

# Summary of Halmos' Naive Set Theory

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# Chapter 1

## Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*,  $\in$ . If  $x \in A$  we say that  $x$  *belongs to*  $A$ ,  $x$  is an *element* of  $A$ , or  $x$  is *contained in*  $A$ . If this does not hold we write  $x \notin A$ .

**Axiom 1.1** (Axiom of Extensionality). *Two sets are equal if and only if they have the same elements.*

**Axiom 1.2** (Axiom of Comprehension, Aussonderungsaxiom). *To every set  $A$  and to every condition  $S(x)$  there corresponds a set  $B$  whose elements are exactly those elements  $x$  of  $A$  for which  $S(x)$  holds.*

**Axiom 1.3.** *A set exists.*

**Axiom 1.4** (Axiom of Pairing). *For any two sets, there exists a set that they both belong to.*

**Axiom 1.5** (Union Axiom). *For every set  $A$ , there exists a set that contains all the elements that belong to at least one element of  $A$ .*

**Definition 1.6** (Subset). Let  $A$  and  $B$  be sets. We say that  $A$  is a *subset* of  $B$ , or  $B$  *includes*  $A$ , and write  $A \subseteq B$  or  $B \supseteq A$ , iff every element of  $A$  is an element of  $B$ .

**Axiom 1.7** (Power Set Axiom). *For any set  $A$ , there exists a set that contains all the subsets of  $A$ .*

## Chapter 2

# The Subset Relation

**Theorem 2.1.** *For any set  $A$ , we have  $A \subseteq A$ .*

PROOF: Every element of  $A$  is an element of  $A$ .  $\square$

**Theorem 2.2.** *For any sets  $A$ ,  $B$  and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ .*

PROOF: If every element of  $A$  is an element of  $B$ , and every element of  $B$  is an element of  $C$ , then every element of  $A$  is an element of  $C$ .  $\square$

**Theorem 2.3.** *For any sets  $A$  and  $B$ , if  $A \subseteq B$  and  $B \subseteq A$  then  $A = B$ .*

PROOF: If every element of  $A$  is an element of  $B$ , and every element of  $B$  is an element of  $A$ , then  $A$  and  $B$  have the same elements, and therefore are equal by the Axiom of Extensionality.  $\square$

**Definition 2.4** (Proper Subset). Let  $A$  and  $B$  be sets. We say that  $A$  is a *proper* subset of  $B$ , or  $B$  *properly* includes  $A$ , and write  $A \subsetneq B$  or  $B \supsetneq A$ , iff  $A \subseteq B$  and  $A \neq B$ .

## Chapter 3

# Comprehension Notation

**Definition 3.1.** Given a set  $A$  and a condition  $S(x)$ , we write  $\{x \in A : S(x)\}$  for the set whose elements are exactly those elements  $x$  of  $A$  for which  $S(x)$  holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality.  $\square$

**Theorem 3.2.** *There is no set that contains every set.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $A$  be a set.

PROVE: There exists a set  $B$  such that  $B \notin A$ .

$\langle 1 \rangle 2.$  LET:  $B = \{x \in A : x \notin x\}$

$\langle 1 \rangle 3.$  If  $B \in A$  then we have  $B \in B$  if and only if  $B \notin B$ .

$\langle 1 \rangle 4.$   $B \notin A$

$\square$

## Chapter 4

# Unordered Pairs

**Theorem 4.1.** *There exists a set with no elements.*

PROOF: Pick a set  $A$  by Axiom 1.3. Then the set  $\{x \in A : x \neq x\}$  has no elements.  $\square$

**Definition 4.2** (Empty Set). The *empty set*  $\emptyset$  is the set with no elements.

**Theorem 4.3.** *For any set  $A$  we have  $\emptyset \subset A$ .*

PROOF: Vacuous.  $\square$

**Definition 4.4** ((Unordered) Pair). For any sets  $a$  and  $b$ , the *(unordered) pair*  $\{a, b\}$  is the set whose elements are just  $a$  and  $b$ .

