

Summary of Halmos' Naive Set Theory

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Chapter 1

Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*, \in . If $x \in A$ we say that x *belongs to* A , x is an *element* of A , or x is *contained in* A . If this does not hold we write $x \notin A$.

Axiom 1.1 (Axiom of Extensionality). *Two sets are equal if and only if they have the same elements.*

Axiom 1.2 (Axiom of Comprehension, Aussonderungsaxiom). *To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

Axiom 1.3 (Axiom of Pairing). *For any two sets, there exists a set that they both belong to.*

Axiom 1.4 (Union Axiom). *For every set A , there exists a set that contains all the elements that belong to at least one element of A .*

Definition 1.5 (Subset). Let A and B be sets. We say that A is a *subset* of B , or B *includes* A , and write $A \subseteq B$ or $B \supseteq A$, iff every element of A is an element of B .

Axiom 1.6 (Power Set Axiom). *For any set A , there exists a set that contains all the subsets of A .*

Axiom 1.7 (Axiom of Infinity). *There exists a set I such that:*

- *I has an element that has no elements*
- *for all $x \in I$, there exists $y \in I$ such that the elements of y are exactly x and the elements of x .*

Chapter 2

The Subset Relation

Theorem 2.1. *For any set A , we have $A \subseteq A$.*

PROOF: Every element of A is an element of A . \square

Theorem 2.2. *For any sets A , B and C , if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

PROOF: If every element of A is an element of B , and every element of B is an element of C , then every element of A is an element of C . \square

Theorem 2.3. *For any sets A and B , if $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

PROOF: If every element of A is an element of B , and every element of B is an element of A , then A and B have the same elements, and therefore are equal by the Axiom of Extensionality. \square

Definition 2.4 (Proper Subset). Let A and B be sets. We say that A is a *proper* subset of B , or B *properly* includes A , and write $A \subsetneq B$ or $B \supsetneq A$, iff $A \subseteq B$ and $A \neq B$.

Chapter 3

Comprehension Notation

Definition 3.1. Given a set A and a condition $S(x)$, we write $\{x \in A : S(x)\}$ for the set whose elements are exactly those elements x of A for which $S(x)$ holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality. \square

Theorem 3.2. *There is no set that contains every set.*

PROOF:

$\langle 1 \rangle 1.$ LET: A be a set.

PROVE: There exists a set B such that $B \notin A$.

$\langle 1 \rangle 2.$ LET: $B = \{x \in A : x \notin x\}$

$\langle 1 \rangle 3.$ If $B \in A$ then we have $B \in B$ if and only if $B \notin B$.

$\langle 1 \rangle 4.$ $B \notin A$

\square

Chapter 4

Unordered Pairs

Theorem 4.1. *There exists a set with no elements.*

PROOF: Immediate from the Axiom of Infinity. \square

Definition 4.2 (Empty Set). The *empty set* \emptyset is the set with no elements.

Theorem 4.3. *For any set A we have $\emptyset \subset A$.*

PROOF: Vacuous. \square

Definition 4.4 ((Unordered) Pair). For any sets a and b , the *(unordered) pair* $\{a, b\}$ is the set whose elements are just a and b .

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality. \square

Definition 4.5 (Singleton). For any set a , the *singleton* $\{a\}$ is defined to be $\{a, a\}$.

Chapter 5

Unions

Definition 5.1 (Union). For any set \mathcal{C} , the *union* of \mathcal{C} , $\bigcup \mathcal{C}$, is the set whose elements are the elements of the elements of \mathcal{C} .

We write $\bigcup_{X \in \mathcal{A}} t[X]$ for $\bigcup \{t[X] \mid X \in \mathcal{A}\}$.

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality. \square

Proposition 5.2.

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of \emptyset . \square

Proposition 5.3. *For any set A , we have $\bigcup \{A\} = A$.*

PROOF: For any x , we have x is an element of an element of $\{A\}$ if and only if x is an element of A . \square

Definition 5.4. We write $A \cup B$ for $\bigcup \{A, B\}$.

Proposition 5.5. *For any set A , we have $A \cup \emptyset = A$.*

PROOF: $x \in A \cup \emptyset$ iff $x \in A$ or $x \in \emptyset$, iff $x \in A$. \square

Proposition 5.6 (Idempotence). *For any set A , we have $A \cup A = A$.*

PROOF: $x \in A$ or $x \in A$ is equivalent to $x \in A$. \square

Proposition 5.7. *For any sets A and B , we have $A \subseteq B$ if and only if $A \cup B = B$.*

PROOF: For any x , the statement "if $x \in A$ then $x \in B$ " is equivalent to " $x \in A$ or $x \in B$ if and only if $x \in B$ ". \square

Proposition 5.8. *For any sets a and b , we have $\{a\} \cup \{b\} = \{a, b\}$.*

PROOF: Immediate from definitions. \square

Chapter 6

Intersections

Definition 6.1 (Intersection). For any sets A and B , the *intersection* $A \cap B$ is defined to be $\{x \in A : x \in B\}$.

Proposition 6.2. For any set A , we have $A \cap \emptyset = \emptyset$.

PROOF: There is no x such that $x \in A$ and $x \in \emptyset$. \square

Proposition 6.3. For any set A , we have

$$A \cap A = A .$$

PROOF: We have $x \in A$ and $x \in A$ if and only if $x \in A$. \square

Proposition 6.4. For any sets A and B , we have $A \subseteq B$ if and only if $A \cap B = A$.

PROOF: For any x , the statement "if $x \in A$ then $x \in B$ " is equivalent to " $x \in A$ and $x \in B$ if and only if $x \in A$ ". \square

Proposition 6.5. For any sets A , B and C , we have $C \subseteq A$ if and only if $(A \cap B) \cup C = A \cap (B \cup C)$.

PROOF: The statement "if $x \in C$ then $x \in A$ " is equivalent to the statement " $((x \in A \wedge x \in B) \vee x \in C) \Leftrightarrow (x \in A \wedge (x \in B \vee x \in C))$ ". \square

Definition 6.6 (Disjoint). Two sets A and B are *disjoint* if and only if $A \cap B = \emptyset$.

Definition 6.7 (Pairwise Disjoint). Let A be a set. We say the elements of A are *pairwise disjoint* if and only if, for all $x, y \in A$, if $x \cap y \neq \emptyset$ then $x = y$.

Definition 6.8 (Intersection). For any nonempty set \mathcal{C} , the *intersection* of \mathcal{C} , $\bigcap \mathcal{C}$, is the set that contains exactly those sets that belong to every element of \mathcal{C} .

We write $\bigcap_{X \in \mathcal{A}} t[X]$ for $\bigcap \{t[X] \mid X \in \mathcal{A}\}$.

PROOF:

⟨1⟩1. LET: \mathcal{C} be a nonempty set.

⟨1⟩2. There exists a set I whose elements are exactly the sets that belong to every element of \mathcal{C} .

PROOF: Pick $A \in \mathcal{C}$, and take $I = \{x \in A : \forall X \in \mathcal{C}. x \in X\}$.

⟨1⟩3. For any sets I, J , if the elements of I and J are exactly the sets that belong to every element of \mathcal{C} then $I = J$.

PROOF: Axiom of Extensionality.

□

Chapter 7

Unordered Triples

Definition 7.1 ((Unordered) Triple). Given sets a_1, \dots, a_n , define the (*unordered*) *n*-tuple $\{a_1, \dots, a_n\}$ to be

$$\{a_1, \dots, a_n\} := \{a_1\} \cup \dots \cup \{a_n\} \ .$$

Chapter 8

Relative Complements

Definition 8.1 (Relative Complement). For any sets A and B , the *difference* or *relative complement* $A - B$ is defined to be

$$A - B := \{x \in A : x \notin B\} .$$

Proposition 8.2. For any sets A and E , we have $A \subseteq E$ if and only if

$$E - (E - A) = A$$

PROOF:

$\langle 1 \rangle 1$. LET: A and E be sets.

$\langle 1 \rangle 2$. If $A \subseteq E$ then $E - (E - A) = A$

$\langle 2 \rangle 1$. ASSUME: $A \subseteq E$

$\langle 2 \rangle 2$. $E - (E - A) \subseteq A$

PROOF: If $x \in E$ and $x \notin E - A$ then $x \in A$.

$\langle 2 \rangle 3$. $A \subseteq E - (E - A)$

PROOF: If $x \in A$ then $x \in E$ and $x \notin E - A$.

