C1 Set Theory

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1 Primitive Notions

Let there be *sets*.

Let there be a binary relation called *membership*, $\in$. When $x \in y$ holds, we say x is a *member* or *element* of y. We write $x \notin y$ iff x is not a member of y.

2 The Axioms

Axiom 1 (Extensionality). *If two sets have exactly the same members, then they are equal.*

As a consequence of this axiom, we may identify a set A with the class $\{x : x \in A\}$. The use of the symbols $\in$ and $=$ is consistent.

Definition 2. We say that a class $\mathbf{A}$ *is a set* iff there exists a set A such that $A = \mathbf{A}$. That is, the class $\{x : P(x)\}$ is a set iff

$$\exists A. \forall x (x \in A \leftrightarrow P(x)) \ .$$

Otherwise, $\mathbf{A}$ is a *proper class*.

Definition 3 (Subset). If A is a set and $\mathbf{B}$ is a class, we say A is a *subset* of $\mathbf{B}$ iff $A \subseteq \mathbf{B}$.

Axiom 4 (Empty Set). *The empty class is a set, called the* empty set*.*

Axiom 5 (Pairing). *For any objects a and b, the class $\{a, b\}$ is a set, called a pair set.*

Definition 6 (Union). For any class of sets $\mathbf{A}$, the *union* $\bigcup \mathbf{A}$ is the class $\{x : \exists A \in \mathbf{A}. x \in A\}$.

We write $\bigcup_{P[x_1,\ldots,x_n]} t[x_1, \ldots, x_n]$ for $\bigcup \{t[x_1, \ldots, x_n] : P[x_1, \ldots, x_n]\}$.

Proposition 7. *If $\mathbf{A} \subseteq \mathbf{B}$ then $\bigcup \mathbf{A} \subseteq \bigcup \mathbf{B}$.*

PROOF: Easy. $\square$

Axiom 8 (Union). *For any set A, the union $\bigcup A$ is a set.*

Proposition 9. *For any sets A and B, the class $A \cup B$ is a set.*

PROOF: It is $\bigcup\{A, B\}$. □

Proposition Schema 10. *For any objects a_1, ..., a_n, the class $\{a_1, \ldots, a_n\}$ is a set.*

PROOF: By repeated application of the Pairing and Union axioms. □

Definition 11 (Power Set). For any set A, the *power set* of A, $\mathcal{P}A$, is the class of all subsets of A.

Axiom 12 (Power Set). *For any set A, the class $\mathcal{P}A$ is a set.*

Axiom 13 (Subset, Aussonderung). *For any class $\mathbf{A}$ and set B, if $\mathbf{A} \subseteq B$ then $\mathbf{A}$ is a set.*

Proposition 14. *For any set A and class $\mathbf{B}$, the intersection $A \cap \mathbf{B}$ is a set.*

PROOF: By the Subset Axiom since it is a subclass of A. □

Proposition 15. *For any set A and class $\mathbf{B}$, the relative complement $A - \mathbf{B}$ is a set.*

PROOF: By the Subset Axiom since it is a subclass of A. □

Theorem 16. *The universal class $\mathbf{V}$ is a proper class.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $\mathbf{V}$ is a set.
$\langle 1 \rangle 2$. LET: $R = \{x : x \notin x\}$
$\langle 1 \rangle 3$. R is a set.
  PROOF: By the Subset Axiom.
$\langle 1 \rangle 4$. $R \in R$ if and only if $R \notin R$
$\langle 1 \rangle 5$. Q.E.D.
  PROOF: This is a contradiction.
□

Definition 17 (Intersection). For any class of sets $\mathbf{A}$, the *intersection* $\bigcap \mathbf{A}$ is the class $\{x : \forall A \in \mathbf{A}.x \in A\}$.
  We write $\bigcap_{P[x_1,\ldots,x_n]} t[x_1, \ldots, x_n]$ for $\bigcap\{t[x_1, \ldots, x_n] : P[x_1, \ldots, x_n]\}$.

