + \beta_n b'_n$ then

$$u \otimes v = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j (b_i \otimes b'_j)$$

 The result follows since the vectors of the form $u \otimes v$ span $U \otimes V$.
□

Corollary 48.5.1. *If U and V are finite dimensional vector spaces over K then*

$$\dim(U \otimes V) = (\dim U)(\dim V) \enspace .$$

Proposition 48.6. $\mathbf{Vect}_K$ *is a symmetric monoidal category under $\otimes$.*

48.1 Dual Spaces

Definition 48.7. Given vector spaces U and V over K, we make $\mathbf{Vect}_K[U, V]$ into a K-vector space by defining:

$$(S + T)(u) = S(u) + T(u)$$
$$(\alpha T)(u) = \alpha T(u)$$

Definition 48.8 (Dual Space)**.** Given a vector space V over K, the *dual space* V^* is

$$V^* = \mathbf{Vect}_K[V, K] \enspace .$$

An element of V^* is called a *linear functional* on V.

Proposition 48.9. *The natural transformation*

$$\eta_V : V \to V^{**}$$

defined by

$$\eta_V(v)(f) = f(v) \ .$$

is a natural isomorphism if V is finite dimensional.

PROOF:
$\langle 1\rangle 1$. For all $v \in V$ we have $\eta_V(v) \in V^{**}$.
PROOF:

$$\begin{aligned}\eta_V(v)(f + \lambda g) &= (f + \lambda g)(v) \\ &= f(v) + \lambda g(v) \\ &= \eta_V(v)(f) + \lambda \eta_V(v)(g)\end{aligned}$$

$\langle 1\rangle 2$. η_V is linear.
PROOF:

$$\begin{aligned}\eta_V(u + \lambda v)(f) &= f(u + \lambda v) \\ &= f(u) + \lambda f(v) \\ &= \eta_V(u)(f) + \lambda \eta_V(v)(f)\end{aligned}$$

$\langle 1\rangle 3$. η_V is natural in V.
$\langle 2\rangle 1$. LET: $T : U \to V$ be a linear transformation.
$\langle 2\rangle 2$. $\eta_V \circ T = \mathbf{Vect}_K[\mathbf{Vect}_K[T, \mathrm{id}_K], \mathrm{id}_K] \circ \eta_U$
PROOF:

$$\begin{aligned}\eta_V(T(u))(f) &= f(T(u)) \\ (\mathbf{Vect}_K[\mathbf{Vect}_K[T, \mathrm{id}_K], \mathrm{id}_K](\eta_U(u)))(f) &= \eta_U(u)(\mathbf{Vect}_K[T, \mathrm{id}_K](f)) \\ &= \eta_U(u)(f \circ T) \\ &= f(T(u))\end{aligned}$$

$\langle 1\rangle 4$. $\ker \eta_V = \{0\}$
PROOF:

$$\begin{aligned}v \in \ker \eta_V &\Leftrightarrow \forall f \in V^*. f(v) = 0 \\ &\Leftrightarrow v = 0\end{aligned}$$

using the fact that V is finite dimensional.
$\langle 1\rangle 5$. $\mathrm{im}\, \eta_V = V^{**}$
$\langle 2\rangle 1$. PICK a basis $\{|e_1\rangle, \ldots, |e_n\rangle\}$ for V.
$\langle 2\rangle 2$. LET: $f \in V^{**}$
$\langle 2\rangle 3$. For $i = 1, \ldots, n$,
LET: $\alpha_i = f(\langle e_i|)$
$\langle 2\rangle 4$. LET: $v = \sum_i \alpha_i |e_i\rangle$
$\langle 2\rangle 5$. $f = \eta_V(v)$
PROOF:

$$\begin{aligned}f(\langle e_i|) &= \alpha_i \\ &= \langle e_i| (v) \\ &= \eta_V(v)(\langle e_i|)\end{aligned}$$

