

Mathematics

Robin Adams

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Contents

I	Category Theory	5
1	Foundations	7
2	Categories	9
2.1	Preorders	10
2.2	Monomorphisms and Epimorphisms	10
2.3	Sections and Retractions	12
2.4	Isomorphisms	13
2.5	Initial and Terminal Objects	13
3	Functors	15
3.1	Comma Categories	15
II	Group Theory	17
4	Semigroups	19
5	Monoids	21
6	Groups	23
6.1	Order of an Element	26
6.2	Generators	29
7	Group Homomorphisms	31
7.1	Subgroups	33
7.2	Kernel	34
7.3	Inner Automorphisms	35
7.4	Direct Products	36
7.5	Free Groups	36
7.6	Normal Subgroups	39
7.7	Quotient Groups	40
7.8	Cosets	44
7.9	Congruence	48
7.10	Cyclic Groups	49

7.11	Commutator Subgroup	49
7.12	Presentations	49
7.13	Index of a Subgroup	50
7.14	Cokernels	51
7.15	Cayley Graphs	52
8	Abelian Groups	53
8.1	The Category of Abelian Groups	57
8.2	Free Abelian Groups	58
8.3	Cokernels	61
9	Group Actions	63
9.1	Group Actions	63
9.2	Category of G -Sets	66
III	Ring Theory	69
10	Rngs	71
10.1	Commutative Rngs	72
11	Rings	75
11.1	Units	76
11.2	Euler's ϕ -function	78
11.3	Nilpotent Elements	80
12	Monoid Rings	81
12.1	Polynomials	81
12.2	Power Series	82
13	Integral Domains	83
14	Unique Factorization Domains	85
15	Principal Ideal Domains	87
16	Euclidean Domains	89
17	Division Rings	91
IV	Field Theory	93
18	Fields	95
V	Linear Algebra	97

Part I

Category Theory

Chapter 1

Foundations

This is a placeholder — I am not sure what foundation I want to use for this project yet. I will try to work in a way which is foundation-independent. What I do could be formalized in ZFC, ETCS, or some other system. I will assume the usual set theoretic constructions as needed. Sets will be defined up to bijection only.

Chapter 2

Categories

Definition 2.1 (Category). A *category* \mathcal{C} consists of:

- A class $|\mathcal{C}|$ of *objects*. We write $A \in \mathcal{C}$ for $A \in |\mathcal{C}|$.
- For any objects A, B , a set $\mathcal{C}[A, B]$ of *morphisms* from A to B . We write $f : A \rightarrow B$ for $f \in \mathcal{C}[A, B]$.
- For any object A , a morphism $\text{id}_A : A \rightarrow A$, the *identity* morphism on A .
- For any morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, a morphism $g \circ f : A \rightarrow C$, the *composite* of f and g .

such that:

Associativity Given $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Left Unit Law For any morphism $f : A \rightarrow B$, we have $\text{id}_B \circ f = f$.

Right Unit Law For any morphism $f : A \rightarrow B$, we have $f \circ \text{id}_A = f$.

Proposition 2.2. *The identity morphism on an object is unique.*

PROOF: If i and j are identity morphisms on A then $i = i \circ j = j$. \square

Example 2.3 (Category of Sets). The *category of sets* **Set** has objects all sets and morphisms all functions.

Definition 2.4 (Endomorphism). In a category \mathcal{C} , an *endomorphism* on an object A is a morphism $A \rightarrow A$. We write $\text{End}_{\mathcal{C}}(A)$ for $\mathcal{C}[A, A]$.

Definition 2.5 (Opposite Category). For any category \mathcal{C} , the *opposite* category \mathcal{C}^{op} is the category with the same objects as \mathcal{C} and

$$\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$$

2.1 Preorders

Definition 2.6 (Preorder). A *preorder* on a set A is a relation \leq on A that is reflexive and transitive.

A *preordered set* is a pair (A, \leq) such that \leq is a preorder on A . We usually write A for the preordered set (A, \leq) .

We identify any preordered set A with the category whose objects are the elements of A , with one morphism $a \rightarrow b$ iff $a \leq b$, and no morphism $a \rightarrow b$ otherwise.

Example 2.7. For any ordinal α , let α be the preorder $\{\beta : \beta < \alpha\}$ under \leq .

Definition 2.8 (Discrete Preorder). We identify any set A with the *discrete* preorder $(A, =)$.

2.2 Monomorphisms and Epimorphisms

Definition 2.9 (Monomorphism). In a category, let $f : A \rightarrow B$. Then f is a *monomorphism* or *monic* iff, for every object X and morphism $x, y : X \rightarrow A$, if $fx = fy$ then $x = y$.

Definition 2.10 (Epimorphism). In a category, let $f : A \rightarrow B$. Then f is a *epimorphism* or *epi* iff, for every object X and morphism $x, y : B \rightarrow X$, if $xf = yf$ then $x = y$.

Proposition 2.11. *The composite of two monomorphism is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ and $g : B \rightarrow C$ be monic.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$

$\langle 1 \rangle 3$. ASSUME: $g \circ f \circ x = g \circ f \circ y$

$\langle 1 \rangle 4$. $f \circ x = f \circ y$

$\langle 1 \rangle 5$. $x = y$

□

Proposition 2.12. *The composite of two epimorphisms is epi.*

PROOF: Dual. □

Proposition 2.13. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is monic then f is monic.*

PROOF: If $f \circ x = f \circ y$ then $gfx = gfy$ and so $x = y$. □

Proposition 2.14. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is epi then g is epi.*

PROOF: Dual. □

Proposition 2.15. *A function is a monomorphism in **Set** iff it is injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is monic then f is injective.

$\langle 2 \rangle 1$. ASSUME: f is monic.

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 4$. LET: $\bar{x}, \bar{y} : 1 \rightarrow A$ be the functions such that $\bar{x}(*) = x$ and $\bar{y}(*) = y$

$\langle 2 \rangle 5$. $f \circ \bar{x} = f \circ \bar{y}$

$\langle 2 \rangle 6$. $\bar{x} = \bar{y}$

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. $x = y$

$\langle 1 \rangle 3$. If f is injective then f is monic.

$\langle 2 \rangle 1$. ASSUME: f is injective.

$\langle 2 \rangle 2$. LET: X be a set and $x, y : X \rightarrow A$.

$\langle 2 \rangle 3$. ASSUME: $f \circ x = f \circ y$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $t \in X$

PROVE: $x(t) = y(t)$

$\langle 2 \rangle 5$. $f(x(t)) = f(y(t))$

$\langle 2 \rangle 6$. $x(t) = y(t)$

PROOF: By $\langle 2 \rangle 1$.

□

Proposition 2.16. *A function is an epimorphism in **Set** iff it is surjective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$

$\langle 1 \rangle 2$. If f is an epimorphism then f is surjective.

$\langle 2 \rangle 1$. ASSUME: f is an epimorphism.

$\langle 2 \rangle 2$. LET: $b \in B$

$\langle 2 \rangle 3$. LET: $x, y : B \rightarrow 2$ be defined by $x(b) = 1$ and $x(t) = 0$ for all other $t \in B$, $y(t) = 0$ for all $t \in B$.

$\langle 2 \rangle 4$. $x \neq y$

$\langle 2 \rangle 5$. $x \circ f \neq y \circ f$

$\langle 2 \rangle 6$. There exists $a \in A$ such that $f(a) = b$.

$\langle 1 \rangle 3$. If f is surjective then f is an epimorphism.

$\langle 2 \rangle 1$. ASSUME: f is surjective.

$\langle 2 \rangle 2$. LET: $x, y : B \rightarrow X$

$\langle 2 \rangle 3$. ASSUME: $x \circ f = y \circ f$

PROVE: $x = y$

$\langle 2 \rangle 4$. LET: $b \in B$

PROVE: $x(b) = y(b)$

$\langle 2 \rangle 5$. PICK $a \in A$ such that $f(a) = b$

$\langle 2 \rangle 6$. $x(f(a)) = y(f(a))$

$\langle 2 \rangle 7$. $x(b) = y(b)$

□

Proposition 2.17. *In a preorder, every morphism is monic and epi.*

PROOF: Immediate from definitions. \square

2.3 Sections and Retractions

Definition 2.18 (Section, Retraction). In a category, let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $r \circ s = \text{id}_B$.

Proposition 2.19. *Every identity morphism is a section and retraction of itself.*

PROOF: Immediate from definitions. \square

Proposition 2.20. *Let $r, r' : A \rightarrow B$ and $s : B \rightarrow A$. If r is a retraction of s and r' is a section of s then $r = r'$.*

PROOF:

$$\begin{aligned} r &= r \circ \text{id}_A \\ &= r \circ s \circ r' \\ &= \text{id}_B \circ r' \\ &= r' \end{aligned} \quad \square$$

Proposition 2.21. *Let $r_1 : A \rightarrow B$, $r_2 : B \rightarrow C$, $s_1 : B \rightarrow A$ and $s_2 : C \rightarrow B$. If r_1 is a retraction of s_1 and r_2 is a retraction of s_2 then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= \text{id}_C \end{aligned} \quad \square$$

Proposition 2.22. *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$. LET: $s : A \rightarrow B$ be a section of $r : B \rightarrow A$.

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$ satisfy $sx = sy$.

$\langle 1 \rangle 3$. $rsx = rsy$

$\langle 1 \rangle 4$. $x = y$

\square

Proposition 2.23. *Every retraction is epi.*

PROOF: Dual. \square

Proposition 2.24. *In Set, every epimorphism has a retraction.*

PROOF: By the Axiom of Choice. \square

Example 2.25. It is not true in general that every monomorphism in any category has a section. nor that every epimorphism in any category has a retraction.

In the category **2**, the morphism $0 \leq 1$ is monic and epi but has no retraction or section.

2.4 Isomorphisms

Definition 2.26 (Isomorphism). In a category \mathcal{C} , a morphism $f : A \rightarrow B$ is an *isomorphism*, denoted $f : A \cong B$, iff there exists a morphism $f^{-1} : B \rightarrow A$, the *inverse* of f , such that $f^{-1} \circ f = \text{id}_A$ and $f \circ f^{-1} = \text{id}_B$.

An *automorphism* on an object A is an isomorphism between A and itself. We write $\text{Aut}_{\mathcal{C}}(A)$ for the set of all automorphisms on A .

Objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Proposition 2.27. *The inverse of an isomorphism is unique.*

PROOF: Proposition 2.20. \square

Proposition 2.28. *For any object A we have $\text{id}_A : A \cong A$ and $\text{id}_A^{-1} = \text{id}_A$.*

PROOF: Since $\text{id}_A \circ \text{id}_A = \text{id}_A$ by the Unit Laws. \square

Proposition 2.29. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Immediate from definitions. \square

Proposition 2.30. *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF: From Proposition 2.21. \square

Definition 2.31 (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

2.5 Initial and Terminal Objects

Definition 2.32 (Initial Object). An object I in a category is *initial* iff, for any object X , there is exactly one morphism $I \rightarrow X$.

Example 2.33. The empty set is the initial object in **Set**.

Definition 2.34 (Terminal Object). An object T in a category is *terminal* iff, for any object X , there is exactly one morphism $X \rightarrow T$.

Example 2.35. Every singleton is terminal in **Set**.

Proposition 2.36. *If I and J are initial in a category, then there exists a unique isomorphism $I \cong J$.*

PROOF:

- $\langle 1 \rangle 1$. LET: i be the unique morphism $I \rightarrow J$.
- $\langle 1 \rangle 2$. LET: i^{-1} be the unique morphism $J \rightarrow I$.
- $\langle 1 \rangle 3$. $i \circ i^{-1} = \text{id}_J$

PROOF: Since there is only one morphism $J \rightarrow J$.

- $\langle 1 \rangle 4$. $i^{-1} \circ i = \text{id}_I$

PROOF: Since there is only one morphism $I \rightarrow I$.
 \square

Proposition 2.37. *If S and T are terminal in a category, then there exists a unique isomorphism $S \cong T$.*

PROOF: Dual. \square

Chapter 3

Functors

Definition 3.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for any morphism $f : A \rightarrow B : \mathcal{C}$, a morphism $Ff : FA \rightarrow FB : \mathcal{D}$

such that:

- $F\text{id}_A = \text{id}_{FA}$
- $F(g \circ f) = Fg \circ Ff$

Definition 3.2 (Identity Functor). For any category \mathcal{C} , the *identity functor* $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is defined by

$$\begin{aligned} 1_{\mathcal{C}}A &= A \\ 1_{\mathcal{C}}f &= f \end{aligned}$$

Definition 3.3 (Constant Functor). Given categories \mathcal{C} , \mathcal{D} and an object $D \in \mathcal{D}$, the *constant functor* $K^{\mathcal{C}}D : \mathcal{C} \rightarrow \mathcal{D}$ is the functor defined by

$$\begin{aligned} K^{\mathcal{C}}DC &= D \\ K^{\mathcal{C}}Df &= \text{id}_D \end{aligned}$$

3.1 Comma Categories

Definition 3.4 (Comma Category). Let $F : \mathcal{C} \rightarrow \mathcal{E}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors. The *comma category* $F \downarrow G$ is the category with:

- objects all pairs (C, D, f) where $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $f : FC \rightarrow GD : \mathcal{E}$

- morphisms $(u, v) : (C, D, f) \rightarrow (C', D', g)$ all pairs $u : C \rightarrow C' : \mathcal{C}$ and $v : D \rightarrow D' : \mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ \downarrow Fu & & \downarrow Gv \\ FC' & \xrightarrow{g} & GD' \end{array}$$

Definition 3.5 (Slice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *slice category* over A , denoted \mathcal{C}/A , is the comma category $1_{\mathcal{C}} \downarrow K^1 A$.

Definition 3.6 (Coslice Category). Let \mathcal{C} be a category and $A \in \mathcal{C}$. The *coslice category* over A , denoted $\mathcal{C} \backslash A$, is the comma category $K^1 A \downarrow 1_{\mathcal{C}}$.

Definition 3.7 (Pointed Sets). The *category of pointed sets* \mathbf{Set}_* is the coslice category $\mathbf{Set} \backslash 1$.

Part II

Group Theory

Chapter 4

Semigroups

Definition 4.1 (Semigroup). A *semigroup* consists of a set S and an associative binary operation \cdot on S .

Chapter 5

Monoids

Definition 5.1 (Monoid). A *monoid* consists of a semigroup M such that there exists $e \in M$, the *unit*, such that, for all $x \in M$, we have $xe = ex = x$.

We identify a monoid M with the category with one object whose morphisms are the elements of M , with composition given by \cdot .

Proposition 5.2. *The identity in a group is unique.*

PROOF: Proposition 2.2.

Chapter 6

Groups

Definition 6.1 (Group). Let \mathcal{C} be a category with finite products. A *group (object)* in \mathcal{C} consists of an object $G \in \mathcal{C}$ and morphisms

$$m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G$$

such that the following diagrams commute.

$$\begin{array}{ccc} G^3 & \xrightarrow{m \times \text{id}_G} & G^2 \\ \downarrow \text{id}_G \times m & & \downarrow m \\ G^2 & \xrightarrow{m} & G \end{array}$$

$$\begin{array}{ccc} 1 \times G & \xrightarrow{e \times \text{id}_G} & G^2 \\ & \searrow \cong & \downarrow m \\ & & G \end{array} \quad \begin{array}{ccc} G \times 1 & \xrightarrow{\text{id}_G \times e} & G^2 \\ & \searrow \cong & \downarrow m \\ & & G \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{\Delta} & G^2 & \xrightarrow{\text{id}_G \times i} & G^2 \\ \downarrow & & & & \downarrow m \\ 1 & \xrightarrow{e} & G & & G \end{array} \quad \begin{array}{ccccc} G & \xrightarrow{\Delta} & G^2 & \xrightarrow{i \times \text{id}_G} & G^2 \\ \downarrow & & & & \downarrow m \\ 1 & \xrightarrow{e} & G & & G \end{array}$$

Definition 6.2 (Group). We write just 'group' for 'group in **Set**'. Thus, a *group* G consists of a set G and a binary operation $\cdot : G^2 \rightarrow G$ such that \cdot is associative, and there exists $e \in G$, the *identity* element of the group, such that:

- For all $x \in G$ we have $xe = ex = x$
- For all $x \in G$, there exists $x^{-1} \in G$, the *inverse* of x , such that $xx^{-1} = x^{-1}x = e$.

The *order* of a group G , denoted $|G|$, is the number of elements in G if G is finite; otherwise we write $|G| = \infty$.

Proposition 6.3. *The inverse of an element is unique.*

PROOF: If i and j are inverses of x then $i = ixj = j$. \square

Example 6.4. • The *trivial* group is $\{e\}$ under $ee = e$.

- \mathbb{Z} is a group under addition
- \mathbb{Q} is a group under addition
- $\mathbb{Q} - \{0\}$ is a group under multiplication
- \mathbb{R} is a group under addition
- $\mathbb{R} - \{0\}$ is a group under multiplication
- \mathbb{C} is a group under addition
- $\mathbb{C} - \{0\}$ is a group under multiplication
- $\{-1, 1\}$ is a group under multiplication
- For any category \mathcal{C} and object $A \in \mathcal{C}$, we have $\text{Aut}_{\mathcal{C}}(A)$ is a group under $gf = f \circ g$.

For A a set, we call $S_A = \text{Aut}_{\text{Set}}(A)$ the *symmetric group* or *group of permutations* of A .

- For $n \geq 3$, the *dihedral group* D_{2n} consists of the set of rigid motions that map the regular n -gon onto itself under composition.
- Let $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ under matrix multiplication.
- The quaternionic group Q_8 is the group

$$\{1, -1, i, -i, j, -j, k, -k\}$$

with multiplication table

1	-1	i	-i	j	-j	k	-k
-1	1	-i	i	-j	j	-k	k
i	-i	-1	1	k	-k	-j	j
-i	i	1	-1	-k	k	j	-j
j	-j	-k	k	-1	1	i	-i
-j	j	k	-k	1	-1	-i	i
k	-k	j	-j	-i	i	-1	1
-k	k	-j	j	i	-i	1	-1

Example 6.5. • The only group of order 1 is the trivial group.

