Solutions Manual for Lawvere and Schanuel
Conceptual Mathematics

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Part I

Preview

Chapter 1

Session 1 — Galileo and Multiplication of Objects

Exercise 1 Many examples — every instance of a product in a category gives an example. I will not list them.

Exercise 2 I am not entirely sure what solution the authors had in mind. Here are some that come to my mind:

Place a spirit level between the two points and see if it reads as level.

Place a smooth plank between the two points and see if a ball placed at one point rolls to the other, or *vice versa*.

Hang a plumbline at each point and see if they form a right angle with the line joining the two points.

Of these, the third is my favourite.

Part II

Part I — The category of sets

Chapter 2

Article I — Sets, maps, composition

Exercise 1 Easy.

Exercise 2 There are 8 maps from A to B.

Exercise 3 There are 27 maps from A to A.

Exercise 4 There are 9 maps from B to A.

Exercise 5 There are 4 maps from B to B.

Exercise 6 There are 10 such maps from A to A.

Exercise 7 There are 3 such maps from B to B.

Exercise 8 There is no such pair of maps.

Exercise 9 There are 12 such pairs of maps.

Chapter 3

Session 3 — Composing maps and counting maps

Exercise 1 (a) and (c) make sense.

Exercise 2 (a) and (c) still make sense.

Part III

Part II — The algebra of composition

Chapter 4

Article II — Isomorphisms

4.1 1. Isomorphisms

Exercise 1

(R) We have $1_A \circ 1_A = 1_A$ by the Identity Laws, so 1_A is an isomorphism with inverse 1_A.

(S) We have $g \circ f = 1_A$ and $f \circ g = 1_B$ (this is what it means for g to be an inverse for f). This says exactly that f is an inverse for g.

(T) Let $f^{-1} : B \to A$ be an inverse for f and $k^{-1} : C \to B$ be an inverse for k. We prove $f^{-1} \circ k^{-1}$ is an inverse for $k \circ f$. We have

$$\begin{aligned} f^{-1} \circ k^{-1} \circ k \circ f &= f^{-1} \circ 1_B \circ f && \text{(definition of inverse)} \\ &= f^{-1} \circ f && \text{(Identity Law)} \\ &= 1_A && \text{(definition of inverse)} \end{aligned}$$

and $k \circ f \circ f^{-1} \circ k^{-1} = 1_C$ similarly.

Exercise 2 We have

$$\begin{aligned} g &= g \circ 1_B && \text{(Identity Law)} \\ &= g \circ f \circ k && (k \text{ is an inverse of } f) \\ &= 1_A \circ k && (g \text{ is an inverse of } f) \\ &= k && \text{(Identity Law)} \end{aligned}$$

Exercise 3

(a) Let $f : A \to B$. Let $h, k : C \to A$.
Suppose $f \circ h = f \circ k$. Then

$$\begin{aligned} f^{-1} \circ f \circ h &= f^{-1} \circ f \circ k \\ \therefore 1_A \circ h &= 1_A \circ k && \text{(Definition of inverse)} \\ \therefore h &= k && \text{(Identity Law)}\square \end{aligned}$$

(b) Let $f : A \to B$. Let $h, k : B \to C$.
Suppose $h \circ f = k \circ f$. Then

$$\begin{aligned} h \circ f \circ f^{-1} &= k \circ f \circ f^{-1} \\ \therefore h \circ 1_B &= k \circ 1_B && \text{(Definition of inverse)} \\ \therefore h &= k && \text{(Identity Law)}\square \end{aligned}$$

(c) Let $A = \{0, 1\}$. Define $f : A \to A$ by $f(0) = 1$ and $f(1) = 0$. Define $h : A \to A$ by $h(x) = 0$ for all x. Define $k : A \to A$ by $k(x) = 1$ for all x.
f is invertible, and is its own inverse.
We have $h \circ f = f \circ k = h$.
We do not have $h = k$.

Exercise 4

(1) This function is invertible with inverse $f^{-1}(x) = (x - 7)/3$.

