Solutions Manual for Enderton *Elements of Set Theory*

Robin Adams

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Contents

Chapter 1

Chapter 1 — Introduction

1.1 Baby Set Theory

Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ — true

- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true

- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — false

- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — true

- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$ — false

- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true

- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — true

- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — false

- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

Exercise 2   We have $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ has an element (namely $\emptyset$) while $\emptyset$ has no elements.

We have $\emptyset \neq \{\{\emptyset\}\}$ because $\{\{\emptyset\}\}$ has an element (namely $\{\emptyset\}$) while $\emptyset$ has no elements.

We have $\{\emptyset\} \neq \{\{\emptyset\}\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$. This last fact is true because $\emptyset \neq \{\emptyset\}$ as we proved in the first paragraph.

Exercise 3   Assume $B \subseteq C$. Let $A \in \mathcal{P}B$; we must show that $A \in \mathcal{P}C$.

We have $A \subseteq B$ (since $A \in \mathcal{P}B$) and $B \subseteq C$. From this it follows that $A \subseteq C$ (every element of A is an element of B; every element of B is an element of C; therefore every element of A is an element of C). Hence $A \in \mathcal{P}C$ as required.

Exercise 4   Since $x \in B$, we have $\{x\} \subseteq B$ and so $\{x\} \in \mathcal{P}B$.

Since $x \in B$ and $y \in B$, we have $\{x, y\} \subseteq B$ and so $\{x, y\} \in \mathcal{P}B$.

From these two facts, it follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$ and so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

1.2   Sets — An Informal View

Exercise 5   We have

$$
\begin{aligned}
V_0 &= A \\
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}A \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= \{\emptyset, \{\emptyset\}\} \\
V_3 &= \mathcal{P}V_2 \\
&= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}
$$

We have $\emptyset \subseteq V_0$ and so $\emptyset \in V_1$. Therefore $\{\emptyset\} \subseteq V_1$ and so $\{\emptyset\} \in V_2$. Hence $\{\{\emptyset\}\} \subseteq V_2$.

We also have $\{\{\emptyset\}\} \nsubseteq V_0$ because $\{\emptyset\}$ is not an atom, and $\{\{\emptyset\}\} \nsubseteq V_1$ since $\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.

Thus the rank of $\{\{\emptyset\}\}$ is 2.

Likewise we have $\emptyset$ and $\{\emptyset\}$ are both subsets of V_1, hence

$$
\emptyset \in V_2, \qquad \{\emptyset\} \in V_2
$$

Thus $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are all subsets of V_2, hence elements of V_3. Therefore,

$$
\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3
$$

Now, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is not a subset of V_0 (because $\emptyset$ is not an atom.) It is not a subset of V_1 ($\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.) It is not a subset of V_2 (we have $\{\emptyset, \{\emptyset\}\} \notin V_2$ since $\{\emptyset\} \notin V_1$).

Therefore the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

Exercise 6

$$V_1 = V_0 \cup \mathcal{P}V_0$$
$$\quad = A \cup \mathcal{P}V_0 \qquad\qquad (\text{since } V_0 = A)$$
$$V_2 = V_1 \cup \mathcal{P}V_1$$
$$\quad = A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1$$
$$\quad = A \cup \mathcal{P}V_1 \qquad\qquad (\text{since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3})$$
$$V_3 = V_2 \cup \mathcal{P}V_2$$
$$\quad = A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2$$
$$\quad = A \cup \mathcal{P}V_2 \qquad\qquad (\text{since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3})$$
$$V_4 = V_3 \cup \mathcal{P}V_3$$
$$\quad = A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3$$
$$\quad = A \cup \mathcal{P}V_3 \qquad\qquad (\text{since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3})$$

