

# Summary of Halmos' Naive Set Theory

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# Chapter 1

## Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*,  $\in$ . If  $x \in A$  we say that  $x$  *belongs to*  $A$ ,  $x$  is an *element* of  $A$ , or  $x$  is *contained in*  $A$ . If this does not hold we write  $x \notin A$ .

**Axiom 1.1** (Axiom of Extensionality). *Two sets are equal if and only if they have the same elements.*

**Axiom 1.2** (Axiom of Comprehension, Aussonderungsaxiom). *To every set  $A$  and to every condition  $S(x)$  there corresponds a set  $B$  whose elements are exactly those elements  $x$  of  $A$  for which  $S(x)$  holds.*

**Axiom 1.3** (Axiom of Pairing). *For any two sets, there exists a set that they both belong to.*

**Axiom 1.4** (Union Axiom). *For every set  $A$ , there exists a set that contains all the elements that belong to at least one element of  $A$ .*

**Definition 1.5** (Subset). Let  $A$  and  $B$  be sets. We say that  $A$  is a *subset* of  $B$ , or  $B$  *includes*  $A$ , and write  $A \subseteq B$  or  $B \supseteq A$ , iff every element of  $A$  is an element of  $B$ .

**Axiom 1.6** (Power Set Axiom). *For any set  $A$ , there exists a set that contains all the subsets of  $A$ .*

**Axiom 1.7** (Axiom of Infinity). *There exists a set  $I$  such that:*

- *$I$  has an element that has no elements*
- *for all  $x \in I$ , there exists  $y \in I$  such that the elements of  $y$  are exactly  $x$  and the elements of  $x$ .*

## Chapter 2

# The Subset Relation

**Theorem 2.1.** *For any set  $A$ , we have  $A \subseteq A$ .*

PROOF: Every element of  $A$  is an element of  $A$ .  $\square$

**Theorem 2.2.** *For any sets  $A$ ,  $B$  and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ .*

PROOF: If every element of  $A$  is an element of  $B$ , and every element of  $B$  is an element of  $C$ , then every element of  $A$  is an element of  $C$ .  $\square$

**Theorem 2.3.** *For any sets  $A$  and  $B$ , if  $A \subseteq B$  and  $B \subseteq A$  then  $A = B$ .*

PROOF: If every element of  $A$  is an element of  $B$ , and every element of  $B$  is an element of  $A$ , then  $A$  and  $B$  have the same elements, and therefore are equal by the Axiom of Extensionality.  $\square$

**Definition 2.4** (Proper Subset). Let  $A$  and  $B$  be sets. We say that  $A$  is a *proper* subset of  $B$ , or  $B$  *properly* includes  $A$ , and write  $A \subsetneq B$  or  $B \supsetneq A$ , iff  $A \subseteq B$  and  $A \neq B$ .

## Chapter 3

# Comprehension Notation

**Definition 3.1.** Given a set  $A$  and a condition  $S(x)$ , we write  $\{x \in A : S(x)\}$  for the set whose elements are exactly those elements  $x$  of  $A$  for which  $S(x)$  holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality.  $\square$

**Theorem 3.2.** *There is no set that contains every set.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $A$  be a set.

PROVE: There exists a set  $B$  such that  $B \notin A$ .

$\langle 1 \rangle 2.$  LET:  $B = \{x \in A : x \notin x\}$

$\langle 1 \rangle 3.$  If  $B \in A$  then we have  $B \in B$  if and only if  $B \notin B$ .

$\langle 1 \rangle 4.$   $B \notin A$

$\square$

## Chapter 4

# Unordered Pairs

**Theorem 4.1.** *There exists a set with no elements.*

PROOF: Immediate from the Axiom of Infinity.  $\square$

**Definition 4.2** (Empty Set). The *empty set*  $\emptyset$  is the set with no elements.

**Theorem 4.3.** *For any set  $A$  we have  $\emptyset \subset A$ .*

PROOF: Vacuous.  $\square$

**Definition 4.4** ((Unordered) Pair). For any sets  $a$  and  $b$ , the *(unordered) pair*  $\{a, b\}$  is the set whose elements are just  $a$  and  $b$ .

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality.  $\square$

**Definition 4.5** (Singleton). For any set  $a$ , the *singleton*  $\{a\}$  is defined to be  $\{a, a\}$ .

# Chapter 5

## Unions

**Definition 5.1** (Union). For any set  $\mathcal{C}$ , the *union* of  $\mathcal{C}$ ,  $\bigcup \mathcal{C}$ , is the set whose elements are the elements of the elements of  $\mathcal{C}$ .

We write  $\bigcup_{X \in \mathcal{A}} t[X]$  for  $\bigcup \{t[X] \mid X \in \mathcal{A}\}$ .

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality.  $\square$

**Proposition 5.2.**

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of  $\emptyset$ .  $\square$

**Proposition 5.3.** *For any set  $A$ , we have  $\bigcup \{A\} = A$ .*

PROOF: For any  $x$ , we have  $x$  is an element of an element of  $\{A\}$  if and only if  $x$  is an element of  $A$ .  $\square$

**Definition 5.4.** We write  $A \cup B$  for  $\bigcup \{A, B\}$ .

**Proposition 5.5.** *For any set  $A$ , we have  $A \cup \emptyset = A$ .*

PROOF:  $x \in A \cup \emptyset$  iff  $x \in A$  or  $x \in \emptyset$ , iff  $x \in A$ .  $\square$

**Proposition 5.6** (Idempotence). *For any set  $A$ , we have  $A \cup A = A$ .*

PROOF:  $x \in A$  or  $x \in A$  is equivalent to  $x \in A$ .  $\square$

**Proposition 5.7.** *For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A \cup B = B$ .*

PROOF: For any  $x$ , the statement "if  $x \in A$  then  $x \in B$ " is equivalent to " $x \in A$  or  $x \in B$  if and only if  $x \in B$ ".  $\square$

**Proposition 5.8.** *For any sets  $a$  and  $b$ , we have  $\{a\} \cup \{b\} = \{a, b\}$ .*

PROOF: Immediate from definitions.  $\square$

## Chapter 6

# Intersections

**Definition 6.1** (Intersection). For any sets  $A$  and  $B$ , the *intersection*  $A \cap B$  is defined to be  $\{x \in A : x \in B\}$ .

**Proposition 6.2.** For any set  $A$ , we have  $A \cap \emptyset = \emptyset$ .

PROOF: There is no  $x$  such that  $x \in A$  and  $x \in \emptyset$ .  $\square$

**Proposition 6.3.** For any set  $A$ , we have

$$A \cap A = A .$$

PROOF: We have  $x \in A$  and  $x \in A$  if and only if  $x \in A$ .  $\square$

**Proposition 6.4.** For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A \cap B = A$ .

PROOF: For any  $x$ , the statement "if  $x \in A$  then  $x \in B$ " is equivalent to " $x \in A$  and  $x \in B$  if and only if  $x \in A$ ".  $\square$

**Proposition 6.5.** For any sets  $A$ ,  $B$  and  $C$ , we have  $C \subseteq A$  if and only if  $(A \cap B) \cup C = A \cap (B \cup C)$ .

PROOF: The statement "if  $x \in C$  then  $x \in A$ " is equivalent to the statement " $((x \in A \wedge x \in B) \vee x \in C) \Leftrightarrow (x \in A \wedge (x \in B \vee x \in C))$ ".  $\square$

**Definition 6.6** (Disjoint). Two sets  $A$  and  $B$  are *disjoint* if and only if  $A \cap B = \emptyset$ .

**Definition 6.7** (Pairwise Disjoint). Let  $A$  be a set. We say the elements of  $A$  are *pairwise disjoint* if and only if, for all  $x, y \in A$ , if  $x \cap y \neq \emptyset$  then  $x = y$ .

**Definition 6.8** (Intersection). For any nonempty set  $\mathcal{C}$ , the *intersection* of  $\mathcal{C}$ ,  $\bigcap \mathcal{C}$ , is the set that contains exactly those sets that belong to every element of  $\mathcal{C}$ .

We write  $\bigcap_{X \in \mathcal{A}} t[X]$  for  $\bigcap \{t[X] \mid X \in \mathcal{A}\}$ .



PROOF:

⟨1⟩1. LET:  $\mathcal{C}$  be a nonempty set.

⟨1⟩2. There exists a set  $I$  whose elements are exactly the sets that belong to every element of  $\mathcal{C}$ .