Proposition 18. *For any nonempty class of sets $\mathbf{A}$, the class $\bigcap \mathbf{A}$ is a set.*

PROOF: Pick $A \in \mathbf{A}$. Then $\bigcap \mathbf{A} \subseteq A$. □

Proposition 19. *If $\mathbf{A} \subseteq \mathbf{B}$ then $\bigcap \mathbf{B} \subseteq \bigcap \mathbf{A}$.*

PROOF: Easy. □

Proposition 20. *For any set A and class of sets $\mathbf{B}$, we have*

$$A \cup \bigcap \mathbf{B} = \bigcap\{A \cup X \mid X \in \mathbf{B}\}$$

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PROOF: Easy. □

Proposition 21. *For any set A and class of sets* **B**, *we have*

$$A \cap \bigcup \mathbf{B} = \bigcup \{A \cap X \mid X \in \mathbf{B}\}$$

PROOF: Easy. □

Proposition 22. *For any set C and class of sets* **A**, *we have*

$$C - \bigcup \mathbf{A} = \bigcap \{C - X \mid X \in \mathbf{A}\} \ .$$

PROOF: Easy. □

Proposition 23. *For any set C and class of sets* **A**, *we have*

$$C - \bigcap \mathbf{A} = \bigcup \{C - X \mid X \in \mathbf{A}\} \ .$$

PROOF: Easy. □

3 Ordered Pairs

Definition 24 (Ordered Pair)**.** For any objects a and b, the *ordered pair* (a, b) is $\{\{a\}, \{a, b\}\}$. We call a its *first coordinate* and b its *second coordinate*.

Theorem 25. *For any objects* (a, b), *we have* $(a, b) = (c, d)$ *if and only if* $a = c$ *and* $b = d$.

PROOF:
⟨1⟩1. If $(a, b) = (c, d)$ then $a = c$ and $b = d$
  ⟨2⟩1. ASSUME: $(a, b) = (c, d)$
  ⟨2⟩2. $a = c$
    PROOF: Since $\{a\} = \bigcap(a, b) = \bigcap(c, d) = \{c\}$.
  ⟨2⟩3. $\{a, b\} = \{c, d\}$
    PROOF: $\{a, b\} = \bigcup(a, b) = \bigcup(c, d) = \{c, d\}$.
  ⟨2⟩4. $b = c$ or $b = d$
  ⟨2⟩5. CASE: $b = c$
    ⟨3⟩1. $a = b$
    ⟨3⟩2. $\{c, d\} = \{a\}$
    ⟨3⟩3. $b = d$
  ⟨2⟩6. CASE: $b = d$
    PROOF: We have $a = c$ and $b = d$ as required.
⟨1⟩2. If $a = c$ and $b = d$ then $(a, b) = (c, d)$
  PROOF: Trivial.
□

Definition 26 (Cartesian Product)**.** The *Cartesian product* of classes **A** and **B** is the class

$$\mathbf{A} \times \mathbf{B} = \{(x, y) : x \in \mathbf{A}, y \in \mathbf{B}\} \ .$$

Lemma 27. *For any objects x and y and set C, if $x \in C$ and $y \in C$ then $(x, y) \in \mathcal{PP}C$.*

PROOF: Easy. □

Corollary 27.1. *For any sets A and B, the Cartesian product $A \times B$ is a set.*

PROOF: By the Subset Axiom applied to $\mathcal{PP}(A \cup B)$. □

Lemma 28. *If $(x, y) \in \mathbf{A}$ then $x, y \in \bigcup\bigcup \mathbf{A}$.*

PROOF: Easy. □

4 Relations

Definition 29 (Relation). A *relation* is a class of ordered pairs. It is *small* iff it is a set.

  When $\mathbf{R}$ is a relation, we write $x\mathbf{R}y$ for $(x, y) \in \mathbf{R}$.

Definition 30 (Domain). The *domain* of a class $\mathbf{R}$ is $\operatorname{dom}\mathbf{R} = \{x : \exists y.(x, y) \in \mathbf{R}\}$.

Definition 31 (Range). The *range* of a class $\mathbf{R}$ is $\operatorname{ran}\mathbf{R} = \{y : \exists x.(x, y) \in \mathbf{R}\}$.