Exercise 7 In Exercise 5 we calculated $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Hence

$$V_4 = \mathcal{P}V_3$$
$$= \{\emptyset,$$
$$\{\emptyset\},$$
$$\{\{\emptyset\}\},$$
$$\{\{\{\emptyset\}\}\},$$
$$\{\{\emptyset, \{\emptyset\}\}\},$$
$$\{\emptyset, \{\emptyset\}\},$$
$$\{\emptyset, \{\{\emptyset\}\}\},$$
$$\{\emptyset, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\{\emptyset\}, \{\{\emptyset\}\}\},$$
$$\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\},$$
$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$
$$\}$$

Chapter 2

Chapter 2 — Axioms and Operations

2.1 Arbitrary Unions and Intersections

Exercise 1 $A \cap B \cap C$ is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

Exercise 2 Take $A = \emptyset$ and $B = \{\emptyset\}$. Then $\bigcup A = \bigcup B = \emptyset$ but $A \neq B$. (There are many other possible answers.)

Exercise 3 Let $b \in A$. We must show that $b \subseteq \bigcup A$.
   Let x be any element of b. We must show that $x \in \bigcup A$. We know that $x \in b$ and $b \in A$, and so $x \in \bigcup A$ by the definition of $\bigcup A$.

Exercise 4 Suppose $A \subseteq B$. Let $x \in \bigcup A$. We must show that $x \in \bigcup B$.
   Pick an element $a \in A$ such that $x \in a$. Then $a \in B$ because $A \subseteq B$. Since we know $x \in a$ and $a \in B$, we know that $x \in \bigcup B$.

Exercise 5 Assume that every member of $\mathcal{A}$ is a subset of B. Let $x \in \bigcup \mathcal{A}$. We must show that $x \in B$.
   Pick $A \in \mathcal{A}$ such that $x \in A$. By our assumption, we have $A \subseteq B$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$ as required.

Exercise 6

   (a) We will show that $\bigcup \mathcal{P}A \subseteq A$ and $A \subseteq \bigcup \mathcal{P}A$.
   To show $\bigcup \mathcal{P}A \subseteq A$: This follows from Exercise 5, since every member of $\mathcal{P}A$ is a subset of A.

To show $A \subseteq \bigcup \mathcal{P}A$: Let $a \in A$. Then we have $a \in \{a\}$ and $\{a\} \in \mathcal{P}A$ so $a \in \bigcup \mathcal{P}A$.

(b) To show $A \subseteq \mathcal{P}\bigcup A$: This holds because every element of A is a subset of $\bigcup A$, as we proved is Exercise 3.

Equality holds if and only if $A = \mathcal{P}X$ for some set X.

Proof: If $A = \mathcal{P}\bigcup A$ then of course $A = \mathcal{P}X$ for some X.

Conversely, if $A = \mathcal{P}X$, then we have

$$\mathcal{P}\bigcup A = \mathcal{P}\bigcup \mathcal{P}X$$
$$= \mathcal{P}X \qquad \text{(by part (a))}$$
$$= A$$

Exercise 7

(a) For any set X,

$$X \in \mathcal{P}A \cap \mathcal{P}B$$
$$\Leftrightarrow X \subseteq A \text{ and } X \subseteq B$$
$$\Leftrightarrow \text{Every member of } X \text{ is a member of } A \text{ and a member of } B$$
$$\Leftrightarrow X \subseteq A \cap B$$
$$\Leftrightarrow X \in \mathcal{P}(A \cap B)$$

(b) Let $X \in \mathcal{P}A \cup \mathcal{P}B$. Then either $X \in \mathcal{P}A$ or $X \in \mathcal{P}B$ (or both). If $X \in \mathcal{P}A$, then we have $X \subseteq A$ and so $X \subseteq A \cup B$ (because $A \subseteq A \cup B$). Similarly if $X \in \mathcal{P}B$ then we have $X \subseteq A \cup B$. So in either case $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

Equality holds if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof: Suppose $A \subseteq B$. Then $\mathcal{P}A \subseteq \mathcal{P}B$ (Chapter 1 Exercise 3) and so $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$. Also $A \cup B = B$ so $\mathcal{P}(A \cup B) = \mathcal{P}B$. Thus $\mathcal{P}A \cup \mathcal{P}B$ and $\mathcal{P}(A \cup B)$ are equal.

