

# Mathematics

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Part I

Category Theory





# Chapter 1

## Foundations

This is a placeholder — I am not sure what foundation I want to use for this project yet. I will try to work in a way which is foundation-independent. What I do could be formalized in ZFC, ETCS, or some other system. I will assume the usual set theoretic constructions as needed. Sets will be defined up to bijection only.



# Chapter 2

## Categories

**Definition 2.1** (Category). A *category*  $\mathcal{C}$  consists of:

- A class  $|\mathcal{C}|$  of *objects*. We write  $A \in \mathcal{C}$  for  $A \in |\mathcal{C}|$ .
- For any objects  $A, B$ , a set  $\mathcal{C}[A, B]$  of *morphisms* from  $A$  to  $B$ . We write  $f : A \rightarrow B$  for  $f \in \mathcal{C}[A, B]$ .
- For any object  $A$ , a morphism  $\text{id}_A : A \rightarrow A$ , the *identity* morphism on  $A$ .
- For any morphisms  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , a morphism  $g \circ f : A \rightarrow C$ , the *composite* of  $f$  and  $g$ .

such that:

**Associativity** Given  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h : C \rightarrow D$ , we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

**Left Unit Law** For any morphism  $f : A \rightarrow B$ , we have  $\text{id}_B \circ f = f$ .

**Right Unit Law** For any morphism  $f : A \rightarrow B$ , we have  $f \circ \text{id}_A = f$ .

**Proposition 2.2.** *The identity morphism on an object is unique.*

PROOF: If  $i$  and  $j$  are identity morphisms on  $A$  then  $i = i \circ j = j$ .  $\square$

**Example 2.3** (Category of Sets). The *category of sets* **Set** has objects all sets and morphisms all functions.

**Definition 2.4** (Endomorphism). In a category  $\mathcal{C}$ , an *endomorphism* on an object  $A$  is a morphism  $A \rightarrow A$ . We write  $\text{End}_{\mathcal{C}}(A)$  for  $\mathcal{C}[A, A]$ .

**Definition 2.5** (Opposite Category). For any category  $\mathcal{C}$ , the *opposite* category  $\mathcal{C}^{\text{op}}$  is the category with the same objects as  $\mathcal{C}$  and

$$\mathcal{C}^{\text{op}}[A, B] = \mathcal{C}[B, A]$$

## 2.1 Preorders

**Definition 2.6** (Preorder). A *preorder* on a set  $A$  is a relation  $\leq$  on  $A$  that is reflexive and transitive.

A *preordered set* is a pair  $(A, \leq)$  such that  $\leq$  is a preorder on  $A$ . We usually write  $A$  for the preordered set  $(A, \leq)$ .

We identify any preordered set  $A$  with the category whose objects are the elements of  $A$ , with one morphism  $a \rightarrow b$  iff  $a \leq b$ , and no morphism  $a \rightarrow b$  otherwise.

**Example 2.7.** For any ordinal  $\alpha$ , let  $\alpha$  be the preorder  $\{\beta : \beta < \alpha\}$  under  $\leq$ .

**Definition 2.8** (Discrete Preorder). We identify any set  $A$  with the *discrete* preorder  $(A, =)$ .

## 2.2 Monomorphisms and Epimorphisms

**Definition 2.9** (Monomorphism). In a category, let  $f : A \rightarrow B$ . Then  $f$  is a *monomorphism* or *monic* iff, for every object  $X$  and morphism  $x, y : X \rightarrow A$ , if  $fx = fy$  then  $x = y$ .

**Definition 2.10** (Epimorphism). In a category, let  $f : A \rightarrow B$ . Then  $f$  is a *epimorphism* or *epi* iff, for every object  $X$  and morphism  $x, y : B \rightarrow X$ , if  $xf = yf$  then  $x = y$ .

**Proposition 2.11.** *The composite of two monomorphism is monic.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be monic.

$\langle 1 \rangle 2$ . LET:  $x, y : X \rightarrow A$

$\langle 1 \rangle 3$ . ASSUME:  $g \circ f \circ x = g \circ f \circ y$

$\langle 1 \rangle 4$ .  $f \circ x = f \circ y$

$\langle 1 \rangle 5$ .  $x = y$

□

**Proposition 2.12.** *The composite of two epimorphisms is epi.*

PROOF: Dual. □

**Proposition 2.13.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $g \circ f$  is monic then  $f$  is monic.*

PROOF: If  $f \circ x = f \circ y$  then  $gfx = gfy$  and so  $x = y$ . □

**Proposition 2.14.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $g \circ f$  is epi then  $g$  is epi.*

PROOF: Dual. □

**Proposition 2.15.** *A function is a monomorphism in **Set** iff it is injective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$

$\langle 1 \rangle 2$ . If  $f$  is monic then  $f$  is injective.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is monic.

$\langle 2 \rangle 2$ . LET:  $x, y \in A$

$\langle 2 \rangle 3$ . ASSUME:  $f(x) = f(y)$

$\langle 2 \rangle 4$ . LET:  $\bar{x}, \bar{y} : 1 \rightarrow A$  be the functions such that  $\bar{x}(*) = x$  and  $\bar{y}(*) = y$

$\langle 2 \rangle 5$ .  $f \circ \bar{x} = f \circ \bar{y}$

$\langle 2 \rangle 6$ .  $\bar{x} = \bar{y}$

PROOF: By  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 7$ .  $x = y$

$\langle 1 \rangle 3$ . If  $f$  is injective then  $f$  is monic.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is injective.

$\langle 2 \rangle 2$ . LET:  $X$  be a set and  $x, y : X \rightarrow A$ .

$\langle 2 \rangle 3$ . ASSUME:  $f \circ x = f \circ y$

PROVE:  $x = y$

$\langle 2 \rangle 4$ . LET:  $t \in X$

PROVE:  $x(t) = y(t)$

$\langle 2 \rangle 5$ .  $f(x(t)) = f(y(t))$

$\langle 2 \rangle 6$ .  $x(t) = y(t)$

PROOF: By  $\langle 2 \rangle 1$ .

□

**Proposition 2.16.** *A function is an epimorphism in **Set** iff it is surjective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$

$\langle 1 \rangle 2$ . If  $f$  is an epimorphism then  $f$  is surjective.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is an epimorphism.

$\langle 2 \rangle 2$ . LET:  $b \in B$

$\langle 2 \rangle 3$ . LET:  $x, y : B \rightarrow 2$  be defined by  $x(b) = 1$  and  $x(t) = 0$  for all other  $t \in B$ ,  $y(t) = 0$  for all  $t \in B$ .

$\langle 2 \rangle 4$ .  $x \neq y$

$\langle 2 \rangle 5$ .  $x \circ f \neq y \circ f$

$\langle 2 \rangle 6$ . There exists  $a \in A$  such that  $f(a) = b$ .

$\langle 1 \rangle 3$ . If  $f$  is surjective then  $f$  is an epimorphism.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is surjective.

$\langle 2 \rangle 2$ . LET:  $x, y : B \rightarrow X$

$\langle 2 \rangle 3$ . ASSUME:  $x \circ f = y \circ f$

PROVE:  $x = y$

$\langle 2 \rangle 4$ . LET:  $b \in B$

PROVE:  $x(b) = y(b)$

$\langle 2 \rangle 5$ . PICK  $a \in A$  such that  $f(a) = b$

$\langle 2 \rangle 6$ .  $x(f(a)) = y(f(a))$

$\langle 2 \rangle 7$ .  $x(b) = y(b)$

□

**Proposition 2.17.** *In a preorder, every morphism is monic and epi.*

PROOF: Immediate from definitions.  $\square$

## 2.3 Sections and Retractions

**Definition 2.18** (Section, Retraction). In a category, let  $r : A \rightarrow B$  and  $s : B \rightarrow A$ . Then  $r$  is a *retraction* of  $s$ , and  $s$  is a *section* of  $r$ , iff  $r \circ s = \text{id}_B$ .

**Proposition 2.19.** *Every identity morphism is a section and retraction of itself.*

PROOF: Immediate from definitions.  $\square$

**Proposition 2.20.** *Let  $r, r' : A \rightarrow B$  and  $s : B \rightarrow A$ . If  $r$  is a retraction of  $s$  and  $r'$  is a section of  $s$  then  $r = r'$ .*

PROOF:

$$\begin{aligned} r &= r \circ \text{id}_A \\ &= r \circ s \circ r' \\ &= \text{id}_B \circ r' \\ &= r' \end{aligned} \quad \square$$

**Proposition 2.21.** *Let  $r_1 : A \rightarrow B$ ,  $r_2 : B \rightarrow C$ ,  $s_1 : B \rightarrow A$  and  $s_2 : C \rightarrow B$ . If  $r_1$  is a retraction of  $s_1$  and  $r_2$  is a retraction of  $s_2$  then  $r_2 \circ r_1$  is a retraction of  $s_1 \circ s_2$ .*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= \text{id}_C \end{aligned} \quad \square$$

**Proposition 2.22.** *Every section is monic.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $s : A \rightarrow B$  be a section of  $r : B \rightarrow A$ .

$\langle 1 \rangle 2$ . LET:  $x, y : X \rightarrow A$  satisfy  $sx = sy$ .

$\langle 1 \rangle 3$ .  $rsx = rsy$

$\langle 1 \rangle 4$ .  $x = y$

$\square$

**Proposition 2.23.** *Every retraction is epi.*

PROOF: Dual.  $\square$

**Proposition 2.24.** *In Set, every epimorphism has a retraction.*

PROOF: By the Axiom of Choice.  $\square$

**Example 2.25.** It is not true in general that every monomorphism in any category has a section. nor that every epimorphism in any category has a retraction.

In the category **2**, the morphism  $0 \leq 1$  is monic and epi but has no retraction or section.

## 2.4 Isomorphisms

**Definition 2.26** (Isomorphism). In a category  $\mathcal{C}$ , a morphism  $f : A \rightarrow B$  is an *isomorphism*, denoted  $f : A \cong B$ , iff there exists a morphism  $f^{-1} : B \rightarrow A$ , the *inverse* of  $f$ , such that  $f^{-1} \circ f = \text{id}_A$  and  $f \circ f^{-1} = \text{id}_B$ .

An *automorphism* on an object  $A$  is an isomorphism between  $A$  and itself. We write  $\text{Aut}_{\mathcal{C}}(A)$  for the set of all automorphisms on  $A$ .

Objects  $A$  and  $B$  are *isomorphic*,  $A \cong B$ , iff there exists an isomorphism between them.

**Proposition 2.27.** *The inverse of an isomorphism is unique.*

PROOF: Proposition 2.20.  $\square$

**Proposition 2.28.** *For any object  $A$  we have  $\text{id}_A : A \cong A$  and  $\text{id}_A^{-1} = \text{id}_A$ .*

PROOF: Since  $\text{id}_A \circ \text{id}_A = \text{id}_A$  by the Unit Laws.  $\square$

**Proposition 2.29.** *If  $f : A \cong B$  then  $f^{-1} : B \cong A$  and  $(f^{-1})^{-1} = f$ .*

PROOF: Immediate from definitions.  $\square$

**Proposition 2.30.** *If  $f : A \cong B$  and  $g : B \cong C$  then  $g \circ f : A \cong C$  and  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .*

PROOF: From Proposition 2.21.  $\square$

**Definition 2.31** (Groupoid). A *groupoid* is a category in which every morphism is an isomorphism.

## 2.5 Initial and Terminal Objects

**Definition 2.32** (Initial Object). An object  $I$  in a category is *initial* iff, for any object  $X$ , there is exactly one morphism  $I \rightarrow X$ .

**Example 2.33.** The empty set is the initial object in **Set**.

**Definition 2.34** (Terminal Object). An object  $T$  in a category is *terminal* iff, for any object  $X$ , there is exactly one morphism  $X \rightarrow T$ .

**Example 2.35.** Every singleton is terminal in **Set**.

**Proposition 2.36.** *If  $I$  and  $J$  are initial in a category, then there exists a unique isomorphism  $I \cong J$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $i$  be the unique morphism  $I \rightarrow J$ .
- $\langle 1 \rangle 2$ . LET:  $i^{-1}$  be the unique morphism  $J \rightarrow I$ .
- $\langle 1 \rangle 3$ .  $i \circ i^{-1} = \text{id}_J$

PROOF: Since there is only one morphism  $J \rightarrow J$ .

- $\langle 1 \rangle 4$ .  $i^{-1} \circ i = \text{id}_I$

PROOF: Since there is only one morphism  $I \rightarrow I$ .  
 $\square$

**Proposition 2.37.** *If  $S$  and  $T$  are terminal in a category, then there exists a unique isomorphism  $S \cong T$ .*

PROOF: Dual.  $\square$



## Chapter 3

# Functors

**Definition 3.1** (Functor). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of:

- for every object  $A \in \mathcal{C}$ , an object  $FA \in \mathcal{D}$
- for any morphism  $f : A \rightarrow B : \mathcal{C}$ , a morphism  $Ff : FA \rightarrow FB : \mathcal{D}$

such that:

- $F\text{id}_A = \text{id}_{FA}$
- $F(g \circ f) = Fg \circ Ff$

**Definition 3.2** (Identity Functor). For any category  $\mathcal{C}$ , the *identity functor*  $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  is defined by

$$\begin{aligned} 1_{\mathcal{C}}A &= A \\ 1_{\mathcal{C}}f &= f \end{aligned}$$

**Definition 3.3** (Constant Functor). Given categories  $\mathcal{C}$ ,  $\mathcal{D}$  and an object  $D \in \mathcal{D}$ , the *constant functor*  $K^{\mathcal{C}}D : \mathcal{C} \rightarrow \mathcal{D}$  is the functor defined by

$$\begin{aligned} K^{\mathcal{C}}DC &= D \\ K^{\mathcal{C}}Df &= \text{id}_D \end{aligned}$$

### 3.1 Comma Categories

**Definition 3.4** (Comma Category). Let  $F : \mathcal{C} \rightarrow \mathcal{E}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$  be functors. The *comma category*  $F \downarrow G$  is the category with:

- objects all pairs  $(C, D, f)$  where  $C \in \mathcal{C}$ ,  $D \in \mathcal{D}$  and  $f : FC \rightarrow GD : \mathcal{E}$

- morphisms  $(u, v) : (C, D, f) \rightarrow (C', D', g)$  all pairs  $u : C \rightarrow C' : \mathcal{C}$  and  $v : D \rightarrow D' : \mathcal{D}$  such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ \downarrow Fu & & \downarrow Gv \\ FC' & \xrightarrow{g} & GD' \end{array}$$

**Definition 3.5** (Slice Category). Let  $\mathcal{C}$  be a category and  $A \in \mathcal{C}$ . The *slice category* over  $A$ , denoted  $\mathcal{C}/A$ , is the comma category  $1_{\mathcal{C}} \downarrow K^1 A$ .

**Definition 3.6** (Coslice Category). Let  $\mathcal{C}$  be a category and  $A \in \mathcal{C}$ . The *coslice category* over  $A$ , denoted  $\mathcal{C} \backslash A$ , is the comma category  $K^1 A \downarrow 1_{\mathcal{C}}$ .

**Definition 3.7** (Pointed Sets). The *category of pointed sets*  $\mathbf{Set}_*$  is the coslice category  $\mathbf{Set} \backslash 1$ .

**Part II**

**Group Theory**



## Chapter 4

# Semigroups

**Definition 4.1** (Semigroup). A *semigroup* consists of a set  $S$  and an associative binary operation  $\cdot$  on  $S$ .



## Chapter 5

# Monoids

**Definition 5.1** (Monoid). A *monoid* consists of a semigroup  $M$  such that there exists  $e \in M$ , the *unit*, such that, for all  $x \in M$ , we have  $xe = ex = x$ .

We identify a monoid  $M$  with the category with one object whose morphisms are the elements of  $M$ , with composition given by  $\cdot$ .

**Proposition 5.2.** *The identity in a group is unique.*

PROOF: Proposition 2.2.





# Chapter 6

## Groups

**Definition 6.1** (Group). Let  $\mathcal{C}$  be a category with finite products. A *group (object)* in  $\mathcal{C}$  consists of an object  $G \in \mathcal{C}$  and morphisms

$$m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G$$

such that the following diagrams commute.

$$\begin{array}{ccc}
 G^3 & \xrightarrow{m \times \text{id}_G} & G^2 \\
 \downarrow \text{id}_G \times m & & \downarrow m \\
 G^2 & \xrightarrow{m} & G
 \end{array}$$
  

$$\begin{array}{ccc}
 1 \times G & \xrightarrow{e \times \text{id}_G} & G^2 \\
 & \searrow \cong & \downarrow m \\
 & & G
 \end{array}
 \qquad
 \begin{array}{ccc}
 G \times 1 & \xrightarrow{\text{id}_G \times e} & G^2 \\
 & \searrow \cong & \downarrow m \\
 & & G
 \end{array}$$
  

$$\begin{array}{ccccc}
 G & \xrightarrow{\Delta} & G^2 & \xrightarrow{\text{id}_G \times i} & G^2 \\
 \downarrow & & & & \downarrow m \\
 1 & \xrightarrow{e} & G & & G
 \end{array}
 \qquad
 \begin{array}{ccccc}
 G & \xrightarrow{\Delta} & G^2 & \xrightarrow{i \times \text{id}_G} & G^2 \\
 \downarrow & & & & \downarrow m \\
 1 & \xrightarrow{e} & G & & G
 \end{array}$$

**Definition 6.2** (Group). We write just 'group' for 'group in **Set**'. Thus, a *group*  $G$  consists of a set  $G$  and a binary operation  $\cdot : G^2 \rightarrow G$  such that  $\cdot$  is associative, and there exists  $e \in G$ , the *identity* element of the group, such that:

- For all  $x \in G$  we have  $xe = ex = x$
- For all  $x \in G$ , there exists  $x^{-1} \in G$ , the *inverse* of  $x$ , such that  $xx^{-1} = x^{-1}x = e$ .

The *order* of a group  $G$ , denoted  $|G|$ , is the number of elements in  $G$  if  $G$  is finite; otherwise we write  $|G| = \infty$ .

**Proposition 6.3.** *The inverse of an element is unique.*

PROOF: If  $i$  and  $j$  are inverses of  $x$  then  $i = ixj = j$ .  $\square$

**Example 6.4.** • The *trivial* group is  $\{e\}$  under  $ee = e$ .

- $\mathbb{Z}$  is a group under addition
- $\mathbb{Q}$  is a group under addition
- $\mathbb{Q} - \{0\}$  is a group under multiplication
- $\mathbb{R}$  is a group under addition
- $\mathbb{R} - \{0\}$  is a group under multiplication
- $\mathbb{C}$  is a group under addition
- $\mathbb{C} - \{0\}$  is a group under multiplication
- $\{-1, 1\}$  is a group under multiplication
- For any category  $\mathcal{C}$  and object  $A \in \mathcal{C}$ , we have  $\text{Aut}_{\mathcal{C}}(A)$  is a group under  $gf = f \circ g$ .

For  $A$  a set, we call  $S_A = \text{Aut}_{\text{Set}}(A)$  the *symmetric group* or *group of permutations* of  $A$ .

- For  $n \geq 3$ , the *dihedral group*  $D_{2n}$  consists of the set of rigid motions that map the regular  $n$ -gon onto itself under composition.
- Let  $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$  under matrix multiplication.
- The quaternionic group  $Q_8$  is the group

$$\{1, -1, i, -i, j, -j, k, -k\}$$

with multiplication table

1	-1	i	-i	j	-j	k	-k
-1	1	-i	i	-j	j	-k	k
i	-i	-1	1	k	-k	-j	j
-i	i	1	-1	-k	k	j	-j
j	-j	-k	k	-1	1	i	-i
-j	j	k	-k	1	-1	-i	i
k	-k	j	-j	-i	i	-1	1
-k	k	-j	j	i	-i	1	-1

**Example 6.5.** • The only group of order 1 is the trivial group.

- The only group of order 2 is  $\mathbb{Z}_2$ .

- The only group of order 3 is  $\mathbb{Z}_3$ .
- There are exactly two groups of order 4:  $\mathbb{Z}_4$  and  $\mathbb{Z}_2 \times \mathbb{Z}_2$  under  $(a, b)(c, d) = (ac, bd)$ .

**Proposition 6.6** (Cancellation). *Let  $G$  be a group. Let  $a, g, h \in G$ . If  $ag = ah$  or  $ga = ha$  then  $g = h$ .*

PROOF: If  $ag = ah$  then  $g = a^{-1}ag = a^{-1}ah = h$ . Similarly if  $ga = ha$ .  $\square$

**Proposition 6.7.** *Let  $G$  be a group and  $g, h \in G$ . Then  $(gh)^{-1} = h^{-1}g^{-1}$ .*

PROOF: Since  $ghh^{-1}g^{-1} = e$ .  $\square$

**Definition 6.8.** Let  $G$  be a group. Let  $g \in G$ . We define  $g^n \in G$  for all  $n \in \mathbb{Z}$  as follows:

$$\begin{aligned} g^0 &= e \\ g^{n+1} &= g^n g & (n \geq 0) \\ g^{-n} &= (g^{-1})^n & (n > 0) \end{aligned}$$

**Proposition 6.9.** *Let  $G$  be a group. Let  $g \in G$  and  $m, n \in \mathbb{Z}$ . Then*

$$g^{m+n} = g^m g^n .$$

PROOF:

$\langle 1 \rangle 1$ . For all  $k \in \mathbb{Z}$  we have  $g^{k+1} = g^k g$

$\langle 2 \rangle 1$ . For all  $k \geq 0$  we have  $g^{k+1} = g^k g$

PROOF: Immediate from definition.

$\langle 2 \rangle 2$ .  $g^{-1+1} = g^{-1} g$

PROOF: Both are equal to  $e$ .

$\langle 2 \rangle 3$ . For all  $k > 1$  we have  $g^{-k+1} = g^{-k} g$

PROOF:

$$\begin{aligned} g^{-k+1} &= (g^{-1})^{k-1} \\ &= (g^{-1})^{k-1} g^{-1} g \\ &= (g^{-1})^k g \\ &= g^{-k} g \end{aligned}$$

$\langle 1 \rangle 2$ . For all  $k \in \mathbb{Z}$  we have  $g^{k-1} = g^k g^{-1}$

PROOF: Substitute  $k = k - 1$  above and multiply by  $g^{-1}$ .

$\langle 1 \rangle 3$ .  $g^{m+0} = g^m g^0$

PROOF: Since  $g^m g^0 = g^m e = g^m$ .

$\langle 1 \rangle 4$ . If  $g^{m+n} = g^m g^n$  then  $g^{m+n+1} = g^m g^{n+1}$

PROOF:

$$\begin{aligned} g^{m+n+1} &= g^{m+n} g & (\langle 1 \rangle 1) \\ &= g^m g^n g \\ &= g^m g^{n+1} & (\langle 1 \rangle 1) \end{aligned}$$

$\langle 1 \rangle 5$ . If  $g^{m+n} = g^m g^n$  then  $g^{m+n-1} = g^m g^{n-1}$

PROOF:

$$g^{m+n-1}g = g^{m+n} \quad (\langle 1 \rangle 1)$$

$$= g^m g^n$$

$$\therefore g^{m+n-1} = g^m g^n g^{-1}$$

$$= g^m g^{n-1} \quad (\langle 1 \rangle 2)$$

□

**Proposition 6.10.** Let  $G$  be a group. Let  $g \in G$  and  $m, n \in \mathbb{Z}$ . Then

$$(g^m)^n = g^{mn} .$$

PROOF:

$\langle 1 \rangle 1$ .  $(g^m)^0 = g^0$

PROOF: Both sides are equal to  $e$ .

$\langle 1 \rangle 2$ . If  $(g^m)^n = g^{mn}$  then  $(g^m)^{n+1} = g^{m(n+1)}$ .

PROOF:

$$(g^m)^{n+1} = (g^m)^n g^m \quad (\text{Proposition 6.9})$$

$$= g^{mn} g^m$$

$$= g^{mn+m}$$

(Proposition 6.9)

$\langle 1 \rangle 3$ . If  $(g^m)^n = g^{mn}$  then  $(g^m)^{n-1} = g^{m(n-1)}$ .

PROOF:

$$(g^m)^n = g^{mn}$$

$$\therefore (g^m)^{n-1} g^m = g^{mn-m} g^m \quad (\text{Proposition 6.9})$$

$$\therefore (g^m)^{n-1} = g^{mn-m} \quad (\text{Cancellation})$$

□

**Definition 6.11** (Commute). Let  $G$  be a group and  $g, h \in G$ . We say  $g$  and  $h$  *commute* iff  $gh = hg$ .

**Definition 6.12.** Let  $G$  be a group. Given  $g \in G$  and  $A \subseteq G$ , we define

$$gA = \{ga : a \in A\}, \quad Ag = \{ag : a \in A\} .$$

Given sets  $A, B \subseteq G$ , we define

$$AB = \{ab : a \in A, b \in B\} .$$

## 6.1 Order of an Element

**Definition 6.13** (Order). Let  $G$  be a group. Let  $g \in G$ . Then  $g$  has *finite order* iff there exists a positive integer  $n$  such that  $g^n = e$ . In this case, the *order* of  $g$ , denoted  $|g|$ , is the least positive integer  $n$  such that  $g^n = e$ .

If  $g$  does not have finite order, we write  $|g| = \infty$ .

**Proposition 6.14.** *Let  $G$  be a group. Let  $g \in G$  and  $n$  be a positive integer. If  $g^n = e$  then  $|g| \mid n$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $n = q|g| + d$  where  $0 \leq d < |g|$

PROOF: Division Algorithm.

$\langle 1 \rangle 2$ .  $g^d = e$

PROOF:

$$\begin{aligned} e &= g^n \\ &= g^{q|g|+d} \\ &= (g^{|g|})^q g^d && \text{(Propositions 6.9, 6.10)} \\ &= e^q g^d \\ &= g^d \end{aligned}$$

$\langle 1 \rangle 3$ .  $d = 0$

PROOF: By minimality of  $|g|$ .

$\langle 1 \rangle 4$ .  $n = q|g|$

□

**Corollary 6.14.1.** *Let  $G$  be a group. Let  $g \in G$  have finite order and  $n \in \mathbb{Z}$ . Then  $g^n = e$  if and only if  $|g| \mid n$ .*

**Proposition 6.15.** *Let  $G$  be a group and  $g \in G$ . Then  $|g| \leq |G|$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: w.l.o.g.  $G$  is finite.

$\langle 1 \rangle 2$ . PICK  $i, j$  with  $0 \leq i < j \leq |G|$  such that  $g^i = g^j$ .

PROOF: Otherwise  $g^0, g^1, \dots, g^{|G|}$  would be  $|G| + 1$  distinct elements of  $G$ .

$\langle 1 \rangle 3$ .  $g^{j-i} = e$

$\langle 1 \rangle 4$ .  $g$  has finite order and  $|g| \leq |G|$

PROOF: Since  $|g| \leq j - i \leq j \leq |G|$ .

□

**Proposition 6.16.** *Let  $G$  be a group. Let  $g \in G$  have finite order. Let  $m \in \mathbb{N}$ . Then*

$$|g^m| = \frac{\text{lcm}(m, |g|)}{m} = \frac{|g|}{\text{gcd}(m, |g|)}$$

PROOF: Since for any integer  $d$  we have

$$g^{md} = e \Leftrightarrow |g| \mid md \quad (\text{Corollary 6.14.1})$$

$$\Leftrightarrow \text{lcm}(m, |g|) \mid md$$

$$\Leftrightarrow \frac{\text{lcm}(m, |g|)}{m} \mid d$$

□

and so  $|g^m| = \frac{\text{lcm}(m, |g|)}{m}$  by Corollary 6.14.1. □

**Corollary 6.16.1.** *If  $g$  has odd order then  $|g^2| = |g|$ .*

**Proposition 6.17.** *Let  $G$  be a group. Let  $g, h \in G$  have finite order. Assume  $gh = hg$ . Then  $|gh|$  has finite order and*

$$|gh| \mid \text{lcm}(|g|, |h|)$$

PROOF: Since  $(gh)^{\text{lcm}(|g|, |h|)} = g^{\text{lcm}(|g|, |h|)} h^{\text{lcm}(|g|, |h|)} = e$ .  $\square$

**Example 6.18.** This example shows that we cannot remove the hypothesis that  $gh = hg$ .

In  $\text{GL}_2(\mathbb{R})$ , take

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then  $|g| = 4$ ,  $|h| = 3$  and  $|gh| = \infty$ .

**Proposition 6.19.** *Let  $G$  be a group and  $g, h \in G$  have finite order. If  $gh = hg$  and  $\gcd(|g|, |h|) = 1$  then  $|gh| = |g||h|$ .*

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } N = |gh|$$

$$\langle 1 \rangle 2. g^N = (h^{-1})^N$$

$$\langle 1 \rangle 3. g^{N|g|} = e$$

$$\langle 1 \rangle 4. |g^N| \mid |g|$$

$$\langle 1 \rangle 5. h^{-N|h|} = e$$

$$\langle 1 \rangle 6. |g^N| \mid |h|$$

$$\langle 1 \rangle 7. |g^N| = 1$$

PROOF: Since  $\gcd(|g|, |h|) = 1$ .

$$\langle 1 \rangle 8. g^N = e$$

$$\langle 1 \rangle 9. |g| \mid N$$

$$\langle 1 \rangle 10. h^{-N} = e$$

$$\langle 1 \rangle 11. |h| \mid N$$

$$\langle 1 \rangle 12. N = |g||h|$$

PROOF: Using Proposition 6.17.

