

Mathematics

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October 26, 2023

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Part I

Set Theory

Chapter 1

Primitive Terms and Axioms

1.1 Primitive Terms

Let there be *sets*.

For any set A , let there be *elements* of A . We write $a \in A$ for: a is an element of A .

For any sets A and B , let there be a set B^A , whose elements are called *functions* from A to B . We write $f : A \rightarrow B$ for $f \in B^A$.

For any function $f : A \rightarrow B$ and element $a \in A$, let there be an element $f(a) \in B$, the *value* of the function f at the *argument* a .

1.2 Injections, Surjections and Bijections

Definition 1.2.1 (Injective). A function $f : A \rightarrow B$ is *injective* or an *injection* iff, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

Definition 1.2.2 (Surjective). A function $f : A \rightarrow B$ is *surjective* or a *surjection* iff, for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Definition 1.2.3 (Bijective). A function $f : A \rightarrow B$ is *bijective* or a *bijection* iff it is injective and surjective.

Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between them.

If we prove there exists a set X such that $P(X)$, and that any two sets that satisfy P are bijective, then we may introduce a constant C and define "Let C be the set such that $P(C)$ ".

1.3 Axioms

Axiom Schema 1.3.1 (Choice). *Let $P[X, Y, x, y]$ be a formula where X and Y are set variables, $x \in X$ and $y \in Y$. Then the following is an axiom.*

Let A and B be sets. Assume that, for all $a \in A$, there exists $b \in B$ such that $P[A, B, a, b]$. Then there exists a function $f : A \rightarrow B$ such that $\forall a \in A. P[A, B, a, f(a)]$.

Axiom 1.3.2 (Extensionality). *Let $f, g : A \rightarrow B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Definition 1.3.3 (Composition). *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. The composite $g \circ f : A \rightarrow C$ is the function such that, for all $a \in A$, we have*

$$(g \circ f)(a) = g(f(a)) .$$

Axiom 1.3.4 (Pairing). *For any sets A and B , there exists a set $A \times B$, the Cartesian product of A and B , and functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ such that, for all $a \in A$ and $b \in B$, there exists a unique $(a, b) \in A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.*

Axiom Schema 1.3.5 (Separation). *For every property $P[X, x]$ where X is a set variable and $x \in X$, the following is an axiom:*

For every set A , there exists a set $S = \{x \in A : P[A, x]\}$ and an injection $i : S \rightarrow A$ such that, for all $x \in A$, we have

$$(\exists y \in S. i(y) = x) \Leftrightarrow P[A, x] .$$

Axiom 1.3.6 (Infinity). *There exists a set \mathbb{N} , an element $0 \in \mathbb{N}$, and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that:*

- $\forall n \in \mathbb{N}. s(n) \neq 0$
- $\forall m, n \in \mathbb{N}. s(m) = s(n) \Rightarrow m = n$.

Axiom Schema 1.3.7 (Collection). *Let $P[X, Y, x]$ be a formula with set variables X and Y and an element variable $x \in X$. Then the following is an axiom.*

For any set A , there exist sets B and Y and functions $p : B \rightarrow A$, and $m : B \times Y \Rightarrow \mathbb{N}$ such that:

- m is injective.
- $\forall b \in B. P[A, \{y \in Y : m(b, y) = 0\}, p(b)]$
- For all $a \in A$, if $\exists Y. P[A, Y, a]$, then there exists $b \in B$ such that $a = p(b)$.

Axiom 1.3.8 (Universe). *There exists a set E , a set U and a function $el : E \rightarrow U$ such that the following holds.*

Let us say that a set A is small iff there exists $u \in U$ such that $A \approx \{e \in E : el(e) = u\}$.

- \mathbb{N} is small.
- For any U -small sets A and B , the set B^A is small.
- For any U -small sets A and B , the set $A \times B$ is small.
- Let $f : A \rightarrow B$ be a function. If B is small and $\{a \in A : f(a) = b\}$ is small for all $b \in B$, then A is small.
- If $p : B \rightarrow A$ is a surjective function such that A is small, then there exists a U -small set C , a surjection $q : C \rightarrow A$, and a function $f : C \rightarrow B$ such that $q = p \circ f$.

Chapter 2

Sets and Functions

2.1 Composition

Proposition 2.1.1. *Given functions $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f .$$

PROOF:

$\langle 1 \rangle 1$. For all $x \in A$ we have $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$.

$\langle 2 \rangle 1$. LET: $x \in A$

$\langle 2 \rangle 2$. $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$

PROOF:

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) \quad (\text{Definition of composition})$$

$$= h(g(f(x))) \quad (\text{Definition of composition})$$

$$= (h \circ g)(f(x)) \quad (\text{Definition of composition})$$

$$= ((h \circ g) \circ f)(x) \quad (\text{Definition of composition})$$

$\langle 1 \rangle 2$. Q.E.D.

PROOF: By the Axiom of Extensionality.

□

2.1.1 Injections

Proposition 2.1.2. *The composite of injective functions is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: A , B and C be sets.

$\langle 1 \rangle 2$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 3$. LET: $g : B \rightarrow C$

$\langle 1 \rangle 4$. ASSUME: g is injective.

$\langle 1 \rangle 5$. ASSUME: f is injective.

$\langle 1 \rangle 6$. LET: $x, y \in A$

$\langle 1 \rangle 7$. ASSUME: $(g \circ f)(x) = (g \circ f)(y)$

PROVE: $x = y$
 $\langle 1 \rangle 8. g(f(x)) = g(f(y))$
 PROOF:

$$\begin{aligned} g(f(x)) &= (g \circ f)(x) && \text{(definition of composition)} \\ &= (g \circ f)(y) && (\langle 1 \rangle 7) \\ &= g(f(y)) && \text{(definition of composition)} \end{aligned}$$

 $\langle 1 \rangle 9. f(x) = f(y)$
 PROOF: $\langle 1 \rangle 4, \langle 1 \rangle 8$
 $\langle 1 \rangle 10. x = y$
 PROOF: $\langle 1 \rangle 5, \langle 1 \rangle 9$
 \square

Proposition 2.1.3. *For functions $f : A \rightarrow B$ and $g : B \rightarrow C$, if $g \circ f$ is injective then f is injective.*

PROOF:
 $\langle 1 \rangle 1.$ LET: A, B and C be sets.
 $\langle 1 \rangle 2.$ LET: $f : A \rightarrow B$
 $\langle 1 \rangle 3.$ LET: $g : B \rightarrow C$
 $\langle 1 \rangle 4.$ ASSUME: $g \circ f$ is injective.
 $\langle 1 \rangle 5.$ LET: $x, y \in A$
 $\langle 1 \rangle 6.$ ASSUME: $f(x) = f(y)$
 $\langle 1 \rangle 7. (g \circ f)(x) = (g \circ f)(y)$
 PROOF:

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{(definition of composition)} \\ &= g(f(y)) && (\langle 1 \rangle 6) \\ &= (g \circ f)(y) && \text{(definition of composition)} \end{aligned}$$

 $\langle 1 \rangle 8. x = y$
 PROOF: $\langle 1 \rangle 4, \langle 1 \rangle 7$
 \square

Proposition 2.1.4. *Let $f : A \rightarrow B$ be injective. For every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.*

PROOF:
 $\langle 1 \rangle 1.$ ASSUME: f is injective.
 $\langle 1 \rangle 2.$ LET: X be a set.
 $\langle 1 \rangle 3.$ LET: $x, y : X \rightarrow A$
 $\langle 1 \rangle 4.$ ASSUME: $f \circ x = f \circ y$
 $\langle 1 \rangle 5. \forall t \in X. x(t) = y(t)$
 $\langle 2 \rangle 1.$ LET: $t \in X$
 $\langle 2 \rangle 2. f(x(t)) = f(y(t))$
 PROOF:

$$\begin{aligned} f(x(t)) &= (f \circ x)(t) && \text{(definition of composition)} \\ &= (f \circ y)(t) && (\langle 1 \rangle 4) \\ &= f(y(t)) && \text{(definition of composition)} \end{aligned}$$

$\langle 2 \rangle 3. x(t) = y(t)$

PROOF: $\langle 1 \rangle 1, \langle 2 \rangle 2$

$\langle 1 \rangle 6. x = y$

PROOF: Axiom of Extensionality, $\langle 1 \rangle 5$

□

We will prove the converse as Proposition 2.5.4.

2.1.2 Surjections

Proposition 2.1.5. *The composite of surjective functions is surjective.*

PROOF:

$\langle 1 \rangle 1.$ LET: A, B and C be sets.

$\langle 1 \rangle 2.$ LET: $f : A \rightarrow B$ and $g : B \rightarrow C$

$\langle 1 \rangle 3.$ ASSUME: g is surjective.

$\langle 1 \rangle 4.$ ASSUME: f is surjective.

$\langle 1 \rangle 5.$ LET: $c \in C$

$\langle 1 \rangle 6.$ PICK $b \in B$ such that $g(b) = c$.

PROOF: $\langle 1 \rangle 3$

$\langle 1 \rangle 7.$ PICK $a \in A$ such that $f(a) = b$.

PROOF: $\langle 1 \rangle 4$

$\langle 1 \rangle 8. (g \circ f)(a) = c$

PROOF:

$$\begin{aligned} (g \circ f)(a) &= g(f(a)) && \text{(definition of composition)} \\ &= g(b) && (\langle 1 \rangle 7) \\ &= c && (\langle 1 \rangle 6) \end{aligned}$$

□

Proposition 2.1.6. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is surjective then g is surjective.*

PROOF:

$\langle 1 \rangle 1.$ LET: A, B and C be sets.

$\langle 1 \rangle 2.$ LET: $f : A \rightarrow B$ and $g : B \rightarrow C$.

$\langle 1 \rangle 3.$ ASSUME: $g \circ f$ is surjective.

$\langle 1 \rangle 4.$ LET: $c \in C$

$\langle 1 \rangle 5.$ PICK $a \in A$ such that $(g \circ f)(a) = c$

PROOF: $\langle 1 \rangle 3$

$\langle 1 \rangle 6. g(f(a)) = c$

PROOF: From $\langle 1 \rangle 5$ and the definition of composition.

$\langle 1 \rangle 7.$ Q.E.D.

PROOF: There exists $b \in B$ such that $g(b) = c$, namely $b = f(a)$.

□

2.1.3 Bijections

Proposition 2.1.7. *The composite of bijections is a bijection.*

PROOF:

- ⟨1⟩1. LET: A, B and C be sets.
- ⟨1⟩2. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$
- ⟨1⟩3. ASSUME: g is bijective.
- ⟨1⟩4. ASSUME: f is bijective.
- ⟨1⟩5. g is injective.
PROOF: From ⟨1⟩3.
- ⟨1⟩6. g is surjective.
PROOF: From ⟨1⟩3.
- ⟨1⟩7. f is injective.
PROOF: From ⟨1⟩4.
- ⟨1⟩8. f is surjective.
PROOF: From ⟨1⟩4.
- ⟨1⟩9. $g \circ f$ is injective.
PROOF: Proposition 2.1.2, ⟨1⟩5, ⟨1⟩7.
- ⟨1⟩10. $g \circ f$ is surjective.
PROOF: Proposition 2.1.5, ⟨1⟩6, ⟨1⟩8.
- ⟨1⟩11. $g \circ f$ is bijective.
PROOF: ⟨1⟩9, ⟨1⟩10

□

2.1.4 Equinumerosity

Proposition 2.1.8.

$$(A \times B)^C \approx A^C \times B^C$$

PROOF: The function that maps f to $(\pi_1 \circ f, \pi_2 \circ f)$ is a bijection. □

Proposition 2.1.9.

$$A^{B \times C} \approx (A^B)^C$$

PROOF: The function Φ such that $\Phi(f)(c)(b) = f(b, c)$ is a bijection. □

2.2 Domination

Definition 2.2.1 (Dominate). Let A and B be sets. We say that B *dominates* A , and write $A \leq B$, iff there exists an injective function $A \rightarrow B$.

Theorem 2.2.2 (Schroeder-Bernstein). *Let A and B be sets. If $A \leq B$ and $B \leq A$ then $A \approx B$.*

PROOF:

- ⟨1⟩1. LET: $f : A \rightarrow B$ and $g : B \rightarrow A$ be injections.
- ⟨1⟩2. Define the subsets A_n of A by

$$\begin{aligned} A_0 &:= A - g(B) \\ A_{n+1} &:= g(f(A_n)) \end{aligned}$$

⟨1⟩3. Define $h : A \rightarrow B$ by

$$h(x) = \begin{cases} f(x) & \text{if } \exists n. x \in A_n \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

⟨1⟩4. h is injective.

⟨2⟩1. LET: $x, y \in A$

⟨2⟩2. ASSUME: $h(x) = h(y)$

⟨2⟩3. CASE: $x \in A_m$ and $y \in A_n$.

PROOF: Then $f(x) = f(y)$ so $x = y$ since f is injective.

⟨2⟩4. CASE: $x \in A_m$ and there is no y such that $y \in A_n$.

⟨3⟩1. $f(x) = g^{-1}(y)$

⟨3⟩2. $y = g(f(x))$

⟨3⟩3. $y \in A_{m+1}$

⟨3⟩4. Q.E.D.

PROOF: This is a contradiction.

⟨2⟩5. CASE: $y \in A_n$ and there is no m such that $x \in A_m$.

PROOF: Similar.

⟨2⟩6. CASE: There is no m such that $x \in A_m$ and there is no n such that $y \in A_n$.

PROOF: Then $g^{-1}(x) = g^{-1}(y)$ and so $x = y$.

⟨1⟩5. h is surjective.

⟨2⟩1. LET: $y \in B$

⟨2⟩2. CASE: $g(y) \in A_n$

⟨3⟩1. $n \neq 0$

⟨3⟩2. PICK $x \in A_{n-1}$ such that $g(y) = g(f(x))$

⟨3⟩3. $y = f(x)$

⟨3⟩4. $y = h(x)$

⟨2⟩3. CASE: There is no n such that $g(y) \in A_n$.

PROOF: Then $h(g(y)) = y$.

□

2.3 Identity Function

Definition 2.3.1 (Identity). For any set A , the *identity* function $\text{id}_A : A \rightarrow A$ is the function defined by $\text{id}_A(a) = a$.

2.3.1 Injections, Surjections, Bijections

Proposition 2.3.2. For any set A , the identity function id_A is a bijection.

PROOF:

⟨1⟩1. LET: A be a set.

⟨1⟩2. id_A is injective.

PROOF: If $\text{id}_A(x) = \text{id}_A(y)$ then $x = y$.

⟨1⟩3. id_A is surjective.

PROOF: For any $y \in A$, there exists $x \in A$ such that $\text{id}_A(x) = y$, namely $x = y$.

□

2.3.2 Composition

Proposition 2.3.3. *Let $f : A \rightarrow B$. Then $\text{id}_B \circ f = f = f \circ \text{id}_A$.*

PROOF: Each is the function that maps a to $f(a)$. □

Proposition 2.3.4. *Let $f : A \rightarrow B$.*

1. *If there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$ then f is injective.*
2. *If f is injective and A is nonempty, then there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$.*

PROOF:

⟨1⟩1. If there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$ then f is injective.

PROOF: If $f(x) = f(y)$ then $x = g(f(x)) = g(f(y)) = y$.

⟨1⟩2. If f is injective and A is nonempty, then there exists $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$.

⟨2⟩1. ASSUME: f is injective and A is nonempty.

⟨2⟩2. PICK $a \in A$

⟨2⟩3. Choose a function $g : B \rightarrow A$ such that $f(g(x)) = x$ if there exists $y \in A$ such that $f(y) = x$, otherwise $g(x) = a$.

⟨2⟩4. LET: $x \in A$

PROVE: $g(f(x)) = x$

⟨2⟩5. $f(g(f(x))) = f(x)$

⟨2⟩6. $g(f(x)) = x$

□

Proposition 2.3.5. *Let $f : A \rightarrow B$. Then f is surjective if and only if there exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$.*

PROOF:

⟨1⟩1. If f is surjective then there exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$.

⟨2⟩1. ASSUME: f is surjective.

⟨2⟩2. PICK $g : B \rightarrow A$ such that, for all $b \in B$, we have $f(g(b)) = b$.

PROOF: Axiom of Choice.

⟨2⟩3. $f \circ g = \text{id}_B$.

⟨1⟩2. If there exists $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$ then f is surjective.

⟨2⟩1. LET: $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$

⟨2⟩2. LET: X be a set.

⟨2⟩3. LET: $h, k : B \rightarrow X$

⟨2⟩4. ASSUME: $h \circ f = k \circ f$

⟨2⟩5. $h = k$

PROOF: $h = h \circ f \circ g = k \circ f \circ g = k$

□

Corollary 2.3.5.1. *Let A and B be sets.*

1. *If there exists a surjective function $A \rightarrow B$ then there exists an injective function $B \rightarrow A$.*
2. *If there exists an injective function $A \rightarrow B$ and A is nonempty then there exists a surjective function $B \rightarrow A$.*

Proposition 2.3.6. *Let $f : A \rightarrow B$. Then f is bijective if and only if there exists a function $f^{-1} : B \rightarrow A$, the inverse of f , such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$, in which case the inverse is unique.*

PROOF:

- ⟨1⟩1. If f is bijective then there exists $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$.
- ⟨2⟩1. ASSUME: f is bijective.
- ⟨2⟩2. PICK $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$
- PROOF: Proposition 2.6.2.
- ⟨2⟩3. $f \circ g \circ f = f$
- ⟨2⟩4. $g \circ f = \text{id}_A$
- PROOF: Proposition 2.1.4.
- ⟨1⟩2. If there exists $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$, then f is bijective.
- ⟨2⟩1. LET: $f^{-1} : B \rightarrow A$ satisfy $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$
- ⟨2⟩2. f is injective.
- PROOF: If $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$.
- ⟨2⟩3. f is surjective.
- PROOF: Proposition 2.6.2.
- ⟨1⟩3. If $g, h : B \rightarrow A$ satisfy $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$ and $f \circ h = \text{id}_B$ and $h \circ f = \text{id}_A$ then $g = h$.
- PROOF: We have $g = g \circ f \circ h = h$.
-

2.4 The Empty Set

Theorem 2.4.1. *There exists a set which has no elements.*

PROOF: Take $\{x \in \mathbb{N} : \perp\}$. □

Theorem 2.4.2. *If E and E' have no elements then $E \approx E'$.*

PROOF:

- ⟨1⟩1. LET: E and E' have no elements.
- ⟨1⟩2. PICK a function $F : E \rightarrow E'$.
- PROOF: Axiom of Choice since vacuously $\forall x \in E. \exists y \in E'. \top$.
- ⟨1⟩3. F is injective.

PROOF: Vacuously, for all $x, y \in E$, if $F(x) = F(y)$ then $x = y$.

$\langle 1 \rangle 4$. F is surjective.

PROOF: Vacuously, for all $y \in E$, there exists $x \in E$ such that $F(x) = y$.

□

Definition 2.4.3 (Empty Set). The *empty set* \emptyset is the set with no elements.

2.5 The Singleton

Theorem 2.5.1. *There exists a set that has exactly one element.*

PROOF: The set $\{x \in \mathbb{N} : x = 0\}$ has exactly one element. □

Theorem 2.5.2. *If A and B both have exactly one element then $A \approx B$.*

PROOF:

$\langle 1 \rangle 1$. LET: A and B both have exactly one element a and b respectively.

$\langle 1 \rangle 2$. LET: $F : A \rightarrow B$ be the function such that, for all $x \in A$, we have
 $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$. F is a bijection.

□

Definition 2.5.3 (Singleton). Let 1 be the set that has exactly one element. Let $*$ be its element.

2.5.1 Injections

Proposition 2.5.4. *Let $f : A \rightarrow B$. Assume that, for every set X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$. Then f is injective.*

PROOF: Take $X = 1$. □

2.6 The Set Two

Definition 2.6.1 (The Set Two). Let $2 = \{x \in \mathbb{N} : x = 0 \vee x = 1\}$.

Proposition 2.6.2. *Let $f : A \rightarrow B$. Then f is surjective if and only if, for any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$.*

PROOF:

$\langle 1 \rangle 1$. If f is surjective then, for any set X and functions $g, h : B \rightarrow X$, if
 $g \circ f = h \circ f$ then $g = h$.

$\langle 2 \rangle 1$. ASSUME: f is surjective.

$\langle 2 \rangle 2$. LET: X be a set.

$\langle 2 \rangle 3$. LET: $g, h : B \rightarrow X$

$\langle 2 \rangle 4$. ASSUME: $g \circ f = h \circ f$

$\langle 2 \rangle 5$. LET: $b \in B$

PROVE: $g(b) = h(b)$

- $\langle 2 \rangle 6$. PICK $a \in A$ such that $f(a) = b$
 $\langle 2 \rangle 7$. $g(b) = h(b)$
 PROOF: $g(b) = g(f(a)) = h(f(a)) = h(b)$
 $\langle 1 \rangle 2$. If, for any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$, then f is surjective.
 $\langle 2 \rangle 1$. ASSUME: For any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$.
 $\langle 2 \rangle 2$. LET: $b \in B$
 $\langle 2 \rangle 3$. LET: $h : B \rightarrow 2$ be the function that maps everything to 1.
 $\langle 2 \rangle 4$. LET: $k : B \rightarrow 2$ be the function that maps b to 0 and everything else to 1.
 $\langle 2 \rangle 5$. $h \neq k$
 $\langle 2 \rangle 6$. $h \circ f \neq k \circ f$
 $\langle 2 \rangle 7$. PICK $a \in A$ such that $h(f(a)) \neq k(f(a))$
 $\langle 2 \rangle 8$. $f(a) = b$
 \square

2.7 Subsets

Definition 2.7.1 (Subset). A *subset* of a set A consists of a set S and an injection $i : S \rightarrow A$. We write $(S, i) \subseteq A$.

We say two subsets (S, i) and (T, j) are *equal*, $(S, i) = (T, j)$, iff there exists a bijection $\phi : S \approx T$ such that $j \circ \phi = i$.

Proposition 2.7.2. For any subset (S, i) of A we have $(S, i) = (S, i)$.

PROOF: We have $\text{id}_S : S \approx S$ and $i \circ \text{id}_S = i$.

Proposition 2.7.3. If $(S, i) = (T, j)$ then $(T, j) = (S, i)$.

PROOF: If $\phi : S \approx T$ and $j \circ \phi = i$ then $\phi^{-1} : T \approx S$ and $i \circ \phi^{-1} = j$. \square

Proposition 2.7.4. If $(R, i) = (S, j)$ and $(S, j) = (T, k)$ then $(R, i) = (T, k)$.

PROOF: If $\phi : R \approx S$ and $j \circ \phi = i$, and $\psi : S \approx T$ and $k \circ \psi = j$, then $\psi \circ \phi : R \approx T$ and $k \circ \psi \circ \phi = i$. \square

Definition 2.7.5 (Membership). Given $(S, i) \subseteq A$ and $a \in A$, we write $a \in (S, i)$ for $\exists s \in S. i(s) = a$.

Proposition 2.7.6. If $a \in (S, i)$ and $(S, i) = (T, j)$ then $a \in (T, j)$.

PROOF: If $i(s) = a$ then $j(\phi(s)) = a$. \square

Definition 2.7.7 (Union). Given subsets S and T of A , the *union* is the subset $\{x \in A : x \in S \vee x \in T\}$.

Definition 2.7.8 (Intersection). Given subsets S and T of A , the *intersection* is the subset $\{x \in A : x \in S \wedge x \in T\}$.

Proposition 2.7.9 (Distributive Law).

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

Proposition 2.7.10 (Distributive Law).

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

Definition 2.7.11. Given a set A , we write \emptyset for the subset $(\emptyset, !)$ where $!$ is the unique function $\emptyset \rightarrow A$.

Proposition 2.7.12.

$$S \cup \emptyset = S$$

Proposition 2.7.13.

$$S \cap \emptyset = \emptyset$$

Definition 2.7.14 (Inclusion). Given subsets (S, i) and (T, j) of a set A , we write $(S, i) \subseteq (T, j)$ iff there exists $f : S \rightarrow T$ such that $j \circ f = i$.

Proposition 2.7.15.

$$\emptyset \subseteq S$$

Definition 2.7.16 (Disjoint). Subsets S and T of A are *disjoint* iff $S \cap T = \emptyset$.

Definition 2.7.17 (Difference). Given subsets S and T of A , the *difference* of S and T is $S - T = \{x \in A : x \in S \wedge x \notin T\}$.

Proposition 2.7.18 (De Morgan's Law).

$$R - (S \cup T) = (R - S) \cap (R - T)$$

Proposition 2.7.19 (De Morgan's Law).

$$R - (S \cap T) = (R - S) \cup (R - T)$$

2.8 Saturated Set

Definition 2.8.1 (Saturated). Let A and B be sets. Let $f : A \rightarrow B$ be surjective. Let $C \subseteq A$. Then C is *saturated* with respect to f iff, for all $x \in C$ and $y \in A$, if $f(x) = f(y)$ then $y \in C$.

2.9 Union

Definition 2.9.1 (Union). Given $\mathcal{A} \in \mathcal{P}\mathcal{P}X$, its *union* is

$$\bigcup \mathcal{A} := \{x \in X : \exists S \in \mathcal{A}. x \in S\} \in \mathcal{P}X .$$

2.9.1 Intersection

Definition 2.9.2 (Intersection). Given $\mathcal{A} \in \mathcal{PP}X$, its *intersection* is

$$\bigcap \mathcal{A} := \{x \in X : \forall S \in \mathcal{A}. x \in S\} \in \mathcal{P}X .$$

2.9.2 Direct Image

Definition 2.9.3 (Direct Image). Let $f : A \rightarrow B$. Let S be a subset of A . The *(direct) image* of S under f is the subset of B given by

$$f(S) := \{f(a) : a \in S\} .$$

Proposition 2.9.4.

1. If $S \subseteq T$ then $f(S) \subseteq f(T)$
2. $f(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f(S)$

Example 2.9.5. It is not true in general that $f(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f(S)$. Take f to be the only function $\{0, 1\} \rightarrow \{0\}$, and $\mathcal{S} = \{\{0\}, \{1\}\}$. Then $f(\bigcap \mathcal{S}) = \emptyset$ but $\bigcap_{S \in \mathcal{S}} f(S) = \{0\}$.

Example 2.9.6. It is not true in general that $f(S - T) = f(S) - f(T)$. Take f to be the only function $\{0, 1\} \rightarrow \{0\}$, $S = \{0\}$ and $T = \{1\}$. Then $f(S - T) = \{0\}$ but $f(S) - f(T) = \emptyset$.

2.10 Inverse Image

Definition 2.10.1 (Inverse Image). Let $f : A \rightarrow B$. Let S be a subset of B . The *inverse image* or *preimage* of S under f is the subset of A given by

$$f^{-1}(S) := \{x \in A : f(x) \in S\} .$$

Proposition 2.10.2. 1. If $S \subseteq T$ then $f^{-1}(S) \subseteq f^{-1}(T)$

2. $f^{-1}(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} f^{-1}(S)$
3. $f^{-1}(\bigcap \mathcal{S}) = \bigcap_{S \in \mathcal{S}} f^{-1}(S)$
4. $f^{-1}(S - T) = f^{-1}(S) - f^{-1}(T)$
5. $S \subseteq f^{-1}(f(S))$. Equality holds if f is injective.
6. $f(f^{-1}(T)) \subseteq T$. Equality holds if f is surjective.
7. $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$

2.10.1 Saturated Sets

Proposition 2.10.3. *Let A and B be sets. Let $f : A \rightarrow B$ be surjective. Let $C \subseteq A$. Then C is saturated if and only if there exists $D \subseteq B$ such that $C = f^{-1}(D)$.*

PROOF:

$\langle 1 \rangle 1$. If C is saturated then there exists $D \subseteq B$ such that $C = f^{-1}(D)$.

$\langle 2 \rangle 1$. ASSUME: C is saturated.

$\langle 2 \rangle 2$. LET: $D = f(C)$

$\langle 2 \rangle 3$. $C \subseteq f^{-1}(D)$

$\langle 3 \rangle 1$. LET: $x \in C$

$\langle 3 \rangle 2$. $f(x) \in D$

PROOF: $\langle 2 \rangle 2$

$\langle 3 \rangle 3$. $x \in f^{-1}(D)$

$\langle 2 \rangle 4$. $f^{-1}(D) \subseteq C$

$\langle 3 \rangle 1$. LET: $x \in f^{-1}(D)$

$\langle 3 \rangle 2$. $f(x) \in D$

$\langle 3 \rangle 3$. PICK $y \in C$ such that $f(x) = f(y)$

PROOF: $\langle 2 \rangle 2$

$\langle 3 \rangle 4$. $x \in C$

PROOF: $\langle 2 \rangle 1$

$\langle 1 \rangle 2$. If there exists $D \subseteq B$ such that $C = f^{-1}(D)$ then C is saturated.

$\langle 2 \rangle 1$. LET: $D \subseteq B$ be such that $C = f^{-1}(D)$.

$\langle 2 \rangle 2$. LET: $x \in C$ and $y \in A$

$\langle 2 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 4$. $f(x) \in D$

$\langle 2 \rangle 5$. $f(y) \in D$

$\langle 2 \rangle 6$. $y \in C$

□

2.11 Relations

Definition 2.11.1 (Relation). Let A and B be sets. A *relation* R between A and B , $R : A \rightarrow B$, is a subset of $A \times B$.

Given $a \in A$ and $b \in B$, we write aRb for $(a, b) \in R$.

A relation *on* a set A is a relation between A and A .

Definition 2.11.2 (Reflexive). A relation R on a set A is *reflexive* iff $\forall a \in A. aRa$.

Definition 2.11.3 (Symmetric). A relation R on a set A is *symmetric* iff, whenever xRy , then yRx .

Definition 2.11.4 (Transitive). A relation R on a set A is *transitive* iff, whenever xRy and yRz , then xRz .

2.11.1 Equivalence Relations

Definition 2.11.5 (Equivalence Relation). A relation R on a set A is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Definition 2.11.6 (Equivalence Class). Let R be an equivalence relation on a set A and $a \in A$. The *equivalence class* of a with respect to R is

$$\{x \in A : xRa\} .$$

Proposition 2.11.7. *Two equivalence classes are either disjoint or equal.*

2.12 Power Set

Definition 2.12.1 (Power Set). The *power set* of a set A is $\mathcal{P}A := 2^A$.

Given $S \in \mathcal{P}A$ and $a \in A$, we write $a \in S$ for $S(a) = 1$.

Definition 2.12.2 (Pairwise Disjoint). Let $P \subseteq \mathcal{P}A$. We say the members of P are *pairwise disjoint* iff, for all $S, T \in P$, if $S \neq T$ then $S \cap T = \emptyset$.

2.12.1 Partitions

Definition 2.12.3 (Partition). Let A be a set. A *partition* of A is a set $P \in \mathcal{P}\mathcal{P}A$ such that:

- $\bigcup P = A$
- Every member of P is nonempty.
- The members of P are pairwise disjoint.

2.13 Cartesian Product

Definition 2.13.1 (Cartesian Product). Let A and B be sets. The *Cartesian product* of A and B , $A \times B$, is the tabulation of the relation $A \looparrowright B$ that holds for all $a \in A$ and $b \in B$. The associated functions $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ are called the *projections*.

Given $a \in A$ and $b \in B$, we write (a, b) for the unique element of $A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

2.14 Quotient Sets

Proposition 2.14.1. *Let \sim be an equivalence relation on X . Then there exists a set X/\sim , the quotient set of X with respect to \sim , and a surjective function $\pi : X \twoheadrightarrow X/\sim$, the canonical projection, such that, for all $x, y \in X$, we have $x \sim y$ if and only if $\pi(x) = \pi(y)$.*

Further, if $p : X \twoheadrightarrow Q$ is another quotient with respect to \sim , then there exists a unique bijection $\phi : X/\sim \approx Q$ such that $\phi \circ \pi = p$.

2.15 Partitions

Definition 2.15.1 (Partition). A *partition* of a set X is a set of pairwise disjoint subsets of X whose union is X .

2.16 Disjoint Union

Theorem 2.16.1. For any sets A and B , there exists a set $A + B$, the disjoint union of A and B , and functions $\kappa_1 : A \rightarrow A + B$ and $\kappa_2 : B \rightarrow A + B$, the injections, such that, for every set X and functions $f : A \rightarrow X$ and $g : B \rightarrow X$, there exists a unique function $[f, g] : A + B \rightarrow X$ such that $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$.

PROOF:

$\langle 1 \rangle 1$. LET: $A + B := \{p \in \mathcal{P}A \times \mathcal{P}B : \exists a \in A. p = (\{a\}, \emptyset) \vee \exists b \in B. p = (\emptyset, \{b\})\}$

Definition 2.16.2 (Restriction). Let $f : A \rightarrow B$ and let (S, i) be a subset of A . The *restriction* of f to S is the function $f \upharpoonright S : S \rightarrow B$ defined by $f \upharpoonright S = f \circ i$.

2.17 Natural Numbers

Theorem 2.17.1 (Principle of Recursive Definition). Let A be a set. Let F be the set of all functions $\{m \in \mathbb{N} : m < n\} \rightarrow A$ for some n . Let $\rho : F \rightarrow A$. Then there exists a unique $g : \mathbb{N} \rightarrow A$ such that, for all $n \in \mathbb{N}$, we have

$$g(n) = \rho(g \upharpoonright \{m \in \mathbb{N} : m < n\}) .$$

PROOF:

$\langle 1 \rangle 1$. Given a subset $B \subseteq \mathbb{N}$, let us say that a function $g : B \rightarrow A$ is *acceptable* iff, for all $n \in B$, we have

$$\forall m < n. m \in B$$

and

$$g(n) = \rho(g \upharpoonright \{m \in \mathbb{N} : m < n\}) .$$

$\langle 1 \rangle 2$. For all $n \in \mathbb{N}$, there exists an acceptable function $\{m \in \mathbb{N} : m < n\} \rightarrow A$.

$\langle 2 \rangle 1$. LET: $P[n]$ be the property: There exists an acceptable function $\{m \in \mathbb{N} : m < n\} \rightarrow A$.

$\langle 2 \rangle 2$. $P[0]$

PROOF: The unique function $\emptyset \rightarrow A$ is acceptable.

$\langle 2 \rangle 3$. For any natural number n , if $P[n]$ then $P[n + 1]$.

$\langle 3 \rangle 1$. ASSUME: $P[n]$

$\langle 3 \rangle 2$. PICK an acceptable $f : \{m \in \mathbb{N} : m < n\} \rightarrow A$.

$\langle 3 \rangle 3$. LET: $g : \{m \in \mathbb{N} : m < n + 1\} \rightarrow A$ be the function

$$g(m) = \begin{cases} f(m) & \text{if } m < n \\ \rho(f) & \text{if } m = n \end{cases}$$

$\langle 3 \rangle 4$. g is acceptable.

- <1>3. If $g : B \rightarrow A$ and $h : C \rightarrow A$ are acceptable, then g and h agree on $B \cap C$.
 <1>4. Define $g : \mathbb{N} \rightarrow A$ by: $g(n) = a$ iff there exists an acceptable $h : \{m \in \mathbb{N} : m < n + 1\}$ such that $h(n) = a$.
 <1>5. g is acceptable.
 <1>6. If $g' : \mathbb{N} \rightarrow A$ is acceptable then $g' = g$.
 \square

2.18 Finite and Infinite Sets

Definition 2.18.1 (Finite). A set A is *finite* iff there exists $n \in \mathbb{N}$ such that $A \approx \{m \in \mathbb{N} : m < n\}$. In this case, we say A has *cardinality* n .

Proposition 2.18.2. Let $n \in \mathbb{N}$. Let A be a set. Let $a_0 \in A$. Then $A \approx \{m \in \mathbb{N} : m < n + 1\}$ if and only if $A - \{a_0\} \approx \{m \in \mathbb{N} : m < n\}$.

Theorem 2.18.3. Let A be a set. Suppose that $A \approx \{m \in \mathbb{N} : m < n\}$. Let B be a proper subset of A . Then $B \not\approx \{m \in \mathbb{N} : m < n\}$ but there exists $m < n$ such that $B \approx \{k \in \mathbb{N} : k < m\}$.

PROOF:

- <1>1. LET: $P[n]$ be the property: for every set A , if $A \approx \{m \in \mathbb{N} : m < n\}$, then for every proper subset B of A , we have $B \not\approx \{m \in \mathbb{N} : m < n\}$ but there exists $m < n$ such that $B \approx \{k \in \mathbb{N} : k < m\}$.
 <1>2. $P[0]$
 PROOF: If $A \approx \{m \in \mathbb{N} : m < 0\}$ then A is empty and so has no proper subset.
 <1>3. For every natural number n , if $P[n]$ then $P[n + 1]$.
 <2>1. LET: n be a natural number.
 <2>2. ASSUME: $P[n]$
 <2>3. LET: A be a set.
 <2>4. ASSUME: $A \approx \{m \in \mathbb{N} : m < n + 1\}$
 <2>5. LET: B be a proper subset of A .
 <2>6. CASE: $B = \emptyset$
 PROOF: Then $B \not\approx \{m \in \mathbb{N} : m < n + 1\}$ but $B \approx \{k \in \mathbb{N} : k < 0\}$.
 <2>7. CASE: $B \neq \emptyset$
 <3>1. PICK $b_0 \in B$
 <3>2. $A - \{b_0\} \approx \{m \in \mathbb{N} : m < n\}$
 <3>3. $B - \{b_0\}$ is a proper subset of $A - \{b_0\}$
 <3>4. $B - \{b_0\} \not\approx \{m \in \mathbb{N} : m < n\}$
 <3>5. $B \approx \{m \in \mathbb{N} : m < n + 1\}$
 <3>6. PICK $m < n$ such that $B - \{b_0\} \approx \{k \in \mathbb{N} : k < m\}$
 <3>7. $m + 1 < n + 1$
 <3>8. $B \approx \{k \in \mathbb{N} : k < m + 1\}$
 \square

Corollary 2.18.3.1. If A is finite then there is no bijection between A and a proper subset of A .

Corollary 2.18.3.2. \mathbb{N} is infinite.

Corollary 2.18.3.3. The cardinality of a finite set is unique.

Corollary 2.18.3.4. A subset of a finite set is finite.

Corollary 2.18.3.5. If A is finite and B is a proper subset of A then $|B| < |A|$.

Corollary 2.18.3.6. Let A be a set. Then the following are equivalent:

1. A is finite.
2. There exists a surjection from an initial segment of \mathbb{N} onto A .
3. There exists an injection from A to an initial segment of \mathbb{N} .

Corollary 2.18.3.7. A finite union of finite sets is finite.

Corollary 2.18.3.8. A finite Cartesian product of finite sets is finite.

Theorem 2.18.4. Let A be a set. The following are equivalent:

1. There exists an injective function $\mathbb{N} \rightarrow A$.
2. There exists a bijection between A and a proper subset of A .
3. A is infinite.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ LET: $f : \mathbb{N} \rightarrow A$ be injective.

$\langle 2 \rangle 2.$ LET: $s : \mathbb{N} \approx \mathbb{N} - \{0\}$ be the function $s(n) = n + 1$.

$\langle 2 \rangle 3.$ $f \circ s \circ f^{-1} : A \approx A - \{f(0)\}$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: Corollary 2.18.3.1.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: Choose a function $f : \mathbb{N} \rightarrow A$ such that $f(n) \in A - \{f(m) : m < n\}$ for all n .

□

2.19 Countable Sets

Definition 2.19.1 (Countable). A set A is *countably infinite* iff $A \approx \mathbb{N}$.

Proposition 2.19.2. $\mathbb{N} \times \mathbb{N}$ is countably infinite.

PROOF: Define $f : \mathbb{N} \times \mathbb{N} \approx \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x\}$ by

$$f(x, y) = (x + y, y)$$

Define $g : \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x\} \approx \mathbb{N}$ by

$$g(x, y) = x(x - 1)/2 + y. \quad \square$$

Proposition 2.19.3. *Every infinite subset of \mathbb{N} is countably infinite.*

PROOF:

$\langle 1 \rangle 1$. LET: C be an infinite subset of \mathbb{N}

$\langle 1 \rangle 2$. Define $h : \mathbb{Z} \rightarrow C$ by recursion thus: $h(n)$ is the smallest element of $C - \{h(m) : m < n\}$.

$\langle 1 \rangle 3$. h is injective.

PROOF: If $m < n$ then $h(m) \neq h(n)$ because $h(n) \in C - \{h(m) : m < n\}$.

$\langle 1 \rangle 4$. h is surjective.

$\langle 2 \rangle 1$. For all $n \in \mathbb{N}$ we have $n \leq h(n)$.

$\langle 2 \rangle 2$. LET: $c \in C$

$\langle 2 \rangle 3$. $c \leq h(c)$

$\langle 2 \rangle 4$. LET: n be least such that $c \leq h(n)$

$\langle 2 \rangle 5$. $c \in C - \{h(m) : m < n\}$

$\langle 2 \rangle 6$. $h(n) \leq c$

$\langle 2 \rangle 7$. $h(n) = c$

□

Definition 2.19.4 (Countable). A set is *countable* iff it is either finite or countably infinite; otherwise it is *uncountable*.

Proposition 2.19.5. *Let B be a nonempty set. Then the following are equivalent.*

1. B is countable.
2. There exists a surjection $\mathbb{N} \twoheadrightarrow B$.
3. There exists an injection $B \hookrightarrow \mathbb{N}$.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: B is countable.

$\langle 2 \rangle 2$. CASE: B is finite.

$\langle 3 \rangle 1$. PICK a natural number n and bijection $f : \{m \in \mathbb{N} : m < n\} \approx B$

$\langle 3 \rangle 2$. PICK $b \in B$

$\langle 3 \rangle 3$. Extend f to a surjection $g : \mathbb{N} \twoheadrightarrow B$ by setting $g(m) = b$ for $m \geq n$.

$\langle 2 \rangle 3$. CASE: B is countably infinite.

PROOF: Then there exists a bijection $\mathbb{N} \approx B$.

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

PROOF: Given a surjection $f : \mathbb{N} \twoheadrightarrow B$, define $g : B \hookrightarrow \mathbb{N}$ by $g(b)$ is the smallest number such that $f(g(b)) = b$.

$\langle 1 \rangle 3$. $3 \Rightarrow 1$

$\langle 2 \rangle 1$. LET: $f : B \hookrightarrow \mathbb{N}$ be injective.

$\langle 2 \rangle 2$. $f(B)$ is countable.

$\langle 2 \rangle 3$. $B \approx f(B)$

$\langle 2 \rangle 4$. B is countable.

□

Corollary 2.19.5.1. *A subset of a countable set is countable.*

Corollary 2.19.5.2. $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

PROOF: The function that maps (m, n) to $2^m 3^n$ is injective. \square

Corollary 2.19.5.3. *The Cartesian product of two countable sets is countable.*

Theorem 2.19.6. *A countable union of countable sets is countable.*

PROOF:

$\langle 1 \rangle 1$. LET: A be a set.

$\langle 1 \rangle 2$. LET: $\mathcal{B} \subseteq \mathcal{P}A$ be a countable set of countable sets such that $\bigcup \mathcal{B} = A$

$\langle 1 \rangle 3$. PICK a surjection $B : \mathbb{N} \twoheadrightarrow \mathcal{B}$

$\langle 1 \rangle 4$. ASSUME: w.l.o.g. each $B(n)$ is nonempty.

$\langle 1 \rangle 5$. For $n \in \mathbb{N}$, PICK a surjective function $g_n : \mathbb{N} \twoheadrightarrow B(n)$

$\langle 1 \rangle 6$. LET: $h : \mathbb{N} \times \mathbb{N} \rightarrow A$ be the function $h(m, n) = g_m(n)$

$\langle 1 \rangle 7$. h is surjective.

\square

Theorem 2.19.7. $2^{\mathbb{N}}$ *is uncountable.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : \mathbb{N} \rightarrow 2^{\mathbb{N}}$

PROVE: f is not surjective.

$\langle 1 \rangle 2$. Define $g : \mathbb{N} \rightarrow 2$ by $g(n) = 1 - f(n)(n)$.

$\langle 1 \rangle 3$. For all $n \in \mathbb{N}$ we have $g(n) \neq f(n)(n)$.

$\langle 1 \rangle 4$. For all $n \in \mathbb{N}$ we have $g \neq f(n)$.