- The only group of order 2 is \mathbb{Z}_2 .

- The only group of order 3 is \mathbb{Z}_3 .
- There are exactly two groups of order 4: \mathbb{Z}_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ under $(a, b)(c, d) = (ac, bd)$.

Proposition 6.6 (Cancellation). *Let G be a group. Let $a, g, h \in G$. If $ag = ah$ or $ga = ha$ then $g = h$.*

PROOF: If $ag = ah$ then $g = a^{-1}ag = a^{-1}ah = h$. Similarly if $ga = ha$. \square

Proposition 6.7. *Let G be a group and $g, h \in G$. Then $(gh)^{-1} = h^{-1}g^{-1}$.*

PROOF: Since $ghh^{-1}g^{-1} = e$. \square

Definition 6.8. Let G be a group. Let $g \in G$. We define $g^n \in G$ for all $n \in \mathbb{Z}$ as follows:

$$\begin{aligned} g^0 &= e \\ g^{n+1} &= g^n g & (n \geq 0) \\ g^{-n} &= (g^{-1})^n & (n > 0) \end{aligned}$$

Proposition 6.9. *Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then*

$$g^{m+n} = g^m g^n .$$

PROOF:

$\langle 1 \rangle 1$. For all $k \in \mathbb{Z}$ we have $g^{k+1} = g^k g$

$\langle 2 \rangle 1$. For all $k \geq 0$ we have $g^{k+1} = g^k g$

PROOF: Immediate from definition.

$\langle 2 \rangle 2$. $g^{-1+1} = g^{-1} g$

PROOF: Both are equal to e .

$\langle 2 \rangle 3$. For all $k > 1$ we have $g^{-k+1} = g^{-k} g$

PROOF:

$$\begin{aligned} g^{-k+1} &= (g^{-1})^{k-1} \\ &= (g^{-1})^{k-1} g^{-1} g \\ &= (g^{-1})^k g \\ &= g^{-k} g \end{aligned}$$

$\langle 1 \rangle 2$. For all $k \in \mathbb{Z}$ we have $g^{k-1} = g^k g^{-1}$

PROOF: Substitute $k = k - 1$ above and multiply by g^{-1} .

$\langle 1 \rangle 3$. $g^{m+0} = g^m g^0$

PROOF: Since $g^m g^0 = g^m e = g^m$.

$\langle 1 \rangle 4$. If $g^{m+n} = g^m g^n$ then $g^{m+n+1} = g^m g^{n+1}$

PROOF:

$$\begin{aligned} g^{m+n+1} &= g^{m+n} g & (\langle 1 \rangle 1) \\ &= g^m g^n g \\ &= g^m g^{n+1} & (\langle 1 \rangle 1) \end{aligned}$$

$\langle 1 \rangle 5$. If $g^{m+n} = g^m g^n$ then $g^{m+n-1} = g^m g^{n-1}$

PROOF:

$$g^{m+n-1} g = g^{m+n} \quad (\langle 1 \rangle 1)$$

$$= g^m g^n$$

$$\therefore g^{m+n-1} = g^m g^n g^{-1}$$

$$= g^m g^{n-1} \quad (\langle 1 \rangle 2)$$

□

Proposition 6.10. Let G be a group. Let $g \in G$ and $m, n \in \mathbb{Z}$. Then

$$(g^m)^n = g^{mn} .$$

PROOF:

$\langle 1 \rangle 1$. $(g^m)^0 = g^0$

PROOF: Both sides are equal to e .

$\langle 1 \rangle 2$. If $(g^m)^n = g^{mn}$ then $(g^m)^{n+1} = g^{m(n+1)}$.

PROOF:

$$(g^m)^{n+1} = (g^m)^n g^m \quad (\text{Proposition 6.9})$$

$$= g^{mn} g^m$$

$$= g^{mn+m} \quad (\text{Proposition 6.9})$$

$\langle 1 \rangle 3$. If $(g^m)^n = g^{mn}$ then $(g^m)^{n-1} = g^{m(n-1)}$.

PROOF:

$$(g^m)^n = g^{mn}$$

$$\therefore (g^m)^{n-1} g^m = g^{mn-m} g^m \quad (\text{Proposition 6.9})$$

$$\therefore (g^m)^{n-1} = g^{mn-m} \quad (\text{Cancellation})$$

□

Definition 6.11 (Commute). Let G be a group and $g, h \in G$. We say g and h *commute* iff $gh = hg$.

Definition 6.12. Let G be a group. Given $g \in G$ and $A \subseteq G$, we define

$$gA = \{ga : a \in A\}, \quad Ag = \{ag : a \in A\} .$$

Given sets $A, B \subseteq G$, we define

$$AB = \{ab : a \in A, b \in B\} .$$

6.1 Order of an Element

Definition 6.13 (Order). Let G be a group. Let $g \in G$. Then g has *finite order* iff there exists a positive integer n such that $g^n = e$. In this case, the *order* of g , denoted $|g|$, is the least positive integer n such that $g^n = e$.

If g does not have finite order, we write $|g| = \infty$.

Proposition 6.14. *Let G be a group. Let $g \in G$ and n be a positive integer. If $g^n = e$ then $|g| \mid n$.*

PROOF:

$\langle 1 \rangle 1$. LET: $n = q|g| + d$ where $0 \leq d < |g|$

PROOF: Division Algorithm.

$\langle 1 \rangle 2$. $g^d = e$

PROOF:

$$\begin{aligned} e &= g^n \\ &= g^{q|g|+d} \\ &= (g^{|g|})^q g^d && \text{(Propositions 6.9, 6.10)} \\ &= e^q g^d \\ &= g^d \end{aligned}$$

$\langle 1 \rangle 3$. $d = 0$

PROOF: By minimality of $|g|$.

$\langle 1 \rangle 4$. $n = q|g|$

□

Corollary 6.14.1. *Let G be a group. Let $g \in G$ have finite order and $n \in \mathbb{Z}$. Then $g^n = e$ if and only if $|g| \mid n$.*

Proposition 6.15. *Let G be a group and $g \in G$. Then $|g| \leq |G|$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: w.l.o.g. G is finite.

$\langle 1 \rangle 2$. PICK i, j with $0 \leq i < j \leq |G|$ such that $g^i = g^j$.

PROOF: Otherwise $g^0, g^1, \dots, g^{|G|}$ would be $|G| + 1$ distinct elements of G .

$\langle 1 \rangle 3$. $g^{j-i} = e$

$\langle 1 \rangle 4$. g has finite order and $|g| \leq |G|$

PROOF: Since $|g| \leq j - i \leq j \leq |G|$.

□

Proposition 6.16. *Let G be a group. Let $g \in G$ have finite order. Let $m \in \mathbb{N}$. Then*

$$|g^m| = \frac{\text{lcm}(m, |g|)}{m} = \frac{|g|}{\text{gcd}(m, |g|)}$$

PROOF: Since for any integer d we have

$$g^{md} = e \Leftrightarrow |g| \mid md \quad (\text{Corollary 6.14.1})$$

$$\Leftrightarrow \text{lcm}(m, |g|) \mid md$$

$$\Leftrightarrow \frac{\text{lcm}(m, |g|)}{m} \mid d$$

□

and so $|g^m| = \frac{\text{lcm}(m, |g|)}{m}$ by Corollary 6.14.1. □

Corollary 6.16.1. *If g has odd order then $|g^2| = |g|$.*

Proposition 6.17. *Let G be a group. Let $g, h \in G$ have finite order. Assume $gh = hg$. Then $|gh|$ has finite order and*

$$|gh| \mid \text{lcm}(|g|, |h|)$$

PROOF: Since $(gh)^{\text{lcm}(|g|, |h|)} = g^{\text{lcm}(|g|, |h|)} h^{\text{lcm}(|g|, |h|)} = e$. \square

Example 6.18. This example shows that we cannot remove the hypothesis that $gh = hg$.

In $\text{GL}_2(\mathbb{R})$, take

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then $|g| = 4$, $|h| = 3$ and $|gh| = \infty$.

Proposition 6.19. *Let G be a group and $g, h \in G$ have finite order. If $gh = hg$ and $\gcd(|g|, |h|) = 1$ then $|gh| = |g||h|$.*

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } N = |gh|$$

$$\langle 1 \rangle 2. g^N = (h^{-1})^N$$

$$\langle 1 \rangle 3. g^{N|g|} = e$$

$$\langle 1 \rangle 4. |g^N| \mid |g|$$

$$\langle 1 \rangle 5. h^{-N|h|} = e$$

$$\langle 1 \rangle 6. |g^N| \mid |h|$$

$$\langle 1 \rangle 7. |g^N| = 1$$

PROOF: Since $\gcd(|g|, |h|) = 1$.

$$\langle 1 \rangle 8. g^N = e$$

$$\langle 1 \rangle 9. |g| \mid N$$

$$\langle 1 \rangle 10. h^{-N} = e$$

$$\langle 1 \rangle 11. |h| \mid N$$

$$\langle 1 \rangle 12. N = |g||h|$$

PROOF: Using Proposition 6.17.

\square

Proposition 6.20. *Let G be a finite group. Assume there is exactly one element $f \in G$ of order 2. Then the product of all the elements of G is f .*

PROOF: Let the elements of G be g_1, g_2, \dots, g_n . Apart from e and f , every element and its inverse are distinct elements of the list. Hence the product of the list is $ef = f$. \square

Proposition 6.21. *Let G be a finite group of order n . Let m be the number of elements of G of order 2. Then $n - m$ is odd.*

PROOF: In the list of all elements that are not of order 2, every element and its inverse are distinct except for e . Hence the list has odd length. \square

Corollary 6.21.1. *If a finite group has even order, then it contains an element of order 2.*

Proposition 6.22. *Let G be a group and $a, g \in G$. Then $|aga^{-1}| = |g|$.*

PROOF: Since

$$\begin{aligned} (aga^{-1})^n = e &\Leftrightarrow ag^na^{-1} = e \\ &\Leftrightarrow g^n = e \end{aligned} \quad \square$$

Proposition 6.23. *Let G be a group and $g, h \in G$. Then $|gh| = |hg|$.*

PROOF: Since $|gh| = |ghgg^{-1}| = |hg|$. \square

Proposition 6.24. *Let G be a group of order n . Let k be relatively prime to n . Then every element in G has the form x^k for some x .*

$\langle 1 \rangle 1$. PICK integers a and b such that $an + bk = 1$.

$\langle 1 \rangle 2$. LET: $g \in G$

$\langle 1 \rangle 3$. $g = (g^b)^k$

PROOF:

$$\begin{aligned} g &= g \cdot (g^n)^{-a} & (g^n = e) \\ &= g^{1-an} \\ &= g^{bk} \end{aligned}$$

\square

6.2 Generators

Definition 6.25 (Generator). Let G be a group and $a \in G$. We say a *generates* the group iff, for all $x \in G$, there exists an integer n such that $x^n = a$.

Example 6.26. $\text{SL}_2(\mathbb{Z})$ is generated by

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

PROOF:

$\langle 1 \rangle 1$. LET: $H = \langle s, t \rangle$

$\langle 1 \rangle 2$. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in H$.

PROOF: It is t^q .

$\langle 1 \rangle 3$. For all $q \in \mathbb{Z}$ we have $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \in H$.

PROOF:

$$\begin{aligned} st^{-q}s^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \end{aligned}$$

⟨1⟩4.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & qa+b \\ c & qc+d \end{pmatrix}$$

⟨1⟩5.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} a+qb & b \\ c+qd & d \end{pmatrix}$$

⟨1⟩6. For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, if c and d are both nonzero, then there exists $N \in H$ such that the bottom row of MN has one entry the same as M and one entry with smaller absolute value.

PROOF: From ⟨1⟩4 and ⟨1⟩5 taking $q = -1$.

⟨1⟩7. For any $M \in \mathrm{SL}_2(\mathbb{Z})$, there exists $N \in H$ such that MN has a zero on the bottom row.

PROOF: Apply ⟨1⟩6 repeatedly.

⟨1⟩8. Any matrix in $\mathrm{SL}_2(\mathbb{Z})$ with a zero on the bottom row is in H .

⟨2⟩1. $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H$

PROOF: ⟨1⟩2

⟨2⟩2. $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \in H$

PROOF: It is $s^2 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ since $s^2 = -I$.

⟨2⟩3. $\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} \in H$

PROOF: It is $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} s$.

⟨2⟩4. $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \in H$

PROOF: It is $s^2 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s$.

⟨1⟩9. Every matrix in $\mathrm{SL}_2(\mathbb{Z})$ is in H .

□

Chapter 7

Group Homomorphisms

Definition 7.1 (Homomorphism). Let G and H be groups. A (group) homomorphism $\phi : G \rightarrow H$ is a function such that, for all $x, y \in G$,

$$\phi(xy) = \phi(x)\phi(y) \ .$$

Proposition 7.2. Let G and H be groups with identities e_G and e_H . Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\phi(e_G) = e_H$.

PROOF: Since $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$ and so $\phi(e_G) = e_H$ by Cancellation. \square

Proposition 7.3. Let $\phi : G \rightarrow H$ be a group homomorphism. For all $x \in G$ we have $\phi(x^{-1}) = \phi(x)^{-1}$.

PROOF: Since $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_H$. \square

Proposition 7.4. Let G, H and K be groups. If $\phi : G \rightarrow H$ and $\psi : H \rightarrow K$ are homomorphisms then $\psi \circ \phi : G \rightarrow K$ is a homomorphism.

PROOF: For $x, y \in G$ we have

$$\psi(\phi(xy)) = \psi(\phi(x)\phi(y)) = \psi(\phi(x))\psi(\phi(y)) \ .$$

Proposition 7.5. Let G be a group. Then $\text{id}_G : G \rightarrow G$ is a group homomorphism.

PROOF: For $x, y \in G$ we have $\text{id}_G(xy) = xy = \text{id}_G(x)\text{id}_G(y)$. \square

Proposition 7.6. Let $\phi : G \rightarrow H$ be a group homomorphism. Let $g \in G$ have finite order. Then $|\phi(g)|$ divides $|g|$.

PROOF: Since $\phi(g)^{|g|} = \phi(g^{|g|}) = e$. \square

Definition 7.7 (Category of Groups). Let **Grp** be the category of groups and group homomorphisms.

Example 7.8. There are 49487365402 groups of order 1024 up to isomorphism.

Proposition 7.9. *A group homomorphism $\phi : G \rightarrow H$ is an isomorphism in **Grp** if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: ϕ is bijective.

PROVE: ϕ^{-1} is a group homomorphism.

$\langle 1 \rangle 2$. LET: $h, h' \in H$

$\langle 1 \rangle 3$. $\phi(\phi^{-1}(hh')) = \phi(\phi^{-1}(h)\phi^{-1}(h'))$

PROOF: Both are equal to hh' .

$\langle 1 \rangle 4$. $\phi^{-1}(hh') = \phi^{-1}(h)\phi^{-1}(h')$

□

Corollary 7.9.1.

$$D_6 \cong C_3$$

PROOF: The canonical homomorphism $D_6 \rightarrow C_3$ is bijective. □

Corollary 7.9.2.

$$(\mathbb{R}, +) \cong (\{x \in \mathbb{R} : x > 0\}, \cdot)$$

PROOF: The function that maps x to e^x is a bijective homomorphism. □

Proposition 7.10. *The trivial group is the zero object in **Grp**.*

PROOF: For any group G , the unique function $G \rightarrow \{e\}$ is a group homomorphism, and the only group homomorphism $\{e\} \rightarrow G$ maps e to e_G . □

Proposition 7.11. *For any groups G and H , the set $G \times H$ under $(g, h)(g', h') = (gg', hh')$ is the product of G and H in **Grp**.*

PROOF:

$\langle 1 \rangle 1$. $G \times H$ is a group.

$\langle 2 \rangle 1$. The multiplication is associative.

PROOF: Since $(g_1, h_1)((g_2, h_2)(g_3, h_3)) = ((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1g_2g_3, h_1h_2h_3)$.

$\langle 2 \rangle 2$. (e_G, e_H) is the identity.

PROOF: Since $(g, h)(e_G, e_H) = (e_G, e_H)(g, h) = (g, h)$.

$\langle 2 \rangle 3$. The inverse of (g, h) is (g^{-1}, h^{-1}) .

PROOF: Since $(g, h)(g^{-1}, h^{-1}) = (g^{-1}, h^{-1})(g, h) = (e_G, e_H)$.

$\langle 1 \rangle 2$. $\pi_1 : G \times H \rightarrow G$ is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$. $\pi_2 : G \times H \rightarrow H$ is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 4$. For any group homomorphism $\phi : K \rightarrow G$ and $\psi : K \rightarrow H$, the function $\langle \phi, \psi \rangle : K \rightarrow G \times H$ where $\langle \phi, \psi \rangle(k) = (\phi(k), \psi(k))$ is a group homomorphism.

PROOF:

$$\begin{aligned} \langle \phi, \psi \rangle(kk') &= (\phi(kk'), \psi(kk')) \\ &= (\phi(k)\phi(k'), \psi(k)\psi(k')) \\ &= (\phi(k), \psi(k))(\phi(k'), \psi(k')) \\ &= \langle \phi, \psi \rangle(k)\langle \phi, \psi \rangle(k') \end{aligned}$$

□

7.1 Subgroups

Definition 7.12 (Subgroup). Let (G, \cdot) and $(H, *)$ be groups such that H is a subset of G . Then H is a *subgroup* of G iff the inclusion $i : H \hookrightarrow G$ is a group homomorphism.

Proposition 7.13. *If $(H, *)$ is a subgroup of (G, \cdot) then $*$ is the restriction of \cdot to H .*

PROOF: Given $x, y \in H$ we have

$$x * y = i(x * y) = i(x) \cdot i(y) = x \cdot y . \quad \square$$

Example 7.14. For any group G we have $\{e\}$ is a subgroup of G .