$\langle 1 \rangle 3$. If $E - (E - A) = A$ then $A \subseteq E$.

PROOF: Since $E - (E - A) \subseteq E$.

□

Proposition 8.3. For any set E we have

$$E - \emptyset = E$$

PROOF: $x \in E$ if and only if $x \in E$ and $x \notin \emptyset$. □

Proposition 8.4. For any set E we have

$$E - E = \emptyset .$$

PROOF: There is no x such that $x \in E$ and $x \notin E$. □

Proposition 8.5. For any sets A and E , we have

$$A \cap (E - A) = \emptyset .$$

PROOF: There is no x such that $x \in A$ and $x \in E - A$. \square

Proposition 8.6. *Let A and E be sets. Then $A \subseteq E$ if and only if*

$$A \cup (E - A) = E .$$

PROOF:

$\langle 1 \rangle 1$. LET: A and E be sets.

$\langle 1 \rangle 2$. If $A \subseteq E$ then $A \cup (E - A) = E$.

$\langle 2 \rangle 1$. ASSUME: $A \subseteq E$

$\langle 2 \rangle 2$. $A \cup (E - A) \subseteq E$

PROOF: If $x \in A$ or $x \in E - A$ then $x \in E$.

$\langle 2 \rangle 3$. $E \subseteq A \cup (E - A)$

PROOF: If $x \in E$ then either $x \in A$ or $x \notin A$. In the latter case, $x \in E - A$.

$\langle 1 \rangle 3$. If $A \cup (E - A) = E$ then $A \subseteq E$

PROOF: Since $A \subseteq A \cup (E - A)$.

\square

Proposition 8.7. *Let A , B and E be sets. Then:*

1. *If $A \subseteq B$ then $E - B \subseteq E - A$.*

2. *If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A , B and E be sets.

$\langle 1 \rangle 2$. If $A \subseteq B$ then $E - B \subseteq E - A$.

PROOF: If $A \subseteq B$, $x \in E$ and $x \notin B$, then we have $x \in E$ and $x \notin A$.

$\langle 1 \rangle 3$. If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.

$\langle 2 \rangle 1$. ASSUME: $A \subseteq E$

$\langle 2 \rangle 2$. ASSUME: $E - B \subseteq E - A$

$\langle 2 \rangle 3$. LET: $x \in A$

$\langle 2 \rangle 4$. $x \in E$

$\langle 2 \rangle 5$. $x \notin E - A$

$\langle 2 \rangle 6$. $x \notin E - B$

$\langle 2 \rangle 7$. $x \in B$

\square

Example 8.8. We cannot remove the hypothesis $A \subseteq E$ in item 2 above. Let $E = \emptyset$, $A = \{\emptyset\}$ and $B = \emptyset$. Then $E - B = E - A = \emptyset$ but $A \not\subseteq B$.

Proposition 8.9 (De Morgan's Law). *For any sets A , B and E , we have $E - (A \cup B) = (E - A) \cap (E - B)$.*

PROOF: $(x \in E \wedge \neg(x \in A \vee x \in B)) \Leftrightarrow (x \in E \wedge x \notin A \wedge x \in E \wedge x \notin B)$. \square

Proposition 8.10 (De Morgan's Law). *For any sets A , B and E , we have $E - (A \cap B) = (E - A) \cup (E - B)$.*

PROOF: $(x \in E \vee \neg(x \in A \wedge x \in B)) \Leftrightarrow (x \in E \wedge x \notin A) \vee (x \in E \wedge x \notin B)$. \square

Proposition 8.11. *For any sets A , B and E , if $A \subseteq E$ then*

$$A - B = A \cap (E - B) .$$

PROOF: If $A \subseteq E$ then we have $(x \in A \wedge x \notin B) \Leftrightarrow (x \in A \wedge x \in E \wedge x \notin B)$. \square

Proposition 8.12. *For any sets A and B , we have $A \subseteq B$ if and only if $A - B = \emptyset$.*

PROOF: Both are equivalent to the statement that there is no x such that $x \in A$ and $x \notin B$. \square

Proposition 8.13. *For any sets A and B , we have*

$$A - (A - B) = A \cap B .$$

PROOF: $(x \in A \wedge \neg(x \in A \wedge x \notin B)) \Leftrightarrow x \in A \wedge x \in B$. \square

Proposition 8.14. *For any sets A , B and C , we have*

$$A \cap (B - C) = (A \cap B) - (A \cap C) .$$

PROOF: $(x \in A \wedge x \in B \wedge x \notin C) \Leftrightarrow (x \in A \wedge x \in B \wedge \neg(x \in A \wedge x \in C))$. \square

Proposition 8.15. *For any sets A , B , C and E , if $(A \cap B) - C \subseteq E$ then we have*

$$A \cap B \subseteq (A \cap C) \cup (B \cap (E - C)) .$$

PROOF:

$\langle 1 \rangle 1$. LET: $x \in A \cap B$

PROVE: $x \in (A \cap C) \cup (B \cap (E - C))$

$\langle 1 \rangle 2$. CASE: $x \in C$

PROOF: Then $x \in A \cap C$.

$\langle 1 \rangle 3$. CASE: $x \notin C$

PROOF: Then $x \in E$ and so $x \in B \cap (E - C)$.

\square

Proposition 8.16. *For any sets A , B , C and E , we have*

$$(A \cup C) \cap (B \cup (E - C)) \subseteq A \cup B .$$

PROOF: The statement $(x \in A \vee x \in C) \wedge (x \in B \vee (x \in E \wedge x \notin C))$ implies $x \in A \vee x \in B$. \square

Proposition 8.17 (De Morgan's Law). *Let E be a set and \mathcal{C} a nonempty set. Then*

$$E - \bigcup_{X \in \mathcal{C}} X = \bigcap_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy. \square

Proposition 8.18 (De Morgan's Law). *Let E be a set and \mathcal{C} a nonempty set. Then*

$$E - \bigcap_{X \in \mathcal{C}} X = \bigcup_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy. \square

Chapter 9

Symmetric Difference

Definition 9.1 (Symmetric Difference). For any sets A and B , the *symmetric difference* $A + B$ is defined to be

$$A + B := (A - B) \cup (B - A) .$$

Proposition 9.2. For any sets A and B , we have

$$A + B = B + A$$

PROOF: From the commutativity of union. \square

Proposition 9.3. For any sets A , B and C , we have

$$A + (B + C) = (A + B) + C .$$

PROOF: Each is the set of all x that belong to either exactly one or all three of A , B and C . \square

Proposition 9.4. For any set A , we have

$$A + \emptyset = A .$$

PROOF:

$$\begin{aligned} A + \emptyset &= (A - \emptyset) \cup (\emptyset - A) \\ &= A \cup \emptyset \\ &= A \end{aligned}$$

\square

Proposition 9.5. For any set A we have

$$A + A = \emptyset .$$

PROOF:

$$\begin{aligned} A + A &= (A - A) \cup (A - A) \\ &= \emptyset \cup \emptyset \\ &= \emptyset \end{aligned}$$

\square

Chapter 10

Power Sets

Definition 10.1 (Power Set). For any set A , the *power set* of A , $\mathcal{P}A$, is the set whose elements are exactly the subsets of A .

PROOF: This exists by the Power Set Axiom and Axiom of Comprehension, and is unique by the Axiom of Extensionality. \square

Proposition 10.2.

$$\mathcal{P}\emptyset = \{\emptyset\}$$

PROOF: The only subset of \emptyset is \emptyset . \square

Proposition 10.3. For any set a , we have

$$\mathcal{P}\{a\} = \{\emptyset, \{a\}\} .$$

PROOF: The only subsets of $\{a\}$ are \emptyset and $\{a\}$. \square

Proposition 10.4. For any sets a and b , we have

$$\mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} .$$

PROOF: The only subsets of $\{a, b\}$ are \emptyset , $\{a\}$, $\{b\}$ and $\{a, b\}$. \square

Proposition 10.5. For any nonempty set \mathcal{C} we have

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}X = \mathcal{P}\left(\bigcap \mathcal{C}\right) .$$

PROOF:

$$x \in \bigcup_{X \in \mathcal{C}} \mathcal{P}X \Leftrightarrow \forall X \in \mathcal{C}. x \subseteq X$$

$$\Leftrightarrow \forall X \in \mathcal{C}. \forall y \in x. y \in X$$

$$\Leftrightarrow \forall y \in x. \forall X \in \mathcal{C}. y \in X$$

$$\Leftrightarrow x \subseteq \bigcap \mathcal{C}$$

\square

Proposition 10.6. *For any set \mathcal{C} we have*

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}X \subseteq \mathcal{P} \bigcup \mathcal{C} .$$

PROOF: If there exists $X \in \mathcal{C}$ such that $x \subseteq X$ then $x \subseteq \bigcup \mathcal{C}$. \square

Proposition 10.7. *For any set E , we have*

$$\bigcap \mathcal{P}E = \emptyset .$$

PROOF: Since $\emptyset \in \mathcal{P}E$. \square

Proposition 10.8. *For any sets E and F , if $E \subseteq F$ then $\mathcal{P}E \subseteq \mathcal{P}F$.*

PROOF: If $E \subseteq F$ and $X \subseteq E$ then $X \subseteq F$. \square

Chapter 11

Ordered Pairs

Definition 11.1 (Ordered Pair). For any sets a and b , the *ordered pair* (a, b) is defined by

$$(a, b) := \{\{a\}, \{a, b\}\} .$$

Proposition 11.2. For any sets a, b, x and y , if $(a, b) = (x, y)$ then $a = x$ and $b = y$.

