

Encyclopaedia of Mathematics and Physics

Robin Adams

Contents

1	Set Theory	5
2	Relations	7
3	Order Theory	9
4	Field Theory	11
4.1	Ordered Fields	13
5	Real Analysis	15
5.1	Construction of the Real Numbers	15
5.2	Properties of the Real Numbers	21
5.2.1	Logarithms	27
5.2.2	Intervals	28
5.2.3	The Cantor Set	28
5.3	The Extended Real Number System	29
6	Complex Analysis	31
6.1	Algebraic Numbers	35
I	Linear Algebra	37
7	Vector Spaces	39
7.1	Convex Sets	39
8	Real Inner Product Spaces	41
8.1	Balls	42
9	Complex Inner Product Spaces	43
9.1	Hilbert Spaces	44
10	Lie Algebras	45
10.1	Lie Algebar Homomorphisms	46

II	Topology	47
11	Metric Spaces	49
11.1	Balls	49
11.2	Limit Points	50
11.3	Closed Sets	50
11.4	Interior Points	50
11.5	Open Sets	51
11.6	Perfect Sets	53
11.7	Bounded Sets	54
11.8	Dense Sets	54
11.9	Closure	54
11.10	Compact Sets	56
11.11	Connected Sets	61
11.12	Separable Spaces	63
11.13	Bases	64
11.14	Condensation Points	64
12	Convergence	67
12.1	Cauchy Sequences	72
12.2	Complete Metric Spaces	73
12.3	Divergent Sequences	74
12.4	Infinite Series	77
12.5	The Number e	81
III	More Algebra	85
13	Lie Groups	87

Chapter 1

Set Theory

Proposition 1.1. *Every infinite subset of a countably infinite set is countable.*

PROOF:

- $\langle 1 \rangle 1$. LET: $i : A \hookrightarrow \mathbb{N}$ be an infinite subset of \mathbb{N} .
- $\langle 1 \rangle 2$. Define $j : \mathbb{N} \rightarrow A$ by: $j(k)$ is the element such that $i(j(k))$ is least such that $i(j(k)) \notin \{i(j(0)), \dots, i(j(k-1))\}$.
- $\langle 1 \rangle 3$. j is a bijection.

□

Proposition 1.2. *A countable union of countable sets is countable.*

PROOF:

- $\langle 1 \rangle 1$. LET: (A_n) be a sequence of countable sets.
- $\langle 1 \rangle 2$. For $n \in \mathbb{N}$, PICK an enumeration $(e_{nm})_m$ of A_n .
- $\langle 1 \rangle 3$. LET: (p_k) be the following enumeration of $\mathbb{N} \times \mathbb{N}$:
 $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \dots$
- $\langle 1 \rangle 4$. $(e_{\pi_1(p_k)\pi_2(p_k)})_k$ is an enumeration of $\bigcup_n A_n$.

□

Theorem 1.3. $2^{\mathbb{N}}$ is uncountable.

PROOF:

- $\langle 1 \rangle 1$. ASSUME: for a contradiction $f : \mathbb{N} \approx 2^{\mathbb{N}}$
- $\langle 1 \rangle 2$. LET: $S = \{n \in \mathbb{N} : n \notin f(n)\}$
- $\langle 1 \rangle 3$. For all n , we have $n \in S \Leftrightarrow n \notin f(n)$
- $\langle 1 \rangle 4$. For all n we have $S \neq f(n)$.
- $\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 1$.

□

Chapter 2

Relations

Definition 2.1 (Antisymmetric). A relation R on a set A is *antisymmetric* iff, whenever xRy and yRx , then $x = y$.

Definition 2.2 (Transitive). A relation R on a type A is *transitive* iff, whenever xRy and yRz , then xRz .

Chapter 3

Order Theory

Definition 3.1 (Linear Order). A *linear order* on a set A is a binary relation \leq on A that is transitive, antisymmetric and:

$$\forall x, y \in A. x \leq y \vee y \leq x .$$

A *linearly ordered set* is a pair (A, \leq) where A is a set and \leq is a binary relation on A .

We write $x < y$ for $x \leq y$ and $x \neq y$.

Definition 3.2 (Upper Bound). Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is an *upper bound* in E iff $\forall x \in E. x \leq u$. We say E is *bounded above* iff it has an upper bound.

The *up-set* of E , denoted $E \uparrow$, is the set of upper bounds of E .

Definition 3.3 (Lower Bound). Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then l is a *lower bound* in E iff $\forall x \in E. l \leq x$. We say E is *bounded below* iff it has a lower bound.

The *down-set* of E , denoted $E \downarrow$, is the set of lower bounds of E .

Definition 3.4 (Supremum). Let S be a linearly ordered set, $u \in S$ and $E \subseteq S$. Then u is the *least upper bound* or *supremum* of E iff u is an upper bound for E and, for any upper bound u' for E , we have $u \leq u'$.

Definition 3.5 (Infimum). Let S be a linearly ordered set, $l \in S$ and $E \subseteq S$. Then l is the *greatest lower bound* or *infimum* of E iff l is a lower bound for E and, for any lower bound l' for E , we have $l' \leq l$.

Definition 3.6 (Least Upper Bound Property). A linearly ordered set S has the *least upper bound property* iff every nonempty subset of S that is bounded above has a least upper bound.

Proposition 3.7. Let S be a linearly ordered set and $E \subseteq S$.

1. If $E \downarrow$ has a supremum l , then l is the infimum of E .

2. If $E \uparrow$ has an infimum u , then U is the supremum of E .

PROOF:

- (1)1. If $E \downarrow$ has a supremum l , then l is the infimum of E .
 (2)1. l is a lower bound for E .
 (3)1. LET: $x \in E$
 (3)2. x is an upper bound for $E \downarrow$.
 PROOF: For all $y \in E \downarrow$ we have $y \leq x$.
 (3)3. $l \leq x$
 (2)2. For any lower bound l' for E , we have $l' \leq l$.
 PROOF: Since l is an upper bound for $E \downarrow$.
 (1)2. If $E \uparrow$ has an infimum u , then u is the supremum of E .
 PROOF: Dual.

□

Corollary 3.7.1. *A linearly ordered set has the least upper bound property if and only if every nonempty set bounded below has an infimum.*

Definition 3.8 (Closed Downwards). Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed downwards* iff, whenever $x \in E$ and $y < x$, then $y \in E$.

Definition 3.9 (Closed Upwards). Let S be a linearly ordered set and $E \subseteq S$. Then E is *closed upwards* iff, whenever $x \in E$ and $x < y$, then $y \in E$.

Definition 3.10 (Greatest). Let S be a linearly ordered set and $u \in S$. Then u is *greatest* in S iff $\forall x \in S. x \leq u$.

Definition 3.11 (Least). Let S be a linearly ordered set and $l \in S$. Then l is *least* in S iff $\forall x \in S. l \leq x$.

Proposition 3.12. *Let \leq be a linear order on a set S and $E \subseteq S$. Then $\leq \cap E^2$ is a linear order on E .*

PROOF: Easy. □

Given a linearly ordered set (S, \leq) and $E \subseteq S$, we write just E for the linearly ordered set $(E, \leq \cap E^2)$.

Definition 3.13 (Lexicographic Order). Let A and B be linearly ordered sets. The *lexicographic order* or *dictionary order* on $A \times B$ is the order defined by

$$(a, b) \leq (a', b') \Leftrightarrow a = a' \vee (a < a' \wedge b \leq b') .$$

Proposition 3.14. *The lexicographic order is a linear order.*

Chapter 4

Field Theory

Definition 4.1 (Field). A *field* F consists of a set F , two operations $+, \cdot : F^2 \rightarrow F$ and an element $0 \in F$ such that:

- $+$ is commutative.
- $+$ is associative.
- $\forall x \in F. x + 0 = x$
- $\forall x \in F. \exists y \in F. x + y = 0$
- \cdot is commutative.
- \cdot is associative.
- There exists $1 \in F$ such that $1 \neq 0$ and $\forall x \in F. x1 = x$ and $\forall x \in F. x \neq 0 \Rightarrow \exists y \in F. xy = 1$
- *Distributive Law* $\forall x, y, z \in F. x(y + z) = xy + xz$

Proposition 4.2. *In any field F , the element 0 is the unique element such that $\forall x \in F. x + 0 = x$.*

PROOF: If 0 and $0'$ both have this property then $0 = 0 + 0' = 0'$. \square

Proposition 4.3. *In any field F , given $x \in F$, there is a unique $y \in F$ such that $x + y = 0$.*

PROOF: If $x + y = x + y' = 0$ then

$$\begin{aligned} y &= y + 0 \\ &= y + x + y' \\ &= 0 + y' \\ &= y' \end{aligned}$$

\square

Definition 4.4. Let F be a field. Let $x \in F$. We denote by $-x$ the unique element of F such that $x + (-x) = 0$.

Given $x, y \in F$, we write $x - y$ for $x + (-y)$.

Proposition 4.5. In any field F , if $x + y = x + z$ then $y = z$.

PROOF: If $x + y = x + z$ we have

$$-x + x + y = -x + x + z$$

$$\therefore 0 + y = 0 + z$$

$$\therefore y = z \quad \square$$

Proposition 4.6. In any field F , we have $-(-x) = x$.

PROOF: Since $x + (-x) = 0$. \square

Proposition 4.7. In any field F , the element 1 such that $\forall x \in F. x1 = x$ is unique.

PROOF: If 1 and $1'$ both have this property then $1 = 1 \cdot 1' = 1'$. \square

Proposition 4.8. In any field F , given $x \in F$ with $x \neq 0$, the element y such that $xy = 1$ is unique.

PROOF: If y and y' both have this property then we have

$$y = y1$$

$$= yxy'$$

$$= 1y'$$

$$= y' \quad \square$$

Definition 4.9. In any field F , if $x \neq 0$, we write x^{-1} for the unique element such that $xx^{-1} = 1$.

We write x/y for xy^{-1} .

Proposition 4.10. In any field F , if $xy = xz$ and $x \neq 0$ then $y = z$.

PROOF:

$$y = 1y$$

$$= x^{-1}xy$$

$$= x^{-1}xz$$

$$= 1z$$

$$= z \quad \square$$

Proposition 4.11. In any field F , if $x \neq 0$ then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.

PROOF: Since $xx^{-1} = 1$. \square

Proposition 4.12. In any field F , we have $x0 = 0$.

PROOF:

$$\begin{aligned}
 x0 + 0 &= x0 \\
 &= x(0 + 0) \\
 &= x0 + x0 \\
 \therefore 0 &= x0 \quad \square
 \end{aligned}$$

Proposition 4.13. *In any field F , if $xy = 0$ then $x = 0$ or $y = 0$.*

PROOF: If $xy = 0$ and $x \neq 0$ then we have $y = x^{-1}xy = x^{-1}0 = 0$. \square

Proposition 4.14. *In any field F , we have $(-x)y = -(xy)$.*

PROOF:

$$\begin{aligned}
 xy + (-x)y &= (x + (-x))y \\
 &= 0y \\
 &= 0 \quad \text{(Proposition 4.12)} \square
 \end{aligned}$$

Corollary 4.14.1. *In any field F , we have $(-x)(-y) = xy$.*

PROOF:

$$\begin{aligned}
 (-x)(-y) &= -(x(-y)) \\
 &= -(-(xy)) \\
 &= xy \quad \text{(Proposition 4.6)} \square
 \end{aligned}$$

Proposition 4.15. *Let K be a field. Let $a, b \in K$. If $a^2 = b^2$ then $a = b$ or $a = -b$.*

PROOF:

$$\begin{aligned}
 a^2 - b^2 &= 0 \\
 \therefore (a - b)(a + b) &= 0
 \end{aligned}$$

Hence either $a - b = 0$ or $a + b = 0$, and the conclusion follows. \square

4.1 Ordered Fields

Definition 4.16 (Ordered Field). An *ordered field* F consists of a field F and a linear order \leq on F such that:

- For all $x, y, z \in F$, if $y < z$ then $x + y < x + z$
- For all $x, y \in F$, if $x > 0$ and $y > 0$ then $xy > 0$.

We call x *positive* iff $x > 0$ and *negative* iff $x < 0$.

Example 4.17. \mathbb{Q} is an ordered field.

Proposition 4.18. *In any ordered field, if x is positive then $-x$ is negative.*

PROOF: If $x > 0$ then $0 = x + (-x) > 0 = (-x) = -x$. \square

Proposition 4.19. *In any ordered field, if $y < z$ and x is positive then $xy < xz$.*

PROOF: If $y < z$ then we have

$$\begin{aligned} 0 &< z - y \\ \therefore 0 &< x(z - y) \\ &= xz - xy \\ \therefore xy &< xz \end{aligned}$$

□

Proposition 4.20. *In any ordered field, if $y < z$ and x is negative then $xy > xz$.*

PROOF:

- <1>1. $-x$ is positive.
- <1>2. $(-x)y < (-x)z$
- <1>3. $-(xy) < -(xz)$
- <1>4. $xz < xy$

□

Proposition 4.21. *In any ordered field, if $x \neq 0$ then $x^2 > 0$.*

PROOF:

- <1>1. If $x > 0$ then $x^2 > 0$.

PROOF: Proposition 4.19.

- <1>2. If $x < 0$ then $x^2 > 0$.

PROOF: Proposition 4.20.

□

Corollary 4.21.1. *In any ordered field, we have $1 > 0$.*

Proposition 4.22. *In any ordered field, if x is positive then x^{-1} is positive.*

PROOF: If $x^{-1} < 0$ then we would have $1 = xx^{-1} < x0 = 0$ contradicting Corollary 4.21.1. □

Proposition 4.23. *In any ordered field, if $0 < x < y$ then $y^{-1} < x^{-1}$.*

PROOF:

- <1>1. ASSUME: $0 < x < y$
- <1>2. x^{-1} and y^{-1} are positive.

PROOF: Proposition 4.22.

- <1>3. $xy^{-1} < yy^{-1} = 1$
- <1>4. $y^{-1} = x^{-1}xy^{-1} < x^{-1}1 = x^{-1}$

□

Lemma 4.24. *Let K be an ordered field. Let $b \in K$ with $b > 1$. Let n be a positive integer. Then*

$$b^n - 1 \geq n(b - 1)$$

PROOF:

$$\begin{aligned} b^n - 1 &= (b - 1)(b^{n-1} + b^{n-2} + \cdots + 1) \\ &\geq (b - 1)(1 + 1 + \cdots + 1) \\ &= n(b - 1) \end{aligned}$$

□

Chapter 5

Real Analysis

5.1 Construction of the Real Numbers

Definition 5.1 (Cut). A *cut* is a subset α of \mathbb{Q} such that:

- $\emptyset \neq \alpha \neq \mathbb{Q}$
- α is closed downwards.
- α has no greatest element.

In this section, we write R for the set of all cuts.

Proposition 5.2. *R is linearly ordered by \subseteq .*

PROOF: The only difficult part is to prove that, for any cuts α and β , either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

$\langle 1 \rangle 1$. ASSUME: $\alpha \not\subseteq \beta$

PROVE: $\beta \subseteq \alpha$

$\langle 1 \rangle 2$. PICK $q \in \alpha$ such that $q \notin \beta$

$\langle 1 \rangle 3$. LET: $r \in \beta$

$\langle 1 \rangle 4$. $q \not\leq r$

$\langle 1 \rangle 5$. $r < q$

$\langle 1 \rangle 6$. $r \in \alpha$

□

Proposition 5.3. *R has the least upper bound property.*

PROOF:

$\langle 1 \rangle 1$. LET: $E \subseteq R$ be nonempty and bounded above.

$\langle 1 \rangle 2$. LET: $s = \bigcup E$

PROVE: s is a cut.

$\langle 1 \rangle 3$. $\emptyset \neq s$

PROOF: Since E is nonempty and every element of E is nonempty.

$\langle 1 \rangle 4$. $s \neq \mathbb{Q}$

- ⟨2⟩1. PICK an upper bound u for E .
- ⟨2⟩2. PICK $q \notin u$
 PROVE: $q \notin s$
- ⟨2⟩3. $\forall \alpha \in E. \alpha \subseteq u$
- ⟨2⟩4. $s \subseteq u$
- ⟨2⟩5. $q \notin s$
- ⟨1⟩5. s is closed downwards.
- ⟨2⟩1. LET: $q \in s$ and $r < q$.
- ⟨2⟩2. PICK $\alpha \in E$ such that $q \in \alpha$.
- ⟨2⟩3. $r \in \alpha$
- ⟨2⟩4. $r \in s$
- ⟨1⟩6. s has no greatest element.
- ⟨2⟩1. LET: $q \in s$
- ⟨2⟩2. PICK $\alpha \in E$ such that $q \in \alpha$.
- ⟨2⟩3. PICK $r \in \alpha$ such that $q < r$.
- ⟨2⟩4. $r \in s$

□

Definition 5.4 (Addition). Given cuts α and β , we define

$$\alpha + \beta = \{q + r : q \in \alpha, r \in \beta\} .$$

Proposition 5.5. *Given cuts α and β , we have $\alpha + \beta$ is a cut.*

PROOF:

- ⟨1⟩1. $\alpha + \beta$ is nonempty.
 PROOF: Since α and β are nonempty.
- ⟨1⟩2. $\alpha + \beta \neq \mathbb{Q}$
 - ⟨2⟩1. PICK $q \in \mathbb{Q} - \alpha$ and $r \in \mathbb{Q} - \beta$.
 PROVE: $q + r \notin \alpha + \beta$
 - ⟨2⟩2. ASSUME: for a contradiction $q + r \in \alpha + \beta$.
 - ⟨2⟩3. PICK $x \in \alpha$ and $y \in \beta$ such that $q + r = x + y$
 - ⟨2⟩4. $x < q$
 - ⟨2⟩5. $y < r$
 - ⟨2⟩6. $x + y < q + r$
 - ⟨2⟩7. Q.E.D.
- PROOF: This is a contradiction.
- ⟨1⟩3. $\alpha + \beta$ is closed downwards.
 - ⟨2⟩1. LET: $q \in \alpha, r \in \beta$ and $x < q + r$
 - ⟨2⟩2. $x - q < r$
 - ⟨2⟩3. $x - q \in \beta$
 - ⟨2⟩4. $x \in \alpha + \beta$
- ⟨1⟩4. $\alpha + \beta$ has no greatest element.
 - ⟨2⟩1. LET: $q \in \alpha$ and $r \in \beta$.
 PROVE: $q + r$ is not greatest in $\alpha + \beta$.
 - ⟨2⟩2. PICK $q' \in \alpha$ with $q < q'$ and $r' \in \beta$ with $r < r'$.
 - ⟨2⟩3. $q + r < q' + r' \in \alpha + \beta$

□

Proposition 5.6. *Addition is commutative and associative on R .*

PROOF: Immediate from definitions and the fact that addition is commutative and associative on \mathbb{Q} . □

Definition 5.7. For any $q \in \mathbb{Q}$, let $q^* = \{r \in \mathbb{Q} : r < q\}$.

