Mathematics

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Chapter 1

Sets and Classes

1.1 Classes

Our language is the language of first-order logic with equality over one primitive binary predicate $\in$. We call all the objects we reason about *sets*. When $a \in b$, we say a is a *member* or *element* of b, or b *contains* a. We write $b \ni a$ for $a \in b$, and $a \notin b$ for $\neg(a \in b)$. We write $\forall x \in a.\phi$ as an abbreviation for $\forall x(x \in a \rightarrow \phi)$, and $\exists x \in a.\phi$ as an abbreviation for $\exists x(x \in a \wedge \phi)$.

We shall speak informally of *classes* as an abbreviation for talking about predicates. A *class* is determined by a unary predicate $\phi[x]$ (possibly with parameters). We write $\{x \mid \phi[x]\}$ or $\{x : \phi[x]\}$ for the class determined by $\phi[x]$. We write 'a is an element of $\{x \mid \phi[x]\}$' or '$a \in \{x \mid \phi[x]\}$' for $\phi[a]$.

We write $\{t[x_1, \ldots, x_n] \mid P[x_1, \ldots, x_n]\}$ for

$$\{y \mid \exists x_1, \ldots, x_n(y = t[x_1, \ldots, x_n] \wedge P[x_1, \ldots, x_n])\} \ .$$

We say two classes $\mathbf{A}$ and $\mathbf{B}$ are *equal*, and write $\mathbf{A} = \mathbf{B}$, iff $\forall x(x \in \mathbf{A} \leftrightarrow x \in \mathbf{B})$.

Proposition Schema 1.1.1. *For any class* $\mathbf{A}$*, the following is a theorem.*

$$\mathbf{A} = \mathbf{A}$$

PROOF: We have $\forall x(x \in \mathbf{A} \Leftrightarrow x \in \mathbf{A})$. $\square$

Proposition Schema 1.1.2. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, the following is a theorem.*

If $\mathbf{A} = \mathbf{B}$ *then* $\mathbf{B} = \mathbf{A}$.

PROOF: If $\forall x(x \in \mathbf{A} \Leftrightarrow x \in \mathbf{B})$ then $\forall x(x \in \mathbf{B} \Leftrightarrow x \in \mathbf{A})$. $\square$

Proposition Schema 1.1.3. *For any classes* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$*, the following is a theorem.*

If $\mathbf{A} = \mathbf{B}$ *and* $\mathbf{B} = \mathbf{C}$ *then* $\mathbf{A} = \mathbf{C}$.

PROOF: If $\forall x(x \in \mathbf{A} \Leftrightarrow x \in \mathbf{B})$ and $\forall x(x \in \mathbf{B} \Leftrightarrow x \in \mathbf{C})$ then $\forall x(x \in \mathbf{A} \Leftrightarrow x \in \mathbf{C})$. $\square$

1.1.1 Subclasses

Definition 1.1.4 (Subclass)**.** We say a class $\mathbf{A}$ is a *subclass* of $\mathbf{B}$, or $\mathbf{B}$ is a *superclass* of $\mathbf{A}$, or $\mathbf{B}$ *includes* $\mathbf{A}$, and write $\mathbf{A} \subseteq \mathbf{B}$ or $\mathbf{B} \supseteq \mathbf{A}$, iff every element of $\mathbf{A}$ is an element of $\mathbf{B}$. Otherwise we write $\mathbf{A} \nsubseteq \mathbf{B}$ or $\mathbf{B} \nsupseteq \mathbf{A}$.

We say $\mathbf{A}$ is a *proper* subclass of $\mathbf{B}$, $\mathbf{B}$ is a *proper* superclass of $\mathbf{A}$, or $\mathbf{B}$ *properly* includes $\mathbf{A}$, and write $\mathbf{A} \subsetneq \mathbf{B}$ or $\mathbf{B} \supsetneq \mathbf{A}$, iff $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$.

Proposition Schema 1.1.5. *For any class* $\mathbf{A}$*, the following is a theorem.*

$$\mathbf{A} \subseteq \mathbf{A}$$

Proof: Every element of $\mathbf{A}$ is an element of $\mathbf{A}$. $\square$

Proposition Schema 1.1.6. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, the following is a theorem.*

If $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{A}$ *then* $\mathbf{A} = \mathbf{B}$*.*

Proof: If every element of $\mathbf{A}$ is an element of $\mathbf{B}$, and every element of $\mathbf{B}$ is an element of $\mathbf{A}$, then $\mathbf{A}$ and $\mathbf{B}$ have exactly the same elements. $\square$

Proposition Schema 1.1.7. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, the following is a theorem.*

If $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{C}$ *then* $\mathbf{A} \subseteq \mathbf{C}$*.*

Proof: If every element of $\mathbf{A}$ is an element of $\mathbf{B}$ and every element of $\mathbf{B}$ is an element of $\mathbf{C}$ then every element of $\mathbf{A}$ is an element of $\mathbf{C}$. $\square$

1.1.2 Constructions of Classes

Definition 1.1.8 (Empty Class)**.** The *empty class* $\emptyset$ is $\{x \mid \bot\}$. Every other class is *nonempty*.

Definition 1.1.9 (Universal Class)**.** The *universal class* $\mathbf{V}$ is $\{x \mid \top\}$.

Definition 1.1.10 (Enumeration)**.** Given objects a_1, …, a_n, we define the class $\{a_1, \ldots, a_n\}$ to be the class $\{x \mid x = a_1 \vee \cdots \vee x = a_n\}$.

Definition 1.1.11 (Intersection)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *intersection* $\mathbf{A} \cap \mathbf{B}$ is $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$.

Definition 1.1.12 (Union)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *union* $\mathbf{A} \cup \mathbf{B}$ is $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$.

Definition 1.1.13 (Relative Complement)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be classes. The *relative complement* of $\mathbf{B}$ in $\mathbf{A}$ is the class $\mathbf{A} - \mathbf{B} := \{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

Definition 1.1.14 (Symmetric Difference)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *symmetric difference* is the class $\mathbf{A} + \mathbf{B} := (\mathbf{A} - \mathbf{B}) \cup (\mathbf{B} - \mathbf{A})$.

Definition 1.1.15 (Pairwise disjoint)**.** Let $\mathbf{A}$ be a class. We say the elements of $\mathbf{A}$ are *pairwise disjoint* iff, for all $x, y \in \mathbf{A}$, if $x \cap y \neq \emptyset$ then $x = y$.

1.2 Sets and the Axiom of Extensionality

Definition 1.2.1 (Axiom of Extensionality)**.** The *Axiom of Extensionality* is the statement: if two sets have exactly the same members, then they are equal.

$$\forall x, y(\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y) \ .$$

When working in a theory with the Axiom of Extensionality, we may identify a set a with the class $\{x \mid x \in a\}$. Our use of the symbols $\in$ and $=$ is consistent. We say a class $\mathbf{A}$ *is a set* iff there exists a set a such that $a = \mathbf{A}$; that is, $\{x \mid \phi[x]\}$ is a set iff $\exists a \forall x(x \in a \leftrightarrow \phi[x])$. Otherwise, $\mathbf{A}$ is a *proper class.*

Definition 1.2.2 (Subset)**.** If A is a set and $A \subseteq \mathbf{B}$, we say A is a *subset* of $\mathbf{B}$.

Definition 1.2.3 (Union)**.** The *union* of a class $\mathbf{A}$ is $\{x \mid \exists X \in \mathbf{A}.x \in X\}$. We write $\bigcup_{P(x)} t(x)$ for $\bigcup\{t(x) \mid P(x)\}$.

Definition 1.2.4 (Intersection)**.** The *intersection* of a class $\mathbf{A}$ is $\{x \mid \forall X \in \mathbf{A}.x \in X\}$. We write $\bigcap_{P(x)} t(x)$ for $\bigcap\{t(x) \mid P(x)\}$.

Definition 1.2.5 (Power Class)**.** For any class $\mathbf{A}$, the *power class* $\mathcal{P}\mathbf{A}$ is $\{X \mid X \subseteq \mathbf{A}\}$.

1.3 The Other Axioms

Definition 1.3.1 (Pairing Axiom)**.** The *Pairing Axiom* is the statement: for any sets a and b, the class $\{a, b\}$ is a set.

$$\forall a \forall b \exists c \forall x(x \in c \Leftrightarrow x = a \vee x = b)$$

Definition 1.3.2 (Union Axiom)**.** The *Union Axiom* is the statement: for any set A, the class $\bigcup A$ is a set.

$$\forall A \exists B \forall x(x \in B \Leftrightarrow \exists y(y \in A \wedge x \in y))$$

Definition 1.3.3 (Comprehension Axiom Scheme)**.** The *Comprehension Axiom Scheme* is the set of sentences of the form, for any class $\mathbf{A}$: If $\mathbf{A}$ is a subclass of a set then $\mathbf{A}$ is a set.

That is, for any property $P[x, y_1, \ldots, y_n]$:

For any sets a_1, …, a_n and B, the class $\{x \in B \mid P[x, a_1, \ldots, a_n]\}$ is a set.

$$\forall a_1, \ldots, a_n, B.\exists C.\forall x(x \in C \Leftrightarrow x \in B \wedge P[x, a_1, \ldots, a_n])$$

Definition 1.3.4 (Replacement Axiom Scheme)**.** The *Replacement Axiom Scheme* is the set of sentences of the form, for some property $P[x, y, z_1, \ldots, z_n]$:

For any sets a_1, …, a_n, B, assume for all $x \in B$ there exists at most one y such that $P[x, y, a_1, \ldots, a_n]$. Then $\{y \mid \exists x \in B.P[x, y, a_1, \ldots, a_n]$ is a set.

$$\forall a_1, \ldots, a_n, B(\forall x \in B.\forall y, y'(P[x, y, a_1, \ldots, a_n] \wedge P[x, y', a_1, \ldots, a_n] \Rightarrow y = y') \Rightarrow$$
$$\exists C \forall y(y \in C \Leftrightarrow \exists x \in B.P[x, y, a_1, \ldots, a_n]))$$

Definition 1.3.5 (Power Set Axiom)**.** The *Power Set Axiom* is the statement: the power class of a set is a set.

$$\forall A \exists B \forall x (x \in B \Leftrightarrow \forall y (y \in x \Rightarrow y \in A))$$

Definition 1.3.6 (Axiom of Infinity)**.** The *Axiom of Infinity* is the statement: there exists a set I such that $\emptyset \in I$ and $\forall x \in I . x \cup \{x\} \in I$.

$$\exists I (\exists e \in I . \forall x . x \notin e \wedge \forall x \in I . \exists y \in I . \forall z (z \in y \Leftrightarrow z \in x \vee z = x))$$

Definition 1.3.7 (Axiom of Choice)**.** The *Axiom of Choice* is the statement: For any set A of pairwise disjoint, nonempty sets, there exists a set C such that, for all $x \in A$, we have $x \cap C$ has exactly one element.

$$\begin{aligned}&\forall A (\forall x \in A . \exists y y \in x \wedge \\ &\forall x, y \in A . \forall z (z \in x \wedge z \in y \Rightarrow x = y) \Rightarrow \\ &\exists C . \forall x \in A . \exists y \forall z (z \in x \wedge z \in C \Leftrightarrow z = y))\end{aligned}$$

Definition 1.3.8 (Axiom of Regularity)**.** The *Axiom of Regularity* is the statement: for any A, if A has a member, then there exists $m \in A$ such that $m \cap A = \emptyset$.

$$\forall A (\exists x . x \in A \Rightarrow \exists m \in A . \neg \exists x (x \in m \wedge x \in A))$$

Definition 1.3.9 (Zermelo Set Theory)**.** *Zermelo set theory* is the theory whose axioms are:

- Extensionality
- Pairing
- Union
- Comprehension
- Power Set
- Infinity
- Choice
- Regularity

We label theorems with Z when they are provable in Zermelo set theory.

Definition 1.3.10 (Zermelo-Fraenkel Set Theory)**.** *Zermelo-Fraenkel set theory* is the theory whose axioms are:

- Extensionality
- Union

- Replacement
- Power Set
- Infinity
- Choice
- Regularity

We label theorems with ZFC when they are provable in Zermelo-Fraenkel set theory.

We label a theorem with FOL if it can be proved in first-order logic, i.e. from no axioms.

1.4 ZFC Extends Z

Proposition 1.4.1 (Z,ZFC). *The empty class $\emptyset$ is a set.*

PROOF: Immediate from the Axiom of Infinity. □

Proposition 1.4.2 (ZFC). *The Axiom of Pairing is a theorem of ZFC.*

PROOF:
$\langle 1\rangle 1$. LET: a, b be sets.
$\langle 1\rangle 2$. LET: $P(x,y)$ be the predicate $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$.
$\langle 1\rangle 3$. For all $x \in \mathcal{P}\mathcal{P}\emptyset$, there exists at most one y such that $P(x,y)$.
  $\langle 2\rangle 1$. LET: $x \in \mathcal{P}\mathcal{P}\emptyset$
  $\langle 2\rangle 2$. LET: y and y' be sets.
  $\langle 2\rangle 3$. ASSUME: $P(x,y)$ and $P(x,y')$
  $\langle 2\rangle 4$. $(x = \emptyset \wedge y = a) \vee (x = \mathcal{P}\emptyset \wedge y = b)$
    PROOF: From $\langle 2\rangle 3$.
  $\langle 2\rangle 5$. $(x = \emptyset \wedge y' = a) \vee (x = \mathcal{P}\emptyset \wedge y' = b)$
    PROOF: From $\langle 2\rangle 3$.
  $\langle 2\rangle 6$. $\emptyset \neq \mathcal{P}\emptyset$
    PROOF: Since $\emptyset \in \mathcal{P}\emptyset$ and $\emptyset \notin \emptyset$.
  $\langle 2\rangle 7$. $y = y'$
$\langle 1\rangle 4$. LET: A be the set $\{y \mid \exists x \in \mathcal{P}\mathcal{P}\emptyset . P(x,y)\}$.
$\langle 1\rangle 5$. $A = \{a, b\}$
□

Proposition Schema 1.4.3 (ZFC). *Every instance of the Comprehension Axiom Scheme is a theorem of ZFC.*

PROOF:
$\langle 1\rangle 1$. LET: $P(x)$ be a predicate.
$\langle 1\rangle 2$. LET: A be a set.
$\langle 1\rangle 3$. LET: $Q(x,y)$ be the predicate $P(x) \wedge y = x$.

$\langle 1\rangle 4.$ For all $x \in A$, there exists at most one y such that $Q(x, y)$.
$\langle 2\rangle 1.$ LET: $x \in A$
$\langle 2\rangle 2.$ LET: y and y' be sets.
$\langle 2\rangle 3.$ ASSUME: $Q(x, y)$ and $Q(x, y')$
$\langle 2\rangle 4.$ $x \in A \wedge P(x) \wedge y = x \wedge y' = x$
PROOF: From $\langle 2\rangle 3$.
$\langle 2\rangle 5.$ $y = y'$
PROOF: From $\langle 2\rangle 4$.
$\langle 1\rangle 5.$ LET: B be the set $\{y \mid \exists x \in A.Q(x, y)\}$
PROOF: This is a set by an Axiom of Replacement and $\langle 1\rangle 4$.
$\langle 1\rangle 6.$ $B = \{y \in A \mid P(y)\}$
PROOF:

$$\begin{aligned} y \in B &\Leftrightarrow \exists x \in A.Q(x, y) && (\langle 1\rangle 5) \\ &\Leftrightarrow \exists x \in A(P(x) \wedge y = x) && (\langle 1\rangle 3) \\ &\Leftrightarrow P(y) \end{aligned}$$

□

Corollary Schema 1.4.3.1 (ZFC). *Every axiom of Z is a theorem of ZFC.*

It follows that every theorem of Z is a theorem of ZFC.

1.5 Consequences of the Axioms

Proposition 1.5.1 (Z). *The union of two sets is a set.*

PROOF: Because $A \cup B = \bigcup\{A, B\}$. □

Proposition Schema 1.5.2 (Z). *For any number n, the following is a theorem: For any sets $a_1, \ldots, a_n$, the class $\{a_1, \ldots, a_n\} = \{x \mid x = a_1 \vee \cdots \vee x = a_n\}$ is a set.*

PROOF: The case $n = 1$ follows from Pairing since $\{a\} = \{a, a\}$.
If we have proved the theorem for n we have $\{a_1, \ldots, a_n, a_{n+1}\} = \{a_1, \ldots, a_n\} \cup \{a_{n+1}\}$. □

Proposition 1.5.3 (Z). *No set is a member of itself.*

PROOF:
$\langle 1\rangle 1.$ LET: x be any set.
$\langle 1\rangle 2.$ PICK $m \in \{x\}$ such that $m \cap \{x\} = \emptyset$.
PROOF: Axiom of Regularity.
$\langle 1\rangle 3.$ $m = x$
$\langle 1\rangle 4.$ $x \cap \{x\} = \emptyset$
$\langle 1\rangle 5.$ $x \notin x$
□

Corollary 1.5.3.1 (Z). *The universal class $\mathbf{V}$ is a proper class.*

PROOF: If $\mathbf{V}$ is a set then $\mathbf{V} \in \mathbf{V}$, contradicting the Proposition. □

Proposition 1.5.4 (Z)**.** *There are no sets a and b such that $a \in b$ and $b \in a$.*

PROOF:
$\langle 1 \rangle 1$. LET: a and b be any sets.
$\langle 1 \rangle 2$. PICK $m \in \{a, b\}$ such that $m \cap \{a, b\} = \emptyset$
$\langle 1 \rangle 3$. CASE: $m = a$
PROOF: Then $b \notin a$.
$\langle 1 \rangle 4$. CASE: $m = b$
PROOF: Then $a \notin b$.
□

Proposition 1.5.5 (Z)**.** *The intersection of a set and a class is a set.*

PROOF: Immediate from Comprehension. □

Proposition 1.5.6 (Z)**.** *The relative complement of a class in a set is a set.*

[Z]

PROOF: Immediate from Comprehension. □

Corollary 1.5.6.1 (Z)**.** *The symmetric difference of two sets is a set.*

Proposition 1.5.7 (Z)**.** *The intersection of a nonempty class is a set.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathbf{A}$ be a nonempty class.
$\langle 1 \rangle 2$. PICK $B \in \mathbf{A}$
$\langle 1 \rangle 3$. $\bigcap \mathbf{A} \subseteq B$
$\langle 1 \rangle 4$. $\bigcap \mathbf{A}$ is a set.
PROOF: By Comprehension.
□

Proposition Schema 1.5.8 (FOL)**.** *For any classes $\mathbf{A}$ and $\mathbf{B}$, the following is a theorem:*
If $\mathbf{A} \subseteq \mathbf{B}$ then $\mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathbf{B}$.

PROOF: Every subset of $\mathbf{A}$ is a subset of $\mathbf{B}$. □

Proposition Schema 1.5.9 (FOL)**.** *For any classes $\mathbf{A}$ and $\mathbf{B}$, the following is a theorem:*
If $\mathbf{A} \subseteq \mathbf{B}$ then $\bigcup \mathbf{A} \subseteq \bigcup \mathbf{B}$.

PROOF: If $x \in X \in \mathbf{A}$ then $x \in X \in \mathbf{B}$. □

Proposition Schema 1.5.10 (Z)**.** *For any class $\mathbf{A}$, the following is a theorem:*

$$\mathbf{A} = \bigcup \mathcal{P}\mathbf{A}$$

PROOF:

$\langle 1\rangle 1.$ $\mathbf{A} \subseteq \bigcup \mathcal{P}\mathbf{A}$
  PROOF: For all $x \in \mathbf{A}$ we have $x \in \{x\} \in \mathcal{P}\mathbf{A}$.
$\langle 1\rangle 2.$ $\bigcup \mathcal{P}\mathbf{A} \subseteq \mathbf{A}$
  $\langle 2\rangle 1.$ LET: $x \in \bigcup \mathcal{P}\mathbf{A}$
  $\langle 2\rangle 2.$ PICK $X \in \mathcal{P}\mathbf{A}$ such that $x \in X$
  $\langle 2\rangle 3.$ $X \subseteq \mathbf{A}$
  $\langle 2\rangle 4.$ $x \in \mathbf{A}$
$\square$

1.6 Transitive Classes

Definition 1.6.1 (Transitive Class)**.** A class $\mathbf{A}$ is a *transitive class* iff whenever $x \in y \in \mathbf{A}$ then $x \in \mathbf{A}$.

Proposition Schema 1.6.2 (FOL)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
The following are equivalent.

1. $\mathbf{A}$ *is a transitive class.*
2. $\bigcup \mathbf{A} \subseteq \mathbf{A}$
3. *Every element of* $\mathbf{A}$ *is a subset of* $\mathbf{A}$*.*
4. $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$

PROOF: Immediate from definitions. $\square$

Proposition Schema 1.6.3 (FOL)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
If $\mathbf{A}$ *is a transitive class then* $\bigcup \mathbf{A}$ *is a transitive class.*

PROOF:
$\langle 1\rangle 1.$ ASSUME: $\mathbf{A}$ is a transitive class.
$\langle 1\rangle 2.$ LET: $x \in y \in \bigcup \mathbf{A}$
$\langle 1\rangle 3.$ $y \in \mathbf{A}$
  PROOF: Since $\bigcup \mathbf{A} \subseteq \mathbf{A}$ by Proposition 1.6.2.
$\langle 1\rangle 4.$ $x \in \bigcup \mathbf{A}$
$\square$

Proposition Schema 1.6.4 (Z)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
We have $\mathbf{A}$ *is a transitive class if and only if* $\mathcal{P}\mathbf{A}$ *is a transitive class.*

PROOF:
$\langle 1\rangle 1.$ If $\mathbf{A}$ is a transitive class then $\mathcal{P}\mathbf{A}$ is a transitive class.
  $\langle 2\rangle 1.$ ASSUME: $\mathbf{A}$ is a transitive class.
  $\langle 2\rangle 2.$ $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$
    PROOF: Proposition 1.6.2.
  $\langle 2\rangle 3.$ $\mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathcal{P}\mathbf{A}$

PROOF: Proposition 1.5.8.
⟨2⟩4. $\mathcal{P}\mathbf{A}$ is a transitive class.
PROOF: Proposition 1.6.2.
⟨1⟩2. If $\mathcal{P}\mathbf{A}$ is a transitive class then $\mathbf{A}$ is a transitive class.
⟨2⟩1. ASSUME: $\mathcal{P}\mathbf{A}$ is a transitive class.
⟨2⟩2. $\bigcup \mathcal{P}\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$
PROOF: Proposition 1.6.2.
⟨2⟩3. $\mathbf{A} \subseteq \mathcal{P}\mathbf{A}$
PROOF: Proposition 1.5.10.
⟨2⟩4. $\mathbf{A}$ is a transitive class.
PROOF: Proposition 1.6.2.
□

Proposition Schema 1.6.5 (FOL). *For any class* $\mathbf{A}$*, the following is a theorem:*
If every member of $\mathbf{A}$ *is a transitive set then* $\bigcup \mathbf{A}$ *is a transitive class.*

PROOF:
⟨1⟩1. ASSUME: Every member of $\mathbf{A}$ is a transitive set.
⟨1⟩2. LET: $x \in y \in \bigcup \mathbf{A}$
⟨1⟩3. PICK $A \in \mathbf{A}$ such that $y \in A$.
⟨1⟩4. $x \in A$
PROOF: Since A is a transitive set.
⟨1⟩5. $x \in \bigcup \mathbf{A}$
□

Proposition Schema 1.6.6 (FOL). *For any class* $\mathbf{A}$*, the following is a theorem:*
If every member of $\mathbf{A}$ *is a transitive set then* $\bigcap \mathbf{A}$ *is a transitive class.*

PROOF:
⟨1⟩1. ASSUME: Every member of $\mathbf{A}$ is a transitive set.
⟨1⟩2. LET: $x \in y \in \bigcap \mathbf{A}$
PROVE: $x \in \bigcap \mathbf{A}$
⟨1⟩3. LET: $A \in \mathbf{A}$
⟨1⟩4. $y \in A$
⟨1⟩5. $x \in A$
PROOF: Since A is a transitive set.
□

Chapter 2

Relations

2.1 Ordered Pairs

Definition 2.1.1 (Ordered Pair)**.** For any sets a and b, the *ordered pair* (a,b) is defined to be $\{\{a\},\{a,b\}\}$.

Theorem 2.1.2 (Z)**.** *For any sets a, b, c, d, we have $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.*

Proof:
⟨1⟩1. If $(a,b) = (c,d)$ then $a = c$ and $b = d$.
  ⟨2⟩1. Assume: $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$
  ⟨2⟩2. $\bigcap\{\{a\},\{a,b\}\} = \bigcap\{\{c\},\{c,d\}\}$
  ⟨2⟩3. $\{a\} = \{c\}$
  ⟨2⟩4. $a = c$
  ⟨2⟩5. $\bigcup\{\{a\},\{a,b\}\} = \bigcup\{\{c\},\{c,d\}\}$
  ⟨2⟩6. $\{a,b\} = \{c,d\}$
  ⟨2⟩7. $b = c$ or $b = d$
  ⟨2⟩8. $a = d$ or $b = d$
  ⟨2⟩9. If $b = c$ and $a = d$ then $b = d$
    Proof: By ⟨2⟩4.
  ⟨2⟩10. $b = d$
    Proof: From ⟨2⟩7, ⟨2⟩8, ⟨2⟩9.
⟨1⟩2. If $a = c$ and $b = d$ then $(a,b) = (c,d)$.
  Proof: First-order logic.
□

Definition 2.1.3 (Cartesian Product)**.** The *Cartesian product* of classes $\mathbf{A}$ and $\mathbf{B}$ is the class $\mathbf{A} \times \mathbf{B} := \{(x,y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$.

Proposition 2.1.4 (Z)**.** *For any sets A and B, the class $A \times B$ is a set.*

Proof: It is a subset of $\mathcal{P}\mathcal{P}(A \cup B)$. □

Proposition Schema 2.1.5 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, the following is a theorem:*

$$\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$$

PROOF:

$$\begin{aligned}(x, y) \in \mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) &\Leftrightarrow x \in \mathbf{A} \wedge (y \in \mathbf{B} \vee y \in \mathbf{C}) \\ &\Leftrightarrow (x \in \mathbf{A} \wedge y \in \mathbf{B}) \vee (x \in \mathbf{A} \wedge y \in \mathbf{C}) \\ &\Leftrightarrow (x, y) \in (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})\end{aligned}$$

□

Proposition Schema 2.1.6 (Z)**.** *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, the following is a theorem:*

If $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ *and* $\mathbf{A}$ *is nonempty then* $\mathbf{B} = \mathbf{C}$*.*

PROOF:

⟨1⟩1. PICK $a \in \mathbf{A}$

⟨1⟩2. For all x we have $x \in \mathbf{B}$ iff $x \in \mathbf{C}$.

PROOF:

$$\begin{aligned}x \in \mathbf{B} &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{B} \\ &\Leftrightarrow (a, x) \in \mathbf{A} \times \mathbf{C} \\ &\Leftrightarrow x \in \mathbf{C}\end{aligned}$$

□

Proposition Schema 2.1.7 (Z)**.** *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, the following is a theorem:*

$$\mathbf{A} \times \bigcup \mathbf{B} = \{(a, b) \mid \exists Y \in \mathbf{B}.(a \in \mathbf{A} \wedge b \in Y)\}$$

PROOF:

$$\begin{aligned}(x, y) \in A \times \bigcup \mathbf{B} &\Leftrightarrow x \in A \wedge \exists Y \in \mathbf{B}. y \in Y \\ &\Leftrightarrow \exists Y \in \mathbf{B}(x \in A \wedge y \in Y)\end{aligned}$$

□

2.2 Relations

Definition 2.2.1 (Relation)**.** A *relation* $\mathbf{R}$ between classes $\mathbf{A}$ and $\mathbf{B}$ is a subclass of $\mathbf{A} \times \mathbf{B}$.

A *(binary) relation on* $\mathbf{A}$ is a relation between $\mathbf{A}$ and $\mathbf{A}$.

We write $x\mathbf{R}y$ for $(x, y) \in \mathbf{R}$.

2.2.1 Identity Functions

Definition 2.2.2 (Identity Function)**.** For any class $\mathbf{A}$, the *identity function* or *diagonal relation* $\mathrm{id}_{\mathbf{A}}$ on $\mathbf{A}$ is

$$\mathrm{id}_{\mathbf{A}} := \{(x, x) \mid x \in \mathbf{A}\} \ .$$

2.2.2 Inverses

Definition 2.2.3 (Inverse)**.** The *inverse* of a relation $\mathbf{R}$ between $\mathbf{A}$ and $\mathbf{B}$ is the relation $\mathbf{R}^{-1}$ between $\mathbf{B}$ and $\mathbf{A}$ defined by

$$b\mathbf{R}^{-1}a \Leftrightarrow a\mathbf{R}b \ .$$

Proposition Schema 2.2.4 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{R}$*, the following is a theorem:*

If $\mathbf{R}$ *is a relation between* $\mathbf{A}$ *and* $\mathbf{B}$*, we have* $(\mathbf{R}^{-1})^{-1} = \mathbf{R}$*.*

Proof:

$$\begin{aligned} x(\mathbf{R}^{-1})^{-1}y &\Leftrightarrow y\mathbf{R}^{-1}x \\ &\Leftrightarrow x\mathbf{R}y \end{aligned} \qquad \square$$

2.2.3 Composition

Definition 2.2.5 (Composition)**.** Let $\mathbf{R}$ be a relation between $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{S}$ be a relation between $\mathbf{B}$ and $\mathbf{C}$. The *composition* $\mathbf{S} \circ \mathbf{R}$ is the relation between $\mathbf{A}$ and $\mathbf{C}$ defined by

$$a(\mathbf{S} \circ \mathbf{R})c \Leftrightarrow \exists b(a\mathbf{R}b \wedge b\mathbf{S}c) \ .$$

Proposition Schema 2.2.6 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{R}$ *and* $\mathbf{S}$*, the following is a theorem:*

If $\mathbf{R}$ *is a relation between* $\mathbf{A}$ *and* $\mathbf{B}$*, and* $\mathbf{S}$ *is a relation between* $\mathbf{B}$ *and* $\mathbf{C}$*, then*

$$(\mathbf{S} \circ \mathbf{R})^{-1} = \mathbf{R}^{-1} \circ \mathbf{S}^{-1} \ .$$

Proof:

$$\begin{aligned} z(\mathbf{S} \circ \mathbf{R})^{-1}x &\Leftrightarrow x(\mathbf{S} \circ \mathbf{R})z \\ &\Leftrightarrow \exists y.(x\mathbf{R}y \wedge y\mathbf{S}z) \\ &\Leftrightarrow \exists y.(y\mathbf{R}^{-1}x \wedge z\mathbf{S}^{-1}y) \\ &\Leftrightarrow z(\mathbf{R}^{-1} \circ \mathbf{S}^{-1})x \end{aligned} \qquad \square$$

2.2.4 Properties of Relaitons

Definition 2.2.7 (Reflexive)**.** Let $\mathbf{R}$ be a binary relation on $\mathbf{A}$. Then $\mathbf{R}$ is *reflexive* on $\mathbf{A}$ iff $\forall x \in \mathbf{A}.(x, x) \in \mathbf{R}$.

Proposition Schema 2.2.8 (Z)**.** *For any classes* $\mathbf{A}$ *and* $\mathbf{R}$*, the following is a theorem:*

If $\mathbf{R}$ *is a reflexive relation on* $\mathbf{A}$ *then so is* $\mathbf{R}^{-1}$*.*

Proof:

$\langle 1 \rangle 1.$ Let: $x \in \mathbf{A}$

$\langle 1 \rangle 2.$ $x\mathbf{R}x$

Proof: Since $\mathbf{R}$ is reflexive.

$\langle 1 \rangle 3.$ $x\mathbf{R}^{-1}x$
□

Definition 2.2.9 (Irreflexive)**.** A relation $\mathbf{R}$ is *irreflexive* iff there is no x such that $(x, x) \in \mathbf{R}$.

Definition 2.2.10 (Symmetric)**.** A relation $\mathbf{R}$ is *symmetric* iff, whenever $x\mathbf{R}y$, then $y\mathbf{R}x$.

Definition 2.2.11 (Antisymmetric)**.** A relation $\mathbf{R}$ is *antisymmetric* iff, whenever $x\mathbf{R}y$ and $y\mathbf{R}x$, then $x = y$.

Proposition Schema 2.2.12 (Z)**.** *For any classes* $\mathbf{A}$ *and* $\mathbf{R}$*, the following is a theorem:*

If $\mathbf{R}$ *is an antisymmetric relation on* $\mathbf{A}$ *then so is* $\mathbf{R}^{-1}$*.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: $x\mathbf{R}^{-1}y$ and $y\mathbf{R}^{-1}x$
$\langle 1 \rangle 2.$ $y\mathbf{R}x$ and $x\mathbf{R}y$
$\langle 1 \rangle 3.$ $x = y$
PROOF: Since $\mathbf{R}$ is antisymmetric.
□

Definition 2.2.13 (Transitive)**.** A relation $\mathbf{R}$ is *transitive* iff, whenever $x\mathbf{R}y$ and $y\mathbf{R}z$, then $x\mathbf{R}z$.

Proposition Schema 2.2.14 (Z)**.** *For any classes* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{R}$*, the following is a theorem:*

If $\mathbf{R}$ *is a transitive relation between* $\mathbf{A}$ *and* $\mathbf{B}$ *then* $\mathbf{R}^{-1}$ *is transitive.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: $(x, y), (y, z) \in \mathbf{R}^{-1}$
$\langle 1 \rangle 2.$ $(y, x), (z, y) \in \mathbf{R}$
$\langle 1 \rangle 3.$ $(z, x) \in \mathbf{R}$
$\langle 1 \rangle 4.$ $(x, z) \in \mathbf{R}^{-1}$
□

Proposition 2.2.15 (Z)**.** *For any relation R on a set A, there exists a smallest transitive relation on A that includes R.*

PROOF: The relation is $\bigcap \{S \in \mathcal{P}A^2 \mid R \subseteq S, S \text{ is transitive}\}$. □

Definition 2.2.16 (Transitive Closure)**.** For any relation R on a set A, the *transitive closure* of R is the smallest transitive relation that includes R.

Definition 2.2.17 (Minimal)**.** Let $\mathbf{R}$ be a relation on $\mathbf{A}$. An element $m \in \mathbf{A}$ is *minimal* iff there is no $x \in \mathbf{A}$ such that $x\mathbf{R}m$.

Definition 2.2.18 (Maximal)**.** Let $\mathbf{R}$ be a relation on $\mathbf{A}$. An element $m \in \mathbf{A}$ is *maximal* iff there is no $x \in \mathbf{A}$ such that $m\mathbf{R}x$.

2.3 n-ary Relations

Definition Schema 2.3.1. For any sets $a_1, \ldots, a_n$, define the *ordered n-tuple* $(a_1, \ldots, a_n)$ by

$$(a_1) := a_1$$
$$(a_1, \ldots, a_n, a_{n+1}) = ((a_1, \ldots, a_n), a_{n+1})$$

Definition Schema 2.3.2. An *n-ary relation on* $\mathbf{A}$ is a class of ordered n-tuples all of whose components are in $\mathbf{A}$.

2.4 Well Founded Relations

Definition 2.4.1 (Well Founded)**.** A relation $\mathbf{R}$ on a class $\mathbf{A}$ is *well founded* iff:

- for all $a \in A$, the class $\{x \in \mathbf{A} \mid x\mathbf{R}a\}$ is a set;
- every nonempty subset of $\mathbf{A}$ has an $\mathbf{R}$-minimal element.

Proposition 2.4.2 (Z)**.** *For any class $\mathbf{A}$, the relation $\{(x, y) \in \mathbf{A}^2 \mid x \in y\}$ is well founded.*

Proof:
$\langle 1 \rangle 1.$ For all $a \in \mathbf{A}$, the class $\{x \in \mathbf{A} \mid x \in a\}$ is a set.
  Proof: It is a subclass of a.
$\langle 1 \rangle 2.$ Every nonempty subset of $\mathbf{A}$ has an $\in$-minimal element.
  $\langle 2 \rangle 1.$ Let: C be a nonempty subset of $\mathbf{A}$
  $\langle 2 \rangle 2.$ Pick $m \in C$ such that $m \cap C = \emptyset$
    Proof: Axiom of Regularity.
  $\langle 2 \rangle 3.$ m is $\in$-minimal in C.
$\square$

Proposition Schema 2.4.3 (Z)**.** *For any classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{R}$, the following is a theorem:*

Assume $\mathbf{R}$ is a well founded relation on $\mathbf{A}$ and $\mathbf{B} \subseteq \mathbf{A}$ is nonempty. Then $\mathbf{B}$ has an $\mathbf{R}$-minimal element.

Proof:
$\langle 1 \rangle 1.$ Pick $b \in \mathbf{B}$
$\langle 1 \rangle 2.$ Let: $S = \{x \in \mathbf{B} \mid x\mathbf{R}b\}$
  Proof: S is a set because it is a subclass of $\{x \in \mathbf{A} \mid x\mathbf{R}b\}$.
$\langle 1 \rangle 3.$ Case: $S = \emptyset$
  Proof: In this case b is an $\mathbf{R}$-minimal element of $\mathbf{B}$.
$\langle 1 \rangle 4.$ Case: $S \neq \emptyset$
  Proof: In this cases S has an $\mathbf{R}$-minimal element, which is an $\mathbf{R}$-minimal element of $\mathbf{B}$.
$\square$

Proposition Schema 2.4.4 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{R}$*, the following is a theorem:*

Assume $\mathbf{R}$ *is a well founded relation on* $\mathbf{B}$ *and* $\mathbf{A} \subseteq \mathbf{B}$*. Then* $\mathbf{R} \cap \mathbf{A}^2$ *is a well founded relation on* $\mathbf{A}$*.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathbf{R}' = \mathbf{R} \cap \mathbf{A}^2$
$\langle 1 \rangle 2$. For all $a \in \mathbf{A}$, the class $\{x \in \mathbf{A} \mid x\mathbf{R}'a\}$ is a set.
PROOF: By Comprehension since it is a subclass of $\{x \in \mathbf{B} \mid x\mathbf{R}a\}$.
$\langle 1 \rangle 3$. Every nonempty subset of $\mathbf{A}$ has an $\mathbf{R}'$-minimal element.
PROOF: It is a nonempty subset of $\mathbf{B}$ and so has an $\mathbf{R}$-minimal element, which is also an $\mathbf{R}'$-minimal element.
□

Theorem Schema 2.4.5 (Transfinite Induction Principle (Z))**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{R}$*, the following is a theorem:*

Assume $\mathbf{R}$ *is a well founded relation on* $\mathbf{A}$ *and* $\mathbf{B} \subseteq \mathbf{A}$*. Assume that, for all* $t \in \mathbf{A}$*,*

$$\{x \in \mathbf{A} \mid x\mathbf{R}t\} \subseteq \mathbf{B} \Rightarrow t \in \mathbf{B} \enspace .$$

Then $\mathbf{B} = \mathbf{A}$*.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $\mathbf{B} \neq \mathbf{A}$
$\langle 1 \rangle 2$. PICK an $\mathbf{R}$-minimal element m of $\mathbf{A} - \mathbf{B}$.
PROOF: Proposition 2.4.3.
$\langle 1 \rangle 3$. $\{x \in \mathbf{A} \mid x\mathbf{R}m\} \subseteq \mathbf{B}$
PROOF: By minimality of m.
$\langle 1 \rangle 4$. $m \in \mathbf{B}$
$\langle 1 \rangle 5$. Q.E.D.
PROOF: This is a contradiction.
□

Theorem 2.4.6 (Z)**.** *The transitive closure of a well founded relation on a set is well founded.*

PROOF:
$\langle 1 \rangle 1$. LET: R be a well founded relation on the set A.
$\langle 1 \rangle 2$. LET: R^t be the transitive closure of R.
$\langle 1 \rangle 3$. For any $x, y \in A$, if xR^ty then there exists $z \in A$ such that zRy.
PROOF: $\{(x, y) \in A^2 \mid \exists z \in A.zRy\}$ is a transitive relation on A that includes R.
$\langle 1 \rangle 4$. LET: B be a nonempty subset of A.
$\langle 1 \rangle 5$. PICK an R-minimal element b of B.
$\langle 1 \rangle 6$. b is R^t-minimal in B.
PROOF: If there exists x such that xR^tb then there exists z such that zRb by $\langle 1 \rangle 3$.
□

Definition 2.4.7 (Initial Segment)**.** Let $\mathbf{R}$ be a relation on $\mathbf{A}$ and $a \in \mathbf{A}$. The *initial segment* up to a is

$$\operatorname{seg} a := \{x \in \mathbf{A} \mid x\mathbf{R}a\} \ .$$

Theorem Schema 2.4.8 (Transfinite Recursion Theorem Schema (ZFC))**.** *For any classes $\mathbf{A}$, $\mathbf{R}$ and any property $G[x, y, z]$, there exists a class $\mathbf{F}$ such that, for any class $\mathbf{F}'$ the following is a theorem:*

Assume that $\mathbf{R}$ is a well-founded relation on $\mathbf{A}$. Assume that, for any f and t, there exists a unique z such that $G[f, t, z]$. Then $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ such that, for all $t \in \mathbf{A}$, we have $\mathbf{F} \restriction \operatorname{seg} t$ is a set and

$$G[\mathbf{F} \restriction \operatorname{seg} t, t, \mathbf{F}(t)] \ .$$

If $\mathbf{F}' : \mathbf{A} \to \mathbf{V}$ satisfies that, for all $t \in \mathbf{A}$, we have $\mathbf{F}' \restriction \operatorname{seg} t$ is a set and $G[\mathbf{F}' \restriction \operatorname{seg} t, t, \mathbf{F}'(t)]$, then $\mathbf{F}' = \mathbf{F}$.

PROOF:

⟨1⟩1. For B a subset of $\mathbf{A}$, let us say a function $v : B \to \mathbf{V}$ is *acceptable* iff, for all $x \in B$, we have $\operatorname{seg} x \subseteq B$ and $G[v \restriction \operatorname{seg} x, x, v(x)]$

⟨1⟩2. LET: $\mathbf{K}$ be the class of all acceptable functions.

⟨1⟩3. LET: $\mathbf{F} = \bigcup \mathbf{K}$

⟨1⟩4. For all $B, C \subseteq \mathbf{A}$, given $v_1 : B \to \mathbf{V}$ and $v_2 : C \to \mathbf{V}$ acceptable and $x \in B \cap C$, we have $v_1(x) = v_2(x)$

  ⟨2⟩1. ASSUME: as transfinite induction hypothesis $\forall y \mathbf{R} x. y \in B \cap C \Rightarrow v_1(y) = v_2(y)$

  ⟨2⟩2. $v_1 \restriction \operatorname{seg} x = v_2 \restriction \operatorname{seg} x$

  ⟨2⟩3. $G[v_1 \restriction \operatorname{seg} x, x, v_1(x)]$

  ⟨2⟩4. $G[v_2 \restriction \operatorname{seg} x, x, v_2(x)]$

  ⟨2⟩5. $v_1(x) = v_2(x)$

⟨1⟩5. $\mathbf{F}$ is a function.

  ⟨2⟩1. ASSUME: $(x, y), (x, z) \in \mathbf{F}$

  ⟨2⟩2. PICK acceptable $v_1 : B \to \mathbf{V}$ and $v_2 : C \to \mathbf{V}$ such that $v_1(x) = y$ and $v_2(x) = z$

  ⟨2⟩3. $y = z$

   PROOF: By ⟨1⟩4.

⟨1⟩6. For all $t \in \operatorname{dom} \mathbf{F}$, we have $\mathbf{F} \restriction \operatorname{seg} t$ is a set and $G[\mathbf{F} \restriction \operatorname{seg} t, t, \mathbf{F}(t)]$

  ⟨2⟩1. LET: $t \in \operatorname{dom} \mathbf{F}$

  ⟨2⟩2. PICK an acceptable $v : A \to \mathbf{V}$ such that $t \in A$

  ⟨2⟩3. For all $y\mathbf{R}x$ we have $v(y) = \mathbf{F}(y)$

  ⟨2⟩4. $\mathbf{F} \restriction \operatorname{seg} x = v \restriction \operatorname{seg} x$

  ⟨2⟩5. $G[v \restriction \operatorname{seg} x, x, v(x)]$

  ⟨2⟩6. $G[\mathbf{F} \restriction \operatorname{seg} x, x, \mathbf{F}(x)]$

⟨1⟩7. $\operatorname{dom} \mathbf{F} = \mathbf{A}$

  ⟨2⟩1. LET: $x \in \mathbf{A}$

  ⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall y \mathbf{R} x. y \in \mathbf{A}$

  ⟨2⟩3. ASSUME: for a contradiction $x \notin \operatorname{dom} \mathbf{F}$

⟨2⟩4. $\mathbf{F} \restriction \operatorname{seg} x$ is a set
PROOF: Axiom of Replacement.
⟨2⟩5. $\mathbf{F} \restriction \operatorname{seg} x$ is acceptable
⟨2⟩6. LET: y be the unique object such that $G[\mathbf{F} \restriction \operatorname{seg} x, x, y]$
⟨2⟩7. $\mathbf{F} \restriction \operatorname{seg} x \cup \{(x, y)\}$ is acceptable
⟨2⟩8. $x \in \operatorname{dom} \mathbf{F}$
⟨2⟩9. Q.E.D.
PROOF: This is a contradiction.
⟨1⟩8. If $\mathbf{F}' : \mathbf{A} \to \mathbf{V}$ satisfies the theorem, then $\mathbf{F}' = \mathbf{F}$.
⟨2⟩1. LET: $x \in \mathbf{A}$
PROVE: $\mathbf{F}'(x) = \mathbf{F}(x)$
⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall y \mathbf{R} x . \mathbf{F}'(y) = \mathbf{F}(y)$
⟨2⟩3. $\mathbf{F} \restriction x = \mathbf{F}' \restriction x$
⟨2⟩4. $G[\mathbf{F} \restriction x, x, \mathbf{F}(x)]$
⟨2⟩5. $G[\mathbf{F}' \restriction x, x, \mathbf{F}'(x)]$
⟨2⟩6. $\mathbf{F}(x) = \mathbf{F}'(x)$
□

Chapter 3

Functions

3.1 Functions

Definition 3.1.1 (Function)**.** A *function* from $\mathbf{A}$ to $\mathbf{B}$ is a relation $\mathbf{F}$ between $\mathbf{A}$ and $\mathbf{B}$ such that, for all $x \in \mathbf{A}$, there is only one y such that $x\mathbf{F}y$. We denote this y by $\mathbf{F}(x)$.

A *binary operation* on a class $\mathbf{A}$ is a function $\mathbf{A}^2 \to \mathbf{A}$.

Definition 3.1.2 (Closed)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{A}$ be a function and $\mathbf{B} \subseteq \mathbf{A}$. Then $\mathbf{B}$ is *closed* under $\mathbf{F}$ iff $\forall x \in \mathbf{B}.\mathbf{F}(x) \in \mathbf{B}$.

Proposition 3.1.3 (Z)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*

$$\mathrm{id}_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$$

Proof: For all $x \in \mathbf{A}$, the only y such that $(x, y) \in \mathrm{id}_{\mathbf{A}}$ is $y = x$. □

Proposition Schema 3.1.4 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{F}$ *and* $\mathbf{G}$*, the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{G} : \mathbf{B} \to \mathbf{C}$*. Then* $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \to \mathbf{C}$ *and, for all* $x \in \mathbf{A}$*, we have*

$$(\mathbf{G} \circ \mathbf{F})(x) = \mathbf{G}(\mathbf{F}(x)) \enspace .$$

Proof:
$\langle 1 \rangle 1.$ $\forall x \in \mathbf{A}.(x, \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G} \circ \mathbf{F})$
  Proof: Because $(x, \mathbf{F}(x)) \in \mathbf{F}$ and $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$.
$\langle 1 \rangle 2.$ If $(x, z) \in \mathbf{F} \circ \mathbf{G}$ then $z = \mathbf{G}(\mathbf{F}(x))$
  $\langle 2 \rangle 1.$ Pick $y \in \mathbf{B}$ such that $x\mathbf{F}y$ and $y\mathbf{G}z$
  $\langle 2 \rangle 2.$ $y = \mathbf{F}(x)$
  $\langle 2 \rangle 3.$ $z = \mathbf{G}(y)$
  $\langle 2 \rangle 4.$ $z = \mathbf{G}(\mathbf{F}(x))$
□

Proposition 3.1.5 (Z)**.** *For any set A there exists a function $F : \mathcal{P}A - \{\emptyset\} \to A$ (a* choice function *for A) such that, for every nonempty $B \subseteq A$, we have $F(B) \in B$.*

PROOF:
$\langle 1 \rangle$1. LET: A be a set.
$\langle 1 \rangle$2. LET: $\mathcal{A} = \{\{B\} \times B \mid B \in \mathcal{P}A - \{\emptyset\}\}$
$\langle 1 \rangle$3. Every member of $\mathcal{A}$ is nonempty.
$\langle 1 \rangle$4. Any two distinct members of $\mathcal{A}$ are disjoint.
$\langle 1 \rangle$5. PICK a set C such that, for all $X \in \mathcal{A}$, we have $C \cap X$ is a singleton.
PROOF: Axiom of Choice.
$\langle 1 \rangle$6. LET: $F = C \cap \bigcup \mathcal{A}$
$\langle 1 \rangle$7. $F : \mathcal{P}A - \{\emptyset\} \to A$
$\langle 2 \rangle$1. F is a function.
$\langle 3 \rangle$1. LET: $(B, b), (B, b') \in F$
$\langle 3 \rangle$2. $(B, b), (B, b') \in \{B\} \times B$
PROOF: Since $(B, b), (B, b') \in \bigcup \mathcal{A}$.
$\langle 3 \rangle$3. $(B, b), (B, b') \in C \cap (\{B\} \times B)$
$\langle 3 \rangle$4. $(B, b) = (B, b')$
PROOF: From $\langle 1 \rangle$5.
$\langle 3 \rangle$5. $b = b'$
$\langle 2 \rangle$2. $\operatorname{dom} F = \mathcal{P}A - \{\emptyset\}$
PROOF:

$$\begin{aligned}
& B \in \operatorname{dom} F \\
\Leftrightarrow & \exists b.(B, b) \in F \\
\Leftrightarrow & \exists b.((B, b) \in \bigcup \mathcal{A} \wedge (B, b) \in C) \\
\Leftrightarrow & \exists b.\exists B' \in \mathcal{P}A - \{\emptyset\}.((B, b) \in \{B'\} \times B' \wedge (B, b) \in C) \\
\Leftrightarrow & B \in \mathcal{P}A - \{\emptyset\} \wedge \exists b \in B.(B, b) \in C \\
\Leftrightarrow & B \in \mathcal{P}A - \{\emptyset\} && (\langle 1 \rangle 5)
\end{aligned}$$

$\langle 2 \rangle$3. $\operatorname{ran} F \subseteq A$
$\langle 1 \rangle$8. For every nonempty $B \subseteq A$ we have $F(B) \in B$
□

Proposition 3.1.6 (Z)**.** *For any relation R between A and B, there exists a function $H : A \to B$ such that $H \subseteq R$ (i.e. $\forall x \in A.xRH(x)$).*

PROOF:
$\langle 1 \rangle$1. LET: R be a relation between A and B.
$\langle 1 \rangle$2. PICK a choice function G for B.
$\langle 1 \rangle$3. Define $H : A \to B$ by $H(x) = G(\{y \mid xRy\})$
$\langle 1 \rangle$4. $H \subseteq R$
□

3.1.1 Injective Functions

Definition 3.1.7 (Injective)**.** A function $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ is *one-to-one*, *injective* or an *injection*, $\mathbf{F} : \mathbf{A} \rightarrowtail \mathbf{B}$, iff, for all $x, y \in \mathbf{A}$, if $\mathbf{F}(x) = \mathbf{F}(y)$, then $x = y$.