\square

Theorem 2.19.8. *For any set A , there is no surjective function $A \rightarrow \mathcal{P}A$.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow \mathcal{P}A$

$\langle 1 \rangle 2$. LET: $S = \{x \in A : x \notin f(x)\}$

$\langle 1 \rangle 3$. For all $a \in A$ we have $S \neq f(a)$

PROOF: We have $a \in S$ if and only if $a \notin f(a)$.

\square

Corollary 2.19.8.1. *For any set A , there is no injective function $\mathcal{P}A \rightarrow A$.*

Chapter 3

Order Theory

Part II

Category Theory

3.1 Relations

Definition 3.1.1 (Reflexive). A relation $R \subseteq A \times A$ is *reflexive* iff, for all $a \in A$, we have $(a, a) \in R$.

Definition 3.1.2 (Antisymmetric). A relation $R \subseteq A \times A$ is *antisymmetric* iff, for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Definition 3.1.3 (Transitive). A relation $R \subseteq A \times A$ is *transitive* iff, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Definition 3.1.4 (Partial Order). A *partial order* on a set A is a relation on A that is reflexive, antisymmetric and transitive.

We say (A, \leq) is a *partially ordered set* or *poset* iff \leq is a partial order on A .

Definition 3.1.5 (Greatest). Let A be a poset and $a \in A$. Then a is the *greatest* element iff $\forall x \in A. x \leq a$.

Definition 3.1.6 (Least). Let A be a poset and $a \in A$. Then a is the *least* element iff $\forall x \in A. a \leq x$.

Definition 3.1.7 (Upper Bound). Let A be a poset, $S \subseteq A$, and $u \in A$. Then u is an *upper bound* for S iff $\forall x \in S. x \leq u$. We say S is *bounded above* iff it has an upper bound.

Definition 3.1.8 (Lower Bound). Let A be a poset, $S \subseteq A$, and $l \in A$. Then l is a *lower bound* for S iff $\forall x \in S. l \leq x$. We say S is *bounded below* iff it has a lower bound.

Definition 3.1.9 (Supremum). Let A be a poset, $S \subseteq A$ and $s \in A$. Then s is the *supremum* or *least upper bound* for S iff s is the least element in the sub-poset of upper bounds for A .

Definition 3.1.10 (Infimum). Let A be a poset, $S \subseteq A$ and $i \in A$. Then i is the *infimum* or *greatest lower bound* for S iff i is the greatest element in the sub-poset of lower bounds for A .

Definition 3.1.11 (Least Upper Bound Property). A poset A has the *least upper bound property* iff every nonempty subset of A that is bounded above has a least upper bound.

Proposition 3.1.12. *Let A be a poset. Then A has the least upper bound property if and only if every nonempty subset of A that is bounded below has a greatest lower bound.*

PROOF:

$\langle 1 \rangle 1$. If A has the least upper bound property then every subset of A that is bounded below has a greatest lower bound.

$\langle 2 \rangle 1$. ASSUME: A has the least upper bound property.

$\langle 2 \rangle 2$. LET: $S \subseteq A$ be nonempty and bounded below.

⟨2⟩3. LET: L be the set of lower bounds of S .

⟨2⟩4. L is nonempty.

PROOF: Because S is bounded below.

⟨2⟩5. L is bounded above.

PROOF: Pick an element $s \in S$. Then s is an upper bound for L .

⟨2⟩6. LET: s be the supremum of L .

⟨2⟩7. s is the greatest lower bound of S .

⟨3⟩1. s is a lower bound of S .

⟨4⟩1. LET: $x \in S$

⟨4⟩2. x is an upper bound for L .

⟨4⟩3. $s \leq x$

⟨3⟩2. For any lower bound l of S we have $l \leq s$.

PROOF: Immediate from ⟨2⟩6.

⟨1⟩2. If every subset of A that is bounded below has a greatest lower bound, then A has the least upper bound property.

PROOF: Dual.

□

3.1.1 Strict Partial Orders

Definition 3.1.13 (Strict Partial Order). A *strict partial order* on a set A is a relation on A that is irreflexive and transitive.

Proposition 3.1.14. 1. If \leq is a partial order on A then $<$ is a strict partial order on A , where $x < y$ iff $x \leq y \wedge x \neq y$.

2. If $<$ is a strict partial order on A then \leq is a partial order on A , where $x \leq y$ iff $x < y \vee x = y$.

3. These two relations are inverses of one another.

3.1.2 Linear Orders

Definition 3.1.15 (Linear Order). A *linear order* on a set A is a partial order \leq on A such that, for all $x, y \in A$, we have $x \leq y$ or $y \leq x$.

A *linearly ordered set* is a pair (X, \leq) such that X is a set and \leq is a linear order on X .

Definition 3.1.16 (Open Interval). Let X be a linearly ordered set and $a, b \in X$. The *open interval* (a, b) is the set

$$\{x \in X : a < x < b\}.$$

Definition 3.1.17 (Immediate Predecessor, Immediate Successor). Let X be a linearly ordered set and $a, b \in X$. Then b is the (*immediate*) *successor* of a , and a is the (*immediate*) *predecessor* of b , iff $a < b$ and there is no x such that $a < x < b$.

Definition 3.1.18 (Dictionary Order). Let A and B be linearly ordered sets. The *dictionary order* on $A \times B$ is the order defined by

$$(a, b) < (a', b') \Leftrightarrow a < a' \vee (a = a' \wedge b < b') .$$

Theorem 3.1.19 (Maximum Principle). *Every poset has a maximal linearly ordered subset.*

PROOF:

$\langle 1 \rangle 1$. LET: (A, \leq) be a poset.

$\langle 1 \rangle 2$. PICK a well ordering \preceq of A .

PROOF: Well Ordering Theorem.

$\langle 1 \rangle 3$. LET: $h : A \rightarrow 2$ be the function defined by \preceq -recursion thus:

$$h(a) = \begin{cases} 1 & \text{if } a \text{ is } \leq\text{-comparable with every } b < a \text{ such that } h(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\langle 1 \rangle 4$. LET: $B = \{x \in A : h(x) = 1\}$

PROVE: B is a maximal subset linearly ordered by \leq .

$\langle 1 \rangle 5$. B is linearly ordered by \leq .

$\langle 2 \rangle 1$. LET: $x, y \in B$

$\langle 2 \rangle 2$. ASSUME: w.l.o.g. $x \preceq y$

$\langle 2 \rangle 3$. y is \leq -comparable with x

$\langle 1 \rangle 6$. For any subset $C \subseteq A$ linearly ordered by \leq , if $B \subseteq C$ then $B = C$.

$\langle 2 \rangle 1$. LET: $x \in C$

$\langle 2 \rangle 2$. x is comparable with every $y \preceq x$ such that $h(y) = 1$

$\langle 2 \rangle 3$. $x \in B$

□

Theorem 3.1.20 (Zorn's Lemma). *Let A be a poset. If every linearly ordered subset of A is bounded above, then A has a maximal element.*

PROOF:

$\langle 1 \rangle 1$. PICK a maximal linearly ordered subset B of A .

PROOF: Maximal Principle

$\langle 1 \rangle 2$. PICK an upper bound c for B .

PROVE: c is maximal.

$\langle 1 \rangle 3$. LET: $x \in A$

$\langle 1 \rangle 4$. ASSUME: $c \leq x$

PROVE: $x = c$

$\langle 1 \rangle 5$. x is an upper bound for B .

$\langle 1 \rangle 6$. $x \in B$

PROOF: By the maximality of B , since $B \cup \{x\}$ is linearly ordered.

$\langle 1 \rangle 7$. $x \leq c$

PROOF: $\langle 1 \rangle 2$

$\langle 1 \rangle 8$. $x = c$

□

Corollary 3.1.20.1 (Kuratowski's Lemma). *Let $\mathcal{A} \subseteq \mathcal{P}X$. Suppose that, for every subset $\mathcal{B} \subseteq \mathcal{A}$ that is linearly ordered by inclusion, we have $\bigcup \mathcal{B} \in \mathcal{A}$. Then \mathcal{A} has a maximal element.*

Definition 3.1.21 (Closed Interval). Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *closed interval* $[a, b]$ is

$$[a, b] := \{x \in X : a \leq x \leq b\} .$$

Definition 3.1.22 (Half-Open Interval). Let X be a linearly ordered set. Let $a, b \in X$ with $a < b$. The *half-open intervals* $(a, b]$ and $[a, b)$ are defined by

$$\begin{aligned} (a, b] &:= \{x \in X : a < x \leq b\} \\ [a, b) &:= \{x \in X : a \leq x < b\} \end{aligned}$$

Definition 3.1.23 (Open Ray). Let X be a linearly ordered set and $a \in X$. The *open rays* $(a, +\infty)$ and $(-\infty, a)$ are defined by:

$$\begin{aligned} (a, +\infty) &:= \{x \in X : a < x\} \\ (-\infty, a) &:= \{x \in X : x < a\} \end{aligned}$$

Definition 3.1.24 (Closed Ray). Let X be a linearly ordered set and $a \in X$. The *closed rays* $[a, +\infty)$ and $(-\infty, a]$ are defined by:

$$\begin{aligned} [a, +\infty) &:= \{x \in X : a \leq x\} \\ (-\infty, a] &:= \{x \in X : x \leq a\} \end{aligned}$$

Definition 3.1.25 (Convex). Let X be a linearly ordered set and $Y \subseteq X$. Then Y is *convex* iff, for all $a, b \in Y$ and $c \in X$, if $a < c < b$ then $c \in Y$.

3.1.3 Sets of Finite Type

Definition 3.1.26 (Finite Type). Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then \mathcal{A} is of *finite type* if and only if, for any $B \subseteq X$, we have $B \in \mathcal{A}$ if and only if every finite subset of B is in \mathcal{A} .

Proposition 3.1.27 (Tukey's Lemma). *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$. If \mathcal{A} is of finite type, then \mathcal{A} has a maximal element.*

PROOF:

$\langle 1 \rangle 1$. For every subset $\mathcal{B} \subseteq \mathcal{A}$ that is linearly ordered by inclusion, we have

$$\bigcup \mathcal{B} \in \mathcal{A}.$$

$\langle 2 \rangle 1$. LET: $\mathcal{B} \subseteq \mathcal{A}$

$\langle 2 \rangle 2$. ASSUME: \mathcal{B} is linearly ordered by inclusion.

$\langle 2 \rangle 3$. Every finite subset of $\bigcup \mathcal{B}$ is in \mathcal{A}

$\langle 2 \rangle 4$. $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Kuratowski's Lemma.

□

3.2 Well Orders

Definition 3.2.1 (Well Ordered Set). A *well ordered set* is a linearly ordered set such that every nonempty subset has a least element.

Proposition 3.2.2. *Any subset of a well ordered set is well ordered.*

Proposition 3.2.3. *The product of two well ordered sets is well ordered under the dictionary order.*

Theorem 3.2.4 (Well Ordering Theorem). *Every set has a well ordering.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a set.

$\langle 1 \rangle 2$. PICK a choice function $c : \mathcal{P}X - \{\emptyset\} \rightarrow X$

$\langle 1 \rangle 3$. Define a *tower* to be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well ordering of T , and

$$\forall x \in T. x = c(X - \{y \in T : y < x\}) .$$

$\langle 1 \rangle 4$. Given two towers, either they are equal or one is a section of the other.

$\langle 2 \rangle 1$. LET: $(T_1, <_1)$ and $(T_2, <_2)$ be towers.

$\langle 2 \rangle 2$. ASSUME: w.l.o.g. there exists a strictly monotone function $h : T_1 \rightarrow T_2$

$\langle 2 \rangle 3$. $h(T_1)$ is either T_2 or a section of T_2

PROOF: Proposition 3.2.11.

$\langle 2 \rangle 4$. $\forall x \in T_1. h(x) = x$

$\langle 3 \rangle 1$. LET: $x \in T_1$

$\langle 3 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall y < x. h(y) = y$

$\langle 3 \rangle 3$. $h(x)$ is the least element of $T_2 - \{h(y) \in T_1 : y < x\}$

$\langle 3 \rangle 4$. $h(x)$ is the least element of $T_2 - \{y \in T_1 : y < x\}$

PROOF: $\langle 3 \rangle 2$

$\langle 3 \rangle 5$. $h(x) = x$

PROOF:

$$h(x) = c(X - \{y \in T_2 : y < h(x)\}) \quad (\langle 1 \rangle 3)$$

$$= c(X - \{y \in T_2 : y < x\}) \quad (\langle 3 \rangle 4)$$

$$= c(X - \{y \in T_1 : y < x\}) \quad (\langle 3 \rangle 2)$$

$$= x \quad (\langle 1 \rangle 3)$$

$\langle 1 \rangle 5$. If $(T, <)$ is a tower and $T \neq X$, then there exists a tower of which $(T, <)$ is a section.

PROOF: Let $T_1 = T \cup \{c(T)\}$ and $<_1$ be the extension of $<$ such that $x <_1 c(T)$ for all $x \in T$.

$\langle 1 \rangle 6$. LET: $\mathbf{T} = \bigcup \{T : \exists R. (T, R) \text{ is a tower}\}$ and $\mathbf{R} = \bigcup \{R : \exists T. (T, R) \text{ is a tower}\}$

$\langle 1 \rangle 7$. (\mathbf{T}, \mathbf{R}) is a tower.

$\langle 2 \rangle 1$. \mathbf{R} is irreflexive.

PROOF: Since for every tower $(T, <)$ we have $<$ is irreflexive.

$\langle 2 \rangle 2$. \mathbf{R} is transitive.

$\langle 3 \rangle 1$. ASSUME: $x \mathbf{R} y$ and $y \mathbf{R} z$

$\langle 3 \rangle 2$. PICK towers $(T_1, <_1)$ and $(T_2, <_2)$ such that $x <_1 y$ and $y <_2 z$

$\langle 3 \rangle 3$. ASSUME: w.l.o.g. $(T_1, <_1)$ is either $(T_2, <_2)$ or a section of $(T_2, <_2)$

- $\langle 3 \rangle 4. x <_2 y <_2 z$
- $\langle 3 \rangle 5. x <_2 z$
- $\langle 3 \rangle 6. x \mathbf{R} z$
- $\langle 2 \rangle 3. \text{ For all } x, y \in \mathbf{T}, \text{ either } x \mathbf{R} y \text{ or } x = y \text{ or } y \mathbf{R} x$
PROOF: There exists a tower that has both x and y .
- $\langle 2 \rangle 4. \text{ Every nonempty subset of } \mathbf{T} \text{ has an } \mathbf{R}\text{-least element.}$
 - $\langle 3 \rangle 1. \text{ LET: } A \subseteq \mathbf{T} \text{ be nonempty.}$
 - $\langle 3 \rangle 2. \text{ PICK } a \in A$
 - $\langle 3 \rangle 3. \text{ PICK a tower } (T, <) \text{ such that } a \in T.$
 - $\langle 3 \rangle 4. \text{ LET: } b \text{ be the } <\text{-least element of } A \cap T$
PROVE: b is \mathbf{R} -least in A .
 - $\langle 3 \rangle 5. \text{ LET: } x \in A$
 - $\langle 3 \rangle 6. \text{ Etc.}$
- $\langle 2 \rangle 5. \forall x \in \mathbf{T}. x = c(X - \{y \in \mathbf{T} : y \mathbf{R} x\})$
- $\langle 1 \rangle 8. \mathbf{T} = X$
- $\langle 1 \rangle 9. \mathbf{R} \text{ is a well ordering of } X.$

Proposition 3.2.5. *There exists a well-ordered set with a largest element Ω such that $(-\infty, \Omega)$ is uncountable but, for all $\alpha < \Omega$, we have $(-\infty, \alpha)$ is countable.*

PROOF:

- $\langle 1 \rangle 1$. PICK an uncountable well ordered set B .
- $\langle 1 \rangle 2$. LET: $C = 2 \times B$ under the dictionary order.
- $\langle 1 \rangle 3$. LET: Ω be the least element of C such that $(-\infty, \Omega)$ is uncountable.
- $\langle 1 \rangle 4$. LET: $A = (-\infty, \Omega]$
- $\langle 1 \rangle 5$. A is a well ordered set with largest element Ω such that $(-\infty, \Omega)$ is uncountable but, for all $\alpha < \Omega$, we have $(-\infty, \alpha)$ is countable.

Proposition 3.2.6. *Every well ordered set has the least upper bound property.*

PROOF: For any subset that is bounded above, the set of upper bounds is nonempty, hence has a least element. \square

Proposition 3.2.7. *In a well ordered set, every element that is not greatest has a successor.*

PROOF: If a is not greatest, then $\{x : x > a\}$ is nonempty, hence has a least element. \square

Theorem 3.2.8 (Transfinite Induction). *Let J be a well ordered set. Let $S \subseteq J$. Assume that, for every $\alpha \in J$, if $\forall x < \alpha. x \in S$ then $\alpha \in S$. Then $S = J$.*

PROOF: Otherwise $J - S$ would be a nonempty subset of J with no least element. \square

Proposition 3.2.9. *Let I be a well ordered set. Let $\{A_i\}_{i \in I}$ be a family of well ordered sets. Define $<$ on $\coprod_{i \in I} A_i$ by: $\kappa_i(a) < \kappa_j(b)$ iff either $i < j$, or $i = j$ and $a < b$ in A_i . Then $<$ well orders $\coprod_{i \in I} A_i$.*

PROOF: Easy. \square

Theorem 3.2.10 (Principle of Transfinite Recursion). *Let J be a well ordered set. Let C be a set. Let \mathcal{F} be the set of all functions from a section of J into C . Let $\rho : \mathcal{F} \rightarrow C$. Then there exists a unique function $h : J \rightarrow C$ such that, for all $\alpha \in J$, we have*

$$h(\alpha) = \rho(h \upharpoonright (-\infty, \alpha)) .$$

PROOF:

$\langle 1 \rangle 1$. For a function h mapping either a section of J or all of J into C , let us say h is *acceptable* iff, for all $x \in \text{dom } h$, we have $(-\infty, x) \subseteq \text{dom } h$ and $h(x) = \rho(h \upharpoonright (-\infty, x))$.

$\langle 1 \rangle 2$. If h and k are acceptable functions then $h(x) = k(x)$ for all x in both domains.

$\langle 2 \rangle 1$. LET: $x \in J$

$\langle 2 \rangle 2$. ASSUME: as transfinite induction hypothesis that, for all $y < x$ and any acceptable functions h and k with $y \in \text{dom } h \cap \text{dom } k$, we have $h(y) = k(y)$

$\langle 2 \rangle 3$. LET: h and k be acceptable functions with $x \in \text{dom } h \cap \text{dom } k$

$\langle 2 \rangle 4$. $h \upharpoonright (-\infty, x) = k \upharpoonright (-\infty, x)$

PROOF: By $\langle 2 \rangle 2$.

$\langle 2 \rangle 5$. $h(x) = k(x)$

PROOF: By $\langle 2 \rangle 3$, each is the least element of the set in $\langle 2 \rangle 4$.

$\langle 1 \rangle 3$. For $\alpha \in J$, if there exists an acceptable function $(-\infty, \alpha) \rightarrow C$, then there exists an acceptable function $(-\infty, \alpha] \rightarrow C$.

$\langle 2 \rangle 1$. LET: $\alpha \in J$

$\langle 2 \rangle 2$. LET: $f : (-\infty, \alpha) \rightarrow C$ be acceptable.

$\langle 2 \rangle 3$. LET: $g : (-\infty, \alpha] \rightarrow C$ be the function given by

$$g(x) = \begin{cases} f(x) & \text{if } x < \alpha \\ \rho(f) & \text{if } x = \alpha \end{cases}$$

$\langle 2 \rangle 4$. g is acceptable.

$\langle 1 \rangle 4$. Let $K \subseteq J$. Assume that, for all $\alpha \in K$, there exists an acceptable function $(-\infty, \alpha) \rightarrow C$. Then there exists an acceptable function $\bigcup_{\alpha \in K} (-\infty, \alpha) \rightarrow C$.

$\langle 2 \rangle 1$. Define $f : \bigcup_{\alpha \in K} (-\infty, \alpha) \rightarrow C$ by: $f(x) = y$ iff there exists $\alpha \in K$ and $g : (-\infty, \alpha) \rightarrow C$ acceptable such that $g(x) = y$.

$\langle 1 \rangle 5$. For every $\beta \in J$, there exists an acceptable function $(-\infty, \beta) \rightarrow C$

$\langle 2 \rangle 1$. LET: $\beta \in J$

$\langle 2 \rangle 2$. ASSUME: as transfinite induction hypothesis that, for all $\alpha < \beta$, there exists an acceptable function $(-\infty, \alpha) \rightarrow C$

$\langle 2 \rangle 3$. CASE: β has a predecessor

$\langle 3 \rangle 1$. LET: α be the predecessor of β .

$\langle 3 \rangle 2$. There exists an acceptable function $(-\infty, \alpha) \rightarrow C$.

$\langle 3 \rangle 3$. There exists an acceptable function $(-\infty, \beta) \rightarrow C$.

PROOF: By $\langle 1 \rangle 3$ since $(-\infty, \beta) = (-\infty, \alpha]$.

$\langle 2 \rangle 4$. CASE: β has no predecessor.

PROOF: The result follows by $\langle 1 \rangle 4$ since $(-\infty, \beta) = \bigcup_{\alpha < \beta} (-\infty, \alpha)$.
 $\langle 1 \rangle 6$. There exists an acceptable function $J \rightarrow C$.
 $\langle 2 \rangle 1$. CASE: J has a greatest element.
 $\langle 3 \rangle 1$. LET: g be greatest.
 $\langle 3 \rangle 2$. There exists an acceptable function $(-\infty, g) \rightarrow C$.
 PROOF: $\langle 1 \rangle 5$
 $\langle 3 \rangle 3$. There exists an acceptable function $J \rightarrow C$.
 PROOF: By $\langle 1 \rangle 3$ since $J = (-\infty, g]$.
 $\langle 2 \rangle 2$. CASE: J has no greatest element.
 PROOF: By $\langle 1 \rangle 4$ since $J = \bigcup_{\alpha \in J} (-\infty, \alpha)$.

□

Corollary 3.2.10.1 (Cardinal Comparability). *Let A and B be sets. Then either $A \leq B$ or $B \leq A$.*

PROOF: Choose well orderings of A and B . Then either there exists a surjection $A \twoheadrightarrow B$, or there exists an injective function $h : A \rightarrow B$ defined by transfinite recursion by $h(x)$ is the least element of $B - h((-\infty, x))$. □

Proposition 3.2.11. *Let J and E be well ordered sets. Let $h : J \rightarrow E$. Then the following are equivalent.*

1. h is strictly monotone and $h(J)$ is either E or a section of E .
2. For all $\alpha \in J$, we have $h(\alpha)$ is the least element of $E - h((-\infty, \alpha))$.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$
 $\langle 2 \rangle 1$. ASSUME: 1
 $\langle 2 \rangle 2$. $h(J)$ is closed downwards.
 $\langle 2 \rangle 3$. LET: $\alpha \in J$
 $\langle 2 \rangle 4$. $h(\alpha) \in E - h((-\infty, \alpha))$
 PROOF: If $\beta < \alpha$ then $h(\beta) < h(\alpha)$.
 $\langle 2 \rangle 5$. For all $y \in E - h((-\infty, \alpha))$ we have $h(\alpha) \leq y$
 $\langle 3 \rangle 1$. ASSUME: for a contradiction $y < h(\alpha)$
 $\langle 3 \rangle 2$. $y \in h(J)$
 $\langle 3 \rangle 3$. PICK $\beta \in J$ such that $h(\beta) = y$
 $\langle 3 \rangle 4$. $h(\beta) < h(\alpha)$
 $\langle 3 \rangle 5$. $\beta < \alpha$
 $\langle 3 \rangle 6$. Q.E.D.

PROOF: This contradicts the fact that $y \notin h((-\infty, \alpha))$.

$\langle 1 \rangle 2$. $2 \Rightarrow 1$
 $\langle 2 \rangle 1$. ASSUME: 2
 $\langle 2 \rangle 2$. h is strictly monotone.
 $\langle 3 \rangle 1$. LET: $\alpha, \beta \in J$ with $\alpha < \beta$
 $\langle 3 \rangle 2$. $h(\alpha) \neq h(\beta)$
 PROOF: Because $h(\beta) \in E - h((-\infty, \beta))$.
 $\langle 3 \rangle 3$. $h(\alpha) \leq h(\beta)$

PROOF: Because $h(\alpha)$ is least in $E - h((-\infty, \alpha))$.

$\langle 3 \rangle 4$. $h(\alpha) < h(\beta)$

$\langle 2 \rangle 3$. $h(J)$ is either E or a section of E .

$\langle 3 \rangle 1$. ASSUME: $h(J) \neq E$

$\langle 3 \rangle 2$. LET: e be least in $E - h(J)$

PROVE: $h(J) = (-\infty, e)$

$\langle 3 \rangle 3$. $h(J) \subseteq (-\infty, e)$

$\langle 4 \rangle 1$. LET: $\alpha \in J$

$\langle 4 \rangle 2$. $h(\alpha) \neq e$

PROOF: $e \notin h(J)$

$\langle 4 \rangle 3$. $h(\alpha) \leq e$

PROOF: Since $h(\alpha)$ is least in $E - h((-\infty, \alpha))$.

$\langle 4 \rangle 4$. $h(\alpha) < e$

$\langle 3 \rangle 4$. $(-\infty, e) \subseteq h(J)$

PROOF: If $e' < e$ then $e' \in h(J)$ by leastness of e .

□

Chapter 4

Category Theory

4.1 Categories

Definition 4.1.1. A *category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*. We write $A \in \mathcal{C}$ for $A \in \text{Ob}(\mathcal{C})$.
- for any objects X and Y , a set $\mathcal{C}[X, Y]$ of *morphisms* from X to Y . We write $f : X \rightarrow Y$ for $f \in \mathcal{C}[X, Y]$.
- for any objects X, Y and Z , a function $\circ : \mathcal{C}[Y, Z] \times \mathcal{C}[X, Y] \rightarrow \mathcal{C}[X, Z]$, called *composition*.

such that:

- Given $f : X \rightarrow Y, g : Y \rightarrow Z$ and $h : Z \rightarrow W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object X , there exists a morphism $\text{id}_X : X \rightarrow X$, the *identity morphism* on X , such that:
 - for any object Y and morphism $f : Y \rightarrow X$ we have $\text{id}_X \circ f = f$
 - for any object Y and morphism $f : X \rightarrow Y$ we have $f \circ \text{id}_X = f$

We write the composite of morphism f_1, \dots, f_n as $f_n \circ \dots \circ f_1$. This is unambiguous thanks to Associativity.

Definition 4.1.2. Let **Set** be the category of small sets and functions.

Definition 4.1.3. Let **LPos** be the category of linearly ordered sets and monotone functions.

Proposition 4.1.4. Any finite linearly ordered set is isomorphic to $\{m \in \mathbb{N} : m < n\}$ for some n .

PROOF:

$\langle 1 \rangle$ 1. Every finite nonempty linearly ordered set has a greatest element.

- $\langle 2 \rangle 1$. LET: $P[n]$ be the property: for any linearly ordered set A , if there exists a bijection $A \approx \{m \in \mathbb{N} : m < n\}$ and A is nonempty then A has a greatest element.
- $\langle 2 \rangle 2$. $P[0]$
 PROOF: Vacuous.
- $\langle 2 \rangle 3$. $\forall n \in \mathbb{N}. P[n] \Rightarrow P[n+1]$
- $\langle 3 \rangle 1$. LET: $n \in \mathbb{N}$
- $\langle 3 \rangle 2$. ASSUME: $P[n]$
- $\langle 3 \rangle 3$. LET: A be a nonempty linearly ordered set.
- $\langle 3 \rangle 4$. LET: $f : A \approx \{m \in \mathbb{N} : m < n+1\}$
- $\langle 3 \rangle 5$. LET: $a = f^{-1}(n)$
- $\langle 3 \rangle 6$. $f \upharpoonright (A - \{a\}) : A - \{a\} \approx \{m \in \mathbb{N} : m < n\}$
- $\langle 3 \rangle 7$. ASSUME: w.l.o.g. a is not greatest in A .
- $\langle 3 \rangle 8$. LET: b be greatest in $A - \{a\}$
 PROOF: $\langle 3 \rangle 2$
- $\langle 3 \rangle 9$. b is greatest in A .
- $\langle 1 \rangle 2$. LET: $P[n]$ be the property: for any linearly ordered set A , if there exists a bijection $A \approx \{m \in \mathbb{N} : m < n\}$ then there exists an isomorphism in **LPos** $A \cong \{m \in \mathbb{N} : m < n\}$.
- $\langle 1 \rangle 3$. $P[0]$
 PROOF: If there exists a bijection $A \approx \emptyset$ then A is empty and so the unique function $A \rightarrow \emptyset$ is an order isomorphism.
- $\langle 1 \rangle 4$. For every natural number n , if $P[n]$ then $P[n+1]$.
- $\langle 2 \rangle 1$. LET: n be a natural number.
- $\langle 2 \rangle 2$. ASSUME: $P[n]$
- $\langle 2 \rangle 3$. LET: A be a linearly ordered set.
- $\langle 2 \rangle 4$. ASSUME: A has $n+1$ elements.
- $\langle 2 \rangle 5$. LET: a be the greatest element in A .
- $\langle 2 \rangle 6$. LET: $f : A - \{a\} \cong \{m \in \mathbb{N} : m < n\}$ be an order isomorphism.
 PROOF: $\langle 2 \rangle 2$
- $\langle 2 \rangle 7$. Define $g : A \rightarrow \{m \in \mathbb{N} : m < n+1\}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ n & \text{if } x = a \end{cases}$$
- $\langle 2 \rangle 8$. g is an order isomorphism.
- $\langle 1 \rangle 5$. $\forall n \in \mathbb{N}. P[n]$
 \square

Corollary 4.1.4.1. *Any finite linearly ordered set is well ordered.*

Proposition 4.1.5. *Let J and E be well ordered sets. Suppose there is a strictly monotone map $J \rightarrow E$. Then J is isomorphic either to E or a section of E .*

PROOF:

- $\langle 1 \rangle 1$. LET: $k : J \rightarrow E$ be strictly monotone.
- $\langle 1 \rangle 2$. ASSUME: w.l.o.g. E is nonempty.
- $\langle 1 \rangle 3$. PICK $e_0 \in E$

⟨1⟩4. LET: $h : J \rightarrow E$ be the function defined by transfinite recursion thus:

$$h(\alpha) = \begin{cases} \text{the least element in } E - h((-\infty, \alpha)) & \text{if } h((-\infty, \alpha)) \neq E \\ e_0 & \text{if } h((-\infty, \alpha)) = E \end{cases}$$

⟨1⟩5. $\forall \alpha \in J, h(\alpha) \leq k(\alpha)$

⟨2⟩1. LET: $\alpha \in J$

⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha, h(\beta) \leq k(\beta)$.

⟨2⟩3. $\forall \beta < \alpha, h(\beta) < k(\alpha)$

⟨2⟩4. $h((-\infty, \alpha)) \neq E$

⟨2⟩5. $h(\alpha)$ is the least element in $E - h((-\infty, \alpha))$.

⟨2⟩6. $k(\alpha) \in E - h((-\infty, \alpha))$

⟨2⟩7. $h(\alpha) \leq k(\alpha)$

⟨1⟩6. $\forall \alpha \in J, h((-\infty, \alpha)) \neq E$

PROOF: For $\beta < \alpha$ we have $h(\beta) \leq k(\beta) < k(\alpha)$ so $k(\alpha) \notin h((-\infty, \alpha))$.

⟨1⟩7. For all $\alpha \in J$, we have $h(\alpha)$ is the least element of $E - h((-\infty, \alpha))$.

⟨1⟩8. h is strictly monotone and $h(J)$ is either E or a section of E .

PROOF: Proposition 3.2.11.

□

Proposition 4.1.6. *If A and B are well ordered sets, then exactly one of the following conditions hold: $A \cong B$, or A is isomorphic to a section of B , or B is isomorphic to a section of A .*

PROOF:

⟨1⟩1. At least one of the conditions holds.

⟨2⟩1. B is isomorphic to either $A + B$ or a section of $A + B$.

⟨2⟩2. CASE: $B \cong A + B$

⟨3⟩1. LET: ϕ be the isomorphism $B \cong A + B$

⟨3⟩2. LET: b_0 be the least element in B .

⟨3⟩3. A is isomorphic to the section $(-\infty, \phi^{-1}(\kappa_2(b_0)))$ of B .

⟨2⟩3. CASE: $a \in A$ and $B \cong (-\infty, \kappa_1(a))$

PROOF: Then B is isomorphic to the section $(-\infty, a)$ of A .

⟨2⟩4. CASE: $b \in B$ and $\phi : B \cong (-\infty, \kappa_2(b))$

⟨3⟩1. CASE: b is least in B .

PROOF: Then $A \cong B$.

⟨3⟩2. CASE: b is not least in B .

⟨4⟩1. LET: b_0 be least in B .

⟨4⟩2. A is isomorphic to the section $(-\infty, \phi^{-1}(\kappa_2(b_0)))$ of B .

⟨1⟩2. At most one of the conditions holds.

PROOF: Since a well ordered set cannot be isomorphic to a section of itself.

□

Theorem 4.1.7. *There exists a well ordered set, unique up to order isomorphism, that is uncountable but such that every section is countable.*

PROOF:

⟨1⟩1. There exists a well ordered set that is uncountable but such that every section is countable.

- $\langle 2 \rangle 1$. PICK a well ordered set A with an element $\Omega \in A$ such that $(-\infty, \Omega)$ is uncountable but $\forall \alpha < \Omega. (-\infty, \alpha)$ is countable.
 $\langle 2 \rangle 2$. LET: $(-\infty, \Omega)$ is uncountable but every section is countable.
 $\langle 1 \rangle 2$. If A and B are uncountable well ordered sets such that every section is countable, then $A \cong B$.

PROOF: Since it cannot be that one of A and B is isomorphic to a section of the other.

□

Definition 4.1.8 (Minimal Uncountable Well Ordered Set). The *minimal uncountable well ordered set* Ω is the well ordered set that is uncountable but such that every section is countable.

We write $\bar{\Omega}$ for the well ordered set $\Omega \cup \{\Omega\}$ where Ω is greatest.

Proposition 4.1.9. *Every countable subset of Ω is bounded above.*

PROOF:

- $\langle 1 \rangle 1$. LET: A be a countable subset of Ω .
 $\langle 1 \rangle 2$. For all $a \in A$ we have $(-\infty, a)$ is countable.
 $\langle 1 \rangle 3$. $\bigcup_{a \in A} (-\infty, a)$ is countable.
 $\langle 1 \rangle 4$. $\bigcup_{a \in A} (-\infty, a) \neq \Omega$
 $\langle 1 \rangle 5$. PICK $x \in \Omega - \bigcup_{a \in A} (-\infty, a)$
 $\langle 1 \rangle 6$. x is an upper bound for A .

□

Proposition 4.1.10. *Ω has no greatest element.*

PROOF: For any $\alpha \in \Omega$ we have $(-\infty, \alpha]$ is countable and hence not the whole of Ω . □

Proposition 4.1.11. *There are uncountably many elements of Ω that have no predecessor.*

PROOF:

- $\langle 1 \rangle 1$. LET: A be the set of all elements of Ω that have no predecessor.
 $\langle 1 \rangle 2$. LET: $f : A \times \mathbb{N} \rightarrow \Omega$ be the function that maps (a, n) to the n th successor of a .
 $\langle 1 \rangle 3$. f is surjective.
 $\langle 2 \rangle 1$. ASSUME: for a contradiction $x \in \Omega$ and there is no element $a \in A$ and $n \in \mathbb{N}$ such that x is the n th successor of a .
 $\langle 2 \rangle 2$. LET: x_n be the n th predecessor of x for $n \in \mathbb{N}$.
 $\langle 2 \rangle 3$. $\{x_n : n \in \mathbb{N}\}$ is a nonempty subset of Ω with no least element.
 $\langle 1 \rangle 4$. $A \times \mathbb{N}$ is uncountable.
 $\langle 1 \rangle 5$. A is uncountable.

□

Definition 4.1.12. We identify a poset (A, \leq) with the category with:

- set of objects A

- for $a, b \in A$, the set of homomorphisms is $\{x \in 1 : a \leq b\}$

Proposition 4.1.13. *A category is a poset iff, for any two objects, there exists at most one morphism between them.*

Proposition 4.1.14. *The identity morphism on an object is unique.*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{C} be a category.

$\langle 1 \rangle 2$. LET: $A \in \mathcal{C}$

$\langle 1 \rangle 3$. LET: $i, j : A \rightarrow A$ be identity morphisms on A .

$\langle 1 \rangle 4$. $i = j$

PROOF:

$$\begin{aligned} i &= i \circ j & (j \text{ is an identity on } A) \\ &= j & (i \text{ is an identity on } A) \end{aligned}$$

□

Proposition 4.1.15. *Let A be a linearly ordered set. Then A is well ordered if and only if it does not contain a subset of order type \mathbb{N}^{op} .*

PROOF:

$\langle 1 \rangle 1$. If A is well ordered then it does not contain a subset of order type \mathbb{N}^{op} .

PROOF: A subset of order type \mathbb{N}^{op} would be a subset with no least element.

$\langle 1 \rangle 2$. If A is not well ordered then it contains a subset of order type \mathbb{N}^{op} .

$\langle 2 \rangle 1$. ASSUME: A is not well ordered.

$\langle 2 \rangle 2$. PICK a nonempty subset S with no least element.

$\langle 2 \rangle 3$. PICK $a_0 \in S$

$\langle 2 \rangle 4$. Extend to a sequence (a_n) in S such that $a_{n+1} < a_n$ for all n .

$\langle 2 \rangle 5$. $\{a_n : n \in \mathbb{N}\}$ has order type \mathbb{N}^{op} .

□

Corollary 4.1.15.1. *Let A be a linearly ordered set. If every countable subset of A is well ordered, then A is well ordered.*

Definition 4.1.16. Given $f : A \rightarrow B$ and an object C , define the function $f^* : \mathcal{C}[B, C] \rightarrow \mathcal{C}[A, C]$ by $f^*(g) = g \circ f$.

Definition 4.1.17. Given $f : A \rightarrow B$ and an object C , define the function $f_* : \mathcal{C}[C, A] \rightarrow \mathcal{C}[C, B]$ by $f_*(g) = f \circ g$.

4.1.1 Monomorphisms

Definition 4.1.18 (Monomorphism). Let $f : A \rightarrow B$. Then f is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for any object X and functions $x, y : X \rightarrow A$, if $f \circ x = f \circ y$ then $x = y$.

4.1.2 Epimorphisms

Definition 4.1.19 (Epimorphism). Let $f : A \rightarrow B$. Then f is *epic* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for any object X and functions $x, y : B \rightarrow X$, if $x \circ f = y \circ f$ then $x = y$.

4.1.3 Sections and Retractions

Definition 4.1.20 (Section, Retraction). Let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $rs = \text{id}_B$.

Proposition 4.1.21. *Let $f : A \rightarrow B$ and $r, s : B \rightarrow A$. If r is a retraction of f and s is a section of f then $r = s$.*

PROOF:

$$\begin{aligned}
 r &= r \text{id}_B && \text{(Unit Law)} \\
 &= rfs && (s \text{ is a section of } f) \\
 &= \text{id}_A s && (r \text{ is a retraction of } f) \\
 &= s && \text{(Unit Law)} \square
 \end{aligned}$$

Proposition 4.1.22. *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $s : B \rightarrow A$ be a section of $r : A \rightarrow B$.

$\langle 1 \rangle 2$. LET: X be an object and $x, y : X \rightarrow B$

$\langle 1 \rangle 3$. ASSUME: $s \circ x = s \circ y$

$\langle 1 \rangle 4$. $x = y$

PROOF: $x = r \circ s \circ x = r \circ s \circ y = y$.

\square

Proposition 4.1.23. *Every retraction is epic.*

PROOF: Dual. \square

4.1.4 Isomorphisms

Definition 4.1.24 (Isomorphism). A morphism $f : A \rightarrow B$ is an *isomorphism*, $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$ that is both a retraction and section of f .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 4.1.25. *The inverse of an isomorphism is unique.*

PROOF: From Proposition 4.1.21. \square

Proposition 4.1.26. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $ff^{-1} = \text{id}_B$ and $f^{-1}f = \text{id}_A$. \square

Isomorphism.

Define the opposite category.

Slice categories

Definition 4.1.27. Let \mathcal{C} be a category and $B \in \mathcal{C}$. The category \mathcal{C}_B^B of objects *over and under* B is the category with:

- objects all triples (X, u, p) such that $u : B \rightarrow X$ and $p : X \rightarrow B$
- morphisms $f : (X, u, p) \rightarrow (Y, u', p')$ all morphisms $f : X \rightarrow Y$ such that $fu = u'$ and $p'f = p$.

Proposition 4.1.28.

$$\mathcal{C}_B^B \cong (\mathcal{C}/B) \backslash \text{id}_B \cong (\mathcal{C} \backslash B) / \text{id}_B$$

$(B, \text{id}_B, \text{id}_B)$ is the zero object in \mathcal{C}_B^B .

4.1.5 Initial Objects

Definition 4.1.29 (Initial Object). An object I is *initial* iff, for any object X , there exists exactly one morphism $I \rightarrow X$.

Proposition 4.1.30. *The empty set is initial in Set.*

PROOF: For any set A , the nowhere-defined function is the unique function $\emptyset \rightarrow A$. \square

Proposition 4.1.31. *If I and I' are initial objects, then there exists a unique isomorphism $I \cong I'$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : I \rightarrow I'$ be the unique morphism $I \rightarrow I'$.

$\langle 1 \rangle 2$. LET: $i^{-1} : I' \rightarrow I$ be the unique morphism $I' \rightarrow I$.

$\langle 1 \rangle 3$. $ii^{-1} = \text{id}_{I'}$

PROOF: There is only one morphism $I' \rightarrow I'$.

$\langle 1 \rangle 4$. $i^{-1}i = \text{id}_I$

PROOF: There is only one morphism $I \rightarrow I$.

\square

4.1.6 Terminal Objects

Definition 4.1.32 (Terminal Object). An object T is *terminal* iff, for any object X , there exists exactly one morphism $X \rightarrow T$.

Proposition 4.1.33. *1 is terminal in Set.*

PROOF: For any set A , the constant function to $*$ is the only function $A \rightarrow 1$. \square

Proposition 4.1.34. *If T and T' are terminal objects, then there exists a unique isomorphism $T \cong T'$.*

PROOF: Dual to Proposition 4.1.31. \square

4.1.7 Zero Objects

Definition 4.1.35 (Zero Object). An object Z is a *zero object* iff it is an initial object and a terminal object.

Definition 4.1.36 (Zero Morphism). Let \mathcal{C} be a category with a zero object Z . Let $A, B \in \mathcal{C}$. The *zero morphism* $A \rightarrow B$ is the unique morphism $A \rightarrow Z \rightarrow B$.

Proposition 4.1.37. *There is no zero object in **Set**.*

PROOF: Since $\emptyset \not\approx 1$. \square

4.1.8 Triads

Definition 4.1.38 (Triad). Let \mathcal{C} be a category. A *triad* consists of objects X, Y, M and morphisms $\alpha : X \rightarrow M, \beta : Y \rightarrow M$. We call M the *codomain* of the triad.

4.1.9 Cotriads

Definition 4.1.39 (Cotriad). Let \mathcal{C} be a category. A *cotriad* consists of objects X, Y, W and morphisms $\xi : W \rightarrow X, \eta : W \rightarrow Y$. We call W the *domain* of the triad.

4.1.10 Pullbacks

Definition 4.1.40 (Pullback). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

is a *pullback* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$.

In this case we also say that η is the *pullback* of β along α .

Proposition 4.1.41. *If $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$ form a pullback of $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$, and $\xi' : W' \rightarrow X$ and $\eta' : W' \rightarrow Y$ also form the pullback of α and β , then there exists a unique isomorphism $\phi : W \cong W'$ such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : W \rightarrow W'$ be the unique morphism such that $\eta'\phi = \eta$ and $\xi'\phi = \xi$.

$\langle 1 \rangle 2$. LET: $\phi^{-1} : W' \rightarrow W$ be the unique morphism such that $\eta\phi^{-1} = \eta'$ and $\xi\phi^{-1} = \xi'$.

$\langle 1 \rangle 3$. $\phi\phi^{-1} = \text{id}_{W'}$

PROOF: Each is the unique $x : W' \rightarrow W'$ such that $\eta'x = \eta'$ and $\xi'x = \xi'$.

$\langle 1 \rangle 4$. $\phi^{-1}\phi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\eta x = \eta$ and $\xi x = \xi$.

