

# Mathematics

Robin Adams

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# Chapter 1

## Primitive Terms and Axioms

### 1.1 Primitive Terms

Let there be *sets*. We write  $A : \text{Set}$  for:  $A$  is a set.

For any set  $A$ , let there be *elements* of  $A$ . We write  $a : \text{El}(A)$  for:  $a$  is an element of  $A$ .

For any sets  $A$  and  $B$ , let there be *functions* from  $A$  to  $B$ . We write  $f : A \rightarrow B$  iff  $f$  is a function from  $A$  to  $B$ .

For any function  $f : A \rightarrow B$  and element  $a : \text{El}(A)$ , let there be an element  $f(a) : \text{El}(B)$ , the *value* of the function  $f$  at the *argument*  $a$ .

### 1.2 Axioms

**Axiom Schema 1.1** (Choice). *Let  $P[X, Y, x, y]$  be a formula where  $X$  and  $Y$  are set variables,  $x : \text{El}(X)$  and  $y : \text{El}(Y)$ . Then the following is an axiom.*

*Let  $A$  and  $B$  be sets. Assume that, for all  $a : \text{El}(A)$ , there exists  $b : \text{El}(B)$  such that  $P[A, B, a, b]$ . Then there exists a function  $f : A \rightarrow B$  such that  $\forall a : \text{El}(A). P[A, B, a, f(a)]$ .*

**Axiom 1.2** (Pairing). *For any sets  $A$  and  $B$ , there exists a set  $A \times B$ , the Cartesian product of  $A$  and  $B$ , and functions  $\pi_1 : A \times B \rightarrow A$  and  $\pi_2 : A \times B \rightarrow B$  such that, for all  $a : \text{El}(A)$  and  $b : \text{El}(B)$ , there exists a unique  $(a, b) : \text{El}(A \times B)$  such that  $\pi_1(a, b) = a$  and  $\pi_2(a, b) = b$ .*

**Definition 1.3** (Injective). A function  $f : A \rightarrow B$  is *injective* or an *injection* iff, for all  $x, y : \text{El}(A)$ , if  $f(x) = f(y)$  then  $x = y$ .

**Axiom Schema 1.4** (Separation). *For every property  $P[X, x]$  where  $X$  is a set variable and  $x : \text{El}(X)$ , the following is an axiom:*

For every set  $A$ , there exists a set  $S = \{x : \text{El}(A) \mid P[A, x]\}$  and an injection  $i : S \rightarrow A$  such that, for all  $x : \text{El}(A)$ , we have

$$(\exists y : S. i(y) = x) \Leftrightarrow P[A, x] .$$

**Axiom 1.5** (Infinity). *There exists a set  $\mathbb{N}$ , an element  $0 : \text{El}(\mathbb{N})$ , and a function  $s : \mathbb{N} \rightarrow \mathbb{N}$  such that:*

- $\forall n : \text{El}(\mathbb{N}). s(n) \neq 0$
- $\forall m, n : \text{El}(\mathbb{N}). s(m) = s(n) \Rightarrow m = n.$

### 1.3 Consequences of the Axioms

#### 1.3.1 Definitions

**Definition 1.6.** Let  $f, g : A \rightarrow B$ . We say  $f$  and  $g$  are *equal*,  $f = g$ , iff  $\forall x : \text{El}(A). f(x) = g(x)$ .

**Definition 1.7** (Surjective). A function  $f : A \rightarrow B$  is *surjective* iff, for all  $y : \text{El}(B)$ , there exists  $x : \text{El}(A)$  such that  $f(x) = y$ .

**Definition 1.8** (Bijective). A function  $f : A \rightarrow B$  is *bijective* or a *bijection* iff it is injective and surjective.

Sets  $A$  and  $B$  are *equinumerous*,  $A \approx B$ , iff there exists a bijection between them.

If we prove there exists a set  $X$  such that  $P(X)$ , and that any two sets that satisfy  $P$  are bijective, then we may introduce a constant  $C$  and define "Let  $C$  be the set such that  $P(C)$ ".

#### 1.3.2 The Empty Set

**Theorem 1.9.** *There exists a set which has no elements.*

PROOF:

$\langle 1 \rangle 1$ . PICK a set  $A$

PROOF: By the Axiom of Infinity, a set exists.

$\langle 1 \rangle 2$ . LET:  $S = \{x : \text{El}(A) \mid \perp\}$  with injection  $i : S \rightarrow A$

PROOF: Axiom of Separation.

$\langle 1 \rangle 3$ .  $S$  has no elements.

□

**Theorem 1.10.** *If  $E$  and  $E'$  have no elements then  $E \approx E'$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $E$  and  $E'$  have no elements.

$\langle 1 \rangle 2$ . PICK a function  $F : E \rightarrow E'$ .

PROOF: Axiom of Choice since vacuously  $\forall x : \text{El}(E). \exists y : \text{El}(E'). \top$ .

$\langle 1 \rangle 3$ .  $F$  is injective.

PROOF: Vacuously, for all  $x, y : \text{El}(E)$ , if  $F(x) = F(y)$  then  $x = y$ .

$\langle 1 \rangle 4$ .  $F$  is surjective.

PROOF: Vacuously, for all  $y : \text{El}(E)$ , there exists  $x : \text{El}(E)$  such that  $F(x) = y$ .

□

**Definition 1.11** (Empty Set). The *empty set*  $\emptyset$  is the set with no elements.

### 1.3.3 The Singleton

**Theorem 1.12.** *There exists a set that has exactly one element.*

PROOF:

$\langle 1 \rangle 1$ . PICK a set  $A$  that has an element.

PROOF: By the Axiom of Infinity, there exists a set that has an element.

$\langle 1 \rangle 2$ . PICK  $a : \text{El}(A)$

$\langle 1 \rangle 3$ . PICK a set  $S$  and injection  $i : S \rightarrow A$  such that, for all  $x : \text{El}(A)$ , there exists  $s : \text{El}(S)$  such that  $s = x$  if and only if  $x = a$

$\langle 1 \rangle 4$ .  $S$  has exactly one element.

□

**Theorem 1.13.** *If  $A$  and  $B$  both have exactly one element then  $A \approx B$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $B$  both have exactly one element  $a$  and  $b$  respectively.

$\langle 1 \rangle 2$ . LET:  $F : A \rightarrow B$  be the function such that, for all  $x : \text{El}(A)$ , we have  
 $(x = a \wedge F(x) = b)$

$\langle 1 \rangle 3$ .  $F$  is a bijection.

□

**Definition 1.14** (Singleton). Let  $1$  be the set that has exactly one element.  
 Let  $*$  be its element.

### 1.3.4 Subsets

**Definition 1.15** (Subset). A *subset* of a set  $A$  is a relation  $1 \rightarrow A$ .

Given  $S : 1 \rightarrow A$  and  $a : \text{El}(A)$ , we write  $a \in S$  for  $*Sa$ .

**Theorem Schema 1.16.** *For any property  $P[X, x]$  where  $X$  is a set variable and  $x : \text{El}(X)$ , the following is a theorem:*

*For any set  $A$ , there exists a set  $B$  and injection  $i : B \rightarrow A$  such that, for all  $x : \text{El}(A)$ , we have  $P[A, x]$  if and only if there exists  $b : \text{El}(B)$  such that  $i(b) = x$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $S : 1 \rightarrow A$  be the relation such that, for all  $e : \text{El}(1)$  and  $a : \text{El}(A)$ , we have  $eSa$  if and only if  $P[A, a]$ .