Proposition 7.15. *Let G be a group. Let H be a subset of G . Then H is a subgroup of G iff H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$.*

PROOF:

$\langle 1 \rangle 1$. If H is a subgroup of G then H is nonempty.

PROOF: Since every group has an identity element and so is nonempty.

$\langle 1 \rangle 2$. If H is a subgroup of G then, for all $x, y \in H$, we have $xy^{-1} \in H$.

PROOF: Easy.

$\langle 1 \rangle 3$. If H is nonempty and, for all $x, y \in H$, we have $xy^{-1} \in H$, then H is a subgroup of G .

$\langle 2 \rangle 1$. ASSUME: H is nonempty.

$\langle 2 \rangle 2$. ASSUME: $\forall x, y \in H. xy^{-1} \in H$

$\langle 2 \rangle 3$. $e \in H$

PROOF: Pick $x \in H$. We have $e = xx^{-1} \in H$.

$\langle 2 \rangle 4$. $\forall x \in H. x^{-1} \in H$

PROOF: Given $x \in H$ we have $x^{-1} = ex^{-1} \in H$.

$\langle 2 \rangle 5$. H is closed under the restriction of \cdot

PROOF: Given $x, y \in H$ we have $xy = x(y^{-1})^{-1} \in H$.

$\langle 2 \rangle 6$. H is a group under the restriction of \cdot

PROOF: Associativity is inherited from G and the existence of an identity element and inverses follows from $\langle 2 \rangle 3$ and $\langle 2 \rangle 4$.

$\langle 2 \rangle 7$. The inclusion $H \hookrightarrow G$ is a group homomorphism.

PROOF: For $x, y \in H$ we have $i(xy) = i(x)i(y) = xy$.

\square

Corollary 7.15.1. *The intersection of a set of subgroups of G is a subgroup of G .*

Corollary 7.15.2. *Let $\phi : G \rightarrow H$ be a group homomorphism. Let K be a subgroup of H . Then $\phi^{-1}(K)$ is a subgroup of G .*

PROOF:

$\langle 1 \rangle 1$. $\phi^{-1}(K)$ is nonempty.

PROOF: Since $e \in \phi^{-1}(K)$.

$\langle 1 \rangle 2$. LET: $x, y \in \phi^{-1}(K)$

- $\langle 1 \rangle 3. \phi(x), \phi(y) \in K$
- $\langle 1 \rangle 4. \phi(x)\phi(y)^{-1} \in K$
- $\langle 1 \rangle 5. \phi(xy^{-1}) \in K$
- $\langle 1 \rangle 6. xy^{-1} \in \phi^{-1}(K)$

□

Corollary 7.15.3. *Let $\phi : G \rightarrow H$ be a group homomorphism. Let K be a subgroup of G . Then $\phi(K)$ is a subgroup of H .*

PROOF:

- $\langle 1 \rangle 1.$ LET: $x, y \in \phi(K)$
- $\langle 1 \rangle 2.$ PICK $a, b \in K$ such that $x = \phi(a)$ and $y = \phi(b)$
- $\langle 1 \rangle 3. xy^{-1} = \phi(ab^{-1})$
- $\langle 1 \rangle 4. xy^{-1} \in \phi(K)$

□

Proposition 7.16. *Let G be a subgroup of \mathbb{Z} . Then there exists $d \geq 0$ such that $G = d\mathbb{Z}$.*

PROOF:

- $\langle 1 \rangle 1.$ ASSUME: w.l.o.g. $G \neq \{0\}$

PROOF: Since $\{0\} = 0\mathbb{Z}$.

- $\langle 1 \rangle 2.$ LET: d be the least positive element of G .

PROVE: $G = d\mathbb{Z}$

PROOF: If $n \in G$ then $-n \in G$ so G must contain a positive element.

- $\langle 1 \rangle 3. G \subseteq d\mathbb{Z}$

- $\langle 2 \rangle 1.$ LET: $n \in G$

- $\langle 2 \rangle 2.$ LET: q and r be the integers such that $n = qd + r$ and $0 \leq r < d$.

- $\langle 2 \rangle 3. r \in G$

PROOF: Since $r = n - qd$.

- $\langle 2 \rangle 4. r = 0$

PROOF: By minimality of d .

- $\langle 2 \rangle 5. n = qd \in d\mathbb{Z}$

- $\langle 1 \rangle 4. d\mathbb{Z} \subseteq G$

□

7.2 Kernel

Definition 7.17 (Kernel). Let $\phi : G \rightarrow H$ be a group homomorphism. The *kernel* of ϕ is

$$\ker \phi = \{g \in G : \phi(g) = e\} .$$

Proposition 7.18. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\ker \phi$ is a subgroup of G .*

PROOF: Corollary 7.15.2. □

Proposition 7.19. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then the inclusion $i : \ker \phi \hookrightarrow G$ is terminal in the category of pairs $(K, \alpha : K \rightarrow G)$ such that $\phi \circ \alpha = 0$.*

PROOF:

$\langle 1 \rangle 1.$ $\phi \circ i = 0$

$\langle 1 \rangle 2.$ For any group K and homomorphism $\alpha : K \rightarrow G$ such that $\phi \circ \alpha = 0$, there exists a unique homomorphism $\beta : K \rightarrow \ker \phi$ such that $i \circ \beta = \alpha$.

□

Proposition 7.20. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then the following are equivalent:*

1. ϕ is monic.
2. $\ker \phi = \{e\}$
3. ϕ is injective.

PROOF:

$\langle 1 \rangle 1.$ $1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: ϕ is monic.

$\langle 2 \rangle 2.$ LET: $i : \ker \phi \hookrightarrow G, j : \{e\} \hookrightarrow \ker \phi \hookrightarrow G$ be the inclusions.

$\langle 2 \rangle 3.$ $\phi \circ i = \phi \circ j$

$\langle 2 \rangle 4.$ $i = j$

$\langle 1 \rangle 2.$ $2 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: $\ker \phi = \{e\}$

$\langle 2 \rangle 2.$ LET: $x, y \in G$

$\langle 2 \rangle 3.$ ASSUME: $\phi(x) = \phi(y)$

$\langle 2 \rangle 4.$ $\phi(xy^{-1}) = e$

$\langle 2 \rangle 5.$ $xy^{-1} \in \ker \phi$

$\langle 2 \rangle 6.$ $xy^{-1} = e$

$\langle 2 \rangle 7.$ $x = y$

$\langle 1 \rangle 3.$ $3 \Rightarrow 1$

PROOF: Easy.

□

Proposition 7.21. *A group homomorphism is an epimorphism if and only if it is surjective.*

7.3 Inner Automorphisms

Proposition 7.22. *Let G be a group and $g \in G$. The function $\gamma_g : G \rightarrow G$ defined by $\gamma_g(a) = gag^{-1}$ is an automorphism on G .*

PROOF:

$\langle 1 \rangle 1.$ γ_g is a homomorphism.

PROOF:

$$\begin{aligned}\gamma_g(ab) &= gabg^{-1} \\ &= gag^{-1}gbg^{-1} \\ &= \gamma_g(a)\gamma_g(b)\end{aligned}$$

<1>2. γ_g is injective.

PROOF: By Cancellation.

<1>3. γ_g is surjective.

PROOF: Given $b \in G$, we have $\gamma_g(g^{-1}bg) = b$.

□

Definition 7.23 (Inner Automorphism). Let G be a group. An *inner automorphism* on G is a function of the form $\gamma_g(a) = gag^{-1}$ for some $g \in G$.

We write $\text{Inn}(G)$ for the set of inner automorphisms of G .

Proposition 7.24. Let G be a group. The function $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$ that maps g to γ_g is a group homomorphism.

PROOF: Since $\gamma_{gh}(a) = ghah^{-1}g^{-1} = \gamma_g(\gamma_h(a))$. □

Corollary 7.24.1. $\text{Inn}(G)$ is a subgroup of $\text{Aut}_{\mathbf{Grp}}(G)$.

7.4 Direct Products

Definition 7.25 (Direct Product). The *direct product* of groups G and H is their product in \mathbf{Grp} .

7.5 Free Groups

Proposition 7.26. Let A be a set. Let \mathcal{F}^A be the category whose objects are pairs (G, j) where G is a group and j is a function $A \rightarrow G$, with morphisms $f : (G, j) \rightarrow (H, k)$ the group homomorphisms $f : G \rightarrow H$ such that $f \circ j = k$. Then \mathcal{F}^A has an initial object.

PROOF:

<1>1. LET: $W(A)$ be the set of words in the alphabet whose elements are the elements of A together with $\{a^{-1} : a \in A\}$.

<1>2. LET: $r : W(A) \rightarrow W(A)$ be the function that, given a word w , removes the first pair of letters of the form aa^{-1} or $a^{-1}a$; if there is no such pair, then $r(w) = w$.

<1>3. Let us say that a word w is a *reduced word* iff $r(w) = w$.

<1>4. For any word w of length n , we have $r^{\lceil \frac{n}{2} \rceil}(w)$ is a reduced word.

PROOF: Since we cannot remove more than $n/2$ pairs of letters from w .

<1>5. LET: $R : W(A) \rightarrow W(A)$ be the function $R(w) = r^{\lceil \frac{n}{2} \rceil}(w)$, where n is the length of w .

<1>6. LET: $F(A)$ be the set of reduced words.

<1>7. Define $\cdot : F(A)^2 \rightarrow F(A)$ by $w \cdot w' = R(ww')$

(1)8. \cdot is associative.

PROOF: Both $w_1 \cdot (w_2 \cdot w_3)$ and $(w_1 \cdot w_2) \cdot w_3$ are equal to $R(w_1 w_2 w_3)$.

(1)9. The empty word is the identity element in $F(A)$

(1)10. The inverse of $a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}$ is $a_n^{\mp 1} \dots a_2^{\mp 1} a_1^{\mp 1}$.

(1)11. LET: $j : A \rightarrow F(A)$ be the function that maps a to the word a of length

(1)12. LET: G be any group and $k : A \rightarrow G$ any function.

(1)13. The only morphism $f : (F(A), j) \rightarrow (G, k)$ in \mathcal{F}^A is $f(a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}) = k(a_1)^{\pm 1} k(a_2)^{\pm 1} \dots k(a_n)^{\pm 1}$.

□

Definition 7.27 (Free Group). For any set A , the *free group* on A is the initial object $(F(A), i)$ in \mathcal{F}^A .

Proposition 7.28. $i : A \rightarrow F(A)$ is injective.

PROOF:

(1)1. LET: $x, y \in A$

(1)2. ASSUME: $x \neq y$

PROVE: $i(x) \neq i(y)$

(1)3. LET: $f : A \rightarrow C_2$ be the function that maps x to 0 and all other elements of A to 1.

(1)4. LET: $\phi : F(A) \rightarrow C_2$ be the group homomorphism such that $f = \phi \circ i$.

(1)5. $f(x) \neq f(y)$

(1)6. $\phi(i(x)) \neq \phi(i(y))$

(1)7. $i(x) \neq i(y)$

□

Proposition 7.29.

$$F(0) \cong \{e\}$$

PROOF: For any set A , the unique group homomorphism $\{e\} \rightarrow A$ makes the following diagram commute.

$$\begin{array}{ccc} \{e\} & \longrightarrow & A \\ \uparrow & \nearrow & \\ \emptyset & & \end{array}$$

Proposition 7.30. The free group on 1 is \mathbb{Z} with the injection mapping 0 to 1.

PROOF: Given any group G and function $a : 1 \rightarrow G$, the required unique homomorphism $\phi : \mathbb{Z} \rightarrow G$ is defined by $\phi(n) = a(0)^n$. □

Proposition 7.31. For any sets A and B , we have that $F(A + B)$ is the coproduct of $F(A)$ and $F(B)$ in **Grp**.

$$\begin{array}{ccccc}
& & G & & \\
& f \nearrow & \uparrow k & \nwarrow g & \\
F(A) & \xrightarrow{\kappa_1} & F(A+B) & \xleftarrow{\kappa_2} & F(B) \\
i_A \uparrow & & j \uparrow & & i_B \uparrow \\
A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
\end{array}$$

PROOF:

- $\langle 1 \rangle 1$. LET: $i_A : A \rightarrow F(A)$, $i_B : B \rightarrow F(B)$, $j : A+B \rightarrow F(A+B)$ be the canonical injections.
 - $\langle 1 \rangle 2$. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
 - $\langle 1 \rangle 3$. LET: G be any group and $f : F(A) \rightarrow G$, $g : F(B) \rightarrow G$ any group homomorphisms.
 - $\langle 1 \rangle 4$. LET: $h : A+B \rightarrow G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.
 - $\langle 1 \rangle 5$. LET: $k : F(A+B) \rightarrow G$ be the unique group homomorphism such that $k \circ j = h$.
 - $\langle 1 \rangle 6$. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
 - $\langle 1 \rangle 7$. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.
-

Definition 7.32 (Subgroup Generated by a Group). Let G be a group and A a subset of G . Let $\phi : F(A) \rightarrow G$ be the unique group homomorphism such that $\phi(a) = a$ for all $a \in A$. The subgroup *generated* by A is

$$\langle A \rangle := \text{im } \phi$$

$$\begin{array}{ccc}
F(A) & \xrightarrow{\phi} & G \\
\uparrow & \nearrow & \\
A & &
\end{array}$$

Proposition 7.33. Let G be a group and A a subset of G . Then $\langle A \rangle$ is the set of all elements of the form $a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$ (where $n \geq 0$) such that $a_1, \dots, a_n \in A$.

PROOF: Immediate from definitions. □

Corollary 7.33.1. Let G be a group and $g \in G$. Then

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\} .$$

Proposition 7.34. Let G be a group and A a subset of G . Then $\langle A \rangle$ is the intersection of all the subgroups of G that include A .

PROOF: Easy. \square

Definition 7.35 (Finitely Generated). Let G be a group. Then G is *finitely generated* iff there exists a finite subset A of G such that $G = \langle A \rangle$.

Proposition 7.36. *Every subgroup of a finitely generated free group is free.*

PROOF: TODO.

Proposition 7.37. *$F(2)$ includes subgroups isomorphic to the free group on arbitrarily many generators.*

PROOF: TODO

Proposition 7.38.

$$[F(2), F(2)] \cong F(\mathbb{Z})$$

PROOF: TODO

7.6 Normal Subgroups

Definition 7.39 (Normal Subgroup). A subgroup N of G is *normal* iff, for all $g \in G$ and $n \in N$, we have $gng^{-1} \in N$.

Example 7.40. Every subgroup of Q_8 is normal.

Proposition 7.41. *Let G be a group and N a subgroup of G . Then the following are equivalent.*

1. N is normal.
2. $\forall g \in G. gNg^{-1} \subseteq N$
3. $\forall g \in G. gNg^{-1} = N$
4. $\forall g \in G. gN \subseteq Ng$
5. $\forall g \in G. gN = Ng$

PROOF:

$\langle 1 \rangle 1. 1 \Leftrightarrow 2$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: If 2 holds then we have $gNg^{-1} \subseteq N$ and $g^{-1}Ng \subseteq N$ hence $N = gNg^{-1}$.

$\langle 1 \rangle 3. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 4. 2 \Leftrightarrow 4$

PROOF: Easy.

$\langle 1 \rangle 5. 3 \Leftrightarrow 5$

PROOF: Easy.

□

Proposition 7.42. *Let $\phi : G \rightarrow H$ be a group homomorphism. Then $\ker \phi$ is a normal subgroup of G .*

PROOF: Given $g \in G$ and $n \in \ker \phi$ we have

$$\begin{aligned}\phi(gng^{-1}) &= \phi(g)\phi(n)\phi(g)^{-1} \\ &= \phi(g)\phi(g)^{-1} \\ &= e\end{aligned}$$

and so $gng^{-1} \in \ker \phi$. □

7.7 Quotient Groups

Definition 7.43. Let G be a group. Let \sim be an equivalence relation on G . Then we say that \sim is *compatible* with the group operation on G iff, for all $a, a', g \in G$, if $a \sim a'$ then $ga \sim ga'$ and $ag \sim a'g$.

Proposition 7.44. *Let G be a group. Let \sim be an equivalence relation on G . Then there exists an operation $\cdot : (G/\sim)^2 \rightarrow G/\sim$ such that*

$$\forall a, b \in G. [a][b] = [ab]$$

iff \sim is compatible with the group operation on G . In this case, G/\sim is a group under \cdot and the canonical function $\pi : G \rightarrow G/\sim$ is a group homomorphism, and is universal with respect to group homomorphisms $\phi : G \rightarrow G'$ such that if $a \sim a'$ then $\phi(a) = \phi(a')$.

PROOF: Easy. □

Definition 7.45 (Quotient Group). Let G be a group. Let \sim be an equivalence relation on G that is compatible with the group operation on G . Then G/\sim is the *quotient group* of G by \sim under $[a][b] = [ab]$.

Proposition 7.46. *Let G be a group and H a subgroup of G . Then H is normal if and only if there exists a group K and homomorphism $\phi : G \rightarrow K$ such that $H = \ker \phi$.*

PROOF: One direction is given by Proposition 7.42. For the other direction, take $K = G/H$ and ϕ to be the canonical map $G \rightarrow G/H$. □

Definition 7.47 (Modular Group). The *modular group* $\text{PSL}_2(\mathbb{Z})$ is $\text{SL}_2(\mathbb{Z})/\{I, -I\}$.

Proposition 7.48. $\text{PSL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

PROOF: By Example 6.26.

Proposition 7.49 (Roger Alperin). $\text{PSL}_2(\mathbb{Z})$ is presented by $(x, y | x^2, y^3)$.

PROOF:

⟨1⟩1. LET: $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

⟨1⟩2. LET: $y = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

⟨1⟩3. Define an action of $\text{PSL}_2(\mathbb{Z})$ on $\mathbb{R} - \mathbb{Q}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} r = \frac{ar + b}{cr + d}.$$

⟨2⟩1. Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z})$ and r irrational we have $\frac{ar+b}{cr+d}$ is irrational.