□

Proposition 4.1.42. *For any morphism $h : A \rightarrow B$, the following diagram is a pullback diagram.*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: Z be an object.

$\langle 1 \rangle 2$. LET: $f : Z \rightarrow B$ and $g : Z \rightarrow A$ satisfy $\text{id}_B f = hg$

$\langle 1 \rangle 3$. $g : Z \rightarrow A$ is the unique morphism such that $\text{id}_A g = g$ and $hg = f$.

□

Proposition 4.1.43. *The pullback of an isomorphism is an isomorphism.*

PROOF:

$\langle 1 \rangle 1$. LET:

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback diagram.

$\langle 1 \rangle 2$. ASSUME: β is an isomorphism.

$\langle 1 \rangle 3$. LET: ξ^{-1} be the unique morphism $X \rightarrow W$ such that $\xi\xi^{-1} = \text{id}_X$ and $\eta\xi^{-1} = \beta^{-1}\alpha$.

PROOF: This exists since $\alpha\text{id}_X = \beta\beta^{-1}\alpha = \alpha$.

$\langle 1 \rangle 4$. $\xi^{-1}\xi = \text{id}_W$

PROOF: Each is the unique $x : W \rightarrow W$ such that $\xi x = \xi$ and $\eta x = \eta$.

□

Proposition 4.1.44. *Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w : A \rightarrow W$ be the unique morphism such that $\xi w = x$ and $\eta w = y$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ is the pullback of β and α in $\mathcal{C} \setminus A$.

PROOF:

$\langle 1 \rangle 1$. LET: $(Z, z) \in \mathcal{C} \backslash A$

$\langle 1 \rangle 2$. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle 3$. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle 4$. $hz = w$

$\langle 2 \rangle 1$. $\xi hz = \xi w$

PROOF:

$$\xi hz = fz \quad (\langle 1 \rangle 3)$$

$$= x \quad (\langle 1 \rangle 2)$$

$$= \xi w$$

$\langle 2 \rangle 2$. $\eta hz = \eta w$

PROOF: Similar.

$\langle 1 \rangle 5$. $h : (Z, z) \rightarrow (W, w)$

□

Proposition 4.1.45. Let $\beta : (Y, y) \rightarrow (M, m)$ and $\alpha : (X, x) \rightarrow (M, m)$ in \mathcal{C}/A . Let

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pullback in \mathcal{C} . Let $w = x\xi : W \rightarrow A$. Then $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ form a pullback of α and β in \mathcal{C}/A .

PROOF:

$\langle 1 \rangle 1$. $\eta : (W, w) \rightarrow (Y, y)$

PROOF:

$$y\eta = m\beta\eta$$

$$= m\alpha\xi$$

$$= x\xi$$

$$= w$$

$\langle 1 \rangle 2$. LET: $(Z, z) \in \mathcal{C}/A$

$\langle 1 \rangle 3$. LET: $f : (Z, z) \rightarrow (X, x)$ and $g : (Z, z) \rightarrow (Y, y)$ satisfy $\alpha f = \beta g$.

$\langle 1 \rangle 4$. LET: $h : Z \rightarrow W$ be the unique morphism such that $\xi h = f$ and $\eta h = g$.

$\langle 1 \rangle 5$. $h : (Z, z) \rightarrow (W, w)$

PROOF:

$$wh = x\xi h$$

$$= xf \quad (\langle 1 \rangle 4)$$

$$= z \quad (\langle 1 \rangle 3)$$

□

Proposition 4.1.46. In **Set**, let $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$. Let $W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$ with inclusion $i : W \rightarrow X \times Y$. Let $\xi = \pi_1 i : W \rightarrow X$ and $\eta = \pi_2 i : W \rightarrow Y$. Then ξ and η form the pullback of α and β .

PROOF:

$\langle 1 \rangle 1.$ $\alpha\xi = \beta\eta$

PROOF: For $w \in W$, if $i(w) = (x, y)$ then $\alpha(\xi(w)) = \alpha(x) = \beta(y) = \beta(\eta(w))$.

$\langle 1 \rangle 2.$ For every set Z and functions $f : Z \rightarrow X$, $g : Z \rightarrow Y$ such that $\alpha f = \beta g$, there exists a unique $h : Z \rightarrow W$ such that $\xi h = f$ and $\eta h = g$

PROOF: For $z \in Z$, let $h(z)$ be the unique element of W such that $i(h(z)) = (f(z), g(z))$.

□

Pullback lemma

4.1.11 Pushouts

Definition 4.1.47 (Pushout). A diagram

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array} \quad (4.1)$$

is a *pushout* iff $\alpha\xi = \beta\eta$ and, for every object Z and morphism $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ such that $f\xi = g\eta$, there exists a unique $h : M \rightarrow Z$ such that $h\alpha = f$ and $h\beta = g$.

We also say that β is the *pushout* of ξ along η .

Proposition 4.1.48. If $\alpha : X \rightarrow M$ and $\beta : Y \rightarrow M$ form a pushout of $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$, and $\alpha' : X \rightarrow M'$ and $\beta' : Y \rightarrow M'$ also form a pushout of ξ and η , then there exists a unique isomorphism $\phi : M \cong M'$ such that $\phi\alpha = \alpha'$ and $\phi\beta = \beta'$.

PROOF: Dual to Proposition 4.1.41. □

Proposition 4.1.49. For any morphism $h : A \rightarrow B$, the following diagram is a pushout diagram.

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \parallel & & \parallel \\ A & \xrightarrow{h} & B \end{array}$$

PROOF: Dual to Proposition 4.1.42.

Proposition 4.1.50. The diagram (4.1) is a pushout in \mathcal{C} iff it is a pullback in \mathcal{C}^{op} .

PROOF: Immediate from definitions. □

Proposition 4.1.51. The pushout of an isomorphism is an isomorphism.

PROOF: Dual to Proposition 4.1.43. \square

Proposition 4.1.52. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in $\mathcal{C} \setminus A$. Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m := \alpha x : A \rightarrow M$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in $\mathcal{C} \setminus A$.

PROOF: Dual to Proposition 4.1.45. \square

Proposition 4.1.53. *Let $\xi : (W, w) \rightarrow (X, x)$ and $\eta : (W, w) \rightarrow (Y, y)$ in \mathcal{C}/A . Let*

$$\begin{array}{ccc} W & \xrightarrow{\xi} & X \\ \eta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & M \end{array}$$

be a pushout in \mathcal{C} . Let $m : M \rightarrow A$ be the unique morphism such that $m\alpha = x$ and $m\beta = y$. Then $\alpha : (X, x) \rightarrow (M, m)$ and $\beta : (Y, y) \rightarrow (M, m)$ is the pushout of ξ and η in \mathcal{C}/A .

PROOF: Dual to Proposition 4.1.44. \square

Proposition 4.1.54. *Set has pushouts.*

PROOF:

$\langle 1 \rangle 1$. LET: $\xi : W \rightarrow X$ and $\eta : W \rightarrow Y$.

$\langle 1 \rangle 2$. LET: \sim be the equivalence relation on $X + Y$ generated by $\xi(w) \sim \eta(w)$ for all $w \in W$

$\langle 1 \rangle 3$. LET: $M = (X + Y)/\sim$ with canonical projection $\pi : X + Y \twoheadrightarrow M$.

$\langle 1 \rangle 4$. LET: $\alpha = \pi \circ \kappa_1 : X \rightarrow M$

$\langle 1 \rangle 5$. LET: $\beta = \pi \circ \kappa_2 : Y \rightarrow M$

$\langle 1 \rangle 6$. LET: Z be any set, $f : X \rightarrow Z$ and $g : Y \rightarrow Z$.

$\langle 1 \rangle 7$. ASSUME: $f\xi = g\eta$

$\langle 1 \rangle 8$. LET: $h : X + Y \rightarrow Z$ be the function defined by $h(x) = f(x)$ and $h(y) = g(y)$ for $x \in X$ and $y \in Y$

$\langle 1 \rangle 9$. h respects \sim

PROOF: For $w \in W$ we have

$$h(\xi(w)) = f(\xi(w)) \quad (\langle 1 \rangle 8)$$

$$= g(\eta(w)) \quad (\langle 1 \rangle 7)$$

$$= h(\eta(w)) \quad (\langle 1 \rangle 8)$$

$\langle 1 \rangle 10$. LET: $\bar{h} : M \rightarrow Z$ be the induced function.

$\langle 1 \rangle 11$. $\bar{h}\alpha = f$

PROOF:

$$\begin{aligned}\bar{h}(\alpha(x)) &= \bar{h}(\pi(\kappa_1(x))) \\ &= h(\kappa_1(x)) \\ &= f(x)\end{aligned}$$

$\langle 1 \rangle 12.$ $\bar{h}\beta = g$

PROOF: Similar.

$\langle 1 \rangle 13.$ For all $k : M \rightarrow Z$, if $k\alpha = f$ and $k\beta = g$ then $k = \bar{h}$.

PROOF:

$$\begin{aligned}k(\pi(\kappa_1(x))) &= k(\alpha(x)) \\ &= f(x) \\ k(\pi(\kappa_2(y))) &= k(\beta(y)) \\ &= g(y) \\ \therefore k \circ \pi &= h \\ \therefore k &= \bar{h}\end{aligned}$$

□

Definition 4.1.55. Let $u : A \rightarrow X$ be an injection. The *pointed set obtained from X by collapsing (A, u)* , denoted $X/(A, u)$, is the pushout

$$\begin{array}{ccc} A & \xrightarrow{\quad} & 1 \\ \downarrow u & & \downarrow * \\ X & \xrightarrow{\quad} & X/(A, u) \end{array}$$

Proposition 4.1.56. In \mathbf{Set}_* , any two morphisms $1 \rightarrow X$ and $1 \rightarrow Y$ have a pushout.

PROOF: The pushout of $a : (1, *) \rightarrow (X, x)$ and $b : (1, *) \rightarrow (Y, y)$ is $(X+Y/\sim, x)$ where \sim is the equivalence relation generated by $x \sim y$. □

Definition 4.1.57 (Wedge). The *wedge* of pointed sets X and Y , $X \vee Y$, is the pushout of the unique morphism $1 \rightarrow X$ and $1 \rightarrow Y$.

Definition 4.1.58 (Smash). Let X and Y be pointed sets. Let $\xi : X \vee Y \rightarrow X$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc} 1 & \xrightarrow{\quad} & X & & \\ \downarrow & & \downarrow & \searrow & \\ Y & \xrightarrow{\quad} & X \vee Y & \xrightarrow{\xi} & X \\ & \searrow 0 & & & \end{array}$$

Let $\eta : X \vee Y \rightarrow Y$ be the unique morphism such that the following diagram

commutes.



Let $\zeta = \langle \xi, \eta \rangle : X \vee Y \rightarrow X \times Y$. The *smash* of X and Y , $X \wedge Y$, is the result of collapsing $X \times Y$ with respect to ζ .

Pushout lemma

4.1.12 Subcategories

Definition 4.1.59 (Subcategory). A *subcategory* \mathcal{C}' of a category \mathcal{C} consists of:

- a subset $\text{Ob}(\mathcal{C}')$ of \mathcal{C}
- for all $A, B \in \text{Ob}(\mathcal{C}')$, a subset $\mathcal{C}'[A, B] \subseteq \mathcal{C}[A, B]$

such that:

- for all $A \in \text{Ob}(\mathcal{C}')$, we have $\text{id}_A \in \mathcal{C}'[A, A]$
- for all $f \in \mathcal{C}'[A, B]$ and $g \in \mathcal{C}'[B, C]$, we have $g \circ f \in \mathcal{C}'[A, C]$.

It is a *full* subcategory iff, for all $A, B \in \text{Ob}(\mathcal{C}')$, we have $\mathcal{C}'[A, B] = \mathcal{C}[A, B]$.

4.1.13 Opposite Category

Definition 4.1.60 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- $\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$
- Given $f \in \mathcal{C}^{\text{op}}[A, B]$ and $g \in \mathcal{C}^{\text{op}}[B, C]$, their composite in \mathcal{C}^{op} is $f \circ g$, where \circ is composition in \mathcal{C} .

Proposition 4.1.61. An object is initial in \mathcal{C} iff it is terminal in \mathcal{C}^{op} .

PROOF: Immediate from definitions. \square

Proposition 4.1.62. An object is terminal in \mathcal{C} iff it is initial in \mathcal{C}^{op} .

PROOF: Immediate from definitions. \square

Corollary 4.1.62.1. If T and T' are terminal objects in \mathcal{C} then there exists a unique isomorphism $T \cong T'$.

4.1.14 Groupoids

Definition 4.1.63 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

4.1.15 Concrete Categories

Definition 4.1.64 (Concrete Category). A *concrete category* \mathcal{C} consists of:

- a set $\text{Ob}(\mathcal{C})$ of *objects*
- for any object $A \in \text{Ob}(\mathcal{C})$, a set $|A|$
- for any objects $A, B \in \text{Ob}(\mathcal{C})$, a set of functions $\mathcal{C}[A, B] \subseteq |B|^{|A|}$

such that:

- for any $f \in \mathcal{C}[A, B]$ and $g \in \mathcal{C}[B, C]$, we have $g \circ f \in \mathcal{C}[A, C]$
- for any object A we have $\text{id}_{|A|} \in \mathcal{C}[A, A]$.

4.1.16 Power of Categories

Definition 4.1.65. Let \mathcal{C} be a category and J a set. The category \mathcal{C}^J is the category with:

- objects all J -indexed families of objects of \mathcal{C}
- morphisms $\{X_j\}_{j \in J} \rightarrow \{Y_j\}_{j \in J}$ all families $\{f_j\}_{j \in J}$ where $f_j : X_j \rightarrow Y_j$

4.1.17 Arrow Category

Definition 4.1.66 (Arrow Category). Let \mathcal{C} be a category. The *arrow category* \mathcal{C}^\rightarrow is the category with:

- objects all triples (A, B, f) where $f : A \rightarrow B$ in \mathcal{C}
- morphisms $(A, B, f) \rightarrow (C, D, g)$ all pairs $(u : A \rightarrow C, v : B \rightarrow D)$ such that $vf = gu$.

4.1.18 Slice Category

Definition 4.1.67 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category under A*, $\mathcal{C}_{\backslash A}$, is the category with:

- objects all pairs (B, f) where $B \in \mathcal{C}$ and $f : A \rightarrow B$
- morphisms $(B, f) \rightarrow (C, g)$ are morphisms $u : B \rightarrow C$ such that $uf = g$.

We identify this with the subcategory of \mathcal{C}^\rightarrow formed by mapping (B, f) to (A, B, f) and u to (id_A, u) .

Proposition 4.1.68. *If $s : (B, f) \rightarrow (C, g)$ in $\mathcal{C} \setminus A$, then any retraction of s in \mathcal{C} is a retraction of s in $\mathcal{C} \setminus A$.*

PROOF:

$\langle 1 \rangle 1$. LET: $r : C \rightarrow B$ be a retraction of s in \mathcal{C} .

$\langle 1 \rangle 2$. $rg = f$

PROOF: $rg = rsf = f$.

$\langle 1 \rangle 3$. $r : (C, g) \rightarrow (B, f)$ in $\mathcal{C} \setminus A$

$\langle 1 \rangle 4$. $rs = \text{id}_{(B, f)}$

PROOF: Because composition is inherited from \mathcal{C} .

□

Proposition 4.1.69. *id_A is the initial object in $\mathcal{C} \setminus A$.*

PROOF: For any $(B, f) \in \mathcal{C} \setminus A$, we have f is the only morphism $A \rightarrow B$ such that $f\text{id}_A = f$. □

Proposition 4.1.70. *If A is terminal in \mathcal{C} then id_A is the zero object in $\mathcal{C} \setminus A$.*

PROOF: For any $(B, f) \in \mathcal{C} \setminus A$, the unique morphism $! : B \rightarrow A$ is the unique morphism such that $!\text{id}_B = f$. □

Definition 4.1.71 (Pointed Sets). The *category of pointed sets* is **Set** \setminus 1.

Definition 4.1.72. Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category over A* , \mathcal{C}/A , is the category with:

- objects all pairs (B, f) with $f : B \rightarrow A$
- morphisms $u : (B, f) \rightarrow (C, g)$ all morphisms $u : B \rightarrow C$ such that $gu = f$.

Proposition 4.1.73. *Let $u : (B, f) \rightarrow (C, g) : \mathcal{C}/A$. Any section of u in \mathcal{C} is a section of u in \mathcal{C}/A .*

PROOF: Dual to Proposition 4.1.68. □

Proposition 4.1.74. *id_A is terminal in \mathcal{C}/A .*

PROOF: Dual to Proposition 4.1.69. □

Proposition 4.1.75. *If A is initial in \mathcal{C} then id_A is the zero object in \mathcal{C}/A .*

PROOF: Dual to Proposition 4.1.70. □

Definition 4.1.76. Let $A \in \mathcal{C}$. The category of objects *over and under* A , written \mathcal{C}_A^A , is the category with:

- objects all triples (X, u, p) where $u : A \rightarrow X$, $p : X \rightarrow A$ and $pu = \text{id}_A$
- morphism $f : (X, u, p) \rightarrow (Y, v, q)$ all morphisms $f : X \rightarrow Y$ such that $fu = v$ and $qf = p$

Proposition 4.1.77. *$(A, \text{id}_A, \text{id}_A)$ is the zero object in \mathcal{C}_A^A .*

PROOF: For any object (X, u, p) , we have p is the unique morphism $(X, u, p) \rightarrow (A, \text{id}_A, \text{id}_A)$, and u is the unique morphism $(A, \text{id}_A, \text{id}_A) \rightarrow (X, u, p)$. \square

Definition 4.1.78 (Fibre Collapsing). Let B be a set. Let $u : (A, a) \rightarrow (X, x)$ in \mathbf{Set}/B . Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow u & & \downarrow j \\ X & \xrightarrow{i} & C \end{array}$$

Let $c : C \rightarrow B$ be the unique morphism such that $cj = \text{id}_B$ and $ci = x$. Then $(C, j, c) \in \mathbf{Set}_B^B$ is called the set over and under B obtained from X by *fibre collapsing* with respect to u . If (A, u) is a subset of X , we denote this set over and under B by $X/_B(A, u)$.

Definition 4.1.79 (Fibre Wedge). Let B be a small set. Let $(X, u_X, p_X), (Y, u_Y, p_Y) \in \mathbf{Set}_B^B$. The *fibre wedge* of X and Y is the pushout of u_X and u_Y :

$$\begin{array}{ccc} B & \xrightarrow{u_X} & X \\ \downarrow u_Y & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \end{array}$$

Definition 4.1.80 (Fibre Smash). Let $X, Y \in \mathbf{Set}_B^B$. Let $\xi : X \vee_B Y \rightarrow X$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \xi & \\ & & X \end{array}$$

0

Let $\eta : X \vee_B Y \rightarrow Y$ be the unique morphism such that the following diagram commutes.

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee_B Y \\ & \searrow \eta & \\ & & Y \end{array}$$

0

Let $\zeta = \langle \xi, \eta \rangle : X \vee_B Y \rightarrow X \times Y$. The *fibre smash* of X and Y , $X \wedge_B Y$, is the result of collapsing $X \times Y$ with respect to ζ .

Proposition 4.1.81. *Set has products and coproducts.*

Proposition 4.1.82. *Let \mathcal{C} be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in \mathcal{C} and $Z \in \mathcal{C}$. Let $\coprod_{\alpha \in I} X_\alpha$ be the coproduct of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[\coprod_{\alpha \in I} X_\alpha, Z] \approx \prod_{\alpha \in I} \mathcal{C}[X_\alpha, Z] .$$

Proposition 4.1.83. *Let \mathcal{C} be a category. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of objects in \mathcal{C} and $Z \in \mathcal{C}$. Let $\prod_{\alpha \in I} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha \in I}$. Then*

$$\mathcal{C}[Z, \prod_{\alpha \in I} X_\alpha] \approx \prod_{\alpha \in I} \mathcal{C}[Z, X_\alpha] .$$

Proposition 4.1.84. *A product in \mathcal{C} constitutes a product in \mathcal{C}/A .*

Proposition 4.1.85. *A coproduct in \mathcal{C} constitutes a product in \mathcal{C}/A .*

4.2 Functors

Definition 4.2.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- a function $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism $f : A \rightarrow B$ in \mathcal{C} , a morphism $Ff : FA \rightarrow FB$ in \mathcal{D}

such that:

- for all $A \in \text{Ob}(\mathcal{C})$ we have $F\text{id}_A = \text{id}_{FA}$
- for any morphism $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathcal{C} , we have $F(g \circ f) = Fg \circ Ff$

Proposition 4.2.2. *Functors preserve isomorphisms.*

PROOF:

$\langle 1 \rangle 1$. LET: $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

$\langle 1 \rangle 2$. LET: $f : A \cong B$ in \mathcal{C}

$\langle 1 \rangle 3$. $Ff^{-1} \circ Ff = \text{id}_{FA}$

PROOF:

$$\begin{aligned} Ff^{-1} \circ Ff &= F(f^{-1} \circ f) \\ &= F\text{id}_A \\ &= \text{id}_{FA} \end{aligned}$$

$\langle 1 \rangle 4$. $Ff \circ Ff^{-1} = \text{id}_{FB}$

PROOF:

$$\begin{aligned} Ff \circ Ff^{-1} &= F(f \circ f^{-1}) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

Definition 4.2.3 (Identity Functor). For any category \mathcal{C} , the *identity* functor on \mathcal{C} is the functor $I_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$\begin{aligned} I_{\mathcal{C}}A &:= A & (A \in \mathcal{C}) \\ I_{\mathcal{C}}f &:= f & (f : A \rightarrow B \text{ in } \mathcal{C}) \end{aligned}$$

Proposition 4.2.4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $r : A \rightarrow B$ is a retraction of $s : B \rightarrow A$ in \mathcal{C} then Fr is a retraction of Fs .

PROOF:

$$\begin{aligned} Fr \circ Fs &= F(r \circ s) \\ &= F\text{id}_B \\ &= \text{id}_{FB} \end{aligned}$$

□

Corollary 4.2.4.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$. If $\phi : A \cong B$ is an isomorphism in \mathcal{C} then $F\phi : FA \cong FB$ is an isomorphism in \mathcal{D} with $(F\phi)^{-1} = F\phi^{-1}$.

Definition 4.2.5 (Composition of Functors). Given functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$, the *composite* functor $GF : \mathcal{C} \rightarrow \mathcal{E}$ is defined by

$$\begin{aligned} (GF)A &= G(FA) & (A \in \mathcal{C}) \\ (GF)f &= G(Ff) & (f : A \rightarrow B : \mathcal{C}) \end{aligned}$$

Definition 4.2.6 (Category of Categories). Let **Cat** be the category of small categories and functors.

Definition 4.2.7 (Isomorphism of Categories). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F is an *isomorphism of categories* iff there exists a functor $F^{-1} : \mathcal{D} \rightarrow \mathcal{C}$, the *inverse* of F , such that $FF^{-1} = I_{\mathcal{D}}$ and $F^{-1}F = I_{\mathcal{C}}$.

Categories \mathcal{C} and \mathcal{D} are *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, iff there exists an isomorphism between them.

Proposition 4.2.8. If A is initial in \mathcal{C} then $\mathcal{C} \setminus A \cong \mathcal{C}$.

PROOF:

⟨1⟩1. Define $F : \mathcal{C} \setminus A \rightarrow \mathcal{C}$ by

$$F(B, f) = B$$

$$F(u : (B, f) \rightarrow (C, g)) = u$$

⟨1⟩2. Define $G : \mathcal{C} \rightarrow \mathcal{C} \setminus A$ by

$$GB = (B, !_B)$$

where $!_B$ is the unique morphism $A \rightarrow B$

$$G(u : B \rightarrow C) = u : (B, !_B) \rightarrow (C, !_C)$$

⟨1⟩3. $FG = \text{id}_{\mathcal{C}}$

⟨1⟩4. $GF = \text{id}_{\mathcal{C} \setminus A}$

PROOF: Since $GF(B, f) = (B, !_B) = (B, f)$ because the morphism $A \rightarrow B$ is unique.

□

Proposition 4.2.9. *If A is terminal in \mathcal{C} then $\mathcal{C}/A \cong \mathcal{C}$.*

PROOF: Dual. \square

Proposition 4.2.10.

$$\mathcal{C}_A^A \cong (\mathcal{C}/A) \backslash (A, \text{id}_A) \cong (\mathcal{C} \backslash A) / (A, \text{id}_A)$$

PROOF:

- $\langle 1 \rangle 1$. Define a functor $F : \mathcal{C}_A^A \rightarrow (\mathcal{C}/A) \backslash (A, \text{id}_A)$.
 $\langle 2 \rangle 1$. Given $A \xrightarrow{u} X \xrightarrow{p} A$ in \mathcal{C}_A^A , let $F(X, u, p) = ((X, p), u)$
 $\langle 2 \rangle 2$. Given $f : (A \xrightarrow{u} X \xrightarrow{p} A) \rightarrow (A \xrightarrow{v} Y \xrightarrow{q} A)$, let $Ff = f$.
 $\langle 1 \rangle 2$. Define a functor $G : (\mathcal{C}/A) \backslash (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.
 $\langle 1 \rangle 3$. Define a functor $H : \mathcal{C}_A^A \rightarrow (\mathcal{C} \backslash A) / (A, \text{id}_A)$.
 $\langle 1 \rangle 4$. Define a functor $K : (\mathcal{C} \backslash A) / (A, \text{id}_A) \rightarrow \mathcal{C}_A^A$.
 \square

Definition 4.2.11 (Forgetful Functor). For any concrete category \mathcal{C} , define the *forgetful* functor $U : \mathcal{C} \rightarrow \mathbf{Set}$ by:

$$\begin{aligned} UA &= |A| \\ Uf &= f \end{aligned}$$

Definition 4.2.12 (Switching Functor). For any category \mathcal{C} , define the *switching* functor $T : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ by

$$\begin{aligned} T(A, B) &= (B, A) \\ T(f, g) &= (g, f) \end{aligned}$$

Definition 4.2.13 (Reduction). Let $\Phi : \mathbf{Set} \rightarrow \mathbf{Set}$ be a functor. The *reduction* of Φ is the functor $\Phi^* : \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ defined by: $\Phi^*(X, a)$ is the collapse of $\Phi(X)$ with respect to $\Phi(a) : \Phi(1) \rightarrow \Phi(X)$.

Definition 4.2.14. Extend the wedge \vee to a functor $\mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ by defining, given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, then $f \vee g$ is the unique morphism that makes the following diagram commute.

$$\begin{array}{ccccc} 1 & \longrightarrow & X & & \\ \downarrow & & \downarrow & \searrow f & \\ Y & \longrightarrow & X \vee Y & & X' \\ & \searrow g & \searrow f \vee g & & \downarrow \\ & & Y' & \longrightarrow & X' \vee Y' \end{array}$$

Definition 4.2.15. Extend smash to a functor $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ as follows. Given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, let $f \wedge g : X \wedge Y \rightarrow X' \wedge Y'$ be the

unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc}
 X \vee Y & \longrightarrow & 1 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 X \times Y & \longrightarrow & X \wedge Y & & \\
 & \searrow & \downarrow & \searrow & \\
 & & X' \vee Y' & \longrightarrow & 1 \\
 & \searrow & \downarrow & \searrow & \\
 & & X' \times Y' & \longrightarrow & X' \wedge Y'
 \end{array}$$

$f \times g$

Definition 4.2.16 (Reduction). Let B be a small set. Let $\Phi_B : \mathbf{Set}/B \rightarrow \mathbf{Set}/B$ be a functor. The *reduction* of Φ_B is the functor $\Phi_B^B : \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ defined as follows.

For $(X, u : B \rightarrow X, p : X \rightarrow B) \in \mathbf{Set}_B^B$, let $\Phi_B^B(X)$ be the set over and under B obtained from $\Phi_B(X)$ by collapsing with respect to $\Phi_B(u) : \Phi_B(B) \rightarrow \Phi_B(X)$.

Definition 4.2.17. Extend \vee_B to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$.

Definition 4.2.18. Extend \wedge_B to a functor $\mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$.

Definition 4.2.19 (Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *faithful* iff, for any objects $A, B \in \mathcal{C}$ and morphisms $f, g : A \rightarrow B : \mathcal{C}$, if $Ff = Fg$ then $f = g$.

Definition 4.2.20 (Full). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *full* iff, for any objects $A, B \in \mathcal{C}$ and morphism $g : FA \rightarrow FB : \mathcal{D}$, there exists $f : A \rightarrow B : \mathcal{C}$ such that $Ff = g$.

Definition 4.2.21 (Fully Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *fully faithful* iff it is full and faithful.

Definition 4.2.22 (Full Embedding). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *full embedding* iff it is fully faithful and injective on objects.

4.3 Natural Transformations

Definition 4.3.1 (Natural Transformation). Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$. A *natural transformation* $\tau : F \Rightarrow G$ is a family of morphisms $\{\tau_X : FX \rightarrow GX\}_{X \in \mathcal{C}}$ such that, for every morphism $f : X \rightarrow Y : \mathcal{C}$, we have $Gf \circ \tau_X = \tau_Y \circ Ff$.

$$\begin{array}{ccc}
 FX & \xrightarrow{Ff} & FY \\
 \tau_X \downarrow & & \downarrow \tau_Y \\
 GX & \xrightarrow{Gf} & GY
 \end{array}$$

Definition 4.3.2 (Natural Isomorphism). A natural transformation $\tau : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$ is a *natural isomorphism*, $\tau : F \cong G$, iff for all $X \in \mathcal{C}$, τ_X is an isomorphism $FX \cong GX$.

Functors F and G are *naturally isomorphic*, $F \cong G$, iff there exists a natural isomorphism between them.

Definition 4.3.3 (Inverse). Let $\tau : F \cong G$. The *inverse* natural isomorphism $\tau^{-1} : G \cong F$ is defined by $(\tau^{-1})_X = \tau_X^{-1}$.

4.4 Bifunctors

Definition 4.4.1 (Commutative). A bifunctor $\square : \mathcal{C}^2 \rightarrow \mathcal{C}$ is *commutative* iff $\square \cong \square \circ T$, where $T : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ is the swap functor.

Proposition 4.4.2. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is commutative.

PROOF: Since the pushout of f and g is the pushout of g and f . \square

Proposition 4.4.3. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is commutative.

PROOF: In the diagram defining $X \wedge Y$, construct the isomorphism between the version with X and Y and the version with Y with X for every object. \square

Proposition 4.4.4. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is commutative.

Proposition 4.4.5. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is commutative.

Definition 4.4.6 (Associative). A bifunctor \square is *associative* iff $\square \circ (\square \times \text{id}) \cong \square \circ (\text{id} \times \square)$.

Proposition 4.4.7. $\vee : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is associative.

PROOF: Since $X \vee (Y \vee Z)$ and $(X \vee Y) \vee Z$ are both the pushout of the unique morphisms $1 \rightarrow X$, $1 \rightarrow Y$ and $1 \rightarrow Z$. \square

Proposition 4.4.8. $\wedge : \mathbf{Set}_* \times \mathbf{Set}_* \rightarrow \mathbf{Set}_*$ is associative.

PROOF: Draw isomorphisms between the diagrams for $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$. \square

Product and coproduct are commutative and associative.

Proposition 4.4.9. $\vee_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is associative.

Proposition 4.4.10. $\wedge_B : \mathbf{Set}_B^B \times \mathbf{Set}_B^B \rightarrow \mathbf{Set}_B^B$ is associative.

Proposition 4.4.11. Let \mathcal{C} be a category with binary coproducts. Let $\square : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ be a bifunctor. Then \square distributes over $+$ iff the canonical morphism

$$(X \square Z) + (Y \square Z) \rightarrow (X + Y) \square Z$$

is an isomorphism for all X, Y, Z .

Proposition 4.4.12. *In a category with binary products and binary coproducts, then \times distributes over $+$.*

Proposition 4.4.13. *In $\mathbf{Set}/*$, we have \times does not distribute over \vee .*

Proposition 4.4.14. *In $\mathbf{Set}/*$, we have \wedge distributes over \vee .*

Proposition 4.4.15. *In \mathbf{Set}/B , we have \times_B distributes over $+_B$.*

Proposition 4.4.16. *In \mathbf{Set}/B^B , we have \wedge_B distributes over \vee_B .*

4.5 Functor Categories

Definition 4.5.1 (Functor Category). Given categories \mathcal{C} and \mathcal{D} , define the *functor category* $\mathcal{C}^{\mathcal{D}}$ to be the category with objects the functors from \mathcal{D} to \mathcal{C} and morphisms the natural transformations.

Definition 4.5.2 (Yoneda Embedding). Let \mathcal{C} be a category. The *Yoneda embedding* $Y : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ is the functor that maps an object A to $\mathcal{C}[-, A]$ and morphisms similarly.

Theorem 4.5.3 (Yoneda Lemma). *Let \mathcal{C} be a category. There exists a natural isomorphism*

$$\phi_{XF} : \mathbf{Set}^{\mathcal{C}^{\text{op}}}[\mathcal{C}[-, X], F] \cong FX$$

that maps $\tau : \mathcal{C}[-, X] \Rightarrow F$ to $\tau_X(\text{id}_X)$.

PROOF:

$\langle 1 \rangle 1$. ϕ is natural in X .

PROOF:

$\langle 2 \rangle 1$. LET: $f : X \rightarrow Y : \mathcal{C}$

$\langle 2 \rangle 2$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$. $Ff(\phi(\tau)) = \phi(\tau \circ \mathcal{C}[-, f])$

PROOF:

$$\begin{aligned} \phi(\tau \circ \mathcal{C}[-, f]) &= \tau_Y(\text{id}_Y \circ f) \\ &= \tau_Y(f) \\ &= \tau_Y(f \circ \text{id}_X) \\ &= Ff(\tau_X(\text{id}_X)) && (\tau \text{ natural}) \\ &= Ff(\phi(\tau)) \end{aligned}$$

$\langle 1 \rangle 2$. ϕ is natural in F .

$\langle 2 \rangle 1$. LET: $\alpha : F \Rightarrow G : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$

$\langle 2 \rangle 2$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 3$. $\alpha_X(\phi(\tau)) = \phi(\alpha \bullet \tau)$

PROOF: $\phi(\alpha \bullet \tau) = \alpha_X(\tau_X(\text{id}_X)) = \alpha_X(\phi(\tau))$

$\langle 1 \rangle 3$. Each ϕ_{XF} is injective.

$\langle 2 \rangle 1$. LET: $\sigma, \tau : \mathcal{C}[-, X] \Rightarrow F$

$\langle 2 \rangle 2$. ASSUME: $\phi(\sigma) = \phi(\tau)$

$\langle 2 \rangle 3$. LET: $f : Y \rightarrow X$

$\langle 2 \rangle 4$. $\sigma_Y(f) = \tau_Y(f)$

PROOF:

$$\begin{aligned}
 \sigma_Y(f) &= \sigma_Y(\text{id}_X \circ f) \\
 &= Ff(\sigma_X(\text{id}_X)) && (\sigma \text{ is natural}) \\
 &= Ff(\tau_X(\text{id}_X)) && (\langle 2 \rangle 2) \\
 &= \tau_Y(\text{id}_X \circ f) && (\tau \text{ is natural}) \\
 &= \tau_Y(f)
 \end{aligned}$$

$\langle 1 \rangle 4$. Each ϕ_{XF} is surjective.

$\langle 2 \rangle 1$. LET: $X \in \mathcal{C}$ and $F : \mathcal{C} \rightarrow \mathcal{D}$

$\langle 2 \rangle 2$. LET: $a \in FX$

$\langle 2 \rangle 3$. LET: $\tau : \mathcal{C}[-, X] \Rightarrow F$ be given by $\tau_Y(g) = Fg(a)$ for $g : Y \rightarrow X$

$\langle 2 \rangle 4$. τ is natural.

$\langle 3 \rangle 1$. LET: $h : Y \rightarrow Z : \mathcal{C}$

PROVE: $Fh \circ \tau_Z = \tau_Y \circ \mathcal{C}[h, \text{id}_X]$

$\langle 3 \rangle 2$. LET: $g : Z \rightarrow X$

$\langle 3 \rangle 3$. $Fh(\tau_Z(g)) = \tau_Y(g \circ h)$

PROOF:

$$\begin{aligned}
 \tau_Y(g \circ h) &= F(g \circ h)(a) \\
 &= Fh(Fg(a)) \\
 &= Fh(\tau_Z(g))
 \end{aligned}$$

$\langle 2 \rangle 5$. $\phi(\tau) = a$

PROOF:

$$\begin{aligned}
 \phi_X(\tau) &= \tau_X(\text{id}_X) \\
 &= F\text{id}_X(a) \\
 &= a
 \end{aligned}$$

□

Corollary 4.5.3.1. *The Yoneda embedding is fully faithful.*

Corollary 4.5.3.2. *Given objects A and B in \mathcal{C} , we have $A \cong B$ if and only if $\mathcal{C}[-, A] \cong \mathcal{C}[-, B]$.*

Chapter 5

The Real Numbers

Theorem 5.0.1. *The following hold in the real numbers:*

1. $x + (y + z) = (x + y) + z$
2. $x(yz) = (xy)z$
3. $x + y = y + x$
4. $xy = yx$
5. $x + 0 = x$
6. $x1 = x$
7. $x + (-x) = 0$
8. *If $x \neq 0$ then $x \cdot (1/x) = 1$*
9. $x(y + z) = xy + xz$
10. *If $x > y$ then $x + z > y + z$.*
11. *If $x > y$ and $z > 0$ then $xz > yz$.*
12. \mathbb{R} *has the least upper bound property.*
13. *If $x < y$ then there exists z such that $x < z < y$.*

Definition 5.0.2 (Subtraction). We write $x - y$ for $x + (-y)$.

Definition 5.0.3. Given real numbers x and y with $y \neq 0$, we write x/y for xy^{-1} .

Theorem 5.0.4. *For any real numbers x and y , if $x + y = x$ then $y = 0$.*

PROOF:

$\langle 1 \rangle$ 1. LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 2$. ASSUME: $x + y = x$

$\langle 1 \rangle 3$. $y = 0$

PROOF:

$$\begin{aligned}
 y &= y + 0 && \text{(Definition of zero)} \\
 &= y + (x + (-x)) && \text{(Definition of } -x) \\
 &= (y + x) + (-x) && \text{(Associativity of Addition)} \\
 &= (x + y) + (-x) && \text{(Commutativity of Addition)} \\
 &= x + (-x) && (\langle 1 \rangle 2) \\
 &= 0 && \text{(Definition of } -x)
 \end{aligned}$$

□

Theorem 5.0.5.

$$\forall x \in \mathbb{R}. 0x = 0$$

PROOF:

$\langle 1 \rangle 1$. LET: $x \in \mathbb{R}$

$\langle 1 \rangle 2$. $xx + 0x = xx$

PROOF:

$$\begin{aligned}
 xx + 0x &= (x + 0)x && \text{(Distributive Law)} \\
 &= xx && \text{(Definition of 0)}
 \end{aligned}$$

$\langle 1 \rangle 3$. $0x = 0$

PROOF: Theorem 5.0.4, $\langle 1 \rangle 2$.

□

Theorem 5.0.6.

$$-0 = 0$$

PROOF: Since $0 + 0 = 0$. □

Theorem 5.0.7.

$$\forall x \in \mathbb{R}. -(-x) = x$$

PROOF: Since $-x + x = 0$. □

Theorem 5.0.8.

$$\forall x, y \in \mathbb{R}. x(-y) = -(xy)$$

PROOF:

$$\begin{aligned}
 x(-y) + xy &= x((-y) + y) && \text{(Distributive Law)} \\
 &= x0 && \text{(Definition of } -y) \\
 &= 0 && \text{(Theorem 5.0.5)} \quad \square
 \end{aligned}$$

Theorem 5.0.9.

$$\forall x \in \mathbb{R}. (-1)x = -x$$

PROOF:

$$\begin{aligned}
 (-1)x &= -(1 \cdot x) && \text{(Theorem 5.0.8)} \\
 &= -x && \text{(Definition of 1)} \quad \square
 \end{aligned}$$

5.0.1 Subtraction

Theorem 5.0.10.

$$\forall x, y, z \in \mathbb{R}. x(y - z) = xy - xz$$

PROOF:

$$\begin{aligned} x(y - z) &= x(y + (-z)) && \text{(Definition of subtraction)} \\ &= xy + x(-z) && \text{(Distributive Law)} \\ &= xy + (-(xz)) && \text{(Theorem 5.0.8)} \\ &= xy - xz && \text{(Definition of subtraction)} \quad \square \end{aligned}$$

Theorem 5.0.11.

$$\forall x, y \in \mathbb{R}. -(x + y) = -x - y$$

PROOF:

$$\begin{aligned} -(x + y) &= (-1)(x + y) && \text{(Theorem 5.0.9)} \\ &= (-1)x + (-1)y && \text{(Distributive Law)} \\ &= -x + (-y) && \text{(Theorem 5.0.9)} \\ &= -x - y && \text{(Definition of subtraction)} \quad \square \end{aligned}$$

Theorem 5.0.12.

$$\forall x, y \in \mathbb{R}. -(x - y) = -x + y$$

PROOF:

$$\begin{aligned} -(x - y) &= -(x + (-y)) && \text{(Definition of subtraction)} \\ &= -x - (-y) && \text{(Theorem 5.0.11)} \\ &= -x + (-(-y)) && \text{(Definition of subtraction)} \\ &= -x + y && \text{(Theorem 5.0.7)} \quad \square \end{aligned}$$

Definition 5.0.13 (Reciprocal). Given $x \in \mathbb{R}$ with $x \neq 0$, the *reciprocal* of x , $1/x$, is the unique real number such that $x \cdot 1/x = 1$.

Theorem 5.0.14. For any real numbers x and y , if $x \neq 0$ and $xy = x$ then $y = 1$.

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 2$. ASSUME: $x \neq 0$

$\langle 1 \rangle 3$. ASSUME: $xy = x$

$\langle 1 \rangle 4$. $y = 1$

PROOF:

$$\begin{aligned} y &= y1 && \text{(Definition of 1)} \\ &= y(x \cdot 1/x) && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\ &= (yx)1/x && \text{(Associativity of Multiplication)} \\ &= (xy)1/x && \text{(Commutativity of Multiplication)} \\ &= x \cdot 1/x && (\langle 1 \rangle 3) \\ &= 1 && \text{(Definition of } 1/x, \langle 1 \rangle 2) \end{aligned}$$

□

Definition 5.0.15 (Quotient). Given real numbers x and y with $y \neq 0$, the quotient x/y is defined by

$$x/y = x \cdot 1/y .$$

Theorem 5.0.16. For any real number x , if $x \neq 0$ then $x/x = 1$.

PROOF: Immediate from definitions. □

Theorem 5.0.17.

$$\forall x \in \mathbb{R}. x/1 = x$$

PROOF:

⟨1⟩1. LET: $x \in \mathbb{R}$

⟨1⟩2. $1/1 = 1$

PROOF: Since $1 \cdot 1 = 1$.

⟨1⟩3. $x/1 = x$

PROOF: Since $x/1 = x \cdot 1/1 = x \cdot 1 = x$.

□

Theorem 5.0.18. For any real numbers x and y , if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

PROOF:

⟨1⟩1. LET: $x, y \in \mathbb{R}$

⟨1⟩2. ASSUME: $xy = 0$ and $x \neq 0$

PROVE: $y = 0$

⟨1⟩3. $y = 0$

PROOF:

$$\begin{aligned} y &= 1y && \text{(Definition of 1)} \\ &= (1/x)xy && \text{(Definition of } 1/x, \langle 1 \rangle 2) \\ &= (1/x)0 && \text{(\langle 1 \rangle 2)} \\ &= 0 && \text{(Theorem 5.0.5)} \end{aligned}$$

□

Theorem 5.0.19. For any real numbers y and z , if $y \neq 0$ and $z \neq 0$ then $(1/y)(1/z) = 1/(yz)$.

PROOF: Since $yz(1/y)(1/z) = 1 \cdot 1 = 1$. □

Corollary 5.0.19.1. For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have $(x/y)(z/w) = (xz)/(yw)$.

Theorem 5.0.20. For any real numbers x, y, z, w with $y \neq 0 \neq w$, we have

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + yz}{yw}$$

PROOF:

$$\begin{aligned} yw \left(\frac{x}{y} + \frac{z}{w} \right) &= yw \frac{x}{y} + yw \frac{z}{w} \\ &= wx + yz \end{aligned} \quad \square$$

Theorem 5.0.21. For any real number x , if $x \neq 0$ then $1/x \neq 0$.