PROOF:

$\langle 1 \rangle 1$. LET: a, b, x and y be sets.

$\langle 1 \rangle 2$. ASSUME: $(a, b) = (x, y)$

$\langle 1 \rangle 3$. $a = x$

PROOF: $\{a\} = \bigcap(a, b) = \bigcap(x, y) = \{x\}$.

$\langle 1 \rangle 4$. $\{a, b\} = \{x, y\}$

$\langle 1 \rangle 5$. CASE: $a = b$

$\langle 2 \rangle 1$. $x = y$

PROOF: Since $\{x, y\} = \{a, b\}$ is a singleton.

$\langle 2 \rangle 2$. $b = y$

PROOF: $b = a = x = y$

$\langle 1 \rangle 6$. CASE: $a \neq b$

$\langle 2 \rangle 1$. $x \neq y$

PROOF: Since $\{x, y\} = \{a, b\}$ is not a singleton.

$\langle 2 \rangle 2$. $b = y$

PROOF: $\{b\} = \{a, b\} - \{a\} = \{x, y\} - \{x\} = \{y\}$.

□

Definition 11.3 (Cartesian Product). For any sets A and B , the *Cartesian product* $A \times B$ is

$$A \times B := \{p \in \mathcal{PP}(A \cup B) : \exists a \in A. \exists b \in B. p = (a, b)\} .$$

Proposition 11.4. For any sets A, B and X , we have

$$(A - B) \times X = (A \times X) - (B \times X) .$$

PROOF: Easy. \square

Proposition 11.5. *For any sets A and B , we have $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.*

PROOF: Easy. \square

Proposition 11.6. *For any sets A , B , X and Y , if $A \subseteq X$ and $B \subseteq Y$ then $A \times B \subseteq X \times Y$. The converse holds assuming $A \neq \emptyset$ and $B \neq \emptyset$.*

PROOF: Easy. \square

Chapter 12

Relations

Definition 12.1 (Relation). A *relation* is a set of ordered pairs.

If R is a relation, we write xRy for $(x, y) \in R$.

Given sets X and Y , a relation *between X and Y* is a subset of $X \times Y$.

Given a set X , a relation *on X* is a relation between X and X .

Definition 12.2 (Domain). The *domain* of a relation R is the set

$$\text{dom } R := \{x \in \bigcup \bigcup R : \exists y. (x, y) \in R\} \ .$$

Definition 12.3 (Range). The *range* of a relation R is the set

$$\text{ran } R := \{y \in \bigcup \bigcup R : \exists x. (x, y) \in R\} \ .$$

Definition 12.4 (Reflexive). Let R be a relation on X . Then R is *reflexive* iff, for all $x \in X$, we have xRx .

Definition 12.5 (Symmetric). Let R be a relation on X . Then R is *symmetric* iff, whenever xRy , then yRx .

Definition 12.6 (Transitive). Let R be a relation on X . Then R is *transitive* iff, whenever xRy and yRz , then xRz .

Definition 12.7 (Equivalence Relation). Let R be a relation on X . Then R is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Definition 12.8 (Partition). Let X be a set. A *partition* of X is a pairwise disjoint set of nonempty subsets of X whose union is X .

Definition 12.9 (Equivalence Class). Let R be an equivalence relation on X . Let $x \in X$. The *equivalence class* of x with respect to R is

$$x/R := \{y \in X : xRy\} \ .$$

We write X/R for the set of all equivalence classes with respect to R .

Definition 12.10 (Induced). Let P be a partition of X . The relation *induced* by P is X/P where $x(X/P)y$ iff there exists $X \in P$ such that $x \in X$ and $y \in X$.

Theorem 12.11. Let R be an equivalence relation on X . Then X/R is a partition of X that induces the relation R .

PROOF: Easy. \square

Theorem 12.12. Let P be a partition of X . Then X/P is an equivalence relation on X , and $P = X/(X/P)$.

PROOF: Easy. \square

Definition 12.13 (Composition). Let R be a relation between X and Y , and S a relation between Y and Z . The *composite* or *relative product* $S \circ R = SR$ is the relation between X and Z defined by

$$x(S \circ R)z \Leftrightarrow \exists y \in Y (xRy \wedge ySz) .$$

Proposition 12.14. Let R be a relation between X and Y , S a relation between Y and Z , and T a relation between Z and W . Then

$$T(SR) = (TS)R .$$

PROOF: Easy. \square

Example 12.15. Composition of relations is not commutative in general. Let $X = \{a, b\}$ where $a \neq b$. Let $R = \{(a, a), (b, a)\}$ and $S = \{(a, b), (b, b)\}$. Then $SR = S$ but $RS = R \neq S$.

Proposition 12.16. A relation R is transitive if and only if $RR \subseteq R$.

PROOF: Easy. \square

Definition 12.17 (Inverse). Let R be a relation between X and Y . The *inverse* or *converse* R^{-1} is the relation between Y and X defined by

$$yR^{-1}x \Leftrightarrow xRy .$$

Proposition 12.18. For any relation R , we have

$$\text{dom } R^{-1} = \text{ran } R .$$

PROOF: Easy. \square

Proposition 12.19. For any relation R , we have

$$\text{ran } R^{-1} = \text{dom } R .$$

PROOF: Easy. \square

Proposition 12.20. *Let R be a relation between X and Y , and S a relation between Y and Z . Then*

$$(SR)^{-1} = R^{-1}S^{-1} .$$

PROOF: Easy. \square

Proposition 12.21. *A relation R is symmetric if and only if $R \subseteq R^{-1}$.*

PROOF: Easy. \square

Definition 12.22 (Identity Relation). For any set X , the *identity relation* I_X on X is

$$I_X = \{(x, x) : x \in X\} .$$

Proposition 12.23. *Let R be a relation between X and Y . Then*

$$I_Y R = R I_X = R .$$

PROOF: Easy. \square

Proposition 12.24. *A relation R on a set X is reflexive if and only if $I_X \subseteq R$.*

PROOF: Easy. \square

Chapter 13

Functions

Definition 13.1 (Function). Let X and Y be sets. A *function, map, mapping, transformation or operator* f from X to Y , $f : X \rightarrow Y$, is a relation f between X and Y such that, for all $x \in X$, there exists a unique $f(x) \in Y$, called the *value* of f at the *argument* x , such that $(x, f(x)) \in f$.

Definition 13.2 (Onto). Let $f : X \rightarrow Y$. We say f maps X *onto* Y iff $\text{ran } f = Y$.

Definition 13.3 (Image). Let $f : X \rightarrow Y$ and $A \subseteq X$. The *image* of A under f is

$$f(A) := \{f(x) : x \in A\} .$$

Definition 13.4 (Inclusion Map). Let Y be a set and $X \subseteq Y$. Then the *inclusion map* $i : X \hookrightarrow Y$ is the function defined by $i(x) = x$ for all $x \in X$.

Proposition 13.5. For any set X , the identity relation I_X is a function $X \rightarrow X$.

PROOF: Easy. \square

Definition 13.6 (Restriction). Let $f : Y \rightarrow Z$ and $X \subseteq Y$. The *restriction* of f to X is the function $f \upharpoonright X : X \rightarrow Z$ defined by

$$(f \upharpoonright X)(x) = f(x) \quad (x \in X) .$$

Given sets X, Y and Z with $X \subseteq Y$, if $f : X \rightarrow Z$ and $g : Y \rightarrow Z$, we say g is an *extension* of f to Y iff $f = g \upharpoonright X$.