Proposition 5.8. *For any $q \in \mathbb{Q}$, we have q^* is a cut.*

PROOF:

⟨1⟩1. $q^* \neq \emptyset$

PROOF: Since $q - 1 \in q^*$.

⟨1⟩2. $q^* \neq \mathbb{Q}$

PROOF: Since $q \notin q^*$.

⟨1⟩3. q^* is closed downwards.

PROOF: Immediate from definition.

⟨1⟩4. q^* has no greatest element.

PROOF: For all $r \in q^*$ we have $r < (q + r)/2 \in q^*$.

□

Proposition 5.9. *For any cut α we have $\alpha + 0^* = \alpha$.*

PROOF:

⟨1⟩1. $\alpha + 0^* \subseteq \alpha$

⟨2⟩1. LET: $q \in \alpha$ and $r \in 0^*$

PROVE: $q + r \in \alpha$

⟨2⟩2. $r < 0$

⟨2⟩3. $q + r < q$

⟨2⟩4. $q + r \in \alpha$

⟨1⟩2. $\alpha \subseteq \alpha + 0^*$

⟨2⟩1. LET: $q \in \alpha$

⟨2⟩2. PICK $r \in \alpha$ such that $q < r$

⟨2⟩3. $q = r + (q - r) \in \alpha + 0^*$

□

Proposition 5.10. *For any cut α , there exists a cut β such that $\alpha + \beta = 0$.*

PROOF:

⟨1⟩1. LET: $\beta = \{p \in \mathbb{Q} : \exists r > 0. -p - r \notin \alpha\}$

⟨1⟩2. β is a cut.

⟨2⟩1. $\beta \neq \emptyset$

⟨3⟩1. PICK $q \notin \alpha$

⟨3⟩2. $-q - 1 \in \beta$

⟨2⟩2. $\beta \neq \mathbb{Q}$

⟨3⟩1. PICK $q \in \alpha$

PROVE: $-q \notin \beta$

⟨3⟩2. ASSUME: for a contradiction $-q \in \beta$

- $\langle 3 \rangle 3$. PICK $r > 0$ such that $q - r \notin \alpha$
- $\langle 3 \rangle 4$. $q - r < q$
- $\langle 3 \rangle 5$. Q.E.D.

PROOF: This contradicts the fact that α is closed downwards.

- $\langle 2 \rangle 3$. β is closed downwards.
 - $\langle 3 \rangle 1$. LET: $p \in \beta$ and $q < p$.
 - $\langle 3 \rangle 2$. PICK $r > 0$ such that $-p - r \notin \alpha$
 - $\langle 3 \rangle 3$. $-p - r < -q - r$
 - $\langle 3 \rangle 4$. $-q - r \notin \alpha$
 - $\langle 3 \rangle 5$. $q \in \beta$
- $\langle 2 \rangle 4$. β has no greatest element.
 - $\langle 3 \rangle 1$. LET: $p \in \beta$
 - $\langle 3 \rangle 2$. PICK $r > 0$ such that $-p - r \notin \alpha$
 - $\langle 3 \rangle 3$. $-(p + r/2) - r/2 \notin \alpha$
 - $\langle 3 \rangle 4$. $p + r/2 \in \beta$
- $\langle 1 \rangle 3$. $\alpha + \beta \subseteq 0^*$
 - $\langle 2 \rangle 1$. LET: $p \in \alpha$ and $q \in \beta$.
 - $\langle 2 \rangle 2$. PICK $r > 0$ such that $-q - r \notin \alpha$.
 - $\langle 2 \rangle 3$. $p < -q - r$
 - $\langle 2 \rangle 4$. $p + q < -r$
 - $\langle 2 \rangle 5$. $p + q < 0$
 - $\langle 2 \rangle 6$. $p + q \in 0^*$
- $\langle 1 \rangle 4$. $0^* \subseteq \alpha + \beta$
 - $\langle 2 \rangle 1$. LET: $v \in 0^*$
 - $\langle 2 \rangle 2$. LET: $w = -v/2$
 - $\langle 2 \rangle 3$. $w > 0$
 - $\langle 2 \rangle 4$. PICK an integer n such that $nw \in \alpha$ and $(n + 1)w \notin \alpha$.
 - $\langle 2 \rangle 5$. LET: $p = -(n + 2)w$
 - $\langle 2 \rangle 6$. $p \in \beta$
 - $\langle 2 \rangle 7$. $v = nw + p$
 - $\langle 2 \rangle 8$. $v \in \alpha + \beta$

□

Proposition 5.11. *Given $\alpha, \beta, \gamma \in R$, if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

PROOF:

- $\langle 1 \rangle 1$. $\alpha + \beta \subseteq \alpha + \gamma$
 PROOF: Immediate from definitions.
- $\langle 1 \rangle 2$. $\alpha + \beta \neq \alpha + \gamma$
 PROOF: If $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$ by cancellation.

□

Definition 5.12. Given cuts α and β , define $\alpha\beta$ by:

$$\alpha\beta = \begin{cases} \{p \in \mathbb{Q} : \exists r \in \alpha. \exists s \in \beta (p \leq rs \wedge r > 0 \wedge s > 0)\} & \text{if } \alpha > 0^* \text{ and } \beta > 0^* \\ (-\alpha)(-\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^* \\ -((-\alpha)\beta) & \text{if } \alpha < 0^* \text{ and } \beta > 0^* \\ -(\alpha(-\beta)) & \text{if } \alpha > 0^* \text{ and } \beta < 0^* \\ 0^* & \text{if } \alpha = 0^* \text{ or } \beta = 0^* \end{cases}$$

Proposition 5.13. For any cuts α and β , we have $\alpha\beta$ is a cut.

PROOF:

(1)1. If $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta$ is a cut.

(2)1. $\alpha\beta \neq \emptyset$

(3)1. PICK $q \in \alpha$ and $r \in \beta$ such that $q, r \notin 0^*$

(3)2. ASSUME: w.l.o.g. $0 < q$ and $0 < r$.

PROOF: Since α and β have no greatest element.

(3)3. $qr \in \alpha\beta$

(2)2. $\alpha\beta \neq \mathbb{Q}$

(3)1. PICK $r \notin \alpha$ and $s \notin \beta$

PROVE: $rs \notin \alpha\beta$

(3)2. ASSUME: for a contradiction $rs \in \alpha\beta$.

(3)3. PICK $r' \in \alpha$ and $s' \in \beta$ such that $rs \leq r's'$ and $r' > 0$ and $s' > 0$.

(3)4. $r' < r$ and $s' < s$

(3)5. $r's' < rs$

(3)6. Q.E.D.

PROOF: This is a contradiction.

(2)3. $\alpha\beta$ is closed downwards.

(3)1. LET: $p \in \alpha\beta$ and $p' < p$

(3)2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$

(3)3. $p' \leq rs$

(3)4. $p' \in \alpha\beta$

(2)4. $\alpha\beta$ has no greatest element.

(3)1. LET: $p \in \alpha\beta$

(3)2. PICK $r \in \alpha$ and $s \in \beta$ such that $p \leq rs$, $r > 0$ and $s > 0$.

(3)3. PICK $r' \in \alpha$ and $s' \in \beta$ with $r < r'$ and $s < s'$.

(3)4. $p < r's' \in \alpha\beta$

(1)2. For any cuts α and β , we have $\alpha\beta$ is a cut.

PROOF: Since if α is a cut then $-\alpha$ is a cut.

□

Proposition 5.14. For any cuts α and β we have $\alpha\beta = \beta\alpha$.

PROOF: Easy from the definitions. □

Proposition 5.15. For any cuts α , β and γ we have

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma .$$

PROOF:

$\langle 1 \rangle 1$. CASE: α , β and γ are all positive.

PROOF: In this case $\alpha(\beta\gamma) = (\alpha\beta)\gamma = \{p \in \mathbb{Q} : \exists r \in \alpha. \exists s \in \beta. \exists t \in \gamma. (p \leq rst \wedge r > 0 \wedge s > 0 \wedge t > 0)\}$.

$\langle 1 \rangle 2$. CASE: One of α , β or γ is 0^* .

PROOF: Then $\alpha(\beta\gamma) = (\alpha\beta)\gamma = 0^*$.

$\langle 1 \rangle 3$. CASE: α and β are positive, γ is negative.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\ &= -(\alpha(\beta(-\gamma))) \\ &= -((\alpha\beta)(-\gamma)) & (\langle 1 \rangle 1) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 4$. CASE: α is positive, β is negative, γ is positive.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-((- \beta)\gamma)) \\ &= -(\alpha((- \beta)\gamma)) \\ &= -((\alpha(-\beta))\gamma) & (\langle 1 \rangle 1) \\ &= -(\alpha(-\beta))\gamma \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 5$. CASE: α is positive, β and γ are negative.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha((- \beta)(- \gamma)) \\ &= (\alpha(-\beta))(-\gamma) & (\langle 1 \rangle 1) \\ &= -(\alpha\beta)(-\gamma) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 6$. CASE: α is negative, β and γ are positive.

PROOF: Similar to $\langle 1 \rangle 3$.

$\langle 1 \rangle 7$. CASE: α is negative, β is positive, γ is negative.

PROOF:

$$\begin{aligned} \alpha(\beta\gamma) &= \alpha(-(\beta(-\gamma))) \\ &= (-\alpha)(\beta(-\gamma)) \\ &= ((-\alpha)\beta)(-\gamma) & (\langle 1 \rangle 1) \\ &= -(\alpha\beta)(-\gamma) \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\langle 1 \rangle 8$. CASE: α and β are negative, γ is positive.

PROOF: Similar to $\langle 1 \rangle 5$.

$\langle 1 \rangle 9$. CASE: α , β and γ are all negative.

PROOF:

$$\begin{aligned}
 \alpha(\beta\gamma) &= \alpha(-(-\beta)(-\gamma)) \\
 &= -((- \alpha)((-\beta)(-\gamma))) \\
 &= -(((- \alpha)(-\beta))(-\gamma)) \quad ((1)1) \\
 &= -((\alpha\beta)(-\gamma)) \\
 &= (\alpha\beta)\gamma
 \end{aligned}$$

□

Proposition 5.16. *For any cut α we have $\alpha 1^* = \alpha$.*

PROOF:

$\langle 1 \rangle 1$. CASE: α is positive.

$\langle 2 \rangle 1$. $\alpha 1^* \subseteq \alpha$

$\langle 2 \rangle 2$. $\alpha \subseteq \alpha 1^*$

$\langle 1 \rangle 2$. CASE: $\alpha = 0^*$

$\langle 1 \rangle 3$. CASE: α is negative.

□

Theorem 5.17. *There exists an ordered field with the least upper bound property.*

Proposition 5.18. *There is no rational p such that $p^2 = 2$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $p^2 = 2$.

$\langle 1 \rangle 2$. PICK integers m, n not both even such that $p = m/n$.

$\langle 1 \rangle 3$. $m^2 = 2n^2$

$\langle 1 \rangle 4$. m is even.

$\langle 1 \rangle 5$. PICK an integer k such that $m = 2k$.

$\langle 1 \rangle 6$. $4k^2 = 2n^2$

$\langle 1 \rangle 7$. $2k^2 = n^2$

$\langle 1 \rangle 8$. n is even.

$\langle 1 \rangle 9$. Q.E.D.

PROOF: $\langle 1 \rangle 2$, $\langle 1 \rangle 4$ and $\langle 1 \rangle 8$ form a contradiction.

□

Theorem 5.19. *Any two complete ordered fields are isomorphic.*

Definition 5.20. Let \mathbb{R} be the complete ordered field. We call its elements *real numbers*.

5.2 Properties of the Real Numbers

Theorem 5.21. \mathbb{Q} is a subfield of \mathbb{R} .

Theorem 5.22 (Archimedean Property). *Let $x, y \in \mathbb{R}$ with $x > 0$. There exists a positive integer n such that $nx > y$.*

PROOF:

- (1)1. LET: $A = \{nx : n \in \mathbb{Z}^+\}$
- (1)2. ASSUME: for a contradiction there is no positive integer n such that $nx > y$.
- (1)3. y is an upper bound for A .
- (1)4. LET: $\alpha = \sup A$
- (1)5. $\alpha - x$ is not an upper bound for A .
- (1)6. PICK a positive integer m such that $\alpha - x < mx$
- (1)7. $\alpha < (m+1)x \in A$
- (1)8. Q.E.D.

PROOF: This contradicts (1)4.

□

Theorem 5.23. \mathbb{Q} is dense in \mathbb{R} .

PROOF:

- (1)1. LET: $x, y \in \mathbb{R}$ with $x < y$
- (1)2. PICK a positive integer n such that $n(y-x) > 1$.
- PROOF: Archimedean property.
- (1)3. PICK a positive integer m_1 such that $m_1 > nx$
- PROOF: Archimedean property.
- (1)4. PICK a positive integer m_2 such that $m_2 > -nx$
- PROOF: Archimedean property.
- (1)5. $-m_2 < nx < m_1$
- (1)6. LET: m be the integer such that $m-1 \leq nx < m$.
- (1)7. $nx < m \leq 1 + nx < ny$
- (1)8. $x < m/n < y$

□

Theorem 5.24. For every real number $x > 0$ and positive integer n , there exists a unique positive real number y such that $y^n = x$.

PROOF:

- (1)1. There exists a real $y > 0$ such that $y^n = x$.
- (2)1. LET: $E = \{t \in \mathbb{R}^+ : t^n < x\}$
- (2)2. LET: $y = \sup E$
- (3)1. $E \neq \emptyset$
- (4)1. LET: $t = x/(x+1)$
- (4)2. $0 < t < 1$
- (4)3. $t^n < t < x$
- (4)4. $t \in E$
- (3)2. $x+1$ is an upper bound for E .
- (4)1. LET: $t > x+1$
- (4)2. $t^n > t > x$
- (4)3. $t \notin E$

⟨2⟩3. $y^n = x$

⟨3⟩1. $y^n \not\leq x$

⟨4⟩1. ASSUME: for a contradiction $y^n < x$.

⟨4⟩2. PICK h such that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y + 1)^{n-1}} .$$

⟨4⟩3. $(y + h)^n - y^n < x - y^n$

PROOF:

$$\begin{aligned} (y + h)^n - y^n &= ((y + h) - y) \sum_{i=0}^{n-1} (y + h)^{n-1-i} y^i \\ &= h \sum_{i=0}^{n-1} (y + h)^{n-1-i} y^i \\ &\leq hn(y + h)^{n-1} \\ &\leq hn(y + 1)^{n-1} \\ &< x - y^n \end{aligned}$$

⟨4⟩4. $(y + h)^n < x$

⟨4⟩5. $y + h \in E$

⟨4⟩6. Q.E.D.

PROOF: This contradicts the fact that y is an upper bound for E .

⟨3⟩2. $y^n \not\geq x$

⟨4⟩1. ASSUME: for a contradiction $y^n > x$

⟨4⟩2. LET:

$$k = \frac{y^n - x}{ny^{n-1}}$$

⟨4⟩3. $0 < k < y$

⟨4⟩4. $y - k$ is an upper bound for E .

⟨5⟩1. LET: $t \geq y - k$

⟨5⟩2. $y^n - t^n \leq y^n - x$

PROOF:

$$\begin{aligned} y^n - t^n &\leq y^n - (y - k)^n \\ &= (y - (y - k)) \sum_{i=0}^{n-1} y^{n-i} (y - k)^i \\ &= k \sum_{i=0}^{n-1} y^{n-i} (y - k)^i \\ &\leq kny^{n-1} \\ &= y^n - x \end{aligned}$$

⟨5⟩3. $t^n \geq x$

⟨5⟩4. $t \notin E$

⟨4⟩5. Q.E.D.

PROOF: This contradicts the fact that y is the least upper bound of E .

⟨1⟩2. If y and y' are positive reals with $y^n = y'^n$ then $y = y'$.

PROOF: Since the function that sends y to y^n is strictly monotone.
 \square

Definition 5.25 (*n*th Root). Given any real number $x > 0$ and positive integer n , the *n*th root of x , denoted $x^{1/n}$, is the unique positive real such that

$$(x^{1/n})^n = x .$$

We write \sqrt{x} for $x^{1/2}$.

Proposition 5.26. Let a and b be positive real numbers and n a positive integer. Then

$$(ab)^{1/n} = a^{1/n}b^{1/n} .$$

PROOF: Since $(a^{1/n}b^{1/n})^n = ab$. \square

Lemma 5.27. Let b be a real number with $b > 1$. Let n be a positive integer. Then

$$b - 1 \geq n(b^{1/n} - 1) .$$

PROOF: From Lemma 4.24. \square

Lemma 5.28. Let b and t be real numbers with $b > 1$ and $t > 1$. For any positive integer n , if $n > \frac{b-1}{t-1}$ then $b^{1/n} < t$.

PROOF:

$$\begin{aligned} b - 1 &\geq n(b^{1/n} - 1) \\ \therefore \frac{b - 1}{n} &\geq b^{1/n} - 1 \\ \therefore t - 1 &> b^{1/n} - 1 \\ \therefore t &> b^{1/n} \end{aligned} \quad \square$$

Lemma 5.29. Let b be a real number with $b > 0$. Let m, n, p, q be integers with $n > 0$ and $q > 0$. Assume $m/n = p/q$. Then

$$(b^m)^{1/n} = (b^p)^{1/q} .$$

PROOF:

$$\langle 1 \rangle 1. (b^m)^{1/n} = (b^{1/n})^m$$

PROOF:

$$\begin{aligned} ((b^{1/n})^m)^n &= ((b^{1/n})^n)^m \\ &= b^m \end{aligned}$$

$$\langle 1 \rangle 2. ((b^m)^{1/n})^q = b^p$$

PROOF:

$$\begin{aligned} ((b^m)^{1/n})^q &= (b^{1/n})^{mq} \\ &= (b^{1/n})^{np} \\ &= b^p \end{aligned}$$

\square

Definition 5.30. For a a positive real and q a rational number, we may therefore define a^q by

$$a^{m/n} = (a^m)^{1/n}$$

for m and n integers with $n > 0$.