Proposition 3.1.8 (Z)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
$\mathrm{id}_\mathbf{A} : \mathbf{A} \to \mathbf{A}$ *is injective.*

PROOF: If $\mathrm{id}_\mathbf{A}(x) = \mathrm{id}_\mathbf{A}(y)$ then immediately $x = y$. □

Proposition Schema 3.1.9 (Z)**.** *For any classes* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{F}$, $\mathbf{G}$*, the following is a theorem:*
Assume $\mathbf{F} : \mathbf{A} \rightarrowtail \mathbf{B}$ *and* $\mathbf{G} : \mathbf{B} \rightarrowtail \mathbf{C}$*. Then* $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \rightarrowtail \mathbf{C}$*.*

PROOF:
⟨1⟩1. LET: $x, y \in \mathbf{A}$
⟨1⟩2. ASSUME: $(\mathbf{G} \circ \mathbf{F})(x) = (\mathbf{G} \circ \mathbf{F})(y)$
⟨1⟩3. $\mathbf{G}(\mathbf{F}(x)) = \mathbf{G}(\mathbf{F}(y))$
⟨1⟩4. $\mathbf{F}(x) = \mathbf{F}(y)$
PROOF: Since $\mathbf{G}$ is injective.
⟨1⟩5. $x = y$
PROOF: Since $\mathbf{F}$ is injective.
□

Proposition 3.1.10 (Z)**.** *Let* $F : A \to B$ *where* A *is nonempty. There exists* $G : B \to A$ *(a* left inverse*) such that* $G \circ F = \mathrm{id}_A$ *if and only if* F *is one-to-one.*

PROOF:
⟨1⟩1. If there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$ then F is one-to-one.
⟨2⟩1. ASSUME: $G : B \to A$ and $G \circ F = I_A$
⟨2⟩2. LET: $x, y \in A$
⟨2⟩3. ASSUME: $F(x) = F(y)$
⟨2⟩4. $x = y$
PROOF: $x = G(F(x)) = G(F(y)) = y$
⟨1⟩2. If F is one-to-one then there exists $G : B \to A$ such that $G \circ F = I_A$.
⟨2⟩1. ASSUME: F is one-to-one.
⟨2⟩2. PICK $a \in A$
⟨2⟩3. LET: $G : B \to A$ be the function defined by: $G(b)$ is the (unique) $x \in A$ such that $F(x) = b$ if there exists such an x, $G(b) = a$ otherwise.
⟨2⟩4. For all $x \in A$ we have $G(F(x)) = x$.
□

3.1.2 Surjective Functions

Definition 3.1.11 (Surjective)**.** Let $F : A \to B$. We say that F is *surjective*, or maps A *onto* B, and write $F : A \twoheadrightarrow B$, iff for all $y \in B$ there exists $x \in A$ such that $F(x) = y$.

Proposition Schema 3.1.12 (Z)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
$\mathrm{id}_\mathbf{A} : \mathbf{A} \to \mathbf{A}$ *is surjective.*

PROOF: For any $y \in \mathbf{A}$ we have $\mathrm{id}_{\mathbf{A}}(y) = y$. □

Proposition Schema 3.1.13 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{F}$ *and* $\mathbf{G}$*, the following is a theorem:*
If $\mathbf{F} : \mathbf{A} \twoheadrightarrow \mathbf{B}$ *and* $\mathbf{G} : \mathbf{B} \twoheadrightarrow \mathbf{C}$*, then* $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \twoheadrightarrow \mathbf{C}$*.*

PROOF:
⟨1⟩1. LET: $c \in \mathbf{C}$
⟨1⟩2. PICK $b \in \mathbf{B}$ such that $\mathbf{G}(b) = c$.
⟨1⟩3. PICK $a \in \mathbf{A}$ such that $\mathbf{F}(a) = b$.
⟨1⟩4. $(\mathbf{G} \circ \mathbf{F})(a) = c$
□

Proposition 3.1.14 (Z)**.** *Let* $F : A \to B$*. There exists* $H : B \to A$ *(a* right inverse*) such that* $F \circ H = \mathrm{id}_B$ *if and only if* F *maps* A *onto* B*.*

PROOF:
⟨1⟩1. If F has a right inverse then F is surjective.
  ⟨2⟩1. ASSUME: F has a right inverse $H : B \to A$.
  ⟨2⟩2. LET: $y \in B$
  ⟨2⟩3. $F(H(y)) = y$
  ⟨2⟩4. There exists $x \in A$ such that $F(x) = y$
⟨1⟩2. If F is surjective then F has a right inverse.
  ⟨2⟩1. ASSUME: F is surjective.
  ⟨2⟩2. PICK a function $H : B \to A$ such that $H \subseteq F^{-1}$
    PROOF: Proposition 3.1.6.
  ⟨2⟩3. $F \circ H = \mathrm{id}_B$
    ⟨3⟩1. LET: $y \in B$
    ⟨3⟩2. $(y, H(y)) \in F^{-1}$
    ⟨3⟩3. $F(H(y)) = y$
□

3.1.3 Bijections

Definition 3.1.15 (Bijection)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$. Then $\mathbf{F}$ is *bijective* or a *bijection*, $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$, iff it is injective and surjective.

Proposition Schema 3.1.16 (Z)**.** *For any class* $\mathbf{A}$*, the following is a theorem:*
The identity function $\mathrm{id}_{\mathbf{A}} : \mathbf{A} \approx \mathbf{A}$ *is a bijection.*

PROOF: Proposition 3.1.8 and 3.1.12. □

Proposition Schema 3.1.17 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{F}$*, the following is a theorem:*
If $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$ *then* $\mathbf{F}^{-1} : \mathbf{B} \approx \mathbf{A}$*.*

PROOF:
⟨1⟩1. $\mathbf{F}^{-1} : \mathbf{B} \to \mathbf{A}$
  ⟨2⟩1. LET: $b \in \mathbf{B}$

⟨2⟩2. PICK $a \in \mathbf{A}$ such that $\mathbf{F}(a) = b$.
PROOF: Since $\mathbf{F}$ is surjective.
⟨2⟩3. $(b, a) \in \mathbf{F}^{-1}$
⟨2⟩4. If $(b, a') \in \mathbf{F}^{-1}$ then $a' = a$.
⟨3⟩1. LET: $a' \in \mathbf{A}$ such that $(b, a') \in \mathbf{F}^{-1}$
⟨3⟩2. $\mathbf{F}(a') = \mathbf{F}(a)$
⟨3⟩3. $a' = a$
PROOF: Since $\mathbf{F}$ is injective.
⟨1⟩2. $\mathbf{F}^{-1}$ is injective.
⟨2⟩1. LET: $x, y \in \mathbf{B}$
⟨2⟩2. ASSUME: $\mathbf{F}^{-1}(x) = \mathbf{F}^{-1}(y)$
⟨2⟩3. $x = y$
PROOF: $x = \mathbf{F}(\mathbf{F}^{-1}(x)) = \mathbf{F}(\mathbf{F}^{-1}(y)) = y$.
⟨1⟩3. $\mathbf{F}^{-1}$ is surjective.
PROOF: For all $a \in \mathbf{A}$ we have $\mathbf{F}^{-1}(\mathbf{F}(a)) = a$.
□

Proposition Schema 3.1.18 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{F}$ *and* $\mathbf{G}$*, the following is a theorem:*
If $\mathbf{F} : \mathbf{A} \approx \mathbf{B}$ *and* $\mathbf{G} : \mathbf{B} \approx \mathbf{C}$ *then* $\mathbf{G} \circ \mathbf{F} : \mathbf{A} \approx \mathbf{C}$.

PROOF: Propositions 3.1.9 and 3.1.13. □

3.1.4 Restrictions

Definition 3.1.19 (Restriction)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$. Let $\mathbf{C} \subseteq \mathbf{A}$. The *restriction* of $\mathbf{F}$ to $\mathbf{C}$, denoted $\mathbf{F} \restriction \mathbf{C}$, is the function

$$\begin{aligned} \mathbf{F} \restriction \mathbf{C} &: \mathbf{C} \to \mathbf{B} \\ (\mathbf{F} \restriction \mathbf{C})(x) &= \mathbf{F}(x) \qquad\qquad (x \in \mathbf{C}) \end{aligned}$$

3.1.5 Images

Definition 3.1.20 (Image)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{C} \subseteq \mathbf{A}$. The *image* of $\mathbf{C}$ under $\mathbf{F}$ is the class

$$\mathbf{F}(\mathbf{C}) := \{\mathbf{F}(x) \mid x \in \mathbf{C}\} \ .$$

Proposition Schema 3.1.21 (Z)**.** *For any classes* $\mathbf{F}$*,* $\mathbf{A}$ *and* $\mathbf{B}$*, the following is a theorem.*
If $\mathbf{F} : \mathbf{A} \to \mathbf{B}$*, then for any subset* $S \subseteq \mathbf{A}$*, the class* $\mathbf{F}(S)$ *is a set.*

PROOF: By an Axiom of Replacement. □

Proposition Schema 3.1.22 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$ *and* $\mathbf{F}$*, the following is a theorem:*
Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C} \subseteq \mathcal{P}\mathbf{A}$*. Then*

$$\mathbf{F}\left(\bigcup \mathbf{C}\right) = \{y \mid \exists X \in \mathbf{C}. y \in \mathbf{F}(X)\}$$

PROOF:

$$\begin{aligned} y \in \mathbf{F}\left(\bigcup \mathbf{C}\right) &\Leftrightarrow \exists x \in \bigcup \mathbf{C}. y = \mathbf{F}(x) \\ &\Leftrightarrow \exists x. \exists X. X \in \mathbf{C} \wedge x \in X \wedge y = \mathbf{F}(x) \\ &\Leftrightarrow \exists X \in \mathbf{C}. y \in \mathbf{F}(X) \end{aligned}$$

∎

Proposition Schema 3.1.23 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{D}$ *and* $\mathbf{F}$*, the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$*. Then*

$$\mathbf{F}(\mathbf{C} \cup \mathbf{D}) = \mathbf{F}(\mathbf{C}) \cup \mathbf{F}(\mathbf{D}) \ .$$

PROOF:

$$\begin{aligned} y \in \mathbf{F}(\mathbf{C} \cup \mathbf{D}) &\Leftrightarrow \exists x \in \mathbf{C} \cup \mathbf{D}. y = \mathbf{F}(x) \\ &\Leftrightarrow \exists x \in \mathbf{C}. y = \mathbf{F}(x) \vee \exists x \in \mathbf{D}. y = \mathbf{F}(x) \\ &\Leftrightarrow y \in \mathbf{F}(\mathbf{C}) \cup \mathbf{F}(\mathbf{D}) \end{aligned}$$

∎

Proposition 3.1.24 (Z)**.** *For any classes* $\mathbf{F}$*,* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$ *and* $\mathbf{D}$*, the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$*. Then*

$$\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B}) \ .$$

Equality holds if $\mathbf{F}$ *is injective.*

PROOF:

$\langle 1 \rangle 1.$ $\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
  $\langle 2 \rangle 1.$ LET: $y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$
  $\langle 2 \rangle 2.$ PICK $x \in \mathbf{A} \cap \mathbf{B}$ such that $y = \mathbf{F}(x)$
  $\langle 2 \rangle 3.$ $y \in \mathbf{F}(\mathbf{A})$
    PROOF: Since $x \in \mathbf{A}$.
  $\langle 2 \rangle 4.$ $y \in \mathbf{F}(\mathbf{B})$
    PROOF: Since $x \in \mathbf{B}$.
$\langle 1 \rangle 2.$ If $\mathbf{F}$ is injective then $\mathbf{F}(\mathbf{A} \cap \mathbf{B}) = \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$.
  $\langle 2 \rangle 1.$ ASSUME: $\mathbf{F}$ is injective.
  $\langle 2 \rangle 2.$ LET: $y \in \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$
  $\langle 2 \rangle 3.$ PICK $x \in \mathbf{A}$ such that $y = \mathbf{F}(x)$
  $\langle 2 \rangle 4.$ PICK $x' \in \mathbf{B}$ such that $y = \mathbf{F}(x')$
  $\langle 2 \rangle 5.$ $x = x'$
    PROOF: $\langle 2 \rangle 1$
  $\langle 2 \rangle 6.$ $x \in \mathbf{A} \cap \mathbf{B}$
  $\langle 2 \rangle 7.$ $y \in \mathbf{F}(\mathbf{A} \cap \mathbf{B})$

∎

Proposition Schema 3.1.25 (Z)**.** *For any classes* $\mathbf{F}$*,* $\mathbf{A}$*,* $\mathbf{B}$*, and* $\mathbf{C}$*, the following is a theorem:*

Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C} \subseteq \mathcal{P}\mathbf{A}$*. Then*

$$\mathbf{F}\left(\bigcap \mathbf{C}\right) \subseteq \bigcap\{\mathbf{F}(X) \mid X \in \mathbf{A}\} \ .$$

Equality holds if $\mathbf{F}$ *is injective and* $\mathbf{A}$ *is nonempty.*

PROOF:
⟨1⟩1. $\mathbf{F}\left(\bigcap \mathbf{A}\right) \subseteq \bigcap\{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
⟨2⟩1. LET: $y \in \mathbf{F}(\bigcap \mathbf{A})$
⟨2⟩2. PICK $x \in \bigcap \mathbf{A}$ such that $y = \mathbf{F}(x)$
⟨2⟩3. LET: $X \in \mathbf{A}$
⟨2⟩4. $x \in X$
⟨2⟩5. $y \in \mathbf{F}(X)$
⟨1⟩2. If $\mathbf{F}$ is injective then $\mathbf{F}\left(\bigcap \mathbf{A}\right) = \bigcap\{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
⟨2⟩1. ASSUME: $\mathbf{F}$ is injective.
⟨2⟩2. ASSUME: $\mathbf{A}$ is nonempty.
⟨2⟩3. LET: $y \in \bigcap\{\mathbf{F}(X) \mid X \in \mathbf{A}\}$
⟨2⟩4. PICK $X_0 \in \mathbf{A}$
⟨2⟩5. PICK $x \in X_0$ such that $(x, y) \in \mathbf{F}$
⟨2⟩6. $x \in \bigcap \mathbf{A}$
⟨3⟩1. LET: $X \in \mathbf{A}$
⟨3⟩2. PICK $x' \in X$ such that $(x', y) \in \mathbf{F}$.
⟨3⟩3. $x = x'$
PROOF: ⟨2⟩1
⟨3⟩4. $x \in X$
⟨2⟩7. $y \in \mathbf{F}(\bigcap \mathbf{A})$
□

Proposition 3.1.26 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{D}$ *and* $\mathbf{F}$*, the following is a theorem:*

Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$*. Then*

$$\mathbf{F}(\mathbf{C}) - \mathbf{F}(\mathbf{D}) \subseteq \mathbf{F}(\mathbf{C} - \mathbf{D}) \ .$$

Equality holds if $\mathbf{F}$ *is injective.*

PROOF:
⟨1⟩1. $\mathbf{F}(\mathbf{C}) - \mathbf{F}(\mathbf{D}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
⟨2⟩1. LET: $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
⟨2⟩2. PICK $x \in \mathbf{A}$ such that $y = \mathbf{F}(x)$
⟨2⟩3. $x \notin \mathbf{B}$
⟨2⟩4. $x \in \mathbf{A} - \mathbf{B}$
⟨2⟩5. $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
⟨1⟩2. If $\mathbf{F}$ is injective then $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) = \mathbf{F}(\mathbf{A} - \mathbf{B})$
⟨2⟩1. ASSUME: $\mathbf{F}$ is injective.
⟨2⟩2. LET: $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
⟨2⟩3. PICK $x \in \mathbf{A} - \mathbf{B}$ such that $y = \mathbf{F}(x)$
⟨2⟩4. $y \in \mathbf{F}(\mathbf{A})$
⟨2⟩5. $y \notin \mathbf{F}(\mathbf{B})$

⟨3⟩1. ASSUME: for a contradiction $y \in \mathbf{F}(\mathbf{B})$
⟨3⟩2. PICK $x' \in \mathbf{B}$ such that $y = \mathbf{F}(x')$
⟨3⟩3. $x = x'$
PROOF: ⟨2⟩1
⟨3⟩4. $x \in \mathbf{B}$
⟨3⟩5. Q.E.D.
PROOF: This contradicts ⟨2⟩3.

□

3.1.6 Inverse Images

Definition 3.1.27 (Inverse Image)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{C} \subseteq \mathbf{B}$. Then the *inverse image* of $\mathbf{C}$ under $\mathbf{F}$ is

$$\mathbf{F}^{-1}(\mathbf{C}) = \{x \in \mathbf{A} \mid \mathbf{F}(x) \in \mathbf{C}\} \enspace .$$

Proposition Schema 3.1.28 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$ *and* $\mathbf{F}$*, the following is a theorem:*
Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C} \subseteq \mathcal{P}\mathbf{B}$*. Then*

$$\mathbf{F}^{-1}\left(\bigcap \mathbf{C}\right) = \bigcap \{\mathbf{F}^{-1}(X) \mid X \in \mathbf{C}\} \enspace .$$

PROOF:

$$\begin{aligned} x \in \mathbf{F}^{-1}\left(\bigcap \mathbf{C}\right) &\Leftrightarrow \mathbf{F}(x) \in \bigcap \mathbf{C} \\ &\Leftrightarrow \forall X \in \mathbf{C}.\mathbf{F}(x) \in X \\ &\Leftrightarrow \forall X \in \mathbf{C}.x \in \mathbf{F}^{-1}(X) \end{aligned}$$

□

Proposition Schema 3.1.29 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{D}$ *and* $\mathbf{F}$*, the following is a theorem:*
Assume $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{C}, \mathbf{D} \subseteq \mathbf{B}$*. Then*

$$\mathbf{F}^{-1}(\mathbf{C} - \mathbf{D}) = \mathbf{F}^{-1}(\mathbf{C}) - \mathbf{F}^{-1}(\mathbf{D}) \enspace .$$

PROOF:

$$\begin{aligned} x \in \mathbf{F}^{-1}(\mathbf{C} - \mathbf{D}) &\Leftrightarrow \mathbf{F}(x) \in \mathbf{C} - \mathbf{D} \\ &\Leftrightarrow \mathbf{F}(x) \in \mathbf{C} \wedge \mathbf{F}(x) \notin \mathbf{D} \\ &\Leftrightarrow x \in \mathbf{F}^{-1}(\mathbf{C}) \wedge x \in \mathbf{F}^{-1}(\mathbf{D}) \\ &\Leftrightarrow x \in \mathbf{F}^{-1}(\mathbf{C}) - \mathbf{F}^{-1}(\mathbf{D}) \end{aligned}$$

□

3.1.7 Function Sets

Proposition 3.1.30 (ZFC)**.** *For any classes* $\mathbf{B}$ *and* $\mathbf{F}$*, the following is a theorem:*
Let A *be a set. If* $\mathbf{F} : A \to \mathbf{B}$ *then* $\mathbf{F}$ *is a set.*

PROOF: By an Axiom of Replacement, we have $R = \{\mathbf{F}(x) \mid x \in A\}$ is a set. Hence $\mathbf{F}$ is a set since $\mathbf{F} \subseteq A \times R$. □

Definition 3.1.31 (Dependent Product Class)**.** Let I be a set and let $\mathbf{H}(i)$ be a class for all $i \in I$. We write $\prod_{i \in I} \mathbf{H}(i)$ for the class of all functions $f : I \to \bigcup_{i \in I} \mathbf{H}(i)$ such that $\forall i \in I . f(i) \in \mathbf{H}(i)$.

We write $\mathbf{B}^I$ for $\prod_{i \in I} \mathbf{B}$ where $\mathbf{B}$ does not depend on I.

Proposition Schema 3.1.32 (ZFC)**.** *Let I be a set. Let $H(i)$ be a set for every $i \in I$. Then $\prod_{i \in I} \mathbf{H}(i)$ is a set.*

PROOF:
$\langle 1 \rangle 1$. $\{\mathbf{H}(i) \mid i \in I\}$ is a set.
PROOF: By an Axiom of Replacement.
$\langle 1 \rangle 2$. $\bigcup_{i \in I} \mathbf{H}(i)$ is a set.
$\langle 1 \rangle 3$. $\prod_{i \in I} \mathbf{H}(i)$ is a set.
PROOF: It is a subset of $\mathcal{P}\left(I \times \bigcup_{i \in I} \mathbf{H}(i)\right)$.
□

Proposition 3.1.33 (Z)**.** *Let I be a set. Let $H(i)$ be a set for all $i \in I$. If $\forall i \in I . H(i) \neq \emptyset$ then $\prod_{i \in I} H(i) \neq \emptyset$.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $\forall i \in I . H(i) \neq \emptyset$
$\langle 1 \rangle 2$. LET: $R = \{(i, x) \mid i \in I, x \in H(i)\}$
$\langle 1 \rangle 3$. PICK a function $f : I \to \bigcup_{i \in I} H(i)$ such that $f \subseteq R$
PROOF: Proposition 3.1.6.
$\langle 1 \rangle 4$. $f \in \prod_{i \in I} H(i)$
□

3.2 Equinumerosity

Definition 3.2.1 (Equinumerous)**.** Sets A and B are *equinumerous*, $A \approx B$, iff there exists a bijection between A and B.

3.3 Domination

Definition 3.3.1 (Dominate)**.** A set A is *dominated* by a set B, $A \preccurlyeq B$, iff there exists an injection $A \to B$.

Proposition 3.3.2 (Z)**.** *Given sets A and B, if $A \neq \emptyset$ or $B = \emptyset$, then we have $A \preccurlyeq B$ iff there exists a surjective function $B \to A$.*

PROOF:
$\langle 1 \rangle 1$. If $A \preccurlyeq B$ and $A \neq \emptyset$ then there exists a surjective function $B \to A$.
$\langle 2 \rangle 1$. ASSUME: $f : A \to B$ be injective.
$\langle 2 \rangle 2$. PICK $a \in A$
$\langle 2 \rangle 3$. LET: $g : B \to A$ be the function defined by $g(b) = f^{-1}(b)$ if $b \in \operatorname{ran} f$, and $g(b) = a$ otherwise.

$\langle 2 \rangle 4$. g is surjective.

$\langle 1 \rangle 2$. If there exists a surjective function $B \to A$ then $A \preccurlyeq B$.

$\langle 2 \rangle 1$. Assume: there exists a surjective function $g : B \to A$

$\langle 2 \rangle 2$. $\forall a \in A . \exists b \in B . g(b) = a$

$\langle 2 \rangle 3$. Choose a function $f : A \to B$ such that $\forall a \in A . g(f(a)) = a$

$\langle 2 \rangle 4$. f is injective.

$\square$

Chapter 4

Equivalence Relations

Definition 4.0.1 (Equivalence Relation)**.** An *equivalence relation* on a class $\mathbf{A}$ is a binary relation on $\mathbf{A}$ that is reflexive, symmetric and transitive.

Proposition 4.0.2 (Z)**.** *Equinumerosity is an equivalence relation on the class of all sets.*

Proof: Propositions 3.1.16, 3.1.17, 3.1.18. □

Definition 4.0.3 (Respects)**.** Let $\mathbf{R}$ be an equivalence relation on $\mathbf{A}$ and $\mathbf{F} : \mathbf{A} \to \mathbf{B}$. Then $\mathbf{F}$ *respects* $\mathbf{A}$ iff, whenever $(x, y) \in \mathbf{R}$, then $\mathbf{F}(x) = \mathbf{F}(y)$.

Definition 4.0.4 (Equivalence Class)**.** Let $\mathbf{R}$ be an equivalence relation on $\mathbf{A}$ and $a \in \mathbf{A}$. The *equivalence class* of a *modulo* $\mathbf{R}$ is

$$[a]_{\mathbf{R}} := \{x \mid a\mathbf{R}x\} \enspace .$$

Proposition Schema 4.0.5 (Z)**.** *For any classes* $\mathbf{A}$ *and* $\mathbf{R}$*, the following is a theorem.*

Assume $\mathbf{R}$ *be an equivalence relation on* $\mathbf{A}$*. Let* $a, b \in \mathbf{A}$*. Then* $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ *if and only if* $a\mathbf{R}b$*.*

Proof:
⟨1⟩1. If $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$ then $a\mathbf{R}b$.
  ⟨2⟩1. Assume: $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$
  ⟨2⟩2. $b\mathbf{R}b$
    Proof: Reflexivity
  ⟨2⟩3. $b \in [b]_{\mathbf{R}}$
  ⟨2⟩4. $b \in [a]_{\mathbf{R}}$
  ⟨2⟩5. $a\mathbf{R}b$
⟨1⟩2. If $a\mathbf{R}b$ then $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$.
  ⟨2⟩1. For all $x, y \in \mathbf{A}$, if $x\mathbf{R}y$ then $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$
    ⟨3⟩1. Let: $x, y \in \mathbf{A}$
    ⟨3⟩2. Assume: $x\mathbf{R}y$

$\langle 3\rangle 3$. LET: $t \in [y]_{\mathbf{R}}$
$\langle 3\rangle 4$. $y\mathbf{R}t$
$\langle 3\rangle 5$. $x\mathbf{R}t$
PROOF: Transitivity, $\langle 3\rangle 2$, $\langle 3\rangle 4$.
$\langle 3\rangle 6$. $t \in [x]_{\mathbf{R}}$
$\langle 2\rangle 2$. ASSUME: $a\mathbf{R}b$
$\langle 2\rangle 3$. $[b]_{\mathbf{R}} \subseteq [a]_{\mathbf{R}}$
PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 2$.
$\langle 2\rangle 4$. $b\mathbf{R}a$
PROOF: Symmetry, $\langle 2\rangle 2$.
$\langle 2\rangle 5$. $[a]_{\mathbf{R}} \subseteq [b]_{\mathbf{R}}$
PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 4$.
$\langle 2\rangle 6$. $[a]_{\mathbf{R}} = [b]_{\mathbf{R}}$
PROOF: $\langle 2\rangle 3$, $\langle 2\rangle 5$.
$\square$

Definition 4.0.6 (Partition)**.** A *partition* Π of a set A is a set of nonempty subsets of A that is disjoint and exhaustive, i.e.

1. no two different sets in Π have any common elements, and
2. each element of A is in some set in Π.

Definition 4.0.7. Let R be an equivalence relation on a set A. The *quotient set* A/R is the set of all equivalence classes.

Theorem 4.0.8 (Z)**.** *Let A be a set and $\mathbf{B}$ a class. Let R be an equivalence relation on A and $F : A \to \mathbf{B}$. Then F respects R if and only if there exists $\hat{F} : A/R \to \mathbf{B}$ such that*

$$\forall a \in A.\hat{F}([a]_R) = F(a) \ .$$

In this case, $\hat{F}$ is unique.

PROOF:
$\langle 1\rangle 1$. If F respects R then there exists $\hat{F} : A/R \to \mathbf{B}$ such that $\forall a \in A.\hat{F}([a]_R) = F(a)$.
$\langle 2\rangle 1$. ASSUME: F respects R.
$\langle 2\rangle 2$. LET: $\hat{F} = \{([a]_R, F(a)) \mid a \in A\}$
$\langle 2\rangle 3$. $\hat{F}$ is a function.
$\langle 3\rangle 1$. ASSUME: $a, a' \in A$ and $[a]_R = [a']_R$
PROVE: $F(a) = F(a')$
$\langle 3\rangle 2$. $(a, a') \in R$
PROOF: Proposition 4.0.5.
$\langle 3\rangle 3$. $F(a) = F(a')$
PROOF: $\langle 2\rangle 1$
$\langle 2\rangle 4$. $\operatorname{dom} \hat{F} = A/R$
$\langle 2\rangle 5$. $\operatorname{ran} \hat{F} \subseteq \mathbf{B}$

⟨2⟩6. $\forall a \in A.\hat{F}([a]_R) = F(a)$

⟨1⟩2. If there exists $\hat{F} : A/R \to \mathbf{B}$ such that $\forall a \in A.\hat{F}([a]_R) = F(a)$ then F respects R.

⟨2⟩1. Assume: $\hat{F} : A/R \to \mathbf{B}$ and $\forall a \in A.\hat{F}([a]_R) = F(a)$

⟨2⟩2. Let: $a, a' \in A$

⟨2⟩3. Assume: $(a, a') \in R$

⟨2⟩4. $[a]_R = [a']_R$

Proof: Proposition 4.0.5.

⟨2⟩5. $F(a) = F(a')$

Proof: ⟨2⟩1

⟨1⟩3. If $G, H : A/R \to \mathbf{B}$ and $\forall a \in A.G([a]_R) = H([a]_R)$ then $G = H$.

□

Proposition 4.0.9 (Z)**.** *Let R be an equivalence relation on a set A. Then A/R is a partition of A.*

Proof:

⟨1⟩1. Every member of A/R is nonempty.

Proof: Since $a \in [a]_R$ by reflexivity.

⟨1⟩2. No two different sets in A/R have any common elements.

⟨2⟩1. Let: $[a]_R, [b]_R \in A/R$

⟨2⟩2. Let: $c \in [a]_R \cap [b]_R$

Prove: $[a]_R = [b]_R$

⟨2⟩3. aRc

Proof: ⟨2⟩2

⟨2⟩4. bRc

Proof: ⟨2⟩2

⟨2⟩5. cRb

Proof: Symmetry, ⟨2⟩4

⟨2⟩6. aRb

Proof: Transitivity, ⟨2⟩3, ⟨2⟩5

⟨2⟩7. $[a]_R = [b]_R$

Proof: Proposition 4.0.5, ⟨2⟩6

⟨1⟩3. Each element of A is in some set in A/R.

Proof: Since $a \in [a]_R$ by reflexivity.

□

Proposition 4.0.10 (Z)**.** *For any partition P of a set A, there exists a unique equivalence relation R on A such that $A/R = P$, namely xRy iff $\exists X \in P(x \in X \wedge y \in X)$.*

Proof: Easy. □

Definition 4.0.11 (Natural Map)**.** Let A be a set and R an equivalence relation on A. The *natural map* $A \to A/R$ is the function that maps $a \in A$ to $[a]_R$.

Chapter 5

Ordering Relations

5.1 Partial Orders

Definition 5.1.1 (Partial Ordering)**.** Let $\mathbf{A}$ be a class. A *partial ordering* on $\mathbf{A}$ is a relation $\mathbf{R}$ on $\mathbf{A}$ that is reflexive, antisymmetric and transitive.

We often write $\leq$ for a partial ordering, and then write $x < y$ for $x \leq y \wedge x \neq y$.

Proposition Schema 5.1.2 (Z)**.** *For any classes $\mathbf{A}$ and $\mathbf{R}$, the following is a theorem:*

If $\mathbf{R}$ is a partial order on $\mathbf{A}$ then so is $\mathbf{R}^{-1}$.

Proof:
⟨1⟩1. $\mathbf{R}^{-1}$ is reflexive.
  Proof: Proposition 2.2.8.
⟨1⟩2. $\mathbf{R}^{-1}$ is antisymmetric.
  Proof: Proposition 2.2.12.
⟨1⟩3. $\mathbf{R}^{-1}$ is transitive.
  ⟨2⟩1. Assume: $x\mathbf{R}^{-1}y$ and $y\mathbf{R}^{-1}z$
  ⟨2⟩2. $y\mathbf{R}x$ and $z\mathbf{R}y$
  ⟨2⟩3. $z\mathbf{R}x$
    Proof: Since $\mathbf{R}$ is transitive.
  ⟨2⟩4. $x\mathbf{R}^{-1}z$
□

Proposition Schema 5.1.3 (Z)**.** *For any classes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{F}$ and $\mathbf{R}$, the following is a theorem:*

Assume $\mathbf{R}$ is a partial order on $\mathbf{B}$ and $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ is injective. Define $\mathbf{S}$ on $\mathbf{A}$ by $x\mathbf{S}y$ iff $\mathbf{F}(x)\mathbf{R}\mathbf{F}(y)$. Then $\mathbf{S}$ is a partial order on $\mathbf{A}$.

Proof:
⟨1⟩1. $\mathbf{S}$ is reflexive.
  Proof: For any $x \in \mathbf{A}$ we have $\mathbf{F}(x)\mathbf{R}\mathbf{F}(x)$.

$\langle 1\rangle 2.$ **S** is antisymmetric.
$\langle 2\rangle 1.$ LET: $x, y \in \mathbf{A}$
$\langle 2\rangle 2.$ ASSUME: $x\mathbf{S}y$ and $y\mathbf{S}x$
$\langle 2\rangle 3.$ $\mathbf{F}(x)\mathbf{R}\mathbf{F}(y)$ and $\mathbf{F}(y)\mathbf{R}\mathbf{F}(x)$
$\langle 2\rangle 4.$ $\mathbf{F}(x) = \mathbf{F}(y)$
PROOF: **R** is antisymmetric.
$\langle 2\rangle 5.$ $x = y$
$\langle 1\rangle 3.$ **S** is transitive.
□

Corollary Schema 5.1.3.1 (Z)**.** *For any classes* **A**, **B** *and* **R**, *the following is a theorem:*

Assume **R** *be a partial order on* **A** *and* $\mathbf{B} \subseteq \mathbf{A}$. *Then* $\mathbf{R} \cap \mathbf{B}^2$ *is a partial order on* **B**.

Definition 5.1.4 (Partially Ordered Set)**.** A *partially ordered set* or *poset* is a pair $(A, \leq)$ where A is a set and $\leq$ is a partial ordering on A. We often write just A for $(A, \leq)$.

If $(A, \leq)$ is a poset and $B \subseteq A$ we write just B for the poset $(B, \leq \cap B^2)$.

Definition 5.1.5 (Strictly Monotone)**.** Let $(A, <_A)$ and $(B, <_B)$ be posets. A function $f : A \to B$ is *strictly monotone* iff, whenever $x <_A y$, then $f(x) <_B f(y)$.

Definition 5.1.6 (Least)**.** Let $\leq$ be a partial order on **A**. An element $m \in \mathbf{A}$ is *least* iff for all $x \in \mathbf{A}$ we have $m \leq x$.

Proposition 5.1.7 (Z)**.** *A partial order has at most one least element.*

PROOF: If m and m' are least then $m \leq m'$ and $m' \leq m$, so $m = m'$. □

Definition 5.1.8 (Greatst)**.** Let $\leq$ be a partial order on **A**. An element $m \in \mathbf{A}$ is *greatest* iff for all $x \in A$ we have $x \leq m$.

Proposition 5.1.9 (Z)**.** *A poset has at most one greatest element.*

PROOF: If m and m' are greatest then $m \leq m'$ and $m' \leq m$, so $m = m'$. □

Definition 5.1.10 (Upper Bound)**.** Let $\leq$ be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Let $u \in \mathbf{A}$. Then u is an *upper bound* for **B** iff $\forall x \in \mathbf{B}. x \leq u$.

Definition 5.1.11 (Lower Bound)**.** Let $\leq$ be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Let $l \in \mathbf{A}$. Then l is a *lower bound* for **B** iff $\forall x \in \mathbf{B}. l \leq x$.

Definition 5.1.12 (Bounded Above)**.** Let $\leq$ be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Then **B** is *bounded above* iff it has an upper bound.

Definition 5.1.13 (Bounded Below)**.** Let $\leq$ be a partial ordering on **A** and $\mathbf{B} \subseteq \mathbf{A}$. Then **B** is *bounded below* iff it has a lower bound.

Definition 5.1.14 (Least Upper Bound)**.** Let $\leq$ be a partial ordering on $\mathbf{A}$ and $\mathbf{B} \subseteq \mathbf{A}$. Let $s \in \mathbf{A}$. Then s is the *least upper bound* or *supremum* of $\mathbf{B}$ iff s is an upper bound for $\mathbf{B}$ and, for every upper bound u for $\mathbf{B}$, we have $s \leq u$.

Definition 5.1.15 (Greatest Lower Bound)**.** Let $\leq$ be a partial ordering on $\mathbf{A}$ and $\mathbf{B} \subseteq \mathbf{A}$. Let $i \in \mathbf{A}$. Then i is the *greatest lower bound* or *infimum* of $\mathbf{B}$ iff i is a lower bound for $\mathbf{B}$ and, for every lower bound l for $\mathbf{B}$, we have $i \leq l$.

Definition 5.1.16 (Complete)**.** A partial order is *complete* iff every nonempty subset bounded above has a supremum, and every nonempty subset bounded below has an infimum.

Definition 5.1.17 (Order Isomorphism)**.** Let A and B be posets. An *order isomorphism* between A and B, $f : A \cong B$, is a bijection $f : A \approx B$ such that f and f^{-1} are monotone.

Theorem 5.1.18 (Knaster Fixed-Point Theorem (Z))**.** *Let A be a complete poset with a greatest and least element. Let $\phi : A \to A$ be monotone. Then there exists $a \in A$ such that $\phi(a) = a$.*

Proof:
$\langle 1\rangle 1.$ Let: $B = \{x \in A \mid x \leq \phi(x)\}$
$\langle 1\rangle 2.$ Let: $a = \sup B$
Proof: B is nonempty because the least element of A is in B, and it is bounded above by the greatest element of A.
$\langle 1\rangle 3.$ For all $b \in B$ we have $b \leq \phi(a)$
$\langle 2\rangle 1.$ Let: $b \in B$
$\langle 2\rangle 2.$ $b \leq \phi(b)$
$\langle 2\rangle 3.$ $b \leq a$
$\langle 2\rangle 4.$ $\phi(b) \leq \phi(a)$
$\langle 2\rangle 5.$ $b \leq \phi(a)$
$\langle 1\rangle 4.$ $a \leq \phi(a)$
$\langle 1\rangle 5.$ $\phi(a) \leq \phi(\phi(a))$
$\langle 1\rangle 6.$ $\phi(a) \in B$
$\langle 1\rangle 7.$ $\phi(a) \leq a$
$\langle 1\rangle 8.$ $\phi(a) = a$
□

Definition 5.1.19 (Dense)**.** Let $\leq$ be a partial order on $\mathbf{A}$ and $\mathbf{B} \subseteq \mathbf{A}$. Then $\mathbf{B}$ is *dense* iff, for all $x, y \in \mathbf{A}$, if $x < y$ then there exists $z \in \mathbf{B}$ such that $x < z < y$.

Proposition 5.1.20 (Z)**.** *Let A be a complete poset with no least element. Let $B \subseteq A$ be dense. Let $\theta : A \to A$ be a monotone map that is the identity on B. Then $\theta = \mathrm{id}_A$.*

Proof:
$\langle 1\rangle 1.$ Let: $a \in A$
Prove: $\theta(a) = a$

$\langle 1 \rangle 2$. LET: $S(a) = \{b \in B \mid b < a\}$
$\langle 1 \rangle 3$. $S(a)$ is nonempty and bounded above.
  $\langle 2 \rangle 1$. $S(a)$ is nonempty.
    $\langle 3 \rangle 1$. PICK $a_1 < a$
      PROOF: Since a is not least.
    $\langle 3 \rangle 2$. There exists $b \in B$ such that $a_1 < b < a$.
  $\langle 2 \rangle 2$. $S(a)$ is bounded above by a.
$\langle 1 \rangle 4$. $\sup S(a) \leq a$
$\langle 1 \rangle 5$. $\sup S(a) = a$
  $\langle 2 \rangle 1$. ASSUME: for a contradiction $\sup S(a) < a$
  $\langle 2 \rangle 2$. PICK $b \in B$ such that $\sup S(a) < b < a$
  $\langle 2 \rangle 3$. $b \in S(a)$
  $\langle 2 \rangle 4$. Q.E.D.
    PROOF: This contradicts the fact that $\sup S(a) < b$.
$\langle 1 \rangle 6$. For all $b \in S(a)$ we have $b \leq \theta(a)$
  $\langle 2 \rangle 1$. LET: $b \in S(a)$
  $\langle 2 \rangle 2$. $b < a$
  $\langle 2 \rangle 3$. $\theta(b) \leq \theta(a)$
  $\langle 2 \rangle 4$. $b \leq \theta(a)$
    PROOF: $\theta(b) = b$
$\langle 1 \rangle 7$. $a \leq \theta(a)$
  PROOF: Since $a = \sup S(a)$ and $\theta(a)$ is an upper bound for $S(a)$.
$\langle 1 \rangle 8$. $a \nless \theta(a)$
  $\langle 2 \rangle 1$. ASSUME: for a contradiction $a < \theta(a)$.
  $\langle 2 \rangle 2$. PICK $b \in B$ such that $a < b < \theta(a)$
  $\langle 2 \rangle 3$. $\theta(a) \leq \theta(b) = b$
  $\langle 2 \rangle 4$. Q.E.D.
    PROOF: This contradicts the fact that $b < \theta(a)$.
$\langle 1 \rangle 9$. $\theta(a) = a$
$\square$

Theorem 5.1.21 (Z)**.** *Let A and P be complete posets with no least or greatest element. Let B be dense in A and Q be dense in P. Every order isomorphism $\phi : B \cong Q$ extends uniquely to an order isomorphism $A \cong P$.*

PROOF:
$\langle 1 \rangle 1$. For $a \in A$, let $S(a) = \{b \in B \mid b < a\}$.
$\langle 1 \rangle 2$. Define $\overline{\phi} : A \to P$ by $\overline{\phi}(a) = \sup \phi(S(a))$.
  $\langle 2 \rangle 1$. $\phi(S(a))$ is nonempty.
    $\langle 3 \rangle 1$. PICK $a_1 < a$
      PROOF: Since a is not least.
    $\langle 3 \rangle 2$. PICK $b \in B$ such that $a_1 < b < a$.
    $\langle 3 \rangle 3$. $\phi(b) \in \phi(S(a))$
  $\langle 2 \rangle 2$. $\phi(S(a))$ is bounded above.
    $\langle 3 \rangle 1$. PICK $a_2 > a$
      PROOF: Since a is not greatest.
    $\langle 3 \rangle 2$. PICK $b \in B$ such that $a < b < a_2$

⟨3⟩3. $\phi(b)$ is an upper bound for $\phi(S(a))$.

⟨1⟩3. $\overline{\phi}$ is monotone.

PROOF: If $a \leq a'$ then $S(a) \subseteq S(a')$ and so $\overline{\phi}(a) \leq \overline{\phi}(a')$.

⟨1⟩4. $\overline{\phi}$ extends ϕ.

⟨2⟩1. LET: $b \in B$

PROVE: $\phi(b) = \sup \phi(S(b))$

⟨2⟩2. $\phi(b)$ is an upper bound for $\phi(S(b))$

⟨2⟩3. LET: u be any upper bound for $\phi(S(b))$

PROVE: $\phi(b) \leq u$

⟨2⟩4. ASSUME: for a contradiction $u < \phi(b)$

⟨2⟩5. PICK $q \in Q$ such that $u < q < \phi(b)$

⟨2⟩6. PICK $b' \in B$ such that $\phi(b') = q$

⟨2⟩7. $b' < b$

⟨2⟩8. $b' \in S(b)$

⟨2⟩9. $q = \phi(b') \leq u$

⟨2⟩10. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩5. LET: $\psi = \phi^{-1}$

⟨1⟩6. LET: $\overline{\psi} : P \to A$ be the function $\overline{\psi}(p) = \sup\{\psi(q) \mid q \in Q, q < p\}$

⟨1⟩7. $\overline{\psi}$ is monotone and extends ψ

PROOF: Similar.

⟨1⟩8. $\overline{\psi} \circ \overline{\phi} : A \to A$ is monotone and the identity on B.

⟨1⟩9. $\overline{\psi} \circ \overline{\phi} = \mathrm{id}_A$

PROOF: Proposition 5.1.20.

⟨1⟩10. $\overline{\phi} \circ \overline{\psi} = \mathrm{id}_B$

PROOF: Proposition 5.1.20.

⟨1⟩11. If $\phi^* : A \cong P$ is any order isomorphism that extends ϕ then $\phi^* = \overline{\phi}$.

⟨2⟩1. LET: $a \in A$

PROVE: $\phi^*(a) = \sup \phi(S(a))$

⟨2⟩2. $\phi^*(a)$ is an upper bound for $\phi(S(a))$

⟨2⟩3. LET: u be any upper bound for $\phi(S(a))$

PROVE: $\phi^*(a) \leq u$

⟨2⟩4. ASSUME: for a contradiction $u < \phi^*(a)$

⟨2⟩5. PICK $q \in Q$ such that $u < q < \phi^*(a)$

⟨2⟩6. PICK $b \in B$ such that $q = \phi(b)$

⟨2⟩7. $b < a$

⟨2⟩8. $b \in S(a)$

⟨2⟩9. $q = \phi(b) \leq u$

⟨2⟩10. Q.E.D.

PROOF: This is a contradiction.

□

Definition 5.1.22 (Initial Segment)**.** Let $\leq$ be a partial order on $\mathbf{A}$ and $t \in A$. The *initial segment* up to t is the class

$$\mathrm{seg}\, t := \{x \in \mathbf{A} \mid x < t\} \ .$$

Definition 5.1.23 (Lexicographic Ordering)**.** Let $\mathbf{R}$ be a partial order on $\mathbf{A}$ and $\mathbf{S}$ a partial order on $\mathbf{B}$. The *lexicographic ordering* $\leq$ on $\mathbf{A} \times \mathbf{B}$ is defined by:

$$(a, b) \leq (a', b') \Leftrightarrow (a\mathbf{R}a' \wedge a \neq a') \vee (a = a' \wedge b\mathbf{S}b') \enspace .$$

Proposition Schema 5.1.24 (Z)**.** *For any classes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{R}$ and $\mathbf{S}$, the following is a theorem:*

If $\mathbf{R}$ is a partial order on $\mathbf{A}$ and $\mathbf{S}$ is a partial order on $\mathbf{B}$ then the lexicographic ordering on $\mathbf{A} \times \mathbf{B}$ is a partial order.

PROOF:
⟨1⟩1. LET: $\leq$ be the lexicographic ordering on $\mathbf{A} \times \mathbf{B}$
⟨1⟩2. $\leq$ is reflexive.
PROOF: For any $a \in \mathbf{A}$ and $b \in \mathbf{B}$ we have $a = a$ and $b\mathbf{S}b$, so $(a, b) \leq (a, b)$.
⟨1⟩3. $\leq$ is antisymmetric.
⟨2⟩1. ASSUME: $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$
⟨2⟩2. $(a\mathbf{R}a' \wedge a \neq a') \vee (a = a' \wedge b\mathbf{S}b')$
⟨2⟩3. $(a'\mathbf{R}a \wedge a' \neq a) \vee (a' = a \wedge b\mathbf{S}b')$
⟨2⟩4. CASE: $a = a'$
PROOF: Then $b\mathbf{S}b'$ and $b'\mathbf{S}b$ hence $b = b'$ and $(a, b) = (a', b')$.
⟨2⟩5. CASE: $a \neq a'$
PROOF: Then $a\mathbf{R}a'$ and $a'\mathbf{R}a$ hence $a = a'$ which is a contradiction.
⟨1⟩4. $\leq$ is transitive.
⟨2⟩1. ASSUME: $(a_1, b_1) \leq (a_2, b_2) \leq (a_3, b_3)$
⟨2⟩2. $(a_1\mathbf{R}a_2 \wedge a_1 \neq a_2) \vee (a_1 = a_2 \wedge b_1\mathbf{S}b_2)$
⟨2⟩3. $(a_2\mathbf{R}a_3 \wedge a_2 \neq a_3) \vee (a_2 = a_3 \wedge b_2\mathbf{S}b_3)$
⟨2⟩4. CASE: $a_1\mathbf{R}a_2, a_1 \neq a_2, a_2\mathbf{R}a_3, a_2 \neq a_3$
⟨3⟩1. $a_1\mathbf{R}a_3$
PROOF: Since $\mathbf{R}$ is transitive.
⟨3⟩2. $a_1 \neq a_3$
PROOF: If $a_1 = a_3$ then $a_1\mathbf{R}a_2$ and $a_2\mathbf{R}a_1$ so $a_1 = a_2$ which is a contradiction.
⟨2⟩5. CASE: $a_1\mathbf{R}a_2, a_1 \neq a_2, a_2 = a_3, b_2\mathbf{S}b_3$
PROOF: Then $a_1\mathbf{R}a_3$ and $a_1 \neq a_3$.
⟨2⟩6. CASE: $a_1 = a_2, b_1\mathbf{S}b_2, a_2\mathbf{R}a_3, a_2 \neq a_3$
PROOF: Then $a_1\mathbf{R}a_3$ and $a_1 \neq a_3$.
⟨2⟩7. CASE: $a_1 = a_2, b_1\mathbf{S}b_2, a_2 = a_3, b_2\mathbf{S}b_3$
PROOF: Then $a_1 = a_3$ and $b_1\mathbf{S}b_3$.
□

5.2 Linear Orders

Definition 5.2.1 (Linear Ordering)**.** Let $\mathbf{A}$ be a class. A *linear ordering* or *total ordering* on $\mathbf{A}$ is a partial ordering $\leq$ on $\mathbf{A}$ that is *total*, i.e.

$$\forall x, y \in \mathbf{A}. x \leq y \vee y \leq x$$

We often use the symbol $<$ for a linear ordering, and then write $x < y$ for $(x, y) \in <$.