Definition 13.7 (Projection). Given sets X and Y , the *projection* maps $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are defined by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y \quad (x \in X, y \in Y) .$$

Definition 13.8 (Canonical Map). Let X be a set and R an equivalence relation on X . The *canonical map* $\pi : X \rightarrow X/R$ is the map defined by $\pi(x) = x/R$.

Definition 13.9 (One-to-One). A function $f : X \rightarrow Y$ is *one-to-one*, or a *one-to-one correspondence*, iff, for all $x, y \in X$, if $f(x) = f(y)$ then $x = y$.

Proposition 13.10. *Let $f : X \rightarrow Y$. Then the following are equivalent:*

1. *f is one-to-one.*
2. *For all $A, B \subseteq X$, we have $f(A \cap B) = f(A) \cap f(B)$.*
3. *For all $A \subseteq X$, we have $f(X - A) \subseteq Y - f(A)$.*

PROOF: Easy. \square

Proposition 13.11. *Let $f : X \rightarrow Y$. Then f maps X onto Y if and only if, for all $A \subseteq X$, we have $Y - f(A) \subseteq f(X - A)$.*

PROOF: Easy. \square

Chapter 14

Families

Definition 14.1 (Family). Let I and X be sets. A *family* of elements of X indexed by I is a function $a : I \rightarrow X$. We write a_i for $a(i)$, and $\{a_i\}_{i \in I}$ for a .

Proposition 14.2 (Generalized Associative Law for Unions). Let $\{I_j\}_{j \in J}$ be a family of sets. Let $K = \bigcup_{j \in J} I_j$. Let $\{A_k\}_{k \in K}$ be a family of sets indexed by K . Then

$$\bigcup_{k \in K} A_k = \bigcup_{j \in J} \bigcup_{i \in I_j} A_i .$$

PROOF: Easy. \square

Proposition 14.3 (Generalized Commutative Law for Unions). Let $\{I_j\}_{j \in J}$ be a family of sets. Let $f : J \rightarrow J$ be a one-to-one correspondence from J onto J . Then

$$\bigcup_{j \in J} I_j = \bigcup_{j \in J} I_{f(j)} .$$

PROOF: Easy. \square

Proposition 14.4 (Generalized Associative Law for Intersections). Let $\{I_j\}_{j \in J}$ be a nonempty family of nonempty sets. Let $K = \bigcup_{j \in J} I_j$. Let $\{A_k\}_{k \in K}$ be a family of sets indexed by K . Then

$$\bigcap_{k \in K} A_k = \bigcap_{j \in J} \bigcap_{i \in I_j} A_i .$$

PROOF: Easy. \square

Proposition 14.5 (Generalized Commutative Law for Intersections). Let $\{I_j\}_{j \in J}$ be a nonempty family of sets. Let $f : J \rightarrow J$ be a one-to-one correspondence from J onto J . Then

$$\bigcap_{j \in J} I_j = \bigcap_{j \in J} I_{f(j)} .$$

PROOF: Easy. \square

Proposition 14.6. *Let B be a set and $\{A_i\}_{i \in I}$ a family of sets. Then*

$$B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i)$$

PROOF: Easy. \square

Proposition 14.7. *Let B be a set and $\{A_i\}_{i \in I}$ a nonempty family of sets. Then*

$$B \cup \bigcap_{i \in I} A_i = \bigcap_{i \in I} (B \cup A_i)$$

PROOF: Easy. \square

Definition 14.8 (Cartesian Product of a Family of Sets). Let $\{A_i\}_{i \in I}$ be a family of sets. The *Cartesian product* $\times_{i \in I} A_i$ is the set of all families $\{a_i\}_{i \in I}$ such that $\forall i \in I. a_i \in A_i$.

We write A^I for $\times_{i \in I} A$.

Definition 14.9 (Projection). Let $\{A_i\}_{i \in I}$ be a family of sets and $i \in I$. The *projection* function $\pi_i : \times_{i \in I} A_i \rightarrow A_i$ is defined by $\pi_i(a) = a_i$.

Proposition 14.10. *Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be families of sets. Then*

$$\left(\bigcup_{i \in I} A_i \right) \times \left(\bigcup_{j \in J} B_j \right) = \bigcup_{i \in I} \bigcup_{j \in J} (A_i \times B_j) .$$

PROOF: Easy. \square

Proposition 14.11. *Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be nonempty families of sets. Then*

$$\left(\bigcap_{i \in I} A_i \right) \times \left(\bigcap_{j \in J} B_j \right) = \bigcap_{i \in I} \bigcap_{j \in J} (A_i \times B_j) .$$

PROOF: Easy. \square

Proposition 14.12. *Let $f : X \rightarrow Y$. Let $\{A_i\}_{i \in I}$ be a family of subsets of X . Then*

$$f \left(\bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} f(A_i) .$$

PROOF: Easy. \square

Example 14.13. It is not true in general that, if $f : X \rightarrow Y$ and $\{A_i\}_{i \in I}$ is a nonempty family of subsets of X , then $f(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f(A_i)$.

Take $X = \{a, b\}$ and $Y = \{c\}$ where $a \neq b$. Take $I = \{i, j\}$ with $i \neq j$. Let $A_i = \{a\}$ and $A_j = \{b\}$. Let f be the unique function $X \rightarrow Y$. Then $f(\bigcap_{i \in I} A_i) = f(\emptyset) = \emptyset$ but $\bigcap_{i \in I} f(A_i) = \{c\}$.

Chapter 15

Inverses and Composites

Definition 15.1 (Inverse). Given a function $f : X \rightarrow Y$, the *inverse* of f is the function $f^{-1} : \mathcal{P}Y \rightarrow \mathcal{P}X$ defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \ .$$

We call $f^{-1}(B)$ the *inverse image* of B under f .

Proposition 15.2. *Let $f : X \rightarrow Y$. Then f maps X onto Y if and only if the inverse image of any nonempty subset of Y is nonempty.*

PROOF: Easy. \square

Proposition 15.3. *Let $f : X \rightarrow Y$. Then f is one-to-one if and only if the inverse image of any singleton subset of Y is a singleton.*

PROOF: Easy. \square

Proposition 15.4. *Let $f : X \rightarrow Y$. Let $B \subseteq Y$. Then*

$$f(f^{-1}(B)) \subseteq B \ .$$

PROOF: Easy. \square

Proposition 15.5. *Let $f : X \rightarrow Y$. Let $A \subseteq X$. Then*

$$A \subseteq f^{-1}(f(A)) \ .$$

Equality holds if f is one-to-one.

PROOF: Easy. \square

Proposition 15.6. *Let $f : X \rightarrow Y$. Let $\{B_i\}_{i \in I}$ be a family of subsets of Y . Then*

$$f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i) \ .$$

PROOF: Easy. \square

Proposition 15.7. *Let $f : X \rightarrow Y$. Let $\{B_i\}_{i \in I}$ be a nonempty family of subsets of Y . Then*

$$f^{-1} \left(\bigcap_{i \in I} B_i \right) = \bigcap_{i \in I} f^{-1}(B_i) .$$

PROOF: Easy. \square

Proposition 15.8. *Let $f : X \rightarrow Y$ and $B \subseteq Y$. Then $f^{-1}(Y - B) = X - f^{-1}(B)$.*

PROOF: Easy. \square

Proposition 15.9. *Let $f : X \rightarrow Y$ be one-to-one. Then the inverse of f as a relation, f^{-1} , is a function $f^{-1} : \text{ran } f \rightarrow X$, and for all $y \in \text{ran } f$, we have $f^{-1}(y)$ is the unique x such that $f(x) = y$.*

PROOF: Easy. \square

Proposition 15.10. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Then $gf : X \rightarrow Z$ and, for all $x \in X$, we have*

$$(g \circ f)(x) = g(f(x)) .$$

PROOF: Easy. \square

Example 15.11. Example 12.15 shows that function composition is not commutative in general.