Proposition 5.31. Let a be a positive real and r, s rational numbers. Then

$$a^{r+s} = a^r a^s .$$

PROOF:

$$\begin{aligned} a^{m/n+p/q} &= a^{(mq+np)/nq} \\ &= (a^{mq+np})^{1/nq} \\ &= (a^{mq})^{1/nq} (a^{np})^{1/nq} \\ &= a^{m/n} a^{p/q} \end{aligned} \quad \square$$

Proposition 5.32. Let $b > 1$ be a real number and q a rational number. Then

$$b^q = \sup\{b^t : t \in \mathbb{Q}, t \leq q\}$$

PROOF: It is the greatest element of this set. \square

Definition 5.33. Let $b > 1$ be a real number and x a real number. Then

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t \leq x\} .$$

Lemma 5.34. Let b, w and y be real numbers with $b > 1$. Assume $b^w < y$. Then there exists a positive integer n such that $b^{w+1/n} < y$.

PROOF:

- $\langle 1 \rangle 1$. LET: $t = yb^{-w}$
- $\langle 1 \rangle 2$. PICK a positive integer n such that $n > \frac{b-1}{t-1}$.
- $\langle 1 \rangle 3$. $b^{1/n} < t$

PROOF: Lemma 5.28.

- $\langle 1 \rangle 4$. $b^{w+1/n} < y$

\square

Lemma 5.35. Let b, w and y be real numbers with $b > 1$. Assume $b^w > y$. Then there exists a positive integer n such that $b^{w-1/n} < y$.

PROOF:

- $\langle 1 \rangle 1$. LET: $t = b^w/y$
- $\langle 1 \rangle 2$. PICK a positive integer n such that $n > \frac{b-1}{t-1}$
- $\langle 1 \rangle 3$. $b^{1/n} < t$

PROOF: Lemma 5.28.

- $\langle 1 \rangle 4$. $y < b^{w-1/n}$

\square

Proposition 5.36. *For b and x real numbers with $b > 1$ we have*

$$b^x = \sup\{b^t : t \in \mathbb{Q}, t < x\} .$$

PROOF:

- $\langle 1 \rangle 1.$ b^x is an upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
- $\langle 1 \rangle 2.$ LET: u be any upper bound for $\{b^t : t \in \mathbb{Q}, t < x\}$.
PROVE: $b^x \leq u$
- $\langle 1 \rangle 3.$ LET: q be a rational number with $q \leq x$.
PROVE: $b^q \leq u$
- $\langle 1 \rangle 4.$ ASSUME: for a contradiction $b^q > u$.
- $\langle 1 \rangle 5.$ PICK a positive integer n such that $b^{q-1/n} > u$.
PROOF: Lemma 5.35.
- $\langle 1 \rangle 6.$ $b^{q-1/n} \leq u$
PROOF: $\langle 1 \rangle 2$
- $\langle 1 \rangle 7.$ Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 4$.

□

Lemma 5.37. *Let A be a set of positive real numbers with supremum $a > 0$ and B a set of positive real numbers with supremum $b > 0$. Then ab is the supremum of $\{xy : x \in A, y \in B\}$.*

PROOF:

- $\langle 1 \rangle 1.$ For all $x \in A$ and $y \in B$ we have $xy \leq ab$.
- $\langle 1 \rangle 2.$ If u is any upper bound for $\{xy : x \in A, y \in B\}$ then $ab \leq u$.
 - $\langle 2 \rangle 1.$ LET: u be an upper bound for $\{xy : x \in A, y \in B\}$.
 - $\langle 2 \rangle 2.$ For all $x \in A$ we have u/x is an upper bound for B .
 - $\langle 2 \rangle 3.$ For all $x \in A$ we have $b \leq u/x$
 - $\langle 2 \rangle 4.$ For all $x \in A$ we have $x \leq u/b$
 - $\langle 2 \rangle 5.$ $a \leq u/b$
 - $\langle 2 \rangle 6.$ $ab \leq u$

□

Proposition 5.38. *Let $b, x, y \in \mathbb{R}$ with $b > 1$. Then*

$$b^{x+y} = b^x b^y .$$

PROOF:

- $\langle 1 \rangle 1.$ For any rational number $q < x + y$, there exist rational numbers $r < x$ and $s < y$ such that $q = r + s$.
 - $\langle 2 \rangle 1.$ $q - x < y$
 - $\langle 2 \rangle 2.$ PICK a rational t such that $q - x < t < y$
 - $\langle 2 \rangle 3.$ $q = t + (q - t)$ and $t < y, q - t < x$
- $\langle 1 \rangle 2.$ $b^x b^y = b^{x+y}$

PROOF:

$$\begin{aligned}
 b^x b^y &= \sup\{b^q b^r : q, r \in \mathbb{Q}, q < x, r < y\} \\
 &= \sup\{b^{q+r} : q, r \in \mathbb{Q}, q < x, r < y\} \\
 &= \sup\{b^q : q \in \mathbb{Q}, q < x + y\} \\
 &= b^{x+y}
 \end{aligned}$$

□

5.2.1 Logarithms

Proposition 5.39. *Let b and y be real numbers with $b > 1$ and $y > 0$. There exists a unique real x such that $b^x = y$.*

PROOF:

⟨1⟩1. LET: $x = \sup\{w : b^w < y\}$

PROVE: $b^x = y$

⟨2⟩1. $\{w : b^w < y\} \neq \emptyset$

PROOF: It contains 0.

⟨2⟩2. $\{w : b^w < y\}$ is bounded above.

⟨3⟩1. LET: n be the least integer such that

$$n \geq \frac{y-1}{b-1}$$

PROOF: Archimedean property.

⟨3⟩2. LET: w be a real number with $b^w < y$

PROVE: $w < n$

⟨3⟩3. $b^w < n(b-1) + 1$

⟨3⟩4. $b^w < b^n$

⟨3⟩5. $w < n$

⟨1⟩2. $b^x \leq y$

⟨2⟩1. ASSUME: for a contradiction $b^x > y$

⟨2⟩2. PICK a positive integer n such that $b^{x-1/n} > y$

PROOF: Lemma 5.35.

⟨2⟩3. PICK w such that $x - 1/n < w$ and $b^w < y$

PROOF: Since $x - 1/n$ is not an upper bound for $\{w : b^w < y\}$.

⟨2⟩4. $b^{x-1/n} < y$

⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩2.

⟨1⟩3. $b^x \geq y$

⟨2⟩1. ASSUME: for a contradiction $b^x < y$.

⟨2⟩2. PICK a positive integer n such that $b^{x+1/n} < y$.

⟨2⟩3. $x + 1/n \leq x$

⟨2⟩4. Q.E.D.

PROOF: This is a contradiction.

□

Definition 5.40 (Logarithm). Let b and y be real numbers with $b > 1$ and $y > 0$. The *logarithm* of y to base b , denoted $\log_b y$, is the unique real number

such that

$$b^{\log_b y} = y .$$

5.2.2 Intervals

Definition 5.41 (Intervals). Let $a, b \in \mathbb{R}$.

The *open interval* (a, b) is $\{x \in \mathbb{R} : a < x < b\}$.

The *closed interval* $[a, b]$ is $\{x \in \mathbb{R} : a \leq x \leq b\}$.

The *half-open intervals* $[a, b)$ and $(a, b]$ are defined by

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

Proposition 5.42. Let (I_n) be a sequence of closed intervals with $I_0 \supseteq I_1 \supseteq \dots$. Then $\bigcap_{n=0}^{\infty} I_n$ is nonempty.

PROOF:

$\langle 1 \rangle 1$. LET: $I_n = [a_n, b_n]$

$\langle 1 \rangle 2$. LET: $x = \sup_n a_n$

PROOF: $\{a_n : n \in \mathbb{N}\}$ is bounded above by b_0 .

$\langle 1 \rangle 3$. $x \in \bigcap_{n=0}^{\infty} I_n$

PROOF: For all n we have $a_n \leq x \leq b_n$ since b_n is an upper bound for $\{a_n : n \in \mathbb{N}\}$.

□

Definition 5.43 (k -cell). Let k be a positive integer. A k -cell is a subset of \mathbb{R}^k of the form

$$\{\vec{x} \in \mathbb{R}^k : \forall i = 1, \dots, k. a_i \leq x_i \leq b_i\}$$

for some real numbers $a_1, \dots, a_k, b_1, \dots, b_k$ with $a_i \leq b_i$ for each i .

Proposition 5.44. Let (I_n) be a sequence of k -cells such that $I_0 \supseteq I_1 \supseteq \dots$. Then $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$.

PROOF:

$\langle 1 \rangle 1$. LET: $I_n = J_{n1} \times \dots \times J_{nk}$ where each J_{ni} is a closed interval.

$\langle 1 \rangle 2$. For $i = 1, \dots, k$, PICK $a_i \in \bigcap_{n=0}^{\infty} J_{ni}$.

$\langle 1 \rangle 3$. $(a_1, \dots, a_k) \in \bigcap_{n=0}^{\infty} I_n$

□

5.2.3 The Cantor Set

Definition 5.45 (Cantor Set). Define a sequence E_n of unions of intervals as follows:

- $E_0 = [0, 1]$
- E_{n+1} is formed from E_n by replacing every interval $[a, b]$ with $[a, (2a+b)/3]$ and $[(a+2b)/3, b]$.

The *Cantor set* is $\bigcap_{n=0}^{\infty} E_n$.

5.3 The Extended Real Number System

Definition 5.46 (Extended Real Number System). The *extended real number system* is the set $\mathbb{R} \cup \{+\infty, -\infty\}$.

We extend the ordering \leq to the extended reals by defining

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

We extend $+$, \cdot and $/$ to partial operations on the extended real by defining:

$$\begin{array}{ll} x + (+\infty) = +\infty & (x \in \mathbb{R}) \\ x + (-\infty) = -\infty & (x \in \mathbb{R}) \\ (+\infty) + x = +\infty & (x \in \mathbb{R}) \\ (+\infty) + (+\infty) \text{ is undefined} & \\ (+\infty) + (-\infty) \text{ is undefined} & \\ (-\infty) + x = -\infty & (x \in \mathbb{R}) \\ (-\infty) + (+\infty) \text{ is undefined} & \\ (-\infty) + (-\infty) \text{ is undefined} & \\ x \cdot (+\infty) = +\infty & (x \in \mathbb{R}) \\ x \cdot (-\infty) = -\infty & (x \in \mathbb{R}) \\ (+\infty) \cdot x = +\infty & (x \in \mathbb{R}) \\ (+\infty) \cdot (+\infty) \text{ is undefined} & \\ (+\infty) \cdot (-\infty) \text{ is undefined} & \\ (-\infty) \cdot x = -\infty & (x \in \mathbb{R}) \\ (-\infty) \cdot (+\infty) \text{ is undefined} & \\ (-\infty) \cdot (-\infty) \text{ is undefined} & \\ x / (+\infty) = 0 & (x \in \mathbb{R}) \\ x / (-\infty) = 0 & (x \in \mathbb{R}) \\ (+\infty) / x \text{ is undefined} & (x \in \mathbb{R}) \\ (+\infty) / (+\infty) \text{ is undefined} & \\ (+\infty) / (-\infty) \text{ is undefined} & \\ (-\infty) / x \text{ is undefined} & (x \in \mathbb{R}) \\ (-\infty) / (+\infty) \text{ is undefined} & \\ (-\infty) / (-\infty) \text{ is undefined} & \end{array}$$

Chapter 6

Complex Analysis

Definition 6.1 (Complex Numbers). A *complex number* is a pair of real numbers. We write \mathbb{C} for the set of complex numbers.

Define $+$ and \cdot on \mathbb{C} by:

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b)(c, d) &= (ac - bd, ad + bc)\end{aligned}$$

Theorem 6.2. *The complex numbers form a field.*

Theorem 6.3. *The function that maps a to $(a, 0)$ is an embedding of \mathbb{R} in \mathbb{C} .*

Definition 6.4.

$$i = (0, 1)$$

Lemma 6.5.

$$(a, b) = a + ib$$

PROOF: Since $(a, 0) + (0, 1)(b, 0) = (a, b)$. \square

Lemma 6.6.

$$i^2 = -1$$

PROOF: Immediate from definitions. \square

Corollary 6.6.1. *There is no linear order on \mathbb{C} that makes \mathbb{C} into an ordered field.*

Definition 6.7 (Complex Conjugate). For any complex number z , the *complex conjugate* \bar{z} is defined by

$$\overline{a + ib} = a - ib \quad (a, b \in \mathbb{R}) .$$

Definition 6.8 (Real Part). For any complex number z , the *real part* of z , denoted $\operatorname{Re}(z)$, is defined by

$$\operatorname{Re}(a + ib) = a \quad (a, b \in \mathbb{R}) .$$

Definition 6.9 (Imaginary Part). For any complex number z , the *imaginary part* of z , denoted $\text{Im}(z)$, is defined by

$$\text{Im}(a + ib) = b \quad (a, b \in \mathbb{R}) .$$

Theorem 6.10. For all $z, w \in \mathbb{C}$ we have

$$\overline{z + w} = \bar{z} + \bar{w} .$$

PROOF:

$$\begin{aligned} \overline{(a + ib) + (c + id)} &= \overline{(a + c) + i(b + d)} \\ &= (a + c) - i(b + d) \\ &= (a - ib) + (c - id) \\ &= \overline{a + ib} + \overline{c + id} \end{aligned} \quad \square$$

Theorem 6.11. For all $z, w \in \mathbb{C}$ we have

$$\overline{zw} = \bar{z} \cdot \bar{w} .$$

PROOF:

$$\begin{aligned} \overline{(a + ib)(c + id)} &= \overline{(ac - bd) + i(ad + bc)} \\ &= (ac - bd) - i(ad + bc) \\ &= (a - ib)(c - id) \\ &= \overline{a + ib} \cdot \overline{c + id} \end{aligned} \quad \square$$

Theorem 6.12. For all $z \in \mathbb{C}$ we have

$$\text{Re}(z) = \frac{1}{2}(z + \bar{z}) .$$

PROOF:

$$\begin{aligned} (a + ib) + \overline{a + ib} &= (a + ib) + (a - ib) \\ &= 2a \\ &= 2 \text{Re}(a + ib) \end{aligned} \quad \square$$

Theorem 6.13. For all $z \in \mathbb{C}$ we have

$$\text{Im}(z) = \frac{1}{2i}(z - \bar{z}) .$$

PROOF:

$$\begin{aligned} (a + ib) - \overline{a + ib} &= (a + ib) - (a - ib) \\ &= 2ib \\ &= 2i \text{Im}(a + ib) \end{aligned} \quad \square$$

Theorem 6.14. For all $z \in \mathbb{C}$ we have $z\bar{z}$ is a non-negative real.

PROOF:

$$\begin{aligned}(a + ib)(\overline{a + ib}) &= (a + ib)(a - ib) \\ &= a^2 + b^2\end{aligned}\quad \square$$

Theorem 6.15. *For any $z \in \mathbb{C}$, if $z\bar{z} = 0$ then $z = 0$.*

PROOF: Let $z = a + ib$. Then $z\bar{z} = a^2 + b^2 = 0$ iff $a = b = 0$. \square

Definition 6.16 (Absolute Value). For $z \in \mathbb{C}$, the *absolute value* of z is

$$|z| = (z\bar{z})^{1/2}.$$

Proposition 6.17. *For x a non-negative real we have $|x| = x$.*

PROOF: Since $|x| = \sqrt{x^2} = x$. \square

Proposition 6.18. *For x a negative real we have $|x| = -x$.*

PROOF: Since $|x| = \sqrt{x^2} = -x$. \square

Theorem 6.19. *For any complex number z we have $|z| \geq 0$.*

PROOF: Immediate from definition. \square

Theorem 6.20. *For any complex number z , if $|z| = 0$ then $z = 0$.*

PROOF: From Theorem 6.15. \square

Theorem 6.21. *For any complex number z we have*

$$|\bar{z}| = |z|.$$

PROOF: Immediate from definitions. \square

Theorem 6.22. *For any complex numbers z and w we have*

$$|zw| = |z||w|.$$

PROOF:

$$\begin{aligned}|zw| &= \sqrt{zw\bar{z}\bar{w}} \\ &= \sqrt{z\bar{z}}\sqrt{w\bar{w}} && \text{(Proposition 5.26)} \\ &= |z||w|\end{aligned}\quad \square$$

Theorem 6.23. *For any complex number z we have*

$$|\operatorname{Re} z| \leq |z|$$

PROOF: Let $z = a + ib$. Then

$$|\operatorname{Re} z| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}. \square$$

Theorem 6.24. *For any complex numbers z and w we have*

$$|z + w| \leq |z| + |w|.$$

PROOF:

$$\begin{aligned}
 |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\
 &= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\
 &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 && \text{(Theorem 6.12)} \\
 &\leq |z|^2 + 2|z\bar{w}| + |w|^2 && \text{(Theorem 6.23)} \\
 &= |z|^2 + 2|z||w| + |w|^2 && \text{(Theorem 6.22)} \\
 &= (|z| + |w|)^2 && \square
 \end{aligned}$$

Theorem 6.25 (Schwarz Inequality). *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be complex numbers. Then*

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2 .$$

PROOF:

$$\langle 1 \rangle 1. \text{ LET: } A = \sum_{j=1}^n |a_j|^2$$

$$\langle 1 \rangle 2. \text{ LET: } B = \sum_{j=1}^n |b_j|^2$$

$$\langle 1 \rangle 3. \text{ LET: } C = \sum_{j=1}^n a_j \bar{b}_j$$

$$\langle 1 \rangle 4. \text{ ASSUME: w.l.o.g. } B > 0$$

PROOF: If $B = 0$ then $b_1 = \dots = b_n = 0$ and both sides of the inequality are 0.

