Summary of Halmos' Naive Set Theory

Robin Adams

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Contents

Chapter 1

Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*, $\in$. If $x \in A$ we say that x *belongs* to A, x is an *element* of A, or x is *contained* in A. If this does not hold we write $x \notin A$.

Axiom 1.1 (Axiom of Extensionality). *Two sets are equal if and only if they have the same elements.*

Axiom 1.2 (Axiom of Comprehension, Aussonderungsaxiom). *To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

Definition 1.3. Given a set A and a condition $S(x)$, we write $\{x \in A : S(x)\}$ for the set whose elements are exactly those elements x of A for which $S(x)$ holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality. $\square$

Axiom 1.4 (Axiom of Pairing). *For any two sets, there exists a set that they both belong to.*

Definition 1.5 ((Unordered) Pair). For any sets a and b, the *(unordered) pair* $\{a, b\}$ is the set whose elements are just a and b.

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality. $\square$

Axiom 1.6 (Union Axiom). *For every set A, there exists a set that contains all the elements that belong to at least one element of A.*

Definition 1.7 (Subset)**.** Let A and B be sets. We say that A is a *subset* of B, or B *includes* A, and write $A \subseteq B$ or $B \supseteq A$, iff every element of A is an element of B.

Axiom 1.8 (Power Set Axiom)**.** *For any set A, there exists a set that contains all the subsets of A.*

Definition 1.9 (Empty)**.** A set is *empty* iff it has no elements; otherwise it is *non-empty*.

Axiom 1.10 (Axiom of Infinity)**.** *There exists a set I such that:*

- *I has an element that is empty*
- *for all $x \in I$, there exists $y \in I$ such that the elements of y are exactly x and the elements of x.*

Definition 1.11 (Ordered Pair)**.** For any sets a and b, the *ordered pair* (a, b) is defined by

$$(a, b) := \{\{a\}, \{a, b\}\} \ .$$

Definition 1.12 (Power Set)**.** For any set A, the *power set* of A, $\mathcal{P}A$, is the set whose elements are exactly the subsets of A.

Proof: This exists by the Power Set Axiom and Axiom of Comprehension, and is unique by the Axiom of Extensionality. □

Definition 1.13 (Cartesian Product)**.** For any sets A and B, the *Cartesian product* $A \times B$ is

$$A \times B := \{p \in \mathcal{P}\mathcal{P}(A \cup B) : \exists a \in A. \exists b \in B. p = (a, b)\} \ .$$

Definition 1.14 (Relation)**.** A *relation* is a set of ordered pairs.

If R is a relation, we write xRy for $(x, y) \in R$.

Given sets X and Y, a relation *between X and Y* is a subset of $X \times Y$.

Given a set X, a relation *on* X is a relation between X and X.

Definition 1.15 (Function)**.** Let X and Y be sets. A *function, map, mapping, transformation* or *operator f from X to Y*, $f : X \to Y$, is a relation f between X and Y such that, for all $x \in X$, there exists a unique $f(x) \in Y$, called the *value* of f at the *argument* x, such that $(x, f(x)) \in f$.

Definition 1.16 (Family)**.** Let I and X be sets. A *family* of elements of X *indexed* by I is a function $a : I \to X$. We write a_i for $a(i)$, and $\{a_i\}_{i \in I}$ for a.

Definition 1.17 (Cartesian Product of a Family of Sets)**.** Let $\{A_i\}_{i \in I}$ be a family of sets. The *Cartesian product* $\bigtimes_{i \in I} A_i$ is the set of all families $\{a_i\}_{i \in I}$ such that $\forall i \in I. a_i \in A_i$.

We write A^I for $\bigtimes_{i \in I} A$.

Axiom 1.18 (Axiom of Choice)**.** *The Cartesian product of a non-empty family of non-empty sets is non-empty.*

Axiom 1.19 (Axiom of substitution)**.** *If $S(a, b)$ is a sentence such that for each a in A the set $\{b : S(a, b)\}$ can be formed, then there exists a function F with domain A such that $F(a) = \{b : S(a, b)\}$ for each a in A.*

Chapter 2

Basic Properties and Operations on Sets

2.1 The Subset Relation

Theorem 2.1. *For any set A, we have $A \subseteq A$.*

PROOF: Every element of A is an element of A. $\square$

Theorem 2.2. *For any sets A, B and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

PROOF: If every element of A is an element of B, and every element of B is an element of C, then every element of A is an element of C. $\square$

Theorem 2.3. *For any sets A and B, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

PROOF: If every element of A is an element of B, and every element of B is an element of A, then A and B have the same elements, and therefore are equal by the Axiom of Extensionality. $\square$

Definition 2.4 (Proper Subset)**.** Let A and B be sets. We say that A is a *proper* subset of B, or B *properly* includes A, and write $A \subsetneq B$ or $B \supsetneq A$, iff $A \subseteq B$ and $A \neq B$.

2.2 Comprehension Notation

Theorem 2.5. *There is no set that contains every set.*

PROOF:
⟨1⟩1. LET: A be a set.
PROVE: There exists a set B such that $B \notin A$.
⟨1⟩2. LET: $B = \{x \in A : x \notin x\}$
⟨1⟩3. If $B \in A$ then we have $B \in B$ if and only if $B \notin B$.
⟨1⟩4. $B \notin A$
$\square$

2.3 The Empty Set

Theorem 2.6. *There exists a set with no elements.*

PROOF: Immediate from the Axiom of Infinity. $\square$

Definition 2.7 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

Theorem 2.8. *For any set A we have $\emptyset \subset A$.*

PROOF: Vacuous. $\square$

2.4 Unordered Pairs

Definition 2.9 (Singleton)**.** For any set a, the *singleton* $\{a\}$ is defined to be $\{a, a\}$.

2.5 Unions

Definition 2.10 (Union)**.** For any set $\mathcal{C}$, the *union* of $\mathcal{C}$, $\bigcup \mathcal{C}$, is the set whose elements are the elements of the elements of $\mathcal{C}$.

We write $\bigcup_{X \in \mathcal{A}} t[X]$ for $\bigcup \{t[X] \mid X \in \mathcal{A}\}$.

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality. $\square$

Proposition 2.11.

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of $\emptyset$. $\square$

Proposition 2.12. *For any set A, we have $\bigcup \{A\} = A$.*

PROOF: For any x, we have x is an element of an element of $\{A\}$ if and only if x is an element of A. $\square$

Definition 2.13. We write $A \cup B$ for $\bigcup \{A, B\}$.

Proposition 2.14. *For any set A, we have $A \cup \emptyset = A$.*

PROOF: $x \in A \cup \emptyset$ iff $x \in A$ or $x \in \emptyset$, iff $x \in A$. $\square$

Proposition 2.15 (Idempotence)**.** *For any set A, we have $A \cup A = A$.*

PROOF: $x \in A$ or $x \in A$ is equivalent to $x \in A$. $\square$

Proposition 2.16. *For any sets A and B, we have $A \subseteq B$ if and only if $A \cup B = B$.*

PROOF: For any x, the statement "if $x \in A$ then $x \in B$" is equivalent to "$x \in A$ or $x \in B$ if and only if $x \in B$". $\square$

Proposition 2.17. *For any sets a and b, we have $\{a\} \cup \{b\} = \{a, b\}$.*

PROOF: Immediate from definitions. $\square$

2.6 Intersections

Definition 2.18 (Intersection)**.** For any sets A and B, the *intersection* $A \cap B$ is defined to be $\{x \in A : x \in B\}$.

Proposition 2.19. *For any set A, we have $A \cap \emptyset = \emptyset$.*

PROOF: There is no x such that $x \in A$ and $x \in \emptyset$. □

Proposition 2.20. *For any set A, we have*

$$A \cap A = A \ .$$

PROOF: We have $x \in A$ and $x \in A$ if and only if $x \in A$. □

Proposition 2.21. *For any sets A and B, we have $A \subseteq B$ if and only if $A \cap B = A$.*

PROOF: For any x, the statement "if $x \in A$ then $x \in B$" is equivalent to "$x \in A$ and $x \in B$ if and only if $x \in A$". □

Proposition 2.22. *For any sets A, B and C, we have $C \subseteq A$ if and only if $(A \cap B) \cup C = A \cap (B \cup C)$.*

PROOF: The statement "if $x \in C$ then $x \in A$" is equivalent to the statement "$((x \in A \wedge x \in B) \vee x \in C) \Leftrightarrow (x \in A \wedge (x \in B \vee x \in C))$". □

Definition 2.23 (Disjoint)**.** Two sets A and B are *disjoint* if and only if $A \cap B = \emptyset$.

Definition 2.24 (Pairwise Disjoint)**.** Let A be a set. We say the elements of A are *pairwise disjoint* if and only if, for all $x, y \in A$, if $x \cap y \neq \emptyset$ then $x = y$.

Definition 2.25 (Intersection)**.** For any nonempty set $\mathcal{C}$, the *intersection* of $\mathcal{C}$, $\bigcap \mathcal{C}$, is the set that contains exactly those sets that belong to every element of $\mathcal{C}$.

We write $\bigcap_{X \in \mathcal{A}} t[X]$ for $\bigcap \{t[X] \mid X \in \mathcal{A}\}$.

PROOF:
⟨1⟩1. LET: $\mathcal{C}$ be a nonempty set.
⟨1⟩2. There exists a set I whose elements are exactly the sets that belong to every element of $\mathcal{C}$.
PROOF: Pick $A \in \mathcal{C}$, and take $I = \{x \in A : \forall X \in \mathcal{C}. x \in X\}$.
⟨1⟩3. For any sets I, J, if the elements of I and J are exactly the sets that belong to every element of $\mathcal{C}$ then $I = J$.
PROOF: Axiom of Extensionality.
□

2.7 Unordered Triples

Definition 2.26 ((Unordered) Triple)**.** Given sets $a_1, \ldots, a_n$, define the *(unordered) n-tuple* $\{a_1, \ldots, a_n\}$ to be

$$\{a_1, \ldots, a_n\} := \{a_1\} \cup \cdots \cup \{a_n\} \ .$$

2.8 Relative Complements

Definition 2.27 (Relative Complement)**.** For any sets A and B, the *difference* or *relative complement* $A - B$ is defined to be

$$A - B := \{x \in A : x \notin B\} \ .$$

Proposition 2.28. *For any sets A and E, we have $A \subseteq E$ if and only if*

$$E - (E - A) = A$$

PROOF:
⟨1⟩1. LET: A and E be sets.
⟨1⟩2. If $A \subseteq E$ then $E - (E - A) = A$
  ⟨2⟩1. ASSUME: $A \subseteq E$
  ⟨2⟩2. $E - (E - A) \subseteq A$
    PROOF: If $x \in E$ and $x \notin E - A$ then $x \in A$.
  ⟨2⟩3. $A \subseteq E - (E - A)$
    PROOF: If $x \in A$ then $x \in E$ and $x \notin E - A$.
⟨1⟩3. If $E - (E - A) = A$ then $A \subseteq E$.
  PROOF: Since $E - (E - A) \subseteq E$.
□

Proposition 2.29. *For any set E we have*

$$E - \emptyset = E$$

PROOF: $x \in E$ if and only if $x \in E$ and $x \notin \emptyset$. □

Proposition 2.30. *For any set E we have*

$$E - E = \emptyset \ .$$

PROOF: There is no x such that $x \in E$ and $x \notin E$. □

Proposition 2.31. *For any sets A and E, we have*

$$A \cap (E - A) = \emptyset \ .$$

PROOF: There is no x such that $x \in A$ and $x \in E - A$. □

Proposition 2.32. *Let A and E be sets. Then $A \subseteq E$ if and only if*

$$A \cup (E - A) = E \ .$$

PROOF:
⟨1⟩1. LET: A and E be sets.
⟨1⟩2. If $A \subseteq E$ then $A \cup (E - A) = E$.
  ⟨2⟩1. ASSUME: $A \subseteq E$
  ⟨2⟩2. $A \cup (E - A) \subseteq E$

PROOF: If $x \in A$ or $x \in E - A$ then $x \in E$.
$\langle 2 \rangle 3.$ $E \subseteq A \cup (E - A)$
PROOF: If $x \in E$ then either $x \in A$ or $x \notin A$. In the latter case, $x \in E - A$.
$\langle 1 \rangle 3.$ If $A \cup (E - A) = E$ then $A \subseteq E$
PROOF: Since $A \subseteq A \cup (E - A)$.
∎

Proposition 2.33. *Let A, B and E be sets. Then:*

1. *If $A \subseteq B$ then $E - B \subseteq E - A$.*
2. *If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.*

PROOF:
$\langle 1 \rangle 1.$ LET: A, B and E be sets.
$\langle 1 \rangle 2.$ If $A \subseteq B$ then $E - B \subseteq E - A$.
PROOF: If $A \subseteq B$, $x \in E$ and $x \notin B$, then we have $x \in E$ and $x \notin A$.
$\langle 1 \rangle 3.$ If $A \subseteq E$ and $E - B \subseteq E - A$ then $A \subseteq B$.
$\langle 2 \rangle 1.$ ASSUME: $A \subseteq E$
$\langle 2 \rangle 2.$ ASSUME: $E - B \subseteq E - A$
$\langle 2 \rangle 3.$ LET: $x \in A$
$\langle 2 \rangle 4.$ $x \in E$
$\langle 2 \rangle 5.$ $x \notin E - A$
$\langle 2 \rangle 6.$ $x \notin E - B$
$\langle 2 \rangle 7.$ $x \in B$
∎

Example 2.34. We cannot remove the hypothesis $A \subseteq E$ in item 2 above. Let $E = \emptyset$, $A = \{\emptyset\}$ and $B = \emptyset$. Then $E - B = E - A = \emptyset$ but $A \nsubseteq B$.

