Mathematics

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September 3, 2023

Contents

Chapter 1

The Foundations

1.1 Primitive Notions and Axioms

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ for 'f is a function from A to B'. We call A the *domain* of A, and B the *codomain*.

Given sets A, B and C, and functions $f : A \to B$ and $g : B \to C$, let there be a function $gf = g \circ f : A \to C$, the *composite* of f and g.

Axiom 1.1 (Associativity)**.** *For any functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \ .$$

Axiom 1.2 (Identity)**.** *For any set A, there exists a function $\mathrm{id}_A : A \to A$, called an* identity *function on A, such that:*

- *for every set B and function $f : A \to B$, we have $f \circ \mathrm{id}_A = f$;*
- *for every set B and function $f : B \to A$, we have $\mathrm{id}_A \circ f = f$.*

Proposition 1.3. *The identity function on a set is unique.*

Proof: If $i, j : A \to A$ are identity functions on A then we have $i = i \circ j = j$. □

Definition 1.4 (Isomorphism)**.** A function $i : A \to B$ is an *isomorphism*, $i : A \cong B$, iff there exists a function $i^{-1} : B \to A$, the *inverse* of i, such that $i^{-1} \circ i = \mathrm{id}_A$ and $i \circ i^{-1} = \mathrm{id}_B$.

Proposition 1.5. *The inverse of an isomorphism is unique.*

Proof: If j and k are inverses of i we have $j = jik = k$. □

Proposition 1.6. *For any set A we have $\mathrm{id}_A : A \cong A$ and ${\mathrm{id}_A}^{-1} = \mathrm{id}_A$.*

Proof: Since $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$. □

Proposition 1.7. *If $i : A \cong B$ then $i^{-1} : B \cong A$ and $(i^{-1})^{-1} = i$.*

PROOF: Since $i \circ i^{-1} = \mathrm{id}_B$ and $i^{-1} \circ i = \mathrm{id}_A$. □

Proposition 1.8. *If $i : A \cong B$ and $j : B \cong C$ then $j \circ i : A \cong C$ and $(j \circ i)^{-1} = i^{-1} \circ j^{-1}$.*

PROOF: Since $j \circ i \circ i^{-1} \circ j^{-1} = \mathrm{id}_C$ and $i^{-1} \circ j^{-1} \circ j \circ i = \mathrm{id}_A$. □

Axiom 1.9 (Terminal Set)**.** *There exists a set 1 such that, for any set A, there exists a unique function $A \to 1$.*

Proposition 1.10. *The terminal set is unique up to unique isomorphism.*

PROOF:
⟨1⟩1. LET: A and B be terminal sets.
⟨1⟩2. LET: i be the unique function $A \to B$.
⟨1⟩3. LET: i^{-1} be the unique function $B \to A$.
⟨1⟩4. $i \circ i^{-1} = \mathrm{id}_B$
PROOF: Since there is only one function $B \to B$.
⟨1⟩5. $i^{-1} \circ i = \mathrm{id}_A$
PROOF: Since there is only one function $A \to A$.
□

Definition 1.11 (Element)**.** For any set A, an *element* of A is a function $1 \to A$.

We write $a \in A$ for $a : 1 \to A$. Given $f : A \to B$ and $a \in A$, we write $f(a)$ for $f \circ a$.

Axiom 1.12 (Extensionality)**.** *Let A and B be sets. Let $f, g : A \to B$. If, for all $x \in A$, we have $f(x) = g(x)$, then $f = g$.*

Axiom 1.13 (Empty Set)**.** *There exists a set with no elements.*

Axiom 1.14 (Products)**.** *Let A and B be sets. There exists a set $A \times B$ and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, the* projections*, such that, for every set X and functions $f : X \to A$, $g : X \to B$, there exists a unique function $\langle f, g\rangle : X \to A \times B$ such that*

$$\pi_1 \circ \langle f, g\rangle = f, \qquad \pi_2 \circ \langle f, g\rangle = g \ .$$