Similarly if $B \subseteq A$ then $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$.

Conversely, suppose $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$. We have $A \cup B \in \mathcal{P}(A \cup B)$, so $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$. If $A \cup B \in \mathcal{P}A$, then we have $B \subseteq A \cup B \subseteq A$. And if $A \cup B \in \mathcal{P}B$, then we have $A \subseteq A \cup B \subseteq B$.

Exercise 8 If A is a set such that every singleton belongs to A, then every set belongs to $\bigcup A$, contradicting Theorem 2A.

Exercise 9 Let $a = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$. Then $a \in B$ but $\mathcal{P}a$ is not a subset of B because $\emptyset \in \mathcal{P}a$ and $\emptyset \notin B$.

Exercise 10   We must show that $\mathcal{P}a \subseteq \mathcal{P} \bigcup B$. So let $X \in \mathcal{P}a$. Then $X \subseteq a$; we must show that $X \subseteq \bigcup B$.

Let $x \in X$; we must show that $x \in \bigcup B$. We have $x \in a$ (because $x \in X$ and $X \subseteq a$) and $a \in B$, hence $x \in \bigcup B$ as required.

2.2   Algebra of Sets

Exercise 11   For any x we have

$$x \in (A \cap B) \cup (A - B) \Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B)$$
$$\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B)$$
$$\Leftrightarrow x \in A$$

Hence $A = (A \cap B) \cup (A - B)$.

For any x we have

$$x \in A \cup (B - A) \Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A)$$
$$\Leftrightarrow x \in A \text{ or } x \in B$$
$$\Leftrightarrow x \in A \cup B$$

Hence $A \cup (B - A) = A \cup B$.

Exercise 12   For any x,

$$x \in C - (A \cap B) \Leftrightarrow x \in C \& \neg(x \in A \& x \in B)$$
$$\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B)$$
$$\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B)$$
$$\Leftrightarrow x \in (C - A) \cup (C - B)$$

Exercise 13   Suppose $A \subseteq B$. Let $x \in C - B$; we must show $x \in C - A$. We have $x \in C$ and $x \notin B$. Therefore $x \notin A$, since every member of A is a member of B. And so we have $x \in C - A$ as required.

Exercise 14   Let $A = \{\emptyset\}$, $B = \emptyset$ and $C = \{\emptyset\}$. Then $A - (B - C) = A - \emptyset = \{\emptyset\}$ while $(A - B) - C = \{\emptyset\} - C = \emptyset$.

Exercise 15

(a)   For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B+C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B+C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have $x \in A \cap (B+C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$.

(b) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$

In every case, we have $x \in A + (B+C) \Leftrightarrow x \in (A+B) + C$.

Exercise 16

$$[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A] = (A \cup B) - A$$
$$= B - A$$

Exercise 17

(a) $\Leftrightarrow$ **(b)**

$A \subseteq B \Leftrightarrow$ Every element of A is an element of B

$ \Leftrightarrow$ There is no element of A that is not an element of B

$ \Leftrightarrow A - B = \emptyset$

(a) $\Rightarrow$ **(c)** Suppose $A \subseteq B$. We have $B \subseteq A \cup B$ from the definition of $A \cup B$; we must prove that $A \cup B \subseteq B$. So let $x \in A \cup B$. Then $x \in A$ or $x \in B$. But in either case $x \in B$, since $x \in A \Rightarrow x \in B$. Thus we have $x \in B$ as required.

(c) $\Rightarrow$ **(a)** We always have $A \subseteq A \cup B$. So if $A \cup B = B$ then we have $A \subseteq B$.