Definition 32 (Field). The *field* of a class $\mathbf{R}$ is $\operatorname{fld}\mathbf{R} = \operatorname{dom}\mathbf{R} \cup \operatorname{ran}\mathbf{R}$.

Proposition 33. *If R is a set then $\operatorname{dom}R$, $\operatorname{ran}R$ and $\operatorname{fld}R$ are sets.*

PROOF: Apply the Subset Axiom to $\bigcup\bigcup R$. □

Definition 34 (Single-Rooted). A class $\mathbf{R}$ is *single-rooted* iff, for all $y \in \operatorname{ran}\mathbf{R}$, there is only one x such that $x\mathbf{R}y$.

Definition 35 (Inverse). The *inverse* of a class $\mathbf{F}$ is the class $\mathbf{F}^{-1} = \{(y, x) \mid (x, y) \in \mathbf{F}\}$.

Theorem 36. *For any class $\mathbf{F}$, we have $\operatorname{dom}\mathbf{F}^{-1} = \operatorname{ran}\mathbf{F}$ and $\operatorname{ran}\mathbf{F}^{-1} = \operatorname{dom}\mathbf{F}$.*

PROOF: Easy. □

Theorem 37. *For a relation $\mathbf{F}$, $(\mathbf{F}^{-1})^{-1} = \mathbf{F}$.*

PROOF: Easy. □

Definition 38 (Composition). The *composition* of classes $\mathbf{F}$ and $\mathbf{G}$ is the class $\mathbf{G} \circ \mathbf{F} = \{(x, z) \mid \exists y.(x, y) \in \mathbf{F} \wedge (y, z) \in \mathbf{G}\}$.

Theorem 39. *For any classes $\mathbf{F}$ and $\mathbf{G}$, $(\mathbf{F} \circ \mathbf{G})^{-1} = \mathbf{G}^{-1} \circ \mathbf{F}^{-1}$.*

PROOF: Easy. □

Definition 40 (Restriction)**.** The *restriction* of the class $\mathbf{F}$ to the class $\mathbf{A}$ is the class $\mathbf{F} \upharpoonright \mathbf{A} = \{(x, y) : x \in A \wedge (x, y) \in \mathbf{F}\}$.

Definition 41 (Image)**.** The *image* of the class $\mathbf{A}$ under the class $\mathbf{F}$ is the class $\mathbf{F}(\mathbf{A}) = \{y : \exists x \in \mathbf{A}.(x, y) \in \mathbf{F}\}$.

Theorem 42.
$$\mathbf{F}(\mathbf{A} \cup \mathbf{B}) = \mathbf{F}(\mathbf{A}) \cup \mathbf{F}(\mathbf{B})$$

PROOF: Easy. $\square$

Theorem 43.
$$\mathbf{F}(\bigcup \mathbf{A}) = \bigcup \{\mathbf{F}(X) : X \in \mathbf{A}\}$$

PROOF: Easy. $\square$

Theorem 44.
$$\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

PROOF: Easy. $\square$

Theorem 45.
$$\mathbf{F}(\bigcap \mathbf{A}) \subseteq \bigcap \{\mathbf{F}(X) : X \in \mathbf{A}\}$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

PROOF: Easy. $\square$

Theorem 46.
$$\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

PROOF: Easy. $\square$

Definition 47 (Reflexive)**.** A binary relation $\mathbf{R}$ on $\mathbf{A}$ is *reflexive* on $\mathbf{A}$ if and only if $\forall x \in \mathbf{A}.x\mathbf{R}x$.

Definition 48 (Symmetric)**.** A binary relation $\mathbf{R}$ is *symmetric* iff, whenever $x\mathbf{R}y$, then $y\mathbf{R}x$.

Definition 49 (Transitive)**.** A binary relation $\mathbf{R}$ is *transitive* iff, whenever $x\mathbf{R}y$ and $y\mathbf{R}z$, then $x\mathbf{R}z$.

5 n-ary Relations

Definition 50. Given objects a, b, c, define the *ordered triple* (a, b, c) to be $((a, b), c)$.