PROOF: Pick  $A \in \mathcal{C}$ , and take  $I = \{x \in A : \forall X \in \mathcal{C}. x \in X\}$ .

⟨1⟩3. For any sets  $I, J$ , if the elements of  $I$  and  $J$  are exactly the sets that belong to every element of  $\mathcal{C}$  then  $I = J$ .

PROOF: Axiom of Extensionality.

□

## Chapter 7

# Unordered Triples

**Definition 7.1** ((Unordered) Triple). Given sets  $a_1, \dots, a_n$ , define the (*unordered*) *n*-tuple  $\{a_1, \dots, a_n\}$  to be

$$\{a_1, \dots, a_n\} := \{a_1\} \cup \dots \cup \{a_n\} \ .$$

## Chapter 8

# Relative Complements

**Definition 8.1** (Relative Complement). For any sets  $A$  and  $B$ , the *difference* or *relative complement*  $A - B$  is defined to be

$$A - B := \{x \in A : x \notin B\} .$$

**Proposition 8.2.** For any sets  $A$  and  $E$ , we have  $A \subseteq E$  if and only if

$$E - (E - A) = A$$

PROOF:

⟨1⟩1. LET:  $A$  and  $E$  be sets.

⟨1⟩2. If  $A \subseteq E$  then  $E - (E - A) = A$

⟨2⟩1. ASSUME:  $A \subseteq E$

⟨2⟩2.  $E - (E - A) \subseteq A$

PROOF: If  $x \in E$  and  $x \notin E - A$  then  $x \in A$ .

⟨2⟩3.  $A \subseteq E - (E - A)$

PROOF: If  $x \in A$  then  $x \in E$  and  $x \notin E - A$ .

⟨1⟩3. If  $E - (E - A) = A$  then  $A \subseteq E$ .

PROOF: Since  $E - (E - A) \subseteq E$ .

□

**Proposition 8.3.** For any set  $E$  we have

$$E - \emptyset = E$$

PROOF:  $x \in E$  if and only if  $x \in E$  and  $x \notin \emptyset$ . □

**Proposition 8.4.** For any set  $E$  we have

$$E - E = \emptyset .$$

PROOF: There is no  $x$  such that  $x \in E$  and  $x \notin E$ . □

**Proposition 8.5.** For any sets  $A$  and  $E$ , we have

$$A \cap (E - A) = \emptyset .$$

PROOF: There is no  $x$  such that  $x \in A$  and  $x \in E - A$ .  $\square$

**Proposition 8.6.** *Let  $A$  and  $E$  be sets. Then  $A \subseteq E$  if and only if*

$$A \cup (E - A) = E .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $E$  be sets.

$\langle 1 \rangle 2$ . If  $A \subseteq E$  then  $A \cup (E - A) = E$ .

$\langle 2 \rangle 1$ . ASSUME:  $A \subseteq E$

$\langle 2 \rangle 2$ .  $A \cup (E - A) \subseteq E$

PROOF: If  $x \in A$  or  $x \in E - A$  then  $x \in E$ .

$\langle 2 \rangle 3$ .  $E \subseteq A \cup (E - A)$

PROOF: If  $x \in E$  then either  $x \in A$  or  $x \notin A$ . In the latter case,  $x \in E - A$ .

$\langle 1 \rangle 3$ . If  $A \cup (E - A) = E$  then  $A \subseteq E$

PROOF: Since  $A \subseteq A \cup (E - A)$ .

$\square$

**Proposition 8.7.** *Let  $A$ ,  $B$  and  $E$  be sets. Then:*

1. *If  $A \subseteq B$  then  $E - B \subseteq E - A$ .*

2. *If  $A \subseteq E$  and  $E - B \subseteq E - A$  then  $A \subseteq B$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$ ,  $B$  and  $E$  be sets.

$\langle 1 \rangle 2$ . If  $A \subseteq B$  then  $E - B \subseteq E - A$ .

PROOF: If  $A \subseteq B$ ,  $x \in E$  and  $x \notin B$ , then we have  $x \in E$  and  $x \notin A$ .

$\langle 1 \rangle 3$ . If  $A \subseteq E$  and  $E - B \subseteq E - A$  then  $A \subseteq B$ .

$\langle 2 \rangle 1$ . ASSUME:  $A \subseteq E$

$\langle 2 \rangle 2$ . ASSUME:  $E - B \subseteq E - A$

$\langle 2 \rangle 3$ . LET:  $x \in A$

$\langle 2 \rangle 4$ .  $x \in E$

$\langle 2 \rangle 5$ .  $x \notin E - A$

$\langle 2 \rangle 6$ .  $x \notin E - B$

$\langle 2 \rangle 7$ .  $x \in B$

$\square$

**Example 8.8.** We cannot remove the hypothesis  $A \subseteq E$  in item 2 above. Let  $E = \emptyset$ ,  $A = \{\emptyset\}$  and  $B = \emptyset$ . Then  $E - B = E - A = \emptyset$  but  $A \not\subseteq B$ .

**Proposition 8.9** (De Morgan's Law). *For any sets  $A$ ,  $B$  and  $E$ , we have  $E - (A \cup B) = (E - A) \cap (E - B)$ .*

PROOF:  $(x \in E \wedge \neg(x \in A \vee x \in B)) \Leftrightarrow (x \in E \wedge x \notin A \wedge x \in E \wedge x \notin B)$ .  $\square$

**Proposition 8.10** (De Morgan's Law). *For any sets  $A$ ,  $B$  and  $E$ , we have  $E - (A \cap B) = (E - A) \cup (E - B)$ .*

PROOF:  $(x \in E \vee \neg(x \in A \wedge x \in B)) \Leftrightarrow (x \in E \wedge x \notin A) \vee (x \in E \wedge x \notin B)$ .  $\square$

**Proposition 8.11.** *For any sets  $A$ ,  $B$  and  $E$ , if  $A \subseteq E$  then*

$$A - B = A \cap (E - B) .$$

PROOF: If  $A \subseteq E$  then we have  $(x \in A \wedge x \notin B) \Leftrightarrow (x \in A \wedge x \in E \wedge x \notin B)$ .  $\square$

**Proposition 8.12.** *For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A - B = \emptyset$ .*

PROOF: Both are equivalent to the statement that there is no  $x$  such that  $x \in A$  and  $x \notin B$ .  $\square$

**Proposition 8.13.** *For any sets  $A$  and  $B$ , we have*

$$A - (A - B) = A \cap B .$$

PROOF:  $(x \in A \wedge \neg(x \in A \wedge x \notin B)) \Leftrightarrow x \in A \wedge x \in B$ .  $\square$

**Proposition 8.14.** *For any sets  $A$ ,  $B$  and  $C$ , we have*

$$A \cap (B - C) = (A \cap B) - (A \cap C) .$$

PROOF:  $(x \in A \wedge x \in B \wedge x \notin C) \Leftrightarrow (x \in A \wedge x \in B \wedge \neg(x \in A \wedge x \in C))$ .  $\square$

**Proposition 8.15.** *For any sets  $A$ ,  $B$ ,  $C$  and  $E$ , if  $(A \cap B) - C \subseteq E$  then we have*

$$A \cap B \subseteq (A \cap C) \cup (B \cap (E - C)) .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in A \cap B$

PROVE:  $x \in (A \cap C) \cup (B \cap (E - C))$

$\langle 1 \rangle 2$ . CASE:  $x \in C$

PROOF: Then  $x \in A \cap C$ .

$\langle 1 \rangle 3$ . CASE:  $x \notin C$

PROOF: Then  $x \in E$  and so  $x \in B \cap (E - C)$ .