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality.  $\square$

**Definition 4.5** (Singleton). For any set  $a$ , the *singleton*  $\{a\}$  is defined to be  $\{a, a\}$ .

## Chapter 5

# Unions and Intersections

**Definition 5.1** (Union). For any set  $\mathcal{C}$ , the *union* of  $\mathcal{C}$ ,  $\bigcup \mathcal{C}$ , is the set whose elements are the elements of the elements of  $\mathcal{C}$ .

We write  $\bigcup_{X \in \mathcal{A}} t[X]$  for  $\bigcup \{t[X] \mid X \in \mathcal{A}\}$ .

PROOF: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality.  $\square$

**Proposition 5.2.**

$$\bigcup \emptyset = \emptyset$$

PROOF: There is no set that is an element of an element of  $\emptyset$ .  $\square$

**Proposition 5.3.** For any set  $A$ , we have  $\bigcup \{A\} = A$ .

PROOF: For any  $x$ , we have  $x$  is an element of an element of  $\{A\}$  if and only if  $x$  is an element of  $A$ .  $\square$

**Definition 5.4.** We write  $A \cup B$  for  $\bigcup \{A, B\}$ .

**Proposition 5.5.** For any set  $A$ , we have  $A \cup \emptyset = A$ .

PROOF:  $x \in A \cup \emptyset$  iff  $x \in A$  or  $x \in \emptyset$ , iff  $x \in A$ .  $\square$

**Proposition 5.6** (Commutativity). For any sets  $A$  and  $B$ , we have  $A \cup B = B \cup A$ .

PROOF:  $x \in A \cup B$  iff  $x \in A$  or  $x \in B$ , iff  $x \in B$  or  $x \in A$ , iff  $x \in B \cup A$ .  $\square$

**Proposition 5.7** (Associativity). For any sets  $A$ ,  $B$  and  $C$ , we have  $A \cup (B \cup C) = (A \cup B) \cup C$ .

PROOF: Each is the set of all  $x$  such that  $x \in A$  or  $x \in B$  or  $x \in C$ .  $\square$

**Proposition 5.8** (Idempotence). For any set  $A$ , we have  $A \cup A = A$ .

PROOF:  $x \in A$  or  $x \in A$  is equivalent to  $x \in A$ .  $\square$

**Proposition 5.9.** *For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A \cup B = B$ .*

PROOF: For any  $x$ , the statement "if  $x \in A$  then  $x \in B$ " is equivalent to " $x \in A$  or  $x \in B$  if and only if  $x \in B$ ".  $\square$

**Proposition 5.10.** *For any sets  $a$  and  $b$ , we have  $\{a\} \cup \{b\} = \{a, b\}$ .*

PROOF: Immediate from definitions.  $\square$

**Definition 5.11** ((Unordered) Triple). Given sets  $a_1, \dots, a_n$ , define the (un-ordered)  $n$ -tuple  $\{a_1, \dots, a_n\}$  to be

$$\{a_1, \dots, a_n\} := \{a_1\} \cup \dots \cup \{a_n\} .$$

**Definition 5.12** (Intersection). For any sets  $A$  and  $B$ , the *intersection*  $A \cap B$  is defined to be  $\{x \in A : x \in B\}$ .

**Proposition 5.13.** *For any set  $A$ , we have  $A \cap \emptyset = \emptyset$ .*

PROOF: There is no  $x$  such that  $x \in A$  and  $x \in \emptyset$ .  $\square$

**Proposition 5.14.** *For any sets  $A$  and  $B$ , we have*

$$A \cap B = B \cap A .$$

PROOF:  $x \in A$  and  $x \in B$  if and only if  $x \in B$  and  $x \in A$ .  $\square$

**Proposition 5.15.** *For any sets  $A$ ,  $B$  and  $C$ , we have*

$$A \cap (B \cap C) = (A \cap B) \cap C .$$

PROOF: Each is the set of all  $x$  such that  $x \in A$  and  $x \in B$  and  $x \in C$ .  $\square$

**Proposition 5.16.** *For any set  $A$ , we have*

$$A \cap A = A .$$

PROOF: We have  $x \in A$  and  $x \in A$  if and only if  $x \in A$ .  $\square$