$\square$

**Proposition 6.20.** *Let  $G$  be a finite group. Assume there is exactly one element  $f \in G$  of order 2. Then the product of all the elements of  $G$  is  $f$ .*

PROOF: Let the elements of  $G$  be  $g_1, g_2, \dots, g_n$ . Apart from  $e$  and  $f$ , every element and its inverse are distinct elements of the list. Hence the product of the list is  $ef = f$ .  $\square$

**Proposition 6.21.** *Let  $G$  be a finite group of order  $n$ . Let  $m$  be the number of elements of  $G$  of order 2. Then  $n - m$  is odd.*

PROOF: In the list of all elements that are not of order 2, every element and its inverse are distinct except for  $e$ . Hence the list has odd length.  $\square$

**Corollary 6.21.1.** *If a finite group has even order, then it contains an element of order 2.*

**Proposition 6.22.** *Let  $G$  be a group and  $a, g \in G$ . Then  $|aga^{-1}| = |g|$ .*

PROOF: Since

$$\begin{aligned} (aga^{-1})^n = e &\Leftrightarrow ag^na^{-1} = e \\ &\Leftrightarrow g^n = e \end{aligned} \quad \square$$

**Proposition 6.23.** *Let  $G$  be a group and  $g, h \in G$ . Then  $|gh| = |hg|$ .*

PROOF: Since  $|gh| = |ghgg^{-1}| = |hg|$ .  $\square$

**Proposition 6.24.** *Let  $G$  be a group of order  $n$ . Let  $k$  be relatively prime to  $n$ . Then every element in  $G$  has the form  $x^k$  for some  $x$ .*

$\langle 1 \rangle 1$ . PICK integers  $a$  and  $b$  such that  $an + bk = 1$ .

$\langle 1 \rangle 2$ . LET:  $g \in G$

$\langle 1 \rangle 3$ .  $g = (g^b)^k$

PROOF:

$$\begin{aligned} g &= g \cdot (g^n)^{-a} & (g^n = e) \\ &= g^{1-an} \\ &= g^{bk} \end{aligned}$$

$\square$

## 6.2 Generators

**Definition 6.25** (Generator). Let  $G$  be a group and  $a \in G$ . We say  $a$  *generates* the group iff, for all  $x \in G$ , there exists an integer  $n$  such that  $x^n = a$ .

**Example 6.26.**  $\text{SL}_2(\mathbb{Z})$  is generated by

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $H = \langle s, t \rangle$

$\langle 1 \rangle 2$ . For all  $q \in \mathbb{Z}$  we have  $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in H$ .

PROOF: It is  $t^q$ .

$\langle 1 \rangle 3$ . For all  $q \in \mathbb{Z}$  we have  $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \in H$ .

PROOF:

$$\begin{aligned} st^{-q}s^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \end{aligned}$$

⟨1⟩4.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & qa+b \\ c & qc+d \end{pmatrix}$$

⟨1⟩5.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} a+qb & b \\ c+qd & d \end{pmatrix}$$

⟨1⟩6. For any  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , if  $c$  and  $d$  are both nonzero, then there exists  $N \in H$  such that the bottom row of  $MN$  has one entry the same as  $M$  and one entry with smaller absolute value.

PROOF: From ⟨1⟩4 and ⟨1⟩5 taking  $q = -1$ .

⟨1⟩7. For any  $M \in \mathrm{SL}_2(\mathbb{Z})$ , there exists  $N \in H$  such that  $MN$  has a zero on the bottom row.

PROOF: Apply ⟨1⟩6 repeatedly.

⟨1⟩8. Any matrix in  $\mathrm{SL}_2(\mathbb{Z})$  with a zero on the bottom row is in  $H$ .

⟨2⟩1.  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H$

PROOF: ⟨1⟩2

⟨2⟩2.  $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \in H$

PROOF: It is  $s^2 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$  since  $s^2 = -I$ .

⟨2⟩3.  $\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} \in H$

PROOF: It is  $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} s$ .

⟨2⟩4.  $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \in H$

PROOF: It is  $s^2 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s$ .

⟨1⟩9. Every matrix in  $\mathrm{SL}_2(\mathbb{Z})$  is in  $H$ .

□



## Chapter 7

# Group Homomorphisms

**Definition 7.1** (Homomorphism). Let  $G$  and  $H$  be groups. A (group) *homomorphism*  $\phi : G \rightarrow H$  is a function such that, for all  $x, y \in G$ ,

$$\phi(xy) = \phi(x)\phi(y) \ .$$

**Proposition 7.2.** Let  $G$  and  $H$  be groups with identities  $e_G$  and  $e_H$ . Let  $\phi : G \rightarrow H$  be a group homomorphism. Then  $\phi(e_G) = e_H$ .

PROOF: Since  $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$  and so  $\phi(e_G) = e_H$  by Cancellation.  $\square$

**Proposition 7.3.** Let  $\phi : G \rightarrow H$  be a group homomorphism. For all  $x \in G$  we have  $\phi(x^{-1}) = \phi(x)^{-1}$ .

PROOF: Since  $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_H$ .  $\square$

**Proposition 7.4.** Let  $G, H$  and  $K$  be groups. If  $\phi : G \rightarrow H$  and  $\psi : H \rightarrow K$  are homomorphisms then  $\psi \circ \phi : G \rightarrow K$  is a homomorphism.

PROOF: For  $x, y \in G$  we have

$$\psi(\phi(xy)) = \psi(\phi(x)\phi(y)) = \psi(\phi(x))\psi(\phi(y)) \ .$$

**Proposition 7.5.** Let  $G$  be a group. Then  $\text{id}_G : G \rightarrow G$  is a group homomorphism.

PROOF: For  $x, y \in G$  we have  $\text{id}_G(xy) = xy = \text{id}_G(x)\text{id}_G(y)$ .  $\square$

**Proposition 7.6.** Let  $\phi : G \rightarrow H$  be a group homomorphism. Let  $g \in G$  have finite order. Then  $|\phi(g)|$  divides  $|g|$ .

PROOF: Since  $\phi(g)^{|g|} = \phi(g^{|g|}) = e$ .  $\square$

**Definition 7.7** (Category of Groups). Let **Grp** be the category of groups and group homomorphisms.

**Example 7.8.** There are 49487365402 groups of order 1024 up to isomorphism.

**Proposition 7.9.** *A group homomorphism  $\phi : G \rightarrow H$  is an isomorphism in **Grp** if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\phi$  is bijective.

PROVE:  $\phi^{-1}$  is a group homomorphism.

$\langle 1 \rangle 2$ . LET:  $h, h' \in H$

$\langle 1 \rangle 3$ .  $\phi(\phi^{-1}(hh')) = \phi(\phi^{-1}(h)\phi^{-1}(h'))$

PROOF: Both are equal to  $hh'$ .

$\langle 1 \rangle 4$ .  $\phi^{-1}(hh') = \phi^{-1}(h)\phi^{-1}(h')$

□

**Corollary 7.9.1.**

$$D_6 \cong C_3$$

PROOF: The canonical homomorphism  $D_6 \rightarrow C_3$  is bijective. □

**Corollary 7.9.2.**

$$(\mathbb{R}, +) \cong (\{x \in \mathbb{R} : x > 0\}, \cdot)$$

PROOF: The function that maps  $x$  to  $e^x$  is a bijective homomorphism. □

**Proposition 7.10.** *The trivial group is the zero object in **Grp**.*

PROOF: For any group  $G$ , the unique function  $G \rightarrow \{e\}$  is a group homomorphism, and the only group homomorphism  $\{e\} \rightarrow G$  maps  $e$  to  $e_G$ . □

**Proposition 7.11.** *For any groups  $G$  and  $H$ , the set  $G \times H$  under  $(g, h)(g', h') = (gg', hh')$  is the product of  $G$  and  $H$  in **Grp**.*

PROOF:

$\langle 1 \rangle 1$ .  $G \times H$  is a group.

$\langle 2 \rangle 1$ . The multiplication is associative.

PROOF: Since  $(g_1, h_1)((g_2, h_2)(g_3, h_3)) = ((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1g_2g_3, h_1h_2h_3)$ .

$\langle 2 \rangle 2$ .  $(e_G, e_H)$  is the identity.

PROOF: Since  $(g, h)(e_G, e_H) = (e_G, e_H)(g, h) = (g, h)$ .

$\langle 2 \rangle 3$ . The inverse of  $(g, h)$  is  $(g^{-1}, h^{-1})$ .

PROOF: Since  $(g, h)(g^{-1}, h^{-1}) = (g^{-1}, h^{-1})(g, h) = (e_G, e_H)$ .

$\langle 1 \rangle 2$ .  $\pi_1 : G \times H \rightarrow G$  is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$ .  $\pi_2 : G \times H \rightarrow H$  is a group homomorphism.

PROOF: Immediate from definitions.

$\langle 1 \rangle 4$ . For any group homomorphism  $\phi : K \rightarrow G$  and  $\psi : K \rightarrow H$ , the function  $\langle \phi, \psi \rangle : K \rightarrow G \times H$  where  $\langle \phi, \psi \rangle(k) = (\phi(k), \psi(k))$  is a group homomorphism.

PROOF:

$$\begin{aligned} \langle \phi, \psi \rangle(kk') &= (\phi(kk'), \psi(kk')) \\ &= (\phi(k)\phi(k'), \psi(k)\psi(k')) \\ &= (\phi(k), \psi(k))(\phi(k'), \psi(k')) \\ &= \langle \phi, \psi \rangle(k)\langle \phi, \psi \rangle(k') \end{aligned}$$

□

## 7.1 Subgroups

**Definition 7.12** (Subgroup). Let  $(G, \cdot)$  and  $(H, *)$  be groups such that  $H$  is a subset of  $G$ . Then  $H$  is a *subgroup* of  $G$  iff the inclusion  $i : H \hookrightarrow G$  is a group homomorphism.

**Proposition 7.13.** *If  $(H, *)$  is a subgroup of  $(G, \cdot)$  then  $*$  is the restriction of  $\cdot$  to  $H$ .*

PROOF: Given  $x, y \in H$  we have

$$x * y = i(x * y) = i(x) \cdot i(y) = x \cdot y. \quad \square$$

**Example 7.14.** For any group  $G$  we have  $\{e\}$  is a subgroup of  $G$ .

**Proposition 7.15.** *Let  $G$  be a group. Let  $H$  be a subset of  $G$ . Then  $H$  is a subgroup of  $G$  iff  $H$  is nonempty and, for all  $x, y \in H$ , we have  $xy^{-1} \in H$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $H$  is a subgroup of  $G$  then  $H$  is nonempty.

PROOF: Since every group has an identity element and so is nonempty.

$\langle 1 \rangle 2$ . If  $H$  is a subgroup of  $G$  then, for all  $x, y \in H$ , we have  $xy^{-1} \in H$ .

PROOF: Easy.

$\langle 1 \rangle 3$ . If  $H$  is nonempty and, for all  $x, y \in H$ , we have  $xy^{-1} \in H$ , then  $H$  is a subgroup of  $G$ .

$\langle 2 \rangle 1$ . ASSUME:  $H$  is nonempty.

$\langle 2 \rangle 2$ . ASSUME:  $\forall x, y \in H. xy^{-1} \in H$

$\langle 2 \rangle 3$ .  $e \in H$

PROOF: Pick  $x \in H$ . We have  $e = xx^{-1} \in H$ .

$\langle 2 \rangle 4$ .  $\forall x \in H. x^{-1} \in H$

PROOF: Given  $x \in H$  we have  $x^{-1} = ex^{-1} \in H$ .

$\langle 2 \rangle 5$ .  $H$  is closed under the restriction of  $\cdot$

PROOF: Given  $x, y \in H$  we have  $xy = x(y^{-1})^{-1} \in H$ .

$\langle 2 \rangle 6$ .  $H$  is a group under the restriction of  $\cdot$

PROOF: Associativity is inherited from  $G$  and the existence of an identity element and inverses follows from  $\langle 2 \rangle 3$  and  $\langle 2 \rangle 4$ .

$\langle 2 \rangle 7$ . The inclusion  $H \hookrightarrow G$  is a group homomorphism.

PROOF: For  $x, y \in H$  we have  $i(xy) = i(x)i(y) = xy$ .

$\square$

**Corollary 7.15.1.** *The intersection of a set of subgroups of  $G$  is a subgroup of  $G$ .*

**Corollary 7.15.2.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Let  $K$  be a subgroup of  $H$ . Then  $\phi^{-1}(K)$  is a subgroup of  $G$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\phi^{-1}(K)$  is nonempty.

PROOF: Since  $e \in \phi^{-1}(K)$ .

$\langle 1 \rangle 2$ . LET:  $x, y \in \phi^{-1}(K)$

- $\langle 1 \rangle 3. \phi(x), \phi(y) \in K$
- $\langle 1 \rangle 4. \phi(x)\phi(y)^{-1} \in K$
- $\langle 1 \rangle 5. \phi(xy^{-1}) \in K$
- $\langle 1 \rangle 6. xy^{-1} \in \phi^{-1}(K)$

□

**Corollary 7.15.3.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Let  $K$  be a subgroup of  $G$ . Then  $\phi(K)$  is a subgroup of  $H$ .*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $x, y \in \phi(K)$
- $\langle 1 \rangle 2.$  PICK  $a, b \in K$  such that  $x = \phi(a)$  and  $y = \phi(b)$
- $\langle 1 \rangle 3. xy^{-1} = \phi(ab^{-1})$
- $\langle 1 \rangle 4. xy^{-1} \in \phi(K)$

□

**Proposition 7.16.** *Let  $G$  be a subgroup of  $\mathbb{Z}$ . Then there exists  $d \geq 0$  such that  $G = d\mathbb{Z}$ .*

PROOF:

- $\langle 1 \rangle 1.$  ASSUME: w.l.o.g.  $G \neq \{0\}$

PROOF: Since  $\{0\} = 0\mathbb{Z}$ .

- $\langle 1 \rangle 2.$  LET:  $d$  be the least positive element of  $G$ .

PROVE:  $G = d\mathbb{Z}$

PROOF: If  $n \in G$  then  $-n \in G$  so  $G$  must contain a positive element.

- $\langle 1 \rangle 3. G \subseteq d\mathbb{Z}$

- $\langle 2 \rangle 1.$  LET:  $n \in G$

- $\langle 2 \rangle 2.$  LET:  $q$  and  $r$  be the integers such that  $n = qd + r$  and  $0 \leq r < d$ .

- $\langle 2 \rangle 3. r \in G$

PROOF: Since  $r = n - qd$ .

- $\langle 2 \rangle 4. r = 0$

PROOF: By minimality of  $d$ .

- $\langle 2 \rangle 5. n = qd \in d\mathbb{Z}$

- $\langle 1 \rangle 4. d\mathbb{Z} \subseteq G$

□

## 7.2 Kernel

**Definition 7.17** (Kernel). Let  $\phi : G \rightarrow H$  be a group homomorphism. The *kernel* of  $\phi$  is

$$\ker \phi = \{g \in G : \phi(g) = e\} .$$

**Proposition 7.18.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Then  $\ker \phi$  is a subgroup of  $G$ .*

PROOF: Corollary 7.15.2. □

**Proposition 7.19.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Then the inclusion  $i : \ker \phi \hookrightarrow G$  is terminal in the category of pairs  $(K, \alpha : K \rightarrow G)$  such that  $\phi \circ \alpha = 0$ .*

PROOF:

$\langle 1 \rangle 1.$   $\phi \circ i = 0$

$\langle 1 \rangle 2.$  For any group  $K$  and homomorphism  $\alpha : K \rightarrow G$  such that  $\phi \circ \alpha = 0$ , there exists a unique homomorphism  $\beta : K \rightarrow \ker \phi$  such that  $i \circ \beta = \alpha$ .

□

**Proposition 7.20.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Then the following are equivalent:*

1.  $\phi$  is monic.
2.  $\ker \phi = \{e\}$
3.  $\phi$  is injective.

PROOF:

$\langle 1 \rangle 1.$   $1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $\phi$  is monic.

$\langle 2 \rangle 2.$  LET:  $i : \ker \phi \hookrightarrow G, j : \{e\} \hookrightarrow \ker \phi \hookrightarrow G$  be the inclusions.

$\langle 2 \rangle 3.$   $\phi \circ i = \phi \circ j$

$\langle 2 \rangle 4.$   $i = j$

$\langle 1 \rangle 2.$   $2 \Rightarrow 3$

$\langle 2 \rangle 1.$  ASSUME:  $\ker \phi = \{e\}$

$\langle 2 \rangle 2.$  LET:  $x, y \in G$

$\langle 2 \rangle 3.$  ASSUME:  $\phi(x) = \phi(y)$

$\langle 2 \rangle 4.$   $\phi(xy^{-1}) = e$

$\langle 2 \rangle 5.$   $xy^{-1} \in \ker \phi$

$\langle 2 \rangle 6.$   $xy^{-1} = e$

$\langle 2 \rangle 7.$   $x = y$

$\langle 1 \rangle 3.$   $3 \Rightarrow 1$

PROOF: Easy.

□

**Proposition 7.21.** *A group homomorphism is an epimorphism if and only if it is surjective.*

## 7.3 Inner Automorphisms

**Proposition 7.22.** *Let  $G$  be a group and  $g \in G$ . The function  $\gamma_g : G \rightarrow G$  defined by  $\gamma_g(a) = gag^{-1}$  is an automorphism on  $G$ .*

PROOF:

$\langle 1 \rangle 1.$   $\gamma_g$  is a homomorphism.

PROOF:

$$\begin{aligned}\gamma_g(ab) &= gabg^{-1} \\ &= gag^{-1}gbg^{-1} \\ &= \gamma_g(a)\gamma_g(b)\end{aligned}$$

<1>2.  $\gamma_g$  is injective.

PROOF: By Cancellation.

<1>3.  $\gamma_g$  is surjective.

PROOF: Given  $b \in G$ , we have  $\gamma_g(g^{-1}bg) = b$ .

□

**Definition 7.23** (Inner Automorphism). Let  $G$  be a group. An *inner automorphism* on  $G$  is a function of the form  $\gamma_g(a) = gag^{-1}$  for some  $g \in G$ .

We write  $\text{Inn}(G)$  for the set of inner automorphisms of  $G$ .

**Proposition 7.24.** Let  $G$  be a group. The function  $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$  that maps  $g$  to  $\gamma_g$  is a group homomorphism.

PROOF: Since  $\gamma_{gh}(a) = ghah^{-1}g^{-1} = \gamma_g(\gamma_h(a))$ . □

**Corollary 7.24.1.**  $\text{Inn}(G)$  is a subgroup of  $\text{Aut}_{\mathbf{Grp}}(G)$ .

## 7.4 Direct Products

**Definition 7.25** (Direct Product). The *direct product* of groups  $G$  and  $H$  is their product in  $\mathbf{Grp}$ .

## 7.5 Free Groups

**Proposition 7.26.** Let  $A$  be a set. Let  $\mathcal{F}^A$  be the category whose objects are pairs  $(G, j)$  where  $G$  is a group and  $j$  is a function  $A \rightarrow G$ , with morphisms  $f : (G, j) \rightarrow (H, k)$  the group homomorphisms  $f : G \rightarrow H$  such that  $f \circ j = k$ . Then  $\mathcal{F}^A$  has an initial object.

PROOF:

<1>1. LET:  $W(A)$  be the set of words in the alphabet whose elements are the elements of  $A$  together with  $\{a^{-1} : a \in A\}$ .

<1>2. LET:  $r : W(A) \rightarrow W(A)$  be the function that, given a word  $w$ , removes the first pair of letters of the form  $aa^{-1}$  or  $a^{-1}a$ ; if there is no such pair, then  $r(w) = w$ .

<1>3. Let us say that a word  $w$  is a *reduced word* iff  $r(w) = w$ .

<1>4. For any word  $w$  of length  $n$ , we have  $r^{\lceil \frac{n}{2} \rceil}(w)$  is a reduced word.

PROOF: Since we cannot remove more than  $n/2$  pairs of letters from  $w$ .

<1>5. LET:  $R : W(A) \rightarrow W(A)$  be the function  $R(w) = r^{\lceil \frac{n}{2} \rceil}(w)$ , where  $n$  is the length of  $w$ .

<1>6. LET:  $F(A)$  be the set of reduced words.

<1>7. Define  $\cdot : F(A)^2 \rightarrow F(A)$  by  $w \cdot w' = R(ww')$

(1)8.  $\cdot$  is associative.

PROOF: Both  $w_1 \cdot (w_2 \cdot w_3)$  and  $(w_1 \cdot w_2) \cdot w_3$  are equal to  $R(w_1 w_2 w_3)$ .

(1)9. The empty word is the identity element in  $F(A)$

(1)10. The inverse of  $a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}$  is  $a_n^{\mp 1} \dots a_2^{\mp 1} a_1^{\mp 1}$ .

(1)11. LET:  $j : A \rightarrow F(A)$  be the function that maps  $a$  to the word  $a$  of length

(1)12. LET:  $G$  be any group and  $k : A \rightarrow G$  any function.

(1)13. The only morphism  $f : (F(A), j) \rightarrow (G, k)$  in  $\mathcal{F}^A$  is  $f(a_1^{\pm 1} a_2^{\pm 1} \dots a_n^{\pm 1}) = k(a_1)^{\pm 1} k(a_2)^{\pm 1} \dots k(a_n)^{\pm 1}$ .

□

**Definition 7.27** (Free Group). For any set  $A$ , the *free group* on  $A$  is the initial object  $(F(A), i)$  in  $\mathcal{F}^A$ .

**Proposition 7.28.**  $i : A \rightarrow F(A)$  is injective.

PROOF:

(1)1. LET:  $x, y \in A$

(1)2. ASSUME:  $x \neq y$

PROVE:  $i(x) \neq i(y)$

(1)3. LET:  $f : A \rightarrow C_2$  be the function that maps  $x$  to 0 and all other elements of  $A$  to 1.

(1)4. LET:  $\phi : F(A) \rightarrow C_2$  be the group homomorphism such that  $f = \phi \circ i$ .

(1)5.  $f(x) \neq f(y)$

(1)6.  $\phi(i(x)) \neq \phi(i(y))$

(1)7.  $i(x) \neq i(y)$

□

**Proposition 7.29.**

$$F(0) \cong \{e\}$$

PROOF: For any set  $A$ , the unique group homomorphism  $\{e\} \rightarrow A$  makes the following diagram commute.

$$\begin{array}{ccc} \{e\} & \longrightarrow & A \\ \uparrow & \nearrow & \\ \emptyset & & \end{array}$$

**Proposition 7.30.** The free group on 1 is  $\mathbb{Z}$  with the injection mapping 0 to 1.

PROOF: Given any group  $G$  and function  $a : 1 \rightarrow G$ , the required unique homomorphism  $\phi : \mathbb{Z} \rightarrow G$  is defined by  $\phi(n) = a(0)^n$ . □

**Proposition 7.31.** For any sets  $A$  and  $B$ , we have that  $F(A + B)$  is the coproduct of  $F(A)$  and  $F(B)$  in **Grp**.

$$\begin{array}{ccccc}
& & G & & \\
& f \nearrow & \uparrow k & \nwarrow g & \\
F(A) & \xrightarrow{\kappa_1} & F(A+B) & \xleftarrow{\kappa_2} & F(B) \\
i_A \uparrow & & j \uparrow & & i_B \uparrow \\
A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
\end{array}$$

PROOF:

- (1)1. LET:  $i_A : A \rightarrow F(A)$ ,  $i_B : B \rightarrow F(B)$ ,  $j : A+B \rightarrow F(A+B)$  be the canonical injections.  
 (1)2. LET:  $\kappa_1, \kappa_2$  be the unique group homomorphisms that make the diagram above commute.  
 (1)3. LET:  $G$  be any group and  $f : F(A) \rightarrow G$ ,  $g : F(B) \rightarrow G$  any group homomorphisms.  
 (1)4. LET:  $h : A+B \rightarrow G$  be the unique function such that  $h \circ k_1 = f \circ i_A$  and  $h \circ k_2 = g \circ i_B$ .  
 (1)5. LET:  $k : F(A+B) \rightarrow G$  be the unique group homomorphism such that  $k \circ j = h$ .  
 (1)6.  $k$  is the unique group homomorphism such that  $k \circ \kappa_1 \circ i_A = f \circ i_A$  and  $k \circ \kappa_2 \circ i_B = g \circ i_B$ .  
 (1)7.  $k$  is the unique group homomorphism such that  $k \circ \kappa_1 = f$  and  $k \circ \kappa_2 = g$ .  
 □

**Definition 7.32** (Subgroup Generated by a Group). Let  $G$  be a group and  $A$  a subset of  $G$ . Let  $\phi : F(A) \rightarrow G$  be the unique group homomorphism such that  $\phi(a) = a$  for all  $a \in A$ . The subgroup *generated* by  $A$  is

$$\langle A \rangle := \text{im } \phi$$

$$\begin{array}{ccc}
F(A) & \xrightarrow{\phi} & G \\
\uparrow & \nearrow & \\
A & & 
\end{array}$$

**Proposition 7.33.** Let  $G$  be a group and  $A$  a subset of  $G$ . Then  $\langle A \rangle$  is the set of all elements of the form  $a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$  (where  $n \geq 0$ ) such that  $a_1, \dots, a_n \in A$ .

PROOF: Immediate from definitions. □

**Corollary 7.33.1.** Let  $G$  be a group and  $g \in G$ . Then

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\}.$$

**Proposition 7.34.** Let  $G$  be a group and  $A$  a subset of  $G$ . Then  $\langle A \rangle$  is the intersection of all the subgroups of  $G$  that include  $A$ .



PROOF: Easy.  $\square$

**Definition 7.35** (Finitely Generated). Let  $G$  be a group. Then  $G$  is *finitely generated* iff there exists a finite subset  $A$  of  $G$  such that  $G = \langle A \rangle$ .

**Proposition 7.36.** *Every subgroup of a finitely generated free group is free.*

PROOF: TODO.

**Proposition 7.37.**  *$F(2)$  includes subgroups isomorphic to the free group on arbitrarily many generators.*

PROOF: TODO

**Proposition 7.38.**

$$[F(2), F(2)] \cong F(\mathbb{Z})$$

PROOF: TODO

## 7.6 Normal Subgroups

**Definition 7.39** (Normal Subgroup). A subgroup  $N$  of  $G$  is *normal* iff, for all  $g \in G$  and  $n \in N$ , we have  $gng^{-1} \in N$ .

**Example 7.40.** Every subgroup of  $Q_8$  is normal.

**Proposition 7.41.** *Let  $G$  be a group and  $N$  a subgroup of  $G$ . Then the following are equivalent.*

1.  $N$  is normal.
2.  $\forall g \in G. gNg^{-1} \subseteq N$
3.  $\forall g \in G. gNg^{-1} = N$
4.  $\forall g \in G. gN \subseteq Ng$
5.  $\forall g \in G. gN = Ng$

PROOF:

$\langle 1 \rangle 1. 1 \Leftrightarrow 2$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: If 2 holds then we have  $gNg^{-1} \subseteq N$  and  $g^{-1}Ng \subseteq N$  hence  $N = gNg^{-1}$ .

$\langle 1 \rangle 3. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 4. 2 \Leftrightarrow 4$

PROOF: Easy.

$\langle 1 \rangle 5. 3 \Leftrightarrow 5$

PROOF: Easy.

□

**Proposition 7.42.** *Let  $\phi : G \rightarrow H$  be a group homomorphism. Then  $\ker \phi$  is a normal subgroup of  $G$ .*

PROOF: Given  $g \in G$  and  $n \in \ker \phi$  we have

$$\begin{aligned}\phi(gng^{-1}) &= \phi(g)\phi(n)\phi(g)^{-1} \\ &= \phi(g)\phi(g)^{-1} \\ &= e\end{aligned}$$

and so  $gng^{-1} \in \ker \phi$ . □

## 7.7 Quotient Groups

**Definition 7.43.** Let  $G$  be a group. Let  $\sim$  be an equivalence relation on  $G$ . Then we say that  $\sim$  is *compatible* with the group operation on  $G$  iff, for all  $a, a', g \in G$ , if  $a \sim a'$  then  $ga \sim ga'$  and  $ag \sim a'g$ .

**Proposition 7.44.** *Let  $G$  be a group. Let  $\sim$  be an equivalence relation on  $G$ . Then there exists an operation  $\cdot : (G/\sim)^2 \rightarrow G/\sim$  such that*

$$\forall a, b \in G. [a][b] = [ab]$$

*iff  $\sim$  is compatible with the group operation on  $G$ . In this case,  $G/\sim$  is a group under  $\cdot$  and the canonical function  $\pi : G \rightarrow G/\sim$  is a group homomorphism, and is universal with respect to group homomorphisms  $\phi : G \rightarrow G'$  such that if  $a \sim a'$  then  $\phi(a) = \phi(a')$ .*

PROOF: Easy. □

**Definition 7.45** (Quotient Group). Let  $G$  be a group. Let  $\sim$  be an equivalence relation on  $G$  that is compatible with the group operation on  $G$ . Then  $G/\sim$  is the *quotient group* of  $G$  by  $\sim$  under  $[a][b] = [ab]$ .

**Proposition 7.46.** *Let  $G$  be a group and  $H$  a subgroup of  $G$ . Then  $H$  is normal if and only if there exists a group  $K$  and homomorphism  $\phi : G \rightarrow K$  such that  $H = \ker \phi$ .*

PROOF: One direction is given by Proposition 7.42. For the other direction, take  $K = G/H$  and  $\phi$  to be the canonical map  $G \rightarrow G/H$ . □

**Definition 7.47** (Modular Group). The *modular group*  $\text{PSL}_2(\mathbb{Z})$  is  $\text{SL}_2(\mathbb{Z})/\{I, -I\}$ .

**Proposition 7.48.**  $\text{PSL}_2(\mathbb{Z})$  is generated by  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ .

PROOF: By Example 6.26.