(a) $\Rightarrow$ **(d)** Suppose $A \subseteq B$. We have $A \cap B \subseteq A$ from the definition of $A \cap B$; we must prove that $A \subseteq A \cap B$. So let $x \in A$. Then $x \in B$ since $A \subseteq B$, hence $x \in A \cap B$ as required.

(d) ⇒ (a) We always have $A \cap B \subseteq B$. So if $A \cap B = A$ then $A \subseteq B$.

Exercise 18 We can make the following 16 sets:

- $\emptyset \ (= A - A)$
- $A - B$
- $A \cap B$
- $B - A$
- $S - (A \cup B)$
- A
- $A + B$
- $S - B$
- B
- $S - (A + B)$
- $S - A$
- $A \cup B$
- $S - (B - A)$
- $S - (A \cap B)$
- $S - (A - B)$

Exercise 19 They are never equal, because for all A, B, we have $\emptyset \in \mathcal{P}(A - B)$ but $\emptyset \notin \mathcal{P}A - \mathcal{P}B$ since $\emptyset \in \mathcal{P}B$.

Exercise 20 Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$.
  We first show $B \subseteq C$. Let $x \in B$; we show $x \in C$. We have $x \in A \cup B = A \cup C$, so either $x \in A$ or $x \in C$. If $x \in C$, we are done. If $x \in A$, then we have $x \in A \cap B = A \cap C$, and so $x \in C$ in this case too.
  We can show $C \subseteq B$ similarly. Hence $B = C$.

Exercise 21 For any x, we have

$$x \in \bigcup(A \cup B) \Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C$$

$$\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C, \text{ or there exists } C \in B \text{ such that } x \in C$$

$$\Leftrightarrow x \in \bigcup A \cup \bigcup B$$

Exercise 22   For any x, we have

$$x \in \bigcap(A \cup B) \Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C$$
$$\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C$$
$$\Leftrightarrow x \in \bigcap A \cap \bigcap B$$

Exercise 23   PROOF:
$\langle 1 \rangle 1.\ A \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 1.$ LET: $x \in A$
  $\langle 2 \rangle 2.$ LET: $X \in \mathcal{B}$
  $\langle 2 \rangle 3.\ x \in A \cup X$
$\langle 1 \rangle 2.\ \bigcap \mathcal{B} \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 1.$ LET: $x \in \bigcap \mathcal{B}$
  $\langle 2 \rangle 2.$ LET: $X \in \mathcal{B}$
  $\langle 2 \rangle 3.\ x \in X$
  $\langle 2 \rangle 4.\ x \in A \cup X$
$\langle 1 \rangle 3.\ \bigcap \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap \mathcal{B}$
  $\langle 2 \rangle 1.$ LET: $x \in \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 2.$ ASSUME: $x \notin A$
      PROVE: $x \in \bigcap \mathcal{B}$
  $\langle 2 \rangle 3.$ LET: $X \in \mathcal{B}$
  $\langle 2 \rangle 4.\ x \in A \cup X$
  $\langle 2 \rangle 5.\ x \in X$
$\square$

Exercise 24

  (a)

$$Y \in \mathcal{P} \bigcap \mathcal{A} \Leftrightarrow Y \subseteq \bigcap \mathcal{A}$$
$$\Leftrightarrow \forall y \in Y. \forall X \in \mathcal{A}. y \in X$$
$$\Leftrightarrow \forall X \in \mathcal{A}. \forall y \in Y. y \in X$$
$$\Leftrightarrow \forall X \in \mathcal{A}. Y \in \mathcal{P} X$$
$$\Leftrightarrow Y \in \bigcap \{\mathcal{P} X \mid X \in \mathcal{A}\}$$

  (b) $\bigcup \{\mathcal{P} X \mid X \in \mathcal{A}\} \subseteq \mathcal{P} \bigcup \mathcal{A}$