Proposition 2.35 (De Morgan's Law)**.** *For any sets A, B and E, we have $E - (A \cup B) = (E - A) \cap (E - B)$.*

PROOF: $(x \in E \wedge \neg(x \in A \vee x \in B)) \Leftrightarrow (x \in E \wedge x \notin A \wedge x \in E \wedge x \notin B)$. ∎

Proposition 2.36 (De Morgan's Law)**.** *For any sets A, B and E, we have $E - (A \cap B) = (E - A) \cup (E - B)$.*

PROOF: $(x \in E \vee \neg(x \in A \wedge x \in B)) \Leftrightarrow (x \in E \wedge x \notin A) \vee (x \in E \wedge x \notin B)$. ∎

Proposition 2.37. *For any sets A, B and E, if $A \subseteq E$ then*

$$A - B = A \cap (E - B) \ .$$

PROOF: If $A \subseteq E$ then we have $(x \in A \wedge x \notin B) \Leftrightarrow (x \in A \wedge x \in E \wedge x \notin B)$. ∎

Proposition 2.38. *For any sets A and B, we have $A \subseteq B$ if and only if $A - B = \emptyset$.*

PROOF: Both are equivalent to the statement that there is no x such that $x \in A$ and $x \notin B$. ∎

Proposition 2.39. *For any sets A and B, we have*

$$A - (A - B) = A \cap B \ .$$

PROOF: $(x \in A \wedge \neg(x \in A \wedge x \notin B)) \Leftrightarrow x \in A \wedge x \in B$. □

Proposition 2.40. *For any sets A, B and C, we have*

$$A \cap (B - C) = (A \cap B) - (A \cap C) \ .$$

PROOF: $(x \in A \wedge x \in B \wedge x \notin C) \Leftrightarrow (x \in A \wedge x \in B \wedge \neg(x \in A \wedge x \in C))$. □

Proposition 2.41. *For any sets A, B, C and E, if $(A \cap B) - C \subseteq E$ then we have*

$$A \cap B \subseteq (A \cap C) \cup (B \cap (E - C)) \ .$$

PROOF:
⟨1⟩1. LET: $x \in A \cap B$
PROVE: $x \in (A \cap C) \cup (B \cap (E - C))$
⟨1⟩2. CASE: $x \in C$
PROOF: Then $x \in A \cap C$.
⟨1⟩3. CASE: $x \notin C$
PROOF: Then $x \in E$ and so $x \in B \cap (E - C)$.
□

Proposition 2.42. *For any sets A, B, C and E, we have*

$$(A \cup C) \cap (B \cup (E - C)) \subseteq A \cup B \ .$$

PROOF: The statement $(x \in A \vee x \in C) \wedge (x \in B \vee (x \in E \wedge x \notin C))$ implies $x \in A \vee x \in B$. □

Proposition 2.43 (De Morgan's Law)**.** *Let E be a set and $\mathcal{C}$ a nonempty set. Then*

$$E - \bigcup \mathcal{C} = \bigcap_{X \in \mathcal{C}} (E - X) \ .$$

PROOF: Easy. □

Proposition 2.44 (De Morgan's Law)**.** *Let E be a set and $\mathcal{C}$ a nonempty set. Then*

$$E - \bigcap \mathcal{C} = \bigcup_{X \in \mathcal{C}} (E - X) \ .$$

PROOF: Easy. □

2.9 Symmetric Difference

Definition 2.45 (Symmetric Difference)**.** For any sets A and B, the *symmetric difference* $A + B$ is defined to be

$$A + B := (A - B) \cup (B - A) \ .$$

Proposition 2.46. *For any sets A and B, we have*

$$A + B = B + A$$

Proof: From the commutativity of union. □

Proposition 2.47. *For any sets A, B and C, we have*

$$A + (B + C) = (A + B) + C \ .$$

Proof: Each is the set of all x that belong to either exactly one or all three of A, B and C. □

Proposition 2.48. *For any set A, we have*

$$A + \varnothing = A \ .$$

Proof:

$$\begin{aligned} A + \varnothing &= (A - \varnothing) \cup (\varnothing - A) \\ &= A \cup \varnothing \\ &= A \end{aligned}$$ □

Proposition 2.49. *For any set A we have*

$$A + A = \varnothing \ .$$

Proof:

$$\begin{aligned} A + A &= (A - A) \cup (A - A) \\ &= \varnothing \cup \varnothing \\ &= \varnothing \end{aligned}$$ □

2.10 Power Sets

Proposition 2.50.

$$\mathcal{P}\varnothing = \{\varnothing\}$$

Proof: The only subset of $\varnothing$ is $\varnothing$. □

Proposition 2.51. *For any set a, we have*

$$\mathcal{P}\{a\} = \{\varnothing, \{a\}\} \ .$$

Proof: The only subsets of $\{a\}$ are $\varnothing$ and $\{a\}$. □

Proposition 2.52. *For any sets a and b, we have*

$$\mathcal{P}\{a, b\} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\} \ .$$

Proof: The only subsets of $\{a, b\}$ are $\varnothing$, $\{a\}$, $\{b\}$ and $\{a, b\}$. □

Proposition 2.53. *For any nonempty set $\mathcal{C}$ we have*

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}X = \mathcal{P}\left(\bigcap \mathcal{C}\right) \enspace .$$

PROOF:

$$\begin{aligned} x \in \bigcup_{X \in \mathcal{C}} \mathcal{P}X &\Leftrightarrow \forall X \in \mathcal{C}.x \subseteq X \\ &\Leftrightarrow \forall X \in \mathcal{C}.\forall y \in x.y \in X \\ &\Leftrightarrow \forall y \in x.\forall X \in mathcalC.y \in X \\ &\Leftrightarrow x \subseteq \bigcap \mathcal{C} \end{aligned}$$

$\square$

Proposition 2.54. *For any set $\mathcal{C}$ we have*

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}X \subseteq \mathcal{P} \bigcup \mathcal{C} \enspace .$$

PROOF: If there exists $X \in \mathcal{C}$ such that $x \subseteq X$ then $x \subseteq \bigcup \mathcal{C}$. $\square$

Proposition 2.55. *For any set E, we have*

$$\bigcap \mathcal{P}E = \emptyset \enspace .$$

PROOF: Since $\emptyset \in \mathcal{P}E$. $\square$

Proposition 2.56. *For any sets E and F, if $E \subseteq F$ then $\mathcal{P}E \subseteq \mathcal{P}F$.*

PROOF: If $E \subseteq F$ and $X \subseteq E$ then $X \subseteq F$. $\square$

Chapter 3

Relations and Functions

3.1 Ordered Pairs

Proposition 3.1. *For any sets a, b, x and y, if $(a,b) = (x,y)$ then $a = x$ and $b = y$.*

PROOF:
$\langle 1 \rangle 1$. LET: a, b, x and y be sets.
$\langle 1 \rangle 2$. ASSUME: $(a,b) = (x,y)$
$\langle 1 \rangle 3$. $a = x$
  PROOF: $\{a\} = \bigcap (a,b) = \bigcap (x,y) = \{x\}$.
$\langle 1 \rangle 4$. $\{a,b\} = \{x,y\}$
$\langle 1 \rangle 5$. CASE: $a = b$
  $\langle 2 \rangle 1$. $x = y$
    PROOF: Since $\{x,y\} = \{a,b\}$ is a singleton.
  $\langle 2 \rangle 2$. $b = y$
    PROOF: $b = a = x = y$
$\langle 1 \rangle 6$. CASE: $a \neq b$
  $\langle 2 \rangle 1$. $x \neq y$
    PROOF: Since $\{x,y\} = \{a,b\}$ is not a singleton.
  $\langle 2 \rangle 2$. $b = y$
    PROOF: $\{b\} = \{a,b\} - \{a\} = \{x,y\} - \{x\} = \{y\}$.
$\square$

Proposition 3.2. *For any sets A, B and X, we have*

$$(A - B) \times X = (A \times X) - (B \times X) \ .$$

PROOF: Easy. $\square$

Proposition 3.3. *For any sets A and B, we have $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.*

PROOF: Easy. $\square$

Proposition 3.4. *For any sets A, B, X and Y, if $A \subseteq X$ and $B \subseteq Y$ then $A \times B \subseteq X \times Y$. The converse holds assuming $A \neq \emptyset$ and $B \neq \emptyset$.*

PROOF: Easy. $\square$

3.2 Relations

Definition 3.5 (Domain)**.** The *domain* of a relation R is the set

$$\operatorname{dom} R := \left\{ x \in \bigcup \bigcup R : \exists y.(x, y) \in R \right\} \ .$$

Definition 3.6 (Range)**.** The *range* of a relation R is the set

$$\operatorname{ran} R := \left\{ y \in \bigcup \bigcup R : \exists x.(x, y) \in R \right\} \ .$$

Definition 3.7 (Reflexive)**.** Let R be a relation on X. Then R is *reflexive* iff, for all $x \in X$, we have xRx.

Definition 3.8 (Symmetric)**.** Let R be a relation on X. Then R is *symmetric* iff, whenever xRy, then yRx.

Definition 3.9 (Antisymmetric)**.** A relation R is *antisymmetric* iff, whenever xRy and yRx, then $x = y$.

Definition 3.10 (Transitive)**.** Let R be a relation on X. Then R is *transitive* iff, whenever xRy and yRz, then xRz.

Definition 3.11 (Identity Relation)**.** For any set X, the *identity relation* I_X on X is

$$I_X = \{(x, x) : x \in X\} \ .$$

3.3 Composition

Definition 3.12 (Composition)**.** Let R be a relation between X and Y, and S a relation between Y and Z. The *composite* or *relative product* $S \circ R = SR$ is the relation between X and Z defined by

$$x(S \circ R)z \Leftrightarrow \exists y \in Y (xRy \wedge ySz) \ .$$

Proposition 3.13. *Let R be a relation between X and Y, S a relation between Y and Z, and T a relation between Z and W. Then*

$$T(SR) = (TS)R \ .$$

PROOF: Easy. $\square$

Example 3.14. Composition of relations is not commutative in general. Let $X = \{a, b\}$ where $a \neq b$. Let $R = \{(a, a), (b, a)\}$ and $S = \{(a, b), (b, b)\}$. Then $SR = S$ but $RS = R \neq S$.

Proposition 3.15. *A relation R is transitive if and only if $RR \subseteq R$.*

PROOF: Easy. $\square$

3.4 Inverses

Definition 3.16 (Inverse)**.** Let R be a relation between X and Y. The *inverse* or *converse* R^{-1} is the relation between Y and X defined by

$$yR^{-1}x \Leftrightarrow xRy \ .$$

Proposition 3.17. *For any relation R, we have*

$$\operatorname{dom} R^{-1} = \operatorname{ran} R \ .$$

PROOF: Easy. □

Proposition 3.18. *For any relation R, we have*

$$\operatorname{ran} R^{-1} = \operatorname{dom} R \ .$$

PROOF: Easy. □

Proposition 3.19. *Let R be a relation between X and Y, and S a relation between Y and Z. Then*

$$(SR)^{-1} = R^{-1}S^{-1} \ .$$

PROOF: Easy. □

Proposition 3.20. *A relation R is symmetric if and only if $R \subseteq R^{-1}$.*

PROOF: Easy. □

Proposition 3.21. *Let R be a relation between X and Y. Then*

$$I_Y R = RI_X = R \ .$$

PROOF: Easy. □

Proposition 3.22. *A relation R on a set X is reflexive if and only if $I_X \subseteq R$.*

PROOF: Easy. □

Proposition 3.23. *Let R be a relation on a set X. Then R is antisymmetric iff $R \cap R^{-1} \subseteq I_X$.*

PROOF: Easy. □

3.5 Equivalence Relations

Definition 3.24 (Equivalence Relation)**.** Let R be a relation on X. Then R is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Definition 3.25 (Partition)**.** Let X be a set. A *partition* of X is a pairwise disjoint set of nonempty subsets of X whose union is X.

Definition 3.26 (Equivalence Class)**.** Let R be an equivalence relation on X. Let $x \in X$. The *equivalence class* of x with respect to R is

$$x/R := \{y \in X : xRy\} \enspace .$$

We write X/R for the set of all equivalence classes with respect to R.

Definition 3.27 (Induced)**.** Let P be a partition of X. The relation *induced* by P is X/P where $x(X/P)y$ iff there exists $X \in P$ such that $x \in X$ and $y \in X$.

Theorem 3.28. *Let R be an equivalence relation on X. Then X/R is a partition of X that induces the relation R.*

Proof: Easy. □

Theorem 3.29. *Let P be a partition of X. Then X/P is an equivalence relation on X, and $P = X/(X/P)$.*

Proof: Easy. □

3.6 Functions

Definition 3.30 (One-to-One)**.** A function $f : X \to Y$ is *one-to-one* or *injective* iff, for all $x, y \in X$, if $f(x) = f(y)$ then $x = y$.

Definition 3.31 (Onto)**.** Let $f : X \to Y$. We say f is *surjective*, or f maps X *onto* Y iff $\operatorname{ran} f = Y$.

Definition 3.32 (Bijective)**.** Let $f : X \to Y$. Then f is *bijective*, or a *bijection*, iff it is injective and surjective.

Definition 3.33 (Image)**.** Let $f : X \to Y$ and $A \subseteq X$. The *image* of A under f is

$$f(A) := \{f(x) : x \in A\} \enspace .$$

Definition 3.34 (Inclusion Map)**.** Let Y be a set and $X \subseteq Y$. Then the *inclusion map* $i : X \hookrightarrow Y$ is the function defined by $i(x) = x$ for all $x \in X$.

Proposition 3.35. *For any set X, the identity relation I_X is a function $X \to X$.*

Proof: Easy. □

Definition 3.36 (Restriction)**.** Let $f : Y \to Z$ and $X \subseteq Y$. The *restriction* of f to X is the function $f \restriction X : X \to Z$ defined by

$$(f \restriction X)(x) = f(x) \qquad (x \in X) \enspace .$$

Given sets X, Y and Z with $X \subseteq Y$, if $f : X \to Z$ and $g : Y \to Z$, we say g is an *extension* of f to Y iff $f = g \restriction X$.

Definition 3.37 (Projection)**.** Given sets X and Y, the *projection* maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are defined by

$$\pi_1(x, y) = x, \qquad \pi_2(x, y) = y \qquad (x \in X, y \in Y) \ .$$

Definition 3.38 (Canonical Map)**.** Let X be a set and R an equivalence relation on X. The *canonical map* $\pi : X \to X/R$ is the map defined by $\pi(x) = x/R$.

