Mathematics

Robin Adams

September 13, 2023

Contents

Chapter 1

Sets and Functions

1.1 Primitive Terms

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B, and call A the *domain* of f and B the *codomain*.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $g \circ f : A \to C$, the *composite* of f and g.

For any set A, let there be a function $\mathrm{id}_A : A \to A$, the *identity* function on A.

Let there be a set 1, the *terminal* set.

For any sets A and B, let there be a set $A \times B$, the *product* of A and B, and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, the *projections*.

Given functions $f : A \to B$ and $g : A \to C$, let there be a function $\langle f, g \rangle : A \to B, C$.

1.2 Definitions Used in the Axioms

Definition 1.1 (Element)**.** For any set A, an *element* of A is a function $1 \to a$. We write $a \in A$ for $a : 1 \to A$.

Given $f : A \to B$ and $a \in A$, we write $f(a)$ for $f \circ a : 1 \to B$.

Definition 1.2 (Injective)**.** A function $f : A \to B$ is *injective* iff, for every set X and functions $x, y : X \to A$, if $fx = fy$ then $x = y$.

Definition 1.3 (Surjective)**.** A function $f : A \to B$ is *surjective* iff, for every element $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Definition 1.4 (Retraction, Section)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of A, iff $r \circ s = \mathrm{id}_B$.

Definition 1.5. Given functions $f : A \to B$ and $g : C \to D$, let $f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle$.

Definition 1.6 (Function Set)**.** Let A and B be sets. A *function set* from A to B consists of a set B^A and function $\epsilon : B^A \times A \to B$ such that, for any set I and function $q : I \times A \to B$, there exists a unique function $\lambda q : I \to B^A$ such that $\epsilon \circ (\lambda q \times \mathrm{id}_A) = q$.

Definition 1.7 (Pullback)**.** Let $p : A \to B$, $q : A \to C$, $f : B \to D$ and $g : C \to D$. Then we say that A, p and q form the *pullback* of f and g if and only if:

- $fp = gq$
- For any set X and functions $x : X \to B$, $y : X \to C$ such that $fx = gy$, there exists a unique function $(x, y) : X \to A$ such that $p(x, y) = x$ and $q(x, y) = y$.

We also say p is the pullback of g along f, or q is the pullback of f along g.

In the case g is injective, we also say A and p form the *inverse image* of g under f.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

1.3 The Axioms

Axiom 1.8 (Associativity)**.** *Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have*

$$h(gf) = (hg)f \quad .$$

Axiom 1.9 (Unit Laws)**.** *For any function $f : A \to B$, we have $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$.*

Axiom 1.10 (Terminal Set)**.** *For any set X, there is exactly one function $X \to 1$.*

Axiom 1.11 (Empty Set)**.** *There exists a set that has no elements.*

Axiom 1.12 (Extensionality)**.** *Let A and B be sets and $f, g : A \to B$. If $\forall a \in A. f(a) = g(a)$ then $f = g$.*

Axiom 1.13 (Products)**.** *Let $f : A \to B$ and $g : A \to C$. Then $\langle f, g \rangle$ is the unique function $A \to B \times C$ such that*

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g \quad .$$

Axiom 1.14 (Function Sets)**.** *Any two sets have a function set.*

Axiom 1.15 (Inverse Images). *Given any function $f : X \to Y$ and element $y \in Y$, then there exists a pullback of f and y.*

Axiom 1.16 (Subset Classifier). *There exists a set 2 and element $\top \in 2$ such that, for any sets A and X and injective function $j : A \rightarrowtail X$, there exists a unique function $\chi : X \to 2$ such that j and the unique function $A \to 1$ form the pullback of $\top$ and χ.*

Axiom 1.17 (Natural Numbers Set). *There exists a set $\mathbb{N}$, an element $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \to \mathbb{N}$ such that, for any set A, element $a \in A$ and function $f : A \to A$, there exists a unique function $r : \mathbb{N} \to A$ such that $r(0) = a$ and $f \circ r = r \circ s$.*

Axiom 1.18 (Choice). *Every surjective function has a section.*

1.4 Isomorphisms

Definition 1.19 (Isomorphism). Let $f : A \to B$. Then f is an *isomorphism* or *bijection*, $f : A \cong B$, iff there exists a function $f^{-1} : B \to A$, the *inverse* of f, such that $f^{-1}f = \mathrm{id}_A$ and $ff^{-1} = \mathrm{id}_B$.

1.5 Subsets

Definition 1.20 (Subset). Let $i : U \to A$. Then we say that (U, i) is a *subset* of A iff i is injective.