**Proposition 7.49** (Roger Alperin).  $\text{PSL}_2(\mathbb{Z})$  is presented by  $(x, y | x^2, y^3)$ .

PROOF:

⟨1⟩1. LET:  $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

⟨1⟩2. LET:  $y = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

⟨1⟩3. Define an action of  $\text{PSL}_2(\mathbb{Z})$  on  $\mathbb{R} - \mathbb{Q}$  by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} r = \frac{ar + b}{cr + d}.$$

⟨2⟩1. Given  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z})$  and  $r$  irrational we have  $\frac{ar+b}{cr+d}$  is irrational.

⟨3⟩1. ASSUME: for a contradiction  $\frac{ar+b}{cr+d} = \frac{p}{q}$  where  $p$  and  $q$  are integers with  $q > 0$ .

⟨3⟩2.  $aqr + bq = cpr + dp$

⟨3⟩3.  $(aq - cp)r = dp - bq$

⟨3⟩4.  $aq = cp = dp - bq = 0$

⟨3⟩5.  $adq - cdp = 0$

⟨3⟩6.  $cdp - cbq = 0$

⟨3⟩7.  $(ad - cb)q = 0$

PROOF: Since  $ad - cb = 1$ .

⟨3⟩8.  $q = 0$

⟨3⟩9. Q.E.D.

PROOF: This contradicts ⟨3⟩1.

⟨2⟩2.  $-Ir = r$

PROOF: Since  $-Ir = \frac{-r}{-1} = r$ .

⟨2⟩3. Given  $A, B \in \text{PSL}_2(\mathbb{Z})$  we have  $A(Br) = (AB)r$ .

PROOF:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[ \begin{pmatrix} e & f \\ g & h \end{pmatrix} r \right] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{er + f}{gr + h} \\ &= \frac{a \frac{er+f}{gr+h} + b}{c \frac{er+f}{gr+h} + d} \\ &= \frac{a(er + f) + b(gr + h)}{c(er + f) + d(gr + h)} \\ &= \frac{(ae + bg)r + (af + bh)}{(ce + dg)r + (cf + dh)} \\ &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} r \\ &= \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right] r \end{aligned}$$

⟨1⟩4.

$$yr = 1 - \frac{1}{r}$$

⟨1⟩5.

$$y^{-1}r = \frac{1}{1 - r}$$

PROOF: Since  $y^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$   
 (1)6.

PROOF: Since  $yx = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ .  $yxr = 1 + r$   
 (1)7.

$y^{-1}xr = \frac{r}{1+r}$   
 PROOF: Since  $y^{-1}x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ .

(1)8. If  $r > -1$  is positive then  $yxr$  is positive.

(1)9. If  $r$  is positive then  $y^{-1}xr$  is positive.

(1)10. If  $r < -1$  then  $y^{-1}xr$  is positive.

(1)11. If  $r$  is negative then  $yr$  is positive.

(1)12. If  $r$  is negative then  $y^{-1}r$  is positive.

(1)13. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)$$

with one or more factors can equal the identity.

PROOF: If the last factor is  $(yx)$ , then the product maps numbers in  $(-1, 0)$  to positive numbers. If the last factor is  $(y^{-1}x)$ , then the product maps numbers  $< -1$  to positive numbers.

(1)14. No product of the form

$$(y^{\pm 1}x)(y^{\pm 1}x) \cdots (y^{\pm 1}x)y^{\pm 1}$$

with one or more factors can equal the identity.

PROOF: The product maps negative numbers to positive numbers.

(1)15.  $\text{PSL}_2(\mathbb{Z})$  is presented by  $(x, y|x^2, y^3)$ .

□

**Corollary 7.49.1.**  $\text{PSL}_2(\mathbb{Z})$  is the coproduct of  $C_2$  and  $C_3$  in **Grp**.

**Theorem 7.50.** Every group homomorphism  $\phi : G \rightarrow H$  may be decomposed as

$$G \longrightarrow G/\ker \phi \xrightarrow{\cong} \text{im } \phi \longrightarrow H$$

PROOF: Easy. □

**Corollary 7.50.1** (First Isomorphism Theorem). Let  $\phi : G \rightarrow H$  be a surjective group homomorphism. Then  $H \cong G/\ker \phi$ .

**Proposition 7.51.** Let  $H_1$  be a normal subgroup of  $G_1$  and  $H_2$  a normal subgroup of  $G_2$ . Then  $H_1 \times H_2$  is a normal subgroup of  $G_1 \times G_2$ , and

$$\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}.$$

PROOF:  $\pi \times \pi : G_1 \times G_2 \twoheadrightarrow G_1/H_1 \times G_2/H_2$  is a surjective homomorphism with kernel  $H_1 \times H_2$ . □

**Example 7.52.**

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

PROOF: Map a real number  $r$  to  $(\cos r, \sin r)$ . The result is a surjective group homomorphism with kernel  $\mathbb{Z}$ .  $\square$

**Proposition 7.53.** *Let  $H$  be a normal subgroup of a group  $G$ . For every subgroup  $K$  of  $G$  that includes  $H$ , we have  $H$  is a normal subgroup of  $K$ , and  $K/H$  is a subgroup of  $G/H$ . The mapping*

$$u : \{\text{subgroups of } G \text{ including } H\} \rightarrow \{\text{subgroups of } G/H\}$$

*with  $u(K) = K/H$  is a poset isomorphism.*

PROOF:

- $\langle 1 \rangle 1$ . If  $K$  is a subgroup of  $G$  that includes  $H$  then  $H$  is normal in  $K$ .
- $\langle 1 \rangle 2$ . If  $K$  is a subgroup of  $G$  that includes  $H$  then  $K/H$  is a subgroup of  $G/H$ .
- $\langle 1 \rangle 3$ . If  $H \subseteq K_1 \subseteq K_2$  then  $K_1/H \subseteq K_2/H$ .
- $\langle 1 \rangle 4$ . If  $K_1/H = K_2/H$  then  $K_1 = K_2$ 
  - $\langle 2 \rangle 1$ . ASSUME:  $K_1/H = K_2/H$
  - $\langle 2 \rangle 2$ .  $K_1 \subseteq K_2$ 
    - $\langle 3 \rangle 1$ . LET:  $k \in K_1$
    - $\langle 3 \rangle 2$ .  $kH \in K_2/H$
    - $\langle 3 \rangle 3$ . PICK  $k' \in K_2$  such that  $kH = k'H$
    - $\langle 3 \rangle 4$ .  $kk'^{-1} \in H$
    - $\langle 3 \rangle 5$ .  $kk'^{-1} \in K_2$
    - $\langle 3 \rangle 6$ .  $k \in K_2$
  - $\langle 2 \rangle 3$ .  $K_2 \subseteq K_1$
- PROOF: Similar.
- $\langle 1 \rangle 5$ . For any subgroup  $L$  of  $G/H$ , there exists a subgroup  $K$  of  $G$  that includes  $H$  such that  $L = K/H$ .
  - $\langle 2 \rangle 1$ . LET:  $L$  be a subgroup of  $G/H$ .
  - $\langle 2 \rangle 2$ . LET:  $K = \{k \in G : kH \in L\}$
  - $\langle 2 \rangle 3$ .  $K$  is a subgroup of  $G$ .
    - PROOF: Given  $k, k' \in K$  we have  $kH, k'H \in L$  hence  $kk'^{-1}H \in L$  and so  $kk'^{-1} \in K$ .
  - $\langle 2 \rangle 4$ .  $H \subseteq K$ 
    - PROOF: For all  $h \in H$  we have  $hH = H \in L$ .
  - $\langle 2 \rangle 5$ .  $L = K/H$ 
    - PROOF: By definition.

$\square$

**Proposition 7.54** (Third Isomorphism Theorem). *Let  $H$  be a normal subgroup of a group  $G$ . Let  $N$  be a subgroup of  $G$  that includes  $H$ . Then  $N/H$  is normal in  $G/H$  if and only if  $N$  is normal in  $G$ , in which case*

$$\frac{G/H}{N/H} \cong \frac{G}{N}$$

PROOF:

- ⟨1⟩1. If  $N/H$  is normal in  $G/H$  then  $N$  is normal in  $G$ .
  - ⟨2⟩1. ASSUME:  $N/H$  is normal in  $G/H$ .
  - ⟨2⟩2. LET:  $g \in G$  and  $n \in N$ .
  - ⟨2⟩3.  $gng^{-1}H \in N/H$
  - ⟨2⟩4. PICK  $n' \in N$  such that  $gng^{-1}H = n'H$
  - ⟨2⟩5.  $gng^{-1}n'^{-1} \in H$
  - ⟨2⟩6.  $gng^{-1}n'^{-1} \in N$
  - ⟨2⟩7.  $gng^{-1} \in N$
- ⟨1⟩2. If  $N$  is normal in  $G$  then  $N/H$  is normal in  $G/H$  and  $(G/H)/(N/H) \cong G/N$ .
  - ⟨2⟩1. ASSUME:  $N$  is normal in  $G$ .
  - ⟨2⟩2. LET:  $\phi : G/H \rightarrow G/N$  be the homomorphism  $\phi(gH) = gN$ 
    - ⟨3⟩1. If  $gH = g'H$  then  $gN = g'N$   
 PROOF: If  $gg'^{-1} \in H$  then  $gg'^{-1} \in N$ .
    - ⟨3⟩2.  $\phi((gH)(g'H)) = \phi(gH)\phi(g'H)$   
 PROOF: Both are  $gg'N$ .
  - ⟨2⟩3.  $\phi$  is surjective.
  - ⟨2⟩4.  $\ker \phi = N/H$
  - ⟨2⟩5.  $(G/H)/(N/H) \cong G/N$   
 PROOF: By the First Isomorphism Theorem.

□

**Proposition 7.55** (Second Isomorphism Theorem). *Let  $H$  and  $K$  be subgroups of a group  $G$ . Assume that  $H$  is normal in  $G$ . Then:*

- 1.  $HK$  is a subgroup of  $G$ , and  $H$  is normal in  $HK$ .
- 2.  $H \cap K$  is normal in  $K$ , and

$$\frac{HK}{H} \cong \frac{K}{H \cap K}.$$

PROOF:

- ⟨1⟩1.  $HK$  is a subgroup of  $G$ .  
 PROOF: Since  $hkh'k' = hh'(h'^{-1}kh')k' \in HK$ .
- ⟨1⟩2.  $H$  is normal in  $HK$ .
- ⟨1⟩3.  $H \cap K$  is normal in  $K$  and  $HK/H \cong K/(H \cap K)$   
 PROOF: The function that maps  $k$  to  $kH$  is a surjective homomorphism  $K \twoheadrightarrow HK/H$  with kernel  $H \cap K$ . Surjectivity follows because  $hkh = hkh^{-1}H$ .

□

See also Proposition 7.70 for a result that holds even if  $H$  is not normal.

## 7.8 Cosets

**Proposition 7.56.** *Let  $G$  be a group. Let  $\sim$  be an equivalence relation on  $G$  such that, for all  $a, b, g \in G$ , if  $a \sim b$  then  $ga \sim gb$ . Let  $H = \{h \in G : h \sim e\}$ .*

Then  $H$  is a subgroup of  $G$  and, for all  $a, b \in G$ , we have

$$a \sim b \Leftrightarrow a^{-1}b \in H \Leftrightarrow aH = bH .$$

PROOF:

$\langle 1 \rangle 1.$   $e \in H$

$\langle 1 \rangle 2.$  For all  $x, y \in H$  we have  $xy^{-1} \in H$ .

$\langle 2 \rangle 1.$  ASSUME:  $x \sim e$  and  $y \sim e$ .

$\langle 2 \rangle 2.$   $e \sim y^{-1}$

PROOF: Since  $yy^{-1} \sim ey^{-1}$ .

$\langle 2 \rangle 3.$   $xy^{-1} \sim e$

PROOF: Since  $xy^{-1} \sim ey^{-1} \sim e$ .

$\langle 1 \rangle 3.$  If  $a \sim b$  then  $a^{-1}b \in H$ .

PROOF: If  $a \sim b$  then  $a^{-1}b \sim a^{-1}a = e$ .

$\langle 1 \rangle 4.$  If  $a^{-1}b \in H$  then  $aH = bH$ .

$\langle 2 \rangle 1.$  ASSUME:  $a^{-1}b \in H$

$\langle 2 \rangle 2.$   $bH \subseteq aH$

PROOF: For any  $h \in H$  we have  $bh = aa^{-1}bh \in aH$ .

$\langle 2 \rangle 3.$   $aH \subseteq bH$

PROOF: Similar since  $b^{-1}a \in H$ .

$\langle 1 \rangle 5.$  If  $aH = bH$  then  $a \sim b$ .

$\langle 2 \rangle 1.$  ASSUME:  $aH = bH$

$\langle 2 \rangle 2.$  PICK  $h \in H$  such that  $a = bh$ .

$\langle 2 \rangle 3.$   $b^{-1}a = h$

$\langle 2 \rangle 4.$   $b^{-1}a \in H$

$\langle 2 \rangle 5.$   $b^{-1}a \sim e$

$\langle 2 \rangle 6.$   $a \sim b$

PROOF:  $a = bb^{-1}a \sim be = b$ .

□

**Definition 7.57** (Coset). Let  $G$  be a group and  $H$  a subgroup of  $G$ . A *left coset* of  $H$  is a set of the form  $aH$  for  $a \in G$ . A *right coset* of  $H$  is a set of the form  $Ha$  for some  $a \in G$ .

We write  $G/H$  for the set of all left cosets of  $H$ , and  $G \backslash H$  for the set of all right cosets of  $H$ .

**Proposition 7.58.**

$$G/H \cong G \backslash H$$

PROOF: The function that maps  $aH$  to  $Ha^{-1}$  is a bijection. □

**Proposition 7.59.** Let  $G$  be a group and  $H$  a subgroup of  $G$ . Define  $\sim_H$  on  $G$  by:  $a \sim b$  iff  $a^{-1}b \in H$ . This defines a one-to-one correspondence between the subgroups of  $G$  and the equivalence relations  $\sim$  on  $G$  such that, for all  $a, b, g \in G$ , if  $a \sim b$ , then  $ga \sim gb$ . The equivalence class of  $a$  is  $aH$ .

PROOF:

$\langle 1 \rangle 1.$  For any subgroup  $H$ , we have  $\sim_H$  is an equivalence relation on  $G$ .

⟨2⟩1.  $\sim$  is reflexive.

PROOF: For any  $a \in G$  we have  $a^{-1}a = e \in H$ .

⟨2⟩2.  $\sim$  is symmetric.

PROOF: If  $a^{-1}b \in H$  then  $b^{-1}a \in H$ .

⟨2⟩3.  $\sim$  is transitive.

PROOF: If  $a^{-1}b \in H$  and  $b^{-1}c \in H$  then  $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$ .

⟨1⟩2. If  $a \sim_H b$  then  $ga \sim_H gb$ .

PROOF: If  $a^{-1}b \in H$  then  $(ga)^{-1}(gb) = a^{-1}g^{-1}gb = a^{-1}b \in H$ .

⟨1⟩3. For any equivalence relation  $\sim$  on  $G$  such that, whenever  $a \sim b$ , then  $ga \sim gb$ , there exists a subgroup  $H$  such that  $\sim = \sim_H$ .

PROOF: Proposition 7.56.

⟨1⟩4. The  $\sim_H$ -equivalence class of  $a$  is  $aH$ .

PROOF:

$$\begin{aligned} a \sim b &\Leftrightarrow a^{-1}b \in H \\ &\Leftrightarrow \exists h \in H. a^{-1}b = h \\ &\Leftrightarrow \exists h \in H. b = ah \\ &\Leftrightarrow b \in aH \end{aligned}$$

□

**Proposition 7.60.** Let  $G$  be a group and  $H$  a subgroup of  $G$ . Define  $\sim_H$  on  $G$  by:  $a \sim b$  iff  $ab^{-1} \in H$ . This defines a one-to-one correspondence between the subgroups of  $G$  and the equivalence relations  $\sim$  on  $G$  such that, for all  $a, b, g \in G$ , if  $a \sim b$ , then  $ag \sim bg$ . The equivalence class of  $a$  is  $Ha$ .

PROOF: Similar. □

**Proposition 7.61.** Let  $G$  be a group and  $H$  be a subgroup of  $G$ . Define  $\sim_L$  and  $\sim_R$  on  $G$  by:

$$a \sim_L b \Leftrightarrow a^{-1}b \in H, \quad a \sim_R b \Leftrightarrow ab^{-1} \in H.$$

Then  $\sim_L = \sim_R$  if and only if  $H$  is normal.

PROOF:

⟨1⟩1. If  $\sim_L = \sim_R$  then  $H$  is normal.

⟨2⟩1. ASSUME:  $\sim_L = \sim_R$

⟨2⟩2. LET:  $h \in H$  and  $g \in G$

⟨2⟩3.  $g \sim_L gh^{-1}$

⟨2⟩4.  $g \sim_R gh^{-1}h$

⟨2⟩5.  $ghg^{-1} \in H$

⟨1⟩2. If  $H$  is normal then  $\sim_L = \sim_R$ .

⟨2⟩1. ASSUME:  $H$  is normal.

⟨2⟩2. If  $a \sim_L b$  then  $a \sim_R b$ .

⟨3⟩1. ASSUME:  $a \sim_L b$

⟨3⟩2.  $a^{-1}b \in H$

⟨3⟩3.  $aa^{-1}ba^{-1} \in H$

⟨3⟩4.  $ba^{-1} \in H$



- $\langle 3 \rangle 5. a \sim_R b$   
 $\langle 2 \rangle 3. \text{ If } a \sim_R b \text{ then } a \sim_L b.$

PROOF: Similar.

□

**Corollary 7.61.1.** *Let  $G$  be a group and  $H$  be a normal subgroup of  $G$ . Define  $\sim$  on  $G$  by  $a \sim b$  iff  $a^{-1}b \in H$ . Then  $G/\sim$  is a group under  $[a][b] = [ab]$ .*

**Definition 7.62** (Quotient Group). Let  $G$  be a group and  $H$  be a normal subgroup of  $G$ . The *quotient group*  $G/H$  is  $G/\sim$  where  $a \sim b$  iff  $a^{-1}b \in H$ , under  $[a][b] = [ab]$  or  $(aH)(bH) = abH$ .

**Corollary 7.62.1.** *Let  $H$  be a normal subgroup of a group  $G$ . For every group homomorphism  $\phi : G \rightarrow G'$  such that  $H \subseteq \ker \phi$ , there exists a unique group homomorphism  $\bar{\phi} : G/H \rightarrow G'$  such that the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ & \searrow \pi & \nearrow \bar{\phi} \\ & G/H & \end{array}$$

**Proposition 7.63.**  $\mathbb{Z}/n\mathbb{Z}$  has exactly  $n$  elements.

PROOF: Every integer is congruent to one of  $0, 1, \dots, n-1$  by the division algorithm, and no two of them are congruent to one another, since if  $0 \leq i < j < n$  then  $0 < j - i < n$ . □

**Proposition 7.64.** *Let  $m$  and  $n$  be integers with  $n > 0$ . The order of  $m$  in  $\mathbb{Z}/n\mathbb{Z}$  is  $\frac{n}{\gcd(m,n)}$ .*

PROOF: By Proposition 6.16 since the order of 1 is  $n$ . □

**Proposition 7.65.** *The integer  $m$  generates  $\mathbb{Z}/n\mathbb{Z}$  if and only if  $\gcd(m, n) = 1$ .*

PROOF: By Proposition 7.64. □

**Corollary 7.65.1.** *If  $p$  is prime then every non-zero element in  $\mathbb{Z}/p\mathbb{Z}$  is a generator.*

**Proposition 7.66.**

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$$

PROOF: Every permutation of  $\{(1, 0), (0, 1), (1, 1)\}$  gives an automorphism of  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . □

**Example 7.67.** Not all monomorphisms split in  $\mathbf{Grp}$ .

Define  $\phi : \mathbb{Z}/3\mathbb{Z} \rightarrow S_3$  by

$$\phi(0) = \text{id}_3, \quad \phi(1) = (1 \ 3 \ 2), \quad \phi(2) = (1 \ 2 \ 3) .$$

Then  $\phi$  is monic but has no retraction.

For if  $r : S_3 \rightarrow \mathbb{Z}/3\mathbb{Z}$  is a retraction, then we would have

$$r(1\ 2) + r(2\ 3) = 1, \quad r(2\ 3) + r(1\ 2) = 2$$

which is impossible.

**Proposition 7.68.** *Let  $G$  be a group,  $H$  a subgroup of  $G$ , and  $g \in G$ . The function that maps  $h$  to  $gh$  is a bijection  $H \cong gH$ .*

PROOF: By Cancellation.  $\square$

**Proposition 7.69.** *Let  $G$  be a group,  $H$  a subgroup of  $G$ , and  $g \in G$ . The function that maps  $h$  to  $hg$  is a bijection  $H \cong Hg$ .*

PROOF: By Cancellation.  $\square$

**Proposition 7.70.** *Let  $H$  and  $K$  be finite subgroups of a group  $G$ . Then*

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : \{hK : h \in H\} \rightarrow H/(H \cap K)$  be the function  $f(hK) = h(H \cap K)$

PROOF: This is well-defined because if  $hK = h'K$  then  $h^{-1}h' \in H \cap K$  so  $h(H \cap K) = h'(H \cap K)$ .

$\langle 1 \rangle 2$ .  $f$  is injective.

PROOF: If  $h(H \cap K) = h'(H \cap K)$  then  $hK = h'K$ .

$\langle 1 \rangle 3$ .  $f$  is surjective.

PROOF: Clear.

$\langle 1 \rangle 4$ .

$$\frac{|HK|}{|K|} = \frac{|H|}{|H \cap K|}$$

$\square$

## 7.9 Congruence

**Definition 7.71** (Congruence). Given integers  $a, b, n$  with  $n$  positive, we say  $a$  is *congruent to  $b$  modulo  $n$* , and write  $a \equiv b \pmod{n}$ , iff  $a + n\mathbb{Z} = b + n\mathbb{Z}$  in  $\mathbb{Z}/n\mathbb{Z}$ .

**Proposition 7.72.** *Given integers  $a, b, n$  with  $n$  positive, we have  $a \equiv b \pmod{n}$  iff  $n \mid a - b$ .*

PROOF: By Proposition 7.56.  $\square$

**Proposition 7.73.** *If  $a \equiv a' \pmod{n}$  and  $b \equiv b' \pmod{n}$  then  $a + b \equiv a' + b' \pmod{n}$ .*

PROOF: If  $n \mid a' - a$  and  $n \mid b' - b$  then  $n \mid (a' + b') - (a + b)$ .  $\square$

**Proposition 7.74.** *If  $a \equiv a' \pmod{n}$  and  $b \equiv b' \pmod{n}$  then  $ab \equiv a'b' \pmod{n}$ .*

PROOF: If  $n \mid a' - a$  and  $n \mid b' - b$  then  $n \mid a'b' - ab = a'(b' - b) + (a' - a)b$ .  $\square$

## 7.10 Cyclic Groups

**Definition 7.75** (Cyclic Group). The *cyclic* groups are  $\mathbb{Z}$  and  $\mathbb{Z}/n\mathbb{Z}$  for positive integers  $n$ .

**Proposition 7.76.** *If  $m$  and  $n$  are positive integers with  $\gcd(m, n) = 1$  then  $C_{mn} \cong C_m \times C_n$ .*

PROOF: The function that maps  $x$  to  $(x \bmod m, x \bmod n)$  is an isomorphism.  $\square$

**Proposition 7.77.** *Let  $G$  be a group and  $g \in G$ . Then  $\langle g \rangle$  is cyclic.*

PROOF: If  $g$  has finite order then  $\langle g \rangle \cong C_{|g|}$ , otherwise  $\langle g \rangle \cong \mathbb{Z}$ .  $\square$

**Proposition 7.78.** *Every finitely generated subgroup of  $\mathbb{Q}$  is cyclic.*

PROOF:

$\langle 1 \rangle$ 1. LET:  $G = \langle a_1/b, \dots, a_n/b \rangle$  where  $a_1, \dots, a_n, b$  are integers with  $b > 0$

$\langle 1 \rangle$ 2. LET:  $a = \gcd(a_1, \dots, a_n)$

$\langle 1 \rangle$ 3.  $G = \langle a/b \rangle$

$\square$

**Corollary 7.78.1.**  $\mathbb{Q}$  is not finitely generated.

## 7.11 Commutator Subgroup

**Definition 7.79** (Commutator Subgroup). Let  $G$  be a group. The *commutator subgroup*  $[G, G]$  is the subgroup generated by the elements of the form  $aba^{-1}b^{-1}$ .

**Proposition 7.80.** *The commutator subgroup is normal.*

PROOF: Since

$$\begin{aligned} & ga_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_nb_na_n^{-1}b_n^{-1}g^{-1} \\ &= (ga_1g^{-1})(gb_1g^{-1})(ga_1g^{-1})^{-1}(gb_1g^{-1})^{-1} \cdots (ga_ng^{-1})(gb_ng^{-1})(ga_ng^{-1})^{-1}(gb_ng^{-1})^{-1}. \end{aligned} \quad \square$$

## 7.12 Presentations

**Definition 7.81** (Presentation). A *presentation* of a group  $G$  is a pair  $(A, R)$  where  $A$  is a set and  $R \subseteq F(A)$  is a set of words such that

$$G \cong F(A)/N(R)$$

where  $N(R)$  is the smallest normal subgroup of  $F(A)$  that includes  $R$ .

**Example 7.82.** • The free group on a set  $A$  is presented by  $(A, \emptyset)$ .

- $S_3$  is presented by  $(x, y | x^2, y^3, xyxy)$ .
- $(a, b | a^2, b^2, (ab)^n)$  is a presentation of  $D_{2n}$ .

- $(x, y \mid x^2y^{-2}, y^4, xyx^{-1}y)$  is a presentation of  $Q_8$ .

**Proposition 7.83** (Word Problem). *Let  $(A, R)$  be a presentation of the group  $G$ . Let  $w_1, w_2 \in F(A)$  be two words. Then it is undecidable in general if  $w_1N(R) = w_2N(R)$  in  $G$ .*

**Definition 7.84** (Finitely Presented). A group is *finitely presented* iff it has a presentation  $(A, R)$  where both  $A$  and  $R$  are finite.

**Proposition 7.85.** *Let  $(A|R)$  be a presentation of  $G$  and  $(A'|R')$  a presentation of  $H$ . Assume w.l.o.g.  $A$  and  $A'$  are disjoint. Then the group  $G * G'$  presented by  $(A \cup A' | R \cup R')$  is the coproduct of  $G$  and  $G'$  in **Grp**.*

$$\begin{array}{ccccc}
 A & \longrightarrow & A \cup A' & \longleftarrow & A' \\
 \downarrow & & \downarrow & & \downarrow \\
 F(A) & \longrightarrow & F(A \cup A') & \longleftarrow & F(A') \\
 \downarrow & & \downarrow & & \downarrow \\
 G & \xrightarrow{\kappa_1} & G * G' & \xleftarrow{\kappa_2} & G'
 \end{array}$$

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\kappa_1 : G \rightarrow G * G'$  and  $\kappa_2 : G' \rightarrow G * G'$  be the unique homomorphisms that make the diagram above commute.
- $\langle 1 \rangle 2$ . LET:  $\phi : G \rightarrow H$  and  $\psi : G' \rightarrow H$  be any homomorphisms.
- $\langle 1 \rangle 3$ . LET:  $[\phi, \psi] : F(A \cup A') \rightarrow H$  be the unique homomorphism such that ...
- $\langle 1 \rangle 4$ .  $R \cup R' \subseteq \ker[\phi, \psi]$
- $\langle 1 \rangle 5$ .  $[\phi, \psi]$  factors uniquely through the morphism  $F(A \cup A') \rightarrow G * G'$

□

## 7.13 Index of a Subgroup

**Definition 7.86** (Index). Let  $G$  be a group and  $H$  a subgroup of  $G$ . The *index* of  $H$  in  $G$ , denoted  $|G : H|$ , is the number of left cosets of  $H$  in  $G$  if this is finite, otherwise  $\infty$ .