PROOF: Axiom of Comprehension.

$\langle 1 \rangle 2$ . LET:  $B$  be the tabulation of  $S$  with projections  $p : B \rightarrow 1$  and  $i : B \rightarrow A$ .  
 PROOF: Axiom of Tabulations.  
 $\langle 1 \rangle 3$ .  $i$  is injective.  
 $\langle 2 \rangle 1$ . LET:  $r, s : \text{El}(B)$   
 $\langle 2 \rangle 2$ . ASSUME:  $i(r) = i(s)$   
 $\langle 2 \rangle 3$ .  $p(r) = p(s)$   
 PROOF: Since 1 has only one element.  
 $\langle 2 \rangle 4$ .  $r = s$   
 PROOF: Axiom of Tabulations.  
 $\langle 1 \rangle 4$ . For all  $x : \text{El}(A)$ , we have  $P[A, x]$  if and only if there exists  $b : \text{El}(B)$  such that  $i(b) = x$ .  
 $\langle 2 \rangle 1$ . LET:  $x : \text{El}(A)$   
 $\langle 2 \rangle 2$ . If  $P[A, x]$  then there exists  $b : \text{El}(B)$  such that  $i(b) = x$   
 $\langle 3 \rangle 1$ . ASSUME:  $P[A, x]$   
 $\langle 3 \rangle 2$ .  $*Sx$   
 PROOF:  $\langle 1 \rangle 1$   
 $\langle 3 \rangle 3$ . There exists  $b : \text{El}(B)$  such that  $p(b) = *$  and  $i(b) = x$   
 PROOF: Axiom of Tabulations.  
 $\langle 2 \rangle 3$ . For all  $b : \text{El}(B)$  we have  $P[A, i(b)]$   
 $\langle 3 \rangle 1$ . LET:  $b : \text{El}(B)$   
 $\langle 3 \rangle 2$ .  $p(b)Si(b)$   
 PROOF: Axiom of Tabulations.  
 $\langle 3 \rangle 3$ .  $P[A, i(b)]$   
 PROOF:  $\langle 1 \rangle 1$

□

## 1.4 Composition

**Definition 1.17** (Composite). Let  $\phi : A \rightharpoonup B$  and  $\psi : B \rightharpoonup C$ . The *composite*  $\psi \circ \phi : A \rightharpoonup C$  is the relation such that  $a(\psi \circ \phi)c$  iff there exists  $b$  such that  $a\phi b$  and  $b\psi c$ .

**Definition 1.18** (Identity). For any set  $A$ , the *identity* function  $\text{id}_A : A \rightarrow A$  is the function defined by  $\text{id}_A(a) = a$ .

**Theorem 1.19.** *Composition of relations is associative, and the identity function is an identity for composition. The composite of functions is a function. The composite of injective functions is injective. The composite of surjective functions is surjective. The composite of bijections is a bijection. A function  $f : A \rightarrow B$  is a bijection iff there exists a function  $f^{-1} : B \rightarrow A$  such that  $f^{-1}f = \text{id}_A$  and  $ff^{-1} = \text{id}_B$ , in which case  $f^{-1}$  is unique.*

## 1.5 Axioms Part Two

**Axiom 1.20** (Power Set). *For any set  $A$ , there exists a set  $\mathcal{P}A$ , the power set of  $A$ , and a relation  $\in : A \rightharpoonup \mathcal{P}A$ , called membership, such that, for any subset*



$S$  of  $A$ , there exists a unique  $\bar{S} \in \mathcal{P}A$  such that, for all  $x \in A$ , we have  $x \in \bar{S}$  if and only if  $x \in S$ .

We usually write just  $S$  for  $\bar{S}$ .

**Axiom Schema 1.21** (Collection). Let  $P[X, Y, x]$  be a formula with set variables  $X$  and  $Y$  and an element variable  $x \in X$ . Then the following is an axiom.

For any set  $A$ , there exists a set  $B$ , a function  $p : B \rightarrow A$ , a set  $Y$  and a relation  $M : B \rightarrowtail Y$  such that:

- $\forall b \in B. P[A, \{y \in Y : bMy\}, p(b)]$
- For all  $a \in A$ , if  $\exists Y. P[A, Y, a]$ , then there exists  $b \in B$  such that  $a = p(b)$ .

**Definition 1.22** (Universe). Let  $E : U \rightarrowtail X$  be a relation. Let us say that a set  $A$  is *small* iff there exists  $u \in U$  such that  $A \approx \{x \in X : uEx\}$ .

Then  $(U, X, E)$  form a *universe* if and only if:

- $\mathbb{N}$  is  $U$ -small.
- For any  $U$ -small sets  $A$  and  $B$  and relation  $R : A \rightarrowtail B$ , the tabulation of  $R$  is  $U$ -small.
- If  $A$  is  $U$ -small then so is  $\mathcal{P}A$
- Let  $f : A \rightarrow B$  be a function. If  $B$  is  $U$ -small and  $f^{-1}(b)$  is  $U$ -small for all  $b \in B$ , then  $A$  is  $U$ -small.
- If  $p : B \twoheadrightarrow A$  is a surjective function such that  $A$  is  $U$ -small, then there exists a  $U$ -small set  $C$ , a surjection  $q : C \twoheadrightarrow A$ , and a function  $f : C \rightarrow B$  such that  $q = pf$ .

**Axiom 1.23** (Universe). *There exists a universe.*

Let  $E : U \rightarrowtail X$  be a universe. We shall say a set is *small* iff it is  $U$ -small, and *large* otherwise.

## 1.6 Cartesian Product

**Definition 1.24** (Cartesian Product). Let  $A$  and  $B$  be sets. The *Cartesian product* of  $A$  and  $B$ ,  $A \times B$ , is the tabulation of the relation  $A \rightarrowtail B$  that holds for all  $a \in A$  and  $b \in B$ . The associated functions  $\pi_1 : A \times B \rightarrow A$  and  $\pi_2 : A \times B \rightarrow B$  are called the *projections*.

Given  $a \in A$  and  $b \in B$ , we write  $(a, b)$  for the unique element of  $A \times B$  such that  $\pi_1(a, b) = a$  and  $\pi_2(a, b) = b$ .

