

Mathematics

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Contents

1	Sets and Functions	5
1.1	Primitive Terms	5
1.2	Definitions Used in the Axioms	5
1.3	The Axioms	6
1.4	Isomorphisms	7
1.5	Subsets	7
1.6	Pullbacks	7
1.7	Functions	7
1.8	The Internal Logic	8
1.9	Functions	8

Chapter 1

Sets and Functions

1.1 Primitive Terms

Let there be *sets*.

Given sets A and B , let there be *functions* from A to B . We write $f : A \rightarrow B$ iff f is a function from A to B , and call A the *domain* of f and B the *codomain*.

Given functions $f : A \rightarrow B$ and $g : B \rightarrow C$, let there be a function $g \circ f : A \rightarrow C$, the *composite* of f and g .

For any set A , let there be a function $\text{id}_A : A \rightarrow A$, the *identity* function on A .

Let there be a set 1 , the *terminal* set.

For any sets A and B , let there be a set $A \times B$, the *product* of A and B , and functions $\pi_1 : A \times B \rightarrow A$, $\pi_2 : A \times B \rightarrow B$, the *projections*.

Given functions $f : A \rightarrow B$ and $g : A \rightarrow C$, let there be a function $\langle f, g \rangle : A \rightarrow B, C$.

1.2 Definitions Used in the Axioms

Definition 1.1 (Element). For any set A , an *element* of A is a function $1 \rightarrow A$. We write $a \in A$ for $a : 1 \rightarrow A$.

Given $f : A \rightarrow B$ and $a \in A$, we write $f(a)$ for $f \circ a : 1 \rightarrow B$.

Definition 1.2 (Injective). A function $f : A \rightarrow B$ is *injective* iff, for every set X and functions $x, y : X \rightarrow A$, if $fx = fy$ then $x = y$.

Definition 1.3 (Surjective). A function $f : A \rightarrow B$ is *surjective* iff, for every element $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Definition 1.4 (Retraction, Section). Let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of A , iff $r \circ s = \text{id}_B$.

Definition 1.5. Given functions $f : A \rightarrow B$ and $g : C \rightarrow D$, let $f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle$.

Definition 1.6 (Function Set). Let A and B be sets. A *function set* from A to B consists of a set B^A and function $\epsilon : B^A \times A \rightarrow B$ such that, for any set I and function $q : I \times A \rightarrow B$, there exists a unique function $\lambda q : I \rightarrow B^A$ such that $\epsilon \circ (\lambda q \times \text{id}_A) = q$.

Definition 1.7 (Pullback). Let $p : A \rightarrow B$, $q : A \rightarrow C$, $f : B \rightarrow D$ and $g : C \rightarrow D$. Then we say that A , p and q form the *pullback* of f and g if and only if:

- $fp = gq$
- For any set X and functions $x : X \rightarrow B$, $y : X \rightarrow C$ such that $fx = gy$, there exists a unique function $(x, y) : X \rightarrow A$ such that $p(x, y) = x$ and $q(x, y) = y$.

We also say p is the pullback of g along f , or q is the pullback of f along g .

In the case g is injective, we also say A and p form the *inverse image* of g under f .

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ q \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

1.3 The Axioms

Axiom 1.8 (Associativity). Given $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, we have

$$h(gf) = (hg)f .$$

Axiom 1.9 (Unit Laws). For any function $f : A \rightarrow B$, we have $\text{id}_B \circ f = f \circ \text{id}_A = f$.

Axiom 1.10 (Terminal Set). For any set X , there is exactly one function $X \rightarrow 1$.

Axiom 1.11 (Empty Set). There exists a set that has no elements.

Axiom 1.12 (Extensionality). Let A and B be sets and $f, g : A \rightarrow B$. If $\forall a \in A. f(a) = g(a)$ then $f = g$.

Axiom 1.13 (Products). Let $f : A \rightarrow B$ and $g : A \rightarrow C$. Then $\langle f, g \rangle$ is the unique function $A \rightarrow B \times C$ such that

$$\pi_1 \circ \langle f, g \rangle = f, \quad \pi_2 \circ \langle f, g \rangle = g .$$

Axiom 1.14 (Function Sets). Any two sets have a function set.

Axiom 1.15 (Inverse Images). *Given any function $f : X \rightarrow Y$ and element $y \in Y$, then there exists a pullback of f and y .*

Axiom 1.16 (Subset Classifier). *There exists a set 2 and element $\top \in 2$ such that, for any sets A and X and injective function $j : A \rightarrow X$, there exists a unique function $\chi : X \rightarrow 2$ such that j and the unique function $A \rightarrow 1$ form the pullback of \top and χ .*

Axiom 1.17 (Natural Numbers Set). *There exists a set \mathbb{N} , an element $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that, for any set A , element $a \in A$ and function $f : A \rightarrow A$, there exists a unique function $r : \mathbb{N} \rightarrow A$ such that $r(0) = a$ and $f \circ r = r \circ s$.*

Axiom 1.18 (Choice). *Every surjective function has a section.*

1.4 Isomorphisms

Definition 1.19 (Isomorphism). Let $f : A \rightarrow B$. Then f is an *isomorphism* or *bijection*, $f : A \cong B$, iff there exists a function $f^{-1} : B \rightarrow A$, the *inverse* of f , such that $f^{-1}f = \text{id}_A$ and $ff^{-1} = \text{id}_B$.

1.5 Subsets

Definition 1.20 (Subset). Let $i : U \rightarrow A$. Then we say that (U, i) is a *subset* of A iff i is injective.