$$\langle 1 \rangle 5. \sum_{j=1}^n |Ba_j - Cb_j|^2 = B(AB - |C|^2)$$

PROOF:

$$\begin{aligned}
 \sum_{j=1}^n |Ba_j - Cb_j|^2 &= \sum_{j=1}^n (Ba_j - Cb_j)(B\bar{a}_j - \bar{C}\bar{b}_j) \\
 &= B^2 \sum_{j=1}^n |a_j|^2 - B\bar{C} \sum_{j=1}^n a_j \bar{b}_j - BC \sum_{j=1}^n \bar{a}_j b_j + |C|^2 \sum_{j=1}^n |b_j|^2 \\
 &= B^2 A - 2B|C|^2 + B|C|^2 \\
 &= B(AB - |C|^2)
 \end{aligned}$$

$$\langle 1 \rangle 6. B(AB - |C|^2) \geq 0$$

$$\langle 1 \rangle 7. AB \geq |C|^2$$

\square

Proposition 6.26. *For any non-zero complex number w , there are exactly two complex numbers z such that $z^2 = w$.*

PROOF:

$$\langle 1 \rangle 1. \text{ There are at most two complex numbers } z \text{ such that } z^2 = w.$$

PROOF: Proposition 4.15.

$$\langle 1 \rangle 2. \text{ There are at least two complex numbers } z \text{ such that } z^2 = w.$$

$$\langle 2 \rangle 1. \text{ LET: } w = u + iv$$

$$\langle 2 \rangle 2. \text{ LET: } a = \sqrt{\frac{|w|+u}{2}}$$

$$\langle 2 \rangle 3. \text{ LET: } b = \sqrt{\frac{|w|-u}{2}}$$

⟨2⟩4. CASE: $v \geq 0$

⟨3⟩1. LET: $z = a + ib$

⟨3⟩2. $z^2 = w$

PROOF:

$$\begin{aligned} z^2 &= (a + ib)^2 \\ &= a^2 - b^2 + 2iab \\ &= u + i\sqrt{|w|^2 - u^2} \\ &= u + iv \\ &= w \end{aligned}$$

⟨3⟩3. $(-z)^2 = w$

⟨2⟩5. CASE: $v \leq 0$

⟨3⟩1. LET: $z = a - ib$

⟨3⟩2. $z^2 = w$

PROOF:

$$\begin{aligned} z^2 &= (a - ib)^2 \\ &= a^2 - b^2 - 2iab \\ &= u - i\sqrt{|w|^2 - u^2} \\ &= u - i|v| \\ &= w \end{aligned}$$

⟨3⟩3. $(-z)^2 = w$

□

6.1 Algebraic Numbers

Definition 6.27 (Algebraic). A complex number z is *algebraic* iff there exist integers a_0, a_1, \dots, a_n not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0 ;$$

otherwise, it is *transcendental*.

Proposition 6.28. *The set of algebraic numbers is countable.*

PROOF: There are countably many finite sequences of integers (a_0, a_1, \dots, a_n) , and for each one, there are only finitely many complex numbers z such that $a_0 z^n + a_1 z^{n-1} + \dots + a_n = 0$. □

Part I

Linear Algebra

Chapter 7

Vector Spaces

7.1 Convex Sets

Definition 7.1 (Convex). Let $E \subseteq \mathbb{R}^k$. Then E is *convex* iff, for all $\vec{x}, \vec{y} \in E$ and $\lambda \in (0, 1)$,

$$\lambda \vec{x} + (1 - \lambda) \vec{y} \in E .$$

Proposition 7.2. *Every k -cell is convex.*

PROOF:

$\langle 1 \rangle 1$. LET: $C = \{\vec{x} \in \mathbb{R}^k : \forall i. a_i \leq x_i \leq b_i\}$ be a k -cell.

$\langle 1 \rangle 2$. LET: $\vec{x}, \vec{y} \in C$ and $\lambda \in (0, 1)$.

PROVE: $\lambda \vec{x} + (1 - \lambda) \vec{y} \in C$

$\langle 1 \rangle 3$. For each i we have $a_i \leq \lambda x_i + (1 - \lambda) y_i \leq b_i$

PROOF: Since $\lambda a_1 + (1 - \lambda) a_i \leq \lambda x_i + (1 - \lambda) y_i \leq \lambda b_i + (1 - \lambda) b_i$.

□

Chapter 8

Real Inner Product Spaces

Definition 8.1 (Inner Product). Given $\vec{x}, \vec{y} \in \mathbb{R}^k$, define the *inner product* $\vec{x} \cdot \vec{y}$ by

$$(x_1, \dots, x_k) \cdot (y_1, \dots, y_k) = x_1 y_1 + \dots + x_k y_k \ .$$

Definition 8.2 (Norm). Define the *norm* of a vector $\vec{x} \in \mathbb{R}^k$ by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} \ .$$

Proposition 8.3.

$$\|\vec{x}\| \geq 0$$

PROOF: Immediate from the definition. \square

Proposition 8.4. If $\|\vec{x}\| = 0$ then $\vec{x} = \vec{0}$.

PROOF: If $\|\vec{x}\| = 0$ then $x_1^2 + \dots + x_n^2 = 0$ so $x_1 = \dots = x_n = 0$. \square

Proposition 8.5. For $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^k$,

$$\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| \ .$$

PROOF: Easy. \square

Proposition 8.6. For $\vec{x}, \vec{y} \in \mathbb{R}^k$, we have

$$\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\| \|\vec{y}\| \ .$$

PROOF: By the Schwarz inequality. \square

Proposition 8.7. For $\vec{x}, \vec{y} \in \mathbb{R}^k$ we have

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \ .$$

PROOF:

$$\begin{aligned}
 \|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
 &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\
 &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Proposition 8.6)} \\
 &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square
 \end{aligned}$$

Corollary 8.7.1. For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ we have

$$\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| .$$

8.1 Balls

Definition 8.8 (Closed Ball). Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *closed ball* with centre \vec{x} and radius r is

$$\{y \in \mathbb{R}^k : \|y - x\| \leq r\} .$$

Proposition 8.9. Every closed ball is convex.

PROOF:

(1)1. LET: B be the closed ball with center \vec{a} and radius r .

(1)2. LET: $\vec{x}, \vec{y} \in B$

(1)3. LET: $\lambda \in (0, 1)$

(1)4. $\lambda\vec{x} + (1 - \lambda)\vec{y} \in B$

PROOF:

$$\begin{aligned}
 \|\lambda\vec{x} + (1 - \lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1 - \lambda)(\vec{y} - \vec{a})\| \\
 &= \lambda\|\vec{x} - \vec{a}\| + (1 - \lambda)\|\vec{y} - \vec{a}\| \\
 &\leq \lambda r + (1 - \lambda)r \\
 &= r && \square
 \end{aligned}$$

\square

Chapter 9

Complex Inner Product Spaces

Definition 9.1 (Inner Product). Let V be a complex vector space. An *inner product* on V is a function $\langle \cdot, \cdot \rangle : V^2 \rightarrow \mathbb{C}$ such that, for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

- $\langle y, x \rangle = \overline{\langle x, y \rangle}$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- $\langle x, x \rangle \geq 0$
- If $\langle x, x \rangle = 0$ then $x = 0$.

An *inner product space* consists of a complex vector space V and an inner product on V .

Definition 9.2 (Norm). Let V be an inner product space and $x \in V$. The *norm* of x is

$$\|x\| = \sqrt{\langle x, x \rangle} .$$

Proposition 9.3. *An inner product space is a metric space under*

$$d(x, y) = \|x - y\| .$$

Definition 9.4 (Bounded). Let V_1 and V_2 be inner product spaces and $T : V_1 \rightarrow V_2$ a linear transformation. Then T is *bounded* iff $\{\|T(x)\| : \|x\| = 1\}$ is bounded above.

Proposition 9.5. *Every linear transformation between finite dimensional inner product spaces is bounded.*

Definition 9.6 (Outer Product). Let V be an inner product space and $|\psi\rangle, |\phi\rangle \in V$. The *outer product* of $|\psi\rangle$ and $|\phi\rangle$ is

$$|\psi\rangle \langle \phi| : V \rightarrow V .$$

9.1 Hilbert Spaces

Definition 9.7 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Theorem 9.8 (Completeness Relation). Let \mathcal{H} be a Hilbert space. Let $\{|e_n\rangle\}_{n \in \mathbb{N}}$ be a countable orthonormal basis for \mathcal{H} . Then

$$\sum_{n=0}^{\infty} |e_n\rangle \langle e_n| = I \quad .$$

PROOF:

$\langle 1 \rangle 1$. LET: $|\psi\rangle \in \mathcal{H}$

$\langle 1 \rangle 2$. LET: $|\psi\rangle = \sum_{n=0}^{\infty} \alpha_n |e_n\rangle$

$\langle 1 \rangle 3$. $\sum_{n=0}^{\infty} \langle e_n | \phi \rangle |e_n\rangle = |\psi\rangle$

PROOF:

$$\begin{aligned} \sum_{n=0}^{\infty} \langle e_n | \phi \rangle |e_n\rangle &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_m \langle e_n | e_m \rangle |e_n\rangle \\ &= \sum_{n=0}^{\infty} \alpha_n |e_n\rangle \\ &= |\psi\rangle \end{aligned}$$

□

□

Definition 9.9 (Separable). A Hilbert space is *separable* iff it has a countable dense orthonormal basis.

Chapter 10

Lie Algebras

Definition 10.1 (Lie Algebra). Let K be a field. A *Lie algebra* \mathcal{L} over K consists of a vector space \mathcal{L} over K and an operation

$$[\cdot, \cdot] : \mathcal{L}^2 \rightarrow \mathcal{L} ,$$

the *Lie bracket* or *commutator*, such that, for all $x, y, z \in \mathcal{L}$ and $\alpha \in K$:

$$\begin{aligned} [x + y, z] &= [x, z] + [y, z] \\ [x, y + z] &= [x, y] + [x, z] \\ [\alpha x, y] &= \alpha[x, y] \\ [x, x] &= 0 \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \end{aligned} \quad \text{(Jacobi identity)}$$

Lemma 10.2. *If K has characteristic 0 then the condition $[x, x] = 0$ can be replaced with $[x, y] = -[y, x]$.*

Proposition 10.3. *The commutator is determined by its values on any basis for \mathcal{L} .*

Example 10.4. \mathbb{R}^3 with the cross product is a real Lie algebra.

Example 10.5. For any $n \geq 0$, we have $GL(n, K)$ is a Lie algebra over K under

$$[A, B] = AB - BA .$$

Definition 10.6 (Linear Lie Algebra). A *linear Lie algebra* over K is a Lie algebra over K that is a subalgebra of $GL(n, K)$ for some n .

Example 10.7 (Special Linear Algebra). The *special Linear algebra* $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \text{tr} = 0\}$ is a real linear Lie algebra.

Example 10.8 (Orthogonal Lie Algebra). The *orthogonal Lie algebra* $SO(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A \text{ is skew-symmetric}\}$ is a real linear Lie algebra.

Example 10.9. Let $u(n)$ be the set of all skew-Hermitian $n \times n$ -matrices as a real Lie algebra.

Let $su(n) = u(n) \cap SL(n, \mathbb{R})$.

Proposition 10.10. $SU(2)$ is spanned by the Pauli matrices

$$\sigma_x = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_y = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which satisfy

$$\begin{aligned} [\sigma_x, \sigma_y] &= \sigma_z \\ [\sigma_y, \sigma_z] &= \sigma_x \\ [\sigma_z, \sigma_x] &= \sigma_y \end{aligned}$$

10.1 Lie Algebar Homomorphisms

Definition 10.11 (Homomorphism). Let L_1 and L_2 be Lie algebras over the same field. A *Lie algebra homomorphism* $\phi : L_1 \rightarrow L_2$ is a linear transformation such that

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in L_1$.

Lemma 10.12. *Every bijective Lie algebra homomorphism is an isomorphism.*

Definition 10.13 (Representation). Let L be a real (complex) Lie algebra. A *representation* of L is a Lie algebra homomorphism $L \rightarrow GL(n, \mathbb{R})$ ($GL(n, \mathbb{C})$) for some n .

Example 10.14. The linear transformation $\mathbb{R}^3 \rightarrow su(2)$ defined by

$$i \mapsto \sigma_x, j \mapsto \sigma_y, k \mapsto \sigma_z$$

is a representation of \mathbb{R}^3 .

Part II

Topology

Chapter 11

Metric Spaces

Definition 11.1 (Metric). A *metric* on a set X is a function $d : X^2 \rightarrow \mathbb{R}$ such that, for all $x, y, z \in X$:

- $d(x, y) \geq 0$
- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- **Triangle Inequality** $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* X consists of a set X and a metric on X .

Example 11.2. \mathbb{R}^k is a metric space under $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$. The triangle inequality is Corollary 8.7.1.

Example 11.3. For any set X , the *discrete* metric on X is defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Proposition 11.4. Let (X, d) be a metric space and Y a subset of X . Then $d \upharpoonright Y^2$ is a metric on Y .

PROOF: Easy. \square

11.1 Balls

Definition 11.5 (Open Ball). Let $\vec{x} \in \mathbb{R}^k$ and $r > 0$. The *open ball* with centre \vec{x} and radius r is

$$\{y \in \mathbb{R}^k : \|y - x\| < r\} .$$

Proposition 11.6. Every open ball in \mathbb{R}^k is convex.

PROOF:

(1)1. LET: B be the open ball with center \vec{a} and radius r .

(1)2. LET: $\vec{x}, \vec{y} \in B$

(1)3. LET: $\lambda \in (0, 1)$

(1)4. $\lambda\vec{x} + (1 - \lambda)\vec{y} \in B$

PROOF:

$$\begin{aligned} \|\lambda\vec{x} + (1 - \lambda)\vec{y} - \vec{a}\| &= \|\lambda(\vec{x} - \vec{a}) + (1 - \lambda)(\vec{y} - \vec{a})\| \\ &= \lambda\|\vec{x} - \vec{a}\| + (1 - \lambda)\|\vec{y} - \vec{a}\| \\ &< \lambda r + (1 - \lambda)r \\ &= r \end{aligned}$$

□

□

11.2 Limit Points

Definition 11.7 (Limit Point). Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is a *limit point* of E iff every open ball with centre p contains a point of E other than p .

Proposition 11.8. Let X be a metric space. Let $E \subseteq X$. Let p be a limit point of E . Then every neighbourhood of p contains infinitely many points of E .

PROOF:

(1)1. ASSUME: for a contradiction N is a neighbourhood of p that contains only finitely many points q_1, \dots, q_n of $E - \{p\}$.

(1)2. LET: $r = \min(q_1, \dots, q_n)$

(1)3. LET: B be the open ball with centre p and radius r .

(1)4. B is a neighbourhood of p that contains no points of E other than p .

□

Corollary 11.8.1. A finite set has no limit points.

Definition 11.9 (Isolated Point). Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *isolated point* of E iff $p \in E$ and p is not a limit point of E .

11.3 Closed Sets

Definition 11.10 (Closed Set). Let X be a metric space. Let $E \subseteq X$. Then E is *closed* iff every limit point of E is a member of E .

11.4 Interior Points

Definition 11.11 (Interior Point). Let X be a metric space. Let $E \subseteq X$ and $p \in X$. Then p is an *interior point* of E iff there exists an open ball B with centre p such that $B \subseteq E$.

Definition 11.12 (Interior). The *interior* of a set E , denoted E° , is the set of all its interior points.

Proposition 11.13. *The interior of E is the largest open set that is included in E .*

PROOF:

- (1)1. LET: I be the interior of E .
- (1)2. I is open.
 - (2)1. LET: $p \in I$
 - (2)2. PICK an open ball B with centre p such that $B \subseteq E$.
 - (2)3. $B \subseteq I$
 - (3)1. LET: $q \in B$
 - (3)2. There exists an open ball B' with centre q such that $B' \subseteq B$.
 - (3)3. There exists an open ball B' with centre q such that $B' \subseteq E$.
 - (3)4. $q \in I$
- (1)3. If J is any open set and $J \subseteq E$ then $J \subseteq I$.
 - (2)1. LET: J be an open set.
 - (2)2. ASSUME: $J \subseteq E$
 - (2)3. For all $p \in J$, there exists an open ball B with centre p such that $B \subseteq J$.
 - (2)4. For all $p \in J$, there exists an open ball B with centre p such that $B \subseteq E$.
 - (2)5. $p \in I$

□

11.5 Open Sets

Definition 11.14 (Open Sets). Let X be a metric space. Let $E \subseteq X$. Then E is *open* iff every point in E is an interior point of E .

Proposition 11.15. *Every open ball is open.*

PROOF:

- (1)1. LET: B be an open ball with centre c and radius r .
- (1)2. LET: $x \in B$
- (1)3. LET: $\epsilon = r - d(x, c)$
- (1)4. LET: B' be the open ball with centre x and radius ϵ .
 - PROVE: $B' \subseteq B$
- (1)5. LET: $y \in B'$
- (1)6. $d(y, c) < r$

PROOF:

$$\begin{aligned}
 d(y, c) &\leq d(y, x) + d(x, c) && \text{(Triangle Inequality)} \\
 &< \epsilon + d(x, c) && ((1)5) \\
 &= r && ((1)3)
 \end{aligned}$$

□

Proposition 11.16. *A set is open if and only if its complement is closed.*

PROOF:

$\langle 1 \rangle 1$. LET: $E \subseteq X$

$\langle 1 \rangle 2$. If E is open then $X - E$ is closed.

$\langle 2 \rangle 1$. ASSUME: E is open.

$\langle 2 \rangle 2$. LET: p be a limit point of $X - E$.

PROVE: $p \in X - E$

$\langle 2 \rangle 3$. ASSUME: for a contradiction $p \in E$.

$\langle 2 \rangle 4$. PICK an open ball B with centre p such that $B \subseteq E$.

$\langle 2 \rangle 5$. B contains a point of $X - E$.

PROOF: $\langle 2 \rangle 2$

$\langle 2 \rangle 6$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 4$.

$\langle 1 \rangle 3$. If $X - E$ is closed then E is open.

$\langle 2 \rangle 1$. ASSUME: $X - E$ is closed.