Proposition 1.15. *If $\pi_1 : P \to A$ and $\pi_2 : P \to B$ form a product of A and B, and $p_1 : Q \to A$ and $p_2 : Q \to B$ form a product of A and B, then there exists a unique isomorphism $i : P \cong Q$ such that $p_1 \circ i = \pi_1$ and $p_2 \circ i = \pi_2$.*

PROOF:
⟨1⟩1. LET: $i : P \to Q$ be the unique function such that $p_1 \circ i = \pi_1$ and $p_2 \circ i = \pi_2$.
⟨1⟩2. LET: $i^{-1} : Q \to P$ be the unique function such that $\pi_1 \circ i^{-1} = p_1$ and $\pi_2 \circ i^{-1} = p_2$
⟨1⟩3. $i \circ i^{-1} = \mathrm{id}_Q$
PROOF: Each is the unique $x : Q \to Q$ such that $p_1 \circ x = p_1$ and $p_2 \circ x = p_2$.

⟨1⟩4. $i^{-1} \circ i = \mathrm{id}_P$
  PROOF: Each is the unique $x : P \to P$ such that $\pi_1 \circ x = \pi_1$ and $\pi_2 \circ x = \pi_2$.
□

Proposition 1.16. *Let $f : A \to B$, $g : B \to C$ and $h : B \to D$. Then*

$$\langle g, h \rangle \circ f = \langle g \circ f, h \circ f \rangle$$

PROOF: Each is the unique x such that $\pi_1 \circ x = g \circ f$ and $\pi_2 \circ x = h \circ f$. □

Definition 1.17. Given functions $f : A \to B$ and $g : C \to D$, define $f \times g : A \times C \to B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle \ .$$

Proposition 1.18. *Let $f : A \to B$, $g : C \to D$, $h : B \to E$ and $k : D \to F$. Then*

$$(h \times k) \circ (f \times g) = (h \circ f) \times (k \circ g) \ .$$

PROOF:

$$\begin{aligned} (h \times k) \circ (f \times g) &= \langle h \circ \pi_1, k \circ \pi_2 \rangle \circ (f \times g) \\ &= \langle h \circ \pi_1 \circ (f \times g), k \circ \pi_2 \circ (f \times g) \rangle \quad \text{(Proposition 1.16)} \\ &= \langle h \circ f \circ \pi_1, k \circ g \circ \pi_2 \rangle \\ &= (h \circ f) \times (k \circ g) \end{aligned}$$

□

Axiom 1.19 (Function Sets)**.** *Let A and B be sets. There exists a set A^B, called the* function set *of A and B, and function $\epsilon : A^B \times B \to A$, the* evaluation map*, such that, for any set X and function $f : X \times B \to A$, there exists a unique function $\lambda f : X \to A^B$ such that*

$$f = \epsilon \circ (\lambda f \times \mathrm{id}_B) \ .$$

Proposition 1.20. *For any sets A and B, if F and G are function sets with evaluation maps $e : F \times B \to A$ and $e' : G \times B \to A$, then there exists a unique isomorphism $i : F \cong G$ such that $e' \circ (i \times \mathrm{id}_B) = e$.*

PROOF:
⟨1⟩1. LET: $i : F \to G$ be the unique function such that $e = e' \circ (i \times \mathrm{id}_B)$.
⟨1⟩2. LET: $i^{-1} : G \to F$ be the unique function such that $e' = e \circ (i^{-1} \times \mathrm{id}_B)$
⟨1⟩3. $i \circ i^{-1} = \mathrm{id}_G$
  PROOF: Each is the unique x such that $e' = e' \circ (x \times \mathrm{id}_B)$.
⟨1⟩4. $i^{-1} \circ i = \mathrm{id}_F$
  PROOF: Each is the unique x such that $e = e \circ (x \times \mathrm{id}_B)$.
□

Definition 1.21 (Inverse Image)**.** Let A, X and Y be sets. Let $f : X \to Y$, $a \in Y$ and $j : A \to X$. Then j is the *inverse image* of a under f if and only if:

- $f \circ j = a \circ !_A$

- for every set I and function $q : I \to X$ such that $f \circ q = a \circ !_I$, there exists a unique $\overline{q} : I \to A$ such that $q = j \circ \overline{q}$.