Proposition Schema 5.2.2 (Trichotomy (Z)). *For any classes* $\mathbf{A}$ *and* $\leq$*, the following is a theorem:*

Assume $\leq$ *be a linear ordering on* $\mathbf{A}$*. For any* $x, y \in \mathbf{A}$*, exactly one of* $x < y$*,* $x = y$*,* $y < x$ *holds.*

PROOF: Immediate from definitions. □

Proposition Schema 5.2.3 (Z). *For any classes* $\mathbf{A}$ *and* $<$*, the following is a theorem:*

Let $<$ *be a transitive relation on* $\mathbf{A}$ *that satisfies trichotomy. Define* $\leq$ *on* $\mathbf{A}$ *by* $x \leq y$ *iff* $x < y$ *or* $x = y$*. Then* $\leq$ *is a linear ordering on* $\mathbf{A}$ *and* $x < y$ *iff* $x \leq y$ *and* $x \neq y$*.*

PROOF:
⟨1⟩1. $\leq$ is reflexive.
  PROOF: By definition we have $\forall x \in \mathbf{A}. x \leq x$.
⟨1⟩2. $\leq$ is antisymmetric.
  ⟨2⟩1. ASSUME: $x \leq y$ and $y \leq x$
  ⟨2⟩2. $x < y$ or $x = y$
  ⟨2⟩3. $y < x$ or $y = x$
  ⟨2⟩4. We cannot have $x < y$ and $y < x$
    PROOF: Trichotomy.
  ⟨2⟩5. $x = y$
⟨1⟩3. $\leq$ is transitive.
  ⟨2⟩1. ASSUME: $x \leq y$ and $y \leq z$
  ⟨2⟩2. $x < y$ or $x = y$
  ⟨2⟩3. $y < z$ or $y = z$
  ⟨2⟩4. CASE: $x < y$ and $y < z$
    PROOF: Then $x < z$ by transitivity, so $x \leq z$.
  ⟨2⟩5. CASE: $x = y$
    PROOF: Then we have $y \leq z$ and so $x \leq z$.
  ⟨2⟩6. CASE: $y = z$
    PROOF: Then we have $x \leq y$ and so $x \leq z$.
⟨1⟩4. $\leq$ is total.
  PROOF: Immediate from trichotomy.
□

Proposition Schema 5.2.4 (Z). *For any classes* $\mathbf{A}$ *and* $\mathbf{R}$*, the following is a theorem:*

If $\mathbf{R}$ *is a linear ordering on* $\mathbf{A}$ *then* $\mathbf{R}^{-1}$ *is also a linear ordering on* $\mathbf{A}$*.*

PROOF:
⟨1⟩1. $\mathbf{R}^{-1}$ is a partial order on $\mathbf{A}$.
  PROOF: Proposition 5.1.2.
⟨1⟩2. $\mathbf{R}^{-1}$ is total.

⟨2⟩1. LET: $x, y \in \mathbf{A}$
⟨2⟩2. $x\mathbf{R}y$ or $y\mathbf{R}x$.
⟨2⟩3. $y\mathbf{R}^{-1}x$ or $x\mathbf{R}^{-1}y$.
□

Proposition Schema 5.2.5 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{F}$*,* $\mathbf{R}$*,* $\mathbf{S}$*, the following is a theorem:*

Assume $\mathbf{R}$ *is a linear order on* $\mathbf{A}$*,* $\mathbf{S}$ *is a partial order on* $\mathbf{B}$*, and* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$*. If* $\mathbf{F}$ *is strictly monotone then it is injective.*

PROOF:
⟨1⟩1. LET: $x, y \in \mathbf{A}$
⟨1⟩2. ASSUME: $x \neq y$
PROVE: $\mathbf{F}(x) \neq \mathbf{F}(y)$
⟨1⟩3. ASSUME: w.l.o.g. $x\mathbf{R}y$
PROOF: $\mathbf{R}$ is total.
⟨1⟩4. $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$ and $\mathbf{F}(x) \neq \mathbf{F}(y)$
PROOF: $\mathbf{F}$ is strictly monotone.
□

Proposition Schema 5.2.6 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\leq$*,* $\preccurlyeq$ *and* $\mathbf{F}$*, the following is a theorem:*

Assume $\leq$ *is a linear order on* $\mathbf{A}$ *and* $\preccurlyeq$ *is a linear order on* $\mathbf{B}$*. Assume* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{F}$ *is strictly monotone. For all* $x, y \in \mathbf{A}$*, if* $\mathbf{F}(x) \prec \mathbf{F}(y)$ *then* $x < y$*.*

PROOF:
⟨1⟩1. $\mathbf{F}(x) \neq \mathbf{F}(y)$ and $\mathbf{F}(y) \not\prec \mathbf{F}(x)$
PROOF: Trichotomy.
⟨1⟩2. $x \neq y$ and $y \not< x$
PROOF: $\mathbf{F}$ is strictly monotone.
⟨1⟩3. $x < y$
PROOF: Trichotomy.
□

Corollary Schema 5.2.6.1 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\leq$*,* $\preccurlyeq$ *and* $\mathbf{F}$*, the following is a theorem:*

Assume $\leq$ *is a linear order on* $\mathbf{A}$ *and* $\preccurlyeq$ *is a linear order on* $\mathbf{B}$*. Assume* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{F}$ *is strictly monotone. Then* $\mathbf{F}$ *is an order isomorphism.*

Proposition Schema 5.2.7 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{F}$ *and* $\mathbf{S}$*, the following is a theorem:*

Assume $\mathbf{S}$ *is a linear order on* $\mathbf{B}$ *and* $\mathbf{F} : \mathbf{A} \rightarrowtail \mathbf{B}$*. Define* $\mathbf{R}$ *on* $\mathbf{A}$ *by* $x\mathbf{R}y$ *if and only if* $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$*. Then* $\mathbf{R}$ *is a linear order on* $\mathbf{A}$*.*

PROOF:
⟨1⟩1. $\mathbf{R}$ is a partial order on $\mathbf{A}$.
PROOF: Proposition 5.1.3.

$\langle 1 \rangle 2$. $\mathbf{R}$ is total.
PROOF: For all $x, y \in \mathbf{A}$ we have $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$ or $\mathbf{F}(y)\mathbf{S}\mathbf{F}(x)$.
$\square$

Corollary Schema 5.2.7.1 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{R}$*, the following is a theorem:*

Assume $\mathbf{R}$ *be a linear order on* $\mathbf{A}$ *and* $\mathbf{B} \subseteq \mathbf{A}$*. Then* $\mathbf{R} \cap \mathbf{B}^2$ *is a linear order on* $\mathbf{B}$*.*

Proposition Schema 5.2.8 (Z)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{R}$ *and* $\mathbf{S}$*, the following is a theorem:*

Assume $\mathbf{R}$ *is a linear order on* $\mathbf{A}$ *and* $\mathbf{S}$ *is a linear order on* $\mathbf{B}$*. Then the lexicographic ordering is a linear order on* $\mathbf{A} \times \mathbf{B}$*.*

PROOF:
$\langle 1 \rangle 1$. LET: $\leq$ be the lexicographic order on $\mathbf{A} \times \mathbf{B}$
$\langle 1 \rangle 2$. $\leq$ is a partial order.
PROOF: Proposition 5.1.24.
$\langle 1 \rangle 3$. $\leq$ is total.
$\langle 2 \rangle 1$. LET: $a, a' \in \mathbf{A}$ and $b, b' \in \mathbf{B}$
$\langle 2 \rangle 2$. CASE: $a\mathbf{R}a'$ and $a \neq a'$
PROOF: Then $(a, b) \leq (a', b')$.
$\langle 2 \rangle 3$. CASE: $a = a'$
PROOF: We have $b\mathbf{S}b'$ or $b'\mathbf{S}b$, so $(a, b) \leq (a', b')$ or $(a', b') \leq (a, b)$.
$\langle 2 \rangle 4$. CASE: $a'\mathbf{R}a$ and $a \neq a'$
PROOF: Then $(a', b') \leq (a, b)$.
$\square$

5.3 Well Orderings

Definition 5.3.1 (Well Ordering)**.** A *well ordering* on a class $\mathbf{A}$ is a well-founded linear ordering on $\mathbf{A}$.

Proposition 5.3.2 (Z)**.** *Let* S *be a well ordering of the set* B *and* $f : A \to B$ *a function. Define* R *on* A *by* xRy *if and only if* $F(x)SF(y)$*. Then* R *well orders* A*.*

PROOF:
$\langle 1 \rangle 1$. R linearly orders A.
PROOF: Proposition 5.2.7.
$\langle 1 \rangle 2$. Every nonempty subset of A has a least element.
$\langle 2 \rangle 1$. LET: C be a nonempty subset of A.
$\langle 2 \rangle 2$. LET: y be the least element of $f(C)$.
$\langle 2 \rangle 3$. PICK $x \in C$ such that $f(x) = y$.
$\langle 2 \rangle 4$. x is least in C.
$\square$

Proposition Schema 5.3.3 (Z). *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{R}$*, the following is a theorem:*

Assume $\mathbf{R}$ *well orders* $\mathbf{B}$ *and* $\mathbf{A} \subseteq \mathbf{B}$*. Then* $\mathbf{R} \cap \mathbf{A}^2$ *well orders* $\mathbf{A}$*.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\mathbf{R}' = \mathbf{R} \cap \mathbf{A}^2$
$\langle 1 \rangle 2.$ $\mathbf{R}'$ linearly orders $\mathbf{A}$.
PROOF: Corollary 5.2.7.1.
$\langle 1 \rangle 3.$ $\mathbf{R}'$ is well founded.
PROOF: Proposition 2.4.4.
□

Proposition Schema 5.3.4 (ZFC). *For any classes* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{F}$ *and* $\mathbf{S}$*, the following is a theorem:*

Assume $\mathbf{S}$ *well orders* $\mathbf{B}$ *and* $\mathbf{F} : \mathbf{A} \rightarrowtail \mathbf{B}$*. Define* $\mathbf{R}$ *on* $\mathbf{A}$ *by* $x\mathbf{R}y$ *if and only if* $\mathbf{F}(x)\mathbf{S}\mathbf{F}(y)$*. Then* $\mathbf{R}$ *well orders* $\mathbf{A}$*.*

PROOF:
$\langle 1 \rangle 1.$ $\mathbf{R}$ linearly orders $\mathbf{A}$.
PROOF: Proposition 5.2.7.
$\langle 1 \rangle 2.$ For all $t \in \mathbf{A}$ we have $\{x \in \mathbf{A} \mid x\mathbf{R}t \wedge x \neq t\}$ is a set.
$\langle 2 \rangle 1.$ LET: $t \in \mathbf{A}$
$\langle 2 \rangle 2.$ LET: $S = \{y \in \mathbf{B} \mid y\mathbf{S}\mathbf{F}(t) \wedge y \neq \mathbf{F}(t)\}$
$\langle 2 \rangle 3.$ LET: $P(x, y)$ be the property $\mathbf{F}(y) = x$
$\langle 2 \rangle 4.$ For all $x \in S$ there exists at most one y such that $P(x, y)$
PROOF: $\mathbf{F}$ is injective.
$\langle 2 \rangle 5.$ LET: $T = \{y \mid \exists x \in S.P(x, y)\}$
PROOF: Axiom of Replacement.
$\langle 2 \rangle 6.$ $T = \{x \in \mathbf{A} \mid x\mathbf{R}t \wedge x \neq t\}$
$\langle 1 \rangle 3.$ Every nonempty subset of $\mathbf{A}$ has a least element.
$\langle 2 \rangle 1.$ LET: S be a nonempty subset of $\mathbf{A}$.
$\langle 2 \rangle 2.$ $\mathbf{F}(S)$ is a nonempty subset of $\mathbf{B}$
PROOF: Axiom of Replacement.
$\langle 2 \rangle 3.$ LET: y be the least element of $\mathbf{F}(S)$.
$\langle 2 \rangle 4.$ PICK $x \in S$ such that $\mathbf{F}(x) = y$.
$\langle 2 \rangle 5.$ x is least in S.
□

Proposition 5.3.5 (Z). *For any well ordered sets* A *and* B*, the lexicographic order well orders* $A \times B$*.*

PROOF:
$\langle 1 \rangle 1.$ $A \times B$ is linearly ordered.
PROOF: Proposition 5.2.8.
$\langle 1 \rangle 2.$ Every nonempty subset of $A \times B$ has a least element.
$\langle 2 \rangle 1.$ LET: S be a nonempty subset of $A \times B$.
$\langle 2 \rangle 2.$ LET: a be the least element of $\{x \in A \mid \exists y \in B.(x, y) \in S\}$.
$\langle 2 \rangle 3.$ LET: b be the least element of $\{y \in B \mid (a, y) \in S\}$.

$\langle 2\rangle 4$. (a, b) is least in S.
$\square$

Definition 5.3.6 (End Extension). Let A and B be well ordered sets. Then B is an *end extension* of A iff $A \subseteq B$ and:

- Whenever $x, y \in A$ then $x \leq_A y$ iff $x \leq_B y$.
- Whenever $x \in A$ and $y \in B - A$ then $x < y$.

Theorem 5.3.7 (Z). *Let $\leq$ be a linear ordering on A. Assume that, for any $B \subseteq A$ such that $\forall t \in A. \operatorname{seg} t \subseteq B \Rightarrow t \in B$, we have $B = A$. Then $\leq$ is a well ordering on A.*

PROOF:
$\langle 1\rangle 1$. LET: $C \subseteq A$ be nonempty.
$\langle 1\rangle 2$. LET: $B = \{t \in A \mid \forall x \in C. t < x\}$
$\langle 1\rangle 3$. $B \cap C = \emptyset$
$\langle 1\rangle 4$. $B \neq A$
$\langle 1\rangle 5$. PICK $t \in A$ such that $\operatorname{seg} t \subseteq B$ and $t \notin B$
$\langle 1\rangle 6$. t is least in C.
$\square$

Proposition Schema 5.3.8 (Z). *For any classes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{F}$, $\mathbf{G}$, $\leq$ and $\preccurlyeq$, the following is a theorem:*

Assume $\leq$ well orders $\mathbf{A}$ and $\preccurlyeq$ well orders $\mathbf{B}$. Assume $\mathbf{F}$ and $\mathbf{G}$ are order isomorphisms between $\mathbf{A}$ and $\mathbf{B}$. Then $\mathbf{F} = \mathbf{G}$.

PROOF:
$\langle 1\rangle 1$. For all $x \in \mathbf{A}$, if $\forall t < x. \mathbf{F}(t) = \mathbf{G}(t)$, then $\mathbf{F}(x) = \mathbf{G}(x)$
$\langle 2\rangle 1$. LET: $x \in \mathbf{A}$
$\langle 2\rangle 2$. ASSUME: $\forall t < x. \mathbf{F}(t) = \mathbf{G}(t)$
$\langle 2\rangle 3$. $\mathbf{F}(\operatorname{seg} x) = \mathbf{G}(\operatorname{seg} x)$
$\langle 2\rangle 4$. $\mathbf{F}(x)$ is the least element of $\mathbf{B} - \mathbf{F}(\operatorname{seg} x)$
$\langle 2\rangle 5$. $\mathbf{G}(x)$ is the least element of $\mathbf{B} - \mathbf{G}(\operatorname{seg} x)$
$\langle 2\rangle 6$. $\mathbf{F}(x) = \mathbf{G}(x)$
$\langle 1\rangle 2$. $\forall x \in \mathbf{A}. \mathbf{F}(x) = \mathbf{G}(x)$
PROOF: Transfinite induction.
$\square$

Theorem 5.3.9 (ZFC). *Let A and B be well ordered sets. Then one of the following holds: $A \cong B$; there exists $b \in B$ such that $A \cong \operatorname{seg} b$; there exists $a \in A$ such that $\operatorname{seg} a \cong B$.*

PROOF:
$\langle 1\rangle 1$. PICK e that is not in A or B.
$\langle 1\rangle 2$. LET: $F : A \to B \cup \{e\}$ be the function defined by transfinite recursion thus:

$$F(t) = \begin{cases} \text{the least element of } B - F(\operatorname{seg} t) & \text{if } B - F(\operatorname{seg} t) \neq \emptyset \\ e & \text{if } B - F(\operatorname{seg} t) = \emptyset \end{cases}$$

$\langle 1 \rangle 3$. CASE: $e \in \operatorname{ran} F$

  $\langle 2 \rangle 1$. LET: t be least such that $F(t) = e$

  $\langle 2 \rangle 2$. $F \upharpoonright \operatorname{seg} t : \operatorname{seg} t \cong B$

$\langle 1 \rangle 4$. CASE: $\operatorname{ran} F = B$

  PROOF: We have $F : A \cong B$

$\langle 1 \rangle 5$. CASE: $\operatorname{ran} F \subsetneq B$

  $\langle 2 \rangle 1$. LET: b be the least element of $B - \operatorname{ran} F$

  $\langle 2 \rangle 2$. $F : A \cong \operatorname{seg} b$

$\square$

Chapter 6

Ordinal Numbers

6.1 Ordinals

Definition 6.1.1 (Ordinal Number)**.** An *ordinal (number)* is a transitive set α that is *well-ordered by* $\in$; that is, such that $\{(x, y) \in \alpha^2 \mid x \in y \vee x = y\}$ well orders α.

Given $x, y \in \alpha$, we write $x < y$ iff $x \in y$, and $x \leq y$ iff $x \in y$ or $x = y$.

Let **On** be the class of ordinal numbers. For $\alpha, \beta \in \mathbf{On}$, we write $\alpha < \beta$ iff $\alpha \in \beta$, and $\alpha \leq \beta$ iff $\alpha < \beta$ or $\alpha = \beta$.

Proposition 6.1.2 (Z)**.** *For any ordinal numbers α and β, if $\alpha \cong \beta$ then $\alpha = \beta$.*

Proof:
$\langle 1 \rangle 1.$ Let: $f : \alpha \cong \beta$
$\langle 1 \rangle 2.$ For all $x \in \alpha$, if $\forall t < x. f(t) = t$ then $f(x) = x$
  $\langle 2 \rangle 1.$ $f(x) \subseteq x$
    $\langle 3 \rangle 1.$ Let: $y \in f(x)$
    $\langle 3 \rangle 2.$ $y \in \beta$
    $\langle 3 \rangle 3.$ Pick $t \in \alpha$ such that $f(t) = y$
      Proof: f is surjective.
    $\langle 3 \rangle 4.$ $f(t) \in f(x)$
    $\langle 3 \rangle 5.$ $t \in x$
      Proof: Since f is an order isomorphism.
    $\langle 3 \rangle 6.$ $f(t) = t$
      Proof: Induction hypothesis.
    $\langle 3 \rangle 7.$ $y = t$
    $\langle 3 \rangle 8.$ $y \in x$
  $\langle 2 \rangle 2.$ $x \subseteq f(x)$
    $\langle 3 \rangle 1.$ Let: $t \in x$
    $\langle 3 \rangle 2.$ $f(t) \in f(x)$
    $\langle 3 \rangle 3.$ $f(t) = t$
    $\langle 3 \rangle 4.$ $t \in f(x)$

$\langle 1\rangle 3.$ $\forall x \in \alpha . f(x) = x$
PROOF: Transfinite induction.
$\langle 1\rangle 4.$ $\alpha = \beta$
PROOF: Since $\beta = \{f(t) \mid t \in \alpha\} = \{t \mid t \in \alpha\} = \alpha$.
□

Theorem 6.1.3 (ZFC). *Every well-ordered set is isomorphic to a unique ordinal.*

PROOF:
$\langle 1\rangle 1.$ For any well-ordered set A, there exists an ordinal α such that $A \cong \alpha$.
$\langle 2\rangle 1.$ LET: A be a well-ordered set.
$\langle 2\rangle 2.$ Define the function E on A by transfinite recursion thus:
$$E(t) = \{E(x) \mid x < t\} \qquad (t \in A) \ .$$
$\langle 2\rangle 3.$ LET: $\alpha = \{E(x) \mid x \in A\}$
$\langle 2\rangle 4.$ α is an ordinal.
$\langle 3\rangle 1.$ α is a transitive set.
$\langle 4\rangle 1.$ LET: $x \in y \in \alpha$
$\langle 4\rangle 2.$ PICK $t \in A$ such that $y = E(t)$
$\langle 4\rangle 3.$ $x \in E(t) = \{E(s) \mid s < t\}$
$\langle 4\rangle 4.$ PICK $s < t$ such that $x = E(s)$
$\langle 4\rangle 5.$ $x \in \alpha$
$\langle 3\rangle 2.$ α is well-ordered by $\in$.
$\langle 4\rangle 1.$ LET: $<= \{(x, y) \in \alpha \mid x \in y\}$
$\langle 4\rangle 2.$ $<$ is transitive.
$\langle 5\rangle 1.$ LET: $x, y, z \in \alpha$ with $x \in y \in z$
$\langle 5\rangle 2.$ PICK $t \in A$ such that $z = E(t)$
$\langle 5\rangle 3.$ PICK $s \in A$ such that $s < t$ and $y = E(s)$
$\langle 5\rangle 4.$ PICK $r \in A$ such that $r < s$ and $x = E(r)$
$\langle 5\rangle 5.$ $r < t$
$\langle 5\rangle 6.$ $x \in z$
$\langle 4\rangle 3.$ $<$ satisfies trichotomy.
$\langle 5\rangle 1.$ LET: $x, y \in \alpha$
$\langle 5\rangle 2.$ PICK $s, t \in A$ such that $E(s) = x$ and $E(t) = y$
$\langle 5\rangle 3.$ Exactly one of $s < t$, $s = t$, $t < s$ holds.
$\langle 5\rangle 4.$ CASE: $s < t$
$\langle 6\rangle 1.$ $x \in y$
$\langle 6\rangle 2.$ $x \neq y$ and $y \notin x$
PROOF: Axiom of Regularity.
$\langle 5\rangle 5.$ CASE: $s = t$
$\langle 6\rangle 1.$ $x = y$
$\langle 6\rangle 2.$ $x \notin y$ and $y \notin x$
PROOF: Axiom of Regularity.
$\langle 5\rangle 6.$ CASE: $t < s$
PROOF: Similar to $\langle 5\rangle 4$.
$\langle 4\rangle 4.$ $\leq$ is a linear order on α.
PROOF: Proposition 5.2.3.

$\langle 4\rangle 5$. Every nonempty subset of α has a least element.
$\langle 5\rangle 1$. LET: S be a nonempty subset of α
$\langle 5\rangle 2$. LET: $T = \{x \in A \mid E(x) \in S\}$
$\langle 5\rangle 3$. LET: t be the least element of T.
PROVE: $E(t)$ is least in S
$\langle 5\rangle 4$. LET: $y \in S$
$\langle 5\rangle 5$. PICK $s \in T$ such that $E(s) = y$
$\langle 5\rangle 6$. $t \leq s$
$\langle 5\rangle 7$. $x \leq y$
$\langle 2\rangle 5$. E is surjective.
PROOF: By definition of α.
$\langle 2\rangle 6$. E is strictly monotone.
PROOF: If $s < t$ then $E(s) \in E(t)$ by definition of $E(t)$.
$\langle 2\rangle 7$. Q.E.D.
PROOF: Corollary 5.2.6.1.
$\langle 1\rangle 2$. For any ordinals α and β, if $\alpha \cong \beta$ then $\alpha = \beta$.
PROOF: Proposition 6.1.2.
□

Proposition 6.1.4 (Z)**.** *The class* **On** *is a transitive class. That is, every element of an ordinal is an ordinal.*

PROOF:
$\langle 1\rangle 1$. LET: α be an ordinal.
$\langle 1\rangle 2$. LET: $\beta \in \alpha$
$\langle 1\rangle 3$. β is a transitive set.
$\langle 2\rangle 1$. LET: $x \in y \in \beta$
$\langle 2\rangle 2$. $y \in \alpha$
PROOF: α is transitive.
$\langle 2\rangle 3$. $x \in \alpha$
PROOF: α is transitive.
$\langle 2\rangle 4$. $x \in \beta$
PROOF: Since $\{(x, y) \in \alpha^2 \mid x \in y\}$ is transitive.
$\langle 1\rangle 4$. β is well ordered by $\in$.
PROOF: By Proposition 5.3.3.
□

Proposition 6.1.5 (ZFC)**.** *Given two ordinal numbers α, β, exactly one of $\alpha \in \beta$, $\alpha = \beta$, $\beta \in \alpha$ holds.*

PROOF:
$\langle 1\rangle 1$. At most one holds.
PROOF: Since every ordinal is a transitive set and we never have $\alpha \in \alpha$.
$\langle 1\rangle 2$. At least one holds.
$\langle 2\rangle 1$. Either $\alpha \cong \beta$ or $\exists t \in \beta.\alpha \cong \operatorname{seg} t$ or $\exists t \in \alpha. \operatorname{seg} t \cong \beta$.
$\langle 2\rangle 2$. CASE: $\alpha \cong \beta$
PROOF: Then $\alpha = \beta$ by Proposition 6.1.2.

$\langle 2\rangle 3$. CASE: There exists $t \in \beta$ such that $\alpha \cong \operatorname{seg} t$
$\langle 3\rangle 1$. t is an ordinal number.
PROOF: Proposition 6.1.4.
$\langle 3\rangle 2$. $t = \operatorname{seg} t$
$\langle 4\rangle 1$. $t \subseteq \operatorname{seg} t$
$\langle 5\rangle 1$. LET: $s \in t$
$\langle 5\rangle 2$. $s \in \beta$
PROOF: β is a transitive set.
$\langle 5\rangle 3$. $s \in \operatorname{seg} t$
$\langle 4\rangle 2$. $\operatorname{seg} t \subseteq t$
PROOF: Immediate from definitions.
$\langle 3\rangle 3$. $\alpha = t$
PROOF: Proposition 6.1.2.
$\langle 3\rangle 4$. $\alpha \in \beta$
$\langle 2\rangle 4$. CASE: There exists $t \in \alpha$ such that $\operatorname{seg} t \cong \beta$
PROOF: $\beta \in \alpha$ similarly.
□

Proposition 6.1.6 (Z). *Any nonempty set S of ordinal numbers has a least element.*

PROOF:
$\langle 1\rangle 1$. PICK $\beta \in S$
$\langle 1\rangle 2$. CASE: $\beta \cap S = \emptyset$
PROOF: Then β is least in S.
$\langle 1\rangle 3$. CASE: $\beta \cap S \neq \emptyset$
PROOF: The least element of $\beta \cap S$ is least in S.
□

Theorem 6.1.7 (ZFC). *The class* **On** *is well ordered by* $\in$*.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathbf{E} = \{(x, y) \in \mathbf{On}^2 \mid x \in y\}$
$\langle 1\rangle 2$. $\mathbf{E}$ is transitive.
PROOF: If $\alpha \in \beta \in \gamma$ then $\alpha \in \gamma$ because every ordinal is a transitive set.
$\langle 1\rangle 3$. $\mathbf{E}$ satisfies trichotomy.
PROOF: Proposition 6.1.5.
$\langle 1\rangle 4$. $\mathbf{E}$ linearly orders **On**.
PROOF: Proposition 5.2.3.
$\langle 1\rangle 5$. $\mathbf{E}$ is well founded.
PROOF: Proposition 2.4.2.
□

Corollary 6.1.7.1 (Burali-Forti Paradox (ZFC)). *The class* **On** *is a proper class.*

PROOF: If it were a set, it would be a transitive set well-ordered by $\in$, and hence a member of itself, contradicting Proposition 1.5.3.

Proposition 6.1.8 (ZFC)**.** *Any transitive set of ordinal numbers is an ordinal number.*

PROOF: It is well-ordered by $\in$ by Proposition 5.3.3 and Theorem 6.1.7. □

Proposition 6.1.9 (Z)**.** *$\emptyset$ is an ordinal number.*

PROOF: Vacuously, it is a transitive set well-ordered by $\in$.

Definition 6.1.10. We define $0 = \emptyset$.

Proposition 6.1.11 (ZFC)**.** *If A is a set of ordinal numbers then $\bigcup A$ is an ordinal number.*

PROOF:
⟨1⟩1. $\bigcup A$ is a transitive set.
  PROOF: Proposition 1.6.3.
⟨1⟩2. $\bigcup A$ is a set of ordinals.
  PROOF: Proposition 6.1.4.
⟨1⟩3. Q.E.D.
  PROOF: Proposition 6.1.8.
□

Corollary 6.1.11.1 (ZFC)**.** *The poset* **On** *is complete.*

PROOF: For any nonempty set A of ordinals, $\bigcup A$ is its supremum. □

Proposition 6.1.12 (ZFC)**.** *Let α be an ordinal and $S \subseteq \alpha$. Then S is well-ordered by $\in$ and the ordinal of $(S, \in)$ is $\leq \alpha$.*

PROOF:
⟨1⟩1. S is well ordered by $\in$.
⟨1⟩2. LET: β be the ordinal of $(S, \in)$
⟨1⟩3. LET: $E : S \approx \beta$ be the unique isomorphism.
⟨1⟩4. $\forall \gamma \in S. E(\gamma) \leq \gamma$
  ⟨2⟩1. LET: $\gamma \in S$
  ⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall \delta < \gamma. E(\delta) \leq \delta$
  ⟨2⟩3. $E(\gamma)$ is the least element of β that is greater than $E(\delta)$ for all $\delta < \gamma$
  ⟨2⟩4. γ is greater than $E(\delta)$ for all $\delta < \gamma$
  ⟨2⟩5. $E(\gamma) \leq \gamma$
⟨1⟩5. $\beta \leq \alpha$
  ⟨2⟩1. $\forall \gamma < \beta. \gamma < \alpha$
    ⟨3⟩1. LET: $\gamma < \beta$
    ⟨3⟩2. PICK $\delta \in S$ such that $E(\delta) = \gamma$
    ⟨3⟩3. $\gamma = E(\delta) \leq \delta < \alpha$
□

Proposition 6.1.13 (ZFC)**.** *Let α be a set. Then the following are equivalent.*

1. *α is an ordinal.*

2. *α is a transitive set and, for all $x, y \in \alpha$, either $x = y$ or $x \in y$ or $y \in x$.*
3. *α is a transitive set of transitive sets.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  PROOF: Immediate from definitions.
⟨1⟩2. $2 \Rightarrow 3$
  ⟨2⟩1. ASSUME: α is a transitive set and, for all $x, y \in \alpha$, either $x = y$ or $x \in y$ or $y \in x$
  ⟨2⟩2. LET: $z \in \alpha$
    PROVE: z is transitive.
  ⟨2⟩3. LET: $x \in y \in z$
  ⟨2⟩4. $y \in \alpha$
  ⟨2⟩5. $x \in \alpha$
  ⟨2⟩6. Either $x = z$ or $x \in z$ or $z \in x$
  ⟨2⟩7. $x \neq z$
    PROOF: We cannot have $x \in y \in x$ by the Axiom of Regularity.
  ⟨2⟩8. $z \notin x$
    PROOF: We cannot have $x \in y \in z \in x$ by the Axiom of Regularity.
⟨1⟩3. $3 \Rightarrow 1$
  ⟨2⟩1. LET: x be a transitive set of transitive sets.
  ⟨2⟩2. ASSUME: as $\in$-induction hypothesis that, for all $y \in x$, if y is a transitive set of transitive sets then y is a transitive set of ordinals.
  ⟨2⟩3. Every element of x is an ordinal.
    ⟨3⟩1. LET: $y \in x$
    ⟨3⟩2. y is transitive.
    ⟨3⟩3. Every element of y is transitive.
      PROOF: Since every element of y is an element of x, because x is transitive.
    ⟨3⟩4. y is an ordinal.
      PROOF: ⟨2⟩2
  ⟨2⟩4. Q.E.D.
    PROOF: Proposition 6.1.8.
□

Lemma 6.1.14 (Z)**.** *Let A and B be well-ordered sets. If B is an end extension of A then the ordinal of A is $\leq$ the ordinal of B.*

PROOF:
⟨1⟩1. LET: α be the ordinal of A and β the ordinal of B.
⟨1⟩2. LET: $E_A : A \cong \alpha$ and $E_B : B \cong \beta$ be the canonical isomorphisms.
⟨1⟩3. $\forall a \in A. E_A(a) = E_B(a)$
  ⟨2⟩1. LET: $a \in A$
  ⟨2⟩2. ASSUME: as transfinite induction hypothesis $\forall x < a. E_A(x) = E_B(x)$
  ⟨2⟩3. $E_A(a)$ is the least ordinal that is greater than $E_A(x)$ for all $x < a$
  ⟨2⟩4. $E_B(a)$ is the least ordinal that is greater than $E_B(x)$ for all $x < b$

⟨2⟩5. $\{x \in A \mid x <_A a\} = \{x \in B \mid x <_B a\}$
⟨2⟩6. $E_A(a) = E_B(a)$
⟨1⟩4. $\alpha \subseteq \beta$
⟨1⟩5. $\alpha \leq \beta$
□

Lemma 6.1.15. *Let $\mathcal{C}$ be a set of well ordered sets such that, for any $A, B \in \mathcal{C}$, we have that one of A and B is an end extension of the other. Let $W = \bigcup \mathcal{C}$ under $x \leq y$ iff there exists $A \in W$ such that $x, y \in A$ and $x \leq y$. Then W is a well ordered set whose ordinal is the supremum of the ordinals of the members of $\mathcal{C}$.*

PROOF:
⟨1⟩1. $\leq$ is reflexive on W.
⟨2⟩1. LET: $x \in W$
⟨2⟩2. PICK $A \in W$ such that $x \in A$.
⟨2⟩3. $x \leq x$
⟨1⟩2. $\leq$ is antisymmetric on W.
⟨2⟩1. LET: $x, y \in W$
⟨2⟩2. ASSUME: $x \leq y$ and $y \leq x$
⟨2⟩3. PICK $A \in W$ such that $x, y \in A$ and $x \leq_A y$, and $B \in W$ such that $x, y \in B$ and $y \leq_B x$
⟨2⟩4. ASSUME: w.l.o.g. B is an end extension of A
⟨2⟩5. $x \leq_B y$ and $y \leq_B x$
⟨2⟩6. $x = y$
⟨1⟩3. $\leq$ is transitive on W.
⟨2⟩1. ASSUME: $x \leq y \leq z$
⟨2⟩2. PICK $A, B \in W$ such that $x \leq_A y$ and $y \leq_B z$
⟨2⟩3. CASE: A is an end extension of B.
⟨3⟩1. $x \leq_A y$ and $y \leq_A z$
⟨3⟩2. $x \leq_A z$
⟨3⟩3. $x \leq z$
⟨2⟩4. CASE: B is an end extension of A.
PROOF: Similar.
⟨1⟩4. $\leq$ is total on W.
⟨2⟩1. LET: $x, y \in W$
⟨2⟩2. PICK $A, B \in \mathcal{C}$ such that $x \in A$ and $y \in B$
⟨2⟩3. ASSUME: w.l.o.g. B is an end extension of A
⟨2⟩4. $x \leq_B y$ or $y \leq_B x$
⟨2⟩5. $x \leq_W y$ or $y \leq_W x$
⟨1⟩5. Every nonempty subset of W has a least element.
⟨2⟩1. LET: S be a nonempty subset of W
⟨2⟩2. PICK $s \in S$
⟨2⟩3. PICK $A \in \mathcal{C}$ such that $s \in A$
⟨2⟩4. LET: a be the $\leq_A$-least element of $S \cap A$
PROVE: a is least in S
⟨2⟩5. LET: $x \in S$

PROVE: $a \leq x$
⟨2⟩6. PICK $B \in \mathcal{C}$ such that $x \in B$
⟨2⟩7. CASE: A is an end extension of B
⟨3⟩1. $a \leq_A x$
⟨3⟩2. $a \leq x$
⟨2⟩8. CASE: B is an end extension of A
⟨3⟩1. CASE: $x \in A$
⟨4⟩1. $a \leq_A x$
⟨4⟩2. $a \leq x$
⟨3⟩2. CASE: $x \in B - A$
⟨4⟩1. $a \leq_B x$
⟨4⟩2. $a \leq x$
⟨1⟩6. For all $A \in \mathcal{C}$, W is an end extension of A.
⟨2⟩1. For all $x, y \in A$, we have $x \leq_A y$ if and only if $x \leq_W y$
⟨3⟩1. LET: $x, y \in A$
⟨3⟩2. If $x \leq_A y$ then $x \leq_W y$
PROOF: Immediate from definitions.
⟨3⟩3. If $x \leq_W y$ then $x \leq_A y$
⟨4⟩1. ASSUME: $x \leq_W y$
⟨4⟩2. PICK $B \in \mathcal{C}$ such that $x \leq_B y$
⟨4⟩3. CASE: A is an end extension of B
PROOF: Then $x \leq_A y$.
⟨4⟩4. CASE: B is an end extension of A
PROOF: Then $x \leq_A y$.
⟨2⟩2. For all $x \in A$ and $y \in W - A$ we have $x < y$
⟨3⟩1. LET: $x \in A$ and $y \in W - A$
⟨3⟩2. PICK $B \in \mathcal{C}$ such that $y \in B$
⟨3⟩3. B is an end extension of A
⟨3⟩4. $x <_B y$
⟨3⟩5. $x <_W y$
⟨1⟩7. For all $A \in \mathcal{C}$, the ordinal of A is $\leq$ the ordinal of W.
PROOF: Lemma 6.1.14.
⟨1⟩8. For any ordinal α, if for all $A \in \mathcal{C}$ the ordinal of A is $\leq \alpha$, then the ordinal of W is $\leq \alpha$.
⟨2⟩1. LET: α be an ordinal.
⟨2⟩2. ASSUME: for all $A \in \mathcal{C}$, the ordinal of A is $\leq \alpha$
⟨2⟩3. LET: β be the ordinal of W
⟨2⟩4. LET: $E : W \approx \beta$ be the canonical isomorphism.
⟨2⟩5. ASSUME: for a contradiction $\alpha < \beta$
⟨2⟩6. LET: $a \in W$ be the element with $E(a) = \alpha$
⟨2⟩7. PICK $A \in \mathcal{C}$ such that $a \in A$
⟨2⟩8. LET: γ be the ordinal of A and $E_A : A \cong \gamma$ be the canonical isomorphism.
⟨2⟩9. For all $x \in A$ we have $E_A(x) = E(x)$
PROOF: Transfinite induction on x.
⟨2⟩10. $E_A(a) = \alpha$

$\langle 2\rangle 11.$ $\alpha < \gamma$
$\langle 2\rangle 12.$ Q.E.D.
PROOF: This contradicts $\langle 2\rangle 2$.
$\square$

6.2 Successors

Definition 6.2.1 (Successor)**.** The *successor* of a set a is the set $a^+ := a \cup \{a\}$.

Proposition 6.2.2 (Z)**.** *A set a is a transitive set if and only if*

$$\bigcup(a^+) = a \ .$$

PROOF:
$\langle 1\rangle 1.$ If a is a transitive set then $\bigcup(a^+) = a$.
$\langle 2\rangle 1.$ ASSUME: a is a transitive set.
$\langle 2\rangle 2.$ $\bigcup(a^+) \subseteq a$
$\langle 3\rangle 1.$ LET: $x \in \bigcup(a^+)$
PROVE: $x \in a$
$\langle 3\rangle 2.$ PICK $y \in a^+$ such that $x \in y$.
$\langle 3\rangle 3.$ $y \in a$ or $y = a$.
$\langle 3\rangle 4.$ CASE: $y \in a$
PROOF: Then $x \in a$ because a is a transitive set.
$\langle 3\rangle 5.$ CASE: $y = a$
PROOF: Then $x \in a$ immediately.
$\langle 2\rangle 3.$ $a \subseteq \bigcup(a^+)$
PROOF: Since $a \in a^+$.
$\langle 1\rangle 2.$ If $\bigcup(a^+) = a$ then a is a transitive set.
$\langle 2\rangle 1.$ ASSUME: $\bigcup(a^+) = a$
$\langle 2\rangle 2.$ $\bigcup a \subseteq a$
PROOF:

$$\begin{aligned}\bigcup a &\subseteq \bigcup(a^+) && \text{(Proposition 1.5.9)}\\ &= a && (\langle 2\rangle 1)\end{aligned}$$

$\langle 2\rangle 3.$ a is a transitive set.
PROOF: Proposition 1.6.2.
$\square$

Proposition 6.2.3. *For any set a, we have a is a transitive set if and only if a^+ is a transitive set.*

PROOF:
$\langle 1\rangle 1.$ If a is a transitive set then a^+ is a transitiveset.
PROOF: If a is a transitive set then $\bigcup(a^+) = a \subseteq a^+$ by Proposition 6.2.2 and so a^+ is a transitive set.
$\langle 1\rangle 2.$ If a^+ is a transitive set then a is a transitive set.
$\langle 2\rangle 1.$ ASSUME: a^+ is a transitive set.

$\langle 2\rangle 2$. LET: $x \in y \in a$
$\langle 2\rangle 3$. $x \in y \in a^+$
$\langle 2\rangle 4$. $x \in a^+$
PROOF: $\langle 2\rangle 1$
$\langle 2\rangle 5$. $x \neq a$
PROOF: From $\langle 2\rangle 2$ and the Axiom of Regularity.
$\langle 2\rangle 6$. $x \in a$
□

Definition 6.2.4. We write 0 for $\emptyset$, 1 for $\emptyset^+$, 2 for $\emptyset^{++}$, etc.

Proposition 6.2.5. *For any set A we have $\mathcal{P}A \approx 2^A$.*

PROOF: The function $H : \mathcal{P}A \to 2^A$ defined by $H(S)(a) = \{\emptyset\}$ if $a \in S$ and $\emptyset$ if $a \notin S$ is a bijection. □

Proposition 6.2.6. *For any ordinal number α we have α^+ is an ordinal number.*

PROOF:
$\langle 1\rangle 1$. α^+ is a transitive set.
PROOF: Proposition 6.2.3.
$\langle 1\rangle 2$. α^+ is well-ordered by $\in$.
$\langle 2\rangle 1$. For all $x, y, z \in \alpha^+$, if $x \in y \in z$ then $x \in z$
$\langle 3\rangle 1$. CASE: $z = \alpha$
PROOF: Then $x \in \alpha$ since α is a transitive set.
$\langle 3\rangle 2$. CASE: $z \in \alpha$
PROOF: Then $x \in z$ since α is well-ordered by $\in$.
$\langle 2\rangle 2$. For all $x, y \in \alpha^+$ we have $x \in y$ or $x = y$ or $y \in x$
$\langle 3\rangle 1$. CASE: $x, y \in \alpha$
PROOF: The result follows because α is well-ordered by $\in$.
$\langle 3\rangle 2$. CASE: $x \in \alpha$, $y = \alpha$
PROOF: Then $x \in y$.
$\langle 3\rangle 3$. CASE: $x = \alpha$, $y \in \alpha$
PROOF: Then $y \in x$.
$\langle 3\rangle 4$. CASE: $x = \alpha$, $y = \alpha$
PROOF: Then $x = y$.
$\langle 2\rangle 3$. Every nonempty subset of α^+ has an $\in$-least element.
$\langle 3\rangle 1$. LET: $S \subseteq \alpha^+$ be nonempty
$\langle 3\rangle 2$. CASE: $S = \{\alpha\}$
PROOF: α is least in S.
$\langle 3\rangle 3$. CASE: $S \neq \{\alpha\}$
$\langle 4\rangle 1$. $S - \{\alpha\}$ is a nonempty subset of α
$\langle 4\rangle 2$. LET: β be least in $S - \{\alpha\}$
$\langle 4\rangle 3$. β is least in S.
□

Proposition 6.2.7. *For ordinals α and β, if $\alpha^+ = \beta^+$ then $\alpha = \beta$.*

PROOF: If $\alpha^+ = \beta^+$ then

$$\begin{aligned} \alpha &= \bigcup(\alpha^+) && \text{(Proposition 6.2.2)} \\ &= \bigcup(\beta^+) \\ &= \beta && \text{(Proposition 6.2.2)} \end{aligned}$$

Proposition 6.2.8. *For ordinals α and β, we have $\alpha < \beta$ if and only if $\alpha^+ < \beta^+$.*

PROOF:

$$\begin{aligned} \alpha < \beta &\Leftrightarrow \alpha^+ \leq \beta \\ &\Leftrightarrow \alpha^+ < \beta^+ \end{aligned} \qquad \square$$

Definition 6.2.9 (Successor Ordinal)**.** An ordinal α is a *successor ordinal* iff $\alpha = \beta^+$ for some β.

Definition 6.2.10 (Limit Ordinal)**.** A *limit ordinal* is an ordinal that is neither 0 nor a successor ordinal.

Proposition 6.2.11. *If λ is a limit ordinal and $\beta < \lambda$ then $\beta^+ < \lambda$.*

PROOF: Since $\beta^+ \leq \lambda$ and $\beta^+ \neq \lambda$. $\square$

6.3 The Well-Ordering Theorem and Zorn's Lemma

Theorem 6.3.1 (Hartogs)**.** *For any set A, there exists an ordinal not dominated by A.*

PROOF:

$\langle 1\rangle 1.$ LET: α be the class of all ordinals β such that $\beta \preccurlyeq A$
  PROVE: α is a set.
$\langle 1\rangle 2.$ LET: $W = \{(B, R) \mid B \subseteq A, R \text{ is a well ordering on } B\}$
$\langle 1\rangle 3.$ α is the class of the ordinals of the elements of W.
  $\langle 2\rangle 1.$ For all $(B, R) \in W$, the ordinal of (B, R) is in α.
    $\langle 3\rangle 1.$ LET: $(B, R) \in W$
    $\langle 3\rangle 2.$ LET: β be the ordinal of (B, R)
    $\langle 3\rangle 3.$ LET: $E : B \cong \beta$ be the canonical isomorphism.
    $\langle 3\rangle 4.$ LET: $i : B \hookrightarrow A$ be the inclusion
    $\langle 3\rangle 5.$ $i \circ E^{-1}$ is an injection $\beta \to A$
    $\langle 3\rangle 6.$ $\beta \in \alpha$
  $\langle 2\rangle 2.$ For all $\beta \in \alpha$, there exists $(B, R) \in W$ such that β is the ordinal number of (B, R).
    $\langle 3\rangle 1.$ LET: $\beta \in \alpha$
    $\langle 3\rangle 2.$ PICK an injection $f : \beta \to A$
    $\langle 3\rangle 3.$ Define $\leq$ on $\operatorname{ran} f$ by $f(x) \leq f(y)$ iff $x \leq y$
    $\langle 3\rangle 4.$ $(\operatorname{ran} f, \leq) \in W$
    $\langle 3\rangle 5.$ β is the ordinal number of $(\operatorname{ran} f, \leq)$

$\langle 1\rangle 4.$ α is a set.
PROOF: By an Axiom of Replacement.
$\langle 1\rangle 5.$ α is an ordinal.
PROOF: It is a transitive set of ordinals.
$\langle 1\rangle 6.$ $\alpha \not\preccurlyeq A$
PROOF: Since $\alpha \notin \alpha$.
□

Theorem 6.3.2 (Numeration Theorem)**.** *Every set is equinumerous with some ordinal.*

PROOF:
$\langle 1\rangle 1.$ LET: A be any set.
$\langle 1\rangle 2.$ PICK an ordinal α not dominated by A.
$\langle 1\rangle 3.$ PICK a choice function G for A.
$\langle 1\rangle 4.$ PICK $e \notin A$
$\langle 1\rangle 5.$ LET: $F : \alpha \to A \cup \{e\}$ by transfinite recursion:

$$F(\gamma) = \begin{cases} G(A - F(\{\delta \mid \delta < \gamma\}) & \text{if } A - F(\{\delta \mid \delta < \gamma\}) \neq \emptyset \\ e & \text{if } A - F(\{\delta \mid \delta < \gamma\}) = \emptyset \end{cases}$$

$\langle 1\rangle 6.$ $e \in \operatorname{ran} F$
$\langle 2\rangle 1.$ ASSUME: for a contradiction $e \notin \operatorname{ran} F$
$\langle 2\rangle 2.$ F is an injection $\alpha \to A$.
$\langle 3\rangle 1.$ LET: $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$
PROVE: $F(\beta) \neq F(\gamma)$
$\langle 3\rangle 2.$ ASSUME: w.l.o.g. $\beta < \gamma$
$\langle 3\rangle 3.$ $F(\gamma) \in A - F(\{\delta \mid \delta < \gamma\})$
$\langle 3\rangle 4.$ $F(\gamma) \notin F(\{\delta \mid \delta < \gamma\})$
$\langle 3\rangle 5.$ $F(\gamma) \neq F(\beta)$
$\langle 2\rangle 3.$ Q.E.D.
PROOF: This contradicts $\langle 1\rangle 2$.
$\langle 1\rangle 7.$ LET: δ be least such that $F(\delta) = e$
$\langle 1\rangle 8.$ $F \restriction \delta : \delta \approx A$

Theorem 6.3.3 (Well-Ordering Theorem)**.** *Any set can be well ordered.*

PROOF:
$\langle 1\rangle 1.$ PICK an ordinal δ and a bijection $F : A \approx \delta$
$\langle 1\rangle 2.$ Define $\leq$ on A by $F(x) \leq F(y)$ iff $x \leq y$ for $x, y \in \delta$
$\langle 1\rangle 3.$ $\leq$ is a well ordering on A.
□

Theorem 6.3.4 (Zorn's Lemma)**.** *Let $\mathcal{A}$ be a set such that, for every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$. Then $\mathcal{A}$ has a maximal element.*

PROOF:
$\langle 1\rangle 1.$ PICK a well ordering $<$ on $\mathcal{A}$.

$\langle 1 \rangle 2$. LET: $F : \mathcal{A} \to 2$ be the function defined by transfinite recursion by:
$$F(A) = \begin{cases} 1 & \text{if } A \text{ includes every set } B < A \text{ for which } F(B) = 1 \\ 0 & \text{otherwise} \end{cases}$$
$\langle 1 \rangle 3$. LET: $\mathcal{C} = \{A \in \mathcal{A} \mid F(A) = 1\}$
PROVE: $\bigcup \mathcal{C}$ is a maximal element of $\mathcal{A}$
$\langle 1 \rangle 4$. For all $A \in \mathcal{A}$, we have $A \in \mathcal{C}$ iff $\forall B < A . B \in \mathcal{C} \Rightarrow B \subseteq A$
$\langle 1 \rangle 5$. $\mathcal{C}$ is a chain.
$\langle 2 \rangle 1$. LET: $A, A' \in \mathcal{C}$
$\langle 2 \rangle 2$. ASSUME: w.l.o.g. $A \leq A'$
$\langle 2 \rangle 3$. $A \subseteq A'$
PROOF: By $\langle 1 \rangle 4$
$\langle 1 \rangle 6$. $\bigcup \mathcal{C} \in \mathcal{A}$
$\langle 1 \rangle 7$. $\bigcup \mathcal{C}$ is maximal in $\mathcal{A}$.
$\langle 2 \rangle 1$. LET: $A \in \mathcal{A}$ and $\bigcup \mathcal{C} \subseteq A$
$\langle 2 \rangle 2$. $A \in \mathcal{C}$
PROOF: By $\langle 1 \rangle 4$ since $\forall B \in \mathcal{C} . B \subseteq A$.
$\langle 2 \rangle 3$. $A \subseteq \bigcup \mathcal{C}$
$\langle 2 \rangle 4$. $A = \bigcup \mathcal{C}$
$\square$

Proposition 6.3.5 (Teichmüller-Tukey Lemma). *Let $\mathcal{A}$ be a nonempty set such that, for every B, we have $B \in \mathcal{A}$ if and only if every finite subset of B is a member of $\mathcal{A}$. Then $\mathcal{A}$ has a maximal element.*

PROOF:
$\langle 1 \rangle 1$. For every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$
$\langle 2 \rangle 1$. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain.
$\langle 2 \rangle 2$. Every finite subset of $\bigcup \mathcal{B}$ is a member of $\mathcal{A}$.
$\langle 3 \rangle 1$. LET: C be a finite subset of $\bigcup \mathcal{B}$.
$\langle 3 \rangle 2$. PICK $B \in \mathcal{B}$ such that $C \subseteq B$.
$\langle 3 \rangle 3$. $B \in \mathcal{A}$
$\langle 3 \rangle 4$. Every finite subset of B is in $\mathcal{A}$.
$\langle 3 \rangle 5$. $C \in \mathcal{A}$
$\langle 2 \rangle 3$. $\bigcup \mathcal{B} \in \mathcal{A}$.
$\langle 1 \rangle 2$. Q.E.D.
PROOF: Zorn's lemma.
$\square$

Theorem Schema 6.3.6. *For any class $\mathbf{A}$, there exists a class $\mathbf{F}$ such that the following is a theorem:*

If $\mathbf{A}$ is a proper class of ordinals, then $\mathbf{F} : \mathbf{On} \to \mathbf{A}$ is an order isomorphism.