$\langle 2 \rangle 2$. LET: $p \in E$

$\langle 2 \rangle 3$. ASSUME: for a contradiction no open ball with centre p is a subset of E .

$\langle 2 \rangle 4$. Every open ball with centre p intersects $X - E$.

$\langle 2 \rangle 5$. p is a limit point of $X - E$.

$\langle 2 \rangle 6$. $p \in X - E$

PROOF: $\langle 2 \rangle 1$

$\langle 2 \rangle 7$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 2$.

□

Corollary 11.16.1. *A set is closed if and only if its complement is open.*

Proposition 11.17. *The union of a set of open sets is open.*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{U} be a set of open sets.

$\langle 1 \rangle 2$. LET: $p \in \bigcup \mathcal{U}$

$\langle 1 \rangle 3$. PICK $U \in \mathcal{U}$ such that $p \in U$.

$\langle 1 \rangle 4$. PICK an open ball B with centre p such that $B \subseteq U$.

$\langle 1 \rangle 5$. $B \subseteq \bigcup \mathcal{U}$

□

Corollary 11.17.1. *The intersection of a set of closed sets is closed.*

Proposition 11.18. *The intersection of two open sets is open.*

PROOF:

$\langle 1 \rangle 1$. LET: U and V be open.

$\langle 1 \rangle 2$. LET: $p \in U \cap V$

$\langle 1 \rangle 3$. PICK open balls B_1 and B_2 with centre p such that $B_1 \subseteq U$ and $B_2 \subseteq V$.

$\langle 1 \rangle 4$. ASSUME: w.l.o.g. the radius of B_1 is \leq the radius of B_2 .

$\langle 1 \rangle 5$. $B_1 \subseteq U \cap V$

□

Corollary 11.18.1. *The union of two closed sets is closed.*

Example 11.19. The intersection of a set of open sets is not necessarily open.

For every positive integer n , we have $(-1/n, 1/n)$ is open in \mathbb{R} , but $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$ is not open.

Theorem 11.20. *Let X be a metric space. Let $Y \subseteq X$ and $E \subseteq Y$. Then E is open in Y if and only if there exists an open subset G of X such that $E = G \cap Y$.*

PROOF:

(1)1. If E is open in Y then there exists an open subset G of X such that $E = G \cap Y$.

(2)1. ASSUME: E is open in Y .

(2)2. For $p \in E$, PICK $r_p > 0$ such that the open ball in Y with centre p and radius r_p is included in E .

(2)3. For $p \in E$,

LET: V_p be the open ball in X with centre p and radius r_p .

(2)4. LET: $G = \bigcup_{p \in E} V_p$

(2)5. G is open in X .

PROOF: Proposition 11.17.

(2)6. $E = G \cap Y$

(3)1. $E \subseteq G \cap Y$

(4)1. LET: $p \in E$

(4)2. $p \in V_p$

(4)3. $p \in G$

(3)2. $G \cap Y \subseteq E$

(4)1. LET: $x \in G \cap Y$

(4)2. PICK $p \in E$ such that $x \in V_p$

(4)3. $d(x, p) < r_p$

(4)4. $x \in E$

(1)2. For any open subset G of X , we have $G \cap Y$ is open in Y .

(2)1. LET: G be an open subset of X .

(2)2. LET: $p \in G \cap Y$

(2)3. PICK $r > 0$ such that the open ball in X with centre p and radius r is included in G .

(2)4. The open ball in Y with centre p and radius r is included in $G \cap Y$.

□

11.6 Perfect Sets

Definition 11.21 (Perfect Set). Let X be a metric space. Let $E \subseteq X$. Then E is *perfect* iff E is closed and every point in E is a limit point of E .

11.7 Bounded Sets

Definition 11.22 (Bounded Set). Let X be a metric space. Let $E \subseteq X$. Then E is *bounded* iff there exists a real number M and $q \in X$ such that, for all $p \in E$, we have $d(p, q) < M$.

Definition 11.23 (Diameter). Let X be a metric space and $E \subseteq X$ be bounded. Then the *diameter* of E is $\sup\{d(x, y) : x, y \in E\}$.

Proposition 11.24. Let X be a metric space. Let $E \subseteq X$ be bounded. Then \overline{E} is bounded and

$$\text{diam } \overline{E} = \text{diam } E .$$

PROOF:

(1)1. $\text{diam } E$ is an upper bound for $\{d(x, y) : x, y \in \overline{E}\}$.

⟨2⟩1. LET: $x, y \in \overline{E}$

⟨2⟩2. For all $\epsilon > 0$ we have $d(x, y) < \text{diam } E + \epsilon$.

⟨3⟩1. LET: $\epsilon > 0$

⟨3⟩2. PICK $x', y' \in E$ such that $d(x', x) < \epsilon/2$ and $d(y', y) < \epsilon/2$

⟨3⟩3. $d(x', y') < \text{diam } E$

⟨3⟩4. $d(x, y) < \text{diam } E + \epsilon$

⟨2⟩3. $d(x, y) \leq \text{diam } E$

(1)2. $\text{diam } \overline{E}$ is an upper bound for $\{d(x, y) : x, y \in E\}$.

PROOF: This follows since $E \subseteq \overline{E}$.

□

11.8 Dense Sets

Definition 11.25 (Dense Set). Let X be a metric space. Let $E \subseteq X$. Then E is *dense* iff every point of X is either a limit point of E or a point of E , or both.

11.9 Closure

Definition 11.26 (Closure). Let X be a metric space. Let $E \subseteq X$. Then the *closure* of E , denoted \overline{E} , is the union of E and the set of limit points of E .

Proposition 11.27. \overline{E} is the smallest closed set that includes E .

PROOF:

(1)1. \overline{E} is closed.

⟨2⟩1. LET: p be a limit point of \overline{E} .

⟨2⟩2. ASSUME: $p \notin E$

PROVE: p is a limit point of E .

⟨2⟩3. LET: B be the open ball with centre p and radius r .

PROVE: B intersects E .

⟨2⟩4. PICK a point $q \in B \cap \overline{E}$.

⟨2⟩5. PICK an open ball B' with centre q such that $B' \subseteq B$.

- ⟨2⟩6. PICK a point $r \in E \cap B'$
- ⟨2⟩7. $r \in E \cap B$
- ⟨1⟩2. If C is closed and $E \subseteq C$ then $\overline{E} \subseteq C$.
 - ⟨2⟩1. ASSUME: C is closed.
 - ⟨2⟩2. ASSUME: $E \subseteq C$
 - ⟨2⟩3. LET: $p \in \overline{E}$
 - ⟨2⟩4. ASSUME: for a contradiction $p \notin C$
 - ⟨2⟩5. p is a limit point of C .
 - ⟨3⟩1. LET: B be an open ball with centre p .
 - ⟨3⟩2. B intersects E .
 - ⟨3⟩3. B intersects C .
 - ⟨3⟩4. B intersects C in a point other than p .
- PROOF: ⟨2⟩3
- ⟨2⟩6. Q.E.D.
- PROOF: This contradicts ⟨2⟩1.

□

Corollary 11.27.1. E is closed if and only if $E = \overline{E}$.

Theorem 11.28. Let E be a nonempty set of real numbers bounded above. Then $\sup E \in \overline{E}$.

PROOF:

- ⟨1⟩1. ASSUME: $\sup E \notin E$
PROVE: $\sup E$ is a limit point of E .
- ⟨1⟩2. LET: B be an open ball with centre $\sup E$ and radius r .
- ⟨1⟩3. There exists $x \in E$ such that $x > \sup E - r$.
- ⟨1⟩4. E intersects B in a point other than p .

□

Proposition 11.29.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:

- ⟨1⟩1. $\overline{A \cup B}$ is a closed set that includes $A \cup B$.
- ⟨1⟩2. If C is a closed set that includes $A \cup B$ then $\overline{A} \cup \overline{B} \subseteq C$.

□

Example 11.30. It is not true in general. that $\overline{\bigcup \mathcal{A}} = \bigcup_{A \in \mathcal{A}} \overline{A}$.

In \mathbb{R} , let $\mathcal{A} = \{\{1/n\} : n \in \mathbb{Z}^+\}$. Then

$$\begin{aligned} \overline{\bigcup \mathcal{A}} &= \{1/n : n \in \mathbb{Z}^+\} \cup \{0\} \\ \bigcup_{A \in \mathcal{A}} \overline{A} &= \{1/n : n \in \mathbb{Z}^+\} \end{aligned}$$

Proposition 11.31.

$$X - E^\circ = \overline{X - E}$$

PROOF:

$$\begin{aligned}
 p \in X - E^\circ &\Leftrightarrow p \notin E^\circ \\
 &\Leftrightarrow \forall B \text{ an open ball with centre } p, B \not\subseteq E \\
 &\Leftrightarrow \forall B \text{ an open ball with centre } p, B \text{ intersects } X - E \\
 &\Leftrightarrow p \in \overline{X - E} \quad \square
 \end{aligned}$$

11.10 Compact Sets

Definition 11.32 (Open Cover). Let X be a metric space. Let $E \subseteq X$. An *open cover* of E is a set \mathcal{U} of open sets such that $E \subseteq \bigcup \mathcal{U}$.

Definition 11.33 (Compact Set). Let X be a metric space. Let $K \subseteq X$. Then K is *compact* iff every open cover of K includes a finite subcover.

Proposition 11.34. *Every finite set is compact.*

PROOF: Easy. \square

Theorem 11.35. *Let X be a metric space. Let $Y \subseteq X$ and $K \subseteq Y$. Then K is compact in Y if and only if K is compact in X .*

PROOF:

- (1) 1. If K is compact in Y then K is compact in X .
 - <2>1. ASSUME: K is compact in Y .
 - <2>2. LET: \mathcal{U} be an open cover of K in X .
 - <2>3. $\{U \cap Y : U \in \mathcal{U}\}$ is an open cover of K in Y .
 - <2>4. PICK a finite subcover $\{U_1 \cap Y, \dots, U_n \cap Y\}$
 - <2>5. $\{U_1, \dots, U_n\}$ is a finite subset of \mathcal{U} that is an open cover of K in X .
- (1) 2. If K is compact in X then K is compact in Y .
 - <2>1. ASSUME: K is compact in X .
 - <2>2. LET: \mathcal{U} be an open cover of K in Y .
 - <2>3. $\{U \text{ open in } X : U \cap Y \in \mathcal{U}\}$ is an open cover of K in X .
 - <2>4. PICK a finite subcover $\{U_1, \dots, U_n\}$.
 - <2>5. $\{U_1 \cap Y, \dots, U_n \cap Y\}$ is a subset of \mathcal{U} that is an open cover of K in Y .

\square

Proposition 11.36. *Every compact set is closed.*

PROOF:

- (1) 1. LET: E be compact.
- (1) 2. LET: $p \in X - E$
 - PROVE: There exists an open ball with centre p that is a subset of $X - E$.
- (1) 3. For all $q \in E$, there exist disjoint open balls B with centre q and B' with centre p .
- (1) 4. The set of open balls B such that there exists a disjoint open ball B' with centre p is an open cover of E .
- (1) 5. PICK a finite subcover $\{B_1, \dots, B_n\}$.

- (1)6. For $i = 1, \dots, n$, PICK an open ball B'_i with centre p such that $B_i \cap B'_i = \emptyset$.
 (1)7. $B'_1 \cap \dots \cap B'_n$ is an open ball with centre p that is a subset of $X - E$.

□

Proposition 11.37. *Every closed subset of a compact set is compact.*

PROOF:

- (1)1. LET: E be compact and $C \subseteq E$ be closed.
 (1)2. LET: \mathcal{U} be an open cover of C .
 (1)3. $\mathcal{U} \cup \{X - C\}$ is an open cover of E .
 (1)4. PICK a finite subcover $\{U_1, \dots, U_n\}$ or $\{U_1, \dots, U_n, X - C\}$.
 (1)5. $\{U_1, \dots, U_n\}$ covers C .

□

Corollary 11.37.1. *The intersection of a compact set and a closed set is compact.*

Proposition 11.38. *Let \mathcal{K} be a nonempty set of compact sets. If every nonempty finite subset of \mathcal{K} has nonempty intersection, then $\bigcap \mathcal{K}$ is nonempty.*

PROOF:

- (1)1. PICK $K \in \mathcal{K}$
 (1)2. ASSUME: $\bigcap \mathcal{K} = \emptyset$
 (1)3. $\{X - K' : K' \in \mathcal{K}\}$ is an open cover of K .
 (1)4. PICK a finite subcover $\{X - K_1, \dots, X - K_n\}$.
 (1)5. There exists $p \in K \cap K_1 \cap \dots \cap K_n$
 (1)6. Q.E.D.

PROOF: (1)4 and (1)5 form a contradiction.

□

Corollary 11.38.1. *Let (K_n) be a sequence of nonempty compact sets such that $K_0 \supseteq K_1 \supseteq \dots$. Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.*

Theorem 11.39. *Let X be a metric space and $E \subseteq X$. Then E is compact if and only if every infinite subset of E has a limit point in E .*

PROOF:

- (1)1. If E is compact then every infinite subset of E has a limit point in E .
 (2)1. ASSUME: E is compact.
 (2)2. LET: $A \subseteq E$ be infinite.
 (2)3. ASSUME: for a contradiction E has no limit point in K .
 (2)4. For all $p \in K$, there exists an open ball B with centre p such that B does not intersect E outside p .
 (2)5. The set of open balls that intersect E in at most one point is an open cover for K .
 (2)6. PICK a finite subcover B_1, \dots, B_n .
 (2)7. E has at most n points.
 (2)8. Q.E.D.

PROOF: This contradicts the fact that E is finite.

- (1)2. If every infinite subset of K has a limit point in K then K is compact.
- (2)1. ASSUME: Every infinite subset of K has a limit point in K .
- (2)2. LET: \mathcal{U} be an open cover of K .
- (2)3. ASSUME: w.l.o.g. \mathcal{U} is countable.
- PROOF: We may replace \mathcal{U} with the set of all open balls B with centres in \mathbb{Q}^2 and rational radius such that there exists $U \in \mathcal{U}$ such that $B \subseteq U$.
- (2)4. PICK an enumeration $\mathcal{U} = \{G_n : n \in \mathbb{N}\}$.
- (2)5. For $n \in \mathbb{N}$,
 LET: $F_n = \bigcup_{i=0}^n G_i$.
- (2)6. For all $n \in \mathbb{N}$, we have $K - F_n \neq \emptyset$.
- PROOF: Since $\{G_0, \dots, G_n\}$ does not cover K .
- (2)7. $\bigcap_{n=0}^{\infty} F_n = \emptyset$
- PROOF: Since $\{G_n : n \in \mathbb{N}\}$ covers K .
- (2)8. For $n \in \mathbb{N}$, PICK $a_n \in K - F_n$
- (2)9. LET: $E = \{a_n : n \in \mathbb{N}\}$
- (2)10. E is infinite.
- (3)1. LET: $n \in \mathbb{N}$
 PROVE: there exists m such that $a_m \notin \{a_0, a_1, \dots, a_n\}$.
- (3)2. For $i = 0, \dots, n$, PICK k_i such that $a_i \in G_{k_i}$.
- (3)3. LET: $m = \max(k_0, \dots, k_n)$
- (3)4. ASSUME: for a contradiction $a_m = a_i$ for some $i = 0, \dots, n$
- (3)5. $a_i \in G_{k_i}$
- (3)6. $a_i \notin F_m$
- (3)7. Q.E.D.
- PROOF: This is a contradiction since $k_i \leq m$.
- (2)11. PICK a limit point l for E in K .
- PROOF: From (2)1.
- (2)12. PICK n such that $l \in G_n$.
- (2)13. PICK an open ball B with centre l such that $B \subseteq G_n$
- (2)14. $B \cap E$ is infinite.
- PROOF: Proposition 11.8.
- (2)15. PICK $m \geq n$ such that $a_m \in B$.
- (2)16. $a_m \in G_n$
- (2)17. Q.E.D.
- PROOF: This is a contradiction since $a_m \notin F_m$.

□

Theorem 11.40 (Heine-Borel). *Let $E \subseteq \mathbb{R}^k$. Then E is compact if and only if it is closed and bounded.*

PROOF:

- (1)1. If E is compact then E is closed.
 PROOF: Proposition 11.36.
- (1)2. If E is compact then E is bounded.
 PROOF: Otherwise $\{(-N, N)^k : N \in \mathbb{Z}^+\}$ would be an open cover of E with no finite subcover.
- (1)3. If E is closed and bounded then E is compact.

- $\langle 2 \rangle 1$. ASSUME: E is closed and bounded.
- $\langle 2 \rangle 2$. PICK \vec{c} and M such that $\forall \vec{x} \in E, \|\vec{x} - \vec{c}\| < M$.
- $\langle 2 \rangle 3$. $E \subseteq \prod_{i=1}^k [c_i - M, c_i + M]$
- $\langle 2 \rangle 4$. E is compact.

PROOF: Proposition 11.37.

□

Corollary 11.40.1 (Weierstrass's Theorem). *Every bounded infinite subset of \mathbb{R}^k has a limit point.*

PROOF: It is a bounded infinite subset of some k -cell and therefore has a limit point in that k -cell. □

Example 11.41. It is not true that, in any metric space, a set is compact if and only if it is closed and bounded.

In \mathbb{Q} , the set $\{p \in \mathbb{Q} : 2 < p^2 < 3\}$ is closed and bounded but not compact.

Theorem 11.42. *Every nonempty perfect set in \mathbb{R}^k is uncountable.*

PROOF:

- $\langle 1 \rangle 1$. LET: P be a nonempty perfect set in \mathbb{R}^k .
- $\langle 1 \rangle 2$. P is infinite.

PROOF: Corollary 11.8.1.