Axiom 1.22 (Inverse Images)**.** *For any sets X and Y, function $f : X \to Y$ and element $a \in Y$, there exists a set $f^{-1}(a)$ and function $j : f^{-1}(a) \to X$ such that j is the inverse image of a under f.*

Proposition 1.23. *If $j : A \to X$ and $k : B \to X$ are inverse images of $a : 1 \to Y$ under $f : X \to Y$, then there exists a unique isomorphism $i : A \cong B$ such that $k \circ i = j$.*

PROOF:
⟨1⟩1. LET: $i : A \to B$ be the unique function such that $j = k \circ i$.
⟨1⟩2. LET: $i^{-1} : B \to A$ be the unique function such that $k = j \circ i^{-1}$
⟨1⟩3. $i \circ i^{-1} = \mathrm{id}_B$
PROOF: Each is the unique x such that $k = k \circ x$.
⟨1⟩4. $i^{-1} \circ i = \mathrm{id}_A$
PROOF: Each is the unique x such that $j = j \circ x$.
□

Definition 1.24 (Injective)**.** A function $f : A \to B$ is *injective*, $f : A \rightarrowtail B$, iff, for every set X and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

Proposition 1.25. *Any function $a : 1 \to A$ is injective.*

PROOF: If $x, y : X \to 1$ satisfy $a \circ x = a \circ y$, then $x = y$ because there is only one function $X \to 1$. □

Definition 1.26 (Surjective)**.** A function $f : A \to B$ is *surjective*, $f : A \twoheadrightarrow B$, iff, for every set X and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

Axiom 1.27 (Subset Classifier)**.** *There exists a set 2, a* subset classifier*, and element $\top \in 2$,* truth*, such that, for any sets A and X and any injective function $f : A \to X$, there exists a unique function $\chi : X \to 2$ such that f is the inverse image of $\top$ under χ.*

Proposition 1.28. *If S and S' are subobject classifiers with truth elements t and t', then there exists a unique isomorphism $i : S \cong S'$ such that $i(t) = t'$.*

PROOF:
⟨1⟩1. LET: $i : S \to S'$ be the characteristic function of t.
⟨1⟩2. LET: $i^{-1} : S' \to S$ be the characteristic function of t'.
⟨1⟩3. $i \circ i^{-1} = \mathrm{id}_{S'}$
PROOF: Each is the characteristic function of t'.
⟨1⟩4. $i^{-1} \circ i = \mathrm{id}_S$
PROOF: Each is the characteristic function of t.
□

Axiom 1.29 (Natural Numbers)**.** *There exists a set $\mathbb{N}$ of* natural numbers*, an element $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \to \mathbb{N}$ such that, for every set X, element $a \in X$ and function $r : X \to X$, there exists a unique function $x : \mathbb{N} \to X$ such that $x(0) = a$ and $x \circ s = r \circ x$.*

Proposition 1.30. *If $N, 0 \in N, s : N \to N$ and $N', 0' \in N', s' : N' \to N'$ are natural number sets, then there exists a unique isomorphism $i : N \cong N'$ such that $i(0) = 0'$ and $s' \circ i = i \circ s$.*

PROOF:
⟨1⟩1. LET: $i : N \to N'$ be the unique function such that $i(0) = 0'$ and $i \circ s = s' \circ i$.
⟨1⟩2. LET: $i^{-1} : N' \to N$ be the unique function such that $i^{-1}(0') = 0$ and $i^{-1} \circ s' = s \circ i^{-1}$
⟨1⟩3. $i \circ i^{-1} = \mathrm{id}_{N'}$
PROOF: Each is the unique x such that $x(0') = 0'$ and $s' \circ x = x \circ s'$.
⟨1⟩4. $i^{-1} \circ i = \mathrm{id}_N$
PROOF: Each is the unique x such that $x(0) = 0$ and $s \circ x = x \circ s$.
□

Definition 1.31 (Retraction, Section)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $r \circ s = \mathrm{id}_B$.

