Mathematics

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Contents

Chapter 1

The Foundations

1.1 Primitive Notions and Axioms

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ for 'f is a function from A to B'. We call A the *domain* of A, and B the *codomain*.

Given sets A, B and C, and functions $f : A \to B$ and $g : B \to C$, let there be a function $gf = g \circ f : A \to C$, the *composite* of f and g.

Axiom 1.1 (Associativity)**.** *For any functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \enspace .$$

Axiom 1.2 (Identity)**.** *For any set A, there exists a function $\mathrm{id}_A : A \to A$, called an* identity *function on A, such that:*

- *for every set B and function $f : A \to B$, we have $f \circ \mathrm{id}_A = f$;*
- *for every set B and function $f : B \to A$, we have $\mathrm{id}_A \circ f = f$.*

Proposition 1.3. *The identity function on a set is unique.*

PROOF: If $i, j : A \to A$ are identity functions on A then we have $i = i \circ j = j$. □

Definition 1.4 (Isomorphism)**.** A function $i : A \to B$ is an *isomorphism*, $i : A \cong B$, iff there exists a function $i^{-1} : B \to A$, the *inverse* of i, such that $i^{-1} \circ i = \mathrm{id}_A$ and $i \circ i^{-1} = \mathrm{id}_B$.

Proposition 1.5. *For any set A we have $\mathrm{id}_A : A \cong A$ and ${\mathrm{id}_A}^{-1} = \mathrm{id}_A$.*

PROOF: Since $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$. □

Proposition 1.6. *If $i : A \cong B$ then $i^{-1} : B \cong A$ and $(i^{-1})^{-1} = i$.*

PROOF: Since $i \circ i^{-1} = \mathrm{id}_B$ and $i^{-1} \circ i = \mathrm{id}_A$. □

Proposition 1.7. *If $i : A \cong B$ and $j : B \cong C$ then $j \circ i : A \cong C$ and $(j \circ i)^{-1} = i^{-1} \circ j^{-1}$.*

PROOF: Since $j \circ i \circ i^{-1} \circ j^{-1} = \mathrm{id}_C$ and $i^{-1} \circ j^{-1} \circ j \circ i = \mathrm{id}_A$. $\square$

Axiom 1.8 (Terminal Set)**.** *There exists a set 1 such that, for any set A, there exists a unique function $A \to 1$.*

Proposition 1.9. *The terminal set is unique up to unique isomorphism.*

PROOF:
⟨1⟩1. LET: A and B be terminal sets.
⟨1⟩2. LET: i be the unique function $A \to B$.
⟨1⟩3. LET: i^{-1} be the unique function $B \to A$.
⟨1⟩4. $i \circ i^{-1} = \mathrm{id}_B$
PROOF: Since there is only one function $B \to B$.
⟨1⟩5. $i^{-1} \circ i = \mathrm{id}_A$
PROOF: Since there is only one function $A \to A$.
$\square$

Definition 1.10 (Element)**.** For any set A, an *element* of A is a function $1 \to A$.

We write $a \in A$ for $a : 1 \to A$. Given $f : A \to B$ and $a \in A$, we write $f(a)$ for $f \circ a$.

Axiom 1.11 (Extensionality)**.** *Let A and B be sets. Let $f, g : A \to B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Axiom 1.12 (Empty Set)**.** *There exists a set with no elements.*

Axiom 1.13 (Products)**.** *Let A and B be sets. There exists a set $A \times B$ and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, the* projections*, such that, for every set X and functions $f : X \to A$, $g : X \to B$, there exists a unique function $\langle f, g \rangle : X \to A \times B$ such that*

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g \ .$$

Proposition 1.14. *If $\pi_1 : P \to A$ and $\pi_2 : P \to B$ form a product of A and B, and $p_1 : Q \to A$ and $p_2 : Q \to B$ form a product of A and B, then there exists a unique isomorphism $i : P \cong Q$ such that $p_1 \circ i = \pi_1$ and $p_2 \circ i = \pi_2$.*

PROOF:
⟨1⟩1. LET: $i : P \to Q$ be the unique function such that $p_1 \circ i = \pi_1$ and $p_2 \circ i = \pi_2$.
⟨1⟩2. LET: $i^{-1} : Q \to P$ be the unique function such that $\pi_1 \circ i^{-1} = p_1$ and $\pi_2 \circ i^{-1} = p_2$
⟨1⟩3. $i \circ i^{-1} = \mathrm{id}_Q$
PROOF: Each is the unique $x : Q \to Q$ such that $p_1 \circ x = p_1$ and $p_2 \circ x = p_2$.
⟨1⟩4. $i^{-1} \circ i = \mathrm{id}_P$
PROOF: Each is the unique $x : P \to P$ such that $\pi_1 \circ x = \pi_1$ and $\pi_2 \circ x = \pi_2$.
$\square$

Definition 1.15. Given functions $f : A \to B$ and $g : C \to D$, define $f \times g : A \times C \to B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle \ .$$