- $\langle 1 \rangle 3$. ASSUME: for a contradiction P is countable.
- $\langle 1 \rangle 4$. PICK an enumeration $P = \{x_n : n \in \mathbb{N}\}$.
- $\langle 1 \rangle 5$. PICK a sequence (V_n) of open balls such that, for all n , we have $\overline{V_{n+1}} \subseteq V_n$ and $x_n \notin \overline{V_{n+1}}$ and $V_n \cap P \neq \emptyset$
 - $\langle 2 \rangle 1$. ASSUME: as induction hypothesis we have picked V_0, \dots, V_{n-1} that satisfy these conditions.
 - $\langle 2 \rangle 2$. PICK $p \in P \cap V_n$ such that $p \neq x_n$

PROOF: We cannot have $P \cap V_n = \{x_n\}$ because then V_n would be a neighbourhood of x_n that only intersects P at x_n .
 - $\langle 2 \rangle 3$. PICK an open ball B with centre p such that $B \subseteq V_n \cap P - \{x_n\}$
 - $\langle 2 \rangle 4$. LET: V_{n+1} be the open ball with centre p and half the radius of B .
 - $\langle 2 \rangle 5$. $\overline{V_{n+1}} \subseteq V_n$

PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq V_n$.
 - $\langle 2 \rangle 6$. $x_n \notin \overline{V_{n+1}}$

PROOF: Since $\overline{V_{n+1}} \subseteq B \subseteq P - \{x_n\}$.
 - $\langle 2 \rangle 7$. $V_{n+1} \cap P \neq \emptyset$

PROOF: Since $p \in V_{n+1} \cap P$.
- $\langle 1 \rangle 6$. For $n \in \mathbb{N}$,

LET: $K_n = \overline{V_n} \cap P$.
- $\langle 1 \rangle 7$. For all $n \in \mathbb{N}$, K_n is compact.

PROOF: By the Heine-Borel Theorem.
- $\langle 1 \rangle 8$. $\bigcap_{n=0}^{\infty} K_n \cap P = \emptyset$

PROOF: Since for each n we have $x_n \notin K_{n+1}$.
- $\langle 1 \rangle 9$. $\bigcap_{n=0}^{\infty} K_n = \emptyset$

PROOF: Since $\bigcap_{n=0}^{\infty} K_n \subseteq P$.

(1)10. Q.E.D.

PROOF: This contradicts Proposition 11.38.

□

Corollary 11.42.1. *For any $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a, b]$ is uncountable.*

Corollary 11.42.2. *\mathbb{R} is uncountable.*

Corollary 11.42.3. *The set of transcendental numbers is uncountable.*

PROOF: Since the set of algebraic numbers is countable. □

Example 11.43. The Cantor set is a perfect set in \mathbb{R} that does not include any open interval.

PROOF:

(1)1. LET: (E_n) be the sequence of unions of closed intervals from the definition of the Cantor set, and C be the Cantor set.

(1)2. $C \neq \emptyset$

PROOF: Since $0 \in C$.

(1)3. C is closed.

PROOF: Each E_n is closed and C is their intersection.

(1)4. Every point of C is a limit point of C .

(2)1. LET: $p \in C$

(2)2. LET: B be an open ball with centre p and radius r .

(2)3. PICK n such that each of the intervals that make up E_n has length $< r/2$.

(2)4. LET: I be the interval in E_n that contains p .

(2)5. $I \subseteq B$

(2)6. The endpoint of I that is not p is in $P \cap B$.

(1)5. C does not include any open interval.

(2)1. LET: (α, β) be any open interval.

(2)2. PICK m such that $3^{-m} < (\beta - \alpha)/6$

(2)3. PICK k such that $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}) \subseteq (\alpha, \beta)$

(2)4. $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}) \subseteq P$

(2)5. $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}) \cap E_m = \emptyset$

(2)6. Q.E.D.

PROOF: This is a contradiction.

□

Corollary 11.43.1. *The Cantor set is uncountable.*

Proposition 11.44. *Let X be a metric space. Let (K_n) be a sequence of compact sets in X such that $K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$. Assume $\text{diam } K_n \rightarrow 0$ as $n \rightarrow \infty$. Then $\bigcap_{n=0}^{\infty} K_n$ is a singleton.*

PROOF:

(1)1. $\bigcap_n K_n \neq \emptyset$

PROOF: Corollary 11.38.1.

- (1)2. $\bigcap_n K_n$ has no more than one point.
- (2)1. ASSUME: for a contradiction $a, b \in \bigcap_n K_n$ with $a \neq b$.
- (2)2. LET: $\epsilon = d(a, b)$
- (2)3. PICK n such that $\text{diam } K_n < \epsilon$
- (2)4. $a, b \in K_n$
- (2)5. Q.E.D.

PROOF: This is a contradiction.

□

11.11 Connected Sets

Definition 11.45 (Separated). Let X be a metric space. Let $A, B \subseteq X$. Then A and B are *separated* iff $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Proposition 11.46. *Any two disjoint open sets are separated.*

PROOF:

- (1)1. LET: A and B be disjoint open sets.
- (1)2. ASSUME: for a contradiction $p \in \overline{A} \cap B$.
- (1)3. B is a neighbourhood of p .
- (1)4. B intersects A .

□

Definition 11.47 (Connected). Let X be a metric space. Let $E \subseteq X$. Then E is *connected* iff E is not the union of two nonempty separated sets.

Theorem 11.48. *A subset E of the real line is connected if and only if it is convex.*

PROOF:

- (1)1. If E is connected then E is convex.
 - (2)1. ASSUME: E is connected.
 - (2)2. LET: $x, y \in E$
 - (2)3. LET: $z \in (x, y)$
 - (2)4. $z \in E$

PROOF: Otherwise $E \cap (-\infty, z)$ and $E \cap (z, +\infty)$ would be a separation of E .
- (1)2. If E is convex then E is connected.
 - (2)1. ASSUME: E is convex.
 - (2)2. ASSUME: for a contradiction $E = A \cup B$ where A and B are nonempty and separated.
 - (2)3. PICK $a \in A$ and $b \in B$.
 - (2)4. ASSUME: w.l.o.g. $a < b$
 - (2)5. LET: $z = \sup(A \cap [a, b])$
 - (2)6. $z \in \overline{A}$
 - (2)7. $z \notin B$

$\langle 2 \rangle 8. z < b$

$\langle 2 \rangle 9. \text{ CASE: } z \in A$

$\langle 3 \rangle 1. z \notin \overline{B}$

$\langle 3 \rangle 2. \text{ PICK } z_1 \in (z, b) \text{ such that } z_1 \notin B$

$\langle 3 \rangle 3. a < z_1 < b$

$\langle 3 \rangle 4. z_1 \notin E$

PROOF: We have $z_1 \notin A$ from $\langle 2 \rangle 5$ since $z_1 \in [a, b]$ and $z_1 > z$, and $z_1 \notin B$ from $\langle 3 \rangle 2$.

$\langle 3 \rangle 5. \text{ Q.E.D.}$

PROOF: This contradicts $\langle 2 \rangle 1$.

$\langle 2 \rangle 10. \text{ CASE: } z \notin A$

PROOF: Then $a < z < b$ and $z \notin E$ contradicting $\langle 2 \rangle 1$.

□

Proposition 11.49. *Every connected metric space with more than one point is uncountable.*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } X \text{ be a connected metric space with more than one points.}$

$\langle 1 \rangle 2. \text{ PICK distinct points } p, q \in X.$

$\langle 1 \rangle 3. \text{ LET: } \epsilon = d(p, q)$

$\langle 1 \rangle 4. \text{ For every } r \in (0, \epsilon), \text{ there exists a point } x \in X \text{ such that } d(p, x) = r.$

PROOF: Otherwise $\{x \in X : d(p, x) < r\}$ and $\{x \in X : d(p, x) > r\}$ would form a separation of X .

□

Proposition 11.50. *The closure of a connected set is connected.*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } X \text{ be a metric space.}$

$\langle 1 \rangle 2. \text{ LET: } E \text{ be a connected subspace of } X.$

$\langle 1 \rangle 3. \text{ ASSUME: for a contradiction } A \text{ and } B \text{ form a separation of } \overline{E}$

PROVE: $A \cap E$ and $B \cap E$ form a separation of E .

$\langle 1 \rangle 4. A \cap E \neq \emptyset$

$\langle 2 \rangle 1. \text{ ASSUME: for a contradiction } A \cap E = \emptyset$

$\langle 2 \rangle 2. E \subseteq B$

$\langle 2 \rangle 3. \overline{E} \subseteq \overline{B}$

$\langle 2 \rangle 4. A \subseteq \overline{B}$

$\langle 2 \rangle 5. A \cap \overline{B} = A \neq \emptyset$

$\langle 2 \rangle 6. \text{ Q.E.D.}$

PROOF: This contradicts $\langle 1 \rangle 3$.

$\langle 1 \rangle 5. B \cap E \neq \emptyset$

PROOF: Similar.

$\langle 1 \rangle 6. \overline{A \cap E} \cap B \cap E = \emptyset$

PROOF: Since $\overline{A \cap E} \cap B \cap E \subseteq \overline{A} \cap B$.

$\langle 1 \rangle 7. A \cap E \cap \overline{B \cap E} = \emptyset$

PROOF: Similar.

□

Example 11.51. The interior of a connected set is not necessarily connected.

Two touching discs in \mathbb{R}^2 form a connected set but the interior is disconnected.

Proposition 11.52. *Every convex set in \mathbb{R}^k is connected.*

PROOF:

⟨1⟩1. LET: E be a convex set in \mathbb{R}^k .

⟨1⟩2. ASSUME: for a contradiction A and B form a separation of E .

⟨1⟩3. PICK $\vec{a} \in A$ and $\vec{b} \in B$.

⟨1⟩4. Define $p : [0, 1] \rightarrow \mathbb{R}^k$ by $p(t) = (1 - t)\vec{a} + t\vec{b}$.

⟨1⟩5. $p^{-1}(A)$ and $p^{-1}(B)$ are separated sets in \mathbb{R} .

⟨1⟩6. PICK $x \in [0, 1]$ such that $x \notin p^{-1}(A)$ and $x \notin p^{-1}(B)$.

PROOF: There exists such an x since $[0, 1]$ is connected.

⟨1⟩7. $p(x) \in E$

PROOF: Since E is convex.

⟨1⟩8. $p(x) \notin A \cup B$

⟨1⟩9. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

11.12 Separable Spaces

Definition 11.53 (Separable). A metric space is *separable* iff it has a countable dense subset.

Example 11.54. \mathbb{R}^k is separable since \mathbb{Q}^k is dense.

Proposition 11.55. *Every compact metric space is separable.*

PROOF:

⟨1⟩1. LET: X be a compact metric space.

⟨1⟩2. For $n \in \mathbb{Z}^+$, pick finitely many points a_{n1}, \dots, a_{nr_n} such that $\{B(a_{ni}, 1/n) : 1 \leq i \leq r_n\}$ covers X .

PROOF: Since $\{B(x, 1/n) : x \in X\}$ covers X .

⟨1⟩3. $\{a_{ni} : n \in \mathbb{Z}^+, 1 \leq i \leq r_n\}$ is dense.

⟨2⟩1. LET: U be an open set and $p \in U$.

⟨2⟩2. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$.

⟨2⟩3. PICK n such that $1/n < \epsilon$.

⟨2⟩4. PICK i such that $p \in B(a_{ni}, 1/n)$

⟨2⟩5. $a_{ni} \in U$

□

11.13 Bases

Definition 11.56 (Basis). A *basis* for a metric space X is a set \mathcal{B} of open sets such that, for every open set U and point $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.

Proposition 11.57. *Every separable metric space has a countable basis.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be a separable metric space.
- $\langle 1 \rangle 2$. PICK a countable dense set D in X .
- $\langle 1 \rangle 3$. LET: $\mathcal{B} = \{B(p, \epsilon) : p \in D, \epsilon \in \mathbb{Q}^+\}$
- PROVE: \mathcal{B} is a basis.
- $\langle 1 \rangle 4$. LET: U be an open set in X and $p \in U$
- $\langle 1 \rangle 5$. PICK $\epsilon > 0$ such that $B(p, \epsilon) \subseteq U$
- $\langle 1 \rangle 6$. PICK $q \in B(p, \epsilon) \cap D$
- $\langle 1 \rangle 7$. PICK a rational δ such that $d(p, q) < \delta < \epsilon$.
- $\langle 1 \rangle 8$. $B(q, \delta) \in \mathcal{B}$ and $B(q, \delta) \subseteq U$.

□

11.14 Condensation Points

Definition 11.58 (Condensation Point). Let X be a metric space, $p \in X$ and $E \subseteq X$. Then p is a *condensation point* of E iff every neighbourhood of p contains uncountably many points in E .

Proposition 11.59. *Let X be a metric space. Let $E \subseteq X$. Let P be the set of condensation points of E . Then P is perfect.*

PROOF:

- $\langle 1 \rangle 1$. P is closed.
- $\langle 2 \rangle 1$. LET: $p \in X - P$
- $\langle 2 \rangle 2$. PICK a neighbourhood U of p that contains only countably many points of E .
- $\langle 2 \rangle 3$. For every $x \in U$, we have that U is a neighbourhood of x that contains only countably many points of E .
- $\langle 2 \rangle 4$. $p \in U \subseteq X - P$
- $\langle 1 \rangle 2$. Every point in P is a limit point of P .

PROOF: Immediate from definitions.

□

Proposition 11.60. *Let X be a metric space with a countable basis. Let $E \subseteq X$ be uncountable. Let P be the set of condensation points of E . Then $E - P$ is countable.*

PROOF:

- $\langle 1 \rangle 1$. PICK a countable basis \mathcal{B} for X .
- $\langle 1 \rangle 2$. LET: $W = \bigcup \{B \in \mathcal{B} : E \cap B \text{ is countable}\}$

- ⟨1⟩3. $P = X - W$
- ⟨2⟩1. $P \subseteq X - W$
 - ⟨3⟩1. ASSUME: for a contradiction $p \in P \cap W$
 - ⟨3⟩2. PICK $B \in \mathcal{B}$ such that $p \in B$ and $E \cap B$ is countable.
 - ⟨3⟩3. $E \cap B$ is uncountable.
 - ⟨3⟩4. Q.E.D.
- PROOF: This is a contradiction.
- ⟨2⟩2. $X - W \subseteq P$
 - ⟨3⟩1. LET: $p \in X - W$
 - ⟨3⟩2. LET: U be a neighbourhood of p .
 - ⟨3⟩3. PICK $B \in \mathcal{B}$ such that $p \in B \subseteq U$.
 - ⟨3⟩4. $E \cap B$ is uncountable.
 - PROOF: Since $p \notin W$.
 - ⟨3⟩5. $E \cap W$ is uncountable.
- ⟨1⟩4. $E - P = E \cap W$
- ⟨1⟩5. $E - P$ is countable.

□

Corollary 11.60.1. *Every closed subset of a metric space with a countable basis is the union of a perfect set and a countable set.*

PROOF:

- ⟨1⟩1. LET: X be a metric space with a countable basis.
- ⟨1⟩2. LET: E be a closed subset of X .
- ⟨1⟩3. LET: P be the set of condensation points of E .
- ⟨1⟩4. $E - P$ is countable.

PROOF: Proposition 11.60.

- ⟨1⟩5. $P \cap E$ is perfect.

- ⟨2⟩1. $P \cap E$ is closed.

PROOF: Proposition 11.59.

- ⟨2⟩2. Every point in $P \cap E$ is a limit point of $P \cap E$.

- ⟨3⟩1. LET: $l \in P \cap E$
- ⟨3⟩2. LET: U be a neighbourhood of l .
- ⟨3⟩3. PICK $x \in P \cap U$
- ⟨3⟩4. U is a neighbourhood of x .
- ⟨3⟩5. U contains uncountably many points of E .
- ⟨3⟩6. U intersects $P \cap E$

PROOF: It cannot be that every point in U and E is not in P since $E - P$ is countable.

□

Corollary 11.60.2. *Let X be a metric space with a countable basis. Then every countable set in X has an isolated point.*

Chapter 12

Convergence

Definition 12.1 (Converge). Let X be a metric space. Let (p_n) be a sequence in X and $l \in X$. Then we say (p_n) *converges* to the *limit* l , and write

$$p_n \rightarrow l \text{ as } n \rightarrow \infty ,$$

iff for every $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon$.

We say (p_n) *diverges* iff it does not converge to any limit.

Proposition 12.2. *A sequence has at most one limit.*

PROOF:

(1)1. ASSUME: $p_n \rightarrow l$ and $p_n \rightarrow m$ as $n \rightarrow \infty$.

(1)2. ASSUME: for a contradiction $l \neq m$.

(1)3. LET: $\epsilon = d(l, m)/2$

(1)4. There exists N such that $\forall n \geq N. d(p_n, l) < \epsilon$ and $d(p_n, m) < \epsilon$

(1)5. $d(l, m) < 2\epsilon$

(1)6. Q.E.D.

PROOF: This is a contradiction.

□

Proposition 12.3. *Every convergent sequence is bounded.*

PROOF:

(1)1. LET: $p_n \rightarrow l$ as $n \rightarrow \infty$

(1)2. PICK N such that $\forall n \geq N. d(p_n, l) < 1$

(1)3. LET: $M = \max(d(p_0, l), \dots, d(p_{N-1}, l), 1)$

(1)4. For all n , we have $d(p_n, l) \leq M$.

□

Proposition 12.4. *If l is a limit point of E , then there exists a sequence in E that converges to l .*

PROOF:

$\langle 1 \rangle 1$. For $n \in \mathbb{Z}^+$, PICK a point $a_n \in E$ such that $d(a_n, l) < 1/n$.

PROOF: Since $B(l, 1/n)$ intersects E .

$\langle 1 \rangle 2$. $a_n \rightarrow l$ as $n \rightarrow \infty$.

□

Corollary 12.4.1. *Every sequence in a compact metric space has a convergent subsequence.*

PROOF: By Theorem 11.39. □

Proposition 12.5. *Assume $s_n \rightarrow s$ and $t_n \rightarrow t$ in \mathbb{C} . Then $s_n + t_n \rightarrow s + t$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\epsilon > 0$

$\langle 1 \rangle 2$. PICK N such that, for all $n \geq N$, we have $d(s_n, s) < \epsilon/2$ and $d(t_n, t) < \epsilon/2$.

$\langle 1 \rangle 3$. For all $n \geq N$ we have $d(s_n + t_n, s + t) < \epsilon$.

□

Lemma 12.6. *If $s_n \rightarrow s$ as $n \rightarrow \infty$ in \mathbb{C} , and $c \in \mathbb{C}$, then $cs_n \rightarrow cs$ as $n \rightarrow \infty$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\epsilon > 0$

$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $c \neq 0$

$\langle 1 \rangle 3$. PICK N such that $\forall n \geq N, |s_n - s| < \epsilon/|c|$.