Axiom 1.32 (Choice)**.** *Every surjective function has a section.*

1.2 Injective and Surjective Functions

Proposition 1.33. *Every section is injective.*

PROOF: Let $r \circ s = \mathrm{id}$. If $s \circ x = s \circ y$ then $x = r \circ s \circ x = r \circ s \circ y = y$. □

Proposition 1.34. *Let $r : A \to B$. Then the following are equivalent.*

1. *r is surjective.*
2. *r has a section.*
3. *For every element $y \in B$, there exists $x \in A$ such that $r(x) = y$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
PROOF: Axiom of Choice.
⟨1⟩2. $2 \Rightarrow 3$
⟨2⟩1. LET: $s : A \to B$ be a section of r.
⟨2⟩2. LET: $y \in B$
⟨2⟩3. LET: $x = s(y)$
⟨2⟩4. $r(x) = y$
⟨1⟩3. $3 \Rightarrow 1$
⟨2⟩1. ASSUME: $\forall y \in B . \exists x \in A . r(x) = y$
⟨2⟩2. LET: X be a set and $f, g : B \to X$
⟨2⟩3. ASSUME: $fr = gr$
⟨2⟩4. For all $y \in B$ we have $f(y) = g(y)$.
⟨3⟩1. LET: $y \in B$

⟨3⟩2. Pick $x \in A$ such that $r(x) = y$.
⟨3⟩3. $f(y) = g(y)$
Proof:

$$\begin{aligned} f(y) &= f(r(x)) && (\langle 3\rangle 2) \\ &= g(r(x)) && (\langle 2\rangle 3) \\ &= g(y) && (\langle 3\rangle 2) \end{aligned}$$

⟨2⟩5. $f = g$
Proof: Axiom of Extensionality.

1.3 Subsets of a Set

Definition 1.35 (Subset)**.** Let $i : X \to A$. We write '(X, i) is a subset of A' for 'i is injective'.

Given subsets $i : X \to A$ and $j : Y \to A$, we write '$(X, i) = (Y, j)$' for 'there exists an isomorphism $k : X \cong Y$ such that $j \circ k = i$.

Proposition 1.36. *Given subsets (X, i), (Y, j) of A, if $(X, i) = (Y, j)$ then the isomorphism $k : X \cong Y$ such that $i \circ k = j$ is unique.*

Proof: Since i is injective. □

Proposition 1.37. *If (X, i) is a subset of A then $(X, i) = (X, i)$.*

Proof: Since $\mathrm{id}_X : X \cong X$ and $i \circ \mathrm{id}_X = i$. □

Proposition 1.38. *Given subsets (X, i), (Y, j) of A, if $(X, i) = (Y, j)$ then $(Y, j) = (X, i)$.*

Proof: If $k : X \cong Y$ and $j \circ k = i$ then $k^{-1} : Y \cong X$ and $i \circ k^{-1} = j$. □

Proposition 1.39. *Given subsets (X, i), (Y, j), (Z, k) of A, if $(X, i) = (Y, j)$ and $(Y, j) = (Z, k)$ then $(X, i) = (Z, k)$.*

Proof: If $f : X \cong Y$ satisfies $j \circ f = i$ and $g : Y \cong Z$ satisfies $k \circ g = j$, then $g \circ f : X \cong Z$ and $k \circ g \circ f = i$. □

Definition 1.40 (Inclusion)**.** Let (X, i) and (Y, j) be subsets of A. We say (X, i) is *included* in (Y, j), and write $(X, i) \subseteq (Y, j)$, iff there exists $k : X \to Y$ such that $j \circ k = i$.