**Proposition 5.17.** *For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A \cap B = A$ .*

PROOF: For any  $x$ , the statement "if  $x \in A$  then  $x \in B$ " is equivalent to " $x \in A$  and  $x \in B$  if and only if  $x \in A$ ".  $\square$

**Definition 5.18** (Disjoint). Two sets  $A$  and  $B$  are *disjoint* if and only if  $A \cap B = \emptyset$ .

**Definition 5.19** (Pairwise Disjoint). Let  $A$  be a set. We say the elements of  $A$  are *pairwise disjoint* if and only if, for all  $x, y \in A$ , if  $x \cap y \neq \emptyset$  then  $x = y$ .



**Proposition 5.20** (Distributive Law). *For any sets  $A$ ,  $B$  and  $C$ , we have*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

PROOF:

$$x \in A \cap (B \cup C) \Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \quad \square$$

**Proposition 5.21** (Distributive Law). *For any sets  $A$ ,  $B$  and  $C$ , we have*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

PROOF:

$$x \in A \cup (B \cap C) \Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \quad \square$$

**Proposition 5.22.** *For any sets  $A$ ,  $B$  and  $C$ , we have  $C \subseteq A$  if and only if  $(A \cap B) \cup C = A \cap (B \cup C)$ .*

PROOF: The statement "if  $x \in C$  then  $x \in A$ " is equivalent to the statement " $((x \in A \wedge x \in B) \vee x \in C) \Leftrightarrow (x \in A \wedge (x \in B \vee x \in C))$ ".  $\square$

**Definition 5.23** (Intersection). For any nonempty set  $\mathcal{C}$ , the *intersection* of  $\mathcal{C}$ ,  $\bigcap \mathcal{C}$ , is the set that contains exactly those sets that belong to every element of  $\mathcal{C}$ .

We write  $\bigcap_{X \in \mathcal{A}} t[X]$  for  $\bigcap \{t[X] \mid X \in \mathcal{A}\}$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C}$  be a nonempty set.

$\langle 1 \rangle 2$ . There exists a set  $I$  whose elements are exactly the sets that belong to every element of  $\mathcal{C}$ .

PROOF: Pick  $A \in \mathcal{C}$ , and take  $I = \{x \in A : \forall X \in \mathcal{C}. x \in X\}$ .

$\langle 1 \rangle 3$ . For any sets  $I$ ,  $J$ , if the elements of  $I$  and  $J$  are exactly the sets that belong to every element of  $\mathcal{C}$  then  $I = J$ .

PROOF: Axiom of Extensionality.

$\square$

## Chapter 6

# Complements and Powers

**Definition 6.1** (Relative Complement). For any sets  $A$  and  $B$ , the *difference* or *relative complement*  $A - B$  is defined to be

$$A - B := \{x \in A : x \notin B\} .$$

**Proposition 6.2.** For any sets  $A$  and  $E$ , we have  $A \subseteq E$  if and only if

$$E - (E - A) = A$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $E$  be sets.

$\langle 1 \rangle 2$ . If  $A \subseteq E$  then  $E - (E - A) = A$

$\langle 2 \rangle 1$ . ASSUME:  $A \subseteq E$

$\langle 2 \rangle 2$ .  $E - (E - A) \subseteq A$

PROOF: If  $x \in E$  and  $x \notin E - A$  then  $x \in A$ .

$\langle 2 \rangle 3$ .  $A \subseteq E - (E - A)$

PROOF: If  $x \in A$  then  $x \in E$  and  $x \notin E - A$ .

$\langle 1 \rangle 3$ . If  $E - (E - A) = A$  then  $A \subseteq E$ .

PROOF: Since  $E - (E - A) \subseteq E$ .

□

**Proposition 6.3.** For any set  $E$  we have

$$E - \emptyset = E$$

PROOF:  $x \in E$  if and only if  $x \in E$  and  $x \notin \emptyset$ . □

**Proposition 6.4.** For any set  $E$  we have

$$E - E = \emptyset .$$

PROOF: There is no  $x$  such that  $x \in E$  and  $x \notin E$ . □

**Proposition 6.5.** For any sets  $A$  and  $E$ , we have

$$A \cap (E - A) = \emptyset .$$

PROOF: There is no  $x$  such that  $x \in A$  and  $x \in E - A$ .  $\square$

**Proposition 6.6.** *Let  $A$  and  $E$  be sets. Then  $A \subseteq E$  if and only if*

$$A \cup (E - A) = E .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $E$  be sets.