Define $(a, b, c, d) = ((a, b, c), d)$, etc.

Define the *1-tuple* (a) to be a.

Definition 51 (n-ary Relation)**.** Given a class $\mathbf{A}$, an *n-ary relation* on $\mathbf{A}$ is a class of ordered n-tuples, all of whose components are in $\mathbf{A}$.

6 Functions

Definition 52 (Function)**.** A *function* is a relation $\mathbf{F}$ such that, for all $x \in$ dom $\mathbf{F}$, there is only one y such that $x\mathbf{F}y$. We call this unique y the *value* of $\mathbf{F}$ at x and denote it by $\mathbf{F}(x)$.

We say $\mathbf{F}$ is a function *from* $\mathbf{A}$ *into* $\mathbf{B}$, or $\mathbf{F}$ *maps* $\mathbf{A}$ into $\mathbf{B}$, and write $\mathbf{F} : \mathbf{A} \to \mathbf{B}$, iff $\mathbf{F}$ is a function, dom $\mathbf{F} = \mathbf{A}$, and ran $\mathbf{F} \subseteq \mathbf{B}$.

If, in addition, ran $\mathbf{F} = \mathbf{B}$, we say $\mathbf{F}$ is a function from $\mathbf{A}$ *onto* $\mathbf{B}$.

Theorem 53. *For a class* $\mathbf{F}$*,* $\mathbf{F}^{-1}$ *is a function if and only if* $\mathbf{F}$ *is single-rooted.*

PROOF: Easy. $\square$

Theorem 54. *A relation* $\mathbf{F}$ *is a function if and only if* $\mathbf{F}^{-1}$ *is single-rooted.*

PROOF: Easy. $\square$

Theorem 55. *For any function* $\mathbf{G}$ *and classes* $\mathbf{A}$ *and* $\mathbf{B}$*,*

$$\mathbf{G}^{-1}(\bigcup \mathbf{A}) = \bigcup \{\mathbf{G}^{-1}(X) : X \in \mathbf{A}\}$$
$$\mathbf{G}^{-1}(\bigcap \mathbf{A}) = \bigcap \{\mathbf{G}^{-1}(X) : X \in \mathbf{A}\} \qquad (\textit{if } \mathbf{A} \neq \emptyset)$$
$$\mathbf{G}^{-1}(\mathbf{A} - \mathbf{B}) = \mathbf{G}^{-1}(\mathbf{A}) - \mathbf{G}^{-1}(\mathbf{B})$$

PROOF: Easy. $\square$

Theorem 56. *Assume that* $\mathbf{F}$ *and* $\mathbf{G}$ *are functions. Then* $\mathbf{F} \circ \mathbf{G}$ *is a function, its domain is* $\{x \in \text{dom } \mathbf{G} : \mathbf{G}(x) \in \text{dom } \mathbf{F}\}$*, and for* x *in its domain,*

$$(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x)) \ .$$

PROOF: Easy. $\square$

Definition 57 (One-to-one)**.** A function $\mathbf{F}$ is *one-to-one* or an *injection* iff it is single-rooted.

Theorem 58. *Let* $\mathbf{F}$ *be a one-to-one function. For* $x \in \text{dom } \mathbf{F}$*,* $\mathbf{F}^{-1}(\mathbf{F}(x)) = x$*.*

PROOF: Easy. $\square$

Theorem 59. *Let* $\mathbf{F}$ *be a one-to-one function. For* $y \in \text{ran } \mathbf{F}$*,* $\mathbf{F}(\mathbf{F}^{-1}(y)) = y$*.*

PROOF: Easy. $\square$

Definition 60 (Identity Function)**.** For any class $\mathbf{A}$, the *identity* function on $\mathbf{A}$ is $\text{id}_{\mathbf{A}} = \{(x,x) \mid x \in \mathbf{A}\}$.

Theorem 61. *Let* $F : A \to B$*. Assume* $A \neq \emptyset$*. Then* F *has a left inverse (i.e. there exists* $G : B \to A$ *such that* $G \circ F = \text{id}_A$*) if and only if* F *is one-to-one.*

PROOF:
$\langle 1 \rangle 1$. If F is one-to-one then F has a left inverse.