$\square$

**Proposition 8.16.** *For any sets  $A$ ,  $B$ ,  $C$  and  $E$ , we have*

$$(A \cup C) \cap (B \cup (E - C)) \subseteq A \cup B .$$

PROOF: The statement  $(x \in A \vee x \in C) \wedge (x \in B \vee (x \in E \wedge x \notin C))$  implies  $x \in A \vee x \in B$ .  $\square$

**Proposition 8.17** (De Morgan's Law). *Let  $E$  be a set and  $\mathcal{C}$  a nonempty set. Then*

$$E - \bigcup_{X \in \mathcal{C}} X = \bigcap_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy.  $\square$

**Proposition 8.18** (De Morgan's Law). *Let  $E$  be a set and  $\mathcal{C}$  a nonempty set. Then*

$$E - \bigcap_{X \in \mathcal{C}} X = \bigcup_{X \in \mathcal{C}} (E - X) .$$

PROOF: Easy.  $\square$

## Chapter 9

# Symmetric Difference

**Definition 9.1** (Symmetric Difference). For any sets  $A$  and  $B$ , the *symmetric difference*  $A + B$  is defined to be

$$A + B := (A - B) \cup (B - A) .$$

**Proposition 9.2.** For any sets  $A$  and  $B$ , we have

$$A + B = B + A$$

PROOF: From the commutativity of union.  $\square$

**Proposition 9.3.** For any sets  $A$ ,  $B$  and  $C$ , we have

$$A + (B + C) = (A + B) + C .$$

PROOF: Each is the set of all  $x$  that belong to either exactly one or all three of  $A$ ,  $B$  and  $C$ .  $\square$

**Proposition 9.4.** For any set  $A$ , we have

$$A + \emptyset = A .$$

PROOF:

$$\begin{aligned} A + \emptyset &= (A - \emptyset) \cup (\emptyset - A) \\ &= A \cup \emptyset \\ &= A \end{aligned}$$

$\square$

**Proposition 9.5.** For any set  $A$  we have

$$A + A = \emptyset .$$

PROOF:

$$\begin{aligned} A + A &= (A - A) \cup (A - A) \\ &= \emptyset \cup \emptyset \\ &= \emptyset \end{aligned}$$

$\square$

# Chapter 10

## Power Sets

**Definition 10.1** (Power Set). For any set  $A$ , the *power set* of  $A$ ,  $\mathcal{P}A$ , is the set whose elements are exactly the subsets of  $A$ .

PROOF: This exists by the Power Set Axiom and Axiom of Comprehension, and is unique by the Axiom of Extensionality.  $\square$

**Proposition 10.2.**

$$\mathcal{P}\emptyset = \{\emptyset\}$$

PROOF: The only subset of  $\emptyset$  is  $\emptyset$ .  $\square$

**Proposition 10.3.** For any set  $a$ , we have

$$\mathcal{P}\{a\} = \{\emptyset, \{a\}\} .$$

PROOF: The only subsets of  $\{a\}$  are  $\emptyset$  and  $\{a\}$ .  $\square$

**Proposition 10.4.** For any sets  $a$  and  $b$ , we have

$$\mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} .$$

PROOF: The only subsets of  $\{a, b\}$  are  $\emptyset$ ,  $\{a\}$ ,  $\{b\}$  and  $\{a, b\}$ .  $\square$

**Proposition 10.5.** For any nonempty set  $\mathcal{C}$  we have

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}X = \mathcal{P}\left(\bigcap \mathcal{C}\right) .$$

PROOF:

$$x \in \bigcup_{X \in \mathcal{C}} \mathcal{P}X \Leftrightarrow \forall X \in \mathcal{C}. x \subseteq X$$

$$\Leftrightarrow \forall X \in \mathcal{C}. \forall y \in x. y \in X$$

$$\Leftrightarrow \forall y \in x. \forall X \in \mathcal{C}. y \in X$$

$$\Leftrightarrow x \subseteq \bigcap \mathcal{C}$$

$\square$

**Proposition 10.6.** *For any set  $\mathcal{C}$  we have*

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}X \subseteq \mathcal{P} \bigcup \mathcal{C} .$$

PROOF: If there exists  $X \in \mathcal{C}$  such that  $x \subseteq X$  then  $x \subseteq \bigcup \mathcal{C}$ .  $\square$

**Proposition 10.7.** *For any set  $E$ , we have*

$$\bigcap \mathcal{P}E = \emptyset .$$

PROOF: Since  $\emptyset \in \mathcal{P}E$ .  $\square$

**Proposition 10.8.** *For any sets  $E$  and  $F$ , if  $E \subseteq F$  then  $\mathcal{P}E \subseteq \mathcal{P}F$ .*

PROOF: If  $E \subseteq F$  and  $X \subseteq E$  then  $X \subseteq F$ .  $\square$



# Chapter 11

## Ordered Pairs

**Definition 11.1** (Ordered Pair). For any sets  $a$  and  $b$ , the *ordered pair*  $(a, b)$  is defined by

$$(a, b) := \{\{a\}, \{a, b\}\} .$$

**Proposition 11.2.** For any sets  $a, b, x$  and  $y$ , if  $(a, b) = (x, y)$  then  $a = x$  and  $b = y$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $a, b, x$  and  $y$  be sets.

$\langle 1 \rangle 2$ . ASSUME:  $(a, b) = (x, y)$

$\langle 1 \rangle 3$ .  $a = x$

PROOF:  $\{a\} = \bigcap (a, b) = \bigcap (x, y) = \{x\}$ .

$\langle 1 \rangle 4$ .  $\{a, b\} = \{x, y\}$

$\langle 1 \rangle 5$ . CASE:  $a = b$

$\langle 2 \rangle 1$ .  $x = y$

PROOF: Since  $\{x, y\} = \{a, b\}$  is a singleton.

$\langle 2 \rangle 2$ .  $b = y$

PROOF:  $b = a = x = y$

$\langle 1 \rangle 6$ . CASE:  $a \neq b$

$\langle 2 \rangle 1$ .  $x \neq y$

PROOF: Since  $\{x, y\} = \{a, b\}$  is not a singleton.

$\langle 2 \rangle 2$ .  $b = y$

PROOF:  $\{b\} = \{a, b\} - \{a\} = \{x, y\} - \{x\} = \{y\}$ .

□

**Definition 11.3** (Cartesian Product). For any sets  $A$  and  $B$ , the *Cartesian product*  $A \times B$  is

$$A \times B := \{p \in \mathcal{PP}(A \cup B) : \exists a \in A. \exists b \in B. p = (a, b)\} .$$

**Proposition 11.4.** For any sets  $A, B$  and  $X$ , we have

$$(A - B) \times X = (A \times X) - (B \times X) .$$

PROOF: Easy.  $\square$

**Proposition 11.5.** *For any sets  $A$  and  $B$ , we have  $A \times B = \emptyset$  if and only if  $A = \emptyset$  or  $B = \emptyset$ .*

PROOF: Easy.  $\square$

**Proposition 11.6.** *For any sets  $A$ ,  $B$ ,  $X$  and  $Y$ , if  $A \subseteq X$  and  $B \subseteq Y$  then  $A \times B \subseteq X \times Y$ . The converse holds assuming  $A \neq \emptyset$  and  $B \neq \emptyset$ .*

PROOF: Easy.  $\square$

# Chapter 12

## Relations

**Definition 12.1** (Relation). A *relation* is a set of ordered pairs.

If  $R$  is a relation, we write  $xRy$  for  $(x, y) \in R$ .

Given sets  $X$  and  $Y$ , a relation *between  $X$  and  $Y$*  is a subset of  $X \times Y$ .

Given a set  $X$ , a relation *on  $X$*  is a relation between  $X$  and  $X$ .

**Definition 12.2** (Domain). The *domain* of a relation  $R$  is the set

$$\text{dom } R := \{x \in \bigcup \bigcup R : \exists y. (x, y) \in R\} .$$

**Definition 12.3** (Range). The *range* of a relation  $R$  is the set

$$\text{ran } R := \{y \in \bigcup \bigcup R : \exists x. (x, y) \in R\} .$$

**Definition 12.4** (Reflexive). Let  $R$  be a relation on  $X$ . Then  $R$  is *reflexive* iff, for all  $x \in X$ , we have  $xRx$ .

**Definition 12.5** (Symmetric). Let  $R$  be a relation on  $X$ . Then  $R$  is *symmetric* iff, whenever  $xRy$ , then  $yRx$ .

**Definition 12.6** (Transitive). Let  $R$  be a relation on  $X$ . Then  $R$  is *transitive* iff, whenever  $xRy$  and  $yRz$ , then  $xRz$ .

**Definition 12.7** (Equivalence Relation). Let  $R$  be a relation on  $X$ . Then  $R$  is an *equivalence relation* iff it is reflexive, symmetric and transitive.

**Definition 12.8** (Partition). Let  $X$  be a set. A *partition* of  $X$  is a pairwise disjoint set of nonempty subsets of  $X$  whose union is  $X$ .

**Definition 12.9** (Equivalence Class). Let  $R$  be an equivalence relation on  $X$ . Let  $x \in X$ . The *equivalence class* of  $x$  with respect to  $R$  is

$$x/R := \{y \in X : xRy\} .$$

We write  $X/R$  for the set of all equivalence classes with respect to  $R$ .