Axiom 1.16 (Function Sets)**.** *Let A and B be sets. There exists a set A^B and function $\epsilon : A^B \times B \to A$ such that, for any set X and function $f : X \times B \to A$, there exists a unique function $\lambda f : X \to A^B$ such that*

$$f = \epsilon \circ \langle \lambda f \circ \pi_1, \pi_2 \rangle \ .$$

Definition 1.17 (Inverse Image)**.** Let A, X and Y be sets. Let $f : X \to Y$, $a \in Y$ and $j : A \to X$. Then j is the *inverse image* of a under f if and only if:

- $f \circ j = a \circ !_A$
- for every set I and function $q : I \to X$ such that $f \circ q = a \circ !_I$, there exists a unique $\overline{q} : I \to A$ such that $q = j \circ \overline{q}$.

Axiom 1.18 (Inverse Images)**.** *For any sets X and Y, function $f : X \to Y$ and element $a \in Y$, there exists a set $f^{-1}(a)$ and function $j : f^{-1}(a) \to X$ such that j is the inverse image of a under f.*

Definition 1.19 (Injective)**.** A function $f : A \to B$ is *injective*, $f : A \rightarrowtail B$, iff, for every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

Definition 1.20 (Surjective)**.** A function $f : A \to B$ is *surjective*, $f : A \twoheadrightarrow B$, iff, for every set X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

Axiom 1.21 (Subset Classifier)**.** *There exists a set 2 and function $\top : 1 \to 2$ such that, for any sets A and X and any injective function $f : A \to X$, there exists a unique function $\chi : X \to 2$ such that f is the inverse image of $\top$ under χ.*

Axiom 1.22 (Natural Numbers)**.** *There exists a set $\mathbb{N}$, an element $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \to \mathbb{N}$ such that, for every set X, element $a \in X$ and function $r : X \to X$, there exists a unique function $x : \mathbb{N} \to X$ such that $x \circ 0 = a$ and $x \circ s = r \circ x$.*

Axiom 1.23 (Choice)**.** *For every surjective function $r : X \to Y$, there exists $s : Y \to X$ such that $r \circ s$ is an identity function on X.*

1.2 Injective and Surjective Functions

Proposition 1.24. *Let $r : A \to B$ and $s : B \to A$. If $r \circ s = \mathrm{id}_B$ then s is injective.*

PROOF: If $s \circ x = s \circ y$ then $x = r \circ s \circ x = r \circ s \circ y = y$. $\square$

1.3 Products

Proposition 1.25. *Let $f : A \to B$, $g : B \to C$ and $h : B \to D$. Then*

$$\langle g, h \rangle \circ f = \langle g \circ f, h \circ f \rangle$$

PROOF: Each is the unique x such that $\pi_1 \circ x = g \circ f$ and $\pi_2 \circ x = h \circ f$. □

1.4 Subsets of a Set

Definition 1.26 (Subset)**.** Let $i : X \to A$. We write '(X, i) is a subset of A' for 'i is injective'.

Given subsets $i : X \to A$ and $j : Y \to A$, we write '$(X, i) = (Y, j)$' for 'there exists an isomorphism $k : X \cong Y$ such that $j \circ k = i$.

Proposition 1.27. *Given subsets (X, i), (Y, j) of A, if $(X, i) = (Y, j)$ then the isomorphism $k : X \cong Y$ such that $i \circ k = j$ is unique.*

PROOF: Since i is injective. □

Proposition 1.28. *If (X, i) is a subset of A then $(X, i) = (X, i)$.*

PROOF: Since $\mathrm{id}_X : X \cong X$ and $i \circ \mathrm{id}_X = i$. □

Proposition 1.29. *Given subsets (X, i), (Y, j) of A, if $(X, i) = (Y, j)$ then $(Y, j) = (X, i)$.*

PROOF: If $k : X \cong Y$ and $j \circ k = i$ then $k^{-1} : Y \cong X$ and $i \circ k^{-1} = j$. □

Proposition 1.30. *Given subsets (X, i), (Y, j), (Z, k) of A, if $(X, i) = (Y, j)$ and $(Y, j) = (Z, k)$ then $(X, i) = (Z, k)$.*

PROOF: If $f : X \cong Y$ satisfies $j \circ f = i$ and $g : Y \cong Z$ satisfies $k \circ g = j$, then $g \circ f : X \cong Z$ and $k \circ g \circ f = i$. □

Definition 1.31 (Inclusion)**.** Let (X, i) and (Y, j) be subsets of A. We say (X, i) is *included* in (Y, j), and write $(X, i) \subseteq (Y, j)$, iff there exists $k : X \to Y$ such that $j \circ k = i$.