⟨2⟩1. ASSUME: F is one-to-one.
⟨2⟩2. $F^{-1} : \operatorname{ran} F \to A$
⟨2⟩3. PICK $a \in A$
⟨2⟩4. Define $G : B \to A$ by:
$$G(x) = \begin{cases} F^{-1}(x) & \text{if } x \in \operatorname{ran} F \\ a & \text{if } x \in B - \operatorname{ran} F \end{cases}$$
⟨2⟩5. $\forall x \in A.G(F(x)) = x$
⟨1⟩2. If F has a left inverse then F is one-to-one.
  ⟨2⟩1. ASSUME: F has a left inverse G.
  ⟨2⟩2. LET: $x, y \in A$ with $F(x) = F(y)$
  ⟨2⟩3. $x = y$
    PROOF: $x = G(F(x)) = G(F(y)) = y$.
□

Definition 62 (Binary Operation). A *binary operation* on a set A is a function from $A \times A$ into A.

7 The Axiom of Choice

Axiom 63 (Choice). *For any relation R there exists a function $H \subseteq R$ with $\operatorname{dom} H = \operatorname{dom} R$.*

Theorem 64. *Let $F : A \to B$. Then F has a right inverse if and only if F maps A onto B.*

PROOF:
⟨1⟩1. If F has a right inverse then F maps A onto B.
  PROOF: If $H : B \to A$ is a right inverse, then for any y in B, we have $y = F(H(y))$.
⟨1⟩2. If F maps A onto B then F has a right inverse.
  ⟨2⟩1. ASSUME: F maps A onto B.
  ⟨2⟩2. PICK a function H with $H \subseteq F^{-1}$ and $\operatorname{dom} H = \operatorname{dom} F^{-1}$
    PROOF: By the Axiom of Choice.
  ⟨2⟩3. $\operatorname{dom} H = B$
    PROOF: $\operatorname{dom} H = \operatorname{dom} F^{-1} = \operatorname{ran} F = B$ by ⟨2⟩1.
  ⟨2⟩4. For all $y \in B$ we have $F(H(y)) = y$
    ⟨3⟩1. LET: $y \in B$
    ⟨3⟩2. $(y, H(y)) \in F^{-1}$
    ⟨3⟩3. $F(H(y)) = y$
□

8 Sets of Functions

Definition 65. Let A be a set and **B** be a class. Then $\mathbf{B}^A$ is the class of all functions $A \to \mathbf{B}$.

9   Dependent Products

Definition 66. Let I be a set and H_i a set for all $i \in I$. Define

$$\prod_{i \in I} H_i = \{f : f \text{ is a function}, \operatorname{dom} f = I, \forall i \in I . f(i) \in H_i\} \ .$$

Theorem 67. *The Axiom of Choice is equivalent to the statement: For any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$*

PROOF:
$\langle 1 \rangle 1$. If the Axiom of Choice is true then, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$.
   $\langle 2 \rangle 1$. ASSUME: The Axiom of Choice.
   $\langle 2 \rangle 2$. LET: I be a set.
   $\langle 2 \rangle 3$. LET: H be a function with domain I.
   $\langle 2 \rangle 4$. ASSUME: $H(i) \neq \emptyset$ for all $i \in I$.
   $\langle 2 \rangle 5$. LET: $R = \{(i, x) : i \in I, x \in H(i)\}$
   $\langle 2 \rangle 6$. PICK a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R$
      PROVE: $F \in \prod_{i \in I} H(i)$
   PROOF: By the Axiom of Choice.
   $\langle 2 \rangle 7$. $\operatorname{dom} H = I$
   PROOF: We have $\operatorname{dom} R = I$ since for all $i \in I$ there exists x such that $x \in H(i)$.
   $\langle 2 \rangle 8$. $\forall i \in I . F(i) \in H(i)$
   PROOF: Since $iRF(i)$.
$\langle 1 \rangle 2$. If, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$, then the Axiom of Choice is true.
   $\langle 2 \rangle 1$. ASSUME: For any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$
   $\langle 2 \rangle 2$. LET: R be a relation
   $\langle 2 \rangle 3$. LET: $I = \operatorname{dom} R$
   $\langle 2 \rangle 4$. Define the function H with domain I by: for $i \in I$, $H(i) = \{y : iRy\}$
   $\langle 2 \rangle 5$. $H(i) \neq \emptyset$ for all $i \in I$
   $\langle 2 \rangle 6$. PICK $F \in \prod_{i \in I} H(i)$
     PROOF: By $\langle 2 \rangle 1$
   $\langle 2 \rangle 7$. F is a function
   $\langle 2 \rangle 8$. $F \subseteq R$
   PROOF: For all $i \in I$ we have $F(i) \in H(i)$ and so $iRF(i)$.
   $\langle 2 \rangle 9$. $\operatorname{dom} F = \operatorname{dom} R$
☐