## 1.7 Quotient Sets

**Proposition 1.25.** *Let  $\sim$  be an equivalence relation on  $X$ . Then there exists a set  $X/\sim$ , the quotient set of  $X$  with respect to  $\sim$ , and a surjective function  $\pi : X \twoheadrightarrow X/\sim$ , the canonical projection, such that, for all  $x, y : \text{El}(X)$ , we have  $x \sim y$  if and only if  $\pi(x) = \pi(y)$ .*

*Further, if  $p : X \twoheadrightarrow Q$  is another quotient with respect to  $\sim$ , then there exists a unique bijection  $\phi : X/\sim \approx Q$  such that  $\phi \circ \pi = p$ .*

## Chapter 2

# Category Theory

### 2.1 Categories

**Definition 2.1.** A *category*  $\mathcal{C}$  consists of:

- a set  $\text{Ob}(\mathcal{C})$  of *objects*
- for any objects  $X$  and  $Y$ , a set  $\text{Mor}(X, Y)$  of *morphisms* from  $X$  to  $Y$ . We write  $f : X \rightarrow Y$  for  $f : \text{El}(\text{Mor}(X, Y))$ .
- for any objects  $X, Y$  and  $Z$ , a function  $\circ : \text{Mor}(Y, Z) \times \text{Mor}(X, Y) \rightarrow \text{Mor}(X, Z)$ , called *composition*.

such that:

- Given  $f : X \rightarrow Y, g : Y \rightarrow Z$  and  $h : Z \rightarrow W$ , we have  $h \circ (g \circ f) = (h \circ g) \circ f$
- For any object  $X$ , there exists a morphism  $\text{id}_X : X \rightarrow X$ , the *identity morphism* on  $X$ , such that:
  - for any object  $Y$  and morphism  $f : Y \rightarrow X$  we have  $\text{id}_X \circ f = f$
  - for any object  $Y$  and morphism  $f : X \rightarrow Y$  we have  $f \circ \text{id}_X = f$

**Definition 2.2.** Let **Set** be the category of small sets and functions.

Define the opposite category.

### 2.2 Functors

**Definition 2.3** (Functor). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of:

- a function  $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$
- for every morphism  $f : A \rightarrow B$  in  $\mathcal{C}$ , a morphism  $Ff : FA \rightarrow FB$  in  $\mathcal{D}$

such that:

- for all  $A : \text{El}(\text{Ob}(C))$  we have  $F\text{id}_A = \text{id}_{FA}$
- for any morphism  $f : A \rightarrow B$  and  $g : B \rightarrow C$  in  $\mathcal{C}$ , we have  $F(g \circ f) = Fg \circ Ff$

Define the identity functor, constant functors.

Functors preserve isomorphisms.

## Chapter 3

# Group Theory

**Definition 3.1.** Let  $\mathbf{Grp}$  be the category of small groups and group homomorphisms.

**Definition 3.2.** We identify any group  $G$  with the category with one object whose morphisms are the elements of  $G$  with composition given by the multiplication in  $G$ .



## Chapter 4

# Linear Algebra

**Definition 4.1.** For any field  $K$ , let  $\mathbf{Vect}_K$  be the category of small vector spaces over  $K$  and linear transformations.

Dual space functor  $\mathbf{Vect}_K^{\mathrm{op}} \rightarrow \mathbf{Vect}_K$ .





# Chapter 5

## Topology

### 5.1 Topological Spaces

**Definition 5.1** (Topological Space). Let  $X$  be a set and  $\mathcal{O} \subseteq \mathcal{P}X$ . Then we say  $(X, \mathcal{O})$  is a *topological space* iff:

- For any  $\mathcal{U} \subseteq \mathcal{O}$  we have  $\bigcup \mathcal{U} \in \mathcal{O}$ .
- For any  $U, V \in \mathcal{O}$  we have  $U \cap V \in \mathcal{O}$ .
- $X \in \mathcal{O}$

We call  $\mathcal{O}$  the *topology* of the topological space, and call its elements *open* sets. We shall often write  $X$  for the topological space  $(X, \mathcal{O})$ .

**Definition 5.2** (Closed Set). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is *closed* iff  $X - A$  is open.

**Proposition 5.3.** *A set  $B$  is open if and only if  $X - B$  is closed.*

**Proposition 5.4.** *Let  $X$  be a set and  $\mathcal{C} \subseteq \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  on  $X$  such that  $\mathcal{C}$  is the set of closed sets if and only if:*

- For any  $\mathcal{D} \subseteq \mathcal{C}$  we have  $\bigcap \mathcal{D} \in \mathcal{C}$
- For any  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .
- $\emptyset \in \mathcal{C}$

*In this case,  $\mathcal{O}$  is unique and is given by  $\mathcal{O} = \{X - C : C \in \mathcal{C}\}$ .*

**Definition 5.5** (Neighbourhood). Let  $X$  be a topological space,  $x \in X$  and  $U \subseteq X$ . Then  $U$  is a *neighbourhood* of  $x$ , and  $x$  is an *interior* point of  $U$ , iff there exists an open set  $V$  such that  $x \in V \subseteq U$ .

**Proposition 5.6.** *A set  $B$  is open if and only if it is a neighbourhood of each of its points.*

**Proposition 5.7.** *Let  $X$  be a set and  $\mathcal{N} : X \rightarrow \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  on  $X$  such that, for all  $x \in X$ , we have  $\mathcal{N}_x$  is the set of neighbourhoods of  $x$ , if and only if:*

- *For all  $x \in X$  and  $N \in \mathcal{N}_x$  we have  $x \in N$*
- *For all  $x \in X$  we have  $X \in \mathcal{N}_x$*
- *For all  $x \in X$ ,  $N \in \mathcal{N}_x$  and  $V \subseteq \mathcal{P}X$ , if  $N \subseteq V$  then  $V \in \mathcal{N}_x$*
- *For all  $x \in X$  and  $M, N \in \mathcal{N}_x$  we have  $M \cap N \in \mathcal{N}_x$*
- *For all  $x \in X$  and  $N \in \mathcal{N}_x$ , there exists  $M \in \mathcal{N}_x$  such that  $M \subseteq N$  and  $\forall y \in M. M \in \mathcal{N}_y$ .*

*In this case,  $\mathcal{O}$  is unique and is given by  $\mathcal{O} = \{U : \forall x \in U. U \in \mathcal{N}_x\}$ .*

**Definition 5.8** (Exterior Point). Let  $X$  be a topological space,  $x \in X$  and  $B \subseteq X$ . Then  $x$  is an *exterior point* of  $B$  iff  $B - X$  is a neighbourhood of  $x$ .

**Definition 5.9** (Boundary Point). Let  $X$  be a topological space,  $x \in X$  and  $B \subseteq X$ . Then  $x$  is a *boundary point* of  $B$  iff it is neither an interior point nor an exterior point of  $B$ .

**Definition 5.10** (Interior). Let  $X$  be a topological space and  $B \subseteq X$ . The *interior* of  $B$ ,  $B^\circ$ , is the set of all interior points of  $B$ .

**Proposition 5.11.** *The interior of  $B$  is the union of all the open sets included in  $B$ .*

**Definition 5.12** (Closure). Let  $X$  be a topological space and  $B \subseteq X$ . The *closure* of  $B$ ,  $\overline{B}$ , is the set of all points that are not exterior points of  $B$ .

