Summary of Halmos' Naive Set Theory

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August 20, 2023

Contents

Chapter 1

Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*, $\in$. If $x \in A$ we say that x *belongs* to A, x is an *element* of A, or x is *contained* in A. If this does not hold we write $x \notin A$.

Axiom 1.1 (Axiom of Extensionality)**.** *Two sets are equal if and only if they have the same elements.*

Axiom 1.2 (Axiom of Comprehension, Aussonderungsaxiom)**.** *To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

Axiom 1.3. *A set exists.*

Axiom 1.4 (Axiom of Pairing)**.** *For any two sets, there exists a set that they both belong to.*

Axiom 1.5 (Union Axiom)**.** *For every set A, there exists a set that contains all the elements that belong to at least one element of A.*

Chapter 2

The Subset Relation

Definition 2.1 (Subset)**.** Let A and B be sets. We say that A is a *subset* of B, or B *includes* A, and write $A \subseteq B$ or $B \supseteq A$, iff every element of A is an element of B.

Theorem 2.2. *For any set A, we have $A \subseteq A$.*

PROOF: Every element of A is an element of A. □

Theorem 2.3. *For any sets A, B and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

PROOF: If every element of A is an element of B, and every element of B is an element of C, then every element of A is an element of C. □

Theorem 2.4. *For any sets A and B, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

PROOF: If every element of A is an element of B, and every element of B is an element of A, then A and B have the same elements, and therefore are equal by the Axiom of Extensionality. □

Definition 2.5 (Proper Subset)**.** Let A and B be sets. We say that A is a *proper* subset of B, or B *properly* includes A, and write $A \subsetneq B$ or $B \supsetneq A$, iff $A \subseteq B$ and $A \neq B$.

Chapter 3

Comprehension Notation

Definition 3.1. Given a set A and a condition $S(x)$, we write $\{x \in A : S(x)\}$ for the set whose elements are exactly those elements x of A for which $S(x)$ holds.

PROOF: This exists by the Axiom of Comprehension and is unique by the Axiom of Extensionality. $\square$

Theorem 3.2. *There is no set that contains every set.*

PROOF:
$\langle 1 \rangle 1.$ LET: A be a set.
PROVE: There exists a set B such that $B \notin A$.
$\langle 1 \rangle 2.$ LET: $B = \{x \in A : x \notin x\}$
$\langle 1 \rangle 3.$ If $B \in A$ then we have $B \in B$ if and only if $B \notin B$.
$\langle 1 \rangle 4.$ $B \notin A$
$\square$

Chapter 4

Unordered Pairs

Theorem 4.1. *There exists a set with no elements.*

PROOF: Pick a set A by Axiom 1.3. Then the set $\{x \in A : x \neq x\}$ has no elements. □

Definition 4.2 (Empty Set)**.** The *empty set* $\emptyset$ is the set with no elements.

Theorem 4.3. *For any set A we have $\emptyset \subset A$.*

PROOF: Vacuous. □

Definition 4.4 ((Unordered) Pair)**.** For any sets a and b, the *(unordered) pair* $\{a, b\}$ is the set whose elements are just a and b.

PROOF: This exists by the Axioms of Pairing and Comprehension, and is unique by the Axiom of Extensionality. □

Definition 4.5 (Singleton)**.** For any set a, the *singleton* $\{a\}$ is defined to be $\{a, a\}$.

Chapter 5

Unions and Intersections

Definition 5.1 (Union)**.** For any set $\mathcal{C}$, the *union* of $\mathcal{C}$, $\bigcup \mathcal{C}$, is the set whose elements are the elements of the elements of $\mathcal{C}$.

Proof: This exists by the Union Axiom and Comprehension Axiom, and is unique by the Axiom of Extensionality. □

Proposition 5.2.

$$\bigcup \emptyset = \emptyset$$

Proof: There is no set that is an element of an element of $\emptyset$. □

Proposition 5.3. *For any set A, we have $\bigcup \{A\} = A$.*

Proof: For any x, we have x is an element of an element of $\{A\}$ if and only if x is an element of A. □

Definition 5.4. We write $A \cup B$ for $\bigcup \{A, B\}$.