**Theorem 7.87** (Lagrange's Theorem). *Let  $G$  be a finite group and  $H$  a subgroup of  $G$ . Then*

$$|G| = |G : H||H|.$$

PROOF:  $G/H$  is a partition of  $G$  into  $|G : H|$  subsets, each of size  $|H|$ . □

**Corollary 7.87.1.** *For  $p$  a prime number, the only group of order  $p$  is  $C_p$ .*

PROOF: Let  $G$  be a group of order  $p$  and  $g \in G$  with  $g \neq e$ . Then  $|\langle g \rangle|$  divides  $p$  and is not 1, hence is  $p$ , that is,  $G = \langle g \rangle$ . □

**Theorem 7.88** (Cauchy's Theorem). *Let  $G$  be a finite group. If  $p$  is prime and  $p \mid |G|$  then  $G$  has a subgroup of order  $p$ .*

**Proposition 7.89.** *Let  $G$  be a group. Let  $K$  be a subgroup of  $G$  and  $H$  a subgroup of  $K$ . If  $|G : H|$ ,  $|G : K|$  and  $|K : H|$  are all finite then*

$$|G : H| = |G : K| |K : H| .$$

PROOF:

- $\langle 1 \rangle 1$ . LET:  $G/K = \{g_1K, g_2K, \dots, g_mK\}$
- $\langle 1 \rangle 2$ . LET:  $K/H = \{k_1H, k_2H, \dots, k_nH\}$
- $\langle 1 \rangle 3$ .  $G/H = \{g_ik_jH : 1 \leq i \leq m, 1 \leq j \leq n\}$ 
  - $\langle 2 \rangle 1$ . LET:  $g \in G$
  - $\langle 2 \rangle 2$ . PICK  $i$  such that  $gK = g_iK$
  - $\langle 2 \rangle 3$ .  $g^{-1}g_i \in K$
  - $\langle 2 \rangle 4$ . PICK  $j$  such that  $g^{-1}g_iH = k_jH$
  - $\langle 2 \rangle 5$ .  $g^{-1}g_ik_j \in H$
  - $\langle 2 \rangle 6$ .  $gH = g_ik_jH$
- $\langle 1 \rangle 4$ . If  $g_ik_jH = g_{i'}k_{j'}H$  then  $i = i'$  and  $j = j'$ .
  - $\langle 2 \rangle 1$ . ASSUME:  $g_ik_jH = g_{i'}k_{j'}H$
  - $\langle 2 \rangle 2$ .  $g_iK = g_{i'}K$
  - $\langle 2 \rangle 3$ .  $i = i'$
  - $\langle 2 \rangle 4$ .  $k_jH = k_{j'}H$
  - $\langle 2 \rangle 5$ .  $j = j'$

□

## 7.14 Cokernels

**Proposition 7.90.** *Let  $\phi : G \rightarrow H$  be a homomorphism between groups. Then there exists a group  $K$  and homomorphism  $\pi : H \rightarrow K$  that is initial with respect to all homomorphism  $\alpha : H \rightarrow L$  such that  $\alpha \circ \phi = 0$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $N$  be the intersection of all the normal subgroups of  $H$  that include  $\text{im } \phi$ .
- $\langle 1 \rangle 2$ . LET:  $K = H/N$  and  $\pi$  be the canonical homomorphism.
- $\langle 1 \rangle 3$ . LET:  $\pi \circ \phi = 0$
- $\langle 1 \rangle 4$ . LET:  $\alpha : H \rightarrow L$  satisfy  $\alpha \circ \phi = 0$
- $\langle 1 \rangle 5$ .  $\text{im } \phi \subseteq \ker \alpha$
- $\langle 1 \rangle 6$ .  $N \subseteq \ker \alpha$
- $\langle 1 \rangle 7$ . There exists a unique  $\bar{\alpha} : H/\text{im } \phi \rightarrow L$  such that  $\bar{\alpha} \circ \pi = \alpha$

□

**Definition 7.91** (Cokernel). For any homomorphism  $\phi : G \rightarrow H$  in **Grp**, the *cokernel* of  $\phi$  is the group  $\text{coker } \phi$  and homomorphism  $\pi : H \rightarrow \text{coker } \phi$  that is initial among homomorphisms  $\alpha : H \rightarrow L$  such that  $\alpha \circ \phi = 0$ .

**Example 7.92.** It is not true that a homomorphism with trivial cokernel is epi. The inclusion  $\langle (1 \ 2) \rangle \hookrightarrow S_3$  has trivial cokernel but is not epi.

## 7.15 Cayley Graphs

**Definition 7.93** (Cayley Graph). Let  $G$  be a finitely generated group. Let  $A$  be a finite set of generators for  $G$ . The *Cayley graph* of  $G$  with respect to  $A$  is the directed graph whose vertices are the elements of  $G$ , with an edge  $g_1 \rightarrow g_2$  labelled by  $a \in A$  iff  $g_2 = g_1 a$ .

**Proposition 7.94.**  $G$  is the free group on  $A$  iff the Cayley graph with respect to  $A$  is a tree.

PROOF: Both are equivalent to saying that the product of two different strings of elements of  $A$  and/or their inverses are not equal.  $\square$

## Chapter 8

# Abelian Groups

**Definition 8.1** (Abelian Group). A group is *Abelian* iff any two elements commute.

In an Abelian group  $G$ , we often denote the group operation by  $+$ , the identity element by  $0$  and the inverse of an element  $g$  by  $-g$ . We write  $ng$  for  $g^n$  ( $g \in G, n \in \mathbb{Z}$ ).

**Example 8.2.** Every group of order  $\leq 4$  is Abelian.

**Example 8.3.** For any positive integer  $n$ , we have  $\mathbb{Z}/n\mathbb{Z}$  is an Abelian group under addition.

**Example 8.4.**  $S_n$  is not Abelian for  $n \geq 3$ . If  $x = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$  and  $y = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$  then  $xy = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$  and  $yx = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ .

**Example 8.5.** There are 42 Abelian groups of order 1024 up to isomorphism.

**Proposition 8.6.** Let  $G$  be a group. If  $g^2 = e$  for all  $g \in G$  then  $G$  is Abelian.

PROOF: For any  $g, h \in G$  we have

$$ghgh = e$$

$$\therefore hgh = g \quad (\text{multiplying on the left by } g)$$

$$\therefore hg = gh \quad (\text{multiplying on the right by } h) \square$$

**Proposition 8.7.** Let  $G$  be a group. Then  $G$  is Abelian if and only if the function that maps  $g$  to  $g^{-1}$  is a group homomorphism.

PROOF:

(1)1. If  $G$  is Abelian then the function that maps  $g$  to  $g^{-1}$  is a group homomorphism.

PROOF: Since  $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$ .

(1)2. If the function that maps  $g$  to  $g^{-1}$  is a group homomorphism then  $G$  is Abelian.

PROOF: Since  $gh = (g^{-1})^{-1}(h^{-1})^{-1} = (g^{-1}h^{-1})^{-1} = hg$ .  
 $\square$

**Proposition 8.8.** *Let  $G$  be a group. Then  $G$  is Abelian if and only if the function that maps  $g$  to  $g^2$  is a group homomorphism.*

PROOF:

$\langle 1 \rangle 1$ . If  $G$  is Abelian then the function that maps  $g$  to  $g^2$  is a group homomorphism.

PROOF: Since  $(gh)^2 = g^2h^2$ .

$\langle 1 \rangle 2$ . If the function that maps  $g$  to  $g^2$  is a group homomorphism then  $G$  is Abelian.

PROOF: Since we have  $(gh)^2 = ghgh = g^2h^2$  and so  $hg = gh$ .

$\square$

**Proposition 8.9.** *Let  $G$  be a group. Then  $G$  is Abelian if and only if the homomorphism  $\gamma : G \rightarrow \text{Aut}_{\mathbf{Grp}}(G)$  is the trivial homomorphism.*

PROOF:

$\langle 1 \rangle 1$ . If  $G$  is Abelian then  $\gamma$  is trivial.

PROOF: Since  $\gamma_g(a) = gag^{-1} = a$ .

$\langle 1 \rangle 2$ . If  $\gamma$  is trivial then  $G$  is Abelian.

PROOF: If  $\gamma_g(a) = gag^{-1} = a$  for all  $g$  and  $a$  then  $ga = ag$  for all  $g, a$ .

$\square$

**Proposition 8.10.** *Let  $G$  be an Abelian group. Let  $g, h \in G$ . If  $g$  has maximal finite order in  $G$ , and  $h$  has finite order, then  $|h| \mid |g|$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $|h| \nmid |g|$ .

$\langle 1 \rangle 2$ . PICK a prime  $p$  such that  $|g| = p^m r$ ,  $|h| = p^n s$  where  $p \nmid r$ ,  $p \nmid s$  and  $m < n$ .

$\langle 1 \rangle 3$ .  $|g^{p^m} h^s| = p^n r$

PROOF: Proposition 6.19.

$\langle 1 \rangle 4$ .  $|g| < |g^{p^m} h^s|$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This contradicts the maximality of  $|g|$ .

$\square$

**Proposition 8.11.** *Given a set  $A$  and an Abelian group  $H$ , the set  $H^A$  is an Abelian group under*

$$(\phi + \psi)(a) = \phi(a) + \psi(a) \quad (\phi, \psi \in H^A, a \in A) .$$

PROOF:

$\langle 1 \rangle 1$ .  $\phi + (\psi + \chi) = (\phi + \psi) + \chi$

$\langle 1 \rangle 2$ .  $\phi + \psi = \psi + \phi$

$\langle 1 \rangle 3$ . LET:  $0 : A \rightarrow H$  be the function  $0(a) = 0$ .

$\langle 1 \rangle 4$ .  $\phi + 0 = 0 + \phi = \phi$



$\langle 1 \rangle 5$ . Given  $\phi : A \rightarrow H$ , define  $-\phi : A \rightarrow H$  by  $(-\phi)(a) = -(\phi(a))$ .

$\langle 1 \rangle 6$ .  $\phi + (-\phi) = (-\phi) + \phi = 0$

□

**Proposition 8.12.** *Given a group  $G$  and an Abelian group  $H$ , the set  $\mathbf{Grp}[G, H]$  is a subgroup of  $H^G$ .*

PROOF:

$\langle 1 \rangle 1$ . Given  $\phi, \psi : G \rightarrow H$  group homomorphisms, we have  $\phi - \psi$  is a group homomorphism.

PROOF:

$$\begin{aligned} (\phi - \psi)(g + g') &= \phi(g + g') - \psi(g + g') \\ &= \phi(g) + \phi(g') - \psi(g) - \psi(g') \\ &= \phi(g) - \psi(g) + \phi(g') - \psi(g') \\ &= (\phi - \psi)(g) + (\phi - \psi)(g') \end{aligned}$$

□

**Proposition 8.13.** *Let  $G$  be a group. The following are equivalent.*

1.  $\text{Inn}(G)$  is cyclic.
2.  $\text{Inn}(G)$  is trivial.
3.  $G$  is Abelian.

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

$\langle 2 \rangle 1$ . ASSUME:  $\text{Inn}(G) = \langle \gamma_g \rangle$

$\langle 2 \rangle 2$ .  $g$  commutes with every element of  $G$

$\langle 3 \rangle 1$ . LET:  $x \in G$

$\langle 3 \rangle 2$ . PICK  $n \in \mathbb{Z}$  such that  $\gamma_x = \gamma_g^n$

$\langle 3 \rangle 3$ .  $\forall y \in G. xyx^{-1} = g^n yg^{-n}$

$\langle 3 \rangle 4$ .  $xgx^{-1} = g$

$\langle 2 \rangle 3$ .  $\gamma_g = \text{id}_G$

$\langle 1 \rangle 2$ .  $2 \Rightarrow 3$

$\langle 2 \rangle 1$ . ASSUME:  $\forall g \in G. \gamma_g = \text{id}_G$

$\langle 2 \rangle 2$ . LET:  $x, y \in G$

$\langle 2 \rangle 3$ .  $\gamma_x(y) = y$

$\langle 2 \rangle 4$ .  $xyx^{-1} = y$

$\langle 2 \rangle 5$ .  $xy = yx$

$\langle 1 \rangle 3$ .  $3 \Rightarrow 2$

PROOF: If  $xy = yx$  for all  $x, y$  then  $\gamma_x(y) = y$  for all  $x, y$ .

$\langle 1 \rangle 4$ .  $2 \Rightarrow 1$

PROOF: Easy.

□

**Corollary 8.13.1.** *If  $\text{Aut}_{\mathbf{Grp}}(G)$  is cyclic then  $G$  is Abelian.*

**Proposition 8.14.** *Every subgroup of an Abelian group is normal.*

PROOF: Let  $G$  be an Abelian group and  $N$  a subgroup of  $G$ . Given  $g \in G$  and  $n \in N$  we have  $gng^{-1} = n \in N$ .  $\square$

**Proposition 8.15.** *For any group  $G$ , the group  $G/[G, G]$  is Abelian.*

PROOF: For any  $g, h \in G$  we have

$$gh(hg)^{-1} \in [G, G]$$

$$\therefore gh[G, G] = hg[G, G] \quad \square$$

**Proposition 8.16.** *Let  $G$  be a finite Abelian group. Let  $p$  be a prime divisor of  $|G|$ . Then  $G$  has an element of order  $p$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: as induction hypothesis the result holds for all groups smaller than  $G$ .

$\langle 1 \rangle 2$ . PICK  $g \in G - \{0\}$ .

$\langle 1 \rangle 3$ . PICK an element  $h \in \langle g \rangle$  with prime order  $q$ .

$\langle 1 \rangle 4$ . CASE:  $q = p$

PROOF:  $h$  is the required element.

$\langle 1 \rangle 5$ . CASE:  $q \neq p$

$\langle 2 \rangle 1$ . PICK  $r \in G$  such that  $r + \langle h \rangle$  has order  $p$  in  $G/\langle h \rangle$ .

PROOF: By induction hypothesis since  $|G/\langle h \rangle| = |G|/q$ .

$\langle 2 \rangle 2$ .  $pr \in \langle h \rangle$

$\langle 2 \rangle 3$ . PICK  $k$  such that  $pr = kh$

$\langle 2 \rangle 4$ .  $pqr = e$

$\langle 2 \rangle 5$ .  $qr$  has order  $p$ .

$\square$

**Corollary 8.16.1.** *For  $n$  an odd integer, any Abelian group of order  $2n$  has exactly one element of order 2.*

PROOF: If  $x$  and  $y$  are distinct elements of order 2 then  $\langle x, y \rangle = \{e, x, y, xy\}$  has size 4 and so  $4 \mid 2n$  which is a contradiction.  $\square$

**Example 8.17.** It is not true that, if  $G$  is a finite group and  $d \mid |G|$ , then  $G$  has an element of order  $d$ . The quaternionic group has no element of order 4.

**Proposition 8.18.** *If  $G$  is a finite Abelian group and  $d \mid |G|$  then  $G$  has a subgroup of size  $d$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: as induction hypothesis the result is true for all  $d' < d$ .

$\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $d \neq 1$ .

$\langle 1 \rangle 3$ . PICK a prime  $p$  such that  $p \mid d$ .

$\langle 1 \rangle 4$ . PICK an element  $g \in G$  of order  $p$ .

$\langle 1 \rangle 5$ .  $d/p \mid |G/\langle g \rangle|$

$\langle 1 \rangle 6$ . PICK a subgroup  $H$  of  $G/\langle g \rangle$  of size  $d/p$ .

$\langle 1 \rangle 7$ .  $\pi^{-1}(H)$  is a subgroup of  $G$  of size  $d$ .

$\square$

**Proposition 8.19.** *Let  $(G, \cdot)$  be a group. Let  $\circ : G^2 \rightarrow G$  be a group homomorphism such that  $(G, \circ)$  is a group. Then  $\circ$  and  $\cdot$  coincide, and  $G$  is Abelian.*

PROOF:

$\langle 1 \rangle 1$ . For all  $g_1, g_2, h_1, h_2 \in G$  we have

$$(g_1 g_2) \circ (h_1 h_2) = (g_1 \circ h_1)(g_2 \circ h_2)$$

$\langle 1 \rangle 2$ .  $e \circ e = e$

PROOF:

$$\begin{aligned} e \circ e &= (ee) \circ (ee) \\ &= (e \circ e)(e \circ e) \end{aligned}$$

Hence  $e \circ e = e$  by Cancellation.

$\langle 1 \rangle 3$ .  $e$  is the identity of  $(G, \circ)$

$\langle 1 \rangle 4$ . For all  $g, h \in G$  we have

$$g \circ h = gh$$

PROOF:

$$\begin{aligned} g \circ h &= (ge) \circ (eh) \\ &= (g \circ e)(e \circ h) \\ &= gh \end{aligned}$$

$\langle 1 \rangle 5$ . For all  $g, h \in G$  we have  $gh = hg$ .

PROOF:

$$\begin{aligned} gh &= (e \circ g)(h \circ e) \\ &= (eh) \circ (ge) \\ &= h \circ g \\ &= hg \end{aligned}$$

□

**Corollary 8.19.1.** *If  $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$  is a group object in **Grp** then  $m$  is the multiplication of  $G$ ,  $e(*)$  is the identity of  $G$ ,  $i(g) = g^{-1}$ , and  $G$  is Abelian.*

*Conversely, if  $(G, m)$  is any Abelian group, then  $(G, m, e, i)$  is a group object in **Grp** where  $e(*) = e$  and  $i(g) = g^{-1}$ .*

## 8.1 The Category of Abelian Groups

**Definition 8.20** (Category of Abelian Groups). Let **Ab** be the full subcategory of **Grp** whose objects are the Abelian groups.

**Proposition 8.21.** *If  $(G, m : G^2 \rightarrow G, e : 1 \rightarrow G, i : G \rightarrow G)$  is a group object in **Ab** then  $m$  is the multiplication of  $G$ ,  $e(*)$  is the identity of  $G$ ,  $i(g) = g^{-1}$ , and  $G$  is Abelian.*

*Conversely, if  $(G, m)$  is any Abelian group, then  $(G, m, e, i)$  is a group object in **Ab** where  $e(*) = e$  and  $i(g) = g^{-1}$ .*

PROOF: Immediate from Corollary 8.19.1. □

**Definition 8.22** (Direct Sum). Given Abelian groups  $G$  and  $H$ , we also call the direct product of  $G$  and  $H$  the *direct sum* and denote it  $G \oplus H$ .

**Proposition 8.23.** *Given Abelian groups  $G$  and  $H$ , the direct sum  $G \oplus H$  is the coproduct of  $G$  and  $H$  in  $\mathbf{Ab}$ .*

PROOF:

- (1)1. LET:  $\kappa_1 : G \rightarrow G \oplus H$  be the group homomorphism  $\kappa_1(g) = (g, e_H)$ .  
 (1)2. LET:  $\kappa_2 : H \rightarrow G \oplus H$  be the group homomorphism  $\kappa_2(h) = (e_G, h)$ .  
 (1)3. Given group homomorphism  $\phi : G \rightarrow K$  and  $\psi : H \rightarrow K$ , define  $[\phi, \psi] : G \oplus H \rightarrow K$  by  $[\phi, \psi](g, h) = \phi(g) + \psi(h)$ .  
 (1)4.  $[\phi, \psi]$  is a group homomorphism.

PROOF:

$$\begin{aligned}
 [\phi, \psi]((g, h) + (g', h')) &= [\phi, \psi](g + g', h + h') \\
 &= \phi(g + g') + \psi(h + h') \\
 &= \phi(g) + \phi(g') + \psi(h) + \psi(h') \\
 &= \phi(g) + \psi(h) + \phi(g') + \psi(h') \\
 &= [\phi, \psi](g, h) + [\phi, \psi](g', h')
 \end{aligned}$$

- (1)5.  $[\phi, \psi] \circ \kappa_1 = \phi$

PROOF:

$$\begin{aligned}
 [\phi, \psi](\kappa_1(g)) &= [\phi, \psi](g, e_H) \\
 &= \phi(g) + \psi(e_H) \\
 &= \phi(g) + e_K \\
 &= \phi(g)
 \end{aligned}$$

- (1)6.  $[\phi, \psi] \circ \kappa_2 = \psi$

PROOF: Similar.

- (1)7. If  $f : G \oplus H \rightarrow K$  is a group homomorphism with  $f \circ \kappa_1 = \phi$  and  $f \circ \kappa_2 = \psi$  then  $f = [\phi, \psi]$ .

PROOF:

$$\begin{aligned}
 f(g, h) &= f((g, e_H) + (e_G, h)) \\
 &= f(\kappa_1(g)) + f(\kappa_2(h)) \\
 &= \phi(g) + \psi(h)
 \end{aligned}$$

□

**Theorem 8.24.** *Every finitely generated Abelian group is a direct sum of cyclic groups.*

PROOF: TODO □

## 8.2 Free Abelian Groups

**Proposition 8.25.** *Let  $A$  be a set. Let  $\mathcal{F}^A$  be the category whose objects are pairs  $(G, j)$  where  $G$  is an Abelian group and  $j$  is a function  $A \rightarrow G$ , with morphisms  $f : (G, j) \rightarrow (H, k)$  the group homomorphisms  $f : G \rightarrow H$  such that  $f \circ j = k$ . Then  $\mathcal{F}^A$  has an initial object.*

PROOF:

- (1)1. LET:  $\mathbb{Z}^{\oplus A}$  be the subgroup of  $\mathbb{Z}^A$  consisting of all functions  $\alpha : A \rightarrow \mathbb{Z}$  such that  $\alpha(a) = 0$  for only finitely many  $a \in A$ .
- (1)2. LET:  $i : A \rightarrow \mathbb{Z}^{\oplus A}$  be the function such that  $i(a)(b) = 1$  if  $a = b$  and 0 if  $a \neq b$ .
- (1)3. LET:  $G$  be any Abelian group and  $j : A \rightarrow G$  any function.
- (1)4. The unique homomorphism  $\phi : \mathbb{Z}^{\oplus A} \rightarrow G$  required is defined by  $\phi(\alpha) = \sum_{a \in A} \alpha(a)j(a)$

□

**Definition 8.26** (Free Abelian Group). For any set  $A$ , the *free Abelian group* on  $A$  is the initial object  $(F^{ab}(A), i)$  in  $\mathcal{F}^A$ .

**Proposition 8.27.** For any sets  $A$  and  $B$ , we have that  $F^{ab}(A + B)$  is the coproduct of  $F^{ab}(A)$  and  $F^{ab}(B)$  in **Grp**.

$$\begin{array}{ccccc}
 & & G & & \\
 & \nearrow f & \uparrow k & \nwarrow g & \\
 F^{ab}(A) & \xrightarrow{\kappa_1} & F^{ab}(A+B) & \xleftarrow{\kappa_2} & F^{ab}(B) \\
 i_A \uparrow & & j \uparrow & & i_B \uparrow \\
 A & \xrightarrow{k_1} & A+B & \xleftarrow{k_2} & B
 \end{array}$$

PROOF:

- (1)1. LET:  $i_A : A \rightarrow F^{ab}(A)$ ,  $i_B : B \rightarrow F^{ab}(B)$ ,  $j : A + B \rightarrow F^{ab}(A + B)$  be the canonical injections.
- (1)2. LET:  $\kappa_1, \kappa_2$  be the unique group homomorphisms that make the diagram above commute.
- (1)3. LET:  $G$  be any group and  $f : F^{ab}(A) \rightarrow G$ ,  $g : F^{ab}(B) \rightarrow G$  any group homomorphisms.
- (1)4. LET:  $h : A + B \rightarrow G$  be the unique function such that  $h \circ k_1 = f \circ i_A$  and  $h \circ k_2 = g \circ i_B$ .
- (1)5. LET:  $k : F^{ab}(A + B) \rightarrow G$  be the unique group homomorphism such that  $k \circ j = h$ .
- (1)6.  $k$  is the unique group homomorphism such that  $k \circ \kappa_1 \circ i_A = f \circ i_A$  and  $k \circ \kappa_2 \circ i_B = g \circ i_B$ .
- (1)7.  $k$  is the unique group homomorphism such that  $k \circ \kappa_1 = f$  and  $k \circ \kappa_2 = g$ .

□

**Proposition 8.28.** For  $A$  and  $B$  finite sets, if  $F^{ab}(A) \cong F^{ab}(B)$  then  $A \cong B$ .

PROOF:

- (1)1. For any set  $C$ , define  $\sim$  on  $F^{ab}(C)$  by:  $f \sim f'$  iff there exists  $g \in F^{ab}(C)$  such that  $f - f' = 2g$ .
- (1)2. For any set  $C$ ,  $\sim$  is an equivalence relation on  $F^{ab}(C)$ .
- (1)3. For any set  $C$ , we have  $F^{ab}(C) / \sim$  is finite if and only if  $C$  is finite, in which case  $|F^{ab}(C) / \sim| = 2^{|C|}$ .

PROOF: There is a bijection between  $F^{\text{ab}}(C)/\sim$  and the finite subsets of  $C$ , which maps  $f$  to  $\{c \in C : f(c) \text{ is odd}\}$ .

$\langle 1 \rangle 4$ . If  $F^{\text{ab}}(A) \cong F^{\text{ab}}(B)$  then  $A \cong B$ .

PROOF: If  $|F^{\text{ab}}(A)/\sim| = |F^{\text{ab}}(B)/\sim|$  then  $2^{|A|} = 2^{|B|}$  and so  $|A| = |B|$ .  $\square$

**Proposition 8.29.** *Let  $G$  be an Abelian group. Then  $G$  is finitely generated if and only if there exists a surjective homomorphism  $\mathbb{Z}^{\oplus n} \rightarrow G$  for some  $n$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $G$  is finitely generated then there exists a surjective homomorphism  $\mathbb{Z}^{\oplus n} \rightarrow G$  for some  $n$ .

PROOF: Let  $G = \langle a_1, \dots, a_n \rangle$ . Define  $\phi : \mathbb{Z}^{\oplus n} \rightarrow G$  by  $\phi(i_1, \dots, i_n) = i_1 \cdot a_1 + \dots + i_n \cdot a_n$ .

$\langle 1 \rangle 2$ . If there exists a surjective homomorphism  $\phi : \mathbb{Z}^{\oplus n} \rightarrow G$  for some  $n$  then  $G$  is finitely generated.

PROOF:  $G$  is generated by  $\phi(1, 0, \dots, 0), \phi(0, 1, 0, \dots, 0), \dots, \phi(0, \dots, 0, 1)$ .  $\square$

**Proposition 8.30.** *Let  $A$  be a set. Let  $i : A \hookrightarrow F(A)$  be the free group on  $A$ . Then  $\pi \circ i : A \rightarrow F(A)/[F(A), F(A)]$  is the free Abelian group on  $A$ .*

$$\begin{array}{ccc}
 & F(A)/[F(A), F(A)] & \\
 \pi \uparrow & \searrow h & \\
 F(A) & \xrightarrow{g} & G \\
 i \uparrow & \nearrow f & \\
 A & & 
 \end{array}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $G$  be an Abelian group and  $f : A \rightarrow G$  a function.

$\langle 1 \rangle 2$ . LET:  $g : F(A) \rightarrow G$  be the unique group homomorphism such that  $g \circ i = f$ .

$\langle 1 \rangle 3$ .  $[F(A), F(A)] \subseteq \ker g$

PROOF: For all  $x, y \in F(A)$  we have  $g(xy x^{-1} y^{-1}) = g(x) + g(y) - g(x) - g(y) = 0$ .

$\langle 1 \rangle 4$ . LET:  $h : F(A)/[F(A), F(A)] \rightarrow G$  be the unique group homomorphism such that  $h \circ \pi = g$ .

$\langle 1 \rangle 5$ .  $h$  is the unique group homomorphism such that  $h \circ \pi \circ i = f$ .  $\square$

**Corollary 8.30.1.** *Let  $A$  and  $B$  be sets. Let  $F(A)$  and  $F(B)$  be the free groups on  $A$  and  $B$  respectively. If  $F(A) \cong F(B)$  then  $A \cong B$ .*

PROOF: Proposition 8.28.  $\square$

### 8.3 Cokernels

**Proposition 8.31.** *Let  $\phi : G \rightarrow H$  be a homomorphism between Abelian groups. Then there exists an Abelian group  $K$  and homomorphism  $\pi : H \rightarrow K$  that is initial with respect to all homomorphism  $\alpha : H \rightarrow L$  such that  $\alpha \circ \phi = 0$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $K = H/\text{im } \phi$  and  $\pi$  be the canonical homomorphism.

$\langle 1 \rangle 2$ . LET:  $\pi \circ \phi = 0$

$\langle 1 \rangle 3$ . LET:  $\alpha : H \rightarrow L$  satisfy  $\alpha \circ \phi = 0$

$\langle 1 \rangle 4$ .  $\text{im } \phi \subseteq \ker \alpha$

$\langle 1 \rangle 5$ . There exists a unique  $\bar{\alpha} : H/\text{im } \phi \rightarrow L$  such that  $\bar{\alpha} \circ \pi = \alpha$

□

**Definition 8.32** (Cokernel). For any homomorphism  $\phi : G \rightarrow H$  in **Ab**, the *cokernel* of  $\phi$  is the Abelian group  $\text{coker } \phi$  and homomorphism  $\pi : H \rightarrow \text{coker } \phi$  that is initial among homomorphisms  $\alpha : H \rightarrow L$  such that  $\alpha \circ \phi = 0$ .

**Proposition 8.33.**  $\pi : H \rightarrow \text{coker } \phi$  is initial among functions  $f : H \rightarrow X$  such that, for all  $x, y \in H$ , if  $x + \text{im } \phi = y + \text{im } \phi$  then  $f(x) = f(y)$ .

PROOF: Easy. □

**Proposition 8.34.** Let  $\phi : G \rightarrow H$  be a homomorphism of Abelian groups. Then the following are equivalent.

- $\phi$  is an epimorphism.
- $\text{coker } \phi$  is trivial.
- $\phi$  is surjective.

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

$\langle 2 \rangle 1$ . ASSUME:  $\phi$  is epi.

$\langle 2 \rangle 2$ . LET:  $\pi : H \rightarrow \text{coker } \phi$  be the canonical homomorphism.

$\langle 2 \rangle 3$ .  $\pi \circ \phi = 0 \circ \phi$

$\langle 2 \rangle 4$ .  $\pi = 0$

$\langle 2 \rangle 5$ .  $\text{coker } \phi = \text{im } \pi$  is trivial.

$\langle 1 \rangle 2$ .  $2 \Rightarrow 3$

PROOF: If  $\text{coker } \phi = H/\text{im } \phi$  is trivial then  $\text{im } \phi = H$ .

$\langle 1 \rangle 3$ .  $3 \Rightarrow 1$

PROOF: If it is surjective then it is epi in **Set**.

□





## Chapter 9

# Group Actions

### 9.1 Group Actions

**Definition 9.1** (Action). Let  $G$  be a group. Let  $A$  be an object of a category  $\mathcal{C}$ . A (left) action of  $G$  on  $A$  is a group homomorphism  $G \rightarrow \text{Aut}_{\mathcal{C}}(A)$ .

It is *faithful* or *effective* iff it is injective.

**Proposition 9.2.** Let  $A$  be a set. An action of the group  $G$  on the set  $A$  is given by a function  $\cdot : G \times A \rightarrow A$  such that

- $\forall a \in A. ea = a$
- $\forall g, h \in G. \forall a \in A. (gh)a = g(ha)$

PROOF: Just unfolding definitions.  $\square$

**Example 9.3.** Left multiplication defines a faithful action of any group on its own underlying set.

In fact, for any subgroup  $H$  of a group  $G$ , left multiplication defines an action of  $G$  on  $G/H$ .

**Corollary 9.3.1** (Cayley's Theorem). Every group  $G$  is a subgroup of a symmetric group, namely  $\text{Aut}_{\text{Set}}(G)$ .