**Definition 12.10** (Induced). Let  $P$  be a partition of  $X$ . The relation *induced* by  $P$  is  $X/P$  where  $x(X/P)y$  iff there exists  $X \in P$  such that  $x \in X$  and  $y \in X$ .

**Theorem 12.11.** Let  $R$  be an equivalence relation on  $X$ . Then  $X/R$  is a partition of  $X$  that induces the relation  $R$ .

PROOF: Easy.  $\square$

**Theorem 12.12.** Let  $P$  be a partition of  $X$ . Then  $X/P$  is an equivalence relation on  $X$ , and  $P = X/(X/P)$ .

PROOF: Easy.  $\square$

**Definition 12.13** (Composition). Let  $R$  be a relation between  $X$  and  $Y$ , and  $S$  a relation between  $Y$  and  $Z$ . The *composite* or *relative product*  $S \circ R = SR$  is the relation between  $X$  and  $Z$  defined by

$$x(S \circ R)z \Leftrightarrow \exists y \in Y (xRy \wedge ySz) .$$

**Proposition 12.14.** Let  $R$  be a relation between  $X$  and  $Y$ ,  $S$  a relation between  $Y$  and  $Z$ , and  $T$  a relation between  $Z$  and  $W$ . Then

$$T(SR) = (TS)R .$$

PROOF: Easy.  $\square$

**Example 12.15.** Composition of relations is not commutative in general. Let  $X = \{a, b\}$  where  $a \neq b$ . Let  $R = \{(a, a), (b, a)\}$  and  $S = \{(a, b), (b, b)\}$ . Then  $SR = S$  but  $RS = R \neq S$ .

**Proposition 12.16.** A relation  $R$  is transitive if and only if  $RR \subseteq R$ .

PROOF: Easy.  $\square$

**Definition 12.17** (Inverse). Let  $R$  be a relation between  $X$  and  $Y$ . The *inverse* or *converse*  $R^{-1}$  is the relation between  $Y$  and  $X$  defined by

$$yR^{-1}x \Leftrightarrow xRy .$$

**Proposition 12.18.** For any relation  $R$ , we have

$$\text{dom } R^{-1} = \text{ran } R .$$

PROOF: Easy.  $\square$

**Proposition 12.19.** For any relation  $R$ , we have

$$\text{ran } R^{-1} = \text{dom } R .$$

PROOF: Easy.  $\square$

**Proposition 12.20.** *Let  $R$  be a relation between  $X$  and  $Y$ , and  $S$  a relation between  $Y$  and  $Z$ . Then*

$$(SR)^{-1} = R^{-1}S^{-1} .$$

PROOF: Easy.  $\square$

**Proposition 12.21.** *A relation  $R$  is symmetric if and only if  $R \subseteq R^{-1}$ .*

PROOF: Easy.  $\square$

**Definition 12.22** (Identity Relation). For any set  $X$ , the *identity relation*  $I_X$  on  $X$  is

$$I_X = \{(x, x) : x \in X\} .$$

**Proposition 12.23.** *Let  $R$  be a relation between  $X$  and  $Y$ . Then*

$$I_Y R = R I_X = R .$$

PROOF: Easy.  $\square$

**Proposition 12.24.** *A relation  $R$  on a set  $X$  is reflexive if and only if  $I_X \subseteq R$ .*

PROOF: Easy.  $\square$

# Chapter 13

## Functions

**Definition 13.1** (Function). Let  $X$  and  $Y$  be sets. A *function*, *map*, *mapping*, *transformation* or *operator*  $f$  from  $X$  to  $Y$ ,  $f : X \rightarrow Y$ , is a relation  $f$  between  $X$  and  $Y$  such that, for all  $x \in X$ , there exists a unique  $f(x) \in Y$ , called the *value* of  $f$  at the *argument*  $x$ , such that  $(x, f(x)) \in f$ .

**Definition 13.2** (Onto). Let  $f : X \rightarrow Y$ . We say  $f$  maps  $X$  *onto*  $Y$  iff  $\text{ran } f = Y$ .

**Definition 13.3** (Image). Let  $f : X \rightarrow Y$  and  $A \subseteq X$ . The *image* of  $A$  under  $f$  is

$$f(A) := \{f(x) : x \in A\} .$$

**Definition 13.4** (Inclusion Map). Let  $Y$  be a set and  $X \subseteq Y$ . Then the *inclusion map*  $i : X \hookrightarrow Y$  is the function defined by  $i(x) = x$  for all  $x \in X$ .

**Proposition 13.5.** *For any set  $X$ , the identity relation  $I_X$  is a function  $X \rightarrow X$ .*

PROOF: Easy.  $\square$

**Definition 13.6** (Restriction). Let  $f : Y \rightarrow Z$  and  $X \subseteq Y$ . The *restriction* of  $f$  to  $X$  is the function  $f \upharpoonright X : X \rightarrow Z$  defined by

$$(f \upharpoonright X)(x) = f(x) \quad (x \in X) .$$

Given sets  $X$ ,  $Y$  and  $Z$  with  $X \subseteq Y$ , if  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$ , we say  $g$  is an *extension* of  $f$  to  $Y$  iff  $f = g \upharpoonright X$ .

**Definition 13.7** (Projection). Given sets  $X$  and  $Y$ , the *projection* maps  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  are defined by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y \quad (x \in X, y \in Y) .$$

**Definition 13.8** (Canonical Map). Let  $X$  be a set and  $R$  an equivalence relation on  $X$ . The *canonical map*  $\pi : X \rightarrow X/R$  is the map defined by  $\pi(x) = x/R$ .

**Definition 13.9** (One-to-One). A function  $f : X \rightarrow Y$  is *one-to-one*, or a *one-to-one correspondence*, iff, for all  $x, y \in X$ , if  $f(x) = f(y)$  then  $x = y$ .

**Proposition 13.10.** Let  $f : X \rightarrow Y$ . Then the following are equivalent:

1.  $f$  is one-to-one.
2. For all  $A, B \subseteq X$ , we have  $f(A \cap B) = f(A) \cap f(B)$ .
3. For all  $A \subseteq X$ , we have  $f(X - A) \subseteq Y - f(A)$ .

PROOF: Easy.  $\square$

**Proposition 13.11.** Let  $f : X \rightarrow Y$ . Then  $f$  maps  $X$  onto  $Y$  if and only if, for all  $A \subseteq X$ , we have  $Y - f(A) \subseteq f(X - A)$ .

PROOF: Easy.  $\square$

# Chapter 14

## Families

**Definition 14.1** (Family). Let  $I$  and  $X$  be sets. A *family* of elements of  $X$  indexed by  $I$  is a function  $a : I \rightarrow X$ . We write  $a_i$  for  $a(i)$ , and  $\{a_i\}_{i \in I}$  for  $a$ .

**Proposition 14.2** (Generalized Associative Law for Unions). Let  $\{I_j\}_{j \in J}$  be a family of sets. Let  $K = \bigcup_{j \in J} I_j$ . Let  $\{A_k\}_{k \in K}$  be a family of sets indexed by  $K$ . Then

$$\bigcup_{k \in K} A_k = \bigcup_{j \in J} \bigcup_{i \in I_j} A_i .$$

PROOF: Easy.  $\square$

**Proposition 14.3** (Generalized Commutative Law for Unions). Let  $\{I_j\}_{j \in J}$  be a family of sets. Let  $f : J \rightarrow J$  be a one-to-one correspondence from  $J$  onto  $J$ . Then

$$\bigcup_{j \in J} I_j = \bigcup_{j \in J} I_{f(j)} .$$

PROOF: Easy.  $\square$

**Proposition 14.4** (Generalized Associative Law for Intersections). Let  $\{I_j\}_{j \in J}$  be a nonempty family of nonempty sets. Let  $K = \bigcup_{j \in J} I_j$ . Let  $\{A_k\}_{k \in K}$  be a family of sets indexed by  $K$ . Then

$$\bigcap_{k \in K} A_k = \bigcap_{j \in J} \bigcap_{i \in I_j} A_i .$$

PROOF: Easy.  $\square$

**Proposition 14.5** (Generalized Commutative Law for Intersections). Let  $\{I_j\}_{j \in J}$  be a nonempty family of sets. Let  $f : J \rightarrow J$  be a one-to-one correspondence from  $J$  onto  $J$ . Then

$$\bigcap_{j \in J} I_j = \bigcap_{j \in J} I_{f(j)} .$$



PROOF: Easy.  $\square$

**Proposition 14.6.** *Let  $B$  be a set and  $\{A_i\}_{i \in I}$  a family of sets. Then*

$$B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i)$$

PROOF: Easy.  $\square$

**Proposition 14.7.** *Let  $B$  be a set and  $\{A_i\}_{i \in I}$  a nonempty family of sets. Then*

$$B \cup \bigcap_{i \in I} A_i = \bigcap_{i \in I} (B \cup A_i)$$

PROOF: Easy.  $\square$

**Definition 14.8** (Cartesian Product of a Family of Sets). Let  $\{A_i\}_{i \in I}$  be a family of sets. The *Cartesian product*  $\times_{i \in I} A_i$  is the set of all families  $\{a_i\}_{i \in I}$  such that  $\forall i \in I. a_i \in A_i$ .