PROOF: Since $x \cdot 1/x = 1 \neq 0$. \square

Theorem 5.0.22. For any real numbers w, z , if $w \neq 0 \neq z$ then $1/(w/z) = z/w$.

PROOF: Since $(z/w)(w/z) = (wz)/(wz) = 1$. \square

Theorem 5.0.23. For any real numbers a, x and y , if $y \neq 0$ then $(ax)/y = a(x/y)$

PROOF: Since $ya(x/y) = ax$. \square

Theorem 5.0.24. For any real numbers x and y , if $y \neq 0$ then $(-x)/y = x/(-y) = -(x/y)$.

PROOF:

$\langle 1 \rangle 1.$ $(-x)/y = -(x/y)$

PROOF: Take $a = -1$ in Theorem 5.0.23.

$\langle 1 \rangle 2.$ $x/(-y) = -(x/y)$

PROOF: Since $(-y)(-(x/y)) = y(x/y) = x$.

\square

Theorem 5.0.25. For any real numbers x, y, z and w , if $x > y$ and $w > z$ then $x + w > y + z$.

PROOF: We have $y + z < x + z < x + w$ by Monotonicity of Addition twice. \square

Corollary 5.0.25.1. For any real numbers x and y , if $x > 0$ and $y > 0$ then $x + y > 0$.

Theorem 5.0.26. For any real numbers x and y , if $x > 0$ and $y > 0$ then $xy > 0$.

PROOF:

$$\begin{aligned} xy &> 0y && \text{(Monotonicity of Multiplication)} \\ &= 0 && \text{(Theorem 5.0.5)} \end{aligned} \quad \square$$

Theorem 5.0.27. For any real number x , we have $x > 0$ iff $-x < 0$.

PROOF:

$\langle 1 \rangle 1.$ If $0 < x$ then $-x < 0$

PROOF: By Monotonicity of Addition adding $-x$ to both sides.

$\langle 1 \rangle 2.$ If $-x < 0$ then $0 < x$

PROOF: By Monotonicity of Addition adding x to both sides.

\square

Theorem 5.0.28. *For any real numbers x and y , we have $x > y$ iff $-x < -y$.*

PROOF:

$\langle 1 \rangle 1$. If $y < x$ then $-x < -y$.

PROOF: By Monotonicity of Addition adding $-x - y$ to both sides.

$\langle 1 \rangle 2$. If $-x < -y$ then $y < x$.

PROOF: By Monotonicity of Addition adding $x + y$ to both sides.

□

Theorem 5.0.29. *For any real numbers x , y and z , if $x > y$ and $z < 0$ then $xz < yz$.*

PROOF:

$\langle 1 \rangle 1$. LET: x , y and z be real numbers.

$\langle 1 \rangle 2$. ASSUME: $x > y$

$\langle 1 \rangle 3$. ASSUME: $z < 0$

$\langle 1 \rangle 4$. $-z > 0$

PROOF: Theorem 5.0.27, $\langle 1 \rangle 3$.

$\langle 1 \rangle 5$. $x(-z) > y(-z)$

PROOF: $\langle 1 \rangle 2$, $\langle 1 \rangle 4$, Monotonicity of Multiplication.

$\langle 1 \rangle 6$. $-(xz) > -(yz)$

PROOF: Theorem 5.0.8, $\langle 1 \rangle 5$.

$\langle 1 \rangle 7$. $xz < yz$

PROOF: Theorem 5.0.27, $\langle 1 \rangle 6$.

□

Theorem 5.0.30. *For any real number x , if $x \neq 0$ then $xx > 0$.*

PROOF:

$\langle 1 \rangle 1$. If $x > 0$ then $xx > 0$

PROOF: By Monotonicity of Multiplication.

$\langle 1 \rangle 2$. If $x < 0$ then $xx > 0$

PROOF: Theorem 5.0.29.

□

Theorem 5.0.31.

$$0 < 1$$

PROOF: By Theorem 5.0.30 since $1 = 1 \cdot 1$. □

Definition 5.0.32 (Positive). A real number x is *positive* iff $x > 0$.

We write \mathbb{R}_+ for the set of positive reals.

Theorem 5.0.33. *For any real numbers x and y , we have xy is positive if and only if x and y are both positive or both negative.*

PROOF: By the Monotonicity of Multiplication and Theorem 5.0.29. □

Corollary 5.0.33.1. *For any real number x , if $x > 0$ then $1/x > 0$.*

PROOF: Since $x \cdot 1/x = 1$ is positive. □

Theorem 5.0.34. For any real numbers x and y , if $x > y > 0$ then $1/x < 1/y$.

PROOF: If $1/y \leq 1/x$ then $1 < 1$ by Monotonicity of Multiplication. \square

Theorem 5.0.35. For any real numbers x and y , if $x < y$ then $x < (x+y)/2 < y$.

PROOF: We have $2x < x+y$ and $x+y < 2y$ by Monotonicity of Addition, hence $x < (x+y)/2 < y$ by Monotonicity of Multiplication since $1/2 > 0$. \square

Corollary 5.0.35.1. \mathbb{R} is a linear continuum.

Definition 5.0.36 (Negative). A real number x is *negative* iff $x < 0$.

We write $\overline{\mathbb{R}_+}$ for the set of nonnegative reals.

Theorem 5.0.37. For every positive real number a , there exists a unique positive real \sqrt{a} such that $\sqrt{a}^2 = a$.

PROOF:

$\langle 1 \rangle 1$. LET: a be a positive real.

$\langle 1 \rangle 2$. For any real numbers x and h , if $0 \leq h < 1$, then

$$(x+h)^2 < x^2 + h(2x+1) .$$

$\langle 2 \rangle 1$. LET: x and h be real numbers.

$\langle 2 \rangle 2$. ASSUME: $0 \leq h < 1$

$\langle 2 \rangle 3$. $(x+h)^2 < x^2 + h(2x+1)$

PROOF:

$$\begin{aligned} (x+h)^2 &= x^2 + 2hx + h^2 \\ &< x^2 + 2hx + h & (\langle 2 \rangle 2) \\ &= x^2 + h(2x+1) \end{aligned}$$

$\langle 1 \rangle 3$. For any real numbers x and h , if $h > 0$ then

$$(x-h)^2 > x^2 - 2hx .$$

$\langle 2 \rangle 1$. LET: x and h be real numbers.

$\langle 2 \rangle 2$. ASSUME: $h > 0$

$\langle 2 \rangle 3$. $(x-h)^2 > x^2 - 2hx$

PROOF:

$$\begin{aligned} (x-h)^2 &= x^2 - 2hx + h^2 \\ &> x^2 - 2hx & (\langle 2 \rangle 2) \end{aligned}$$

$\langle 1 \rangle 4$. For any positive real x , if $x^2 < a$ then there exists $h > 0$ such that

$$(x+h)^2 < a.$$

$\langle 2 \rangle 1$. LET: x be a positive real.

$\langle 2 \rangle 2$. ASSUME: $x^2 < a$

$\langle 2 \rangle 3$. LET: $h = \min((a-x^2)/(2x+1), 1/2)$

$\langle 2 \rangle 4$. $0 < h < 1$

$\langle 2 \rangle 5$. $(x+h)^2 < a$

PROOF:

$$\begin{aligned} (x+h)^2 &< x^2 + h(2x+1) & (\langle 1 \rangle 2) \\ &\leq a \end{aligned}$$

⟨1⟩5. For any positive real x , if $x^2 > a$ then there exists $h > 0$ such that $(x - h)^2 > a$.

⟨2⟩1. LET: x be a positive real.

⟨2⟩2. ASSUME: $x^2 > a$

⟨2⟩3. LET: $h = (x^2 - a)/2x$

⟨2⟩4. $h > 0$

⟨2⟩5. $(x - h)^2 > a$

PROOF:

$$(x - h)^2 > x^2 - 2hx$$

$$= a$$

(⟨2⟩3)

⟨1⟩6. LET: $B = \{x \in \mathbb{R} : x^2 < a\}$

⟨1⟩7. B is bounded above.

PROOF: If $a \geq 1$ then a is an upper bound. If $a < 1$ then 1 is an upper bound.

⟨1⟩8. B contains at least one positive real.

PROOF: If $a \geq 1$ then $1 \in B$. If $a < 1$ then $a \in B$.

⟨1⟩9. LET: $b = \sup B$

⟨1⟩10. $b^2 = a$

⟨2⟩1. $b^2 \geq a$

⟨3⟩1. ASSUME: for a contradiction $b^2 < a$

⟨3⟩2. PICK $h > 0$ such that $(b + h)^2 < a$

PROOF: ⟨1⟩4

⟨3⟩3. $b + h \in B$

⟨3⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩9.

⟨2⟩2. $b^2 \leq a$

⟨3⟩1. ASSUME: for a contradiction $b^2 > a$

⟨3⟩2. PICK $h > 0$ such that $(b - h)^2 > a$

PROOF: ⟨1⟩5

⟨3⟩3. PICK $x \in B$ such that $b - h < x$

PROOF: ⟨1⟩9

⟨3⟩4. $(b - h)^2 < x^2 < a$

⟨3⟩5. Q.E.D.

PROOF: This contradicts ⟨3⟩2

⟨1⟩11. For any positive reals b and c , if $b^2 = c^2$ then $b = c$.

⟨2⟩1. LET: b and c be positive reals.

⟨2⟩2. ASSUME: $b^2 = c^2$

⟨2⟩3. $b^2 - c^2 = 0$

⟨2⟩4. $(b - c)(b + c) = 0$

⟨2⟩5. $b - c = 0$ or $b + c = 0$

⟨2⟩6. $b + c \neq 0$

PROOF: Since $b + c > 0$

⟨2⟩7. $b - c = 0$

⟨2⟩8. $b = c$

□

Theorem 5.0.38. *The set of real numbers is uncountable.*

Definition 5.0.39. We write \mathbb{R}^∞ for the set of sequences in \mathbb{R}^ω that are eventually zero.

Definition 5.0.40 (Hilbert Cube). The *Hilbert cube* is $\prod_{n=0}^\infty [0, 1/(n+1)]$.

Definition 5.0.41 (Unit Ball). The *unit ball* is $B^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

Chapter 6

Integers and Rationals

6.1 Positive Integers

Definition 6.1.1 (Inductive). A set of real numbers A is *inductive* iff $1 \in A$ and $\forall x \in A. x + 1 \in A$.

Definition 6.1.2 (Positive Integer). The set \mathbb{Z}_+ of *positive integers* is the intersection of the set of inductive sets.

Proposition 6.1.3. *Every positive integer is positive.*

PROOF: The set of positive reals is inductive. \square

Proposition 6.1.4. *1 is the least element of \mathbb{Z}_+ .*

PROOF: Since $\{x \in \mathbb{R} : x \geq 1\}$ is inductive. \square

Proposition 6.1.5. *\mathbb{Z}_+ is inductive.*

PROOF: 1 is an element of every inductive set, and for all $x \in \mathbb{R}$, if x is an element of every inductive set then so is $x + 1$. \square

Theorem 6.1.6 (Principle of Induction). *If A is an inductive set of positive integers then $A = \mathbb{Z}_+$.*

PROOF: Immediate from definitions. \square

Theorem 6.1.7 (Well-Ordering Property). *\mathbb{Z}_+ is well ordered.*

PROOF: Construct the obvious order isomorphism $\omega \cong \mathbb{Z}_+$. \square

Theorem 6.1.8 (Archimedean Ordering Property). *The set \mathbb{Z}_+ is unbounded above.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction \mathbb{Z}_+ is bounded above.

⟨1⟩2. LET:

$$s = \sup \mathbb{Z}_+$$

⟨1⟩3. PICK $n \in \mathbb{Z}_+$ such that $s - 1 < n$

⟨1⟩4. $s < n + 1$

⟨1⟩5. Q.E.D.

PROOF: ⟨1⟩2 and ⟨1⟩4 form a contradiction.

□

6.1.1 Exponentiation

Definition 6.1.9. For a a real number and n a positive integer, define the real number a^n recursively as follows:

$$\begin{aligned} a^1 &= a \\ a^{n+1} &= a^n a \end{aligned}$$

Theorem 6.1.10. For all $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}_+$, we have

$$a^n a^m = a^{n+m}$$

PROOF:

⟨1⟩1. LET: $P(m)$ be the property $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. a^n a^m = a^{n+m}$

⟨1⟩2. $P(1)$

PROOF: $a^n a^1 = a^n a = a^{n+1}$.

⟨1⟩3. $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$

⟨2⟩1. LET: m be a positive integer.

⟨2⟩2. ASSUME: $P(m)$

⟨2⟩3. LET: $a \in \mathbb{R}$

⟨2⟩4. LET: $n \in \mathbb{Z}_+$

⟨2⟩5. $a^n a^{m+1} = a^{n+m+1}$

PROOF:

$$\begin{aligned} a^n a^{m+1} &= a^n a^m a \\ &= a^{n+m} a && (\langle 2 \rangle 2) \\ &= a^{n+m+1} \end{aligned}$$

⟨1⟩4. Q.E.D.

PROOF: By induction.

□

Theorem 6.1.11. For all $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}_+$,

$$(a^n)^m = a^{nm}.$$

PROOF:

⟨1⟩1. LET: $P(m)$ be the property $\forall a \in \mathbb{R}. \forall n \in \mathbb{Z}_+. (a^n)^m = a^{nm}$.

⟨1⟩2. $P(1)$

PROOF: $(a^n)^1 = a^n = a^{n \cdot 1}$

⟨1⟩3. $\forall m \in \mathbb{Z}_+. P(m) \Rightarrow P(m+1)$

PROOF:

$$\begin{aligned} (a^n)^{m+1} &= (a^n)^m a^n \\ &= a^{nm} a^n \\ &= a^{nm+n} && (\text{Theorem 6.1.10}) \\ &= a^{n(m+1)} \end{aligned}$$

□

Theorem 6.1.12. *For any real numbers a and b and positive integer m ,*

$$a^m b^m = (ab)^m .$$

PROOF: Induction on m . □

6.2 Integers

Definition 6.2.1 (Integer). The set \mathbb{Z} of *integers* is

$$\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \{-x : x \in \mathbb{Z}_+\} .$$

Proposition 6.2.2. *The sum, difference and product of two integers is an integer.*

PROOF: Easy. □

Example 6.2.3. $1/2$ is not an integer.

Proposition 6.2.4. *For any integer n , there is no integer a such that $n < a < n+1$.*

PROOF:

⟨1⟩1. For any positive integer n , there is no integer a such that $n < a < n+1$.

⟨2⟩1. There is no integer a such that $1 < a < 2$.

⟨3⟩1. There is no positive integer a such that $1 < a < 2$.

⟨4⟩1. We do not have $1 < 1 < 2$.

⟨4⟩2. For any positive integer n , we do not have $1 < n+1 < 2$.

PROOF: Since $n \geq 1$ so $n+1 \geq 2$.

⟨3⟩2. We do not have $1 < 0 < 2$.

⟨3⟩3. For any positive integer a , we do not have $1 < -a < 2$.

PROOF: Since $-a < 0 < 1$.

⟨2⟩2. For any positive integer n , if there is no integer a such that $n < a < n+1$, then there is no integer a such that $n+1 < a < n+2$.

PROOF: If $n+1 < a < n+2$ then $n < a-1 < n+1$.

⟨1⟩2. There is no integer a such that $0 < a < 1$.

PROOF: If $0 < a < 1$ then $1 < a+1 < 2$.

⟨1⟩3. For any positive integer n , there is no integer a such that $-n < a < -n+1$.

PROOF: If $-n < a < -n+1$ then $n-1 < -a < n$.

□

Theorem 6.2.5. *Every nonempty subset of \mathbb{Z} bounded above has a largest element.*

PROOF:

⟨1⟩1. LET: S be a nonempty subset of \mathbb{Z} bounded above.

⟨1⟩2. LET: u be an upper bound for S .

⟨1⟩3. PICK an integer $n > u$

PROOF: Archimedean property.

⟨1⟩4. LET: k be the least positive integer such that $n - k \in S$.

⟨2⟩1. PICK $m \in S$

⟨2⟩2. $n - m$ is a positive integer.

⟨2⟩3. There exists a positive integer k such that $n - k \in S$.

⟨1⟩5. $n - k$ is the greatest element in S .

⟨2⟩1. LET: $m \in S$

⟨2⟩2. $n - m \geq k$

⟨2⟩3. $m \leq n - k$

□

Theorem 6.2.6. *For any real number x , if x is not an integer then there exists a unique integer n such that $n < x < n + 1$.*

PROOF:

⟨1⟩1. $\{n \in \mathbb{Z} : n < x\}$ is a nonempty set of integers bounded above.

⟨2⟩1. PICK $m > -x$

PROOF: Archimedean property.

⟨2⟩2. $-m < x$

⟨2⟩3. $\{n \in \mathbb{Z} : n < x\}$ is nonempty.

⟨1⟩2. LET: n be the greatest integer such that $n < x$

⟨1⟩3. $x < n + 1$

⟨1⟩4. If n' is an integer with $n' < x < n' + 1$ then $n' = n$.

PROOF: We have $n' < n + 1$ so $n' \leq n$, and $n < n' + 1$ so $n \leq n'$.

□

Definition 6.2.7 (Even). An integer n is *even* iff $n/2$ is an integer; otherwise, n is *odd*.

Theorem 6.2.8. *If the integer m is odd then there exists an integer n such that $m = 2n + 1$.*

PROOF:

⟨1⟩1. LET: n be the integer such that $n < m/2 < n + 1$

PROOF: Theorem 6.2.6.

⟨1⟩2. $2n < m < 2n + 2$

⟨1⟩3. $m = 2n + 1$

□

Theorem 6.2.9. *The product of two odd integers is odd.*

PROOF: $(2m + 1)(2n + 1) = 2(2mn + m + n) + 1$. \square

Corollary 6.2.9.1. *If p is an odd integer and n is a positive integer then p^n is an odd integer.*

Definition 6.2.10 (Exponentiation). Extend the definition of exponentiation so a^n is defined for:

- all real numbers a and non-negative integers n
- all non-zero real numbers a and integers n

as follows:

$$\begin{aligned} a^0 &= 1 \\ a^{-n} &= 1/a^n \end{aligned} \quad (n \text{ a positive integer})$$

Theorem 6.2.11 (Laws of Exponents). *For all non-zero reals a and b and integers m and n ,*

$$\begin{aligned} a^n a^m &= a^{n+m} \\ (a^n)^m &= a^{nm} \\ a^m b^m &= (ab)^m \end{aligned}$$

PROOF: Easy. \square

Theorem 6.2.12. \mathbb{Z} is countable.

PROOF: The function that maps an integer n to $2n$ if $n \geq 0$ and $-1 - 2n$ if $n < 0$ is a bijection $\mathbb{Z} \approx \mathbb{N}$. \square

6.3 Rational Numbers

Definition 6.3.1 (Rational Number). The set \mathbb{Q} of *rational numbers* is the set of all real numbers that are the quotient of two integers. A real that is not rational is *irrational*.

Theorem 6.3.2. $\sqrt{2}$ is irrational.

PROOF:

- $\langle 1 \rangle$ 1. For any positive rational a , there exist positive integers m and n not both even such that $a = m/n$.
- $\langle 2 \rangle$ 1. LET: a be a positive rational.
- $\langle 2 \rangle$ 2. LET: n be the least positive integer such that na is a positive integer.
- $\langle 2 \rangle$ 3. LET: $m = na$
- $\langle 2 \rangle$ 4. ASSUME: for a contradiction m and n are both even.
- $\langle 2 \rangle$ 5. $m/2 = (n/2)a$
- $\langle 2 \rangle$ 6. Q.E.D.

PROOF: This contradicts the leastness of n ($\langle 2 \rangle 2$).

$\langle 1 \rangle 2$. ASSUME: for a contradiction $\sqrt{2}$ is rational.

$\langle 1 \rangle 3$. PICK positive integers m and n not both even such that $\sqrt{2} = m/n$.

$\langle 1 \rangle 4$. $m^2 = 2n^2$

$\langle 1 \rangle 5$. m^2 is even.

$\langle 1 \rangle 6$. m is even.

PROOF: Theorem 6.2.9.

$\langle 1 \rangle 7$. LET: $k = m/2$

$\langle 1 \rangle 8$. $4k^2 = 2n^2$

$\langle 1 \rangle 9$. $n^2 = 2k^2$

$\langle 1 \rangle 10$. n^2 is even.

$\langle 1 \rangle 11$. n is even.

PROOF: Theorem 6.2.9.

$\langle 1 \rangle 12$. Q.E.D.

PROOF: $\langle 1 \rangle 3$, $\langle 1 \rangle 6$ and $\langle 1 \rangle 11$ form a contradiction.

□

Theorem 6.3.3. \mathbb{Q} is countably infinite.

PROOF: The function $\mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$ that maps (m, n) to $m/(n+1)$ is a surjection.

□

6.4 Algebraic Numbers

Definition 6.4.1 (Algebraic Number). A real number r is *algebraic* iff there exists a natural number n and rational numbers a_0, a_1, \dots, a_{n-1} such that

$$r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0$$

Otherwise, r is *transcendental*.

Proposition 6.4.2. The set of algebraic numbers is countably infinite.

PROOF: There are countably many finite sequences of rational numbers, and each corresponding polynomial has only finitely many roots. □

Corollary 6.4.2.1. The set of transcendental numbers is uncountable.

Chapter 7

Monoid Theory

Definition 7.0.1 (Monoid). A *monoid* is a category with one object.

Definition 7.0.2. Let \mathcal{C} be a category and $X \in \mathcal{C}$. The monoid $\text{End}_{\mathcal{C}}(X)$ is the set of all morphisms $X \rightarrow X$ under composition.

Proposition 7.0.3. *For any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and $X \in \mathcal{C}$, we have that $F : \text{End}_{\mathcal{C}}(X) \rightarrow \text{End}_{\mathcal{D}}(FX)$ is a monoid homomorphism.*

PROOF: Since $F\text{id}_X = \text{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$. \square

Chapter 8

Group Theory

8.1 Category of Small Groups

Definition 8.1.1. Let **Grp** be the category of small groups and group homomorphisms.

Definition 8.1.2. We identify any group G with the category with one object whose morphisms are the elements of G with composition given by the multiplication in G .

Proposition 8.1.3. *The trivial group is a zero object in **Grp**.*

PROOF: Easy. \square

The zero morphism $G \rightarrow H$ maps every element in G to e .

Definition 8.1.4. Let \mathcal{C} be a category and $X \in \mathcal{C}$. We write $\text{Aut}_{\mathcal{C}}(X)$ for the set of all isomorphisms $X \cong X$ under composition.

Proposition 8.1.5. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $X \in \mathcal{C}$. Then $F : \text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(FX)$ is a group homomorphism.*

PROOF: Since $F \text{id}_X = \text{id}_{FX}$, $F(g \circ f) = Fg \circ Ff$, and $Ff^{-1} = (Ff)^{-1}$. \square

Proposition 8.1.6. **Grp** has products.

Definition 8.1.7 (Free Product). The product of a family of groups in **Grp** is called the *free product*.

Proposition 8.1.8. **Ab** has products given by direct sums.

Definition 8.1.9 (Left Coset). Let G be a group and H a subgroup of G . The *left cosets* of H are the sets of the form

$$xH := \{xh : h \in H\}$$

We write G/H for the set of left cosets of H in G .

Proposition 8.1.10. *Let G be a group and H a subgroup of G . Then G/H is a partition of G .*

PROOF:

$\langle 1 \rangle 1$. $\bigcup (G/H) = G$

PROOF: Since $x = xe$ and so $x \in xH$.

$\langle 1 \rangle 2$. Any two distinct left cosets of H are disjoint.

PROOF: Since if $z \in xH$ and $z \in yH$ then $xH = yH = zH$.

□

Definition 8.1.11. Let G be a group. Let A and B be subsets of G . Then

$$AB := \{ab : a \in A, b \in B\} .$$

Definition 8.1.12. Let G be a group. Let A be a subset of G . Then

$$A^{-1} := \{a^{-1} : a \in A\} .$$

Chapter 9

Ring Theory

Definition 9.0.1. Let **Ring** be the concrete category of rings and ring homomorphisms.

Definition 9.0.2 (Spectrum). Let R be a commutative ring. The *spectrum* of R , $\text{spec } R$, is the set of all prime ideals of R .

Definition 9.0.3 (Zariski Topology). Let R be a commutative ring. The *Zariski topology* on $\text{spec } R$ is the topology where the closed sets are the sets of the form

$$VE := \{p \in \text{spec } R : E \subseteq p\}$$

for any $E \in \mathcal{P}R$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{C} = \{VE : E \in \mathcal{P}R\}$

$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{A} \in \mathcal{C}$

$\langle 2 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{C}$

$\langle 2 \rangle 2$. LET: $E = \bigcup \{E' \in \mathcal{P}R : VE' \in \mathcal{A}\}$

PROVE: $VE = \bigcap \mathcal{A}$

$\langle 2 \rangle 3$. For all $p \in \text{spec } R$, if $E \subseteq p$ then $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 1$. LET: $p \in \text{spec } R$

$\langle 3 \rangle 2$. ASSUME: $E \subseteq p$

$\langle 3 \rangle 3$. LET: $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$

$\langle 3 \rangle 4$. $E' \subseteq E$

$\langle 3 \rangle 5$. $E' \subseteq p$

$\langle 3 \rangle 6$. $p \in VE'$

$\langle 2 \rangle 4$. For all $p \in \text{spec } R$, if $p \in \bigcap \mathcal{A}$ then $E \subseteq p$

$\langle 3 \rangle 1$. LET: $p \in \bigcap \mathcal{A}$

$\langle 3 \rangle 2$. For all $E' \in \mathcal{P}R$ with $VE' \in \mathcal{A}$ we have $E' \subseteq p$

$\langle 3 \rangle 3$. $E \subseteq p$

$\langle 1 \rangle 3$. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

PROOF: Since $VE \cup VE' = V(E \cap E')$

$\langle 1 \rangle 4. \emptyset \in \mathcal{C}$

$\langle 2 \rangle 1. VR = \emptyset$

PROOF: If $p \in VR$ then $R \subseteq p$ contradicting the fact that p is a prime ideal.

□

Definition 9.0.4. For any ring R , let $R - \mathbf{Mod}$ be the category of small R -modules and R -module homomorphisms.

Proposition 9.0.5. $R - \mathbf{Mod}$ has products and coproducts.

Chapter 10

Field Theory

Proposition 10.0.1. *Field does not have binary products.*

PROOF: There cannot be a field K with field homomorphisms $K \rightarrow \mathbb{Z}_2$ and $K \rightarrow \mathbb{Z}_3$, because its characteristic would be both 2 and 3. \square

Chapter 11

Linear Algebra

Definition 11.0.1 (Span). Let V be a vector space and $A \subseteq V$. The *span* of A is the set of all linear combinations of elements of A .

Definition 11.0.2 (Independent). Let V be a vector space and $A \subseteq V$. Then A is *linearly independent* iff, whenever

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

where $v_1, \dots, v_n \in A$, then

$$\alpha_1 = \cdots = \alpha_n = 0 \text{ .}$$

Proposition 11.0.3. *Let V be a vector space, $A \subseteq V$ and $v \in V$. If A is linearly independent and $v \notin \text{span } A$, then $A \cup \{v\}$ is independent.*

PROOF:

$\langle 1 \rangle 1$. LET: $\alpha_1 v_1 + \cdots + \alpha_n v_n + \beta v = 0$ where $v_1, \dots, v_n \in A$

$\langle 1 \rangle 2$. $\beta = 0$

PROOF: Otherwise $v = (\alpha_1/\beta)v_1 + \cdots + (\alpha_n/\beta)v_n \in \text{span } A$.

$\langle 1 \rangle 3$. $\alpha_1 = \cdots = \alpha_n = 0$

PROOF: Since A is linearly independent.

□

Theorem 11.0.4. *Every vector space has a basis.*

PROOF:

$\langle 1 \rangle 1$. LET: V be a vector space.

$\langle 1 \rangle 2$. PICK a maximal linearly independent set \mathcal{B} .

PROOF: By Tukey's Lemma.

$\langle 1 \rangle 3$. $\text{span } \mathcal{B} = V$

PROOF: Proposition 11.0.3.

□

Definition 11.0.5. For any field K , we write \mathbf{Vect}_K for $K - \mathbf{Mod}$.

Dual space functor $\mathbf{Vect}_K^{\text{op}} \rightarrow \mathbf{Vect}_K$.

Chapter 12

Topology

12.1 Topological Spaces

Definition 12.1.1 (Topological Space). Let X be a set and $\mathcal{O} \subseteq \mathcal{P}X$. Then we say (X, \mathcal{O}) is a *topological space* iff:

- For any $\mathcal{U} \subseteq \mathcal{O}$ we have $\bigcup \mathcal{U} \in \mathcal{O}$.
- For any $U, V \in \mathcal{O}$ we have $U \cap V \in \mathcal{O}$.
- $X \in \mathcal{O}$

We call \mathcal{O} the *topology* of the topological space, and call its elements *open* sets. We shall often write X for the topological space (X, \mathcal{O}) .

Definition 12.1.2 (Discrete Topology). For any set X , the power set $\mathcal{P}X$ is called the *discrete* topology on X .

Proposition 12.1.3. *For any set X , the discrete topology on X is a topology on X .*

Definition 12.1.4 (Indiscrete Topology). For any set X , the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Proposition 12.1.5. *For any set X , the indiscrete topology on X is a topology on X .*

Definition 12.1.6 (Cofinite Topology). For any set X , the *cofinite* topology is $\{X - U : U \subseteq X \text{ is finite}\}$.

Definition 12.1.7 (Cocountable Topology). For any set X , the *cocountable* topology is $\{X - U : U \subseteq X \text{ is countable}\}$.

Definition 12.1.8 (Sierpiński Two-Point Space). The *Sierpiński two-point space* is $\{0, 1\}$ under the topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Proposition 12.1.9. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

Proposition 12.1.10. *The intersection of a set of topologies on a set X is a topology on X .*

Definition 12.1.11 (Closed Set). *Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X - A$ is open.*

Proposition 12.1.12. *A set B is open if and only if $X - B$ is closed.*

Proposition 12.1.13. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that \mathcal{C} is the set of closed sets if and only if:*

- For any $\mathcal{D} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{D} \in \mathcal{C}$
- For any $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.
- $\emptyset \in \mathcal{C}$

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$.

Theorem 12.1.14. *Let X be a set. Let $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology on X such that \mathcal{C} is the set of closed sets if and only if:*

1. $\emptyset \in \mathcal{C}$
2. $\forall \mathcal{A} \subseteq \mathcal{C}. \bigcap \mathcal{A} \in \mathcal{C}$
3. $\forall C, D \in \mathcal{C}. C \cup D \in \mathcal{C}$

In this case, the topology is unique, and is $\{X - C : C \in \mathcal{C}\}$.

PROOF: Straightforward.

Theorem 12.1.15. *There are infinitely many primes.*

Furstenberg's proof:

PROOF:

- $\langle 1 \rangle 1$. For $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$,
 LET: $S(a, b) := \{an + b : n \in \mathbb{N}\}$
- $\langle 1 \rangle 2$. LET: \mathcal{T} be the topology generated by the basis $\{S(a, b) : a \in \mathbb{Z} - \{0\}, b \in \mathbb{Z}\}$
- $\langle 2 \rangle 1$. For every $n \in \mathbb{Z}$, there exist a, b such that $n \in S(a, b)$.
 PROOF: $n \in S(n, 0)$
- $\langle 2 \rangle 2$. If $n \in S(a_1, b_1) \cap S(a_2, b_2)$ then there exist a_3, b_3 such that $n \in S(a_3, b_3) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$
- $\langle 3 \rangle 1$. LET: $d = \text{lcm}(a_1, a_2)$
 PROVE: $S(d, n) \subseteq S(a_1, b_1) \cap S(a_2, b_2)$
- $\langle 3 \rangle 2$. LET: $d = a_1k = a_2l$
- $\langle 3 \rangle 3$. LET: $n = a_1c + b_1 = a_2d + b_2$
- $\langle 3 \rangle 4$. LET: $z \in \mathbb{Z}$
 PROVE: $dz + n \in S(a_1, b_1) \cap S(a_2, b_2)$

⟨3⟩5. $dz + n \in S(a_1, b_1)$

PROOF:

$$\begin{aligned} dz + n &= a_1 kz + a_1 c + b_1 \\ &= a_1(kz + c) + b_1 \end{aligned}$$

⟨3⟩6. $dz + n \in S(a_2, b_2)$

PROOF: Similar.

⟨1⟩3. For all $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$ we have $S(a, b)$ is closed.

⟨2⟩1. LET: $a \in \mathbb{Z} - \{0\}$ and $b \in \mathbb{Z}$

⟨2⟩2. LET: $n \in \mathbb{Z} - S(a, b)$

⟨2⟩3. $n \in S(a, n) \subseteq \mathbb{Z} - S(a, b)$

⟨3⟩1. LET: $x \in S(a, n)$

⟨3⟩2. ASSUME: for a contradiction $x \in S(a, b)$

⟨3⟩3. PICK m such that $x = am + b$

⟨3⟩4. PICK l such that $x = al + n$

⟨3⟩5. $n = a(m - l) + b$

⟨3⟩6. $n \in S(a, b)$

⟨3⟩7. Q.E.D.

PROOF: This contradicts ⟨2⟩2.

⟨1⟩4.

$$\mathbb{Z} - \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$$

PROOF: Since every integer except 1 and -1 is divisible by a prime.

⟨1⟩5. No nonempty finite set is open.

⟨2⟩1. LET: U be a nonempty open set

⟨2⟩2. PICK $n \in U$

⟨2⟩3. There exist a, b such that $n \in S(a, b) \subseteq U$

⟨2⟩4. U is infinite.

⟨1⟩6. $\mathbb{Z} - \{1, -1\}$ is not closed.

⟨1⟩7. $\bigcup_{p \text{ prime}} S(p, 0)$ is not closed.

⟨1⟩8. The union of finitely many closed sets is closed.

⟨1⟩9. There are infinitely many primes.

□

Proposition 12.1.16. *In a discrete topological space, every set is closed.*

PROOF: Immediate from definitions. □

Proposition 12.1.17. *In a linearly ordered set under the order topology, every closed interval and closed ray is closed.*

PROOF:

⟨1⟩1. LET: X be a linearly ordered set under the order topology.

⟨1⟩2. Every closed interval in X is closed.

PROOF: Since $X - [a, b] = (-\infty, a) \cup (b, +\infty)$.

⟨1⟩3. Every closed ray in X is closed.

PROOF: Since $X - [a, +\infty) = (-\infty, a)$ and $X - (-\infty, a] = (a, +\infty)$.

□

Proposition 12.1.18. *Let X be a topological space and Y a subspace of X . Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set B in X such that $A = B \cap Y$.*

PROOF:

$$\begin{aligned}
 A \text{ is closed in } Y &\Leftrightarrow Y - A \text{ is open in } Y \\
 &\Leftrightarrow \exists U \text{ open in } X. Y - A = U \cap Y \\
 &\Leftrightarrow \exists C \text{ closed in } X. Y - A = Y - C \\
 &\Leftrightarrow \exists C \text{ closed in } X. A = Y \cap C \quad \square
 \end{aligned}$$

Proposition 12.1.19. *Let X be a topological space and Y a subspace of X . Let $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X .*

PROOF:

$\langle 1 \rangle 1$. PICK C closed in X such that $A = C \cap Y$.

$\langle 1 \rangle 2$. A is closed in X .

PROOF: It is the intersection of two closed sets in X .

\square

Definition 12.1.20 (Neighbourhood). Let X be a topological space, $Sx \in X$ and $U \subseteq X$. Then U is a *neighbourhood* of x , and x is an *interior* point of U , iff there exists an open set V such that $x \in V \subseteq U$.

Proposition 12.1.21. *A set B is open if and only if it is a neighbourhood of each of its points.*

Proposition 12.1.22. *Let X be a set and $\mathcal{N} : X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} on X such that, for all $x \in X$, we have \mathcal{N}_x is the set of neighbourhoods of x , if and only if:*

- For all $x \in X$ and $N \in \mathcal{N}_x$ we have $x \in N$
- For all $x \in X$ we have $X \in \mathcal{N}_x$
- For all $x \in X$, $N \in \mathcal{N}_x$ and $V \subseteq \mathcal{P}X$, if $N \subseteq V$ then $V \in \mathcal{N}_x$
- For all $x \in X$ and $M, N \in \mathcal{N}_x$ we have $M \cap N \in \mathcal{N}_x$
- For all $x \in X$ and $N \in \mathcal{N}_x$, there exists $M \in \mathcal{N}_x$ such that $M \subseteq N$ and $\forall y \in M. M \in \mathcal{N}_y$.

In this case, \mathcal{O} is unique and is given by $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$.

Definition 12.1.23 (Exterior Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is an *exterior point* of B iff $B - X$ is a neighbourhood of x .

Definition 12.1.24 (Boundary Point). Let X be a topological space, $x \in X$ and $B \subseteq X$. Then x is a *boundary point* of B iff it is neither an interior point nor an exterior point of B .

Definition 12.1.25 (Interior). Let X be a topological space and $B \subseteq X$. The *interior* of B , B° , is the set of all interior points of B .

Proposition 12.1.26. *The interior of B is the union of all the open sets included in B .*

Definition 12.1.27 (Closure). Let X be a topological space and $B \subseteq X$. The *closure* of B , \overline{B} , is the set of all points that are not exterior points of B .

Proposition 12.1.28. *The closure of B is the intersection of all the closed sets that include B .*

Proposition 12.1.29. *A set B is open iff $X - B = \overline{X - B}$.*

Proposition 12.1.30 (Kuratowski Closure Axioms). *Let X be a set and $- : \mathcal{P}X \rightarrow \mathcal{P}X$. Then there exists a topology \mathcal{O} such that, for all $B \subseteq X$, \overline{B} is the closure of B , if and only if:*

- $\overline{\emptyset} = \emptyset$
- For all $A \subseteq X$ we have $A \subseteq \overline{A}$
- For all $A \subseteq X$ we have $\overline{\overline{A}} = \overline{A}$
- For all $A, B \subseteq X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$

In this case, \mathcal{O} is unique and is defined by $\mathcal{O} = \{U : X - U = \overline{X - U}\}$.

Definition 12.1.31 (Finer, Coarser). Let \mathcal{T} and \mathcal{T}' be topologies on the set X . Then \mathcal{T} is *coarser*, *smaller* or *weaker* than \mathcal{T}' , or \mathcal{T}' is *finer*, *larger* or *weaker* than \mathcal{T} , iff $\mathcal{T} \subseteq \mathcal{T}'$.

12.2 Topological Disjoint Union

Definition 12.2.1 (Coproduct Topology). Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The *coproduct topology* on $\coprod_{\alpha \in A} X_\alpha$ is

$$\mathcal{T} = \left\{ \coprod_{\alpha \in A} U_\alpha : \{U_\alpha\}_{\alpha \in A} \text{ is a family with } U_\alpha \text{ open in } X_\alpha \text{ for all } \alpha \right\}.$$

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

PROOF:

$$\bigcup_{i \in I} \coprod_{\alpha \in A} U_{i\alpha} = \coprod_{\alpha \in A} \bigcup_{i \in I} U_{i\alpha}$$

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF:

$$\coprod_{\alpha \in A} U_\alpha \cap \coprod_{\alpha \in A} V_\alpha = \coprod_{\alpha \in A} (U_\alpha \cap V_\alpha)$$

$\langle 1 \rangle 3$. $\coprod_{\alpha \in A} X_\alpha \in \mathcal{T}$

PROOF: Since every X_α is open in X_α .

□

Proposition 12.2.2. *The coproduct topology is the finest topology on $\coprod_{\alpha \in A} X_\alpha$ such that every injection $\kappa_\alpha : X_\alpha \rightarrow \coprod_{\alpha \in A} X_\alpha$ is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $P = \coprod_{\alpha \in A} X_\alpha$

$\langle 1 \rangle 2$. LET: \mathcal{T}_c be the coproduct topology.

$\langle 1 \rangle 3$. LET: \mathcal{T} be any topology on P

$\langle 1 \rangle 4$. For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T}_c)$ is continuous.

$\langle 2 \rangle 1$. LET: $\alpha \in A$

$\langle 2 \rangle 2$. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with each U_α open in X_α .

$\langle 2 \rangle 3$. For all $\alpha \in A$, we have $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha)$ is open in X_α .

PROOF: Since $\kappa_\alpha^{-1}(\coprod_{\alpha \in A} U_\alpha) = U_\alpha$.

$\langle 1 \rangle 5$. If, for all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$ is continuous, then $\mathcal{T} \subseteq \mathcal{T}_c$.

$\langle 2 \rangle 1$. ASSUME: For all $\alpha \in A$, the injection $\kappa_\alpha : X_\alpha \rightarrow (P, \mathcal{T})$ is continuous.

$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 3$. For all $\alpha \in a$, we have $\kappa_\alpha^{-1}(U)$ is open in X_α .

$\langle 2 \rangle 4$. $U = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(U) \in \mathcal{T}_c$

□

Theorem 12.2.3. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The coproduct topology is the unique topology on $\coprod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : \coprod_{\alpha \in A} X_\alpha \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $X = \coprod_{\alpha \in A} X_\alpha$

$\langle 1 \rangle 2$. LET: \mathcal{T}_c be the coproduct topology.

$\langle 1 \rangle 3$. For every topological space Z and function $f : (X, \mathcal{T}_c) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.

$\langle 2 \rangle 1$. LET: Z be a topological space.

$\langle 2 \rangle 2$. LET: $f : X \rightarrow Z$

$\langle 2 \rangle 3$. If f is continuous then $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.

PROOF: Because the composite of two continuous functions is continuous.

$\langle 2 \rangle 4$. If $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous then f is continuous.

$\langle 3 \rangle 1$. ASSUME: $\forall \alpha \in A, f \circ \kappa_\alpha$ is continuous.

$\langle 3 \rangle 2$. LET: U be open in Z

$\langle 3 \rangle 3$. For all $\alpha \in A$ we have $\kappa_\alpha^{-1}(f^{-1}(U))$ is open in X_α

$\langle 3 \rangle 4$. $f^{-1}(U) = \coprod_{\alpha \in A} \kappa_\alpha^{-1}(f^{-1}(U))$

$\langle 3 \rangle 5$. $f^{-1}(U)$ is open in X

- $\langle 1 \rangle 4$. For any topology \mathcal{T} on X , if for every topological space Z and function $f : (X, \mathcal{T}) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous, then $\mathcal{T} = \mathcal{T}_c$.
 $\langle 2 \rangle 1$. LET: \mathcal{T} be a topology on X .
 $\langle 2 \rangle 2$. ASSUME: For every topological space Z and function $f : (X, \mathcal{T}) \rightarrow Z$, we have f is continuous if and only if $\forall \alpha \in A. f \circ \kappa_\alpha$ is continuous.
 $\langle 2 \rangle 3$. $\mathcal{T} \subseteq \mathcal{T}_c$
 $\langle 3 \rangle 1$. For all $\alpha \in A$ we have $\kappa_\alpha : X_\alpha \rightarrow (X, \mathcal{T})$ is continuous.
 PROOF: From $\langle 2 \rangle 1$ since id_X is continuous.
 $\langle 3 \rangle 2$. $\mathcal{T} \subseteq \mathcal{T}_c$
 PROOF: Proposition 12.2.2.
 $\langle 2 \rangle 4$. $\mathcal{T}_c \subseteq \mathcal{T}$
 $\langle 3 \rangle 1$. LET: $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}_c)$ be the identity function.
 $\langle 3 \rangle 2$. $f \circ \kappa_\alpha$ is continuous for all α .
 $\langle 3 \rangle 3$. f is continuous.
 PROOF: $\langle 2 \rangle 1$
 $\langle 3 \rangle 4$. $\mathcal{T}_c \subseteq \mathcal{T}$

□

12.3 Bases

Definition 12.3.1 (Basis). Let X be a topological space. A *basis* for the topology on X is a set of open sets \mathcal{B} such that every open set is the union of a subset of \mathcal{B} . The elements of \mathcal{B} are called *basic open neighbourhoods* of their elements.

Proposition 12.3.2. Let X be a set. The set of all one-element subsets of X is a basis for the discrete topology on X .

Proposition 12.3.3. Let X be a topological space. Let \mathcal{B} be a basis for the topology on X . Then the topology on X is the coarsest topology that includes \mathcal{B} .