**Example 9.4.** Conjugation  $g * h = ghg^{-1}$  is an action of any group on its own underlying set.

**Definition 9.5** (Transitive). An action of a group  $G$  on a set  $A$  is *transitive* iff, for all  $a, b \in A$ , there exists  $g \in G$  such that  $ga = b$ .

**Example 9.6.** Left multiplication of a group  $G$  is a transitive action of  $G$  on  $G$ .

**Definition 9.7** (Orbit). Given an action of a group  $G$  on a set  $A$  and  $a \in A$ , the *orbit* of  $a$  is

$$\text{O}_G(a) := \{ga : g \in G\} .$$

**Proposition 9.8.** *Given an action of a group  $G$  on a set  $A$ , the orbits form a partition of  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . Every element of  $A$  is in some orbit.

PROOF: Since  $a \in O_G(a)$ .

$\langle 1 \rangle 2$ . Distinct orbits are disjoint.

$\langle 2 \rangle 1$ . LET:  $a \in O_G(b) \cap O_G(c)$

$\langle 2 \rangle 2$ . PICK  $g, h \in G$  such that  $a = gb = hc$ .

$\langle 2 \rangle 3$ .  $O_G(b) \subseteq O_G(c)$

PROOF: For all  $k \in G$  we have  $kb = kg^{-1}hc$ .

$\langle 2 \rangle 4$ .  $O_G(c) \subseteq O_G(b)$

PROOF: Similar.

□

**Proposition 9.9.** *Given an action of a group  $G$  on a set  $A$  and  $a \in A$ , the action is transitive on  $O_G(a)$ .*

PROOF:

$\langle 1 \rangle 1$ . The restriction of the action is an action on  $O_G(a)$ .

PROOF: Since  $g(ha) = (gh)a$ , the action maps  $O_G(a)$  to itself.

$\langle 1 \rangle 2$ . The restricted action is transitive.

PROOF: Given  $ga, ha \in O_G(a)$ , we have  $ha = (hg^{-1})(ga)$ .

□

**Definition 9.10** (Stabilizer Subgroup). Given an action of a group  $G$  on a set  $A$  and  $a \in A$ , the *stabilizer subgroup* of  $a$  is

$$\text{Stab}_G(a) := \{g \in G : ga = a\} .$$

**Proposition 9.11.** *Stabilizer subgroups are subgroups.*

PROOF: If  $g, h \in \text{Stab}_G(a)$  then  $gh^{-1}a = a$  so  $gh^{-1} \in \text{Stab}_G(a)$ . □

**Proposition 9.12.** *Let  $G$  act on a set  $A$ . Let  $a \in A$  and  $g \in G$ . Then*

$$\text{Stab}_G(ga) = g\text{Stab}_G(a)g^{-1} .$$

PROOF:

$$h \in \text{Stab}_G(ga) \Leftrightarrow hga = ga$$

$$\Leftrightarrow g^{-1}hga = a$$

$$\Leftrightarrow g^{-1}hg \in \text{Stab}_G(a)$$

$$\Leftrightarrow h \in g\text{Stab}_G(a)g^{-1}$$

□

**Corollary 9.12.1.** *Let  $G$  be an action on a set  $A$  and  $a \in A$ . If  $\text{Stab}_G(a)$  is normal in  $G$ , then for any  $b \in O_G(a)$  we have  $\text{Stab}_G(a) = \text{Stab}_G(b)$ .*

**Definition 9.13** (Free). An action of a group  $G$  on a set  $A$  is *free* iff, whenever  $ga = a$ , then  $g = e$ .

**Example 9.14.** The action of left multiplication is free.

**Proposition 9.15.** *Let  $G$  be a group. Let  $H$  be a subgroup of  $G$  of finite index  $n$ . Then  $H$  includes a subgroup  $K$  that is normal in  $G$  and such that  $|G : K|$  divides  $\gcd(|G|, n!)$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\sigma : G \rightarrow \text{Aut}_{\text{Set}}(G/H)$  be the action of left multiplication.

$\langle 1 \rangle 2$ . LET:  $K = \ker \sigma$

$\langle 1 \rangle 3$ .  $K \subseteq H$

$\langle 2 \rangle 1$ . LET:  $g \in K$

$\langle 2 \rangle 2$ .  $\sigma(g)(H) = H$

$\langle 2 \rangle 3$ .  $gH = H$

$\langle 2 \rangle 4$ .  $g \in H$

$\langle 1 \rangle 4$ .  $K$  is normal in  $G$ .

PROOF: Proposition 7.42.

$\langle 1 \rangle 5$ .  $|G : K| \mid |G|$

PROOF: Lagrange's Theorem.

$\langle 1 \rangle 6$ .  $|G : K| \mid n!$

PROOF: Since  $G/K$  is a subgroup of  $\text{Aut}_{\text{Set}}(G/H)$ .

□

**Corollary 9.15.1.** *Let  $G$  be a finite group. Let  $H$  be a subgroup of  $G$  of index  $p$  where  $p$  is the smallest prime that divides  $|G|$ . Then  $H$  is normal in  $G$ .*

PROOF:

$\langle 1 \rangle 1$ . PICK a subgroup  $K$  of  $H$  normal in  $G$  such that  $|G : K|$  divides  $\gcd(|G|, p!)$ .

$\langle 1 \rangle 2$ .  $|G : K|$  divides  $p$ .

$\langle 1 \rangle 3$ .  $|G : H| |H : K|$  divides  $p$ .

$\langle 1 \rangle 4$ .  $|H : K| = 1$

$\langle 1 \rangle 5$ .  $H = K$

$\langle 1 \rangle 6$ .  $H$  is normal.

□

**Corollary 9.15.2.** *Any subgroup of index 2 is normal.*

**Proposition 9.16.** *Let  $G$  be a group with finite set of generators  $A$ . Then left multiplication defines a free action of  $G$  on its Cayley graph.*

PROOF: Easy since if  $g_2 = g_1 a$  then  $hg_2 = hg_1 a$ . □

**Corollary 9.16.1.** *A free group acts freely on a tree.*

**Theorem 9.17.** *If a group  $G$  acts freely on a tree then  $G$  is free.*

**Corollary 9.17.1.** *Every subgroup of the free group on a finite set is free.*

PROOF: If  $H$  is a subgroup of  $F(A)$  then left multiplication defines a free action of  $H$  on the Cayley graph of  $F(A)$ , which is a tree. □

## 9.2 Category of $G$ -Sets

**Definition 9.18.** Given a group  $G$ , let  $G - \mathbf{Set}$  be the category with:

- objects all pairs  $(A, \rho)$  such that  $A$  is a set and  $\rho : G \times A \rightarrow A$  is an action of  $G$  on  $A$ ;
- morphisms  $f : (A, \rho) \rightarrow (B, \sigma)$  are functions  $f : A \rightarrow B$  that are  $(G-)$ equivariant, i.e.

$$\forall g \in G. \forall a \in A. f(\rho(g, a)) = \sigma(g, f(a)) .$$

**Proposition 9.19.** *A  $G$ -equivariant function  $f : A \rightarrow B$  is an isomorphism in  $G - \mathbf{Set}$  if and only if it is bijective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$  be  $G$ -equivariant and bijective.

PROVE:  $f^{-1}$  is  $G$ -equivariant.

$\langle 1 \rangle 2$ . LET:  $g \in G$  and  $b \in B$

$\langle 1 \rangle 3$ .  $f^{-1}(gb) = gf^{-1}(b)$

PROOF:

$$\begin{aligned} f(f^{-1}(gb)) &= gb \\ &= gf(f^{-1}(b)) \\ &= f(gf^{-1}(b)) \end{aligned}$$

□

**Proposition 9.20.** *Let  $G$  be a group and  $A$  a transitive  $G$ -set. Let  $a \in A$ . Then  $A$  is isomorphic to  $G/\text{Stab}_G(a)$  under left multiplication.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : G/\text{Stab}_G(a) \rightarrow A$  be the function  $f(g\text{Stab}_G(a)) = ga$ .

$\langle 2 \rangle 1$ . ASSUME:  $g\text{Stab}_G(a) = h\text{Stab}_G(a)$

PROVE:  $ga = ha$

$\langle 2 \rangle 2$ .  $g^{-1}h \in \text{Stab}_G(a)$

$\langle 2 \rangle 3$ .  $g^{-1}ha = a$

$\langle 2 \rangle 4$ .  $ha = ga$

$\langle 1 \rangle 2$ .  $f$  is  $G$ -equivariant.

PROOF: Since  $f(gh\text{Stab}_G(a)) = gha = gf(h\text{Stab}_G(a))$ .

$\langle 1 \rangle 3$ .  $f$  is injective.

PROOF: If  $ga = ha$  then  $g^{-1}h \in \text{Stab}_G(a)$  so  $g\text{Stab}_G(a) = h\text{Stab}_G(a)$ .

$\langle 1 \rangle 4$ .  $f$  is surjective.

PROOF: Since for all  $b \in A$  there exists  $g \in G$  such that  $ga = b$ .

□

**Corollary 9.20.1.** *If  $O$  is an orbit of the action of a finite group  $G$  on a set  $A$ , then  $O$  is finite and  $|O|$  divides  $|G|$ .*

**Corollary 9.20.2.** *Let  $H$  be a subgroup of  $G$  and  $g \in G$ . Then*

$$G/H \cong G/(gHg^{-1})$$

*in  $G - \mathbf{Set}$ .*

PROOF: Taking  $A = G/H$  and  $a = gH$ .  $\square$

**Proposition 9.21.** *Given a family of  $G$ -sets  $\{A_i\}_{i \in I}$ , we have  $\prod_{i \in I} A_i$  is their product in  $G - \mathbf{Set}$  under*

$$g\{a_i\}_{i \in I} = \{ga_i\}_{i \in I}.$$

PROOF: Easy.  $\square$

**Proposition 9.22.** *Given a family of  $G$ -sets  $\{A_i\}_{i \in I}$ , we have  $\coprod_{i \in I} A_i$  is their product in  $G - \mathbf{Set}$  under*

$$g(i, a_i) = (i, ga_i).$$

PROOF: Easy.  $\square$

**Proposition 9.23.** *Every finite  $G$ -set is a coproduct of  $G$ -sets of the form  $G/H$ .*

PROOF: If  $O(a_1), \dots, O(a_n)$  are the orbits of the  $G$ -set  $A$ , then  $G$  is the coproduct of  $G/\text{Stab}_G(a_1), \dots, G/\text{Stab}_G(a_n)$ .  $\square$

**Proposition 9.24.** *For any group  $G$  we have  $G \cong \text{Aut}_{G-\mathbf{Set}}(G)$  (considering  $G$  as a  $G$ -set under left multiplication).*

PROOF:

$\langle 1 \rangle 1$ . Define  $\phi : G \rightarrow \text{Aut}_{G-\mathbf{Set}}(G)$  by  $\phi(g)(g') = g'g^{-1}$ .

$\langle 2 \rangle 1$ . LET:  $g \in G$

PROVE:  $\lambda g' \in G.g'g^{-1}$  is an automorphism of  $G$  in  $G - \mathbf{Set}$ .

$\langle 2 \rangle 2$ .  $\phi(g)$  is  $G$ -equivariant.

PROOF: Since  $\phi(g)(h_1h_2) = h_1h_2g^{-1} = h_1\phi(g)(h_2)$ .

$\langle 2 \rangle 3$ .  $\phi(g)$  is injective.

PROOF: By Cancellation.

$\langle 2 \rangle 4$ .  $\phi(g)$  is surjective.

PROOF: For any  $h \in G$  we have  $h = \phi(g)(hg)$ .

$\langle 1 \rangle 2$ .  $\phi$  is a group homomorphism.

PROOF:  $\phi(g_1g_2)(h) = hg_2^{-1}g_1^{-1} = \phi(g_1)(\phi(g_2)(h))$ .

$\langle 1 \rangle 3$ .  $\phi$  is injective.

PROOF: If  $\phi(g) = \phi(g')$  then  $g = \phi(g)(e) = \phi(g')(e) = g'$ .

$\langle 1 \rangle 4$ .  $\phi$  is surjective.

$\langle 2 \rangle 1$ . LET:  $\sigma \in \text{Aut}_{G-\mathbf{Set}}(G)$

$\langle 2 \rangle 2$ . LET:  $g = \sigma(e)$

PROVE:  $\sigma = \phi(g^{-1})$

$\langle 2 \rangle 3$ .  $\sigma(h) = hg$

PROOF:  $\sigma(h) = \sigma(hg) = h\sigma(e) = hg$ .

$\square$



**Part III**

**Ring Theory**





# Chapter 10

## Rngs

**Definition 10.1** (Ring). A *rng* consists of a set  $R$  and binary operations  $+, \cdot : R^2 \rightarrow R$  such that:

- $(R, +)$  is an Abelian group
- $\cdot$  is associative.
- The *distributive properties* hold: for all  $r, s, t \in R$  we have

$$(r + s)t = rt + st, \quad r(s + t) = rs + rt .$$

**Example 10.2.**     • The *zero rng* is  $\{0\}$ .

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  are rngs.
- $2\mathbb{Z}$  is a rng.
- Given a rng  $R$  and natural number  $n$ , then the set  $\mathfrak{gl}_n(R)$  of all  $n \times n$  matrices with entries in  $R$  is a rng under matrix addition and matrix multiplication.
- For any set  $S$ , the power set  $\mathcal{P}S$  is a rng under  $A + B = (A \cup B) - (A \cap B)$  and  $AB = A \cap B$ .
- Given a rng  $R$  and a set  $S$ , then  $R^S$  is a rng under  $(f + g)(s) = f(s) + g(s)$  and  $(fg)(s) = f(s)g(s)$  for all  $f, g \in R^S$  and  $s \in S$ .
- The set  $\mathfrak{sl}_n(\mathbb{R}) = \{M \in \mathfrak{gl}_n(\mathbb{R}) : \text{tr } M = 0\}$  is a rng.
- The set  $\mathfrak{sl}_n(\mathbb{C}) = \{M \in \mathfrak{gl}_n(\mathbb{C}) : \text{tr } M = 0\}$  is a rng.
- $\mathbb{Z}/n\mathbb{Z}$  is a rng.

- The ring  $\mathbb{H}$  of *quaternions* is  $\mathbb{R}^4$  under the following operations, where we write  $(a, b, c, d)$  as  $a + bi + cj + dk$ :

$$\begin{aligned}
 (a + bi + cj + dk) + (a' + b'i + c'j + d'k) &= (a + a') + (b + b')i \\
 &\quad + (c + c')j + (d + d')k \\
 (a + bi + cj + dk)(a' + b'i + c'j + d'k) &= (aa' - bb' - cc' - dd') \\
 &\quad + (ab' + ba' + cd' - dc')i \\
 &\quad + (ac' - bd' + ca' + db')j \\
 &\quad + (ad' + bc' - cb' + da')k
 \end{aligned}$$

- For any Abelian group  $G$ , the set  $\text{End}_{\mathbf{Ab}}(G)$  is a ring under pointwise addition and composition.

**Proposition 10.3.** *In any rng  $R$  we have*

$$\forall x \in R. x0 = 0x = 0 \text{ .}$$

PROOF:

$$\begin{aligned}
 x0 &= x(0 + 0) \\
 &= x0 + x0
 \end{aligned}$$

and so  $x0 = 0$  by Cancellation. Similarly  $0x = 0$ .  $\square$

**Definition 10.4** (Zero Divisor). Let  $R$  be a rng and  $a \in R$ .

Then  $a$  is a *left-zero-divisor* iff there exists  $b \in R - \{0\}$  such that  $ab = 0$ .

The element  $a$  is a *right-zero-divisor* iff there exists  $b \in R - \{0\}$  such that  $ba = 0$ .

**Example 10.5.**  $0$  is a left- and right-zero-divisor in every non-zero rng.

The zero rng is the only ring with no zero-divisors.

**Proposition 10.6.** *Let  $R$  be a rng and  $a \in R$ . Then  $a$  is not a left-zero-divisor if and only if left multiplication by  $a$  is an injective function  $R \rightarrow R$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $a$  is not a left-zero-divisor then left multiplication by  $a$  is injective.

$\langle 2 \rangle 1$ . ASSUME:  $a$  is not a left-zero-divisor.

$\langle 2 \rangle 2$ . LET:  $ab = ac$

$\langle 2 \rangle 3$ .  $a(b - c) = 0$

$\langle 2 \rangle 4$ .  $b - c = 0$

$\langle 2 \rangle 5$ .  $b = c$

$\langle 1 \rangle 2$ . If  $a$  is a left-zero-divisor then left multiplication by  $a$  is not injective.

$\langle 2 \rangle 1$ . PICK  $b \neq 0$  such that  $ab = 0$ .

$\langle 2 \rangle 2$ .  $ab = a0$  but  $b \neq 0$

$\square$

## 10.1 Commutative Rings

**Definition 10.7** (Commutative). A rng  $R$  is *commutative* iff  $\forall x, y \in R. xy = yx$ .

**Example 10.8.** • The zero rng is commutative.

- $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are commutative.
- $2\mathbb{Z}$  is commutative.
- $\mathfrak{gl}_2(\mathbb{R})$  is not commutative.
- For any set  $S$ , the rng  $\mathcal{P}S$  is commutative.
- If  $R$  is commutative then  $R^S$  is commutative.

## 10.2 Rng Homomorphisms

**Definition 10.9.** Let  $R$  and  $S$  be rngs. A *rng homomorphism*  $\phi : R \rightarrow S$  is a function such that, for all  $x, y \in R$ , we have

$$\begin{aligned}\phi(x + y) &= \phi(x) + \phi(y) \\ \phi(xy) &= \phi(x)\phi(y)\end{aligned}$$

Let **Rng** be the category of rngs and rng homomorphisms.

## 10.3 Quaternions

**Definition 10.10** (Norm). The *norm* of a quaternion is defined by

$$N(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2 .$$



# Chapter 11

## Rings

**Definition 11.1** (Ring). A *ring*  $R$  is a rng such that there exists  $1 \in R$ , the *multiplicative identity*, such that

$$\forall x \in R. x1 = 1x = x \text{ .}$$

**Example 11.2.**     • The zero rng is a ring with  $1 = 0$ .

- $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are rngs.
- $2\mathbb{Z}$  is not a ring.
- If  $R$  is a ring then  $\mathfrak{gl}_n(R)$  is a ring.
- For any set  $S$ , the rng  $\mathcal{P}S$  is a ring with  $1 = S$ .
- If  $R$  is a ring then  $R^S$  is a ring.
- $\mathfrak{sl}_n(\mathbb{R})$  is not a ring for  $n > 0$ .
- $\mathfrak{sl}_n(\mathbb{C})$  is not a ring for  $n > 0$ .
- $\mathfrak{so}_n(\mathbb{R}) = \{M \in \mathfrak{sl}_n(\mathbb{R}) : M + M^T = 0\}$  is not a ring.
- $\mathbb{Z}/n\mathbb{Z}$  is a ring.

**Proposition 11.3.** *In any ring  $R$ , if  $0 = 1$  then  $R$  is the zero ring.*

PROOF: For any  $x \in R$  we have  $x = 1x = 0x = 0$ .  $\square$

**Proposition 11.4.** *In any ring we have  $(-1)x = -x$ .*

PROOF: Since

$$\begin{aligned} x + (-1)x &= 1x + (-1)x \\ &= (1 + (-1))x \\ &= 0x \\ &= 0 \end{aligned}$$

$\square$

## 11.1 Units

**Definition 11.5** (Left-Unit, Right-Unit). Let  $R$  be a ring and  $a \in R$ . Then  $a$  is a *left-unit* iff there exists  $b \in R$  such that  $ab = 1$ . The element  $a$  is a *right-unit* iff there exists  $b \in R$  such that  $ba = 1$ .

An element is a *unit* iff it is a left-unit and a right-unit.

**Proposition 11.6.** *Let  $R$  be a ring and  $a \in R$ . Then  $a$  is a left-unit iff left multiplication by  $a$  is a surjective function  $R \rightarrow R$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $a$  is a left-unit then left multiplication by  $a$  is surjective.

$\langle 2 \rangle 1$ . PICK  $b \in R$  such that  $ab = 1$ .

$\langle 2 \rangle 2$ . For all  $c \in R$  we have  $c = a(bc)$ .

$\langle 1 \rangle 2$ . If left multiplication by  $a$  is surjective then  $a$  is a left-unit.

PROOF: Immediate.

□

**Proposition 11.7.** *Let  $R$  be a ring and  $a \in R$ . Then  $a$  is a right-unit iff right multiplication by  $a$  is a surjective function  $R \rightarrow R$ .*

PROOF: Similar. □

**Proposition 11.8.** *No left-unit is a right-zero-divisor.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $ab = 1$  and  $ca = 0$  where  $c \neq 0$ .

$\langle 1 \rangle 2$ .  $c = 0$

PROOF:

$$0 = 0b$$

$$= cab$$

$$= c1$$

$$= c$$

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: This is a contradiction.

□

**Proposition 11.9.** *No right-unit is a left-zero-divisor.*

PROOF: Similar. □

**Proposition 11.10.** *The inverse of a unit is unique.*

PROOF: If  $ba = 1$  and  $ac = 1$  then  $b = bac = c$ . □

**Proposition 11.11.** *The units of a ring form a group under multiplication.*

PROOF:

$\langle 1 \rangle 1$ . If  $a$  and  $b$  are units then  $ab$  is a unit.

PROOF: We have  $b^{-1}a^{-1}ab = 1$  and  $abb^{-1}a^{-1} = 1$ .

⟨1⟩2. 1 is a unit.

PROOF: Since  $1 \cdot 1 = 1$ .

⟨1⟩3. If  $a$  is a unit then its inverse is a unit.

PROOF: Immediate from definitions.

□

**Definition 11.12** (Group of Units). For any ring  $R$ , we write  $R^*$  for the group of the units of  $R$  under multiplication.

**Example 11.13.** The quaternionic group is a subgroup of  $\mathbb{H}^*$ .

**Example 11.14.** The norm is a group homomorphism  $\mathbb{H}^* \rightarrow \mathbb{R}^+$  where  $\mathbb{R}^+$  is the group of positive real numbers under multiplication with kernel isomorphic to  $\text{SU}_2(\mathbb{C})$ . The isomorphism maps a quaternion  $a + bi + cj + dk$  to

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

**Theorem 11.15** (Fermat's Little Theorem). *Let  $p$  be a prime number and  $a$  any integer. Then  $a^p \equiv a \pmod{p}$ .*

PROOF: If  $p \mid a$  then  $a^p \equiv a \equiv 0 \pmod{p}$ . Otherwise, we have  $a^{p-1} \equiv 1 \pmod{p}$  by applying Lagrange's Theorem to  $(\mathbb{Z}/p\mathbb{Z})^*$ . □

**Example 11.16.** It is not true that, if  $n \mid |G|$ , then  $G$  has a subgroup of order  $n$ . The group  $A_4$  has order 12 but no subgroup of order 6.

**Proposition 11.17.** *If  $p$  is prime then  $(\mathbb{Z}/p\mathbb{Z})^*$  is cyclic.*

PROOF:

⟨1⟩1. LET:  $g$  be an element of maximal order in  $(\mathbb{Z}/p\mathbb{Z})^*$ .

⟨1⟩2. For all  $h \in (\mathbb{Z}/p\mathbb{Z})^*$  we have  $h^{|g|} = 1$ .

PROOF: Proposition 8.10.

⟨1⟩3. There are at most  $|g|$  elements  $x$  such that  $x^{|g|} = 1$  in  $\mathbb{Z}/p\mathbb{Z}$

⟨1⟩4.  $p - 1 \leq |g|$

⟨1⟩5.  $|g| = p - 1$

⟨1⟩6.  $g$  generates  $(\mathbb{Z}/p\mathbb{Z})^*$ .

□

**Example 11.18.**  $(\mathbb{Z}/12\mathbb{Z})^*$  is not cyclic. Its elements are 1, 5, 7 and 11 with orders 1, 2, 2 and 2.

**Theorem 11.19** (Wilson's Theorem). *A positive integer  $p$  is prime if and only if  $(p - 1)! \equiv 1 \pmod{p}$ .*

⟨1⟩1. If  $p$  is prime then  $(p - 1)! \equiv 1 \pmod{p}$ .

⟨2⟩1. ASSUME:  $p$  is prime.

⟨2⟩2.  $(p - 1)!$  is the product of all the elements of  $(\mathbb{Z}/p\mathbb{Z})^*$

⟨2⟩3. The only element of  $(\mathbb{Z}/p\mathbb{Z})^*$  with order 2 is  $-1$ .

⟨2⟩4.  $(p - 1)! \equiv -1 \pmod{p}$

PROOF: Proposition 6.20.

$\langle 1 \rangle 2$ . If  $(p-1)! \equiv -1 \pmod{p}$  then  $p$  is prime.

$\langle 2 \rangle 1$ . ASSUME:  $(p-1)! \equiv -1 \pmod{p}$

$\langle 2 \rangle 2$ . LET:  $d$  be a proper divisor of  $p$ .

PROVE:  $d = 1$

$\langle 2 \rangle 3$ .  $d \mid (p-1)!$

$\langle 2 \rangle 4$ .  $d \mid 1$

PROOF: Since  $d \mid p \mid (p-1)! + 1$ .

$\langle 2 \rangle 5$ .  $d = 1$

□

**Proposition 11.20.** *If  $p$  and  $q$  are distinct odd primes then  $(\mathbb{Z}/pq\mathbb{Z})^*$  is not cyclic.*

PROOF:

$\langle 1 \rangle 1$ .  $|(\mathbb{Z}/pq\mathbb{Z})^*| = (p-1)(q-1)$

$\langle 1 \rangle 2$ . LET:  $g \in (\mathbb{Z}/pq\mathbb{Z})^*$

PROVE:  $g$  does not have order  $(p-1)(q-1)$

$\langle 1 \rangle 3$ .  $g^{(p-1)(q-1)/2} \equiv 1 \pmod{p}$

$\langle 1 \rangle 4$ .  $g^{(p-1)(q-1)/2} \equiv 1 \pmod{q}$

$\langle 1 \rangle 5$ .  $pq \mid g^{(p-1)(q-1)/2} - 1$

$\langle 1 \rangle 6$ .  $g^{(p-1)(q-1)/2} \equiv 1 \pmod{pq}$

$\langle 1 \rangle 7$ .  $|g| \mid (p-1)(q-1)/2$

□

**Proposition 11.21.** *For any prime  $p$ , we have  $\text{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : \text{Aut}_{\mathbf{Grp}}(C_p) \rightarrow (\mathbb{Z}/p\mathbb{Z})^*$  be the function  $\phi(\alpha) = \alpha(1)$ .

PROOF:  $\alpha(1)$  has order  $p$  in  $C_p$  and so is coprime with  $p$ .

$\langle 1 \rangle 2$ .  $\phi$  is a homomorphism.

PROOF:  $\phi(\alpha \circ \beta) = \alpha(\beta(1)) = \alpha(\beta(1)1) = \beta(1)\alpha(1) = \phi(\alpha)\phi(\beta)$

$\langle 1 \rangle 3$ .  $\phi$  is injective.

PROOF: If  $\phi(\alpha) = \phi(\beta)$  then for any  $n$  we have  $\alpha(n) = n\alpha(1) = n\phi(\alpha) = n\phi(\beta) = n\beta(1) = \beta(n)$ .

$\langle 1 \rangle 4$ .  $\phi$  is surjective.

PROOF: For any  $r \in (\mathbb{Z}/p\mathbb{Z})^*$  we have  $r = \phi(\alpha)$  where  $\alpha(n) = nr \pmod{p}$ .

$\langle 1 \rangle 5$ .  $(\mathbb{Z}/p\mathbb{Z})^* \cong C_{p-1}$

□

## 11.2 Euler's $\phi$ -function

**Proposition 11.22.** *For  $n$  a positive integer, we have  $(\mathbb{Z}/n\mathbb{Z})^* = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$ .*



PROOF:

$$\begin{aligned} m \in (\mathbb{Z}/n\mathbb{Z})^* &\Leftrightarrow \exists a.am \equiv 1 \pmod{n} \\ &\Leftrightarrow \exists a, b.am + bn = 1 \\ &\Leftrightarrow \gcd(m, n) = 1 \quad \square \end{aligned}$$

**Definition 11.23** (Euler's Totient Function). For  $n$  a positive integer, let  $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$ .

**Proposition 11.24.** *If  $n$  is an odd positive integer then  $\phi(2n) = \phi(n)$ .*

PROOF:

(1)1. LET:  $n$  be an odd positive integer.

(1)2. For any integer  $m$ , if  $\gcd(m, n) = 1$  then  $\gcd(2m + n, 2n) = 1$

PROOF: For  $p$  a prime, if  $p \mid 2m + n$  and  $p \mid 2n$  then  $p \neq 2$  (since  $2m + n$  is odd) so  $p \mid n$  and hence  $p \mid m$ , which is a contradiction.

(1)3. For any integer  $r$ , if  $\gcd(r, 2n) = 1$  then  $\gcd(\frac{r+n}{2}, n) = 1$

PROOF: If  $p \mid n$  and  $p \mid \frac{r+n}{2}$  then  $p \mid r + n$  so  $p \mid r$  which is a contradiction.

(1)4. The function that maps  $m$  to  $2m + n$  is a bijection between  $(\mathbb{Z}/n\mathbb{Z})^*$  and  $(\mathbb{Z}/2n\mathbb{Z})^*$ .

□

**Theorem 11.25.** *For any positive integer  $n$  we have*

$$\sum_{m>0, m|n} \phi(m) = n.$$

PROOF:

(1)1. Define  $\chi : \{0, 1, \dots, n-1\} \rightarrow \{(m, d) : m > 0, m \mid n, d \text{ generates } \langle n/m \rangle\}$   
by:  $\chi(x) = (\gcd(x, n), x)$ .

(1)2.  $\chi$  is injective.