□

48.2 Eigenvalues and Eigenvectors

Definition 48.10 (Eigenvalue, Eigenvector)**.** Let V be a vector space over K and $T : V \to V$ a linear transformation. Then $v \in V$ is an *eigenvector* of T with *eigenvalue* λ iff

$$T(v) = \lambda v \ .$$

Proposition 48.11. *For $\lambda \in K$, the set of all eigenvectors with eigenvalue λ forms a subspace of V.*

Proof: If u and v are λ-eigenvectors then

$$\begin{aligned} T(u + \alpha v) &= T(u) + \alpha T(v) \\ &= \lambda u + \alpha \lambda v \\ &= \lambda(u + \alpha v) \end{aligned} \qquad \square$$

Definition 48.12 (Eigenspace)**.** For $\lambda \in K$, the *eigenspace* of λ is the subspace of all eigenvectors with eigenvalue λ.

Definition 48.13 (Degenerate)**.** We say $\lambda \in K$ is a *degenerate* eigenvalue iff its eigenspace has dimension > 1, and *non-degenerate* iff its eigenspace has dimension 1.

48.3 Commutators and Anticommutators

Definition 48.14 (Commutator)**.** The *commutator* of linear transformations $S, T : V \to V$ is $[S, T] = ST - TS$.

Definition 48.15 (Anticommutator)**.** The *antycommutator* of linear transformations $S, T : V \to V$ is $\{S, T\} = ST + TS$.

Chapter 49

Inner Product Spaces

Definition 49.1 (Orthogonal)**.** Vectors u and v are *orthogonal* iff $\langle u \mid v\rangle = 0$.

Definition 49.2 (Norm)**.** The *norm* of a vector v is defined by

$$\|v\| = \langle v \mid v\rangle \ .$$

Proposition 49.3. *Let V and W be complex inner product spaces and $T : V \to W$ linear. If V is finite dimensional then T is bounded.*

PROOF:
$\langle 1\rangle$1. PICK an orthonormal basis $|v_1\rangle, \ldots, |v_n\rangle$.
$\langle 1\rangle$2. LET: $B = \max(\|T\,|v_1\rangle\,\|, \ldots, \|T\,|v_n\rangle\,\|)$
$\langle 1\rangle$3. For all $v \in V$ we have $\|Tv\| \leq B\|v\|$
PROOF:

$$\begin{aligned}\|T(\alpha_1 v_1 + \cdots + \alpha_n v_n)\| &= \|\alpha_1 T v_1 + \cdots + \alpha_n T v_n\| \\ &\leq |\alpha_1|\|Tv_1\| + \cdots + |\alpha_n|\|Tv_n\| \\ &\leq B(|\alpha_1| + \cdots + |\alpha_n|) \\ &= B\|\alpha_1 v_1 + \cdots + \alpha_n v_n\|\end{aligned}$$

$\square$

Definition 49.4 (Bra)**.** Given a vector $|v\rangle \in V$, define the *bra* $\langle v| \in V^*$ by:

$$(\langle v|)(|u\rangle) = \langle v|u\rangle \ .$$

Proposition 49.5.

$$\begin{aligned}\langle u + v| &= \langle u| + \langle v| \\ \langle \alpha v| &= \overline{\alpha}\,\langle v|\end{aligned}$$

Proposition 49.6 (Schwarz Inequality)**.** *Let V be an inner product space and $|\alpha\rangle\,, |\beta\rangle \in V$. Then*

$$\|\alpha\|^2\|\beta\|^2 \geq |\langle \alpha \mid \beta\rangle|^2 \ .$$

PROOF:

⟨1⟩1. For all $\lambda \in \mathbb{C}$ we have

$$(\langle\alpha| + \overline{\lambda}\,\langle\beta|)(|\alpha\rangle + \lambda\,|\beta\rangle) \geq 0 \ .$$

⟨1⟩2. $\langle\alpha \mid \alpha\rangle\langle\beta \mid \beta\rangle - |\langle\alpha \mid \beta\rangle|^2 \geq 0$

PROOF: Taking $\lambda = -\langle\beta \mid \alpha\rangle\langle\beta \mid \beta\rangle$ in ⟨1⟩1.

□

Chapter 50

Hilbert Spaces

Definition 50.1 (Hilbert Space)**.** A *Hilbert space* is a complete complex inner product space.