Proposition 3.39. *Let $f : X \to Y$. Then the following are equivalent:*

1. *f is one-to-one.*
2. *For all $A, B \subseteq X$, we have $f(A \cap B) = f(A) \cap f(B)$.*
3. *For all $A \subseteq X$, we have $f(X - A) \subseteq Y - f(A)$.*

Proof: Easy. □

Proposition 3.40. *Let $f : X \to Y$. Then f maps X onto Y if and only if, for all $A \subseteq X$, we have $Y - f(A) \subseteq f(X - A)$.*

Proof: Easy. □

3.7 Families

Proposition 3.41 (Generalized Associative Law for Unions)**.** *Let $\{I_j\}_{j\in J}$ be a family of sets. Let $K = \bigcup_{j\in J} I_j$. Let $\{A_k\}_{k\in K}$ be a family of sets indexed by K. Then*

$$\bigcup_{k\in K} A_k = \bigcup_{j\in J} \bigcup_{i\in I_j} A_i \ .$$

Proof: Easy. □

Proposition 3.42 (Generalized Commutative Law for Unions)**.** *Let $\{I_j\}_{j\in J}$ be a family of sets. Let $f : J \to J$ be a one-to-one correspondence from J onto J. Then*

$$\bigcup_{j\in J} I_j = \bigcup_{j\in J} I_{f(j)} \ .$$

Proof: Easy. □

Proposition 3.43 (Generalized Associative Law for Intersections)**.** *Let $\{I_j\}_{j\in J}$ be a nonempty family of nonempty sets. Let $K = \bigcup_{j\in J} I_j$. Let $\{A_k\}_{k\in K}$ be a family of sets indexed by K. Then*

$$\bigcap_{k\in K} A_k = \bigcap_{j\in J} \bigcap_{i\in I_j} A_i \ .$$

Proof: Easy. □

Proposition 3.44 (Generalized Commutative Law for Intersections)**.** *Let* $\{I_j\}_{j\in J}$ *be a nonempty family of sets. Let* $f : J \to J$ *be a one-to-one correspondence from* J *onto* J*. Then*

$$\bigcap_{j\in J} I_j = \bigcap_{j\in J} I_{f(j)} \quad .$$

Proof: Easy. □

Proposition 3.45. *Let* B *be a set and* $\{A_i\}_{i\in I}$ *a family of sets. Then*

$$B \cap \bigcup_{i\in I} A_i = \bigcup_{i\in I}(B \cap A_i)$$

Proof: Easy. □

Proposition 3.46. *Let* B *be a set and* $\{A_i\}_{i\in I}$ *a nonempty family of sets. Then*

$$B \cup \bigcap_{i\in I} A_i = \bigcap_{i\in I}(B \cup A_i)$$

Proof: Easy. □

Definition 3.47 (Projection)**.** Let $\{A_i\}_{i\in I}$ be a family of sets and $i \in I$. The *projection* function $\pi_i : \bigtimes_{i\in I} A_i \to A_i$ is defined by $\pi_i(a) = a_i$.

Proposition 3.48. *Let* $\{A_i\}_{i\in I}$ *and* $\{B_j\}_{j\in J}$ *be families of sets. Then*

$$\left(\bigcup_{i\in I} A_i\right) \times \left(\bigcup_{j\in J} B_j\right) = \bigcup_{i\in I}\bigcup_{j\in J}(A_i \times B_j) \quad .$$

Proof: Easy. □

Proposition 3.49. *Let* $\{A_i\}_{i\in I}$ *and* $\{B_j\}_{j\in J}$ *be nonempty families of sets. Then*

$$\left(\bigcap_{i\in I} A_i\right) \times \left(\bigcap_{j\in J} B_j\right) = \bigcap_{i\in I}\bigcap_{j\in J}(A_i \times B_j) \quad .$$

Proof: Easy. □

Proposition 3.50. *Let* $f : X \to Y$*. Let* $\{A_i\}_{i\in I}$ *be a family of subsets of* X*. Then*

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i) \quad .$$

Proof: Easy. □

Example 3.51. It is not true in general that, if $f : X \to Y$ and $\{A_i\}_{i\in I}$ is a nonempty family of subsets of X, then $f\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f(A_i)$.

Take $X = \{a, b\}$ and $Y = \{c\}$ where $a \neq b$. Take $I = \{i, j\}$ with $i \neq j$. Let $A_i = \{a\}$ and $A_j = \{b\}$. Let f be the unique function $X \to Y$. Then $f\left(\bigcap_{i\in I} A_i\right) = f(\emptyset) = \emptyset$ but $\bigcap_{i\in I} f(A_i) = \{c\}$.

3.8 Inverses and Composites of Functions

Definition 3.52 (Inverse)**.** Given a function $f : X \to Y$, the *inverse* of f is the function $f^{-1} : \mathcal{P}Y \to \mathcal{P}X$ defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \ .$$

We call $f^{-1}(B)$ the *inverse image* of B under f.

Proposition 3.53. *Let $f : X \to Y$. Then f maps X onto Y if and only if the inverse image of any nonempty subset of Y is nonempty.*

PROOF: Easy. □

Proposition 3.54. *Let $f : X \to Y$. Then f is one-to-one if and only if the inverse image of any singleton subset of Y is a singleton.*

PROOF: Easy. □

Proposition 3.55. *Let $f : X \to Y$. Let $B \subseteq Y$. Then*

$$f(f^{-1}(B)) \subseteq B \ .$$

PROOF: Easy. □

Proposition 3.56. *Let $f : X \to Y$. Let $A \subseteq X$. Then*

$$A \subseteq f^{-1}(f(A)) \ .$$

Equality holds if f is one-to-one.

PROOF: Easy. □

Proposition 3.57. *Let $f : X \to Y$. Let $\{B_i\}_{i \in I}$ be a family of subsets of Y. Then*

$$f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i) \ .$$

PROOF: Easy. □

Proposition 3.58. *Let $f : X \to Y$. Let $\{B_i\}_{i \in I}$ be a nonempty family of subsets of Y. Then*

$$f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i) \ .$$

PROOF: Easy. □

Proposition 3.59. *Let $f : X \to Y$ and $B \subseteq Y$. Then $f^{-1}(Y - B) = X - f^{-1}(B)$.*

PROOF: Easy. □

Proposition 3.60. *Let $f : X \to Y$ be one-to-one. Then the inverse of f as a relation, f^{-1}, is a function $f^{-1} : \operatorname{ran} f \to X$, and for all $y \in \operatorname{ran} f$, we have $f^{-1}(y)$ is the unique x such that $f(x) = y$.*

PROOF: Easy. ∎

Proposition 3.61. *Let $f : X \to Y$ and $g : Y \to Z$. Then $gf : X \to Z$ and, for all $x \in X$, we have*

$$(g \circ f)(x) = g(f(x)) \enspace .$$

PROOF: Easy. ∎

Example 3.62. Example 3.14 shows that function composition is not commutative in general.

Proposition 3.63. *Let $f : X \to Y$ and $g : Y \to Z$. Then*

$$(gf)^{-1} = f^{-1}g^{-1} : \mathcal{P}Z \to \mathcal{P}X \enspace .$$

PROOF: Easy. ∎

Proposition 3.64. *Let $f : X \to Y$ and $g : Y \to X$. If $gf = I_X$ then f is one-to-one and g maps Y onto X.*

PROOF: Easy. ∎

3.9 Choice Functions

Definition 3.65 (Choice Function)**.** A *choice function* for a set X is a function $f : \mathcal{P}X - \{\emptyset\} \to X$ such that $f(S) \in S$ for all S.

Proposition 3.66. *Every set has a choice function.*

PROOF: Given a nonempty set X, apply the Axiom of Choice to the family $\{S\}_{S \in \mathcal{P}X - \{\emptyset\}}$. ∎

Proposition 3.67. *For any relation R, there exists a function $f \subseteq R$ such that* $\operatorname{dom} f = \operatorname{dom} R$.

PROOF:
⟨1⟩1. LET: R be a relation.
⟨1⟩2. PICK a choice function g for $\operatorname{ran} R$.
⟨1⟩3. LET: $f : \operatorname{dom} R \to \operatorname{ran} R$ be the function $f(x) = g(\{y \in \operatorname{ran} R : xRy\})$
⟨1⟩4. $f \subseteq R$ and $\operatorname{dom} f = \operatorname{dom} R$.
∎

Proposition 3.68. *If $\mathcal{C}$ is a set of pairwise disjoint nonempty sets , then there exists a set A such that, for all $C \in \mathcal{C}$, we have $A \cap C$ is a singleton.*

PROOF:
⟨1⟩1. LET: f be a choice function for $\bigcup \mathcal{C}$
⟨1⟩2. LET: $A = \{f(C) : C \in \mathcal{C}\}$
⟨1⟩3. For all $C \in \mathcal{C}$ we have $A \cap C = \{f(C)\}$
∎

Chapter 4

Equivalence

Definition 4.1 (Equivalent)**.** Sets E and F are *equivalent*, $E \sim F$, iff there exists a one-to-one correspondence between them.

Proposition 4.2. *For any set X, equivalence is an equivalence relation on $\mathcal{P}X$.*

PROOF: Easy.

Theorem 4.3 (Schröder-Bernstein)**.** *Let X and Y be sets. If there exist injective functions $X \to Y$ and $Y \to X$, then $X \sim Y$.*

PROOF:

⟨1⟩1. LET: $f : X \to Y$ and $g : Y \to X$ be one-to-one.

⟨1⟩2. ASSUME: w.l.o.g. $X \cap Y = \varnothing$

⟨1⟩3. For $x \in X$, let us say that x is the *parent* of $f(x)$; and for $y \in Y$, let us say that y is the *parent* of $g(y)$.

⟨1⟩4. For $z \in X \cup Y$, let the set of *descendants* of z be the intersection of all the subsets S of $X \cup Y$ such that $z \in S$ and, if $t \in S$ and t is the parent of u then $u \in S$.

⟨1⟩5. LET: X_X be the set of all elements of X that are descendants of the elements of X that have no parent.

⟨1⟩6. LET: X_Y be the set of all elements of X that are descendants of the elements of Y that have no parent.

⟨1⟩7. LET: $X_\infty = X - X_X - X_Y$

⟨1⟩8. LET: Y_X be the set of all elements of Y that are descendants of the elements of X that have no parent.

⟨1⟩9. LET: Y_Y be the set of all elements of Y that are descendants of the elements of Y that have no parent.

⟨1⟩10. LET: $Y_\infty = Y - Y_X - Y_Y$

⟨1⟩11. $f \restriction X_X : X_X \sim Y_X$

⟨1⟩12. $g \restriction Y_Y : Y_Y \sim X_Y$

⟨1⟩13. $f \restriction X_\infty : X_\infty \sim Y_\infty$

⟨1⟩14. Define $h : X \to Y$ by $h(x) = g^{-1}(x)$ if $x \in X_Y$, and $f(x)$ if not.

$\langle 1 \rangle 15.$ $h : X \sim Y$
□

Theorem 4.4 (Cantor)**.** *For any set X we have $X \nsim \mathcal{P}X$.*

PROOF: If $f : X \to \mathcal{P}X$ then $\{x \in X : x \notin f(x)\}$ is a subset of X not in $\operatorname{ran} f$. □

Chapter 5

Order

Definition 5.1 (Partial Order)**.** A *partial order* on a set X is a relation on X that is reflexive, antisymmetric and transitive.

A *partially ordered set* or *poset* is a pair $(X, \leqslant)$ such that $\leqslant$ is a partial order on X. We write X for the poset $(X, \leqslant)$.

Given a partial order $\leqslant$, we write $\geqslant$ for the inverse of $\leqslant$.

We write $x < y$ or $y > x$ for $x \leqslant y \wedge x \neq y$. When this holds, we say x is *less than* y, *smaller* than y, or a *predecessor* of y; and y is *greater than* x, *larger* than x, or a *successor* of x.

Proposition 5.2. *For any set X, the relation $\subseteq$ is a partial order on $\mathcal{P}X$.*

Proof: Easy. $\square$

Proposition 5.3. *In a poset, we never have $x < y$ and $y < x$.*

Proof: We would then have $x \leqslant y$ and $y \leqslant x$ hence $x = y$ by antisymmetry. But if $x < y$ or $y < x$ then $x \neq y$. $\square$

Proposition 5.4. *The relation $<$ is transitive.*

Proof:
$\langle 1 \rangle 1$. Assume: $x < y$ and $y < z$
$\langle 1 \rangle 2$. $x \leqslant y$ and $y \leqslant z$
$\langle 1 \rangle 3$. $x \leqslant z$
Proof: Since $\leqslant$ is transitive.
$\langle 1 \rangle 4$. $x \neq z$
Proof: By Proposition 5.3.
$\square$

Proposition 5.5. *Let $<$ be a transitive relation on X such that we never have $x < y$ and $y < x$. Define $\leqslant$ by: $x \leqslant y$ iff $x < y$ or $x = y$. Then $\leqslant$ is a partial order on X.*

Proof:

⟨1⟩1. $\leqslant$ is reflexive.
PROOF: By definition.
⟨1⟩2. $\leqslant$ is asymmetric.
PROOF: If $x \leqslant y$ and $y \leqslant x$, we must have $x = y$, because otherwise we would have $x < y$ and $y < x$.
⟨1⟩3. $\leqslant$ is transitive.
⟨2⟩1. LET: $x \leqslant y$ and $y \leqslant z$
⟨2⟩2. CASE: $x = y$
PROOF: We have $y \leqslant z$ so $x \leqslant z$.
⟨2⟩3. CASE: $y = z$
PROOF: We have $x \leqslant y$ so $x \leqslant z$.
⟨2⟩4. CASE: $x < y$ and $y < z$
PROOF: We have $x < z$ by transitivity, so $x \leqslant z$.
$\square$

Definition 5.6 ((Strict) Initial Segment)**.** Let X be a poset and $a \in X$. The *(strict) initial segment* determined by a is

$$s(a) := \{x \in X : x < a\} \ .$$

Definition 5.7 (Weak Initial Segment)**.** Let X be a poset and $a \in X$. The *weak initial segment* determined by a is

$$\overline{s}(a) := \{x \in X : x \leqslant a\} \ .$$

Definition 5.8 (Immediate Successor)**.** Let X be a poset and $x, y \in X$. Then y is the *immediate successor* of x, and x is the *immediate predecessor* of y, iff $x < y$ and there is no z such that $x < z < y$.

Definition 5.9 (Least)**.** Let X be a partial order and $a \in X$. Then a is *least* in X iff $\forall x \in X. a \leqslant x$.

Proposition 5.10. *A poset has at most one least element.*

PROOF: If a and b are least then $a \leqslant b$ and $b \leqslant a$, hence $a = b$. $\square$

Definition 5.11 (Greatest)**.** Let X be a partial order and $a \in X$. Then a is *greatest* in X iff $\forall x \in X. x \leqslant a$.

Proposition 5.12. *A poset has at most one greatest element.*

PROOF: If a and b are greatest then $a \leqslant b$ and $b \leqslant a$, hence $a = b$. $\square$

Definition 5.13 (Minimal)**.** Let X be a poset and $a \in X$. Then a is *minimal* iff there is no $x \in X$ such that $x < a$.