⟨3⟩1. ASSUME: for a contradiction $\frac{ar+b}{cr+d} = \frac{p}{q}$ where p and q are integers with $q > 0$.

⟨3⟩2. $aqr + bq = cpr + dp$

⟨3⟩3. $(aq - cp)r = dp - bq$

⟨3⟩4. $aq = cp = dp - bq = 0$

⟨3⟩5. $adq - cdp = 0$

⟨3⟩6. $cdp - cbq = 0$

⟨3⟩7. $(ad - cb)q = 0$

PROOF: Since $ad - cb = 1$.

⟨3⟩8. $q = 0$

⟨3⟩9. Q.E.D.

PROOF: This contradicts ⟨3⟩1.

⟨2⟩2. $-Ir = r$

PROOF: Since $-Ir = \frac{-r}{-1} = r$.

⟨2⟩3. Given $A, B \in \text{PSL}_2(\mathbb{Z})$ we have $A(Br) = (AB)r$.

PROOF:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} r \right] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{er + f}{gr + h} \\ &= \frac{a \frac{er+f}{gr+h} + b}{c \frac{er+f}{gr+h} + d} \\ &= \frac{a(er + f) + b(gr + h)}{c(er + f) + d(gr + h)} \\ &= \frac{(ae + bg)r + (af + bh)}{(ce + dg)r + (cf + dh)} \\ &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} r \\ &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right] r \end{aligned}$$

⟨1⟩4.

$$yr = 1 - \frac{1}{r}$$

⟨1⟩5.

$$y^{-1}r = \frac{1}{1 - r}$$

PROOF: Since $y^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

(1)6.

PROOF: Since $yx = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. $yxr = 1 + r$

(1)7.

$$y^{-1}xr = \frac{r}{1+r}$$

PROOF: Since $y^{-1}x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

(1)8. If $r > -1$ is positive then yxr is positive.

(1)9. If r is positive then $y^{-1}xr$ is positive.

(1)10. If $r < -1$ then $y^{-1}xr$ is positive.

(1)11. If r is negative then yr is positive.

(1)12. If r is negative then $y^{-1}r$ is positive.

(1)13. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)$$

with one or more factors can equal the identity.

PROOF: If the last factor is (yx) , then the product maps numbers in $(-1, 0)$ to positive numbers. If the last factor is $(y^{-1}x)$, then the product maps numbers < -1 to positive numbers.

(1)14. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)y^{\pm 1}$$

with one or more factors can equal the identity.

PROOF: The product maps negative numbers to positive numbers.

(1)15. $\text{PSL}_2(\mathbb{Z})$ is presented by $(x, y|x^2, y^3)$.

□

Corollary 7.49.1. $\text{PSL}_2(\mathbb{Z})$ is the coproduct of C_2 and C_3 in **Grp**.

Theorem 7.50. Every group homomorphism $\phi : G \rightarrow H$ may be decomposed as

$$G \longrightarrow G/\ker \phi \xrightarrow{\cong} \text{im } \phi \longrightarrow H$$

PROOF: Easy. □

Corollary 7.50.1 (First Isomorphism Theorem). Let $\phi : G \rightarrow H$ be a surjective group homomorphism. Then $H \cong G/\ker \phi$.

Proposition 7.51. Let H_1 be a normal subgroup of G_1 and H_2 a normal subgroup of G_2 . Then $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$, and

$$\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}.$$

PROOF: $\pi \times \pi : G_1 \times G_2 \twoheadrightarrow G_1/H_1 \times G_2/H_2$ is a surjective homomorphism with kernel $H_1 \times H_2$. □

Example 7.52.

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

PROOF: Map a real number r to $(\cos r, \sin r)$. The result is a surjective group homomorphism with kernel \mathbb{Z} . \square

Proposition 7.53. *Let H be a normal subgroup of a group G . For every subgroup K of G that includes H , we have H is a normal subgroup of K , and K/H is a subgroup of G/H . The mapping*

$$u : \{\text{subgroups of } G \text{ including } H\} \rightarrow \{\text{subgroups of } G/H\}$$

with $u(K) = K/H$ is a poset isomorphism.

PROOF:

- $\langle 1 \rangle 1$. If K is a subgroup of G that includes H then H is normal in K .
- $\langle 1 \rangle 2$. If K is a subgroup of G that includes H then K/H is a subgroup of G/H .
- $\langle 1 \rangle 3$. If $H \subseteq K_1 \subseteq K_2$ then $K_1/H \subseteq K_2/H$.
- $\langle 1 \rangle 4$. If $K_1/H = K_2/H$ then $K_1 = K_2$
 - $\langle 2 \rangle 1$. ASSUME: $K_1/H = K_2/H$
 - $\langle 2 \rangle 2$. $K_1 \subseteq K_2$
 - $\langle 3 \rangle 1$. LET: $k \in K_1$
 - $\langle 3 \rangle 2$. $kH \in K_2/H$
 - $\langle 3 \rangle 3$. PICK $k' \in K_2$ such that $kH = k'H$
 - $\langle 3 \rangle 4$. $kk'^{-1} \in H$
 - $\langle 3 \rangle 5$. $kk'^{-1} \in K_2$
 - $\langle 3 \rangle 6$. $k \in K_2$
 - $\langle 2 \rangle 3$. $K_2 \subseteq K_1$
- PROOF: Similar.
- $\langle 1 \rangle 5$. For any subgroup L of G/H , there exists a subgroup K of G that includes H such that $L = K/H$.
 - $\langle 2 \rangle 1$. LET: L be a subgroup of G/H .
 - $\langle 2 \rangle 2$. LET: $K = \{k \in G : kH \in L\}$
 - $\langle 2 \rangle 3$. K is a subgroup of G .
 - PROOF: Given $k, k' \in K$ we have $kH, k'H \in L$ hence $kk'^{-1}H \in L$ and so $kk'^{-1} \in K$.
 - $\langle 2 \rangle 4$. $H \subseteq K$
 - PROOF: For all $h \in H$ we have $hH = H \in L$.
 - $\langle 2 \rangle 5$. $L = K/H$
 - PROOF: By definition.

\square

Proposition 7.54 (Third Isomorphism Theorem). *Let H be a normal subgroup of a group G . Let N be a subgroup of G that includes H . Then N/H is normal in G/H if and only if N is normal in G , in which case*

$$\frac{G/H}{N/H} \cong \frac{G}{N}$$

PROOF:

- ⟨1⟩1. If N/H is normal in G/H then N is normal in G .
 - ⟨2⟩1. ASSUME: N/H is normal in G/H .
 - ⟨2⟩2. LET: $g \in G$ and $n \in N$.
 - ⟨2⟩3. $gng^{-1}H \in N/H$
 - ⟨2⟩4. PICK $n' \in N$ such that $gng^{-1}H = n'H$
 - ⟨2⟩5. $gng^{-1}n'^{-1} \in H$
 - ⟨2⟩6. $gng^{-1}n'^{-1} \in N$
 - ⟨2⟩7. $gng^{-1} \in N$
- ⟨1⟩2. If N is normal in G then N/H is normal in G/H and $(G/H)/(N/H) \cong G/N$.
 - ⟨2⟩1. ASSUME: N is normal in G .
 - ⟨2⟩2. LET: $\phi : G/H \rightarrow G/N$ be the homomorphism $\phi(gH) = gN$
 - ⟨3⟩1. If $gH = g'H$ then $gN = g'N$
 PROOF: If $gg'^{-1} \in H$ then $gg'^{-1} \in N$.
 - ⟨3⟩2. $\phi((gH)(g'H)) = \phi(gH)\phi(g'H)$
 PROOF: Both are $gg'N$.
 - ⟨2⟩3. ϕ is surjective.
 - ⟨2⟩4. $\ker \phi = N/H$
 - ⟨2⟩5. $(G/H)/(N/H) \cong G/N$
 PROOF: By the First Isomorphism Theorem.

□

Proposition 7.55 (Second Isomorphism Theorem). *Let H and K be subgroups of a group G . Assume that H is normal in G . Then:*

- 1. HK is a subgroup of G , and H is normal in HK .
- 2. $H \cap K$ is normal in K , and

$$\frac{HK}{H} \cong \frac{K}{H \cap K}.$$

PROOF:

- ⟨1⟩1. HK is a subgroup of G .
 PROOF: Since $hkh'k' = hh'(h'^{-1}kh')k' \in HK$.
- ⟨1⟩2. H is normal in HK .
- ⟨1⟩3. $H \cap K$ is normal in K and $HK/H \cong K/(H \cap K)$
 PROOF: The function that maps k to kH is a surjective homomorphism $K \twoheadrightarrow HK/H$ with kernel $H \cap K$. Surjectivity follows because $hkh = hkh^{-1}H$.

□

See also Proposition 7.70 for a result that holds even if H is not normal.

7.8 Cosets

Proposition 7.56. *Let G be a group. Let \sim be an equivalence relation on G such that, for all $a, b, g \in G$, if $a \sim b$ then $ga \sim gb$. Let $H = \{h \in G : h \sim e\}$.*

Then H is a subgroup of G and, for all $a, b \in G$, we have

$$a \sim b \Leftrightarrow a^{-1}b \in H \Leftrightarrow aH = bH .$$

PROOF:

$\langle 1 \rangle 1.$ $e \in H$

$\langle 1 \rangle 2.$ For all $x, y \in H$ we have $xy^{-1} \in H$.

$\langle 2 \rangle 1.$ ASSUME: $x \sim e$ and $y \sim e$.

$\langle 2 \rangle 2.$ $e \sim y^{-1}$

PROOF: Since $yy^{-1} \sim ey^{-1}$.

$\langle 2 \rangle 3.$ $xy^{-1} \sim e$

PROOF: Since $xy^{-1} \sim ey^{-1} \sim e$.

$\langle 1 \rangle 3.$ If $a \sim b$ then $a^{-1}b \in H$.

PROOF: If $a \sim b$ then $a^{-1}b \sim a^{-1}a = e$.

$\langle 1 \rangle 4.$ If $a^{-1}b \in H$ then $aH = bH$.

$\langle 2 \rangle 1.$ ASSUME: $a^{-1}b \in H$

$\langle 2 \rangle 2.$ $bH \subseteq aH$

PROOF: For any $h \in H$ we have $bh = aa^{-1}bh \in aH$.

$\langle 2 \rangle 3.$ $aH \subseteq bH$

PROOF: Similar since $b^{-1}a \in H$.

$\langle 1 \rangle 5.$ If $aH = bH$ then $a \sim b$.

$\langle 2 \rangle 1.$ ASSUME: $aH = bH$

$\langle 2 \rangle 2.$ PICK $h \in H$ such that $a = bh$.

$\langle 2 \rangle 3.$ $b^{-1}a = h$

$\langle 2 \rangle 4.$ $b^{-1}a \in H$

$\langle 2 \rangle 5.$ $b^{-1}a \sim e$

$\langle 2 \rangle 6.$ $a \sim b$

PROOF: $a = bb^{-1}a \sim be = b$.

□

Definition 7.57 (Coset). Let G be a group and H a subgroup of G . A *left coset* of H is a set of the form aH for $a \in G$. A *right coset* of H is a set of the form Ha for some $a \in G$.

We write G/H for the set of all left cosets of H , and $G \backslash H$ for the set of all right cosets of H .

Proposition 7.58.

$$G/H \cong G \backslash H$$

PROOF: The function that maps aH to Ha^{-1} is a bijection. □

Proposition 7.59. Let G be a group and H a subgroup of G . Define \sim_H on G by: $a \sim b$ iff $a^{-1}b \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations \sim on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ga \sim gb$. The equivalence class of a is aH .

PROOF:

$\langle 1 \rangle 1.$ For any subgroup H , we have \sim_H is an equivalence relation on G .

⟨2⟩1. \sim is reflexive.

PROOF: For any $a \in G$ we have $a^{-1}a = e \in H$.

⟨2⟩2. \sim is symmetric.

PROOF: If $a^{-1}b \in H$ then $b^{-1}a \in H$.

⟨2⟩3. \sim is transitive.

PROOF: If $a^{-1}b \in H$ and $b^{-1}c \in H$ then $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$.

⟨1⟩2. If $a \sim_H b$ then $ga \sim_H gb$.

PROOF: If $a^{-1}b \in H$ then $(ga)^{-1}(gb) = a^{-1}g^{-1}gb = a^{-1}b \in H$.

⟨1⟩3. For any equivalence relation \sim on G such that, whenever $a \sim b$, then $ga \sim gb$, there exists a subgroup H such that $\sim = \sim_H$.

PROOF: Proposition 7.56.

⟨1⟩4. The \sim_H -equivalence class of a is aH .

PROOF:

$$\begin{aligned} a \sim b &\Leftrightarrow a^{-1}b \in H \\ &\Leftrightarrow \exists h \in H. a^{-1}b = h \\ &\Leftrightarrow \exists h \in H. b = ah \\ &\Leftrightarrow b \in aH \end{aligned}$$

□

Proposition 7.60. Let G be a group and H a subgroup of G . Define \sim_H on G by: $a \sim b$ iff $ab^{-1} \in H$. This defines a one-to-one correspondence between the subgroups of G and the equivalence relations \sim on G such that, for all $a, b, g \in G$, if $a \sim b$, then $ag \sim bg$. The equivalence class of a is Ha .

PROOF: Similar. □

Proposition 7.61. Let G be a group and H be a subgroup of G . Define \sim_L and \sim_R on G by:

$$a \sim_L b \Leftrightarrow a^{-1}b \in H, \quad a \sim_R b \Leftrightarrow ab^{-1} \in H.$$

Then $\sim_L = \sim_R$ if and only if H is normal.

PROOF:

⟨1⟩1. If $\sim_L = \sim_R$ then H is normal.

⟨2⟩1. ASSUME: $\sim_L = \sim_R$

⟨2⟩2. LET: $h \in H$ and $g \in G$

⟨2⟩3. $g \sim_L gh^{-1}$

⟨2⟩4. $g \sim_R gh^{-1}h$

⟨2⟩5. $ghg^{-1} \in H$

⟨1⟩2. If H is normal then $\sim_L = \sim_R$.

⟨2⟩1. ASSUME: H is normal.

⟨2⟩2. If $a \sim_L b$ then $a \sim_R b$.

⟨3⟩1. ASSUME: $a \sim_L b$

⟨3⟩2. $a^{-1}b \in H$

⟨3⟩3. $aa^{-1}ba^{-1} \in H$

⟨3⟩4. $ba^{-1} \in H$

- $\langle 3 \rangle 5. a \sim_R b$
 $\langle 2 \rangle 3. \text{ If } a \sim_R b \text{ then } a \sim_L b.$

PROOF: Similar.

□

Corollary 7.61.1. *Let G be a group and H be a normal subgroup of G . Define \sim on G by $a \sim b$ iff $a^{-1}b \in H$. Then G/\sim is a group under $[a][b] = [ab]$.*

Definition 7.62 (Quotient Group). Let G be a group and H be a normal subgroup of G . The *quotient group* G/H is G/\sim where $a \sim b$ iff $a^{-1}b \in H$, under $[a][b] = [ab]$ or $(aH)(bH) = abH$.

Corollary 7.62.1. *Let H be a normal subgroup of a group G . For every group homomorphism $\phi : G \rightarrow G'$ such that $H \subseteq \ker \phi$, there exists a unique group homomorphism $\bar{\phi} : G/H \rightarrow G'$ such that the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ & \searrow \pi & \nearrow \bar{\phi} \\ & G/H & \end{array}$$

Proposition 7.63. $\mathbb{Z}/n\mathbb{Z}$ has exactly n elements.

PROOF: Every integer is congruent to one of $0, 1, \dots, n-1$ by the division algorithm, and no two of them are congruent to one another, since if $0 \leq i < j < n$ then $0 < j - i < n$. □

Proposition 7.64. *Let m and n be integers with $n > 0$. The order of m in $\mathbb{Z}/n\mathbb{Z}$ is $\frac{n}{\gcd(m,n)}$.*

PROOF: By Proposition 6.16 since the order of 1 is n . □

Proposition 7.65. *The integer m generates $\mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(m, n) = 1$.*

PROOF: By Proposition 7.64. □

Corollary 7.65.1. *If p is prime then every non-zero element in $\mathbb{Z}/p\mathbb{Z}$ is a generator.*

Proposition 7.66.

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$$

PROOF: Every permutation of $\{(1, 0), (0, 1), (1, 1)\}$ gives an automorphism of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

Example 7.67. Not all monomorphisms split in \mathbf{Grp} .

Define $\phi : \mathbb{Z}/3\mathbb{Z} \rightarrow S_3$ by

$$\phi(0) = \text{id}_3, \quad \phi(1) = (1 \ 3 \ 2), \quad \phi(2) = (1 \ 2 \ 3) .$$

Then ϕ is monic but has no retraction.

For if $r : S_3 \rightarrow \mathbb{Z}/3\mathbb{Z}$ is a retraction, then we would have

$$r(1\ 2) + r(2\ 3) = 1, \quad r(2\ 3) + r(1\ 2) = 2$$

which is impossible.

Proposition 7.68. *Let G be a group, H a subgroup of G , and $g \in G$. The function that maps h to gh is a bijection $H \cong gH$.*

PROOF: By Cancellation. \square

Proposition 7.69. *Let G be a group, H a subgroup of G , and $g \in G$. The function that maps h to hg is a bijection $H \cong Hg$.*

PROOF: By Cancellation. \square

Proposition 7.70. *Let H and K be finite subgroups of a group G . Then*

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

PROOF:

$\langle 1 \rangle 1$. LET: $f : \{hK : h \in H\} \rightarrow H/(H \cap K)$ be the function $f(hK) = h(H \cap K)$

PROOF: This is well-defined because if $hK = h'K$ then $h^{-1}h' \in H \cap K$ so $h(H \cap K) = h'(H \cap K)$.

$\langle 1 \rangle 2$. f is injective.

PROOF: If $h(H \cap K) = h'(H \cap K)$ then $hK = h'K$.

$\langle 1 \rangle 3$. f is surjective.

PROOF: Clear.