We write  $A^I$  for  $\times_{i \in I} A$ .

**Definition 14.9** (Projection). Let  $\{A_i\}_{i \in I}$  be a family of sets and  $i \in I$ . The *projection* function  $\pi_i : \times_{i \in I} A_i \rightarrow A_i$  is defined by  $\pi_i(a) = a_i$ .

**Proposition 14.10.** *Let  $\{A_i\}_{i \in I}$  and  $\{B_j\}_{j \in J}$  be families of sets. Then*

$$\left( \bigcup_{i \in I} A_i \right) \times \left( \bigcup_{j \in J} B_j \right) = \bigcup_{i \in I} \bigcup_{j \in J} (A_i \times B_j) .$$

PROOF: Easy.  $\square$

**Proposition 14.11.** *Let  $\{A_i\}_{i \in I}$  and  $\{B_j\}_{j \in J}$  be nonempty families of sets. Then*

$$\left( \bigcap_{i \in I} A_i \right) \times \left( \bigcap_{j \in J} B_j \right) = \bigcap_{i \in I} \bigcap_{j \in J} (A_i \times B_j) .$$

PROOF: Easy.  $\square$

**Proposition 14.12.** *Let  $f : X \rightarrow Y$ . Let  $\{A_i\}_{i \in I}$  be a family of subsets of  $X$ . Then*

$$f \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} f(A_i) .$$

PROOF: Easy.  $\square$

**Example 14.13.** It is not true in general that, if  $f : X \rightarrow Y$  and  $\{A_i\}_{i \in I}$  is a nonempty family of subsets of  $X$ , then  $f(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f(A_i)$ .

Take  $X = \{a, b\}$  and  $Y = \{c\}$  where  $a \neq b$ . Take  $I = \{i, j\}$  with  $i \neq j$ . Let  $A_i = \{a\}$  and  $A_j = \{b\}$ . Let  $f$  be the unique function  $X \rightarrow Y$ . Then  $f(\bigcap_{i \in I} A_i) = f(\emptyset) = \emptyset$  but  $\bigcap_{i \in I} f(A_i) = \{c\}$ .

## Chapter 15

# Inverses and Composites

**Definition 15.1** (Inverse). Given a function  $f : X \rightarrow Y$ , the *inverse* of  $f$  is the function  $f^{-1} : \mathcal{P}Y \rightarrow \mathcal{P}X$  defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \ .$$

We call  $f^{-1}(B)$  the *inverse image* of  $B$  under  $f$ .

**Proposition 15.2.** *Let  $f : X \rightarrow Y$ . Then  $f$  maps  $X$  onto  $Y$  if and only if the inverse image of any nonempty subset of  $Y$  is nonempty.*

PROOF: Easy.  $\square$

**Proposition 15.3.** *Let  $f : X \rightarrow Y$ . Then  $f$  is one-to-one if and only if the inverse image of any singleton subset of  $Y$  is a singleton.*

PROOF: Easy.  $\square$

**Proposition 15.4.** *Let  $f : X \rightarrow Y$ . Let  $B \subseteq Y$ . Then*

$$f(f^{-1}(B)) \subseteq B \ .$$

PROOF: Easy.  $\square$

**Proposition 15.5.** *Let  $f : X \rightarrow Y$ . Let  $A \subseteq X$ . Then*

$$A \subseteq f^{-1}(f(A)) \ .$$

*Equality holds if  $f$  is one-to-one.*

PROOF: Easy.  $\square$

**Proposition 15.6.** *Let  $f : X \rightarrow Y$ . Let  $\{B_i\}_{i \in I}$  be a family of subsets of  $Y$ . Then*

$$f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i) \ .$$

PROOF: Easy.  $\square$

**Proposition 15.7.** *Let  $f : X \rightarrow Y$ . Let  $\{B_i\}_{i \in I}$  be a nonempty family of subsets of  $Y$ . Then*

$$f^{-1} \left( \bigcap_{i \in I} B_i \right) = \bigcap_{i \in I} f^{-1}(B_i) .$$

PROOF: Easy.  $\square$

**Proposition 15.8.** *Let  $f : X \rightarrow Y$  and  $B \subseteq Y$ . Then  $f^{-1}(Y - B) = X - f^{-1}(B)$ .*

PROOF: Easy.  $\square$

**Proposition 15.9.** *Let  $f : X \rightarrow Y$  be one-to-one. Then the inverse of  $f$  as a relation,  $f^{-1}$ , is a function  $f^{-1} : \text{ran } f \rightarrow X$ , and for all  $y \in \text{ran } f$ , we have  $f^{-1}(y)$  is the unique  $x$  such that  $f(x) = y$ .*

PROOF: Easy.  $\square$

**Proposition 15.10.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . Then  $gf : X \rightarrow Z$  and, for all  $x \in X$ , we have*

$$(g \circ f)(x) = g(f(x)) .$$

PROOF: Easy.  $\square$

**Example 15.11.** Example 12.15 shows that function composition is not commutative in general.

**Proposition 15.12.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . Then*

$$(gf)^{-1} = f^{-1}g^{-1} : \mathcal{P}Z \rightarrow \mathcal{P}X .$$

PROOF: Easy.  $\square$

**Proposition 15.13.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$ . If  $gf = I_X$  then  $f$  is one-to-one and  $g$  maps  $Y$  onto  $X$ .*

PROOF: Easy.  $\square$

# Chapter 16

## Numbers

**Definition 16.1** (Successor). The *successor* of a set  $x$ ,  $x^+$ , is defined by

$$x^+ := x \cup \{x\} .$$

**Definition 16.2.** We define

$$0 = \emptyset$$

$$1 = 0^+$$

$$2 = 1^+$$

etc.

**Definition 16.3** (Characteristic Function). Let  $X$  be a set and  $A \subseteq X$ . The *characteristic function* of  $A$  is the function  $\chi_A : X \rightarrow 2$  defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

**Theorem 16.4.** Let  $X$  be a set. The function  $\chi : \mathcal{P}X \rightarrow 2^X$  that maps a subset  $A$  of  $X$  to  $\chi_A$  is a one-to-one correspondence.

PROOF: Easy.  $\square$

**Definition 16.5.** The set  $\omega$  of *natural numbers* is the set such that:

- $0 \in \omega$
- For all  $n \in \omega$  we have  $n^+ \in \omega$
- For any set  $X$ , if  $0 \in X$  and  $\forall n \in X. n^+ \in X$  then  $\omega \subseteq X$

PROOF: To show this exists, pick a set  $A$  such that  $0 \in A$  and  $\forall n \in A. n^+ \in A$  (by the Axiom of Infinity), and let  $\omega = \bigcap \{X \in \mathcal{P}A : 0 \in X \wedge \forall n \in X. n^+ \in X\}$ .  
 $\square$

**Definition 16.6** (Sequence). A *finite sequence* is a family whose index set is a natural number. An *infinite sequence* is a family whose index set is  $\omega$ .

Given a finite sequence of sets  $\{A_i\}_{i \in n^+}$ , we write  $\bigcup_{i=0}^n A_i$  for  $\bigcup_{i \in n^+} A_i$ . Given an infinite sequence of sets  $\{A_i\}_{i \in \omega}$ , we write  $\bigcup_{i=0}^{\infty} A_i$  for  $\bigcup_{i \in \omega} A_i$ .

We make similar definitions for  $\bigcap$  and  $\times$ .