Definition 5.14 (Maximal)**.** Let X be a poset and $a \in X$. Then a is *maximal* iff there is no $x \in X$ such that $a < x$.

Definition 5.15 (Lower Bound)**.** Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is a *lower bound* for E iff $\forall x \in E. a \leqslant x$.

Definition 5.16 (Upper Bound)**.** Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is an *upper bound* for E iff $\forall x \in E.x \leqslant a$.

Definition 5.17 (Greatest Lower Bound, Infimum)**.** Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is the *greatest lower bound* or *infimum* for E iff a is the greatest element in the set of lower bounds for E.

Definition 5.18 (Least Upper Bound, Supremum)**.** Let X be a poset. Let $E \subseteq X$ and $a \in X$. Then a is the *least upper bound* or *supremum* for E iff a is the least element in the set of upper bounds for E.

Definition 5.19 (Total Order)**.** A partial order $\leqslant$ on a set X is a *total order*, *simple order* or *linear order* iff, for all $x, y \in X$, either $x \leqslant y$ or $y \leqslant x$. We then call the poset $(X, \leqslant)$ a *linearly ordered set* or a *chain*.

Proposition 5.20. *Let R be a partial order on X. Then R is total if and only if $X^2 \subseteq R \cup R^{-1}$.*

Proof: Easy. □

Proposition 5.21. *For any set X, the relation $\subseteq$ is a total order on X iff X is either $\varnothing$ or a singleton.*

Proof: Easy. □

Theorem 5.22 (Zorn's Lemma)**.** *Let X be a poset such that every chain in X has an upper bound. Then X has a maximal element.*

Proof:
⟨1⟩1. Pick a choice function f for X.
⟨1⟩2. Let: $\mathcal{X}$ be the set of chains in X.
⟨1⟩3. For all $A \in \mathcal{X}$,
Let: $\hat{A} = \{x \in X : A \cup \{x\} \in \mathcal{X}\}$
⟨1⟩4. Let: $g : \mathcal{X} \to \mathcal{X}$ be the function

$$g(A) = \begin{cases} A \cup \{f(\hat{A} - A)\} & \text{if } \hat{A} - A \neq \varnothing \\ A & \text{if } \hat{A} - A = \varnothing \end{cases}$$

⟨1⟩5. For $\mathcal{T} \subseteq \mathcal{X}$, let us say $\mathcal{T}$ is a *tower* iff:

- $\varnothing \in \mathcal{T}$
- $\forall A \in \mathcal{T}.g(A) \in \mathcal{T}$
- For every chain $\mathcal{C}$ in $\mathcal{T}$, we have $\bigcup \mathcal{C} \in \mathcal{T}$

⟨1⟩6. Let: $\mathcal{T}_0$ be the intersection of the set of all towers.
Proof: The set of all towers is nonempty since $\mathcal{X}$ is a tower.
⟨1⟩7. Let: $A = \bigcup \mathcal{T}_0$
⟨1⟩8. A is a chain in X.
⟨2⟩1. $\mathcal{T}_0$ is a chain under $\subseteq$
⟨3⟩1. Given $C \in \mathcal{T}_0$, let us say that C is *comparable* iff, for all $A \in \mathcal{T}_0$, either $A \subseteq C$ or $C \subseteq A$.

⟨3⟩2. For all $A, C \in \mathcal{T}_0$, if C is comparable and $A \subsetneq C$ then $g(A) \subseteq C$.
Proof: Since $g(A) - A$ has at most one element, so if $A \subsetneq C \subseteq g(A)$ then $C = g(A)$.
⟨3⟩3. For $C \in \mathcal{T}_0$ comparable,
Let: $\mathcal{U}_C = \{A \in \mathcal{T}_0 : A \subseteq C \vee g(C) \subseteq A\}$
⟨3⟩4. For $C \in \mathcal{T}_0$ comparable, $\mathcal{U}_C$ is a tower.
⟨4⟩1. Let: $C \in \mathcal{T}_0$ be comparable
⟨4⟩2. $\emptyset \in \mathcal{U}_C$
Proof: Since $\emptyset \subseteq C$.
⟨4⟩3. $\forall A \in \mathcal{U}_C . g(A) \in \mathcal{U}_C$
Proof: By ⟨1⟩8.
⟨4⟩4. For every chain $\mathcal{C} \subseteq \mathcal{U}_C$ we have $\bigcup \mathcal{C} \in \mathcal{U}_C$
⟨5⟩1. Let: $\mathcal{C} \subseteq \mathcal{U}_C$ be a chain.
⟨5⟩2. Case: $\exists A \in \mathcal{C} . g(C) \subseteq A$
Proof: Then $g(C) \subseteq \bigcup \mathcal{C}$
⟨5⟩3. Case: $\forall A \in \mathcal{C} . A \subseteq C$
Proof: Then $\bigcup \mathcal{C} \subseteq C$.
⟨3⟩5. For $C \in \mathcal{T}_0$ comparable, $\mathcal{U}_C = \mathcal{T}_0$.
⟨3⟩6. For $C \in \mathcal{T}_0$ comparable we have $g(C)$ is comparable.
Proof: Since for all $A \in \mathcal{T}_0$ either $A \subseteq C \subseteq g(C)$ or $g(C) \subseteq A$.
⟨3⟩7. The set of comparable sets in $\mathcal{T}_0$ is a tower.
⟨4⟩1. $\emptyset$ is comparable.
Proof: $\forall A \in \mathcal{T}_0 . \emptyset \subseteq A$
⟨4⟩2. For all $C \in \mathcal{T}_0$, if A is comparable then $g(C)$ is comparable.
Proof: ⟨3⟩6
⟨4⟩3. For every chain $\mathcal{C} \subseteq \mathcal{T}_0$ of comparable sets, we have $\bigcup \mathcal{C}$ is comparable.
⟨5⟩1. Let: $\mathcal{C} \subseteq \mathcal{T}_0$ be a chain of comparable sets.
⟨5⟩2. Let: $A \in \mathcal{T}_0$
⟨5⟩3. Case: there exists $C \in \mathcal{C}$ such that $A \subseteq C$
Proof: Then $A \subseteq \bigcup \mathcal{C}$.
⟨5⟩4. Case: for all $C \in \mathcal{C}$ we have $C \subseteq A$
Proof: Then $\bigcup \mathcal{C} \subseteq A$.
⟨3⟩8. Every set in $\mathcal{T}_0$ is comparable.
⟨2⟩2. Let: $x, y \in A$
⟨2⟩3. Pick $A, C \in \mathcal{T}_0$ such that $x \in A$ and $y \in C$
⟨2⟩4. Assume: w.l.o.g. $A \subseteq C$
⟨2⟩5. $x, y \in C$
⟨2⟩6. $x \leqslant y$ or $y \leqslant x$
Proof: Since $C \in \mathcal{X}$ so C is a chain.
⟨1⟩9. Pick an upper bound u for A.
⟨1⟩10. $A \in \mathcal{T}_0$
Proof: Since $\mathcal{T}_0$ is a chain in $\mathcal{T}_0$ so $\bigcup \mathcal{T}_0 \in \mathcal{T}_0$.
⟨1⟩11. $g(A) \in \mathcal{T}_0$
⟨1⟩12. $g(A) \subseteq A$
⟨1⟩13. $g(A) = A$

⟨1⟩14. $\hat{A} - A = \varnothing$
⟨1⟩15. $u \in A$
  PROOF: Since $A \cup \{u\}$ is a chain so $u \in \hat{A}$ and therefore $u \in A$.
⟨1⟩16. u is maximal in X.
  ⟨2⟩1. LET: $x \in X$
  ⟨2⟩2. ASSUME: $u \leqslant x$
  ⟨2⟩3. $A \cup \{x\}$ is a chain.
  ⟨2⟩4. $x \in A$
  ⟨2⟩5. $x \leqslant u$
  ⟨2⟩6. $x = u$
□

Definition 5.23 (Cofinal)**.** Let X be a poset and $A \subseteq X$. Then A is *cofinal* iff, for all $x \in X$, there exists $a \in A$ such that $x \leqslant a$.

Definition 5.24 (Similar)**.** Two posets X and Y are *similar*, $X \cong Y$ iff there exists an order preserving one-to-one correspondence f between them. We write $f : X \cong Y$ and call f a *similarity.*

Proposition 5.25. *Let X and Y be posets. Let f be a one-to-one correspondence between X and Y. Then f is a similarity if and only if, for all $x, y \in X$, we have $x < y$ iff $f(x) < f(y)$.*

PROOF: Easy. □

Proposition 5.26. *For any poset X we have $I_X : X \cong X$.*

PROOF: Easy. □

Proposition 5.27. *If $f : X \cong Y$ then $f^{-1} : Y \cong X$.*

PROOF: Easy. □

Proposition 5.28. *If $f : X \cong Y$ and $g : Y \cong Z$ then $g \circ f : X \cong Z$.*

PROOF: Easy. □

Corollary 5.28.1. *For any set E, similarity is an equivalence relation on the set of all posets that are subsets of E.*

5.1 Well Orderings

Definition 5.29 (Well Ordered Set)**.** A poset X is *well ordered*, and its ordering is a *well ordering*, iff every nonempty subset of X has a least element.

Proposition 5.30. *Every well ordered set is totally ordered.*

PROOF: For all x and y we have $\{x, y\}$ has a least element, so $x \leqslant y$ or $y \leqslant x$. □

Theorem 5.31 (Transfinite Induction)**.** *Let X be a well ordered set. Let $S \subseteq X$ satisfy:*

$$\forall x \in X(\forall y < x.y \in S) \Rightarrow x \in S \ .$$

Then $S = X$.

PROOF: We have $X - S$ has no least element, so $X - S = \emptyset$. ∎

Definition 5.32 (Continuation)**.** Let A and B be well ordered sets. Then B is a *continuation* of A iff there exists $b \in B$ such that $A = s(b)$ and the order on A is the restriction of the order on B to A.

Proposition 5.33. *Let $\mathcal{C}$ be a set of well ordered sets that is totally ordered under continuation. Then there exists a unique well ordering on $\bigcup \mathcal{C}$ such that $\bigcup \mathcal{C}$ is a continuation of every element of $\mathcal{C}$.*

PROOF: Define $\leqslant$ on $\bigcup \mathcal{C}$ by: $x \leqslant y$ iff there exists $C \in \mathcal{C}$ such that $x, y \in C$ and $x \leqslant y$ in C. ∎

Proposition 5.34. *Every totally ordered set has a cofinal well ordered subset.*

PROOF:
⟨1⟩1. LET: X be a totally ordered set.
⟨1⟩2. LET: $\mathcal{C}$ be the poset of all well ordered subsets of X under continuation.
⟨1⟩3. Every chain in $\mathcal{C}$ has an upper bound.
  PROOF: Proposition 5.33.
⟨1⟩4. PICK a maximal element C of $\mathcal{C}$
    PROVE: C is cofinal
  PROOF: Zorn's Lemma
⟨1⟩5. LET: $x \in X$
⟨1⟩6. We cannot have $\forall c \in C.c < x$
  PROOF: Then $C \cup \{x\}$ would be a larger chain.
⟨1⟩7. $\exists c \in C.x \leqslant c$
∎

Theorem 5.35 (Well Ordering Theorem)**.** *Every set can be well ordered.*

PROOF:
⟨1⟩1. LET: X be a set.
⟨1⟩2. LET: $\mathcal{W}$ be the poset of all well ordered subsets of X under continuation.
⟨1⟩3. Every chain in $\mathcal{W}$ has an upper bound.
  PROOF: Proposition 5.33.
⟨1⟩4. PICK a maximal $M \in \mathcal{W}$
  PROOF: Zorn's Lemma
⟨1⟩5. $M = X$
  PROOF: If $x \in X - M$ then $M \cup \{x\}$ with x as the greatest element is a continuation of M.
∎

Theorem 5.36 (Transfinite Recursion). *Let W be a well ordered set and X a set. Let S be the set of all functions f such that* $\operatorname{ran} f \subseteq X$*, and there exists $a \in W$ such that* $\operatorname{dom} f = s(a)$*. Then there exists a unique function $U : W \to X$ such that*

$$\forall a \in W. U(a) = f(U \restriction s(a)) \ .$$

PROOF:
⟨1⟩1. Let us say that a subset $A \subseteq W \times X$ is *f-closed* iff, whenever $a \in W$ and $t : s(a) \to X$ satisfies $\forall c < a.(c, t(c)) \in A$, then $(a, f(t)) \in A$.
⟨1⟩2. LET: U be the intersection of the set of f-closed subsets of $W \times X$
PROOF: This set is nonempty since $W \times X$ is f-closed.
⟨1⟩3. U is f-closed.
⟨1⟩4. U is a function.
⟨2⟩1. LET: $P(a)$ be the property: there is at most one $x \in X$ such that $(a, x) \in U$
⟨2⟩2. LET: $a \in W$
⟨2⟩3. ASSUME: as transfinite induction hypothesis $\forall c < a.P(c)$
⟨2⟩4. LET: $(a, x), (a, y) \in U$
⟨2⟩5. $x = f(U \restriction c)$
PROOF: If not then $U - \{(a, x)\}$ would be f-closed.
⟨2⟩6. $y = f(U \restriction c)$
⟨2⟩7. $x = y$
⟨1⟩5. $\operatorname{dom} U = W$
⟨2⟩1. LET: $a \in W$
⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall c < a.c \in \operatorname{dom} U$
⟨2⟩3. $(a, f(U \restriction s(a))) \in U$
⟨1⟩6. If $U' : W \to X$ and $\forall a \in W.U'(a) = f(U' \restriction s(a))$, then $U' = U$.
PROOF: Prove $U'(a) = U(a)$ by transfinite induction on a.
□

Proposition 5.37. *Let X be a well ordered set and f a similarity between X and a subset of X. Then, for all $a \in X$, we have $a \leqslant f(a)$.*

PROOF:
⟨1⟩1. LET: $a \in X$
⟨1⟩2. ASSUME: as transfinite induction hypothesis $\forall c < a.c \leqslant f(c)$
⟨1⟩3. ASSUME: for a contradiction $f(a) < a$
⟨1⟩4. $f(a) \leqslant f(f(a))$
PROOF: ⟨1⟩2
⟨1⟩5. $f(f(a)) < f(a)$
PROOF: From ⟨1⟩3 since f is a similarity.
⟨1⟩6. Q.E.D.
PROOF: This is a contradiction.
□