Definition 1.21. Let (U, i) and (V, j) be subsets of A. Then we say (U, i) and (V, j) are *equal*, and write $(U, i) = (V, j)$, iff there exists an isomorphism $\phi : U \cong V$ such that $j\phi = i$.

1.6 Intersections

Definition 1.22 (Intersection). Let (U, i) and (V, j) be subsets of a set A. Let $p : W \to U$ and $q : W \to V$ form the pullback of i under j. Then the *intersection* of (U, i) and (V, j) is defined to be $(W, ip) = (W, jq)$.

1.7 Pullbacks

1.8 Functions

Proposition 1.23. *Let $f : A \to B$. Then f is injective if and only if, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.*

PROOF:
$\langle 1 \rangle 1$. If f is injective then, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

PROOF: Immediate from the definition of injective.
⟨1⟩2. If $\forall x, y \in A.f(x) = f(y) \Rightarrow x = y$ then f is injective.
⟨2⟩1. ASSUME: $\forall x, y \in A.f(x) = f(y) \Rightarrow x = y$
⟨2⟩2. LET: X be a set and $s, t : X \to A$
⟨2⟩3. ASSUME: $fs = ft$
⟨2⟩4. $\forall x \in X.s(x) = t(x)$
⟨3⟩1. LET: $x \in X$
⟨3⟩2. $f(s(x)) = f(t(x))$
PROOF: ⟨2⟩3
⟨3⟩3. $s(x) = t(x)$
PROOF: ⟨2⟩1
⟨2⟩5. $s = t$
PROOF: Axiom of Extensionality
□

1.9 The Internal Logic

Proposition 1.24. *Let $i : U \rightarrowtail A$ be injective. Let $\chi : A \to 2$ be its characteristic function. Then, for all $a \in A$, we have $\chi(a) = \top$ if and only if there exists $u \in U$ such that $i(u) = a$.*

PROOF:
⟨1⟩1. If $\chi(a) = \top$ then there exists $u \in U$ such that $i(u) = a$.
PROOF: If $\chi \circ a = \top = \top \circ !_1$ then there exists a unique $u : 1 \to U$ such that $i \circ u = a$ and $!_U \circ u = !_1$.
⟨1⟩2. For all $u \in U$ we have $\chi(i(u)) = \top$.
PROOF: Since $\chi \circ i = \top \circ !_U$.
□

Proposition 1.25. *Subsets of a set A are equal if and only if they have the same characteristic function.*

PROOF: Follows from the fact that pullbacks are unique up to isomorphism and the uniqueness of the characteristic function. □

Proposition 1.26. *There are exactly two subsets of 1.*

PROOF:
⟨1⟩1. PICK a set E with no elements.
⟨1⟩2. $!_E : E \to 1$ is injective.
PROOF: Vacuously, $\forall x, y \in E.!_E(x) = !_E(y) \Rightarrow x = y$.
⟨1⟩3. $(E, !_E) \neq (1, \mathrm{id}_1)$
PROOF: Since there cannot be an isomorphism $1 \cong E$.
⟨1⟩4. For any subsets (U, i) and (V, j) of 1, if $(U, i) \neq (U, i) \cap (V, j)$ then $(U, i) = (1, \mathrm{id}_1)$
⟨2⟩1. LET: (U, i) and (V, j) be subsets of 1.
⟨2⟩2. LET: $p : W \to U$ and $q : W \to V$ form the intersection of (U, i) and (V, j)

⟨2⟩3. ASSUME: $(U, i) \neq (W, k)$
⟨2⟩4. LET: $(U, \mathrm{id}_U) \neq (W, p)$ as subsets of U.
⟨2⟩5. LET: $\chi_U, \chi_W : U \to 2$ be the characteristic functions of (U, id_U) and (W, p) respectively.
⟨2⟩6. $\chi_U \neq \chi_W$
⟨2⟩7. PICK $x \in U$
PROOF: By the Axiom of Extensionality, there exists $x \in U$ such that $\chi_U(x) \neq \chi_W(x)$.
⟨2⟩8. $ix = \mathrm{id}_1$
⟨2⟩9. $x : 1 \cong U$
⟨2⟩10. $(U, i) = (1, \mathrm{id}_1)$
⟨1⟩5. For any subset (U, i) of 1, either $(U, i) = (E, !_E)$ or $(U, i) = (1, \mathrm{id}_1)$.
⟨2⟩1. LET: (U, i) be a subset of 1.
⟨2⟩2. ASSUME: $(U, i) \neq (E, !_E)$
⟨2⟩3. $(U, i) \neq (U, i) \cap (E, !_E)$ or $(E, !_E) \neq (U, i) \cap (E, !_E)$
⟨2⟩4. $(U, i) = (1, \mathrm{id}_1)$ or $(E, !_E) = (1, \mathrm{id}_1)$
PROOF: ⟨1⟩4
⟨2⟩5. $(U, i) = (1, \mathrm{id}_1)$
PROOF: ⟨1⟩3
□