(2) This function is invertible with inverse $g^{-1}(x) = \sqrt{x}$.

(3) This function is not invertible because $h(1) = h(-1) = 1$.

(4) This function is not invertible because $k(1) = k(-1) = 1$.

(5) This function is not invertible because there is no x such that $l(x) = 2$.

4.2 2 — General division problems: Determination and choice

Exercise 5 There are 6 maps f such that $g \circ f = 1_{\{0,1\}}$; we can map 0 to any of b, p or q, and 1 to either of r or s.

Given any one of these maps f, there are 8 maps g such that $g \circ f = 1_{\{0,1\}}$. We must map $f(0)$ to 0, $f(1)$ to 1, and the other three elements to any of 0 or 1.

Exercise 6 If $r : B \to A$ is a section for f, then we take $t = g \circ r$. We have $t \circ f = g \circ r \circ f = g \circ 1_A = g$.

Exercise 7 Let $s : B \to A$ be a section for f. Let T be any set and $t_1, t_2 : T \to B$. Suppose $t_1 \circ f = t_2 \circ f$. Then

$$\begin{aligned} t_1 \circ f \circ s &= t_2 \circ f \circ s \\ \therefore t_1 \circ 1_B &= t_2 \circ 1_B \\ \therefore t_1 &= t_2 \end{aligned}$$

Exercise 8 If $s_1 : B \to A$ is a section for $r_1 : A \to B$ and $s_2 : C \to B$ is a section for $r_2 : B \to C$, then $s_1 \circ s_2$ is a section for $r_2 \circ r_1$ since

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ 1_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= 1_C \end{aligned}$$

Exercise 9 We have

$$\begin{aligned} e \circ e &= f \circ r \circ f \circ r \\ &= f \circ 1 \circ r && (r \text{ is a retraction of } f) \\ &= f \circ r \\ &= e \end{aligned}$$

Exercise 10 From the proof of Proposition 3, $f^{-1} \circ g^{-1}$ is both a section and a retraction for $g \circ f$.

Exercise 11 Set $f(\mathit{Fatima}) = \mathit{coffee}$, $f(\mathit{Omer}) = \mathit{tea}$ and $f(\mathit{Alysia}) = \mathit{cocoa}$. Then f is an isomorphism.

There is no isomorphism $g : A \to C$. For if $g(\mathit{Fatima}) = \mathit{true}$ then $g(\mathit{Omer})$ must be false, and then it is impossible to choose a value for $g(\mathit{Alysia})$ without having $g(\mathit{Alysia}) = g(\mathit{Fatima})$ or $g(\mathit{Alysia}) = g(\mathit{Omer})$. Similarly if $g(\mathit{Fatima}) = \mathit{false}$ then $g(\mathit{Omer})$ must be true, and then again we cannot choose a value for $g(\mathit{Alysia})$.

Chapter 5

Session 4 — Division of Maps: Isomorphisms

5.1 4. A small zoo of isomorphisms in other categories

Exercise 1 We have $h(d(x)) = h(2x) = x$ and $d(h(x)) = d(x/2) = x$ for any x.

Exercise 2 $f(odd) = negative$ and $f(even) = positive$

Exercise 3

(a) This is not an isomorphism because $p(0 + 0) = 1$ but $p(0) + p(0) = 2$

(b) This is not an isomorphism because it is not surjective; there is no x such that $sq(x) = -1$.

(c) This is not an isomorphism because it is not injective. We have $sq(1) = sq(-1) = 1$.

(d) This is an isomorphism; it is bijective and $-(x + y) = (-x) + (-y)$.

(e) This is not an isomorphism because $m(1\times1) = -1$ but $m(1)\times m(1) = 1$.

(f) This is not a well-defined map because $c(-1) = -1 \notin \mathbb{R}_{>0}$.

Chapter 6

Session 5 — Division of Maps: Sections and Retractions

6.1 1. Determination Problems

Exercise 1

(a) Suppose $h = g \circ f$ and $fa_1 = fa_2$. Then $ha_1 = g(fa_1) = g(fa_2) = ha_2$.