$\langle 1 \rangle 4$.

$$\frac{|HK|}{|K|} = \frac{|H|}{|H \cap K|}$$

\square

7.9 Congruence

Definition 7.71 (Congruence). Given integers a, b, n with n positive, we say a is *congruent to b modulo n* , and write $a \equiv b \pmod{n}$, iff $a + n\mathbb{Z} = b + n\mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$.

Proposition 7.72. *Given integers a, b, n with n positive, we have $a \equiv b \pmod{n}$ iff $n \mid a - b$.*

PROOF: By Proposition 7.56. \square

Proposition 7.73. *If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then $a + b \equiv a' + b' \pmod{n}$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid (a' + b') - (a + b)$. \square

Proposition 7.74. *If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then $ab \equiv a'b' \pmod{n}$.*

PROOF: If $n \mid a' - a$ and $n \mid b' - b$ then $n \mid a'b' - ab = a'(b' - b) + (a' - a)b$. \square

7.10 Cyclic Groups

Definition 7.75 (Cyclic Group). The *cyclic* groups are \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$ for positive integers n .

Proposition 7.76. *If m and n are positive integers with $\gcd(m, n) = 1$ then $C_{mn} \cong C_m \times C_n$.*

PROOF: The function that maps x to $(x \bmod m, x \bmod n)$ is an isomorphism. \square

Proposition 7.77. *Let G be a group and $g \in G$. Then $\langle g \rangle$ is cyclic.*

PROOF: If g has finite order then $\langle g \rangle \cong C_{|g|}$, otherwise $\langle g \rangle \cong \mathbb{Z}$. \square

Proposition 7.78. *Every finitely generated subgroup of \mathbb{Q} is cyclic.*

PROOF:

$\langle 1 \rangle$ 1. LET: $G = \langle a_1/b, \dots, a_n/b \rangle$ where a_1, \dots, a_n, b are integers with $b > 0$

$\langle 1 \rangle$ 2. LET: $a = \gcd(a_1, \dots, a_n)$

$\langle 1 \rangle$ 3. $G = \langle a/b \rangle$

\square

Corollary 7.78.1. \mathbb{Q} is not finitely generated.

7.11 Commutator Subgroup

Definition 7.79 (Commutator Subgroup). Let G be a group. The *commutator subgroup* $[G, G]$ is the subgroup generated by the elements of the form $aba^{-1}b^{-1}$.

Proposition 7.80. *The commutator subgroup is normal.*

PROOF: Since

$$\begin{aligned} & ga_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_nb_na_n^{-1}b_n^{-1}g^{-1} \\ &= (ga_1g^{-1})(gb_1g^{-1})(ga_1g^{-1})^{-1}(gb_1g^{-1})^{-1} \cdots (ga_ng^{-1})(gb_ng^{-1})(ga_ng^{-1})^{-1}(gb_ng^{-1})^{-1}. \end{aligned} \quad \square$$

7.12 Presentations

Definition 7.81 (Presentation). A *presentation* of a group G is a pair (A, R) where A is a set and $R \subseteq F(A)$ is a set of words such that

$$G \cong F(A)/N(R)$$

where $N(R)$ is the smallest normal subgroup of $F(A)$ that includes R .

Example 7.82. • The free group on a set A is presented by (A, \emptyset) .

- S_3 is presented by $(x, y | x^2, y^3, xyxy)$.
- $(a, b | a^2, b^2, (ab)^n)$ is a presentation of D_{2n} .

- $(x, y \mid x^2y^{-2}, y^4, xyx^{-1}y)$ is a presentation of Q_8 .

Proposition 7.83 (Word Problem). *Let (A, R) be a presentation of the group G . Let $w_1, w_2 \in F(A)$ be two words. Then it is undecidable in general if $w_1N(R) = w_2N(R)$ in G .*

Definition 7.84 (Finitely Presented). A group is *finitely presented* iff it has a presentation (A, R) where both A and R are finite.

Proposition 7.85. *Let $(A|R)$ be a presentation of G and $(A'|R')$ a presentation of H . Assume w.l.o.g. A and A' are disjoint. Then the group $G * G'$ presented by $(A \cup A' | R \cup R')$ is the coproduct of G and G' in **Grp**.*

$$\begin{array}{ccccc}
 A & \longrightarrow & A \cup A' & \longleftarrow & A' \\
 \downarrow & & \downarrow & & \downarrow \\
 F(A) & \longrightarrow & F(A \cup A') & \longleftarrow & F(A') \\
 \downarrow & & \downarrow & & \downarrow \\
 G & \xrightarrow{\kappa_1} & G * G' & \xleftarrow{\kappa_2} & G'
 \end{array}$$

PROOF:

- $\langle 1 \rangle 1$. LET: $\kappa_1 : G \rightarrow G * G'$ and $\kappa_2 : G' \rightarrow G * G'$ be the unique homomorphisms that make the diagram above commute.
- $\langle 1 \rangle 2$. LET: $\phi : G \rightarrow H$ and $\psi : G' \rightarrow H$ be any homomorphisms.
- $\langle 1 \rangle 3$. LET: $[\phi, \psi] : F(A \cup A') \rightarrow H$ be the unique homomorphism such that ...
- $\langle 1 \rangle 4$. $R \cup R' \subseteq \ker[\phi, \psi]$
- $\langle 1 \rangle 5$. $[\phi, \psi]$ factors uniquely through the morphism $F(A \cup A') \rightarrow G * G'$

□

7.13 Index of a Subgroup

Definition 7.86 (Index). Let G be a group and H a subgroup of G . The *index* of H in G , denoted $|G : H|$, is the number of left cosets of H in G if this is finite, otherwise ∞ .

Theorem 7.87 (Lagrange's Theorem). *Let G be a finite group and H a subgroup of G . Then*

$$|G| = |G : H| |H| .$$

PROOF: G/H is a partition of G into $|G : H|$ subsets, each of size $|H|$. □

Corollary 7.87.1. *For p a prime number, the only group of order p is C_p .*

PROOF: Let G be a group of order p and $g \in G$ with $g \neq e$. Then $|\langle g \rangle|$ divides p and is not 1, hence is p , that is, $G = \langle g \rangle$. □

Theorem 7.88 (Cauchy's Theorem). *Let G be a finite group. If p is prime and $p \mid |G|$ then G has a subgroup of order p .*

Proposition 7.89. *Let G be a group. Let K be a subgroup of G and H a subgroup of K . If $|G : H|$, $|G : K|$ and $|K : H|$ are all finite then*

$$|G : H| = |G : K| |K : H| .$$

PROOF:

- $\langle 1 \rangle 1$. LET: $G/K = \{g_1K, g_2K, \dots, g_mK\}$
- $\langle 1 \rangle 2$. LET: $K/H = \{k_1H, k_2H, \dots, k_nH\}$
- $\langle 1 \rangle 3$. $G/H = \{g_ik_jH : 1 \leq i \leq m, 1 \leq j \leq n\}$
- $\langle 2 \rangle 1$. LET: $g \in G$
- $\langle 2 \rangle 2$. PICK i such that $gK = g_iK$
- $\langle 2 \rangle 3$. $g^{-1}g_i \in K$
- $\langle 2 \rangle 4$. PICK j such that $g^{-1}g_iH = k_jH$
- $\langle 2 \rangle 5$. $g^{-1}g_ik_j \in H$
- $\langle 2 \rangle 6$. $gH = g_ik_jH$
- $\langle 1 \rangle 4$. If $g_ik_jH = g_{i'}k_{j'}H$ then $i = i'$ and $j = j'$.
- $\langle 2 \rangle 1$. ASSUME: $g_ik_jH = g_{i'}k_{j'}H$
- $\langle 2 \rangle 2$. $g_iK = g_{i'}K$
- $\langle 2 \rangle 3$. $i = i'$
- $\langle 2 \rangle 4$. $k_jH = k_{j'}H$
- $\langle 2 \rangle 5$. $j = j'$

□

7.14 Cokernels

Proposition 7.90. *Let $\phi : G \rightarrow H$ be a homomorphism between groups. Then there exists a group K and homomorphism $\pi : H \rightarrow K$ that is initial with respect to all homomorphism $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.*

PROOF:

- $\langle 1 \rangle 1$. LET: N be the intersection of all the normal subgroups of H that include $\text{im } \phi$.
- $\langle 1 \rangle 2$. LET: $K = H/N$ and π be the canonical homomorphism.
- $\langle 1 \rangle 3$. LET: $\pi \circ \phi = 0$
- $\langle 1 \rangle 4$. LET: $\alpha : H \rightarrow L$ satisfy $\alpha \circ \phi = 0$
- $\langle 1 \rangle 5$. $\text{im } \phi \subseteq \ker \alpha$
- $\langle 1 \rangle 6$. $N \subseteq \ker \alpha$
- $\langle 1 \rangle 7$. There exists a unique $\bar{\alpha} : H/\text{im } \phi \rightarrow L$ such that $\bar{\alpha} \circ \pi = \alpha$

□

Definition 7.91 (Cokernel). For any homomorphism $\phi : G \rightarrow H$ in **Grp**, the *cokernel* of ϕ is the group $\text{coker } \phi$ and homomorphism $\pi : H \rightarrow \text{coker } \phi$ that is initial among homomorphisms $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.

Example 7.92. It is not true that a homomorphism with trivial cokernel is epi. The inclusion $\langle (1 \ 2) \rangle \hookrightarrow S_3$ has trivial cokernel but is not epi.

7.15 Cayley Graphs

Definition 7.93 (Cayley Graph). Let G be a finitely generated group. Let A be a finite set of generators for G . The *Cayley graph* of G with respect to A is the directed graph whose vertices are the elements of G , with an edge $g_1 \rightarrow g_2$ labelled by $a \in A$ iff $g_2 = g_1 a$.

Proposition 7.94. G is the free group on A iff the Cayley graph with respect to A is a tree.

PROOF: Both are equivalent to saying that the product of two different strings of elements of A and/or their inverses are not equal. \square

Chapter 8

Abelian Groups

Definition 8.1 (Abelian Group). A group is *Abelian* iff any two elements commute.

In an Abelian group G , we often denote the group operation by $+$, the identity element by 0 and the inverse of an element g by $-g$. We write ng for g^n ($g \in G, n \in \mathbb{Z}$).

Example 8.2. Every group of order ≤ 4 is Abelian.

Example 8.3. For any positive integer n , we have $\mathbb{Z}/n\mathbb{Z}$ is an Abelian group under addition.

Example 8.4. S_n is not Abelian for $n \geq 3$. If $x = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ then $xy = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $yx = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$.

Example 8.5. There are 42 Abelian groups of order 1024 up to isomorphism.

Proposition 8.6. Let G be a group. If $g^2 = e$ for all $g \in G$ then G is Abelian.

PROOF: For any $g, h \in G$ we have

$$ghgh = e$$

$$\therefore hgh = g \quad (\text{multiplying on the left by } g)$$

$$\therefore hg = gh \quad (\text{multiplying on the right by } h) \square$$

Proposition 8.7. Let G be a group. Then G is Abelian if and only if the function that maps g to g^{-1} is a group homomorphism.

PROOF:

(1)1. If G is Abelian then the function that maps g to g^{-1} is a group homomorphism.

PROOF: Since $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$.

(1)2. If the function that maps g to g^{-1} is a group homomorphism then G is Abelian.

PROOF: Since $gh = (g^{-1})^{-1}(h^{-1})^{-1} = (g^{-1}h^{-1})^{-1} = hg$.
 \square

Proposition 8.8. *Let G be a group. Then G is Abelian if and only if the function that maps g to g^2 is a group homomorphism.*

PROOF:

$\langle 1 \rangle 1$. If G is Abelian then the function that maps g to g^2 is a group homomorphism.

PROOF: Since $(gh)^2 = g^2h^2$.

$\langle 1 \rangle 2$. If the function that maps g to g^2 is a group homomorphism then G is Abelian.

PROOF: Since we have $(gh)^2 = ghgh = g^2h^2$ and so $hg = gh$.

\square

Proposition 8.9. *Let G be a group. Then G is Abelian if and only if the homomorphism $\gamma : G \rightarrow \text{Aut}_{\text{Grp}}(G)$ is the trivial homomorphism.*

PROOF:

$\langle 1 \rangle 1$. If G is Abelian then γ is trivial.

PROOF: Since $\gamma_g(a) = gag^{-1} = a$.

$\langle 1 \rangle 2$. If γ is trivial then G is Abelian.

PROOF: If $\gamma_g(a) = gag^{-1} = a$ for all g and a then $ga = ag$ for all g, a .

\square

Proposition 8.10. *Let G be an Abelian group. Let $g, h \in G$. If g has maximal finite order in G , and h has finite order, then $|h| \mid |g|$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $|h| \nmid |g|$.

$\langle 1 \rangle 2$. PICK a prime p such that $|g| = p^m r$, $|h| = p^n s$ where $p \nmid r$, $p \nmid s$ and $m < n$.

$\langle 1 \rangle 3$. $|g^{p^m} h^s| = p^n r$

PROOF: Proposition 6.19.

$\langle 1 \rangle 4$. $|g| < |g^{p^m} h^s|$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts the maximality of $|g|$.

\square

Proposition 8.11. *Given a set A and an Abelian group H , the set H^A is an Abelian group under*

$$(\phi + \psi)(a) = \phi(a) + \psi(a) \quad (\phi, \psi \in H^A, a \in A) .$$

PROOF:

$\langle 1 \rangle 1$. $\phi + (\psi + \chi) = (\phi + \psi) + \chi$

$\langle 1 \rangle 2$. $\phi + \psi = \psi + \phi$

$\langle 1 \rangle 3$. LET: $0 : A \rightarrow H$ be the function $0(a) = 0$.

$\langle 1 \rangle 4$. $\phi + 0 = 0 + \phi = \phi$

$\langle 1 \rangle 5$. Given $\phi : A \rightarrow H$, define $-\phi : A \rightarrow H$ by $(-\phi)(a) = -(\phi(a))$.

$\langle 1 \rangle 6$. $\phi + (-\phi) = (-\phi) + \phi = 0$

□

Proposition 8.12. *Given a group G and an Abelian group H , the set $\mathbf{Grp}[G, H]$ is a subgroup of H^G .*

PROOF:

$\langle 1 \rangle 1$. Given $\phi, \psi : G \rightarrow H$ group homomorphisms, we have $\phi - \psi$ is a group homomorphism.

PROOF:

$$\begin{aligned} (\phi - \psi)(g + g') &= \phi(g + g') - \psi(g + g') \\ &= \phi(g) + \phi(g') - \psi(g) - \psi(g') \\ &= \phi(g) - \psi(g) + \phi(g') - \psi(g') \\ &= (\phi - \psi)(g) + (\phi - \psi)(g') \end{aligned}$$

□

Proposition 8.13. *Let G be a group. The following are equivalent.*

1. $\text{Inn}(G)$ is cyclic.
2. $\text{Inn}(G)$ is trivial.
3. G is Abelian.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: $\text{Inn}(G) = \langle \gamma_g \rangle$

$\langle 2 \rangle 2$. g commutes with every element of G

$\langle 3 \rangle 1$. LET: $x \in G$

$\langle 3 \rangle 2$. PICK $n \in \mathbb{Z}$ such that $\gamma_x = \gamma_g^n$

$\langle 3 \rangle 3$. $\forall y \in G. xyx^{-1} = g^n yg^{-n}$

$\langle 3 \rangle 4$. $xgx^{-1} = g$

$\langle 2 \rangle 3$. $\gamma_g = \text{id}_G$

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

$\langle 2 \rangle 1$. ASSUME: $\forall g \in G. \gamma_g = \text{id}_G$

$\langle 2 \rangle 2$. LET: $x, y \in G$

$\langle 2 \rangle 3$. $\gamma_x(y) = y$

$\langle 2 \rangle 4$. $xyx^{-1} = y$

$\langle 2 \rangle 5$. $xy = yx$

$\langle 1 \rangle 3$. $3 \Rightarrow 2$

PROOF: If $xy = yx$ for all x, y then $\gamma_x(y) = y$ for all x, y .

$\langle 1 \rangle 4$. $2 \Rightarrow 1$

PROOF: Easy.

□

Corollary 8.13.1. *If $\text{Aut}_{\mathbf{Grp}}(G)$ is cyclic then G is Abelian.*

Proposition 8.14. *Every subgroup of an Abelian group is normal.*

PROOF: Let G be an Abelian group and N a subgroup of G . Given $g \in G$ and $n \in N$ we have $gng^{-1} = n \in N$. \square

Proposition 8.15. *For any group G , the group $G/[G, G]$ is Abelian.*

PROOF: For any $g, h \in G$ we have

$$gh(hg)^{-1} \in [G, G]$$

$$\therefore gh[G, G] = hg[G, G] \quad \square$$

Proposition 8.16. *Let G be a finite Abelian group. Let p be a prime divisor of $|G|$. Then G has an element of order p .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: as induction hypothesis the result holds for all groups smaller than G .

$\langle 1 \rangle 2$. PICK $g \in G - \{0\}$.

$\langle 1 \rangle 3$. PICK an element $h \in \langle g \rangle$ with prime order q .

$\langle 1 \rangle 4$. CASE: $q = p$

PROOF: h is the required element.

$\langle 1 \rangle 5$. CASE: $q \neq p$

$\langle 2 \rangle 1$. PICK $r \in G$ such that $r + \langle h \rangle$ has order p in $G/\langle h \rangle$.

PROOF: By induction hypothesis since $|G/\langle h \rangle| = |G|/q$.

$\langle 2 \rangle 2$. $pr \in \langle h \rangle$

$\langle 2 \rangle 3$. PICK k such that $pr = kh$

$\langle 2 \rangle 4$. $pqr = e$

$\langle 2 \rangle 5$. qr has order p .

\square

Corollary 8.16.1. *For n an odd integer, any Abelian group of order $2n$ has exactly one element of order 2.*

PROOF: If x and y are distinct elements of order 2 then $\langle x, y \rangle = \{e, x, y, xy\}$ has size 4 and so $4 \mid 2n$ which is a contradiction. \square

Example 8.17. It is not true that, if G is a finite group and $d \mid |G|$, then G has an element of order d . The quaternionic group has no element of order 4.