Proposition 5.38. *Let X and Y be well ordered sets. Then there is at most one similarity between them.*

PROOF:
$\langle 1\rangle 1.$ LET: $f, g : X \cong Y$
PROVE: $\forall a \in X . f(a) = g(a)$
$\langle 1\rangle 2.$ LET: $a \in X$
$\langle 1\rangle 3.$ ASSUME: as transfinite induction hypothesis $\forall c < a . f(c) = g(c)$
$\langle 1\rangle 4.$ $f(a)$ is the least element of $Y - \{f(c) : c < a\}$
$\langle 1\rangle 5.$ $g(a)$ is the least element of $Y - \{g(c) : c < a\}$
$\langle 1\rangle 6.$ $f(a) = g(a)$
$\square$

Proposition 5.39. *A well ordered set is not similar to any of its initial segments.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a well ordered set.
$\langle 1\rangle 2.$ ASSUME: for a contradiction $f : X \cong s(a)$ for some $a \in X$
$\langle 1\rangle 3.$ $f(a) < a$
$\langle 1\rangle 4.$ Q.E.D.
PROOF: This contradicts Proposition 5.37.
$\square$

Theorem 5.40 (Comparability Theorem)**.** *Given well ordered sets X and Y, either $X \cong Y$, or X is similar to an initial segment of Y, or Y is similar to an initial segment of X.*

PROOF:
$\langle 1\rangle 1.$ LET: $X_0 = \{a \in X : \exists b \in Y . s(a) \cong s(b)\}$
$\langle 1\rangle 2.$ LET: $U : X_0 \to Y$ be the function: for $a \in X_0$, we have $U(a)$ is the unique element in Y such that $s(a) \cong s(U(a))$
$\langle 1\rangle 3.$ LET: $Y_0 = \operatorname{ran} U$
$\langle 1\rangle 4.$ Either $X_0 = X$ or there exists $a \in X$ such that $X_0 = s(a)$
$\langle 2\rangle 1.$ ASSUME: $X_0 \neq X$
$\langle 2\rangle 2.$ LET: a be the least element of $X - X_0$
$\langle 2\rangle 3.$ LET: $x \in X_0$
PROVE: $x < a$
$\langle 2\rangle 4.$ PICK $f : s(x) \cong s(U(x))$
$\langle 2\rangle 5.$ ASSUME: for a contradiction $a < x$
$\langle 2\rangle 6.$ $f \restriction s(a) : s(a) \cong s(f(a))$
$\langle 2\rangle 7.$ $a \in X_0$
$\langle 2\rangle 8.$ Q.E.D.
PROOF: This is a contradiction.
$\langle 1\rangle 5.$ Either $Y_0 = Y$ or there exists $b \in Y$ such that $Y_0 = s(b)$
PROOF: Similar.
$\langle 1\rangle 6.$ CASE: $X_0 = X$ and $Y_0 = Y$
PROOF: Then $U : X \cong Y$.
$\langle 1\rangle 7.$ CASE: $X_0 = X$ and $Y_0 \neq Y$
PROOF: Then $U : X \cong s(b)$ where $Y_0 = s(b)$.

$\langle 1\rangle 8$. CASE: $X_0 \neq X$ and $Y_0 = Y$

PROOF: Then $U : s(a) \cong Y$ where $X_0 = s(a)$.

$\langle 1\rangle 9$. CASE: $X_0 \neq X$ and $Y_0 \neq Y$

$\langle 2\rangle 1$. LET: $X_0 = s(a)$ and $Y_0 = s(b)$

$\langle 2\rangle 2$. $U : s(a) \cong s(b)$

$\langle 2\rangle 3$. $a \in X_0$

$\langle 2\rangle 4$. Q.E.D.

PROOF: This is a contradiction.

□

Corollary 5.40.1. *Let X be a well ordered set. Then any subset A of X is either similar to X or to an initial segment of X.*

PROOF: We cannot have X is similar to an initial segment of A, say $f : X \cong \{x \in A : x < a\}$, because then we would have $f(a) < a$ contradicting Proposition 5.37. □

Corollary 5.40.2. *For any sets X and Y, either there exists an injective function $X \to Y$, or there exists an injective function $Y \to X$.*

PROOF: Using the Well Ordering Theorem. □

Chapter 6

Natural Numbers

6.1 Natural Numbers

Definition 6.1 (Successor)**.** The *successor* of a set x, x^+, is defined by

$$x^+ := x \cup \{x\} \ .$$

Definition 6.2. We define

$$\begin{aligned} 0 &= \emptyset \\ 1 &= 0^+ \\ 2 &= 1^+ \end{aligned}$$

etc.

Definition 6.3 (Characteristic Function)**.** Let X be a set and $A \subseteq X$. The *characteristic function* of A is the function $\chi_A : X \to 2$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Theorem 6.4. *Let X be a set. The function $\chi : \mathcal{P}X \to 2^X$ that maps a subset A of X to χ_A is a one-to-one correspondence.*

Proof: Easy. □

Definition 6.5. The set ω of *natural numbers* is the set such that:

- $0 \in \omega$
- For all $n \in \omega$ we have $n^+ \in \omega$
- For any set X, if $0 \in X$ and $\forall n \in X . n^+ \in X$ then $\omega \subseteq X$

PROOF: To show this exists, pick a set A such that $0 \in A$ and $\forall n \in A.n^+ \in A$ (by the Axiom of Infinity), and let $\omega = \bigcap\{X \in \mathcal{P}A : 0 \in X \wedge \forall n \in X.n^+ \in X\}$.
□

Definition 6.6 (Sequence)**.** A *finite sequence* is a family whose index set is a natural number. An *infinite sequence* is a family whose index set is ω.

Given a finite sequence of sets $\{A_i\}_{i \in n^+}$, we write $\bigcup_{i=0}^n A_i$ for $\bigcup_{i \in n^+} A_i$. Given an infinite sequence of sets $\{A_i\}_{i \in \omega}$, we write $\bigcup_{i=0}^\infty A_i$ for $\bigcup_{i \in \omega} A_i$.

We make similar definitions for $\bigcap$ and $\times$.

Proposition 6.7. *For any natural numbers m and n, if $m \in n$ then $m^+ \in n^+$.*

PROOF:
⟨1⟩1. LET: $P(n)$ be the property $\forall m \in n.m^+ \in n^+$
⟨1⟩2. $P(0)$
  PROOF: Vacuous.
⟨1⟩3. For any natural number n, if $P(n)$ then $P(n^+)$.
  ⟨2⟩1. LET: n be a natural number.
  ⟨2⟩2. ASSUME: $P(n)$
  ⟨2⟩3. LET: $m \in n^+$
  ⟨2⟩4. $m \in n$ or $m = n$
  ⟨2⟩5. $m^+ \in n^+$ or $m^+ = n^+$
    PROOF: ⟨2⟩2
  ⟨2⟩6. CASE: $m^+ \in n^{++}$
□

Theorem 6.8 (Principle of Mathematical Induction)**.** *For any subset S of ω, if $0 \in S$ and $\forall n \in S.n^+ \in S$, then $S = \omega$.*

PROOF: From the definition of ω. □

Proposition 6.9.

$$\forall n \in \omega.\forall x \in n.n \nsubseteq x$$

PROOF:
⟨1⟩1. $\forall x \in 0.0 \nsubseteq x$
  PROOF: Vacuous.
⟨1⟩2. For any natural number n, if $\forall x \in n.n \nsubseteq x$ then $\forall x \in n^+.n^+ \nsubseteq x$.
  ⟨2⟩1. LET: n be a natural number.
  ⟨2⟩2. ASSUME: $\forall x \in n.n \nsubseteq x$
  ⟨2⟩3. LET: $x \in n^+$
  ⟨2⟩4. ASSUME: for a contradiction $n^+ \subseteq x$
  ⟨2⟩5. $x \in n$ or $x = n$
  ⟨2⟩6. CASE: $x \in n$
    PROOF: Then we have $n \subseteq n^+ \subseteq x$ contradicting ⟨2⟩2.
  ⟨2⟩7. CASE: $x = n$
    PROOF: Then we have $n \in n^+ \subseteq x = n$ and $n \subseteq n$ contradicting ⟨2⟩2.
□

Corollary 6.9.1. *For any natural number n we have $n \notin n$.*

Corollary 6.9.2. *For any natural number n we have $n \neq n^+$.*

Definition 6.10 (Transitive Set)**.** A set E is a *transitive* set iff, whenever $x \in y \in E$, then $x \in E$.

Proposition 6.11. *Every natural number is a transitive set.*

PROOF:
$\langle 1 \rangle 1$. 0 is a transitive set.
  PROOF: Vacuously, if $x \in y \in 0$ then $x \in 0$.
$\langle 1 \rangle 2$. For any natural number n, if n is a transitive set, then n^+ is a transitive set.
  $\langle 2 \rangle 1$. LET: n be a natural number.
  $\langle 2 \rangle 2$. ASSUME: n is a transitive set.
  $\langle 2 \rangle 3$. LET: $x \in y \in n^+$
  $\langle 2 \rangle 4$. $y \in n$ or $y = n$
  $\langle 2 \rangle 5$. CASE: $y \in n$
    $\langle 3 \rangle 1$. $x \in n$
      PROOF: $\langle 2 \rangle 2$, $\langle 2 \rangle 3$, $\langle 2 \rangle 5$.
    $\langle 3 \rangle 2$. $x \in n^+$
  $\langle 2 \rangle 6$. CASE: $y = n$
    $\langle 3 \rangle 1$. $x \in n$
      PROOF: $\langle 2 \rangle 3$, $\langle 2 \rangle 6$
    $\langle 3 \rangle 2$. $x \in n^+$
□

Proposition 6.12. *For any natural numbers m and n, if $m^+ = n^+$ then $m = n$.*

PROOF:
$\langle 1 \rangle 1$. LET: m and n be natural numbers.
$\langle 1 \rangle 2$. ASSUME: $m^+ = n^+$
$\langle 1 \rangle 3$. $m \in m^+ = n^+$
$\langle 1 \rangle 4$. $m \in n$ or $m = n$
$\langle 1 \rangle 5$. $n \in n^+ = m^+$
$\langle 1 \rangle 6$. $n \in m$ or $n = m$
$\langle 1 \rangle 7$. We cannot have $m \in n$ and $n \in m$
  $\langle 2 \rangle 1$. ASSUME: for a contradiction $m \in n$ and $n \in m$
  $\langle 2 \rangle 2$. $m \in m$
    PROOF: Since m is a transitive set (Proposition 6.11).
  $\langle 2 \rangle 3$. Q.E.D.
    PROOF: This contradicts Proposition 6.9.
$\langle 1 \rangle 8$. $m = n$
□

Theorem 6.13 (Recursion Theorem)**.** *Let X be a set. Let $a \in X$. Let $f : X \to X$. There exists a function $u : \omega \to X$ such that $u(0) = a$ and, for all $n \in \omega$, we have $u(n^+) = f(u(n))$.*

PROOF:
⟨1⟩1. LET: $\mathcal{C} = \{A \in \mathcal{P}(\omega \times X) : (0, a) \in A \wedge \forall n \in \omega . \forall x \in X.(n, x) \in A \Rightarrow (n^+, f(x)) \in A\}$
⟨1⟩2. $\mathcal{C} \neq \emptyset$
PROOF: $\omega \times X \in \mathcal{C}$
⟨1⟩3. LET: $u = \bigcap \mathcal{C}$
⟨1⟩4. $u \in \mathcal{C}$
⟨1⟩5. u is a function.
⟨2⟩1. LET: $P(n)$ be the property: $\forall x, y \in X.(n, x) \in u \wedge (n, y) \in u \Rightarrow x = y$
⟨2⟩2. $P(0)$
⟨3⟩1. $\forall x \in X.(0, x) \in u \Rightarrow x = a$
PROOF: If $(0, x) \in u$ and $x \neq a$ then $u - \{(0, x)\} \in \mathcal{C}$ and so $u - \{(0, x)\} \subseteq u$, which is impossible.
⟨2⟩3. For every natural number n, if $P(n)$ then $P(n^+)$.
⟨3⟩1. LET: n be a natural number.
⟨3⟩2. ASSUME: $P(n)$
⟨3⟩3. LET: $x, y \in X$
⟨3⟩4. ASSUME: $(n^+, x), (n^+, y) \in u$
⟨3⟩5. PICK $x', y' \in X$ such that $(n, x') \in u$, $(n, y') \in u$ and $f(x') = x$ and $f(y') = y$
PROOF: If no such x' exists then $u - \{(n^+, x)\} \in \mathcal{C}$ and so $u - \{(n^+, x)\} \subseteq u$ which is impossible. Similarly for y'.
⟨3⟩6. $x' = y'$
PROOF: ⟨3⟩2
⟨3⟩7. $x = y$
□

Proposition 6.14. *For any natural number n, either $n = 0$ or there exists a natural number m such that $n = m^+$.*

PROOF: Easy induction on n. □

Proposition 6.15. *ω is a transitive set.*

PROOF:
⟨1⟩1. LET: $P(n)$ be the property $\forall x \in n.x \in \omega$
⟨1⟩2. $P(0)$
PROOF: Vacuous.
⟨1⟩3. For any natural number n, if $P(n)$ then $P(n^+)$.
⟨2⟩1. LET: n be a natural number.
⟨2⟩2. ASSUME: $P(n)$
⟨2⟩3. LET: $x \in n^+$
⟨2⟩4. $x \in n$ or $x = n$
⟨2⟩5. CASE: $x \in n$
PROOF: Then $x \in \omega$ by ⟨2⟩2.
⟨2⟩6. CASE: $x = n$
PROOF: Then $x \in \omega$ by ⟨2⟩1.

□

Proposition 6.16. *For any natural number n and any nonempty subset $E \subseteq n$, there exists $k \in E$ such that $\forall m \in E.k = m \vee k \in m$.*

PROOF:
⟨1⟩1. LET: $P(n)$ be the property: for any nonempty subset $E \subseteq n$, there exists $k \in E$ such that $\forall m \in E.k = m \vee k \in m$
⟨1⟩2. $P(0)$
PROOF: Vacuous as there is no nonempty subset of 0.
⟨1⟩3. For any natural number n, if $P(n)$ then $P(n^+)$.
⟨2⟩1. LET: n be a natural number.
⟨2⟩2. ASSUME: $P(n)$
⟨2⟩3. LET: E be a nonempty subset of n^+
⟨2⟩4. CASE: $E - \{n\} = \varnothing$
PROOF: Then $E = \{n\}$ so take $k = n$.
⟨2⟩5. CASE: $E - \{n\} \neq \varnothing$
⟨3⟩1. PICK $k \in E - \{n\}$ such that $\forall m \in E - \{n\}.k = m \vee k \in m$
PROOF: By ⟨2⟩2.
⟨3⟩2. $\forall m \in E.k = m \vee k \in m$
PROOF: Since $k \in n$.