Proposition 12.3.4. Let X and Y be topological spaces. Let \mathcal{B} be a basis for the topology on X and \mathcal{C} a basis for the topology on Y . Then

$$\{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Definition 12.3.5 (Order Topology). Let X be a linearly ordered set. The *order topology* on X is the topology generated by the open interval (a, b) as well as the open rays $(a, +\infty)$ and $(-\infty, b)$ for $a, b \in X$.

The *standard topology* on \mathbb{R} is the order topology.

Proposition 12.3.6. Let X be a linearly ordered set. Then the order topology is generated by the basis consisting of:

- all open intervals (a, b)
- all intervals of the form $[\perp, b)$ where \perp is the least element of X , if any
- all intervals of the form $(a, \top]$ where \top is the greatest element of X , if any.

Proposition 12.3.7. *Let X be a linearly ordered set. The open rays in X form a subbasis for the order topology.*

Definition 12.3.8 (Lower Limit Topology). The *lower limit topology*, *Sorgenfrey topology*, *uphill topology* or *half-open topology* is the topology on \mathbb{R} generated by the basis consisting of all half-open intervals $[a, b)$.

We write \mathbb{R}_l for \mathbb{R} under the lower limit topology.

Definition 12.3.9 (K -topology). Let $K = \{1/n : n \in \mathbb{Z}_+\}$. The K -topology on \mathbb{R} is the topology generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) - K$.

We write \mathbb{R}_K for \mathbb{R} under the K -topology.

Definition 12.3.10 (Ordered Square). The *ordered square* I_o^2 is the set $[0, 1]^2$ under the order topology induced by the dictionary order.

Proposition 12.3.11. *Let X be a linearly ordered set under the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology.*

PROOF:

$\langle 1 \rangle 1$. The order topology is coarser than the subspace topology.

$\langle 2 \rangle 1$. For all $a \in Y$, the open ray $\{y \in Y : a < y\}$ is open in the subspace topology.

PROOF: It is $(a, +\infty) \cap Y$.

$\langle 2 \rangle 2$. For all $a \in Y$, the open ray $\{y \in Y : y < a\}$ is open in the subspace topology.

PROOF: It is $(-\infty, a) \cap Y$.

$\langle 1 \rangle 2$. The subspace topology is coarser than the order topology.

$\langle 2 \rangle 1$. For all $a \in X$, the set $(-\infty, a) \cap Y$ is open in the order topology.

$\langle 3 \rangle 1$. CASE: $a \in Y$

PROOF: Then $(-\infty, a) \cap Y = \{y \in Y : y < a\}$ is an open ray in Y .

$\langle 3 \rangle 2$. CASE: a is an upper bound for Y

PROOF: Then $(-\infty, a) \cap Y = Y$.

$\langle 3 \rangle 3$. CASE: a is a lower bound for Y

PROOF: Then $(-\infty, a) \cap Y = \emptyset$.

$\langle 3 \rangle 4$. Q.E.D.

PROOF: These are the only three cases because Y is convex.

$\langle 2 \rangle 2$. For all $a \in X$, the set $(a, +\infty) \cap Y$ is open in the order topology.

PROOF: Similar.

□

Example 12.3.12. We cannot remove the hypothesis that the set Y is convex.

Let $X = \mathbb{R}$ and $Y = [0, 1) \cup \{2\}$. Then $\{2\}$ is open in the subspace topology but not in the order topology on Y .

Proposition 12.3.13. *Let X be a topological space. Let \mathcal{B} be a basis for the topology on X and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proposition 12.3.14. *Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Assume that, for every open set U and element $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then \mathcal{B} is a basis for the topology on X .*

Proposition 12.3.15. *Let X be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is a basis for a topology on X if and only if:*

1. $\bigcup \mathcal{B} = X$
2. For all $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

In this case, the topology is unique and is the set of all unions of subsets of \mathcal{B} . We call it the topology generated by \mathcal{B} .

Proposition 12.3.16. *Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' , respectively, on X . Then \mathcal{T}' is finer than \mathcal{T} if and only if, for every $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Corollary 12.3.16.1. *The topologies of \mathbb{R}_l and \mathbb{R}_K are strictly finer than the standard topology on \mathbb{R} but are not comparable to one another.*

12.3.1 Subspaces

Proposition 12.3.17. *Let X be a topological space. Let Y be a subspace of X . Let \mathcal{B} be a basis for the topology on X . Then $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for the topology on Y .*

PROOF:

$\langle 1 \rangle 1$. For all $B \in \mathcal{B}$ we have $B \cap Y$ is open in Y .

PROOF: Since B is open in X .

$\langle 1 \rangle 2$. For any open set V in Y and $y \in V$, there exists $B \in \mathcal{B}$ such that $y \in B \cap Y \subseteq V$.

$\langle 2 \rangle 1$. LET: V be open in Y .

$\langle 2 \rangle 2$. LET: $y \in V$

$\langle 2 \rangle 3$. PICK U open in X such that $V = U \cap Y$.

$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$.

$\langle 2 \rangle 5$. $y \in B \cap Y \subseteq V$

□

12.3.2 Product Topology

Proposition 12.3.18. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. For all $i \in I$, let \mathcal{B}_i be a basis for the topology on X_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i : \text{for finitely many } i \in I \text{ we have } B_i \in \mathcal{B}_i, \text{ and } B_i = X_i \text{ for all other } i\}$ is a basis for the product topology on $\prod_{i \in I} X_i$.*

PROOF:

$\langle 1 \rangle 1$. Every $B \in \mathcal{B}$ is open in the product topology.

PROOF: Since every element of \mathcal{B}_i is open in X_i .

$\langle 1 \rangle 2$. For any open set U in the product topology and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 1$. LET: U be a set open in the box topology.

$\langle 2 \rangle 2$. LET: $x \in U$

$\langle 2 \rangle 3$. PICK a family $\{U_i\}_{i \in I}$ where U_i is open in X_i for $i = i_1, \dots, i_n$, and $U_i = X_i$ for all other i , such that $x \in \prod_{i \in I} U_i \subseteq U$

$\langle 2 \rangle 4$. For $i = i_1, \dots, i_n$, choose $B_i \in \mathcal{B}_i$ such that $x_i \in B_i \subseteq U_i$. Let $B_i = X_i$ for all other i .

$\langle 2 \rangle 5$. $\prod_{i \in I} B_i \in \mathcal{B}$

$\langle 2 \rangle 6$. $x \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} U_i \subseteq U$

□

12.4 Subbases

Definition 12.4.1 (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a set \mathcal{S} of open sets such that every open set is a union of finite intersections of \mathcal{S} .

Proposition 12.4.2. *Let X be a set and $\mathcal{S} \subseteq X$. Then \mathcal{S} is a subbasis for a topology on X if and only if $\bigcup \mathcal{S} = X$, in which case the topology is unique and is the set of all unions of finite intersections of elements of \mathcal{S} .*

Proposition 12.4.3. *Let X be a topological space. Let \mathcal{S} be a subbasis for the topology on X . Then the topology on X is the coarsest topology that includes \mathcal{S} .*

Proposition 12.4.4. *Let X and Y be topological spaces. Then*

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

PROOF:

$\langle 1 \rangle 1$. Every element of \mathcal{S} is open.

PROOF: Since $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$.

$\langle 1 \rangle 2$. Every open set is a union of finite intersections of elements of \mathcal{S} .

PROOF: Since, for U open in X and V open in Y , we have $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$.

□

Definition 12.4.5 (Space with Basepoint). A *space with basepoint* is a pair (X, x) where X is a topological space and $x \in X$.

12.5 Neighbourhood Bases

Definition 12.5.1 (Neighbourhood Basis). Let X be a topological space and $x_0 \in X$. A *neighbourhood basis* of x_0 is a set \mathcal{U} of neighbourhoods of x_0 such that every neighbourhood of x_0 includes an element of \mathcal{U} .

12.6 First Countable Spaces

Definition 12.6.1 (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

Proposition 12.6.2. \mathbb{R}_l is first countable.

PROOF: For any $x \in \mathbb{R}$ we have $\{[x, x + 1/n) : n \in \mathbb{Z}_+\}$ is a countable local basis.
□

Proposition 12.6.3. The ordered square is first countable.

PROOF:

⟨1⟩1. Every point (a, b) with $0 < b < 1$ has a countable local basis.

PROOF: The set of all intervals $((a, q), (a, r))$ where q and r are rational and $0 \leq q < b < r \leq 1$ is a countable local basis.

⟨1⟩2. Every point $(a, 0)$ has a countable local basis with $a > 0$.

PROOF: The set of all intervals $((q, 0), (a, r))$ where q and r are rational with $0 \leq q < a$ and $0 < r \leq 1$ is a countable local basis.

⟨1⟩3. Every point $(a, 1)$ has a countable local basis with $a < 1$.

PROOF: The set of all intervals $((a, q), (r, 1))$ with q and r rational and $0 \leq q < 1, a < r \leq 1$ is a countable local basis.

⟨1⟩4. $(0, 0)$ has a countable local basis.

PROOF: The set of all intervals $[(0, 0), (0, r))$ with r rational and $0 < r \leq 1$ is a countable local basis.

⟨1⟩5. $(1, 1)$ has a countable local basis.

PROOF: The set of all intervals $((1, q), (1, 1])$ with q rational and $0 \leq q < 1$ is a countable local basis.

□

12.7 Second Countable Spaces

Definition 12.7.1 (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

\mathbb{R}^n is second countable.

An uncountable discrete space is first countable but not second countable.

Proposition 12.7.2. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces such that no X_λ is indiscrete. If Λ is uncountable, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not first countable.*

PROOF:

- $\langle 1 \rangle 1$. For all $\lambda \in \Lambda$, PICK U_λ open in X_λ such that $\emptyset \neq U_\lambda \neq X_\lambda$.
- $\langle 1 \rangle 2$. For all $\lambda \in \Lambda$, PICK $x_\lambda \in U_\lambda$.
- $\langle 1 \rangle 3$. ASSUME: for a contradiction B is a countable neighbourhood basis for $(x_\lambda)_{\lambda \in \Lambda}$.
- $\langle 1 \rangle 4$. PICK $\lambda \in \Lambda$ such that, for all $U \in B$, we have $\pi_\lambda(U) = X_\lambda$.
- $\langle 1 \rangle 5$. There is no $U \in B$ such that $U \subseteq \pi_\lambda^{-1}(U_\lambda)$.
- $\langle 1 \rangle 6$. Q.E.D.

PROOF: This is a contradiction.

□

12.8 Interior

Definition 12.8.1 (Interior). Let X be a topological space. Let $A \subseteq X$. The *interior* of A , A° , is the union of all the open sets included in A .

12.9 Closure

Definition 12.9.1 (Closure). Let X be a topological space. Let $A \subseteq X$. The *closure* of A , \overline{A} , is the intersection of all the closed sets that include A .

Proposition 12.9.2. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every open set that contains x intersects A .*

PROOF:

- $x \in \overline{A} \Leftrightarrow$ for every closed set C , if $A \subseteq C$ then $x \in C$
- \Leftrightarrow for every open set U , if $A \subseteq X - U$ then $x \in X - U$
- \Leftrightarrow for every open set U , if $A \cap U = \emptyset$ then $x \notin U$
- \Leftrightarrow for every open set U , if $x \in U$ then A intersects U □

Proposition 12.9.3. *Let X be a topological space. Let $A \subseteq B \subseteq X$. Then $\overline{A} \subseteq \overline{B}$.*

PROOF: Since every closed set that includes B is a closed set that includes A . □

Proposition 12.9.4. *Let X be a topological space. Let $A, B \subseteq X$. Then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

PROOF:

- $\langle 1 \rangle 1$. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
 PROOF: Since $\overline{A \cup B}$ is a closed set that includes $A \cup B$.
- $\langle 1 \rangle 2$. $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$
 PROOF: Since $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ by Proposition 12.9.3.

□

Proposition 12.9.5. *Let X be a topological space. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then*

$$\bigcup \{\bar{A} : A \in \mathcal{A}\} \subseteq \overline{\bigcup \mathcal{A}}.$$

PROOF: For all $A \in \mathcal{A}$ we have $\bar{A} \subseteq \overline{\bigcup \mathcal{A}}$ by Proposition 12.9.3. □

Example 12.9.6. The converse does not always hold. In \mathbb{R} , let $\mathcal{A} = \{\{x\} : 0 < x < 1\}$. Then $\bigcup \{\bar{A} : A \in \mathcal{A}\} = (0, 1)$ but $\overline{\bigcup \mathcal{A}} = [0, 1]$.

Proposition 12.9.7. *Let X be a topological space. Let $\mathcal{A} \subseteq \mathcal{P}X$. Then $\overline{\bigcap \mathcal{A}} \subseteq \bigcap \{\bar{A} : A \in \mathcal{A}\}$.*

PROOF: Since $\overline{\bigcap \mathcal{A}} \subseteq \bar{A}$ for all $A \in \mathcal{A}$ by Proposition 12.9.3. □

Example 12.9.8. The converse does not always hold. In \mathbb{R} , if A is the set of all rational numbers and B is the set of all irrational numbers then $\bigcap A \cap B = \emptyset$ but $\bigcap A \cap \bigcap B = \mathbb{R}$.

12.9.1 Bases

Proposition 12.9.9. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let \mathcal{B} be a basis for the topology on X . Then $x \in \bar{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .*

PROOF:

⟨1⟩1. If $x \in \bar{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

PROOF: Proposition 12.9.2 since every element of \mathcal{B} is open.

⟨1⟩2. If, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A , then $x \in \bar{A}$.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

⟨2⟩2. LET: U be an open set that contains x .

⟨2⟩3. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

⟨2⟩4. B intersects A .

PROOF: ⟨2⟩1

⟨2⟩5. U intersects A .

□

12.9.2 Subspaces

Proposition 12.9.10. *Let X be a topological space. Let Y be a subspace of X . Let $A \subseteq Y$. Let \bar{A} be the closure of A in X . Then the closure of A in Y is $\bar{A} \cap Y$.*

PROOF:

⟨1⟩1. $\bar{A} \cap Y$ is the closed in Y .

PROOF: Since \bar{A} is closed in X .

⟨1⟩2. For any closed set B in Y , if $A \subseteq B$ then $\bar{A} \cap Y \subseteq B$.

- ⟨2⟩1. LET: B be closed in Y .
- ⟨2⟩2. ASSUME: $A \subseteq B$
- ⟨2⟩3. PICK C closed in X such that $B = C \cap Y$.
- ⟨2⟩4. $A \subseteq C$
- ⟨2⟩5. $\overline{A} \subseteq C$
- ⟨2⟩6. $\overline{A} \cap Y \subseteq B$

□

12.9.3 Product Topology

Proposition 12.9.11. *Let X and Y be topological spaces. Let $A \subseteq X$ and $B \subseteq Y$. Then $\overline{A \times B} = \overline{A} \times \overline{B}$.*

PROOF:

- ⟨1⟩1. $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$

PROOF: Since $\overline{A} \times \overline{B}$ is a closed set that includes $A \times B$ by Proposition 12.20.2.

- ⟨1⟩2. $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$

- ⟨2⟩1. LET: $x \in \overline{A}$ and $y \in \overline{B}$.

- ⟨2⟩2. LET: U be an open set that contains (x, y) .

- ⟨2⟩3. PICK open sets V in X and W in Y such that $(x, y) \in V \times W \subseteq U$.

- ⟨2⟩4. V intersects A and W intersects B .

- ⟨2⟩5. U intersects $A \times B$.

□

12.9.4 Interior

Proposition 12.9.12. *Let X be a topological space and $A \subseteq X$. Then*

$$X - A^\circ = \overline{X - A}$$

PROOF:

$$\begin{aligned}
 X - A^\circ &= X - \bigcup \{U \text{ open in } X : U \subseteq A\} \\
 &= \bigcap \{X - U : U \text{ open in } X, U \subseteq A\} && \text{(De Morgan's Law)} \\
 &= \bigcap \{C : C \text{ closed in } X, X - A \subseteq C\} \\
 &= \overline{X - A}
 \end{aligned}$$

□

Proposition 12.9.13. *Let X be a topological space and $A \subseteq X$. Then*

$$X - \overline{A} = (X - A)^\circ$$

PROOF: Dual. □

12.10 Boundary

Definition 12.10.1 (Boundary). Let X be a topological space. Let $A \subseteq X$. The *boundary* of A is

$$\partial A := \overline{A} \cap \overline{X - A}.$$

Proposition 12.10.2. *Let X be a topological space. Let $A \subseteq X$. Then*

$$A^\circ \cap \partial A = \emptyset .$$

PROOF:

- $\langle 1 \rangle 1. A^\circ \subseteq A$
- $\langle 1 \rangle 2. X - A \subseteq X - A^\circ$
- $\langle 1 \rangle 3. \overline{X - A} \subseteq X - A^\circ$
- $\langle 1 \rangle 4. \partial A \subseteq X - A^\circ$

□

Proposition 12.10.3. *Let X be a topological space. Let $A \subseteq X$. Then*

$$\overline{A} = A^\circ \cup \partial A$$

- $\langle 1 \rangle 1. A^\circ \subseteq \overline{A}$

PROOF: Since $A^\circ \subseteq A \subseteq \overline{A}$.

- $\langle 1 \rangle 2. \partial A \subseteq \overline{A}$

PROOF: Definition of ∂A .

- $\langle 1 \rangle 3. \overline{A} \subseteq A^\circ \cup \partial A$

- $\langle 2 \rangle 1. \text{ LET: } x \in \overline{A}$

- $\langle 2 \rangle 2. \text{ ASSUME: } x \notin A^\circ$

PROVE: $x \in \partial A$

- $\langle 2 \rangle 3. x \in \overline{X - A}$

PROOF: Since $\overline{X - A} = X - A^\circ$.

- $\langle 2 \rangle 4. x \in \partial A$

PROOF: Since $\partial A = \overline{A} \cap \overline{X - A}$.

□

Proposition 12.10.4. *Let X be a topological space. Let $A \subseteq X$. Then $\partial A = \emptyset$ if and only if A is both open and closed.*

PROOF:

- $\langle 1 \rangle 1. \text{ If } \partial A = \emptyset \text{ then } A \text{ is open and closed.}$

- $\langle 2 \rangle 1. \text{ ASSUME: } \partial A = \emptyset$

- $\langle 2 \rangle 2. \overline{A} = A^\circ$

PROOF: Proposition 12.10.3.

- $\langle 2 \rangle 3. \overline{A} = A = A^\circ$

- $\langle 1 \rangle 2. \text{ If } A \text{ is open and closed then } \partial A = \emptyset.$

PROOF: If A is open and closed then

$$\begin{aligned} \partial A &= \overline{A} \cap \overline{X - A} \\ &= \overline{A} \cap (X - A^\circ) \\ &= A \cap (X - A) \\ &= \emptyset \end{aligned}$$

□

Proposition 12.10.5. *Let X be a topological space. Let $U \subseteq X$. Then U is open if and only if $\partial U = \overline{U} - U$.*

PROOF:

⟨1⟩1. If U is open then $\partial U = \overline{U} - U$

PROOF: If U is open then

$$\begin{aligned}\partial U &= \overline{U} \cap \overline{X - U} \\ &= \overline{U} \cap (X - U^\circ) \\ &= \overline{U} - U^\circ \\ &= \overline{U} - U\end{aligned}$$

⟨1⟩2. If $\partial U = \overline{U} - U$ then U is open.

⟨2⟩1. ASSUME: $\partial U = \overline{U} - U$

⟨2⟩2. $\overline{U} - U^\circ = \overline{U} - U$

⟨2⟩3. $U \subseteq U^\circ$

⟨2⟩4. $U = U^\circ$

□

12.11 Limit Points

Definition 12.11.1 (Limit Point). Let X be a topological space, $x \in X$ and $A \subseteq X$. Then x is a *limit point*, *cluster point* or *point of accumulation* of A iff every neighbourhood of x intersects $A - \{x\}$.

Proposition 12.11.2. Let X be a topological space. Let $A \subseteq X$. Let A' be the set of limit points of A . Then

$$\overline{A} = A \cup A'$$

PROOF:

⟨1⟩1. $\overline{A} \subseteq A \cup A'$

⟨2⟩1. LET: $x \in \overline{A}$

⟨2⟩2. ASSUME: $x \notin A$

PROVE: $x \in A'$

⟨2⟩3. LET: U be a neighbourhood of x .

⟨2⟩4. PICK $y \in U \cap A$

PROOF: Proposition 12.9.2.

⟨2⟩5. $y \neq x$

⟨1⟩2. $A \subseteq \overline{A}$

PROOF: Immediate from the definition of \overline{A} .

⟨1⟩3. $A' \subseteq \overline{A}$

PROOF: From Proposition 12.9.2.

□

Corollary 12.11.2.1. A set is closed if and only if it contains all its limit points.

12.12 Continuous Functions

Definition 12.12.1 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* iff, for every open set V in Y , the inverse image $f^{-1}(V)$ is open in X .

Proposition 12.12.2. *The composite of two continuous functions is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous.

$\langle 1 \rangle 2$. LET: U be open in Z .

$\langle 1 \rangle 3$. $g^{-1}(U)$ is open in Y .

$\langle 1 \rangle 4$. $\inf f(g^{-1}(U))$ is open in X .

□

Proposition 12.12.3. 1. id_X is continuous

2. If $f : X \rightarrow Y$ is continuous and $X_0 \subseteq X$ then $f \upharpoonright X_0 : X_0 \rightarrow Y$ is continuous.

3. If $f : X + Y \rightarrow Z$, then f is continuous iff $f \circ \kappa_1 : X \rightarrow Z$ and $f \circ \kappa_2 : Y \rightarrow Z$ are continuous.

4. If $f : Z \rightarrow X \times Y$, then f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Proposition 12.12.4. Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then the following are equivalent.

1. f is continuous.

2. For all $A \subseteq X$ we have $f(\overline{A}) \subseteq \overline{f(A)}$.

3. For every closed B in Y , we have $f^{-1}(B)$ is closed in X .

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: f is continuous.

$\langle 2 \rangle 2$. LET: $A \subseteq X$

$\langle 2 \rangle 3$. LET: $x \in \overline{A}$

PROVE: $f(x) \in \overline{f(A)}$

$\langle 2 \rangle 4$. LET: V be a neighbourhood of $f(x)$.

PROVE: V intersects $f(A)$.

$\langle 2 \rangle 5$. $f^{-1}(V)$ is a neighbourhood of x .

$\langle 2 \rangle 6$. PICK $y \in f^{-1}(V) \cap A$

$\langle 2 \rangle 7$. $f(y) \in V \cap f(A)$

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

$\langle 2 \rangle 1$. ASSUME: 2

$\langle 2 \rangle 2$. LET: B be closed in Y

$\langle 2 \rangle 3$. LET: $A = f^{-1}(B)$

PROVE: $\overline{A} = A$

$\langle 2 \rangle 4$. $f(A) \subseteq B$

$\langle 2 \rangle 5$. $\overline{A} \subseteq A$

$\langle 3 \rangle 1$. LET: $x \in \overline{A}$

$\langle 3 \rangle 2$. $f(x) \in B$

PROOF:

$$\begin{aligned} f(x) &\in f(\overline{A}) \\ &\subseteq \overline{f(A)} && (\langle 2 \rangle 1) \\ &\subseteq \overline{B} && (\langle 2 \rangle 4) \\ &= B && (\langle 2 \rangle 2) \end{aligned}$$

$\langle 1 \rangle 3$. $3 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: 3

$\langle 2 \rangle 2$. LET: V be open in Y .

$\langle 2 \rangle 3$. $f^{-1}(Y - V)$ is closed in X .

$\langle 2 \rangle 4$. $X - f^{-1}(V)$ is closed in X .

$\langle 2 \rangle 5$. $f^{-1}(V)$ is open in X .

□

Proposition 12.12.5. *Let X and Y be topological spaces. Any constant function $X \rightarrow Y$ is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $b \in Y$

$\langle 1 \rangle 2$. LET: $f : X \rightarrow Y$ be the constant function with value b .

$\langle 1 \rangle 3$. LET: $V \subseteq Y$ be open.

$\langle 1 \rangle 4$. $f^{-1}(V)$ is either \emptyset or X .

$\langle 1 \rangle 5$. $f^{-1}(V)$ is open.

□

Proposition 12.12.6. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{B} be a basis for Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .*

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF: Since every element of \mathcal{B} is open in Y .

$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X , then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

$\langle 2 \rangle 2$. LET: U be open in Y .

$\langle 2 \rangle 3$. LET: $x \in f^{-1}(U)$

$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq U$.

$\langle 2 \rangle 5$. $x \in f^{-1}(B) \subseteq f^{-1}(U)$

□

Proposition 12.12.7. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{S} be a subbasis for the topology on Y . Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .*

PROOF:

⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

PROOF: Immediate from definitions.

⟨1⟩2. If, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X , then f is continuous.

⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

⟨2⟩2. For all $V_1, \dots, V_n \in \mathcal{S}$ we have $f^{-1}(V_1 \cap \dots \cap V_n)$ is open in X .

PROOF: Since $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$.

⟨2⟩3. Q.E.D.

PROOF: By Proposition 12.12.6 since the set of all finite intersections of elements of \mathcal{S} forms a basis for the topology on Y .

□

Proposition 12.12.8. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Then f is continuous if and only if, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.*

PROOF:

⟨1⟩1. If f is continuous then, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.

⟨2⟩1. ASSUME: f is continuous.

⟨2⟩2. LET: $x \in \mathbb{R}$

⟨2⟩3. LET: $\epsilon > 0$

⟨2⟩4. $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open in X .

⟨2⟩5. PICK a, b such that $x \in (a, b) \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$.

⟨2⟩6. LET: $\delta = \min(x - a, b - x)$

⟨2⟩7. LET: $y \in \mathbb{R}$

⟨2⟩8. ASSUME: $|y - x| < \delta$

⟨2⟩9. $y \in (a, b)$

⟨2⟩10. $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$

⟨2⟩11. $|f(y) - f(x)| < \epsilon$

⟨1⟩2. If, for all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$, then f is continuous.

⟨2⟩1. ASSUME: For all $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.

⟨2⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.

⟨3⟩1. LET: $a \in \mathbb{R}$

⟨3⟩2. LET: $x \in f^{-1}((a, +\infty))$

⟨3⟩3. LET: $\epsilon = f(x) - a$

⟨3⟩4. PICK $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$

⟨3⟩5. $x \in (x - \delta, x + \delta) \subseteq f^{-1}((a, +\infty))$

⟨2⟩3. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.

PROOF: Similar.

⟨2⟩4. Q.E.D.

PROOF: Proposition 12.12.8.

□

Definition 12.12.9 (Continuity at a Point). Let X and Y be topological spaces.

Let $f : X \rightarrow Y$. Let $a \in X$. Then f is *continuous at a* iff, for every neighbourhood V of $f(a)$, there exists a neighbourhood U of a such that $f(U) \subseteq V$.

Proposition 12.12.10. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is continuous if and only if f is continuous at every point in X .*

$\langle 1 \rangle 1$. If f is continuous then f is continuous at every point in X .

$\langle 2 \rangle 1$. ASSUME: f is continuous.

$\langle 2 \rangle 2$. LET: $a \in X$

$\langle 2 \rangle 3$. LET: V be a neighbourhood of $f(a)$

$\langle 2 \rangle 4$. LET: $U = f^{-1}(V)$

$\langle 2 \rangle 5$. U is a neighbourhood of a .

$\langle 2 \rangle 6$. $f(U) \subseteq V$

$\langle 1 \rangle 2$. If f is continuous at every point in X then f is continuous.

$\langle 2 \rangle 1$. ASSUME: f is continuous at every point in X .

$\langle 2 \rangle 2$. LET: V be open in Y .

$\langle 2 \rangle 3$. LET: $x \in f^{-1}(V)$

$\langle 2 \rangle 4$. V is a neighbourhood of $f(x)$

$\langle 2 \rangle 5$. PICK a neighbourhood U of x such that $f(U) \subseteq V$

$\langle 2 \rangle 6$. $x \in U \subseteq f^{-1}(V)$

□

Definition 12.12.11 (Homeomorphism). Let X and Y be topological spaces. A *homeomorphism* between X and Y is a bijection $f : X \approx Y$ such that f and f^{-1} are continuous.

Proposition 12.12.12. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is a homeomorphism iff f is bijective and, for all $U \subseteq X$, we have $f(U)$ is open if and only if U is open.*

PROOF: Immediate from definitions. □

Definition 12.12.13 (Topological Property). A property P of topological spaces is a *topological* property iff, for any topological spaces X and Y , if $P[X]$ and $X \cong Y$ then $P[Y]$.

Definition 12.12.14 (Retraction). Let X be a topological space and A a subspace of X . A continuous function $\rho : X \rightarrow A$ is a *retraction* iff $\rho \upharpoonright A = \text{id}_A$. We say A is a *retract* of X iff there exists a retraction.

Definition 12.12.15. Let **Top** be the category of small topological spaces and continuous functions.

Proposition 12.12.16. \emptyset is initial in **Top**.

Proposition 12.12.17. 1 is terminal in **Top**.

Forgetful functor **Top** \rightarrow **Set**.

Basepoint preserving continuous functor.

Proposition 12.12.18. *Let (X, \mathcal{T}) be a topological space. Let S be the Sierpiński two-point space. Define $\Phi : \mathcal{T} \rightarrow \mathbf{Top}[X, S]$ by $\Phi(U)(x) = 1$ iff $x \in U$. Then Φ is a bijection.*

PROOF:

$\langle 1 \rangle 1$. For all $U \in \mathcal{T}$ we have $\Phi(U)$ is continuous.

$\langle 2 \rangle 1$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 2$. $\Phi(U)(\{1\})$ is open.

PROOF: Since $\Phi(U)(\{1\}) = U$.

$\langle 1 \rangle 2$. Φ is injective.

PROOF: If $\Phi(U) = \Phi(V)$ then we have $\forall x (x \in U \Leftrightarrow \Phi(U)(x) = 1 \Leftrightarrow \Phi(V)(x) = 1 \Leftrightarrow x \in V)$.

$\langle 1 \rangle 3$. Φ is surjective.

PROOF: Given $f : X \rightarrow S$ continuous we have $\Phi(f^{-1}(1)) = f$.

□

12.12.1 Paths

Definition 12.12.19 (Path). A *path* in a topological space X is a continuous function $[0, 1] \rightarrow X$.

12.12.2 Loops

Definition 12.12.20 (Loop). A *loop* in a topological space X is a path $\alpha : [0, 1] \rightarrow X$ such that $\alpha(0) = \alpha(1)$.

12.13 Convergence

Definition 12.13.1 (Convergence). Let X be a topological space. Let (x_n) be a sequence in X . A point $a \in X$ is a *limit* of the sequence iff, for every neighbourhood U of a , there exists n_0 such that $\forall n \geq n_0. x_n \in U$.

Proposition 12.13.2. *If $f : X \rightarrow Y$ is continuous and $x_n \rightarrow l$ in X then $f(x_n) \rightarrow f(l)$ in Y .*

Example 12.13.3. The converse does not hold.

Let X be the set of all continuous functions $[0, 1] \rightarrow [-1, 1]$ under the product topology. Let $i : X \rightarrow L^2([0, 1])$ be the inclusion.

If $f_n \rightarrow f$ then $i(f_n) \rightarrow i(f)$ — Lebesgue convergence theorem.

We prove that i is not continuous.

Assume for a contradiction i is continuous. Choose a neighbourhood K of 0 in X such that $\forall \phi \in K. \int \phi^2 < 1/2$. Let $K = \prod_{\lambda \in [0, 1]} U_\lambda$ where $U_\lambda = [-1, 1]$ except for $\lambda = \lambda_1, \dots, \lambda_n$. Let ϕ be the function that is 0 at $\lambda_1, \dots, \lambda_n$ and 1 everywhere else. Then $\phi \in K$ but $\int \phi^2 = 1$.

Proposition 12.13.4. *The converse does hold for first countable spaces. If $f : X \rightarrow Y$ where X is first countable, and Y is a topological space, and whenever $x_n \rightarrow x$ then $f(x_n) \rightarrow f(x)$, then f is continuous.*

Proposition 12.13.5. *If (s_n) is an increasing sequence of real numbers bounded above, then (s_n) converges.*

PROOF:

$\langle 1 \rangle 1$. LET: s be the supremum of $\{s_n : n \in \mathbb{N}\}$.

PROVE: $s_n \rightarrow s$ as $n \rightarrow \infty$.

$\langle 1 \rangle 2$. LET: $\epsilon > 0$

$\langle 1 \rangle 3$. PICK N such that $s_N > s - \epsilon$.

$\langle 1 \rangle 4$. $\forall n \geq N. s - \epsilon \leq s_n \leq s$

$\langle 1 \rangle 5$. $\forall n \geq N. |s_n - s| < \epsilon$

□

12.13.1 Closure

Proposition 12.13.6. *Let X be a topological space. Let $A \subseteq X$. Let (a_n) be a sequence in A and $l \in X$. If $a_n \rightarrow l$ as $n \rightarrow \infty$, then $l \in \overline{A}$.*

PROOF:

$\langle 1 \rangle 1$. LET: U be a neighbourhood of l .

$\langle 1 \rangle 2$. PICK N such that $\forall n \geq N. a_n \in U$

$\langle 1 \rangle 3$. $a_N \in A \cap U$

□

12.13.2 Continuous Functions

Proposition 12.13.7. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$ be continuous. Let $x_n \rightarrow x$ as $n \rightarrow \infty$ in X . Then $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$ in Y .*

PROOF:

$\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(x)$.

$\langle 1 \rangle 2$. PICK N such that $\forall n \geq N. x_n \in f^{-1}(V)$

$\langle 1 \rangle 3$. $\forall n \geq N. f(x_n) \in V$

□

12.13.3 Infinite Series

Definition 12.13.8 (Series). Let (a_n) be a sequence of real numbers. We say that the infinite series $\sum_{n=0}^{\infty} a_n$ converges to s , and write

$$\sum_{n=0}^{\infty} a_n = s$$

iff $\sum_{n=0}^N a_n \rightarrow s$ as $N \rightarrow \infty$.

12.14 Strong Continuity

Definition 12.14.1 (Strong Continuity). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is *strongly continuous* iff, for every $V \subseteq Y$, we have V is open in Y if and only if $f^{-1}(V)$ is open in X .

Proposition 12.14.2. Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is strongly continuous if and only if, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X .

PROOF:

$$\begin{aligned} f \text{ is continuous} &\Leftrightarrow \forall V \subseteq Y (V \text{ is open in } Y \Leftrightarrow f^{-1}(V) \text{ is open in } X) \\ &\Leftrightarrow \forall C \subseteq Y (Y - C \text{ is open in } Y \Leftrightarrow f^{-1}(Y - C) \text{ is open in } X) \\ &\Leftrightarrow \forall C \subseteq Y (C \text{ is closed in } Y \Leftrightarrow f^{-1}(C) \text{ is closed in } X) \quad \square \end{aligned}$$

12.15 Subspaces

Definition 12.15.1 (Subspace). Let X be a topological space, Y a set, and $f : Y \rightarrow X$. The *subspace topology* on Y induced by f is $\mathcal{T} = \{f^{-1}(U) : U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

- $\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
PROOF: Since $\bigcup \mathcal{U} = f^{-1}(\bigcup \{V : f^{-1}(V) \in \mathcal{U}\})$.
 $\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF: Since $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.
 $\langle 1 \rangle 3$. $Y \in \mathcal{T}$
PROOF: Since $Y = f^{-1}(X)$.
 \square

Proposition 12.15.2. Let X be a topological space, Y a set and $f : Y \rightarrow X$ a function. Then the subspace topology on Y is the coarsest topology such that f is continuous.

PROOF: Immediate from definition. \square

Proposition 12.15.3 (Local Formulation of Continuity). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let \mathcal{U} be a set of open subspaces of X such that $X = \bigcup \mathcal{U}$. If $f|_U : U \rightarrow Y$ is continuous for all $U \in \mathcal{U}$, then f is continuous.

PROOF:

- $\langle 1 \rangle 1$. LET: $x \in X$
PROVE: f is continuous at x .
 $\langle 1 \rangle 2$. LET: V be a neighbourhood of $f(x)$.
 $\langle 1 \rangle 3$. PICK $U \in \mathcal{U}$ such that $x \in U$.
 $\langle 1 \rangle 4$. PICK W open in U such that $x \in W$ and $f(W) \subseteq V$.

⟨1⟩5. W is open in X .

□

Definition 12.15.4 (Unit Circle). The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ as a subspace of \mathbb{R}^2 .

Example 12.15.5. The *unit sphere* S^2 is $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ as a subspace of \mathbb{R}^3 .

Theorem 12.15.6. Let X be a topological space and (Y, i) a subset of X . Then the subspace topology on Y is the unique topology such that, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f : Z \rightarrow X$ is continuous.

PROOF:

⟨1⟩1. If we give Y the subspace topology then, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.

⟨2⟩1. Given Y the subspace topology.

⟨2⟩2. LET: Z be a topological space.

⟨2⟩3. LET: $f : Z \rightarrow Y$

⟨2⟩4. If f is continuous then $i \circ f$ is continuous.

PROOF: Since i is continuous.

⟨2⟩5. If $i \circ f$ is continuous then f is continuous.

⟨3⟩1. ASSUME: $i \circ f$ is continuous.

⟨3⟩2. LET: U be open in Y .

⟨3⟩3. $f^{-1}(i^{-1}(i(U)))$ is open in Z .

⟨3⟩4. $f^{-1}(U)$ is open in Z .

⟨1⟩2. If, for every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.

⟨2⟩1. ASSUME: For every topological space Z and function $f : Z \rightarrow Y$, we have f is continuous if and only if $i \circ f$ is continuous.

⟨2⟩2. i is continuous.

⟨2⟩3. For every open set U in X , we have $i^{-1}(U)$ is open in Y

⟨2⟩4. LET: Z be the set Y under the subspace topology and $f : Z \rightarrow Y$ the identity function.

⟨2⟩5. $i \circ f$ is continuous.

⟨2⟩6. f is continuous.

⟨2⟩7. Every set open in Y is open in Z .

□

Proposition 12.15.7. Let X be a topological space, Y a subspace of X and $U \subseteq Y$. If Y is open in X and U is open in Y then U is open in X .

PROOF:

⟨1⟩1. PICK V open in X such that $U = V \cap Y$

⟨1⟩2. U is open in X .

PROOF: It is the intersection of two open sets in X .

□

Proposition 12.15.8. *Let Y be a subspace of X and $A \subseteq Y$. Then the subspace topology on A as a subspace of Y is the same as the subspace topology on A as a subspace of X .*

PROOF:

⟨1⟩1. LET: \mathcal{T}_Y be the subspace topology on A as a subspace of Y .

⟨1⟩2. LET: \mathcal{T}_X be the subspace topology on A as a subspace of X .

⟨1⟩3. LET: $U \subseteq A$

⟨1⟩4. $U \in \mathcal{T}_Y \Leftrightarrow U \in \mathcal{T}_X$

PROOF:

$$U \in \mathcal{T}_Y \Leftrightarrow \exists V \text{ open in } Y. U = V \cap A$$

$$\Leftrightarrow \exists V. \exists W \text{ open in } X. (V = Y \cap W \wedge U = V \cap A)$$

$$\Leftrightarrow \exists W \text{ open in } X. U = Y \cap W \cap A$$

$$\Leftrightarrow \exists W \text{ open in } X. U = W \cap A$$

$$\Leftrightarrow U \in \mathcal{T}_X$$

□

Proposition 12.15.9. *Let X be a topological space. Let \mathcal{B} be a basis for the topology on X . Let $Y \subseteq X$. Then $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the topology on Y .*

PROOF:

⟨1⟩1. Every element of \mathcal{B}' is open.

PROOF: For all $B \in \mathcal{B}$, we have B is open in X , so $B \cap Y$ is open in Y .

⟨1⟩2. For any open set V in Y and $y \in V$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq V$

⟨2⟩1. LET: V be open in Y .

⟨2⟩2. LET: $y \in V$

⟨2⟩3. PICK U open in X such that $V = U \cap Y$.

⟨2⟩4. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$

⟨2⟩5. $B \cap Y \in \mathcal{B}'$ and $y \in B \cap Y \subseteq V$

□

12.15.1 Product Topology

Proposition 12.15.10. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology on $\prod_{i \in I} Y_i$ as a subspace of $\prod_{i \in I} X_i$.*

PROOF:

⟨1⟩1. Given $\prod_{i \in I} Y_i$ the subspace topology.

⟨1⟩2. LET: $\iota : \prod_{i \in I} Y_i$ be the inclusion.

⟨1⟩3. LET: Z be any topological space.

⟨1⟩4. LET: $f : Z \rightarrow \prod_{i \in I} Y_i$

⟨1⟩5. f is continuous if and only if, for all $i \in I$, we have $\pi_i \circ f$ is continuous.

PROOF:

f is continuous $\Leftrightarrow \iota \circ f : Z \rightarrow \prod_{i \in I} X_i$ is continuous (Theorem 12.15.6)

$\Leftrightarrow \forall i \in I. \pi_i \circ \iota \circ f : Z \rightarrow X_i$ is continuous (Theorem 12.20.4)

$\Leftrightarrow \forall i \in I. \iota_i \circ \pi_i \circ f : Z \rightarrow X_i$ is continuous

$\Leftrightarrow \forall i \in I. \pi_i \circ f : Z \rightarrow Y_i$ is continuous (Theorem 12.15.6)

where ι_i is the inclusion $Y_i \rightarrow X_i$.

□

12.16 Embedding

Definition 12.16.1 (Embedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *embedding* iff f is injective and the topology on X is the subspace induced by f .

Proposition 12.16.2. *Every embedding is continuous.*

PROOF: Theorem 12.15.6. □

Proposition 12.16.3. *Let X and Y be topological spaces. Let $b \in Y$. The function $\kappa : X \rightarrow X \times Y$ that maps x to (x, b) is an embedding.*

PROOF:

⟨1⟩1. For all U open in X , we have $U = \kappa^{-1}(V)$ for some V open in $X \times Y$.

PROOF: Take $V = U \times Y$.

⟨1⟩2. For all V open in $X \times Y$ we have $\kappa^{-1}(V)$ is open in X .

PROOF: Since $\pi_1 \circ \kappa = \text{id}_X$ and $\pi_2 \circ \kappa$ (which is the constant function with value b) are both continuous, hence κ is continuous.

□

12.17 Open Maps

Definition 12.17.1 (Open Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *open map* iff, for all U open in X , we have $f(U)$ is open in Y .

Proposition 12.17.2. *Let X and Y be topological spaces. The projections $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are open maps.*

PROOF:

⟨1⟩1. π_1 is an open map.

⟨2⟩1. LET: U be open in $X \times Y$.

⟨2⟩2. LET: $x \in \pi_1(U)$

⟨2⟩3. PICK y such that $(x, y) \in U$

⟨2⟩4. PICK V and W open in X and Y respectively such that $(x, y) \in V \times W \subseteq U$

- $\langle 2 \rangle 5. x \in V \subseteq \pi_1(U)$
 $\langle 1 \rangle 2. \pi_2$ is an open map.

PROOF: Similar.

□

12.17.1 Subspaces

Proposition 12.17.3. *Let X and Y be topological spaces. Let $p : X \rightarrow Y$ be an open map. Let A be an open set in X . Then $p|_A : A \rightarrow p(A)$ is an open map.*

PROOF:

- $\langle 1 \rangle 1.$ LET: U be open in A .
 $\langle 1 \rangle 2.$ U is open in X .

PROOF: Proposition 12.15.7.

- $\langle 1 \rangle 3.$ $p(U)$ is open in Y .
 $\langle 1 \rangle 4.$ $p(U)$ is open in $p(A)$.

PROOF: Since $p(U) = p(U) \cap p(A)$.

□

12.18 Locally Finite

Definition 12.18.1 (Locally Finite). Let X be a topological space. Let $\{A_i\}_{i \in I}$ be a family of subsets of X . Then $\{A_i\}_{i \in I}$ is *locally finite* iff, for every $x \in X$, there exist only finitely many $i \in I$ such that $x \in A_i$.