(1)3.  $\chi$  is surjective.

PROOF: Given  $(m, d)$  such that  $d$  generates  $\langle n/m \rangle$  we have  $\chi(d) = (m, d)$ .

(1)4.  $n = \sum_{m>0, m|n} \phi(m)$

PROOF: Since  $\langle n/m \rangle \cong C_m$  and so has  $\phi(m)$  generators.

□

**Proposition 11.26.** *For any positive integers  $a$  and  $n$ , we have  $n \mid \phi(a^n - 1)$ .*

PROOF: Since the order of  $a$  is  $n$  in  $(\mathbb{Z}/(a^n - 1)\mathbb{Z})^*$ . □

**Theorem 11.27** (Euler's Theorem). *For any coprime integers  $a$  and  $n$  we have  $a^{\phi(n)} \equiv a \pmod{n}$ .*

PROOF: Immediate from Lagrange's Theorem. □

**Proposition 11.28.**

$$|\text{Aut}_{\mathbf{Grp}}(C_n)| = \phi(n)$$

PROOF: An automorphism  $\alpha$  is determined by  $\alpha(1)$  which is any element of order  $n$ , and  $g$  has order  $n$  iff  $\gcd(g, n) = 1$ . □

**Example 11.29.**

$$\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}) \cong C_2$$

PROOF: The only automorphisms are the identity and multiplication by -1.  $\square$

### 11.3 Nilpotent Elements

**Definition 11.30** (Nilpotent). Let  $R$  be a ring and  $a \in R$ . Then  $a$  is *nilpotent* iff there exists  $n$  such that  $a^n = 0$ .

**Proposition 11.31.** *Let  $R$  be a ring and  $a, b \in R$ . If  $a$  and  $b$  are nilpotent and  $ab = ba$  then  $a + b$  is nilpotent.*

PROOF:

$\langle 1 \rangle 1$ . PICK  $m$  and  $n$  such that  $a^m = b^n = 0$ .

$\langle 1 \rangle 2$ .  $(a + b)^{m+n} = 0$

PROOF: Since  $(a + b)^{m+n} = \sum_k \binom{m+n}{k} a^k b^{m+n-k}$  and every term in this sum is 0 since, for every  $k$ , either  $k \geq m$  or  $m + n - k \geq n$ .

$\square$

**Proposition 11.32.**  *$m$  is nilpotent in  $\mathbb{Z}/n\mathbb{Z}$  if and only if  $m$  is divisible by all the prime factors of  $n$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $m$  is nilpotent then  $m$  is divisible by all the prime factors of  $n$ .

$\langle 2 \rangle 1$ . ASSUME:  $m^a \equiv 0 \pmod{n}$

$\langle 2 \rangle 2$ . For every prime  $p$ , if  $p \mid n$  then  $p \mid m^a$ .

$\langle 2 \rangle 3$ . For every prime  $p$ , if  $p \mid n$  then  $p \mid m$ .

$\langle 1 \rangle 2$ . If  $m$  is divisible by all the prime factors of  $n$  then  $m$  is nilpotent in  $\mathbb{Z}/n\mathbb{Z}$ .

$\langle 2 \rangle 1$ . ASSUME:  $m$  is divisible by all the prime factors of  $n$ .

$\langle 2 \rangle 2$ . LET:  $a$  be the largest number such that  $p^a \mid n$  for some prime  $p$ .

$\langle 2 \rangle 3$ . For every prime  $p$  that divides  $n$  we have  $p^a \mid m^a$

$\langle 2 \rangle 4$ .  $n \mid m^a$

$\langle 2 \rangle 5$ .  $m^a \equiv 0 \pmod{n}$

$\langle 2 \rangle 6$ .  $m$  is nilpotent in  $\mathbb{Z}/n\mathbb{Z}$ .

$\square$

## Chapter 12

# Ring Homomorphisms

**Definition 12.1** (Ring Homomorphism). Let  $R$  and  $S$  be rings. A *ring homomorphism*  $\phi : R \rightarrow S$  is a rng homomorphism such that  $\phi(1) = 1$ .

**Proposition 12.2.** *The zero-ring is terminal in **Ring**.*

PROOF: Easy.  $\square$

**Proposition 12.3.** *The ring  $\mathbb{Z}$  is initial in **Ring**.*

PROOF: Easy.  $\square$

**Proposition 12.4.** *Let  $R$  and  $S$  be rings and  $\phi : R \rightarrow S$  be a rng homomorphism. If  $\phi$  is surjective, then  $\phi$  is a ring homomorphism.*

PROOF:

$\langle 1 \rangle 1$ . PICK  $a \in R$  such that  $\phi(a) = 1$

$\langle 1 \rangle 2$ .  $\phi(1) = 1$

PROOF:

$$\begin{aligned}\phi(1) &= \phi(1)\phi(a) \\ &= \phi(1a) \\ &= \phi(a) \\ &= 1\end{aligned}$$

$\square$

**Example 12.5.** For any set  $S$  we have  $\mathcal{P}S \cong (\mathbb{Z}/2\mathbb{Z})^S$  in **Ring** with the isomorphism

$$\begin{aligned}\phi : \mathcal{P}S &\cong (\mathbb{Z}/2\mathbb{Z})^S \\ \phi(A)(s) &= \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}\end{aligned}$$

**Example 12.6.** The function  $\mathbb{H} \rightarrow \mathfrak{gl}_4(\mathbb{R})$  that maps  $a + bi + cj + dk$  to

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

is a monomorphism in **Ring**, as is the function  $\mathbb{H} \rightarrow \mathfrak{sl}_2(\mathbb{C})$  that maps  $a + bi + cj + dk$  to

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

**Proposition 12.7.** *Ring homomorphisms preserve units.*

PROOF: If  $uv = 1$  then  $\phi(u)\phi(v) = 1$ .  $\square$

**Proposition 12.8.** *Let  $\phi : R \rightarrow S$  be a ring homomorphism. Then the following are equivalent.*

1.  $\phi$  is a monomorphism.
2.  $\ker \phi = \{0\}$
3.  $\phi$  is injective.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $\phi$  is a monomorphism.

$\langle 2 \rangle 2.$  LET:  $r \in \ker \phi$

$\langle 2 \rangle 3.$  LET:  $\text{ev}_r : \mathbb{Z}[x] \rightarrow R$  be the unique ring homomorphism such that  $\text{ev}_r(x) = r$ .

$\langle 2 \rangle 4.$  LET:  $\text{ev}_0 : \mathbb{Z}[x] \rightarrow R$  be the unique ring homomorphism such that  $\text{ev}_0(x) = 0$ .

$\langle 2 \rangle 5.$   $\phi \circ \text{ev}_r = \phi \circ \text{ev}_0$

$\langle 2 \rangle 6.$   $\text{ev}_r = \text{ev}_0$

$\langle 2 \rangle 7.$   $r = 0$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

PROOF: Proposition 7.20.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: Easy.

$\square$

**Example 12.9.** It is not true that every epimorphism in **Ring** is surjective. The inclusion  $\mathbb{Z} \hookrightarrow \mathbb{Q}$  is an epimorphism but not surjective.

The same example shows that a ring homomorphism may be a monomorphism and an epimorphism but not be an isomorphism.

**Example 12.10.**

$$\text{End}_{\mathbf{Ab}}(\mathbb{Z}) \cong \mathbb{Z}$$

The isomorphism maps any group endomorphism  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$  to  $\phi(1)$ .

**Example 12.11.** The group of units of  $\text{End}_{\mathbf{Ab}}(G)$  is  $\text{Aut}_{\mathbf{Ab}}(G)$ .

**Example 12.12.** Let  $R$  be a ring. Then the function  $\lambda : R \rightarrow \text{End}_{\mathbf{Ab}}(R)$  defined by

$$\lambda(a)(b) = ab$$

is a ring monomorphism.

PROOF: Easy.  $\square$

## 12.1 Products

**Proposition 12.13.** *Let  $R$  and  $S$  be rings. Then  $R \times S$  is a ring under componentwise addition and multiplication, and this ring is the product of  $R$  and  $S$  in **Ring**.*

PROOF: Easy.  $\square$



# Chapter 13

## Subrings

**Definition 13.1** (Subring). Let  $S$  be a ring. A *subring* of  $S$  is a ring  $R$  such that  $R$  is a subset of  $S$  and the inclusion  $R \hookrightarrow S$  is a ring homomorphism.

**Proposition 13.2.** *Let  $R$  and  $S$  be rings. Then  $R$  is a subring of  $S$  if and only if  $R$  is a subset of  $S$ , the unit  $1$  of  $S$  is an element of  $R$ , and the operations of  $R$  are the restrictions of the operations of  $S$  to  $R$ .*

PROOF: Easy.  $\square$

**Corollary 13.2.1.** *The zero ring is not a subring of any non-zero ring.*

**Proposition 13.3.** *Let  $\phi : R \rightarrow S$  be a ring homomorphism. Then  $\phi(R)$  is a subring of  $S$ .*

PROOF: Easy.  $\square$

### 13.1 Centralizer

**Definition 13.4** (Centralizer). Let  $R$  be a ring and  $a \in R$ . The *centralizer* of  $a$  is  $\{r \in R : ar = ra\}$ .

**Proposition 13.5.** *The centralizer of  $a$  is a subring of  $R$ .*

PROOF: Easy.  $\square$

### 13.2 Center

**Definition 13.6** (Center). The *center* of a ring  $R$  is  $\{x \in R : \forall y \in R. xy = yx\}$ .

**Proposition 13.7.** *The center of a ring is a subring.*

PROOF: Easy.  $\square$

**Proposition 13.8.** *Let  $R$  be a ring. The center of  $\text{End}_{\mathbf{Ab}}(R)$  is isomorphic to the center of  $R$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\lambda : R \rightarrow \text{End}_{\mathbf{Ab}}(R)$  be left multiplication.

$\langle 1 \rangle 2$ .  $\lambda$  maps  $Z(R)$  to  $Z(\text{End}_{\mathbf{Ab}}(R))$ .

$\langle 2 \rangle 1$ . LET:  $a \in Z(R)$

$\langle 2 \rangle 2$ . LET:  $\phi \in \text{End}_{\mathbf{Ab}}(R)$

PROVE:  $\lambda(a) \circ \phi = \phi \circ \lambda(a)$

$\langle 2 \rangle 3$ . LET:  $x \in R$

$\langle 2 \rangle 4$ .  $a + \phi(x) = \phi(a + x)$

$\langle 1 \rangle 3$ .  $\lambda(Z(R)) = Z(\text{End}_{\mathbf{Ab}}(R))$

$\langle 2 \rangle 1$ . LET:  $\phi \in Z(\text{End}_{\mathbf{Ab}}(R))$

$\langle 2 \rangle 2$ . For all  $r \in R$ ,

LET:  $\mu_r \in \text{End}_{\mathbf{Ab}}(R)$  be right multiplication by  $r$ .

$\langle 2 \rangle 3$ . For all  $r \in R$  we have  $\phi \circ \mu_r = \mu_r \circ \phi$ .

$\langle 2 \rangle 4$ . For all  $r, x \in R$  we have  $\phi(xr) = \phi(x)r$

$\langle 2 \rangle 5$ . For all  $r \in R$  we have  $\phi(r) = \phi(1)r$

$\langle 2 \rangle 6$ .  $\phi = \lambda(\phi(1))$

□

**Corollary 13.8.1.** *If  $R$  is a commutative ring then  $R$  is isomorphic to the center of  $\text{End}_{\mathbf{Ab}}(R)$ .*

**Example 13.9.** For  $n$  a positive integer we have  $\mathbb{Z}/n\mathbb{Z} \cong \text{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$ .

Since, for any  $\phi \in \text{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$  we have  $\phi(m) = m\phi(1)$  and so the whole of  $\text{End}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z})$  is the image of  $\lambda$ .



## Chapter 14

# Monoid Rings

**Definition 14.1** (Monoid Ring). Let  $R$  be a ring and  $M$  a monoid. Define  $R[M]$  to be the ring whose elements are the families  $\{a_m\}_{m \in M}$  such that  $a_m = 0$  for all but finitely many  $m \in M$ , written

$$\sum_{m \in M} a_m m \text{ ,}$$

under

$$\begin{aligned} \sum_m a_m m + \sum_m b_m m &= \sum_m (a_m + b_m) m \\ \left( \sum_m a_m m \right) \left( \sum_m b_m m \right) &= \sum_{m \in M} \sum_{m_1 m_2 = m} a_{m_1} b_{m_2} m \end{aligned}$$

**Example 14.2.** Ring homomorphisms do not necessarily preserve zero-divisors. The canonical homomorphism  $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z}$  maps the non-zero-divisor 2 to a zero-divisor.

### 14.1 Polynomials

**Definition 14.3** (Polynomial). Let  $R$  be a ring. The ring of *polynomials*  $R[x]$  is  $R[\mathbb{N}]$ . We write

$$\sum_n a_n x^n \text{ for } \sum_n a_n n \text{ .}$$

Concretely, a *polynomial* in  $R$  is a sequence  $(a_n)$  in  $R$  such that there exists  $N$  such that  $\forall n \geq N. a_n = 0$ . We write the polynomial as

$$\sum_{n=0}^{N-1} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1} \text{ .}$$

We write  $R[x]$  for the set of all polynomials in  $R$ .

Define addition and multiplication on  $R[x]$  by

$$\begin{aligned}\sum_n a_n x^n + \sum_n b_n x^n &= \sum_n (a_n + b_n) x^n \\ \left( \sum_n a_n x^n \right) \left( \sum_n b_n x^n \right) &= \sum_n \sum_{i+j=n} a_i b_j x^n\end{aligned}$$

A *constant* is a polynomial of the form  $a + 0x + 0x^2 + \cdots$  for some  $a \in R$ . We write  $R[x_1, \dots, x_n]$  for  $R[x_1][x_2] \cdots [x_n]$ .

**Proposition 14.4.** *For any ring  $R$ , the set of polynomials  $R[x]$  is a ring.*

PROOF: Easy.  $\square$

**Definition 14.5** (Degree). The *degree* of a polynomial  $\sum_n a_n x^n$  is the largest integer  $d$  such that  $a_d \neq 0$ . We take the degree of the zero polynomial to be  $-\infty$ .

**Proposition 14.6.** *Let  $R$  be a ring and  $f, g \in R[x]$  be nonzero polynomials. Then*

$$\deg(f + g) \leq \max(\deg f, \deg g) .$$

PROOF: If  $a_n + b_n \neq 0$  then  $a_n \neq 0$  or  $b_n \neq 0$ .  $\square$

**Proposition 14.7.** *The function  $i : n \rightarrow \mathbb{Z}[x_1, \dots, x_n]$  that maps  $k$  to  $x_k$  is initial in the category with:*

- *objects all pairs  $j : n \rightarrow R$  where  $R$  is a commutative ring and  $j$  a function*
- *morphisms  $\phi : (j_1, R_1) \rightarrow (j_2, R_2)$  are ring homomorphisms  $\phi : R_1 \rightarrow R_2$  such that  $\phi \circ j_1 = j_2$ .*

PROOF: The unique morphism  $(i, \mathbb{Z}[x_1, \dots, x_n]) \rightarrow (j, R)$  maps a polynomial  $p$  to  $p(j(0), j(1), \dots, j(n-1))$ .  $\square$

**Proposition 14.8.** *Let  $\alpha : R \rightarrow S$  be a ring homomorphism. Let  $s \in S$  commute with  $\alpha(r)$  for all  $r \in R$ . Then there exists a unique ring homomorphism  $\bar{\alpha} : R[x] \rightarrow S$  such that  $\bar{\alpha}(x) = s$  and the following diagram commutes:*

$$\begin{array}{ccc} R[x] & \xrightarrow{\bar{\alpha}} & S \\ \uparrow & \nearrow \alpha & \\ R & & \end{array}$$

PROOF: The map  $\bar{\alpha}$  is given by

$$\bar{\alpha}(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = \alpha(a_0) + \alpha(a_1)s + \alpha(a_2)s^2 + \cdots + \alpha(a_n)s^n .$$

$\square$

**Definition 14.9.** Let  $R$  be a commutative ring. Given a polynomial  $p \in R[x]$ , the *polynomial function*  $p : R \rightarrow R$  is the function given by:  $p(r) = \alpha_r(p)$ , where  $\alpha_r : R[x] \rightarrow R$  is the unique ring homomorphism such that the following diagram commutes.

$$\begin{array}{ccc} R[x] & \xrightarrow{\alpha_r} & R \\ \uparrow x & \nearrow r & \\ 1 & & \end{array}$$

**Proposition 14.10.**  $\mathbb{Z}[x, y]$  is the coproduct of  $\mathbb{Z}[x]$  and  $\mathbb{Z}[y]$  in the category of commutative rings.

PROOF: Given ring homomorphisms  $f : \mathbb{Z}[x] \rightarrow R$  and  $g : \mathbb{Z}[y] \rightarrow R$ , the required morphism  $\mathbb{Z}[x, y] \rightarrow R$  maps  $p(x, y)$  to  $p(f(x), g(y))$ .  $\square$

**Example 14.11.**  $\mathbb{Z}[x, y]$  is not the coproduct of  $\mathbb{Z}[x]$  and  $\mathbb{Z}[y]$  in **Ring**. Given  $f : \mathbb{Z}[x] \rightarrow R$  and  $g : \mathbb{Z}[y] \rightarrow R$  with  $f(x) \neq g(y)$ , the mediating morphism  $\mathbb{Z}[x, y] \rightarrow R$  cannot exist since it must map  $xy$  to both  $f(x)g(y)$  and  $g(y)f(x)$ .  $\square$

**Definition 14.12.** A polynomial is *monic* iff its last non-zero coefficient is 1.

**Proposition 14.13.** A monic polynomial is not a left- or right-zero-divisor.

PROOF: Easy.  $\square$

**Proposition 14.14.** Let  $R$  be a ring. Let  $f, g \in R[x]$  with  $f$  monic. Then there exist unique polynomials  $q, r \in R[x]$  with  $\deg r < \deg f$  such that

$$g = qf + r .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $d = \deg f$

$\langle 1 \rangle 2$ . For all  $a \in R$  and  $n > d$ , there exists  $h \in R[x]$  with  $\deg h < n$  such that

$$ax^n = ax^{n-d}f + h .$$

PROOF: Take  $h = ax^n - ax^{n-d}f$ .

$\langle 1 \rangle 3$ . For all  $a \in R$  and  $n > d$ , there exists  $q, h \in R[x]$  with  $\deg h \leq d$  such that

$$ax^n = qf + h .$$

PROOF: Repeating  $\langle 1 \rangle 2$  by induction.

$\langle 1 \rangle 4$ . LET:  $g = \sum_{i=0}^n a_i x^i$

$\langle 1 \rangle 5$ . For  $i > d$ , PICK  $q_i h_i \in R[x]$  with  $\deg h < \deg f$  such that  $a_i x^i = q_i f + h_i$

$\langle 1 \rangle 6$ .  $g = (\sum_{i=d+1}^n q_i) f + \sum_{i=d+1}^n h_i$

$\langle 1 \rangle 7$ .  $q$  and  $r$  are unique.

PROOF: If  $q_1 f + r_1 = q_2 f + r_2$  then  $r_1 - r_2 = (q_2 - q_1)f$  and so  $r_1 - r_2 = (q_2 - q_1)f = 0$  since  $\deg(r_1 - r_2) < \deg f$ .

$\square$

## 14.2 Laurent Polynomials

**Definition 14.15** (Laurent Polynomial). Let  $R$  be a ring. The ring of *Laurent polynomials* is the group ring  $R[\mathbb{Z}]$ . We write  $\sum_{n \in \mathbb{Z}} a_n x^n$  for  $\sum_n a_n n$ .

### 14.3 Power Series

**Definition 14.16** (Power Series). Let  $R$  be a ring. A *power series* in  $R$  is a sequence  $(a_n)$  in  $R$ . We write the power series as

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

We write  $R[[x]]$  for the set of all power series in  $R$ .

Define addition and multiplication on  $R[[x]]$  by

$$\begin{aligned} \sum_n a_n x^n + \sum_n b_n x^n &= \sum_n (a_n + b_n) x^n \\ \left( \sum_n a_n x^n \right) \left( \sum_n b_n x^n \right) &= \sum_n \sum_{i+j=n} a_i b_j x^n \end{aligned}$$

**Proposition 14.17.** *For any ring  $R$ , the set of power series  $R[[x]]$  is a ring.*

PROOF: Easy.  $\square$

**Proposition 14.18.** *A power series  $\sum_n a_n x^n$  is a unit in  $R[[x]]$  if and only if  $a_0$  is a unit in  $R$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $\sum_n a_n x^n$  is a unit then  $a_0$  is a unit.

$\langle 2 \rangle 1$ . LET:  $\sum_n b_n x^n$  be the inverse of  $\sum_n a_n x^n$ .

$\langle 2 \rangle 2$ .  $a_0 b_0 = b_0 a_0 = 1$

$\langle 1 \rangle 2$ . If  $a_0$  is a unit then  $\sum_n a_n x^n$  is a unit.

PROOF: Define the sequence  $(b_n)$  in  $R$  by

$$b_n = -a_0^{-1} \sum_{i=1}^n a_i b_{n-i}$$

Then  $\sum_n b_n x^n$  is the inverse of  $\sum_n a_n x^n$ .

$\square$

# Chapter 15

## Ideals

**Definition 15.1** (Left-Ideal). Let  $R$  be a ring.

A subgroup  $I$  of  $R$  is a *left-ideal* iff, for all  $r \in R$ , we have  $rI \subseteq I$ .

A subgroup  $I$  of  $R$  is a *right-ideal* iff, for all  $r \in R$ , we have  $Ir \subseteq I$ .

A subgroup  $I$  of  $R$  is a *(two-sided) ideal* iff it is a left-ideal and a right-ideal.

**Example 15.2.** Let  $R$  be a ring and  $a \in R$ . Then  $Ra$  is a left-ideal and  $aR$  is a right-ideal.

In particular,  $\{0\}$  is always a two-sided ideal.

**Example 15.3.** Let  $S$  be a set and  $T \subseteq S$ . Then  $\{X \in \mathcal{P}S : X \subseteq T\}$  is an ideal in  $\mathcal{P}S$ .

**Proposition 15.4.** Let  $S$  be a finite set. Then every ideal in  $\mathcal{P}S$  is of the form  $\{X \in \mathcal{P}S : X \subseteq T\}$  for some  $T \subseteq S$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $I$  be an ideal in  $\mathcal{P}S$ .

$\langle 1 \rangle 2$ . LET:  $T = \bigcup I$

$\langle 1 \rangle 3$ . For all  $i \in T$  we have  $\{i\} \in I$ .

$\langle 2 \rangle 1$ . LET:  $i \in T$

$\langle 2 \rangle 2$ . PICK  $X \in I$  such that  $i \in X$

$\langle 2 \rangle 3$ .  $\{i\} = \{i\} \cap X \in I$

$\langle 1 \rangle 4$ . For all  $X \subseteq T$  we have  $X \in I$ .

PROOF: If  $X = \{x_1, \dots, x_n\}$  then  $X = \{x_1\} + \dots + \{x_n\} \in I$ .

□

**Example 15.5.** If  $S$  is an infinite set, then there is always an ideal in  $\mathcal{P}S$  that is not of the form  $\{X \in \mathcal{P}S : X \subseteq T\}$  for some  $T \subseteq S$ , namely the set of all finite subsets of  $S$ .

**Proposition 15.6.** Let  $\phi : R \twoheadrightarrow S$  be a surjective ring homomorphism. Let  $J$  be an ideal in  $R$ . Then  $\phi(J)$  is an ideal in  $S$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $j \in J$  and  $s \in S$   
 PROVE:  $s\phi(j), \phi(j)s \in \phi(J)$   
 $\langle 1 \rangle 2$ . PICK  $r \in R$  such that  $\phi(r) = s$   
 $\langle 1 \rangle 3$ .  $rj, jr \in J$   
 $\langle 1 \rangle 4$ .  $s\phi(j), \phi(j)s \in \phi(J)$   
 $\square$

**Example 15.7.** We cannot remove the hypothesis that  $\phi$  is surjective.

Let  $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$  be the inclusion. Then  $i(2\mathbb{Z}) = 2\mathbb{Z}$  is not an ideal in  $\mathbb{Q}$ .

**Proposition 15.8.** Let  $\phi : R \rightarrow S$  be a ring homomorphism and  $I$  a (left-, right-)ideal in  $S$ . Then  $\phi^{-1}I$  is a (left-, right-)ideal in  $R$ .

PROOF: Easy.  $\square$

**Corollary 15.8.1.** Let  $\phi : R \rightarrow S$  be a ring homomorphism. Then  $\ker \phi$  is an ideal in  $R$ .

**Definition 15.9** (Quotient Ring). Let  $I$  be an ideal in  $R$ . The *quotient ring*  $R/I$  is the quotient group  $R/I$  under

$$(a + I)(b + I) = ab + I.$$

This is well-defined as, if  $a + I = a' + I$  and  $b + I = b' + I$  then

$$\begin{aligned}
 a - a' &\in I \\
 b - b' &\in I \\
 \therefore ab - a'b &\in I \\
 a'b - a'b' &\in I \\
 \therefore ab - a'b' &\in I
 \end{aligned}$$

**Proposition 15.10.** Let  $I$  be an ideal in  $R$ . Then the canonical group homomorphism  $\pi : R \rightarrow R/I$  is a ring homomorphism.

PROOF: By construction.  $\square$

**Proposition 15.11.** Let  $I$  be an ideal in a ring  $R$ . For every ring homomorphism  $\phi : R \rightarrow S$  such that  $I \subseteq \ker \phi$ , there exists a unique ring homomorphism  $\bar{\phi} : R/I \rightarrow S$  such that the following diagram commutes.

$$\begin{array}{ccc}
 R & \xrightarrow{\phi} & S \\
 \searrow \pi & & \nearrow \bar{\phi} \\
 & R/I &
 \end{array}$$

PROOF: Easy.  $\square$

**Corollary 15.11.1.** Every ring homomorphism  $\phi : R \rightarrow S$  decomposes as follows.

$$\begin{array}{ccccc}
 & & \phi & & \\
 & \searrow & & \nearrow & \\
 R & \twoheadrightarrow & R/\ker \phi & \xrightarrow{\cong} & \text{im } \phi & \hookrightarrow & S
 \end{array}$$

**Corollary 15.11.2** (First Isomorphism Theorem). *Let  $\phi : R \rightarrow S$  be a surjective ring homomorphism. Then*

$$S \cong R/\ker \phi .$$

**Theorem 15.12** (Third Isomorphism Theorem). *Let  $I$  and  $J$  be ideals in  $R$  with  $I \subseteq J$ . Then  $J/I$  is an ideal in  $R/I$ , and*

$$\frac{R/I}{J/I} \cong R/J$$

PROOF: Since the function  $R/I \rightarrow R/J$  that maps  $r + I$  to  $r + J$  is a surjective ring homomorphism with kernel  $J/I$ .  $\square$

**Corollary 15.12.1.** *Let  $\phi : R \rightarrow S$  be a surjective ring homomorphism. Let  $J$  be an ideal in  $R$ . Then*

$$\frac{S}{\phi(J)} \cong \frac{R}{\ker \phi + J}$$

**Proposition 15.13.** *Let  $R$  be a ring and  $J$  an ideal in  $\mathfrak{gl}_n(R)$ . Let  $A \in \mathfrak{gl}_n(R)$ . Then  $A \in J$  if and only if the matrices obtained by placing any entry of  $A$  in any position and zeros elsewhere all belong to  $J$ .*

PROOF: Each such matrix can be obtained by pre- and post-multiplying  $A$  by matrices which have a single 1 and 0s elsewhere. Conversely,  $A$  is a sum of such matrices.  $\square$

**Corollary 15.13.1.** *Let  $R$  be a ring. Let  $J$  be an ideal in  $\mathfrak{gl}_n(R)$ . Let  $I$  be the set of all entries of elements of  $J$ . Then  $I$  is an ideal in  $R$ , and  $J$  is the set of all matrices whose entries are in  $I$ .*

**Proposition 15.14.** *Let  $R$  be a ring. Let  $\{I_\alpha\}_{\alpha \in A}$  be a family of ideals in  $R$ . Let*

$$\sum_{\alpha \in A} I_\alpha = \left\{ \sum_{\alpha \in A} r_\alpha : \forall \alpha, r_\alpha \in I_\alpha, r_\alpha = 0 \text{ for all but finitely many } \alpha \in A \right\} .$$

*Then  $\sum_{\alpha \in A} I_\alpha$  is an ideal, and is the smallest ideal that includes every  $I_\alpha$ .*

PROOF: Easy.  $\square$

**Proposition 15.15.** *The intersection of a set of ideals is an ideal.*

PROOF: Easy.  $\square$

## 15.1 Characteristic

**Definition 15.16** (Characteristic). The *characteristic* of a ring  $R$  is the non-negative integer  $n$  such that  $n\mathbb{Z}$  is the kernel of the unique ring homomorphism  $\mathbb{Z} \rightarrow R$ .

**Proposition 15.17.** *Let  $R$  be a ring. If the unit 1 has finite order in  $R$ , then its order is the characteristic of  $R$ ; otherwise, the characteristic of  $R$  is 0.*

PROOF: Easy.  $\square$

**Example 15.18.** The zero ring is the only ring with characteristic 1.

## 15.2 Nilradical

**Definition 15.19** (Nilradical). Let  $R$  be a commutative ring. The *nilradical* of  $R$  is the set of all nilpotent elements.

**Proposition 15.20.** *Let  $R$  be a commutative ring. The nilradical of  $R$  is an ideal in  $R$ .*

PROOF: If  $a^n = 0$  then for any  $b$  we have  $(ba)^n = 0$ .  $\square$

**Example 15.21.** We cannot remove the assumption that  $R$  is commutative. In  $\mathfrak{gl}_2(\mathbb{R})$  we have that  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  is nilpotent but  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$  is not.