(b) No. Take $A = C = \emptyset$ and $B = \{*\}$. Let $f : A \to B$ and $h : A \to C$ be the unique such maps. Vacuously, if $fa_1 = fa_2$ then $ha_1 = ha_2$. But there is no map $g : B \to C$.

6.2 3. Choice Problems

Exercise 2

(a) Suppose $g \circ f = h$. Let $a \in A$. Let $b = f(a)$. Then $h(a) = g(f(a)) = g(b)$.

(b) This is equivalent to the Axiom of Choice.

6.3 5. Stacking or Sorting

Exercise 3 I'm not going to draw all of them, but there are 8 of them.

Chapter 7

Session 9 — Retracts and Idempotents

7.1 1. Retracts and Comparisons

Exercise 1 If A is empty, then the nowhere-defined function is a map $A \to B$.
If B has a point, say b, then the constant map with value b is a map $A \to B$.

7.2 2. Idempotents as records of retracts

Exercise 3 Suppose $s : A \to B$, $r : B \to A$ and $s' : A' \to B$, $r' : B \to A'$ are splittings of $e : B \to B$. Let

$$\begin{aligned} f &= r' \circ s & &: A \to A' \\ f^{-1} &= r \circ s' & &: A' \to A \end{aligned}$$

Then we have

$$\begin{aligned} f \circ f^{-1} &= r' \circ s \circ r \circ s' \\ &= r' \circ e \circ s' \\ &= r' \circ s' \circ r' \circ s' \\ &= 1 \\ f^{-1} \circ f &= r \circ s' \circ r' \circ s \\ &= r \circ e \circ s \\ &= r \circ s \circ r \circ s \\ &= 1 \end{aligned}$$

Chapter 8

Quiz

Question 1 Let $A = \{*\}$ and $B = \{0, 1\}$. Define $f : A \to B$ by $f(*) = 0$. Then the unique function $r : B \to A$ is a retraction for f (since $r(f(*)) = *$) but not a section for f (since $f(s(1)) = 0$). Therefore there is no section for f, since there is only one map $B \to A$.

Question 2

(a) Yes: if $pqp = p$ then $pqpq = pq$

(b) Yes: if $pqp = p$ then $qpqp = qp$

Question 2* Let $q' = qpq$ Then we have

$$\begin{aligned} pq'p &= pqpqp \\ &= pqp \\ &= p \\ q'pq' &= qpqpqpq \\ &= qpqpq \\ &= qpq \\ &= q' \end{aligned}$$

Question 1* Take $A = B = \mathbb{N}$ and define $f : A \to B$ by $f(x) = 2x$. Then f has a retraction r given by

$$r(y) = \begin{cases} y/2 & \text{if } y \text{ is even} \\ 0 & \text{if } y \text{ is odd} \end{cases}$$

It has no section since it is not surjective (Article II Proposition 1).

Chapter 9

Summary / quiz on pairs of 'opposed' maps

Question 1 Given two maps f, g with domains and codomains as above, we can always form the composites $g \circ f$ and $f \circ g$. All we can say about $g \circ f$ and $f \circ g$ as maps in themselves is that they are endomaps.

Question 2 If we know that g is a retraction for f, that means $g \circ f$ is actually the identity map 1_A; then we can prove that $f \circ g$ is not only an endomap, but actually an idempotent. The latter means that the equation $f \circ g \circ f \circ g = f \circ g$ is true.

Question 3 If we even know that f is an isomorphism *and* that $g \circ f = 1_A$, then $f \circ g$ is not only an idempotent, but is the identity map 1_B. If, moreover, s is a map for which $f \circ s = 1_B$, we can conclude that $s = g$.

Question 4 Going back to 0, i.e. assuming no equations, but only the domain and codomain statements about f and g, the composite $f \circ g \circ f$ could be different from f. Likewise $f \circ g \circ f \circ g$ could be different from $f \circ g$.