$\langle 1 \rangle 2$ . If  $A \subseteq E$  then  $A \cup (E - A) = E$ .

$\langle 2 \rangle 1$ . ASSUME:  $A \subseteq E$

$\langle 2 \rangle 2$ .  $A \cup (E - A) \subseteq E$

PROOF: If  $x \in A$  or  $x \in E - A$  then  $x \in E$ .

$\langle 2 \rangle 3$ .  $E \subseteq A \cup (E - A)$

PROOF: If  $x \in E$  then either  $x \in A$  or  $x \notin A$ . In the latter case,  $x \in E - A$ .

$\langle 1 \rangle 3$ . If  $A \cup (E - A) = E$  then  $A \subseteq E$

PROOF: Since  $A \subseteq A \cup (E - A)$ .

$\square$

**Proposition 6.7.** *Let  $A$ ,  $B$  and  $E$  be sets. Then:*

1. *If  $A \subseteq B$  then  $E - B \subseteq E - A$ .*

2. *If  $A \subseteq E$  and  $E - B \subseteq E - A$  then  $A \subseteq B$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$ ,  $B$  and  $E$  be sets.

$\langle 1 \rangle 2$ . If  $A \subseteq B$  then  $E - B \subseteq E - A$ .

PROOF: If  $A \subseteq B$ ,  $x \in E$  and  $x \notin B$ , then we have  $x \in E$  and  $x \notin A$ .

$\langle 1 \rangle 3$ . If  $A \subseteq E$  and  $E - B \subseteq E - A$  then  $A \subseteq B$ .

$\langle 2 \rangle 1$ . ASSUME:  $A \subseteq E$

$\langle 2 \rangle 2$ . ASSUME:  $E - B \subseteq E - A$

$\langle 2 \rangle 3$ . LET:  $x \in A$

$\langle 2 \rangle 4$ .  $x \in E$

$\langle 2 \rangle 5$ .  $x \notin E - A$

$\langle 2 \rangle 6$ .  $x \notin E - B$

$\langle 2 \rangle 7$ .  $x \in B$

$\square$

**Example 6.8.** We cannot remove the hypothesis  $A \subseteq E$  in item 2 above. Let  $E = \emptyset$ ,  $A = \{\emptyset\}$  and  $B = \emptyset$ . Then  $E - B = E - A = \emptyset$  but  $A \not\subseteq B$ .

**Proposition 6.9** (De Morgan's Law). *For any sets  $A$ ,  $B$  and  $E$ , we have  $E - (A \cup B) = (E - A) \cap (E - B)$ .*

PROOF:  $(x \in E \wedge \neg(x \in A \vee x \in B)) \Leftrightarrow (x \in E \wedge x \notin A \wedge x \in E \wedge x \notin B)$ .  $\square$

**Proposition 6.10** (De Morgan's Law). *For any sets  $A$ ,  $B$  and  $E$ , we have  $E - (A \cap B) = (E - A) \cup (E - B)$ .*

PROOF:  $(x \in E \vee \neg(x \in A \wedge x \in B)) \Leftrightarrow (x \in E \wedge x \notin A) \vee (x \in E \wedge x \notin B)$ .  $\square$

**Proposition 6.11.** *For any sets  $A$ ,  $B$  and  $E$ , if  $A \subseteq E$  then*

$$A - B = A \cap (E - B) .$$

PROOF: If  $A \subseteq E$  then we have  $(x \in A \wedge x \notin B) \Leftrightarrow (x \in A \wedge x \in E \wedge x \notin B)$ .  $\square$

**Proposition 6.12.** *For any sets  $A$  and  $B$ , we have  $A \subseteq B$  if and only if  $A - B = \emptyset$ .*

PROOF: Both are equivalent to the statement that there is no  $x$  such that  $x \in A$  and  $x \notin B$ .  $\square$