Proposition 1.32. *For any subsets (X, i), (Y, j) of A, if $(X, i) = (Y, j)$ then $(X, i) \subseteq (Y, j)$.*

PROOF: Immediate from definitions. □

Corollary 1.32.1. *For any subset (X, i) of A we have $(X, i) \subseteq (X, i)$.*

Proposition 1.33. *For any subsets (X, i), (Y, j), (Z, k) of A, if $(X, i) \subseteq (Y, j)$ and $(Y, j) \subseteq (Z, k)$, then $(X, i) \subseteq (Z, k)$.*

PROOF: If $f : X \to Y$ satisfies $j \circ f = i$ and $g : Y \to Z$ satisfies $k \circ g = j$, then $g \circ f : X \to Z$ and $k \circ g \circ f = i$. □

Corollary 1.33.1. *Inclusion is well defined. That is, if $(X, i) = (X', i')$, $(Y, j) = (Y', j')$ and $(X, i) \subseteq (Y, j)$ then $(X', i') \subseteq (Y', j')$.*

Proposition 1.34. *For any subsets (X, i) and (Y, j) of A, if $(X, i) \subseteq (Y, j)$ and $(Y, j) \subseteq (X, i)$ then $(X, i) = (Y, j)$.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : X \to Y$ satisfy $j \circ f = i$.
$\langle 1 \rangle 2$. LET: $g : Y \to X$ satisfy $i \circ g = j$.
$\langle 1 \rangle 3$. $g \circ f = \mathrm{id}_X$
  PROOF: Since $i \circ g \circ f = i$ and i is injective.
$\langle 1 \rangle 4$. $f \circ g = \mathrm{id}_Y$
  PROOF: Since $j \circ f \circ g = j$ and j is injective.
$\langle 1 \rangle 5$. $f : X \cong Y$ and $j \circ f = i$.
$\langle 1 \rangle 6$. $(X, i) = (Y, j)$
□

1.5 Equalizers

Proposition 1.35. *For any set A, the function $\langle \mathrm{id}_A, \mathrm{id}_A \rangle : A \to A \times A$ is injective.*

PROOF: Since $\pi_1 \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle = \mathrm{id}_A$. □

Proposition 1.36. *Given sets A and B and functions $f, g : A \to B$, there exists a set E and function $e : E \to A$, called the* equalizer *of f and g, such that:*

- $f \circ e = g \circ e$
- *for any set X and function $x : X \to A$, if $f \circ x = g \circ x$ then there exists a unique $\overline{x} : X \to E$ such that $x = e \circ \overline{x}$.*

$$\begin{tikzcd}
X \arrow[dr, "\overline{x}"] \arrow[drr, "f \circ x"] \arrow[ddr, "x"'] & & & \\
& E \arrow[r, "m"] \arrow[d, "e"'] & B \arrow[r, "!"] \arrow[d, "\langle \mathrm{id}_B, \mathrm{id}_B \rangle"'] & 1 \arrow[d, "\top"] \\
& A \arrow[r, "\langle f, g \rangle"'] & B \times B \arrow[r, "eq"'] & 2
\end{tikzcd}$$

PROOF:
$\langle 1 \rangle 1$. LET: $eq : B \times B \to 2$ be the characteristic function of $\langle \mathrm{id}_B, \mathrm{id}_B \rangle : B \to B \times B$
  PROOF: By the Axiom of the Subset Classifier.
$\langle 1 \rangle 2$. LET: $e : E \to A$ be the inverse image of $\top$ under $eq \circ \langle f, g \rangle$

PROOF: By the Axiom of Inverse Images.

$\langle 1 \rangle 3.$ $f \circ e = g \circ e$

$\langle 2 \rangle 1.$ $eq \circ \langle f, g \rangle \circ e = \top$

$\langle 2 \rangle 2.$ LET: $m : E \to B$ be the unique function such that $\langle \mathrm{id}_B, \mathrm{id}_B \rangle \circ m = \langle f, g \rangle \circ e$

$\langle 2 \rangle 3.$ $\langle m, m \rangle = \langle f \circ e, g \circ e \rangle$

$\langle 2 \rangle 4.$ $f \circ e = g \circ e = m$

$\langle 1 \rangle 4.$ For any set X and function $x : X \to A$, if $f \circ x = g \circ x$ then there exists a unique $\overline{x} : X \to E$ such that $x = e \circ \overline{x}$.

$\langle 2 \rangle 1.$ LET: X be a set.

$\langle 2 \rangle 2.$ LET: $x : X \to A$

$\langle 2 \rangle 3.$ ASSUME: $f \circ x = g \circ x$

$\langle 2 \rangle 4.$ $\langle f, g \rangle \circ x = \langle \mathrm{id}_B, \mathrm{id}_B \rangle \circ f \circ x$

$\langle 2 \rangle 5.$ $eq \circ \langle f, g \rangle \circ x = \top \circ !_X$

PROOF:

$$\begin{aligned} eq \circ \langle f, g \rangle \circ x &= eq \circ \langle \mathrm{id}_B, \mathrm{id}_B \rangle \circ f \circ x \\ &= \top \circ !_B \circ f \circ x \\ &= \top \circ !_X \end{aligned}$$

$\langle 2 \rangle 6.$ There exists a unique $\overline{x} : X \to E$ such that $e \circ \overline{x} = x$

PROOF: From $\langle 1 \rangle 2$.

$\square$