□

Chapter 7

Ordinal Numbers

Definition 7.1 (Ordinal (Number))**.** An *ordinal (number)* is a well ordered set α such that $\forall \xi \in \alpha . s(\xi) = \xi$.

Given ordinals α, β, we write $\alpha < \beta$ iff $\alpha \in \beta$.

Proposition 7.2. *Every natural number is an ordinal.*

PROOF: Easy. □

Proposition 7.3. *ω is an ordinal.*

PROOF: Easy. □

Proposition 7.4. *If α is an ordinal number then so is α^+.*

PROOF: Easy. □

Proposition 7.5. *Let α be an ordinal and $\eta, \xi \in \alpha$. Then $\eta < \xi$ if and only if $\eta \in \xi$.*

PROOF: Easy. □

Proposition 7.6. *Every ordinal is a transitive set.*

PROOF: Easy. □

Proposition 7.7. *Every element of an ordinal is an ordinal.*

PROOF: Easy. □

Proposition 7.8. *Similar ordinals are equal.*

PROOF:
$\langle 1 \rangle 1.$ LET: α, β be ordinals.
$\langle 1 \rangle 2.$ LET: $f : \alpha \cong \beta$ be a similarity.
PROVE: $\forall \xi \in \alpha . f(\xi) = \xi$
$\langle 1 \rangle 3.$ LET: $\xi \in \alpha$

$\langle 1\rangle 4.$ ASSUME: as transfinite induction hypothesis $\forall \eta < \xi . f(\eta) = \eta$
$\langle 1\rangle 5.$ $f(\xi) \subseteq \xi$
$\langle 2\rangle 1.$ LET: $\eta \in f(\xi)$
$\langle 2\rangle 2.$ PICK $\zeta \in \alpha$ such that $f(\zeta) = \eta$
$\langle 2\rangle 3.$ $\zeta \in \xi$
PROOF: Since $f(\zeta) \in f(\xi)$ and f is a similarity.
$\langle 2\rangle 4.$ $f(\zeta) = \zeta$
PROOF: $\langle 1\rangle 4$
$\langle 2\rangle 5.$ $\eta = \zeta$
PROOF: $\langle 2\rangle 2$, $\langle 2\rangle 4$
$\langle 2\rangle 6.$ $\eta \in \xi$
PROOF: $\langle 2\rangle 3$, $\langle 2\rangle 5$
$\langle 1\rangle 6.$ $\xi \subseteq f(\xi)$
$\langle 2\rangle 1.$ LET: $\eta \in \xi$
$\langle 2\rangle 2.$ $\eta = f(\eta) \in f(\xi)$
$\langle 1\rangle 7.$ $f(\xi) = \xi$
$\square$

Proposition 7.9. *Let α and β be ordinals. Then the following are equivalent.*

1. $\alpha \in \beta$
2. $\alpha \subsetneq \beta$
3. *β is a continuation of α.*

PROOF:
$\langle 1\rangle 1.$ $1 \Rightarrow 3$
PROOF: If $\alpha \in \beta$ then $\alpha = s(\alpha)$.
$\langle 1\rangle 2.$ $3 \Rightarrow 2$
PROOF: Immediate from definitions.
$\langle 1\rangle 3.$ $2 \Rightarrow 1$
$\langle 2\rangle 1.$ LET: γ be the least element of β such that $\gamma \notin \alpha$
$\langle 2\rangle 2.$ $\alpha \subseteq \gamma$
$\langle 3\rangle 1.$ LET: $\eta \in \alpha$
$\langle 3\rangle 2.$ $\eta \subseteq \alpha$
$\langle 3\rangle 3.$ $\gamma \notin \eta$
$\langle 3\rangle 4.$ $\eta \in \gamma$ or $\eta = \gamma$
$\langle 3\rangle 5.$ $\eta \neq \gamma$
PROOF: Since $\eta \in \alpha$ and $\gamma \notin \alpha$.
$\langle 3\rangle 6.$ $\eta \in \gamma$
$\langle 2\rangle 3.$ $\gamma \subseteq \alpha$
PROOF: For all $\eta \in \gamma$ we have $\eta \in \alpha$ by leastness of γ.
$\langle 2\rangle 4.$ $\gamma = \alpha$
$\langle 2\rangle 5.$ $\alpha \in \beta$
$\square$

Proposition 7.10. *For any ordinal numbers α and β, either $\alpha = \beta$, or $\alpha < \beta$, or $\beta < \alpha$.*

PROOF:
$\langle 1 \rangle 1.$ Either $\alpha = \beta$, or α is similar to an initial segment of β, or β is similar to an initial segment of α.
$\langle 1 \rangle 2.$ CASE: α is similar to an initial segment of β.
$\langle 2 \rangle 1.$ PICK $\eta \in \beta$ such that $\alpha \sim s(\eta)$
$\langle 2 \rangle 2.$ $\alpha \sim \eta$
$\langle 2 \rangle 3.$ $\alpha = \eta$
PROOF: Proposition 7.8.
$\langle 2 \rangle 4.$ $\alpha \in \beta$
$\langle 1 \rangle 3.$ CASE: β is similar to an initial segment of α.
PROOF: Then $\beta \in \alpha$ similarly.
$\square$

Proposition 7.11. *Every set of ordinals is well ordered by* $<$.

PROOF:
$\langle 1 \rangle 1.$ LET: E be a set of ordinals.
$\langle 1 \rangle 2.$ LET: A be a nonempty subset of E.
$\langle 1 \rangle 3.$ PICK $\alpha \in A$
$\langle 1 \rangle 4.$ CASE: $\alpha \cap A = \varnothing$
PROOF: Then α is least in A.
$\langle 1 \rangle 5.$ CASE: $\alpha \cap A \neq \varnothing$
PROOF: Then $\alpha \cap A$ has a least element, which is least in A.
$\square$

Definition 7.12 (Limit Ordinal)**.** A *limit ordinal* is an ordinal number that is not 0 and not α^+ for any ordinal α.

Proposition 7.13. *For any set* E *of ordinal numbers,* $\bigcup E$ *is an ordinal and is the supremum of* E.

PROOF: Proposition 5.33. $\square$

Theorem 7.14 (Burali-Forti Paradox)**.** *There is no set whose members are exactly the ordinal numbers.*

PROOF: For any set of ordinals E, we have $(\bigcup E)^+$ is an ordinal that is not in E. $\square$

Theorem 7.15 (Counting Theorem)**.** *Every well ordered set is similar to a unique ordinal.*

PROOF:
$\langle 1 \rangle 1.$ LET: X be a well ordered set.
$\langle 1 \rangle 2.$ There exists an ordinal α such that $X \cong \alpha$.
$\langle 2 \rangle 1.$ For all $a \in X$, there exists a unique ordinal α such that $s(a) \cong \alpha$
$\langle 3 \rangle 1.$ LET: $a \in X$
$\langle 3 \rangle 2.$ ASSUME: as transfinite induction hypothesis that, for all $b < a$, there exists a unique ordinal β such that $s(b) \cong \beta$

$\langle 3\rangle 3$. Let: $\alpha = \{\beta : \beta \text{ is an ordinal } \wedge \exists b < a.s(b) \cong \beta\}$
Proof: This is a set by the Axiom of Substitution.
$\langle 3\rangle 4$. α is an ordinal
$\langle 4\rangle 1$. Let: $\gamma \in \beta \in \alpha$
$\langle 4\rangle 2$. Pick $b < a$ and $f : s(b) \cong \beta$
$\langle 4\rangle 3$. Pick $c < b$ such that $f(c) = \gamma$
$\langle 4\rangle 4$. $f \restriction s(c) : s(c) \cong \gamma$
$\langle 3\rangle 5$. $s(a) \cong \alpha$
Proof: The function $f : s(a) \to \alpha$ defined by $f(b)$ is the ordinal such that $s(b) \cong f(b)$ is a similarity.
$\langle 3\rangle 6$. α is unique.
Proof: Proposition 7.8.
$\langle 2\rangle 2$. Let: $\alpha = \{\beta : \beta \text{ is an ordinal } \wedge \exists a \in X.s(a) \cong \beta\}$
Proof: This is a set by the Axiom of Substitution.
$\langle 2\rangle 3$. α is an ordinal.
Proof: Similar.
$\langle 2\rangle 4$. $X \cong \alpha$
Proof: Similar.
$\langle 1\rangle 3$. For any ordinals α and β, if $X \cong \alpha$ and $X \cong \beta$ then $\alpha = \beta$.
Proof: Proposition 7.8.
□

7.1 Order on the Natural Numbers

Proposition 7.16. *For natural numbers m, n and k, if $m < n$ then $m + k < n + k$.*

Proof:
$\langle 1\rangle 1$. Let: $m, n \in \omega$
$\langle 1\rangle 2$. Assume: $m < n$
$\langle 1\rangle 3$. $m + 0 < n + 0$
$\langle 1\rangle 4$. $\forall k \in \omega.m + k < n + k \Rightarrow m + k^+ < n + k^+$
Proof: By Proposition 6.7.
□

Proposition 7.17. *For natural numbers m, n and k, if $m < n$ and $k \neq 0$ then $mk < nk$.*

Proof:
$\langle 1\rangle 1$. Let: $m, n \in \omega$
$\langle 1\rangle 2$. Assume: $m < n$
$\langle 1\rangle 3$. $m1 < n1$
$\langle 1\rangle 4$. For all $k \in \omega$, if $k \neq 0$ and $mk < nk$ then $m(k + 1) < n(k + 1)$

Proof:

$$\begin{aligned} m(k+1) &= mk + m \\ &< mk + n && \text{(Proposition 7.16)} \\ &< nk + n && \text{(Proposition 7.16)} \\ &= n(k+1) \end{aligned}$$

∎

Proposition 7.18. *Let n be a natural number. Let X be a proper subset of n. Then there exists $m < n$ such that $X \sim m$.*

Proof:

⟨1⟩1. Let: $P(n)$ be the property: for every proper subset $X \subsetneq n$, there exists $m < n$ such that $X \sim m$.

⟨1⟩2. $P(0)$

Proof: Vacuous.

⟨1⟩3. $\forall n \in \omega . P(n) \Rightarrow P(n+1)$

⟨2⟩1. Let: $n \in \omega$

⟨2⟩2. Assume: $P(n)$

⟨2⟩3. Let: X be a proper subset of $n+1$

⟨2⟩4. Case: $X - \{n\} = n$

Proof: Then $X = n$ so $X \sim n < n+1$.

⟨2⟩5. Case: $X - \{n\} \subsetneq n$

⟨3⟩1. Pick $m < n$ such that $X - \{n\} \sim m$

⟨3⟩2. $X \sim m$ or $X \sim m+1$

Proof: If $n \in X$ then $X \sim m+1$. If $n \notin X$ then $X \sim m$.

∎

Proposition 7.19. *For every natural number n, we have n is not equivalent to a proper subset of n.*

Proof:

⟨1⟩1. Let: $P(n)$ be the property: every one-to-one function $n \to n$ is onto.

⟨1⟩2. $P(0)$

Proof: The only function $0 \to 0$ is $\emptyset$.

⟨1⟩3. $\forall n \in \omega . P(n) \Rightarrow P(n+1)$

⟨2⟩1. Let: $n \in \omega$

⟨2⟩2. Assume: $P(n)$

⟨2⟩3. Assume: $f : n+1 \to n+1$ is one-to-one.

⟨2⟩4. Let: $g : n \to n$ be the function

$$g(k) = \begin{cases} f(k) & \text{if } f(k) < n \\ f(n) & \text{if } f(k) = n \end{cases}$$

Proof: If $k < n$ and $f(k) = n$ then $f(n) < n$ since f is one-to-one.

⟨2⟩5. g is one-to-one.

⟨3⟩1. Let: $k, l < n$

⟨3⟩2. Assume: $g(k) = g(l)$

⟨3⟩3. Case: $f(k) < n$ and $f(l) < n$

PROOF: Then $f(k) = g(k) = g(l) = f(l)$ so $k = l$ since f is one-to-one.

$\langle 3 \rangle 4$. CASE: $f(k) < n$ and $f(l) = n$

PROOF: Then $f(k) = g(k) = g(l) = f(n)$ contradicting the fact that f is one-to-one.

$\langle 3 \rangle 5$. CASE: $f(k) = n$ and $f(l) < n$

PROOF: Similar.

$\langle 3 \rangle 6$. CASE: $f(k) = n$ and $f(l) = n$

PROOF: Then $k = l$ since f is one-to-one.

$\langle 2 \rangle 6$. g maps n onto n.

PROOF: $\langle 2 \rangle 2$

$\langle 2 \rangle 7$. f maps $n + 1$ onto $n + 1$.

$\langle 3 \rangle 1$. LET: $l < n + 1$

$\langle 3 \rangle 2$. CASE: $l < n$

$\langle 4 \rangle 1$. PICK $k < n$ such that $g(k) = l$

$\langle 4 \rangle 2$. $f(k) = l$ or $f(n) = l$

$\langle 3 \rangle 3$. CASE: $l = n$

$\langle 4 \rangle 1$. CASE: $f(n) = n$

PROOF: Then $l \in \operatorname{ran} f$ as required.

$\langle 4 \rangle 2$. CASE: $f(n) < n$

$\langle 5 \rangle 1$. PICK $k < n$ such that $g(k) = f(n)$

$\langle 5 \rangle 2$. $f(k) = n$

$\square$

Corollary 7.19.1. *Equivalent natural numbers are equal.*

Definition 7.20 (Lexicographical Order)**.** The *lexicographical* order on $\omega \times \omega$ is the relation S defined by $(a, b)S(x, y)$ iff $a < x$ or ($a = x$ and $b < y$).

Proposition 7.21. *The lexicographical order is a well ordering on $\omega \times \omega$.*

PROOF: Easy. $\square$

7.2 Finite Sets

Definition 7.22 (Finite)**.** A set is *finite* iff it is equivalent to a natural number; otherwise, it is *infinite*.