Definition 50.2 (Separable)**.** A Hilbert space is *separable* iff there exists a countable orthonormal basis that is dense.

Definition 50.3 (Bra)**.** Let $\mathcal{H}$ be a Hilbert space and $|\psi\rangle \in \mathcal{H}$. We define the *bra* $\langle\psi| : \mathcal{H} \to \mathbb{C}$ to be the linear functional

$$\langle\psi| \left(|\phi\rangle\right) = \langle\psi|\phi\rangle \ .$$

Proposition 50.4.

$$\langle\psi + \phi| = \langle\psi| + \langle\phi|$$

Proof: Since $\langle\psi + \phi \mid \chi\rangle = \langle\psi \mid \chi\rangle + \langle\phi \mid \chi\rangle$. □

Proposition 50.5.

$$\langle\alpha\psi| = \overline{\alpha} \langle\psi|$$

Proof: Since $\langle\alpha\psi \mid \chi\rangle = \overline{\alpha}\langle\psi \mid \chi\rangle$. □

Definition 50.6 (Outer Product)**.** Let $\mathcal{H}$ be a Hilbert space and $|\phi\rangle, |\psi\rangle \in \mathcal{H}$. Define the *outer product*

$$|\psi\rangle \langle\phi| : \mathcal{H} \to \mathcal{H}$$

to be the linear transformation

$$(|\psi\rangle \langle\phi|)(|\chi\rangle) = \langle\phi \mid \chi\rangle |\psi\rangle \ .$$

Proposition 50.7 (Completeness Relation)**.** *If $\{|e_1\rangle, \ldots, |e_n\rangle\}$ is an orthonormal basis for $\mathcal{H}$, then*

$$\sum_{i=1}^{n} |e_i\rangle \langle e_i| = \mathrm{id}_{\mathcal{H}} \ .$$

PROOF:

$$\left(\sum_{i=1}^{n} |e_i\rangle \langle e_i|\right) \sum_{j=1}^{n} \alpha_j |e_j\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_j \langle e_i \mid e_j\rangle |e_j\rangle$$
$$= \sum_{j=1}^{n} \alpha_j |e_j\rangle \qquad \square$$

Proposition 50.8. *Given a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$, there exists a unique linear operator $T^\dagger : \mathcal{H} \to \mathcal{H}$ such that, for all $|\phi\rangle, |\psi\rangle \in \mathcal{H}$,*

$$\langle T\psi \mid \phi\rangle = \langle \psi \mid T^\dagger \mid \phi\rangle$$

Definition 50.9 (Adjoint)**.** Given a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$, the *adjoint* of T is the linear operator $T^\dagger : \mathcal{H} \to \mathcal{H}$ such that, for all $|\phi\rangle, |\psi\rangle \in \mathcal{H}$,

$$\langle T\psi \mid \phi\rangle = \langle \psi \mid T^\dagger \mid \phi\rangle$$

Proposition 50.10.

$$(S \circ T)^\dagger = T^\dagger \circ S^\dagger$$

PROOF:

$$\langle \psi \mid (S \circ T)^\dagger \mid \phi\rangle = \langle S(T\psi) \mid \phi\rangle$$
$$= \langle T\psi \mid S^\dagger \phi\rangle$$
$$= \langle T\psi \mid T^\dagger(S^\dagger \phi)\rangle \qquad \square$$

Proposition 50.11.

$$\overline{\langle \alpha \mid T \mid \beta\rangle} = \langle \beta \mid T^\dagger \mid \alpha\rangle$$

PROOF: Immediate from definitions. $\square$

Definition 50.12 (Hermitian)**.** A linear operator $T : \mathcal{H} \to \mathcal{H}$ is *Hermitian* or *self-adjoint* iff T is bounded and $T^\dagger = T$.