**Proposition 5.13.** *The closure of  $B$  is the intersection of all the closed sets that include  $B$ .*

**Proposition 5.14.** *A set  $B$  is open iff  $X - B = \overline{X - B}$ .*

**Proposition 5.15** (Kuratowski Closure Axioms). *Let  $X$  be a set and  $- : \mathcal{P}X \rightarrow \mathcal{P}X$ . Then there exists a topology  $\mathcal{O}$  such that, for all  $B \subseteq X$ ,  $\overline{B}$  is the closure of  $B$ , if and only if:*

- $\overline{\emptyset} = \emptyset$
- *For all  $A \subseteq X$  we have  $A \subseteq \overline{A}$*
- *For all  $A \subseteq X$  we have  $\overline{\overline{A}} = \overline{A}$*
- *For all  $A, B \subseteq X$  we have  $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

*In this case,  $\mathcal{O}$  is unique and is defined by  $\mathcal{O} = \{U : X - U = \overline{X - U}\}$ .*

### 5.1.1 Subspaces

**Definition 5.16** (Subspace). Let  $X$  be a topological space and  $X_0 \subseteq X$ . The *subspace topology* on  $X_0$  is  $\{U \cap X_0 : U \text{ is open in } X\}$ .

**Example 5.17.** The *unit sphere*  $S^2$  is  $\{x \in \mathbb{R}^3 : \|x\| = 1\}$  as a subspace of  $\mathbb{R}^3$ .

### 5.1.2 Topological Disjoint Union

**Definition 5.18.** Let  $X$  and  $Y$  be topological spaces. The *disjoint union* is  $X + Y$  where  $U \subseteq X + Y$  is open if and only if  $\kappa_1^{-1}(U)$  is open in  $X$  and  $\kappa_2^{-1}(U)$  is open in  $Y$ .

### 5.1.3 Product Topology

**Definition 5.19** (Product Topology). Let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be a family of topological spaces. The *product topology* on  $\prod_{\lambda \in \Lambda} X_\lambda$  is the coarsest topology such that every projection onto  $X_\lambda$  is continuous.

### 5.1.4 Bases

**Definition 5.20** (Basis). Let  $X$  be a topological space. A *basis* for the topology on  $X$  is a set of open sets  $\mathcal{B}$  such that every open set is the union of a subset of  $\mathcal{B}$ .

### 5.1.5 Subbases

**Definition 5.21** (Subbasis). Let  $X$  be a topological space. A *subbasis* for the topology on  $X$  is a subset  $\mathcal{S} \subseteq \mathcal{P}X$  such that every open set is a union of finite intersections of  $\mathcal{S}$ .

**Definition 5.22** (Space with Basepoint). A *space with basepoint* is a pair  $(X, x)$  where  $X$  is a topological space and  $x \in \text{El}(X)$ .

### 5.1.6 Countability Axioms

**Definition 5.23** (Neighbourhood Basis). Let  $X$  be a topological space and  $x_0 \in \text{El}(X)$ . A *neighbourhood basis* of  $x_0$  is a set  $\mathcal{U}$  of neighbourhoods of  $x_0$  such that every neighbourhood of  $x_0$  includes an element of  $\mathcal{U}$ .

**Definition 5.24** (First Countable). A topological space is *first countable* iff every point has a countable neighbourhood basis.

**Definition 5.25** (Second Countable). A topological space is *second countable* iff it has a countable basis.

Every second countable space is first countable.

A subspace of a first countable space is first countable.

A subspace of a second countable space is second countable.

$\mathbb{R}^n$  is second countable.

An uncountable discrete space is first countable but not second countable.

**Proposition 5.26.** *Let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be a family of topological spaces such that no  $X_\lambda$  is indiscrete. If  $\Lambda$  is uncountable, then  $\prod_{\lambda \in \Lambda} X_\lambda$  is not first countable.*

PROOF:

$\langle 1 \rangle 1$ . For all  $\lambda : \text{El}(\Lambda)$ , PICK  $U_\lambda$  open in  $X_\lambda$  such that  $\emptyset \neq U_\lambda \neq X_\lambda$ .

$\langle 1 \rangle 2$ . For all  $\lambda : \text{El}(\Lambda)$ , PICK  $x_\lambda \in U_\lambda$ .

$\langle 1 \rangle 3$ . ASSUME: for a contradiction  $B$  is a countable neighbourhood basis for  $(x_\lambda)_{\lambda \in \Lambda}$ .

$\langle 1 \rangle 4$ . PICK  $\lambda \in \Lambda$  such that, for all  $U \in B$ , we have  $\pi_\lambda(U) = X_\lambda$

$\langle 1 \rangle 5$ . There is no  $U \in B$  such that  $U \subseteq \pi_\lambda^{-1}(U_\lambda)$

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This is a contradiction.

□

## 5.2 Continuous Functions

**Definition 5.27** (Continuous). Let  $X$  and  $Y$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous* iff, for every open set  $V$  in  $Y$ , the inverse image  $f^{-1}(V)$  is open in  $X$ .

**Proposition 5.28.** 1.  $\text{id}_X$  is continuous

2. The composite of two continuous functions is continuous.

3. If  $f : X \rightarrow Y$  is continuous and  $X_0 \subseteq X$  then  $f|_{X_0} : X_0 \rightarrow Y$  is continuous.

4. If  $f : X + Y \rightarrow Z$ , then  $f$  is continuous iff  $f \circ \kappa_1 : X \rightarrow Z$  and  $f \circ \kappa_2 : Y \rightarrow Z$  are continuous.

5. If  $f : Z \rightarrow X \times Y$ , then  $f$  is continuous iff  $\pi_1 \circ f$  and  $\pi_2 \circ f$  are continuous.

**Definition 5.29** (Homeomorphism). Let  $X$  and  $Y$  be topological spaces. A *homeomorphism* between  $X$  and  $Y$  is a bijection  $f : X \approx Y$  such that  $f$  and  $f^{-1}$  are continuous.

**Definition 5.30** (Retraction). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A continuous function  $\rho : X \rightarrow A$  is a *retraction* iff  $\rho|_A = \text{id}_A$ . We say  $A$  is a *retract* of  $X$  iff there exists a retraction.

**Definition 5.31.** Let **Top** be the category of small topological spaces and continuous functions.

Forgetful functor **Top**  $\rightarrow$  **Set**.

Basepoint preserving continuous functor.

## 5.3 Convergence

**Definition 5.32** (Convergence). Let  $X$  be a topological space. Let  $(x_n)$  be a sequence in  $X$ . A point  $a \in \text{El}(X)$  is a *limit* of the sequence iff, for every neighbourhood  $U$  of  $a$ , there exists  $n_0$  such that  $\forall n \geq n_0. x_n \in U$ .

Convergence in a product space is pointwise convergence.

If  $f : X \rightarrow Y$  is continuous and  $x_n \rightarrow l$  in  $X$  then  $f(x_n) \rightarrow f(l)$  in  $Y$ .

**Example 5.33.** The converse does not hold.

Let  $X$  be the set of all continuous functions  $[0, 1] \rightarrow [-1, 1]$  under the product topology. Let  $i : X \rightarrow L^2([0, 1])$  be the inclusion.

If  $f_n \rightarrow f$  then  $i(f_n) \rightarrow i(f)$  — Lebesgue convergence theorem.

We prove that  $i$  is not continuous.

Assume for a contradiction  $i$  is continuous. Choose a neighbourhood  $K$  of 0 in  $X$  such that  $\forall \phi \in K. \int \phi^2 < 1/2$ . Let  $K = \prod_{\lambda \in [0, 1]} U_\lambda$  where  $U_\lambda = [-1, 1]$  except for  $\lambda = \lambda_1, \dots, \lambda_n$ . Let  $\phi$  be the function that is 0 at  $\lambda_1, \dots, \lambda_n$  and 1 everywhere else. Then  $\phi \in K$  but  $\int \phi^2 = 1$ .