PROOF:
$\langle 1 \rangle 1$. Define $\mathbf{F} : \mathbf{On} \to \mathbf{A}$ by transfinite recursion as follows: $\mathbf{F}(\alpha)$ is the least element of $\mathbf{A}$ that is different from $\mathbf{F}(\beta)$ for all $\beta < \alpha$.
$\langle 1 \rangle 2$. For all $\alpha, \beta \in \mathbf{On}$, if $\alpha < \beta$ then $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$

PROOF: We have $\mathbf{F}(\alpha) \neq \mathbf{F}(\beta)$ by the definition of $\mathbf{F}(\beta)$, and $\mathbf{F}(\beta) \not< \mathbf{F}(\alpha)$ by the leastness of $\mathbf{F}(\alpha)$.

$\langle 1 \rangle 3$. $\mathbf{F}$ is surjective.

$\langle 2 \rangle 1$. LET: $\alpha \in \mathbf{A}$

$\langle 2 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall \beta \in \mathbf{A}$, if $\beta < \alpha$ then there exists γ such that $\beta = \mathbf{F}(\gamma)$.

$\langle 2 \rangle 3$. LET: $\gamma = \{\delta \in \mathbf{On} \mid \mathbf{F}(\delta) < \alpha\}$

$\langle 2 \rangle 4$. γ is a set.

PROOF: Axiom of Replacement applied to α.

$\langle 2 \rangle 5$. γ is a transitive set.

PROOF: If $\mathbf{F}(\delta) < \alpha$ and $\epsilon < \delta$ then $\mathbf{F}(\epsilon) < \alpha$ by $\langle 1 \rangle 2$.

$\langle 2 \rangle 6$. γ is an ordinal.

PROOF: Proposition 6.1.8.

$\langle 2 \rangle 7$. $\mathbf{F}(\gamma) = \alpha$

$\langle 3 \rangle 1$. $\mathbf{F}(\gamma)$ is the least element of $\mathbf{A}$ different from $\mathbf{F}(\delta)$ for all $\delta < \gamma$

$\langle 3 \rangle 2$. $\mathbf{F}(\gamma)$ is the least element of $\mathbf{A}$ different from x for all $x \in \mathbf{A}$ with $x < \alpha$

$\langle 3 \rangle 3$. $\mathbf{F}(\gamma) = \alpha$

$\square$

6.4 Ordinal Operations

Definition 6.4.1 (Ordinal Operation)**.** An *ordinal operation* is a function $\mathbf{On} \to \mathbf{On}$.

Definition 6.4.2 (Continuous)**.** An ordinal operation $\mathbf{T} : \mathbf{On} \to \mathbf{On}$ is *continuous* iff, for every limit ordinal λ, we have $\mathbf{T}(\lambda) = \bigcup_{\alpha < \lambda} \mathbf{T}(\alpha)$.

Definition 6.4.3 (Normal)**.** An ordinal operation is *normal* iff it is continuous and strictly monotone.

Proposition Schema 6.4.4. *For any class* $\mathbf{T}$*, the following is a theorem.*

If $\mathbf{T}$ *is a continuous ordinal operation and* $\forall \gamma . \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$*, then* $\mathbf{T}$ *is normal.*

PROOF:

$\langle 1 \rangle 1$. LET: $P[\beta]$ be the property $\forall \gamma < \beta . \mathbf{T}(\gamma) < \mathbf{T}(\beta)$

$\langle 1 \rangle 2$. $P[0]$

PROOF: Vacuous.

$\langle 1 \rangle 3$. For any ordinal γ, if $P[\gamma]$ then $P[\gamma^+]$

$\langle 2 \rangle 1$. ASSUME: $P[\gamma]$

$\langle 2 \rangle 2$. LET: $\delta < \gamma^+$

$\langle 2 \rangle 3$. CASE: $\delta < \gamma$

PROOF: Then $\mathbf{T}(\delta) < \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$.

$\langle 2 \rangle 4$. CASE: $\delta = \gamma$

PROOF: Then $\mathbf{T}(\delta) = \mathbf{T}(\gamma) < \mathbf{T}(\gamma^+)$.

$\langle 1 \rangle 4$. For any limit ordinal λ, if $\forall \gamma < \lambda . P[\gamma]$ then $P[\lambda]$.

⟨2⟩1. ASSUME: $\forall\gamma < \lambda.P[\gamma]$
⟨2⟩2. LET: $\delta < \lambda$
⟨2⟩3. $\mathbf{T}(\delta) < \mathbf{T}(\lambda)$
PROOF:

$$\begin{aligned}\mathbf{T}(\delta) &< \mathbf{T}(\delta^+)\\ &\leq \bigcup_{\epsilon<\lambda}\mathbf{T}(\epsilon)\\ &= \mathbf{T}(\lambda)\end{aligned}$$

□

Proposition Schema 6.4.5. *For any class* $\mathbf{T}$*, the following is a theorem:*
Assume $\mathbf{T}$ *is a normal ordinal operation. For every ordinal* α*, we have* $\alpha \leq \mathbf{T}(\alpha)$*.*

PROOF:
⟨1⟩1. LET: γ be an ordinal.
⟨1⟩2. ASSUME: as induction hypothesis $\forall\delta < \gamma.\mathbf{T}(\delta) \geq \delta$
⟨1⟩3. For all $\delta < \gamma$ we have $\delta < \mathbf{T}(\gamma)$
PROOF: $\mathbf{T}$ is strictly monotone.
⟨1⟩4. $\gamma \leq \mathbf{T}(\gamma)$
□

Proposition Schema 6.4.6. *For any class* $\mathbf{T}$*, the following is a theorem:*
Assume $\mathbf{T}$ *is a normal ordinal operation. For any ordinal* $\beta \geq \mathbf{T}(0)$*, there exists a greatest ordinal* γ *such that* $\mathbf{T}(\gamma) \leq \beta$*.*

PROOF:
⟨1⟩1. There exists γ such that $\mathbf{T}(\gamma) > \beta$
⟨2⟩1. For all γ we have $\mathbf{T}(\gamma) \geq \gamma$
PROOF: Proposition 6.4.5.
⟨2⟩2. $\mathbf{T}(\beta^+) > \beta$
⟨1⟩2. LET: δ be least such that $\mathbf{T}(\delta) > \beta$
⟨1⟩3. δ is a successor ordinal.
⟨2⟩1. $\delta \neq 0$
PROOF: Since $\mathbf{T}(0) \leq \beta$.
⟨2⟩2. δ is not a limit ordinal.
⟨3⟩1. ASSUME: for a contradiction δ is a limit ordinal.
⟨3⟩2. $\beta < \bigcup_{\epsilon<\delta}\mathbf{T}(\epsilon)$
PROOF: $\mathbf{T}$ is continuous.
⟨3⟩3. There exists $\epsilon < \delta$ such that $\beta < \mathbf{T}(\epsilon)$
⟨3⟩4. Q.E.D.
PROOF: This contradicts the minimality of δ.
⟨1⟩4. LET: $\delta = \gamma^+$
⟨1⟩5. γ is greatest such that $\mathbf{T}(\gamma) \leq \beta$
□

Theorem Schema 6.4.7. *For any class* $\mathbf{T}$*, the following is a theorem:*

Assume that $\mathbf{T}$ *is a normal ordinal operation. For any nonempty set of ordinals* S*, we have*

$$\mathbf{T}(\sup S) = \sup_{\alpha \in S} \mathbf{T}(\alpha) \ .$$

PROOF:
$\langle 1 \rangle 1.$ $\forall \alpha \in S . \mathbf{T}(\alpha) \leq \mathbf{T}(\sup S)$
PROOF: Since $\mathbf{T}$ is monotone.
$\langle 1 \rangle 2.$ For any ordinal β, if $\forall \alpha \in S . \mathbf{T}(\alpha) \leq \beta$, then $\mathbf{T}(\sup S) \leq \beta$
$\langle 2 \rangle 1.$ LET: β be an ordinal.
$\langle 2 \rangle 2.$ LET: $\gamma = \sup S$
$\langle 2 \rangle 3.$ ASSUME: $\forall \alpha \in S . \mathbf{T}(\alpha) \leq \beta$
$\langle 2 \rangle 4.$ CASE: γ is 0 or a successor ordinal
PROOF: Then we must have $\gamma \in S$ so $\mathbf{T}(\gamma) \leq \beta$ from $\langle 2 \rangle 3$.
$\langle 2 \rangle 5.$ CASE: γ is a limit ordinal
$\langle 3 \rangle 1.$ $\mathbf{T}(\gamma) = \sup_{\alpha < \gamma} \mathbf{T}(\alpha)$
PROOF: $\mathbf{T}$ is continuous.
$\langle 3 \rangle 2.$ ASSUME: for a contradiction $\beta < \mathbf{T}(\gamma)$
$\langle 3 \rangle 3.$ PICK $\alpha < \gamma$ such that $\beta < \mathbf{T}(\alpha)$
PROOF: $\langle 3 \rangle 1$, $\langle 3 \rangle 2$
$\langle 3 \rangle 4.$ PICK $\alpha' \in S$ such that $\alpha < \alpha'$
PROOF: $\langle 2 \rangle 2$, $\langle 3 \rangle 3$
$\langle 3 \rangle 5.$ $\beta < \mathbf{T}(\alpha') \leq \beta$
PROOF: $\mathbf{T}$ is strictly monotone, $\langle 3 \rangle 3$, $\langle 3 \rangle 4$, $\langle 2 \rangle 3$.
$\langle 3 \rangle 6.$ Q.E.D.
PROOF: This is a contradiction.
□

Proposition 6.4.8. *For any classes* $\mathbf{A}$ *and* $\mathbf{T}$*, the following is a theorem:*

Assume $\mathbf{A}$ *is a proper class of ordinals such that, for every set* $S \subseteq \mathbf{A}$*, we have* $\bigcup S \in \mathbf{A}$*. Assume* $\mathbf{T}$ *is the unique order isomorphism* $\mathbf{On} \cong \mathbf{A}$*. Then* $\mathbf{T}$ *is normal.*

PROOF:
$\langle 1 \rangle 1.$ $\mathbf{T}$ is strictly monotone.
PROOF: Since it is an order isomorphism.
$\langle 1 \rangle 2.$ $\mathbf{T}$ is continuous.
$\langle 2 \rangle 1.$ LET: λ be a limit ordinal.
$\langle 2 \rangle 2.$ $\mathbf{T}'(\lambda)$ is the least member of $\mathbf{A}$ that is greater than $\mathbf{T}'(\alpha)$ for all $\alpha < \lambda$
$\langle 2 \rangle 3.$ $\mathbf{T}'(\lambda) = \sup_{\alpha < \lambda} \mathbf{T}'(\alpha)$
□

Proposition Schema 6.4.9. *For any class* $\mathbf{T}$*, the following is a theorem:*

If $\mathbf{T}$ *is a normal ordinal operation, then for any limit ordinal* λ*, we have* $\mathbf{T}(\lambda)$ *is a limit ordinal.*

PROOF:
$\langle 1 \rangle 1.$ $\mathbf{T}(\lambda) \neq 0$

PROOF: Since $0 \leq \mathbf{T}(0) < \mathbf{T}(\lambda)$.
$\langle 1 \rangle 2$. $\mathbf{T}(\lambda)$ is not a successor ordinal.
$\langle 2 \rangle 1$. ASSUME: for a contradiction $\mathbf{T}(\lambda) = \alpha^+$
$\langle 2 \rangle 2$. $\alpha < \mathbf{T}(\lambda) = \sup_{\beta < \lambda} \mathbf{T}(\beta)$
$\langle 2 \rangle 3$. PICK $\beta < \lambda$ such that $\alpha < \mathbf{T}(\beta)$
$\langle 2 \rangle 4$. $\alpha^+ \leq \mathbf{T}(\beta) < \mathbf{T}(\lambda)$
$\langle 2 \rangle 5$. Q.E.D.
PROOF: This is a contradiction.
$\square$

6.5 Ordinal Arithmetic

6.5.1 Addition

Definition 6.5.1. Let A and B be disjoint well-ordered sets. The *concatenation* of A and B is the set $A \cup B$ under the relation:

- if $a, a' \in A$ then $a \leq a'$ iff $a \leq a'$ in A
- if $b, b' \in B$ then $b \leq b'$ iff $b \leq b'$ in B
- if $a \in A$ and $b \in B$ then $a \leq b$ and $b \not\leq a$.

Proposition 6.5.2. *If A and B are disjoint well-ordered sets, then their concatenation is well-ordered.*

PROOF:
$\langle 1 \rangle 1$. $\leq$ is reflexive.
PROOF: For all $a \in A$ we have $a \leq a$, and for all $b \in B$ we have $b \leq b$.
$\langle 1 \rangle 2$. $\leq$ is antisymmetric.
$\langle 2 \rangle 1$. ASSUME: $x \leq y \leq x$
$\langle 2 \rangle 2$. CASE: $x, y \in A$
PROOF: Then $x = y$ since the order on A is antisymmetric.
$\langle 2 \rangle 3$. CASE: $x \in A$ and $y \in B$
PROOF: This is impossible as it would imply $y \not\leq x$.
$\langle 2 \rangle 4$. CASE: $x \in B$ and $y \in A$
PROOF: This is impossible as it would imply $x \not\leq y$.
$\langle 2 \rangle 5$. CASE: $x, y \in B$
PROOF: Then $x = y$ since the order on B is antisymmetric.
$\langle 1 \rangle 3$. $\leq$ is transitive.
$\langle 2 \rangle 1$. ASSUME: $x \leq y \leq z$
$\langle 2 \rangle 2$. CASE: $x, z \in A$
PROOF: In this case $y \in A$ since $y \leq z$, and so $x \leq z$ since the order on A is transitive.
$\langle 2 \rangle 3$. CASE: $x \in A$ and $z \in B$
PROOF: Then $x \leq z$ immediately.
$\langle 2 \rangle 4$. CASE: $x \in B$ and $z \in A$

PROOF: This is impossible because we have $y \notin A$ since $x \leq y$ and $y \notin B$ since $y \leq z$.

⟨2⟩5. CASE: $x, z \in B$

PROOF: In this case $y \in B$ since $x \leq y$, and so $x \leq z$ since the order on B is transitive.

⟨1⟩4. $\leq$ is total.

⟨2⟩1. LET: $x, y \in A \cup B$

⟨2⟩2. CASE: $x, y \in A$

PROOF: Then $x \leq y$ or $y \leq x$ because the order on A is total.

⟨2⟩3. CASE: $x \in A$ and $y \in B$

PROOF: Then $x \leq y$.

⟨2⟩4. CASE: $x \in B$ and $y \in A$

PROOF: Then $y \leq x$.

⟨2⟩5. CASE: $x, y \in B$

PROOF: Then $x \leq y$ or $y \leq x$ because the order on B is total.

⟨1⟩5. Every nonempty subset of $A \cup B$ has a least element.

⟨2⟩1. LET: S be a nonempty subset of $A \cup B$

⟨2⟩2. CASE: $S \cap A = \emptyset$

PROOF: Then $S \subseteq B$ and so S has a least element.

⟨2⟩3. CASE: $S \cap A \neq \emptyset$

PROOF: The least element of $S \cap A$ is the least element of S.

□

Definition 6.5.3 (Ordinal Addition)**.** Let α and β be ordinal numbers. Then $\alpha + \beta$ is the ordinal number of the concatenation of A and B, where A is any well ordered set with ordinal α and B is any well ordered set with ordinal β.

Theorem 6.5.4 (Associative Law for Addition)**.** *For any ordinals ρ, σ and τ, we have*

$$\rho + (\sigma + \tau) = (\rho + \sigma) + \tau \ .$$

PROOF: Given disjoint well ordered sets A, B and C, the concatenation of A with (the concatenation of B and C) is the same as the concatenation of (the concatenation of A and B) and C. □

Theorem 6.5.5. *For any ordinal ρ we have*

$$\rho + 0 = 0 + \rho = \rho \ .$$

PROOF: For any well ordered set A, the concatenation of A with $\emptyset$ is A, and the concatenation of $\emptyset$ with A is A. □

Theorem 6.5.6. *For any ordinal α we have $\alpha + 1 = \alpha^+$.*

PROOF: Since α^+ is the concatenation of α and $\{\alpha\}$. □

Theorem 6.5.7. *For any ordinal α, the operation that maps β to $\alpha + \beta$ is normal.*

PROOF:

$\langle 1 \rangle 1$. For any limit ordinal λ, we have $\alpha + \lambda = \sup_{\beta<\lambda}(\alpha + \beta)$.
$\langle 2 \rangle 1$. LET: λ be a limit ordinal.
$\langle 2 \rangle 2$. $(\{0\} \times \alpha) \cup (\{1\} \times \lambda) = \bigcup_{\beta \in \lambda}((\{0\} \times \alpha) \cup (\{1\} \times \beta))$, where the order on the right hand side is as in Lemma 6.1.15.
PROOF:

$$\begin{aligned}(\{0\} \times \alpha) \cup (\{1\} \times \lambda) &= (\{0\} \times \alpha) \cup (\{1\} \times \bigcup_{\beta<\lambda} \beta) \\ &= (\{0\} \times \alpha) \cup \bigcup_{\beta<\lambda} (\{1\} \times \beta) \\ &= \bigcup_{\beta<\lambda} ((\{0\} \times \alpha) \cup (\{1\} \times \beta))\end{aligned}$$

$\langle 1 \rangle 2$. For any ordinal β we have $\alpha + \beta < \alpha + \beta^+$
PROOF: Since $\alpha + \beta^+ = \alpha + \beta + 1 = (\alpha + \beta)^+$.
□

Corollary 6.5.7.1. *For any ordinals α, β, γ, we have $\beta < \gamma$ if and only if $\alpha + \beta < \alpha + \gamma$.*

Corollary 6.5.7.2 (Left Cancellation for Addition)**.** *For any ordinals α, β and γ, if $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$.*

Theorem 6.5.8. *For any ordinals α, β, γ, if $\beta \leq \gamma$ then $\beta + \alpha \leq \gamma + \alpha$.*

PROOF: Transfinite induction on α. □

Theorem 6.5.9 (Subtraction Theorem)**.** *Let α and β be ordinals with $\alpha \leq \beta$. Then there exists a unique ordinal δ such that $\alpha + \delta = \beta$.*

PROOF:
$\langle 1 \rangle 1$. For all ordinals α and β with $\alpha \leq \beta$, there exists δ such that $\alpha + \delta = \beta$
$\langle 2 \rangle 1$. LET: α and β be ordinals with $\alpha \leq \beta$
$\langle 2 \rangle 2$. LET: δ be the greatest ordinal such that $\alpha + \delta \leq \beta$
PROOF: Proposition 6.4.6.
$\langle 2 \rangle 3$. $\alpha + \delta = \beta$
PROOF: If $\alpha + \delta < \beta$ then $\alpha + \delta + 1 \leq \beta$ contradicting the greatestness of δ.
$\langle 1 \rangle 2$. Q.E.D.
PROOF: Uniqueness follows from the Left Cancellation Law.
□

6.5.2 Multiplication

Definition 6.5.10 (Ordinal Multiplication)**.** Let α and β be ordinal numbers. Then $\alpha\beta$ is the ordinal number of $A \times B$ under the lexicographic order, where A is any well ordered set with ordinal α and B is any well ordered set with ordinal β.

This is well defined by Proposition 5.3.5.

Theorem 6.5.11 (Associative Law). *For any ordinals ρ, σ and τ, we have*

$$\rho(\sigma\tau) = (\rho\sigma)\tau \ .$$

Proof: Let A, B and C be well ordered sets with ordinals ρ, σ and τ. Then both $\rho(\sigma\tau)$ and $(\rho\sigma)\tau$ are the ordinal of $A \times B \times C$ under
$(a, b, c) \leq (a', b', c') \Leftrightarrow a \leq a' \vee (a = a' \wedge b \leq b') \vee (a = a' \wedge b = b' \wedge c \leq c')$. □

Theorem 6.5.12 (Left Distributive Law). *For any ordinals ρ, σ and τ, we have*

$$\rho(\sigma + \tau) = \rho\sigma + \rho\tau$$

Proof: Let A, B and C be well ordered sets with ordinals ρ, σ and τ and with $B \cap C = \emptyset$. Then both $\rho(\sigma + \tau)$ and $\rho\sigma + \rho\tau$ are the ordinal of $A \times (B \cup C)$ under the lexicographic ordering. □

Theorem 6.5.13. *For any ordinal ρ we have $\rho 0 = 0\rho = 0$.*

Proof: For any well ordered set A we have $A \times \emptyset = \emptyset \times A = \emptyset$. □

Theorem 6.5.14. *For any ordinal ρ we have $\rho 1 = 1\rho = \rho$.*

Proof: Easy. □

Theorem 6.5.15. *For any ordinals ρ and σ, if $\rho\sigma = 0$ then $\rho = 0$ or $\sigma = 0$.*

Proof: If $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$. □

Theorem 6.5.16. *For any non-zero ordinal α, the operation that maps β to $\alpha\beta$ is normal.*

Proof:
⟨1⟩1. For any limit ordinal λ, we have $\alpha\lambda = \bigcup_{\beta<\lambda} \alpha\beta$
  ⟨2⟩1. Let: λ be a limit ordinal
  ⟨2⟩2. $\alpha \times \lambda = \bigcup_{\beta<\lambda}(\alpha \times \beta)$ as well-ordered sets
⟨1⟩2. For any ordinal β we have $\alpha\beta < \alpha\beta^+$
  Proof: $\alpha\beta^+ = \alpha\beta + \alpha > \alpha\beta$
□

Corollary 6.5.16.1. *For any ordinals α, β, γ, if $\alpha \neq 0$ then $\beta < \gamma$ if and only if $\alpha\beta < \alpha\gamma$.*

Corollary 6.5.16.2 (Left Cancellation for Multiplication). *For any ordinals α, β, γ, if $\alpha \neq 0$ and $\alpha\beta = \alpha\gamma$ then $\beta = \gamma$.*

Theorem 6.5.17. *For any ordinals α, β and γ, if $\beta \leq \gamma$ then $\beta\alpha \leq \gamma\alpha$.*

Proof: Transfinite induction on α. □

Theorem 6.5.18 (Division Theorem). *Let α and δ be ordinal numbers with $\delta \neq 0$. Then there exist unique ordinals β and γ with $\gamma < \delta$ and*

$$\alpha = \delta\beta + \gamma \ .$$

PROOF:
⟨1⟩1. For any ordinal numbers α and δ with $\delta \neq 0$, there exist ordinals β and γ such that $\gamma < \delta$ and $\alpha = \delta\beta + \gamma$
⟨2⟩1. LET: α and δ be ordinals with $\delta \neq 0$
⟨2⟩2. LET: β be the greatest ordinal such that $\delta\beta \leq \alpha$
PROOF: Proposition 6.4.6.
⟨2⟩3. There exists an ordinal γ such that $\alpha = \delta\beta + \gamma$
PROOF: Subtraction Theorem
⟨1⟩2. For any ordinals δ, β, β', γ, γ', if $\delta\beta + \gamma = \delta\beta' + \gamma'$ and $\delta \neq 0$ and $\gamma, \gamma' < \delta$ then $\beta = \beta'$ and $\gamma = \gamma'$
⟨2⟩1. LET: δ, β, β', γ, γ' be ordinals.
⟨2⟩2. ASSUME: $\delta \neq 0$ and $\delta\beta + \gamma = \delta\beta' + \gamma'$
⟨2⟩3. $\beta = \beta'$
⟨3⟩1. $\beta \not< \beta'$
PROOF: If $\beta < \beta'$ then

$$\begin{aligned} \delta\beta' + \gamma' &\geq \delta\beta' \\ &\geq \delta(\beta + 1) \\ &= \delta\beta + \delta \\ &> \delta\beta + \gamma \end{aligned}$$

⟨3⟩2. $\beta' \not< \beta$
PROOF: Similar.
⟨2⟩4. $\gamma = \gamma'$
PROOF: By Cancellation.
□

6.5.3 Exponentiation

Definition 6.5.19. Given ordinals α and β, define the ordinal α^β as follows:

$$\begin{aligned} 0^\alpha &:= 0 && (\alpha > 0) \\ \alpha^0 &:= 1 \\ \alpha^{\beta^+} &:= \alpha^\beta \alpha && (\alpha > 0) \\ \alpha^\lambda &:= \sup_{\beta < \lambda} \alpha^\beta && (\alpha > 0, \lambda \text{ a limit ordinal}) \end{aligned}$$

Theorem 6.5.20. *Let α be an ordinal ≥ 2. The operation that maps β to α^β is normal.*

PROOF:
⟨1⟩1. For λ a limit ordinal we have $\alpha^\lambda = \sup_{\beta<\lambda} \alpha^\beta$
PROOF: By definition.
⟨1⟩2. For any ordinal β we have $\alpha^\beta < \alpha^{\beta^+}$
PROOF: We have $\alpha^{\beta^+} = \alpha^\beta \alpha > \alpha^\beta$ by Theorem 6.5.16 since $\alpha > 1$ and $\alpha^\beta \neq 0$.
□

Corollary 6.5.20.1. *For any ordinals α, β, γ, if $\alpha \geq 2$ then $\beta < \gamma$ if and only if $\alpha^\beta < \alpha^\gamma$.*

Corollary 6.5.20.2 (Cancellation for Exponentiation)**.** *For any ordinals α, β, γ, if $\alpha \geq 2$ and $\alpha^\beta = \alpha^\gamma$ then $\beta = \gamma$.*

Theorem 6.5.21. *For any ordinals α, β and γ, if $\beta \leq \gamma$ then $\beta^\alpha \leq \gamma^\alpha$.*

Proof: Transfinite induction on α.

Theorem 6.5.22 (Logarithm Theorem)**.** *Let α and β be ordinal numbers with $\alpha \neq 0$ and $\beta > 1$. Then there exist unique ordinals γ, δ and ρ such that*

$$\alpha = \beta^\gamma \delta + \rho, \qquad 0 \neq \delta < \beta, \qquad \rho < \beta^\gamma \ .$$

Proof:
⟨1⟩1. For any ordinals α and β with $\alpha \neq 0$ and $\beta > 1$, there exist ordinals γ, δ, ρ such that

$$\alpha = \beta^\gamma \delta + \rho, \qquad 0 \neq \delta < \beta, \qquad \rho < \beta^\gamma \ .$$

⟨2⟩1. Let: α and β be ordinals with $\alpha \neq 0$ and $\beta > 1$.
⟨2⟩2. Let: γ be the greatest ordinal such that $\beta^\gamma \leq \alpha$.
Proof: Proposition 6.4.6.
⟨2⟩3. Let: δ and ρ be the unique ordinals with $\rho < \beta^\gamma$ such that $\alpha = \beta^\gamma\delta + \rho$.
Proof: By the Division Theorem.
⟨2⟩4. $\delta \neq 0$
Proof: If $\delta = 0$ then $\alpha = \beta^\gamma 0 + \rho = \rho < \beta^\gamma \leq \alpha$ which is a contradiction.
⟨2⟩5. $\delta < \beta$
Proof: If $\beta \leq \delta$ then $\alpha \geq \beta^\gamma\delta \geq \beta^\gamma\beta = \beta^{\gamma+1}$, contradicting the greatestness of γ.
⟨1⟩2. If $\beta^\gamma\delta + \rho = \beta^{\gamma'}\delta' + \rho'$ with $\beta > 1$, $0 \neq \delta < \beta$, $0 \neq \delta' < \beta$, $\rho < \beta^\gamma$ and $\rho' < \beta^{\gamma'}$, then $\gamma = \gamma'$, $\delta = \delta'$ and $\rho = \rho'$.
⟨2⟩1. Let: $\alpha = \beta^\gamma\delta + \rho = \beta^{\gamma'}\delta' + \rho'$
⟨2⟩2. $\beta^\gamma \leq \alpha < \beta^{\gamma+1}$
⟨2⟩3. $\beta^{\gamma'} \leq \alpha < \beta^{\gamma'+1}$
⟨2⟩4. $\beta^\gamma < \beta^{\gamma'+1}$ and $\beta^{\gamma'} < \beta^{\gamma+1}$
⟨2⟩5. $\gamma < \gamma' + 1$ and $\gamma' < \gamma + 1$
⟨2⟩6. $\gamma = \gamma'$
⟨2⟩7. $\delta = \delta'$ and $\rho = \rho'$
Proof: By the Division Theorem.
□

Theorem 6.5.23. *For any ordinal numbers α, β, γ, we have*

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma \ .$$

Proof:
⟨1⟩1. Let: $P[\gamma]$ be the property: for any ordinals α and β we have $\alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma$
⟨1⟩2. $P[0]$

PROOF:

$$\begin{aligned}\alpha^{\beta+0} &= \alpha^\beta \\ &= \alpha^\beta 1 \\ &= \alpha^\beta \alpha^0\end{aligned}$$

⟨1⟩3. For all γ, if $P[\gamma]$ then $P[\gamma+1]$

PROOF:

$$\begin{aligned}\alpha^{\beta+\gamma+1} &= \alpha^{\beta+\gamma}\alpha \\ &= \alpha^\beta\alpha^\gamma\alpha && \text{(induction hypothesis)} \\ &= \alpha^\beta\alpha^{\gamma+1}\end{aligned}$$

⟨1⟩4. For any limit ordinal λ, if $\forall\gamma < \lambda.P[\gamma]$ then $P[\lambda]$.

⟨2⟩1. LET: λ be a limit ordinal

⟨2⟩2. ASSUME: $\forall\gamma < \lambda.P[\gamma]$

⟨2⟩3. LET: α and β be any ordinals.

⟨2⟩4. CASE: $\alpha = 0$

PROOF: We have $\alpha^{\beta+\lambda} = \alpha^\beta\alpha^\lambda = 0$.

⟨2⟩5. CASE: $\alpha = 1$

PROOF: We have $\alpha^{\beta+\lambda} = \alpha^\beta\alpha^\lambda = 1$.

⟨2⟩6. CASE: $\alpha > 1$

PROOF:

$$\begin{aligned}\alpha^{\beta+\lambda} &= \alpha^{\sup_{\gamma<\lambda}(\beta+\gamma)} \\ &= \sup_{\gamma<\lambda}\alpha^{\beta+\gamma} && \text{(Theorem 6.4.7)} \\ &= \sup_{\gamma<\lambda}\alpha^\beta\alpha^\gamma && (\langle 2\rangle 2) \\ &= \alpha^\beta\sup_{\gamma<\lambda}\alpha^\gamma && \text{(Theorem 6.4.7)} \\ &= \alpha^\beta\alpha^\lambda\end{aligned}$$

□

Theorem 6.5.24. *For any ordinal numbers α, β and γ, we have*

$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma} \enspace .$$

PROOF:

⟨1⟩1. LET: $P[\gamma]$ be the property: For any ordinals α and β, we have $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$

⟨1⟩2. $P[0]$

PROOF:

$$\begin{aligned}(\alpha^\beta)^0 &= 1 \\ &= \alpha^{\beta 0}\end{aligned}$$

⟨1⟩3. $\forall\gamma \in \mathbf{On}.P[\gamma] \Rightarrow P[\gamma+1]$

PROOF:

$$\begin{aligned}(\alpha^\beta)^{\gamma+1} &= (\alpha^\beta)^\gamma \alpha^\beta \\ &= \alpha^{\beta\gamma}\alpha^\beta \\ &= \alpha^{\beta\gamma+\beta} \\ &= \alpha^{\beta(\gamma+1)}\end{aligned}$$

$\langle 1\rangle 4$. For any limit ordinal λ, if $\forall \gamma < \lambda . P[\gamma]$ then $P[\lambda]$.

$\langle 2\rangle 1$. LET: λ be a limit ordinal.

$\langle 2\rangle 2$. ASSUME: $\forall \gamma < \lambda . P[\gamma]$

$\langle 2\rangle 3$. LET: α and β be any ordinals.

$\langle 2\rangle 4$. CASE: $\alpha = 0$ and $\beta = 0$

PROOF:

$$\begin{aligned}(0^\beta)^\lambda &= 1^\lambda \\ &= 1 \\ &= 0^0 \\ &= 0^{0\lambda}\end{aligned}$$

$\langle 2\rangle 5$. CASE: $\alpha = 0$ and $\beta \neq 0$

PROOF: $(0^\beta)^\lambda = 0^{\beta\lambda} = 0$.

$\langle 2\rangle 6$. CASE: $\alpha = 1$

PROOF: $(1^\beta)^\lambda = 1^{\beta\lambda} = 1$

$\langle 2\rangle 7$. CASE: $\alpha > 1$

PROOF:

$$\begin{aligned}(\alpha^\beta)^\lambda &= \sup_{\gamma<\lambda} (\alpha^\beta)^\gamma \\ &= \sup_{\gamma<\lambda} \alpha^{\beta\gamma} \\ &= \alpha^{\sup_{\gamma<\lambda} \beta\gamma} \\ &= \alpha^{\beta\lambda}\end{aligned}$$

$\square$

6.6 Sequences

¡

Definition 6.6.1 (Sequence)**.** Given an ordinal α and class $\mathbf{A}$, an *α-sequence* in $\mathbf{A}$ is a function $a : \alpha \to \mathbf{A}$. We write a_β for $a(\beta)$, and $(a_\beta)_{\beta<\alpha}$ for a.

Definition 6.6.2 (Strictly Increasing)**.** A sequence (a_β) of ordinals is *strictly increasing* iff, whenever $\beta < \gamma$, then $a_\beta < a_\gamma$.

Definition 6.6.3 (Subsequence)**.** Let $(a_\beta)_{\beta<\gamma}$ be a sequence in $\mathbf{A}$. A *subsequence* of (a_β) is a sequence of the form $(a_{\beta_\xi})_{\xi<\delta}$ where $(\beta_\xi)_{\xi<\delta}$ is a strictly increasing sequence in γ.

Definition 6.6.4 (Convergence)**.** Let $(a_\beta)_{\beta<\gamma}$ be a sequence of ordinals and λ an ordinal. Then (a_β) *converges* to the *limit* λ iff $\lambda = \sup_{\beta<\gamma} a_\beta$.

Lemma 6.6.5. *Let* $(a_\beta)_{\beta<\gamma}$ *be a sequence of ordinals. Then there is a strictly increasing subsequence* $(a_{\beta_\xi})_{\xi<\delta}$ *such that* $\sup_{\xi<\delta} a_{\beta_\xi} = \sup_{\beta<\gamma} a_\beta$.

PROOF: Define β_ξ by transfinite recursion as follows. β_ξ is the least β such that $a_\beta > a_{\beta_\zeta}$ for all $\zeta < \xi$ if there is such an a_β; if not, the sequence ends. □

6.7 Strict Supremum

Definition 6.7.1 (Strict Supremum)**.** For any set S of ordinals, define the *strict supremum* of S, $\operatorname{ssup} S$, to be the least ordinal greater than every member of S.

Chapter 7

Cardinal Numbers

7.1 Cardinal Numbers

Definition 7.1.1 (Cardinality)**.** For any set A, the *cardinality* or *cardinal number* $|A|$ of A is the least ordinal equinumerous with A.

Let **Card** be the class of all cardinal numbers.

Proposition 7.1.2. *For any sets A and B, we have $A \approx B$ iff $|A| = |B|$.*

PROOF: Easy. ∎

Definition 7.1.3 (Addition)**.** Given cardinal numbers κ and λ, we define $\kappa+\lambda$ to be $|A \cup B|$ where A and B are disjoint sets of cardinality κ and λ respectively.

We prove this is well-defined.

PROOF:
$\langle 1 \rangle 1$. ASSUME: $A \approx A'$, $B \approx B'$, and $A \cap B = A' \cap B' = \emptyset$
$\langle 1 \rangle 2$. PICK bijections $f : A \approx A'$ and $g : B \approx B'$
$\langle 1 \rangle 3$. The function $A \cup B \to A' \cup B'$ that maps $a \in A$ to $f(a)$ and $b \in B$ to $g(b)$ is a bijection.
∎

Proposition 7.1.4. *For any cardinal number κ, we have $\kappa + 0 = \kappa$.*

PROOF: Let A and B be disjoint sets of cardinality κ and 0. Then $B = \emptyset$ so $A \cup B = A$ and so $|A \cup B| = \kappa$. ∎

Theorem 7.1.5 (Associative Law for Addition)**.** *For any cardinal numbers κ, λ, μ we have $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$.*

PROOF: Since $A \cup (B \cup C) = (A \cup B) \cup C$. ∎

Proposition 7.1.6. *For any cardinal numbers κ and λ we have $\kappa + \lambda = \lambda + \kappa$.*

PROOF: Since $A \cup B = B \cup A$. ∎

Definition 7.1.7 (Multiplication)**.** For κ and λ cardinal numbers, we define $\kappa\lambda$ to be the cardinal number of $A \times B$, where $|A| = \kappa$ and $|B| = \lambda$.

We prove this is well-defined.

PROOF: If $f : A \approx A'$ and $g : B \approx B'$ then the function that maps (a, b) to $(f(a), g(b))$ is a bijection $A \times B \approx A' \times B'$. □

Proposition 7.1.8. *For any cardinal number κ we have $\kappa \cdot 0 = 0$.*

PROOF: Since $A \times \emptyset = \emptyset$. □

Proposition 7.1.9. *For any cardinal number κ we have $\kappa \cdot 1 = \kappa$.*

PROOF: The function that maps (a, e) to a is a bijection $A \times \{e\} \approx A$. □

Theorem 7.1.10 (Distributive Law)**.** *For any cardinal numbers κ, λ and μ, we have $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$.*

PROOF: Since $A \times (B \cup C) = (A \times B) \cup (A \times C)$. □

Theorem 7.1.11 (Associative Law for Multiplication)**.** *For any cardinal numbers κ, λ and μ, we have $\kappa(\lambda\mu) = (\kappa\lambda)\mu$.*

PROOF: Since $A \times (B \times C) \approx (A \times B) \times C$. □

Theorem 7.1.12 (Commutative Law for Multiplication)**.** *For any cardinal numbers κ and λ, we have $\kappa\lambda = \lambda\kappa$.*

PROOF: Since $A \times B \approx B \times A$. □

Theorem 7.1.13. *For any cardinal numbers κ and λ, if $\kappa\lambda = 0$ then $\kappa = 0$ or $\lambda = 0$.*

PROOF: if $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$. □

Definition 7.1.14 (Exponentiation)**.** Given cardinal numbers κ and λ, we define κ^λ to be $|A^B|$, where $|A| = \kappa$ and $|B| = \lambda$.

We prove this is well-defined.

PROOF:If $f : A \approx A'$ and $g : B \approx B'$, then the function that maps $h : B \to A$ to $f \circ h \circ g^{-1}$ is a bijection $A^B \approx A'^{B'}$. □

Proposition 7.1.15. *For any cardinal numbers κ, λ and μ,*

$$\kappa^{\lambda+\mu} = (\kappa^\lambda)^\mu$$

PROOF: The function that maps $f : A \times B \to C$ to $\lambda a \in A.\lambda b \in B.f(a, b)$ is a bijection $A^{B \times C} \approx (A^B)^C$. □

Proposition 7.1.16. *For any cardinal numbers κ, λ and μ,*

$$(\kappa\lambda)^\mu = \kappa^\mu\lambda^\mu \quad .$$

PROOF: The function $f : A^C \times B^C \to (A \times B)^C$ with $f(g,h)(c) = (g(c), h(c))$ is a bijection. □

Proposition 7.1.17. *For any cardinal numbers κ, λ and μ, we have*

$$\kappa^{\lambda+\mu} = \kappa^\lambda \kappa^\mu \ .$$

PROOF: If $B \cap C = \emptyset$, then $f : A^B \times A^C \to A^{B \cup C}$ given by $f(g,h)(b) = g(b)$ and $f(g,h)(c) = h(c)$ is a bijection. □

Proposition 7.1.18. *For any cardinal number κ, we have $\kappa^0 = 1$.*

PROOF: For any set A, we have $A^\emptyset = \{\emptyset\}$. □

Proposition 7.1.19. *For any cardinal number κ, we have $\kappa^1 = \kappa$.*

PROOF: For any sets A and B, if $B = \{b\}$ then the function $f : A \to A^B$ with $f(a)(b) = a$ is a bijection. □

Proposition 7.1.20. *For any non-zero cardinal number κ we have $0^\kappa = 0$.*

PROOF: If A is nonempty then there is no function $A \to \emptyset$. □

Proposition 7.1.21. *For any set A we have $|\mathcal{P}A| = 2^{|A|}$.*

PROOF: The function $f : \mathcal{P}A \to 2^A$ where $f(X)(a) = 0$ if $a \notin X$ and $f(X)(a) = 1$ if $a \in X$. □

Theorem 7.1.22 (König)**.** *Let I be a set. Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets. Assume that $\forall i \in I. |A_i| < |B_i|$. Then $\left|\bigcup_{i \in I} A_i\right| < \left|\prod_{i \in I} B_i\right|$.*

PROOF:
⟨1⟩1. For all $i \in I$, choose an injection $f_i : A_i \rightarrowtail B_i$
⟨1⟩2. For all $i \in I$, choose $b_i \in B_i - f_i(A_i)$
⟨1⟩3. $\left|\bigcup_{i \in I} A_i\right| \leq \left|\prod_{i \in I} B_i\right|$
  ⟨2⟩1. Define $g : \bigcup_{i \in I} A_i \to \prod_{i \in I} B_i$ by

$$g(i,a)(j) = \begin{cases} f_i(a) & \text{if } i = j \\ b_j & \text{otherwise} \end{cases}$$

  ⟨2⟩2. g is injective.
⟨1⟩4. $\left|\bigcup_{i \in I} A_i\right| \neq \left|\prod_{i \in I} B_i\right|$
  ⟨2⟩1. LET: $h : \bigcup_{i \in I} A_i \to \prod_{i \in I} B_i$
     PROVE: h is not surjective.
  ⟨2⟩2. For $i \in I$, PICK $c_i \in B_i - \{h(i,a)(i) \mid i \in I\}$
  ⟨2⟩3. $c \in \prod_{i \in I} B_i$
  ⟨2⟩4. $c \notin \operatorname{ran} h$
□

Corollary 7.1.22.1. *For any cardinal number κ we have $\kappa < 2^\kappa$.*

7.2 Ordering on Cardinal Numbers

Definition 7.2.1. Given cardinal numbers κ and λ, we have $\kappa \leq \lambda$ iff $A \preccurlyeq B$, where $|A| = \kappa$ and $|B| = \lambda$.

PROOF:
⟨1⟩1. LET: $|A| = \kappa$ and $|B| = \lambda$
⟨1⟩2. PICK bijections $f : A \approx \kappa$ and $g : B \approx \lambda$
⟨1⟩3. If $\kappa \leq \lambda$ then $A \preccurlyeq B$
PROOF: Let $i : \kappa \hookrightarrow \lambda$ be the inclusion. Then $g^{-1} \circ i \circ f$ is an injection $A \to B$.
⟨1⟩4. If $A \preccurlyeq B$ then $\kappa \leq \lambda$
⟨2⟩1. ASSUME: $A \preccurlyeq B$
⟨2⟩2. PICK an injection $h : A \rightarrowtail B$
⟨2⟩3. $g(h(A)) \subseteq B$ is well-ordered by $\in$
⟨2⟩4. LET: γ be the ordinal number of $(g(h(A)), \in)$
⟨2⟩5. $\gamma \leq \lambda$
PROOF: Proposition 6.1.12.
⟨2⟩6. $\kappa \leq \gamma$
PROOF: By the leastness of κ, since A is equinumerous with γ.
⟨2⟩7. $\kappa \leq \lambda$
□

Corollary 7.2.1.1. *There is no largest cardinal number.*

Proposition 7.2.2. *For any cardinal numbers κ, λ, μ, if $\kappa \leq \lambda$ then $\kappa + \mu \leq \lambda + \mu$.*

PROOF: If $f : A \to B$ is injective, and $A \cap C = B \cap C = \emptyset$, then the function $A \cup C \to B \cup C$ that maps a to $f(a)$ and maps c to c is an injection. □

Proposition 7.2.3. *For any cardinal numbers κ, λ, μ, if $\kappa \leq \lambda$ then $\kappa\mu \leq \lambda\mu$.*

PROOF: If $f : A \to B$ is injective, then the function $A \times C \to B \times C$ that maps (a, c) to $(f(a), c)$ is injective. □

Proposition 7.2.4. *For any cardinal numbers κ, λ, μ, if $\kappa \leq \lambda$ then $\kappa^\mu \leq \lambda^\mu$.*

PROOF: Given an injection $f : A \to B$, the function that maps $A^C \to B^C$ that maps g to $f \circ g$ is an injection. □

Proposition 7.2.5. *For any cardinal numbers κ, λ, μ, if $\kappa \leq \lambda$ and μ and κ are not both 0, then $\mu^\kappa \leq \mu^\lambda$.*

PROOF:
⟨1⟩1. LET: A, B and C be sets with A and C not both empty.
⟨1⟩2. LET: $f : A \to B$ be an injection.
PROVE: $C^A \preccurlyeq C^B$
⟨1⟩3. CASE: $C = \emptyset$
PROOF: Then $A \neq \emptyset$ so $C^A = \emptyset \preccurlyeq C^B$.
⟨1⟩4. CASE: $C \neq \emptyset$

⟨2⟩1. Pick $c \in C$
⟨2⟩2. Let: $g : C^A \to C^B$ be the function $g(h)(f(a)) = h(a)$, $g(h)(b) = c$ if $b \notin f(A)$
⟨2⟩3. g is an injection.
□

Proposition 7.2.6. *Let $\mathcal{A}$ be a set such that $\forall X \in \mathcal{A}.|X| \leq \kappa$. Then*

$$\left|\bigcup \mathcal{A}\right| \leq |\mathcal{A}|\kappa \ .$$

Proof:
⟨1⟩1. For $X \in \mathcal{A}$, choose a surjection $f_X : \kappa \to X$.
⟨1⟩2. Define $g : \mathcal{A} \times \kappa \to \bigcup \mathcal{A}$ by $g(X, \alpha) = f_X(\alpha)$
⟨1⟩3. g is surjective.
□

Lemma 7.2.7. *The union of a set of cardinal numbers is a cardinal number.*

Proof:
⟨1⟩1. Let: A be a set of cardinal numbers.
Prove: $\bigcup A$ is the smallest ordinal equinumerous with $\bigcup A$
⟨1⟩2. Let: $\alpha < \bigcup A$
Prove: $\alpha \not\approx \bigcup A$
⟨1⟩3. Pick $\kappa \in A$ such that $\alpha < \kappa$
⟨1⟩4. $\alpha \prec \kappa$
⟨1⟩5. $\alpha \prec \bigcup A$
□

Chapter 8

Natural Numbers

8.1 Inductive Sets

Definition 8.1.1 (Inductive)**.** A set I is *inductive* iff $0 \in I$ and $\forall x \in I.x^+ \in I$.

Definition 8.1.2 (Natural Number)**.** A *natural number* is a set that belongs to every inductive set.

Theorem 8.1.3. *The class* $\mathbb{N}$ *of natural numbers is a set.*

PROOF:
$\langle 1 \rangle 1.$ PICK an inductive set I.
  PROOF: Axiom of Infinity.
$\langle 1 \rangle 2.$ $\mathbb{N} \subseteq I$
$\square$

Theorem 8.1.4. $\mathbb{N}$ *is inductive, and is a subset of every other inductive set.*

PROOF:
$\langle 1 \rangle 1.$ $\mathbb{N}$ is inductive.
  $\langle 2 \rangle 1.$ $0 \in \mathbb{N}$
    PROOF: Since 0 is a member of every inductive set.
  $\langle 2 \rangle 2.$ $\forall n \in \mathbb{N}.n^+ \in \mathbb{N}$
    $\langle 3 \rangle 1.$ LET: $n \in \mathbb{N}$
    $\langle 3 \rangle 2.$ LET: I be any inductive set.
        PROVE: $n^+ \in I$
    $\langle 3 \rangle 3.$ $n \in I$
      PROOF: $\langle 3 \rangle 1$, $\langle 3 \rangle 2$
    $\langle 3 \rangle 4.$ $n^+ \in I$
      PROOF: $\langle 3 \rangle 2$, $\langle 3 \rangle 3$
$\langle 1 \rangle 2.$ $\mathbb{N}$ is a subset of every inductive set.
  PROOF: Immediate from definitions.
$\square$

Corollary 8.1.4.1 (Induction Principle for $\mathbb{N}$)**.** *Any inductive subset of* $\mathbb{N}$ *coincides with* $\mathbb{N}$*.*

Theorem 8.1.5. *Every natural number except 0 is the successor of some natural number.*

PROOF: Trivially by induction. □

Proposition 8.1.6. *Every natural number is an ordinal.*

PROOF: By induction. □

Proposition 8.1.7. $\mathbb{N}$ *is a transitive set.*

PROOF:
$\langle 1\rangle 1.$ $0 \subseteq \mathbb{N}$
$\langle 1\rangle 2.$ $\forall n \in \mathbb{N}. n \subseteq \mathbb{N} \Rightarrow n^+ \subseteq \mathbb{N}$
$\langle 1\rangle 3.$ $\forall n \in \mathbb{N}. n \subseteq \mathbb{N}$
  PROOF: From $\langle 1\rangle 1$ and $\langle 1\rangle 2$ by induction.
□

Corollary 8.1.7.1. $\mathbb{N}$ *is an ordinal.*

Definition 8.1.8. We define $\omega = \mathbb{N}$.

Proposition 8.1.9 (Dependent Choice)**.** *Let A be a nonempty set and R a relation on A such that $\forall x \in A. \exists y \in A.(y, x) \in R$. Then there exists a function $f : \mathbb{N} \to A$ such that $\forall n \in \mathbb{N}.(f(n+1), f(n)) \in R$.*

PROOF:
$\langle 1\rangle 1.$ PICK a choice function F for A.
$\langle 1\rangle 2.$ PICK $a \in A$
$\langle 1\rangle 3.$ Define $f : \mathbb{N} \to A$ by $f(0) = a$ and $f(n+1) = F(\{y \in A \mid (y, f(n)) \in R\})$.
□

Theorem Schema 8.1.10. *For any classes* $\mathbf{A}$ *and* $\mathbf{R}$*, the following is a theorem:*

Assume $\mathbf{R}$ is a relation on $\mathbf{A}$ and, for all $a \in \mathbf{A}$, the class $\{x \in \mathbf{A} \mid x\mathbf{R}a\}$ is a set. Then $\mathbf{R}$ is well founded if and only if there does not exist a function $f : \mathbb{N} \to \mathbf{A}$ such that $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$.