Proposition 50.13. *If T is Hermitian then*

$$\overline{\langle \alpha \mid T \mid \beta} = \langle \beta \mid T \mid \alpha\rangle$$

PROOF: Proposition 50.11. $\square$

Theorem 50.14. *The eigenvalues of a Hermitian operator are real.*

PROOF:
$\langle 1\rangle 1$. LET: T be Hermitian.
$\langle 1\rangle 2$. LET: $T|v\rangle = \alpha |v\rangle$
$\langle 1\rangle 3$. ASSUME: w.l.o.g. $\| |v\rangle \| = 1$
$\langle 1\rangle 4$. $\langle v \mid T \mid v\rangle = \alpha$
$\langle 1\rangle 5$. $\alpha = \overline{\alpha}$

Proof:

$$\begin{aligned}\overline{\alpha} &= \overline{\langle v \mid T \mid v \rangle} \\ &= \langle v \mid T^\dagger \mid v \rangle \\ &= \langle v \mid T \mid v \rangle \\ &= \alpha\end{aligned}$$

□

Theorem 50.15. *Let T be a Hermitian operator. Then eigenvectors of T with different eigenvalues are orthogonal.*

Proof:
$\langle 1 \rangle 1$. Let: $T |u\rangle = \alpha |u\rangle$ and $T |v\rangle = \beta |v\rangle$ where $\alpha \neq \beta$.
$\langle 1 \rangle 2$. $\alpha \langle u \mid v \rangle = \beta \langle u \mid v \rangle$
Proof:

$$\begin{aligned}\alpha \langle u \mid v \rangle &= \langle u \mid T \mid v \rangle \\ &= \overline{\langle v \mid T \mid u \rangle} \\ &= \beta \overline{\langle v \mid u \rangle} && (\beta \text{ is real}) \\ &= \beta \langle u \mid v \rangle\end{aligned}$$

$\langle 1 \rangle 3$. $\langle u \mid v \rangle = 0$
□

Definition 50.16 (Projection)**.** A linear operator $P : \mathcal{H} \to \mathcal{H}$ is a *projection* iff $P = P^2 = P^\dagger$.

Definition 50.17 (Unitary)**.** A linear operator $U : \mathcal{H} \to \mathcal{H}$ is *unitary* iff $U^\dagger U = \mathrm{id}_{\mathcal{H}}$.

Definition 50.18 (Trace)**.** Let $\{|e_1\rangle, \ldots, |e_n\rangle\}$ be an orthonormal basis for $\mathcal{H}$. The *trace* of a linear operator $T : \mathcal{H} \to \mathcal{H}$ is

$$\mathrm{Tr}(T) = \sum_{i=1}^{n} \langle e_i \mid T \mid e_i \rangle \ .$$

Definition 50.19 (Positive Semidefinite)**.** A Hermitian operator $T : \mathcal{H} \to \mathcal{H}$ is *positive semidefinite* iff, for all $\psi \in \mathcal{H}$, we have

$$\langle \psi \mid T \mid \psi \rangle \geq 0 \ .$$

Definition 50.20 (Outer Product)**.** Given vectors $|\alpha\rangle, |\beta\rangle \in H$, the *outer product* $|\beta\rangle \langle \alpha| : H \to H$ is the linear transformation defined by

$$(|\beta\rangle \langle \alpha|)(|\gamma\rangle) = \langle \alpha \mid \gamma \rangle |\beta\rangle \ .$$

Proposition 50.21.

$$(|\beta\rangle \langle \alpha|)^\dagger = |\alpha\rangle \langle \beta|$$

PROOF:
$$\begin{aligned}\langle\psi \mid \alpha\rangle\langle\beta \mid \phi\rangle &= \langle\langle\alpha \mid \psi\rangle\beta \mid \phi\rangle \\ &= \langle(|\beta\rangle\,\langle\alpha|)(|\psi\rangle) \mid \phi\rangle \qquad \square\end{aligned}$$

Definition 50.22 (Expectation Value)**.** Let $A : H \to H$ be a linear transformation and $|\alpha\rangle \in H$ be normalized. The *expectation* value of A with respect to $|\alpha\rangle$ is
$$\langle A\rangle_{|\alpha\rangle} = \langle\alpha \mid A \mid \alpha\rangle \ .$$