Theorem 12.18.2 (Pasting Lemma). *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Let $\{A_i\}_{i \in I}$ be a locally finite family of closed subspaces of X such that $X = \bigcup_{i \in I} A_i$. If $f|_{A_i} : A_i \rightarrow Y$ is continuous for all $i \in I$, then f is continuous.*

PROOF:

- $\langle 1 \rangle 1.$ LET: B be closed in Y .
 $\langle 1 \rangle 2.$ LET: $A = f^{-1}(B)$
 PROVE: A is closed in X .
 $\langle 1 \rangle 3.$ $A = \bigcup_{i \in I} f|_{A_i}^{-1}(B)$
 $\langle 1 \rangle 4.$ LET: $x \in X - A$
 PROVE: There exists a neighbourhood U' of x such that $U' \subseteq X - A$.
 $\langle 1 \rangle 5.$ PICK a neighbourhood U of x such that U intersects A_i for only finitely many $i \in I$.
 $\langle 1 \rangle 6.$ LET: i_1, \dots, i_n be the elements of I such that U intersects A_{i_1}, \dots, A_{i_n} .
 $\langle 1 \rangle 7.$ For $j = 1, \dots, n$,
 LET: $S_j = f|_{A_{i_j}}^{-1}(B)$
 $\langle 1 \rangle 8.$ For $j = 1, \dots, n$, we have S_j is closed in X .
 $\langle 1 \rangle 9.$ For $j = 1, \dots, n$, we have $x \notin S_j$.
 $\langle 1 \rangle 10.$ LET: $U' = U \cap \bigcap_{j=1}^n (X - S_j)$

⟨1⟩11. U' is a neighbourhood of x .

⟨1⟩12. $U' \subseteq X - A$

□

12.19 Closed Maps

Definition 12.19.1 (Closed Map). Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is a *closed map* iff, for every closed set C in X , we have $f(C)$ is closed in Y .

12.20 Product Topology

Definition 12.20.1 (Product Topology). Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. The *product topology* on $\prod_{\lambda \in \Lambda} X_\lambda$ is the coarsest topology such that every projection onto X_λ is continuous.

12.20.1 Closed Sets

Proposition 12.20.2. Let X and Y be topological spaces. Let A be a closed set in X and B a closed set in Y . Then $A \times B$ is closed in $X \times Y$.

PROOF: Since $(X \times Y) - (A \times B) = ((X - A) \times Y) \cup (X \times (Y - B))$. □

Proposition 12.20.3. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The product topology on $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis $\mathcal{B} = \{\prod_{\alpha \in A} U_\alpha : \text{for all } \alpha \in A, U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in A\}$.

PROOF:

⟨1⟩1. \mathcal{B} is a basis for a topology.

⟨1⟩2. LET: \mathcal{T} be the topology generated by \mathcal{B} .

⟨1⟩3. LET: \mathcal{T}_p be the product topology.

⟨1⟩4. $\mathcal{T} \subseteq \mathcal{T}_p$

⟨2⟩1. LET: $B \in \mathcal{B}$

⟨2⟩2. LET: $B = \prod_{\alpha \in A} U_\alpha$ with each U_α open in X_α and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$

⟨2⟩3. $B = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$

⟨2⟩4. $B \in \mathcal{T}_p$

⟨1⟩5. $\mathcal{T}_p \subseteq \mathcal{T}$

⟨2⟩1. For every $\alpha \in A$ we have π_α is continuous.

PROOF: Since $\pi^{-1}(U)$ is open for every U open in X_α .

□

Theorem 12.20.4. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Then the product topology on $\prod_{\alpha \in A} X_\alpha$ is the unique topology such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f : Z \rightarrow X_\alpha$ is continuous.

PROOF:

- ⟨1⟩1. If we give $\prod_{\alpha \in A} X_\alpha$ the product topology, then for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
- ⟨2⟩1. Give $\prod_{\alpha \in A} X_\alpha$ the product topology.
- ⟨2⟩2. LET: Z be a topological space.
- ⟨2⟩3. LET: $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$
- ⟨2⟩4. If f is continuous then, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
- PROOF: Since the composite of two continuous functions is continuous.
- ⟨2⟩5. If, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then f is continuous.
- ⟨3⟩1. ASSUME: For all $\alpha \in A$ we have $\pi_\alpha \circ f$ is continuous.
- ⟨3⟩2. LET: $\{U_\alpha\}_{\alpha \in A}$ be a family with U_α open in X_α such that $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \dots, \alpha_n$.
- ⟨3⟩3. For all α we have $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ is open in Z .
- ⟨3⟩4. $f^{-1}(\prod_\alpha U_\alpha)$ is open in Z
- PROOF: Since $f^{-1}(\prod_\alpha U_\alpha) = f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \dots \cap f^{-1}(\pi_{\alpha_n}^{-1}(U_{\alpha_n}))$.
- ⟨1⟩2. If \mathcal{T} is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous, then \mathcal{T} is the product topology.
- ⟨2⟩1. ASSUME: \mathcal{T} is a topology on $\prod_{\alpha \in A} X_\alpha$ such that, for every topological space Z and function $f : Z \rightarrow \prod_{\alpha \in A} X_\alpha$, we have f is continuous if and only if, for all $\alpha \in A$, we have $\pi_\alpha \circ f$ is continuous.
- ⟨2⟩2. LET: \mathcal{T}_p be the product topology.
- ⟨2⟩3. $\mathcal{T} \subseteq \mathcal{T}_p$
- ⟨3⟩1. LET: $Z = (\prod_\alpha X_\alpha, \mathcal{T}_p)$
- ⟨3⟩2. LET: $f : Z \rightarrow \prod_\alpha X_\alpha$ be the identity function
- ⟨3⟩3. For all α we have $\pi_\alpha \circ f$ is continuous.
- ⟨3⟩4. f is continuous.
- PROOF: ⟨2⟩1
- ⟨3⟩5. Every set open in \mathcal{T} is open in \mathcal{T}_p
- ⟨2⟩4. $\mathcal{T}_p \subseteq \mathcal{T}$
- ⟨3⟩1. $\text{id}_{\prod_\alpha X_\alpha}$ is continuous.
- ⟨3⟩2. For all α we have π_α is continuous.
- PROOF: ⟨2⟩1
- ⟨3⟩3. $\mathcal{T}_p \subseteq \mathcal{T}$
- PROOF: Since \mathcal{T}_p is the coarsest topology such that every π_α is continuous.

□

Example 12.20.5. It is not true that, for any function $f : \prod_{\alpha \in A} X_\alpha \rightarrow Y$, if f is continuous in every variable separately then f is continuous.

Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

Then f is continuous in x and in y , but is not continuous.

Proposition 12.20.6. *Let $\{X_i\}_{i \in I}$ be a nonempty family of topological spaces. The product topology on $\prod_{i \in I} X_i$ is the topology generated by the subbasis $\{\pi_i^{-1}(U) : i \in I, U \text{ is open in } X_i\}$.*

PROOF:

$\langle 1 \rangle 1.$ $\{\pi_i^{-1}(U) : i \in I, U \text{ is open in } X_i\}$ is a subbasis for a topology on $\prod_{i \in I} X_i$.

$\langle 2 \rangle 1.$ PICK $i_0 \in I$

$\langle 2 \rangle 2.$ $\prod_{i \in I} X_i = \pi_{i_0}^{-1}(X_{i_0})$

$\langle 1 \rangle 2.$ The topology generated by this subbasis is the product topology.

PROOF: Since the basis in Proposition 12.20.3 is the set of all finite intersections of elements of this subbasis.

□

12.20.2 Closure

Proposition 12.20.7. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

$\langle 1 \rangle 1.$ $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

$\langle 2 \rangle 1.$ LET: $x \in \prod_{i \in I} \overline{A_i}$

$\langle 2 \rangle 2.$ For any family $\{U_i\}_{i \in I}$ where each U_i is open in X_i , and $U_i = X_i$ for all but finitely many $i \in I$, if $x \in \prod_{i \in I} U_i$ then $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$.

$\langle 3 \rangle 1.$ LET: $\{U_i\}_{i \in I}$ be a family where each U_i is open in X_i , and $U_i = X_i$ for all but finitely many i .

$\langle 3 \rangle 2.$ ASSUME: $x \in \prod_{i \in I} \overline{A_i}$

$\langle 3 \rangle 3.$ For all $i \in I$ we have U_i intersects A_i

PROOF: Since $\pi_i(x) \in \overline{A_i}$ and U_i is a neighbourhood of $\pi_i(x)$.

$\langle 3 \rangle 4.$ $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$

$\langle 2 \rangle 3.$ $x \in \overline{\prod_{i \in I} A_i}$

PROOF: Proposition 12.9.9.

$\langle 1 \rangle 2.$ $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

$\langle 2 \rangle 1.$ LET: $x \in \overline{\prod_{i \in I} A_i}$

$\langle 2 \rangle 2.$ LET: $i \in I$

PROVE: $\pi_i(x) \in \overline{A_i}$

$\langle 2 \rangle 3.$ LET: U be a neighbourhood of $\pi_i(x)$ in X_i

$\langle 2 \rangle 4.$ $\pi_i^{-1}(U)$ is a neighbourhood of x in $\prod_{i \in I} X_i$

$\langle 2 \rangle 5.$ PICK $y \in \pi_i^{-1}(U) \cap \prod_{i \in I} A_i$

$\langle 2 \rangle 6.$ $\pi_i(y) \in U \cap A_i$

□

12.20.3 Convergence

Proposition 12.20.8. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (x_n) be a sequence of points in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $x_n \rightarrow l$ as $n \rightarrow \infty$ if*

and only if, for all $i \in I$, we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.

PROOF:

$\langle 1 \rangle 1$. If $x_n \rightarrow l$ as $n \rightarrow \infty$ then, for all $i \in I$, we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.

PROOF: Proposition 12.13.2.

$\langle 1 \rangle 2$. If, for all $i \in I$, we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$, then $x_n \rightarrow l$ as $n \rightarrow \infty$.

$\langle 2 \rangle 1$. ASSUME: For all $i \in I$ we have $\pi_i(x_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.

$\langle 2 \rangle 2$. LET: U be a neighbourhood of l .

$\langle 2 \rangle 3$. PICK $i_1, \dots, i_n \in I$ and open sets U_j in X_{i_j} for $j = 1, \dots, n$ such that $l \in \pi_{i_1}^{-1}(U_1) \cap \dots \cap \pi_{i_n}^{-1}(U_n) \subseteq U$

$\langle 2 \rangle 4$. For $j = 1, \dots, n$ we have $\pi_{i_j}(l) \in U_j$

$\langle 2 \rangle 5$. PICK N such that, for all $m \geq N$, we have $\pi_{i_j}(x_m) \in U_j$

$\langle 2 \rangle 6$. $\forall m \geq N. x_m \in U$

□

12.21 Quotient Spaces

Definition 12.21.1 (Quotient Topology). Let X be a topological space, S a set, and $\pi : X \twoheadrightarrow S$ be a surjection. The *quotient topology* on S induced by π is $\mathcal{T} = \{U \in \mathcal{P}S : \pi^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.

PROOF: Since $\pi^{-1}(\bigcup \mathcal{U}) = \bigcup \{\pi^{-1}(U) : U \in \mathcal{U}\}$.

$\langle 1 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

PROOF: Since $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$.

$\langle 1 \rangle 3$. $X \in \mathcal{T}$

PROOF: Since $X = \pi^{-1}(Y)$.

□

Proposition 12.21.2. Let X be a topological space, S a set and $\pi : X \twoheadrightarrow S$ a surjection. Then the quotient topology on S is the finest topology such that π is continuous.

PROOF: Immediate from definitions. □

Theorem 12.21.3. Let X be a topological space, let S be a set, and let $\pi : X \twoheadrightarrow S$ be surjective. Then the quotient topology on S is the unique topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.

PROOF:

$\langle 1 \rangle 1$. If S is given the quotient topology, then for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.

$\langle 2 \rangle 1$. Give S the quotient topology.

- $\langle 2 \rangle 2$. LET: Z be a topological space.
 $\langle 2 \rangle 3$. LET: $f : S \rightarrow Z$
 $\langle 2 \rangle 4$. If f is continuous then $f \circ \pi$ is continuous.
 PROOF: The composite of two continuous functions is continuous.
 $\langle 2 \rangle 5$. If $f \circ \pi$ is continuous then f is continuous.
 $\langle 3 \rangle 1$. ASSUME: $f \circ \pi$ is continuous.
 $\langle 3 \rangle 2$. LET: U be open in Z .
 $\langle 3 \rangle 3$. $\pi^{-1}(f^{-1}(U))$ is open in X .
 $\langle 3 \rangle 4$. $f^{-1}(U)$ is open in S .
 $\langle 1 \rangle 2$. If S is given a topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous, then that topology is the quotient topology.
 $\langle 2 \rangle 1$. Give S a topology such that, for every topological space Z and function $f : S \rightarrow Z$, we have f is continuous if and only if $f \circ \pi$ is continuous.
 $\langle 2 \rangle 2$. LET: $U \subseteq S$
 $\langle 2 \rangle 3$. If $\pi^{-1}(U)$ is open in X then U is open in S .
 $\langle 3 \rangle 1$. LET: Z be S under the quotient topology induced by π .
 $\langle 3 \rangle 2$. LET: $f : S \rightarrow Z$ be the identity function.
 $\langle 3 \rangle 3$. $f \circ \pi$ is continuous.
 $\langle 3 \rangle 4$. f is continuous.
 PROOF: $\langle 2 \rangle 1$
 $\langle 3 \rangle 5$. U is open in Z .
 $\langle 3 \rangle 6$. U is open in X .
 $\langle 2 \rangle 4$. If U is open in S then $\pi^{-1}(U)$ is open in X .
 PROOF: Since π is continuous (taking $Z = S$ and $f = \text{id}_S$ in $\langle 2 \rangle 1$).

□

12.21.1 Quotient Maps

Definition 12.21.4 (Quotient Map). Let X and S be topological spaces and $\pi : X \rightarrow S$. Then π is a *quotient map* iff π is surjective and the topology on S is the quotient topology induced by π .

Proposition 12.21.5. Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Then f is a quotient map if and only if f is surjective and strongly continuous.

PROOF: Immediate from definition. □

Proposition 12.21.6. Let X and Y be topological spaces. Let $p : X \twoheadrightarrow Y$ be surjective. Then the following are equivalent.

1. p is a quotient map.
2. p is continuous and maps saturated open sets to open sets.
3. p is continuous and maps saturated closed sets to closed sets.

PROOF:

- ⟨1⟩1. $1 \Rightarrow 2$
 - ⟨2⟩1. ASSUME: p is a quotient map.
 - ⟨2⟩2. p is continuous.
 - ⟨2⟩3. p maps saturated open sets to open sets.
 - ⟨3⟩1. LET: $U \subseteq X$ be a saturated open set.
 - ⟨3⟩2. $p^{-1}(p(U)) = U$
 - ⟨3⟩3. $p^{-1}(p(U))$ is open in X .
 - ⟨3⟩4. $p(U)$ is open in Y .
- ⟨1⟩2. $2 \Rightarrow 3$
 - ⟨2⟩1. ASSUME: p is continuous and maps saturated open sets to open sets.
 - ⟨2⟩2. LET: C be a saturated closed set in X .
 - ⟨2⟩3. $X - C$ is a saturated open set.
 - ⟨2⟩4. $Y - p(C)$ is open.
 - ⟨2⟩5. $p(C)$ is closed.
- ⟨1⟩3. $3 \Rightarrow 1$
 - ⟨2⟩1. ASSUME: p is continuous and maps closed sets to closed sets.
 - ⟨2⟩2. LET: $C \subseteq Y$
 - ⟨2⟩3. ASSUME: $p^{-1}(C)$ is closed in X .
PROVE: C is closed in Y .
 - ⟨2⟩4. $p^{-1}(C)$ is saturated.
 - ⟨2⟩5. $p(p^{-1}(C))$ is closed.
 - ⟨2⟩6. C is closed.

□

Corollary 12.21.6.1. *Let X and Y be topological spaces. Let $p : X \rightarrow Y$ be continuous and surjective. If p is either an open map or a closed map, then p is a quotient map.*

Example 12.21.7. The converse does not hold.

Let $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \vee y = 0\}$. Then the first projection $\pi_1 : A \rightarrow \mathbb{R}$ is a quotient map that is neither an open map nor a closed map.

PROOF:

- ⟨1⟩1. π_1 is a quotient map.
 - ⟨2⟩1. LET: $U \subseteq \mathbb{R}$
 - ⟨2⟩2. If U is open then $\pi_1^{-1}(U)$ is open.
PROOF: Since $\pi_1^{-1}(U) = (U \times \mathbb{R}) \cap A$.
 - ⟨2⟩3. If $\pi_1^{-1}(U)$ is open then U is open.
 - ⟨3⟩1. ASSUME: $\pi_1^{-1}(U)$ is open.
 - ⟨3⟩2. LET: $x \in U$
 - ⟨3⟩3. $(x, 0) \in \pi_1^{-1}(U)$
 - ⟨3⟩4. PICK open neighbourhoods V of x and W of 0 such that $V \times W \subseteq \pi_1^{-1}(U)$
 - ⟨3⟩5. $V \subseteq U$
 - PROOF: For all $x' \in V$ we have $(x', 0) \in V \times W \subseteq \pi_1^{-1}(U)$.
- ⟨1⟩2. π_1 is not an open map.
 - PROOF: $\pi_1(((-1, 1) \times (1, 2)) \cap A) = [0, 1)$ which is not open in \mathbb{R} .

⟨1⟩3. π_1 is not a closed map.

PROOF: $\pi_1(\{(x, 1/x) \in \mathbb{R}^2 : x > 0\}) = (0, +\infty)$ is not closed in \mathbb{R} .

□

Corollary 12.21.7.1. *Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be families of topological spaces and $p_i : X_i \rightarrow Y_i$ for all $i \in I$.*

1. *If every p_i is an open quotient map, then $\prod_{i \in I} p_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$ is an open quotient map.*
2. *If every p_i is a closed quotient map, then $\prod_{i \in I} p_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$ is a closed quotient map.*

Example 12.21.8. The product of two quotient maps is not necessarily a quotient map.

Let Y be the quotient space of \mathbb{R}_K obtained by collapsing the set K to a point. Let $p : \mathbb{R}_K \rightarrow Y$ be the quotient map. Then $q \times q : \mathbb{R}_K^2 \rightarrow Y^2$ is not a quotient map.

PROOF:

⟨1⟩1. LET: $\Delta = \{(y, y) : y \in Y\}$

⟨1⟩2. Y is not Hausdorff.

⟨2⟩1. LET: $*_K \in Y$ be the point such that $q(K) = \{*_K\}$

⟨2⟩2. ASSUME: for a contradiction U and V are disjoint neighbourhoods of 0 and $*_K$

⟨2⟩3. $q^{-1}(U)$ and $q^{-1}(V)$ are disjoint open sets with $0 \in q^{-1}(U)$ and $K \subseteq q^{-1}(V)$

⟨2⟩4. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩3. Δ is not closed in Y^2 .

⟨1⟩4. $(q \times q)^{-1}(\Delta)$ is closed in \mathbb{R}_K^2

PROOF: It is $\{(x, x) : x \in \mathbb{R}\} \cup K^2$.

□

Proposition 12.21.9. *Let $\pi : X \rightarrow S$ be a quotient map. Let Z be a topological space. Let $f : X \rightarrow Z$ be continuous. Then there exists a continuous map $g : S \rightarrow Z$ such that $f = g \circ \pi$ if and only if, for all $s \in S$, we have f is constant on $\pi^{-1}(s)$.*

PROOF: From Theorem 12.21.3. □

Proposition 12.21.10. *Let Z be a topological space. Define $\pi : [0, 1] \rightarrow S^1$ by $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$. Given any continuous function $f : S^1 \rightarrow Z$, we have $f \circ \pi$ is a loop in Z . This defines a bijection between $\mathbf{Top}[S^1, Z]$ and the set of loops in Z .*

PROOF: Since π is a quotient map. □

Definition 12.21.11 (Projective Space). The *projective space* \mathbb{RP}^n is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by \sim where $x \sim \lambda x$ for all $x \in \mathbb{R}^{n+1} - \{0\}$ and $\lambda \in \mathbb{R}$.

Definition 12.21.12 (Torus). The *torus* T is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$.

Definition 12.21.13 (Möbius Band). The *Möbius band* is the quotient of $[0, 1]^2$ by \sim where $(0, y) \sim (1, 1 - y)$.

Definition 12.21.14 (Klein Bottle). The *Klein bottle* is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$.

Proposition 12.21.15. \mathbb{RP}^2 is the quotient of $[0, 1]^2$ by \sim where $(x, 0) \sim (1 - x, 1)$ and $(0, y) \sim (1, 1 - y)$.

PROOF:TODO

Example 12.21.16. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and $\{Y_i\}_{i \in I}$ a family of sets. Let $q_i : X_i \rightarrow Y_i$ be a surjective function for all $i \in I$. Give each Y_i the quotient topology. It is not true in general that the product topology on $\prod_{i \in I} Y_i$ is the same as the quotient topology induced by $\prod_{i \in I} q_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$.

PROOF:

- <1>1. LET: $X^* = \mathbb{R} - \mathbb{Z}_+ + \{b\}$ be the quotient space obtained from \mathbb{R} by identifying the subset \mathbb{Z}_+ to the point b .
- <1>2. LET: $p : \mathbb{R} \rightarrow X^*$ be the quotient map.
PROVE: $p \times \text{id}_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$ is not a quotient map.
- <1>3. For $n \in \mathbb{Z}_+$,
LET: $c_n = \sqrt{2}/n$
- <1>4. For $n \in \mathbb{Z}_+$,
LET: $U_n = \{(x, y) \in \mathbb{Q} \times \mathbb{R} : n - 1/4 < x < n + 1/4 \text{ and } ((y > x + c_n - n \text{ and } y > -x + c_n + n) \text{ or } (y < x + c_n - n \text{ and } y < -x + c_n + n))\}$
- <1>5. For all $n \in \mathbb{Z}_+$, U_n is open in $\mathbb{R} \times \mathbb{Q}$
- <1>6. For all $n \in \mathbb{Z}_+$ we have $\{n\} \times \mathbb{Q} \subseteq U_n$
- <1>7. LET: $U = \bigcup_{n \in \mathbb{Z}_+} U_n$
- <1>8. U is open in $\mathbb{R} \times \mathbb{Q}$.
- <1>9. U is saturated with respect to $p \times \text{id}_{\mathbb{Q}}$.
- <1>10. LET: $U' = (p \times \text{id}_{\mathbb{Q}})(U)$
- <1>11. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$.

Proposition 12.21.17. Let X and Y be topological spaces. Let \sim be an equivalence relation on X . Let $\phi : Y \rightarrow X/\sim$.

Assume that, for all $y \in Y$, there exists a neighbourhood U of y and a continuous function $\Phi : U \rightarrow X$ such that $\pi \circ \Phi = \phi|_U$. Then ϕ is continuous.

Proposition 12.21.18. A quotient of a connected space is connected.

Proposition 12.21.19. A quotient of a path connected space is path connected.

Proposition 12.21.20. Let X be a topological space and \sim an equivalence relation on X . If X/\sim is Hausdorff then every equivalence class of \sim is closed in X .

Definition 12.21.21. Let X be a topological space and $A_1, \dots, A_r \subseteq X$. Then $X/A_1, \dots, A_r$ is the quotient space of X with respect to \sim where $x \sim y$ iff $x = y$ or $\exists i(x \in A_i \wedge y \in A_i)$.

Definition 12.21.22 (Cone). Let X be a topological space. The *cone over X* is the space $(X \times [0, 1])/(X \times \{1\})$.

Definition 12.21.23 (Suspension). Let X be a topological space. The *suspension* of X is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

Definition 12.21.24 (Wedge Product). Let $x_0 \in X$ and $y_0 \in Y$. The *wedge product* $X \vee Y$ is $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ as a subspace of $X \times Y$.

Definition 12.21.25 (Smash Product). Let $x_0 \in X$ and $y_0 \in Y$. The *smash product* $X \wedge Y$ is $(X \times Y)/(X \vee Y)$.

Example 12.21.26. $D^n/S^{n-1} \cong S^n$

PROOF:

$\langle 1 \rangle 1$. LET: $\phi : D^n/S^{n-1} \rightarrow S^n$ be the function induced by the map $D^n \rightarrow S^n$ that maps the radii of D^n onto the meridians of S^n from the north to the south pole.

$\langle 1 \rangle 2$. ϕ is a bijection.

$\langle 1 \rangle 3$. ϕ is a homeomorphism.

PROOF: Since D^n/S^{n-1} is compact and S^n is Hausdorff.

□

12.22 Box Topology

Definition 12.22.1 (Box Topology). Let $\{X_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $X = \prod_{i \in I} X_i$ is the topology generated by the basis $\mathcal{B} = \{\prod_{i \in I} U_i : \{U_i\}_{i \in I} \text{ is a family with each } U_i \text{ an open set in } X_i\}$.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. $\bigcup \mathcal{B} = X$

PROOF: Since $\prod_{i \in I} X_i \in \mathcal{B}$.

$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

$\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$

$\langle 2 \rangle 2$. LET: $x \in B_1 \cap B_2$

$\langle 2 \rangle 3$. PICK a family $\{U_i\}_{i \in I}$ such that $B_1 = \prod_{i \in I} U_i$.

$\langle 2 \rangle 4$. PICK a family $\{V_i\}_{i \in I}$ such that $B_2 = \prod_{i \in I} V_i$.

$\langle 2 \rangle 5$. LET: $B_3 = \prod_{i \in I} (U_i \cap V_i)$

$\langle 2 \rangle 6$. $x \in B_3 \subseteq B_1 \cap B_2$

□

Proposition 12.22.2. *The box topology is finer than the product topology.*

PROOF: Immediate from definitions. \square

Proposition 12.22.3. *On a finite family of topological spaces, the box topology and the product topology are the same.*

PROOF: Immediate from definitions. \square

Proposition 12.22.4. *The box topology is strictly finer than the product topology on the Hilbert cube.*

PROOF: The set $\prod_{n=0}^{\infty} (0, 1/(n+1)^2)$ is open in the box topology but not in the product topology. \square

12.22.1 Bases

Proposition 12.22.5. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. For all $i \in I$, let \mathcal{B}_i be a basis for the topology on X_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i : \forall i \in I, B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} X_i$.*

PROOF:

$\langle 1 \rangle 1$. For every family $\{B_i\}_{i \in I}$ where $\forall i \in I, B_i \in \mathcal{B}_i$, we have $\prod_{i \in I} B_i$ is open in the box topology.

PROOF: Since each B_i is open in X_i .

$\langle 1 \rangle 2$. For any open set U in the box topology and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 1$. LET: U be a set open in the box topology.

$\langle 2 \rangle 2$. LET: $x \in U$

$\langle 2 \rangle 3$. PICK a family $\{U_i\}_{i \in I}$ where each U_i is open in X_i such that $x \in \prod_{i \in I} U_i \subseteq U$

$\langle 2 \rangle 4$. For $i \in I$, choose $B_i \in \mathcal{B}_i$ such that $x_i \in B_i \subseteq U_i$.

$\langle 2 \rangle 5$. $\prod_{i \in I} B_i \in \mathcal{B}$

$\langle 2 \rangle 6$. $x \in \prod_{i \in I} B_i \subseteq \prod_{i \in I} U_i \subseteq U$

\square

12.22.2 Subspaces

Proposition 12.22.6. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let Y_i be a subspace of X_i for all $i \in I$. Then the box topology on $\prod_{i \in I} Y_i$ is the same as the subspace topology that $\prod_{i \in I} Y_i$ inherits as a subspace of $\prod_{i \in I} X_i$ under the box topology.*

PROOF: A basis for the box topology is

$$\begin{aligned} & \left\{ \prod_{i \in I} V_i : V_i \text{ open in } X_i \right\} \\ &= \left\{ \prod_{i \in I} (U_i \cap Y_i) : U_i \text{ open in } X_i \right\} \\ &= \left\{ \prod_{i \in I} U_i \cap \prod_{i \in I} Y_i : U_i \text{ open in } X_i \right\} \end{aligned}$$

which is a basis for the subspace topology by Proposition 12.3.17. \square

12.22.3 Closure

Proposition 12.22.7. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

- $\langle 1 \rangle 1.$ $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
 $\langle 2 \rangle 1.$ LET: $x \in \prod_{i \in I} \overline{A_i}$
 $\langle 2 \rangle 2.$ For any family $\{U_i\}_{i \in I}$ where each U_i is open in X_i , if $x \in \prod_{i \in I} U_i$ then $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$.
 $\langle 3 \rangle 1.$ LET: $\{U_i\}_{i \in I}$ be a family where each U_i is open in X_i .
 $\langle 3 \rangle 2.$ ASSUME: $x \in \prod_{i \in I} U_i$
 $\langle 3 \rangle 3.$ For all $i \in I$ we have U_i intersects A_i
PROOF: Since $\pi_i(x) \in U_i$ and U_i is a neighbourhood of $\pi_i(x)$.
 $\langle 3 \rangle 4.$ $\prod_{i \in I} U_i$ intersects $\prod_{i \in I} A_i$
 $\langle 2 \rangle 3.$ $x \in \overline{\prod_{i \in I} A_i}$
PROOF: Proposition 12.9.9.
 $\langle 1 \rangle 2.$ $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
 $\langle 2 \rangle 1.$ LET: $x \in \overline{\prod_{i \in I} A_i}$
 $\langle 2 \rangle 2.$ LET: $i \in I$
PROVE: $\pi_i(x) \in \overline{A_i}$
 $\langle 2 \rangle 3.$ LET: U be a neighbourhood of $\pi_i(x)$ in X_i
 $\langle 2 \rangle 4.$ $\pi_i^{-1}(U)$ is a neighbourhood of x in $\prod_{i \in I} X_i$
 $\langle 2 \rangle 5.$ PICK $y \in \pi_i^{-1}(U) \cap \prod_{i \in I} A_i$
 $\langle 2 \rangle 6.$ $\pi_i(y) \in U \cap A_i$

\square

12.23 Separations

Definition 12.23.1 (Separation). Let X be a topological space. A *separation* of X is a pair (U, V) of disjoint nonempty open subsets in X such that $U \cup V = X$.

Subspaces

Proposition 12.23.2. *Let X be a topological space and Y a subspace of X . Then a separation of Y is a pair (A, B) of disjoint nonempty subsets of Y , neither of which contains a limit point of the other, such that $A \cup B = Y$.*

PROOF: Since the following are equivalent:

- Neither of A and B contains a limit point of the other.

- A contains all its own limit points in Y , and B contains all its own limit points in Y .
- A and B are closed in Y .

□

12.24 Connected Spaces

Definition 12.24.1 (Connected). A topological space is *connected* iff it has no separation.

Example 12.24.2. Any indiscrete space is connected.

Example 12.24.3. The rationals are disconnected. For any irrational a , we have $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of \mathbb{Q} .

Proposition 12.24.4. *A topological space X is connected if and only if the only sets that are both open and closed are \emptyset and X .*

PROOF: Since (U, V) is a separation of X iff U is both open and closed and $V = X - U$. □

12.24.1 Continuous Functions

Proposition 12.24.5. *The continuous image of a connected space is connected.*

PROOF:

- ⟨1⟩1. LET: X and Y be topological spaces.
- ⟨1⟩2. LET: $f : X \rightarrow Y$ be a surjective continuous function.
- ⟨1⟩3. LET: (C, D) be a separation of Y .
- ⟨1⟩4. $(f^{-1}(C), f^{-1}(D))$ is a separation of X .

□

12.24.2 Subspaces

Proposition 12.24.6. *Let X be a topological space. Let (C, D) be a separation of X . Let Y be a connected subspace of X . Then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $(Y \cap C, Y \cap D)$ would be a separation of Y . □

Proposition 12.24.7. *Let X be a topological space. The union of a set of connected subspaces of X that have a point in common is connected.*

PROOF:

- ⟨1⟩1. LET: \mathcal{A} be a set of connected subspaces of X that have a point in common.
- ⟨1⟩2. PICK a such that $a \in \bigcap \mathcal{A}$
- ⟨1⟩3. ASSUME: for a contradiction (A, B) is a separation of $\bigcup \mathcal{A}$.
- ⟨1⟩4. ASSUME: w.l.o.g. $a \in A$

⟨1⟩5. For all $Y \in \mathcal{A}$ we have $Y \subseteq A$

PROOF: Proposition 12.24.5.

⟨1⟩6. $B = \emptyset$

⟨1⟩7. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 12.24.8. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are connected, then X is connected.*

Proposition 12.24.9. *If X and Y are nonempty topological spaces, then $X \times Y$ is connected if and only if X and Y are connected.*

Definition 12.24.10 (Path-connected). A topological space X is *path-connected* iff, for any points $a, b \in X$, there exists a continuous function $\alpha : [0, 1] \rightarrow X$, called a *path*, such that $\alpha(0) = a$ and $\alpha(1) = b$.

Proposition 12.24.11. *The continuous image of a path connected space is path connected.*

Proposition 12.24.12. *Let X be a topological space and $A, B \subseteq X$. If $X = A \cup B$, $A \cap B \neq \emptyset$, and A and B are path connected, then X is path connected.*

Proposition 12.24.13. *If X and Y are nonempty topological spaces, then $X \times Y$ is path connected if and only if X and Y are path connected.*

12.25 T_1 Spaces

Definition 12.25.1 (T_1). A topological space is T_1 iff every one-point set is closed.

Proposition 12.25.2. *A topological space is T_1 iff every finite set is closed.*

PROOF: Since the union of finitely many closed sets is closed. □

Proposition 12.25.3. *Let X be a topological space. Then X is T_1 if and only if, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x .*

PROOF:

⟨1⟩1. If X is T_1 then, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x .

⟨2⟩1. ASSUME: X is T_1 .

⟨2⟩2. LET: $x, y \in X$

⟨2⟩3. ASSUME: $x \neq y$

⟨2⟩4. $X - \{y\}$ is a neighbourhood of x that does not contain y .

⟨2⟩5. $X - \{x\}$ is a neighbourhood of y that does not contain x .

- <1>2. If, for all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x , then X is T_1 .
 <2>1. ASSUME: For all $x, y \in X$, if $x \neq y$ then there exists a neighbourhood of x that does not contain y , and there exists a neighbourhood of y that does not contain x .
 <2>2. LET: $x \in X$
 PROVE: $\{x\}$ is closed.
 <2>3. LET: $y \in X - \{x\}$
 <2>4. PICK a neighbourhood U of y that does not contain x .
 <2>5. $y \in U \subseteq X - \{x\}$

□

12.25.1 Limit Points

Proposition 12.25.4. *Let X be a T_1 space. Let $A \subseteq X$ and $l \in X$. Then l is a limit point of A if and only if every neighbourhood of l contains infinitely many points of A .*

PROOF:

- <1>1. If l is a limit point of A then every neighbourhood of l contains infinitely many points of A .
 <2>1. ASSUME: l is a limit point of A .
 <2>2. LET: U be a neighbourhood of l .
 <2>3. ASSUME: for a contradiction $U \cap A - \{l\}$ is finite.
 <2>4. $U \cap A - \{l\}$ is closed.
 PROOF: Since X is T_1 .
 <2>5. $U - (A - \{l\})$ is a neighbourhood of l .
 <2>6. $U - (A - \{l\})$ intersects A .
 <2>7. Q.E.D.
 <1>2. If every neighbourhood of l contains infinitely many points of A then l is a limit point of A .

PROOF: Immediate from definitions.

□

12.26 Hausdorff Spaces

Definition 12.26.1 (Hausdorff). A topological space is a *Hausdorff* space or a T_2 space iff any two distinct points have disjoint neighbourhoods.

Proposition 12.26.2. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

- <1>1. LET: X be a Hausdorff space.
 <1>2. LET: (a_n) be a sequence in X and $l, m \in X$
 <1>3. ASSUME: $a_n \rightarrow l$ and $a_n \rightarrow m$

- ⟨1⟩4. ASSUME: for a contradiction $l \neq m$
- ⟨1⟩5. PICK disjoint open sets U and V with $l \in U$ and $m \in V$
- ⟨1⟩6. PICK M, N such that $\forall n \geq M. a_n \in U$ and $\forall n \geq N. a_n \in V$
- ⟨1⟩7. $a_{\max(M,N)} \in U \cap V$
- ⟨1⟩8. Q.E.D.

PROOF: This contradicts the fact that $U \cap V = \emptyset$.

□

Example 12.26.3. We cannot weaken the hypothesis from being Hausdorff to being T_1 .

In the cofinite topology on any infinite set, every sequence converges to every point.

Proposition 12.26.4. *Any linearly ordered set is Hausdorff under the order topology.*

PROOF:

- ⟨1⟩1. LET: X be a linearly ordered set under the order topology.
- ⟨1⟩2. LET: $a, b \in X$ with $a \neq b$.
- ⟨1⟩3. ASSUME: w.l.o.g. $a < b$.
- ⟨1⟩4. CASE: There exists $c \in X$ such that $a < c < b$.
 - ⟨2⟩1. LET: $U = (-\infty, c)$
 - ⟨2⟩2. LET: $V = (c, +\infty)$
 - ⟨2⟩3. U and V are disjoint open sets with $a \in U$ and $b \in V$
- ⟨1⟩5. CASE: There is no $c \in X$ such that $a < c < b$.
 - ⟨2⟩1. LET: $U = (-\infty, b)$
 - ⟨2⟩2. LET: $V = (a, +\infty)$
 - ⟨2⟩3. U and V are disjoint open sets with $a \in U$ and $b \in V$

□

Proposition 12.26.5. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space.
- ⟨1⟩2. LET: Y be a subspace of X .
- ⟨1⟩3. LET: $a, b \in Y$ with $a \neq b$.
- ⟨1⟩4. PICK disjoint open sets U and V in X with $a \in U$ and $b \in V$.
- ⟨1⟩5. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y with $a \in U \cap Y$ and $b \in V \cap Y$.

□

Proposition 12.26.6. *The disjoint union of two Hausdorff spaces is Hausdorff.*

Proposition 12.26.7. *Let A be a topological space and B a Hausdorff space. Let $f, g : A \rightarrow B$ be continuous. Let $X \subseteq A$ be dense. If f and g agree on X , then $f = g$.*

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction $a \in A$ and $f(a) \neq g(a)$.
- ⟨1⟩2. PICK disjoint neighbourhoods U and V of $f(a)$ and $g(a)$ respectively.

⟨1⟩3. PICK $x \in f^{-1}(U) \cap g^{-1}(V)$

⟨1⟩4. $f(x) = g(x) \in U \cap V$

⟨1⟩5. Q.E.D.

PROOF: This is a contradiction.

□

12.26.1 Product Topology

Proposition 12.26.8. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.

⟨1⟩2. LET: $x, y \in \prod_{i \in I} X_i$ with $x \neq y$.

⟨1⟩3. PICK $i \in I$ such that $\pi_i(x) \neq \pi_i(y)$

⟨1⟩4. PICK disjoint open sets U and V in X_i such that $\pi_i(x) \in U$ and $\pi_i(y) \in V$.

⟨1⟩5. $x \in \pi_i^{-1}(U)$ and $y \in \pi_i^{-1}(V)$.

□

12.26.2 Box Topology

Proposition 12.26.9. *The box product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.

⟨1⟩2. LET: $x, y \in \prod_{i \in I} X_i$ with $x \neq y$.

⟨1⟩3. PICK $i \in I$ such that $\pi_i(x) \neq \pi_i(y)$

⟨1⟩4. PICK disjoint open sets U and V in X_i such that $\pi_i(x) \in U$ and $\pi_i(y) \in V$.

⟨1⟩5. $x \in \pi_i^{-1}(U)$ and $y \in \pi_i^{-1}(V)$.

□

12.26.3 T_1 Spaces

Proposition 12.26.10. *Every Hausdorff space is T_1 .*

PROOF:

⟨1⟩1. LET: X be a Hausdorff space.

⟨1⟩2. LET: $a \in X$

PROVE: $X - \{a\}$ is open.

⟨1⟩3. LET: $x \in X - \{a\}$

⟨1⟩4. PICK disjoint open sets U and V with $a \in U$ and $x \in V$

⟨1⟩5. $x \in V \subseteq X - U \subseteq X - \{a\}$

□

Example 12.26.11. The converse does not hold. If X is an infinite set under the cofinite topology, then X is T_1 but not Hausdorff.

Proposition 12.26.12. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$ be uniformly continuous. Let \hat{X} and \hat{Y} be the completions of X and Y . Then f extends uniquely to a continuous map $\hat{X} \rightarrow \hat{Y}$.*

PROOF: The extension maps $\lim_{n \rightarrow \infty} x_n$ to $\lim_{n \rightarrow \infty} f(x_n)$. \square

Proposition 12.26.13. *Let X be a topological space. Then X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in X^2 .*

PROOF:

Δ is closed

$\Leftrightarrow X^2 - \Delta$ is open

$\Leftrightarrow \forall x, y \in X ((x, y) \notin \Delta \Rightarrow \exists V, W \text{ open in } X (x \in V \wedge y \in W \wedge V \times W \subseteq X^2 - \Delta))$

$\Leftrightarrow \forall x, y \in X (x \neq y \Rightarrow \exists V, W \text{ open in } X (x \in V \wedge y \in W \wedge V \cap W = \emptyset))$

$\Leftrightarrow X$ is Hausdorff \square

12.27 Separable Spaces

Definition 12.27.1 (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

12.28 Sequential Compactness

Definition 12.28.1 (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

12.29 Compactness

Definition 12.29.1 (Compact). A topological space is *compact* iff every open cover has a finite subcover.

Proposition 12.29.2. *Let X be a compact topological space. Let P be a set of open sets such that, for all $U, V \in P$, we have $U \cup V \in P$. Assume that every point has an open neighbourhood in P . Then $X \in P$.*

PROOF:

$\langle 1 \rangle$ 1. P is an open cover of X

$\langle 1 \rangle$ 2. PICK a finite subcover $U_1, \dots, U_n \in P$

$\langle 1 \rangle$ 3. $X = U_1 \cup \dots \cup U_n \in P$

\square

Corollary 12.29.2.1. *Let f be a compact space and $f : X \rightarrow \mathbb{R}$ be locally bounded. Then f is bounded.*

PROOF: Take $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$. \square

Proposition 12.29.3. *The continuous image of a compact space is compact.*

Proposition 12.29.4. *A closed subspace of a compact space is compact.*

Proposition 12.29.5. *Let X and Y be nonempty spaces. Then the following are equivalent.*

1. X and Y are compact.
2. $X + Y$ is compact.
3. $X \times Y$ is compact.

Proposition 12.29.6. *A compact subspace of a Hausdorff space is closed.*

Proposition 12.29.7. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 12.29.8. *A first countable compact space is sequentially compact.*

12.30 Gluing

Definition 12.30.1 (Gluing). Let X and Y be topological spaces, $X_0 \subseteq X$ and $\phi : X_0 \rightarrow Y$ a continuous map. Then $Y \cup_\phi X$ is the quotient space $(X + Y)/\sim$, where \sim is the equivalence relation generated by $x \sim \phi(x)$ for all $x \in X_0$.

Proposition 12.30.2. *Y is a subspace of $Y \cup_\phi X$.*

Definition 12.30.3. Let X be a topological space and $\alpha : X \cong X$ a homeomorphism. Then $(X \times [0, 1])/\alpha$ is the quotient space of $X \times [0, 1]$ by the equivalence relation generated by $(x, 0) \sim (\alpha(x), 1)$ for all $x \in X$.

Definition 12.30.4 (Möbius Strip). The *Möbius strip* is $([-1, 1] \times [0, 1])/\alpha$ where $\alpha(x) = -x$.

Definition 12.30.5 (Klein Bottle). The *Klein bottle* is $(S^1 \times [0, 1])/\alpha$ where $\alpha(z) = \bar{z}$.

Proposition 12.30.6. *Let M be the Möbius strip and K the Klein bottle. Then $M \cup_{\text{id}_M} M \cong K$.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$ be the function that maps $\kappa_1(\theta, t)$ to $(e^{\pi i \theta / 2}, t)$ and $\kappa_2(\theta, t)$ to $(-e^{-\pi i \theta / 2}, t)$.

$\langle 1 \rangle 2$. f induces a bijection $M \cup_{\text{id}_M} M \approx K$

$\langle 1 \rangle 3$. f is a homeomorphism.

\square

12.31 Homogeneous Spaces

Definition 12.31.1 (Homogeneous). A topological space X is *homogeneous* iff, for any $x, y \in X$, there exists a homeomorphism $f : X \cong X$ such that $f(x) = y$.

12.32 Regular Spaces

Definition 12.32.1 (Regular). A topological space X is *regular* iff it is T_1 and, for every closed set A and point $x \notin A$, there exist disjoint open sets U and V with $A \subseteq U$ and $x \in V$.

Chapter 13

Metric Spaces

Definition 13.0.1 (Metric Space). Let X be a set and $d : X^2 \rightarrow \mathbb{R}$. We say (X, d) is a *metric space* iff:

- For all $x, y \in X$ we have $d(x, y) \geq 0$
- For all $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$
- For all $x, y \in X$ we have $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$

We call d the *metric* of the metric space (X, d) . We often write X for the metric space (X, d) .