PROOF:
$\langle 1 \rangle 1.$ LET: $Y \in \bigcup \{\mathcal{P} X \mid X \in \mathcal{A}\}$
$\langle 1 \rangle 2.$ PICK $X \in \mathcal{A}$ such that $Y \in \mathcal{P} X$
$\langle 1 \rangle 3.\ Y \subseteq X$
$\langle 1 \rangle 4.\ Y \subseteq \bigcup \mathcal{A}$
$\langle 1 \rangle 5.\ Y \in \mathcal{P} \bigcup \mathcal{A}$

Equality holds if and only if $\bigcup \mathcal{A} \in \mathcal{A}$.

$\langle 1 \rangle 1$. If $\bigcup \{ \mathcal{P}X \mid X \in \mathcal{A} \} = \mathcal{P} \bigcup \mathcal{A}$ then $\bigcup \mathcal{A} \in \mathcal{A}$
  $\langle 2 \rangle 1$. ASSUME: $\bigcup \{ \mathcal{P}X \mid X \in \mathcal{A} \} = \mathcal{P} \bigcup \mathcal{A}$
  $\langle 2 \rangle 2$. $\bigcup \mathcal{A} \in \bigcup \{ \mathcal{P}X \mid X \in \mathcal{A} \}$
  $\langle 2 \rangle 3$. PICK $X \in \mathcal{A}$ such that $\bigcup \mathcal{A} \in \mathcal{P}X$
  $\langle 2 \rangle 4$. $X = \bigcup \mathcal{A}$
$\langle 1 \rangle 2$. If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\bigcup \{ \mathcal{P}X \mid X \in \mathcal{A} \} = \mathcal{P} \bigcup \mathcal{A}$
  PROOF: If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\mathcal{P} \bigcup \mathcal{A} \in \{ \mathcal{P}X \mid X \in \mathcal{A} \}$.
☐

Exercise 25 We have $A \cup \bigcup \mathcal{B} = \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$ if and only if $A = \emptyset$ or $\mathcal{B} \neq \emptyset$

$\langle 1 \rangle 1$. If $A \cup \bigcup \mathcal{B} = \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$ then $A = \emptyset$ or $\mathcal{B} \neq \emptyset$
  PROOF: If $A \cup \bigcup \mathcal{B} = \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$ and $\mathcal{B} = \emptyset$ then
$$A \cup \bigcup \emptyset = \bigcup \emptyset$$
$$\therefore A = \emptyset$$
$\langle 1 \rangle 2$. If $A = \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$
  PROOF: Both sides are equal to $\bigcup \mathcal{B}$
$\langle 1 \rangle 3$. If $\mathcal{B} \neq \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$
  $\langle 2 \rangle 1$. ASSUME: $\mathcal{B} \neq \emptyset$
  $\langle 2 \rangle 2$. $A \cup \bigcup \mathcal{B} \subseteq \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$
    $\langle 3 \rangle 1$. LET: $x \in A \cup \bigcup \mathcal{B}$
      PROVE: $x \in \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$
    $\langle 3 \rangle 2$. CASE: $x \in A$
      $\langle 4 \rangle 1$. PICK $X \in \mathcal{B}$
      PROOF: By $\langle 2 \rangle 1$
      $\langle 4 \rangle 2$. $x \in A \cup X$
    $\langle 3 \rangle 3$. CASE: $x \in \bigcup \mathcal{B}$
      $\langle 4 \rangle 1$. PICK $X \in \mathcal{B}$ such that $x \in X$
      $\langle 4 \rangle 2$. $x \in A \cup X$
  $\langle 2 \rangle 3$. $\bigcup \{ A \cup X \mid X \in \mathcal{B} \} \subseteq A \cup \bigcup \mathcal{B}$
    $\langle 3 \rangle 1$. LET: $x \in \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$
    $\langle 3 \rangle 2$. PICK $X \in \mathcal{B}$ such that $x \in A \cup X$
    $\langle 3 \rangle 3$. $X \subseteq \bigcup \mathcal{B}$
    $\langle 3 \rangle 4$. $A \cup X \subseteq A \cup \bigcup \mathcal{B}$
    $\langle 3 \rangle 5$. $x \in A \cup \bigcup \mathcal{B}$

2.3 Review Exercises

Exercise 26 Sets A, B, D and F are all equal to each other. Sets C, E and G are equal to each other. None of the first list is equal to any of the second list.