Proposition 50.23. *The expectation value of a Hermitian operator is real.*

PROOF: Since $\langle\alpha \mid A \mid \alpha\rangle = \overline{\langle\alpha \mid A \mid \alpha\rangle}$. $\square$

Proposition 50.24. *The expectation value of an anti-Hermitian operator is purely imaginary.*

PROOF: If A is anti-Hermitian then
$$\begin{aligned}\overline{\langle\alpha \mid A \mid \alpha\rangle} &= \langle\alpha \mid A^\dagger \mid \alpha\rangle \\ &= -\langle\alpha \mid A \mid \alpha\rangle \qquad \square\end{aligned}$$

Definition 50.25 (Dispersion)**.** Let A be a Hermitian operator on H and $|\alpha\rangle \in H$ be normalized. Define
$$\Delta A = A - \langle A\rangle_{|\alpha\rangle} \ .$$
The *dispersion*, *variance* or *mean square deviation* of A is $\langle(\Delta A)^2\rangle_{|\alpha\rangle}$.

Proposition 50.26.
$$\langle(\Delta A)^2\rangle_{|\alpha\rangle} = \langle A^2\rangle_{|\alpha\rangle} - \langle A\rangle^2_{|\alpha\rangle}$$

PROOF:
$$\begin{aligned}\langle(\Delta A)^2\rangle &= \langle A^2 - 2\langle A\rangle A + \langle A\rangle^2\rangle \\ &= \langle A^2\rangle - 2\langle A\rangle^2 + \langle A\rangle^2 \\ &= \langle A^2\rangle - \langle A\rangle^2 \qquad \square\end{aligned}$$

Theorem 50.27 (Uncertainty Relation)**.** *Let A and B be Hermitian operators on H and $|\alpha\rangle \in H$ be normalized. Then*
$$\langle(\Delta A)^2\rangle_{|\alpha\rangle}\langle(\Delta B)^2\rangle_{|\alpha\rangle} \geq \frac{1}{4}|\langle[A, B]\rangle_{|\alpha\rangle}|^2 \ .$$

PROOF:
⟨1⟩1. $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle\Delta A\Delta B\rangle|^2$
PROOF:
$$\begin{aligned}\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle &= \langle\Delta A\alpha \mid \Delta A\alpha\rangle\langle\Delta B\alpha \mid \Delta B\alpha\rangle \\ &\geq |\langle\Delta A\alpha \mid \Delta B\alpha\rangle|^2 && \text{(Schwarz inequality)} \\ &= |\langle\Delta A\Delta B\rangle|\end{aligned}$$
⟨1⟩2. $\Delta A\Delta B = \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\}$

$\langle 1\rangle 3.$ $[\Delta A, \Delta B] = [A, B]$
PROOF:

$$\begin{aligned}[\Delta A, \Delta B] &= \Delta A \Delta B - \Delta B \Delta A \\ &= (AB - \langle A\rangle B - \langle B\rangle A + \langle A\rangle\langle B\rangle) \\ &- (BA - \langle A\rangle B - \langle B\rangle A + \langle A\rangle\langle B\rangle) \\ &= AB - BA \\ &= [A, B]\end{aligned}$$

$\langle 1\rangle 4.$ $[A, B]$ is anti-Hermitian.
PROOF:

$$\begin{aligned}[A, B]^\dagger &= (AB - BA)^\dagger \\ &= B^\dagger A^\dagger - A^\dagger B^\dagger \\ &= BA - AB \\ &= -[A, B]\end{aligned}$$

$\langle 1\rangle 5.$ $\langle \Delta A \Delta B\rangle = \frac{1}{2}\langle [A, B]\rangle + \frac{1}{2}\langle\{\Delta A, \Delta B\}\rangle$
$\langle 1\rangle 6.$ $\langle [A, B]\rangle$ is purely imaginary.
$\langle 1\rangle 7.$ $\{\Delta A, \Delta B\}$ is Hermitian.
PROOF:

$$\begin{aligned}\{\Delta A, \Delta B\}^\dagger &= (\Delta A \Delta B + \Delta B \Delta A)^\dagger \\ &= \Delta B \Delta A + \Delta A \Delta B \\ &= \{\Delta A, \Delta B\}\end{aligned}$$