Definition 13.0.2 (Discrete Metric). On any set X , define the *discrete* metric by $d(x, y) = 0$ if $x = y$, 1 if $x \neq y$.

Definition 13.0.3 (Standard Metric). The *standard metric* on \mathbb{R} is defined by $d(x, y) = |x - y|$.

Definition 13.0.4 (Square Metric). The *square metric* ρ on \mathbb{R}^n is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1$. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have $\rho(\vec{x}, \vec{y}) \geq 0$.

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have $\rho(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$.

PROOF:

$$\begin{aligned} \rho(\vec{x}, \vec{y}) = 0 &\Leftrightarrow \max(|x_1 - y_1|, \dots, |x_n - y_n|) = 0 \\ &\Leftrightarrow |x_1 - y_1| = \dots = |x_n - y_n| = 0 \\ &\Leftrightarrow x_1 = y_1 \wedge \dots \wedge x_n = y_n \\ &\Leftrightarrow \vec{x} = \vec{y} \end{aligned}$$

⟨1⟩3. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$.

PROOF: Immediate from definition.

⟨1⟩4. For all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ we have $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$.

PROOF:

$$\begin{aligned} & \max(|x_1 - z_1|, \dots, |x_n - z_n|) \\ & \leq \max(|x_1 - y_1| + |y_1 - z_1|, \dots, |x_n - y_n| + |y_n - z_n|) \\ & \leq \max(|x_1 - y_1|, \dots, |x_n - y_n|) + \max(|y_1 - z_1|, \dots, |y_n - z_n|) \\ & = \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z}) \end{aligned}$$

□

13.0.1 Balls

Definition 13.0.5 (Ball). Let X be a metric space. Let $x \in X$ and $r > 0$. The *ball* with *centre* x and *radius* r is

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

Definition 13.0.6 (Metric Topology). Let (X, d) be a metric space. The *metric topology* on X is the topology generated by the basis consisting of the balls.

We prove this is a basis for a topology.

PROOF:

⟨1⟩1. Every point is a member of some ball.

PROOF: Since $x \in B(x, 1)$.

⟨1⟩2. If B_1 and B_2 are balls and $x \in B_1 \cap B_2$, then there exists a ball B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.

⟨2⟩1. LET: $x \in B(a, \epsilon_1) \cap B(b, \epsilon_2)$

⟨2⟩2. LET: $\epsilon = \min(\epsilon_1 - d(x, a), \epsilon_2 - d(x, b))$

PROVE: $x \in B(x, \epsilon) \subseteq B(a, \epsilon_1) \cap B(b, \epsilon_2)$

⟨2⟩3. $B(x, \epsilon) \subseteq B(a, \epsilon_1)$

⟨3⟩1. LET: $y \in B(x, \epsilon)$

⟨3⟩2. $d(y, a) < \epsilon_1$

PROOF:

$$d(y, a) \leq d(y, x) + d(x, a) \quad (\text{Triangle Inequality})$$

$$< \epsilon + d(x, a) \quad (\langle 3 \rangle 1)$$

$$\leq \epsilon_1 \quad (\langle 2 \rangle 2)$$

⟨2⟩4. $B(x, \epsilon) \subseteq B(b, \epsilon_2)$

PROOF: Similar.

□

Proposition 13.0.7. *The discrete metric on a set X induces the discrete topology.*

PROOF: Since $B(x, 1/2) = \{x\}$ for all $x \in X$. □

Proposition 13.0.8. *The standard metric on \mathbb{R} induces the standard topology.*

PROOF:

$\langle 1 \rangle 1$. Every ball is open in the standard topology.

PROOF: Since $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$.

$\langle 1 \rangle 2$. Every open ray is open in the metric topology.

PROOF: If $x \in (a, +\infty)$ then $x \in B(x, x - a) \subseteq (a, +\infty)$. Similarly for $(-\infty, a)$.

□

Proposition 13.0.9. *The square metric on \mathbb{R}^n induces the product topology.*

PROOF:

$\langle 1 \rangle 1$. For any real numbers $a_1, \dots, a_n, b_1, \dots, b_n$ with $a_1 < b_1, \dots, a_n < b_n$, we have $(a_1, b_1) \times \dots \times (a_n, b_n)$ is open in the metric topology.

$\langle 2 \rangle 1$. LET: $\vec{x} \in (a_1, b_1) \times \dots \times (a_n, b_n)$

$\langle 2 \rangle 2$. LET: $\epsilon = \min(x_1 - a_1, b_1 - x_1, \dots, x_n - a_n, b_n - x_n)$

$\langle 2 \rangle 3$. $B(\vec{x}, \epsilon) \subseteq (a_1, b_1) \times \dots \times (a_n, b_n)$

$\langle 1 \rangle 2$. For any $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B(\vec{a}, \epsilon)$ is open in the product topology.

PROOF: Since $B(\vec{a}, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \dots \times (a_n - \epsilon, a_n + \epsilon)$.

□

Proposition 13.0.10. *Addition is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

$\langle 1 \rangle 2$. LET: $\delta = \epsilon/2$

$\langle 1 \rangle 3$. LET: $(x', y') \in \mathbb{R}^2$ with $\rho((x, y), (x', y')) < \delta$

$\langle 1 \rangle 4$. $|x - x'|, |y - y'| < \delta$

$\langle 1 \rangle 5$. $|(x + y) - (x' + y')| < \epsilon$

PROOF:

$$\begin{aligned} |(x + y) - (x' + y')| &\leq |x - x'| + |y - y'| \\ &< \delta + \delta && (\langle 1 \rangle 4) \\ &= \epsilon && (\langle 1 \rangle 2) \end{aligned}$$

□

Proposition 13.0.11. *Multiplication is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

$\langle 1 \rangle 2$. LET: $\delta = \min(\epsilon/(|x| + |y| + 1), 1)$

$\langle 1 \rangle 3$. LET: $(x', y') \in \mathbb{R}^2$ with $\rho((x, y), (x', y')) < \delta$

$\langle 1 \rangle 4$. $|x - x'|, |y - y'| < \delta$

$\langle 1 \rangle 5$. $|xy - x'y'| < \epsilon$

PROOF:

$$\begin{aligned} |xy - x'y'| &= |xy - xy' + xy' - x'y - xy + x'y + xy' - x'y'| \\ &\leq |xy - xy'| + |xy - x'y| + |xy - x'y - xy' + xy'y| = |x||y - y'| + |x - x'||y| + |x - x'||y - y'| \\ &< |x|\delta + |y|\delta + \delta^2 && (\langle 1 \rangle 4) \\ &\leq |x|\delta + |y|\delta + \delta && (\langle 1 \rangle 2) \\ &= (|x| + |y| + 1)\delta \\ &\leq \epsilon && (\langle 1 \rangle 2) \end{aligned}$$

□

Corollary 13.0.11.1. *The unit circle S^1 is a closed subset of \mathbb{R}^2 .*

PROOF: The function f that maps (x, y) to $x^2 + y^2$ is continuous, and $S^1 = f^{-1}(\{1\})$. □

Corollary 13.0.11.2. *The unit ball B^2 is a closed subset of \mathbb{R}^2 .*

PROOF: The function f that maps (x, y) to $x^2 + y^2$ is continuous, and $B^2 = f^{-1}([0, 1])$. □

Proposition 13.0.12. *Let (a_n) and (b_n) be sequences of real numbers. Let $c, s, t \in \mathbb{R}$. Assume*

$$\sum_{n=0}^{\infty} a_n = s \text{ and } \sum_{n=0}^{\infty} b_n = t .$$

Then

$$\sum_{n=0}^{\infty} (ca_n + b_n) = cs + t .$$

PROOF:

$$\sum_{n=0}^N (ca_n + b_n) = c \sum_{n=0}^N a_n + \sum_{n=0}^N b_n \rightarrow cs + t \text{ as } n \rightarrow \infty \quad \square$$

Proposition 13.0.13 (Comparison Test). *Let (a_n) and (b_n) be sequences of real numbers. Assume $|a_n| \leq b_n$ for all n . Assume $\sum_{n=0}^{\infty} b_n$ converges. Then $\sum_{n=0}^{\infty} a_n$ converges.*

PROOF:

⟨1⟩1. For all n ,

LET: $c_n = |a_n| + a_n$

⟨1⟩2. $\sum_{n=0}^{\infty} |a_n|$ converges.

PROOF: Since $(\sum_{n=0}^N |a_n|)_N$ is an increasing sequence of real numbers bounded above by $\sum_{n=0}^{\infty} b_n$.

⟨1⟩3. $\sum_{n=0}^{\infty} c_n$ converges.

PROOF: Since $(\sum_{n=0}^N c_n)_N$ is an increasing sequence of real numbers bounded above by $2 \sum_{n=0}^{\infty} a_n$.

⟨1⟩4. $\sum_{n=0}^{\infty} a_n$ converges.

PROOF: Since $a_n = c_n - |a_n|$.

□

Proposition 13.0.14. *Let X be a metric space. Let $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:

⟨1⟩1. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

⟨2⟩1. ASSUME: U is open.

⟨2⟩2. LET: $x \in U$

⟨2⟩3. PICK a ball $B(a, \delta)$ such that $x \in B(a, \delta) \subseteq U$

⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$

PROVE: $B(x, \epsilon) \subseteq U$

⟨2⟩5. LET: $y \in B(x, \epsilon)$

⟨2⟩6. $y \in B(a, \delta)$

PROOF:

$$d(a, y) \leq d(a, x) + d(x, y) \quad (\text{Triangle Inequality})$$

$$< d(a, x) + \epsilon \quad (\langle 2 \rangle 5)$$

$$= \delta$$

⟨2⟩7. $y \in U$

PROOF: ⟨2⟩3

⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.

PROOF: Immediate from definition of the metric topology.

□

Proposition 13.0.15. *Let X be a metric space. Let $a, b, c \in X$. Then*

$$|d(a, b) - d(a, c)| \leq d(b, c) .$$

PROOF:

⟨1⟩1. $d(a, b) - d(a, c) \leq d(b, c)$

PROOF: Triangle Inequality.

⟨1⟩2. $d(a, c) - d(a, b) \leq d(b, c)$

PROOF: Triangle Inequality.

□

Proposition 13.0.16. *Let (X, d) be a metric space. Then the metric topology on X is the coarsest topology such that $d : X^2 \rightarrow \mathbb{R}$ is continuous.*

PROOF:

⟨1⟩1. d is continuous with respect to the metric topology.

⟨2⟩1. LET: $(a, b) \in X^2$

⟨2⟩2. LET: V be a neighbourhood of $d(a, b)$.

⟨2⟩3. PICK $\epsilon > 0$ such that $(d(a, b) - \epsilon, d(a, b) + \epsilon) \subseteq V$.

⟨2⟩4. LET: $U = B(a, \epsilon/2) \times B(b, \epsilon/2)$

⟨2⟩5. LET: $(x, y) \in U$

⟨2⟩6. $|d(x, y) - d(a, b)| < \epsilon$

PROOF:

$$\begin{aligned} |d(x, y) - d(a, b)| &\leq |d(x, y) - d(a, y)| + |d(a, y) - d(a, b)| \\ &\leq d(a, x) + d(b, y) \quad (\text{Proposition 13.0.15}) \\ &< \epsilon \end{aligned}$$

⟨2⟩7. $d(x, y) \in V$

⟨1⟩2. If \mathcal{T} is a topology on X with respect to which d is continuous then \mathcal{T} is finer than the metric topology.

⟨2⟩1. LET: \mathcal{T} be a topology on X with respect to which d is continuous.

⟨2⟩2. LET: $a \in X$ and $\epsilon > 0$.

- PROVE: $B(a, \epsilon) \in \mathcal{T}$
- $\langle 2 \rangle 3$. LET: $x \in B(a, \epsilon)$
- $\langle 2 \rangle 4$. $(a, x) \in d^{-1}((0, \epsilon))$
- $\langle 2 \rangle 5$. PICK $U, V \in \mathcal{T}$ such that $(a, x) \in U \times V \subseteq d^{-1}((0, \epsilon))$
- $\langle 2 \rangle 6$. $x \in V \subseteq B(a, \epsilon)$

□

Proposition 13.0.17. *Let d and d' be two metrics on the same set X . Let \mathcal{T} and \mathcal{T}' be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

- $\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.
- $\langle 2 \rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
- $\langle 2 \rangle 2$. LET: $x \in X$ and $\epsilon > 0$
- $\langle 2 \rangle 3$. $x \in B_d(x, \epsilon) \in \mathcal{T}'$
- $\langle 2 \rangle 4$. There exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
- PROOF: Proposition 13.0.14.
- $\langle 1 \rangle 2$. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$.
- $\langle 2 \rangle 1$. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.
- $\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
- $\langle 2 \rangle 3$. For all $x \in U$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$
- $\langle 3 \rangle 1$. LET: $x \in U$
- $\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$
- PROOF: Proposition 13.0.14.
- $\langle 3 \rangle 3$. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.
- PROOF: $\langle 2 \rangle 1$
- $\langle 3 \rangle 4$. $B_{d'}(x, \delta) \subseteq U$
- $\langle 2 \rangle 4$. $U \in \mathcal{T}'$
- PROOF: Proposition 13.0.14.

□

Definition 13.0.18 (Metrizability). A topological space is *metrizable* iff there exists a metric that induces its topology.

Proposition 13.0.19. \mathbb{R}^2 under the dictionary order is metrizable.

PROOF:

- $\langle 1 \rangle 1$. LET: $d : (\mathbb{R}^2)^2 \rightarrow \mathbb{R}$ be defined by
- $$d((x_1, y_1), (x_2, y_2)) = \begin{cases} \min(|y_2 - y_1|, 1) & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$
- $\langle 1 \rangle 2$. d is a metric.

- $\langle 2 \rangle 1$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) \geq 0$.
 PROOF: Immediate from definition.
 $\langle 2 \rangle 2$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) = 0$ iff $x = y$.
 PROOF: Immediate from definition.
 $\langle 2 \rangle 3$. For all $x, y \in \mathbb{R}^2$ we have $d(x, y) = d(y, x)$.
 PROOF: Immediate from definition.
 $\langle 2 \rangle 4$. For all $x, y, z \in \mathbb{R}^2$ we have $d(x, z) \leq d(x, y) + d(y, z)$.
 PROOF: Easy.
 $\langle 1 \rangle 3$. The metric topology induced by d is finer than the order topology.
 $\langle 2 \rangle 1$. LET: $a, b \in \mathbb{R}^2$
 $\langle 2 \rangle 2$. LET: $x \in (a, b)$
 $\langle 2 \rangle 3$. CASE: $\pi_1(a) = \pi_1(x) = \pi_1(b)$
 $\langle 3 \rangle 1$. LET: $\epsilon = \min(\pi_2(x) - \pi_2(a), \pi_2(b) - \pi_2(x))$
 $\langle 3 \rangle 2$. $B(x, \epsilon) \subseteq (a, b)$
 $\langle 2 \rangle 4$. CASE: $\pi_1(a) = \pi_1(x) < \pi_1(b)$
 $\langle 3 \rangle 1$. LET: $\epsilon = \pi_2(x) - \pi_2(a)$
 $\langle 3 \rangle 2$. $B(x, \epsilon) \subseteq (a, b)$
 $\langle 2 \rangle 5$. CASE: $\pi_1(a) < \pi_1(x) = \pi_1(b)$
 PROOF: Similar.
 $\langle 2 \rangle 6$. CASE: $\pi_1(a) < \pi_1(x) < \pi_1(b)$
 PROOF: Then $B(x, \epsilon) \subseteq (a, b)$.
 $\langle 1 \rangle 4$. The order topology is finer than the metric topology.
 PROOF: Since $B((a, b), \epsilon) = ((a, b - \epsilon), (a, b + \epsilon))$ if $\epsilon \leq 1$, and \mathbb{R}^2 if $\epsilon > 1$.
 \square

Every metrizable space is first countable.

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

13.0.2 Subspaces

Proposition 13.0.20. *Let (X, d) be a metric space and $Y \subseteq X$. Then $d|Y^2$ is a metric on Y that induces the subspace topology.*

PROOF:

- $\langle 1 \rangle 1$. LET: $d' = d|Y^2 : Y^2 \rightarrow \mathbb{R}$
 $\langle 1 \rangle 2$. d' is a metric.
 PROOF: Each of the axioms follows from the axiom in X .
 $\langle 1 \rangle 3$. The metric topology induced by d' is finer than the subspace topology.
 $\langle 2 \rangle 1$. LET: U be open in X
 PROVE: $U \cap Y$ is open in the d' -topology.
 $\langle 2 \rangle 2$. LET: $y \in U \cap Y$
 $\langle 2 \rangle 3$. PICK $\epsilon > 0$ such that $B_d(y, \epsilon) \subseteq U$
 $\langle 2 \rangle 4$. $B_{d'}(y, \epsilon) \subseteq U \cap Y$
 $\langle 1 \rangle 4$. The subspace topology is finer than the metric topology induced by d' .
 $\langle 2 \rangle 1$. LET: $y \in Y$ and $\epsilon > 0$
 PROVE: $B_{d'}(y, \epsilon)$ is open in the subspace topology.

$\langle 2 \rangle 2. B_{d'}(y, \epsilon) = B_d(y, \epsilon) \cap Y$
 \square

13.0.3 Convergence

Proposition 13.0.21 (Sequence Lemma). *Let X be a metric space. Let $A \subseteq X$. Let $l \in \bar{A}$. Then there exists a sequence in A that converges to l .*

PROOF:

$\langle 1 \rangle 1.$ For $n \in \mathbb{N}$, PICK $a_n \in B(l, 1/(n+1)) \cap A$.
 $\langle 1 \rangle 2.$ $a_n \rightarrow l$ as $n \rightarrow \infty$.
 \square

Corollary 13.0.21.1. \mathbb{R}^ω under the box topology is not first countable.

PROOF:

$\langle 1 \rangle 1.$ LET: A be the set of all sequences of positive reals.
 $\langle 1 \rangle 2.$ $0 \in \bar{A}$
 $\langle 1 \rangle 3.$ LET: (a_n) be a sequence in A
 PROVE: (a_n) does not converge to 0.
 $\langle 1 \rangle 4.$ For all $n \in \mathbb{N}$,
 LET: $a_n = (x_{nm})$
 $\langle 1 \rangle 5.$ LET: $B' = \prod_{n=0}^{\infty} (-x_{nn}, x_{nn})$
 $\langle 1 \rangle 6.$ B' is open in the box topology.
 $\langle 1 \rangle 7.$ $0 \in B'$
 $\langle 1 \rangle 8.$ For all n we have $a_n \notin B'$
 \square

Corollary 13.0.21.2. If J is an uncountable set then \mathbb{R}^J under the product topology is not first countable.

PROOF:

$\langle 1 \rangle 1.$ LET: $A = \{x \in \mathbb{R}^J : \pi_j(x) = 1 \text{ for all but finitely many } j \in J\}$
 $\langle 1 \rangle 2.$ $0 \in \bar{A}$
 $\langle 1 \rangle 3.$ LET: (a_n) be a sequence in A .
 PROVE: (a_n) does not converge to 0.
 $\langle 1 \rangle 4.$ For $n \in \mathbb{N}$,
 LET: $J_n = \{j \in J : \pi_j(a_n) \neq 1\}$
 $\langle 1 \rangle 5.$ $\bigcup_{n \in \mathbb{N}} J_n$ is countable.
 $\langle 1 \rangle 6.$ PICK $\beta \in J - \bigcup_{n \in \mathbb{N}} J_n$
 $\langle 1 \rangle 7.$ $\forall n \in \mathbb{N}. \pi_\beta(a_n) = 1$
 $\langle 1 \rangle 8.$ LET: $U = \pi_\beta^{-1}((-1, 1))$
 $\langle 1 \rangle 9.$ $0 \in U$
 $\langle 1 \rangle 10.$ $\forall n \in \mathbb{N}. a_n \notin U$
 $\langle 1 \rangle 11.$ (a_n) does not converge to 0.
 \square

13.0.4 Continuous Functions

Proposition 13.0.22. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:

- ⟨1⟩1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
- ⟨2⟩1. ASSUME: f is continuous.
- ⟨2⟩2. LET: $x \in X$
- ⟨2⟩3. LET: $\epsilon > 0$
- ⟨2⟩4. $x \in f^{-1}(B(f(x), \epsilon))$
- ⟨2⟩5. There exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$.
- ⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.
- ⟨2⟩1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
- ⟨2⟩2. LET: V be open in Y
- ⟨2⟩3. LET: $x \in f^{-1}(V)$
- ⟨2⟩4. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
- ⟨2⟩5. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
- ⟨2⟩6. $B(x, \delta) \subseteq f^{-1}(V)$

□

Proposition 13.0.23. *Let X be a metrizable space and Y a topological space. Let $f : X \rightarrow Y$. Assume that, for every sequence (x_n) in X and $l \in X$, if $x_n \rightarrow l$ as $n \rightarrow \infty$ then $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$. Then f is continuous.*

PROOF:

- ⟨1⟩1. LET: $A \subseteq X$
PROVE: $\overline{f(A)} \subseteq f(\overline{A})$
- ⟨1⟩2. LET: $l \in \overline{A}$
PROVE: $f(l) \in \overline{f(A)}$
- ⟨1⟩3. PICK a sequence (x_n) in A such that $x_n \rightarrow l$ as $n \rightarrow \infty$.
- ⟨1⟩4. $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$.
- ⟨1⟩5. $f(l) \in \overline{f(A)}$

□

Proposition 13.0.24. *The function $i : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ that maps x to x^{-1} is continuous.*

PROOF:

- ⟨1⟩1. LET: $a, b \in \mathbb{R}$ with $a < b$
PROVE: $i^{-1}((a, b))$ is open.
- ⟨1⟩2. CASE: $0 < a$
PROOF: $i^{-1}((a, b)) = (b^{-1}, a^{-1})$
- ⟨1⟩3. CASE: $a = 0$

PROOF: $i^{-1}((a, b)) = (b^{-1}, +\infty)$
 $\langle 1 \rangle 4$. CASE: $a < 0 < b$
 PROOF: $i^{-1}((a, b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$
 $\langle 1 \rangle 5$. CASE: $b = 0$
 PROOF: $i^{-1}((a, b)) = (-\infty, a^{-1})$
 $\langle 1 \rangle 6$. CASE: $b < 0$
 PROOF: $i^{-1}((a, b)) = (b^{-1}, a^{-1})$
 \square

Proposition 13.0.25. *Subtraction is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF: Since $a - b = a + (-1)b$ and both addition and multiplication are continuous. \square

Proposition 13.0.26. *Division is a continuous function $\mathbb{R} \times (\mathbb{R} - \{0\}) \rightarrow \mathbb{R}$.*

PROOF: Since both multiplication and the function that maps x to x^{-1} are continuous. \square

13.0.5 First Countable Spaces

Proposition 13.0.27. *Every metrizable space is first countable.*

PROOF: For any point x , the set $\{B(x, 1/n) : n \in \mathbb{Z}_+\}$ is a countable basis at x .
 \square

Corollary 13.0.27.1. \mathbb{R}^ω under the box topology is not metrizable.

Corollary 13.0.27.2. *If J is an uncountable set then \mathbb{R}^J under the product topology is not metrizable.*

13.0.6 Hausdorff Spaces

Proposition 13.0.28. *Every metric space is Hausdorff.*

PROOF:
 $\langle 1 \rangle 1$. LET: X be a metric space.
 $\langle 1 \rangle 2$. LET: $x, y \in X$ with $x \neq y$.
 $\langle 1 \rangle 3$. LET: $\epsilon = d(x, y)$
 $\langle 1 \rangle 4$. $B(x, \epsilon/2)$ and $B(y, \epsilon/2)$ are disjoint neighbourhoods of x and y .
 \square

13.0.7 Bounded Sets

Definition 13.0.29 (Bounded). Let X be a metric space. Let $A \subseteq X$. Then A is *bounded* iff there exists M such that $\forall x, y \in A. d(x, y) \leq M$. Its *diameter* is then defined to be

$$\text{diam } A := \sup\{d(x, y) : x, y \in A\} .$$

13.0.8 Uniform Convergence

Definition 13.0.30 (Uniform Convergence). Let X be a set and Y a metric space. Let (f_n) be a sequence of functions $X \rightarrow Y$, and $f : X \rightarrow Y$. Then (f_n) converges uniformly to f iff, for all $\epsilon > 0$, there exists N such that

$$\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon .$$

Example 13.0.31. For $n \in \mathbb{N}$ define $f_n : [0, 1] \rightarrow \mathbb{R}$ by $f_n(x) = x^n$. Define $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = 0$ for $x < 1$, $f(1) = 1$. Then f_n converges pointwise to f , but does not converge uniformly to f .

We prove that, for all N , there exists $n \geq N$ and $x \in [0, 1]$ such that $|x^n - f(x)| \geq 1/2$. Take $n = N$ and x to be the N th root of $3/4$.

Example 13.0.32. For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} .$$

Then for all $x \in \mathbb{R}$ we have $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$, but (f_n) does not converge uniformly to 0.

We prove that, for all N , there exists $n \geq N$ and $x \in \mathbb{R}$ such that $|f_n(x)| \geq 1/2$. Take $n = N$ and $x = 1/N$. We have $f_N(1/N) = 1$.

Theorem 13.0.33 (Uniform Limit Theorem). *Let X be a topological space and Y a metric space. Let (f_n) be a sequence of functions $X \rightarrow Y$, and $f : X \rightarrow Y$. If every f_n is continuous and (f_n) converges uniformly to f , then f is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: V be open in Y .

$\langle 1 \rangle 2$. LET: $x_0 \in f^{-1}(V)$

PROVE: There exists a neighbourhood U of x_0 such that $f(U) \subseteq V$.

$\langle 1 \rangle 3$. LET: $y_0 = f(x_0)$

$\langle 1 \rangle 4$. PICK $\epsilon > 0$ such that $B(y_0, \epsilon) \subseteq V$.

$\langle 1 \rangle 5$. PICK N such that $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/3$.

$\langle 1 \rangle 6$. PICK a neighbourhood U of x_0 such that $f_N(U) \subseteq B(f_N(x_0), \epsilon/3)$.

PROVE: $f(U) \subseteq V$

$\langle 1 \rangle 7$. LET: $y \in U$

$\langle 1 \rangle 8$. $d(f(y), y_0) < \epsilon$

PROOF:

$$d(f(y), y_0) \leq d(f(y), f_N(y)) + d(f_N(y), f_N(x_0)) + d(f_N(x_0), y_0)$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3$$

($\langle 1 \rangle 5$, $\langle 1 \rangle 6$)

$$= \epsilon$$

$\langle 1 \rangle 9$. $f(y) \in V$

PROOF: $\langle 1 \rangle 4$

□

Proposition 13.0.34. *Let X be a topological space. Let Y be a metric space. Let f_n be a sequence of functions $X \rightarrow Y$ and $f : X \rightarrow Y$. Let x_n be a sequence of points in X and $l \in X$. If f_n converges uniformly to f , x_n converges to l , and f is continuous, then $f_n(x_n)$ converges to $f(l)$.*

PROOF:

<1>1. f is continuous.

<1>2. LET: $\epsilon > 0$

<1>3. PICK $\delta > 0$ such that $\forall y \in X. d(y, l) < \delta \Rightarrow d(f(y), f(l)) < \epsilon/2$

<1>4. PICK N such that $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/2$ and $\forall n \geq N. d(x_n, l) < \delta$

<1>5. For all $n \geq N$ we have $d(f_n(x_n), f(l)) < \epsilon$

PROOF:

$$\begin{aligned} d(f_n(x_n), f(l)) &\leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(l)) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

Theorem 13.0.35 (Weierstrass M -Test). *Let X be a set. Let (f_n) be a sequence of functions $X \rightarrow \mathbb{R}$. Let (M_n) be a sequence of real numbers. For $n \in \mathbb{N}$, let*

$$s_n(x) = \sum_{i=0}^n f_i(x) \ .$$

Assume that $\forall n \in \mathbb{N}. \forall x \in X. |f_n(x)| \leq M_n$. Assume that $\sum_{n=0}^{\infty} M_n$ converges. Then (s_n) uniformly converges to s where $s(x) = \sum_{n=0}^{\infty} f_n(x)$.

PROOF:

<1>1. For all $x \in X$ we have $\sum_{n=0}^{\infty} f_n(x)$ converges.

PROOF: By the Comparison Test.

<1>2. For $n \in \mathbb{N}$,

LET: $r_n = \sum_{i=n+1}^{\infty} M_i$.

<1>3. For all $k, n \in \mathbb{N}$ and $x \in X$, if $k > n$ then $|s_k(x) - s_n(x)| \leq r_n$.

PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\ &\leq \sum_{i=n+1}^k |f_i(x)| \\ &\leq \sum_{i=n+1}^k M_i \\ &\leq \sum_{i=n+1}^{\infty} M_i \\ &= r_n \end{aligned}$$

⟨1⟩4. For all $n \in \mathbb{N}$ we have $|s(x) - s_n(x)| \leq r_n$.

PROOF: Taking the limit $k \rightarrow \infty$ in ⟨1⟩3.

⟨1⟩5. (s_n) converges uniformly to s .

PROOF: We have $\bar{\rho}(s_n, s) \leq r_n$ and so $\bar{\rho}(s_n, s) \rightarrow 0$ as $n \rightarrow \infty$ by the Sandwich Theorem.

□

13.0.9 Standard Bounded Metric

Definition 13.0.36 (Standard Bounded Metric). Let (X, d) be a metric space. The *standard bounded metric* corresponding to d is

$$\bar{d}(x, y) := \min(d(x, y), 1) .$$

Proposition 13.0.37. *The standard bounded metric associated with d induces the same topology as d .*

PROOF:

⟨1⟩1. LET: (X, d) be a metric space.

⟨1⟩2. Every d -ball is open under the topology induced by \bar{d} .

⟨2⟩1. LET: $a \in X$ and $\epsilon > 0$

⟨2⟩2. LET: $x \in B_d(a, \epsilon)$

⟨2⟩3. LET: $\delta = \min(\epsilon - d(a, x), 1/2)$

⟨2⟩4. $B_{\bar{d}}(x, \delta) \subseteq B_d(a, \epsilon)$

⟨1⟩3. Every \bar{d} -ball is open under the topology induced by d .

PROOF: Since $B_{\bar{d}}(a, \epsilon) = B_d(a, \epsilon)$ if $\epsilon \leq 1$, and X if $\epsilon > 1$.

□

13.0.10 Product Spaces

Proposition 13.0.38. *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

⟨1⟩1. LET: (X_n, d_n) be a sequence of metric spaces.

⟨1⟩2. For $n \in \mathbb{N}$,

LET: \bar{d}_n be the standard bounded metric associated with d_n .

⟨1⟩3. LET: $X = \prod_{n \in \mathbb{N}} X_n$

⟨1⟩4. Define $D : X^2 \rightarrow \mathbb{R}$ by $D(x, y) = \sup_{n \in \mathbb{N}} \bar{d}_n(\pi_n(x), \pi_n(y)) / (n + 1)$.

⟨1⟩5. D is a metric on X .

⟨2⟩1. For all $x, y \in X$ we have $D(x, y) \geq 0$.

⟨2⟩2. For all $x, y \in X$ we have $D(x, y) = 0$ iff $x = y$.

⟨2⟩3. For all $x, y \in X$ we have $D(x, y) = D(y, x)$.

⟨2⟩4. For all $x, y, z \in X$ we have $D(x, z) \leq D(x, y) + D(y, z)$.

⟨1⟩6. The product topology is finer than the metric topology induced by D .

⟨2⟩1. LET: $a \in X$ and $\epsilon > 0$.

⟨2⟩2. LET: $x \in B(a, \epsilon)$

- $\langle 2 \rangle 3$. LET: $\delta = \epsilon - D(a, x)$
 $\langle 2 \rangle 4$. PICK $N \in \mathbb{N}$ such that $1/(N+1) < \delta$
 $\langle 2 \rangle 5$. $x \in \prod_{n=0}^N B_{d_n}(\pi_n(a), n\delta) \times \prod_{n=N+1}^{\infty} B(a, \epsilon)$
 $\langle 1 \rangle 7$. The metric topology induced by D is finer than the product topology.
 $\langle 2 \rangle 1$. LET: $n \in \mathbb{N}$ and U be an open set in X_n .
 PROVE: $\pi_n^{-1}(U)$ is open in the metric topology.
 $\langle 2 \rangle 2$. LET: $x \in \pi_n^{-1}(U)$
 $\langle 2 \rangle 3$. PICK $\epsilon > 0$ such that $B_{d_n}(\pi_n(x), \epsilon) \subseteq U$
 $\langle 2 \rangle 4$. $B(x, \epsilon/(n+1)) \subseteq \pi_n^{-1}(U)$
 \square

13.0.11 Uniform Metric

Definition 13.0.39 (Uniform Metric). Let J be a nonempty set. The *uniform metric* $\bar{\rho}$ on \mathbb{R}^J is defined by

$$\bar{\rho}(x, y) = \sup_{j \in J} \bar{d}(x_j, y_j)$$

where \bar{d} is the standard bounded metric associated with the standard metric on \mathbb{R} .

The topology it induces is called the *uniform topology*.

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1$. For all $x, y \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, y) \geq 0$.

PROOF: Pick $j_0 \in J$. Then

$$\begin{aligned}
 \bar{\rho}(x, y) &= \sup_j \bar{d}(x_j, y_j) \\
 &\geq \bar{d}(x_{j_0}, y_{j_0}) \\
 &\geq 0
 \end{aligned}$$

$\langle 1 \rangle 2$. For all $x, y \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, y) = 0$ iff $x = y$.

PROOF:

$$\begin{aligned}
 \bar{\rho}(x, y) = 0 &\Leftrightarrow \sup_j \bar{d}(x_j, y_j) = 0 \\
 &\Leftrightarrow \forall j. \bar{d}(x_j, y_j) = 0 \\
 &\Leftrightarrow \forall j. x_j = y_j \\
 &\Leftrightarrow x = y
 \end{aligned}$$

$\langle 1 \rangle 3$. For all $x, y \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, y) = \bar{\rho}(y, x)$.

PROOF:

$$\begin{aligned}
 \bar{\rho}(x, y) &= \sup_j \bar{d}(x_j, y_j) \\
 &= \sup_j \bar{d}(y_j, x_j) \\
 &= \bar{\rho}(y, x)
 \end{aligned}$$

$\langle 1 \rangle 4$. For all $x, y, z \in \mathbb{R}^\omega$ we have $\bar{\rho}(x, z) \leq \bar{\rho}(x, y) + \bar{\rho}(y, z)$.

PROOF:

$$\begin{aligned}
 \bar{\rho}(x, z) &= \sup_j \bar{d}(x_j, z_j) \\
 &\leq \sup_j (\bar{d}(x_j, y_j) + \bar{d}(y_j, z_j)) \\
 &\leq \sup_j \bar{d}(x_j, y_j) + \sup_j \bar{d}(y_j, z_j) \\
 &= \bar{\rho}(x, y) + \bar{\rho}(y, z)
 \end{aligned}$$

□

Proposition 13.0.40. *The uniform topology is finer than the product topology. It is strictly finer iff J is infinite.*

PROOF:

⟨1⟩1. The uniform topology is finer than the product topology.

⟨2⟩1. LET: U be open in \mathbb{R} and $j \in J$

PROVE: $\pi_j^{-1}(U)$ is open in the uniform topology.

⟨2⟩2. LET: $x \in \pi_j^{-1}(U)$

⟨2⟩3. $\pi_j(x) \in U$

⟨2⟩4. PICK $\epsilon > 0$ such that $B_{\bar{d}}(\pi_j(x), \epsilon) \subseteq U$

⟨2⟩5. $B_{\bar{\rho}}(x, \epsilon) \subseteq \pi_j^{-1}(U)$

⟨1⟩2. If J is finite then the uniform topology is equal to the product topology.

PROOF: In \mathbb{R}^n , the uniform topology is the square topology.

⟨1⟩3. If J is infinite then the uniform topology is not equal to the product topology.

PROOF: If J is infinite then $B(0, 1)$ is not open in the product topology.

□

Proposition 13.0.41. *The uniform topology is coarser than the box topology. It is strictly coarser iff J is infinite.*

PROOF:

⟨1⟩1. The uniform topology is coarser than the box topology.

⟨2⟩1. LET: U be open in the uniform topology.

PROVE: U is open in the box topology.

⟨2⟩2. LET: $x \in U$

⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$

⟨2⟩4. $\prod_{j \in J} (x_j - \epsilon, x_j + \epsilon) \subseteq U$

⟨1⟩2. If J is finite then the uniform topology is equal to the box topology.

PROOF: On \mathbb{R}^n , the uniform metric is the square metric.

⟨1⟩3. If J is infinite then the uniform topology is not equal to the box topology.

⟨2⟩1. ASSUME: J is infinite.

⟨2⟩2. PICK a sequence (j_n) of distinct elements in J .

⟨2⟩3. LET: $U = \prod_j U_j$ where $J_{j_n} = (-1/(n+1), 1/(n+1))$ for $n \in \mathbb{N}$ and $J_j = (-1, 1)$ for all other j .

⟨2⟩4. U is not open in the uniform topology.

□

Proposition 13.0.42. *The uniform topology on \mathbb{R}^∞ is strictly finer than the product topology.*

PROOF: The set of all sequences $(x_n) \in \mathbb{R}^\infty$ such that $\forall n. |x_n| < 1$ is open in the uniform topology but not in the product topology. \square

Proposition 13.0.43. *The uniform topology on \mathbb{R}^∞ is strictly coarser than the box topology.*

PROOF: The set of sequences $(x_n) \in \mathbb{R}^\infty$ such that $\forall n. |x_n| < 1/n$ is open in the box topology but not in the uniform topology. \square

Proposition 13.0.44. *The uniform topology on the Hilbert cube is the same as the product topology.*

PROOF:

$\langle 1 \rangle 1$. LET: (x_n) be in the Hilbert cube H and $\epsilon > 0$.

PROVE: $B((x_n), \epsilon) \cap H$ is open in the product topology.

$\langle 1 \rangle 2$. PICK N such that $1/N < \epsilon$

$\langle 1 \rangle 3$. $B((x_n), \epsilon) = (\prod_{n=0}^N (x_n - \epsilon, x_n + \epsilon) \times \prod_{n=N+1}^\infty [0, 1/(n+1)]) \cap H$
 \square

Corollary 13.0.44.1. *The uniform topology on the Hilbert cube is strictly finer than the box topology.*

Proposition 13.0.45. *Let X be a set and Y a metric space. Let (f_n) be a sequence of functions $X \rightarrow Y$, and $f : X \rightarrow Y$. Then (f_n) converges uniformly to f iff (f_n) converges to f in Y^X under the uniform topology.*

PROOF:

$\langle 1 \rangle 1$. If (f_n) converges uniformly to f then (f_n) converges to f in Y^X under the uniform topology.

$\langle 2 \rangle 1$. ASSUME: (f_n) converges uniformly to f .

$\langle 2 \rangle 2$. LET: $\epsilon > 0$

$\langle 2 \rangle 3$. PICK N such that $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon/2$

$\langle 2 \rangle 4$. $\forall n \geq N. \bar{\rho}(f_n, f) \leq \epsilon/2$

$\langle 2 \rangle 5$. $\forall n \geq N. \bar{\rho}(f_n, f) < \epsilon$

$\langle 1 \rangle 2$. If (f_n) converges to f in Y^X under the uniform topology then (f_n) converges uniformly to f .

$\langle 2 \rangle 1$. ASSUME: (f_n) converges to f in Y^X under the uniform topology.

$\langle 2 \rangle 2$. LET: $\epsilon > 0$

$\langle 2 \rangle 3$. PICK N such that $\forall n \geq N. \bar{\rho}(f_n, f) < \epsilon$

$\langle 2 \rangle 4$. $\forall n \geq N. \forall x \in X. d(f_n(x), f(x)) < \epsilon$
 \square

13.0.12 Products

Definition 13.0.46 (Euclidean Metric). Let X and Y be metric spaces. The Euclidean metric on $X \times Y$ is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}.$$

We write $X \times Y$ for the set $X \times Y$ under this metric.

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1$. $d((x_1, y_1), (x_2, y_2)) \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. $d((x_1, y_1), (x_2, y_2)) = 0$ iff $(x_1, y_1) = (x_2, y_2)$

PROOF: $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = 0$ iff $d(x_1, x_2) = d(y_1, y_2) = 0$ iff $x_1 = x_2$ and $y_1 = y_2$.

$\langle 1 \rangle 3$. $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$

PROOF: Since $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2} = \sqrt{d(x_2, x_1)^2 + d(y_2, y_1)^2}$.

$\langle 1 \rangle 4$. The triangle inequality holds.

PROOF:

$$\begin{aligned}
 & (d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)))^2 \\
 &= d((x_1, y_1), (x_2, y_2))^2 + 2d((x_1, y_1), (x_2, y_2))d((x_2, y_2), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3))^2 \\
 &= d(x_1, x_2)^2 + d(y_1, y_2)^2 + 2\sqrt{(d(x_1, x_2)^2 + d(y_1, y_2)^2)(d(x_2, x_3)^2 + d(y_2, y_3)^2)} + d(x_2, x_3)^2 + d(y_2, y_3)^2 \\
 &\geq d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(y_1, y_2)^2 + d(y_2, y_3)^2 + 2(d(x_1, x_2)d(x_2, x_3) + d(y_1, y_2)d(y_2, y_3)) \\
 &\quad \text{(Cauchy-Schwarz)} \\
 &= (d(x_1, x_2) + d(x_2, x_3))^2 + (d(y_1, y_2) + d(y_2, y_3))^2 \\
 &\geq d(x_1, x_3)^2 + d(y_1, y_3)^2 \\
 &= d((x_1, y_1), (x_3, y_3))^2
 \end{aligned}$$

□

Proposition 13.0.47. *Let X and Y be metric spaces. The Euclidean metric on $X \times Y$ induces the product topology on $X \times Y$.*

PROOF:

$\langle 1 \rangle 1$. Every open ball is open in the product topology.

$\langle 2 \rangle 1$. LET: $(x, y) \in B((a, b), \epsilon)$

PROVE: $B(x, \sqrt{\epsilon}) \times B(y, \sqrt{\epsilon}) \subseteq B((a, b), \epsilon)$

$\langle 2 \rangle 2$. LET: $x' \in B(x, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$ and $y' \in B(y, \sqrt{(\epsilon - d((x, y), (a, b)))^2/2})$

PROVE: $d((x', y'), (a, b)) < \epsilon$

$\langle 2 \rangle 3$. $d((x', y'), (x, y)) < \epsilon - d((x, y), (a, b))$

PROOF:

$$\begin{aligned}
 d((x', y'), (x, y)) &= \sqrt{d(x', x)^2 + d(y', y)^2} \\
 &< \sqrt{(\epsilon - d((x, y), (a, b)))^2/2 + (\epsilon - d((x, y), (a, b)))^2/2} \\
 &= \epsilon - d((x, y), (a, b))
 \end{aligned}$$

$\langle 2 \rangle 4$. $d((x', y'), (a, b)) < \epsilon$

PROOF:

$$d((x', y'), (a, b)) \leq d((x', y'), (x, y)) + d((x, y), (a, b)) \quad \text{(Triangle Inequality)}$$

$$< \epsilon$$

($\langle 2 \rangle 3$)

$\langle 1 \rangle 2$. If U is open in X and V is open in Y then $U \times V$ is open under the Euclidean metric.

- ⟨2⟩1. LET: $(x, y) \in U \times V$
- ⟨2⟩2. PICK $\delta, \epsilon > 0$ such that $B(x, \delta) \subseteq U$ and $B(y, \epsilon) \subseteq V$
 PROVE: $(B((x, y), \min(\delta, \epsilon))) \subseteq U \times V$
- ⟨2⟩3. LET: $(x', y') \in B((x, y), \min(\delta, \epsilon))$
- ⟨2⟩4. $d(x', x) < \delta$
 - ⟨3⟩1. $d((x', y'), (x, y)) < \min(\delta, \epsilon)$
 - ⟨3⟩2. $d(x', x)^2 + d(y', y)^2 < \delta^2$
 - ⟨3⟩3. $d(x', x)^2 < \delta^2$
- ⟨2⟩5. $d(y', y) < \epsilon$
 PROOF: Similar.
- ⟨2⟩6. $(x', y') \in U \times V$

□

Proposition 13.0.48. *The square metric on \mathbb{R}^n induces the product topology.*

PROOF:

- ⟨1⟩1. LET: d be the Euclidean metric on \mathbb{R}^n and ρ the square metric.
- ⟨1⟩2. For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_d(x, \delta) \subseteq B_\rho(x, \epsilon)$
 PROOF: If $d(x, y) < \epsilon$ then $\rho(x, y) < \epsilon$.
- ⟨1⟩3. For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_\rho(x, \delta) \subseteq B_d(x, \epsilon)$
 PROOF: If $\rho(x, y) < \epsilon/\sqrt{n}$ then $d(x, y) < \epsilon$.
- ⟨1⟩4. d and ρ induce the same topology.
 PROOF: Proposition 13.0.17.