Proposition 1.41. *For any subsets (X, i), (Y, j) of A, if $(X, i) = (Y, j)$ then $(X, i) \subseteq (Y, j)$.*

Proof: Immediate from definitions. □

Corollary 1.41.1. *For any subset (X, i) of A we have $(X, i) \subseteq (X, i)$.*

Proposition 1.42. *For any subsets (X, i), (Y, j), (Z, k) of A, if $(X, i) \subseteq (Y, j)$ and $(Y, j) \subseteq (Z, k)$, then $(X, i) \subseteq (Z, k)$.*

PROOF: If $f : X \to Y$ satisfies $j \circ f = i$ and $g : Y \to Z$ satisfies $k \circ g = j$, then $g \circ f : X \to Z$ and $k \circ g \circ f = i$. □

Corollary 1.42.1. *Inclusion is well defined. That is, if* $(X, i) = (X', i')$, $(Y, j) = (Y', j')$ *and* $(X, i) \subseteq (Y, j)$ *then* $(X', i') \subseteq (Y', j')$.

Proposition 1.43. *For any subsets* (X, i) *and* (Y, j) *of* A*, if* $(X, i) \subseteq (Y, j)$ *and* $(Y, j) \subseteq (X, i)$ *then* $(X, i) = (Y, j)$.

PROOF:
⟨1⟩1. LET: $f : X \to Y$ satisfy $j \circ f = i$.
⟨1⟩2. LET: $g : Y \to X$ satisfy $i \circ g = j$.
⟨1⟩3. $g \circ f = \mathrm{id}_X$
PROOF: Since $i \circ g \circ f = i$ and i is injective.
⟨1⟩4. $f \circ g = \mathrm{id}_Y$
PROOF: Since $j \circ f \circ g = j$ and j is injective.
⟨1⟩5. $f : X \cong Y$ and $j \circ f = i$.
⟨1⟩6. $(X, i) = (Y, j)$
□

1.4 Equalizers

Proposition 1.44. *For any set* A*, the function* $\langle \mathrm{id}_A, \mathrm{id}_A \rangle : A \to A \times A$ *is injective.*

PROOF: Since $\pi_1 \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle = \mathrm{id}_A$. □

Proposition 1.45. *Given sets* A *and* B *and functions* $f, g : A \to B$*, there exists a set* E *and function* $e : E \to A$*, called the* equalizer *of* f *and* g*, such that:*

- $f \circ e = g \circ e$
- *for any set* X *and function* $x : X \to A$*, if* $f \circ x = g \circ x$ *then there exists a unique* $\overline{x} : X \to E$ *such that* $x = e \circ \overline{x}$.

$$\begin{tikzcd}
X \arrow[dr, "\overline{x}"] \arrow[drr, "f\circ x"] \arrow[ddr, "x"'] & & & \\
& E \arrow[r, "m"] \arrow[d, "e"] & B \arrow[r, "!"] \arrow[d, "\langle \mathrm{id}_B, \mathrm{id}_B \rangle"'] & 1 \arrow[d, "\top"] \\
& A \arrow[r, "\langle f, g \rangle"'] & B \times B \arrow[r, "eq"'] & 2
\end{tikzcd}$$

PROOF:
⟨1⟩1. LET: $eq : B \times B \to 2$ be the characteristic function of $\langle \mathrm{id}_B, \mathrm{id}_B \rangle : B \to B \times B$