Proposition 15.12. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Then*

$$(gf)^{-1} = f^{-1}g^{-1} : \mathcal{P}Z \rightarrow \mathcal{P}X .$$

PROOF: Easy. \square

Proposition 15.13. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$. If $gf = I_X$ then f is one-to-one and g maps Y onto X .*

PROOF: Easy. \square

Chapter 16

Numbers

Definition 16.1 (Successor). The *successor* of a set x , x^+ , is defined by

$$x^+ := x \cup \{x\} .$$

Definition 16.2. We define

$$0 = \emptyset$$

$$1 = 0^+$$

$$2 = 1^+$$

etc.

Definition 16.3 (Characteristic Function). Let X be a set and $A \subseteq X$. The *characteristic function* of A is the function $\chi_A : X \rightarrow 2$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Theorem 16.4. Let X be a set. The function $\chi : \mathcal{P}X \rightarrow 2^X$ that maps a subset A of X to χ_A is a one-to-one correspondence.

PROOF: Easy. \square

Definition 16.5. The set ω of *natural numbers* is the set such that:

- $0 \in \omega$
- For all $n \in \omega$ we have $n^+ \in \omega$
- For any set X , if $0 \in X$ and $\forall n \in X. n^+ \in X$ then $\omega \subseteq X$

PROOF: To show this exists, pick a set A such that $0 \in A$ and $\forall n \in A. n^+ \in A$ (by the Axiom of Infinity), and let $\omega = \bigcap \{X \in \mathcal{P}A : 0 \in X \wedge \forall n \in X. n^+ \in X\}$.
 \square

Definition 16.6 (Sequence). A *finite sequence* is a family whose index set is a natural number. An *infinite sequence* is a family whose index set is ω .

Given a finite sequence of sets $\{A_i\}_{i \in n^+}$, we write $\bigcup_{i=0}^n A_i$ for $\bigcup_{i \in n^+} A_i$. Given an infinite sequence of sets $\{A_i\}_{i \in \omega}$, we write $\bigcup_{i=0}^{\infty} A_i$ for $\bigcup_{i \in \omega} A_i$.

We make similar definitions for \bigcap and \times .

Chapter 17

The Peano Axioms

Theorem 17.1 (Principle of Mathematical Induction). *For any subset S of ω , if $0 \in S$ and $\forall n \in S. n^+ \in S$, then $S = \omega$.*

PROOF: From the definition of ω . \square

Proposition 17.2.

$$\forall n \in \omega. \forall x \in n. n \not\subseteq x$$

PROOF:

$\langle 1 \rangle 1.$ $\forall x \in 0. 0 \not\subseteq x$

PROOF: Vacuous.

$\langle 1 \rangle 2.$ For any natural number n , if $\forall x \in n. n \not\subseteq x$ then $\forall x \in n^+. n^+ \not\subseteq x$.

$\langle 2 \rangle 1.$ LET: n be a natural number.

$\langle 2 \rangle 2.$ ASSUME: $\forall x \in n. n \not\subseteq x$

$\langle 2 \rangle 3.$ LET: $x \in n^+$

$\langle 2 \rangle 4.$ ASSUME: for a contradiction $n^+ \subseteq x$

$\langle 2 \rangle 5.$ $x \in n$ or $x = n$

$\langle 2 \rangle 6.$ CASE: $x \in n$

PROOF: Then we have $n \subseteq n^+ \subseteq x$ contradicting $\langle 2 \rangle 2$.

$\langle 2 \rangle 7.$ CASE: $x = n$

PROOF: Then we have $n \in n^+ \subseteq x = n$ and $n \subseteq n$ contradicting $\langle 2 \rangle 2$.

\square

Corollary 17.2.1. *For any natural number n we have $n \notin n$.*

Corollary 17.2.2. *For any natural number n we have $n \neq n^+$.*

Definition 17.3 (Transitive Set). A set E is a *transitive* set iff, whenever $x \in y \in E$, then $x \in E$.

Proposition 17.4. *Every natural number is a transitive set.*

PROOF:

$\langle 1 \rangle 1.$ 0 is a transitive set.

PROOF: Vacuously, if $x \in y \in 0$ then $x \in 0$.

$\langle 1 \rangle 2$. For any natural number n , if n is a transitive set, then n^+ is a transitive set.

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. ASSUME: n is a transitive set.

$\langle 2 \rangle 3$. LET: $x \in y \in n^+$

$\langle 2 \rangle 4$. $y \in n$ or $y = n$

$\langle 2 \rangle 5$. CASE: $y \in n$

$\langle 3 \rangle 1$. $x \in n$

PROOF: $\langle 2 \rangle 2$, $\langle 2 \rangle 3$, $\langle 2 \rangle 5$.

$\langle 3 \rangle 2$. $x \in n^+$

$\langle 2 \rangle 6$. CASE: $y = n$

$\langle 3 \rangle 1$. $x \in n$

PROOF: $\langle 2 \rangle 3$, $\langle 2 \rangle 6$

$\langle 3 \rangle 2$. $x \in n^+$

□

Proposition 17.5. *For any natural numbers m and n , if $m^+ = n^+$ then $m = n$.*

PROOF:

$\langle 1 \rangle 1$. LET: m and n be natural numbers.

$\langle 1 \rangle 2$. ASSUME: $m^+ = n^+$

$\langle 1 \rangle 3$. $m \in m^+ = n^+$

$\langle 1 \rangle 4$. $m \in n$ or $m = n$

$\langle 1 \rangle 5$. $n \in n^+ = m^+$

$\langle 1 \rangle 6$. $n \in m$ or $n = m$

$\langle 1 \rangle 7$. We cannot have $m \in n$ and $n \in m$

$\langle 2 \rangle 1$. ASSUME: for a contradiction $m \in n$ and $n \in m$

$\langle 2 \rangle 2$. $m \in m$

PROOF: Since m is a transitive set (Proposition 17.4).

$\langle 2 \rangle 3$. Q.E.D.

PROOF: This contradicts Proposition 17.2.

$\langle 1 \rangle 8$. $m = n$

□

Theorem 17.6 (Recursion Theorem). *Let X be a set. Let $a \in X$. Let $f : X \rightarrow X$. There exists a function $u : \omega \rightarrow X$ such that $u(0) = a$ and, for all $n \in \omega$, we have $u(n^+) = f(u(n))$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{C} = \{A \in \mathcal{P}(\omega \times X) : (0, a) \in A \wedge \forall n \in \omega. \forall x \in X. (n, x) \in A \Rightarrow (n^+, f(x)) \in A\}$

$\langle 1 \rangle 2$. $\mathcal{C} \neq \emptyset$

PROOF: $\omega \times X \in \mathcal{C}$

$\langle 1 \rangle 3$. LET: $u = \bigcap \mathcal{C}$

$\langle 1 \rangle 4$. $u \in \mathcal{C}$

$\langle 1 \rangle 5$. u is a function.

$\langle 2 \rangle 1$. LET: $P(n)$ be the property: $\forall x, y \in X. (n, x) \in u \wedge (n, y) \in u \Rightarrow x = y$
 $\langle 2 \rangle 2$. $P(0)$
 $\langle 3 \rangle 1$. $\forall x \in X. (0, x) \in u \Rightarrow x = a$
 PROOF: If $(0, x) \in u$ and $x \neq a$ then $u - \{(0, x)\} \in \mathcal{C}$ and so $u - \{(0, x)\} \subseteq u$, which is impossible.
 $\langle 2 \rangle 3$. For every natural number n , if $P(n)$ then $P(n^+)$.
 $\langle 3 \rangle 1$. LET: n be a natural number.
 $\langle 3 \rangle 2$. ASSUME: $P(n)$
 $\langle 3 \rangle 3$. LET: $x, y \in X$
 $\langle 3 \rangle 4$. ASSUME: $(n^+, x), (n^+, y) \in u$
 $\langle 3 \rangle 5$. PICK $x', y' \in X$ such that $(n, x') \in u, (n, y') \in u$ and $f(x') = x$ and $f(y') = y$
 PROOF: If no such x' exists then $u - \{(n^+, x)\} \in \mathcal{C}$ and so $u - \{(n^+, x)\} \subseteq u$ which is impossible. Similarly for y' .
 $\langle 3 \rangle 6$. $x' = y'$
 PROOF: $\langle 3 \rangle 2$
 $\langle 3 \rangle 7$. $x = y$

□

Proposition 17.7. *For any natural number n , either $n = 0$ or there exists a natural number m such that $n = m^+$.*

PROOF: Easy induction on n . □

Proposition 17.8. *ω is a transitive set.*

PROOF:

$\langle 1 \rangle 1$. LET: $P(n)$ be the property $\forall x \in n. x \in \omega$
 $\langle 1 \rangle 2$. $P(0)$
 PROOF: Vacuous.
 $\langle 1 \rangle 3$. For any natural number n , if $P(n)$ then $P(n^+)$.
 $\langle 2 \rangle 1$. LET: n be a natural number.
 $\langle 2 \rangle 2$. ASSUME: $P(n)$
 $\langle 2 \rangle 3$. LET: $x \in n^+$
 $\langle 2 \rangle 4$. $x \in n$ or $x = n$
 $\langle 2 \rangle 5$. CASE: $x \in n$
 PROOF: Then $x \in \omega$ by $\langle 2 \rangle 2$.
 $\langle 2 \rangle 6$. CASE: $x = n$
 PROOF: Then $x \in \omega$ by $\langle 2 \rangle 1$.