Exercise 27 Take $A = \{\{0\}, \{1\}\}$ and $B = \{\{1\}\}$. Then $A \cap B = \{\{1\}\}$ and

$$\bigcap A \cap \bigcap B = \emptyset \cap \{1\}$$
$$= \emptyset$$
$$\bigcap (A \cap B) = \bigcap \{\{1\}\}$$
$$= \{1\}$$

Exercise 28

$$\bigcup \{\{3, 4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\} = \{3, 4, \{3\}, \{4\}\}$$

Exercise 29

(a) $\emptyset$

(b) We have

$$\{\emptyset\} \subseteq \mathcal{P}\{\emptyset\}$$
$$\therefore \mathcal{P}\{\emptyset\} \subseteq \mathcal{PP}\{\emptyset\}$$
$$\{\emptyset\} \subseteq \mathcal{PP}\{\emptyset\}$$
$$\therefore \mathcal{P}\{\emptyset\} \subseteq \mathcal{PPP}\{\emptyset\}$$
$$\therefore \bigcap \{\mathcal{PPP}\{\emptyset\}, \mathcal{PP}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} = \mathcal{PPP}\{\emptyset\} \cap \mathcal{PP}\{\emptyset\} \cap \mathcal{P}\{\emptyset\}$$
$$= \mathcal{P}\{\emptyset\}$$
$$= \{\emptyset, \{\emptyset\}\}$$

Exercise 30

(a) $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$

(b) $\{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d) $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

Exercise 31

(a) $\{1, 2, 3, \emptyset\}$

(b) $\emptyset$

(c) $\emptyset$

(d) $\emptyset$

Exercise 32

(a) $a \cup b$

(b) a

(c)

$$\bigcap\bigcup S \cup \left(\bigcup\bigcup S - \bigcup\bigcap S\right) = (a \cap b) \cup ((a \cup b) - a)$$
$$= (a \cap b) \cup (b - a)$$
$$= b$$

Exercise 33   When $a \neq b$:

$$\bigcup\left(\bigcup S - \bigcap S\right) = \bigcup(\{a, b\} - \{a\})$$
$$= \bigcup\{b\}$$
$$= b$$

When $a = b$:

$$\bigcup\left(\bigcup S - \bigcap S\right) = \bigcup(\{a, b\} - \{a\})$$
$$= \bigcup\emptyset$$
$$= \emptyset$$

Exercise 34   For any set S, we have

$$\emptyset \subseteq \mathcal{P}S$$
$$\therefore \emptyset \in \mathcal{P}\mathcal{P}S$$
$$\emptyset \subseteq S$$
$$\therefore \emptyset \in \mathcal{P}S$$
$$\therefore \{\emptyset\} \subseteq \mathcal{P}S$$
$$\therefore \{\emptyset\} \in \mathcal{P}\mathcal{P}S$$
$$\therefore \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}\mathcal{P}S$$
$$\therefore \{\emptyset, \{\emptyset\}\} \in \mathcal{P}\mathcal{P}\mathcal{P}S$$

Exercise 35 Assume $\mathcal{P}A = \mathcal{P}B$. Then we have

$$A \in \mathcal{P}A$$
$$\therefore A \in \mathcal{P}B$$
$$\therefore A \subseteq B$$
$$B \in \mathcal{P}B$$
$$\therefore B \in \mathcal{P}A$$
$$\therefore B \subseteq A$$
$$\therefore A = B$$