**Proposition 5.34.** *The converse does hold for first countable spaces. If  $f : X \rightarrow Y$  where  $X$  is first countable, and  $Y$  is a topological space, and whenever  $x_n \rightarrow x$  then  $f(x_n) \rightarrow f(x)$ , then  $f$  is continuous.*

## 5.4 Connected Spaces

**Definition 5.35** (Connected). A topological space is *connected* iff it is not the union of two nonempty open disjoint subsets.

**Proposition 5.36.** *The continuous image of a connected space is connected.*

**Proposition 5.37.** *Let  $X$  be a topological space and  $A, B \subseteq X$ . If  $X = A \cup B$ ,  $A \cap B \neq \emptyset$ , and  $A$  and  $B$  are connected, then  $X$  is connected.*

**Proposition 5.38.** *If  $X$  and  $Y$  are nonempty topological spaces, then  $X \times Y$  is connected if and only if  $X$  and  $Y$  are connected.*

**Definition 5.39** (Path-connected). A topological space  $X$  is *path-connected* iff, for any points  $a, b \in X$ , there exists a continuous function  $\alpha : [0, 1] \rightarrow X$ , called a *path*, such that  $\alpha(0) = a$  and  $\alpha(1) = b$ .

**Proposition 5.40.** *The continuous image of a path connected space is path connected.*

**Proposition 5.41.** *Let  $X$  be a topological space and  $A, B \subseteq X$ . If  $X = A \cup B$ ,  $A \cap B \neq \emptyset$ , and  $A$  and  $B$  are path connected, then  $X$  is path connected.*

**Proposition 5.42.** *If  $X$  and  $Y$  are nonempty topological spaces, then  $X \times Y$  is path connected if and only if  $X$  and  $Y$  are path connected.*

## 5.5 Hausdorff Spaces

**Definition 5.43** (Hausdorff). A topological space is a *Hausdorff* space or a  $T_2$  space iff any two distinct points have disjoint neighbourhoods.

**Proposition 5.44.** *In a Hausdorff space, a sequence has at most one limit.*

**Proposition 5.45.** 1. *Every subspace of a Hausdorff space is Hausdorff.*

2. *The disjoint union of two Hausdorff spaces is Hausdorff.*

3. *The product of two Hausdorff spaces is Hausdorff.*

**Proposition 5.46.** *Let  $A$  be a topological space and  $B$  a Hausdorff space. Let  $f, g : A \rightarrow B$  be continuous. Let  $X \subseteq A$  be dense. If  $f$  and  $g$  agree on  $X$ , then  $f = g$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $a \in A$  and  $f(a) \neq g(a)$ .

$\langle 1 \rangle 2$ . PICK disjoint neighbourhoods  $U$  and  $V$  of  $f(a)$  and  $g(a)$  respectively.

$\langle 1 \rangle 3$ . PICK  $x \in f^{-1}(U) \cap g^{-1}(V)$

$\langle 1 \rangle 4$ .  $f(x) = g(x) \in U \cap V$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 5.47.** *Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$  be uniformly continuous. Let  $\hat{X}$  and  $\hat{Y}$  be the completions of  $X$  and  $Y$ . Then  $f$  extends uniquely to a continuous map  $\hat{X} \rightarrow \hat{Y}$ .*

PROOF: The extension maps  $\lim_{n \rightarrow \infty} x_n$  to  $\lim_{n \rightarrow \infty} f(x_n)$ . □

## 5.6 Separable Spaces

**Definition 5.48** (Separable). A topological space is *separable* iff it has a countable dense subset.

Every second countable space is separable.

## 5.7 Sequential Compactness

**Definition 5.49** (Sequentially Compact). A topological space is *sequentially compact* iff every sequence has a convergent subsequence.

## 5.8 Compactness

**Definition 5.50** (Compact). A topological space is *compact* iff every open cover has a finite subcover.

**Proposition 5.51.** *Let  $X$  be a compact topological space. Let  $P$  be a set of open sets such that, for all  $U, V \in P$ , we have  $U \cup V \in P$ . Assume that every point has an open neighbourhood in  $P$ . Then  $X \in P$ .*

PROOF:

- $\langle 1 \rangle 1$ .  $P$  is an open cover of  $X$   
 $\langle 1 \rangle 2$ . PICK a finite subcover  $U_1, \dots, U_n \in P$   
 $\langle 1 \rangle 3$ .  $X = U_1 \cup \dots \cup U_n \in P$   
 $\square$

**Corollary 5.51.1.** *Let  $f$  be a compact space and  $f : X \rightarrow \mathbb{R}$  be locally bounded. Then  $f$  is bounded.*

PROOF: Take  $P = \{U \text{ open in } X : f \text{ is bounded on } U\}$ .  $\square$

**Proposition 5.52.** *The continuous image of a compact space is compact.*

**Proposition 5.53.** *A closed subspace of a compact space is compact.*

**Proposition 5.54.** *Let  $X$  and  $Y$  be nonempty spaces. Then the following are equivalent.*

1.  $X$  and  $Y$  are compact.
2.  $X + Y$  is compact.
3.  $X \times Y$  is compact.

**Proposition 5.55.** *A compact subspace of a Hausdorff space is closed.*

**Proposition 5.56.** *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

**Proposition 5.57.** *A first countable compact space is sequentially compact.*

## 5.9 Quotient Spaces

**Definition 5.58** (Quotient Space). Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . The *quotient topology* on  $X/\sim$  is defined by:  $U : \text{El}(\mathcal{P}X)$  is open in  $X/\sim$  if and only if  $\pi^{-1}(U)$  is open in  $X$ .

**Proposition 5.59.** *Let  $X$  and  $Y$  be topological spaces. Let  $\sim$  be an equivalence relation on  $X$ . Let  $f : X/\sim \rightarrow Y$ . Then  $f$  is continuous if and only if  $f \circ \pi$  is continuous.*

**Proposition 5.60.** *Let  $X$  and  $Y$  be topological spaces. Let  $\sim$  be an equivalence relation on  $X$ . Let  $\phi : Y \rightarrow X/\sim$ .*

*Assume that, for all  $y \in Y$ , there exists a neighbourhood  $U$  of  $y$  and a continuous function  $\Phi : U \rightarrow X$  such that  $\pi \circ \Phi = \phi|U$ . Then  $\phi$  is continuous.*

**Proposition 5.61.** *A quotient of a connected space is connected.*

**Proposition 5.62.** *A quotient of a path connected space is path connected.*

**Proposition 5.63.** *Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . If  $X/\sim$  is Hausdorff then every equivalence class of  $\sim$  is closed in  $X$ .*

**Definition 5.64.** Let  $X$  be a topological space and  $A_1, \dots, A_r \subseteq X$ . Then  $X/A_1, \dots, A_r$  is the quotient space of  $X$  with respect to  $\sim$  where  $x \sim y$  iff  $x = y$  or  $\exists i(x \in A_i \wedge y \in A_i)$ .

**Definition 5.65** (Cone). Let  $X$  be a topological space. The *cone over  $X$*  is the space  $(X \times [0, 1])/(X \times \{1\})$ .