□

Proposition 17.9. *For any natural number n and any nonempty subset $E \subseteq n$, there exists $k \in E$ such that $\forall m \in E. k = m \vee k \in m$.*

PROOF:

$\langle 1 \rangle 1$. LET: $P(n)$ be the property: for any nonempty subset $E \subseteq n$, there exists $k \in E$ such that $\forall m \in E. k = m \vee k \in m$
 $\langle 1 \rangle 2$. $P(0)$

PROOF: Vacuous as there is no nonempty subset of 0.

⟨1⟩3. For any natural number n , if $P(n)$ then $P(n^+)$.

⟨2⟩1. LET: n be a natural number.

⟨2⟩2. ASSUME: $P(n)$

⟨2⟩3. LET: E be a nonempty subset of n^+

⟨2⟩4. CASE: $E - \{n\} = \emptyset$
PROOF: Then $E = \{n\}$ so take $k = n$.

⟨2⟩5. CASE: $E - \{n\} \neq \emptyset$

⟨3⟩1. PICK $k \in E - \{n\}$ such that $\forall m \in E - \{n\}. k = m \vee k \in m$
PROOF: By ⟨2⟩2.

⟨3⟩2. $\forall m \in E. k = m \vee k \in m$
PROOF: Since $k \in n$.

□

Chapter 18

Arithmetic

Definition 18.1 (Addition). Define *addition* $+$ on ω by recursion thus:

$$\begin{aligned}m + 0 &= m \\m + n^+ &= (m + n)^+\end{aligned}$$

Proposition 18.2. *For all $m, n, p \in \omega$ we have*

$$m + (n + p) = (m + n) + p .$$

PROOF:

$\langle 1 \rangle 1$. LET: $P(p)$ be the property $\forall m, n \in \omega. m + (n + p) = (m + n) + p$

$\langle 1 \rangle 2$. $P(0)$

PROOF: $m + (n + 0) = m + n = (m + n) + 0$.

$\langle 1 \rangle 3$. $\forall p \in \omega. P(p) \Rightarrow P(p^+)$

$\langle 2 \rangle 1$. LET: $p \in \omega$

$\langle 2 \rangle 2$. ASSUME: $P(p)$

$\langle 2 \rangle 3$. LET: $m, n \in \omega$

$\langle 2 \rangle 4$. $m + (n + p^+) = (m + n) + p^+$

PROOF:

$$\begin{aligned}m + (n + p^+) &= m + (n + p)^+ \\&= (m + (n + p))^+ \\&= ((m + n) + p)^+ \\&= (m + n) + p^+\end{aligned}$$

□

Proposition 18.3. *For all $m, n \in \omega$, we have*

$$m + n = n + m .$$

PROOF:

$\langle 1 \rangle 1$. LET: $P(m)$ be the property $\forall n \in \omega. m + n = n + m$

$\langle 1 \rangle 2. P(0)$
 $\langle 2 \rangle 1. \text{ LET: } Q(n) \text{ be the property } 0 + n = n + 0$
 $\langle 2 \rangle 2. Q(0)$
 PROOF: Trivial.
 $\langle 2 \rangle 3. \forall n \in \omega. Q(n) \Rightarrow Q(n^+)$
 $\langle 3 \rangle 1. \text{ LET: } n \in \omega$
 $\langle 3 \rangle 2. \text{ ASSUME: } Q(n)$
 $\langle 3 \rangle 3. 0 + n^+ = n^+ + 0$
 PROOF:

$$\begin{aligned}
 0 + n^+ &= (0 + n)^+ \\
 &= (n + 0)^+ & (\langle 3 \rangle 2) \\
 &= n^+ \\
 &= n^+ + 0
 \end{aligned}$$

$\langle 1 \rangle 3. \forall m \in \omega. P(m) \Rightarrow P(m^+)$
 $\langle 2 \rangle 1. \text{ LET: } m \in \omega$
 $\langle 2 \rangle 2. \text{ ASSUME: } P(m)$
 $\langle 2 \rangle 3. \text{ LET: } Q(n) \text{ be the property } m^+ + n = n + m^+$
 $\langle 2 \rangle 4. Q(0)$
 PROOF: $\langle 1 \rangle 2$
 $\langle 2 \rangle 5. \forall n \in \omega. Q(n) \Rightarrow Q(n^+)$
 $\langle 3 \rangle 1. \text{ LET: } n \in \omega$
 $\langle 3 \rangle 2. \text{ ASSUME: } Q(n)$
 $\langle 3 \rangle 3. Q(n^+)$
 PROOF:

$$\begin{aligned}
 m^+ + n^+ &= (m^+ + n)^+ \\
 &= (n + m^+)^+ & (\langle 3 \rangle 2) \\
 &= (n + m)^{++} \\
 &= (m + n)^{++} & (\langle 2 \rangle 2) \\
 &= (m + n^+)^+ \\
 &= (n^+ + m)^+ & (\langle 2 \rangle 2) \\
 &= n^+ + m^+
 \end{aligned}$$

□

Definition 18.4 (Multiplication). Define *multiplication* \cdot on ω by

$$\begin{aligned}
 m0 &= 0 \\
 mn^+ &= mn + m
 \end{aligned}$$

Proposition 18.5. For all $m, n, p \in \omega$, we have

$$m(n + p) = mn + mp .$$

PROOF:

$\langle 1 \rangle 1. \text{ LET: } P(p) \text{ be the statement } \forall m, n \in \omega. m(n + p) = mn + mp$

⟨1⟩2. $P(0)$

PROOF:

$$\begin{aligned} m(n+0) &= mn \\ &= mn + 0 \\ &= mn + m0 \end{aligned}$$

⟨1⟩3. $\forall p \in \omega. P(p) \Rightarrow P(p^+)$

⟨2⟩1. LET: $p \in \omega$

⟨2⟩2. ASSUME: $P(p)$

⟨2⟩3. LET: $m, n \in \omega$

⟨2⟩4. $m(n+p^+) = mn + mp^+$

PROOF:

$$\begin{aligned} m(n+p^+) &= m(n+p)^+ \\ &= m(n+p) + m \\ &= (mn+mp) + m && (\langle 2 \rangle 2) \\ &= mn + (mp+m) && (\text{Proposition 18.2}) \\ &= mn + mp^+ \end{aligned}$$

□

Proposition 18.6. *For all $m, n, p \in \omega$ we have*

$$m(np) = (mn)p .$$

PROOF:

⟨1⟩1. LET: $P(p)$ be the statement $\forall m, n \in \omega. m(np) = (mn)p$

⟨1⟩2. $P(0)$

PROOF:

$$\begin{aligned} m(n0) &= m0 \\ &= 0 \\ &= (mn)0 \end{aligned}$$

⟨1⟩3. $\forall p \in \omega. P(p) \Rightarrow P(p^+)$

⟨2⟩1. LET: $p \in \omega$

⟨2⟩2. ASSUME: $P(p)$

⟨2⟩3. LET: $m, n \in \omega$

⟨2⟩4. $m(np^+) = (mn)p^+$

PROOF:

$$\begin{aligned} m(np^+) &= m(np+n) \\ &= m(np) + mn && (\text{Proposition 18.5}) \\ &= (mn)p + mn && (\langle 2 \rangle 2) \\ &= (mn)p^+ \end{aligned}$$

□

Proposition 18.7. *For all $m, n \in \omega$, we have*

$$mn = nm .$$

PROOF:

$\langle 1 \rangle 1$. LET: $P(m)$ be the statement $\forall n \in \omega. mn = nm$

$\langle 1 \rangle 2$. $P(0)$

$\langle 2 \rangle 1$. LET: $Q(n)$ be the statement $0n = n0$

$\langle 2 \rangle 2$. $Q(0)$

PROOF: Trivial.