$\langle 1\rangle 8.$ $\langle\{\Delta A, \Delta B\}\rangle$ is real.
$\langle 1\rangle 9.$ $|\langle \Delta A \Delta B\rangle|^2 = \frac{1}{4}|\langle [A, B]\rangle|^2 + \frac{1}{4}|\langle\{\Delta A, \Delta B\}\rangle|^2$
PROOF: From $\langle 1\rangle 5$, $\langle 1\rangle 6$ and $\langle 1\rangle 8$.
$\langle 1\rangle 10.$

$$\langle (\Delta A)^2\rangle\langle (\Delta B)^2\rangle \geq \frac{1}{4}|\langle [A, B]\rangle|^2 \ .$$

PROOF: From $\langle 1\rangle 1$ and $\langle 1\rangle 9$.
□

Definition 50.28 (Unitary)**.** Let H be a Hilbert space and $U : H \to H$ be a bounded linear transformation. Then U is *unitary* iff $U^\dagger U = I$ and $UU^\dagger = I$.

Proposition 50.29. *Let $\{|e_1\rangle, \ldots, |e_n\rangle\}$ and $\{|f_1\rangle, \ldots, |f_n\rangle\}$ be orthonormal bases. Then the operator U such that $U|e_i\rangle = |f_i\rangle$ for all i is unitary.*

PROOF:
$\langle 1\rangle 1.$ $U = \sum_i |f_i\rangle\langle e_i|$
$\langle 1\rangle 2.$ $U^\dagger U = I$
PROOF:

$$\begin{aligned}U^\dagger U &= \sum_{i,j} |e_i\rangle\langle f_i \mid f_j\rangle\langle e_j| \\ &= \sum_i |e_i\rangle\langle e_i| \\ &= I & \text{(Completeness Relation)}\end{aligned}$$

$\langle 1 \rangle 3.$ $UU^\dagger = I$
PROOF: Similar.
$\square$

Chapter 51

Lie Algebras

Definition 51.1 (Lie Algebra)**.** Let K be a field. A *Lie algebra* $\mathcal{L}$ over K is a vector space over K with an operation

$$[\ ,\ ] : \mathcal{L}^2 \to \mathcal{L}\ ,$$

the *Lie bracket* or *commutator*, such that, for all $\alpha \in K$ and $x, y, z \in \mathcal{L}$:

$$\begin{aligned}
[x + y, z] &= [x, z] + [y, z] \\
[x, y + z] &= [x, y] + [x, z] \\
[\alpha x, y] &= \alpha[x, y] \\
[x, x] &= 0 \\
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0
\end{aligned}$$

The last equation is called the *Jacobi identity*.

Proposition 51.2. *If K is either $\mathbb{R}$ or $\mathbb{C}$, then the condition $[x, x] = 0$ is equivalent to the* skew-symmetry condition*:*

$$[x, y] = -[y, x]\ .$$

Example 51.3. $\mathbb{R}^3$ is a Lie algebra under the cross product.

Example 51.4. $\mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{gl}_n(\mathbb{C})$ are Lie algebras under

$$[A, B] = AB - BA\ .$$

A sub-Lie algebra of one of these is called a *linear* Lie algebra.

Example 51.5. $\mathfrak{sl}_n(\mathbb{R})$ is a linear Lie algebra.

Example 51.6. $\mathfrak{so}_n(\mathbb{R})$ is a linear Lie algebra.

Example 51.7. The set $u(n)$ of skew-Hermitian $n \times n$ matrices is a real linear Lie algebra.

Example 51.8. $su(n) = \{M \in u(n) : \operatorname{tr} U = 0\}$ is a sub-Lie algebra of $u(n)$.

Definition 51.9 (Pauli matrices)**.** The *Pauli matrices* are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Proposition 51.10. *The Pauli matrices span $su(2)$.*

Proposition 51.11.

$$[\sigma_x, \sigma_y] = \sigma_z$$
$$[\sigma_y, \sigma_z] = \sigma_x$$
$$[\sigma_z, \sigma_x] = \sigma_y$$

Corollary 51.11.1. *Any two of the Pauli matrices generate $su(2)$ as a Lie algebra.*

51.1 Lie Algebra Homomorphisms

Definition 51.12 (Lie Algebra Homomorphism)**.** A *Lie algebra homomorphism* is a linear transformation that preserves the Lie bracket.