**Definition 5.66** (Suspension). Let  $X$  be a topological space. The *suspension* of  $X$  is the space

$$\Sigma X := (X \times [-1, 1])/(X \times \{-1\}), (X \times \{1\})$$

**Definition 5.67** (Wedge Product). Let  $x_0 \in X$  and  $y_0 \in Y$ . The *wedge product*  $X \vee Y$  is  $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$  as a subspace of  $X \times Y$ .

**Definition 5.68** (Smash Product). Let  $x_0 \in X$  and  $y_0 \in Y$ . The *smash product*  $X \wedge Y$  is  $(X \times Y)/(X \vee Y)$ .

**Example 5.69.**  $D^n/S^{n-1} \cong S^n$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : D^n/S^{n-1} \rightarrow S^n$  be the function induced by the map  $D^n \rightarrow S^n$  that maps the radii of  $D^n$  onto the meridians of  $S^n$  from the north to the south pole.

$\langle 1 \rangle 2$ .  $\phi$  is a bijection.

$\langle 1 \rangle 3$ .  $\phi$  is a homeomorphism.

PROOF: Since  $D^n/S^{n-1}$  is compact and  $S^n$  is Hausdorff.

□

## 5.10 Gluing

**Definition 5.70** (Gluing). Let  $X$  and  $Y$  be topological spaces,  $X_0 \subseteq X$  and  $\phi : X_0 \rightarrow Y$  a continuous map. Then  $Y \cup_\phi X$  is the quotient space  $(X + Y)/\sim$ , where  $\sim$  is the equivalence relation generated by  $x \sim \phi(x)$  for all  $x : \text{El}(X)$ .

**Proposition 5.71.**  *$Y$  is a subspace of  $Y \cup_\phi X$ .*

**Definition 5.72.** Let  $X$  be a topological space and  $\alpha : X \cong X$  a homeomorphism. Then  $(X \times [0, 1])/\alpha$  is the quotient space of  $X \times [0, 1]$  by the equivalence relation generated by  $(x, 0) \sim (\alpha(x), 1)$  for all  $x : \text{El}(X)$ .

**Definition 5.73** (Möbius Strip). The *Möbius strip* is  $([-1, 1] \times [0, 1])/\alpha$  where  $\alpha(x) = -x$ .



**Definition 5.74** (Klein Bottle). The *Klein bottle* is  $(S^1 \times [0, 1])/\alpha$  where  $\alpha(z) = \bar{z}$ .

**Proposition 5.75.** Let  $M$  be the Möbius strip and  $K$  the Klein bottle. Then  $M \cup_{\text{id}_M} M \cong K$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : ([-1, 1] \times [0, 1]) + ([-1, 1] \times [0, 1]) \rightarrow S^1 \times [0, 1]$  be the function that maps  $\kappa_1(\theta, t)$  to  $(e^{i\theta/2}, t)$  and  $\kappa_2(\theta, t)$  to  $(-e^{-i\theta/2}, t)$ .

$\langle 1 \rangle 2$ .  $f$  induces a bijection  $M \cup_{\text{id}_M} M \approx K$

$\langle 1 \rangle 3$ .  $f$  is a homeomorphism.

□

## 5.11 Metric Spaces

**Definition 5.76** (Metric Space). Let  $X$  be a set and  $d : X^2 \rightarrow \mathbb{R}$ . We say  $(X, d)$  is a *metric space* iff:

- For all  $x, y \in X$  we have  $d(x, y) \geq 0$
- For all  $x, y \in X$  we have  $d(x, y) = 0$  iff  $x = y$
- For all  $x, y \in X$  we have  $d(x, y) = d(y, x)$
- (*Triangle Inequality*) For all  $x, y, z \in X$  we have  $d(x, z) \leq d(x, y) + d(y, z)$

We call  $d$  the *metric* of the metric space  $(X, d)$ . We often write  $X$  for the metric space  $(X, d)$ .

**Definition 5.77** (Topology of a Metric Space). Let  $(X, d)$  be a metric space. The topology *induced* by the metric  $d$  is defined by: for  $V \subseteq X$ , we have  $V$  is open if and only if, for all  $x \in V$ , there exists  $\epsilon > 0$  such that  $\{y \in X : d(x, y) < \epsilon\} \subseteq V$ .

**Definition 5.78** (Metrizable). A topological space is *metrizable* iff there exists a metric that induces its topology.

**Proposition 5.79.** Every metrizable space is Hausdorff.

Every metrizable space is first countable.

A metric space is compact if and only if it is sequentially compact.

A metric space is separable if and only if it is second countable.

## 5.12 Complete Metric Spaces

**Definition 5.80** (Complete). A metric space is *complete* iff every Cauchy sequence converges.

**Example 5.81.**  $\mathbb{R}$  is complete.

**Proposition 5.82.** *The product of two complete metric spaces is complete.*

**Proposition 5.83.** *Every compact metric space is complete.*

**Proposition 5.84.** *Let  $X$  be a complete metric space and  $A \subseteq X$ . Then  $A$  is complete if and only if  $A$  is closed.*

**Definition 5.85** (Completion). Let  $X$  be a metric space. A *completion* of  $X$  is a complete metric space  $\hat{X}$  and injection  $i : X \rightarrow \hat{X}$  such that:

- The metric on  $X$  is the restriction of the metric on  $\hat{X}$
- $X$  is dense in  $\hat{X}$ .

**Proposition 5.86.** *Let  $i_1 : X \rightarrow Y_1$  and  $i_2 : X \rightarrow Y_2$  be completions of  $X$ . Then there exists a unique isometry  $\phi : Y_1 \cong Y_2$  such that  $\phi \circ i_1 = i_2$ .*

PROOF: Define  $\phi(\lim_{n \rightarrow \infty} i_1(x_n)) = \lim_{n \rightarrow \infty} i_2(x_n)$ .  $\square$

**Theorem 5.87.** *Every metric space has a completion.*

PROOF: Let  $\hat{X}$  be the set of Cauchy sequences in  $X$  quotiented by  $\sim$  where  $(x_n) \sim (y_n)$  if and only if  $d(x_n, y_n) \rightarrow 0$ .  $\square$

## 5.13 Manifolds

**Definition 5.88** (Manifold). An  $n$ -dimensional manifold is a second countable Hausdorff space locally homeomorphic to  $\mathbb{R}^n$ .

## Chapter 6

# Homotopy Theory

### 6.1 Homotopies

**Definition 6.1** (Homotopy). Let  $X$  and  $Y$  be topological spaces. Let  $f, g : X \rightarrow Y$  be continuous. A *homotopy* between  $f$  and  $g$  is a continuous function  $h : X \times [0, 1] \rightarrow Y$  such that

- $\forall x : \text{El}(X) . h(x, 0) = f(x)$
- $\forall x : \text{El}(X) . h(x, 1) = g(x)$

We say  $f$  and  $g$  are *homotopic*,  $f \simeq g$ , iff there exists a homotopy between them.

Let  $[X, Y]$  be the set of all homotopy classes of functions  $X \rightarrow Y$ .

**Proposition 6.2.** Let  $f, f' : X \rightarrow Y$  and  $g, g' : Y \rightarrow Z$  be continuous. If  $f \simeq f'$  and  $g \simeq g'$  then  $g \circ f \simeq g' \circ f'$ .