# Chapter 17

## The Peano Axioms

**Theorem 17.1** (Principle of Mathematical Induction). *For any subset  $S$  of  $\omega$ , if  $0 \in S$  and  $\forall n \in S. n^+ \in S$ , then  $S = \omega$ .*

PROOF: From the definition of  $\omega$ .  $\square$

**Proposition 17.2.**

$$\forall n \in \omega. \forall x \in n. n \not\subseteq x$$

PROOF:

$\langle 1 \rangle 1.$   $\forall x \in 0. 0 \not\subseteq x$

PROOF: Vacuous.

$\langle 1 \rangle 2.$  For any natural number  $n$ , if  $\forall x \in n. n \not\subseteq x$  then  $\forall x \in n^+. n^+ \not\subseteq x$ .

$\langle 2 \rangle 1.$  LET:  $n$  be a natural number.

$\langle 2 \rangle 2.$  ASSUME:  $\forall x \in n. n \not\subseteq x$

$\langle 2 \rangle 3.$  LET:  $x \in n^+$

$\langle 2 \rangle 4.$  ASSUME: for a contradiction  $n^+ \subseteq x$

$\langle 2 \rangle 5.$   $x \in n$  or  $x = n$

$\langle 2 \rangle 6.$  CASE:  $x \in n$

PROOF: Then we have  $n \subseteq n^+ \subseteq x$  contradicting  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 7.$  CASE:  $x = n$

PROOF: Then we have  $n \in n^+ \subseteq x = n$  and  $n \subseteq n$  contradicting  $\langle 2 \rangle 2$ .

$\square$

**Corollary 17.2.1.** *For any natural number  $n$  we have  $n \notin n$ .*

**Corollary 17.2.2.** *For any natural number  $n$  we have  $n \neq n^+$ .*

**Definition 17.3** (Transitive Set). A set  $E$  is a *transitive* set iff, whenever  $x \in y \in E$ , then  $x \in E$ .

**Proposition 17.4.** *Every natural number is a transitive set.*

PROOF:

$\langle 1 \rangle 1.$   $0$  is a transitive set.

PROOF: Vacuously, if  $x \in y \in 0$  then  $x \in 0$ .

$\langle 1 \rangle 2$ . For any natural number  $n$ , if  $n$  is a transitive set, then  $n^+$  is a transitive set.

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $n$  is a transitive set.

$\langle 2 \rangle 3$ . LET:  $x \in y \in n^+$

$\langle 2 \rangle 4$ .  $y \in n$  or  $y = n$

$\langle 2 \rangle 5$ . CASE:  $y \in n$

$\langle 3 \rangle 1$ .  $x \in n$

PROOF:  $\langle 2 \rangle 2$ ,  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 5$ .

$\langle 3 \rangle 2$ .  $x \in n^+$

$\langle 2 \rangle 6$ . CASE:  $y = n$

$\langle 3 \rangle 1$ .  $x \in n$

PROOF:  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 6$

$\langle 3 \rangle 2$ .  $x \in n^+$

□

**Proposition 17.5.** *For any natural numbers  $m$  and  $n$ , if  $m^+ = n^+$  then  $m = n$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $m$  and  $n$  be natural numbers.

$\langle 1 \rangle 2$ . ASSUME:  $m^+ = n^+$

$\langle 1 \rangle 3$ .  $m \in m^+ = n^+$

$\langle 1 \rangle 4$ .  $m \in n$  or  $m = n$

$\langle 1 \rangle 5$ .  $n \in n^+ = m^+$

$\langle 1 \rangle 6$ .  $n \in m$  or  $n = m$

$\langle 1 \rangle 7$ . We cannot have  $m \in n$  and  $n \in m$

$\langle 2 \rangle 1$ . ASSUME: for a contradiction  $m \in n$  and  $n \in m$

$\langle 2 \rangle 2$ .  $m \in m$

PROOF: Since  $m$  is a transitive set (Proposition 17.4).

$\langle 2 \rangle 3$ . Q.E.D.

PROOF: This contradicts Proposition 17.2.

$\langle 1 \rangle 8$ .  $m = n$

□

**Theorem 17.6** (Recursion Theorem). *Let  $X$  be a set. Let  $a \in X$ . Let  $f : X \rightarrow X$ . There exists a function  $u : \omega \rightarrow X$  such that  $u(0) = a$  and, for all  $n \in \omega$ , we have  $u(n^+) = f(u(n))$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C} = \{A \in \mathcal{P}(\omega \times X) : (0, a) \in A \wedge \forall n \in \omega. \forall x \in X. (n, x) \in A \Rightarrow (n^+, f(x)) \in A\}$

$\langle 1 \rangle 2$ .  $\mathcal{C} \neq \emptyset$

PROOF:  $\omega \times X \in \mathcal{C}$

$\langle 1 \rangle 3$ . LET:  $u = \bigcap \mathcal{C}$

$\langle 1 \rangle 4$ .  $u \in \mathcal{C}$

$\langle 1 \rangle 5$ .  $u$  is a function.

$\langle 2 \rangle 1$ . LET:  $P(n)$  be the property:  $\forall x, y \in X. (n, x) \in u \wedge (n, y) \in u \Rightarrow x = y$   
 $\langle 2 \rangle 2$ .  $P(0)$   
 $\langle 3 \rangle 1$ .  $\forall x \in X. (0, x) \in u \Rightarrow x = a$   
 PROOF: If  $(0, x) \in u$  and  $x \neq a$  then  $u - \{(0, x)\} \in \mathcal{C}$  and so  $u - \{(0, x)\} \subseteq u$ , which is impossible.  
 $\langle 2 \rangle 3$ . For every natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .  
 $\langle 3 \rangle 1$ . LET:  $n$  be a natural number.  
 $\langle 3 \rangle 2$ . ASSUME:  $P(n)$   
 $\langle 3 \rangle 3$ . LET:  $x, y \in X$   
 $\langle 3 \rangle 4$ . ASSUME:  $(n^+, x), (n^+, y) \in u$   
 $\langle 3 \rangle 5$ . PICK  $x', y' \in X$  such that  $(n, x') \in u, (n, y') \in u$  and  $f(x') = x$  and  $f(y') = y$   
 PROOF: If no such  $x'$  exists then  $u - \{(n^+, x)\} \in \mathcal{C}$  and so  $u - \{(n^+, x)\} \subseteq u$  which is impossible. Similarly for  $y'$ .  
 $\langle 3 \rangle 6$ .  $x' = y'$   
 PROOF:  $\langle 3 \rangle 2$   
 $\langle 3 \rangle 7$ .  $x = y$

□

**Proposition 17.7.** *For any natural number  $n$ , either  $n = 0$  or there exists a natural number  $m$  such that  $n = m^+$ .*

PROOF: Easy induction on  $n$ . □

**Proposition 17.8.**  *$\omega$  is a transitive set.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the property  $\forall x \in n. x \in \omega$   
 $\langle 1 \rangle 2$ .  $P(0)$   
 PROOF: Vacuous.  
 $\langle 1 \rangle 3$ . For any natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .  
 $\langle 2 \rangle 1$ . LET:  $n$  be a natural number.  
 $\langle 2 \rangle 2$ . ASSUME:  $P(n)$   
 $\langle 2 \rangle 3$ . LET:  $x \in n^+$   
 $\langle 2 \rangle 4$ .  $x \in n$  or  $x = n$   
 $\langle 2 \rangle 5$ . CASE:  $x \in n$   
 PROOF: Then  $x \in \omega$  by  $\langle 2 \rangle 2$ .  
 $\langle 2 \rangle 6$ . CASE:  $x = n$   
 PROOF: Then  $x \in \omega$  by  $\langle 2 \rangle 1$ .

□

**Proposition 17.9.** *For any natural number  $n$  and any nonempty subset  $E \subseteq n$ , there exists  $k \in E$  such that  $\forall m \in E. k = m \vee k \in m$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the property: for any nonempty subset  $E \subseteq n$ , there exists  $k \in E$  such that  $\forall m \in E. k = m \vee k \in m$   
 $\langle 1 \rangle 2$ .  $P(0)$



PROOF: Vacuous as there is no nonempty subset of 0.

$\langle 1 \rangle 3$ . For any natural number  $n$ , if  $P(n)$  then  $P(n^+)$ .

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$

$\langle 2 \rangle 3$ . LET:  $E$  be a nonempty subset of  $n^+$

$\langle 2 \rangle 4$ . CASE:  $E - \{n\} = \emptyset$   
PROOF: Then  $E = \{n\}$  so take  $k = n$ .

