# Mathematics

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# Contents

# Chapter 1

# Primitive Notions and Axioms

Let there be *sets*.

Given sets $A$ and $B$, let there be *functions* from $A$ to $B$. We write $f : A \to B$ for '$f$ is a function from $A$ to $B$'. We call $A$ the *domain* of $A$, and $B$ the *codomain*.

Given sets $A$, $B$ and $C$, and functions $f : A \to B$ and $g : B \to C$, let there be a function $gf = g \circ f : A \to C$, the *composite* of $f$ and $g$.

**Axiom 1.1** (Associativity)**.** *For any functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f \quad .$$

**Axiom 1.2** (Identity)**.** *For any set $A$, there exists a function $\mathrm{id}_A : A \to A$, called an* identity *function on $A$, such that:*

- *for every set $B$ and function $f : A \to B$, we have $f \circ \mathrm{id}_A = f$;*
- *for every set $B$ and function $f : B \to A$, we have $\mathrm{id}_A \circ f = f$.*

**Proposition 1.3.** *The identity function on a set is unique.*

Proof: If $i, j : A \to A$ are identity functions on $A$ then we have $i = i \circ j = j$. $\square$

**Definition 1.4** (Isomorphism)**.** A function $i : A \to B$ is an *isomorphism*, $i : A \cong B$, iff there exists a function $i^{-1} : B \to A$, the *inverse* of $i$, such that $i^{-1} \circ i = \mathrm{id}_A$ and $i \circ i^{-1} = \mathrm{id}_B$.

**Axiom 1.5** (Terminal Set)**.** *There exists a set 1 such that, for any set $A$, there exists a unique function $A \to 1$.*

**Proposition 1.6.** *The terminal set is unique up to unique isomorphism.*

Proof:

⟨1⟩1. LET: $A$ and $B$ be terminal sets.
⟨1⟩2. LET: $i$ be the unique function $A \to B$.
⟨1⟩3. LET: $i^{-1}$ be the unique function $B \to A$.
⟨1⟩4. $i \circ i^{-1} = \mathrm{id}_B$
PROOF: Since there is only one function $B \to B$.
⟨1⟩5. $i^{-1} \circ i = \mathrm{id}_A$
PROOF: Since there is only one function $A \to A$.
□

**Definition 1.7** (Element)**.** For any set $A$, an *element* of $A$ is a function $1 \to A$. We write $a \in A$ for $a : 1 \to A$.

**Axiom 1.8** (Empty Set)**.** *There exists a set 0 with no elements.*

**Axiom 1.9** (Extensionality)**.** *Let $A$ and $B$ be sets. Let $f, g : A \to B$. If, for all $x : 1 \to A$, we have $f \circ x = g \circ x$, then $f = g$.*

**Axiom 1.10** (Products)**.** *Let $A$ and $B$ be sets. There exists a set $A \times B$ and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ such that, for every set $X$ and functions $f : X \to A$, $g : X \to B$, there exists a unique function $\langle f, g \rangle : X \to A \times B$ such that*

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g \ .$$

**Axiom 1.11** (Function Sets)**.** *Let $A$ and $B$ be sets. There exists a set $A^B$ and function $\epsilon : A^B \times B \to A$ such that, for any set $X$ and function $f : X \times B \to A$, there exists a unique function $\lambda f : X \to A^B$ such that*

$$f = \epsilon \circ \langle \lambda f \circ \pi_1, \pi_2 \rangle \ .$$

**Definition 1.12** (Inverse Image)**.** Let $A$, $X$ and $Y$ be sets. Let $f : X \to Y$, $a \in Y$ and $j : A \to X$. Then $j$ is the *inverse image* of $a$ under $f$ if and only if:

- $f \circ j = a \circ !_A$
- for every set $I$ and function $q : I \to X$ such that $f \circ q = a \circ !_I$, there exists a unique $\overline{q} : I \to A$ such that $q = j \circ \overline{q}$.

**Axiom 1.13** (Inverse Images)**.** *For any sets $X$ and $Y$, function $f : X \to Y$ and element $a \in Y$, there exists a set $f^{-1}(a)$ and function $j : f^{-1}(a) \to X$ such that $j$ is the inverse image of $a$ under $f$.*

**Definition 1.14** (Injective)**.** A function $f : A \to B$ is *injective* iff, for every set $X$ and functions $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

**Definition 1.15** (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for every set $X$ and functions $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

**Axiom 1.16** (Subset Classifier)**.** *There exists a set 2 and function $\top : 1 \to 2$ such that, for every injective function $f : A \to X$, there exists a unique function $\chi : X \to 2$ such that $f$ is the inverse image of $\top$ under $\chi$.*

**Axiom 1.17** (Natural Numbers)**.** *There exists a set* $\mathbb{N}$*, an element* $0 \in \mathbb{N}$ *and a function* $s : \mathbb{N} \to \mathbb{N}$ *such that, for every set* $X$*, element* $a \in X$ *and function* $r : X \to X$*, there exists a unique function* $x : \mathbb{N} \to X$ *such that* $x \circ 0 = a$ *and* $x \circ s = r \circ x$*.*

**Axiom 1.18** (Choice)**.** *For every surjective function* $r : X \to Y$*, there exists* $s : Y \to X$ *such that* $r \circ s$ *is an identity function on* $X$*.*