PROOF:
$\langle 1\rangle 1.$ If there exists a function $f : \mathbb{N} \to \mathbf{A}$ such that $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$ then $\mathbf{R}$ is not well founded.
  PROOF: $f(\mathbb{N})$ is a nonempty subset of $\mathbf{A}$ with no $\mathbf{R}$-minimal element.
$\langle 1\rangle 2.$ If $\mathbf{R}$ is not well founded then there exists a function $f : \mathbb{N} \to \mathbf{A}$ such that $\forall n \in \mathbb{N}. f(n+1)\mathbf{R}f(n)$.
  $\langle 2\rangle 1.$ ASSUME: $\mathbf{R}$ is not well founded.
  $\langle 2\rangle 2.$ PICK a nonempty subset $B \subseteq \mathbf{A}$ that has no $\mathbf{R}$-minimal element.
  $\langle 2\rangle 3.$ $\forall x \in B. \exists y \in B. y\mathbf{R}x$

⟨2⟩4. Choose a function $g : B \to B$ such that $\forall x \in B.g(x)\mathbf{R}x$
⟨2⟩5. Pick $b \in B$
⟨2⟩6. Define $f : \mathbb{N} \to \mathbf{A}$ recursively by $f(0) = b$ and $\forall n \in \mathbb{N}.f(n+1) = g(f(n))$
⟨2⟩7. $\forall n \in \mathbb{N}.f(n+1)\mathbf{R}f(n)$
□

8.2 Cardinality

Definition 8.2.1 (Finite)**.** A set is *finite* iff it is equinumerous to some natural number; otherwise it is *infinite*.

Theorem 8.2.2 (Pigeonhole Principle)**.** *No natural number is equinumerous to a proper subset of itself.*

Proof:
⟨1⟩1. Let: $P(n)$ be the property: any one-to-one function $n \to n$ is surjective.
⟨1⟩2. $P(0)$
Proof: The only function $0 \to 0$ is injective.
⟨1⟩3. For every natural number n, if $P(n)$ then $P(n+1)$.
⟨2⟩1. Assume: $P(n)$
⟨2⟩2. Let: f be a one-to-one function $n+1 \to n+1$
⟨2⟩3. $f \upharpoonright n$ is a one-to-one function $n \to n+1$
⟨2⟩4. Case: $n \notin ranf$
⟨3⟩1. $f \upharpoonright n : n \to n$
⟨3⟩2. $\mathrm{ran}(f \upharpoonright n) = n$
⟨3⟩3. $f(n) = n$
Proof: ⟨2⟩1.
⟨3⟩4. $\mathrm{ran}\, f = n+1$
⟨2⟩5. Case: $n \in \mathrm{ran}\, f$
⟨3⟩1. Pick $p \in n$ such that $f(p) = n$
⟨3⟩2. Let: $\hat{f} : n \to n$ be the function

$$\hat{f}(p) = f(n)$$
$$\hat{f}(x) = f(x) \qquad (x \neq p)$$

⟨3⟩3. $\hat{f}$ is one-to-one
⟨3⟩4. $\mathrm{ran}\, \hat{f} = n$
Proof: ⟨2⟩1
⟨3⟩5. $\mathrm{ran}\, f = n+1$
⟨1⟩4. For every natural number n, $P(n)$.
□

Corollary 8.2.2.1. *No finite set is equinumerous to a proper subset of itself.*

Corollary 8.2.2.2. *Every natural number is a cardinal number.*

Proof: For any natural number n, we have that n is the least ordinal such that $n \approx n$. □

Corollary 8.2.2.3. *$\mathbb{N}$ is a cardinal number.*

Corollary 8.2.2.4. *$\mathbb{N}$ is infinite.*

PROOF: The function that maps n to $n+1$ is a bijection between $\mathbb{N}$ and $\mathbb{N}-\{0\}$. $\square$

Corollary 8.2.2.5. *If C is a proper subset of a natural number n, then there exists $m < n$ such that $C \approx m$.*

PROOF: By Proposition 6.1.12. $\square$

Corollary 8.2.2.6. *Any subset of a finite set is finite.*

Proposition 8.2.3. *For any natural numbers m and n we have $m+n$ (cardinal addition) is a natural number.*

PROOF: Induction on n. $\square$

Corollary 8.2.3.1. *The union of two finite sets is finite.*

Corollary 8.2.3.2. *The union of a finite set of finite sets is finite.*

PROOF: By induction on the number of elements. $\square$

Proposition 8.2.4. *For natural numbers m and n, the cardinal sum $m + n$ is equal to the ordinal sum $m + n$.*

PROOF: Induction on n. $\square$

Proposition 8.2.5. *For any natural numbers m and n, we have mn (cardinal multiplication) is a natural number.*

Corollary 8.2.5.1. *If A and B are finite sets then $A \times B$ is finite.*

Proposition 8.2.6. *For natural numbers m and n, the cardinal product mn is equal to the ordinal product mn.*

PROOF: Induction on n. $\square$

Proposition 8.2.7. *For any natural numbers m and n we have m^n (cardinal exponentiation) is a natural number.*

PROOF: Induction on n.

Corollary 8.2.7.1. *If A and B are finite sets then A^B are finite.*

Proposition 8.2.8. *For natural numbers m and n, the cardinal exponentiation m^n and the ordinal exponentiation m^n agree.*

PROOF: Induction on n. $\square$

Proposition 8.2.9. $\mathbb{N}^2 \approx \mathbb{N}$

PROOF: The function $J : \mathbb{N}^2 \to \mathbb{N}$ defined by $J(m, n) = ((m + n)^2 + 3m + n)/2$ is a bijection. □

Proposition 8.2.10. *For any infinite cardinal κ we have $\aleph_0 \leq \kappa$.*

PROOF:
⟨1⟩1. LET: A be an infinite set.
PROVE: $\mathbb{N} \preccurlyeq A$
⟨1⟩2. PICK a choice function F for A.
⟨1⟩3. Define $h : \mathbb{N} \to \{X \in \mathcal{P}A \mid X \text{ is finite}\}$ by

$$h(0) = \emptyset$$
$$h(n + 1) = h(n) \cup \{F(A - \{h(m) \mid m < n\})\}$$

⟨1⟩4. Define $g : \mathbb{N} \to A$ by $g(n) = F(A - \{h(m) \mid m < n\})$
⟨1⟩5. g is injective.
PROOF: If $m < n$ then $g(m) \neq g(n)$.
□

Theorem Schema 8.2.11 (König's Lemma)**.** *For any classes* **A** *and* **R***, the following is a theorem:*

Assume **R** *is a well founded relation on* **A** *such that, for all $y \in \mathbf{A}$, the class $\{x \in \mathbf{A} \mid x\mathbf{R}y\}$ is a finite set. Let $\mathbf{R}^t$ be the transitive closure of* **R***. Then,for all $y \in \mathbf{A}$, the class $\{x \in \mathbf{A} \mid x\mathbf{R}^t y\}$ is a finite set.*

PROOF:
⟨1⟩1. LET: $y \in \mathbf{A}$
⟨1⟩2. ASSUME: as transfinite induction hypothesis $\forall x\mathbf{R}y.\{z \in \mathbf{A} \mid z\mathbf{R}^t x\}$ is a finite set.
⟨1⟩3. $\{x \mid x\mathbf{R}^t y\} = \bigcup_{x\mathbf{R}y}(\{x\} \cup \{z \mid z\mathbf{R}^t x\}$
⟨1⟩4. $\{x \mid x\mathbf{R}^t y\}$ is finite.
PROOF: Corollary 8.2.3.2.
□

8.3 Countable Sets

Definition 8.3.1 (Countable)**.** A set A is *countable* iff $|A| \leq \aleph_0$.

Theorem 8.3.2. *The union of a countable set of countable sets is countable.*

PROOF: Proposition 7.2.6. □

8.4 Arithmetic

Definition 8.4.1 (Even)**.** A natural number n is *even* iff there exists $m \in \mathbb{N}$ such that $n = 2m$.

Definition 8.4.2 (Odd)**.** A natural number n is *odd* iff there exists $p \in \mathbb{N}$ such that $n = 2p + 1$.

Proposition 8.4.3. *Every natural number is either even or odd.*

PROOF:
⟨1⟩1. 0 is even.
  PROOF: $0 = 2 \times 0$.
⟨1⟩2. For any natural number n, if n is either even or odd then n^+ is either even or odd.
  PROOF:
  ⟨2⟩1. LET: $n \in \mathbb{N}$
  ⟨2⟩2. If n is even then n^+ is odd.
    PROOF: If $n = 2p$ then $n^+ = 2p + 1$.
  ⟨2⟩3. If n is odd then n^+ is even.
    PROOF: If $n = 2p + 1$ then $n^+ = 2(p + 1)$.
□

Proposition 8.4.4. *No natural number is both even and odd.*

PROOF:
⟨1⟩1. 0 is not odd.
  PROOF: For any p we have $2p + 1 = (2p)^+ \neq 0$.
⟨1⟩2. For any natural number n, if n is not both even and odd, then n^+ is not both even and odd.
  ⟨2⟩1. LET: n be a natural number.
  ⟨2⟩2. If n^+ is even then n is odd.
    ⟨3⟩1. ASSUME: n^+ is even.
    ⟨3⟩2. PICK p such that $n^+ = 2p$
    ⟨3⟩3. $p \neq 0$
      PROOF: Since $n^+ \neq 0$.
    ⟨3⟩4. PICK q such that $p = q^+$
      PROOF: Theorem 8.1.5.
    ⟨3⟩5. $n^+ = 2q + 2$
      PROOF: ⟨3⟩2, ⟨3⟩4.
    ⟨3⟩6. $n = 2q + 1$
      PROOF: Proposition 6.2.7, ⟨3⟩5
    ⟨3⟩7. n is odd.
  ⟨2⟩3. If n^+ is odd then n is even.
    ⟨3⟩1. ASSUME: n^+ is odd.
    ⟨3⟩2. PICK p such that $n^+ = 2p + 1$
    ⟨3⟩3. $n = 2p$
      PROOF: Proposition 6.2.7, ⟨3⟩2
    ⟨3⟩4. n is even.
□

Proposition 8.4.5. *Let m, n, p, q be natural numbres. Assume $m + n = p + q$. Then $m < p$ if and only if $q < n$.*

PROOF:
⟨1⟩1. If $m < p$ then $q < n$.

PROOF: If $m < p$ and $n \leq q$ then $m + n < p + q$.
$\langle 1 \rangle 2$. If $q < n$ then $m < p$.
PROOF: Similar.
□

Proposition 8.4.6. *Let m, n, p and q be natural numbers. Assume $n < m$ and $q < p$. Then*

$$mq + np < mp + nq \ .$$

PROOF:
$\langle 1 \rangle 1$. PICK positive natural numbers a and b such that $m = n + a$ and $p = q + b$.
$\langle 1 \rangle 2$. $mp + nq > mq + np$
PROOF:

$$\begin{aligned} mp + nq &= (n + a)(q + b) + nq \\ &= 2nq + nb + aq + ab \\ mq + np &= (n + a)q + n(q + b) \\ &= 2nq + aq + nb \\ \therefore mp + nq &= mq + np + ab \\ &> mq + np \end{aligned}$$

□

8.5 Sequences

Definition 8.5.1 (Sequence)**.** Let A be a set. A *finite sequence* in A is a function $a : n \to A$ for some natural number n; we write it as $(a(0), a(1), \ldots, a(n-1))$. An *(infinite) sequence* in A is a function $\mathbb{N} \to A$.

We write A^* for the set of all finite sequences in A.

Proposition 8.5.2. *If A is countable then A^* is countable.*

PROOF: For any n, the set A^n is countable, and A^* is equinumerous with $\bigcup_n A^n$.
□

8.6 Transitive Closure of a Set

Proposition 8.6.1. *For any set A, there exists a unique transitive set C such that:*

- $A \subseteq C$
- *For any transitive set X, if $A \subseteq X$ then $C \subseteq X$*

PROOF:
$\langle 1 \rangle 1$. Define a function $F : \mathbb{N} \to \mathbf{V}$ by

$$\begin{aligned} F(0) &= A \\ F(n+1) &= A \cup \bigcup (F(0) \cup \cdots \cup F(n)) \end{aligned}$$

$\langle 1 \rangle 2$. For all $n \in \mathbb{N}$ and $a \in F(n)$ we have $a \subseteq F(n+1)$
  PROOF: $a \in F(0) \cup \cdots \cup F(n)$ so $a \subseteq \bigcup(F(0) \cup \cdots \cup F(n)) \subseteq F(n+1)$.
$\langle 1 \rangle 3$. LET: $C = \bigcup_{n \in \mathbb{N}} F(n)$
$\langle 1 \rangle 4$. C is transitive.
  $\langle 2 \rangle 1$. LET: $x \in y \in C$
  $\langle 2 \rangle 2$. PICK $n \in \mathbb{N}$ such that $y \in F(n)$
  $\langle 2 \rangle 3$. $y \subseteq F(n+1)$
    PROOF: $\langle 1 \rangle 2$
  $\langle 2 \rangle 4$. $x \in F(n+1)$
  $\langle 2 \rangle 5$. $x \in C$
$\langle 1 \rangle 5$. $A \subseteq C$
  PROOF: Since $F(0) = A$.
$\langle 1 \rangle 6$. For any transitive set X, if $A \subseteq X$ then $C \subseteq X$
  $\langle 2 \rangle 1$. LET: X be a transitive set
  $\langle 2 \rangle 2$. ASSUME: $A \subseteq X$
  $\langle 2 \rangle 3$. For all $n \in \mathbb{N}$ we have $F(n) \subseteq X$.
    $\langle 3 \rangle 1$. $F(0) \subseteq X$
      PROOF: $\langle 2 \rangle 2$
    $\langle 3 \rangle 2$. For all $n \in \mathbb{N}$, if $F(n) \subseteq X$, then $F(n+1) \subseteq X$.
      $\langle 4 \rangle 1$. LET: $n \in \mathbb{N}$
      $\langle 4 \rangle 2$. ASSUME: $\forall m < n.F(m) \subseteq X$
      $\langle 4 \rangle 3$. $F(0) \cup \cdots \cup F(n) \subseteq X$
      $\langle 4 \rangle 4$. $\bigcup(F(0) \cup \cdots \cup F(n)) \subseteq X$
        PROOF: Since X is transitive.
      $\langle 4 \rangle 5$. $F(n+1) \subseteq X$
  $\langle 2 \rangle 4$. $C \subseteq X$
$\langle 1 \rangle 7$. Let D be a transitive set such that $A \subseteq D$ and, for any transitive set X, if $A \subseteq X$ then $D \subseteq X$. Then $D = C$.
  PROOF: We have $C \subseteq D$ and $D \subseteq C$.
$\square$

8.7 The Veblen Fixed Point Theorem

Theorem Schema 8.7.1 (Veblen Fixed Point Theorem)**.** *For any class* $\mathbf{T}$*, the following is a theorem:*

Assume $\mathbf{T}$ *is a normal ordinal operation. For every ordinal* β*, there exists* $\gamma \geq \beta$ *such that* $\mathbf{T}(\gamma) = \gamma$*.*

PROOF:
$\langle 1 \rangle 1$. LET: β be an ordinal.
$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $\beta < \mathbf{T}(\beta)$
  PROOF: We have $\beta \leq \mathbf{T}(\beta)$ by Proposition 6.4.5, and if $\beta = \mathbf{T}(\beta)$ we take $\gamma := \beta$.

⟨1⟩3. Define $f : \mathbb{N} \to \mathbf{On}$ by recursion thus:

$$f(0) = \beta$$
$$f(n^+) = \mathbf{T}(f(n))$$

⟨1⟩4. LET: $\gamma = \sup_{n \in \mathbb{N}} f(n)$

⟨1⟩5. $\beta \leq \gamma$

PROOF: Since $\beta = f(0)$.

⟨1⟩6. $\mathbf{T}(\gamma) = \gamma$

⟨2⟩1. $\mathbf{T}(\gamma) \leq \gamma$

PROOF:

$$\begin{aligned}
\mathbf{T}(\gamma) &= \sup_{n \in \mathbb{N}} \mathbf{T}(f(n)) && \text{(Theorem 6.4.7)} \\
&= \sup_{n \in \mathbb{N}} f(n^+) && (\langle 1 \rangle 3) \\
&\leq \sup_{n \in \mathbb{N}} f(n) \\
&= \gamma
\end{aligned}$$

⟨2⟩2. $\gamma \leq \mathbf{T}(\gamma)$

PROOF:Proposition 6.4.5.

□

Definition 8.7.2 (Derived Operation)**.** Let $\mathbf{T}$ be a normal ordinal operation. The *derived* operation $\mathbf{T}' : \mathbf{On} \to \mathbf{V}$ is the unique order isomorphism between $\mathbf{On}$ and the fixed points of $\mathbf{T}$.

Proposition Schema 8.7.3. *For any class* $\mathbf{T}$*, the following is a theorem:*

If $\mathbf{T}$ *is a normal ordinal operation, then the derived operation is normal.*

PROOF:

⟨1⟩1. For any set S of fixed points of $\mathbf{T}$, we have $\bigcup S$ is a fixed point of $\mathbf{T}$

⟨2⟩1. LET: S be a set of fixed points of $\mathbf{T}$.

⟨2⟩2. $\mathbf{T}(\sup S) = \sup S$

PROOF:

$$\begin{aligned}
\mathbf{T}(\sup S) &= \sup_{\alpha \in S} \mathbf{T}(\alpha) && \text{(Theorem 6.4.7)} \\
&= \sup_{\alpha \in S} \alpha && (\langle 2 \rangle 1) \\
&= \sup S
\end{aligned}$$

⟨1⟩2. Q.E.D.

PROOF: Proposition 6.4.8.

□

8.8 Cantor Normal Form

Theorem 8.8.1. *For any ordinal* α*, there exist a unique sequence of nonzero natural numbers* $(n_1, \ldots, n_k)$ *and sequence of ordinals* $(\gamma_1, \ldots, \gamma_k)$ *such that*

$$\gamma_k < \gamma_{k-1} < \cdots < \gamma_1$$

and

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_k} n_k \ .$$

PROOF:
⟨1⟩1. For any ordinal α, there exist a sequence of nonzero natural numbers $(n_1, \ldots, n_k)$ and sequence of ordinals $(\gamma_1, \ldots, \gamma_k)$ such that

$$\gamma_k < \gamma_{k-1} < \cdots < \gamma_1$$

and

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_k} n_k \ .$$

⟨2⟩1. LET: α be an ordinal
⟨2⟩2. ASSUME: as an induction hypothesis that, for all $\beta < \alpha$, the theorem holds.
⟨2⟩3. ASSUME: w.l.o.g. $\alpha \neq 0$
⟨2⟩4. LET: γ_1, n_1, ρ_1 be the unique ordinals such that $0 \neq n_1 < \omega$, $\rho_1 < \omega^{\gamma_1}$, and $\alpha = \omega^{\gamma_1} n_1 + \rho_1$
⟨2⟩5. LET: $(\gamma_2, \ldots, \gamma_k)$ and $(n_2, \ldots, n_k)$ be sequences such that $\gamma_k < \gamma_{k-1} < \cdots < \gamma_2$ and $\rho_1 = \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_k} n_k$
⟨2⟩6. $\gamma_2 < \gamma_1$
PROOF: Since $\omega^{\gamma_2} \leq \rho_1 < \omega^{\gamma_1}$
⟨1⟩2. If

$$\gamma_k < \gamma_{k-1} < \cdots < \gamma_1$$
$$\gamma'_k < \gamma'_{k-1} < \cdots < \gamma'_1$$

and

$$\omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_k} n_k = \omega^{\gamma'_1} n'_1 + \omega^{\gamma'_2} n'_2 + \cdots + \omega^{\gamma'_k} n'_k$$

then $\gamma_i = \gamma'_i$ for all i and $n_i = n'_i$ for all i
PROOF: Prove by induction on i using the Logarithm Theorem.
□

Definition 8.8.2 (Cantor Normal Form)**.** For any ordinal α, the *Cantor normal form* of α is the expression $\alpha = \omega^{\gamma_1} n_1 + \cdots + \omega^{\gamma_k} n_k$ such that $n_1, \ldots, n_k$ are nonzero natural numbers and $\gamma_k < \gamma_{k-1} < \cdots < \gamma_1$.

Chapter 9

The Cumulative Hierarchy

Definition 9.0.1 (Cumulative Hierarchy)**.** Define the function $V : \mathbf{On} \to \mathbf{V}$ by transfinite recursion thus:

$$V_\alpha = \bigcup_{\beta < \alpha} \mathcal{P} V_\beta$$

Proposition 9.0.2. *For all $\alpha \in \mathbf{On}$, V_α is a transitive set.*

PROOF:
$\langle 1 \rangle 1$. LET: $\alpha \in \mathbf{On}$
$\langle 1 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha . V_\beta$ is a transitive set.
$\langle 1 \rangle 3$. For all $\beta < \alpha$, $\mathcal{P} V_\beta$ is a transitive set.
PROOF: Proposition 1.6.4.
$\langle 1 \rangle 4$. V_α is a transitive set.
PROOF: Proposition 1.6.3.
$\square$

Proposition 9.0.3. *For any ordinals α and β, if $\beta < \alpha$ then $V_\beta \subseteq V_\alpha$.*

PROOF: Since $V_\beta \in \mathcal{P} V_\beta \subseteq V_\alpha$ and V_α is a transitive set. $\square$

Theorem 9.0.4.

1. $V_0 = \emptyset$
2. $\forall \alpha \in \mathbf{On} . V_{\alpha^+} = \mathcal{P} V_\alpha$
3. *For any limit ordinal λ, $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$.*

PROOF:
$\langle 1 \rangle 1$. $V_0 = \emptyset$
PROOF: Immediate from definition.
$\langle 1 \rangle 2$. $\forall \alpha \in \mathbf{On} . V_{\alpha^+} = \mathcal{P} V_\alpha$
PROOF:

⟨2⟩1. LET: $\alpha \in \mathbf{On}$
⟨2⟩2. For all $\beta < \alpha$ we have $\mathcal{P}V_\beta \subseteq \mathcal{P}V_\alpha$
PROOF: Propositions 1.5.8 and 9.0.3.
⟨2⟩3. $V_{\alpha^+} = \mathcal{P}V_\alpha$

$$\begin{aligned} V_{\alpha^+} &= \bigcup_{\beta<\alpha^+} \mathcal{P}V_\beta \\ &= \bigcup_{\beta<\alpha} \mathcal{P}V_\beta \cup \mathcal{P}V_\alpha \\ \mathcal{P}V_\alpha \end{aligned}$$

∎

⟨1⟩3. For any limit ordinal λ, $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$
PROOF:
⟨2⟩1. $V_\lambda \subseteq \bigcup_{\alpha<\lambda} V_\alpha$
PROOF:

$$\begin{aligned} V_\lambda &= \bigcup_{\alpha<\lambda} \mathcal{P}V_\alpha \\ &= \bigcup_{\alpha<\lambda} V_{\alpha^+} && (\langle 1\rangle 2) \\ &\subseteq \bigcup_{\alpha<\lambda} V_\alpha \end{aligned}$$

⟨2⟩2. $\bigcup_{\alpha<\lambda} V_\alpha \subseteq V_\lambda$
PROOF: Proposition 9.0.3.
∎

Proposition 9.0.5. *For every set A, there exists an ordinal α such that $A \in V_\alpha$.*

PROOF:
⟨1⟩1. Let us say a set A is *grounded* iff there exists an ordinal α such that $A \in V_\alpha$.
⟨1⟩2. For any set A, if every element of A is grounded, then A is grounded.
⟨2⟩1. LET: A be a set.
⟨2⟩2. $S = \{\alpha \mid \exists a \in A.\alpha$ is the least ordinal such that $a \in V_\alpha\}$
PROOF: S is a set by an Axiom of Replacement.
⟨2⟩3. LET: $\beta = \sup S$
⟨2⟩4. $A \subseteq V_\beta$
⟨3⟩1. LET: $a \in A$
⟨3⟩2. LET: α be the least ordinal such that $a \in V_\beta$
⟨3⟩3. $\alpha \in S$
⟨3⟩4. $\alpha \leq \beta$
⟨3⟩5. $a \in V_\beta$
⟨2⟩5. $A \in V_{\beta+}$
⟨1⟩3. ASSUME: for a contradiction there exists an ungrounded set.
⟨1⟩4. PICK a transitive set B that has an ungrounded member.
PROOF: Pick a transitive set c, and take B to be the transitive closure of $\{c\}$.
⟨1⟩5. LET: $A = \{x \in B \mid x$ is ungrounded$\}$

$\langle 1 \rangle 6$. PICK $m \in A$ such that $m \cap A = \emptyset$
PROOF: Axiom of Regularity.
$\langle 1 \rangle 7$. Every member of m is grounded.
$\langle 2 \rangle 1$. ASSUME: for a contradiction $x \in m$ is ungrounded.
$\langle 2 \rangle 2$. $x \in B$
PROOF: Since B is transitive ($\langle 1 \rangle 4$).
$\langle 2 \rangle 3$. $x \in A$
PROOF: $\langle 1 \rangle 5$
$\langle 2 \rangle 4$. Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 6$.
$\langle 1 \rangle 8$. m is grounded.
PROOF: $\langle 1 \rangle 2$
$\langle 1 \rangle 9$. Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 6$.
$\square$

Definition 9.0.6 (Rank)**.** The *rank* of a set A is the least ordinal α such that $A \in V_{\alpha^+}$.

Proposition 9.0.7. *For any set A we have*

$$\operatorname{rank} A = \bigcup_{a \in A} (\operatorname{rank} a)^+$$

PROOF:
$\langle 1 \rangle 1$. LET: $\alpha = \bigcup_{a \in A} (\operatorname{rank} a)^+$
$\langle 1 \rangle 2$. $A \subseteq V_\alpha$
$\langle 2 \rangle 1$. LET: $a \in A$
$\langle 2 \rangle 2$. $a \in V_{(\operatorname{rank} a)^+}$
$\langle 2 \rangle 3$. $a \in V_\alpha$
$\langle 1 \rangle 3$. $A \in V_{\alpha^+}$
$\langle 1 \rangle 4$. If $A \subseteq V_\beta$ then $\alpha \leq \beta$
$\langle 2 \rangle 1$. ASSUME: $A \subseteq V_\beta$
$\langle 2 \rangle 2$. For all $a \in A$ we have $(\operatorname{rank} a)^+ \leq \beta$
PROOF: Since $a \in V_\beta$.
$\langle 2 \rangle 3$. $\alpha \leq \beta$
$\square$

Corollary 9.0.7.1. *For any sets a and b, if $a \in b$ then* $\operatorname{rank} a < \operatorname{rank} b$*.*

Proposition 9.0.8. *For any ordinal number α we have* $\operatorname{rank} \alpha = \alpha$*.*

PROOF:
$\langle 1 \rangle 1$. LET: α be an ordinal.
$\langle 1 \rangle 2$. ASSUME: as transfinite induction hypothesis $\forall \beta < \alpha . \operatorname{rank} \beta = \beta$
$\langle 1 \rangle 3$. $\operatorname{rank} \alpha = \bigcup_{\beta < \alpha} \beta^+$

PROOF:

$$\operatorname{rank}\alpha = \bigcup_{\beta<\alpha} (\operatorname{rank}\beta)^+$$
$$= \bigcup_{\beta<\alpha} \beta^+$$

⟨1⟩4. $\bigcup_{\beta<\alpha} \beta^+ \leq \alpha$

PROOF: Since for all $\beta < \alpha$ we have $\beta^+ \leq \alpha$.

⟨1⟩5. $\alpha \leq \bigcup_{\beta<\alpha} \beta^+$

⟨2⟩1. LET: $\gamma = \bigcup_{\beta<\alpha} \beta^+$

⟨2⟩2. ASSUME: for a contradiction $\gamma < \alpha$

⟨2⟩3. $\gamma^+ \leq \bigcup_{\beta<\alpha} \beta^+ = \gamma$

⟨2⟩4. Q.E.D.

PROOF: This is a contradiction.

□

Definition 9.0.9 (Hereditarily Finite)**.** A set is *hereditarily finite* iff it is in V_ω.

Chapter 10

Models of Set Theory

Definition 10.0.1 (Relativization)**.** Let σ be a sentence in the language of set theory and $\mathbf{M}$ a class. The *relativization* of σ to $\mathbf{M}$ is the sentence $\sigma^{\mathbf{M}}$ formed by replacing every quantifier $\forall x$ with $\forall x \in \mathbf{M}$, and $\exists x$ with $\exists x \in \mathbf{M}$.

We write '$\mathbf{M}$ is a model of σ' for the sentence $\sigma^{\mathbf{M}}$.

Theorem Schema 10.0.2. *For any class* $\mathbf{M}$*, the following is a theorem:*

If $\mathbf{M}$ *is a transitive class, then* $\mathbf{M}$ *is a model of the Axiom of Extensionality.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: $\mathbf{M}$ is a transitive class.
PROVE: $\forall x, y \in \mathbf{M}(\forall z \in \mathbf{M}(z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$
$\langle 1 \rangle 2.$ LET: $x, y \in \mathbf{M}$
$\langle 1 \rangle 3.$ ASSUME: $\forall z \in \mathbf{M}(z \in x \Leftrightarrow z \in y)$
$\langle 1 \rangle 4.$ $\forall z(z \in x \Leftrightarrow z \in y)$
PROOF: Since $z \in x \Rightarrow z \in \mathbf{M}$ and $z \in y \Rightarrow z \in \mathbf{M}$ by $\langle 1 \rangle 1$.
$\langle 1 \rangle 5.$ $x = y$
□

Theorem 10.0.3. *If* α *is a non-zero ordinal then* V_α *is a model of the statement: The empty class is a set.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\alpha \neq 0$
PROVE: $\exists x \in V_\alpha . \forall y \in V_\alpha . y \notin x$
$\langle 1 \rangle 2.$ $\emptyset \in V_\alpha$
$\langle 1 \rangle 3.$ $\forall y \in V_\alpha . y \notin \emptyset$
□

Theorem 10.0.4. *For any limit ordinal* λ*, we have* V_λ *is a model of the statement: for any sets* a *and* b*, the class* $\{a, b\}$ *is a set.*

PROOF:
$\langle 1 \rangle 1.$ LET: λ be a limit ordinal.

PROVE: $\forall a, b \in V_\lambda . \exists c \in V_\lambda . \forall x \in V_\lambda (x \in c \Leftrightarrow x = a \vee x = b)$
$\langle 1 \rangle 2$. LET: $a, b \in V_\lambda$
$\langle 1 \rangle 3$. PICK $\alpha, \beta < \lambda$ such that $a \in V_\alpha$ and $b \in V_\beta$
$\langle 1 \rangle 4$. ASSUME: w.l.o.g. $\alpha \leq \beta$
$\langle 1 \rangle 5$. $a, b \in V_\beta$
$\langle 1 \rangle 6$. $\{a, b\} \in V_{\beta+1}$
$\langle 1 \rangle 7$. $\{a, b\} \in V_\lambda$
$\langle 1 \rangle 8$. $\forall x \in V_\lambda (x \in \{a, b\} \Leftrightarrow x = a \vee x = b)$
□

Theorem 10.0.5. *For any ordinal α, we have V_α is a model of the Union Axiom.*

PROOF:
$\langle 1 \rangle 1$. LET: α be an ordinal.
PROVE: $\forall a \in V_\alpha . \exists b \in V_\alpha . \forall x \in V_\alpha (x \in b \Leftrightarrow \exists y \in V_\alpha (x \in y \wedge y \in a))$
$\langle 1 \rangle 2$. LET: $a \in V_\alpha$
$\langle 1 \rangle 3$. PICK $\beta < \alpha$ such that $a \subseteq V_\beta$
$\langle 1 \rangle 4$. $\bigcup a \subseteq V_\beta$
PROOF: V_β is a transitive set.
$\langle 1 \rangle 5$. $\bigcup a \in V_\alpha$
$\langle 1 \rangle 6$. $\forall x \in V_\alpha (x \in \bigcup a \Leftrightarrow \exists y \in V_\alpha (x \in y \wedge y \in a))$
PROOF: V_α is a transitive set.
□

Theorem 10.0.6. *For any limit ordinal λ, we have V_λ is a model of the Power Set Axiom.*

PROOF:
$\langle 1 \rangle 1$. LET: λ be a limit ordinal.
PROVE: $\forall a \in V_\lambda . \exists b \in V_\lambda . \forall x \in V_\lambda (x \in b \Leftrightarrow \forall y \in V_\lambda (y \in x \Rightarrow y \in a))$
$\langle 1 \rangle 2$. LET: $a \in V_\lambda$
$\langle 1 \rangle 3$. PICK $\alpha < \lambda$ such that $a \in V_\alpha$
$\langle 1 \rangle 4$. $\mathcal{P}a \in V_{\alpha+1}$
$\langle 1 \rangle 5$. $\mathcal{P}a \in V_\lambda$
$\langle 1 \rangle 6$. $\forall x \in V_\lambda (x \in \mathcal{P}a \Leftrightarrow \forall y \in V_\lambda (y \in x \Rightarrow y \in a))$
□

Theorem Schema 10.0.7. *For any property $P[x, y_1, \ldots, y_n]$, the following is a theorem:*

For any ordinal α, the set V_α is a model of the statement: for any sets a_1, $\ldots$, a_n, B, the class $\{x \in B \mid P[x, a_1, \ldots, a_n]\}$ is a set.

PROOF:
$\langle 1 \rangle 1$. LET: α be an ordinal.
$\langle 1 \rangle 2$. LET: $a_1, \ldots, a_n, B \in V_\alpha$
$\langle 1 \rangle 3$. LET: $C = \{x \in B \mid P[x, a_1, \ldots, a_n]^{V_\alpha}\}$
$\langle 1 \rangle 4$. $C \in V_\alpha$

⟨1⟩5. $\forall x \in V_\alpha (x \in C \Leftrightarrow x \in B \wedge P[x, a_1, \ldots, a_n]^{V_\alpha})$
□

Theorem 10.0.8. *For any ordinal $\alpha > \omega$, we have: V_α is a model of the Axiom of Infinity.*

Proof:
⟨1⟩1. Let: $\alpha > \omega$
⟨1⟩2. $\mathbb{N} \in V_\alpha$
⟨1⟩3. $\exists e \in V_\alpha (e \in \mathbb{N} \wedge \forall x \in V_\alpha . x \notin e)$
⟨1⟩4. $\forall x \in V_\alpha (x \in \mathbb{N} \Rightarrow \exists y \in V_\alpha \forall z \in V_\alpha (z \in y \Leftrightarrow z \in x \vee z = x))$
□

Theorem 10.0.9. *For any ordinal α, we have V_α is a model of the Axiom of Choice.*

Proof:
⟨1⟩1. Let: α be an ordinal.
⟨1⟩2. Let: $A \in V_\alpha$
⟨1⟩3. Assume: $\forall x \in V_\alpha (x \in A \Rightarrow \exists y \in V_\alpha . y \in A)$
⟨1⟩4. Assume: $\forall x, y, z \in V_\alpha (x \in A \wedge y \in A \wedge z \in x \wedge z \in y \Rightarrow x = y)$
⟨1⟩5. A is a set of pairwise disjoint nonempty sets.
⟨1⟩6. Pick c such that, for all $x \in A$, $x \cap c = \emptyset$
⟨1⟩7. $c \cap \bigcup A \in V_\alpha$
⟨1⟩8. $\forall x \in V_\alpha (x \in A \Rightarrow \exists y \in V_\alpha \forall z \in V_\alpha (z \in x \wedge z \in c \cap \bigcup A \Leftrightarrow z = y))$
□

Theorem 10.0.10. *For any ordinal α, we have V_α is a model of the Axiom of Regularity.*

Proof:
⟨1⟩1. Let: α be an ordinal.
⟨1⟩2. Let: $A \in V_\alpha$
⟨1⟩3. Assume: $\exists x \in V_\alpha . x \in A$
⟨1⟩4. Pick $m \in A$ of least rank.
⟨1⟩5. $m \in V_\alpha$
⟨1⟩6. $\neg \exists x \in V_\alpha (x \in m \wedge x \in A)$
□

Theorem Schema 10.0.11. *For any axiom α of Zermelo set theory, the following is a theorem:*
For any limit ordinal $\lambda > \omega$, we have V_λ is a model of α.

Proof: Theorems 10.0.2, 10.0.3, 10.0.4, 10.0.5, 10.0.6, 10.0.7, 10.0.8, 10.0.9, 10.0.10. □

Corollary Schema 10.0.11.1. *for any axiom α of Zermelo set theory, the following is a theorem:*
$V_{\omega 2}$ is a model of α.

Lemma 10.0.12. *There exists a well-ordered structure in $V_{\omega 2}$ whose ordinal is not in $V_{\omega 2}$.*

PROOF: Take the well-ordered set $\mathbb{N} \times \{0, 1\}$ whose ordinal is $\omega 2$. □

Corollary Schema 10.0.12.1. *There exists an instance α of the Axiom Schema of Replacement such that the following is a theorem:*
$V_{\omega 2}$ is not a model of α.

Chapter 11

Infinite Cardinals

11.1 Arithmetic of Infinite Cardinals

Proposition 11.1.1. *For any infinite cardinal κ we have $\kappa\kappa = \kappa$.*

Proof:
$\langle 1\rangle 1.$ Pick a set B with $|B| = \kappa$
$\langle 1\rangle 2.$ Let: $\mathcal{H} = \{f \mid f = \emptyset \vee \exists A \subseteq B.(A \text{ is infinite} \wedge f : A \times A \approx A\}$
$\langle 1\rangle 3.$ For any chain $\mathcal{C} \subseteq \mathcal{H}$ we have $\bigcup \mathcal{C} \in \mathcal{H}$
  $\langle 2\rangle 1.$ Let: $\mathcal{C} \subseteq \mathcal{H}$ be a chain.
  $\langle 2\rangle 2.$ Assume: w.l.o.g. $\mathcal{C}$ has a nonempty element.
  $\langle 2\rangle 3.$ $\bigcup \mathcal{C}$ is a function.
    $\langle 3\rangle 1.$ Assume: $(x, y), (x, z) \in \bigcup \mathcal{C}$
    $\langle 3\rangle 2.$ Pick $f, g \in \mathcal{C}$ such that $f(x) = y$ and $g(x) = z$
    $\langle 3\rangle 3.$ Assume: w.l.o.g. $f \subseteq g$
    $\langle 3\rangle 4.$ $y = z$
  $\langle 2\rangle 4.$ $\bigcup \mathcal{C}$ is injective.
    Proof: Similar.
  $\langle 2\rangle 5.$ Let: $A = \operatorname{ran} \bigcup \mathcal{C}$
  $\langle 2\rangle 6.$ A is infinite.
    $\langle 3\rangle 1.$ Pick a nonzero $f \in \mathcal{C}$
    $\langle 3\rangle 2.$ Let: A' be the infinite subset of B such that $f : A'^2 \approx A'$
    $\langle 3\rangle 3.$ $A' \subseteq A$
  $\langle 2\rangle 7.$ $\operatorname{dom} \bigcup \mathcal{C} = A^2$
    $\langle 3\rangle 1.$ Let: $x, y \in A$
    $\langle 3\rangle 2.$ Pick $f, g \in \mathcal{C}$ such that $x \in \operatorname{ran} f$ and $y \in \operatorname{ran} g$
    $\langle 3\rangle 3.$ Assume: w.l.o.g. $f \subseteq g$
    $\langle 3\rangle 4.$ Let: A' be the infinite subset of B such that $g : A'^2 \approx A'$
    $\langle 3\rangle 5.$ $x, y \in A'$
    $\langle 3\rangle 6.$ $(x, y) \in \operatorname{dom} g$
    $\langle 3\rangle 7.$ $(x, y) \in \operatorname{dom} \bigcup \mathcal{C}$
  $\langle 2\rangle 8.$ $\bigcup \mathcal{C} \in \mathcal{H}$

$\langle 1\rangle 4.$ Pick a maximal $f_0 \in \mathcal{H}$
$\langle 1\rangle 5.$ $f_0 \neq \emptyset$
  $\langle 2\rangle 1.$ Pick a countably infinite subset A of B.
    Proof: Proposition 8.2.10.
  $\langle 2\rangle 2.$ Pick a bijection $f : A^2 \approx A$
    Proof: Proposition 8.2.9.
  $\langle 2\rangle 3.$ $\emptyset \subseteq f \in \mathcal{H}$
  $\langle 2\rangle 4.$ $\emptyset$ is not maximal in $\mathcal{H}$
$\langle 1\rangle 6.$ Let: A_0 be the infinite subset of B such that $f_0 : A_0^2 \approx A_0$
$\langle 1\rangle 7.$ Let: $\lambda = |A_0|$
$\langle 1\rangle 8.$ λ is infinite.
$\langle 1\rangle 9.$ $\lambda^2 = \lambda$
$\langle 1\rangle 10.$ $\lambda = \kappa$
  $\langle 2\rangle 1.$ Assume: for a contradiction $\lambda < \kappa$
  $\langle 2\rangle 2.$ $\lambda \leq |B - A_0|$
  $\langle 2\rangle 3.$ Pick a subset $D \subseteq B - A_0$ with $|D| = \lambda$
  $\langle 2\rangle 4.$ $(A_0 \cup D)^2 = A_0^2 \cup (A_0 \times D) \cup (D \times A_0) \cup D^2$
  $\langle 2\rangle 5.$ Let: $C = (A_0 \times D) \cup (D \times A_0) \cup D^2$
  $\langle 2\rangle 6.$ $|C| = \lambda$
    Proof:

$$\begin{aligned}
|(A_0 \times D) \cup (D \times A_0) \cup D^2| &= \lambda^2 + \lambda^2 + \lambda^2 \\
&= \lambda + \lambda + \lambda && (\langle 1\rangle 9) \\
&= 3\lambda \\
&\leq \lambda \cdot \lambda \\
&= \lambda && (\langle 1\rangle 9)
\end{aligned}$$

  $\langle 2\rangle 7.$ Pick a bijection $g : C \approx D$
  $\langle 2\rangle 8.$ $f_0 \cup g : (A_0 \cup D)^2 \approx A_0 \cup D$
  $\langle 2\rangle 9.$ Q.E.D.
    Proof: This contradicts the maximality of f_0.
□

Theorem 11.1.2 (Absorption Law of Cardinal Arithmetic)**.** *Let κ and λ be nonzero cardinal numbers such that at least one is infinite. Then*

$$\kappa + \lambda = \kappa\lambda = \max(\kappa, \lambda)$$

Proof:
$\langle 1\rangle 1.$ Assume: w.l.o.g. $\lambda \leq \kappa$
$\langle 1\rangle 2.$ $\kappa + \lambda = \kappa\lambda = \kappa$

PROOF:

$$\begin{aligned}\kappa &\leq \kappa + \lambda \\ &\leq \kappa + \kappa \\ &= 2\kappa \\ &\leq \kappa\lambda \\ &\leq \kappa\kappa \\ &= \kappa && \text{(Proposition 11.1.1)}\end{aligned}$$

∎

11.2 Alephs

Definition 11.2.1 (Aleph)**.** Let $\aleph$ be the unique order isomorphism between **On** and the class of infinite cardinals.

Proposition 11.2.2. *The operation $\aleph$ is normal.*

PROOF: Proposition 6.4.8 and Lemma 7.2.7. ∎

Definition 11.2.3 (Continuum Hypothesis)**.** The *continuum hypothesis* is the statement that $\aleph_1 = 2^{\aleph_0}$.

Definition 11.2.4 (Generalised Continuum Hypothesis)**.** The *generalised continuum hypothesis* is the statement that, for all α, $\aleph_{\alpha^+} = 2^{\aleph_\alpha}$.

11.3 Beths

Definition 11.3.1 (Beth)**.** Define the operation $\beth : \mathbf{On} \to \mathbf{Card}$ by transfinite recursion as follows:

$$\begin{aligned}\beth_0 &:= \aleph_0 \\ \beth_{\alpha^+} &:= 2^{\beth_\alpha} \\ \beth_\lambda &:= \bigcup_{\alpha<\lambda} \beth_\alpha && (\lambda \text{ a limit ordinal})\end{aligned}$$

Proposition 11.3.2. *$\beth$ is a normal operation.*

PROOF: It is continuous by definition, and $\beth_\alpha < \beth_{\alpha^+}$ by Cantor's Theorem. ∎

Proposition 11.3.3. *The continuum hypothesis is equivalent to the statement $\beth_1 = \aleph_1$.*

The generalised continuum hypothesis is equivalent to the statement $\beth = \alpha$.

PROOF: Immediate from definitions. ∎

Lemma 11.3.4. *For any ordinal number α, we have $|V_{\omega+\alpha}| = \beth_\alpha$.*

PROOF:

$\langle 1 \rangle 1.$ $|V_\omega| = \beth_0$

PROOF: Since V_ω is the union of $\aleph_0$ finite sets of increasing size.

$\langle 1 \rangle 2.$ For any ordinal α, if $|V_{\omega+\alpha}| = \beth_\alpha$ then $|V_{\omega+\alpha+1}| = \beth_{\alpha+1}$

PROOF: Since $V_{\omega+\alpha+1} = \mathcal{P}V_{\omega+\alpha}$.

$\langle 1 \rangle 3.$ For any limit ordinal λ, if $\forall \alpha < \lambda. |V_{\omega+\alpha}| = \beth_\alpha$ then $|V_{\omega+\lambda}| = \beth_\lambda$.

PROOF:

$$\begin{aligned} |V_{\omega+\lambda}| &= \left| \bigcup_{\alpha<\lambda} V_{\omega+\alpha} \right| \\ &= \sup_{\alpha<\lambda} |V_{\omega+\alpha}| \\ &= \sup_{\alpha<\lambda} \beth_\alpha \\ &= \beth_\lambda \end{aligned}$$

□

11.4 Cofinality

Definition 11.4.1 (Cofinal)**.** Let λ be a limit ordinal and S a set of ordinals smaller than λ. Then S is *cofinal* in λ if and only if $\lambda = \sup S$.

Definition 11.4.2 (Cofinality)**.** For any ordinal α, define the *cofinality* of α, $\mathrm{cf}\,\alpha$, as follows:

- $\mathrm{cf}\,0 = 0$
- For any ordinal α, $\mathrm{cf}\,\alpha^+ = 1$
- For any limit ordinal λ, $\mathrm{cf}\,\lambda$ is the smallest cardinal such that there exists a set S of ordinals cofinal in λ with $|S| = \mathrm{cf}\,\lambda$.

Definition 11.4.3 (Regular)**.** A cardinal κ is *regular* iff $\mathrm{cf}\,\kappa = \kappa$; otherwise it is *singular*.

Proposition 11.4.4. *$\aleph_0$ is regular.*

PROOF: $\aleph_0$ is not the supremum of $< \aleph_0$ smaller ordinals, because a finite union of finite ordinals is finite. □

Proposition 11.4.5. *For every ordinal α, $\aleph_{\alpha+1}$ is regular.*

PROOF: If S is a set of ordinals with $|S| < \aleph_{\alpha+1}$ and $\forall \beta \in S. \beta < \aleph_{\alpha+1}$, then we have $|S| \leq \aleph_\alpha$ and $\forall \beta \in S. \beta \leq \aleph_\alpha$, hence

$$\begin{aligned} \left| \bigcup S \right| &\leq \aleph_\alpha^2 && \text{(Proposition 7.2.6)} \\ &= \aleph_\alpha && \text{(Proposition 11.1.1)} \end{aligned}$$ □

Proposition Schema 11.4.6. *For any class* $\mathbf{T}$*, the following is a theorem.*

Assume $\mathbf{T} : \mathbf{On} \to \mathbf{On}$ *is a normal operation. For any limit ordinal λ we have* $\mathrm{cf}\,\mathbf{T}(\lambda) = \mathrm{cf}\,\lambda$.

PROOF:
$\langle 1 \rangle 1.$ $\mathrm{cf}\,\mathbf{T}(\lambda) \leq \mathrm{cf}\,\lambda$
  $\langle 2 \rangle 1.$ PICK a set S of ordinals $< \lambda$ with $|S| = \mathrm{cf}\,\lambda$ and $\sup S = \lambda$
  $\langle 2 \rangle 2.$ $\mathbf{T}(\lambda) = \sup_{\alpha \in S} \mathbf{T}(\alpha)$
    PROOF: Theorem 6.4.7.
$\langle 1 \rangle 2.$ $\mathrm{cf}\,\lambda \leq \mathrm{cf}\,\mathbf{T}(\lambda)$
  $\langle 2 \rangle 1.$ PICK a set A of ordinals $< \mathbf{T}(\lambda)$ such that $|A| = \mathrm{cf}\,\mathbf{T}(\lambda)$ and $\sup A = \mathbf{T}(\lambda)$
  $\langle 2 \rangle 2.$ LET: $B = \{\gamma < \lambda \mid \exists \alpha \in A . |\alpha| = \mathbf{T}(\gamma)\}$
  $\langle 2 \rangle 3.$ $|B| \leq |A| = \mathrm{cf}\,\mathbf{T}(\lambda)$
    PROVE: $\sup B = \lambda$
  $\langle 2 \rangle 4.$ $\forall \alpha \in A . |\alpha| \leq \mathbf{T}(\sup B)$
  $\langle 2 \rangle 5.$ $\forall \alpha \in A . \alpha < \mathbf{T}(\sup B + 1)$
  $\langle 2 \rangle 6.$ $\aleph_\lambda = \sup A \leq \mathbf{T}(\sup B + 1)$
  $\langle 2 \rangle 7.$ $\lambda \leq \sup B + 1$
  $\langle 2 \rangle 8.$ $\lambda \leq \sup B$
    PROOF: λ is a limit ordinal.
  $\langle 2 \rangle 9.$ $\sup B = \lambda$
□

Corollary 11.4.6.1. $\aleph_\omega$ *is singular.*

PROOF: $\mathrm{cf}\,\aleph_\omega = \mathrm{cf}\,\aleph_0 = \aleph_0$. □

Corollary 11.4.6.2. *The operation* cf *is not strictly monotone or continuous.*

PROOF: $\mathrm{cf}\,\aleph_\omega < \mathrm{cf}\,\aleph_1$ □

Definition 11.4.7 (Weakly Inaccessible)**.** A cardinal is *weakly inaccessible* iff it is $\aleph_\lambda$ for some limit ordinal λ and regular.