Exercise 36

(a)

$$x \in A - (A \cap B) \Leftrightarrow x \in A \;\&\; \neg(x \in A \;\&\; x \in B)$$
$$\Leftrightarrow x \in A \;\&\; x \notin B$$
$$\Leftrightarrow x \in A - B$$

(b)

$$x \in A - (A - B) \Leftrightarrow x \in A \;\&\; \neg(x \in A \;\&\; x \notin B)$$
$$\Leftrightarrow x \in A \;\&\; x \in B$$
$$\Leftrightarrow x \in A \cap B$$

Exercise 37

(a)

$$x \in (A \cup B) - C \Leftrightarrow (x \in A \text{ or } x \in B) \;\&\; x \notin C$$
$$\Leftrightarrow (x \in A \;\&\; x \notin C) \text{ or } (x \in B \;\&\; x \notin C)$$
$$\Leftrightarrow x \in (A - C) \cup (B - C)$$

(b)

$$x \in A - (B - C) \Leftrightarrow x \in A \;\&\; \neg(x \in B \;\&\; x \notin C)$$
$$\Leftrightarrow x \in A \;\&\; (x \notin B \text{ or } x \in C)$$
$$\Leftrightarrow (x \in A \;\&\; x \notin B) \text{ or } (x \in A \;\&\; x \in C)$$
$$\Leftrightarrow x \in (A - B) \cup (A \cap C)$$

(c)

$$x \in (A - B) - C \Leftrightarrow x \in A \;\&\; x \notin B \;\&\; x \notin C$$
$$\Leftrightarrow x \in A \;\&\; \neg(x \in B \vee x \in C)$$
$$\Leftrightarrow x \in A - (B \cup C)$$

Exercise 38

(a)   If every element of A is an element of C, and every element of B is an element of C, then everything that is an element of either A or B is an element of C.

(b)   If every element of C is an element of A, and every element of C is an element of B, then every element of C is an element of both A and B.

Chapter 3

Chapter 3 — Relations and Functions

3.1 Ordered Pairs

Exercise 1   We have $\langle 0, 1, 0 \rangle^* = \langle 0, 1, 1 \rangle^* = \{\{0\}, \{0, 1\}\}$.

Exercise 2

(a)

$$
\begin{aligned}
& z \in A \times (B \cup C) \\
\Leftrightarrow\, & \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ (y \in B \text{ or } y \in C)) \\
\Leftrightarrow\, & \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in B) \text{ or } (z = (x, y) \ \& \ x \in A \ \& \ y \in C) \\
\Leftrightarrow\, & z \in (A \times B) \cup (A \times C)
\end{aligned}
$$

(b)
$\langle 1 \rangle 1.$ Assume: $A \times B = A \times C$ and $A \neq \emptyset$
$\langle 1 \rangle 2.$ Pick $a \in A$
$\langle 1 \rangle 3.$ For all x, $x \in B \Leftrightarrow x \in C$
    Proof: $x \in B$ iff $(a, x) \in A \times B$ iff $(a, x) \in A \times C$ iff $x \in C$.
☐

Exercise 3

$$
\begin{aligned}
& z \in A \times \bigcup \mathcal{B} \\
\Leftrightarrow\, & \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ \exists X \in \mathcal{B}.y \in X) \\
\Leftrightarrow\, & \exists X \in \mathcal{B}.\exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in X) \\
\Leftrightarrow\, & z \in \bigcup \{A \times X : X \in \mathcal{B}\}
\end{aligned}
$$

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Exercise 4   If every ordered pair belongs to A then every set belongs to $\bigcup\bigcup A$ contradicting Theorem 2A.

Exercise 5

(a)   Apply a Subset Axiom to $\mathcal{P}(A \times B)$: we have $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A.z = \{x\} \times B\}$.