**Definition 6.3.** Let **HTop** be the category whose objects are the small topological spaces and whose morphisms are the homotopy classes of continuous functions.

**Definition 6.4.** A functor  $F : \mathbf{Top} \rightarrow \mathcal{C}$  is *homotopy invariant* iff, for any topological spaces  $X, Y$  and continuous functions  $f, g : X \rightarrow Y$ , if  $f \simeq g$  then  $Hf = Hg$ .

Basepoint-preserving homotopy.

### 6.2 Homotopy Equivalence

**Definition 6.5** (Homotopy Equivalence). Let  $X$  and  $Y$  be topological spaces. A *homotopy equivalence* between  $X$  and  $Y$ ,  $f : X \simeq Y$ , is a continuous function  $f : X \rightarrow Y$  such that there exists a continuous function  $g : Y \rightarrow X$ , the *homotopy inverse* to  $f$ , such that  $g \circ f \simeq \text{id}_X$  and  $f \circ g \simeq \text{id}_Y$ .

**Definition 6.6** (Contractible). A topological space  $X$  is *contractible* iff  $X \simeq 1$ .

**Example 6.7.**  $\mathbb{R}^n$  is contractible.

**Example 6.8.**  $D^n$  is contractible.

**Definition 6.9** (Deformation Retract). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A retraction  $\rho : X \rightarrow A$  is a *deformation retraction* iff  $i \circ \rho \simeq \text{id}_X$ , where  $i$  is the inclusion  $A \hookrightarrow X$ . We say  $A$  is a *deformation retract* of  $X$  iff there exists a deformation retraction.

**Definition 6.10** (Strong Deformation Retract). Let  $X$  be a topological space and  $A$  a subspace of  $X$ . A *strong deformation retraction*  $\rho : X \rightarrow A$  is a continuous function such that there exists a homotopy  $h : X \times [0, 1] \rightarrow X$  between  $i \circ \rho$  and  $\text{id}_X$  such that, for all  $a : \text{El}(X)$  and  $t : \text{El}([0, 1])$ , we have  $h(a, t) = a$ .

We say  $A$  is a *strong deformation retract* of  $X$  iff a strong deformation retraction exists.

**Example 6.11.**  $\{0\}$  is a strong deformation retract of  $\mathbb{R}^n$  and of  $D^n$ .

**Example 6.12.**  $S^1$  is a strong deformation retract of the torus  $S^1 \times D^2$ .

**Example 6.13.**  $S^{n-1}$  is a strong deformation retract of  $D^n - \{0\}$ .

**Example 6.14.** For any topological space  $X$ , the singleton consisting of the vertex is a strong deformation retract of the cone over  $X$ .

## Chapter 7

# Topological Groups

**Definition 7.1** (Topological Group). A *topological group* is a group  $G$  with a topology such that the function  $G^2 \rightarrow G$  that maps  $(x, y)$  to  $xy^{-1}$  is continuous.

**Example 7.2.**  $GL(n, \mathbb{R})$  and  $GL(n, \mathbb{C})$  are topological groups.

**Proposition 7.3.** *Any subgroup of a topological group is a topological group under the subspace topology.*

**Definition 7.4** (Homogeneous Space). A *homogeneous space* is a topological space of the form  $G/H$ , where  $G$  is a topological group and  $H$  is a normal subgroup of  $G$ , under the quotient topology.

**Proposition 7.5.** *Let  $G$  be a topological group and  $H$  a normal subgroup of  $G$ . Then  $G/H$  is Hausdorff if and only if  $H$  is closed.*

PROOF: See Bourbaki, N., General Topology. III.12  $\square$

### 7.1 Continuous Actions

**Definition 7.6** (Continuous Action). Let  $G$  be a topological group and  $X$  a topological space. A *continuous action* of  $G$  on  $X$  is a continuous function  $\cdot : G \times X \rightarrow X$  such that:

- $\forall x : \text{El}(X) . ex = x$
- $\forall g, h : \text{El}(G) . \forall x : \text{El}(X) . g(hx) = (gh)x$

A  $G$ -space consists of a topological space  $X$  and a continuous action of  $G$  on  $X$ .

**Definition 7.7** (Orbit). Let  $X$  be a  $G$ -space and  $x \in X$ . The *orbit* of  $x$  is  $\{gx : g \in G\}$ .

The *orbit space*  $X/G$  is the set of all orbits under the quotient topology.

**Proposition 7.8.** *Define an action of  $SO(2)$  on  $S^2$  by  $g(x_1, x_2, x_3) = (g(x_1, x_2), x_3)$ . Then  $S^2/SO(2) \cong [-1, 1]$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f_3 : S^2/SO(2) \rightarrow [-1, 1]$  be the function induced by  $\pi_3 : S^2 \rightarrow [-1, 1]$

$\langle 1 \rangle 2$ .  $f_3$  is bijective.

$\langle 1 \rangle 3$ .  $S^2/SO(2)$  is compact.

PROOF: It is the continuous image of  $S^2$  which is compact.

$\langle 1 \rangle 4$ .  $[-1, 1]$  is Hausdorff.

$\langle 1 \rangle 5$ .  $f_3$  is a homeomorphism.

□

**Definition 7.9** (Stabilizer). Let  $X$  be a  $G$ -space and  $x \in X$ . The *stabilizer* of  $x$  is  $G_x := \{g \in G \mid gx = x\}$ .

**Proposition 7.10.** *The function that maps  $gG_x$  to  $gx$  is a continuous bijection from  $G/G_x$  to  $Gx$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $gG_x = hG_x$  then  $gx = hx$ .

$\langle 2 \rangle 1$ . ASSUME:  $gG_x = hG_x$

$\langle 2 \rangle 2$ .  $g^{-1}h \in G_x$

$\langle 2 \rangle 3$ .  $g^{-1}hx = x$

$\langle 2 \rangle 4$ .  $gx = hx$

$\langle 1 \rangle 2$ . If  $gx = hx$  then  $gG_x = hG_x$ .

PROOF: Similar.

$\langle 1 \rangle 3$ . The function is continuous.

PROOF: Proposition 5.59.

□

## Chapter 8

# Topological Vector Spaces

**Definition 8.1** (Topological Vector Space). Let  $K$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . A *topological vector space* over  $K$  consists of a vector space  $E$  over  $K$  and a topology on  $E$  such that:

- Subtraction is a continuous function  $E^2 \rightarrow E$
- Multiplication is a continuous function  $K \times E \rightarrow E$

**Proposition 8.2.** *Every topological vector space is a topological group under addition.*

PROOF: Immediate from the definition.  $\square$

**Theorem 8.3.** *The usual topology on a finite dimensional vector space over  $K$  is the only one that makes it into a Hausdorff topological vector space.*

PROOF: See Bourbaki. Elements de Mathematique, Livre V: Espaces Vectoriels Topologiques, Th. 2, p. 18  $\square$

**Proposition 8.4.** *Let  $E$  be a topological vector space and  $E_0$  a subspace of  $E$ . Then  $\overline{E_0}$  is a subspace of  $E$ .*

**Definition 8.5.** Let  $E$  be a topological vector space. The topological space associated with  $E$  is  $E/\overline{\{0\}}$ .