10   Equivalence Relations

Definition 68 (Equivalence Relation)**.** An *equivalence relation* on **A** is a binary relation on **A** that is reflexive on **A**, symmetric and transitive.

Theorem 69. *If* $\mathbf{R}$ *is a symmetric and transitive relation then* $\mathbf{R}$ *is an equivalence relation on* fld $\mathbf{R}$*.*

PROOF:
⟨1⟩1. LET: $x \in$ fld $\mathbf{R}$
⟨1⟩2. PICK y such that either $x\mathbf{R}y$ or $y\mathbf{R}x$
⟨1⟩3. $x\mathbf{R}y$ and $y\mathbf{R}x$
   PROOF: Since $\mathbf{R}$ is symmetric.
⟨1⟩4. $x\mathbf{R}x$
   PROOF: Since $\mathbf{R}$ is transitive.
☐

Definition 70 (Equivalence Class)**.** If $\mathbf{R}$ is an equivalence relation and $x \in$ fld $\mathbf{R}$, the *equivalence class* of x modulo $\mathbf{R}$ is

$$[x]_{\mathbf{R}} = \{t : x\mathbf{R}t\} \ .$$

Lemma 71. *Assume that* $\mathbf{R}$ *is an equivalence relation on* $\mathbf{A}$ *and that* x *and* y *belong to* $\mathbf{A}$*. Then*
$$[x]_{\mathbf{R}} = [y]_{\mathbf{R}} \text{ iff } x\mathbf{R}y \ .$$

PROOF:
⟨1⟩1. If $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ then $x\mathbf{R}y$
  ⟨2⟩1. ASSUME: $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$
  ⟨2⟩2. $y \in [y]_{\mathbf{R}}$
     PROOF: Since $\mathbf{R}$ is reflexive on $\mathbf{A}$.
  ⟨2⟩3. $y \in [x]_{\mathbf{R}}$
  ⟨2⟩4. $x\mathbf{R}y$
⟨1⟩2. If $x\mathbf{R}y$ then $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$
  ⟨2⟩1. ASSUME: $x\mathbf{R}y$
  ⟨2⟩2. $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$
    ⟨3⟩1. LET: $z \in [y]_{\mathbf{R}}$
    ⟨3⟩2. $y\mathbf{R}z$
    ⟨3⟩3. $x\mathbf{R}z$
      PROOF: Since $\mathbf{R}$ is transitive.
    ⟨3⟩4. $z \in [x]_{\mathbf{R}}$
  ⟨2⟩3. $y\mathbf{R}x$
     PROOF: Since $\mathbf{R}$ is symmetric.
  ⟨2⟩4. $[x]_{\mathbf{R}} \subseteq [y]_{\mathbf{R}}$
     PROOF: Similar.
☐

Definition 72 (Partition)**.** A *partition* of a set A is a set $P \subseteq \mathcal{P}A$ such that:

- Every member of P is nonempty.

- Any two distinct members of P are disjoint.