Corollary 1.26.1. *There are exactly two elements of 2.*

Definition 1.27 (Falsehood)**.** Let *falsehood* $\bot$ be the element of 2 that is not $\top$.

Corollary 1.27.1. *2 is the coproduct of 1 and 1 with injections $\top$ and $\bot$.*

1.10 Functions

Proposition 1.28. *Let $f : A \to B$, $g : B \to C$ and $a \in A$. Then*

$$(g \circ f)(a) = g(f(a)) \ .$$

PROOF: Immediate from the Axiom of Associativity. □

Proposition 1.29. *For any set A, any function $1 \to A$ is injective.*

PROOF: Since there is only one function $X \to 1$ for any set X. □

Proposition 1.30. *Let $f : A \to B$. Then the following are equivalent:*

1. *f is surjective.*
2. *f is a retraction (i.e. f has a section).*
3. *For any set X and functions $x, y : B \to X$, if $xf = yf$ then $x = y$.*

PROOF:

$\langle 1\rangle$1. $1 \Rightarrow 2$
PROOF: Immediate from the Axiom of Choice.
$\langle 1\rangle$2. $2 \Rightarrow 3$
$\langle 2\rangle$1. LET: $s : B \to A$ be a section of f.
$\langle 2\rangle$2. LET: X be a set and $x, y : B \to X$ satisfy $xf = yf$.
$\langle 2\rangle$3. $x = y$
PROOF: $x = xfs = yfs = y$
$\langle 1\rangle$3. $3 \Rightarrow 1$
$\langle 2\rangle$1. ASSUME: 3
$\langle 2\rangle$2. LET: $b \in B$
$\langle 2\rangle$3. ASSUME: for a contradiction $\forall a \in A.f(a) \neq b$
$\langle 2\rangle$4. LET: $\psi_1 : B \to 2$ be the characteristic function of b.
$\langle 2\rangle$5. LET: $\psi_2 = \bot \circ !_B : B \to 2$
$\langle 2\rangle$6. $\forall x \in A.\psi_1(f(x)) = \psi_2(f(x))$
$\langle 3\rangle$1. LET: $x \in A$
$\langle 3\rangle$2. $\psi_1(f(x)) \neq \top$
PROOF: Proposition 1.24, $\langle 2\rangle$3, $\langle 2\rangle$4.
$\langle 3\rangle$3. $\psi_1(f(x)) = \bot$
$\langle 3\rangle$4. $\psi_1(f(x)) = \psi_2(f(x))$
$\langle 2\rangle$7. $\psi_1 \circ f = \psi_2 \circ f$
PROOF: Axiom of Extensionality
$\langle 2\rangle$8. $\psi_1 = \psi_2$
PROOF: $\langle 2\rangle$1
$\langle 2\rangle$9. $\psi_1(b) \neq \psi_2(b)$
PROOF: Since $\psi_1(b) = \top$ and $\psi_2(b) = \bot$.
$\langle 2\rangle$10. Q.E.D.
PROOF: This is a contradiction
□

Corollary 1.30.1. *A function is bijective iff it is injective and surjective.*

1.11 Equalizers

Theorem 1.31. *Any two functions $f, g : A \to B$ have an equalizer.*

PROOF: Take the inverse image of $\delta_B = \langle \mathrm{id}_B, \mathrm{id}_B \rangle : B \rightarrowtail B^2$ and $\langle f, g \rangle : A \to B^2$. □

1.12 The Empty Set

Theorem 1.32. *If E is a set with no elements, then E has no proper subsets.*

PROOF: A proper subset of E would give a proper subset of 1 that is different from $(E, !_E)$. □

Theorem 1.33. *If E is a set with no elements, then for any set X there exists exactly one function $E \to X$.*