Proposition 7.23. *No finite set is equivalent to one of its proper subsets.*

PROOF: From Proposition 7.19. $\square$

Proposition 7.24. *ω is infinite.*

PROOF: Since the function that maps n to $n+1$ is a one-to-one correspondence between ω and $\omega - \{0\}$. $\square$

Proposition 7.25. *Every subset of a finite set is finite.*

PROOF: Proposition 7.18. $\square$

Definition 7.26 (Number of Elements)**.** For any finite set E, the *number of elements* in E, $\sharp(E)$, is the unique natural number such that $E \sim \sharp(E)$.

Proposition 7.27. *Let E and F be finite sets. If $E \subseteq F$ then $\sharp(E) \leqslant \sharp(F)$.*

PROOF: Proposition 7.18. ☐

Proposition 7.28. *Let E and F be disjoint finite sets. Then $E \cup F$ is finite and $\sharp(E \cup F) = \sharp(E) \cup \sharp(F)$.*

PROOF:
⟨1⟩1. LET: $P(n)$ be the statement: $n \in \omega$ and for any $m \in \omega$, if $E \sim m$, $F \sim n$ and $E \cap F = \emptyset$, then $E \cup F \sim m + n$
⟨1⟩2. $P(0)$
⟨2⟩1. LET: $m \in \omega$
⟨2⟩2. LET: $E \sim m$ and $F \sim 0$
⟨2⟩3. $F = \emptyset$
⟨2⟩4. $E \cup F = E \sim m = m + 0$
⟨1⟩3. $\forall n \in \omega . P(n) \Rightarrow P(n+1)$
⟨2⟩1. LET: $n \in \omega$
⟨2⟩2. ASSUME: $P(n)$
⟨2⟩3. LET: $m \in \omega$
⟨2⟩4. LET: $E \sim m$ and $F \sim n + 1$
⟨2⟩5. ASSUME: $E \cap F = \emptyset$
⟨2⟩6. PICK $f \in F$
⟨2⟩7. $F - \{f\} \sim n$
⟨2⟩8. $E \cap (F - \{f\}) = \emptyset$
⟨2⟩9. $E \cup (F - \{f\}) \sim m + n$
PROOF: ⟨2⟩2
⟨2⟩10. $E \cup F \sim m + n + 1$
☐

Corollary 7.28.1. *The union of two finite sets is finite.*

PROOF: Since, if E and F are finite, then $E \cup F = (E - F) \cup (E \cap F) \cup (F - E)$ and these are finite and disjoint. ☐

Proposition 7.29. *If E and F are finite sets then $E \times F$ is finite and $\sharp(E \times F) = \sharp(E)\sharp(F)$.*

PROOF:
⟨1⟩1. LET: $P(n)$ be the statement: $n \in \omega$ and for all $m \in \omega$, if $E \sim m$ and $F \sim n$ then $E \times F \sim mn$
⟨1⟩2. $P(0)$
PROOF: If $F \sim 0$ then $F = \emptyset$ so $E \times F = \emptyset \sim 0$.
⟨1⟩3. $\forall n \in \omega . P(n) \Rightarrow P(n+1)$
⟨2⟩1. LET: $n \in \omega$
⟨2⟩2. ASSUME: $P(n)$
⟨2⟩3. LET: $m \in \omega$

⟨2⟩4. Assume: $E \sim m$ and $F \sim n+1$
⟨2⟩5. Pick $f \in F$
⟨2⟩6. $F - \{f\} \sim n$
⟨2⟩7. $E \times (F - \{f\}) \sim mn$
⟨2⟩8. $E \times F = (E \times (F - \{f\})) \cup (E \times \{f\})$
⟨2⟩9. $E \times \{f\} \sim m$
⟨2⟩10. $E \times F \sim mn + m$
Proof: Proposition 7.28.
□

Proposition 7.30. *For any finite sets E and F, we have E^F is finite and $\sharp(E^F) = \sharp(E)^{\sharp(F)}$.*

Proof:
⟨1⟩1. Let: $P(n)$ be the property: $n \in \omega$ and for all $m \in \omega$, if $E \sim m$ and $F \sim n$ then $E^F \sim m^n$
⟨1⟩2. $P(0)$
Proof: Since $E^{\varnothing} = \{\varnothing\} \sim 1$
⟨1⟩3. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$
⟨2⟩1. Let: $n \in \omega$
⟨2⟩2. Assume: $P(n)$
⟨2⟩3. Let: $m \in \omega$
⟨2⟩4. Let: $E \sim m$ and $F \sim n+1$
⟨2⟩5. Pick $f \in F$
⟨2⟩6. $F - \{f\} \sim n$
⟨2⟩7. Let: $\phi : E^F \to E^{F-\{f\}} \times E$ be the function $\phi(g) = (g \restriction (F - \{f\}), g(f))$
⟨2⟩8. ϕ is a one-to-one correspondence
⟨2⟩9. $\sharp(E^F) = m^{n+1}$
Proof:

$$\begin{aligned}
\sharp(E^F) &= \sharp(E^{F-\{f\}} \times E) & \\
&= \sharp(E^{F-\{f\}})\sharp(E) & \text{(Proposition 7.29)} \\
&= m^n m & (\langle 2\rangle 2,\ \langle 2\rangle 4) \\
&= m^{n+1}
\end{aligned}$$

□

Corollary 7.30.1. *If E is finite then $\mathcal{P}E$ is finite and $\sharp(\mathcal{P}E) = 2^{\sharp(E)}$.*

Proposition 7.31. *The union of a finite set of finite sets is finite.*

Proof:
⟨1⟩1. Let: $P(n)$ be the property: for any set E, if $E \sim n$ and every element of E is finite, then $\bigcup E$ is finite.
⟨1⟩2. $P(0)$
Proof: Since $\bigcup \varnothing = \varnothing$ is finite.
⟨1⟩3. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$
⟨2⟩1. Let: n be a natural number.

⟨2⟩2. Assume: $P(n)$
⟨2⟩3. Let: $E \sim n+1$
⟨2⟩4. Pick $X \in E$
⟨2⟩5. $E - \{X\} \sim n$
⟨2⟩6. $\bigcup(E - \{X\})$ is finite.
Proof: ⟨2⟩2
⟨2⟩7. $\bigcup E = \bigcup(E - \{X\}) \cup X$
⟨2⟩8. $\bigcup E$ is finite.
Proof: Corollary 7.28.1.
□

Proposition 7.32. *Every nonempty finite set of natural numbers has a greatest element.*

Proof:
⟨1⟩1. Let: $P(n)$ be the property: for every $E \subseteq \mathbb{N}$, if $E \sim n$ then E has a greatest element.
⟨1⟩2. $P(1)$
Proof: Since k is the greatest element of $\{k\}$.
⟨1⟩3. $\forall n \geqslant 1. P(n) \Rightarrow P(n+1)$
⟨2⟩1. Let: $n \geqslant 1$
⟨2⟩2. Assume: $P(n)$
⟨2⟩3. Assume: $E \subseteq \omega$ and $E \sim n+1$
⟨2⟩4. Pick $k \in E$
⟨2⟩5. Let: l be the greatest element of $E - \{k\}$
⟨2⟩6. Either k or l is greatest in E.
□

Proposition 7.33. *Every infinite set has a subset equivalent to ω.*

Proof:
⟨1⟩1. Let: X be an infinite set.
⟨1⟩2. Pick a choice function f for X.
⟨1⟩3. Let: $\mathcal{C}$ be the set of all finite subsets of X.
⟨1⟩4. For all $A \in \mathcal{C}$ we have $X - A \in \operatorname{dom} f$.
Proof: For all $A \in \mathcal{C}$ we have $X - A \neq \emptyset$.
⟨1⟩5. Let: $U : \omega \to \mathcal{C}$ be the function defined recursively by $U(0) = \emptyset$ and $U(n+1) = U(n) \cup \{f(X - U(n))\}$ for all $n \in \omega$.
⟨1⟩6. Let: $v : \omega \to X$ be the function $v(n) = f(X - U(n))$
Prove: v is one-to-one.
⟨1⟩7. $\forall n \in \omega. v(n) \notin U(n)$
Proof: Since $v(n) = f(X - U(n)) \in X - U(n)$.
⟨1⟩8. $\forall n \in \omega. v(n) \in U(n+1)$
⟨1⟩9. $\forall m, n \in \omega. n \leqslant m \Rightarrow U(n) \subseteq U(m)$
Proof: Since $U(n) \subseteq U(n+1)$ for all n.
⟨1⟩10. $\forall m, n \in \omega. n < m \Rightarrow v(n) \neq v(m)$
Proof: Since $v(n) \in U(m)$ and $v(m) \notin U(m)$.

□

Corollary 7.33.1. *A set is infinite if and only if it is equivalent to a proper subset.*

7.3 Ordinal Arithmetic

Definition 7.34 (Addition)**.** Let I be a well ordered set and $(\alpha_i)_{i \in I}$ be a sequence of ordinals. Choose a well ordered set A_i such that $A_i \cong \alpha_i$ for each $i \in I$, and assume the sets A_i are pairwise disjoint. The *sum* $\sum_{i \in I} \alpha_i$ is the ordinal of the well ordered set $\bigcup_{i \in I} A_i$, where:

- for $x, y \in A_i$, we have $x <_{\bigcup_{i \in I} A_i} y$ if and only if $x <_{A_i} y$
- for $x \in A_i$ and $y \in A_j$ with $i \neq j$, we have $x <_{\bigcup_{i \in I} A_i} y$ iff $i <_I j$

We write $\alpha + \beta$ for $\sum_{i \in 2} \gamma_i$ where $\gamma_0 = \alpha$ and $\gamma_1 = \beta$.

Proposition 7.35.

$$\begin{aligned} \alpha + 0 &= \alpha \\ 0 + \alpha &= \alpha \\ \alpha + 1 &= \alpha^+ \\ \alpha + (\beta + \gamma) &= (\alpha + \beta) + \gamma \end{aligned}$$

Proof: Easy. □

Proposition 7.36. *For any ordinals α and β, we have $\alpha < \beta$ if and only if there exists $\gamma \neq 0$ such that $\beta = \alpha + \gamma$.*

Proof: Easy. □

Proposition 7.37.

$$1 + \omega = \omega$$

Proof: Easy. □

Definition 7.38 (Multiplication)**.** Given ordinals α and β, the *product* $\alpha\beta$ is the ordinal of $\alpha \times \beta$ under the *reverse lexicographic order*: $(a, b) < (c, d)$ iff $b < d$ or ($b = d$ and $a < c$).

Proposition 7.39.

$$\begin{aligned} \alpha 0 &= 0 \\ 0\alpha &= 0 \\ \alpha 1 &= \alpha \\ 1\alpha &= \alpha \\ \alpha(\beta\gamma) &= (\alpha\beta)\gamma \\ \alpha(\beta + \gamma) &= \alpha\beta + \alpha\gamma \end{aligned}$$

PROOF: Easy. ∎

Proposition 7.40. *For ordinals α and β, if $\alpha\beta = 0$ then $\alpha = 0$ or $\beta = 0$.*

PROOF: Easy. ∎

Example 7.41. The commutative law fails:

$$2\omega = \omega \neq \omega 2$$

PROOF: Easy. ∎

Example 7.42. The right distributive law fails:

$$(1+1)\omega = \omega \neq 1\omega + 1\omega = \omega 2$$

Definition 7.43 (Exponentiation)**.** Given ordinals α and β, define the ordinal α^β by

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta \alpha \\ \alpha^\lambda &= \bigcup_{\beta<\lambda} \alpha^\beta && (\lambda \text{ a limit ordinal}) \end{aligned}$$

Proposition 7.44.

$$\begin{aligned} 0^\alpha &= 0 && (\alpha \geqslant 1) \\ 1^\gamma &= 1 \\ \alpha^{\beta+\gamma} &= \alpha^\beta \alpha^\gamma \\ \alpha^{\beta\gamma} &= (\alpha^\beta)^\gamma \end{aligned}$$

PROOF: Easy. ∎

Example 7.45. $(\alpha\beta)^\gamma$ is different from $\alpha^\gamma\beta^\gamma$ in general:

$$(2 \cdot 2)^\omega = \omega \neq 2^\omega 2^\omega = \omega^2 \enspace .$$

7.4 Arithmetic on the Natural Numbers

Proposition 7.46. *For all $m, n \in \omega$, we have*

$$m + n = n + m \enspace .$$

PROOF:
⟨1⟩1. LET: $P(m)$ be the property $\forall n \in \omega . m + n = n + m$
⟨1⟩2. $P(0)$
  ⟨2⟩1. LET: $Q(n)$ be the property $0 + n = n + 0$
  ⟨2⟩2. $Q(0)$

PROOF: Trivial.

⟨2⟩3. $\forall n \in \omega . Q(n) \Rightarrow Q(n^+)$

⟨3⟩1. LET: $n \in \omega$

⟨3⟩2. ASSUME: $Q(n)$

⟨3⟩3. $0 + n^+ = n^+ + 0$

PROOF:

$$\begin{aligned} 0 + n^+ &= (0+n)^+ \\ &= (n+0)^+ && (\langle 3\rangle 2) \\ &= n^+ \\ &= n^+ + 0 \end{aligned}$$

⟨1⟩3. $\forall m \in \omega . P(m) \Rightarrow P(m^+)$

⟨2⟩1. LET: $m \in \omega$

⟨2⟩2. ASSUME: $P(m)$

⟨2⟩3. LET: $Q(n)$ be the property $m^+ + n = n + m^+$

⟨2⟩4. $Q(0)$

PROOF: ⟨1⟩2

⟨2⟩5. $\forall n \in \omega . Q(n) \Rightarrow Q(n^+)$

⟨3⟩1. LET: $n \in \omega$

⟨3⟩2. ASSUME: $Q(n)$

⟨3⟩3. $Q(n^+)$

PROOF:

$$\begin{aligned} m^+ + n^+ &= (m^+ + n)^+ \\ &= (n + m^+)^+ && (\langle 3\rangle 2) \\ &= (n+m)^{++} \\ &= (m+n)^{++} && (\langle 2\rangle 2) \\ &= (m+n^+)^+ \\ &= (n^+ + m)^+ && (\langle 2\rangle 2) \\ &= n^+ + m^+ \end{aligned}$$

□

Proposition 7.47. *For all $m, n \in \omega$, we have*

$$mn = nm \ .$$

PROOF:

⟨1⟩1. LET: $P(m)$ be the statement $\forall n \in \omega . mn = nm$

⟨1⟩2. $P(0)$

⟨2⟩1. LET: $Q(n)$ be the statement $0n = n0$

⟨2⟩2. $Q(0)$

PROOF: Trivial.