$\langle 1 \rangle 4$. $\forall n \geq N, |cs_n - cs| < \epsilon$

□

Proposition 12.7. *If $s_n \rightarrow s$ and $t_n \rightarrow t$ in \mathbb{C} then $s_n t_n \rightarrow st$.*

PROOF:

$\langle 1 \rangle 1$. $(s_n - s)(t_n - t) \rightarrow 0$ as $n \rightarrow \infty$

$\langle 2 \rangle 1$. LET: $\epsilon > 0$

$\langle 2 \rangle 2$. PICK N such that, for all $n \geq N$, we have $|s_n - s| < \sqrt{\epsilon}$ and $|t_n - t| < \sqrt{\epsilon}$.

$\langle 2 \rangle 3$. For all $n \geq N$ we have $|(s_n - s)(t_n - t)| < \epsilon$

$\langle 1 \rangle 2$. $s_n t_n - st \rightarrow 0$ as $n \rightarrow \infty$

PROOF:

$$s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$$

$$\rightarrow 0$$

$$\text{as } n \rightarrow \infty$$

□

Proposition 12.8. *If $s_n \rightarrow s$ as $n \rightarrow \infty$ in \mathbb{C} , and every s_n and s is nonzero, then $1/s_n \rightarrow 1/s$ as $n \rightarrow \infty$.*

PROOF:

$\langle 1 \rangle 1$. PICK m such that, for all $n \geq m$, we have $|s_n - s| < \frac{1}{2}|s|$.

$\langle 1 \rangle 2$. $\forall n \geq m, |s_n| > \frac{1}{2}|s|$

$\langle 1 \rangle 3$. LET: $\epsilon > 0$

$\langle 1 \rangle 4$. PICK $N > m$ such that, for all $n \geq N$, we have

$$|s_n - s| < \frac{1}{2}|s|^2\epsilon .$$

(1)5. For all $n \geq N$, we have

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| < \epsilon .$$

PROOF:

$$\begin{aligned} \left| \frac{1}{s_n} - \frac{1}{s} \right| &= \frac{|s_n - s|}{|s_n||s|} \\ &< \frac{|s|^2 \epsilon}{2|s_n||s|} \\ &= \frac{|s| \epsilon}{2|s_n|} \\ &< \epsilon \end{aligned}$$

□

Theorem 12.9. Let (\vec{x}_n) be a sequence in \mathbb{R}^k and $\vec{l} \in \mathbb{R}^k$. Then $\vec{x}_n \rightarrow \vec{l}$ as $n \rightarrow \infty$ iff, for $i = 1, \dots, k$, we have $\pi_i(\vec{x}_n) \rightarrow \pi_i(\vec{l})$ as $n \rightarrow \infty$.

PROOF:

(1)1. If $\vec{x}_n \rightarrow \vec{l}$ then $\pi_i(\vec{x}_n) \rightarrow \pi_i(l)$.

(2)1. $\|\vec{x}_n - \vec{l}\| \rightarrow 0$ as $n \rightarrow \infty$.

(2)2. $\sqrt{\sum_{i=1}^k (\pi_i(\vec{x}_n) - \pi_i(l))^2} \rightarrow 0$ as $n \rightarrow \infty$.

(2)3. $\sum_{i=1}^k (\pi_i(\vec{x}_n) - \pi_i(l))^2 \rightarrow 0$ as $n \rightarrow \infty$.

(2)4. $(\pi_i(\vec{x}_n) - \pi_i(l))^2 \rightarrow 0$ as $n \rightarrow \infty$

(2)5. $\pi_i(\vec{x}_n) - \pi_i(l) \rightarrow 0$ as $n \rightarrow \infty$.

(1)2. If $\pi_i(\vec{x}_n) \rightarrow \pi_i(\vec{l})$ for every i then $\vec{x}_n \rightarrow l$.

(2)1. ASSUME: $\pi_i(\vec{x}_n) \rightarrow \pi_i(\vec{l})$ for every i .

(2)2. $\vec{x}_n \rightarrow \vec{l}$

PROOF:

$$\begin{aligned} \|\vec{x}_n - \vec{l}\|^2 &= \sum_{i=1}^k (\pi_i(\vec{x}_n) - \pi_i(\vec{l}))^2 \\ &\rightarrow 0 \end{aligned}$$

□

Corollary 12.9.1. If $\vec{x}_n \rightarrow \vec{x}$ and $\vec{y}_n \rightarrow \vec{y}$ in \mathbb{R}^k , then $\vec{x}_n + \vec{y}_n \rightarrow \vec{x} + \vec{y}$.

Corollary 12.9.2. If $\beta_n \rightarrow \beta$ in \mathbb{R} and $\vec{x}_n \rightarrow \vec{l}$ in \mathbb{R}^k , then $\beta_n \vec{x}_n \rightarrow \beta \vec{l}$.

Proposition 12.10. If $\vec{x}_n \rightarrow \vec{x}$ and $\vec{y}_n \rightarrow \vec{y}$ in \mathbb{R}^k , then $\vec{x}_n \cdot \vec{y}_n \rightarrow \vec{x} \cdot \vec{y}$.

PROOF:

$$\begin{aligned} \vec{x}_n \cdot \vec{y}_n &= \sum_{i=1}^k \pi_i(\vec{x}_n) \pi_i(\vec{y}_n) \\ &\rightarrow \sum_{i=1}^k \pi_i(\vec{x}) \pi_i(\vec{y}) \\ &= \vec{x} \cdot \vec{y} \end{aligned}$$

□

Proposition 12.11. *Let (p_n) be a sequence in the metric space X . The set E^* of all limits of convergent subsequences is a closed set.*

PROOF:

- $\langle 1 \rangle 1$. ASSUME: w.l.o.g. $\{p_n : n \in \mathbb{N}\}$ is infinite.
- $\langle 1 \rangle 2$. LET: q be a limit point of E^* .
 PROVE: $q \in E^*$
- $\langle 1 \rangle 3$. PICK an integer n_0 such that $q \neq p_{n_0}$.
- $\langle 1 \rangle 4$. Extend a strictly increasing sequence of integers (n_i) such that, for all i , we have $d(q, p_{n_i}) \leq 2^i d(q, p_{n_0})$.
 $\langle 2 \rangle 1$. ASSUME: as induction hypothesis we have picked $n_0 < n_1 < \cdots < n_i$ such that, for $0 \leq j \leq i$, we have $d(q, p_{n_j}) \leq 2^j d(q, p_{n_0})$.
 $\langle 2 \rangle 2$. PICK $x \in E^*$ such that $d(x, q) < 2^{-(i+2)} \delta$
 $\langle 2 \rangle 3$. There exists a subsequence of (p_n) that converges to x .
 $\langle 2 \rangle 4$. There exists $n_{i+1} > n_i$ such that $d(p_{n_{i+1}}, x) < 2^{-(i+2)} \delta$.
 $\langle 2 \rangle 5$. $d(p_{n_{i+1}}, q) < 2^{-(i+1)} \delta$
- $\langle 1 \rangle 5$. $p_{n_i} \rightarrow q$ as $i \rightarrow \infty$.
- $\langle 1 \rangle 6$. $q \in E^*$

□

Theorem 12.12. *Every monotonically increasing sequence in \mathbb{R} that is bounded above converges to its supremum.*

PROOF:

- $\langle 1 \rangle 1$. LET: (s_n) be a monotonically increasing sequence with supremum s .
- $\langle 1 \rangle 2$. LET: $\epsilon > 0$
- $\langle 1 \rangle 3$. PICK N such that $|s_N - s| < \epsilon$
- $\langle 1 \rangle 4$. For all $n \geq N$, we have $s - \epsilon < s - s_N \leq s - s_n \leq s$.
- $\langle 1 \rangle 5$. $\forall n \geq N, |s_n - s| < \epsilon$

□

Theorem 12.13. *Every monotonically decreasing sequence in \mathbb{R} that is bounded below converges to its infimum.*

PROOF: Similar. □

Proposition 12.14 (Sandwich Theorem). *Let (a_n) , (b_n) and (c_n) be sequences of real numbers and $l \in \mathbb{R}$. Assume $\forall n, a_n \leq b_n \leq c_n$ and $a_n \rightarrow l$ and $c_n \rightarrow l$. Then $b_n \rightarrow l$.*

PROOF:

- $\langle 1 \rangle 1$. LET: $\epsilon > 0$
- $\langle 1 \rangle 2$. PICK N such that, for all $n \geq N$, we have $|a_n - l| < \epsilon$ and $|c_n - l| < \epsilon$.
- $\langle 1 \rangle 3$. $\forall n \geq N, |b_n - l| < \epsilon$

□

Theorem 12.15. *For any real $p > 0$ we have*

$$\frac{1}{(n+1)^p} \rightarrow 0$$

as $n \rightarrow \infty$.

PROOF:

- $\langle 1 \rangle 1$. LET: $\epsilon > 0$
 - $\langle 1 \rangle 2$. PICK N such that $N > (1/\epsilon)^{1/p}$.
 - $\langle 1 \rangle 3$. LET: $n \geq N$
 - $\langle 1 \rangle 4$. $1/n^p < \epsilon$
-

Theorem 12.16. *For any real $p > 0$ we have*

$$p^{\frac{1}{n+1}} \rightarrow 1$$

as $n \rightarrow \infty$.

PROOF:

- $\langle 1 \rangle 1$. CASE: $p > 1$

- $\langle 2 \rangle 1$. For $n \in \mathbb{N}$

LET: $x_n = p^{\frac{1}{n+1}} - 1$.

- $\langle 2 \rangle 2$. $\forall n \in \mathbb{N}. x_n > 0$

- $\langle 2 \rangle 3$. $\forall n \in \mathbb{N}$.

$$1 + (n+1)x_n \leq p.$$

PROOF: Since $1 + (n+1)x_n \leq (1+x_n)^{n+1}$ by the Binomial Theorem.

- $\langle 2 \rangle 4$. $\forall n \in \mathbb{N}$.

$$0 < x_n \leq \frac{p-1}{n+1}.$$

- $\langle 2 \rangle 5$. $x_n \rightarrow 0$ as $n \rightarrow \infty$.

PROOF: Sandwich Theorem.

- $\langle 1 \rangle 2$. CASE: $p = 1$

PROOF: Trivial.

- $\langle 1 \rangle 3$. CASE: $p < 1$

PROOF: Then $p^{1/(n+1)} = 1/((1/p)^{1/(n+1)}) \rightarrow 1/1 = 1$ by $\langle 1 \rangle 1$.

□

Theorem 12.17.

$$(n+1)^{1/(n+1)} \rightarrow 1 \text{ as } n \rightarrow \infty$$

PROOF:

- $\langle 1 \rangle 1$. For $n \in \mathbb{N}$,

LET: $x_n = (n+1)^{1/(n+1)} - 1$.

- $\langle 1 \rangle 2$. $\forall n \in \mathbb{N}. x_n \geq 0$

- $\langle 1 \rangle 3$. $\forall n \in \mathbb{N}$

$$n+1 \geq \frac{n(n+1)}{2} x_n^2.$$

PROOF: Since $(1+x_n)^{n+1} \geq \frac{n(n+1)}{2} x_n^2$ by the Binomial Theorem.

- $\langle 1 \rangle 4$. $\forall n \geq 1$

$$0 \leq x_n \leq \sqrt{\frac{2}{n}}$$

- $\langle 1 \rangle 5$. $x_n \rightarrow 0$ as $n \rightarrow \infty$.

PROOF: Sandwich Theorem.

□

Theorem 12.18. *Let p and α be real numbers with $p > 0$. Then*

$$\frac{n^\alpha}{(1+p)^n} \rightarrow 0 \text{ as } n \rightarrow \infty .$$

PROOF:

(1)1. PICK a positive integer k such that $k > \alpha$.

PROOF: Archimedean Property.

(1)2. $\forall n > 2k$

$$(1+p)^n > \frac{n^k p^k}{2^k k!} .$$

$$\begin{aligned} (1+p)^n &> \binom{n}{k} p^k && \text{(Binomial Theorem)} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k!} p^k \\ &> \frac{n^k p^k}{2^k k!} && (n > 2k \text{ so if } n-k < i \leq n \text{ then } i > n/2) \end{aligned}$$

(1)3. $\forall n > 2k$

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k} .$$

(1)4. $n^{\alpha-k} \rightarrow 0$ as $n \rightarrow \infty$

PROOF: Theorem 12.15.

(1)5. $\frac{n^\alpha}{(1+p)^n} \rightarrow 0$ as $n \rightarrow \infty$.

PROOF: Sandwich Theorem.

□

Corollary 12.18.1. *For any real number x with $|x| < 1$ we have $x^n \rightarrow 0$ as $n \rightarrow \infty$.*

PROOF: Taking $\alpha = 0$. □

12.1 Cauchy Sequences

Definition 12.19 (Cauchy Sequence). Let (p_n) be a sequence in the metric space X . Then (p_n) is a *Cauchy sequence* iff, for every $\epsilon > 0$, there exists N such that, for all $m, n \geq N$, we have $d(p_m, p_n) < \epsilon$.

Proposition 12.20. *Let (p_n) be a sequence in the metric space X and let $E_N = \{p_n : n \geq N\}$ for all N . Then (p_n) is a Cauchy sequence if and only if $\text{diam } E_N \rightarrow 0$ as $N \rightarrow \infty$.*

PROOF: Immediate from definitions. □

Theorem 12.21. *Every convergent sequence is Cauchy.*

PROOF:

- (1)1. LET: (p_n) be a convergent sequence with limit l .
- (1)2. LET: $\epsilon > 0$
- (1)3. PICK N such that, for all $n \geq N$, we have $d(p_n, l) < \epsilon/2$
- (1)4. $\forall m, n \geq N. d(p_m, p_n) < \epsilon$

□

12.2 Complete Metric Spaces

Definition 12.22 (Complete Metric Space). A metric space is *complete* iff every Cauchy sequence converges.

Theorem 12.23. *Every compact metric space is complete.*

PROOF:

- (1)1. LET: X be a compact metric space.
- (1)2. LET: (p_n) be a Cauchy sequence in X .
- (1)3. For $N \in \mathbb{N}$,
LET: $E_N = \{p_n : n \geq N\}$.
- (1)4. $\text{diam } \overline{E_N} \rightarrow 0$ as $N \rightarrow \infty$.
- (1)5. For all N , every $\overline{E_N}$ is compact.
PROOF: Proposition 11.37.
- (1)6. For all N we have $\overline{E_N} \supseteq \overline{E_{N+1}}$.
- (1)7. LET: l be the unique point in $\bigcap_{N=0}^{\infty} \overline{E_N}$
PROVE: $p_n \rightarrow l$ as $n \rightarrow \infty$.
PROOF: Proposition 11.44.
- (1)8. LET: $\epsilon > 0$
- (1)9. PICK N_0 such that $\forall N \geq N_0. \text{diam } \overline{E_N} < \epsilon$.
- (1)10. $\forall q \in E_N. d(l, q) < \epsilon$
- (1)11. $\forall n \geq N. d(l, p_n) < \epsilon$

□

Corollary 12.23.1. *Let X be a metric space. If every closed bounded set in X is compact, then X is complete.*

PROOF:

- (1)1. LET: S be a Cauchy sequence in X .
- (1)2. S is bounded.
- (1)3. \overline{S} is closed and bounded.
- (1)4. \overline{S} is compact.
- (1)5. S is a Cauchy sequence in \overline{S} .
- (1)6. S converges.

□

Corollary 12.23.2. *For every natural number k , we have \mathbb{R}^k is complete.*

Corollary 12.23.3. *Every closed subspace of a complete metric space is complete.*

12.3 Divergent Sequences

Definition 12.24. Let (s_n) be a sequence in \mathbb{R} . Then we say s_n *diverges to* $+\infty$, and write

$$s_n \rightarrow +\infty \text{ as } n \rightarrow \infty ,$$

iff for every real number M , there exists an integer N such that

$$\forall n \geq N. s_n \geq M .$$

We say s_n *diverges to* $-\infty$, and write

$$s_n \rightarrow -\infty \text{ as } n \rightarrow \infty ,$$

iff for every real number M , there exists an integer N such that

$$\forall n \geq N. s_n \leq M .$$

Definition 12.25 (Limit Supremum, Limit Infimum). Let (s_n) be a sequence in \mathbb{R} . Let E be the set of all $l \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that there exists a subsequence of (s_n) that converges to l .

The *limit supremum* of (s_n) , denoted

$$\limsup_{n \rightarrow \infty} s_n ,$$

is the supremum of E in the extended reals.

The *limit infimum* of (s_n) , denoted

$$\liminf_{n \rightarrow \infty} s_n ,$$

is the infimum of E in the extended reals.

PROOF: The set E is always nonempty because: if (s_n) is unbounded above then $+\infty \in E$; if it is unbounded below then $-\infty \in E$; and if it is bounded above and below then there is a real number in E by Corollary 12.4.1. \square

Theorem 12.26. Let (s_n) be a sequence in \mathbb{R} . Then there exists a subsequence of (s_n) that converges or diverges to $\limsup_{n \rightarrow \infty} s_n$

PROOF:

(1)1. CASE: $\limsup_n s_n = +\infty$

PROOF: (s_n) is unbounded above and so has a subsequence that diverges to $+\infty$.

(1)2. CASE: $\limsup_n s_n \in \mathbb{R}$

PROOF: Then $\limsup s_n$ is in the set of limits of subsequences of (s_n) by Proposition 12.11.

(1)3. CASE: $\limsup_n s_n = -\infty$

PROOF: (s_n) is unbounded below and so has a subsequence that diverges to $-\infty$.

□

Theorem 12.27. *Let (s_n) be a sequence in \mathbb{R} . Then there exists a subsequence of (s_n) that converges or diverges to $\liminf_{n \rightarrow \infty} s_n$*

PROOF: Similar. □

Theorem 12.28. *Let (s_n) be a sequence in \mathbb{R} . If $x > \limsup_n s_n$, then there exists N such that $\forall n \geq N, s_n < x$.*

PROOF: If not, we could choose a subsequence of (s_n) that converges to a value $\geq x$, contradicting the definition of $\limsup_n s_n$. □

Theorem 12.29. *Let (s_n) be a sequence in \mathbb{R} . If $x < \liminf_n s_n$, then there exists N such that $\forall n \geq N, s_n > x$.*

PROOF: Similar. □

Theorem 12.30. *Let (s_n) be a sequence in \mathbb{R} . Let s^* be an extended real such that:*

- *There exists a subsequence of (s_n) that converges or diverges to s^* .*
- *For any $x > s^*$, there exists N such that $\forall n \geq N, s_n < x$.*

Then $s^ = \limsup_n s_n$.*

PROOF:

(1)1. LET: E be the set of subsequential limits of (s_n) .