(b)

$$z \in \bigcup C$$
$$\Leftrightarrow \exists x \in A.z \in \{x\} \times B$$
$$\Leftrightarrow \exists x \in A.\exists y \in B.z = (x, y)$$
$$\Leftrightarrow z \in A \times B$$

3.2   Relations

Exercise 6   If $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$ then A is a set of ordered pairs, i.e. a relation.

Conversely, suppose A is a relation. Let $z \in A$. Then z is an ordered pair; let $z = (x, y)$. We have $x \in \operatorname{dom} A$ and $y \in \operatorname{ran} A$ and so $z \in \operatorname{dom} A \times \operatorname{ran} A$ as required.

Exercise 7   We have $\operatorname{fld} R \subseteq \bigcup\bigcup R$ by Lemma 3D.

Conversely, let $x \in \bigcup\bigcup R$. Pick a and b such that $x \in a$, $a \in b$ and $b \in R$. Then b is an ordered pair; let $b = (y, z)$. We have $a = \{y\}$ or $\{y, z\}$, hence $x = y$ or $x = z$. In either case, $x \in \operatorname{fld} R$.

Exercise 8

(a)

$$x \in \operatorname{dom} \bigcup \mathcal{A}$$
$$\Leftrightarrow \exists y.\exists R \in \mathcal{A}.(x, y) \in R$$
$$\Leftrightarrow \exists R \in \mathcal{A}.\exists y.(x, y) \in R$$
$$\Leftrightarrow x \in \bigcup\{\operatorname{dom} R : R \in \mathcal{A}\}$$

(b)

$$y \in \mathrm{ran} \bigcup \mathcal{A}$$
$$\Leftrightarrow \exists x. \exists R \in \mathcal{A}.(x,y) \in R$$
$$\Leftrightarrow \exists R \in \mathcal{A}.\exists x.(x,y) \in R$$
$$\Leftrightarrow y \in \bigcup \{\mathrm{ran}\, R : R \in \mathcal{A}\}$$

Exercise 9   Assume $\mathcal{A}$ is nonempty. We have $\mathrm{dom} \bigcap \mathcal{A} \subseteq \bigcap \{\mathrm{dom}\, R : R \in \mathcal{A}\}$.

PROOF:

$$x \in \mathrm{dom} \bigcap \mathcal{A}$$
$$\Leftrightarrow \exists y.\forall R \in \mathcal{A}.(x,y) \in R$$
$$\Rightarrow \forall R \in \mathcal{A}.\exists y.(x,y) \in R$$
$$\Leftrightarrow x \in \bigcap \{\mathrm{dom}\, R : R \in \mathcal{A}\}$$

Equality holds iff the middle '$\Rightarrow$' can be reversed, i.e. iff for all x, if $\forall R \in \mathcal{A}.\exists y.(x,y) \in R$ then $\exists y.\forall R \in \mathcal{A}.(x,y) \in R$. I haven't found a simpler condition than this. The condition does not always hold, for example if $\mathcal{A} = \{\{(1,2)\}, \{(1,3)\}\}$ then $\mathrm{dom} \bigcap \mathcal{A} = \emptyset$ while $\bigcap \{\mathrm{dom}\, R : R \in \mathcal{A}\} = \{1\}$.

Similarly, $\mathrm{ran} \bigcap \mathcal{A} \subseteq \bigcap \{\mathrm{ran}\, R : R \in \mathcal{A}\}$, and equality holds iff, for any y, if $\forall R \in \mathcal{A}.\exists x.(x,y) \in R$ then $\exists x.\forall R \in \mathcal{A}.(x,y) \in R$.

3.3 n-ary Relations

Exercise 10   This follows from the equations at the top of page 42. An ordered 4-tuple $\langle a, b, c, d \rangle$ is also an ordered 1-tuple (because every set is), and the ordered pair $\langle \langle a, b, c \rangle, d \rangle$, and the ordered triple $\langle \langle a, b \rangle, c, d \rangle$.