Proposition 8.18. *If G is a finite Abelian group and $d \mid |G|$ then G has a subgroup of size d .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: as induction hypothesis the result is true for all $d' < d$.

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $d \neq 1$.

$\langle 1 \rangle 3$. PICK a prime p such that $p \mid d$.

$\langle 1 \rangle 4$. PICK an element $g \in G$ of order p .

$\langle 1 \rangle 5$. $d/p \mid |G/\langle g \rangle|$

$\langle 1 \rangle 6$. PICK a subgroup H of $G/\langle g \rangle$ of size d/p .

$\langle 1 \rangle 7$. $\pi^{-1}(H)$ is a subgroup of G of size d .

\square

Proposition 8.19. *Let (G, \cdot) be a group. Let $\circ : G^2 \rightarrow G$ be a group homomorphism such that (G, \circ) is a group. Then \circ and \cdot coincide, and G is Abelian.*

PROOF:

$\langle 1 \rangle 1$. For all $g_1, g_2, h_1, h_2 \in G$ we have

$$(g_1 g_2) \circ (h_1 h_2) = (g_1 \circ h_1)(g_2 \circ h_2)$$

$\langle 1 \rangle 2$. $e \circ e = e$

PROOF:

$$\begin{aligned} e \circ e &= (ee) \circ (ee) \\ &= (e \circ e)(e \circ e) \end{aligned}$$

Hence $e \circ e = e$ by Cancellation.

$\langle 1 \rangle 3$. e is the identity of (G, \circ)

$\langle 1 \rangle 4$. For all $g, h \in G$ we have

$$g \circ h = gh$$

PROOF:

$$\begin{aligned} g \circ h &= (ge) \circ (eh) \\ &= (g \circ e)(e \circ h) \\ &= gh \end{aligned}$$

$\langle 1 \rangle 5$. For all $g, h \in G$ we have $gh = hg$.

PROOF:

$$\begin{aligned} gh &= (e \circ g)(h \circ e) \\ &= (eh) \circ (ge) \\ &= h \circ g \\ &= hg \end{aligned}$$

□

Corollary 8.19.1. *If $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$ is a group object in **Grp** then m is the multiplication of G , $e(*)$ is the identity of G , $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Grp** where $e(*) = e$ and $i(g) = g^{-1}$.*

8.1 The Category of Abelian Groups

Definition 8.20 (Category of Abelian Groups). Let **Ab** be the full subcategory of **Grp** whose objects are the Abelian groups.

Proposition 8.21. *If $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$ is a group object in **Ab** then m is the multiplication of G , $e(*)$ is the identity of G , $i(g) = g^{-1}$, and G is Abelian.*

*Conversely, if (G, m) is any Abelian group, then (G, m, e, i) is a group object in **Ab** where $e(*) = e$ and $i(g) = g^{-1}$.*

PROOF: Immediate from Corollary 8.19.1. □

Definition 8.22 (Direct Sum). Given Abelian groups G and H , we also call the direct product of G and H the *direct sum* and denote it $G \oplus H$.

Proposition 8.23. *Given Abelian groups G and H , the direct sum $G \oplus H$ is the coproduct of G and H in \mathbf{Ab} .*

PROOF:

- (1)1. LET: $\kappa_1 : G \rightarrow G \oplus H$ be the group homomorphism $\kappa_1(g) = (g, e_H)$.
 (1)2. LET: $\kappa_2 : H \rightarrow G \oplus H$ be the group homomorphism $\kappa_2(h) = (e_G, h)$.
 (1)3. Given group homomorphism $\phi : G \rightarrow K$ and $\psi : H \rightarrow K$, define $[\phi, \psi] : G \oplus H \rightarrow K$ by $[\phi, \psi](g, h) = \phi(g) + \psi(h)$.
 (1)4. $[\phi, \psi]$ is a group homomorphism.

PROOF:

$$\begin{aligned}
 [\phi, \psi]((g, h) + (g', h')) &= [\phi, \psi](g + g', h + h') \\
 &= \phi(g + g') + \psi(h + h') \\
 &= \phi(g) + \phi(g') + \psi(h) + \psi(h') \\
 &= \phi(g) + \psi(h) + \phi(g') + \psi(h') \\
 &= [\phi, \psi](g, h) + [\phi, \psi](g', h')
 \end{aligned}$$

- (1)5. $[\phi, \psi] \circ \kappa_1 = \phi$

PROOF:

$$\begin{aligned}
 [\phi, \psi](\kappa_1(g)) &= [\phi, \psi](g, e_H) \\
 &= \phi(g) + \psi(e_H) \\
 &= \phi(g) + e_K \\
 &= \phi(g)
 \end{aligned}$$

- (1)6. $[\phi, \psi] \circ \kappa_2 = \psi$

PROOF: Similar.

- (1)7. If $f : G \oplus H \rightarrow K$ is a group homomorphism with $f \circ \kappa_1 = \phi$ and $f \circ \kappa_2 = \psi$ then $f = [\phi, \psi]$.

PROOF:

$$\begin{aligned}
 f(g, h) &= f((g, e_H) + (e_G, h)) \\
 &= f(\kappa_1(g)) + f(\kappa_2(h)) \\
 &= \phi(g) + \psi(h)
 \end{aligned}$$

□

Theorem 8.24. *Every finitely generated Abelian group is a direct sum of cyclic groups.*

PROOF: TODO □

8.2 Free Abelian Groups

Proposition 8.25. *Let A be a set. Let \mathcal{F}^A be the category whose objects are pairs (G, j) where G is an Abelian group and j is a function $A \rightarrow G$, with morphisms $f : (G, j) \rightarrow (H, k)$ the group homomorphisms $f : G \rightarrow H$ such that $f \circ j = k$. Then \mathcal{F}^A has an initial object.*

PROOF:

- (1)1. LET: $\mathbb{Z}^{\oplus A}$ be the subgroup of \mathbb{Z}^A consisting of all functions $\alpha : A \rightarrow \mathbb{Z}$ such that $\alpha(a) = 0$ for only finitely many $a \in A$.
- (1)2. LET: $i : A \rightarrow \mathbb{Z}^{\oplus A}$ be the function such that $i(a)(b) = 1$ if $a = b$ and 0 if $a \neq b$.
- (1)3. LET: G be any Abelian group and $j : A \rightarrow G$ any function.
- (1)4. The unique homomorphism $\phi : \mathbb{Z}^{\oplus A} \rightarrow G$ required is defined by $\phi(\alpha) = \sum_{a \in A} \alpha(a)j(a)$

□

Definition 8.26 (Free Abelian Group). For any set A , the *free Abelian group* on A is the initial object $(F^{ab}(A), i)$ in \mathcal{F}^A .

Proposition 8.27. For any sets A and B , we have that $F^{ab}(A + B)$ is the coproduct of $F^{ab}(A)$ and $F^{ab}(B)$ in **Grp**.

$$\begin{array}{ccccc}
 & & G & & \\
 & \nearrow f & \uparrow k & \nwarrow g & \\
 F^{ab}(A) & \xrightarrow{\kappa_1} & F^{ab}(A+B) & \xleftarrow{\kappa_2} & F^{ab}(B) \\
 i_A \uparrow & & j \uparrow & & i_B \uparrow \\
 A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
 \end{array}$$

PROOF:

- (1)1. LET: $i_A : A \rightarrow F^{ab}(A)$, $i_B : B \rightarrow F^{ab}(B)$, $j : A + B \rightarrow F^{ab}(A + B)$ be the canonical injections.
- (1)2. LET: κ_1, κ_2 be the unique group homomorphisms that make the diagram above commute.
- (1)3. LET: G be any group and $f : F^{ab}(A) \rightarrow G$, $g : F^{ab}(B) \rightarrow G$ any group homomorphisms.
- (1)4. LET: $h : A + B \rightarrow G$ be the unique function such that $h \circ k_1 = f \circ i_A$ and $h \circ k_2 = g \circ i_B$.
- (1)5. LET: $k : F^{ab}(A + B) \rightarrow G$ be the unique group homomorphism such that $k \circ j = h$.
- (1)6. k is the unique group homomorphism such that $k \circ \kappa_1 \circ i_A = f \circ i_A$ and $k \circ \kappa_2 \circ i_B = g \circ i_B$.
- (1)7. k is the unique group homomorphism such that $k \circ \kappa_1 = f$ and $k \circ \kappa_2 = g$.

□

Proposition 8.28. For A and B finite sets, if $F^{ab}(A) \cong F^{ab}(B)$ then $A \cong B$.

PROOF:

- (1)1. For any set C , define \sim on $F^{ab}(C)$ by: $f \sim f'$ iff there exists $g \in F^{ab}(C)$ such that $f - f' = 2g$.
- (1)2. For any set C , \sim is an equivalence relation on $F^{ab}(C)$.
- (1)3. For any set C , we have $F^{ab}(C) / \sim$ is finite if and only if C is finite, in which case $|F^{ab}(C) / \sim| = 2^{|C|}$.

PROOF: There is a bijection between $F^{\text{ab}}(C)/\sim$ and the finite subsets of C , which maps f to $\{c \in C : f(c) \text{ is odd}\}$.

$\langle 1 \rangle 4$. If $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$ then $A \cong B$.

PROOF: If $|F^{\text{ab}}(A)/\sim| = |F^{\text{ab}}(B)/\sim|$ then $2^{|A|} = 2^{|B|}$ and so $|A| = |B|$. \square

Proposition 8.29. *Let G be an Abelian group. Then G is finitely generated if and only if there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \rightarrow G$ for some n .*

PROOF:

$\langle 1 \rangle 1$. If G is finitely generated then there exists a surjective homomorphism $\mathbb{Z}^{\oplus n} \rightarrow G$ for some n .

PROOF: Let $G = \langle a_1, \dots, a_n \rangle$. Define $\phi : \mathbb{Z}^{\oplus n} \rightarrow G$ by $\phi(i_1, \dots, i_n) = i_1 \cdot a_1 + \dots + i_n \cdot a_n$.

$\langle 1 \rangle 2$. If there exists a surjective homomorphism $\phi : \mathbb{Z}^{\oplus n} \rightarrow G$ for some n then G is finitely generated.

PROOF: G is generated by $\phi(1, 0, \dots, 0), \phi(0, 1, 0, \dots, 0), \dots, \phi(0, \dots, 0, 1)$. \square

Proposition 8.30. *Let A be a set. Let $i : A \hookrightarrow F(A)$ be the free group on A . Then $\pi \circ i : A \rightarrow F(A)/[F(A), F(A)]$ is the free Abelian group on A .*

$$\begin{array}{ccc}
 & F(A)/[F(A), F(A)] & \\
 \pi \uparrow & \searrow h & \\
 F(A) & \xrightarrow{g} & G \\
 i \uparrow & \nearrow f & \\
 A & &
 \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: G be an Abelian group and $f : A \rightarrow G$ a function.

$\langle 1 \rangle 2$. LET: $g : F(A) \rightarrow G$ be the unique group homomorphism such that $g \circ i = f$.

$\langle 1 \rangle 3$. $[F(A), F(A)] \subseteq \ker g$

PROOF: For all $x, y \in F(A)$ we have $g(xy x^{-1} y^{-1}) = g(x) + g(y) - g(x) - g(y) = 0$.

$\langle 1 \rangle 4$. LET: $h : F(A)/[F(A), F(A)] \rightarrow G$ be the unique group homomorphism such that $h \circ \pi = g$.

$\langle 1 \rangle 5$. h is the unique group homomorphism such that $h \circ \pi \circ i = f$. \square

Corollary 8.30.1. *Let A and B be sets. Let $F(A)$ and $F(B)$ be the free groups on A and B respectively. If $F(A) \cong F(B)$ then $A \cong B$.*

PROOF: Proposition 8.28. \square

8.3 Cokernels

Proposition 8.31. *Let $\phi : G \rightarrow H$ be a homomorphism between Abelian groups. Then there exists an Abelian group K and homomorphism $\pi : H \rightarrow K$ that is initial with respect to all homomorphism $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.*

PROOF:

$\langle 1 \rangle 1$. LET: $K = H/\text{im } \phi$ and π be the canonical homomorphism.

$\langle 1 \rangle 2$. LET: $\pi \circ \phi = 0$

$\langle 1 \rangle 3$. LET: $\alpha : H \rightarrow L$ satisfy $\alpha \circ \phi = 0$

$\langle 1 \rangle 4$. $\text{im } \phi \subseteq \ker \alpha$

$\langle 1 \rangle 5$. There exists a unique $\bar{\alpha} : H/\text{im } \phi \rightarrow L$ such that $\bar{\alpha} \circ \pi = \alpha$

□

Definition 8.32 (Cokernel). For any homomorphism $\phi : G \rightarrow H$ in **Ab**, the *cokernel* of ϕ is the Abelian group $\text{coker } \phi$ and homomorphism $\pi : H \rightarrow \text{coker } \phi$ that is initial among homomorphisms $\alpha : H \rightarrow L$ such that $\alpha \circ \phi = 0$.

Proposition 8.33. $\pi : H \rightarrow \text{coker } \phi$ is initial among functions $f : H \rightarrow X$ such that, for all $x, y \in H$, if $x + \text{im } \phi = y + \text{im } \phi$ then $f(x) = f(y)$.

PROOF: Easy. □

Proposition 8.34. Let $\phi : G \rightarrow H$ be a homomorphism of Abelian groups. Then the following are equivalent.

- ϕ is an epimorphism.
- $\text{coker } \phi$ is trivial.
- ϕ is surjective.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: ϕ is epi.

$\langle 2 \rangle 2$. LET: $\pi : H \rightarrow \text{coker } \phi$ be the canonical homomorphism.

$\langle 2 \rangle 3$. $\pi \circ \phi = 0 \circ \phi$

$\langle 2 \rangle 4$. $\pi = 0$

$\langle 2 \rangle 5$. $\text{coker } \phi = \text{im } \pi$ is trivial.

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

PROOF: If $\text{coker } \phi = H/\text{im } \phi$ is trivial then $\text{im } \phi = H$.

$\langle 1 \rangle 3$. $3 \Rightarrow 1$

PROOF: If it is surjective then it is epi in **Set**.

□

Chapter 9

Group Actions

9.1 Group Actions

Definition 9.1 (Action). Let G be a group. Let A be an object of a category \mathcal{C} . A (left) action of G on A is a group homomorphism $G \rightarrow \text{Aut}_{\mathcal{C}}(A)$.

It is *faithful* or *effective* iff it is injective.

Proposition 9.2. Let A be a set. An action of the group G on the set A is given by a function $\cdot : G \times A \rightarrow A$ such that

- $\forall a \in A. ea = a$
- $\forall g, h \in G. \forall a \in A. (gh)a = g(ha)$

PROOF: Just unfolding definitions. \square

Example 9.3. Left multiplication defines a faithful action of any group on its own underlying set.

In fact, for any subgroup H of a group G , left multiplication defines an action of G on G/H .

Corollary 9.3.1 (Cayley's Theorem). Every group G is a subgroup of a symmetric group, namely $\text{Aut}_{\text{Set}}(G)$.

Example 9.4. Conjugation $g * h = ghg^{-1}$ is an action of any group on its own underlying set.

Definition 9.5 (Transitive). An action of a group G on a set A is *transitive* iff, for all $a, b \in A$, there exists $g \in G$ such that $ga = b$.

Example 9.6. Left multiplication of a group G is a transitive action of G on G .

Definition 9.7 (Orbit). Given an action of a group G on a set A and $a \in A$, the *orbit* of a is

$$\text{O}_G(a) := \{ga : g \in G\} .$$

Proposition 9.8. *Given an action of a group G on a set A , the orbits form a partition of A .*

PROOF:

$\langle 1 \rangle 1$. Every element of A is in some orbit.

PROOF: Since $a \in O_G(a)$.

$\langle 1 \rangle 2$. Distinct orbits are disjoint.

$\langle 2 \rangle 1$. LET: $a \in O_G(b) \cap O_G(c)$

$\langle 2 \rangle 2$. PICK $g, h \in G$ such that $a = gb = hc$.

$\langle 2 \rangle 3$. $O_G(b) \subseteq O_G(c)$

PROOF: For all $k \in G$ we have $kb = kg^{-1}hc$.

$\langle 2 \rangle 4$. $O_G(c) \subseteq O_G(b)$

PROOF: Similar.

□

Proposition 9.9. *Given an action of a group G on a set A and $a \in A$, the action is transitive on $O_G(a)$.*

PROOF:

$\langle 1 \rangle 1$. The restriction of the action is an action on $O_G(a)$.

PROOF: Since $g(ha) = (gh)a$, the action maps $O_G(a)$ to itself.

$\langle 1 \rangle 2$. The restricted action is transitive.

PROOF: Given $ga, ha \in O_G(a)$, we have $ha = (hg^{-1})(ga)$.

□

Definition 9.10 (Stabilizer Subgroup). Given an action of a group G on a set A and $a \in A$, the *stabilizer subgroup* of a is

$$\text{Stab}_G(a) := \{g \in G : ga = a\} .$$

Proposition 9.11. *Stabilizer subgroups are subgroups.*

PROOF: If $g, h \in \text{Stab}_G(a)$ then $gh^{-1}a = a$ so $gh^{-1} \in \text{Stab}_G(a)$. □

Proposition 9.12. *Let G act on a set A . Let $a \in A$ and $g \in G$. Then*

$$\text{Stab}_G(ga) = g\text{Stab}_G(a)g^{-1} .$$

PROOF:

$$h \in \text{Stab}_G(ga) \Leftrightarrow hga = ga$$

$$\Leftrightarrow g^{-1}hga = a$$

$$\Leftrightarrow g^{-1}hg \in \text{Stab}_G(a)$$

$$\Leftrightarrow h \in g\text{Stab}_G(a)g^{-1}$$

□

Corollary 9.12.1. *Let G be an action on a set A and $a \in A$. If $\text{Stab}_G(a)$ is normal in G , then for any $b \in O_G(a)$ we have $\text{Stab}_G(a) = \text{Stab}_G(b)$.*

Definition 9.13 (Free). An action of a group G on a set A is *free* iff, whenever $ga = a$, then $g = e$.