⟨2⟩3. $\forall n \in \omega . Q(n) \Rightarrow Q(n^+)$

⟨3⟩1. LET: $n \in \omega$

⟨3⟩2. ASSUME: $Q(n)$

⟨3⟩3. $Q(n^+)$

Proof:

$$\begin{aligned} 0n^+ &= 0n + 0 \\ &= 0n \\ &= n0 && (\langle 3\rangle 2) \\ &= 0 \\ &= n^+0 \end{aligned}$$

⟨1⟩3. $\forall m \in \omega . P(m) \Rightarrow P(m^+)$

⟨2⟩1. Let: $m \in \omega$

⟨2⟩2. Assume: $P(m)$

⟨2⟩3. Let: $Q(n)$ be the statement $m^+n = nm^+$

⟨2⟩4. $Q(0)$

Proof: ⟨1⟩2

⟨2⟩5. $\forall n \in \omega . Q(n) \Rightarrow Q(n^+)$

⟨3⟩1. Let: $n \in \omega$

⟨3⟩2. Assume: $Q(n)$

⟨3⟩3. $Q(n^+)$

Proof:

$$\begin{aligned} m^+n^+ &= m^+n + m^+ \\ &= (m^+n + m)^+ \\ &= (nm^+ + m)^+ && (\langle 3\rangle 2) \\ &= (nm + n + m)^+ \\ &= (mn + m + n)^+ && (\langle 2\rangle 2, \text{Proposition 7.46}) \\ &= (mn^+ + n)^+ \\ &= (n^+m + n)^+ && (\langle 2\rangle 2) \\ &= n^+m + n^+ \\ &= n^+m^+ \end{aligned}$$

□

Chapter 8

Countable Sets

Definition 8.1 (Countable)**.** A set A is *countable* or *denumerable* iff there exists an injective function $A \to \omega$.

Definition 8.2 (Countably Infinite)**.** A set is *countably infinite* iff it is similar to ω.

Proposition 8.3. *Every subset of a countable set is countable.*

PROOF: Easy. □

Proposition 8.4. *Let X be a set. If there exists a function from ω onto X, then X is countable.*

PROOF:
$\langle 1 \rangle 1$. LET: f be a function from ω onto X.
$\langle 1 \rangle 2$. Choose a function $g : X \to \omega$ such that, for all $x \in X$, we have $f(g(x)) = x$.
$\langle 1 \rangle 3$. g is one-to-one.
□

Proposition 8.5. *$\omega \times \omega$ is countable.*

PROOF: The sequence

$$(0,0),(0,1),(1,0),(0,2),(1,1),(2,0),\ldots$$

is an enumeration of $\omega \times \omega$. □

Corollary 8.5.1. *A countable union of countable sets is countable.*

PROOF:
$\langle 1 \rangle 1$. LET: A be a countable set of countable sets.
$\langle 1 \rangle 2$. PICK a surjection $f : \omega \to A$
$\langle 1 \rangle 3$. For $n \in \omega$, PICK a surjection $g_n : \omega \to f(n)$
$\langle 1 \rangle 4$. PICK a surjection $h : \omega \to \omega \times \omega$
$\langle 1 \rangle 5$. $\lambda n \in \omega . g_{\pi_1(h(n))}(\pi_2(h(n)))$ is a surjection $\omega \to \bigcup A$
□

Corollary 8.5.2. *The Cartesian product of two countable sets is countable.*

Corollary 8.5.3. *For any countable set A, the set of all finite subsets of A is countable.*

PROOF: Prove by induction on n that the set of all subsets of size n is countable. The set of all finite subsets is then the union of these. $\square$

Proposition 8.6. *$\mathcal{P}\omega$ is uncountable.*

PROOF: Cantor's Theorem. $\square$

Chapter 9

Cardinal Numbers

Definition 9.1 (Cardinal Number)**.** A *cardinal number* or *initial ordinal* is an ordinal α such that, for all $\beta < \alpha$, we have $\beta \not\sim \alpha$.

Definition 9.2 (Cardinality)**.** For any set X, the *cardinality* of X, $\mathrm{card}\, X$, is the least ordinal that is equivalent to X.

Proposition 9.3. *Given sets X and Y, we have $X \sim Y$ if and only if* $\mathrm{card}\, X = \mathrm{card}\, Y$.

Proof: Easy. □

Proposition 9.4. *For sets X and Y, we have* $\mathrm{card}\, X \leqslant \mathrm{card}\, Y$ *if and only if there exists an injective function $X \to Y$.*

Proof: Easy. □

Proposition 9.5. *Every natural number is a cardinal. ω is a cardinal.*

Proof: Easy. □

Proposition 9.6. *Every inifinite cardinal is a limit ordinal.*

Proof: For α infinite we have $f : \alpha^+ \sim \alpha$ where $f(\alpha) = 0$ and $f(\beta) = \beta^+$ for all other β. □

9.1 Cardinal Arithmetic

Definition 9.7 (Addition)**.** Given a family of cardinal numbers $\{\kappa_i\}_{i \in I}$, let $\sum_{i \in I} \kappa_i$ be $\mathrm{card} \bigcup_{i \in I} A_i$, where $\{A_i\}_{i \in I}$ is a pairwise disjoint family of sets with $\mathrm{card}\, A_i = \kappa_i$ for all i.

We write $\kappa + \lambda$ for $\sum_{i \in 2} \kappa_i$ where $\kappa_0 = \kappa$ and $\kappa_1 = \lambda$.

Proposition 9.8.

$$\kappa + \lambda = \lambda + \kappa$$
$$\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$$

PROOF: Easy. □

Proposition 9.9. *Cardinal addition agrees with ordinal addition on the natural numbers.*

PROOF: Easy induction. □

Proposition 9.10. *If $\kappa \leqslant \kappa'$ then $\kappa + \lambda \leqslant \kappa' + \lambda$.*

PROOF: Easy. □

Proposition 9.11. *If κ is an infinite cardinal number then $\kappa + \kappa = \kappa$.*

PROOF:
⟨1⟩1. LET: A be an infinite set.
PROVE: $A \times 2 \sim A$
⟨1⟩2. LET: $\mathcal{F}$ be the set of all functions f such that there exists $X \subseteq A$ such that $f : X \times 2 \sim X$.
⟨1⟩3. $\mathcal{F}$ is non-empty.
PROOF: Pick a subset $X \subseteq A$ such that $X \sim \omega$, and a bijection $X \times 2 \sim X$.
⟨1⟩4. $\mathcal{F}$ is partially ordered by extension.
⟨1⟩5. Every chain in $\mathcal{F}$ has an upper bound.
PROOF: If $\mathcal{C} \subseteq \mathcal{F}$ is a chain then $\bigcup \mathcal{C} \in \mathcal{F}$.
⟨1⟩6. PICK $f \in \mathcal{F}$ maximal.
⟨1⟩7. PICK $X \subseteq A$ such that $f : X \times 2 \sim X$
⟨1⟩8. $X - A$ is finite.
⟨2⟩1. ASSUME: for a contradiction $X - A$ is infinite.
⟨2⟩2. PICK $Y \subseteq X - A$ such that $Y \sim \omega$.
⟨2⟩3. PICK $g : Y \times 2 \sim Y$
⟨2⟩4. $f \cup g : (X \cup Y) \times 2 \sim X \cup Y$
⟨2⟩5. Q.E.D.
PROOF: This contradicts the maximality of f.
⟨1⟩9. $\operatorname{card} A + \operatorname{card} A = \operatorname{card} A$
PROOF:

$$\begin{aligned} 2 \operatorname{card} A &= 2(\operatorname{card} X + \operatorname{card}(A - X)) \\ &= 2 \operatorname{card} X + 2 \operatorname{card}(A - X) \\ &= \operatorname{card} X + 2 \operatorname{card}(A - X) && (\langle 1 \rangle 7) \\ &= \operatorname{card} X && (\langle 1 \rangle 8) \\ &= \operatorname{card} X + \operatorname{card}(A - X) && (\langle 1 \rangle 8) \\ &= \operatorname{card} A \end{aligned}$$

□

Corollary 9.11.1. *For any cardinals κ and λ that are not both finite, we have*

$$\kappa + \lambda = \max(\kappa, \lambda) \ .$$

Definition 9.12 (Multiplication)**.** Given a family of cardinal numbers $\{\kappa_i\}_{i \in I}$, let $\prod_{i \in I} \kappa_i = \operatorname{card} \bigtimes_{i \in I} \kappa_i$.

We write $\kappa\lambda$ for $\prod_{i \in 2} \kappa_i$ where $\kappa_0 = \kappa$ and $\kappa_1 = \lambda$.

Proposition 9.13.

$$\kappa\lambda = \lambda\kappa$$
$$\kappa(\lambda\mu) = (\kappa\lambda)\mu$$
$$\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$$

Proposition 9.14. *Cardinal multiplication agrees with ordinal multiplication on the natural numbers.*

PROOF: Easy induction. □

Proposition 9.15. *If $\kappa \leqslant \kappa'$ then $\kappa\lambda \leqslant \kappa'\lambda$.*

PROOF: Easy. □

Proposition 9.16. *Let $\{\kappa_i\}_{i\in I}$ and $\{\lambda_i\}_{i\in I}$ be families of cardinal numbers with the same index set. If $\kappa_i < \lambda_i$ for all i, then $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$.*

PROOF:
⟨1⟩1. Choose a one-to-one function $f_i : \kappa_i \to \lambda_i$ for each $i \in I$
⟨1⟩2. $\sum_{i\in I} \kappa_i \leqslant \prod_{i\in I} \lambda_i$
PROOF: Define $g : \sum_{i\in I} \kappa_i \to \prod_{i\in I} \lambda_i$ by

$$g(i,\eta)(j) = \begin{cases} f_i(\eta) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

⟨1⟩3. There is no surjective function $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$
⟨2⟩1. LET: $h : \sum_i \kappa_i \to \prod_i \lambda_i$
⟨2⟩2. Choose $t(i) < \lambda_i$ for each $i \in I$ such that, for all $\eta < \kappa_i$, we have $t(i) \neq h(i,\eta)(i)$.
PROOF: Since the function that maps η to $h(i,\eta)(i)$ cannot be surjective $\kappa_i \to \lambda_i$.
⟨2⟩3. For all $i \in I$ and $\eta < \kappa_i$ we have $h \neq t(i,\eta)$.
□

Proposition 9.17. *If κ is an infinite cardinal then $\kappa\kappa = \kappa$.*

PROOF:
⟨1⟩1. LET: A be an infinite set.
⟨1⟩2. LET: $\mathcal{F}$ be the set of all functions f such that there exists $X \subseteq A$ such that $f : X \times X \sim X$
⟨1⟩3. $\mathcal{F}$ is nonempty.
PROOF: Pick a countably infinite $X \subseteq A$. Then $X \times X \sim X$.
⟨1⟩4. $\mathcal{F}$ is partially ordered by extension.
⟨1⟩5. Every chain in $\mathcal{F}$ has an upper bound.
⟨1⟩6. PICK $f \in \mathcal{F}$ maximal.
⟨1⟩7. PICK $X \subseteq A$ such that $f : X \times X \sim X$.
⟨1⟩8. $\operatorname{card} X = \operatorname{card} A$
⟨2⟩1. ASSUME: for a contradiction $\operatorname{card} X < \operatorname{card} A$
⟨2⟩2. $\operatorname{card} A = \operatorname{card}(A - X)$

PROOF: Corollary 9.11.1.

⟨2⟩3. $\operatorname{card} X < \operatorname{card}(A - X)$

⟨2⟩4. PICK $Y \subseteq A - X$ such that $Y \sim X$

⟨2⟩5. PICK $g : (X \times Y) \cup (Y \times X) \cup (Y \times Y) \sim Y$

PROOF:

$$\begin{aligned}
(X \times Y) \cup (Y \times X) \cup (Y \times Y) &\sim 3 \times X \times X && (\langle 2\rangle 4) \\
&\sim 3 \times X && (\langle 1\rangle 7) \\
&\sim X && (\text{Corollary 9.11.1}) \\
&\sim Y && (\langle 2\rangle 4)
\end{aligned}$$

⟨2⟩6. $f \cup g : (X \cup Y) \times (X \cup Y) \sim X \cup Y$

⟨2⟩7. Q.E.D.

PROOF: This contradicts the maximality of f.

□

Corollary 9.17.1. *If κ and λ are non-zero cardinals that are not both finite, then*

$$\kappa\lambda = \max(\kappa, \lambda) \ .$$

Definition 9.18 (Exponentiation)**.** Given cardinal numbers κ and λ, let κ^λ be the cardinality of the set of all functions $\lambda \to \kappa$.

Proposition 9.19.

$$\begin{aligned}
\kappa^{\lambda+\mu} &= \kappa^\lambda \kappa^\mu \\
(\kappa\lambda)^\mu &= \kappa^\mu \lambda^\mu \\
\kappa^{\lambda\mu} &= (\kappa^\lambda)^\mu
\end{aligned}$$

PROOF: Easy. □

Proposition 9.20. *Cardinal exponentiation and ordinal exponentiation agree on the natural numbers.*

PROOF: Easy. □

Proposition 9.21.

$$\operatorname{card} \mathcal{P}X = 2^{\operatorname{card} X}$$

PROOF: Define $\chi : \mathcal{P}X \sim 2^X$ to be the function that maps S to the function $\chi_S : X \to 2$ where $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \notin S$. □

Proposition 9.22. *For any infinite cardinal κ we have $\kappa < 2^\kappa$.*

PROOF: Proposition 9.16. □

Proposition 9.23. *If $\kappa \leqslant \lambda$ then $\kappa^\mu \leqslant \lambda^\mu$.*

PROOF: Easy. □

9.2 Alephs

Definition 9.24 (Aleph)**.** Define the cardinal $\aleph_\alpha$ for every ordinal α as follows: $\aleph_\alpha$ is the least infinite cardinal greater than $\aleph_\beta$ for all $\beta < \alpha$.

Proposition 9.25.

$$\aleph_0 = \omega$$

PROOF: Easy. □

Definition 9.26 (Continuum Hypothesis)**.** The *continuum hypothesis* is the statement $\aleph_1 = 2^{\aleph_0}$.

Definition 9.27 (Generalized Continuum Hypothesis)**.** The *generalized continuum hypothesis* is the statement: for every ordinal α we have $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$.