PROOF: By the Axiom of the Subset Classifier.
$\langle 1 \rangle 2$. LET: $e : E \to A$ be the inverse image of $\top$ under $eq \circ \langle f, g \rangle$
PROOF: By the Axiom of Inverse Images.
$\langle 1 \rangle 3$. $f \circ e = g \circ e$
$\langle 2 \rangle 1$. $eq \circ \langle f, g \rangle \circ e = \top$
$\langle 2 \rangle 2$. LET: $m : E \to B$ be the unique function such that $\langle \mathrm{id}_B, \mathrm{id}_B \rangle \circ m = \langle f, g \rangle \circ e$
$\langle 2 \rangle 3$. $\langle m, m \rangle = \langle f \circ e, g \circ e \rangle$
$\langle 2 \rangle 4$. $f \circ e = g \circ e = m$
$\langle 1 \rangle 4$. For any set X and function $x : X \to A$, if $f \circ x = g \circ x$ then there exists a unique $\overline{x} : X \to E$ such that $x = e \circ \overline{x}$.
$\langle 2 \rangle 1$. LET: X be a set.
$\langle 2 \rangle 2$. LET: $x : X \to A$
$\langle 2 \rangle 3$. ASSUME: $f \circ x = g \circ x$
$\langle 2 \rangle 4$. $\langle f, g \rangle \circ x = \langle \mathrm{id}_B, \mathrm{id}_B \rangle \circ f \circ x$
$\langle 2 \rangle 5$. $eq \circ \langle f, g \rangle \circ x = \top \circ !_X$
PROOF:

$$\begin{aligned} eq \circ \langle f, g \rangle \circ x &= eq \circ \langle \mathrm{id}_B, \mathrm{id}_B \rangle \circ f \circ x \\ &= \top \circ !_B \circ f \circ x \\ &= \top \circ !_X \end{aligned}$$

$\langle 2 \rangle 6$. There exists a unique $\overline{x} : X \to E$ such that $e \circ \overline{x} = x$
PROOF: From $\langle 1 \rangle 2$.
□

1.5 Pullbacks

Proposition 1.46. *Let $f : A \to C$ and $g : B \to C$. Then there exists a set P and functions $p : P \to A$, $q : P \to B$ such that:*

- $f \circ p = g \circ q$
- *For any set X and functions $x : X \to A$, $y : X \to B$ such that $f \circ x = g \circ y$, there exists a unique function $(x, y) : X \to P$ such that $p \circ (x, y) = x$ and $q \circ (x, y) = y$.*

$$\begin{tikzcd}
E \arrow[dr, "e"] \arrow[drr, "p"] \arrow[ddr, "q"'] & & \\
& A \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

PROOF:
$\langle 1 \rangle 1$. LET: $e : P \to A \times B$ be the equalizer of $f \circ \pi_1, g \circ \pi_2 : A \times B \to C$.

$\langle 1 \rangle 2$. LET: $p = \pi_1 \circ e : E \to A$ and $q = \pi_2 \circ e : E \to B$.
$\langle 1 \rangle 3$. $f \circ p = g \circ q$
$\langle 1 \rangle 4$. For any set X and functions $x : X \to A$, $y : X \to B$ such that $f \circ x = g \circ y$, there exists a unique function $(x, y) : X \to P$ such that $p \circ (x, y) = x$ and $q \circ (x, y) = y$.
  $\langle 2 \rangle 1$. LET: X be a set.
  $\langle 2 \rangle 2$. LET: $x : X \to A$ and $y : X \to B$
  $\langle 2 \rangle 3$. ASSUME: $f \circ x = g \circ y$
  $\langle 2 \rangle 4$. $f \circ \pi_1 \circ \langle x, y \rangle = g \circ \pi_2 \circ \langle x, y \rangle$
  $\langle 2 \rangle 5$. LET: $(x, y) : X \to E$ be the unique morphism such that $e \circ (x, y) = \langle x, y \rangle$
  $\langle 2 \rangle 6$. (x, y) is unique such that $\pi_1 \circ e \circ (x, y) = x$ and $\pi_2 \circ e \circ (x, y) = y$
  $\langle 2 \rangle 7$. (x, y) is unique such that $p \circ (x, y) = x$ and $q \circ (x, y) = y$.

□