Example 9.14. The action of left multiplication is free.

Proposition 9.15. *Let G be a group. Let H be a subgroup of G of finite index n . Then H includes a subgroup K that is normal in G and such that $|G : K|$ divides $\gcd(|G|, n!)$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\sigma : G \rightarrow \text{Aut}_{\text{Set}}(G/H)$ be the action of left multiplication.

$\langle 1 \rangle 2$. LET: $K = \ker \sigma$

$\langle 1 \rangle 3$. $K \subseteq H$

$\langle 2 \rangle 1$. LET: $g \in K$

$\langle 2 \rangle 2$. $\sigma(g)(H) = H$

$\langle 2 \rangle 3$. $gH = H$

$\langle 2 \rangle 4$. $g \in H$

$\langle 1 \rangle 4$. K is normal in G .

PROOF: Proposition 7.42.

$\langle 1 \rangle 5$. $|G : K| \mid |G|$

PROOF: Lagrange's Theorem.

$\langle 1 \rangle 6$. $|G : K| \mid n!$

PROOF: Since G/K is a subgroup of $\text{Aut}_{\text{Set}}(G/H)$.

□

Corollary 9.15.1. *Let G be a finite group. Let H be a subgroup of G of index p where p is the smallest prime that divides $|G|$. Then H is normal in G .*

PROOF:

$\langle 1 \rangle 1$. PICK a subgroup K of H normal in G such that $|G : K|$ divides $\gcd(|G|, p!)$.

$\langle 1 \rangle 2$. $|G : K|$ divides p .

$\langle 1 \rangle 3$. $|G : H| |H : K|$ divides p .

$\langle 1 \rangle 4$. $|H : K| = 1$

$\langle 1 \rangle 5$. $H = K$

$\langle 1 \rangle 6$. H is normal.

□

Corollary 9.15.2. *Any subgroup of index 2 is normal.*

Proposition 9.16. *Let G be a group with finite set of generators A . Then left multiplication defines a free action of G on its Cayley graph.*

PROOF: Easy since if $g_2 = g_1 a$ then $hg_2 = hg_1 a$. □

Corollary 9.16.1. *A free group acts freely on a tree.*

Theorem 9.17. *If a group G acts freely on a tree then G is free.*

Corollary 9.17.1. *Every subgroup of the free group on a finite set is free.*

PROOF: If H is a subgroup of $F(A)$ then left multiplication defines a free action of H on the Cayley graph of $F(A)$, which is a tree. □

9.2 Category of G -Sets

Definition 9.18. Given a group G , let $G - \mathbf{Set}$ be the category with:

- objects all pairs (A, ρ) such that A is a set and $\rho : G \times A \rightarrow A$ is an action of G on A ;
- morphisms $f : (A, \rho) \rightarrow (B, \sigma)$ are functions $f : A \rightarrow B$ that are $(G-)$ equivariant, i.e.

$$\forall g \in G. \forall a \in A. f(\rho(g, a)) = \sigma(g, f(a)) .$$

Proposition 9.19. *A G -equivariant function $f : A \rightarrow B$ is an isomorphism in $G - \mathbf{Set}$ if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ be G -equivariant and bijective.

PROVE: f^{-1} is G -equivariant.

$\langle 1 \rangle 2$. LET: $g \in G$ and $b \in B$

$\langle 1 \rangle 3$. $f^{-1}(gb) = gf^{-1}(b)$

PROOF:

$$\begin{aligned} f(f^{-1}(gb)) &= gb \\ &= gf(f^{-1}(b)) \\ &= f(gf^{-1}(b)) \end{aligned}$$

□

Proposition 9.20. *Let G be a group and A a transitive G -set. Let $a \in A$. Then A is isomorphic to $G/\text{Stab}_G(a)$ under left multiplication.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : G/\text{Stab}_G(a) \rightarrow A$ be the function $f(g\text{Stab}_G(a)) = ga$.

$\langle 2 \rangle 1$. ASSUME: $g\text{Stab}_G(a) = h\text{Stab}_G(a)$

PROVE: $ga = ha$

$\langle 2 \rangle 2$. $g^{-1}h \in \text{Stab}_G(a)$

$\langle 2 \rangle 3$. $g^{-1}ha = a$

$\langle 2 \rangle 4$. $ha = ga$

$\langle 1 \rangle 2$. f is G -equivariant.

PROOF: Since $f(gh\text{Stab}_G(a)) = gha = gf(h\text{Stab}_G(a))$.

$\langle 1 \rangle 3$. f is injective.

PROOF: If $ga = ha$ then $g^{-1}h \in \text{Stab}_G(a)$ so $g\text{Stab}_G(a) = h\text{Stab}_G(a)$.

$\langle 1 \rangle 4$. f is surjective.

PROOF: Since for all $b \in A$ there exists $g \in G$ such that $ga = b$.

□

Corollary 9.20.1. *If O is an orbit of the action of a finite group G on a set A , then O is finite and $|O|$ divides $|G|$.*

Corollary 9.20.2. *Let H be a subgroup of G and $g \in G$. Then*

$$G/H \cong G/(gHg^{-1})$$

in $G - \mathbf{Set}$.

PROOF: Taking $A = G/H$ and $a = gH$. \square

Proposition 9.21. *Given a family of G -sets $\{A_i\}_{i \in I}$, we have $\prod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g\{a_i\}_{i \in I} = \{ga_i\}_{i \in I}.$$

PROOF: Easy. \square

Proposition 9.22. *Given a family of G -sets $\{A_i\}_{i \in I}$, we have $\coprod_{i \in I} A_i$ is their product in $G - \mathbf{Set}$ under*

$$g(i, a_i) = (i, ga_i).$$

PROOF: Easy. \square

Proposition 9.23. *Every finite G -set is a coproduct of G -sets of the form G/H .*

PROOF: If $O(a_1), \dots, O(a_n)$ are the orbits of the G -set A , then G is the coproduct of $G/\text{Stab}_G(a_1), \dots, G/\text{Stab}_G(a_n)$. \square

Proposition 9.24. *For any group G we have $G \cong \text{Aut}_{G-\mathbf{Set}}(G)$ (considering G as a G -set under left multiplication).*

PROOF:

$\langle 1 \rangle 1$. Define $\phi : G \rightarrow \text{Aut}_{G-\mathbf{Set}}(G)$ by $\phi(g)(g') = g'g^{-1}$.

$\langle 2 \rangle 1$. LET: $g \in G$

PROVE: $\lambda g' \in G.g'g^{-1}$ is an automorphism of G in $G - \mathbf{Set}$.

$\langle 2 \rangle 2$. $\phi(g)$ is G -equivariant.

PROOF: Since $\phi(g)(h_1h_2) = h_1h_2g^{-1} = h_1\phi(g)(h_2)$.

$\langle 2 \rangle 3$. $\phi(g)$ is injective.

PROOF: By Cancellation.

$\langle 2 \rangle 4$. $\phi(g)$ is surjective.

PROOF: For any $h \in G$ we have $h = \phi(g)(hg)$.

$\langle 1 \rangle 2$. ϕ is a group homomorphism.

PROOF: $\phi(g_1g_2)(h) = hg_2^{-1}g_1^{-1} = \phi(g_1)(\phi(g_2)(h))$.

$\langle 1 \rangle 3$. ϕ is injective.

PROOF: If $\phi(g) = \phi(g')$ then $g = \phi(g)(e) = \phi(g')(e) = g'$.

$\langle 1 \rangle 4$. ϕ is surjective.

$\langle 2 \rangle 1$. LET: $\sigma \in \text{Aut}_{G-\mathbf{Set}}(G)$

$\langle 2 \rangle 2$. LET: $g = \sigma(e)$

PROVE: $\sigma = \phi(g^{-1})$

$\langle 2 \rangle 3$. $\sigma(h) = hg$

PROOF: $\sigma(h) = \sigma(hg) = h\sigma(e) = hg$.

\square

Part III

Ring Theory

Chapter 10

Rngs

Definition 10.1 (Ring). A *rng* consists of a set R and binary operations $+, \cdot : R^2 \rightarrow R$ such that:

- $(R, +)$ is an Abelian group
- \cdot is associative.
- The *distributive properties* hold: for all $r, s, t \in R$ we have

$$(r + s)t = rt + st, \quad r(s + t) = rs + rt .$$

Example 10.2. • The *zero rng* is $\{0\}$.

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} are rngs.
- $2\mathbb{Z}$ is a rng.
- Given a rng R and natural number n , then the set $\mathfrak{gl}_n(R)$ of all $n \times n$ matrices with entries in R is a rng under matrix addition and matrix multiplication.
- For any set S , the power set $\mathcal{P}S$ is a rng under $A + B = (A \cup B) - (A \cap B)$ and $AB = A \cap B$.
- Given a rng R and a set S , then R^S is a rng under $(f + g)(s) = f(s) + g(s)$ and $(fg)(s) = f(s)g(s)$ for all $f, g \in R^S$ and $s \in S$.
- The set $\mathfrak{sl}_n(\mathbb{R}) = \{M \in \mathfrak{gl}_n(\mathbb{R}) : \text{tr } M = 0\}$ is a rng.
- The set $\mathfrak{sl}_n(\mathbb{C}) = \{M \in \mathfrak{gl}_n(\mathbb{C}) : \text{tr } M = 0\}$ is a rng.
- $\mathbb{Z}/n\mathbb{Z}$ is a rng.

- The ring \mathbb{H} of *quaternions* is \mathbb{R}^4 under the following operations, where we write (a, b, c, d) as $a + bi + cj + dk$:

$$\begin{aligned}
 (a + bi + cj + dk) + (a' + b'i + c'j + d'k) &= (a + a') + (b + b')i \\
 &\quad + (c + c')j + (d + d')k \\
 (a + bi + cj + dk)(a' + b'i + c'j + d'k) &= (aa' - bb' - cc' - dd') \\
 &\quad + (ab' + ba' + cd' - dc')i \\
 &\quad + (ac' - bd' + ca' + db')j \\
 &\quad + (ad' + bc' - cb' + da')k
 \end{aligned}$$

Proposition 10.3. *In any rng R we have*

$$\forall x \in R. x0 = 0x = 0 \text{ .}$$

PROOF:

$$\begin{aligned}
 x0 &= x(0 + 0) \\
 &= x0 + x0
 \end{aligned}$$

and so $x0 = 0$ by Cancellation. Similarly $0x = 0$. \square

Definition 10.4 (Zero Divisor). Let R be a rng and $a \in R$.

Then a is a *left-zero-divisor* iff there exists $b \in R - \{0\}$ such that $ab = 0$.

The element a is a *right-zero-divisor* iff there exists $b \in R - \{0\}$ such that $ba = 0$.

Example 10.5. 0 is a left- and right-zero-divisor in every non-zero rng.

The zero rng is the only ring with no zero-divisors.

Proposition 10.6. *Let R be a rng and $a \in R$. Then a is not a left-zero-divisor if and only if left multiplication by a is an injective function $R \rightarrow R$.*

PROOF:

$\langle 1 \rangle 1$. If a is not a left-zero-divisor then left multiplication by a is injective.

$\langle 2 \rangle 1$. ASSUME: a is not a left-zero-divisor.

$\langle 2 \rangle 2$. LET: $ab = ac$

$\langle 2 \rangle 3$. $a(b - c) = 0$

$\langle 2 \rangle 4$. $b - c = 0$

$\langle 2 \rangle 5$. $b = c$

$\langle 1 \rangle 2$. If a is a left-zero-divisor then left multiplication by a is not injective.

$\langle 2 \rangle 1$. PICK $b \neq 0$ such that $ab = 0$.

$\langle 2 \rangle 2$. $ab = a0$ but $b \neq 0$

\square

10.1 Commutative Rngs

Definition 10.7 (Commutative). A rng R is *commutative* iff $\forall x, y \in R. xy = yx$.

Example 10.8. • The zero rng is commutative.

- \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are commutative.
- $2\mathbb{Z}$ is commutative.
- $\mathfrak{gl}_2(\mathbb{R})$ is not commutative.
- For any set S , the rng $\mathcal{P}S$ is commutative.
- If R is commutative then R^S is commutative.

Chapter 11

Rings

Definition 11.1 (Ring). A *ring* R is a rng such that there exists $1 \in R$, the *multiplicative identity*, such that

$$\forall x \in R. x1 = 1x = x \text{ .}$$

Example 11.2. • The zero rng is a ring with $1 = 0$.

- \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are rngs.
- $2\mathbb{Z}$ is not a ring.
- If R is a ring then $\mathfrak{gl}_n(R)$ is a ring.
- For any set S , the rng $\mathcal{P}S$ is a ring with $1 = S$.
- If R is a ring then R^S is a ring.
- $\mathfrak{sl}_n(\mathbb{R})$ is not a ring for $n > 0$.
- $\mathfrak{sl}_n(\mathbb{C})$ is not a ring for $n > 0$.
- $\mathfrak{so}_n(\mathbb{R}) = \{M \in \mathfrak{sl}_n(\mathbb{R}) : M + M^T = 0\}$ is not a ring.
- $\mathbb{Z}/n\mathbb{Z}$ is a ring.

Proposition 11.3. *In any ring R , if $0 = 1$ then R is the zero ring.*

PROOF: For any $x \in R$ we have $x = 1x = 0x = 0$. \square

Proposition 11.4. *In any ring we have $(-1)x = -x$.*

PROOF: Since

$$\begin{aligned} x + (-1)x &= 1x + (-1)x \\ &= (1 + (-1))x \\ &= 0x \\ &= 0 \end{aligned}$$

\square

11.1 Units

Definition 11.5 (Left-Unit, Right-Unit). Let R be a ring and $a \in R$. Then a is a *left-unit* iff there exists $b \in R$ such that $ab = 1$. The element a is a *right-unit* iff there exists $b \in R$ such that $ba = 1$.

An element is a *unit* iff it is a left-unit and a right-unit.

Proposition 11.6. *Let R be a ring and $a \in R$. Then a is a left-unit iff left multiplication by a is a surjective function $R \rightarrow R$.*

PROOF:

$\langle 1 \rangle 1$. If a is a left-unit then left multiplication by a is surjective.

$\langle 2 \rangle 1$. PICK $b \in R$ such that $ab = 1$.

$\langle 2 \rangle 2$. For all $c \in R$ we have $c = a(bc)$.

$\langle 1 \rangle 2$. If left multiplication by a is surjective then a is a left-unit.

PROOF: Immediate.

□

Proposition 11.7. *Let R be a ring and $a \in R$. Then a is a right-unit iff right multiplication by a is a surjective function $R \rightarrow R$.*

PROOF: Similar. □

Proposition 11.8. *No left-unit is a right-zero-divisor.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $ab = 1$ and $ca = 0$ where $c \neq 0$.

$\langle 1 \rangle 2$. $c = 0$

PROOF:

$$0 = 0b$$

$$= cab$$

$$= c1$$

$$= c$$

$\langle 1 \rangle 3$. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 11.9. *No right-unit is a left-zero-divisor.*

PROOF: Similar. □

Proposition 11.10. *The inverse of a unit is unique.*

PROOF: If $ba = 1$ and $ac = 1$ then $b = bac = c$. □

Proposition 11.11. *The units of a ring form a group under multiplication.*

PROOF:

$\langle 1 \rangle 1$. If a and b are units then ab is a unit.

PROOF: We have $b^{-1}a^{-1}ab = 1$ and $abb^{-1}a^{-1} = 1$.

⟨1⟩2. 1 is a unit.

PROOF: Since $1 \cdot 1 = 1$.

⟨1⟩3. If a is a unit then its inverse is a unit.

PROOF: Immediate from definitions.

□

Definition 11.12 (Group of Units). For any ring R , we write R^* for the group of the units of R under multiplication.

Example 11.13. The quaternionic group is a subgroup of \mathbb{H}^* .

Theorem 11.14 (Fermat's Little Theorem). *Let p be a prime number and a any integer. Then $a^p \equiv a \pmod{p}$.*

PROOF: If $p \mid a$ then $a^p \equiv a \equiv 0 \pmod{p}$. Otherwise, we have $a^{p-1} \equiv 1 \pmod{p}$ by applying Lagrange's Theorem to $(\mathbb{Z}/p\mathbb{Z})^*$. □

Example 11.15. It is not true that, if $n \mid |G|$, then G has a subgroup of order n . The group A_4 has order 12 but no subgroup of order 6.

Proposition 11.16. *If p is prime then $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.*

PROOF:

⟨1⟩1. LET: g be an element of maximal order in $(\mathbb{Z}/p\mathbb{Z})^*$.

⟨1⟩2. For all $h \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $h^{|g|} = 1$.

PROOF: Proposition 8.10.

⟨1⟩3. There are at most $|g|$ elements x such that $x^{|g|} = 1$ in $\mathbb{Z}/p\mathbb{Z}$

⟨1⟩4. $p - 1 \leq |g|$

⟨1⟩5. $|g| = p - 1$

⟨1⟩6. g generates $(\mathbb{Z}/p\mathbb{Z})^*$.

□

Example 11.17. $(\mathbb{Z}/12\mathbb{Z})^*$ is not cyclic. Its elements are 1, 5, 7 and 11 with orders 1, 2, 2 and 2.

Theorem 11.18 (Wilson's Theorem). *A positive integer p is prime if and only if $(p - 1)! \equiv -1 \pmod{p}$.*

⟨1⟩1. If p is prime then $(p - 1)! \equiv -1 \pmod{p}$.

⟨2⟩1. ASSUME: p is prime.

⟨2⟩2. $(p - 1)!$ is the product of all the elements of $(\mathbb{Z}/p\mathbb{Z})^*$

⟨2⟩3. The only element of $(\mathbb{Z}/p\mathbb{Z})^*$ with order 2 is -1 .

⟨2⟩4. $(p - 1)! \equiv -1 \pmod{p}$

PROOF: Proposition 6.20.