Lemma 11.4.8. *Let λ be a limit ordinal. Then there exists a strictly increasing* $\mathrm{cf}\,\lambda$*-sequence that converges to λ.*

PROOF:
$\langle 1 \rangle 1.$ PICK a set S of ordinals $< \lambda$ with $|S| = \mathrm{cf}\,\lambda$ and $\sup S = \lambda$
$\langle 1 \rangle 2.$ PICK a bijection $a : \mathrm{cf}\,\lambda \approx S$
$\langle 1 \rangle 3.$ PICK a strictly increasing subsequence $(b_\delta)_{\delta < \beta}$ of a that converges to λ.
  PROOF: Lemma 6.6.5.
$\langle 1 \rangle 4.$ $\beta = \mathrm{cf}\,\lambda$
  PROOF: By minimiality of $\mathrm{cf}\,\lambda$.
□

Corollary 11.4.8.1. *Let λ be a limit ordinal. Then* $\mathrm{cf}\,\lambda$ *is the least ordinal such that there exists a strictly increasing* $\mathrm{cf}\,\lambda$*-sequence that converges to λ.*

Proposition 11.4.9. *For any ordinal λ,* $\mathrm{cf}\,\lambda$ *is a regular cardinal.*

PROOF:

⟨1⟩1. LET: λ be an ordinal.
⟨1⟩2. ASSUME: w.l.o.g. λ is a limit ordinal.
⟨1⟩3. PICK a strictly increasing sequence $(a_\alpha)_{\alpha<\mathrm{cf}\,\lambda}$ that converges to λ.
⟨1⟩4. LET: S be a set of ordinals $< \mathrm{cf}\,\lambda$ such that $|S| = \mathrm{cf}\,\mathrm{cf}\,\lambda$ and $\sup S = \mathrm{cf}\,\lambda$.
⟨1⟩5. LET: $a(S) = \{a_\alpha \mid \alpha \in S\}$
⟨1⟩6. $a(S)$ is cofinal in λ.
  ⟨2⟩1. LET: $\beta < \lambda$
  ⟨2⟩2. PICK $\gamma < \mathrm{cf}\,\lambda$ such that $\beta < a_\gamma$
  ⟨2⟩3. PICK $\delta \in S$ such that $\gamma < \delta$
  ⟨2⟩4. $a_\delta \in a(S)$ and $\beta < a_\gamma < a_\delta$
⟨1⟩7. $\mathrm{cf}\,\lambda \leq \mathrm{cf}\,\mathrm{cf}\,\lambda$
  PROOF: Since $a(S)$ is a set of ordinals $< \lambda$ with $|a(S)| = \mathrm{cf}\,\mathrm{cf}\,\lambda$ and $\sup a(S) = \lambda$.
⟨1⟩8. $\mathrm{cf}\,\mathrm{cf}\,\lambda = \mathrm{cf}\,\lambda$
□

Theorem 11.4.10. *Let λ be an infinite cardinal. Then* $\mathrm{cf}\,\lambda$ *is the least cardinal such that λ can be partitioned into* $\mathrm{cf}\,\lambda$ *sets, each of cardinality $< \lambda$.*

PROOF:
⟨1⟩1. λ can be partitioned into $\mathrm{cf}\,\lambda$ sets, each of cardinality $< \lambda$
  ⟨2⟩1. PICK a strictly increasing sequence of ordinlas $(a_\alpha)_{\alpha<\mathrm{cf}\,\lambda}$ that converges to λ
  ⟨2⟩2. $\{\{\beta \mid a_\alpha \leq \beta < a_{\alpha+1}\} \mid \alpha < \mathrm{cf}\,\lambda\}$ is a partition of λ into $\mathrm{cf}\,\lambda$ sets, each of cardinality $< \lambda$
⟨1⟩2. If λ can be partitioned into κ sets, each of cardinality $< \lambda$, then $\mathrm{cf}\,\lambda \leq \kappa$.
  ⟨2⟩1. LET: $\mathcal{A}$ be a partition of λ into sets of cardinality $< \lambda$
  ⟨2⟩2. LET: $\kappa = |P|$
  ⟨2⟩3. PICK a bijection $A : \kappa \approx P$
  ⟨2⟩4. $\lambda = \bigcup_{\xi<\kappa} A(\xi)$
  ⟨2⟩5. For all $\xi < \kappa$ we have $|A(\xi)| < \lambda$
  ⟨2⟩6. LET: $\mu = \sup_{\xi<\kappa} |A(\xi)|$
  ⟨2⟩7. $\mu \leq \lambda$
  ⟨2⟩8. For all $\xi < \kappa$ we have $|A(\xi)| \leq \mu$
  ⟨2⟩9. $\lambda \leq \mu\kappa$
    PROOF: Proposition 7.2.6.
  ⟨2⟩10. ASSUME: w.l.o.g. $\kappa < \lambda$
    PROOF: If $\lambda \leq \kappa$ then $\mathrm{cf}\,\lambda \leq \kappa$ since $\mathrm{cf}\,\lambda \leq \lambda$.
  ⟨2⟩11. $\lambda = \mu$
    PROOF:
$$\begin{aligned} \lambda &\leq \mu\kappa && (\langle 2\rangle 9) \\ &\leq \lambda\lambda && (\langle 2\rangle 7,\ \langle 2\rangle 10) \\ &= \lambda && (\text{Proposition 11.1.1}) \end{aligned}$$
  ⟨2⟩12. $\{|A(\xi)| \mid \xi < \kappa\}$ is a set of $\leq \kappa$ ordinals all $< \lambda$ whose supremum is λ
  ⟨2⟩13. $\mathrm{cf}\,\lambda \leq \kappa$
□

Theorem 11.4.11 (König). *For any infinite cardinal κ we have $\kappa < \mathrm{cf}\, 2^\kappa$.*

PROOF:
$\langle 1\rangle 1$. ASSUME: for a contradiction $\mathrm{cf}\, 2^\kappa \leq \kappa$
$\langle 1\rangle 2$. LET: $S = 2^\kappa$
$\langle 1\rangle 3$. PICK a partition $\{A_\xi \mid \xi < \kappa\}$ of S^κ with $\forall \xi < \kappa . |A_\xi| < 2^\kappa$.
PROOF: Theorem 11.4.10.
$\langle 1\rangle 4$. $\forall \xi < \kappa . \{g(\xi) \mid g \in A_\xi\} \subsetneq S$
PROOF: We do not have equality because $|\{g(\xi) \mid g \in A_\xi\}| \leq |A_\xi| < 2^\kappa$.
$\langle 1\rangle 5$. For all $\xi < \kappa$, choose $s_\xi \in S - \{g(\xi) \mid g \in A_\xi\}$
$\langle 1\rangle 6$. $s \in S^\kappa$
$\langle 1\rangle 7$. For all $\xi < \kappa$ we have $s \notin A_\xi$
PROOF: Since for all $\xi < \kappa$ and $g \in A_\xi$ we have $s_\xi(\xi) \neq g(\xi)$.
$\langle 1\rangle 8$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 3$.
□

Corollary 11.4.11.1.

$$2^{\aleph_0} \neq \aleph_\omega$$

Proposition 11.4.12. *For any ordinal α, we have $\mathrm{cf}\,\alpha$ is the least cardinal such that α is the strict supremum of $\mathrm{cf}\,\alpha$ smaller ordinals.*

PROOF:
$\langle 1\rangle 1$. CASE: $\alpha = 0$
PROOF: Since $0 = \mathrm{ssup}\,\emptyset$.
$\langle 1\rangle 2$. CASE: $\alpha = \beta^+$
PROOF: Since $\beta^+ = \mathrm{ssup}\{\beta\}$.
$\langle 1\rangle 3$. CASE: α is a limit ordinal.
$\langle 2\rangle 1$. There exists a set S of ordinals $< \alpha$ such that $|S| = \mathrm{cf}\,\alpha$ and $\alpha = \mathrm{ssup}\, S$.
$\langle 3\rangle 1$. PICK a set S of ordinals $< \alpha$ such that $|S| = \mathrm{cf}\,\alpha$ and $\sup S = \alpha$
PROVE: $\alpha = \mathrm{ssup}\, S$
$\langle 3\rangle 2$. $\forall \beta \in S . \beta < \alpha$
$\langle 3\rangle 3$. For any ordinal γ, if $\forall \beta \in S . \beta < \gamma$ then $\alpha \leq \gamma$
$\langle 2\rangle 2$. If T is a set of ordinals $< \alpha$ such that $\alpha = \mathrm{ssup}\, T$, then $\mathrm{cf}\,\alpha \leq |T|$.
$\langle 3\rangle 1$. LET: T be a set of ordinals $< \alpha$ such that $\alpha = \mathrm{ssup}\, T$
$\langle 3\rangle 2$. $\alpha = \sup T$
$\langle 4\rangle 1$. For all $\beta \in T$ we have $\beta \leq \alpha$
$\langle 4\rangle 2$. LET: μ be any upper bound for T
PROVE: $\alpha \leq \mu$
$\langle 4\rangle 3$. $\alpha \leq \mu + 1$
PROOF: Since $\forall \beta \in T . \beta < \mu + 1$.
$\langle 4\rangle 4$. $\alpha \neq \mu + 1$
PROOF: Since α is a limit ordinal.
$\langle 4\rangle 5$. $\alpha < \mu + 1$
$\langle 4\rangle 6$. $\alpha \leq \mu$
$\langle 3\rangle 3$. $\mathrm{cf}\,\alpha \leq |T|$
□

11.5 Inaccessible Cardinals

Definition 11.5.1 (Inaccessible Cardinal)**.** A cardinal number κ is *inaccessible* iff:

- $\kappa > \aleph_0$
- $\forall \lambda < \kappa . 2^\lambda < \kappa$ (cardinal exponentiation)
- κ is regular.

Any inaccessible cardinal is weakly inaccessible.

PROOF:
⟨1⟩1. LET: $\kappa = \aleph_\lambda$ be weakly inaccessible.
PROVE: λ is a limit ordinal.
⟨1⟩2. $\lambda \neq 0$
⟨1⟩3. ASSUME: for a contradiction $\lambda = \beta + 1$
⟨1⟩4. $\aleph_\beta < \kappa$
⟨1⟩5. $2^{\aleph_\beta} < \kappa$
⟨1⟩6. $\aleph_{\beta+1} < \kappa$
PROOF: Since $\aleph_{\beta+1} \leq 2^{\aleph_\beta}$.
⟨1⟩7. Q.E.D.
PROOF: This contradicts ⟨1⟩3.
□

Proposition 11.5.2. *If the Generalized Continuum Hypothesis is true, then every weakly inaccessible cardinal is inaccessible.*

PROOF:
⟨1⟩1. ASSUME: The Generalized Continuum Hypothesis.
⟨1⟩2. LET: $\kappa = \aleph_\lambda$ be weakly inaccessible.
⟨1⟩3. $\kappa > \aleph_0$
PROOF: $\lambda > 0$ because λ is a limit ordinal.
⟨1⟩4. For all $\mu < \kappa$ we have $2^\mu < \kappa$
⟨2⟩1. LET: $\mu < \kappa$
⟨2⟩2. LET: $\mu = \aleph_\alpha$
⟨2⟩3. $\alpha < \lambda$
⟨2⟩4. $\alpha + 1 < \lambda$
PROOF: λ is a limit ordinal.
⟨2⟩5. $2^\mu < \kappa$
PROOF:

$$\begin{aligned}
2^\mu &= 2^{\aleph_\alpha} && (\langle 2\rangle 2) \\
&= 2^{\beth_\alpha} && (\langle 1\rangle 1) \\
&= \beth_{\alpha+1} \\
&= \aleph_{\alpha+1} && (\langle 1\rangle 1) \\
&< \aleph_\lambda && (\langle 2\rangle 4) \\
&= \kappa && (\langle 1\rangle 2)
\end{aligned}$$

$\langle 1 \rangle 5.$ κ is regular.
PROOF: $\langle 1 \rangle 2$
□

Lemma 11.5.3. *Let κ be an inaccessible cardinal. For every ordinal $\alpha < \kappa$ we have $\beth_\alpha < \kappa$.*

PROOF:
$\langle 1 \rangle 1.$ $\beth_0 < \kappa$
PROOF: Since $\kappa > \aleph_0$.
$\langle 1 \rangle 2.$ For any ordinal α, if $\beth_\alpha < \kappa$ then $\beth_{\alpha+1} < \kappa$.
PROOF: Since $\beth_{\alpha+1} = 2^{\beth_\alpha} < \kappa$.
$\langle 1 \rangle 3.$ For any limit ordinal λ, if $\forall \alpha < \lambda . \beth_\alpha < \kappa$ and $\lambda < \kappa$ then $\beth_\lambda < \kappa$.
PROOF: By regularity of κ, since $\beth_\lambda$ is the union of $|\lambda|$ cardinals all $< \kappa$.
□

Lemma 11.5.4. *Let κ be an inaccessible cardinal. For all $A \in V_\kappa$ we have $|A| < \kappa$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $A \in V_\kappa$
$\langle 1 \rangle 2.$ PICK $\alpha < \kappa$ such that $A \in V_\alpha$
$\langle 1 \rangle 3.$ $A \subseteq V_\alpha$
$\langle 1 \rangle 4.$ $|A| \leq |V_\alpha| \leq \beth_\alpha < \kappa$
□

Theorem Schema 11.5.5. *For every axiom α of ZFC, the following is a theorem:*

For any inaccessible cardinal κ, we have V_κ is a model of α.

PROOF: For every axiom except the Replacement Axioms, we have Corollary 10.0.11.1.
For an Axiom of Replacement using the property $P[x, y, z_1, \ldots, z_n]$, we reason as follows:
$\langle 1 \rangle 1.$ LET: κ be an inaccessible cardinal
PROVE:

$$\begin{gathered}\forall a_1, \ldots, a_n, B \in V_\kappa (\forall x \in B . \forall y, y' \in V_\kappa \\ (P[x, y, a_1, \ldots, a_n]^{V_\kappa} \wedge P[x, y', a_1, \ldots, a_n]^{V_\kappa} \Rightarrow y = y') \Rightarrow \\ \exists C \in V_\kappa \forall y \in V_\kappa (y \in C \Leftrightarrow \exists x \in B . P[x, y, a_1, \ldots, a_n]^{V_\kappa}))\end{gathered}$$

$\langle 1 \rangle 2.$ LET: $a_1, \ldots, a_n, B \in V_\kappa$
$\langle 1 \rangle 3.$ ASSUME: for all $x \in B$, there exists at most one $y \in V_\kappa$ such that $P[x, y, a_1, \ldots, a_n]^{V_\kappa}$.
$\langle 1 \rangle 4.$ LET: $F = \{(x, y) \in B \times V_\kappa \mid P[x, y, a_1, \ldots, a_n]^{V_\kappa}\}$
$\langle 1 \rangle 5.$ LET: $C = \operatorname{ran} F$
PROVE: $C \in V_\kappa$
$\langle 1 \rangle 6.$ LET: $S = \{\operatorname{rank} F(x) \mid x \in \operatorname{dom} F\}$
$\langle 1 \rangle 7.$ $|S| < \kappa$
PROOF: Since $|S| \leq |\operatorname{dom} F| \leq |B| < \kappa$.

$\langle 1 \rangle 8.$ $\forall \alpha \in S . \alpha < \kappa$
 Proof: Since $F(x) \in V_\kappa$ for all $x \in \operatorname{dom} F$.
$\langle 1 \rangle 9.$ $\sup S < \kappa$
 Proof: Since κ is regular.
$\langle 1 \rangle 10.$ $\operatorname{rank} C \leq \sup S + 1$
$\langle 1 \rangle 11.$ $\operatorname{rank} C < \kappa$
$\langle 1 \rangle 12.$ $C \in V_\kappa$
$\square$

Chapter 12

Group Theory

12.1 Groups

Definition 12.1.1 (Group)**.** A *group* G consists of a set G and a function $\cdot : G^2 \to G$ such that:

1. $\cdot$ is associative
2. There exists $e \in G$ such that $\forall x \in G. xe = x$ and $\forall x \in G. \exists y \in G. xy = e$.

Proposition 12.1.2. *The inverse of an element in a group is unique.*

Proof:
⟨1⟩1. Assume: b and b' are inverses of a.
⟨1⟩2. $b = b'$
Proof:

$$\begin{aligned} b &= be \\ &= bab' \\ &= eb' \\ &= b' \end{aligned}$$

□

Definition 12.1.3. We write x^{-1} for the inverse of x.

Proposition 12.1.4. *In any group, if $ab = ac$ then $b = c$.*

Proof:

$$\begin{aligned} b &= eb \\ &= a^{-1}ab \\ &= a^{-1}ac \\ &= ec \\ &= c \end{aligned}$$

□

12.2 Abelian Groups

Definition 12.2.1 (Abelian group)**.** An *Abelian group* is a group whose multiplication is commutative.

We may say we are writing an Abelian group *additively*, meaning we write $a + b$ for ab, 0 for e and $-a$ for a^{-1}. In this case we write $a - b$ for ab^{-1}.

Chapter 13

Ring Theory

13.1 Rings

Definition 13.1.1 (Commutative Ring)**.** A *commutative ring* consists of a set R and two binary operations $+, \cdot$ on R such that:

- D is an Abelian group under $+$. Let us write 0 for its identity element.
- $\cdot$ is commutative and associative, and distributes over $+$.
- $\cdot$ has an identity element 1 that is different from 0.

Proposition 13.1.2. *In any commutative ring,* $0x = 0$.

PROOF:

$$\begin{aligned} (0+0)x &= 0x \\ \therefore 0x + 0x &= 0x + 0 \\ \therefore 0x &= 0 && \text{(Proposition 12.1.4)} \square \end{aligned}$$

Proposition 13.1.3. *In any commutative ring,* $(-a)b = -(ab)$.

PROOF:

$$\begin{aligned} ab + (-a)b &= (a + (-a))b \\ &= 0b \\ &= 0 && \text{(Proposition 13.1.2)} \square \end{aligned}$$

13.2 Ordered Rings

Definition 13.2.1 (Ordered Commutative Ring)**.** An *ordered commutative ring* consists of a commutative ring R with a linear order $<$ on R such that:

- for all $x, y, z \in R$, we have $x < y$ if and only if $x + z < y + z$.

- for all $x, y, z \in R$, if $0 < z$ then we have $x < y$ if and only if $xz < yz$.

Proposition 13.2.2. *In any ordered commutative ring,* $0 < 1$.

PROOF: If $1 < 0$ then we have $0 < -1$ and so $0 < (-1)(-1) = 1$, which is a contradiction. □

Proposition 13.2.3. *The ordering on an ordered commutative ring is dense; that is, if $x < y$ then there exists z such that $x < z < y$.*

PROOF: Take $z = (x + y)/2$. □

13.3 Integral Domains

Definition 13.3.1 (Integral Domain)**.** An *integral domain* is a commutative ring such that, for all $a, b \in D$, if $ab = 0$ then $a = 0$ or $b = 0$.

Proposition 13.3.2. *In any integral domain, if $ab = ac$ and $a \neq 0$ then $b = c$.*

PROOF: We have $a(b - c) = 0$ and $a \neq 0$ so $b - c = 0$ hence $b = c$. □

Definition 13.3.3 (Ordered Integral Domain)**.** An *ordered integral domain* is an ordered commutative ring that is an integral domain.

Chapter 14

Field Theory

14.1 Fields

Definition 14.1.1 (Field)**.** A *field* F is a commutative ring such that $0 \neq 1$ and, for all $x \in F$, if $x \neq 0$ then there exists $y \in F$ such that $xy = 1$.

Proposition 14.1.2. *Every field is an integral domain.*

PROOF: If $ab = 0$ and $a \neq 0$ then $b = a^{-1}ab = 0$. $\square$

Proposition 14.1.3. *In any field F, we have $F - \{0\}$ is an Abelian group under multiplication.*

PROOF: Immediate from the definition. $\square$

Definition 14.1.4 (Field of Fractions)**.** Let D be an integral domain. The *field of fractions* of D is the quotient set $F = (D \times (D - \{0\}))/\sim$ where

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

under

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)]$$
$$[(a, b)][(c, d)] = [(ac, bd)]$$

We prove this is a field.

PROOF:
⟨1⟩1. $\sim$ is an equivalence relation on $D \times (D - \{0\})$.
  PROOF:
  ⟨2⟩1. $\sim$ is reflexive.
    PROOF: We always have $ab = ba$.
  ⟨2⟩2. $\sim$ is symmetric.
    PROOF: If $ad = bc$ then $cb = da$.

⟨2⟩3. $\sim$ is transitive.
⟨3⟩1. ASSUME: $(a,b) \sim (c,d) \sim (e,f)$
⟨3⟩2. $ad = bc$ and $cf = de$
⟨3⟩3. $adf = bde$
PROOF: $adf = bcf = bde$
⟨3⟩4. $af = be$
PROOF: Proposition 13.3.2.
□
⟨1⟩2. Addition is well-defined.
PROOF:
⟨2⟩1. If $b \neq 0$ and $d \neq 0$ then $bd \neq 0$.
PROOF: Since D is an integral domain.
⟨2⟩2. If $ab' = a'b$ and $cd' = c'd$ then $(ad + bc)b'd' = (a'd' + b'c')bd$.
PROOF:

$$\begin{aligned}(ad+bc)b'd' &= ab'dd' + bb'cd' \\ &= a'bdd' + bb'c'd \\ &= (a'd' + b'c')bd\end{aligned}$$

□
⟨1⟩3. Multiplication is well-defined.
PROOF:
⟨2⟩1. If $b \neq 0$ and $d \neq 0$ then $bd \neq 0$.
PROOF: Since D is an integral domain.
⟨2⟩2. If $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$ then $[(ac,bd)] = [(a'c',b'd')]$.
PROOF: If $ab' = a'b$ and $cd' = c'd$ then $acb'd' = a'c'bd$.
□
⟨1⟩4. Addition is commutative.
PROOF: $[(a,b)] + [(c,d)] = [(ad+bc,bd)] = [(cb+da,db)] = [(c,d)] + [(a,b)]$ □
⟨1⟩5. Addition is associative.
PROOF:

$$\begin{aligned}[(a,b)] + ([(c,d)] + [(e,f)]) &= [(a,b)] + [(cf+de,df)] \\ &= [(adf + bcf + bde, bdf)] \\ &= [(ad+bc,bd)] + [(e,f)] \\ &= ([(a,b)] + [(c,d)]) + [(e,f)]\end{aligned}$$ □

⟨1⟩6. For any $x \in F$ we have $x + [(0,1)] = x$
PROOF: $[(a,b)] + [(0,1)] = [(a \cdot 1 + b \cdot 0, b \cdot 1)] = [(a,b)]$ □
⟨1⟩7. For any $x \in F$, there exists $y \in F$ such that $x + y = [(0,1)]$.
PROOF: $[(a,b)] + [(-a,b)] = [(ab - ab, b^2)] = [(0,b^2)] = [(0,1)]$ □
⟨1⟩8. Multiplication is commutative.
PROOF: $[(a,b)][(c,d)] = [(c,d)][(a,b)] = [(ac,bd)]$. □
⟨1⟩9. Multiplication is assocative.
PROOF: $[(a,b)]([(c,d)][(e,f)]) = ([(a,b)][(c,d)])[(e,f)] = [(ace,bdf)]$. □
⟨1⟩10. For any $x \in F$ we have $x[(1,1)] = x$
PROOF: $[(a,b)][(1,1)] = [(a,b)]$ □
⟨1⟩11. For any non-zero $x \in F$, there exists $y \in F$ such that $xy = [(1,1)]$.

PROOF:
⟨2⟩1. LET: $[(a, b)] \in \mathbb{Q}$
⟨2⟩2. ASSUME: $[(a, b)] \neq [(0, 1)]$
⟨2⟩3. $a \neq 0$
⟨2⟩4. $[(a, b)][(b, a)] = [(1, 1)]$
□
□

Definition 14.1.5. For any field F, let $N(F)$ be the intersection of all the subsets $S \subseteq F$ such that $1 \in S$ and $\forall x \in S.x + 1 \in S$.

Definition 14.1.6 (Characteristic Zero)**.** A field F has *characteristic 0* iff $0 \notin N(F)$.

Proposition 14.1.7. *In a field F with characteristic 0, the function $n : \mathbb{N} \to N(F)$ defined by*

$$n(0) = 1$$
$$n(x + 1) = n(x) + 1$$

is a bijection.

PROOF:
⟨1⟩1. n is injective.
⟨2⟩1. ASSUME: for a contradiction $n(i) = n(j)$ with $i \neq j$
⟨2⟩2. ASSUME: w.l.o.g. $i < j$
⟨2⟩3. $n(j - i) = 0$
⟨2⟩4. Q.E.D.
PROOF: This contradicts the fact that F has characteristic 0.
⟨1⟩2. n is surjective.
PROOF: Since $\operatorname{ran} n$ is a subset of F that includes 1 and is closed under $+1$.
□

Definition 14.1.8. In any field F, let

$$I(F) = N(F) \cup \{0\} \cup \{-x \mid x \in N(F)\}$$

Definition 14.1.9. In any field F, let

$$Q(F) = \{x/y \mid x, y \in I(F), y \neq 0\}$$

Proposition 14.1.10. *$Q(F)$ is the smallest subfield of F.*

PROOF: $Q(F)$ is closed under $+$ and $\cdot$, and any subset of F closed under $+$ and $\cdot$ that contains 0 and 1 must include $Q(F)$. □

Theorem 14.1.11. *Let F and G be fields of characteristic 0. Then there exists a unique field isomorphism between $Q(F)$ and $Q(G)$.*

PROOF:

$\langle 1\rangle$1. LET: $\phi : N(F) \to N(G)$ be the unique function such that $\phi(1) = 1$ and $\forall x \in N(F).\phi(x+1) = \phi(x) + 1$.
$\langle 1\rangle$2. ϕ is a bijection.
PROOF: Similar to Proposition 14.1.7.
$\langle 1\rangle$3. $\forall x, y \in N(F).\phi(x+y) = \phi(x) + \phi(y)$
PROOF: Induction on y.
$\langle 1\rangle$4. $\forall x, y \in N(F).\phi(xy) = \phi(x)\phi(y)$
PROOF: Induction on y.
$\langle 1\rangle$5. Extend ϕ to a bijection $I(F) \cong I(G)$ such that $\forall x, y \in I(F).\phi(x+y) = \phi(x) + \phi(y)$ and $\forall x, y \in I(F).\phi(xy) = \phi(x)\phi(y)$
$\langle 2\rangle$1. Define $\phi(0) = 0$ and $\phi(-x) = -\phi(x)$ for $x \in N(F)$
$\langle 3\rangle$1. $0 \notin N(F)$
$\langle 3\rangle$2. For all $x \in N(F)$ we have $-x \notin N(F)$
PROOF: Then we would have $x + -x = 0 \in N(F)$.
$\langle 3\rangle$3. For all $x \in N(F)$ we have $-x \neq 0$
$\langle 2\rangle$2. For all $x, y \in I(F)$ we have $\phi(x+y) = \phi(x) + \phi(y)$
PROOF: Case analysis on x and y.
$\langle 2\rangle$3. For all $x, y \in I(F)$ we have $\phi(xy) = \phi(x)\phi(y)$
PROOF: Case analysis on x and y.
$\langle 1\rangle$6. Extend ϕ to a bijection $Q(F) \cong Q(G)$ such that $\forall x, y \in Q(F).\phi(x+y) = \phi(x) + \phi(y)$ and $\forall x, y \in Q(F).\phi(xy) = \phi(x)\phi(y)$
$\langle 2\rangle$1. Define $\phi(x/y) = \phi(x)/\phi(y)$
$\langle 1\rangle$7. ϕ is unique.
$\langle 2\rangle$1. LET: θ satisfy the theorem.
$\langle 2\rangle$2. For all $x \in N(F)$ we have $\theta(x) = \phi(x)$
$\langle 2\rangle$3. For all $x \in I(F)$ we have $\theta(x) = \phi(x)$
$\langle 2\rangle$4. For all $x \in Q(F)$ we have $\theta(x) = \phi(x)$
$\square$

14.2 Ordered Fields

Definition 14.2.1 (Ordered Field)**.** An *ordered field* is an ordered commutative ring that is a field.

Proposition 14.2.2. *Every ordered field F has characteristic 0.*

PROOF: We have $0 < n$ for all $n \in N(F)$. $\square$

Proposition 14.2.3. *Let F be a field of characteristic 0. Then there exists a unique relation $<$ on $Q(F)$ that makes $Q(F)$ into an ordered field.*

PROOF: Easy. $\square$

Corollary 14.2.3.1. *Let F and G be ordered fields. Let ϕ be the unique field isomorphism between $Q(F)$ and $Q(G)$. Then ϕ is an ordered field isomorphism.*

Definition 14.2.4 (Archimedean)**.** An ordered field F is *Archimedean* iff

$$\forall x \in F.\exists n \in N(F).n > x \enspace .$$

Proposition 14.2.5. *Let F be an Archimedean ordered field. Let $x, y \in F$ with $x > 0$. Then there exists $n \in N(F)$ such that $nx > y$.*

PROOF: Pick $n > y/x$. □

Proposition 14.2.6. *Let F be an Archimedean ordered field. For all $x, y \in F$, if $x < y$, then there exists $r \in Q(F)$ such that $x < r < y$.*

PROOF:
⟨1⟩1. CASE: $x > 0$
  ⟨2⟩1. PICK $n \in N(F)$ such that $n(y - x) > 1$
    PROOF: Proposition 14.2.5.
  ⟨2⟩2. $ny > 1 + nx$
  ⟨2⟩3. LET: m be the least element of $N(F)$ such that $m > nx$.
  ⟨2⟩4. $m - 1 \leq nx$
  ⟨2⟩5. $nx < m < ny$
  ⟨2⟩6. $x < m/n < y$
⟨1⟩2. CASE: $x \leq 0$
  ⟨2⟩1. PICK $k \in N(F)$ such that $k > -x$
  ⟨2⟩2. $0 < x + k < y + k$
  ⟨2⟩3. PICK $r \in Q(F)$ such that $x + k < r < y + k$
    PROOF: ⟨1⟩1
  ⟨2⟩4. $x < r - k < y$

Definition 14.2.7 (Complete)**.** An ordered field F is *complete* iff every nonempty subset of F bounded above has a least upper bound.

Proposition 14.2.8. *Every complete ordered field is Archimedean.*

PROOF:
⟨1⟩1. LET: F be a complete ordered field.
⟨1⟩2. LET: $x \in F$
⟨1⟩3. ASSUME: for a contradiction there is no member of $N(F)$ greater than x.
⟨1⟩4. x is an upper bound for $N(F)$.
⟨1⟩5. LET: $y = \sup N(F)$
⟨1⟩6. PICK $n \in N(F)$ such that $y - 1 < n$
⟨1⟩7. $y < n + 1$
⟨1⟩8. Q.E.D.
  PROOF: This is a contradiction.
□

Proposition 14.2.9. *Let F be a complete ordered field and $a \in F$ be nonnegative. Then there exists $b \in F$ such that $b^2 = a$.*

PROOF:
⟨1⟩1. LET: $B = \{x \in F \mid 0 \leq x \leq 1 + a\}$
⟨1⟩2. LET: $\phi : B \to B$ be the function

$$\phi(x) = x + \frac{1}{2(1+a)}(a - x^2) \ .$$

⟨1⟩3. ϕ is strictly monotone.
  ⟨2⟩1. LET: $0 \leq x < y \leq 1 + a$
  ⟨2⟩2. $1 - \frac{x+y}{2(1+a)} > 0$
  ⟨2⟩3. $\phi(y) - \phi(x) = (y - x)(1 - \frac{x+y}{2(1+a)}) > 0$
  ⟨2⟩4. $\phi(x) < \phi(y)$
⟨1⟩4. PICK $b \in B$ such that $\phi(b) = b$.
  PROOF: Knaster Fixed-Point Theorem.
⟨1⟩5. $b^2 = a$
□

Theorem 14.2.10 (Uniqueness of the Complete Ordered Field)**.** *If F and G are complete ordered fields, then there exists a unique bijection $\phi : F \cong G$ such that, for all $x, y \in F$,*

$$\phi(x + y) = \phi(x) + \phi(y)$$
$$\phi(xy) = \phi(x)\phi(y)$$

This bijection also satisfies: for all $x, y \in F$,

$$x < y \Leftrightarrow \phi(x) < \phi(y) \quad .$$

PROOF:
⟨1⟩1. PICK a bijection $\phi : Q(F) \cong Q(G)$ such that, for all $x, y \in Q(F)$,
$$\phi(x + y) = \phi(x) + \phi(y)$$
$$\phi(xy) = \phi(x)\phi(y)$$
$$x < y \Leftrightarrow \phi(x) < \phi(y)$$
  PROOF: Corollary 14.2.3.1.
⟨1⟩2. $Q(F)$ intersects every interval in F.
  PROOF: Proposition 14.2.6.
⟨1⟩3. $Q(G)$ intersects every interval in G.
  PROOF: Proposition 14.2.6.
⟨1⟩4. PICK an order isomorphism $\psi : F \cong G$ that extends ϕ.
  PROOF: Theorem 5.1.21.
⟨1⟩5. $\forall x, y \in F.\psi(x + y) = \psi(x) + \psi(y)$
  ⟨2⟩1. LET: $x, y \in F$
  ⟨2⟩2. $\psi(x) + \psi(y) \not< \psi(x + y)$
    ⟨3⟩1. ASSUME: for a contradiction $\psi(x) + \psi(y) < \psi(x + y)$
    ⟨3⟩2. PICK $r' \in Q(G)$ such that $\psi(x) < r' < \psi(x + y) - \psi(y)$
    ⟨3⟩3. PICK $s' \in Q(G)$ such that $\psi(y) < s' < \psi(x + y) - r'$
    ⟨3⟩4. $r' + s' < \psi(x + y)$
    ⟨3⟩5. PICK $r, s \in Q(F)$ such that $\phi(r) = r'$ and $\phi(s) = s'$
    ⟨3⟩6. $\phi(r + s) = r' + s'$
    ⟨3⟩7. $\psi(x) < \psi(r)$
    ⟨3⟩8. $\psi(y) < \psi(s)$
    ⟨3⟩9. $\psi(x + y) > \psi(r + s)$
    ⟨3⟩10. $x < r$

⟨3⟩11. $y < s$
⟨3⟩12. $x + y > r + s$
⟨3⟩13. Q.E.D.
PROOF: This is a contradiction.
⟨2⟩3. $\psi(x + y) \not< \psi(x) + \psi(y)$
PROOF: Similar.
⟨1⟩6. $\forall x, y \in F.\psi(xy) = \psi(x)\psi(y)$
⟨2⟩1. LET: $x, y \in F$
⟨2⟩2. CASE: x and y are positive.
⟨3⟩1. $\psi(x)\psi(y) \not< \psi(xy)$
⟨4⟩1. ASSUME: for a contradiction $\psi(x)\psi(y) < \psi(xy)$
⟨4⟩2. PICK $r' \in Q(G)$ such that $\psi(x) < r' < \psi(xy)/\psi(y)$
⟨4⟩3. PICK $s' \in Q(G)$ such that $\psi(y) < s' < \psi(xy)/r'$
⟨4⟩4. $r's' < \psi(xy)$
⟨4⟩5. PICK $r, s \in Q(F)$ such that $\phi(r) = r'$ and $\phi(s) = s'$
⟨4⟩6. $\phi(rs) = r's'$
⟨4⟩7. $x < r$, $y < s$ and $rs < xy$
⟨4⟩8. Q.E.D.
PROOF: This is a contradiction.
⟨3⟩2. $\psi(xy) \not< \psi(x)\psi(y)$
PROOF: Similar.
⟨2⟩3. CASE: x and y are not both positive.
PROOF: Follows from ⟨2⟩2 since $\psi(-x) = -\psi(x)$ by ⟨1⟩5.
⟨1⟩7. For any field isomorphism $\theta : F \cong G$, we have $\theta = \psi$.
⟨2⟩1. $\theta \upharpoonright Q(F) = \phi$
PROOF: Theorem 14.1.11.
⟨2⟩2. θ is strictly monotone.
⟨3⟩1. LET: $x, y \in F$ with $x < y$
⟨3⟩2. $y - x > 0$
⟨3⟩3. PICK $z \in F$ such that $z^2 = y - x$
⟨3⟩4. $\theta(z)^2 = \theta(y) - \theta(x)$
⟨3⟩5. $\theta(y) - \theta(x) > 0$
⟨3⟩6. $\theta(x) < \theta(y)$
⟨2⟩3. $\theta = \psi$
PROOF: By the uniqueness of ψ.
$\square$

Chapter 15

Number Systems

15.1 The Integers

Definition 15.1.1. The set of *integers* $\mathbb{Z}$ is the quotient set $\mathbb{N}^2/\sim$, where $(m,n) \sim (p,q)$ iff $m+q = n+p$.

We prove $\sim$ is an equivalence relation on $\mathbb{N}^2$.

PROOF:
⟨1⟩1. $\sim$ is reflexive.
  PROOF: For all $m, n \in \mathbb{N}$ we have $m+n = n+m$.
⟨1⟩2. $\sim$ is symmetric.
  PROOF: If $m+q = n+p$ then $p+n = q+m$.
⟨1⟩3. $\sim$ is transitive.
  ⟨2⟩1. ASSUME: $(m,n) \sim (p,q) \sim (r,s)$
  ⟨2⟩2. $m+q = n+p$ and $p+s = q+r$
  ⟨2⟩3. $m+q+s = n+q+r$
  ⟨2⟩4. $m+s = n+r$
    PROOF: By cancellation.
□

Definition 15.1.2 (Addition)**.** Define *addition* $+$ on $\mathbb{Z}$ by $[(m,n)]+[(p,q)] = [(m+p, n+q)]$.

We prove this is well-defined.

PROOF: If $m+n' = n+m'$ and $p+q' = q+p'$ then $m+p+n'+q' = n+q+m'+p'$.
□

Proposition 15.1.3. *Addition on $\mathbb{Z}$ is commutative.*

PROOF: $[(m,n)]+[(p,q)] = [(m+p, n+q)] = [(p+m, q+n)] = [(p,q)]+[(m,n)]$.
□

Proposition 15.1.4. *Addition on $\mathbb{Z}$ is associative.*

PROOF: $[(m,n)] + ([(p,q)] + [(r,s)]) = [(m+p+r, n+q+s)] = ([(m,n)] + [(p,q)]) + [(r,s)]$. ∎

Proposition 15.1.5. *Given natural numbers m and n, we have $[(m,0)] = [(n,0)]$ iff $m = n$.*

PROOF: Immediate from definitions. ∎

Definition 15.1.6. We identify any natural number n with the integer $[(n,0)]$.

Proposition 15.1.7. *Addition on integers agrees with addition on natural numbers.*

PROOF: Since $[(m,0)] + [(n,0)] = [(m+n,0)]$. ∎

Proposition 15.1.8. *For all $a \in \mathbb{Z}$ we have $a + 0 = a$.*

PROOF: $[(m,n)] + [(0,0)] = [(m+0, n+0)] = [(m,n)]$. ∎

Proposition 15.1.9. *For all $a \in \mathbb{Z}$, there exists $b \in \mathbb{Z}$ such that $a + b = 0$.*

PROOF: $[(m,n)] + [(n,m)] = [(m+n, m+n)] = [(0,0)]$ ∎

Proposition 15.1.10. *The integers form an Abelian group under addition.*

PROOF: Proposition 15.1.3, 15.1.4, 15.1.8, 15.1.9. ∎

Definition 15.1.11. Define multiplication $\cdot$ on $\mathbb{Z}$ by: $[(m,n)][(p,q)] = [(mp+nq, mq+np)]$.

We prove this is well defined.

PROOF:
$\langle 1 \rangle 1$. ASSUME: $m + n' = n + m'$ and $p + q' = q + p'$
PROVE: $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$
$\langle 1 \rangle 2$. $mp + n'p = np + m'p$
$\langle 1 \rangle 3$. $nq + m'q = mq + n'q$
$\langle 1 \rangle 4$. $m'p + m'q' = m'q + m'p'$
$\langle 1 \rangle 5$. $n'q + n'p' = n'p + n'q'$
$\langle 1 \rangle 6$. $mp + n'p + nq + m'q + m'p + m'q' + n'q + n'p' = np + m'p + mq + n'q + m'q + m'p' + n'p + n'q'$
$\langle 1 \rangle 7$. $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$
PROOF: By cancellation.
∎

Proposition 15.1.12. *Multiplication on integers agrees with multiplication on natural numbers.*

PROOF: Since $[(m,0)][(n,0)] = [(mn+0, m0+n0)] = [(mn,0)]$. ∎

Proposition 15.1.13. *Multiplication on $\mathbb{Z}$ is commutative.*

PROOF: $[(m,n)][(p,q)] = [(mp+nq, mq+np)] = [(pm+qn, pn+qm)] = [(p,q)][(m,n)]$. ∎

Proposition 15.1.14. *Multiplication on $\mathbb{Z}$ is associative.*

PROOF:

$$\begin{aligned}[(m,n)]([(p,q)][(r,s)]) &= [(m,n)][(pr+qs,ps+qr)] \\ &= [(mpr+mqs+nps+nqr,mps+mqr+npr+nqs)] \\ &= [(mp+nq,mq+np)][(r,s)] \\ &= ([(m,n)][(p,q)])[(r,s)]\end{aligned}$$ ∎

Proposition 15.1.15. *Multiplication distributes over addition.*

PROOF:

$$\begin{aligned}[(m,n)]([(p,q)]+[(r,s)]) &= [(m,n)][(p+r,q+s)] \\ &= [(mp+mr+nq+ns,np+nr+mq+ms)] \\ [(m,n)][(p,q)]+[(m,n)][(r,s)] &= [(mp+nq,mq+np)]+[(mr+ns,ms+nr)] \\ &= [(mp+nq+mr+ns,mq+np+ms+nr)]\end{aligned}$$ ∎

Proposition 15.1.16. *For any integer a we have $a1 = a$.*

PROOF: Since $[(m,n)][(1,0)] = [(m1+n0,m0+n1)] = [(m,n)]$. ∎

Proposition 15.1.17. *For any integeres a and b, if $ab = 0$ then $a = 0$ or $b = 0$.*

PROOF:
⟨1⟩1. ASSUME: $[(m,n)][(p,q)] = [(0,0)]$
⟨1⟩2. $mp + nq = mq + np$
⟨1⟩3. ASSUME: $[(m,n)] \neq [(0,0)]$
⟨1⟩4. $m \neq n$
PROVE: $p = q$
⟨1⟩5. CASE: $m < n$
⟨2⟩1. $p \not< q$
PROOF: If $p < q$ then $mq + np < mp + nq$ by Proposition 8.4.6.
⟨2⟩2. $q \not< p$
PROOF: If $q < p$ then $mp + nq < mq + np$ by Proposition 8.4.6.
⟨2⟩3. $p = q$
PROOF: By trichotomy.
⟨1⟩6. CASE: $n < m$
PROOF: Similar.
∎

Proposition 15.1.18. *The integers $\mathbb{Z}$ form an integral domain.*

PROOF: Propositions 15.1.13, 15.1.14, 15.1.15, 15.1.16, 15.1.17, 15.1.10. ∎

Definition 15.1.19. Define $<$ on $\mathbb{Z}$ by $[(m,n)] < [(p,q)]$ if and only if $m+q < n+p$.

We prove this is well-defined.

PROOF:

⟨1⟩1. ASSUME: $m + n' = n + m'$ and $p + q' = q + p'$.
PROVE: $m + q < n + p$ if and only if $m' + q' < n' + p'$
⟨1⟩2. $m + q < n + p$ if and only if $m' + q' < n' + p'$
PROOF:

$$\begin{aligned} m + q < n + p &\Leftrightarrow m + n' + q < n + n' + p && \text{(Corollary 6.5.7.1)} \\ &\Leftrightarrow m' + n + q < n + n' + p \\ &\Leftrightarrow m' + q < n' + p && \text{(Corollary 6.5.7.1)} \\ &\Leftrightarrow m' + q + p' < n' + p + p' && \text{(Corollary 6.5.7.1)} \\ &\Leftrightarrow m' + q' + p < n' + p + p' \\ &\Leftrightarrow m' + q' < n' + p' && \text{(Corollary 6.5.7.1)}\square \end{aligned}$$

Proposition 15.1.20. *The ordering on the integers agrees with the ordering on the natural numbers.*

PROOF: We have $[(m, 0)] < [(n, 0)]$ iff $m < n$. $\square$

Proposition 15.1.21. $<$ *is a linear order on* $\mathbb{Z}$.

PROOF:
⟨1⟩1. $<$ is irreflexive.
PROOF: We never have $m + n < m + n$.
⟨1⟩2. $<$ is transitive.
⟨2⟩1. ASSUME: $[(m, n)] < [(p, q)] < [(r, s)]$
⟨2⟩2. $m + q < n + p$ and $p + s < q + r$
⟨2⟩3. $m + q + s < n + q + r$
PROOF: $m + q + s < n + p + s < n + q + r$
⟨2⟩4. $m + s < n + r$
PROOF: Corollary 6.5.7.1.
⟨1⟩3. $<$ is total.
PROOF: Given natural numbers m, n, p and q, either $m + q < n + p$, or $m + q = n + p$, or $n + p < m + q$.
$\square$

Definition 15.1.22 (Positive)**.** An integer a is *positive* iff $a > 0$.

Theorem 15.1.23. *For any integers a, b and c, we have $a < b$ if and only if $a + c < b + c$.*

PROOF:
⟨1⟩1. If $a < b$ then $a + c < b + c$.
⟨2⟩1. LET: $a = [(m, n)]$, $b = [(p, q)]$ and $c = [(r, s)]$.
⟨2⟩2. ASSUME: $a < b$
⟨2⟩3. $m + q < n + p$
⟨2⟩4. $m + r + q + s < n + r + p + s$
⟨2⟩5. $[(m + r, n + s)] < [(p + r, q + s)]$
⟨2⟩6. $a + c < b + c$

$\langle 1\rangle 2$. If $a + c < b + c$ then $a < b$.
  PROOF: From $\langle 1\rangle 1$ and Proposition 5.2.6.
□

Proposition 15.1.24. *Let a, b and c be integers. If $0 < c$, then $a < b$ if and only if $ac < bc$.*

PROOF:
$\langle 1\rangle 1$. LET: $c = [(r, s)]$
$\langle 1\rangle 2$. ASSUME: $0 < c$
$\langle 1\rangle 3$. $s < r$
$\langle 1\rangle 4$. For all integers a and b, if $a < b$ then $ac < bc$
  $\langle 2\rangle 1$. LET: $a = [(m, n)]$, $b = [(p, q)]$.
  $\langle 2\rangle 2$. ASSUME: $a < b$
  $\langle 2\rangle 3$. $m + q < n + p$
  $\langle 2\rangle 4$. $(m + q)r + (p + n)s < (m + q)s + (p + n)r$
    PROOF: Proposition 8.4.6, $\langle 1\rangle 3$, $\langle 2\rangle 3$.
  $\langle 2\rangle 5$. $mr + ns + ps + qr < ms + nr + pr + qs$
  $\langle 2\rangle 6$. $[(mr + ns, ms + nr)] < [(pr + qs, ps + qr)]$
  $\langle 2\rangle 7$. $ac < bc$
$\langle 1\rangle 5$. For all integers a and b, if $ac < bc$ then $a < b$
  PROOF: From $\langle 1\rangle 4$ and Proposition 5.2.6.
□

Proposition 15.1.25. *Let a be a positive integer. For any integer b, there exists $k \in \mathbb{N}$ such that $b < ak$.*

PROOF:
$\langle 1\rangle 1$. CASE: $b \leq 0$
  PROOF: Take $k = 1$.
$\langle 1\rangle 2$. CASE: $b > 0$
  PROOF: Take $k = b + 1$.
□

15.2 The Rationals

Definition 15.2.1 (Rational Numbers)**.** The set $\mathbb{Q}$ of *rational numbers* is the field of fractions over the integers.

Proposition 15.2.2. *For any integers a and b, we have $[(a, 1)] = [(b, 1)]$ iff $a = b$.*

PROOF: Immediate from definitions. □

Henceforth we identify any integer a with the rational number $[(a, 1)]$.

Proposition 15.2.3. *Addition on the rationals agrees with addition on the integers.*

PROOF: $[(a,1)] + [(b,1)] = [(a \cdot 1 + b \cdot 1, 1 \cdot 1)] = [(a+b,1)]$. □

Proposition 15.2.4. *Multiplication on the rationals agrees with multiplication on the integers.*

PROOF: $[(a,1)][(b,1)] = [(ab,1)]$ □

Definition 15.2.5. Define the ordering $<$ on the rationals by: if b and d are positive, then $[(a,b)] < [(c,d)]$ iff $ad < bc$.

We prove this is well-defined.

PROOF:
⟨1⟩1. For any rational q, there exist integers a, b with b positive such that $q = [(a,b)]$.
PROOF: Since $[(a,b)] = [(-a,-b)]$, and if $b \neq 0$ then one of b and $-b$ is positive.
⟨1⟩2. If b, b', d and d' are positive, $[(a,b)] = [(a',b')]$, and $[(c,d)] = [(c',d')]$, then $ad < bc$ iff $a'd' < b'c'$.
PROOF:
⟨2⟩1. If $ad < bc$ then $a'd' < b'c'$.
⟨3⟩1. ASSUME: $ad < bc$
⟨3⟩2. $ab'd < bb'c$
⟨3⟩3. $a'bd < bb'c$
⟨3⟩4. $a'd < b'c$
⟨3⟩5. $a'dd' < b'cd'$
⟨3⟩6. $a'dd' < b'c'd$
⟨3⟩7. $a'd' < b'c'$
⟨2⟩2. If $a'd' < b'c'$ then $ad < bc$.
PROOF: Similar.
□

Proposition 15.2.6. *The ordering on the rationals agrees with the ordering on the integers.*

PROOF: We have $[(a,1)] < [(b,1)]$ if and only if $a < b$. □

Proposition 15.2.7. *The relation $<$ is a linear ordering on $\mathbb{Q}$.*

PROOF:
⟨1⟩1. $<$ is irreflexive.
PROOF: We never have $ab < ab$.
⟨1⟩2. $<$ is transitive.
⟨2⟩1. ASSUME: $[(a,b)] < [(c,d)] < [(e,f)]$ where b, d and f are positive.
⟨2⟩2. $ad < bc$ and $cf < de$
⟨2⟩3. $adf < bde$
PROOF: $adf < bcf < bde$
⟨2⟩4. $af < be$
⟨1⟩3. $<$ is total.
PROOF: For any integers a, b, c, d, we have $ad < bc$ or $ad = bc$ or $bc < ad$.

∎

Proposition 15.2.8. *For any rationals r, s and t, we have $r < s$ if and only if $r + t < s + t$.*

PROOF:
⟨1⟩1. LET: a, b, c, d, e, f be integers with b, d and f positive.
⟨1⟩2. $[(a, b)] + [(e, f)] < [(c, d)] + [(e, f)]$ if and only if $[(a, b)] < [(c, d)]$.
PROOF:

$$\begin{aligned}
[(a,b)] + [(e,f)] < [(c,d)] + [(e,f)] &\Leftrightarrow [(af + be, bf)] < [(cf + de, df)] \\
&\Leftrightarrow (af + be)df < (cf + de)bf \\
&\Leftrightarrow afdf + bedf < cfbf + debf \\
&\Leftrightarrow afdf < cfbf \\
&\Leftrightarrow ad < bc \\
&\Leftrightarrow [(a,b)] < [(c,d)]
\end{aligned}$$

∎

Corollary 15.2.8.1. *For any rational r, we have $r < 0$ if and only if $0 < -r$.*

Definition 15.2.9 (Absolute Value)**.** For any rational r, the *absolute value* of r is defined by

$$|r| := \begin{cases} -r & \text{if } 0 < -r \\ r & \text{otherwise} \end{cases}$$

Proposition 15.2.10. *For any rationals r, s and t, if t is positive then $r < s$ iff $rt < st$.*

PROOF:
⟨1⟩1. LET: $r = [(a, b)]$, $s = [(c, d)]$ and $t = [(e, f)]$ where b, d and f are positive.
⟨1⟩2. ASSUME: $0 < t$
⟨1⟩3. $e > 0$
⟨1⟩4. $rt < st$ iff $r < s$
PROOF:

$$\begin{aligned}
rt < st &\Leftrightarrow [(ae, bf)] < [(ce, df)] \\
&\Leftrightarrow aedf < cebf \\
&\Leftrightarrow ad < bc \\
&\Leftrightarrow r < s
\end{aligned}$$

∎

Corollary 15.2.10.1. *The rationals form an ordered field.*

Proposition 15.2.11. *Let p be a positive rational. For any rational number r, there exists $k \in \mathbb{N}$ such that $r < pk$.*

PROOF:
⟨1⟩1. LET: $p = a/b$ and $r = c/d$ where a, b and d are positive.