$\langle 2 \rangle 3$. $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$. LET: $n \in \omega$

$\langle 3 \rangle 2$. ASSUME: $Q(n)$

$\langle 3 \rangle 3$. $Q(n^+)$

PROOF:

$$\begin{aligned}
 0n^+ &= 0n + 0 \\
 &= 0n \\
 &= n0 & (\langle 3 \rangle 2) \\
 &= 0 \\
 &= n^+0
 \end{aligned}$$

$\langle 1 \rangle 3$. $\forall m \in \omega. P(m) \Rightarrow P(m^+)$

$\langle 2 \rangle 1$. LET: $m \in \omega$

$\langle 2 \rangle 2$. ASSUME: $P(m)$

$\langle 2 \rangle 3$. LET: $Q(n)$ be the statement $m^+n = nm^+$

$\langle 2 \rangle 4$. $Q(0)$

PROOF: $\langle 1 \rangle 2$

$\langle 2 \rangle 5$. $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$. LET: $n \in \omega$

$\langle 3 \rangle 2$. ASSUME: $Q(n)$

$\langle 3 \rangle 3$. $Q(n^+)$

PROOF:

$$\begin{aligned}
 m^+n^+ &= m^+n + m^+ \\
 &= (m^+n + m)^+ \\
 &= (nm^+ + m)^+ & (\langle 3 \rangle 2) \\
 &= (nm + n + m)^+ \\
 &= (mn + m + n)^+ & (\langle 2 \rangle 2, \text{Proposition 18.2, Proposition 18.3}) \\
 &= (mn^+ + n)^+ \\
 &= (n^+m + n)^+ & (\langle 2 \rangle 2) \\
 &= n^+m + n^+ \\
 &= n^+m^+
 \end{aligned}$$

□

Definition 18.8 (Exponentiation). Define *exponentiation* on ω by recursion:

$$\begin{aligned}
 m^0 &= 1 \\
 m^{n^+} &= m^n m
 \end{aligned}$$

Proposition 18.9. *For all $m, n, p \in \omega$ we have*

$$m^{n+p} = m^n m^p .$$

PROOF:

$$\langle 1 \rangle 1. m^{n+0} = m^n m^0$$

PROOF:

$$\begin{aligned} m^{n+0} &= m^n \\ &= m^n 1 \\ &= m^n m^0 \end{aligned}$$

$$\langle 1 \rangle 2. \text{ If } m^{n+p} = m^n m^p \text{ then } m^{n+p^+} = m^n m^{p^+}$$

PROOF:

$$\begin{aligned} m^{n+p^+} &= m^{n+p} m \\ &= m^n m^p m \\ &= m^n m^{p^+} \end{aligned}$$

□

Proposition 18.10. *For all $m, n, p \in \omega$ we have*

$$(m^n)^p = m^{np} .$$

PROOF:

$$\langle 1 \rangle 1. (m^n)^0 = m^{n0}$$

PROOF: Both are equal to 1.

$$\langle 1 \rangle 2. \text{ If } (m^n)^p = m^{np} \text{ then } (m^n)^{p^+} = m^{np^+}$$

PROOF:

$$\begin{aligned} (m^n)^{p^+} &= (m^n)^p m^n \\ &= m^{np} m^n \\ &= m^{np+n} && \text{(Proposition 18.9)} \\ &= m^{np^+} \end{aligned}$$

□

Proposition 18.11. *For any natural numbers m and n , if $m \in n$ then $m^+ \in n^+$.*

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } P(n) \text{ be the property } \forall m \in n. m^+ \in n^+$$

$$\langle 1 \rangle 2. P(0)$$

PROOF: Vacuous.

$$\langle 1 \rangle 3. \text{ For any natural number } n, \text{ if } P(n) \text{ then } P(n^+).$$

$$\langle 2 \rangle 1. \text{ LET: } n \text{ be a natural number.}$$

$$\langle 2 \rangle 2. \text{ ASSUME: } P(n)$$

$$\langle 2 \rangle 3. \text{ LET: } m \in n^+$$

$$\langle 2 \rangle 4. m \in n \text{ or } m = n$$

$$\langle 2 \rangle 5. m^+ \in n^+ \text{ or } m^+ = n^+$$

PROOF: $\langle 2 \rangle 2$

□ $\langle 2 \rangle 6$. CASE: $m^+ \in n^{++}$

Proposition 18.12. *For any natural numbers m and n , either $m \in n$ or $m = n$ or $n \in m$.*

PROOF:

$\langle 1 \rangle 1$. LET: $P(n)$ be the property: for all $m \in \omega$, either $m \in n$ or $m = n$ or $n \in m$

$\langle 1 \rangle 2$. $P(0)$

$\langle 2 \rangle 1$. LET: $Q(m)$ be the property: either $m = 0$ or $0 \in m$

$\langle 2 \rangle 2$. $Q(0)$

PROOF: Since $0 = 0$.

$\langle 2 \rangle 3$. For all $m \in \omega$, if $Q(m)$ then $Q(m^+)$

PROOF: If $m = 0$ or $0 \in m$ then $0 \in m^+$.

$\langle 1 \rangle 3$. For any natural number n , if $P(n)$ then $P(n^+)$

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. ASSUME: $P(n)$

$\langle 2 \rangle 3$. LET: $m \in \omega$

$\langle 2 \rangle 4$. $m \in n$ or $m = n$ or $n \in m$

PROOF: $\langle 2 \rangle 2$

$\langle 2 \rangle 5$. CASE: $m \in n$ or $m = n$

PROOF: Then $m \in n^+$.

$\langle 2 \rangle 6$. CASE: $n \in m$

$\langle 3 \rangle 1$. PICK p such that $m = p^+$

$\langle 3 \rangle 2$. $n \in p$ or $n = p$

$\langle 3 \rangle 3$. CASE: $n \in p$

PROOF: Then $n^+ \in p^+ = m$ by Proposition 18.11.

$\langle 3 \rangle 4$. CASE: $n = p$

PROOF: Then $m = n^+$.

□

Corollary 18.12.1 (Trichotomy). *For any natural numbers m and n , exactly one of $m \in n$, $m = n$, $n \in m$ holds.*

PROOF:

$\langle 1 \rangle 1$. We never have $m \in n$ and $m = n$.

PROOF: By Corollary 17.2.1.

$\langle 1 \rangle 2$. We never have $m \in n$ and $n \in m$.

PROOF: Since m is a transitive set this would imply $m \in m$ contradicting Corollary 17.2.1.

$\langle 1 \rangle 3$. We never have $m = n$ and $n \in m$.

PROOF: By Corollary 17.2.1.

□

Proposition 18.13. *For any natural numbers m and n , we have $m \in n$ if and only if $m \subsetneq n$.*

PROOF:

$\langle 1 \rangle 1$. LET: m and n be natural numbers.
 $\langle 1 \rangle 2$. If $m \in n$ then $m \subsetneq n$.
 PROOF: Since n is a transitive set, and $m \neq n$ by Corollary 17.2.1.
 $\langle 1 \rangle 3$. If $m \subsetneq n$ then $m \in n$.
 $\langle 2 \rangle 1$. ASSUME: $m \subsetneq n$
 $\langle 2 \rangle 2$. $n \notin m$
 PROOF: Proposition 17.2.
 $\langle 2 \rangle 3$. $m \neq n$
 $\langle 2 \rangle 4$. $m \in n$
 PROOF: Trichotomy.
 \square

Definition 18.14. Given natural numbers m and n , we write $m < n$ iff $m \in n$.
 We write $m \leq n$ iff $m < n \vee m = n$.

Proposition 18.15. For natural numbers m and n , if $m \leq n$ and $n \leq m$ then $m = n$.

PROOF: We cannot have $m < n$ and $n < m$ by trichotomy. \square

Proposition 18.16. For natural numbers m , n and k , if $m < n$ then $m + k < n + k$.

PROOF:
 $\langle 1 \rangle 1$. LET: $m, n \in \omega$
 $\langle 1 \rangle 2$. ASSUME: $m < n$
 $\langle 1 \rangle 3$. $m + 0 < n + 0$
 $\langle 1 \rangle 4$. $\forall k \in \omega. m + k < n + k \Rightarrow m + k^+ < n + k^+$
 PROOF: By Proposition 18.11.
 \square

Proposition 18.17. For natural numbers m , n and k , if $m < n$ and $k \neq 0$ then $mk < nk$.