⟨1⟩2. If $(p - 1)! \equiv -1 \pmod{p}$ then p is prime.

⟨2⟩1. ASSUME: (

$(p - 1)! \equiv -1 \pmod{p}$)

⟨2⟩2. LET: d be a proper divisor of p .

PROVE: $d = 1$

$\langle 2 \rangle 3. d \mid (p-1)!$

$\langle 2 \rangle 4. d \mid 1$

PROOF: Since $d \mid p \mid (p-1)! + 1$.

$\langle 2 \rangle 5. d = 1$

□

Proposition 11.19. *If p and q are distinct odd primes then $(\mathbb{Z}/pq\mathbb{Z})^*$ is not cyclic.*

PROOF:

$\langle 1 \rangle 1. |(\mathbb{Z}/pq\mathbb{Z})^*| = (p-1)(q-1)$

$\langle 1 \rangle 2. \text{LET: } g \in (\mathbb{Z}/pq\mathbb{Z})^*$

PROVE: g does not have order $(p-1)(q-1)$

$\langle 1 \rangle 3. g^{(p-1)(q-1)/2} \equiv 1 \pmod{p}$

$\langle 1 \rangle 4. g^{(p-1)(q-1)/2} \equiv 1 \pmod{q}$

$\langle 1 \rangle 5. pq \mid g^{(p-1)(q-1)/2} - 1$

$\langle 1 \rangle 6. g^{(p-1)(q-1)/2} \equiv 1 \pmod{pq}$

$\langle 1 \rangle 7. |g| \mid (p-1)(q-1)/2$

□

Proposition 11.20. *For any prime p , we have $\text{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$.*

PROOF:

$\langle 1 \rangle 1. \text{LET: } \phi : \text{Aut}_{\mathbf{Grp}}(C_p) \rightarrow (\mathbb{Z}/p\mathbb{Z})^*$ be the function $\phi(\alpha) = \alpha(1)$.

PROOF: $\alpha(1)$ has order p in C_p and so is coprime with p .

$\langle 1 \rangle 2. \phi$ is a homomorphism.

PROOF: $\phi(\alpha \circ \beta) = \alpha(\beta(1)) = \alpha(\beta(1)1) = \beta(1)\alpha(1) = \phi(\alpha)\phi(\beta)$

$\langle 1 \rangle 3. \phi$ is injective.

PROOF: If $\phi(\alpha) = \phi(\beta)$ then for any n we have $\alpha(n) = n\alpha(1) = n\phi(\alpha) = n\phi(\beta) = n\beta(1) = \beta(n)$.

$\langle 1 \rangle 4. \phi$ is surjective.

PROOF: For any $r \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $r = \phi(\alpha)$ where $\alpha(n) = nr \pmod{p}$.

$\langle 1 \rangle 5. (\mathbb{Z}/p\mathbb{Z})^* \cong C_{p-1}$

□

11.2 Euler's ϕ -function

Proposition 11.21. *For n a positive integer, we have $(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$.*

PROOF:

$$m \in (\mathbb{Z}/n\mathbb{Z})^* \Leftrightarrow \exists a. am \equiv 1 \pmod{n}$$

$$\Leftrightarrow \exists a, b. am + bn = 1$$

$$\Leftrightarrow \gcd(m, n) = 1$$

□

Definition 11.22 (Euler's Totient Function). For n a positive integer, let $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$.

Proposition 11.23. *If n is an odd positive integer then $\phi(2n) = \phi(n)$.*

PROOF:

$\langle 1 \rangle 1$. LET: n be an odd positive integer.

$\langle 1 \rangle 2$. For any integer m , if $\gcd(m, n) = 1$ then $\gcd(2m + n, 2n) = 1$

PROOF: For p a prime, if $p \mid 2m + n$ and $p \mid 2n$ then $p \neq 2$ (since $2m + n$ is odd) so $p \mid n$ and hence $p \mid m$, which is a contradiction.

$\langle 1 \rangle 3$. For any integer r , if $\gcd(r, 2n) = 1$ then $\gcd(\frac{r+n}{2}, n) = 1$

PROOF: If $p \mid n$ and $p \mid \frac{r+n}{2}$ then $p \mid r + n$ so $p \mid r$ which is a contradiction.

$\langle 1 \rangle 4$. The function that maps m to $2m + n$ is a bijection between $(\mathbb{Z}/n\mathbb{Z})^*$ and $(\mathbb{Z}/2n\mathbb{Z})^*$.

□

Theorem 11.24. *For any positive integer n we have*

$$\sum_{m>0, m \mid n} \phi(m) = n .$$

PROOF:

$\langle 1 \rangle 1$. Define $\chi : \{0, 1, \dots, n-1\} \rightarrow \{(m, d) : m > 0, m \mid n, d \text{ generates } \langle n/m \rangle\}$
by: $\chi(x) = (\gcd(x, n), x)$.

$\langle 1 \rangle 2$. χ is injective.

$\langle 1 \rangle 3$. χ is surjective.

PROOF: Given (m, d) such that d generates $\langle n/m \rangle$ we have $\chi(d) = (m, d)$.

$\langle 1 \rangle 4$. $n = \sum_{m>0, m \mid n} \phi(m)$

PROOF: Since $\langle n/m \rangle \cong C_m$ and so has $\phi(m)$ generators.

□

Proposition 11.25. *For any positive integers a and n , we have $n \mid \phi(a^n - 1)$.*

PROOF: Since the order of a is n in $(\mathbb{Z}/(a^n - 1)\mathbb{Z})^*$. □

Theorem 11.26 (Euler's Theorem). *For any coprime integers a and n we have $a^{\phi(n)} \equiv a \pmod{n}$.*

PROOF: Immediate from Lagrange's Theorem. □

Proposition 11.27.

$$|\text{Aut}_{\mathbf{Grp}}(C_n)| = \phi(n)$$

PROOF: An automorphism α is determined by $\alpha(1)$ which is any element of order n , and g has order n iff $\gcd(g, n) = 1$. □

Example 11.28.

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}) \cong C_2$$

PROOF: The only automorphisms are the identity and multiplication by -1. □

11.3 Nilpotent Elements

Definition 11.29 (Nilpotent). Let R be a ring and $a \in R$. Then a is *nilpotent* iff there exists n such that $a^n = 0$.

Proposition 11.30. *Let R be a ring and $a, b \in R$. If a and b are nilpotent and $ab = ba$ then $a + b$ is nilpotent.*

PROOF:

$\langle 1 \rangle 1$. PICK m and n such that $a^m = b^n = 0$.

$\langle 1 \rangle 2$. $(a + b)^{m+n} = 0$

PROOF: Since $(a + b)^{m+n} = \sum_k \binom{m+n}{k} a^k b^{m+n-k}$ and every term in this sum is 0 since, for every k , either $k \geq m$ or $m + n - k \geq n$.

□

Proposition 11.31. *m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$ if and only if m is divisible by all the prime factors of n .*

PROOF:

$\langle 1 \rangle 1$. If m is nilpotent then m is divisible by all the prime factors of n .

$\langle 2 \rangle 1$. ASSUME: $m^a \equiv 0 \pmod{n}$

$\langle 2 \rangle 2$. For every prime p , if $p \mid n$ then $p \mid m^a$.

$\langle 2 \rangle 3$. For every prime p , if $p \mid n$ then $p \mid m$.

$\langle 1 \rangle 2$. If m is divisible by all the prime factors of n then m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

$\langle 2 \rangle 1$. ASSUME: m is divisible by all the prime factors of n .

$\langle 2 \rangle 2$. LET: a be the largest number such that $p^a \mid n$ for some prime p .

$\langle 2 \rangle 3$. For every prime p that divides n we have $p^a \mid m^a$

$\langle 2 \rangle 4$. $n \mid m^a$

$\langle 2 \rangle 5$. $m^a \equiv 0 \pmod{n}$

$\langle 2 \rangle 6$. m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

□

Chapter 12

Monoid Rings

Definition 12.1 (Monoid Ring). Let R be a ring and M a monoid. Define $R[M]$ to be the ring whose elements are the families $\{a_m\}_{m \in M}$ such that $a_m = 0$ for all but finitely many $m \in M$, written

$$\sum_{m \in M} a_m m ,$$

under

$$\begin{aligned} \sum_m a_m m + \sum_m b_m m &= \sum_m (a_m + b_m) m \\ \left(\sum_m a_m m \right) \left(\sum_m b_m m \right) &= \sum_{m \in M} \sum_{m_1 m_2 = m} a_{m_1} b_{m_2} m \end{aligned}$$

12.1 Polynomials

Definition 12.2 (Polynomial). Let R be a ring. The ring of *polynomials* $R[x]$ is $R[\mathbb{N}]$. We write

$$\sum_n a_n x^n \text{ for } \sum_n a_n n .$$

Concretely, a *polynomial* in R is a sequence (a_n) in R such that there exists N such that $\forall n \geq N. a_n = 0$. We write the polynomial as

$$\sum_{n=0}^{N-1} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1} .$$

We write $R[x]$ for the set of all polynomials in R .

Define addition and multiplication on $R[x]$ by

$$\begin{aligned}\sum_n a_n x^n + \sum_n b_n x^n &= \sum_n (a_n + b_n) x^n \\ \left(\sum_n a_n x^n \right) \left(\sum_n b_n x^n \right) &= \sum_n \sum_{i+j=n} a_i b_j x^n\end{aligned}$$

A *constant* is a polynomial of the form $a + 0x + 0x^2 + \cdots$ for some $a \in R$.

Proposition 12.3. *For any ring R , the set of polynomials $R[x]$ is a ring.*

PROOF: Easy. \square

Definition 12.4 (Degree). The *degree* of a polynomial $\sum_n a_n x^n$ is the largest integer d such that $a_d \neq 0$. We take the degree of the zero polynomial to be $-\infty$.

Proposition 12.5. *Let R be a ring and $f, g \in R[x]$ be nonzero polynomials. Then*

$$\deg(f + g) \leq \max(\deg f, \deg g) .$$

PROOF: If $a_n + b_n \neq 0$ then $a_n \neq 0$ or $b_n \neq 0$. \square

12.2 Power Series

Definition 12.6 (Power Series). Let R be a ring. A *power series* in R is a sequence (a_n) in R . We write the power series as

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

We write $R[[x]]$ for the set of all power series in R .

Define addition and multiplication on $R[[x]]$ by

$$\begin{aligned}\sum_n a_n x^n + \sum_n b_n x^n &= \sum_n (a_n + b_n) x^n \\ \left(\sum_n a_n x^n \right) \left(\sum_n b_n x^n \right) &= \sum_n \sum_{i+j=n} a_i b_j x^n\end{aligned}$$

Proposition 12.7. *For any ring R , the set of power series $R[[x]]$ is a ring.*

PROOF: Easy. \square

Chapter 13

Integral Domains

Definition 13.1 (Integral Domain). An *integral domain* is a non-trivial commutative ring with no nonzero zero-divisors.

Example 13.2. \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are integral domains.

Proposition 13.3. $\mathbb{Z}/n\mathbb{Z}$ is an integral domain if and only if n is prime.

PROOF:

$$\begin{aligned} n \text{ is prime} &\Leftrightarrow \forall a, b \in \mathbb{Z} (n \mid ab \Rightarrow n \mid a \vee n \mid b) \\ &\Leftrightarrow \forall a, b \in \mathbb{Z}/n\mathbb{Z} (ab \cong 0(\bmod n) \Rightarrow a \cong 0(\bmod n) \vee b \cong 0(\bmod n)) \\ &\Leftrightarrow \mathbb{Z}/n\mathbb{Z} \text{ is an integral domain} \quad \square \end{aligned}$$

Proposition 13.4. In an integral domain, if $x^2 = 1$ then $x = \pm 1$.

PROOF: We have $x^2 - 1 = (x - 1)(x + 1) = 0$ so $x - 1 = 0$ or $x + 1 = 0$. \square

Proposition 13.5. Let R be an integral domain and $f, g \in R[x]$. Then

$$\deg(fg) = \deg f + \deg g$$

PROOF:

$\langle 1 \rangle 1$. LET: $f = \sum_n a_n x^n$ and $g = \sum_n b_n x^n$.

$\langle 1 \rangle 2$. LET: $d = \deg f$ and $e = \deg g$.

$\langle 1 \rangle 3$. The $d + e$ th term of fg is

$$a_d b_e x^{d+e}$$

which is non-zero.

$\langle 1 \rangle 4$. For $n > d + e$ the n th term of fg is 0.

\square

Corollary 13.5.1. If R is an integral domain then $R[x]$ is an integral domain.

Example 13.6. We cannot remove the hypothesis that R is an integral domain. In $(\mathbb{Z}/6\mathbb{Z})[x]$ we have

$$\begin{aligned}\deg(1 + 2x) &= 1 \\ \deg(1 + 3x) &= 1 \\ \deg((1 + 2x)(1 + 3x)) &= \deg(1 + 5x + 6x^2) \\ &= \deg(1 + 5x) \\ &= 1\end{aligned}$$

Chapter 14

Unique Factorization Domains

Example 14.1. \mathbb{Z} is a UFD.

Chapter 15

Principal Ideal Domains

Example 15.1. \mathbb{Z} is a PID.

Chapter 16

Euclidean Domains

Example 16.1. \mathbb{Z} is a Euclidean domain.

Chapter 17

Division Rings

Definition 17.1 (Division Ring). A *division ring* is a ring in which every nonzero element is a two-sided unit.

Example 17.2. The quaternions form a division ring, with the inverse of a non-zero element $a + bi + cj + dk$ being

$$\frac{1}{a^2 + b^2 + c^2 + d^2}(a - bi - cj - dk) .$$

Example 17.3. For any ring R , the ring of polynomials $R[x]$ is not a division ring, since x has no inverse.

Part IV

Field Theory

Chapter 18

Fields

Definition 18.1 (Field). A *field* is a non-trivial commutative division ring.

Example 18.2. \mathbb{Q} , \mathbb{R} and \mathbb{C} are fields.

Proposition 18.3. *Every field is an integral domain.*

PROOF: By Propositions 11.8 and 11.9. \square

Example 18.4. The converse does not hold: \mathbb{Z} is an integral domain but not a field.

Proposition 18.5. *Every finite integral domain is a field.*

PROOF: In a finite integral domain, multiplication by any non-zero element is injective, hence surjective. \square

Corollary 18.5.1. *For any positive integer n , the following are equivalent:*

- n is prime.
- $\mathbb{Z}/n\mathbb{Z}$ is an integral domain.
- $\mathbb{Z}/n\mathbb{Z}$ is a field.

Theorem 18.6 (Wedderburn's Little Theorem). *Every finite division ring is a field.*

Definition 18.7. For any prime p and positive integer r , define a multiplication on $(\mathbb{Z}/p\mathbb{Z})^r$ that makes this group into a field by:

Part V

Linear Algebra

Definition 18.8. Let $\text{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ real matrices.
 $\text{GL}_n(\mathbb{R})$ acts on \mathbb{R}^n by matrix multiplication.

Definition 18.9. Let $\text{GL}_n(\mathbb{C})$ be the group of invertible $n \times n$ complex matrices.
 $\text{GL}_n(\mathbb{C})$ acts on \mathbb{C}^n by matrix multiplication.

Definition 18.10. Let $\text{SL}_n(\mathbb{R}) = \{M \in \text{GL}_n(\mathbb{R}) : \det M = 1\}$.

Proposition 18.11. $\text{SL}_n(\mathbb{R})$ is a normal subgroup of $\text{GL}_n(\mathbb{R})$.

PROOF: If $\det M = 1$ then $\det(AMA^{-1}) = (\det A)(\det M)(\det A)^{-1} = 1$. \square

Proposition 18.12.

$$\text{GL}_n(\mathbb{R})/\text{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$$

Definition 18.13. Let $\text{SL}_n(\mathbb{C}) = \{M \in \text{GL}_n(\mathbb{C}) : \det M = 1\}$.

Definition 18.14. Let $\text{O}_n(\mathbb{R}) = \{M \in \text{GL}_n(\mathbb{R}) : MM^T = M^T M = I_n\}$.

Proposition 18.15. The action of $\text{O}_n(\mathbb{R})$ on \mathbb{R}^n preserves lengths and angles.

Definition 18.16. Let $\text{SO}_n(\mathbb{R}) = \{M \in \text{O}_n(\mathbb{R}) : \det M = 1\}$.

Definition 18.17. Let $\text{U}_n(\mathbb{C}) = \{M \in \text{GL}_n(\mathbb{C}) : MM^\dagger = M^\dagger M = I_n\}$.

Definition 18.18. Let $\text{SU}_n(\mathbb{C}) = \{M \in \text{U}_n(\mathbb{C}) : \det M = 1\}$.

Proposition 18.19. Every matrix in $\text{SU}_2(\mathbb{C})$ can be written in the form

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 + c^2 + d^2 = 1$.

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SU}_2(\mathbb{C})$$

$$\langle 1 \rangle 2. M^{-1} = M^\dagger$$

$$\langle 1 \rangle 3. \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

$$\langle 1 \rangle 4. \text{ LET: } \alpha = a + bi \text{ and } \beta = c + di.$$

$$\langle 1 \rangle 5. \delta = \bar{\alpha} = a - bi$$

$$\langle 1 \rangle 6. \gamma = -\bar{\beta} = -c + di$$

$$\langle 1 \rangle 7. \det M = a^2 + b^2 + c^2 + d^2 = 1$$

\square

Corollary 18.19.1. $\text{SU}_2(\mathbb{C})$ is simply connected.

Corollary 18.19.2.

$$\text{SO}_3(\mathbb{R}) \cong \text{SU}_2(\mathbb{C})/\{I, -I\}$$

PROOF: The function that maps $\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$ to $\begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(ac + bd) \\ 2(ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(cd - ab) \\ 2(bd - ac) & 2(ab + cd) & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$

is a surjective homomorphism with kernel $\{I, -I\}$. \square

Corollary 18.19.3. The fundamental group of $\text{SO}_3(\mathbb{R})$ is C_2 .