□

13.1 Isometric Embeddings

Definition 13.1.1 (Isometric Embedding). Let X and Y be metric spaces. Let $f : X \rightarrow Y$. Then f is an *isometric embedding* of X in Y iff, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 13.1.2. *Every isometric embedding is an embedding.*

PROOF:

- ⟨1⟩1. LET: X and Y be metric spaces.
- ⟨1⟩2. LET: $f : X \rightarrow Y$ be an isometric embedding.
- ⟨1⟩3. f is injective.
- ⟨1⟩4. The subspace topology induced by f is finer than the metric topology.
 - ⟨2⟩1. LET: $x \in X$ and $\epsilon > 0$
 PROVE: $B(x, \epsilon)$ is open in the subspace topology.
 - ⟨2⟩2. $B(x, \epsilon) = f^{-1}(B(f(x), \epsilon))$
- ⟨1⟩5. The metric topology is finer than the subspace topology induced by f .
 - ⟨2⟩1. LET: V be open in Y
 PROVE: $f^{-1}(V)$ is open in X
 - ⟨2⟩2. LET: $x \in f^{-1}(V)$
 - ⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
 - ⟨2⟩4. $B(x, \epsilon) \subseteq f^{-1}(V)$

□

13.2 Complete Metric Spaces

Definition 13.2.1 (Complete). A metric space is *complete* iff every Cauchy sequence converges.

Example 13.2.2. \mathbb{R} is complete.

Proposition 13.2.3. *The product of two complete metric spaces is complete.*

Proposition 13.2.4. *Every compact metric space is complete.*

Proposition 13.2.5. *Let X be a complete metric space and $A \subseteq X$. Then A is complete if and only if A is closed.*

Definition 13.2.6 (Completion). Let X be a metric space. A *completion* of X is a complete metric space \hat{X} and injection $i : X \rightarrow \hat{X}$ such that:

- The metric on X is the restriction of the metric on \hat{X}
- X is dense in \hat{X} .

Proposition 13.2.7. *Let $i_1 : X \rightarrow Y_1$ and $i_2 : X \rightarrow Y_2$ be completions of X . Then there exists a unique isometry $\phi : Y_1 \cong Y_2$ such that $\phi \circ i_1 = i_2$.*

PROOF: Define $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$. \square

Theorem 13.2.8. *Every metric space has a completion.*

PROOF: Let \hat{X} be the set of Cauchy sequences in X quotiented by \sim where $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \rightarrow 0$. \square

13.3 Manifolds

Definition 13.3.1 (Manifold). An *n -dimensional manifold* is a second countable Hausdorff space locally homeomorphic to \mathbb{R}^n .

Chapter 14

Homotopy Theory

14.1 Homotopies

Definition 14.1.1 (Homotopy). Let X and Y be topological spaces. Let $f, g : X \rightarrow Y$ be continuous. A *homotopy* between f and g is a continuous function $h : X \times [0, 1] \rightarrow Y$ such that

- $\forall x \in X. h(x, 0) = f(x)$
- $\forall x \in X. h(x, 1) = g(x)$

We say f and g are *homotopic*, $f \simeq g$, iff there exists a homotopy between them.

Let $[X, Y]$ be the set of all homotopy classes of functions $X \rightarrow Y$.

Proposition 14.1.2. Let $f, f' : X \rightarrow Y$ and $g, g' : Y \rightarrow Z$ be continuous. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.

Definition 14.1.3. Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

A *homotopy functor* is a functor $\mathbf{Top} \rightarrow \mathcal{C}$ that factors through the canonical functor $\mathbf{Top} \rightarrow \mathbf{HTop}$.

Definition 14.1.4. A functor $F : \mathbf{Top} \rightarrow \mathcal{C}$ is *homotopy invariant* iff, for any topological spaces X, Y and continuous functions $f, g : X \rightarrow Y$, if $f \simeq g$ then $Hf = Hg$.

Basepoint-preserving homotopy.

14.2 Homotopy Equivalence

Definition 14.2.1 (Homotopy Equivalence). Let X and Y be topological spaces. A *homotopy equivalence* between X and Y , $f : X \simeq Y$, is a continuous function $f : X \rightarrow Y$ such that there exists a continuous function $g : Y \rightarrow X$, the *homotopy inverse* to f , such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.

Definition 14.2.2 (Contractible). A topological space X is *contractible* iff $X \simeq 1$.

Example 14.2.3. \mathbb{R}^n is contractible.

Example 14.2.4. D^n is contractible.

Definition 14.2.5 (Deformation Retract). Let X be a topological space and A a subspace of X . A retraction $\rho : X \rightarrow A$ is a *deformation retraction* iff $i \circ \rho \simeq \text{id}_X$, where i is the inclusion $A \hookrightarrow X$. We say A is a *deformation retract* of X iff there exists a deformation retraction.

Definition 14.2.6 (Strong Deformation Retract). Let X be a topological space and A a subspace of X . A *strong deformation retraction* $\rho : X \rightarrow A$ is a continuous function such that there exists a homotopy $h : X \times [0, 1] \rightarrow X$ between $i \circ \rho$ and id_X such that, for all $a \in X$ and $t \in [0, 1]$, we have $h(a, t) = a$.

We say A is a *strong deformation retract* of X iff a strong deformation retraction exists.

Example 14.2.7. $\{0\}$ is a strong deformation retract of \mathbb{R}^n and of D^n .

Example 14.2.8. S^1 is a strong deformation retract of the torus $S^1 \times D^2$.

Example 14.2.9. S^{n-1} is a strong deformation retract of $D^n - \{0\}$.

Example 14.2.10. For any topological space X , the singleton consisting of the vertex is a strong deformation retract of the cone over X .

Chapter 15

Simplicial Complexes

Definition 15.0.1 (Simplex). A k -dimensional simplex or k -simplex in \mathbb{R}^n is the convex hull $s(x_0, \dots, x_k)$ of $k + 1$ points in general position.

Definition 15.0.2 (Face). A *sub-simplex* or *face* of $s(x_0, \dots, x_k)$ is the convex hull of a subset of $\{x_0, \dots, x_k\}$.

Definition 15.0.3 (Simplicial Complex). A *simplicial complex* in \mathbb{R}^n is a set K of simplices such that:

- for every simplex s in K , every face of s is in K .
- The intersection of two simplices $s_1, s_2 \in K$ is either empty or is a face of both s_1 and s_2 .
- K is locally finite, i.e. every point of \mathbb{R}^n has a neighbourhood that only intersects finitely many elements of K .

The topological space *underlying* K is $|K| = \bigcup K$ as a subspace of \mathbb{R}^n .

15.1 Cell Decompositions

Definition 15.1.1 (n -cell). An n -cell is a topological space homeomorphic to \mathbb{R}^n .

Definition 15.1.2 (Cell Decomposition). Let X be a topological space. A *cell decomposition* of X is a partition of X into subspaces that are n -cells.

Definition 15.1.3 (n -skeleton). Given a cell decomposition of X , the n -skeleton X^n is the union of all the cells of dimension $\leq n$.

15.2 CW-complexes

Definition 15.2.1 (CW-Complex). A *CW-complex* consists of a topological space X and a cell decomposition \mathcal{E} of X such that:

1. *Characteristic Maps* For every n -cell $e \in \mathcal{E}$, there exists a continuous map $\Phi_e : D^n \rightarrow X$ such that $\Phi_e((D^n)^\circ) = e$, the corestriction $\Phi_e : (D^n)^\circ \approx e$ is a homeomorphism, and $\Phi_e(S^n)$ is the union of all the cells in \mathcal{E} of dimension $< n$.
2. *Closure Finiteness* For all $e \in \mathcal{E}$, we have \bar{e} intersects only finitely many other cells in \mathcal{E} .
3. *Weak Topology* Given $A \subseteq X$, we have A is closed iff for all $e \in \mathcal{E}$, $A \cap \bar{e}$ is closed.

Proposition 15.2.2. *If a cell decomposition \mathcal{E} satisfies the Characteristic Maps axiom, then for every n -cell $e \in \mathcal{E}$ we have $\bar{e} = \Phi_e(D^n)$. Therefore \bar{e} is compact and $\bar{e} - e = \Phi_e(S^{n-1}) \subseteq X^{n-1}$.*

PROOF:

$\langle 1 \rangle 1.$ $e \subseteq \Phi_e(D^n) \subseteq \bar{e}$

PROOF:

$$\begin{aligned}
 e &= \Phi_e((D^n)^\circ) \\
 &\subseteq \Phi_e(D^n) \\
 &= \Phi_e(\overline{(D^n)^\circ}) \\
 &\subseteq \overline{\Phi_e((D^n)^\circ)} \\
 &= \bar{e}
 \end{aligned}$$

$\langle 1 \rangle 2.$ $\Phi_e(D^n)$ is compact.

PROOF: Because D^n is compact.

$\langle 1 \rangle 3.$ $\Phi_e(D^n)$ is closed.

$\langle 1 \rangle 4.$ $\Phi_e(D^n) = \bar{e}$

□

Chapter 16

Topological Groups

16.1 Topological Groups

Definition 16.1.1 (Topological Group). A *topological group* is a group G with a topology such that the function $G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.

Example 16.1.2. \mathbb{Z} is a topological group under addition.

PROOF: The function that sends (x, y) to xy^{-1} is continuous because the topology on \mathbb{Z} is discrete. \square

Example 16.1.3. \mathbb{R} is a topological group under addition.

PROOF: From Propositions 13.0.10 and 13.0.11. \square

Example 16.1.4. \mathbb{R}_+ is a topological group under multiplication.

PROOF: From Propositions 13.0.11 and 13.0.24. \square

Example 16.1.5. S^1 as a subspace of \mathbb{C} is a topological group under multiplication.

PROOF:

$\langle 1 \rangle 1$. LET: $f : S^1 \rightarrow S^1$ be the function $f(x, y) = xy^{-1}$

$\langle 1 \rangle 2$. LET: U be an open set in S^1

PROVE: $f^{-1}(U)$ is open in $(S^1)^2$

$\langle 1 \rangle 3$. LET: $(x, y) \in f^{-1}(U)$

$\langle 1 \rangle 4$. $xy^{-1} \in U$

$\langle 1 \rangle 5$. LET: $x = e^{i\phi}$ and $y = e^{i\psi}$

$\langle 1 \rangle 6$. $xy^{-1} = e^{i(\phi-\psi)} \in U$

$\langle 1 \rangle 7$. PICK $\epsilon > 0$ such that, for all t , if $|\phi - \psi - t| < \epsilon$ then $e^{it} \in U$

$\langle 1 \rangle 8$. $(x, y) \in \{e^{it} : |\phi - t| < \epsilon/2\} \times \{e^{it} : |\psi - t| < \epsilon/2\} \subseteq f^{-1}(U)$

\square

Example 16.1.6. $GL(n, \mathbb{R})$ is a topological group considered as a subspace of \mathbb{R}^{n^2} .

PROOF: Since the calculations for matrix multiplication and inverse are compositions of continuous functions. \square

Example 16.1.7. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.

Proposition 16.1.8. *Let G be a group with a topology. Then G is a topological group if and only if the functions $m : G^2 \rightarrow G$ that sends (x, y) to xy and the function $i : G \rightarrow G$ that sends x to x^{-1} are continuous.*

PROOF:

$\langle 1 \rangle 1$. If G is a topological group then i is continuous.

PROOF: Since $x^{-1} = ex^{-1}$.

$\langle 1 \rangle 2$. If G is a topological group then m is continuous.

PROOF: Since $xy = x(y^{-1})^{-1}$.

$\langle 1 \rangle 3$. If m and i are continuous then G is a topological group.

PROOF: Since $xy^{-1} = m(x, i(y))$.

\square

Proposition 16.1.9. *Let G be a topological group. Let $\alpha \in G$. The function that maps x to αx is a homeomorphism between G and itself.*

PROOF:

$\langle 1 \rangle 1$. For any $\alpha \in G$, the function that maps x to αx is continuous.

PROOF: From the definition of topological group.

$\langle 1 \rangle 2$. For any $\alpha \in G$, the function that maps x to αx is a homeomorphism between G and itself.

PROOF: Its inverse is the function that maps x to $\alpha^{-1}x$.

\square

Corollary 16.1.9.1. *Every topological group is homogeneous.*

Proposition 16.1.10. *Let G be a topological group. Let $\alpha \in G$. The function that maps x to $x\alpha$ is a homeomorphism between G and itself.*

PROOF: Similar. \square

16.1.1 Subgroups

Proposition 16.1.11. *Any subgroup of a topological group is a topological group under the subspace topology.*

PROOF: Since the restriction of continuous functions is continuous. \square

Proposition 16.1.12. *Let G be a topological group and H a subgroup of G . Then \overline{H} is a topological group under the subspace topology.*

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \overline{H}$

PROVE: $xy^{-1} \in \overline{H}$

$\langle 1 \rangle 2$. LET: U be a neighbourhood of xy^{-1} .

PROVE: U intersects H .

⟨1⟩3. LET: $f : G^2 \rightarrow G$ be the function that maps (x, y) to xy^{-1} .

⟨1⟩4. $f^{-1}(U)$ is a neighbourhood of (x, y)

⟨1⟩5. PICK neighbourhoods V of x and W of y such that $V \times W \subseteq f^{-1}(U)$.

⟨1⟩6. PICK elements $x' \in V \cap H$ and $y' \in W \cap H$

⟨1⟩7. $x'y'^{-1} \in U \cap H$

□

16.1.2 Left Cosets

Proposition 16.1.13. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. Let $\alpha \in G$. Define $f_\alpha : G/H \rightarrow G/H$ by*

$$f_\alpha(xH) = \alpha xH .$$

Then f_α is a homeomorphism.

PROOF:

⟨1⟩1. For all $\alpha \in G$ we have f_α is well defined.

⟨2⟩1. LET: $x, y \in G$

⟨2⟩2. ASSUME: $xH = yH$

PROVE: $\alpha xH = \alpha yH$

⟨2⟩3. $x^{-1}y \in H$

⟨2⟩4. $x^{-1}\alpha^{-1}\alpha y \in H$

⟨2⟩5. $\alpha xH = \alpha yH$

⟨1⟩2. For all $\alpha \in G$ we have f_α is injective.

⟨2⟩1. LET: $x, y \in G$

⟨2⟩2. ASSUME: $\alpha xH = \alpha yH$

PROVE: $xH = yH$

⟨2⟩3. $\alpha x^{-1}\alpha y \in H$

⟨2⟩4. $x^{-1}y \in H$

⟨2⟩5. $xH = yH$

⟨1⟩3. For all $\alpha \in G$ we have f_α is surjective.

PROOF: For all $x \in G$ we have $xH = f_\alpha(\alpha^{-1}xH)$.

⟨1⟩4. For all $\alpha \in G$ we have f_α is continuous.

⟨2⟩1. LET: V be open in G/H

⟨2⟩2. $\pi^{-1}(f_\alpha^{-1}(V))$ is open in G .

PROOF: It is $g_\alpha^{-1}(\pi^{-1}(V))$ where $g_\alpha : V \rightarrow V$ is the homeomorphism

$$g_\alpha(x) = \alpha x.$$

⟨2⟩3. $f_\alpha^{-1}(V)$ is open in G/H .

⟨1⟩5. For all $\alpha \in G$ we have f_α^{-1} is continuous.

PROOF: It is $f_{\alpha^{-1}}$.

□

Corollary 16.1.13.1. *Let G be a topological group and H a subgroup of G . Then G/H is a homogeneous space.*

Proposition 16.1.14. *Let G be a T_1 topological group and H a closed subgroup of G . Then G/H is T_1 .*

PROOF:

$\langle 1 \rangle 1$. LET: $x \in G$

PROVE: xH is closed.

$\langle 1 \rangle 2$. $\pi^{-1}(xH)$ is closed in G .

PROOF: It is $f_x(H)$ and f_x is a homeomorphism.

$\langle 1 \rangle 3$. xH is closed in G/H .

□

Proposition 16.1.15. *Let G be a topological group and H a subgroup of G . Then the canonical map $\pi : G \rightarrow G/H$ is an open map.*

PROOF:

$\langle 1 \rangle 1$. LET: U be open in G .

$\langle 1 \rangle 2$. $\forall h \in H. Uh$ is open in G .

PROOF: Since the function that maps g to gh is an automorphism of G .

$\langle 1 \rangle 3$. UH is open in G

PROOF: It is $\bigcup_{h \in H} Uh$.

$\langle 1 \rangle 4$. $UH = \pi^{-1}(\pi(U))$

PROOF:

$$\begin{aligned} \pi^{-1}(\pi(U)) &= \{x \in G : \exists y \in U. xH = yH\} \\ &= \{x \in G : \exists y \in U. x^{-1}y \in H\} \\ &= \{x \in G : \exists y \in U. \exists h \in H. y^{-1}x = h\} \\ &= \{x \in G : \exists y \in U. \exists h \in H. x = yh\} \\ &= UH \end{aligned}$$

$\langle 1 \rangle 5$. $\pi^{-1}(\pi(U))$ is open in G .

$\langle 1 \rangle 6$. $\pi(U)$ is open in G/H .

□

Proposition 16.1.16. *Let G be a topological group. Let H be a normal subgroup of G . Then G/H is a topological group.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : G^2 \rightarrow G$ be the map $f(x, y) = xy^{-1}$

$\langle 1 \rangle 2$. LET: $g : (G/H)^2 \rightarrow G/H$ be the map $g(xH, yH) = xy^{-1}H$

$\langle 1 \rangle 3$. $g \circ (\pi \times \pi) = \pi \circ f : G^2 \rightarrow G/H$

$\langle 1 \rangle 4$. $g \circ (\pi \times \pi)$ is continuous.

PROOF: Since π and f are continuous.

$\langle 1 \rangle 5$. π is an open quotient map.

PROOF: Proposition 16.1.15.

$\langle 1 \rangle 6$. $\pi \times \pi$ is an open quotient map.

PROOF: Corollary 12.21.7.1.

$\langle 1 \rangle 7$. g is continuous.

PROOF: Theorem 12.21.3.

□

16.1.3 Homogeneous Spaces

Definition 16.1.17 (Homogeneous Space). A *homogeneous space* is a topological space of the form G/H , where G is a topological group and H is a normal subgroup of G , under the quotient topology.

Proposition 16.1.18. *Let G be a topological group and H a normal subgroup of G . Then G/H is Hausdorff if and only if H is closed.*

PROOF: See Bourbaki, N., General Topology. III.12 \square

16.2 Symmetric Neighbourhoods

Definition 16.2.1 (Symmetric Neighbourhood). Let G be a topological group. Let V be a neighbourhood of e . Then V is *symmetric* iff $V = V^{-1}$.

Proposition 16.2.2. *Let G be a topological group. Let U be a neighbourhood of e . Then there exists a symmetric neighbourhood V of e such that $VV \subseteq U$.*

PROOF:

- $\langle 1 \rangle 1$. PICK a neighbourhood V' of e such that $V'V' \subseteq U$.
 - $\langle 2 \rangle 1$. LET: $m : G^2 \rightarrow G$ be the function $m(x, y) = xy$
 - $\langle 2 \rangle 2$. $m^{-1}(U)$ is open in G^2
 - $\langle 2 \rangle 3$. $(e, e) \in m^{-1}(U)$
 - $\langle 2 \rangle 4$. PICK neighbourhoods V_1, V_2 of e such that $V_1 \times V_2 \subseteq m^{-1}(U)$
 - $\langle 2 \rangle 5$. LET: $V' = V_1 \cap V_2$
 - $\langle 1 \rangle 2$. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
 - $\langle 2 \rangle 1$. LET: $f : G^2 \rightarrow G$ be the function $m(x, y) = xy^{-1}$
 - $\langle 2 \rangle 2$. $f^{-1}(V')$ is open in G^2
 - $\langle 2 \rangle 3$. $(e, e) \in m^{-1}(V')$
 - $\langle 2 \rangle 4$. PICK neighbourhoods W_1, W_2 of e such that $W_1 \times W_2 \subseteq f^{-1}(V')$
 - $\langle 2 \rangle 5$. LET: $W = W_1 \cap W_2$
 - $\langle 1 \rangle 3$. LET: $V = WW^{-1}$
 - $\langle 1 \rangle 4$. V is a neighbourhood of e .
 - $\langle 1 \rangle 5$. V is symmetric.
 - $\langle 1 \rangle 6$. $VV \subseteq U$
- \square

Proposition 16.2.3. *Every T_1 topological group is regular.*

PROOF:

- $\langle 1 \rangle 1$. LET: G be a T_1 topological group.
 - $\langle 1 \rangle 2$. LET: A be a closed set in G and $x \in G - A$.
 - $\langle 1 \rangle 3$. $G - Ax^{-1}$ is a neighbourhood of e .
 - $\langle 1 \rangle 4$. PICK a symmetric neighbourhood V of e such that $VV \subseteq G - Ax^{-1}$.
 - $\langle 1 \rangle 5$. LET: $U = VA$ and $U' = Vx$
 - $\langle 1 \rangle 6$. U and U' are disjoint open sets with $A \subseteq U$ and $x \in U'$.
- \square

Proposition 16.2.4. *Let G be a T_1 topological group. Let H be a closed subgroup of G . Then G/H is regular.*

PROOF:

- ⟨1⟩1. LET: A be a closed set in G/H and $xH \in G/H - A$.
- ⟨1⟩2. $G - \pi^{-1}(A)x^{-1}$ is a neighbourhood of e .
- ⟨1⟩3. PICK a symmetric neighbourhood V of e such that $VV \subseteq G - \pi^{-1}(A)x^{-1}$.
- ⟨1⟩4. LET: $U = \pi(V)A$ and $U' = \pi(V)(xH)$.
- ⟨1⟩5. U and U' are disjoint open sets with $A \subseteq U$ and $xH \in U'$
 - ⟨2⟩1. ASSUME: for a contradiction $U \cap U' \neq \emptyset$.
 - ⟨2⟩2. PICK $v_1, v_2 \in V$ and $a \in G$ such that $aH \in A$ and $v_1aH = v_2xH$.
 - ⟨2⟩3. $a^{-1}v_1^{-1}v_2x \in H$
 - ⟨2⟩4. $v_1^{-1}v_2 \in \pi^{-1}(A)x^{-1}$
 - ⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩3.

□

16.3 Continuous Actions

Definition 16.3.1 (Continuous Action). Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous function $\cdot : G \times X \rightarrow X$ such that:

- $\forall x \in X. ex = x$
- $\forall g, h \in G. \forall x \in X. g(hx) = (gh)x$

A G -space consists of a topological space X and a continuous action of G on X .

Definition 16.3.2 (Orbit). Let X be a G -space and $x \in X$. The *orbit* of x is $\{gx : g \in G\}$.

The *orbit space* X/G is the set of all orbits under the quotient topology.

Proposition 16.3.3. *Define an action of $SO(2)$ on S^2 by*

$$g(x_1, x_2, x_3) = (g(x_1, x_2), x_3) \ .$$

Then $S^2/SO(2) \cong [-1, 1]$.

PROOF:

- ⟨1⟩1. LET: $f_3 : S^2/SO(2) \rightarrow [-1, 1]$ be the function induced by $\pi_3 : S^2 \rightarrow [-1, 1]$
 - ⟨1⟩2. f_3 is bijective.
 - ⟨1⟩3. $S^2/SO(2)$ is compact.
- PROOF: It is the continuous image of S^2 which is compact.
- ⟨1⟩4. $[-1, 1]$ is Hausdorff.
 - ⟨1⟩5. f_3 is a homeomorphism.

□

Definition 16.3.4 (Stabilizer). Let X be a G -space and $x \in X$. The *stabilizer* of x is $G_x := \{g \in G : gx = x\}$.

Proposition 16.3.5. *The function that maps gG_x to gx is a continuous bijection from G/G_x to Gx .*

PROOF:

⟨1⟩1. If $gG_x = hG_x$ then $gx = hx$.

⟨2⟩1. ASSUME: $gG_x = hG_x$

⟨2⟩2. $g^{-1}h \in G_x$

⟨2⟩3. $g^{-1}hx = x$

⟨2⟩4. $gx = hx$

⟨1⟩2. If $gx = hx$ then $gG_x = hG_x$.

PROOF: Similar.

⟨1⟩3. The function is continuous.

PROOF: Theorem 12.21.3.

□

Chapter 17

Topological Vector Spaces

Definition 17.0.1 (Topological Vector Space). Let K be either \mathbb{R} or \mathbb{C} . A *topological vector space* over K consists of a vector space E over K and a topology on E such that:

- Subtraction is a continuous function $E^2 \rightarrow E$
- Multiplication is a continuous function $K \times E \rightarrow E$

Proposition 17.0.2. *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition. \square

Theorem 17.0.3. *The usual topology on a finite dimensional vector space over K is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18 \square

Proposition 17.0.4. *Let E be a topological vector space and E_0 a subspace of E . Then $\overline{E_0}$ is a subspace of E .*

Definition 17.0.5. Let E be a topological vector space. The topological space associated with E is $E/\overline{\{0\}}$.

17.1 Cauchy Sequences

Definition 17.1.1 (Cauchy Sequence). Let E be a topological vector space. A sequence (x_n) in E is a *Cauchy sequence* iff, for every neighbourhood U of 0, there exists n_0 such that $\forall m, n \geq n_0, x_n - x_m \in U$.

Definition 17.1.2 (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

17.2 Seminorms

Definition 17.2.1 (Seminorm). Let E be a vector space over K . A *seminorm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ such that:

1. $\forall x \in E, \|x\| \geq 0$
2. $\forall \alpha \in K, \forall x \in E, \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality* $\forall x, y \in E, \|x + y\| \leq \|x\| + \|y\|$

Example 17.2.2. The function that maps (x_1, \dots, x_n) to $|x_i|$ is a seminorm on \mathbb{R}^n .

Definition 17.2.3. Let E be a vector space over K . Let Λ be a set of seminorms on E . The topology *generated* by Λ is the topology generated by the subbasis consisting of all sets of the form $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$ for $\epsilon > 0$, $\lambda \in \Lambda$ and $x \in E$.

Proposition 17.2.4. E is a topological vector space under this topology. It is Hausdorff iff, for all $x \in E$, if $\forall \lambda \in \Lambda, \lambda(x) = 0$ then $x = 0$.

17.3 Fréchet Spaces

Definition 17.3.1 (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

Proposition 17.3.2. Let E be a pre-Fréchet space whose topology is generated by the family of seminorms $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$. Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

Definition 17.3.3 (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

17.4 Normed Spaces

Definition 17.4.1 (Normed Space). Let E be a vector space over K . A *norm* on E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ is a seminorm such that, $\forall x \in E, \|x\| = 0 \Leftrightarrow x = 0$.

A *normed space* consists of a vector space with a norm.

Proposition 17.4.2. If E is a normed space then $d(x, y) = \|x - y\|$ is a metric on E that makes E into a topological vector space. The two definitions of Cauchy sequence agree on E .

Definition 17.4.3 (p -norm). For any $p \geq 1$, the p -norm on \mathbb{R}^n is defined by

$$\|\vec{x}\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1$. For all $\vec{x} \in \mathbb{R}^n$ we have $\|\vec{x}\|_p \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. For all $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ we have $\|\alpha \vec{x}\|_p = |\alpha| \|\vec{x}\|_p$

PROOF:

$$\begin{aligned} \|\alpha(x_1, \dots, x_n)\| &= \|(\alpha x_1, \dots, \alpha x_n)\| \\ &= \left(\sum_{i=1}^n (\alpha x_i)^p \right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|\vec{x}\|_p \end{aligned}$$

$\langle 1 \rangle 3$. The triangle inequality holds.

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\ &= \sum_{i=1}^n |x_i + y_i| |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^n (|x_i| + |y_i|) |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} \\ &\leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} \quad (\text{Hölder's Inequality}) \\ &= (\|\vec{x}\|_p + \|\vec{y}\|_p) \|\vec{x} + \vec{y}\|_p^{p-1} \end{aligned}$$

Assuming w.l.o.g. $\|\vec{x} + \vec{y}\|_p^{p-1} \neq 0$ (using ??) we have $\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$.

$\langle 1 \rangle 4$. For any $\vec{x} \in \mathbb{R}^n$, we have $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.

PROOF: $\sum_{i=1}^n x_i^p = 0$ iff $x_1 = \dots = x_n = 0$.

□

Proposition 17.4.4. The p -norm on \mathbb{R}^n induces the product topology.

PROOF:

⟨1⟩1. LET: d be the metric induced by the p -norm and ρ the square metric on \mathbb{R}^n .

⟨1⟩2. The metric topology is finer than the product topology.

⟨2⟩1. LET: $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$

⟨2⟩2. LET: $\delta = \epsilon/n^{\frac{1}{p}}$

PROVE: $B_\rho(\vec{x}, \delta) \subseteq B_d(\vec{x}, \epsilon)$

⟨2⟩3. LET: $\vec{y} \in B_\rho(\vec{x}, \delta)$

⟨2⟩4. $\forall i. |x_i - y_i| < \delta$

⟨2⟩5. $d(\vec{x}, \vec{y}) < \epsilon$

PROOF:

$$\begin{aligned} d(\vec{x}, \vec{y}) &= \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}} \\ &< \left(\sum_{i=1}^n \delta^p \right)^{\frac{1}{p}} && (\langle 2 \rangle 4) \\ &= n^{\frac{1}{p}} \delta \\ &= \epsilon && (\langle 2 \rangle 2) \end{aligned}$$

⟨1⟩3. The product topology is finer than the metric topology.

⟨2⟩1. LET: $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$

⟨2⟩2. LET: $\vec{y} \in B_d(\vec{x}, \epsilon)$

⟨2⟩3. $d(\vec{x}, \vec{y}) < \epsilon$

⟨2⟩4. $\sum_{i=1}^n |x_i - y_i|^p < \epsilon^p$

⟨2⟩5. $\forall i. |x_i - y_i|^p < \epsilon^p$

⟨2⟩6. $\forall i. |x_i - y_i| < \epsilon$

⟨2⟩7. $\rho(\vec{x}, \vec{y}) < \epsilon$

□

Definition 17.4.5 (Sup-norm). The *sup-norm* on \mathbb{R}^n is defined by

$$\|(x_1, \dots, x_n)\|_\infty := \max(|x_1|, \dots, |x_n|) .$$

Proposition 17.4.6. The 2-norm on \mathbb{R}^n induces the standard metric.

PROOF: Immediate from definitions. □

Definition 17.4.7. For $p \geq 1$, the normed space l_p is the set of all sequences (x_n) in \mathbb{R} such that $\sum_{n=1}^\infty x_n^p$ converges, under

$$\|(x_n)\|_p := \left(\sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}} .$$

Proposition 17.4.8. The spaces l_p for $p \geq 1$ are all homeomorphic.

PROOF: See Kadets, Mikhail Iosifovich. 1967. Proof of the topological equivalence of all separable infinite-dimensional banach spaces. Functional Analysis and Its Applications 1 (1): 53–62. <http://dx.doi.org/10.1007/BF01075865>.

Proposition 17.4.9. *The metric topology on l_2 is strictly finer than the uniform topology.*

PROOF:

⟨1⟩1. LET: d be the metric induced by the l^2 -norm and $\bar{\rho}$ the uniform topology.

⟨1⟩2. The metric topology is finer than the uniform topology.

⟨2⟩1. LET: $x \in l_2$

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. LET: $\delta = \epsilon/2$

⟨2⟩4. LET: $y \in B_d(x, \delta)$

⟨2⟩5. $\sum_{n=0}^{\infty} (x_n - y_n)^2 < \delta^2$

⟨2⟩6. $\forall n. (x_n - y_n)^2 < \delta^2$

⟨2⟩7. $\forall n. |x_n - y_n| < \delta$

⟨2⟩8. $\forall n. \bar{d}(x_n, y_n) < \delta$

⟨2⟩9. $\bar{\rho}(x, y) \leq \delta$

⟨2⟩10. $\bar{\rho}(x, y) < \epsilon$

⟨2⟩11. $y \in B_{\bar{\rho}}(x, \epsilon)$

⟨1⟩3. The metric topology is not the same as the uniform topology.

⟨2⟩1. ASSUME: for a contradiction $B_d(0, 1)$ is open in the uniform topology.

⟨2⟩2. PICK $\epsilon > 0$ such that $B_{\bar{\rho}}(0, \epsilon) \subseteq B_d(0, 1)$

⟨2⟩3. PICK an integer N such that $1/N < \epsilon^2/4$

⟨2⟩4. LET: (x_n) be the sequence with $x_n = \epsilon/2$ for $n < N$ and $x_n = 0$ for $n \geq N$

⟨2⟩5. $(x_n) \in l_2$

⟨2⟩6. $(x_n) \in B_{\bar{\rho}}(0, \epsilon)$

PROOF: Since $\bar{\rho}((x_n), 0) = \epsilon/2$.

⟨2⟩7. $d((x_n), 0) > 1$

PROOF:

$$\begin{aligned} d((x_n), 0)^2 &= \sum_{n=0}^{\infty} x_n^2 \\ &= N\epsilon^2/4 \\ &> 1 \end{aligned}$$

□

Proposition 17.4.10. *The metric topology on l_2 is strictly coarser than the box topology.*

PROOF:

⟨1⟩1. The box topology is finer than the metric topology.

⟨2⟩1. LET: $(x_n) \in l_2$ and $\epsilon > 0$.

⟨2⟩2. LET: $(y_n) \in B((x_n), \epsilon)$

⟨2⟩3. PICK a sequence of real numbers (δ_n) such that $\sum_{n=0}^{\infty} \delta_n^2 < (\epsilon - d((x_n), (y_n)))^2$

⟨2⟩4. LET: $U = \prod_n (y_n - \delta_n, y_n + \delta_n)$

PROVE: $U \subseteq B((x_n), \epsilon)$

⟨2⟩5. LET: $(z_n) \in U$

⟨2⟩6. $d((z_n), (y_n)) < \epsilon - d((x_n), (y_n))$

PROOF:

$$\begin{aligned} d((z_n), (y_n))^2 &= \sum_{n=0}^{\infty} (z_n - y_n)^2 \\ &< \sum_{n=0}^{\infty} \delta_n^2 \\ &< (\epsilon - d((x_n), (y_n)))^2 \end{aligned}$$

$$\langle 2 \rangle 7. d((z_n), (x_n)) < \epsilon$$

$\langle 1 \rangle 2.$ The box topology is not equal to the metric topology.

$\langle 2 \rangle 1.$ LET: $U = \prod_n (-1/n, 1/n)$

$\langle 2 \rangle 2.$ ASSUME: for a contradiction U is open in the metric topology.

$\langle 2 \rangle 3.$ PICK $\epsilon > 0$ such that $B(0, \epsilon) \subseteq U$

$\langle 2 \rangle 4.$ PICK N such that $1/N < \epsilon/2$.

$\langle 2 \rangle 5.$ LET: (x_n) be the sequence with $x_N = \epsilon/2$ and $x_n = 0$ for all other n .

$\langle 2 \rangle 6.$ $d((x_n), 0) = \epsilon/2$

$\langle 2 \rangle 7.$ $(x_n) \notin U$

□

Proposition 17.4.11. *The l^2 -topology on \mathbb{R}^∞ is strictly finer than the uniform topology.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: for a contradiction $B_d(0, 1) \cap \mathbb{R}^\infty$ is open in the uniform topology.

$\langle 1 \rangle 2.$ PICK $\epsilon > 0$ such that $B_{\bar{\rho}}(0, \epsilon) \cap \mathbb{R}^\infty \subseteq B_d(0, 1) \cap \mathbb{R}^\infty$

$\langle 1 \rangle 3.$ PICK an integer N such that $1/N < \epsilon^2/4$

$\langle 1 \rangle 4.$ LET: (x_n) be the sequence with $x_n = \epsilon/2$ for $n < N$ and $x_n = 0$ for $n \geq N$

$\langle 1 \rangle 5.$ $(x_n) \in \mathbb{R}^\infty$

$\langle 1 \rangle 6.$ $(x_n) \in B_{\bar{\rho}}(0, \epsilon)$

PROOF: Since $\bar{\rho}((x_n), 0) = \epsilon/2$.

$\langle 1 \rangle 7.$ $d((x_n), 0) > 1$

PROOF:

$$\begin{aligned} d((x_n), 0)^2 &= \sum_{n=0}^{\infty} x_n^2 \\ &= N\epsilon^2/4 \\ &> 1 \end{aligned}$$

□

Proposition 17.4.12. *The l^2 -topology on \mathbb{R}^∞ is strictly coarser than the box topology.*

$\langle 1 \rangle 1.$ LET: $U = \prod_n (-1/n, 1/n) \cap \mathbb{R}^\infty$

$\langle 1 \rangle 2.$ ASSUME: for a contradiction U is open in the metric topology.

$\langle 1 \rangle 3.$ PICK $\epsilon > 0$ such that $B(0, \epsilon) \cap \mathbb{R}^\infty \subseteq U \cap \mathbb{R}^\infty$

$\langle 1 \rangle 4.$ PICK N such that $1/N < \epsilon/2$.

⟨1⟩5. LET: (x_n) be the sequence with $x_N = \epsilon/2$ and $x_n = 0$ for all other n .

⟨1⟩6. $d((x_n), 0) = \epsilon/2$

⟨1⟩7. $(x_n) \notin U$

□

Proposition 17.4.13. *The l^2 -topology on the Hilbert cube the same as the product topology.*

PROOF:

⟨1⟩1. For every $(x_n) \in H$ and $\epsilon > 0$, there exists a neighbourhood U of (x_n) in the product topology such that $U \subseteq B((x_n), \epsilon)$.

⟨2⟩1. LET: $(x_n) \in H$

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. PICK N such that $\sum_{i=N+1}^{\infty} 1/i^2 < \epsilon^2/2$

⟨2⟩4. LET: $B' = (\prod_{i=0}^N (x_i - \epsilon/\sqrt{2N}, x_i + \epsilon/\sqrt{2N}) \times \prod_{i=N+1}^{\infty} [0, 1/(i+1)]) \cap H$

PROVE: $B' \subseteq B((x_n), \epsilon)$

⟨2⟩5. LET: $(y_n) \in B'$

⟨2⟩6. $d((x_n), (y_n)) < \epsilon$

PROOF:

$$\begin{aligned} d((x_n), (y_n))^2 &= \sum_{i=0}^{\infty} |x_n - y_n|^2 \\ &< \sum_{i=0}^N \epsilon^2/2N + \sum_{i=N+1}^{\infty} 1/(i+1)1/(i+1)^2 \\ &< \epsilon^2/2 + \epsilon^2/2 \\ &= \epsilon^2 \end{aligned}$$

⟨1⟩2. The product topology is finer than the l^2 -topology.

⟨2⟩1. LET: $(x_n) \in H$ and $\epsilon > 0$

PROVE: $B((x_n), \epsilon) \cap H$ is open in the product topology.

⟨2⟩2. LET: $(y_n) \in B((x_n), \epsilon)$

⟨2⟩3. PICK a neighbourhood U of (y_n) in the product topology such that

$U \subseteq B((y_n), \epsilon - d((x_n), (y_n)))$

⟨2⟩4. $U \subseteq B((x_n), \epsilon)$

□

Definition 17.4.14. Let l_{∞} be the set of all bounded sequences in \mathbb{R} under

$$\|(x_n)\| := \sup_n |x_n|$$

Proposition 17.4.15. *For all $p \geq 1$ we have l_p is not homeomorphic to l_{∞} .*

Proposition 17.4.16. *Let $\| \cdot \|$ be a seminorm on the vector space E . Then $\| \cdot \|$ defines a norm on $E/\{0\}$.*

Proposition 17.4.17. *Let E and F be normed spaces. Any continuous linear map $E \rightarrow F$ is uniformly continuous.*

Definition 17.4.18. For $p \geq 1$, let $\mathcal{L}^p(\mathbb{R}^n)$ be the vector space of all Lebesgue-measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $|f|^p$ is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on $\mathcal{L}^p(\mathbb{R}^n)$. Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n) / \{0\} .$$

17.5 Inner Product Spaces

Definition 17.5.1 (Inner Product). Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n .$$

Proposition 17.5.2.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF:

$$\begin{aligned} \vec{x} \cdot (\vec{y} + \vec{z}) &= x_1(y_1 + z_1) + \cdots + x_n(y_n + z_n) \\ &= x_1 y_1 + x_1 z_1 + \cdots + x_n y_n + x_n z_n \\ &= \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z} \end{aligned}$$

□

Proposition 17.5.3. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\| .$$

PROOF:

- <1>1. ASSUME: w.l.o.g. $\vec{x} \neq \vec{0} \neq \vec{y}$
- <1>2. LET: $a = 1/\|\vec{x}\|$
- <1>3. LET: $b = 1/\|\vec{y}\|$
- <1>4. $\|a\vec{x} + b\vec{y}\| \geq 0$
- <1>5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
- <1>6. $ab\vec{x} \cdot \vec{y} \geq -1$
- <1>7. $\|a\vec{x} - b\vec{y}\| \geq 0$
- <1>8. $ab\vec{x} \cdot \vec{y} \leq 1$
- <1>9. $|\vec{x} \cdot \vec{y}| \leq 1/ab$

□

Proposition 17.5.4. Let $(x_n), (y_n)$ be sequences of real numbers. If $\sum_{n=0}^{\infty} x_n^2$ and $\sum_{n=0}^{\infty} y_n^2$ converge then $\sum_{n=0}^{\infty} |x_n y_n|$ converges.

PROOF:

$$\begin{aligned} \sum_{n=0}^N |x_n y_n| &\leq \sqrt{\sum_{n=0}^N x_n^2 \sum_{n=0}^N y_n^2} && \text{(Proposition 17.5.3)} \\ &\leq \sqrt{\sum_{n=0}^{\infty} x_n^2 \sum_{n=0}^{\infty} y_n^2} \end{aligned}$$

□

Proposition 17.5.5. *If E is an inner product space then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on E .*

17.6 Banach Spaces

Definition 17.6.1 (Banach Space). A *Banach space* is a complete normed space.

Example 17.6.2. For any topological space X , the set $C(X)$ of bounded continuous functions $X \rightarrow \mathbb{R}$ is a Banach space under $\|f\| = \sup_{x \in X} |f(x)|$.

Proposition 17.6.3. *The completion of a normed space is a Banach space.*

Proposition 17.6.4. *Let E and F be normed spaces. Let $f : E \rightarrow F$ be a continuous linear map. Then the extension to the completions $\hat{E} \rightarrow \hat{F}$ is linear.*

Proposition 17.6.5. *$L^p(\mathbb{R}^n)$ is a Banach space.*

Proposition 17.6.6. *$C(\mathbb{R})$ is first countable but not second countable.*

PROOF: For every sequence of 0s and 1s $s = (s_n)$, let f_s be a continuous bounded function whose value at n is s_n . Then the set of all f_s is an uncountable discrete set in $C(\mathbb{R})$. Hence $C(\mathbb{R})$ is not second countable.

It is first countable because it is metrizable. \square

17.7 Hilbert Spaces

Definition 17.7.1 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Example 17.7.2. The set of *square-integrable functions* is the set of Lebesgue integrable functions $[-\pi, \pi] \rightarrow \mathbb{R}$ quotiented by: $f \sim g$ iff $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$ has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

Proposition 17.7.3. *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

17.8 Locally Convex Spaces

Definition 17.8.1 (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

Proposition 17.8.2. *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Proposition 17.8.3. *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18. \square

Example 17.8.4. Let E be an infinite dimensional Hilbert space. Let E' be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map $E \rightarrow \mathbb{R}$ is continuous as a map $E' \rightarrow \mathbb{R}$. Then E is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

Definition 17.8.5 (Thom Space). Let E be a vector bundle with a Riemannian metric, $DE = \{x \in E : \|x\| \leq 1\}$ its disc bundle and $SE := \{v \in E : \|v\| = 1\}$ its sphere bundle. The *Thom space* of E is the quotient space DE/SE .