## 15.3 Principal Ideals

**Definition 15.22** (Principal Ideal). Let  $R$  be a commutative ring and  $a \in R$ . The *principal ideal* generated by  $a$  is  $(a) = Ra = aR$ .

**Example 15.23.**  $\{0\} = (0)$  and  $R = \{1\}$  are principal ideals.

**Definition 15.24.** Let  $R$  be a commutative ring and  $\{a_\alpha\}_{\alpha \in A}$  be a family of elements of  $R$ . The *ideal generated by the elements  $a_\alpha$*  is

$$(a_\alpha)_{\alpha \in A} := \sum_{\alpha \in A} (a_\alpha) \ .$$

An ideal is *finitely generated* iff it is generated by a finite family of elements.

**Definition 15.25.** Let  $R$  be a commutative ring and  $I, J$  be ideals in  $R$ . Then  $IJ$  is the ideal generated by  $\{ij\}_{i \in I, j \in J}$ .

**Proposition 15.26.**

$$IJ \subseteq I \cap J$$



PROOF: Easy.  $\square$

**Proposition 15.27.** *Let  $R$  be a commutative ring. Let  $I$  and  $J$  be ideals in  $R$ . If  $I + J = R$  then  $IJ = I \cap J$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $r \in I \cap J$

$\langle 1 \rangle 2$ . PICK  $i \in I$  and  $j \in J$  such that  $i + j = 1$ .

$\langle 1 \rangle 3$ .  $ri, rj \in IJ$

$\langle 1 \rangle 4$ .  $r = ri + rj \in IJ$

$\square$

**Proposition 15.28.** *Let  $R$  be a commutative ring. Let  $f \in R[x]$  be a monic polynomial of degree  $d$ . Then the function*

$$\phi : R[x] \rightarrow R^{\oplus d}$$

*that sends a polynomial  $g$  to the remainder of the division of  $g$  by  $f$  induces an isomorphism of Abelian groups*

$$\frac{R[x]}{(f(x))} \cong R^{\oplus d}.$$

PROOF: It is clearly a group homomorphism; it is surjective since it maps any polynomial of degree  $< d$  to itself, and its kernel is  $(f(x))$  since these are the polynomials with remainder 0.  $\square$

**Corollary 15.28.1.** *Let  $R$  be a commutative ring and  $a \in R$ . Then we have*

$$\frac{R[x]}{(x - a)} \cong R$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : R[x] \rightarrow R$  be evaluation at  $a$ .

$\langle 1 \rangle 2$ .  $\phi(g)$  is the remainder when dividing  $g$  by  $x - a$ .

PROOF: If  $g = (x - a)q + r$  then  $g(a) = (a - a)q(a) + r = r$ .

$\langle 1 \rangle 3$ .  $\phi$  induces a group isomorphism  $R[x]/(x - a) \cong R$

PROOF: By the theorem.

$\langle 1 \rangle 4$ . This isomorphism is a ring isomorphism.

PROOF: Since evaluation at  $a$  is a ring homomorphism.

$\square$

**Example 15.29.** We have

$$\frac{\mathbb{R}[x]}{(x^2 + 1)} \cong \mathbb{C}$$

as rings.

## 15.4 Maximal Ideals

**Definition 15.30** (Maximal Ideal). Let  $R$  be a ring and  $I$  an ideal in  $R$ . Then  $I$  is a *maximal ideal* iff  $I \neq R$  and, whenever  $J$  is an ideal with  $I \subseteq J$ , then either  $I = J$  or  $J = R$ .



## Chapter 16

# Integral Domains

**Definition 16.1** (Integral Domain). An *integral domain* is a non-trivial commutative ring with no nonzero zero-divisors.

**Example 16.2.**  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are integral domains.

**Proposition 16.3.**  $\mathbb{Z}/n\mathbb{Z}$  is an integral domain if and only if  $n$  is prime.

PROOF:

$$\begin{aligned} n \text{ is prime} &\Leftrightarrow \forall a, b \in \mathbb{Z} (n \mid ab \Rightarrow n \mid a \vee n \mid b) \\ &\Leftrightarrow \forall a, b \in \mathbb{Z}/n\mathbb{Z} (ab \cong 0(\bmod n) \Rightarrow a \cong 0(\bmod n) \vee b \cong 0(\bmod n)) \\ &\Leftrightarrow \mathbb{Z}/n\mathbb{Z} \text{ is an integral domain} \quad \square \end{aligned}$$

**Proposition 16.4.** In an integral domain, if  $x^2 = 1$  then  $x = \pm 1$ .

PROOF: We have  $x^2 - 1 = (x - 1)(x + 1) = 0$  so  $x - 1 = 0$  or  $x + 1 = 0$ .  $\square$

**Proposition 16.5.** Let  $R$  be an integral domain and  $f, g \in R[x]$ . Then

$$\deg(fg) = \deg f + \deg g$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $f = \sum_n a_n x^n$  and  $g = \sum_n b_n x^n$ .

$\langle 1 \rangle 2$ . LET:  $d = \deg f$  and  $e = \deg g$ .

$\langle 1 \rangle 3$ . The  $d + e$ th term of  $fg$  is

$$a_d b_e x^{d+e}$$

which is non-zero.

$\langle 1 \rangle 4$ . For  $n > d + e$  the  $n$ th term of  $fg$  is 0.

$\square$

**Corollary 16.5.1.** Let  $R$  be a ring. Then  $R[x]$  is an integral domain if and only if  $R$  is an integral domain.

**Proposition 16.6.** Let  $R$  be a ring. Then  $R[[x]]$  is an integral domain if and only if  $R$  is an integral domain.

PROOF:

$\langle 1 \rangle 1$ . If  $R[[x]]$  is an integral domain then  $R$  is an integral domain.

PROOF: Easy.

$\langle 1 \rangle 2$ . If  $R$  is an integral domain then  $R[[x]]$  is an integral domain.

$\langle 2 \rangle 1$ . ASSUME:  $R$  is an integral domain.

$\langle 2 \rangle 2$ . LET:  $(\sum_n a_n x^n)(\sum_n b_n x^n) = 0$

$\langle 2 \rangle 3$ .  $a_0 b_0 = 0$

$\langle 2 \rangle 4$ .  $a_0 = 0$  or  $b_0 = 0$

$\langle 2 \rangle 5$ . ASSUME: w.l.o.g.  $b_0 \neq 0$

PROVE: For all  $n$  we have  $a_n = 0$

$\langle 2 \rangle 6$ . ASSUME: as induction hypothesis  $a_0 = a_1 = \cdots = a_{n-1} = 0$

$\langle 2 \rangle 7$ .  $\sum_{i=0}^n a_i b_{n-i} = 0$

$\langle 2 \rangle 8$ .  $a_n b_0 = 0$

$\langle 2 \rangle 9$ .  $a_n = 0$

□

**Proposition 16.7.** *Let  $R$  be a ring and  $S$  an integral domain. Every ring homomorphism  $\phi : R \rightarrow S$  is a ring homomorphism.*

PROOF:

$$\begin{aligned}\phi(1) &= \phi(1 \cdot 1) \\ &= \phi(1)\phi(1)\end{aligned}$$

and so  $\phi(1) = 1$  by Cancellation. □

**Proposition 16.8.** *The characteristic of an integral domain is either 0 or a prime number.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $D$  be an integral domain.

$\langle 1 \rangle 2$ . LET:  $n$  be the characteristic of  $D$

$\langle 1 \rangle 3$ . ASSUME:  $n \neq 0$

$\langle 1 \rangle 4$ . ASSUME:  $n = ab$

$\langle 1 \rangle 5$ .  $ab = 0$  in  $D$

$\langle 1 \rangle 6$ .  $a = 0$  or  $b = 0$  in  $D$

$\langle 1 \rangle 7$ .  $n \mid a$  or  $n \mid b$

$\langle 1 \rangle 8$ . One of  $a, b$  is 1 and the other is  $n$ .

□

## 16.1 Prime Ideals

**Definition 16.9** (Prime Ideal). Let  $I$  be an ideal in a commutative ring  $R$ . Then  $I$  is a *prime ideal* iff  $R/I$  is an integral domain.

**Example 16.10.** Let  $R$  be a commutative ring and  $a \in R$ . Then  $(x - a)$  is a prime ideal in  $R$  iff  $R$  is an integral domain.

**Proposition 16.11.** *Let  $R$  be a commutative ring and  $I$  a proper ideal in  $R$ . Then  $I$  is prime iff, whenever  $ab \in I$ , then  $a \in I$  or  $b \in I$ .*

PROOF: The condition is the same as saying that, if  $(a + I)(b + I) = I$ , then  $a + I = I$  or  $b + I = I$ .  $\square$

**Definition 16.12** (Spectrum). The *spectrum* of a commutative ring  $R$ ,  $\text{Spec } R$ , is the set of prime ideals.

**Proposition 16.13.** Let  $\phi : R \rightarrow S$  be a ring homomorphism. If  $I$  is a prime ideal in  $S$  then  $\phi^{-1}(I)$  is a prime ideal in  $R$ .

PROOF: If  $ab \in \phi^{-1}(I)$  then  $\phi(a)\phi(b) \in I$  so either  $\phi(a) \in I$  or  $\phi(b) \in I$ , i.e. either  $a \in \phi^{-1}(I)$  or  $b \in \phi^{-1}(I)$ .  $\square$

**Proposition 16.14.** Let  $R$  be a commutative ring. Suppose there exists a prime ideal  $P$  in  $R$  such that the only zero-divisor in  $P$  is 0. Then  $R$  is an integral domain.

PROOF:

$\langle 1 \rangle$ 1. ASSUME:  $ab = 0$  in  $R$

$\langle 1 \rangle$ 2.  $ab \in P$

$\langle 1 \rangle$ 3.  $a \in P$  or  $b \in P$

$\langle 1 \rangle$ 4.  $a = 0$  or  $b = 0$

$\square$

**Proposition 16.15.** Let  $R$  be a commutative ring. The nilradical of  $R$  is included in every prime ideal of  $R$ .

PROOF: Let  $P$  be a prime ideal. If  $a^n = 0$  then  $a^n \in P$  hence  $a \in P$ .  $\square$

**Definition 16.16** (Krull Dimension). The (*Krull*) *dimension* of a commutative ring  $R$  is the length of the longest chain of prime ideals in  $R$ .

**Example 16.17.**  $\mathbb{Z}[x]$  has Krull dimension 2.



## Chapter 17

# Unique Factorization Domains

**Example 17.1.**  $\mathbb{Z}$  is a UFD.





## Chapter 18

# Noetherian Rings

**Definition 18.1** (Noetherian Ring). A commutative ring is *Noetherian* iff every ideal is finitely generated.

**Proposition 18.2.** *The homomorphic image of a Noetherian ring is Noetherian.*

PROOF:

⟨1⟩1. LET:  $R$  be a Noetherian ring,  $S$  be a commutative ring, and  $\phi : R \rightarrow S$  a surjective ring homomorphism.

⟨1⟩2. LET:  $I$  be an ideal in  $S$ .

⟨1⟩3. LET:  $\phi^{-1}(I) = (a_1, \dots, a_n)$

⟨1⟩4.  $I = (\phi(a_1), \dots, \phi(a_n))$

□



## Chapter 19

# Principal Ideal Domains

**Definition 19.1** (Principal Ideal Domain). A commutative ring is a *principal ideal domain* (PID) iff every ideal is principal.

**Example 19.2.**  $\mathbb{Z}$  is a PID by Proposition 7.16.

**Example 19.3.**  $\mathbb{Z}[x]$  is not a PID. The ideal  $(2, x)$  is not principal.

**Proposition 19.4.** *Every PID is Noetherian.*

PROOF: Trivial.  $\square$

**Proposition 19.5.** *Every nonzero prime ideal in a PID is maximal.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $R$  be a PID.
- $\langle 1 \rangle 2$ . LET:  $I$  be a nonzero prime ideal in  $R$ .
- $\langle 1 \rangle 3$ . PICK  $a \in R$  such that  $I = (a)$ .
- $\langle 1 \rangle 4$ . LET:  $J$  be an ideal such that  $I \subseteq J$
- $\langle 1 \rangle 5$ . PICK  $b \in R$  such that  $J = (b)$ .
- $\langle 1 \rangle 6$ . PICK  $t \in R$  such that  $a = bt$ .
- $\langle 1 \rangle 7$ .  $b \in I$  or  $t \in I$
- $\langle 1 \rangle 8$ . CASE:  $b \in I$   
PROOF: Then  $J \subseteq I$  so  $I = J$ .
- $\langle 1 \rangle 9$ . CASE:  $t \in I$ 
  - $\langle 2 \rangle 1$ . PICK  $s \in R$  such that  $t = as$ .
  - $\langle 2 \rangle 2$ .  $a = ast$
  - $\langle 2 \rangle 3$ .  $st = 1$   
PROOF: Since  $R$  is an integral domain.
  - $\langle 2 \rangle 4$ .  $1 \in I$
  - $\langle 2 \rangle 5$ .  $I = R$

$\square$

**Corollary 19.5.1.** *Any PID has Krull dimension 1.*



## Chapter 20

# Euclidean Domains

**Example 20.1.**  $\mathbb{Z}$  is a Euclidean domain.



## Chapter 21

# Division Rings

**Definition 21.1** (Division Ring). A *division ring* is a ring in which every nonzero element is a two-sided unit.

**Example 21.2.** The quaternions form a division ring, with the inverse of a non-zero element  $a + bi + cj + dk$  being

$$\frac{1}{a^2 + b^2 + c^2 + d^2}(a - bi - cj - dk) .$$

**Example 21.3.** For any ring  $R$ , the ring of polynomials  $R[x]$  is not a division ring, since  $x$  has no inverse.

**Proposition 21.4.** *Every centralizer in a division ring is a division ring.*

PROOF: If  $ar = ra$  then  $ra^{-1} = a^{-1}r$ .  $\square$

**Proposition 21.5.** *A non-trivial ring  $R$  is a division ring if and only if its only left-ideals and right-ideals are  $\{0\}$  and  $R$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $R$  is a division ring then the only left-ideals and right-ideals are  $\{0\}$  and  $R$ .

$\langle 2 \rangle 1$ . ASSUME:  $R$  is a division ring.

$\langle 2 \rangle 2$ . The only left-ideals are  $\{0\}$  and  $R$ .

$\langle 3 \rangle 1$ . LET:  $I$  be a left-ideal that is not  $\{0\}$ .

PROVE:  $I = R$

$\langle 3 \rangle 2$ . PICK  $a \in I - \{0\}$

$\langle 3 \rangle 3$ . PICK a left inverse  $b$  for  $a$

$\langle 3 \rangle 4$ .  $1 \in I$

PROOF: Since  $1 = ba$ .

$\langle 3 \rangle 5$ .  $I = R$

PROOF: For any  $r \in R$  we have  $r = r1 \in I$ .

$\langle 2 \rangle 3$ . The only right-ideals are  $\{0\}$  and  $R$ .

PROOF: Similar.

$\langle 1 \rangle 2$ . If the only left-ideals and right-ideals are  $\{0\}$  and  $R$  then  $R$  is a division ring.

□

**Proposition 21.6.** *Let  $K$  be a division ring and  $R$  a non-trivial ring. Every ring homomorphism  $K \rightarrow R$  is injective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : K \rightarrow R$  be a ring homomorphism.

PROVE:  $\ker \phi = \{0\}$

$\langle 1 \rangle 2$ . LET:  $x \in \ker \phi$

$\langle 1 \rangle 3$ . ASSUME: for a contradiction  $x \neq 0$ .

$\langle 1 \rangle 4$ .  $\phi(xx^{-1}) = 1$

$\langle 1 \rangle 5$ .  $0 = 1$

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts the assumption that  $R$  is non-trivial.

□



## Chapter 22

# Simple Rings

**Definition 22.1** (Simple Ring). A non-trivial ring is *R simple* iff its only two-sided ideals are  $\{0\}$  and  $R$ .

**Example 22.2.** For any simple ring  $R$  we have  $\mathfrak{gl}_n(R)$  is simple, by Corollary 15.13.1.

**Proposition 22.3.** *Let  $R$  be a ring and  $I$  an ideal in  $R$ . Then  $I$  is maximal iff  $R/I$  is simple.*

PROOF:

$$\begin{aligned} R/I \text{ is simple} &\Leftrightarrow \text{the only ideals in } R/I \text{ are } \{I\} \text{ and } R/I \\ &\Leftrightarrow \text{the only ideals in } R \text{ that include } I \text{ are } I \text{ and } R \\ &\Leftrightarrow I \text{ is maximal} \end{aligned}$$

□



## Chapter 23

# Reduced Rings

**Definition 23.1** (Reduced Ring). A ring is *reduced* iff it has no non-zero nilpotent elements.

**Proposition 23.2.** *Let  $R$  be a commutative ring. Let  $N$  be its nilradical. Then  $R/N$  is reduced.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $r + N$  be nilpotent.
- $\langle 1 \rangle 2$ . PICK  $n$  such that  $(r + N)^n = N$
- $\langle 1 \rangle 3$ .  $r^n \in N$
- $\langle 1 \rangle 4$ . PICK  $k$  such that  $(r^n)^k = 0$
- $\langle 1 \rangle 5$ .  $r^{nk} = 0$
- $\langle 1 \rangle 6$ .  $r \in N$
- $\langle 1 \rangle 7$ .  $r + N = N$

□

**Proposition 23.3.** *Let  $R$  be a commutative ring. Let  $I$  and  $J$  be ideals in  $R$ . If  $R/IJ$  is reduced then  $IJ = I \cap J$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $r \in I \cap J$   
PROVE:  $r \in IJ$
- $\langle 1 \rangle 2$ .  $r^2 \in IJ$
- $\langle 1 \rangle 3$ .  $(r + IJ)^2 = IJ$
- $\langle 1 \rangle 4$ .  $r + IJ = IJ$

PROOF: Since  $R/IJ$  is reduced.

- $\langle 1 \rangle 5$ .  $r \in IJ$

□



## Chapter 24

# Boolean Rings

**Definition 24.1** (Boolean). A ring is *Boolean* iff  $a^2 = a$  for every element  $a$ .

**Example 24.2.** For any set  $S$ , the ring  $\mathcal{P}S$  is Boolean.

**Proposition 24.3.** *Every non-trivial Boolean ring has characteristic 2.*

PROOF: We have  $4 = 2$  and so  $2 = 0$ .  $\square$

**Proposition 24.4.** *Every Boolean ring is commutative.*

PROOF:

$$\begin{aligned}(a+b)^2 &= a+b \\ \therefore a^2 + ab + ba + b^2 &= a+b \\ \therefore a + ab + ba + b &= a+b \\ \therefore ab + ba &= 0 \\ \therefore ab &= -ba \\ &= ba \quad (\text{Proposition 24.3})\end{aligned}$$

**Example 24.5.** The only Boolean integral domain is  $\mathbb{Z}/2\mathbb{Z}$ . For, if  $D$  is a Boolean integral domain and  $x \in D$ , we have  $x^2 = x$ , so  $x^2 - x = x(x-1) = 0$  and so  $x = 0$  or  $x = 1$ , i.e.  $D = \{0, 1\}$ .

**Proposition 24.6.** *Every Boolean ring has Krull dimension 0.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $R$  be a Boolean ring.

$\langle 1 \rangle 2$ . LET:  $I$  be a prime ideal in  $R$ .

PROVE:  $I$  is maximal.

$\langle 1 \rangle 3$ . LET:  $J$  be an ideal with  $I \subsetneq J$

$\langle 1 \rangle 4$ . PICK  $a \in J$  with  $a \notin I$

$\langle 1 \rangle 5$ .  $a^2 - a = 0 \in I$

$\langle 1 \rangle 6$ .  $a(a-1) \in I$

$$\langle 1 \rangle 7. \ a - 1 \in I$$

$$\langle 1 \rangle 8. \ a - 1 \in J$$

$$\langle 1 \rangle 9. \ 1 \in J$$

$$\langle 1 \rangle 10. \ J = R$$

□

# Chapter 25

## Modules

**Definition 25.1** (Left Module). Let  $R$  be a ring and  $M$  an Abelian group. A *left-action* of  $R$  on  $M$  is a ring homomorphism

$$R \rightarrow \text{End}_{\mathbf{Ab}}(M) \quad .$$

A *left  $R$ -module* consists of an Abelian group  $M$  and a left-action of  $R$  on  $M$ .

**Proposition 25.2.** *Let  $R$  be a ring and  $M$  an Abelian group. Let  $\cdot : R \times M \rightarrow M$ . Then  $\cdot$  defines a left-action of  $R$  on  $M$  if and only if, for all  $r, s \in R$  and  $m, n \in M$ :*

- $r(m + n) = rm + rn$
- $(r + s)m = rm + sm$
- $(rs)m = r(sm)$
- $1m = m$

PROOF: Immediate from definitions.  $\square$

**Proposition 25.3.** *In any  $R$ -module  $M$  we have  $0m = 0$  for all  $m \in M$ .*

PROOF: Since  $0m = (0 + 0)m = 0m + 0m$  and so  $0m = 0$  by cancellation in  $M$ .  $\square$

**Proposition 25.4.** *In any  $R$ -module  $M$  we have  $(-1)m = -m$  for all  $m \in M$ .*

PROOF: Since  $m + (-1)m = 1m + (-1)m = (1 + (-1))m = 0m = 0$ .  $\square$

**Proposition 25.5.** *Every Abelian group is a  $\mathbb{Z}$ -module in exactly one way.*

PROOF: Since  $\mathbb{Z}$  is initial in **Ring**.  $\square$

**Definition 25.6** (Right Module). Let  $R$  be a ring. A *right  $R$ -module* consists of an Abelian group  $M$  and a function  $\cdot : M \times R \rightarrow M$  such that, for all  $r, s \in R$  and  $m, n \in M$ :

- $(m + n)r = mr + nr$
- $m(r + s) = mr + ms$
- $m(rs) = (mr)s$
- $m1 = m$

## 25.1 Homomorphisms

**Definition 25.7** (Homomorphism of Left-Modules). Let  $R$  be a ring. Let  $M$  and  $N$  be left- $R$ -modules. A *homomorphism of left- $R$ -modules*  $\phi : M \rightarrow N$  is a group homomorphism such that, for all  $r \in R$  and  $m \in M$ , we have  $\phi(rm) = r\phi(m)$ .

Let  $R - \mathbf{Mod}$  be the category of left- $R$ -modules and left- $R$ -module homomorphisms.

**Example 25.8.**

$$\mathbb{Z} - \mathbf{Mod} \cong \mathbf{Ab}$$

**Example 25.9.** The trivial group  $0$  is the zero object in  $R - \mathbf{Mod}$ .

**Proposition 25.10.** *Every bijective  $R$ -module homomorphism is an isomorphism.*

PROOF: Easy.  $\square$

**Proposition 25.11.** *Let  $R$  be a ring. Let  $M$  be an  $R$ -module. Then*

$$M \cong R - \mathbf{Mod}[R, M]$$

*as  $R$ -modules.*

PROOF: The isomorphism maps  $m$  to the function  $\lambda r.rm$ . Its inverse maps an  $R$ -module homomorphism  $\alpha$  to  $\alpha(1)$ .  $\square$

**Proposition 25.12.** *Let  $R$  be a commutative ring. Let  $M$  be an  $R$ -module. Then there is a bijection between the set of  $R[x]$ -module structures on  $M$  that extend the given  $R$ -module structure and  $\text{End}_{R - \mathbf{Mod}}(M)$ .*

PROOF:

- (1)1. LET:  $\alpha : R \rightarrow \text{End}_{\mathbf{Ab}}(M)$  be the given  $R$ -module structure on  $M$ .
- (1)2. An  $R[x]$ -module structure on  $M$  that extends  $\alpha$  is a ring homomorphism  $\beta : R[x] \rightarrow \text{End}_{\mathbf{Ab}}(M)$  such that  $\beta \circ i = \alpha$ , where  $i$  is the inclusion  $R \rightarrow R[x]$ .
- (1)3. There is a bijection between the  $R[x]$ -module structures on  $M$  that extend  $\alpha$  and the elements  $s \in \text{End}_{\mathbf{Ab}}(M)$  that commute with  $\alpha(r)$  for all  $r \in R$ .

PROOF: By the universal property for polynomials.

- (1)4. There is a bijection between the  $R[x]$ -module structures on  $M$  that extend  $\alpha$  and the  $R$ -module homomorphisms  $(M, \alpha) \rightarrow (M, \alpha)$ .



□

**Proposition 25.13.** *Let  $R$  be a commutative ring. Let  $M$  and  $N$  be  $R$ -modules. Then  $R - \mathbf{Mod}[M, N]$  is an  $R$ -module under*

$$\begin{aligned}(\phi + \psi)(m) &= \phi(m) + \psi(m) \\ (r\phi)(m) &= r\phi(m)\end{aligned}$$

PROOF: Easy. □

**Proposition 25.14.** *Let  $R$  be an integral domain. Let  $I$  be a nonzero principal ideal of  $R$ . Then  $I \cong R$  in  $R - \mathbf{Mod}$ .*

PROOF:

⟨1⟩1. PICK  $a \in R$  such that  $I = (a)$ .

⟨1⟩2. LET:  $\phi : R \rightarrow I$  be the map  $\phi(r) = ra$ .

⟨1⟩3.  $\phi$  is an  $R$ -module homomorphism.

PROOF: Since  $(r + s)a = ra + sa$  and  $(rs)a = r(sa)$ .

⟨1⟩4.  $\phi$  is surjective.

⟨1⟩5.  $\phi$  is injective.

PROOF: If  $ra = sa$  then  $(r - s)a = 0$  so  $r - s = 0$  and  $r = s$ .

⟨1⟩6.  $\phi : R \cong I$

□

## 25.2 Submodules

**Definition 25.15** (Submodule). Let  $M$  be a left- $R$ -module and  $N \subseteq M$ . Then  $N$  is a *submodule* of  $M$  iff  $N$  is a subgroup of  $M$  and  $\forall r \in R, \forall n \in N, rn \in N$ .

**Proposition 25.16.** *Let  $R$  be a ring and  $I \subseteq R$ . Then  $I$  is a left-ideal in  $R$  iff  $I$  is a submodule of  $R$  as an  $R$ -module.*

PROOF: Immediate from definitions. □

**Proposition 25.17.** *Let  $R$  be a ring. Let  $M$  and  $N$  be left- $R$ -modules and  $\phi : M \rightarrow N$  an  $R$ -module homomorphism. Then  $\ker \phi$  is a submodule of  $M$  and  $\text{im } \phi$  is a submodule of  $N$ .*

PROOF: Easy. □

**Proposition 25.18.** *Let  $R$  be a commutative ring. Let  $M$  be a left- $R$ -module. Let  $r \in R$ . Then  $rM = \{rm : m \in M\}$  is a submodule of  $M$ .*

PROOF: Easy. □

**Proposition 25.19.** *Let  $R$  be a ring. Let  $M$  be a left- $R$ -module. Let  $I$  be a left-ideal in  $R$ . Then  $IM = \{rm : r \in I, m \in M\}$  is a submodule of  $M$ .*

PROOF:

- $\langle 1 \rangle 1$ .  $IM$  is a subgroup of  $M$ .  
 $\langle 2 \rangle 1$ . LET:  $r, s \in I$  and  $m, n \in M$ .  
 PROVE:  $rm + sn \in IM$   
 $\langle 2 \rangle 2$ .  $rm + sn = r(m - n) + (s - r)n$   
 $\langle 1 \rangle 2$ . For all  $r \in R$  and  $x \in IM$  we have  $rx \in IM$ .  
 $\square$

### 25.3 Quotient Modules

**Definition 25.20** (Quotient Module). Let  $R$  be a ring. Let  $M$  be a left- $R$ -module. Let  $N$  be a submodule of  $M$ . Then the *quotient module*  $M/N$  is the quotient group  $M/N$  under

$$r(m + N) = rm + N \ .$$

**Proposition 25.21.** Let  $R$  be a ring. Let  $M$  and  $P$  be left- $R$ -modules. Let  $N$  be a submodule of  $M$ . Let  $\phi : M \rightarrow P$  be an  $R$ -module homomorphism. If  $N \subseteq \ker \phi$ , then there exists a unique  $R$ -module homomorphism  $\bar{\phi} : M/N \rightarrow P$  such that the following diagram commutes.

$$\begin{array}{ccc}
 M & \xrightarrow{\phi} & P \\
 & \searrow & \nearrow \bar{\phi} \\
 & M/N &
 \end{array}$$

PROOF: Easy.  $\square$

**Theorem 25.22.** Every  $R$ -module homomorphism  $\phi : M \rightarrow M'$  may be decomposed as:

$$M \longrightarrow M/\ker \phi \xrightarrow{\cong} \text{im } \phi \longrightarrow N$$

PROOF: Easy.  $\square$

**Corollary 25.22.1** (First Isomorphism Theorem). Let  $\phi : M \rightarrow M'$  be a surjective  $R$ -module homomorphism. Then

$$M' \cong \frac{M}{\ker \phi} \ .$$

**Proposition 25.23** (Second Isomorphism Theorem). Let  $R$  be a ring. Let  $M$  be a left- $R$ -module. Let  $N$  and  $P$  be submodules of  $M$ . Then  $N + P$  is a submodule of  $M$ ,  $N \cap P$  is a submodule of  $P$ , and

$$\frac{N + P}{N} \cong \frac{P}{N \cap P}$$

PROOF: The function that maps  $P$  to  $p + N$  is a surjective homomorphism  $P \rightarrow (N + P)/N$  with kernel  $N \cap P$ .  $\square$

**Proposition 25.24** (Third Isomorphism Theorem). *Let  $R$  be a ring. Let  $M$  be a left- $R$ -module. Let  $N$  be a submodule of  $M$  and  $P$  a submodule of  $N$ . Then  $N/P$  is a submodule of  $M/P$  and*

$$\frac{M/P}{N/P} \cong \frac{M}{N}$$

PROOF: The canonical map  $M \rightarrow M/N$  induces a surjective homomorphism  $M/P \rightarrow M/N$  which has kernel  $N/P$ .  $\square$

**Proposition 25.25.** *Let  $R$  be a ring. Let  $M$  be a left- $R$ -module. The sum and intersection of a family of submodules of  $M$  are submodules of  $M$ .*

PROOF: Easy.  $\square$

## 25.4 Products

**Proposition 25.26.**  $R - \mathbf{Mod}$  has products.

PROOF: Given a family  $\{M_\alpha\}_{\alpha \in A}$  of left- $R$ -modules, we make  $\prod_{\alpha \in A} M_\alpha$  into a left- $R$ -module by

$$(f + g)(\alpha) = f(\alpha) + g(\alpha)$$

$$(rf)(\alpha) = rf(\alpha)$$

$\square$

## 25.5 Coproducts

**Proposition 25.27.**  $R - \mathbf{Mod}$  has coproducts.

PROOF: Given a family  $\{M_\alpha\}_{\alpha \in A}$  of left- $R$ -modules, take  $\bigoplus_{\alpha \in A} M_\alpha$  to be  $\{f \in \prod_{\alpha \in A} M_\alpha : f(\alpha) = 0 \text{ for all but finitely many } \alpha \in A\}$ .  $\square$

## 25.6 Direct Sum

**Definition 25.28** (Direct Sum). Let  $R$  be a ring. Let  $M$  and  $N$  be left- $R$ -modules. Then the direct sum  $M \oplus N$  is an  $R$ -module under

$$r(m, n) = (rm, rn) .$$

**Proposition 25.29.**  $M \oplus N$  is the biproduct of  $M$  and  $N$  in  $R - \mathbf{Mod}$ .

PROOF: Easy.  $\square$

**Example 25.30.** Infinite products and coproducts are in general different. We have  $\mathbb{Z}^{\mathbb{N}} \not\cong \mathbb{Z}^{\oplus \mathbb{N}}$  since  $\mathbb{Z}^{\mathbb{N}}$  is uncountable but  $\mathbb{Z}^{\oplus \mathbb{N}}$  is countable.