### 8.1 Cauchy Sequences

**Definition 8.6** (Cauchy Sequence). Let  $E$  be a topological vector space. A sequence  $(x_n)$  in  $E$  is a *Cauchy sequence* iff, for every neighbourhood  $U$  of 0, there exists  $n_0$  such that  $\forall m, n \geq n_0. x_n - x_m \in U$ .

**Definition 8.7** (Complete Topological Vector Space). A topological vector space is *complete* iff every Cauchy sequence converges.

## 8.2 Seminorms

**Definition 8.8** (Seminorm). Let  $E$  be a vector space over  $K$ . A *seminorm* on  $E$  is a function  $\| \cdot \| : E \rightarrow \mathbb{R}$  such that:

1.  $\forall x : \text{El}(E) . \|x\| \geq 0$
2.  $\forall \alpha : \text{El}(K) . \forall x : \text{El}(E) . \|\alpha x\| = |\alpha| \|x\|$
3. *Triangle Inequality*  $\forall x, y : \text{El}(E) . \|x + y\| \leq \|x\| + \|y\|$

**Example 8.9.** The function that maps  $(x_1, \dots, x_n)$  to  $|x_i|$  is a seminorm on  $\mathbb{R}^n$ .

**Definition 8.10.** Let  $E$  be a vector space over  $K$ . Let  $\Lambda$  be a set of seminorms on  $E$ . The topology *generated* by  $\Lambda$  is the topology generated by the subbasis consisting of all sets of the form  $B_\epsilon^\lambda(x) = \{y \in E : \lambda(y - x) < \epsilon\}$  for  $\epsilon > 0$ ,  $\lambda \in \Lambda$  and  $x : \text{El}(E)$ .

**Proposition 8.11.**  $E$  is a topological vector space under this topology. It is Hausdorff iff, for all  $x : \text{El}(E)$ , if  $\forall \lambda \in \Lambda . \lambda(x) = 0$  then  $x = 0$ .

## 8.3 Fréchet Spaces

**Definition 8.12** (Pre-Fréchet Space). A *pre-Fréchet space* is a Hausdorff topological vector space whose topology is generated by a countable set of seminorms.

**Proposition 8.13.** Let  $E$  be a pre-Fréchet space whose topology is generated by the family of seminorms  $\{\| \cdot \|_n : n \in \mathbb{Z}^+\}$ . Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

is a metric that induces the same topology. The two definitions of Cauchy sequence agree.

**Definition 8.14** (Fréchet Space). A *Fréchet space* is a complete pre-Fréchet space.

## 8.4 Normed Spaces

**Definition 8.15** (Normed Space). Let  $E$  be a vector space over  $K$ . A *norm* on  $E$  is a function  $\| \cdot \| : E \rightarrow \mathbb{R}$  is a seminorm such that,  $\forall x \in E . \|x\| = 0 \Leftrightarrow x = 0$ .

A *normed space* consists of a vector space with a norm.

**Proposition 8.16.** If  $E$  is a normed space then  $d(x, y) = \|x - y\|$  is a metric on  $E$  that makes  $E$  into a topological vector space. The two definitions of Cauchy sequence agree on  $E$ .



**Proposition 8.17.** *Let  $\| \cdot \|$  be a seminorm on the vector space  $E$ . Then  $\| \cdot \|$  defines a norm on  $E/\{0\}$ .*

**Proposition 8.18.** *Let  $E$  and  $F$  be normed spaces. Any continuous linear map  $E \rightarrow F$  is uniformly continuous.*

**Definition 8.19.** For  $p \geq 1$ , let  $\mathcal{L}^p(\mathbb{R}^n)$  be the vector space of all Lebesgue-measurable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $|f|^p$  is Lebesgue-integrable. Then

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p dx}$$

defines a seminorm on  $\mathcal{L}^p(\mathbb{R}^n)$ . Let

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\{0\}.$$

## 8.5 Inner Product Spaces

**Proposition 8.20.** *If  $E$  is an inner product space then  $\|x\| = \sqrt{\langle x, x \rangle}$  is a norm on  $E$ .*

## 8.6 Banach Spaces

**Definition 8.21** (Banach Space). A *Banach space* is a complete normed space.

**Example 8.22.** For any topological space  $X$ , the set  $C(X)$  of bounded continuous functions  $X \rightarrow \mathbb{R}$  is a Banach space under  $\|f\| = \sup_{x \in X} |f(x)|$ .

**Proposition 8.23.** *The completion of a normed space is a Banach space.*

**Proposition 8.24.** *Let  $E$  and  $F$  be normed spaces. Let  $f : E \rightarrow F$  be a continuous linear map. Then the extension to the completions  $\hat{E} \rightarrow \hat{F}$  is linear.*

**Proposition 8.25.**  $L^p(\mathbb{R}^n)$  is a Banach space.

**Proposition 8.26.**  $C(\mathbb{R})$  is first countable but not second countable.

PROOF: For every sequence of 0s and 1s  $s = (s_n)$ , let  $f_s$  be a continuous bounded function whose value at  $n$  is  $s_n$ . Then the set of all  $f_s$  is an uncountable discrete set in  $C(\mathbb{R})$ . Hence  $C(\mathbb{R})$  is not second countable.

It is first countable because it is metrizable.  $\square$

## 8.7 Hilbert Spaces

**Definition 8.27** (Hilbert Space). A *Hilbert space* is a complete inner product space.

**Example 8.28.** The set of *square-integrable functions* is the set of Lebesgue integrable functions  $[-\pi, \pi] \rightarrow \mathbb{R}$  quotiented by:  $f \sim g$  iff  $\{x \in [-\pi, \pi] : f(x) \neq g(x)\}$  has measure 0. This is a Hilbert space under

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx .$$

**Proposition 8.29.** *The completion of an inner product space is a Hilbert space.*

An infinite dimensional Hilbert space with the weak topology is not first countable.

## 8.8 Locally Convex Spaces

**Definition 8.30** (Locally Convex Space). A topological vector space is *locally convex* iff every neighbourhood of 0 includes a convex neighbourhood of 0.

**Proposition 8.31.** *A topological vector space is locally convex if and only if its topology is generated by a set of seminorms.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18.  $\square$

**Proposition 8.32.** *A locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.*

PROOF: See Köthe, G. Topological Vector Spaces 1. Section 18.  $\square$

**Example 8.33.** Let  $E$  be an infinite dimensional Hilbert space. Let  $E'$  be the same vector space under the *weak topology*, the coarsest topology such that every continuous linear map  $E \rightarrow \mathbb{R}$  is continuous as a map  $E' \rightarrow \mathbb{R}$ . Then  $E$  is locally convex Hausdorff but not metrizable.

Proof: See Dieudonne, J. A., Treatise on Analysis, Vol. II, New York and London: Academic Press, 1970, p. 76.

**Definition 8.34** (Thom Space). Let  $E$  be a vector bundle with a Riemannian metric,  $DE = \{x : \text{El}(E) \mid \|x\| \leq 1\}$  its disc bundle and  $SE := \{v : \text{El}(E) \mid \|v\| = 1\}$  its sphere bundle. The *Thom space* of  $E$  is the quotient space  $DE/SE$ .