Let $\mathbf{Lie}_K$ be the category of Lie algebras over K.

Proposition 51.13. *The forgetful functor $\mathbf{Lie}_K \to \mathbf{Set}$ reflects isomorphisms; i.e. bijective Lie homomorphisms are isomorphisms.*

Example 51.14. $su(2)$ is isomorphic to $\mathbb{R}^3$ under the cross product, with the isomorphism given by

$$\sigma_x \mapsto \vec{i}, \sigma_y \mapsto \vec{j}, \sigma_z \mapsto \vec{k}$$

Chapter 52

Lie Groups

Definition 52.1 (Lie Group)**.** A *Lie group* is a group object in the category of analytic differentiable manifolds.

Example 52.2. $GL(n, \mathbb{C})$ is a Lie group.

Example 52.3. $U(n)$ is a Lie subgroup of $GL(n, \mathbb{C})$.

Example 52.4. $SU(n)$ is a Lie subgroup of $U(n)$.

Proposition 52.5. *The forgetful functor* **LieGroup** $\to$ **Set** *reflects isomorphisms.*

Definition 52.6 (Tangent Vector at the Identity)**.** Let G be a Lie group. Let N be a neighbourhood of e. The set of *tangent vectors at the identity* at N is the set of curves $\gamma : \mathbb{R} \to N$ such that $\gamma(0) = e$ quotiented by: $\gamma_1 = \gamma_2$ iff, for every smooth function $f : N \to \mathbb{R}$,

$$(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0) \ \ .$$

Proposition 52.7. *The function that maps γ to*

$$\lambda f.(f \circ \gamma)'(0)$$

is a bijection between the tangent vectors at the identity and the set of derivations on the space of the smooth functions $N \to \mathbb{R}$.

Definition 52.8 (Tangent Space)**.** The *tangent space* is the set of tangent vectors at the identity under the vector space structure inherited from the space of derivations on $C^\infty[N, \mathbb{R}]$.

Proposition 52.9. *The dimension of the tangent space is the same as the dimension of G.*

Part IX

Topology

Proposition 52.10. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Then there is a product $\prod_i X_i$ in* **Top** *which consists of the set $\prod_i X_i$ under the topology generated by ${\pi_i}^{-1}(U)$ for all $i \in I$ and U open in X_i.*

Part X

Measure Theory

Definition 52.11 (σ-algebra)**.** Let X be a set. A *σ-algebra* on X is a nonempty set $\Sigma \subseteq \mathcal{P}X$ that is closed under complement, countable union, and countable intersection.

A *measurable space* consists of a set with a σ-algebra.

Definition 52.12 (Measure)**.** Let (X, σ) be a measurable space. A *measure* on (X, σ) is a function $\mu : \Sigma \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that:

- $\mu(\emptyset) = 0$
- For any countable set of pairwise disjoint sets $\{E_n : n \in \mathbb{N}\}$ in Σ,

$$\mu\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \mu(E_n) \ .$$

Part XI

Quantum Theory

Associated with any physical system S is a Hilbert space H_S such that the states of S correspond to the non-zero vectors in H_S quotiented by: $|\alpha\rangle$ and $c\,|\alpha\rangle$ represent the same physical state ($c \neq 0$).

Associated with any *observable* property P of the system S is a Hermitian operator $T_P : H_S \to H_S$.

If we measure the property P, then the value measured is an eigevalue of T_P, and after the measurement the state is an eigenvector of T_P with that eigenvalue.

Example 52.13. A *spin*-1/2 *system* has a 2-dimensional Hilbert space with basis $\{|+\rangle, |-\rangle\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$.