$\langle 1\rangle 2$. PICK $k \in \mathbb{N}$ such that $bc < adk$
PROOF: Proposition 15.1.25.
$\langle 1\rangle 3$. $r < pk$
□

Proposition 15.2.12. $\mathbb{Q} \approx \mathbb{N}$

PROOF: Arrange the rationals in order $0/1$, $1/1$, $1/2$, $0/2$, $-1/2$, $-1/1$, $-2/1$, $-2/2$, $-2/3$, $-1/3$, $0/3$, $1/3$, $2/3$, etc. then remove all duplicates. □

15.3 The Real Numbers

Definition 15.3.1 (Cauchy Sequence)**.** A *Cauchy sequence* is a sequence (q_n) of rationals such that, for every positive rational ϵ, there exists $k \in \mathbb{N}$ such that $\forall m, n > k. |q_m - q_n| < \epsilon$.

Definition 15.3.2 (Dedekind Cut)**.** A *Dedekind cut* is a set $x \subseteq \mathbb{Q}$ such that:

1. $\emptyset \neq x \neq \mathbb{Q}$
2. x is closed downwards.
3. x has no greatest member.

The set $\mathbb{R}$ of *real numbers* is the set of Dedekind cuts.

Proposition 15.3.3. *For any rational q, we have $\{r \in \mathbb{Q} \mid r < q\} \in \mathbb{R}$.*

PROOF:
$\langle 1\rangle 1$. LET: $q \in \mathbb{Q}$
$\langle 1\rangle 2$. LET: $q\downarrow = \{r \mid r < q\}$
$\langle 1\rangle 3$. $q\downarrow \neq \emptyset$
PROOF: We have $q - 1 \in q\downarrow$.
$\langle 1\rangle 4$. $q\downarrow \neq \mathbb{Q}$
PROOF: Since $q \notin q\downarrow$.
$\langle 1\rangle 5$. $q\downarrow$ is closed downwards.
PROOF: Trivial.
$\langle 1\rangle 6$. $q\downarrow$ has no greatest element.
PROOF: For all $r \in q\downarrow$ we have $r < (q + r)/2 \in q\downarrow$.
□

Proposition 15.3.4. *For rationals q and r, we have $q = r$ if and only if $\{s \in \mathbb{Q} \mid s < q\} = \{s \in \mathbb{Q} \mid s < r\}$.*

PROOF:
$\langle 1\rangle 1$. LET: $q\downarrow = \{s \in \mathbb{Q} \mid s < q\}$
$\langle 1\rangle 2$. LET: $r\downarrow = \{s \in \mathbb{Q} \mid s < r\}$
$\langle 1\rangle 3$. If $q = r$ then $q\downarrow = r\downarrow$
PROOF: Trivial.

$\langle 1\rangle 4.$ If $q < r$ then $q\downarrow \neq r\downarrow$
  PROOF: We have $q \in r\downarrow$ and $q \notin q\downarrow$.
$\langle 1\rangle 5.$ If $r < q$ then $q\downarrow \neq r\downarrow$
  PROOF: We have $r \in q\downarrow$ and $q \notin q\downarrow$.
$\square$

Henceforth we identify a rational q with the real number $\{r \in \mathbb{Q} \mid r < q\}$.

Definition 15.3.5. Define the ordering $<$ on $\mathbb{R}$ by: $x < y$ iff $x \subsetneq y$.

Proposition 15.3.6. *The ordering on the reals agrees with the ordering on the rationals.*

PROOF:
$\langle 1\rangle 1.$ LET: $q, r \in \mathbb{Q}$
$\langle 1\rangle 2.$ LET: $q\downarrow = \{s \in \mathbb{Q} \mid s < q\}$.
$\langle 1\rangle 3.$ LET: $r\downarrow = \{s \in \mathbb{Q} \mid s < r\}$.
    PROVE: $q < r$ iff $q\downarrow \subsetneq r\downarrow$
$\langle 1\rangle 4.$ If $q < r$ then $q\downarrow \subsetneq r\downarrow$
  $\langle 2\rangle 1.$ ASSUME: $q < r$
  $\langle 2\rangle 2.$ $q\downarrow \subseteq r\downarrow$
    PROOF: If $s < q$ then $s < r$.
  $\langle 2\rangle 3.$ $q\downarrow \neq r\downarrow$
    PROOF: Proposition 15.3.4.
$\langle 1\rangle 5.$ If $q\downarrow \subsetneq r\downarrow$ then $q < r$
  $\langle 2\rangle 1.$ ASSUME: $q\downarrow \subsetneq r\downarrow$
  $\langle 2\rangle 2.$ PICK $s \in r\downarrow$ such that $s \notin q\downarrow$
  $\langle 2\rangle 3.$ $q \leq s < r$
$\square$

Proposition 15.3.7. *The ordering $<$ is a linear ordering on $\mathbb{R}$.*

PROOF:
$\langle 1\rangle 1.$ $<$ is irreflexive.
  PROOF: No set is a proper subset of itself.
$\langle 1\rangle 2.$ $<$ is transitive.
  PROOF: Since the relationship $\subsetneq$ is transitive on the class of all sets.
$\langle 1\rangle 3.$ $<$ is total.
  $\langle 2\rangle 1.$ LET: x, y be Dedekind cuts.
  $\langle 2\rangle 2.$ ASSUME: $x \nsubseteq y$
    PROVE: $y \subsetneq x$
  $\langle 2\rangle 3.$ PICK $q \in x$ such that $q \notin y$
  $\langle 2\rangle 4.$ LET: $r \in y$
    PROVE: $r \in x$
  $\langle 2\rangle 5.$ $q \nleq r$
    PROOF: Since y is closed downwards.
  $\langle 2\rangle 6.$ $r < q$
  $\langle 2\rangle 7.$ $r \in x$

PROOF: Since x is closed downwards.
□

Proposition 15.3.8. *Any bounded nonempty subset of $\mathbb{R}$ has a least upper bound.*

PROOF:
⟨1⟩1. LET: A be a bounded nonempty subset of $\mathbb{R}$.
⟨1⟩2. $\bigcup A$ is a Dedekind cut.
⟨2⟩1. $\bigcup A \neq \emptyset$
⟨3⟩1. PICK$x \in A$
⟨3⟩2. PICK $q \in x$
⟨3⟩3. $q \in \bigcup A$
⟨2⟩2. $\bigcup A \neq \mathbb{Q}$
⟨3⟩1. PICK an upper bound u for A
⟨3⟩2. PICK $q \notin u$
PROVE: $q \notin \bigcup A$
⟨3⟩3. ASSUME: for a contradiction $q \in \bigcup A$
⟨3⟩4. PICK $x \in A$ such that $q \in x$
⟨3⟩5. $x \leq u$
⟨3⟩6. $q \in u$
⟨3⟩7. Q.E.D.
PROOF: This is a contradiction.
⟨2⟩3. $\bigcup A$ is closed downwards.
⟨3⟩1. LET: $q \in \bigcup A$ and $r < q$
⟨3⟩2. PICK $x \in A$ such that $q \in x$
⟨3⟩3. $r \in x$
⟨3⟩4. $r \in \bigcup A$
⟨2⟩4. $\bigcup A$ has no greatest element.
⟨3⟩1. LET: $q \in \bigcup A$
⟨3⟩2. PICK $x \in A$ such that $q \in x$
⟨3⟩3. PICK $r \in x$ such that $q < r$
⟨3⟩4. $r \in \bigcup A$
⟨1⟩3. $\bigcup A$ is an upper bound for A.
PROOF: For all $x \in A$ we have $x \subseteq \bigcup A$.
⟨1⟩4. For any upper bound u for $\bigcup A$ we have $\bigcup A \leq u$.
PROOF: If $\forall x \in A.x \subseteq u$ we have $\bigcup A \subseteq u$.
□

Definition 15.3.9 (Addition)**.** Define *addition* $+$ on the reals by

$$x + y := \{q + r \mid q \in x, r \in y\} \ .$$

We prove this is well-defined.

PROOF:
⟨1⟩1. LET: $x, y \in \mathbb{R}$
PROVE: $X + y$ is a Dedekind cut.

$\langle 1 \rangle 2.$ $x + y \neq \emptyset$
PROOF: Pick $q \in x$ and $r \in y$; then $q + r \in x + y$.
$\langle 1 \rangle 3.$ $x + y \neq \mathbb{Q}$
$\langle 2 \rangle 1.$ PICK $q \notin x$ and $r \notin y$
PROVE: $q + r \notin x + y$
$\langle 2 \rangle 2.$ ASSUME: for a contradiction $q + r \in x + y$
$\langle 2 \rangle 3.$ PICK $q' \in x$ and $r' \in y$ such that $q + r = q' + r'$
$\langle 2 \rangle 4.$ $q' < q$ and $r' < r$
$\langle 2 \rangle 5.$ $q' + r' < q + r$
$\langle 2 \rangle 6.$ Q.E.D.
PROOF: This is a contradiction.
$\langle 1 \rangle 4.$ $x + y$ is closed downwards.
$\langle 2 \rangle 1.$ LET: $q \in x$ and $r \in y$
$\langle 2 \rangle 2.$ LET: $s < q + r$
PROVE: $s \in x + y$
$\langle 2 \rangle 3.$ $s - r < q$
$\langle 2 \rangle 4.$ $s - r \in x$
$\langle 2 \rangle 5.$ $s = (s - r) + r \in x + y$
$\langle 1 \rangle 5.$ $x + y$ has no greatest element.
$\langle 2 \rangle 1.$ LET: $q \in x$ and $r \in y$
PROVE: There exists $s \in x + y$ such that $q + r < s$
$\langle 2 \rangle 2.$ PICK $q' \in x$ and $r' \in y$ such that $q < q'$ and $r < r'$
$\langle 2 \rangle 3.$ $q + r < q' + r' \in x + y$
$\square$

Proposition 15.3.10. *Addition on the reals agrees with addition on the rationals.*

PROOF:
$\langle 1 \rangle 1.$ LET: $q, r \in \mathbb{Q}$
$\langle 1 \rangle 2.$ $q \downarrow + r \downarrow \subseteq (q + r) \downarrow$
PROOF: If $s_1 < q$ and $s_2 < r$ then $s_1 + s_2 < q + r$.
$\langle 1 \rangle 3.$ $(q + r) \downarrow \subseteq q \downarrow + r \downarrow$
$\langle 2 \rangle 1.$ LET: $s < q + r$
$\langle 2 \rangle 2.$ $s - r < q$
$\langle 2 \rangle 3.$ PICK t such that $s - r < t < q$
$\langle 2 \rangle 4.$ $s - t < r$
$\langle 2 \rangle 5.$ $s = t + (s - t) \in q \downarrow + r \downarrow$
$\square$

Proposition 15.3.11. *Addition is associative.*

PROOF:
$$\begin{aligned} x + (y + z) &= \{q + r \mid q \in x, r \in y + z\} \\ &= \{q + s_1 + s_2 \mid q \in x, s_1 \in y, s_2 \in z\} \\ &= \{r + s_2 \mid r \in x + y, s_2 \in z\} \\ &= (x + y) + z \end{aligned}$$ $\square$

Proposition 15.3.12. *Addition is commutative.*

PROOF:
$$\begin{aligned} x + y &= \{q + r \mid q \in x, r \in y\} \\ &= \{r + q \mid r \in y, q \in x\} \\ &= y + x \end{aligned}$$
□

Proposition 15.3.13. *For any $x \in \mathbb{R}$ we have $x + 0 = x$.*

PROOF:
⟨1⟩1. $x + 0 \subseteq x$
PROOF: If $q \in x$ and $r < 0$ then $q + r < q$ so $q + r \in x$.
⟨1⟩2. $x \subseteq x + 0$
⟨2⟩1. LET: $q \in x$
⟨2⟩2. PICK $r \in x$ such that $q < r$.
PROOF: x has no greatest element.
⟨2⟩3. $q - r < 0$
⟨2⟩4. $q = r + (q - r) \in x + 0$
□

Definition 15.3.14. For $x \in \mathbb{R}$, define $-x := \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$.

Proposition 15.3.15. *For all $x \in \mathbb{R}$ we have $-x \in \mathbb{R}$.*

PROOF:
⟨1⟩1. LET: $x \in \mathbb{R}$
⟨1⟩2. $-x \neq \emptyset$
⟨2⟩1. PICK $s \notin x$
⟨2⟩2. $-s - 1 \in -x$
⟨1⟩3. $-x \neq \mathbb{Q}$
⟨2⟩1. PICK $s \in x$
PROVE: $-s \notin -x$
⟨2⟩2. ASSUME: for a contradiction $-s \in -x$
⟨2⟩3. PICK $r > -s$ such that $-r \notin x$
⟨2⟩4. $-r < s$
⟨2⟩5. Q.E.D.
PROOF: This contradicts the fact that x is closed downwards.
⟨1⟩4. $-x$ is closed downwards.
PROOF: Immediate from definition.
⟨1⟩5. $-x$ has no greatest element.
⟨2⟩1. LET: $q \in -x$
⟨2⟩2. PICK $r > q$ such that $-r \notin x$
⟨2⟩3. PICK s such that $q < s < r$
⟨2⟩4. $s \in -x$
□

Lemma 15.3.16. *Let p be a positive rational number. For any real number x, there exists a rational $q \in x$ such that $p + q \notin x$.*

PROOF:
$\langle 1\rangle 1.$ PICK $q_0 \in x$
$\langle 1\rangle 2.$ There exists $k \in \mathbb{N}$ such that $q_0 + kp \notin x$
  $\langle 2\rangle 1.$ PICK $q_1 \notin x$
  $\langle 2\rangle 2.$ PICK $k \in \mathbb{N}$ such that $q_1 - q_0 < pk$
    PROOF: Proposition 15.2.11.
  $\langle 2\rangle 3.$ $q_1 < q_0 + kp$
  $\langle 2\rangle 4.$ $q_0 + kp \notin x$
$\langle 1\rangle 3.$ LET: k be the least natural number such that $q_0 + kp \notin x$
$\langle 1\rangle 4.$ $k \neq 0$
  PROOF: $\langle 1\rangle 1$
$\langle 1\rangle 5.$ LET: $q = q_0 + (k-1)p$
$\langle 1\rangle 6.$ $q \in x$ and $q + p \notin x$.
□

Proposition 15.3.17. *For every real x we have $x + (-x) = 0$.*

PROOF:
$\langle 1\rangle 1.$ LET: x be a real number.
$\langle 1\rangle 2.$ $x + (-x) \subseteq 0$
  $\langle 2\rangle 1.$ LET: $q_1 \in x$ and $q_2 \in -x$
  $\langle 2\rangle 2.$ PICK $r > q_2$ such that $-r \notin x$
  $\langle 2\rangle 3.$ $q_1 < -r$
  $\langle 2\rangle 4.$ $r < -q_1$
  $\langle 2\rangle 5.$ $q_2 < -q_1$
  $\langle 2\rangle 6.$ $q_1 + q_2 < 0$
$\langle 1\rangle 3.$ $0 \subseteq x + (-x)$
  $\langle 2\rangle 1.$ LET: $p < 0$
  $\langle 2\rangle 2.$ $0 < -p$
  $\langle 2\rangle 3.$ PICK $q \in x$ such that $q - p/2 \notin x$
    PROOF: Lemma 15.3.16.
  $\langle 2\rangle 4.$ LET: $s = p/2 - q$
  $\langle 2\rangle 5.$ $-s \notin x$
  $\langle 2\rangle 6.$ $p - q < s$
  $\langle 2\rangle 7.$ $p - q \in -x$
  $\langle 2\rangle 8.$ $p \in x + (-x)$
□

Corollary 15.3.17.1. *The reals form an Abelian group under addition.*

Proposition 15.3.18. *For any reals x, y and z, we have $x < y$ if and only if $x + z < y + z$.*

PROOF:
$\langle 1\rangle 1.$ $\forall x, y, z \in \mathbb{R}. x \leq y \Rightarrow x + z \leq y + z$
  $\langle 2\rangle 1.$ LET: $x, y, z \in \mathbb{R}$
  $\langle 2\rangle 2.$ ASSUME: $x \leq y$
  $\langle 2\rangle 3.$ For all $q \in x$ and $r \in z$ we have $q + r \in y + z$

$\langle 1\rangle 2.$ $\forall x, y, z \in \mathbb{R}. x + z = y + z \Leftrightarrow x = y$
PROOF: Proposition 12.1.4.
$\langle 1\rangle 3.$ $\forall x, y, z \in \mathbb{R}. x < y \Rightarrow x + z < y + z$
$\langle 1\rangle 4.$ Q.E.D.
PROOF: Proposition 5.2.6.
$\square$

Definition 15.3.19 (Absolute Value)**.** The *absolute value* of a real number x is defined to be

$$|x| = \begin{cases} x & \text{if } 0 \leq x \\ -x & \text{if } x < 0 \end{cases}$$

Definition 15.3.20 (Multiplication)**.** Define *multiplication* $\cdot$ on $\mathbb{R}$ as follows:

- If x and y are non-negative then
$$xy = 0 \cup \{rs \mid 0 \leq r \in x \wedge 0 \leq s \in y\} \enspace .$$
- If x and y are both negative then
$$xy = (-x)(-y) \enspace .$$
- If one of x and y is negative and one is non-negative then
$$xy = -(|x||y|) \enspace .$$

We prove this is well-defined.

PROOF:
$\langle 1\rangle 1.$ LET: x and y be non-negative reals.
PROVE: xy is real.
$\langle 1\rangle 2.$ $xy \neq \emptyset$
PROOF: Since $-1 \in xy$.
$\langle 1\rangle 3.$ $xy \neq \mathbb{Q}$
$\langle 2\rangle 1.$ PICK $r \notin x$ and $s \notin y$
PROVE: $rs \notin xy$
$\langle 2\rangle 2.$ $0 \leq r$ and $0 \leq s$
PROOF: Since $0 \subseteq x$ and $0 \subseteq y$.
$\langle 2\rangle 3.$ ASSUME: for a contradiction $rs \in xy$
$\langle 2\rangle 4.$ PICK r' and s' such that $0 \leq r' \in x$, $0 \leq s' \in y$ and $rs = r's'$
$\langle 2\rangle 5.$ $r' < r$
$\langle 2\rangle 6.$ $s' < s$
$\langle 2\rangle 7.$ $r's' < rs$
$\langle 2\rangle 8.$ Q.E.D.
PROOF: This is a contradiction.
$\langle 1\rangle 4.$ xy is closed downwards.
$\langle 2\rangle 1.$ LET: $q \in xy$ and $r < q$

$\langle 2\rangle 2$. CASE: $q \in 0$
PROOF: Then $r < q < 0$ so $r \in xy$
$\langle 2\rangle 3$. CASE: $q = s_1 s_2$ where $0 \leq s_1 \in x$ and $0 \leq s_2 \in y$
$\langle 3\rangle 1$. ASSUME: w.l.o.g. $0 \leq r$
$\langle 3\rangle 2$. $0 < s_1$ and $0 < s_2$
$\langle 3\rangle 3$. $r/s_2 < s_1$
$\langle 3\rangle 4$. $r/s_2 \in x$
$\langle 3\rangle 5$. $r = (r/s_2)s_2 \in xy$
$\langle 1\rangle 5$. xy has no greatest element.
$\langle 2\rangle 1$. LET: $q \in xy$
$\langle 2\rangle 2$. CASE: $q \in 0$
PROOF: $q < q/2 \in 0$
$\langle 2\rangle 3$. CASE: $q = rs$ where $0 \leq r \in x$ and $0 \leq s \in y$
$\langle 3\rangle 1$. PICK r' and s' with $r < r' \in x$ and $s < s' \in y$
$\langle 3\rangle 2$. $q < r's' \in xy$
□

Proposition 15.3.21. *Multiplication is commutative.*

PROOF: Immediate from definition. □

Proposition 15.3.22. *Multiplication is associative.*

PROOF:
$\langle 1\rangle 1$. For non-negative reals x, y and z, we have $x(yz) = (xy)z$
PROOF: It computes to $0 \cup \{qrs \mid 0 \leq q \in x, 0 \leq r \in y, 0 \leq s \in z\}$.
$\langle 1\rangle 2$. For all reals x, y and z, we have $x(yz) = (xy)z$
PROOF: It is equal to $|x||y||z|$ if an even number of them are negative, and $-(|x||y||z|)$ otherwise.
□

Proposition 15.3.23. *Multiplication distributes over addition.*

PROOF:
$\langle 1\rangle 1$. For all non-negative reals x, y and z, we have $x(y+z) = xy + xz$
$\langle 2\rangle 1$. LET: x, y and z be non-negative reals.
$\langle 2\rangle 2$. $x(y+z) \subseteq xy + xz$
$\langle 3\rangle 1$. LET: $q \in x(y+z)$
$\langle 3\rangle 2$. CASE: $q < 0$
PROOF: Then we have $q/2 \in xy$ and $q/2 \in xz$ so $q \in xy + xz$.
$\langle 3\rangle 3$. CASE: $q = rs$ where $0 \leq r \in x$ and $0 \leq s \in y + z$
$\langle 4\rangle 1$. PICK $s_1 \in y$ and $s_2 \in z$ such that $s = s_1 + s_2$
$\langle 4\rangle 2$. $rs_1 \in xy$
PROOF: If $s_1 < 0$ then $rs_1 < 0$ so $rs_1 \in xy$. If $0 \leq s_1$ then we also have $rs_1 \in xy$.
$\langle 4\rangle 3$. $rs_2 \in xz$
PROOF: Similar.
$\langle 4\rangle 4$. $q \in xy + xz$

PROOF: Since $q = rs_1 + rs_2$.

$\langle 2\rangle 3.$ $xy + xz \subseteq x(y+z)$

$\langle 3\rangle 1.$ LET: $q \in xy$ and $r \in xz$.

PROVE: $q + r \in x(y+z)$

$\langle 3\rangle 2.$ CASE: $q < 0$ and $r < 0$

PROOF: Then $q + r < 0$ so $q + r \in x(y+z)$.

$\langle 3\rangle 3.$ CASE: $q < 0$ and $r = r_1 r_2$ where $0 \leq r_1 \in x$ and $0 \leq r_2 \in z$

$\langle 4\rangle 1.$ $q + r < r$

$\langle 4\rangle 2.$ $q + r \in xz$

$\langle 4\rangle 3.$ ASSUME: w.l.o.g. $0 \leq q + r$

PROOF: Otherwise $q + r \in x(y+z)$ immediately.

$\langle 4\rangle 4.$ PICK s_1, s_2 with $0 \leq s_1 \in x$, $0 \leq s_2 \in y$ and $q + r = s_1 s_2$

$\langle 4\rangle 5.$ $s_2 \in y + z$

PROOF: Since $0 \in z$ so $s_2 = s_2 + 0 \in y + z$.

$\langle 4\rangle 6.$ $q + r \in x(y+z)$

$\langle 3\rangle 4.$ CASE: $q = q_1 q_2$ where $0 \leq q_1 \in x$ and $0 \leq q_2 \in y$ and $r < 0$

PROOF: Similar.

$\langle 3\rangle 5.$ CASE: $q = q_1 q_2$ where $0 \leq q_1 \in x$ and $0 \leq q_2 \in y$ and $r = r_1 r_2$ where $0 \leq r_1 \in x$ and $0 \leq r_2 \in z$

$\langle 4\rangle 1.$ ASSUME: w.l.o.g. $q_1 \leq r_1$

$\langle 4\rangle 2.$ $q + r \leq r_1(q_2 + r_2) \in x(y+z)$

$\langle 1\rangle 2.$ For any negative real x and non-negative reals y and z, we have $x(y+z) = xy + xz$

PROOF:

$$\begin{aligned}
x(y+z) = -(-x)(y+z) &= -((-x)y + (-x)z) && (\langle 1\rangle 1) \\
&= -((-x)y) - ((-x)z) \\
&= xy + xz
\end{aligned}$$

$\langle 1\rangle 3.$ For any non-negative real x and reals y and z with one negative and one non-negative, we have $x(y+z) = xy + xz$

$\langle 2\rangle 1.$ ASSUME: w.l.o.g. y is negative and z is non-negative.

$\langle 2\rangle 2.$ CASE: $0 \leq y + z$

PROOF:

$$\begin{aligned}
xy + xz &= xy + x(-y + y + z) \\
&= -(x(-y)) + x(-y + y + z) \\
&= -(x(-y)) + x(-y) + x(y+z) && (\langle 1\rangle 1) \\
&= x(y+z)
\end{aligned}$$

$\langle 2\rangle 3.$ CASE: $y + z < 0$

$\langle 3\rangle 1.$ $-y - z > 0$

$\langle 3\rangle 2.$ $-y = z - y - z$

$\langle 3\rangle 3.$ $xy + xz = x(y+z)$

PROOF:

$$\begin{aligned} xy + xz &= -(x(-y)) + xz \\ &= -(x(z - y - z)) + xz \\ &= -(xz + x(-y - z)) + xz && (\langle 1\rangle 1) \\ &= -xy - x(-y - z) + xz \\ &= -x(-y - z) \\ &= x(y + z) \end{aligned}$$

⟨1⟩4. For any non-negative real x and negative reals y and z, we have $x(y+z) = xy + xz$

PROOF:

$$\begin{aligned} x(y + z) &= -x(-y - z) \\ &= -(x(-y) + x(-z)) && (\langle 1\rangle 1) \\ &= -x(-y) - x(-z) \\ &= xy + xz \end{aligned}$$

⟨1⟩5. For any negative real x and reals y and z with one negative and one non-negative, we have $x(y + z) = xy + xz$

⟨2⟩1. ASSUME: w.l.o.g. y is negative and z is non-negative.

⟨2⟩2. CASE: $0 \leq y + z$

PROOF:

$$\begin{aligned} x(y + z) &= -((-x)(y + z)) \\ &= -((-x)y + (-x)z) && (\langle 1\rangle 3) \\ &= -((-x)y) - ((-x)z) \\ &= (-x)(-y) - ((-x)z) \\ &= xy + xz \end{aligned}$$

⟨2⟩3. CASE: $y + z < 0$

PROOF:

$$\begin{aligned} x(y + z) &= (-x)(-y - z) \\ &= (-x)(-y) + (-x)(-z) && (\langle 1\rangle 3) \\ &= xy + xz \end{aligned}$$

⟨1⟩6. For any negative reals x, y and z, we have $x(y + z) = xy + xz$

PROOF:

$$\begin{aligned} x(y + z) &= (-x)(-y - z) \\ &= (-x)(-y) + (-x)(-z) && (\langle 1\rangle 1) \\ &= xy + xz \end{aligned}$$

□

Proposition 15.3.24. *For any real x we have $x1 = x$.*

PROOF:

⟨1⟩1. CASE: $0 \leq x$

⟨2⟩1. $x1 \subseteq x$

⟨3⟩1. LET: $q \in x1$

$\langle 3\rangle 2$. CASE: $q < 0$
PROOF: Then $q \in x$ because $0 \leq x$.
$\langle 3\rangle 3$. $q = rs$ where $0 \leq r \in x$ and $0 \leq s < 1$
PROOF: Then $q < r$ so $q \in x$.
$\langle 2\rangle 2$. $x \subseteq x1$
$\langle 3\rangle 1$. LET: $q \in x$
$\langle 3\rangle 2$. ASSUME: w.l.o.g. $0 \leq q$
$\langle 3\rangle 3$. PICK r such that $q < r \in x$
$\langle 3\rangle 4$. $0 \leq q/r < 1$
$\langle 3\rangle 5$. $q = r(q/r) \in x1$
$\langle 1\rangle 2$. CASE: $x < 0$
PROOF:

$$\begin{aligned} x1 &= -((-x)1) \\ &= -(-x) && (\langle 1\rangle 1) \\ &= x \end{aligned}$$

□

Lemma 15.3.25. *Let $x \in \mathbb{R}$ and c be a positive rational. Then there exists $a \in x$ and a non-least rational upper bound b for x such that $b - a = c$.*

PROOF:
$\langle 1\rangle 1$. PICK $a_1 \in x$ such that if x has a rational supremum s then $a_1 > s - c$
$\langle 1\rangle 2$. There exists a natural number n such that $a_1 + nc$ is an upper bound for x.
$\langle 2\rangle 1$. PICK a non-least upper bound b_1 for x.
$\langle 2\rangle 2$. PICK a natural number n such that $nc > b_1 - a_1$
PROOF: Proposition 15.2.11.
$\langle 2\rangle 3$. $a_1 + nc > b_1$
$\langle 2\rangle 4$. $a_1 + nc$ is an upper bound for x.
$\langle 1\rangle 3$. LET: k be the least natural number such that $a_1 + kc$ is an upper bound for x.
$\langle 1\rangle 4$. $a_1 + (k-1)c \in x$
$\langle 1\rangle 5$. $a_1 + kc$ is not the supremum of x.
$\langle 2\rangle 1$. ASSUME: for a contradiction $a_1 + kc$ is the supremum of x.
$\langle 2\rangle 2$. $a_1 > a_1 + (k-1)c$
PROOF: $\langle 1\rangle 1$
$\langle 2\rangle 3$. Q.E.D.
PROOF: This is a contradiction.
$\langle 1\rangle 6$. LET: $a = a_1 + (k-1)c$
$\langle 1\rangle 7$. LET: $b = a_1 + kc$
$\langle 1\rangle 8$. $b - a = c$
□

Proposition 15.3.26. *For any non-zero real x, there exists a real y such that $xy = 1$.*

PROOF:

⟨1⟩1. CASE: $0 < x$
⟨2⟩1. LET: $y = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{u^{-1} \mid u$ is an upper bound for x but not the supremum of $x\}$
⟨2⟩2. y is a real number.
⟨3⟩1. $y \neq \emptyset$
PROOF: Since $0 \in y$.
⟨3⟩2. $y \neq \mathbb{Q}$
⟨4⟩1. PICK $q \in x$ such that $0 < q$
⟨4⟩2. $q^{-1} \notin y$
⟨3⟩3. y is closed downwards.
⟨4⟩1. LET: $q \in y$ and $r < q$
PROVE: $r \in y$
⟨4⟩2. ASSUME: w.l.o.g. $0 < r$
⟨4⟩3. q^{-1} is a non-least upper bound for x.
⟨4⟩4. $q^{-1} < r^{-1}$
⟨4⟩5. r^{-1} is a non-least upper bound for x.
⟨4⟩6. $r \in y$
⟨3⟩4. y has no greatest element.
⟨4⟩1. LET: $q \in y$
PROVE: There exists $r \in y$ such that $q < r$
⟨4⟩2. CASE: $q \leq 0$
⟨5⟩1. PICK a non-least upper bound u for x.
⟨5⟩2. $q < u^{-1} \in x$
⟨4⟩3. CASE: $q = u^{-1}$ where u is a non-least upper bound for x.
⟨5⟩1. PICK a non-least upper bound v with $v < u$
⟨5⟩2. $u^{-1} < v^{-1} \in y$
⟨2⟩3. $0 < y$
⟨2⟩4. $xy \subseteq 1$
⟨3⟩1. LET: $q \in xy$
⟨3⟩2. ASSUME: w.l.o.g. $0 < q$
⟨3⟩3. PICK $0 < r \in x$ and $0 < s \in y$ such that $q = rs$
⟨3⟩4. s^{-1} is a non-least upper bound for x
⟨3⟩5. $r < s^{-1}$
⟨3⟩6. $rs < 1$
⟨2⟩5. $1 \subseteq xy$
⟨3⟩1. LET: $q < 1$
PROVE: $q \in xy$
⟨3⟩2. ASSUME: w.l.o.g. $0 < q$
⟨3⟩3. PICK a_1 with $0 < a_1 \in x$
⟨3⟩4. $(1 - q)a_1 > 0$
⟨3⟩5. PICK $a \in x$ and a non-least upper bound w of x such that $w - a = (1 - q)a_1$
PROOF: Lemma 15.3.25.
⟨3⟩6. $w - a < (1 - q)w$
⟨3⟩7. $qw < a$
⟨3⟩8. $w < a/q$
⟨3⟩9. a/q is a non-least upper bound for x

⟨3⟩10. $q/a \in y$
⟨3⟩11. $q \in xy$
⟨1⟩2. CASE: $x < 0$
⟨2⟩1. PICK y such that $(-x)y = 1$
PROOF: ⟨1⟩1
⟨2⟩2. $x(-y) = 1$
□

Proposition 15.3.27. *For real numbers x, y and z, if $0 < z$ then $x < y$ if and only if $xz < yz$.*

PROOF:
⟨1⟩1. For any real numbers x, y and z, if $0 < z$ and $x < y$ then $xz < yz$
⟨2⟩1. LET: x, y and z be real numbers.
⟨2⟩2. ASSUME: $0 < z$ and $x < y$.
⟨2⟩3. $y = x + (y - x)$
⟨2⟩4. $y - x > 0$
⟨2⟩5. $(y - x)z > 0$
⟨2⟩6. $yz > xz$
PROOF:

$$\begin{aligned} yz &= (x + (y - x))z \\ &= xz + (y - x)z \\ &> xz \end{aligned}$$

⟨1⟩2. For any real numbers x, y and z, if $0 < z$ and $xz < yz$ then $x < y$
PROOF: Proposition 5.2.6.
□

Corollary 15.3.27.1. *The real numbers form a complete ordered field.*

Proposition 15.3.28.

$$(0, 1) \approx \mathbb{R}$$

PROOF: The function $f(x) = (2x - 1)/(x - x^2)$ is a bijection between $(0, 1)$ and $\mathbb{R}$. □

Proposition 15.3.29.

$$|\mathbb{R}| = 2^{\aleph_0}$$

PROOF:
⟨1⟩1. $(0, 1) \preccurlyeq 2^{\mathbb{N}}$
PROOF: The function H where $H(x)(n)$ is the nth binary digit of the binary expansion of x is an injection.
⟨1⟩2. $2^{\mathbb{N}} \preccurlyeq \mathbb{R}$
PROOF: Map f to the real number in $[0, 1/9]$ whose $n + 1$st decimal digit is $f(n)$.
□

Proposition 15.3.30. *The set of algebraic numbers is countable.*

Proof: There are countably many integer polynomials, each with finitely many roots. □

Corollary 15.3.30.1. *There are uncountably many transcendental numbers.*

Proposition 15.3.31. *Let A be a set of disks in the plane, no two of which intersect. Then A is countable.*

Proof: Every circle includes a point with rational coordinates. Define $f : \{q \in \mathbb{Q}^2 \mid \exists C \in A. q \in C\} \to A$ by $f(q) = C$ iff $q \in C$. Then f is surjective. □

Proposition 15.3.32. *There exists an uncountable set of circles in the plane that do not intersect.*

Proof: The set of all circles with origin O is uncountable. □

Chapter 16

Real Analysis

Theorem 16.0.1 (Weierstrass). *Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous. For every $\epsilon > 0$, there exists a polynomial p such that $\forall x \in [a, b]. |f(x) - p(x)| < \epsilon$.*

Theorem 16.0.2 (Bolzano-Weierstrass). *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Chapter 17

Complex Analysis

Theorem 17.0.1 (Hölder's Inequality)**.** *Let p and q be real numbers with $p > 1$, $q > 1$ and $1/p + 1/q = 1$. If $(x_n) \in l^p$ and $(y_n) \in l^q$ then*

$$\sum_{n=0}^{\infty} |x_n y_n| \leq \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=0}^{\infty} |y_n|^q \right)^{1/q}$$

PROOF:
$\langle 1 \rangle 1$. LET: p and q be real numbers with $p > 1$ and $q > 1$
$\langle 1 \rangle 2$. ASSUME: $1/p + 1/q = 1$
$\langle 1 \rangle 3$. LET: $(x_n) \in l^p$
$\langle 1 \rangle 4$. LET: $(y_n) \in l^q$
$\langle 1 \rangle 5$. ASSUME: w.l.o.g. $x_0 \neq 0$ and $y_0 \neq 0$
$\langle 1 \rangle 6$. For all $x \in [0, 1]$, we have

$$x^{1/p} \leq \frac{1}{p} x + \frac{1}{q} \quad .$$

PROOF:
$\langle 2 \rangle 1$. LET: $f : [0, 1] \to \mathbb{R}$ be the function

$$f(x) = \frac{1}{p} x + \frac{1}{q} - x^{1/p} \quad .$$

$\langle 2 \rangle 2$. $f'(x) = \frac{1}{p} - \frac{1}{p} x^{-1/q}$ for $x \in (0, 1]$
$\langle 2 \rangle 3$. $f'(x) < 0$ for $x \in (0, 1]$
$\langle 2 \rangle 4$. $f(1) = 1/p + 1/q - 1 = 0$
$\langle 2 \rangle 5$. $f(x) \geq 0$ for all $x \in [0, 1]$
$\langle 1 \rangle 7$. For all non-negative reals a and b, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad .$$

$\langle 2 \rangle 1$. LET: a and b be non-negative reals.
$\langle 2 \rangle 2$. CASE: $a^p \leq b^q$
$\langle 3 \rangle 1$. $0 \leq a^p / b^q \leq 1$
$\langle 3 \rangle 2$.

$$ab^{-q/p} \leq \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q}$$

PROOF: Taking $x = a^p/b^q$ in $\langle 1\rangle 6$.

$\langle 3\rangle 3$.

$$ab^{1-q} \leq \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q}$$

PROOF: $-q/p = 1 - q$ from $\langle 1\rangle 2$.

$\langle 3\rangle 4$.

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$\langle 2\rangle 3$. CASE: $b^q \leq a^p$

PROOF: Similar.

$\langle 1\rangle 8$. For $j = 1, \ldots, n$, we have

$$\frac{|x_j|}{\left(\sum_{k=0}^n |x_k|^p\right)^{1/p}} \frac{|y_j|}{\left(\sum_{k=0}^n |y_k|^q\right)^{1/q}} \leq \frac{1}{p}\frac{|x_j|^p}{\sum_{k=0}^n |x_k|^p} + \frac{1}{q}\frac{|y_j|^q}{\sum_{k=0}^n |y_k|^q}$$

PROOF: From $\langle 1\rangle 7$ with

$$a = \frac{|x_j|}{\left(\sum_{k=0}^n |x_k|^p\right)^{1/p}} \text{ and } b = \frac{|y_j|}{\left(\sum_{k=0}^n |y_k|^q\right)^{1/q}} .$$

$\langle 1\rangle 9$.

$$\frac{\sum_{j=0}^n |x_j||y_j|}{\left(\sum_{k=0}^n |x_k|^p\right)^{1/p}\left(\sum_{k=0}^n |y_k|^q\right)^{1/q}} \leq 1$$

PROOF:

$$\begin{aligned} \frac{\sum_{j=0}^n |x_j||y_j|}{\left(\sum_{k=0}^n |x_k|^p\right)^{1/p}\left(\sum_{k=0}^n |y_k|^q\right)^{1/q}} &\leq \frac{1}{p} + \frac{1}{q} && \text{(Taking the sum } j = 0 \text{ to } n \text{ in } \langle 1\rangle 8) \\ &= 1 && (\langle 1\rangle 2) \end{aligned}$$

$\langle 1\rangle 10$. Q.E.D.

PROOF: Taking the limit $n \to \infty$ in $\langle 1\rangle 9$.

□

Theorem 17.0.2 (Minkowski's Inequality)**.** *Let p be a real number, $p \geq 1$. Let $(x_n), (y_n) \in l^p$. Then*

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}$$

PROOF:

$\langle 1\rangle 1$. LET: p be a real number with $p \geq 1$

$\langle 1\rangle 2$. ASSUME: w.l.o.g. $p > 1$

PROOF: The case $p = 1$ is just the Triangle Inequality.

$\langle 1\rangle 3$. LET: q be the real such that $1/p + 1/q = 1$

$\langle 1\rangle 4$.

$$\begin{aligned} \sum_{n=0}^{\infty} |x_n + y_n|^p \leq & \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)}\right)^{1/q} \\ & + \left(\sum_{n=0}^{\infty} |y_n|^p\right)^{1/p} \left(\sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)}\right)^{1/q} \end{aligned}$$

PROOF:

$$\begin{aligned}\sum_{n=0}^{\infty} |x_n + y_n|^p &= \sum_{n=0}^{\infty} |x_n + y_n||x_n + y_n|^{p-1} \\ &\leq \sum_{n=0}^{\infty} |x_n||x_n + y_n|^{p-1} + \sum_{n=0}^{\infty} |y_n||x_n + y_n|^{p-1} \quad \text{(Triangle Inequality)} \\ &\leq \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)}\right)^{1/q} \\ &+ \left(\sum_{n=0}^{\infty} |y_n|^p\right)^{1/p} \left(\sum_{n=0}^{\infty} |x_n + y_n|^{q(p-1)}\right)^{1/q} \quad \text{(Hölder's Inequality)}\end{aligned}$$

$\langle 1 \rangle 5.$

$$\sum_{n=0}^{\infty} |x_n + y_n|^p \leq \left\{ \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=0}^{\infty} |y_n|^p\right)^{1/p} \right\} \left(\sum_{n=0}^{\infty} |x_n + y_n|^p\right)^{1/q}$$

$\langle 1 \rangle 6.$ Q.E.D.

□

Chapter 18

Topology

18.1 Topological Spaces

Definition 18.1.1 (Topology)**.** Let X be a set. A *topology* on X is a set $\mathcal{T} \subseteq \mathcal{P}X$, whose elements are called *open sets*, such that:

- $X \in \mathcal{T}$
- $\forall \mathcal{U} \subseteq \mathcal{T}. \bigcup \mathcal{U} \in \mathcal{T}$
- $\forall U, V \in \mathcal{T}. U \cap V \in \mathcal{T}$

A *topological space* is a pair $(X, \mathcal{T})$ such that X is a set and $\mathcal{T}$ is a topology on X. We refer to the elements of X as *points*.

An *open neighbourhood* of a point x is an open set U such that $x \in U$. We write $\mathcal{T}_x$ for the set of all open neighbourhoods of x.

Definition 18.1.2 (Closed Set)**.** In a topological space X, a set C is *closed* iff $X - C$ is open.

Definition 18.1.3 (Discrete Topology)**.** The *discrete topology* on a set X is $\mathcal{P}X$.

Definition 18.1.4 (Indiscrete Topology)**.** The *indiscrete topology* or *trivial topology* on a set X is $\{\emptyset, X\}$.

Definition 18.1.5 (Finer, Coarser)**.** Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. Then $\mathcal{T}$ is *finer*, *larger* or *stronger* than $\mathcal{T}'$, and $\mathcal{T}'$ is *coarser*, *smaller* or *weaker* than $\mathcal{T}$, iff $\mathcal{T}' \subseteq \mathcal{T}$.

Definition 18.1.6 (Basis)**.** Let X be a set. A *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$, whose elements we call *basic open neighbourhoods*, such that:

- $\bigcup \mathcal{B} = X$
- $\forall A, B \in \mathcal{B}. \forall x \in A \cap B. \exists C \in \mathcal{B}. x \in C \subseteq A \cap B$.

The topology *generated* by $\mathcal{B}$ is the coarsest topology that includes $\mathcal{B}$.

Proposition 18.1.7. *The topology generated by* $\mathcal{B}$ *is* $\{U \in \mathcal{P}X \mid \forall x \in U.\exists B \in \mathcal{B}.x \in B \subseteq U\}$

18.2 Continuous Functions

Definition 18.2.1 (Continuous)**.** Let X and Y be topological spaces. Let $f : X \to Y$. Then f is *continuous* iff, for every open set U in Y, the set $f^{-1}(U)$ is open in X.

18.3 Metric Spaces

Definition 18.3.1 (Metric)**.** Let X be a set. A *metric* on a set X is a function $d : X^2 \to \mathbb{R}$ such that:

- $\forall x, y \in X.d(x, y) \geq 0$
- $\forall x, y \in X.d(x, y) = d(y, x)$
- *Triangle Inequality* $\forall x, y, z \in X.d(x, y) + d(y, z) \geq d(x, z)$
- $\forall x, y \in X.d(x, y) = 0$ iff $x = y$.

A *metric space* is a pair (X, d) such that d is a metric on X.

Definition 18.3.2 (Open Ball)**.** In a metric space X, let $c \in X$ and $r > 0$. The *open ball* with *centre* c and *radius* r is

$$B(c, r) := \{x \in X \mid d(x, c) < r\} \ .$$

Proposition 18.3.3. *In a metric space, the set of open balls forms a basis for a topology.*

Definition 18.3.4 (Metric Topology)**.** Given a metric space X, the *metric topology* on X is the topology generated by the basis of open balls.

A topological space $(X, \mathcal{T})$ is *metrizable* iff there exists a metric d on X such that $\mathcal{T}$ is the metric topology induced by d.

We identify a metric space with this topological space.

Proposition 18.3.5. *If* d *is a metric on* X *and* $Y \subseteq X$ *then* $d \restriction Y^2$ *is a metric on* Y*.*

We write just Y for the metric space $(Y, d \restriction Y^2)$.

Chapter 19

Linear Algebra

19.1 Vector Spaces

Definition 19.1.1 (Vector Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *vector space* over K consists of:

- a set V, whose elements are called *vectors*;
- an operation $+ : V^2 \to V$, *addition*;
- an operation $\cdot : K \times V \to V$, *scalar multiplication*

such that:

- V is an Abelian group under $+$
- $\forall \alpha, \beta \in K. \forall x \in V. \alpha(\beta x) = (\alpha\beta)x$
- $\forall \alpha, \beta \in K. \forall x \in V. (\alpha + \beta)x = \alpha x + \beta x$
- $\forall \alpha \in K. \forall x, y \in V. \alpha(x + y) = \alpha x + \alpha y$
- $\forall x \in V. 1x = x$

We call the elements of K *scalars*. A *real vector space* is a vector space over $\mathbb{R}$, and a *complex vector space* is a vector space over $\mathbb{C}$.

Proposition 19.1.2. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. For any $\lambda \in K$ we have $\lambda 0 = 0$.*

Proof:

$$\begin{aligned} \lambda 0 &= \lambda(0 + 0) \\ &= \lambda 0 + \lambda 0 \\ \therefore 0 &= \lambda 0 \end{aligned} \qquad \square$$

Proposition 19.1.3. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. Let $\lambda \in K$ and $x \in V$. If $\lambda x = 0$ then either $\lambda = 0$ or $x = 0$.*

PROOF: If $\lambda \neq 0$ then $x = 1x = \lambda^{-1}\lambda x = \lambda^{-1}0 = 0$. ∎

Proposition 19.1.4. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. For any $x \in V$ we have $0x = 0$.*

PROOF:

$$\begin{aligned} 0x &= (0+0)x \\ &= 0x + 0x \\ \therefore 0 &= 0x \end{aligned}$$ ∎

Proposition 19.1.5. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. For any $x \in V$, we have $(-1)x = -x$.*

PROOF:

$$\begin{aligned} x + (-1)x &= 1x + (-1)x \\ &= (1 + (-1))x \\ &= 0x \\ &= 0 \\ \therefore (-1)x &= -x \end{aligned}$$ ∎

Proposition 19.1.6. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Then K is a vector space over K under addition and multiplication in K.*

PROOF: Easy. ∎

Proposition 19.1.7. *$\mathbb{C}$ is a vector space over $\mathbb{R}$.*

PROOF: Easy. ∎

Proposition 19.1.8. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let $\{V_i\}_{i \in I}$ be a family of vector spaces over K. Then $\prod_{i \in I} V_i$ is a vector space under*

$$\begin{aligned} (f+g)(i) &= f(i) + g(i) && (f, g \in \prod_{i \in I} V_i, x \in X) \\ (\lambda f)(x) &= \lambda f(x) && (\lambda \in K, f \in \prod_{i \in I} V_i, x \in X) \end{aligned}$$

PROOF: Easy. ∎

19.2 Subspaces

Definition 19.2.1 (Vector Subspace)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. A *vector subspace* of V is a subset $U \subseteq V$ such that, for all $\alpha, \beta \in K$ and $x, y \in U$, we have $\alpha x + \beta y \in U$.

It is a *proper* subspace iff $U \neq V$.

Proposition 19.2.2. *If U is a subspace of V then U is a vector space under the restrictions of $+$ and $\cdot$ to U.*

PROOF: Easy. □

Proposition 19.2.3. *V is a subspace of V.*

PROOF: Easy. □

Proposition 19.2.4. *If U is a subspace of V and V is a subspace of W then U is a subspace of W.*

PROOF: Easy. □

Definition 19.2.5. Let Ω be a topological space. Then $\mathcal{C}(\Omega)$ is the complex vector space of all continuous functions from Ω to $\mathbb{C}$. This is a subspace of $\mathbb{C}^\Omega$.

Definition 19.2.6. Let $n, k \in \mathbb{N}$. Let Ω be an open subset of $\mathbb{R}^n$. Then $\mathcal{C}^k(\Omega)$ is the complex vector space of all functions $\Omega \to \mathbb{C}$ that have all continuous partial derivatives of order k. This is a subspace of $\mathcal{C}(\Omega)$. If $l > k$ then $\mathcal{C}^l(\Omega)$ is a subpase of $\mathcal{C}^k(\Omega)$.

Definition 19.2.7. Let $n \in \mathbb{N}$. Let Ω be an open subset of $\mathbb{R}^n$. Then $\mathcal{C}^\infty(\Omega)$ is the complex vector space of all infinitely differentiable functions $\Omega \to \mathbb{C}$. This is a subspace of $\mathcal{C}^k(\Omega)$ for all k.

Definition 19.2.8. Let $n \in \mathbb{N}$. Let Ω be an open subset of $\mathbb{R}^n$. Then $\mathcal{P}(\Omega)$ is the complex vector space of all complex polynomials of n variables, considered as functions $\Omega \to \mathbb{C}$. This is a subspace of $\mathcal{C}^\infty(\Omega)$.

Proposition 19.2.9. *The space of all convergent sequences in $\mathbb{C}$ is a subspace of the space of all bounded sequences in $\mathbb{C}$, which is a subspace of $\mathbb{C}^\mathbb{N}$.*

PROOF: Easy. □

Definition 19.2.10. Let p be a real number, $p \geq 1$. Let l^p be the set of all complex sequences (z_n) such that $\sum_{n=1}^\infty |z_n|^p < \infty$.

