

Summary of Halmos' Naive Set Theory

Robin Adams

August 20, 2023

Contents

1	Primitive Terms and Axioms	2
2	The Subset Relation	3
3	Comprehension Notation	4
4	Unordered Pairs	5

Chapter 1

Primitive Terms and Axioms

Let there be *sets*. We assume that everything is a set.

Let there be a binary relation of *membership*, \in . If $x \in A$ we say that x *belongs to* A , x is an *element* of A , or x is *contained* in A . If this does not hold we write $x \notin A$.

Axiom 1.1 (Axiom of Extensionality). *Two sets are equal if and only if they have the same elements.*

Axiom 1.2 (Axiom of Specification, Aussonderungsaxiom). *To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

Axiom 1.3. *A set exists.*

Chapter 2

The Subset Relation

Definition 2.1 (Subset). Let A and B be sets. We say that A is a *subset* of B , or B *includes* A , and write $A \subseteq B$ or $B \supseteq A$, iff every element of A is an element of B .

Theorem 2.2. *For any set A , we have $A \subseteq A$.*

PROOF: Every element of A is an element of A . \square

Theorem 2.3. *For any sets A , B and C , if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

PROOF: If every element of A is an element of B , and every element of B is an element of C , then every element of A is an element of C . \square

Theorem 2.4. *For any sets A and B , if $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

PROOF: If every element of A is an element of B , and every element of B is an element of A , then A and B have the same elements, and therefore are equal by the Axiom of Extensionality. \square

Definition 2.5 (Proper Subset). Let A and B be sets. We say that A is a *proper subset* of B , or B *properly includes* A , and write $A \subsetneq B$ or $B \supsetneq A$, iff $A \subseteq B$ and $A \neq B$.

Chapter 3

Comprehension Notation

Definition 3.1. Given a set A and a condition $S(x)$, we write $\{x \in A : S(x)\}$ for the set whose elements are exactly those elements x of A for which $S(x)$ holds.

PROOF: This exists by the Axiom of Specification and is unique by the Axiom of Extensionality. \square

Theorem 3.2. *There is no set that contains every set.*

PROOF:

$\langle 1 \rangle 1.$ LET: A be a set.

PROVE: There exists a set B such that $B \notin A$.

$\langle 1 \rangle 2.$ LET: $B = \{x \in A : x \notin x\}$

$\langle 1 \rangle 3.$ If $B \in A$ then we have $B \in B$ if and only if $B \notin B$.

$\langle 1 \rangle 4.$ $B \notin A$

\square

Chapter 4

Unordered Pairs

Theorem 4.1. *There exists a set with no elements.*

PROOF: Pick a set A by Axiom 1.3. Then the set $\{x \in A : x \neq x\}$ has no elements. \square

Definition 4.2 (Empty Set). The *empty set* \emptyset is the set with no elements.