We have the observables

$$\begin{aligned}
S_x &= \frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|) \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
S_y &= \frac{\hbar}{2}(-i\,|+\rangle\langle-| + i\,|-\rangle\langle+|) \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
S_z &= \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|) \\
&= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
S_+ &= \hbar\,|+\rangle\langle-| \\
&= \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= S_x + iS_y \\
S_- &= \hbar\,|-\rangle\langle+| \\
&= \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= S_x - iS_y \\
\vec{S}^2 &= S_x^2 + S_y^2 + S_z^2 \\
&= \frac{3}{4}\hbar^2 I
\end{aligned}$$

The eigenstates of S_x are $\frac{1}{\sqrt{2}}\,|+\rangle + \frac{1}{\sqrt{2}}\,|-\rangle$ with eigenvalue $\hbar/2$, and $\frac{1}{\sqrt{2}}\,|+\rangle - \frac{1}{\sqrt{2}}\,|-\rangle$ with eigenvalue $-\hbar/2$.

The eigenstates of S_y are $\frac{1}{\sqrt{2}}\,|+\rangle + \frac{i}{\sqrt{2}}\,|-\rangle$ with eigenvalue $\hbar/2$ and $\frac{1}{\sqrt{2}}\,|+\rangle - \frac{i}{\sqrt{2}}\,|-\rangle$ with eigenvalue $-\hbar/2$.

The eigenstates of S_z are $|+\rangle$ with eigenvalue $\hbar/2$ and $|-\rangle$ with eigenvalue $-\hbar/2$.

For $a, b, c \in \{x, y, z\}$ we have

$$\begin{aligned} [S_a, S_b] &= i\epsilon_{abc}\hbar S_c \\ \{S_a, S_b\} &= \frac{\hbar}{2}\delta_{ab} \\ [\vec{S}^2, S_a] &= 0 \end{aligned}$$

where $\epsilon_{abc} = 1$ if a, b, c are a cyclic permutation of x, y, z; $\epsilon_{abc} = -1$ if a, b, c are a cyclic permutation of y, x, z; and 0 if a, b and c are not all distinct.

Definition 52.14 (Compatible Observables)**.** Observables A and B are *compatible* iff $[A, B] = 0$; otherwise they are *incompatible*.

Theorem 52.15. *Suppose that A and B are compatible observables, and the eigenvalues of A are non-degenerate. Assume the eigenvectors of A span H. Pick an orthonormal basis of eigenvectors of A. With respect to this basis, the matrix of B is diagonal.*

PROOF:
$\langle 1\rangle 1.$ LET: $|u\rangle$ and $|v\rangle$ be distinct basis elements with A-eigenvalues α and β.
$\langle 1\rangle 2.$ $\langle u \mid [A, B] \mid v\rangle = 0$
$\langle 1\rangle 3.$ $(\alpha - \beta)\langle u \mid B \mid v\rangle = 0$
PROOF:

$$\begin{aligned} \langle u \mid [A, B] \mid v\rangle &= \langle u \mid AB \mid v\rangle - \langle u \mid BA \mid v\rangle \\ &= \langle Au \mid B \mid v\rangle - \langle u \mid B \mid Av\rangle \qquad (A \text{ is Hermitian}) \\ &= \alpha\langle u \mid B \mid v\rangle - \beta\langle u \mid B \mid v\rangle \end{aligned}$$

$\langle 1\rangle 4.$ $\alpha \neq \beta$
PROOF: Since the α and β eigenspaces are non-degenerate.
$\langle 1\rangle 5.$ $\langle u \mid B \mid v\rangle = 0$
□

Proposition 52.16. *Let $\mathcal{H}$ be a finite dimensional Hilbert space. Let S and T be compatible observables. Suppose there exists an orthonormal basis $\{|e_1\rangle, \ldots, |e_n\rangle\}$ of eigenvectors of T. If $|e_i\rangle$ is a λ-eigenvector of T, then it is a $\langle e_i \mid S \mid e_i\rangle$-eigenvector of S.*

Corollary 52.16.1. *Let $\mathcal{H}$ be a finite dimensional Hilbert space. Let S and T be compatible observables. Then every orthonormal basis of eigenvectors of T is an orthonormal basis of eigenvectors of S.*