Proposition 19.2.11. *For p a real number ≥ 1, we have that l^p is a subspace of $\mathbb{C}^\mathbb{N}$.*

PROOF:
$\langle 1 \rangle 1.$ For all $(x_n), (y_n) \in l^p$, we have $(x_n + y_n) \in l^p$.
PROOF: From Minkowski's Inequality.
$\langle 1 \rangle 2.$ For all $\lambda \in \mathbb{C}$ and $(x_n) \in l^p$ we have $(\lambda x_n) \in l^p$
PROOF:

$$\sum_{n=1}^\infty |\lambda x_n|^p = |\lambda|^p \sum_{n=1}^\infty |x_n|^p < \infty$$

□

Definition 19.2.12 (Linear Combination)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. Let $x, x_1, \ldots, x_n \in V$. Then x is a *linear combination* of $x_1, \ldots, x_n$ iff there exist $\alpha_1, \ldots, \alpha_n \in K$ such that

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n \ .$$

Definition 19.2.13 (Linearly Independent)**.** A finite set of vectors $\{x_1, \ldots, x_n\}$ is *linearly independent* iff, whenever $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$, then $\alpha_1 = \cdots = \alpha_n = 0$.

A set of vectors is *linearly independent* iff every finite subset is linearly independent; otherwise, it is *linearly dependent*.

Definition 19.2.14 (Span)**.** Let $\mathcal{A}$ be a set of vectors. The *span* of $\mathcal{A}$, $\operatorname{span} \mathcal{A}$, is the set of all linear combinations of elements of $\mathcal{A}$.

Proposition 19.2.15. $\operatorname{span} \mathcal{A}$ *is the smallest subspace of V that includes $\mathcal{A}$.*

PROOF: Easy. □

Definition 19.2.16 (Basis)**.** A *basis* for V is a linearly independent set of vectors $\mathcal{B}$ such that $\operatorname{span} \mathcal{B} = V$.

Definition 19.2.17 (Finite Dimensional)**.** A vector space is *finite dimensional* iff it has a finite basis; otherwise it is *infinite dimensional*.

Proposition 19.2.18. *In a finite dimensional vector space, any two bases have the same number of elements.*

Definition 19.2.19 (Dimension)**.** The *dimension* of a finite dimensional vector space V, $\dim V$, is the number of elements in any basis.

Proposition 19.2.20.

$$\dim K^n = n$$

PROOF: The standard basis is the set of vectors with one coordinate 1 and all others 0. □

Proposition 19.2.21. *The dimension of $\mathbb{C}^n$ as a real vector space is $2n$.*

19.3 Linear Transformations

Definition 19.3.1 (Linear Transformation)**.** Let U and V be vector spaces over K. Let $T : U \to V$. Then T is a *linear transformation* iff

$$\forall \alpha, \beta \in K. \forall x, y \in U. T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \ .$$

Definition 19.3.2 (Null Space)**.** Let U and V be vector spaces over K and $T : U \to V$. The *null space* of T is

$$\mathcal{N}(T) := \{x \in U \mid T(x) = 0\} \ .$$

19.4 Normed Spaces

Definition 19.4.1 (Norm)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *norm* on a vector space V over K is a function $\| \| : V \to \mathbb{R}$ such that:

1. $\forall x \in V. \|x\| = 0 \Rightarrow x = 0$
2. $\forall \lambda \in K. \forall x \in V. \|\lambda x\| = |\lambda| \|x\|$
3. *Triangle Inequality* $\forall x, y \in V. \|x + y\| \leq \|x\| + \|y\|$

Proposition 19.4.2. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a normed space over K. Define $d : V^2 \to \mathbb{R}$ by $d(x, y) = \|x - y\|$. Then d is a metric on V.*

Proof: Easy. □

We identify any normed space V with this metric space.

Proposition 19.4.3. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. Let $\| \|$ be a norm on V. Then*

$$\|0\| = 0 \ .$$

Proof:

$$\begin{aligned} \|0\| &= \|0 \cdot 0\| && \text{(Proposition 19.1.4)} \\ &= |0| \|0\| && \text{(Axiom 2 for a norm)} \\ &= 0 && \square \end{aligned}$$

Proposition 19.4.4. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. Let $\| \|$ be a norm on V. Let $x \in V$. Then*

$$\|x\| \geq 0 \ .$$

Proof:

$$\begin{aligned} 0 &= \|0\| && \text{(Proposition 19.4.3)} \\ &= \|x - x\| \\ &\leq \|x\| + \| - x\| && \text{(Triangle Inequality)} \\ &= \|x\| + \|x\| && \text{(Axiom 2 for a norm)} \\ &= 2\|x\| && \square \end{aligned}$$

Proposition 19.4.5. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. Let $\| \|$ be a norm on V. Let $x, y \in V$. Then*

$$|\|x\| - \|y\|| \leq \|x - y\| \ .$$

Proof:
$\langle 1 \rangle 1.$ $\|x\| - \|y\| \leq \|x - y\|$
Proof: $\|x\| \leq \|x - y\| + \|y\|$ by the Triangle Inequality.
$\langle 1 \rangle 2.$ $\|y\| - \|x\| \leq \|x - y\|$

PROOF:

$$\begin{aligned} \|x\| + \|x - y\| &= \|x\| + \|y - x\| && \text{(Axiom 2 for a norm)} \\ &\leq \|y\| && \text{(Triangle Inequality)} \end{aligned}$$

□

Definition 19.4.6 (Euclidean Norm)**.** The *Euclidean norm* on $\mathbb{C}^n$ is defined by

$$\|(z_1, \ldots, z_n)\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$$

Proposition 19.4.7. *Define* $\| \ \| : \mathbb{C}^n \to \mathbb{R}$ *by*

$$\|(z_1, \ldots, z_n)\| = |z_1| + \cdots + |z_n|$$

Then this defines a norm on $\mathbb{C}^n$*.*

PROOF: Easy. □

Proposition 19.4.8. *Define* $\| \ \| : \mathbb{C}^n \to \mathbb{R}$ *by*

$$\|(z_1, \ldots, z_n)\| = \max(|z_1|, \ldots, |z_n|)$$

Then this defines a norm on $\mathbb{C}^n$*.*

PROOF: Easy. □

Proposition 19.4.9. *Let* Ω *be a closed bounded subset of* $\mathbb{R}^n$*. Define* $\| \ \| : \mathcal{C}(\Omega) \to \mathbb{R}$ *by* $\|f\| = \max_{x \in \Omega} |f(x)|$*. Then* $\| \ \|$ *defines a norm on* $\mathcal{C}(\Omega)$*.*

PROOF: Easy. □

Proposition 19.4.10. *Let* p *be a real number,* $p \geq 1$*. Define* $\| \ \| : l^p \to \mathbb{R}$ *by*

$$\|(z_n)\| = \left(\sum_{n=0}^{\infty} |z_n|^p \right)^{1/p} .$$

Then this defines a norm on l^p*.*

PROOF: Easy. The triangle inequality is Minkowski's Inequality. □

Definition 19.4.11 (Normed Space)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. A *normed space* over K consists of a vector space V over K and a norm on V.

We shall write simply:

- K^n for the normed space K^n under the Euclidean norm
- $\mathcal{C}(\Omega)$ for the normed space $\mathcal{C}(\Omega)$ under the norm $\|f\| = \max_{x \in \Omega} |f(x)|$, called the *uniform convergence norm*
- l^p for the normed space l^p under the norm $\|(z_n)\| = \left(\sum_{n=0}^{\infty} |z_n|^p\right)^{1/p}$.

Proposition 19.4.12. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a normed space over K. If $x_n \to l$ as $n \to \infty$ in V and $\lambda_n \to \lambda$ as $n \to \infty$ in K, then $\lambda_n x_n \to \lambda l$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. LET: $\epsilon > 0$
$\langle 1 \rangle 2$. LET: $K = |\lambda| + \epsilon / 2\|l\|$
$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $|\lambda_n - \lambda| < \epsilon / 2\|l\|$ and $\|x_n - l\| < \epsilon / (2K)$
$\langle 1 \rangle 4$. For all $n \geq N$ we have $|\lambda_n| < K$
$\langle 1 \rangle 5$. $\|\lambda_n x_n - \lambda l\| < \epsilon$
PROOF:

$$\begin{aligned} \|\lambda_n x_n - \lambda l\| &\leq \|\lambda_n x_n - \lambda_n l\| + \|\lambda_n l - \lambda l\| \\ &= |\lambda_n| \|x_n - l\| + |\lambda_n - \lambda| \|l\| \\ &< K \frac{\epsilon}{2K} + \frac{\epsilon}{2\|l\|} \|l\| \\ &= \epsilon \end{aligned}$$

□

Proposition 19.4.13. *In a normed space, if $x_n \to l$ and $y_n \to m$ then $x_n + y_n \to l + m$*

PROOF:

$$\begin{aligned} \|(x_n + y_n) - (l + m)\| &\leq \|x_n - l\| + \|y_n - m\| \\ &\to 0 \end{aligned}$$

□

Definition 19.4.14 (Uniform Convergence)**.** Let Ω be a closed bounded set in $\mathbb{R}^n$. Let (f_n) be a sequence in $\mathcal{C}(\Omega)$ and $f \in \mathcal{C}(\Omega)$. Then (f_n) *converges uniformly* to f if and only if, for every $\epsilon > 0$, there exists N such that $\forall x \in \Omega . \forall n \geq N . |f_n(x) - f(x)| < \epsilon$.

Proposition 19.4.15. *(f_n) converges uniformly to f iff (f_n) converges to f under the uniform convergence norm.*

PROOF: Easy. □

Proposition 19.4.16. *There is no norm on $\mathcal{C}([0,1])$ that induces pointwise convergence.*

PROOF:
$\langle 1 \rangle 1$. LET: $\| \ \|$ be any norm on $\mathcal{C}([0,1])$
$\langle 1 \rangle 2$. For $n \in \mathbb{N}$, define $g_n \in \mathcal{C}([0,1])$ by

$$g_n(t) = \begin{cases} 2^n t & \text{if } 0 \leq t \leq 2^{-n} \\ 2 - 2^n t & \text{if } 2^{-n} \leq t \leq 2^{1-n} \\ 0 & \text{otherwise} \end{cases}$$

$\langle 1 \rangle 3$. For all $n \in \mathbb{N}$ we have $\|g_n\| \neq 0$
$\langle 1 \rangle 4$. For $n \in \mathbb{N}$,

LET: $f_n = g_n/\|g_n\|$
⟨1⟩5. For all $n \in \mathbb{N}$, $\|f_n\| = 1$
⟨1⟩6. f_n does not converge to 0
⟨1⟩7. $f_n \to 0$ as $n \to \infty$ pointwise.
□

Definition 19.4.17 (Equivalent Norms)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector spaces over K. Then two norms $\| \|_1$ and $\| \|_2$ are *equivalent* if and only if, for any sequence (x_n) in V and $l \in V$, we have $x_n \to l$ under $\| \|_1$ if and only if $x_n \to l$ under $\| \|_2$.

Proposition 19.4.18. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector spaces over K. Let $\| \|_1$ and $\| \|_2$ be norms on V. Then $\| \|_1$ and $\| \|_2$ are equivalent if and only if there exist positive reals α and β such that, for all $x \in V$,*

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 \tag{19.1}$$

PROOF:
⟨1⟩1. If $\| \|_1$ and $\| \|_2$ are equivalent then (19.1) holds.
⟨2⟩1. ASSUME: $\| \|_1$ and $\| \|_2$ are equivalent.
⟨2⟩2. There exists $\alpha > 0$ such that, for all $x \in V$, we have $\alpha\|x\|_1 \leq \|x\|_2$
⟨3⟩1. ASSUME: for a contradiction $\forall \alpha > 0. \exists x \in V. \alpha\|x\|_1 > \|x\|_2$
⟨3⟩2. For $n \in \mathbb{Z}^+$, choose $x_n \in V$ such that $1/n\|x_n\|_1 > \|x_n\|_2$
⟨3⟩3. For $n \in \mathbb{Z}^+$,
LET:
$$y_n = \frac{1}{\sqrt{n}} \frac{x_n}{\|x_n\|_2}$$
⟨3⟩4. $\|y_n\|_2 \to 0$ as $n \to \infty$
⟨3⟩5. For all $n \in \mathbb{Z}^+$, $\|y_n\|_1 > \sqrt{n}$
⟨3⟩6. $\|y_n\| \not\to 0$ as $n \to \infty$
⟨2⟩3. There exists $\beta > 0$ such that, for all $x \in V$, we have $\|x\|_2 \leq \beta\|x\|_1$
PROOF: Similar.
⟨1⟩2. If (19.1) holds then $\| \|_1$ and $\| \|_2$ are equivalent.
⟨2⟩1. ASSUME: (19.1) holds.
⟨2⟩2. LET: (x_n) be a sequence in V and $l \in V$
⟨2⟩3. If $x_n \to l$ under $\| \|_1$ then $x_n \to l$ under $\| \|_2$.
⟨3⟩1. ASSUME: $x_n \to l$ und $\| \|_1$.
⟨3⟩2. LET: $\epsilon > 0$
⟨3⟩3. PICK N such that $\forall n \geq N. \|x_n - l\| < \epsilon/\beta$
⟨3⟩4. LET: $n \geq N$
⟨3⟩5. $\|x_n - l\|_2 < \epsilon$
PROOF:
$$\begin{aligned}\|x_n - l\|_2 &\leq \beta\|x_n - l\|_1 \\ &< \epsilon\end{aligned}$$
⟨2⟩4. If $x_n \to l$ under $\| \|_2$ then $x_n \to l$ under $\| \|_1$.
PROOF: Similar.
□

Proposition 19.4.19. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a normed space over K. If $x_1, \ldots, x_n \in V$ are linearly independent, then there exists $c > 0$ such that, for all $\alpha_1, \ldots, \alpha_n \in K$,*

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|) \ .$$

PROOF:
$\langle 1 \rangle 1$. LET: $B = \{(\beta_1, \ldots, \beta_n) \in K^n \mid |\beta_1| + \cdots + |\beta_n| = 1\}$
$\langle 1 \rangle 2$. LET: $f : B \to \mathbb{R}$ be the function

$$f(\beta_1, \ldots, \beta_n) = \|\beta_1 x_1 + \cdots + \beta_n x_n\| \ .$$

$\langle 1 \rangle 3$. LET: c be the minimum value in $f(B)$
PROOF: f is continuous and B is compact.
$\langle 1 \rangle 4$. $c > 0$
PROOF: We never have $f(\beta_1, \ldots, \beta_n) = 0$ by linear independence.
$\langle 1 \rangle 5$. LET: $\alpha_1, \ldots, \alpha_n \in K$
$\langle 1 \rangle 6$. ASSUME: w.l.o.g. $\alpha_1, \ldots, \alpha_n$ are not all zero.
$\langle 1 \rangle 7$. For $i = 1, \ldots, n$,
LET: $\beta_i = \alpha_i / (|\alpha_1| + \cdots + |\alpha_n|)$
$\langle 1 \rangle 8$. $(\beta_1, \ldots, \beta_n) \in B$
$\langle 1 \rangle 9$. $f(\beta_1, \ldots, \beta_n) \geq c$
$\langle 1 \rangle 10$. $\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|)$
□

Theorem 19.4.20. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a finite dimensional vector space over K. Then any two norms on V are equivalent.*

PROOF:
$\langle 1 \rangle 1$. PICK a basis $\{e_1, \ldots, e_n\}$ for V
$\langle 1 \rangle 2$. LET: $\| \ \|_0 : V \to \mathbb{R}$ be the function

$$\|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 = |\alpha_1| + \cdots + |\alpha_n| \ .$$

$\langle 1 \rangle 3$. $\| \ \|_0$ is a norm.
$\langle 2 \rangle 1$. $\forall x \in V. \|x\|_0 = 0 \Rightarrow x = 0$
PROOF: If $|\alpha_1| + \cdots |\alpha_n| = 0$ then $\alpha_1 = \cdots = \alpha_n = 0$.
$\langle 2 \rangle 2$. $\forall \lambda \in K. \forall x \in V. \|\lambda x\| = |\lambda| \|x\|$
PROOF:

$$\begin{aligned} \|\lambda(\alpha_1 e_1 + \cdots + \alpha_n e_n)\|_0 &= \|\lambda \alpha_1 e_1 + \cdots + \lambda \alpha_n e_n\|_0 \\ &= |\lambda \alpha_1| + \cdots + |\lambda \alpha_n| \\ &= |\lambda|(|\alpha_1| + \cdots + |\alpha_n|) \\ &= |\lambda| \|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \end{aligned}$$

$\langle 2 \rangle 3$. The triangle inequality holds.
PROOF:

$$\begin{aligned} \|(\alpha_1 e_1 + \cdots + \alpha_n e_n) + (\beta_1 e_1 + \cdots + \beta_n e_n)\| &= \|(\alpha_1 + \beta_1) e_1 + \cdots + (\alpha_n + \beta_n) e_n\| \\ &= |\alpha_1 + \beta_1| + \cdots + |\alpha_n + \beta_n| \\ &\leq (|\alpha_1| + \cdots + |\alpha_n|) + (|\beta_1| + \cdots + |\beta_n|) \\ &= \|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 + \|\beta_1 e_1 + \cdots + \beta_n e_n\|_0 \end{aligned}$$

⟨1⟩4. LET: $\| \ \|$ be any norm on V.
PROVE: $\| \ \|$ is equivalent to $\| \ \|_0$.
⟨1⟩5. For all $\alpha_1, \ldots, \alpha_n \in K$,
$$\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \leq \max(\|e_1\|, \ldots, \|e_n\|)(|\alpha_1| + \cdots |\alpha_n|)$$
⟨2⟩1. LET: $\alpha_1, \ldots, \alpha_n \in K$
⟨2⟩2. $\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \leq \max(\|e_1\|, \ldots, \|e_n\|)(|\alpha_1| + \cdots |\alpha_n|)$
PROOF:
$$\begin{aligned}\|\alpha_1 e_1 + \cdots + \alpha_n e_n\| &\leq |\alpha_1|\|e_1\| + \cdots + |\alpha_n|\|e_n\| \\ &\leq (|\alpha_1| + \cdots + |\alpha_n|)\max(\|e_1\|, \ldots, \|e_n\|)\end{aligned}$$
□
⟨1⟩6. LET: $\beta = \max(\|e_1\|, \ldots, \|e_n\|)$
⟨1⟩7. For all $x \in V$,
$$\|x\| \leq \beta\|x\|_0 \ .$$
⟨1⟩8. There exists $\alpha > 0$ such that, for all $x \in V$,
$$\alpha\|x\|_0 \leq \|x\| \ .$$
PROOF: Proposition 19.4.19.
□

Definition 19.4.21 (Open Ball)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. Let $x \in V$ and let r be a positive real number. The *open ball* with *centre* x and *radius* r is
$$B(x, r) := \{y \in V \mid \|x - y\| < r\} \ .$$

Definition 19.4.22 (Closed Ball)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. Let $x \in V$ and let r be a positive real number. The *closed ball* with *centre* x and *radius* r is
$$\overline{B(x, r)} := \{y \in V \mid \|x - y\| \leq r\} \ .$$

Definition 19.4.23 (Sphere)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over K. Let $x \in V$ and let r be a positive real number. The *sphere* with *centre* x and *radius* r is
$$S(x, r) := \{y \in V \mid \|x - y\| = r\} \ .$$

Theorem 19.4.24. *Let V be a normed space. Then $\{U \in \mathcal{P}V \mid \forall x \in U.\exists\epsilon > 0.B(x, \epsilon) \subseteq U\}$ is a topology on V.*

We identify a normed space V with this topological space.

Proposition 19.4.25. *Every open ball is open.*

Proposition 19.4.26. *Every closed ball is closed.*

Proposition 19.4.27. *Every sphere is closed.*

Proposition 19.4.28. *The union of two closed sets is closed.*

Proposition 19.4.29. *The intersection of a nonempty set of closed sets is closed.*

Proposition 19.4.30. *In a normed space V, both $\emptyset$ and V are closed.*

Proposition 19.4.31. *Let V be a normed space and $C \subseteq V$. Then C is closed iff, for every sequence (x_n) in C and $l \in V$, if $x_n \to l$ then $l \in C$.*

PROOF:
$\langle 1 \rangle 1$. If C is closed then, for every sequence (x_n) in C and $l \in V$, if $x_n \to l$ then $l \in C$.
  $\langle 2 \rangle 1$. ASSUME: C is closed.
  $\langle 2 \rangle 2$. LET: (x_n) be a sequence in C.
  $\langle 2 \rangle 3$. LET: $l \in V$
  $\langle 2 \rangle 4$. ASSUME: $x_n \to l$ as $n \to \infty$
  $\langle 2 \rangle 5$. ASSUME: for a contradiction $l \notin C$
  $\langle 2 \rangle 6$. PICK $\epsilon > 0$ such that $B(l, \epsilon) \subseteq V - C$
  $\langle 2 \rangle 7$. PICK N such that $\forall n \geq N. \|x_n - l\| < \epsilon$
  $\langle 2 \rangle 8$. $x_N \in C$
  $\langle 2 \rangle 9$. $\|x_N - l\| < \epsilon$
  $\langle 2 \rangle 10$. Q.E.D.
    PROOF: This is a contradiction.
$\langle 1 \rangle 2$. If, for every sequence (x_n) in C and $l \in V$, if $x_n \to l$ then $l \in C$, then C is closed.
  $\langle 2 \rangle 1$. ASSUME: for every sequence (x_n) in C and $l \in V$, if $x_n \to l$ then $l \in C$.
  $\langle 2 \rangle 2$. LET: $x \in V - C$
  $\langle 2 \rangle 3$. ASSUME: for a contradiction there is no $\epsilon > 0$ such that $B(x, \epsilon) \subseteq V - C$
  $\langle 2 \rangle 4$. For $n \in \mathbb{Z}^+$, PICK $x_n \in B(x, \epsilon) \cap C$
  $\langle 2 \rangle 5$. $x_n \to x$ as $n \to \infty$
  $\langle 2 \rangle 6$. Q.E.D.
    PROOF: This is a contradiction.
$\square$

Definition 19.4.32 (Closure)**.** Let V be a normed space and $A \subseteq V$. The *closure* of A, $\mathrm{cl}\, A$, is the intersection of all the closed sets that include A.

Theorem 19.4.33. *Let V be a normed space and $S \subseteq V$. Then the closure of S is the set of all limits of convergent sequences in S.*

PROOF:
$\langle 1 \rangle 1$. For all $l \in \mathrm{cl}\, S$, there exists a sequence (x_n) that converges to l.
  $\langle 2 \rangle 1$. LET: $l \in \mathrm{cl}\, S$
  $\langle 2 \rangle 2$. For $n \in \mathbb{N}$, PICK $x_n \in B(l, 1/(n+1)) \cap S$
    $\langle 3 \rangle 1$. ASSUME: for a contradiction $B(l, 1/(n+1))$ does not intersect S.
    $\langle 3 \rangle 2$. $V - B(l, 1/(n+1))$ is a closed set that includes S.
    $\langle 3 \rangle 3$. $\mathrm{cl}\, S \subseteq V - B(l, 1/(n+1))$
    $\langle 3 \rangle 4$. $l \notin B(l, 1/(n+1))$
    $\langle 3 \rangle 5$. Q.E.D.
      PROOF: This is a contradiction.
  $\langle 2 \rangle 3$. $x_n \to l$
$\langle 1 \rangle 2$. For every sequence (x_n) in S, if $x_n \to l$ then $l \in \mathrm{cl}\, S$.

Proof: Proposition 19.4.31.
□

Definition 19.4.34 (Dense)**.** Let V be a normed space and $S \subseteq V$. Then S is *dense* iff $\mathrm{cl}\, S = V$.

Proposition 19.4.35. *In $\mathcal{C}([a,b])$, the set of polynomials is dense.*

Proof: By the Weierstrass Theorem. □

Proposition 19.4.36. *For any real $p \geq 1$, the set of all sequences with only finitely many non-zero terms is dense in l^p.*

Proof:
$\langle 1\rangle 1.$ Let: $p \geq 1$
$\langle 1\rangle 2.$ Let: $(z_n) \in l^p$
$\langle 1\rangle 3.$ Let: $\epsilon > 0$
$\langle 1\rangle 4.$ Pick N such that $\forall n \geq N . |z_n| < \epsilon/2$
$\langle 1\rangle 5.$ Let: (y_n) be the sequence with $y_n = z_n$ for $n < N$, and $y_n = 0$ for $n \geq N$
$\langle 1\rangle 6.$ $\|(z_n) - (y_n)\| \leq \epsilon/2$
$\langle 1\rangle 7.$ $\|(z_n) - (y_n)\| < \epsilon$
□

Theorem 19.4.37. *Let V be a normed space. Let $S \subseteq V$. Then the following are equivalent:*

1. *S is dense.*
2. *For all $x \in V$, there exists a sequence (x_n) in S such that $x_n \to x$.*
3. *Every nonempty open subset of V intersects S.*

Proof:
$\langle 1\rangle 1.$ $1 \Leftrightarrow 2$
Proof: Theorem 19.4.33.
$\langle 1\rangle 2.$ $1 \Leftrightarrow 3$
Proof:

$$\begin{aligned} S \text{ is dense} &\Leftrightarrow \text{the only closed set that includes } S \text{ is } V \\ &\Leftrightarrow \text{the only open set that does not intersect } S \text{ is empty} \end{aligned}$$

□

Definition 19.4.38 (Compact)**.** Let V be a normed space and $S \subseteq V$. Then S is *compact* if and only if every sequence in S has a subsequence that converges to a limit in S.

Proposition 19.4.39. *Every compact set is closed.*

Proof:
$\langle 1\rangle 1.$ Let: V be a normed space.
$\langle 1\rangle 2.$ Let: $C \subseteq V$ be compact.

$\langle 1\rangle 3.$ LET: (x_n) be a sequence in C that converges to $l \in V$.
$\langle 1\rangle 4.$ PICK a subsequence (y_n) of (x_n) that converges to $m \in C$.
$\langle 1\rangle 5.$ $l = m$
$\langle 1\rangle 6.$ $l \in C$
$\langle 1\rangle 7.$ Q.E.D.
PROOF: Proposition 19.4.31.
□

Definition 19.4.40 (Bounded)**.** Let V be a normed space and $S \subseteq V$. Then S is *bounded* iff there exists $r > 0$ such that $S \subseteq B(0, r)$.

Proposition 19.4.41. *In $\mathbb{R}^n$ and $\mathbb{C}^n$, the compact sets are the closed bounded sets.*

PROOF:
$\langle 1\rangle 1.$ LET: $C \subseteq K^n$
$\langle 1\rangle 2.$ If C is compact then C is closed.
PROOF: Proposition 19.4.39.
$\langle 1\rangle 3.$ If C is compact then C is bounded.
$\langle 2\rangle 1.$ ASSUME: C is compact.
$\langle 2\rangle 2.$ ASSUME: for a contradiction C is not bounded.
$\langle 2\rangle 3.$ For $n \in \mathbb{N}$, PICK $x_n \in C$ with $\|x_n\| > n + 1$.
$\langle 2\rangle 4.$ PICK a convergent subsequence (x_{n_r}) that converges to $l \in C$
$\langle 2\rangle 5.$ $\|x_{n_r}\| \to \|l\|$
$\langle 2\rangle 6.$ $\|x_{n_r}\| \to +\infty$
$\langle 2\rangle 7.$ Q.E.D.
PROOF: This is a contradiction.
$\langle 1\rangle 4.$ If C is closed and bounded then C is compact.
PROOF: By the Bolzano-Weierstrass Theorem.
□

Proposition 19.4.42. *Let V be a normed space and $S \subseteq V$. Then S is bounded if and only if, for every sequence (x_n) in S and every sequence (λ_n) in K, if $\lambda_n \to 0$ then $\|\lambda_n x_n\| \to 0$.*

PROOF:
$\langle 1\rangle 1.$ If S is bounded then, for every sequence (x_n) in S and every sequence (λ_n) in K, if $\lambda_n \to 0$ then $\|\lambda_n x_n\| \to 0$.
$\langle 2\rangle 1.$ ASSUME: S is bounded.
$\langle 2\rangle 2.$ PICK $r > 0$ such that $S \subseteq B(0, r)$.
$\langle 2\rangle 3.$ LET: (x_n) be a sequence in S.
$\langle 2\rangle 4.$ LET: (λ_n) be a sequence in K.
$\langle 2\rangle 5.$ ASSUME: $\lambda_n \to 0$
$\langle 2\rangle 6.$ LET: $\epsilon > 0$
$\langle 2\rangle 7.$ PICK N such that $\forall n \geq N. |\lambda_n| < \epsilon / r$
$\langle 2\rangle 8.$ $\forall n \geq N. \|\lambda_n x_n\| < \epsilon$
$\langle 1\rangle 2.$ If S is unbounded then there exists a sequence (x_n) in S and (λ_n) in K such that $\lambda_n \to 0$ and $\|\lambda_n x_n\| \not\to 0$.

⟨2⟩1. S is unbounded.
⟨2⟩2. For $n \in \mathbb{N}$, Pick $x_n \in S$ such that $\|x_n\| > n$.
⟨2⟩3. For $n \in \mathbb{N}$,
Let: $\lambda_n = 1/n$ if $n > 0$, 1 if $n = 0$
⟨2⟩4. $\lambda_n \to 0$
⟨2⟩5. $\|\lambda_n x_n\| > 1$ for all $n > 1$
□

Proposition 19.4.43. *In $\mathcal{C}([0,1])$, the unit ball $\overline{B(0,1)}$ is closed and bounded but not compact.*

Proof:
⟨1⟩1. $\overline{B(0,1)}$ is closed.
Proof: Proposition 19.4.26.
⟨1⟩2. $\overline{B(0,1)}$ is bounded.
Proof: $\overline{B(0,1)} \subseteq B(0,2)$.
⟨1⟩3. $\overline{B(0,1)}$ is not compact.
⟨2⟩1. For $n \in \mathbb{N}$,
Let: $x_n : [0,1] \to \mathbb{R}$ be the function $x_n(t) = t^n$.
⟨2⟩2. For $n \in \mathbb{N}$, we have $x_n \in \overline{B(0,1)}$.
⟨2⟩3. No subsequence of (x_n) converges.
□

Theorem 19.4.44 (Riesz's Lemma)**.** *Let X be a closed proper subspace of a normed space V. For every $\epsilon \in (0,1)$, there exists $x_\epsilon \in V$ such that $\|x_\epsilon\| = 1$ and $\forall x \in X. \|x_\epsilon - x\| \geq \epsilon$.*

Proof:
⟨1⟩1. Pick $z \in E - X$
⟨1⟩2. Let: $d = \inf_{x \in X} \|z - x\|$
⟨1⟩3. $d > 0$
⟨2⟩1. Pick- $\epsilon > 0$ such that $B(z, \epsilon) \subseteq E - X$
⟨2⟩2. $d \geq \epsilon$
⟨1⟩4. For all $\epsilon \in (0,1)$, choose $y_\epsilon \in X$ such that
$$d \leq \|z - y_\epsilon\| < d/\epsilon \ .$$
⟨1⟩5. For $\epsilon \in (0,1)$,
Let:
$$x_\epsilon = \frac{z - y_\epsilon}{\|z - y_\epsilon\|} \ .$$
⟨1⟩6. For all $x \in X$ we have $\|x_\epsilon - x\| > \epsilon$
Proof:
$$\begin{aligned}
\|x_\epsilon - x\| &= \|\frac{z - y_\epsilon}{\|z - y_\epsilon\|} - x\| \\
&= \frac{1}{\|z - y_\epsilon\|}\|z - y_\epsilon - \|z - y_\epsilon\|x\| \qquad (y_\epsilon + \|z - y_\epsilon\|x \in X) \\
&\geq \frac{1}{\|z - y_\epsilon\|}d \\
&> \epsilon
\end{aligned}$$

□

Theorem 19.4.45. *Let V be a normed space. Then V is finite dimensional if and only if $\overline{B(0,1)}$ is compact.*

PROOF:
⟨1⟩1. If V is finite dimensional then $\overline{B(0,1)}$ is compact.
  ⟨2⟩1. ASSUME: V is finite dimensional.
  ⟨2⟩2. PICK a basis $\{e_1, \ldots, e_n\}$.
  ⟨2⟩3. ASSUME: w.l.o.g. $\forall \alpha_1, \ldots, \alpha_n \in K. \|\alpha_1 e_1 + \cdots + \alpha_n e_n\| = |\alpha_1| + \cdots + |\alpha_n|$
  ⟨2⟩4. LET: (x_m) be a sequence in $\overline{B(0,1)}$
  ⟨2⟩5. For $m \in \mathbb{N}$,
    LET: $x_m = \alpha_{m1} e_1 + \cdots + \alpha_{mn} e_n$
  ⟨2⟩6. For $m \in \mathbb{N}$ and $i = 1, \ldots, n$, we have $|\alpha_{mi}| \leq 1$
  ⟨2⟩7. For $i = 1, \ldots, n$, PICK a convergent subsequence $(\alpha_{m_r i})$ of (α_{mi}) in $\mathbb{C}$ that converges to l_i
   PROOF: Since $\overline{B(0,1)}$ is compact in K.
  ⟨2⟩8. x_m converges to $l_1 e_1 + \cdots + l_n e_n$
   PROOF:

$$\begin{aligned} \|x_m - l_1 e_1 - \cdots - l_n e_n\| &= \|(\alpha_{m1} - l_1) e_1 + \cdots + (\alpha_{mn} - l_n) e_n\| \\ &= |\alpha_{m1} - l_1| + \cdots + |\alpha_{mn} - l_n| \\ &\to 0 \qquad \text{as } m \to \infty \end{aligned}$$

⟨1⟩2. If V is infinite dimensional then $\overline{B(0,1)}$ is not compact.
  ⟨2⟩1. ASSUME: V is infinite dimensional.
  ⟨2⟩2. Choose a sequence (x_n) such that $\|x_n\| = 1$ and $\|x_m - x_n\| \geq 1/2$ for all $m \neq n$.
   ⟨3⟩1. ASSUME: as induction hypothesis there exists a sequence $(x_0, x_1, \ldots, x_n)$ such that $\|x_i\| = 1$ and $\|x_i - x_j\| \geq 1/2$ for $i \neq j$
   ⟨3⟩2. PICK x_{n+1} such that $\|x_{n+1}\| = 1$ and $\|x_{n_1} - x\| \geq 1/2$ for $x \in \{x_1, \ldots, x_n\}$.
  ⟨2⟩3. ASSUME: for a contradiction (x_{n_r}) is a subsequence that converges to l
  ⟨2⟩4. For all r we have $1/2 \leq \|x_{n_r} - l\| + \|x_{n_{r+1}} - l\|$
   PROOF:

$$\begin{aligned} 1/2 &\leq \|x_{n_r} - x_{n_{r+1}}\| && (\langle 2\rangle 2) \\ &\leq \|x_{n_r} - l\| + \|x_{n_{r+1}} - l\| && \text{(Triangle Inequality)} \end{aligned}$$

  ⟨2⟩5. Q.E.D.
   PROOF: This is a contradiction.

□

19.5 Banach Spaces

Definition 19.5.1 (Cauchy Sequence)**.** A sequence (x_n) in a normed space is a *Cauchy sequence* iff, for every $\epsilon > 0$, there exists N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$.

Theorem 19.5.2. *Let V be a normed space. Let (x_n) be a sequence in V. Then the following are equivalent.*

1. *(x_n) is Cauchy.*
2. *For every pair of strictly increasing sequences of natural numbers (p_n) and (q_n), we have $\|x_{p_n} - x_{q_n}\| \to 0$ as $n \to \infty$.*
3. *For every strictly increasing sequence of natural numbers (p_n), we have $\|x_{p_{n+1}} - x_{p_n}\| \to 0$ as $n \to \infty$.*

PROOF:
$\langle 1\rangle 1.$ $1 \Rightarrow 2$
  $\langle 2\rangle 1.$ ASSUME: (x_n) is Cauchy.
  $\langle 2\rangle 2.$ LET: (p_n) and (q_n) be a pair of increasing sequences of natural numbers.
  $\langle 2\rangle 3.$ LET: $\epsilon > 0$
  $\langle 2\rangle 4.$ PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$
  $\langle 2\rangle 5.$ $\forall n \geq N. p_n, q_n \geq N$
  $\langle 2\rangle 6.$ $\forall n \geq N. \|x_{p_n} - x_{q_n}\| < \epsilon$
$\langle 1\rangle 2.$ $2 \Rightarrow 3$
  PROOF: Trivial.
$\langle 1\rangle 3.$ $2 \Rightarrow 1$
  $\langle 2\rangle 1.$ ASSUME: (x_n) is not Cauchy.
  $\langle 2\rangle 2.$ PICK $\epsilon > 0$ such that, for all N, there exist $m, n \geq N$ such that $\|x_m - x_n\| \geq \epsilon$
  $\langle 2\rangle 3.$ PICK strictly increasing sequences of natural numbers (p_n) and (q_n) such that, for all n, $\|x_{p_n} - x_{q_n}\| \geq \epsilon$
    $\langle 3\rangle 1.$ ASSUME: as induction hypothesis we have chosen $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ strictly increasing such that $\forall i. \|x_{p_i} - x_{q_i}\| \geq \epsilon$
    $\langle 3\rangle 2.$ PICK $p_{n+1}, q_{n+1} \geq \max(p_n, q_n)$ such that $\|x_{p_{n+1}} - x_{q_{n+1}}\| \geq \epsilon$
  $\langle 2\rangle 4.$ 2 is false.
$\langle 1\rangle 4.$ $3 \Rightarrow 2$
  $\langle 2\rangle 1.$ ASSUME: (p_n) and (q_n) are strictly increasonig sequences such that $\|x_{p_n} - x_{q_n}\| \not\to 0$ as $n \to \infty$.
  $\langle 2\rangle 2.$ PICK $\epsilon > 0$ such that, for all N, there exists $n \geq N$ such that $\|x_{p_n} - x_{q_n}\| \geq \epsilon$
  $\langle 2\rangle 3.$ Choose a strictly increasing sequence (r_n) such that, for all n, we have $\|x_{r_{2n}} - x_{r_{2n+1}}\| \geq \epsilon$
    $\langle 3\rangle 1.$ ASSUME: as induction hypothesis we have chosen $(r_0, r_1, \ldots, r_{2n+1})$ such that, for $i = 0, 1, \ldots, n$, we have $\|x_{r_{2i}} - x_{r_{2i+1}}\| \geq \epsilon$
    $\langle 3\rangle 2.$ PICK $i, j \geq r_{2n+1}$ such that $\|x_i - x_j\| \geq \epsilon$
    $\langle 3\rangle 3.$ Set $r_{2n+2} = \min(i, j)$ and $r_{2n+3} = \max(i, j)$
  $\langle 2\rangle 4.$ 3 is false.
$\square$

Proposition 19.5.3. *Every convergent sequence is a Cauchy sequence.*

PROOF:

⟨1⟩1. LET: (x_n) be a convergent sequence in a normed space V with limit l.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK N such that $\forall n \geq N. \|x_n - l\| < \epsilon/2$
⟨1⟩4. $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$
□

Proposition 19.5.4. *Let $\mathcal{P}([0,1])$ be the space of polynomials on $[0,1]$ under the norm of uniform convergence. For $n \in \mathbb{N}$, let $P_n = 1+x+x^2/2!+\cdots+x^n/n!$. Then (P_n) is Cauchy but does not converge, since e^x is not a polynomial.*

PROOF: Easy. □

Proposition 19.5.5. *If (x_n) is a Cauchy sequence in a normed space V, then $(\|x_n\|)$ converges in $\mathbb{R}$.*

PROOF:
⟨1⟩1. $(\|x_n\|)$ is Cauchy.
⟨2⟩1. LET: $\epsilon > 0$
⟨2⟩2. PICK N such that $\forall m, n \geq N. \|x_m - x_n\| < \epsilon$
⟨2⟩3. $\forall m, n \geq N. |\|x_m\| - \|x_n\|| < \epsilon$
PROOF: Proposition 19.4.5.
⟨1⟩2. Q.E.D.
PROOF: Since $\mathbb{R}$ is complete.
□

Corollary 19.5.5.1. *Every Cauchy sequence is bounded.*

Definition 19.5.6 (Banach space)**.** A normed space is *complete* or a *Banach space* iff every Cauchy sequence converges.

Proposition 19.5.7. *For all $p \geq 1$, the space l^p in complete.*

PROOF:
⟨1⟩1. LET: (a_n) be a Cauchy sequence in l^p.
⟨1⟩2. For $n \in \mathbb{N}$,
LET: $a_n = (\alpha_{n0}, \alpha_{n1}, \ldots)$
⟨1⟩3. For all $\epsilon > 0$, there exists N such that $\forall m, n \geq N$

$$\sum_{k=0}^{\infty} |\alpha_{mk} - \alpha_{nk}|^p < \epsilon \ .$$

PROOF: ⟨1⟩1
⟨1⟩4. For all $k \in \mathbb{N}$ and $\epsilon > 0$, there exists N such that $\forall m, n \geq N$

$$|\alpha_{mk} - \alpha_{nk}| < \epsilon \ .$$

PROOF: ⟨1⟩3
⟨1⟩5. For all $k \in \mathbb{N}$, the sequence $(\alpha_{nk})_n$ converges in $\mathbb{C}$.
PROOF: $\mathbb{C}$ is complete.
⟨1⟩6. For $k \in \mathbb{N}$,
LET:

$$\alpha_k = \lim_{n \to \infty} \alpha_{nk} \ .$$

$\langle 1\rangle 7$. LET: $a = (\alpha_k)_k$
$\langle 1\rangle 8$. For all $\epsilon > 0$, there exists N such that $\forall n \geq N$

$$\sum_{k=0}^{\infty} |\alpha_k - \alpha_{nk}|^p < \epsilon \ .$$

PROOF: Take the limit $m \to \infty$ in $\langle 1\rangle 4$.
$\langle 1\rangle 9$. $a \in l^p$
$\langle 2\rangle 1$. PICK N such that $\forall n \geq N. \sum_{k=0}^{\infty} |\alpha_k - \alpha_{nk}|^p < 1$
$\langle 2\rangle 2$. $a - a_N \in l^p$
$\langle 2\rangle 3$. Q.E.D.
PROOF: Since l^p is closed under $+$.
$\langle 1\rangle 10$. $a_n \to a$ as $n \to \infty$.
PROOF: Immediate from $\langle 1\rangle 8$.
□

Proposition 19.5.8. *For any real number a, b with $a < b$, the space $\mathcal{C}([a, b])$ is complete.*

PROOF:
$\langle 1\rangle 1$. LET: (f_n) be a Cauchy sequence in $\mathcal{C}([a, b])$
$\langle 1\rangle 2$. For all $\epsilon > 0$, there exists N such that, for all $m, n \geq N$ and $x \in [a, b]$,

$$|f_m(x) - f_n(x)| < \epsilon \ .$$

$\langle 1\rangle 3$. For all $x \in [a, b]$, $(f_n(x))_n$ is Cauchy.
$\langle 1\rangle 4$. LET: $f : [a, b] \to \mathbb{C}$ be the function

$$f(x) = \lim_{n\to\infty} f_n(x) \ .$$

$\langle 1\rangle 5$. For all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in [a, b]$,

$$|f_n(x) - f(x)| < \epsilon \ .$$

PROOF: Take the limit $m \to \infty$ in $\langle 1\rangle 2$.
$\langle 1\rangle 6$. f is continuous.
$\langle 2\rangle 1$. LET: $x_0 \in [a, b]$
$\langle 2\rangle 2$. LET: $\epsilon > 0$
$\langle 2\rangle 3$. PICK N such that, for all $n \geq N$ and $y \in [a, b]$, we have

$$|f_n(y) - f(y)| < \epsilon/3 \ .$$

PROOF: $\langle 1\rangle 5$
$\langle 2\rangle 4$. PICK $\delta > 0$ such that, for all $y \in [a, b]$ with $|x_0 - y| < \delta$, we have $|f_N(x_0) - f_N(y)| < \epsilon/3$
$\langle 2\rangle 5$. For all $y \in [a, b]$, if $|x_0 - y| < \delta$ then $|f(x_0) - f(y)| < \epsilon$
PROOF:

$$\begin{aligned} &|f(x_0) - f(y)| \\ \leq &|f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(y)| + |f_N(y) - f(y)| && \text{(Triangle Inequality)} \\ < &\epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 2\rangle 3,\ \langle 2\rangle 4) \\ = &\epsilon \end{aligned}$$

$\langle 1\rangle 7$. $f_n \to f$ as $n \to \infty$.
PROOF: Immediate from $\langle 1\rangle 5$.
□

Definition 19.5.9 (Convergent Series)**.** Let (x_n) be a sequence in a normed space V. We say the series $\sum_{n=0}^{\infty} x_n$ is *convergent* iff the sequence $(\sum_{n=0}^{N} x_n)_N$ is convergent, and then we write $\sum_{n=0}^{\infty} x_n = l$ for $\lim_{N\to\infty} \sum_{n=0}^{N} x_n = l$.

Definition 19.5.10 (Absolutely Convergent Series)**.** Let (x_n) be a sequence in a normed space V. We say the series $\sum_{n=0}^{\infty} x_n$ is *absolutely convergent* iff the series $\sum_{n=0}^{\infty} \|x_n\|$ converges in $\mathbb{R}$.

Theorem 19.5.11. *A normed space is complete if and only if every absolutely convergent series is convergent.*

PROOF:
$\langle 1\rangle 1$. LET: V be a normed space.
$\langle 1\rangle 2$. If V is complete then every absolutely convergent series is convergent.
  $\langle 2\rangle 1$. ASSUME: V is complete.
  $\langle 2\rangle 2$. LET: (x_n) be a sequence in V.
  $\langle 2\rangle 3$. ASSUME: $\sum_{n=0}^{\infty} \|x_n\|$ converges.
    PROVE: $(\sum_{n=0}^{N} x_n)_N$ is Cauchy.
  $\langle 2\rangle 4$. LET: $\epsilon > 0$
  $\langle 2\rangle 5$. PICK N such that $\sum_{n=N}^{\infty} \|x_n\| < \epsilon$
  $\langle 2\rangle 6$. LET: $m > n \geq N$
  $\langle 2\rangle 7$. $\| \sum_{k=0}^{m} x_k - \sum_{k=0}^{n} x_k \| < \epsilon$
   PROOF:

$$\begin{aligned} \| \sum_{k=0}^{m} x_k - \sum_{k=0}^{n} x_k \| &= \| \sum_{k=n+1}^{m} x_k \| \\ &\leq \sum_{k=n+1}^{m} \|x_k\| \\ &\leq \sum_{k=n+1}^{\infty} \|x_k\| \\ &< \epsilon \end{aligned}$$

$\langle 1\rangle 3$. If every absolutely convergent series is convergent then V is complete.
  $\langle 2\rangle 1$. ASSUME: Every absolutely convergent series is convergent.
  $\langle 2\rangle 2$. LET: (x_n) be a Cauchy sequence in V.
  $\langle 2\rangle 3$. Choose an increasing sequence of natural numbers (p_k) such that, for all $m, n \geq p_k$, we have

$$\|x_m - x_n\| < 2^{-k} \quad .$$

  $\langle 2\rangle 4$. $\sum_{k=0}^{\infty} \|x_{p_{k+1}} - x_{p_k}\|$ is absolutely convergent.
  $\langle 2\rangle 5$. $\sum_{k=0}^{\infty} \|x_{p_{k+1}} - x_{p_k}\|$ is convergent.
  $\langle 2\rangle 6$. (x_{p_k}) converges.
  $\langle 2\rangle 7$. LET: $l = \lim_{k\to\infty} x_{p_k}$
  $\langle 2\rangle 8$. $x_n \to l$ as $n \to \infty$.
   PROOF:

$$\begin{aligned} \|x_n - l\| &\leq \|x_n - x_{p_n}\| + \|x_{p_n} - l\| \\ &\to 0 \qquad \text{as } n \to \infty \text{(Theorem 19.5.2)} \end{aligned}$$

□

Proposition 19.5.12. *A closed subspace of a Banach space is a Banach space.*

PROOF:
⟨1⟩1. LET: V be a Banach space.
⟨1⟩2. LET: U be a closed subspace of V.
⟨1⟩3. LET: (x_n) be a Cauchy sequence in U.
⟨1⟩4. (x_n) is a Cauchy sequence in V.
⟨1⟩5. LET: l be the limit of (x_n) in V.
⟨1⟩6. $l \in U$
PROOF: Proposition 19.4.31.
□

Definition 19.5.13 (Completion)**.** Let V be a normed space. A *completion* of V consists of a Banach space W and a function $\phi : V \to W$ such that:

- ϕ is injective.
- ϕ is a linear transformation.
- ϕ preserves the norm.
- $\phi(V)$ is dense in W.

Definition 19.5.14 (Equivalent Cauchy Sequences)**.** Let V be a normed space. Let (x_n) and (y_n) be Cauchy sequences in V. Then (x_n) and (y_n) are *equivalent*, $(x_n) \sim (y_n)$, iff $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Proposition 19.5.15. *Equivalence is an equivalence relation on the set of Cauchy sequences.*

Proposition 19.5.16. *If $(x_n) \sim (y_n)$ then $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\|$.*

Theorem 19.5.17. *Let V be a normed space. Let W be the quotient set of all Cauchy sequences modulo $\sim$. Define $+$, $\cdot$ and $\| \; \|$ on W by*

$$\begin{aligned}
[(x_n)] + [(y_n)] &= [(x_n + y_n)] \\
\lambda[(x_n)] &= [(\lambda x_n)] \\
\|[(x_n)]\| &= \lim_{n\to\infty} \|x_n\|
\end{aligned}$$

Define $\phi : V \to W$ by $\phi(x)$ is the constant sequence (x). Then $\phi : V \to W$ is the completion of V.

PROOF:
⟨1⟩1. $+$, $\cdot$ and $\| \; \|$ are well defined.
⟨1⟩2. W is a normed space.
⟨1⟩3. $\phi(V)$ is dense in W.
PROOF: For any $[(x_n)] \in W$ we have $[(x_n)] = \lim_{n\to\infty} \phi(x_n)$.
⟨1⟩4. W is complete.

⟨2⟩1. LET: (X_n) be a Cauchy sequence in W.
⟨2⟩2. For all $n \in \mathbb{N}$, PICK $x_n \in V$ such that $\|\phi(x_n) - X_n\| < 1/n$
PROOF: ⟨1⟩3
⟨2⟩3. For all m, n we have $\|x_n - x_m\| \leq \|X_n - X_m\| + 1/n + 1/m$
PROOF:

$$\begin{aligned}
\|x_n - x_m\| &= \|\phi(x_n) - \phi(x_m)\| \\
&\leq \|\phi(x_n) - X_n\| + \|X_n - X_m\| + \|\phi(x_m) - X_m\| \\
&\leq \|X_n - X_m\| + 1/n + 1/m
\end{aligned}$$

⟨2⟩4. (x_n) is a Cauchy sequence in V.
⟨2⟩5. LET: $X = [(x_n)]$
PROVE: $X_n \to X$ as $n \to \infty$
⟨2⟩6. $\|X_n - X\| \to 0$ as $n \to \infty$
PROOF:

$$\begin{aligned}
\|X_n - X\| &\leq \|X_n - \phi(x_n)\| + \|\phi(x_n) - X\| \\
&< \|\phi(x_n) - X\| + 1/n \\
&\to 0
\end{aligned}$$

⟨1⟩5. ϕ is injective.
⟨1⟩6. ϕ is a linear transformation.
⟨1⟩7. ϕ preserves the norm.
□