PROOF:
 $\langle 1 \rangle 1$. LET: $m, n \in \omega$
 $\langle 1 \rangle 2$. ASSUME: $m < n$
 $\langle 1 \rangle 3$. $m1 < n1$
 $\langle 1 \rangle 4$. For all $k \in \omega$, if $k \neq 0$ and $mk < nk$ then $m(k + 1) < n(k + 1)$
 PROOF:

$$\begin{aligned}
 m(k + 1) &= mk + m \\
 &< mk + n && \text{(Proposition 18.16)} \\
 &< nk + n && \text{(Proposition 18.16)} \\
 &= n(k + 1)
 \end{aligned}$$

\square

Proposition 18.18. For any nonempty set of natural numbers E , there exists $k \in E$ such that $\forall m \in E. k \leq m$.

PROOF:

- $\langle 1 \rangle 1$. LET: $E \subseteq \omega$
- $\langle 1 \rangle 2$. ASSUME: there is no $k \in E$ such that $\forall m \in E. k \leq m$.
PROVE: $E = \emptyset$
- $\langle 1 \rangle 3$. $\forall n \in \omega. n \notin E$
 - $\langle 2 \rangle 1$. LET: $P(n)$ be the property: $\forall m < n. m \notin E$
 - $\langle 2 \rangle 2$. $P(0)$
PROOF: Vacuous.
 - $\langle 2 \rangle 3$. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$
 - $\langle 3 \rangle 1$. LET: $n \in \omega$
 - $\langle 3 \rangle 2$. ASSUME: $\forall m < n. m \notin E$
 - $\langle 3 \rangle 3$. $n \notin E$
PROOF: From $\langle 1 \rangle 2$.
 - $\langle 3 \rangle 4$. $\forall m < n+1. m \notin E$

□

Definition 18.19 (Equivalent). Sets E and F are *equivalent*, $E \sim F$, iff there exists a one-to-one correspondence between them.

Proposition 18.20. For any set X , equivalence is an equivalence relation on $\mathcal{P}X$.

PROOF: Easy.

Proposition 18.21. Let n be a natural number. Let X be a proper subset of n . Then there exists $m < n$ such that $X \sim m$.

PROOF:

- $\langle 1 \rangle 1$. LET: $P(n)$ be the property: for every proper subset $X \subsetneq n$, there exists $m < n$ such that $X \sim m$.
- $\langle 1 \rangle 2$. $P(0)$
PROOF: Vacuous.
- $\langle 1 \rangle 3$. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$
 - $\langle 2 \rangle 1$. LET: $n \in \omega$
 - $\langle 2 \rangle 2$. ASSUME: $P(n)$
 - $\langle 2 \rangle 3$. LET: X be a proper subset of $n+1$
 - $\langle 2 \rangle 4$. CASE: $X - \{n\} = n$
PROOF: Then $X = n$ so $X \sim n < n+1$.
 - $\langle 2 \rangle 5$. CASE: $X - \{n\} \subsetneq n$
 - $\langle 3 \rangle 1$. PICK $m < n$ such that $X - \{n\} \sim m$
 - $\langle 3 \rangle 2$. $X \sim m$ or $X \sim m+1$
PROOF: If $n \in X$ then $X \sim m+1$. If $n \notin X$ then $X \sim m$.

□

Proposition 18.22. For every natural number n , we have n is not equivalent to a proper subset of n .

PROOF:

⟨1⟩1. LET: $P(n)$ be the property: every one-to-one function $n \rightarrow n$ is onto.

⟨1⟩2. $P(0)$

PROOF: The only function $0 \rightarrow 0$ is \emptyset .

⟨1⟩3. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

⟨2⟩1. LET: $n \in \omega$

⟨2⟩2. ASSUME: $P(n)$

⟨2⟩3. ASSUME: $f : n+1 \rightarrow n+1$ is one-to-one.

⟨2⟩4. LET: $g : n \rightarrow n$ be the function

$$g(k) = \begin{cases} f(k) & \text{if } f(k) < n \\ f(n) & \text{if } f(k) = n \end{cases}$$

PROOF: If $k < n$ and $f(k) = n$ then $f(n) < n$ since f is one-to-one.

⟨2⟩5. g is one-to-one.

⟨3⟩1. LET: $k, l < n$

⟨3⟩2. ASSUME: $g(k) = g(l)$

⟨3⟩3. CASE: $f(k) < n$ and $f(l) < n$

PROOF: Then $f(k) = g(k) = g(l) = f(l)$ so $k = l$ since f is one-to-one.

⟨3⟩4. CASE: $f(k) < n$ and $f(l) = n$

PROOF: Then $f(k) = g(k) = g(l) = f(n)$ contradicting the fact that f is one-to-one.

⟨3⟩5. CASE: $f(k) = n$ and $f(l) < n$

PROOF: Similar.

⟨3⟩6. CASE: $f(k) = n$ and $f(l) = n$

PROOF: Then $k = l$ since f is one-to-one.

⟨2⟩6. g maps n onto n .

PROOF: ⟨2⟩2

⟨2⟩7. f maps $n+1$ onto $n+1$.

⟨3⟩1. LET: $l < n+1$

⟨3⟩2. CASE: $l < n$

⟨4⟩1. PICK $k < n$ such that $g(k) = l$

⟨4⟩2. $f(k) = l$ or $f(n) = l$

⟨3⟩3. CASE: $l = n$

⟨4⟩1. CASE: $f(n) = n$

PROOF: Then $l \in \text{ran } f$ as required.

⟨4⟩2. CASE: $f(n) < n$

⟨5⟩1. PICK $k < n$ such that $g(k) = f(n)$

⟨5⟩2. $f(k) = n$

□

Corollary 18.22.1. *Equivalent natural numbers are equal.*

Definition 18.23 (Finite). A set is *finite* iff it is equivalent to a natural number; otherwise, it is *infinite*.

Proposition 18.24. *No finite set is equivalent to one of its proper subsets.*

PROOF: From Proposition 18.22. □

Proposition 18.25. ω is infinite.

PROOF: Since the function that maps n to $n + 1$ is a one-to-one correspondence between ω and $\omega - \{0\}$. \square

Proposition 18.26. *Every subset of a finite set is finite.*

PROOF: Proposition 18.21. \square

Definition 18.27 (Number of Elements). For any finite set E , the *number of elements* in E , $\sharp(E)$, is the unique natural number such that $E \sim \sharp(E)$.

Proposition 18.28. *Let E and F be finite sets. If $E \subseteq F$ then $\sharp(E) \leq \sharp(F)$.*

PROOF: Proposition 18.21. \square

Proposition 18.29. *Let E and F be disjoint finite sets. Then $E \cup F$ is finite and $\sharp(E \cup F) = \sharp(E) + \sharp(F)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $P(n)$ be the statement: $n \in \omega$ and for any $m \in \omega$, if $E \sim m$, $F \sim n$ and $E \cap F = \emptyset$, then $E \cup F \sim m + n$

$\langle 1 \rangle 2$. $P(0)$

$\langle 2 \rangle 1$. LET: $m \in \omega$

$\langle 2 \rangle 2$. LET: $E \sim m$ and $F \sim 0$

$\langle 2 \rangle 3$. $F = \emptyset$

$\langle 2 \rangle 4$. $E \cup F = E \sim m = m + 0$

$\langle 1 \rangle 3$. $\forall n \in \omega. P(n) \Rightarrow P(n + 1)$

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: $P(n)$

$\langle 2 \rangle 3$. LET: $m \in \omega$

$\langle 2 \rangle 4$. LET: $E \sim m$ and $F \sim n + 1$

$\langle 2 \rangle 5$. ASSUME: $E \cap F = \emptyset$

$\langle 2 \rangle 6$. PICK $f \in F$

$\langle 2 \rangle 7$. $F - \{f\} \sim n$

$\langle 2 \rangle 8$. $E \cap (F - \{f\}) = \emptyset$

$\langle 2 \rangle 9$. $E \cup (F - \{f\}) \sim m + n$

PROOF: $\langle 2 \rangle 2$

$\langle 2 \rangle 10$. $E \cup F \sim m + n + 1$

\square

Corollary 18.29.1. *The union of two finite sets is finite.*

PROOF: Since, if E and F are finite, then $E \cup F = (E - F) \cup (E \cap F) \cup (F - E)$ and these are finite and disjoint. \square