## 25.7 Kernels and Cokernels

**Proposition 25.31.** *Let  $R$  be a ring. Let  $\phi : M \rightarrow N$  be a left- $R$ -module homomorphism. Then  $\ker \phi \hookrightarrow M$  is terminal in the category of left- $R$ -module homomorphisms  $\alpha : P \rightarrow M$  such that  $\phi \circ \alpha = 0$ .*

PROOF: Easy.  $\square$

**Proposition 25.32.** *Let  $R$  be a ring. Let  $\phi : M \rightarrow N$  be a left- $R$ -module homomorphism. Then  $N \twoheadrightarrow \operatorname{coker} \phi$  is initial in the category of left- $R$ -module homomorphisms  $\alpha : N \rightarrow P$  such that  $\alpha \circ \phi = 0$ .*

PROOF: Easy.  $\square$

**Proposition 25.33.** *Let  $R$  be a ring. Let  $\phi : M \rightarrow N$  be a left- $R$ -module homomorphism. Then the following are equivalent.*

- $\phi$  is a monomorphism.
- $\ker \phi$  is trivial.
- $\phi$  is injective.

PROOF: Easy.  $\square$

**Proposition 25.34.** *Let  $R$  be a ring. Let  $\phi : M \rightarrow N$  be a left- $R$ -module homomorphism. Then the following are equivalent.*

- $\phi$  is an epimorphism.
- $\operatorname{coker} \phi$  is trivial.
- $\phi$  is surjective.

PROOF: Easy.  $\square$

**Proposition 25.35.** *Every monomorphism in  $R - \mathbf{Mod}$  is the kernel of some homomorphism.*

PROOF: If  $\phi : M \rightarrow N$  is a monomorphism then it is the kernel of  $N \twoheadrightarrow N/\operatorname{im} \phi$ .  $\square$

**Proposition 25.36.** *Every epimorphism in  $R - \mathbf{Mod}$  is the cokernel of some homomorphism.*

PROOF: If  $\phi : M \rightarrow N$  is epi then it is the cokernel of  $\ker \phi \hookrightarrow M$ .  $\square$

**Example 25.37.** Monomorphisms do not split in  $R - \mathbf{Mod}$ . Multiplication by 2 is a monomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}$  but has no left inverse.

**Example 25.38.** Epimorphisms do not split in  $R - \mathbf{Mod}$ . The canonical map  $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$  is an epimorphism without a right inverse.

## 25.8 Free Modules

**Proposition 25.39.** *Let  $R$  be a ring and  $A$  a set. Then there exists a left- $R$ -module  $F^R(A)$  and function  $j : A \rightarrow F^R(A)$  such that, for any left- $R$ -module  $M$  and function  $f : A \rightarrow M$ , there exists a unique left- $R$ -module homomorphism  $\bar{f} : F^R(A) \rightarrow M$  such that the following diagram commutes.*

$$\begin{array}{ccc} F^R(A) & \xrightarrow{\bar{f}} & M \\ \uparrow j & \nearrow f & \\ A & & \end{array}$$

PROOF:

(1)1. LET:  $R^{\oplus A} = \{\alpha : A \rightarrow R : \alpha(a) = 0 \text{ for all but finitely many } a \in A\}$   
under the operations

$$\begin{aligned} (\alpha + \beta)(a) &= \alpha(a) + \beta(a) \\ (r\alpha)(a) &= r\alpha(a) \end{aligned}$$

(1)2.  $R^{\oplus A}$  is a left- $R$ -module.

(1)3. LET:  $j : A \rightarrow R^{\oplus A}$  be the function

$$j(a)(a') = \begin{cases} 1 & \text{if } a = a' \\ 0 & \text{if } a \neq a' \end{cases}$$

(1)4. LET:  $M$  be any left- $R$ -module.

(1)5. LET:  $f : A \rightarrow M$  be a function.

(1)6. LET:  $\bar{f} : R^{\oplus A} \rightarrow M$  be the function

$$\bar{f}(\alpha) = \sum_{a \in A, \alpha(a) \neq 0} \alpha(a)f(a)$$

(1)7.  $\bar{f}$  is a left- $R$ -module homomorphism.

(1)8.  $\bar{f} \circ j = f$

(1)9.  $\bar{f}$  is unique.

**Definition 25.40.** We call  $j : A \rightarrow F^R(A)$  the *free* left- $R$ -module over  $A$ .

**Proposition 25.41.**  $j$  is injective.

PROOF: By the proof of the previous proposition.  $\square$

## 25.9 Generators

**Definition 25.42** (Submodule Generated by a Set). Let  $R$  be a ring. Let  $M$  be a left- $R$ -module. Let  $A$  be a subset of  $M$ . Let  $\phi_A : F^R(A) \rightarrow M$  be the unique left- $R$ -module homomorphism such that the following diagram commutes.

$$\begin{array}{ccc} F^R(A) & \xrightarrow{\phi_A} & M \\ \uparrow & \nearrow & \\ A & & \end{array}$$

The submodule of  $M$  *generated* by  $A$ , denoted  $\langle A \rangle$ , is defined to be  $\text{im } \phi_A$ .

**Definition 25.43** (Finitely Generated). Let  $R$  be a ring. Let  $M$  be a left- $R$ -module. Then  $M$  is *finitely generated* iff there exists a finite set  $A \subseteq M$  such that  $M = \langle A \rangle$ .

## 25.10 Projections

**Definition 25.44** (Projection). Let  $R$  be a ring. Let  $M$  be a left- $R$ -module. Let  $p : M \rightarrow M$  be a left- $R$ -module homomorphism. Then  $p$  is a *projection* iff  $p^2 = p$ .

**Proposition 25.45.** *Let  $R$  be a ring. Let  $M$  be a left- $R$ -module. Let  $p : M \rightarrow M$  be a projection. Then*

$$M \cong \ker p \oplus \text{im } p .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : M \rightarrow \ker p \oplus \text{im } p$  be the map  $\phi(m) = (m - p(m), p(m))$

$\langle 1 \rangle 2$ .  $\phi$  is a left- $R$ -module homomorphism.

$\langle 1 \rangle 3$ .  $\phi$  is injective.

$\langle 1 \rangle 4$ .  $\phi$  is surjective.

□

## Chapter 26

# Simple Modules

**Definition 26.1** (Simple Module). Let  $R$  be a ring. An  $R$ -module  $M$  is *simple* or *irreducible* iff its only submodules are  $\{0\}$  and  $M$ .

**Proposition 26.2** (Schur's Lemma). *Let  $R$  be a ring. Let  $M$  and  $N$  be simple  $R$ -modules. Let  $\phi : M \rightarrow N$  be an  $R$ -module homomorphism. Then either  $\phi = 0$  or  $\phi$  is an isomorphism.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\phi \neq 0$

$\langle 1 \rangle 2$ .  $\ker \phi = 0$

PROOF: Since  $\ker \phi$  is a submodule of  $M$  that is not  $M$ .

$\langle 1 \rangle 3$ .  $\operatorname{im} \phi = N$

PROOF: Since  $\operatorname{im} \phi$  is a submodule of  $N$  that is not  $\{0\}$ .

□





## Chapter 27

# Noetherian Modules

**Definition 27.1** (Noetherian Module). Let  $R$  be a ring. A left- $R$ -module is *Noetherian* iff every submodule is finitely generated.

**Proposition 27.2.** *Let  $R$  be a ring. Let  $M$  be a left- $R$ -module and  $N$  a submodule of  $M$ . Then  $M$  is Noetherian if and only if  $N$  and  $M/N$  are Noetherian.*

PROOF:

⟨1⟩1. If  $M$  is Noetherian then  $N$  is Noetherian.

PROOF: Every submodule of  $N$  is a submodule of  $M$ , hence finitely generated.

⟨1⟩2. If  $M$  is Noetherian then  $M/N$  is Noetherian.

⟨2⟩1. ASSUME:  $M$  is Noetherian.

⟨2⟩2. LET:  $\pi : M \rightarrow M/N$  be the canonical epimorphism.

⟨2⟩3. LET:  $P$  be a submodule of  $M/N$ .

⟨2⟩4. PICK  $a_1, \dots, a_n \in M$  that generate  $\pi^{-1}(P)$ .

⟨2⟩5.  $a_1 + N, \dots, a_n + N$  generate  $P$ .

⟨1⟩3. If  $N$  and  $M/N$  are Noetherian then  $M$  is Noetherian.

⟨2⟩1. ASSUME:  $N$  and  $M/N$  are Noetherian.

⟨2⟩2. LET:  $P$  be a submodule of  $M$ .

⟨2⟩3. PICK  $a_1, \dots, a_m \in P$  such that  $a_1 + N, \dots, a_m + N$  generate  $\pi(P)$ .

⟨2⟩4. PICK  $b_1, \dots, b_n \in M$  that generated  $P \cap N$ .

PROVE:  $a_1, \dots, a_m, b_1, \dots, b_n$  generate  $P$ .

⟨2⟩5. LET:  $p \in P$

⟨2⟩6. PICK  $r_1, \dots, r_m \in R$  such that  $p + N = r_1 a_1 + \dots + r_m a_m + N$

⟨2⟩7.  $p - r_1 a_1 - \dots - r_m a_m \in P \cap N$

⟨2⟩8. PICK  $s_1, \dots, s_n \in R$  such that  $p - r_1 a_1 - \dots - r_m a_m = s_1 b_1 + \dots + s_n b_n$

⟨2⟩9.  $p = r_1 a_1 + \dots + r_m a_m + s_1 b_1 + \dots + s_n b_n$

□

**Corollary 27.2.1.** *If  $R$  is a Noetherian ring then  $R^{\oplus n}$  is a Noetherian left- $R$ -module.*

PROOF: The proof is by induction on  $n$ . The case  $n = 1$  is immediate.

The induction step holds since  $R^{\oplus(n+1)}/R^{\oplus n} \cong R$ . □

**Corollary 27.2.2.** *If  $R$  is a Noetherian ring and  $M$  is a finitely generated left- $R$ -module then  $M$  is Noetherian.*

PROOF: There is a surjective homomorphism  $R^{\oplus n} \twoheadrightarrow M$  for some  $n$ , so  $M$  is a quotient of  $R^{\oplus n}$ .  $\square$

# Chapter 28

## Algebras

**Definition 28.1** (Algebra). Let  $R$  be a commutative ring. An  $R$ -algebra consists of a ring  $S$  and a ring homomorphism  $\alpha : R \rightarrow S$  such that  $\alpha(R)$  is included in the center of  $S$ . We write  $rs$  for  $\alpha(r)s$ .

**Proposition 28.2.** Let  $R$  be a commutative ring and  $S$  a ring. Let  $\cdot : R \times S \rightarrow S$ . Then there exists  $\alpha : R \rightarrow S$  that makes  $S$  into an  $R$ -algebra such that

$$rs = \alpha(r)s \quad (r \in R, s \in S)$$

iff  $S$  is an  $R$ -module under  $\cdot$  and, for all  $r_1, r_2 \in R$  and  $s_1, s_2 \in S$ ,

$$(r_1 s_1)(r_2 s_2) = (r_1 r_2)(s_1 s_2) .$$

PROOF: Immediate from definitions.  $\square$

**Example 28.3.** Let  $R$  be a commutative ring. Then  $R$  is an  $R$ -algebra under multiplication.

**Example 28.4.** Let  $R$  be a commutative ring and  $I$  an ideal in  $R$ . Then  $R/I$  is an  $R$ -algebra.

**Example 28.5.** Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Then  $\text{End}_{R\text{-Mod}}(M)$  is an  $R$ -algebra under composition.

**Example 28.6.** Let  $R$  be a commutative ring. Then  $\mathfrak{gl}_n(R)$  is an  $R$ -algebra under matrix multiplication.

**Definition 28.7** (Algebra Homomorphism). Let  $R$  be a commutative ring. Let  $S$  and  $T$  be  $R$ -algebras. An  $R$ -algebra homomorphism  $\phi : S \rightarrow T$  is a ring homomorphism such that, for all  $r \in R$  and  $s \in S$ , we have  $\phi(rs) = r\phi(s)$ .

Let  $R\text{-Alg}$  be the category of  $R$ -algebras and  $R$ -algebra homomorphisms.

**Example 28.8.**

$$\mathbb{Z}\text{-Alg} \cong \mathbf{Ring}$$

**Example 28.9.** Let  $R$  be a commutative ring. Then  $R[x_1, \dots, x_n]$ , and any quotient ring of  $R[x_1, \dots, x_n]$ , is a commutative  $R$ -algebra.

**Example 28.10.**  $R$  is the initial object in  $R\text{-Alg}$ .

## 28.1 Rees Algebra

**Definition 28.11** (Rees Algebra). Let  $R$  be a commutative ring. Let  $I$  be an ideal in  $R$ . The *Rees algebra* is the direct sum

$$\text{Rees}_R(I) = \bigoplus_{j \geq 0} I^j$$

under the multiplication

$$\begin{aligned} (r_0, r_1, r_2, r_3, \dots)(s_0, s_1, s_2, \dots) &= (r_0s_0, r_1s_0 + r_0s_1, r_2s_0 + r_1s_1 + r_0s_2, \dots) \\ r(r_0, r_1, r_2, \dots) &= (rr_0, rr_1, rr_2, \dots) \end{aligned}$$

**Proposition 28.12.** *Let  $R$  be a commutative ring. Let  $a \in R$  be a non-zero-divisor. Then  $R[x]$  is the Rees algebra of  $(a)$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : R[x] \rightarrow \text{Rees}_R((a))$  be the function  $\phi(r_0 + r_1x + r_2x^2 + \dots) = (r_0, r_1a, r_2a^2, \dots)$ .

$\langle 1 \rangle 2$ .  $\phi$  is an  $R$ -algebra homomorphism.

$\langle 1 \rangle 3$ .  $\phi$  is injective.

$\langle 2 \rangle 1$ . LET:  $\phi(r_0 + r_1x + r_2x^2 + \dots) = \phi(s_0 + s_1x + s_2x^2 + \dots)$

$\langle 2 \rangle 2$ . For all  $n$  we have  $r_na^n = s_na^n$

$\langle 2 \rangle 3$ .  $(r_n - s_n)a^n = 0$

$\langle 2 \rangle 4$ .  $r_n - s_n = 0$

PROOF: Since  $a$  is not a zero-divisor.

$\langle 2 \rangle 5$ .  $r_n = s_n$

$\langle 1 \rangle 4$ .  $\phi$  is surjective.

□

**Proposition 28.13.** *Let  $R$  be a commutative ring. Let  $a \in R$  be a non-zero-divisor. Let  $I$  be an ideal of  $R$ . Then  $\text{Rees}_R(I) \cong \text{Rees}_R(aI)$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\phi : \text{Rees}_R(I) \rightarrow \text{Rees}_R(aI)$  be the function  $\phi(r_0, r_1, r_2, \dots) = (r_0, ar_1, a^2r_2, \dots)$ .

$\langle 1 \rangle 2$ .  $\phi$  is an  $R$ -algebra homomorphism.

$\langle 1 \rangle 3$ .  $\phi$  is injective.

$\langle 1 \rangle 4$ .  $\phi$  is surjective.

□

## 28.2 Free Algebras

**Proposition 28.14.** *Let  $R$  be a ring. Then  $R[x_1, \dots, x_n]$  is the free commutative  $R$ -algebra on  $\{1, \dots, n\}$ .*

PROOF: Easy. □

**Proposition 28.15.** *Let  $R$  be a ring and  $A$  a set. Let  $A^*$  be the free monoid on  $A$ . Then the monoid ring  $R[A^*]$  is the free  $R$ -algebra on  $A$ .*

PROOF: Easy. □

## Chapter 29

# Algebras of Finite Type

**Definition 29.1** (Algebra of Finite Type). Let  $R$  be a ring. Let  $S$  be an  $R$ -algebra. Then  $R$  is of *finite type* iff  $S$  is a finitely generated  $R$ -algebra.

**Proposition 29.2.** *Let  $R$  be a Noetherian ring. Let  $S$  be a finite-type  $R$ -algebra. Then  $S$  is a Noetherian ring.*



## Chapter 30

# Finite Algebras

**Definition 30.1** (Finite Algebra). Let  $R$  be a ring. Let  $S$  be an  $R$ -algebra. Then  $S$  is a *finite*  $R$ -algebra iff it is a finitely generated left- $R$ -module.

**Proposition 30.2.** *Let  $R$  be a ring. Every finite  $R$ -algebra is of finite type.*

PROOF: If  $S$  is generated by  $a_1, \dots, a_n$  as an  $R$ -module, then it is generated by  $a_1, \dots, a_n$  as an  $R$ -algebra.  $\square$

**Example 30.3.** The converse does not hold.  $R[x]$  is of finite type but is not finite.





## Chapter 31

# Division Algebras

**Definition 31.1** (Division Algebra). Let  $R$  be a commutative ring. A *division  $R$ -algebra* is an  $R$ -algebra that is a division ring.

**Example 31.2.** Let  $R$  be a commutative ring. Let  $M$  be a simple  $R$ -algebra. Then  $\text{End}_{R\text{-Mod}}(M)$  is a division algebra. For if  $\phi \circ \psi = 0$  then  $\phi$  and  $\psi$  cannot both be isomorphisms, hence  $\phi = 0$  or  $\psi = 0$  by Schur's Lemma.



**Part IV**

**Field Theory**



# Chapter 32

## Fields

**Definition 32.1** (Field). A *field* is a non-trivial commutative division ring.

**Example 32.2.**  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are fields.

**Proposition 32.3.** *Every field is an integral domain.*

PROOF: By Propositions 11.8 and 11.9.  $\square$

**Example 32.4.** The converse does not hold:  $\mathbb{Z}$  is an integral domain but not a field.

**Proposition 32.5.** *Every finite integral domain is a field.*

PROOF: In a finite integral domain, multiplication by any non-zero element is injective, hence surjective.  $\square$

**Corollary 32.5.1.** *For any positive integer  $n$ , the following are equivalent:*

- $n$  is prime.
- $\mathbb{Z}/n\mathbb{Z}$  is an integral domain.
- $\mathbb{Z}/n\mathbb{Z}$  is a field.

**Theorem 32.6** (Wedderburn's Little Theorem). *Every finite division ring is a field.*

**Proposition 32.7.** *Every subring of a field is an integral domain.*

PROOF: Easy.  $\square$

**Proposition 32.8.** *The center of a division ring is a field.*

PROOF:

$\langle 1 \rangle$ 1. LET:  $R$  be a division ring.

$\langle 1 \rangle$ 2. LET:  $Z$  be the center of  $R$ .

$\langle 1 \rangle$ 3.  $Z$  is non-trivial.

PROOF: Since  $1 \in Z$ .

$\langle 1 \rangle 4$ .  $Z$  is commutative.

$\langle 1 \rangle 5$ .  $Z$  is a division ring.

$\langle 2 \rangle 1$ . LET:  $a \in Z$

$\langle 2 \rangle 2$ .  $a^{-1} \in Z$

$\langle 3 \rangle 1$ . LET:  $x \in R$

$\langle 3 \rangle 2$ .  $ax = xa$

$\langle 3 \rangle 3$ .  $xa^{-1} = a^{-1}x$

□

**Definition 32.9.** For any prime  $p$  and positive integer  $r$ , define a multiplication on  $(\mathbb{Z}/p\mathbb{Z})^r$  that makes this group into a field by:

**Proposition 32.10.** *A commutative ring is a field if and only if it is simple.*

PROOF: Proposition 21.5. □

**Corollary 32.10.1.** *Every field has Krull dimension 0.*

**Proposition 32.11.** *Let  $K$  be a field. Then  $K[x]$  is a PID, and every non-zero ideal in  $K[x]$  is generated by a unique monic polynomial.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $I$  be a non-zero ideal in  $K[x]$

$\langle 1 \rangle 2$ . PICK a monic polynomial  $f \in K[x]$  of minimal degree.

PROVE:  $I = (f)$

$\langle 1 \rangle 3$ . LET:  $g \in I$

$\langle 1 \rangle 4$ . PICK polynomials  $q, r$  with  $\deg r < \deg f$  such that  $g = qf + r$

$\langle 1 \rangle 5$ .  $r \in I$

$\langle 1 \rangle 6$ .  $r = 0$

$\langle 1 \rangle 7$ .  $g \in (f)$

□

**Proposition 32.12.** *Let  $R$  be a commutative ring and  $I$  an ideal in  $R$ . Then  $I$  is maximal iff  $R/I$  is a field.*

PROOF: From Proposition 22.3. □

**Example 32.13.** Let  $R$  be a commutative ring and  $a \in R$ . Then  $(x - a)$  is a maximal ideal in  $R[x]$  iff  $R$  is a field, since  $R[x]/(x - a) \cong R$ .

**Example 32.14.** The ideal  $(2, x)$  is a maximal ideal in  $\mathbb{Z}[x]$ , since  $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}/2\mathbb{Z}$ .

**Proposition 32.15.** *Every maximal ideal in a commutative ring is a prime ideal.*

PROOF: Since every field is an integral domain. □

**Proposition 32.16.** *Let  $R$  be a commutative ring and  $I$  an ideal in  $R$ . If  $I$  is a prime ideal and  $R/I$  is finite then  $I$  is a maximal ideal.*

PROOF: Since every finite integral domain is a field.  $\square$

**Proposition 32.17.** *Let  $R$  be a commutative ring and  $I$  a proper ideal in  $R$ . Then  $I$  is maximal iff, whenever  $J$  is an ideal and  $I \subseteq J$ , then  $I = J$  or  $J = R$ .*

**Example 32.18.** The inverse image of a maximal ideal under a homomorphism is not necessarily maximal.

Let  $i : \mathbb{Z}[x] \rightarrow \mathbb{Q}[x]$  be the inclusion. Then  $(x)$  is maximal in  $\mathbb{Q}[x]$  but its inverse image  $(x)$  is not maximal in  $\mathbb{Z}[x]$ .

**Definition 32.19** (Maximal Spectrum). Let  $R$  be a commutative ring. The *maximal spectrum* of  $R$  is the set of all maximal ideals in  $R$ .

**Proposition 32.20.** *Let  $K$  be a field. The Krull dimension of  $K[x_1, \dots, x_n]$  is  $n$ .*

**Theorem 32.21** (Hilbert's Nullstellensatz). *Let  $K$  be a field and  $L$  a subfield of  $K$ . If  $K$  is an  $L$ -algebra of finite type, then  $K$  is a finite  $L$ -algebra.*

**Proposition 32.22.** *Let  $K$  be a subfield of  $L$ . Then  $L$  is a  $K$ -algebra under multiplication.*

PROOF: Easy.  $\square$





## Chapter 33

# Algebraically Closed Fields

**Definition 33.1** (Algebraically Closed). A field  $K$  is *algebraically closed* iff, for every  $f \in K[x]$  that is not constant, there exists  $r \in K$  such that  $f(r) = 0$ .

**Theorem 33.2.**  $\mathbb{C}$  is algebraically closed.

**Proposition 33.3.** Let  $K$  be an algebraically closed field. Let  $I$  be an ideal in  $K[x]$ . Then  $I$  is maximal if and only if  $I = (x - c)$  for some  $c \in K$ .

PROOF:

$\langle 1 \rangle 1$ . If  $I$  is maximal then there exists  $c \in K$  such that  $I = (x - c)$ .

$\langle 2 \rangle 1$ . ASSUME:  $I$  is maximal.

$\langle 2 \rangle 2$ . PICK  $f$  monic of minimal degree such that  $f \in I$ .

$\langle 2 \rangle 3$ .  $f$  is not constant.

PROOF: Otherwise  $f = 1$  and  $I = K[x]$ .

$\langle 2 \rangle 4$ . PICK  $c \in K$  such that  $f(c) = 0$

$\langle 2 \rangle 5$ .  $x - c \mid f$

$\langle 2 \rangle 6$ .  $I \subseteq (x - c)$

$\langle 2 \rangle 7$ .  $I = (x - c)$

$\langle 1 \rangle 2$ . For all  $c \in K$  we have  $(x - c)$  is maximal.

PROOF: Example 32.13.

□



**Part V**

**Linear Algebra**



## Chapter 34

# Vector Spaces

**Definition 34.1** (Vector Space). Let  $K$  be a field. A  $K$ -vector space is a  $K$ -module. A *linear map* is a homomorphism of  $K$ -modules. We write  $K - \mathbf{Vect}$  for  $K - \mathbf{Mod}$ .

**Definition 34.2.** Let  $\mathrm{GL}_n(\mathbb{R})$  be the group of invertible  $n \times n$  real matrices.  $\mathrm{GL}_n(\mathbb{R})$  acts on  $\mathbb{R}^n$  by matrix multiplication.

**Definition 34.3.** Let  $\mathrm{GL}_n(\mathbb{C})$  be the group of invertible  $n \times n$  complex matrices.  $\mathrm{GL}_n(\mathbb{C})$  acts on  $\mathbb{C}^n$  by matrix multiplication.

**Definition 34.4.** Let  $\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : \det M = 1\}$ .

**Proposition 34.5.**  $\mathrm{SL}_n(\mathbb{R})$  is a normal subgroup of  $\mathrm{GL}_n(\mathbb{R})$ .

PROOF: If  $\det M = 1$  then  $\det(AMA^{-1}) = (\det A)(\det M)(\det A)^{-1} = 1$ .  $\square$

**Proposition 34.6.**

$$\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$$

**Definition 34.7.** Let  $\mathrm{SL}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : \det M = 1\}$ .

**Definition 34.8.** Let  $\mathrm{O}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : MM^T = M^T M = I_n\}$ .

**Proposition 34.9.** The action of  $\mathrm{O}_n(\mathbb{R})$  on  $\mathbb{R}^n$  preserves lengths and angles.

**Definition 34.10.** Let  $\mathrm{SO}_n(\mathbb{R}) = \{M \in \mathrm{O}_n(\mathbb{R}) : \det M = 1\}$ .

**Definition 34.11.** Let  $\mathrm{U}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : MM^\dagger = M^\dagger M = I_n\}$ .

**Definition 34.12.** Let  $\mathrm{SU}_n(\mathbb{C}) = \{M \in \mathrm{U}_n(\mathbb{C}) : \det M = 1\}$ .

**Proposition 34.13.** Every matrix in  $\mathrm{SU}_2(\mathbb{C})$  can be written in the form

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

for some  $a, b, c, d \in \mathbb{R}$  with  $a^2 + b^2 + c^2 + d^2 = 1$ .

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SU}_2(\mathbb{C})$$

$$\langle 1 \rangle 2. M^{-1} = M^\dagger$$

$$\langle 1 \rangle 3. \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

$$\langle 1 \rangle 4. \text{ LET: } \alpha = a + bi \text{ and } \beta = c + di.$$

$$\langle 1 \rangle 5. \delta = \bar{\alpha} = a - bi$$

$$\langle 1 \rangle 6. \gamma = -\bar{\beta} = -c + di$$

$$\langle 1 \rangle 7. \det M = a^2 + b^2 + c^2 + d^2 = 1$$

□

**Corollary 34.13.1.**  $\text{SU}_2(\mathbb{C})$  is simply connected.

**Corollary 34.13.2.**

$$\text{SO}_3(\mathbb{R}) \cong \text{SU}_2(\mathbb{C}) / \{I, -I\}$$

$$\text{PROOF: The function that maps } \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} \text{ to } \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(ad + bc) \\ 2(ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(bc - ad) \\ 2(bd - ac) & 2(ab + cd) & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$$

is a surjective homomorphism with kernel  $\{I, -I\}$ . □

**Corollary 34.13.3.** The fundamental group of  $\text{SO}_3(\mathbb{R})$  is  $C_2$ .