$\langle 2 \rangle 5$ . CASE:  $E - \{n\} \neq \emptyset$

$\langle 3 \rangle 1$ . PICK  $k \in E - \{n\}$  such that  $\forall m \in E - \{n\}. k = m \vee k \in m$   
PROOF: By  $\langle 2 \rangle 2$ .

$\langle 3 \rangle 2$ .  $\forall m \in E. k = m \vee k \in m$   
PROOF: Since  $k \in n$ .

□

## Chapter 18

# Arithmetic

**Definition 18.1** (Addition). Define *addition*  $+$  on  $\omega$  by recursion thus:

$$\begin{aligned}m + 0 &= m \\m + n^+ &= (m + n)^+\end{aligned}$$

**Proposition 18.2.** *For all  $m, n, p \in \omega$  we have*

$$m + (n + p) = (m + n) + p .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(p)$  be the property  $\forall m, n \in \omega. m + (n + p) = (m + n) + p$

$\langle 1 \rangle 2$ .  $P(0)$

PROOF:  $m + (n + 0) = m + n = (m + n) + 0$ .

$\langle 1 \rangle 3$ .  $\forall p \in \omega. P(p) \Rightarrow P(p^+)$

$\langle 2 \rangle 1$ . LET:  $p \in \omega$

$\langle 2 \rangle 2$ . ASSUME:  $P(p)$

$\langle 2 \rangle 3$ . LET:  $m, n \in \omega$

$\langle 2 \rangle 4$ .  $m + (n + p^+) = (m + n) + p^+$

PROOF:

$$\begin{aligned}m + (n + p^+) &= m + (n + p)^+ \\&= (m + (n + p))^+ \\&= ((m + n) + p)^+ \\&= (m + n) + p^+\end{aligned}$$

□

**Proposition 18.3.** *For all  $m, n \in \omega$ , we have*

$$m + n = n + m .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(m)$  be the property  $\forall n \in \omega. m + n = n + m$

$\langle 1 \rangle 2. P(0)$   
 $\langle 2 \rangle 1. \text{ LET: } Q(n) \text{ be the property } 0 + n = n + 0$   
 $\langle 2 \rangle 2. Q(0)$   
 PROOF: Trivial.  
 $\langle 2 \rangle 3. \forall n \in \omega. Q(n) \Rightarrow Q(n^+)$   
 $\langle 3 \rangle 1. \text{ LET: } n \in \omega$   
 $\langle 3 \rangle 2. \text{ ASSUME: } Q(n)$   
 $\langle 3 \rangle 3. 0 + n^+ = n^+ + 0$   
 PROOF:  

$$\begin{aligned}
 0 + n^+ &= (0 + n)^+ \\
 &= (n + 0)^+ & (\langle 3 \rangle 2) \\
 &= n^+ \\
 &= n^+ + 0
 \end{aligned}$$
 $\langle 1 \rangle 3. \forall m \in \omega. P(m) \Rightarrow P(m^+)$   
 $\langle 2 \rangle 1. \text{ LET: } m \in \omega$   
 $\langle 2 \rangle 2. \text{ ASSUME: } P(m)$   
 $\langle 2 \rangle 3. \text{ LET: } Q(n) \text{ be the property } m^+ + n = n + m^+$   
 $\langle 2 \rangle 4. Q(0)$   
 PROOF:  $\langle 1 \rangle 2$   
 $\langle 2 \rangle 5. \forall n \in \omega. Q(n) \Rightarrow Q(n^+)$   
 $\langle 3 \rangle 1. \text{ LET: } n \in \omega$   
 $\langle 3 \rangle 2. \text{ ASSUME: } Q(n)$   
 $\langle 3 \rangle 3. Q(n^+)$   
 PROOF:  

$$\begin{aligned}
 m^+ + n^+ &= (m^+ + n)^+ \\
 &= (n + m^+)^+ & (\langle 3 \rangle 2) \\
 &= (n + m)^{++} \\
 &= (m + n)^{++} & (\langle 2 \rangle 2) \\
 &= (m + n^+)^+ \\
 &= (n^+ + m)^+ & (\langle 2 \rangle 2) \\
 &= n^+ + m^+
 \end{aligned}$$

□

**Definition 18.4** (Multiplication). Define *multiplication*  $\cdot$  on  $\omega$  by

$$\begin{aligned}
 m0 &= 0 \\
 mn^+ &= mn + m
 \end{aligned}$$

**Proposition 18.5.** For all  $m, n, p \in \omega$ , we have

$$m(n + p) = mn + mp .$$

PROOF:

$\langle 1 \rangle 1. \text{ LET: } P(p) \text{ be the statement } \forall m, n \in \omega. m(n + p) = mn + mp$

$\langle 1 \rangle 2. P(0)$

PROOF:

$$\begin{aligned} m(n+0) &= mn \\ &= mn + 0 \\ &= mn + m0 \end{aligned}$$

$\langle 1 \rangle 3. \forall p \in \omega. P(p) \Rightarrow P(p^+)$

$\langle 2 \rangle 1. \text{ LET: } p \in \omega$

$\langle 2 \rangle 2. \text{ ASSUME: } P(p)$

$\langle 2 \rangle 3. \text{ LET: } m, n \in \omega$

$\langle 2 \rangle 4. m(n+p^+) = mn + mp^+$

PROOF:

$$\begin{aligned} m(n+p^+) &= m(n+p)^+ \\ &= m(n+p) + m \\ &= (mn + mp) + m && (\langle 2 \rangle 2) \\ &= mn + (mp + m) && (\text{Proposition 18.2}) \\ &= mn + mp^+ \end{aligned}$$

□

**Proposition 18.6.** *For all  $m, n, p \in \omega$  we have*

$$m(np) = (mn)p .$$

PROOF:

$\langle 1 \rangle 1. \text{ LET: } P(p)$  be the statement  $\forall m, n \in \omega. m(np) = (mn)p$

$\langle 1 \rangle 2. P(0)$

PROOF:

$$\begin{aligned} m(n0) &= m0 \\ &= 0 \\ &= (mn)0 \end{aligned}$$

$\langle 1 \rangle 3. \forall p \in \omega. P(p) \Rightarrow P(p^+)$

$\langle 2 \rangle 1. \text{ LET: } p \in \omega$

$\langle 2 \rangle 2. \text{ ASSUME: } P(p)$

$\langle 2 \rangle 3. \text{ LET: } m, n \in \omega$

$\langle 2 \rangle 4. m(np^+) = (mn)p^+$

PROOF:

$$\begin{aligned} m(np^+) &= m(np+n) \\ &= m(np) + mn && (\text{Proposition 18.5}) \\ &= (mn)p + mn && (\langle 2 \rangle 2) \\ &= (mn)p^+ \end{aligned}$$

□

**Proposition 18.7.** *For all  $m, n \in \omega$ , we have*

$$mn = nm .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(m)$  be the statement  $\forall n \in \omega. mn = nm$

$\langle 1 \rangle 2$ .  $P(0)$

$\langle 2 \rangle 1$ . LET:  $Q(n)$  be the statement  $0n = n0$

$\langle 2 \rangle 2$ .  $Q(0)$

PROOF: Trivial.

$\langle 2 \rangle 3$ .  $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$ . LET:  $n \in \omega$

$\langle 3 \rangle 2$ . ASSUME:  $Q(n)$

$\langle 3 \rangle 3$ .  $Q(n^+)$

PROOF:

$$\begin{aligned}
 0n^+ &= 0n + 0 \\
 &= 0n \\
 &= n0 & (\langle 3 \rangle 2) \\
 &= 0 \\
 &= n^+0
 \end{aligned}$$

$\langle 1 \rangle 3$ .  $\forall m \in \omega. P(m) \Rightarrow P(m^+)$

$\langle 2 \rangle 1$ . LET:  $m \in \omega$

$\langle 2 \rangle 2$ . ASSUME:  $P(m)$

$\langle 2 \rangle 3$ . LET:  $Q(n)$  be the statement  $m^+n = nm^+$

$\langle 2 \rangle 4$ .  $Q(0)$

PROOF:  $\langle 1 \rangle 2$

$\langle 2 \rangle 5$ .  $\forall n \in \omega. Q(n) \Rightarrow Q(n^+)$

$\langle 3 \rangle 1$ . LET:  $n \in \omega$

$\langle 3 \rangle 2$ . ASSUME:  $Q(n)$

$\langle 3 \rangle 3$ .  $Q(n^+)$

PROOF:

$$\begin{aligned}
 m^+n^+ &= m^+n + m^+ \\
 &= (m^+n + m)^+ \\
 &= (nm^+ + m)^+ & (\langle 3 \rangle 2) \\
 &= (nm + n + m)^+ \\
 &= (mn + m + n)^+ & (\langle 2 \rangle 2, \text{Proposition 18.2, Proposition 18.3}) \\
 &= (mn^+ + n)^+ \\
 &= (n^+m + n)^+ & (\langle 2 \rangle 2) \\
 &= n^+m + n^+ \\
 &= n^+m^+
 \end{aligned}$$