PROOF:
⟨1⟩1. LET: E be a set with no elements.
⟨1⟩2. LET: X be a set.
⟨1⟩3. There exists a function $E \to X$.
⟨2⟩1. LET: $t : 1 \to 2^X$ be the name of the characteristic function of $\mathrm{id}_X : X \to X$.
⟨2⟩2. LET: $\sigma : X \to 2^X$ be the lambda of the characteristic function of $\delta = \langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \to X \times X$.
⟨2⟩3. LET: $p : P \to E$ and $q : P \to X$ be the pullback of $t \circ !_E$ and σ.
PROOF: $t \circ !_E$ is vacuously injective.
⟨2⟩4. p is injective.
PROOF: It is the pullback of the injective function σ.
⟨2⟩5. p is bijective.
⟨2⟩6. $q \circ p^{-1} : E \to X$
⟨1⟩4. For any functions $f, g : E \to X$ we have $f = g$.
⟨2⟩1. LET: $f, g : E \to X$
⟨2⟩2. LET: $m : M \to E$ be the pullback of f and g.
⟨2⟩3. $(M, m) = (E, \mathrm{id}_E)$
PROOF: Since E has no proper subsets.
⟨2⟩4. $m : M \cong E$
⟨2⟩5. $f = g$
□

Corollary 1.33.1. *If E and E' are sets with no elements then there exists a unique isomorphism $E \cong E'$.*

Definition 1.34 (Empty Set)**.** Let the *empty set* $\emptyset$ be the set with no elements.

Theorem 1.35. *For any set A, if there exists a function $A \to \emptyset$ then $A \cong \emptyset$.*

PROOF: If $f : A \to \emptyset$ then A has no elements, because for any $a \in A$ we have $f(a) \in \emptyset$. □

1.13 Universal Quantification

Definition 1.36. For any set A, let $t_A : 1 \to 2^A$ be the name of the characteristic function of $\top \circ !_A : A \to 2$. Define *universal quantification* $\forall_A : 2^A \to 2$ to be the characteristic function of t_A.

1.14 Intersection

Theorem 1.37. *Let X be a set. There exists a function $\bigcap : 2^{2^X} \to 2^X$ such that, for all $S \in 2^{2^X}$ and $a \in X$, we have*

$$\epsilon(\bigcap S, a) = \top \Leftrightarrow \forall A \in 2^X . (\epsilon(S, A) = \top \Rightarrow \epsilon(A, a) = \top)$$

PROOF:

⟨1⟩1. Let: X be a set.
⟨1⟩2. Let: $\phi_2 : X \to 2^{2^X}$ be the lambda of $\epsilon : 2^X \times X \to 2$
⟨1⟩3. Let: F be the function

$$2^{2^X} \times X \xrightarrow{\langle \mathrm{id}_{2^{2^X}}, \phi_2 \rangle} 2^{2^X} \times 2^{2^X} \xrightarrow{\cong} (2 \times 2)^{2^X} \xrightarrow{\Rightarrow} 2^{2^X} \xrightarrow{\forall} 2$$

⟨1⟩4. Let: $\bigcap$ be the lambda
□

1.15 Union

Theorem 1.38. *Any two subsets of a set have a union.*

Proof:
⟨1⟩1. Let: A and B be subsets of X
⟨1⟩2. Let: $\chi_A \in 2^X$ be the name of the characteristic function of A.
⟨1⟩3. Let: $t_X \in 2^X$ be the name of $\top \circ !_X : X \to 2$
⟨1⟩4. Let: C be the pullback of t_X and $\chi_A \Rightarrow - : 2^X \to 2^X$
⟨1⟩5. Let: D be the pullback of t_X and $\chi_B \Rightarrow -$
⟨1⟩6. $\bigcap(C \cap D)$ is the union of A and B.
□

Theorem 1.39. *Any two sets have a coproduct.*

Proof:
⟨1⟩1. Let: X and Y be sets.
⟨1⟩2. Let: $\sigma_X : X \to 2^X$ be the lambda of the characteristic function of $\langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \to X \times X$
⟨1⟩3. Let: $\chi_0 : 1 \to Y$ be the characteristic function of the unique function $\varnothing \to Y$
⟨1⟩4. Let: $i_X = \langle \sigma_X, \chi_0 \circ !_X \rangle : X \to 2^X \times 2^Y$
⟨1⟩5. Let: $i_Y : Y \to 2^X \times 2^Y$ be defined similarly.
⟨1⟩6. i_X and i_Y are monic.
⟨1⟩7. $\varnothing$ is the pullback of i_X and i_Y (i.e. $(X, i_X) \cap (Y, i_Y) = \varnothing$).
⟨1⟩8. Let: $j : Z \to 2^X \times 2^Y$ be the union of i_X and i_Y
⟨1⟩9. Z is the coproduct of X and Y.
□