Definition 1.21. Let (U, i) and (V, j) be subsets of A . Then we say (U, i) and (V, j) are *equal*, and write $(U, i) = (V, j)$, iff there exists an isomorphism $\phi : U \cong V$ such that $j\phi = i$.

1.6 Pullbacks

1.7 Functions

Proposition 1.22. *Let $f : A \rightarrow B$. Then f is injective if and only if, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.*

PROOF:

$\langle 1 \rangle 1$. If f is injective then, for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

PROOF: Immediate from the definition of injective.

$\langle 1 \rangle 2$. If $\forall x, y \in A. f(x) = f(y) \Rightarrow x = y$ then f is injective.

$\langle 2 \rangle 1$. ASSUME: $\forall x, y \in A. f(x) = f(y) \Rightarrow x = y$

$\langle 2 \rangle 2$. LET: X be a set and $s, t : X \rightarrow A$

$\langle 2 \rangle 3$. ASSUME: $fs = ft$

$\langle 2 \rangle 4$. $\forall x \in X. s(x) = t(x)$

$\langle 3 \rangle 1.$ LET: $x \in X$
 $\langle 3 \rangle 2.$ $f(s(x)) = f(t(x))$
 PROOF: $\langle 2 \rangle 3$
 $\langle 3 \rangle 3.$ $s(x) = t(x)$
 PROOF: $\langle 2 \rangle 1$
 $\langle 2 \rangle 5.$ $s = t$
 PROOF: Axiom of Extensionality
 \square

1.8 The Internal Logic

Proposition 1.23. *Let $i : U \rightarrowtail A$ be injective. Let $\chi : A \rightarrow 2$ be its characteristic function. Then, for all $a \in A$, we have $\chi(a) = \top$ if and only if there exists $u \in U$ such that $i(u) = a$.*

PROOF:

$\langle 1 \rangle 1.$ If $\chi(a) = \top$ then there exists $u \in U$ such that $i(u) = a$.
 PROOF: If $\chi \circ a = \top = \top \circ !_1$ then there exists a unique $u : 1 \rightarrow U$ such that $i \circ u = a$ and $!_U \circ u = !_1$.
 $\langle 1 \rangle 2.$ For all $u \in U$ we have $\chi(i(u)) = \top$.
 PROOF: Since $\chi \circ i = \top \circ !_U$.
 \square

Proposition 1.24. *Subsets of a set A are equal if and only if they have the same characteristic function.*

PROOF: Follows from the fact that pullbacks are unique up to isomorphism and the uniqueness of the characteristic function. \square

Proposition 1.25. *There are exactly two subsets of 1.*

PROOF:

$\langle 1 \rangle 1.$ PICK a set E with no elements.
 $\langle 1 \rangle 2.$ $!_E : E \rightarrow 1$ is injective.
 PROOF: Vacuously, $\forall x, y \in E. !_E(x) = !_E(y) \Rightarrow x = y$.
 $\langle 1 \rangle 3.$ $(E, !_E) \neq (1, \text{id}_1)$
 $\langle 1 \rangle 4.$ For any subset (U, i) of 1, either $(U, i) = (E, !_E)$ or $(U, i) = (1, \text{id}_1)$
 \square

1.9 Functions

Proposition 1.26. *Let $f : A \rightarrow B$, $g : B \rightarrow C$ and $a \in A$. Then*

$$(g \circ f)(a) = g(f(a)) .$$

PROOF: Immediate from the Axiom of Associativity. \square

Proposition 1.27. *For any set A , any function $1 \rightarrow A$ is injective.*

PROOF: Since there is only one function $X \rightarrow 1$ for any set X . \square

Proposition 1.28. *Let $f : A \rightarrow B$. Then the following are equivalent:*

1. f is surjective.
2. f is a retraction (i.e. f has a section).
3. For any set X and functions $x, y : B \rightarrow X$, if $xf = yf$ then $x = y$.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

PROOF: Immediate from the Axiom of Choice.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$ LET: $s : B \rightarrow A$ be a section of f .

$\langle 2 \rangle 2.$ LET: X be a set and $x, y : B \rightarrow X$ satisfy $xf = yf$.

$\langle 2 \rangle 3. x = y$

PROOF: $x = xfs = yfs = y$

$\langle 1 \rangle 3. 3 \Rightarrow 1$

$\langle 2 \rangle 1.$ ASSUME: 3

$\langle 2 \rangle 2.$ LET: $b \in B$

$\langle 2 \rangle 3.$ ASSUME: for a contradiction $\forall a \in A. f(a) \neq b$

$\langle 2 \rangle 4.$ LET: $\psi_1 : B \rightarrow 2$ be the characteristic function of b .

$\langle 2 \rangle 5.$ LET: $\psi_0 = \perp \circ !_B : B \rightarrow 2$

$\langle 2 \rangle 6. \forall x \in A. \psi_1(f(x)) = \psi_2(f(x))$

$\langle 3 \rangle 1.$ LET: $x \in A$

$\langle 3 \rangle 2. \psi_1(f(x)) \neq \top$

PROOF: Proposition 1.23, $\langle 2 \rangle 3$, $\langle 2 \rangle 4$.

$\langle 3 \rangle 3. \psi_1(f(x)) = \perp$

$\langle 3 \rangle 4. \psi_1(f(x)) = \psi_2(f(x))$

$\langle 2 \rangle 7. \psi_1 \circ f = \psi_2 \circ f$

$\langle 2 \rangle 8. \psi_1 = \psi_2$

$\langle 2 \rangle 9. \psi_1(a) \neq \psi_2(a)$

$\langle 2 \rangle 10.$ Q.E.D.

\square