□

**Definition 18.8** (Exponentiation). Define *exponentiation* on  $\omega$  by recursion:

$$\begin{aligned}
 m^0 &= 1 \\
 m^{n^+} &= m^n m
 \end{aligned}$$

**Proposition 18.9.** *For all  $m, n, p \in \omega$  we have*

$$m^{n+p} = m^n m^p .$$

PROOF:

$$\langle 1 \rangle 1. m^{n+0} = m^n m^0$$

PROOF:

$$\begin{aligned} m^{n+0} &= m^n \\ &= m^n 1 \\ &= m^n m^0 \end{aligned}$$

$$\langle 1 \rangle 2. \text{ If } m^{n+p} = m^n m^p \text{ then } m^{n+p^+} = m^n m^{p^+}$$

PROOF:

$$\begin{aligned} m^{n+p^+} &= m^{n+p} m \\ &= m^n m^p m \\ &= m^n m^{p^+} \end{aligned}$$

□

**Proposition 18.10.** *For all  $m, n, p \in \omega$  we have*

$$(m^n)^p = m^{np} .$$

PROOF:

$$\langle 1 \rangle 1. (m^n)^0 = m^{n0}$$

PROOF: Both are equal to 1.

$$\langle 1 \rangle 2. \text{ If } (m^n)^p = m^{np} \text{ then } (m^n)^{p^+} = m^{np^+}$$

PROOF:

$$\begin{aligned} (m^n)^{p^+} &= (m^n)^p m^n \\ &= m^{np} m^n \\ &= m^{np+n} && \text{(Proposition 18.9)} \\ &= m^{np^+} \end{aligned}$$

□

**Proposition 18.11.** *For any natural numbers  $m$  and  $n$ , if  $m \in n$  then  $m^+ \in n^+$ .*

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } P(n) \text{ be the property } \forall m \in n. m^+ \in n^+$$

$$\langle 1 \rangle 2. P(0)$$

PROOF: Vacuous.

$$\langle 1 \rangle 3. \text{ For any natural number } n, \text{ if } P(n) \text{ then } P(n^+).$$

$$\langle 2 \rangle 1. \text{ LET: } n \text{ be a natural number.}$$

$$\langle 2 \rangle 2. \text{ ASSUME: } P(n)$$

$$\langle 2 \rangle 3. \text{ LET: } m \in n^+$$

$$\langle 2 \rangle 4. m \in n \text{ or } m = n$$

$$\langle 2 \rangle 5. m^+ \in n^+ \text{ or } m^+ = n^+$$

PROOF:  $\langle 2 \rangle 2$

□  $\langle 2 \rangle 6$ . CASE:  $m^+ \in n^{++}$

**Proposition 18.12.** *For any natural numbers  $m$  and  $n$ , either  $m \in n$  or  $m = n$  or  $n \in m$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(n)$  be the property: for all  $m \in \omega$ , either  $m \in n$  or  $m = n$  or  $n \in m$

$\langle 1 \rangle 2$ .  $P(0)$

$\langle 2 \rangle 1$ . LET:  $Q(m)$  be the property: either  $m = 0$  or  $0 \in m$

$\langle 2 \rangle 2$ .  $Q(0)$

PROOF: Since  $0 = 0$ .

$\langle 2 \rangle 3$ . For all  $m \in \omega$ , if  $Q(m)$  then  $Q(m^+)$

PROOF: If  $m = 0$  or  $0 \in m$  then  $0 \in m^+$ .

$\langle 1 \rangle 3$ . For any natural number  $n$ , if  $P(n)$  then  $P(n^+)$

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$

$\langle 2 \rangle 3$ . LET:  $m \in \omega$

$\langle 2 \rangle 4$ .  $m \in n$  or  $m = n$  or  $n \in m$

PROOF:  $\langle 2 \rangle 2$

$\langle 2 \rangle 5$ . CASE:  $m \in n$  or  $m = n$

PROOF: Then  $m \in n^+$ .

$\langle 2 \rangle 6$ . CASE:  $n \in m$

$\langle 3 \rangle 1$ . PICK  $p$  such that  $m = p^+$

$\langle 3 \rangle 2$ .  $n \in p$  or  $n = p$

$\langle 3 \rangle 3$ . CASE:  $n \in p$

PROOF: Then  $n^+ \in p^+ = m$  by Proposition 18.11.

$\langle 3 \rangle 4$ . CASE:  $n = p$

PROOF: Then  $m = n^+$ .

□

**Corollary 18.12.1** (Trichotomy). *For any natural numbers  $m$  and  $n$ , exactly one of  $m \in n$ ,  $m = n$ ,  $n \in m$  holds.*

PROOF:

$\langle 1 \rangle 1$ . We never have  $m \in n$  and  $m = n$ .

PROOF: By Corollary 17.2.1.

$\langle 1 \rangle 2$ . We never have  $m \in n$  and  $n \in m$ .

PROOF: Since  $m$  is a transitive set this would imply  $m \in m$  contradicting Corollary 17.2.1.

$\langle 1 \rangle 3$ . We never have  $m = n$  and  $n \in m$ .

PROOF: By Corollary 17.2.1.

□

**Proposition 18.13.** *For any natural numbers  $m$  and  $n$ , we have  $m \in n$  if and only if  $m \subsetneq n$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $m$  and  $n$  be natural numbers.  
 $\langle 1 \rangle 2$ . If  $m \in n$  then  $m \subsetneq n$ .  
 PROOF: Since  $n$  is a transitive set, and  $m \neq n$  by Corollary 17.2.1.  
 $\langle 1 \rangle 3$ . If  $m \subsetneq n$  then  $m \in n$ .  
 $\langle 2 \rangle 1$ . ASSUME:  $m \subsetneq n$   
 $\langle 2 \rangle 2$ .  $n \notin m$   
 PROOF: Proposition 17.2.  
 $\langle 2 \rangle 3$ .  $m \neq n$   
 $\langle 2 \rangle 4$ .  $m \in n$   
 PROOF: Trichotomy.  
 $\square$

**Definition 18.14.** Given natural numbers  $m$  and  $n$ , we write  $m < n$  iff  $m \in n$ .  
 We write  $m \leq n$  iff  $m < n \vee m = n$ .

**Proposition 18.15.** For natural numbers  $m$  and  $n$ , if  $m \leq n$  and  $n \leq m$  then  $m = n$ .

PROOF: We cannot have  $m < n$  and  $n < m$  by trichotomy.  $\square$

**Proposition 18.16.** For natural numbers  $m$ ,  $n$  and  $k$ , if  $m < n$  then  $m + k < n + k$ .

PROOF:  
 $\langle 1 \rangle 1$ . LET:  $m, n \in \omega$   
 $\langle 1 \rangle 2$ . ASSUME:  $m < n$   
 $\langle 1 \rangle 3$ .  $m + 0 < n + 0$   
 $\langle 1 \rangle 4$ .  $\forall k \in \omega. m + k < n + k \Rightarrow m + k^+ < n + k^+$   
 PROOF: By Proposition 18.11.  
 $\square$

**Proposition 18.17.** For natural numbers  $m$ ,  $n$  and  $k$ , if  $m < n$  and  $k \neq 0$  then  $mk < nk$ .

PROOF:  
 $\langle 1 \rangle 1$ . LET:  $m, n \in \omega$   
 $\langle 1 \rangle 2$ . ASSUME:  $m < n$   
 $\langle 1 \rangle 3$ .  $m1 < n1$   
 $\langle 1 \rangle 4$ . For all  $k \in \omega$ , if  $k \neq 0$  and  $mk < nk$  then  $m(k + 1) < n(k + 1)$   
 PROOF:

$$\begin{aligned}
 m(k + 1) &= mk + m \\
 &< mk + n && \text{(Proposition 18.16)} \\
 &< nk + n && \text{(Proposition 18.16)} \\
 &= n(k + 1)
 \end{aligned}$$

$\square$