(1)2. s^* is an upper bound for E .

(2)1. LET: $x \in E$

(2)2. ASSUME: for a contradiction $x > s^*$.

(2)3. $s^* \in \mathbb{R}$

(2)4. LET: $y = x$ if $x \in \mathbb{R}$, or $s^* + 1$ if $x = +\infty$

(2)5. There exists N such that $\forall n \geq N, s_n < y$.

(2)6. Q.E.D.

PROOF: This contradicts the fact that some subsequence of (s_n) converges or diverges to x .

(1)3. If u is an upper bound for E then $s^* \leq u$.

□

Theorem 12.31. *Let (s_n) be a sequence in \mathbb{R} . Let s^* be an extended real such that:*

- *There exists a subsequence of (s_n) that converges or diverges to s^* .*
- *For any $x < s^*$, there exists N such that $\forall n \geq N, s_n > x$.*

Then $s^ = \liminf_n s_n$.*

PROOF: Similar. □

Proposition 12.32. *Let (s_n) be a sequence of real numbers and $l \in \mathbb{R}$. Then (s_n) converges to l iff $\limsup_n s_n = \liminf_n s_n = l$.*

PROOF:

(1)1. If (s_n) converges to l then $\limsup_n s_n = \liminf_n s_n = l$.

PROOF: If (s_n) converges to l then every subsequence of (s_n) converges to l .

(1)2. If $\limsup_n s_n = \liminf_n s_n = l$ then (s_n) converges to l .

(2)1. ASSUME: $\limsup_n s_n = \liminf_n s_n = l$

(2)2. For all $\epsilon > 0$, there exists N such that $\forall n \geq N. l - \epsilon < s_n < l + \epsilon$.

PROOF: Theorem 12.30 and 12.31.

(2)3. $s_n \rightarrow l$ as $n \rightarrow \infty$.

□

Theorem 12.33. *Let (s_n) and (t_n) be sequences of real numbers and $N \in \mathbb{N}$. Assume $\forall n \geq N. s_n \leq t_n$. Then*

$$\liminf_{n \rightarrow \infty} s_n \leq \liminf_{n \rightarrow \infty} t_n .$$

PROOF:

(1)1. For any subsequence (t_{n_r}) of (t_n) that converges or diverges to $\pm\infty$, we have $\liminf_n s_n \leq \lim_r t_{n_r}$

(2)1. LET: (t_{n_r}) be a subsequence of (t_n) with limit l .

(2)2. PICK m such that a subsequence of (s_{n_r}) has limit m .

(2)3. $\forall r. s_{n_r} \leq t_{n_r}$

(2)4. $m \leq l$

(2)5. $\liminf_n s_n \leq l$

(1)2. $\liminf_n s_n \leq \liminf_n t_n$

□

Theorem 12.34. *Let (s_n) and (t_n) be sequences of real numbers and $N \in \mathbb{N}$. Assume $\forall n \geq N. s_n \leq t_n$. Then*

$$\limsup_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} t_n .$$

PROOF: Similar. □

Theorem 12.35. *For any sequence (c_n) of positive real numbers, we have*

$$\limsup_{n \rightarrow \infty} c_n^{1/n} \leq \limsup_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} .$$

PROOF:

(1)1. LET: $\alpha = \limsup_n c_{n+1}/c_n$

(1)2. ASSUME: w.l.o.g. $\alpha < +\infty$

(1)3. For all $\beta > \alpha$ we have $\limsup_{n \rightarrow \infty} c_n^{1/n} \leq \beta$.

(2)1. LET: $\beta > \alpha$

(2)2. PICK N such that, for all $n \geq N$, we have

$$\frac{c_{n+1}}{c_n} \leq \beta .$$

PROOF: Theorem 12.28.

⟨2⟩3. For all $k \geq 0$ we have

$$c_{N+k+1} \leq \beta c_{N+k} .$$

⟨2⟩4. For all $n \geq N$ we have

$$c_n \leq c_N \beta^{-N} \beta^n .$$

PROOF: Induction on n .

⟨2⟩5. For all $n \geq N$ we have

$$c_n^{1/n} \leq (c_N \beta^{-N})^{1/n} \beta .$$

⟨2⟩6.

$$\limsup_{n \rightarrow \infty} c_n^{1/n} \leq \beta$$

PROOF:

$$\limsup_{n \rightarrow \infty} c_n^{1/n} \leq \limsup_{n \rightarrow \infty} (c_N \beta^{-N})^{1/n} \beta \quad (\text{Theorem 12.34})$$

$$= \beta \quad (\text{Theorem 12.16})$$

⟨1⟩4.

$$\limsup_{n \rightarrow \infty} c_n^{1/n} \leq \alpha$$

□

Theorem 12.36. For any sequence (c_n) of positive real numbers, we have

$$\liminf_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n \rightarrow \infty} c_n^{1/n} .$$

PROOF: Similar. □

12.4 Infinite Series

Definition 12.37. Let (a_n) be a sequence of complex numbers and $s \in \mathbb{C}$. We say the *infinite series* $\sum_{n=0}^{\infty} a_n$ *converges* to s , and write

$$\sum_{n=0}^{\infty} a_n = s$$

iff

$$\sum_{n=0}^N a_n \rightarrow s \text{ as } N \rightarrow \infty .$$

If $(\sum_{n=0}^N a_n)$ diverges, we say the infinite series $\sum_{n=0}^{\infty} a_n$ diverges.

Theorem 12.38. Let (a_n) be a sequence of complex numbers. Then $\sum_{n=0}^{\infty} a_n$ converges if and only if, for all $\epsilon > 0$, there exists N such that, for all $m, n \geq N$,

$$\left| \sum_{k=m}^n a_k \right| \leq \epsilon .$$

PROOF: This is what it means for $(\sum_{k=0}^n a_k)$ to be a Cauchy sequence. □

Corollary 12.38.1. *If $\sum_{n=0}^{\infty} a_n$ converges then $a_n \rightarrow 0$ as $n \rightarrow \infty$.*

Theorem 12.39. *A series of nonnegative reals converges if and only if its partial sums form a bounded sequence.*

PROOF: Its partial sums are a monotonically increasing sequence, and so converge if and only if they are bounded above. \square

Theorem 12.40 (Comparison Test). *Let (a_n) be a sequence of complex numbers and (c_n) a sequence of real numbers. If there exists N such that $\forall n \geq N, |a_n| \leq c_n$, and if $\sum_n c_n$ converges, then $\sum_n a_n$ converges.*

PROOF:

$\langle 1 \rangle 1$. LET: $\epsilon > 0$

$\langle 1 \rangle 2$. PICK N such that $\forall n \geq N, |a_n| \leq c_n$ and $\forall m, n \geq N, \sum_{k=m}^n c_k < \epsilon$.

$\langle 1 \rangle 3$. $\forall m, n \geq N, |\sum_{k=m}^n a_k| \leq \epsilon$

\square

Corollary 12.40.1. *Let (a_n) and (d_n) be sequences of real numbers. If there exists N such that $\forall n \geq N, a_n \geq d_n \geq 0$, and if $\sum_n d_n$ diverges, then $\sum_n a_n$ diverges.*

Theorem 12.41 (Geometric Series). *For x a real number with $0 \leq x < 1$ we have*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

PROOF: Since $\sum_{n=0}^N x^n = \frac{1-x^{N+1}}{1-x} \rightarrow \frac{1}{1-x}$ as $n \rightarrow \infty$. \square

Theorem 12.42. *For x a real number with $x \geq 1$ we have $\sum_{n=0}^{\infty} x^n$ diverges.*

PROOF: If $x = 1$ then $\sum_{n=0}^N x^n = N + 1$. If $x > 1$ then $\sum_{n=0}^N x^n = \frac{1-x^{N+1}}{1-x}$. Both of these sequences diverge. \square

Theorem 12.43. *Let (a_n) be a monotonically decreasing sequence of nonnegative real numbers. Then $\sum_n a_n$ converges if and only if $\sum_n 2^n a_{2^n}$ converges.*

PROOF:

$\langle 1 \rangle 1$. For $N \in \mathbb{N}$,

LET: $s_N = \sum_{n=0}^N a_n$.

$\langle 1 \rangle 2$. For $N \in \mathbb{N}$,

LET: $t_N = \sum_{n=0}^N 2^n a_{2^n}$.

$\langle 1 \rangle 3$. For natural number N and k with $N < 2^k$ we have $s_N \leq a_0 + t_{k-1}$.

PROOF:

$$\begin{aligned}
 s_N &\leq \sum_{n=0}^{2^k-1} a_n \\
 &= a_0 + \sum_{i=0}^{k-1} \sum_{n=2^i}^{2^{i+1}-1} a_n \\
 &\leq a_0 + \sum_{i=0}^{k-1} 2^i a_{2^i} \\
 &= a_0 + t_{k-1}
 \end{aligned}$$

(1)4. For natural number N and k with $N > 2^k$ we have $t_k < 2s_N$.

PROOF:

$$\begin{aligned}
 s_N &\geq \sum_{n=1}^{2^k} a_n \\
 &\geq \sum_{i=0}^k \sum_{n=2^i}^{2^{i+1}-1} a_n \\
 &\geq \sum_{i=0}^k 2^i a_{2^{i+1}} \\
 &= (1/2)t_k
 \end{aligned}$$

(1)5. (s_N) converges if and only if (t_k) converges.

□

Theorem 12.44. *If p is a real number with $p > 1$ then $\sum_n 1/n^p$ converges.*

PROOF: Since

$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}$$

which converges since $2^{1-p} < 1$. □

Theorem 12.45. *If p is a real number with $p \leq 1$ then $\sum_n 1/n^p$ diverges.*

PROOF: If $p \leq 0$ then $1/n^p$ does not converge to 0.

If $0 < p \leq 1$ we have

$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}$$

which diverges since $2^{1-p} \geq 1$. □

Theorem 12.46. *Let p be a real number. The series*

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converges if and only if $p > 1$.

PROOF:

$$\begin{aligned} 2^k \frac{1}{2^k (\ln 2^k)^p} &= \frac{1}{(k \ln 2)^p} \\ &= \frac{1}{(\ln 2)^p} \cdot \frac{1}{k^p} \end{aligned}$$

and this series converges iff $\sum_k \frac{1}{k^p}$ converges iff $p > 1$. \square

Theorem 12.47 (Root Test). *Let $(a_n)_{n \geq 1}$ be a sequence of complex numbers. Let $\alpha = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$.*

1. *If $\alpha < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $\alpha > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.*

PROOF:

(1)1. If $\alpha < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.

$\langle 2 \rangle 1$. ASSUME: $\alpha < 1$

$\langle 2 \rangle 2$. PICK β such that $\alpha < \beta < 1$

$\langle 2 \rangle 3$. PICK N such that $\forall n \geq N, |a_n|^{1/n} < \beta$

PROOF: Theorem 12.28.

$\langle 2 \rangle 4$. $\forall n \geq N, |a_n| < \beta^n$

$\langle 2 \rangle 5$. $\sum_{n=1}^{\infty} \beta^n$ converges.

PROOF: Theorem 12.40.

$\langle 2 \rangle 6$. $\sum_{n=1}^{\infty} a_n$ converges.

PROOF: Comparison Test.

(1)2. If $\alpha > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

$\langle 2 \rangle 1$. ASSUME: $\alpha > 1$

$\langle 2 \rangle 2$. There exists a sequence of positive integers (n_k) such that $|a_{n_k}|^{1/n_k} \rightarrow \alpha$ as $k \rightarrow \infty$.

PROOF: Theorem 12.26.

$\langle 2 \rangle 3$. There are infinitely many n such that $|a_n| > 1$.

$\langle 2 \rangle 4$. $a_n \not\rightarrow 0$ as $n \rightarrow \infty$.

$\langle 2 \rangle 5$. $\sum_{n=1}^{\infty} a_n$ diverges.

PROOF: Corollary 12.37.1.

\square

Example 12.48. If $a_n = 1/n$ then $|a_n|^{1/n} \rightarrow 1$ and $\sum_n a_n$ diverges.

If $a_n = 1/n^2$ then $|a_n|^{1/n} \rightarrow 1$ and $\sum_n a_n$ converges.

Theorem 12.49 (Ratio Test). *Let $(a_n)_{n \geq 0}$ be a sequence of complex numbers.*

1. *If*

$$\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

then $\sum_{n=0}^{\infty} a_n$ converges.

2. *If there exists N such that $\forall n \geq N, \left| \frac{a_{n+1}}{a_n} \right| > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.*

PROOF:

(1)1. If $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ then $\sum_{n=0}^{\infty} a_n$ converges.

(2)1. ASSUME: $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

(2)2. PICK $\beta < 1$ and N such that $\forall n \geq N$

$$\left| \frac{a_{n+1}}{a_n} \right| < \beta .$$

PROOF: Theorem 12.28.

(2)3. For all $n > N$ we have

$$|a_n| < |a_N| \beta^{-N} \beta^n .$$

PROOF: By induction on n .

(2)4. $\sum_n \beta^n$ converges.

PROOF: Theorem 12.40.

(2)5. $\sum_n a_n$ converges.

PROOF: Comparison Test.

(1)2. If there exists N such that $\forall n \geq N, \left| \frac{a_{n+1}}{a_n} \right| > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.

PROOF: Since $a_n \not\rightarrow 0$ as $n \rightarrow \infty$.

□

Example 12.50. If $a_n = 1/n$ then $a_{n+1}/a_n \rightarrow 1$ and $\sum_n a_n$ diverges.

If $a_n = 1/n^2$ then $a_{n+1}/a_n \rightarrow 1$ and $\sum_n a_n$ converges.

12.5 The Number e

Lemma 12.51. The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

PROOF:

$$\begin{aligned} \sum_{n=0}^N \frac{1}{n!} &\leq 1 + \sum_{n=1}^N \frac{1}{2^{n-1}} \\ &< 3 \end{aligned}$$

□

Definition 12.52. The number e is

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} .$$

Theorem 12.53.

$$\left(1 + \frac{1}{n} \right)^n \rightarrow e \text{ as } n \rightarrow \infty$$

PROOF:

(1)1. For $n \in \mathbb{N}$,

$$\text{LET: } s_n = \sum_{k=0}^n \frac{1}{k!}$$

(1)2. For $n \in \mathbb{Z}^+$,

$$\text{LET: } t_n = \left(1 + \frac{1}{n} \right)^n$$

(1)3. For $n \in \mathbb{Z}^+$ we have

$$t_n = \sum_{k=0}^n \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) .$$

PROOF:

$$\begin{aligned} t_n &= \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} && \text{(Binomial Theorem)} \\ &= \sum_{k=0}^n \frac{\prod_{i=0}^{k-1} (n-i)}{k!} \frac{1}{n^k} \\ &= \sum_{k=0}^n \frac{1}{k!} \prod_{i=0}^{k-1} \frac{n-i}{n} \\ &= \sum_{k=0}^n \frac{1}{k!} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) \\ &= \sum_{k=0}^n \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \end{aligned}$$

(1)4. For $n \in \mathbb{Z}^+$ we have $t_n \leq s_n$.

(1)5. $\limsup_{n \rightarrow \infty} t_n \leq e$

(1)6. For $m, n \in \mathbb{Z}^+$ with $n \geq m$ we have

$$t_n \geq \sum_{k=0}^m \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) .$$

(1)7. For $m \in \mathbb{Z}^+$ we have

$$\liminf_{n \rightarrow \infty} t_n \geq \sum_{k=0}^m \frac{1}{k!} .$$

(1)8. For $m \in \mathbb{Z}^+$ we have

$$s_m \leq \liminf_{n \rightarrow \infty} t_n .$$

(1)9.

$$e \leq \liminf_{n \rightarrow \infty} t_n$$

(1)10. $t_n \rightarrow e$ as $n \rightarrow \infty$.

PROOF: From (1)5 and (1)9.

□

Theorem 12.54. e is irrational.

PROOF:

(1)1. ASSUME: for a contradiction $e = p/q$ where p and q are positive integers.

(1)2. For $n \in \mathbb{N}$,

$$\text{LET: } s_n = \sum_{k=0}^n \frac{1}{k!} .$$

(1)3. For $n \in \mathbb{Z}^+$ we have

$$0 < e - s_n < \frac{1}{n!n} .$$

PROOF:

$$\begin{aligned} e - s_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &< \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \frac{1}{(n+1)^k} \\ &= \frac{1}{n!n} \end{aligned}$$

(1)4.

$$0 < q!(e - s_q) < \frac{1}{q}$$

(1)5. $q!e$ is an integer.

(1)6. $q!(e - s_q)$ is an integer.

(1)7. There exists an integer between 0 and 1.

(1)8. Q.E.D.

PROOF: This is a contradiction.

□

Theorem 12.55. e is transcendental.

PROOF: See I. M. Niven. *Irrational Numbers* p. 25. □

Part III

More Algebra

Chapter 13

Lie Groups

Definition 13.1 (Lie Group). A *Lie group* G is a group G that is also an analytic differentiable manifold such that the group operation and inverse operation are analytic.

A *homomorphism of Lie groups* is a group homomorphism that is an analytic function.

Lemma 13.2. *Every bijective Lie group homomorphism is an isomorphism.*

Definition 13.3 (Unitary Group). The *unitary group* $U(n)$ is the Lie group of all $n \times n$ unitary matrices.

Definition 13.4 (Special Unitary Group). The *special unitary group* $SU(n)$ is the Lie group of all $n \times n$ unitary matrices with determinant 1.

Definition 13.5 (Lie Subgroup). Let G be a Lie group. A *Lie subgroup* of G is a subgroup that is also an analytic submanifold of G .

Example 13.6. $U(n)$ and $SU(n)$ are Lie subgroups of $GL(n, \mathbb{C})$.