**Proposition 6.13.** *For any sets  $A$  and  $B$ , we have*

$$A - (A - B) = A \cap B .$$

PROOF:  $(x \in A \wedge \neg(x \in A \wedge x \notin B)) \Leftrightarrow x \in A \wedge x \in B$ .  $\square$

**Proposition 6.14.** *For any sets  $A$ ,  $B$  and  $C$ , we have*

$$A \cap (B - C) = (A \cap B) - (A \cap C) .$$

PROOF:  $(x \in A \wedge x \in B \wedge x \notin C) \Leftrightarrow (x \in A \wedge x \in B \wedge \neg(x \in A \wedge x \in C))$ .  $\square$

**Proposition 6.15.** *For any sets  $A$ ,  $B$ ,  $C$  and  $E$ , if  $(A \cap B) - C \subseteq E$  then we have*

$$A \cap B \subseteq (A \cap C) \cup (B \cap (E - C)) .$$

PROOF:

(1)1. LET:  $x \in A \cap B$

PROVE:  $x \in (A \cap C) \cup (B \cap (E - C))$

(1)2. CASE:  $x \in C$

PROOF: Then  $x \in A \cap C$ .

(1)3. CASE:  $x \notin C$

PROOF: Then  $x \in E$  and so  $x \in B \cap (E - C)$ .

$\square$

**Proposition 6.16.** *For any sets  $A$ ,  $B$ ,  $C$  and  $E$ , we have*

$$(A \cup C) \cap (B \cup (E - C)) \subseteq A \cup B .$$

PROOF: The statement  $(x \in A \vee x \in C) \wedge (x \in B \vee (x \in E \wedge x \notin C))$  implies  $x \in A \vee x \in B$ .  $\square$

**Definition 6.17** (Symmetric Difference). For any sets  $A$  and  $B$ , the *symmetric difference*  $A + B$  is defined to be

$$A + B := (A - B) \cup (B - A) .$$

**Proposition 6.18.** *For any sets  $A$  and  $B$ , we have*

$$A + B = B + A$$

PROOF: From the commutativity of union.  $\square$

**Proposition 6.19.** *For any sets  $A$ ,  $B$  and  $C$ , we have*

$$A + (B + C) = (A + B) + C .$$

PROOF: Each is the set of all  $x$  that belong to either exactly one or all three of  $A$ ,  $B$  and  $C$ .  $\square$

**Proposition 6.20.** *For any set  $A$ , we have*

$$A + \emptyset = A .$$

PROOF:

$$\begin{aligned} A + \emptyset &= (A - \emptyset) \cup (\emptyset - A) \\ &= A \cup \emptyset \\ &= A \end{aligned} \quad \square$$

**Proposition 6.21.** *For any set  $A$  we have*

$$A + A = \emptyset .$$

PROOF:

$$\begin{aligned} A + A &= (A - A) \cup (A - A) \\ &= \emptyset \cup \emptyset \\ &= \emptyset \end{aligned} \quad \square$$

**Definition 6.22** (Power Set). For any set  $A$ , the *power set* of  $A$ ,  $\mathcal{P}A$ , is the set whose elements are exactly the subsets of  $A$ .

PROOF: This exists by the Power Set Axiom and Axiom of Comprehension, and is unique by the Axiom of Extensionality.  $\square$

**Proposition 6.23.**

$$\mathcal{P}\emptyset = \{\emptyset\}$$

PROOF: The only subset of  $\emptyset$  is  $\emptyset$ .  $\square$

**Proposition 6.24.** *For any set  $a$ , we have*

$$\mathcal{P}\{a\} = \{\emptyset, \{a\}\} .$$

PROOF: The only subsets of  $\{a\}$  are  $\emptyset$  and  $\{a\}$ .  $\square$

**Proposition 6.25.** *For any sets  $a$  and  $b$ , we have*

$$\mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} .$$

PROOF: The only subsets of  $\{a, b\}$  are  $\emptyset$ ,  $\{a\}$ ,  $\{b\}$  and  $\{a, b\}$ .  $\square$