- $A = \bigcup P$

Theorem 73. *Let R be an equivalence relation on the set A. Then the set of all equivalence classes is a partition of A.*

PROOF:

$\langle 1 \rangle 1$. Every equivalence class is nonempty.
  PROOF: For any $x \in A$ we have $x \in [x]_R$.
$\langle 1 \rangle 2$. Any two distinct equivalence classes are disjoint.
  $\langle 2 \rangle 1$. LET: $x, y \in A$
  $\langle 2 \rangle 2$. ASSUME: $z \in [x]_R \cap [y]_R$
        PROVE: $[x]_R = [y]_R$
  $\langle 2 \rangle 3$. xRy
    $\langle 3 \rangle 1$. xRz
    $\langle 3 \rangle 2$. yRz
    $\langle 3 \rangle 3$. zRy
      PROOF: By $\langle 3 \rangle 2$ and symmetry.
    $\langle 3 \rangle 4$. xRy
      PROOF: By $\langle 3 \rangle 1$, $\langle 3 \rangle 3$ and transitivity.
  $\langle 2 \rangle 4$. $[x]_R = [y]_R$
    PROOF: By Lemma 3N.
$\langle 1 \rangle 3$. A is the union of all the equivalence classes.
  PROOF: For any $x \in A$ we have $x \in [x]_R$.
$\square$

Definition 74 (Quotient Set). If R is an equivalence relation on the set A, then the *quotient set* A/R is the set of all equivalence classes, and the *natural map* or *canonical map* $\phi : A \to A/R$ is defined by $\phi(x) = [x]_R$.

Theorem 75. *Assume that R is an equivalence relation on A and that $F : A \to B$. Assume that F is* compatible *with R; that is, whenever xRy, then $F(x) = F(y)$. Then there exists a unique $\overline{F} : A/R \to B$ such that $F = \overline{F} \circ \phi$.*

PROOF: The unique such $\overline{F}$ is $\{([x], F(x)) : x \in A\}$. $\square$

11 Linear Orders

Definition 76 (Linear Ordering). Let $\mathbf{A}$ be a class. A *linear ordering* or *total ordering* on $\mathbf{A}$ is a relation $\mathbf{R}$ on $\mathbf{A}$ such that:

- $\mathbf{R}$ is transitive.

- $\mathbf{R}$ satisfies *trichotomy* on $\mathbf{A}$; i.e. for any $x, y \in \mathbf{A}$, exactly one of

$$x\mathbf{R}y, x = y, y\mathbf{R}x$$

  holds.

Theorem 77. *Let $\mathbf{R}$ be a linear ordering on $\mathbf{A}$.*

  1. There is no x such that $x\mathbf{R}x$.

2. For distinct x and y in $\mathbf{A}$, either $x\mathbf{R}y$ or $y\mathbf{R}x$.

PROOF: Immediate from trichotomy. □

12 Natural Numbers

Definition 78 (Successor). The *successor* of a set a is the set $a^+ = a \cup \{a\}$.

Definition 79 (Inductive). A class $\mathbf{A}$ is *inductive* iff $\emptyset \in \mathbf{A}$ and $\forall a \in \mathbf{A}.a^+ \in \mathbf{A}$.

Axiom 80 (Infinity). *There exists an inductive set.*

Definition 81 (Natural Number). A *natural number* is a set that belongs to every inductive set.

We write ω for the class of all natural numbers.

Theorem 82. *The class ω is a set.*

PROOF: Pick an inductive set I (by the Axiom of Infinity), then apply a Subset Axiom to I. □

Theorem 83. *The set ω is inductive, and is a subset of every inductive set.*

PROOF: Easy. □

Corollary 83.1 (Proof by Induction). *Any inductive subclass of ω is equal to ω.*

Theorem 84. *Every natural number except 0 is the successor of some natural number.*

PROOF: Easy proof by induction. □

Definition 85 (Peano System). A *Peano system* is a triple $\langle N, S, e \rangle$ consisting of a set N, a function $S : N \to N$ and an element $e \in N$ such that:

1. $e \notin \operatorname{ran} S$

2. S is one-to-one

3. Any subset $A \subseteq N$ that contains e and is closed under S equals N.

Definition 86 (Transitive Set). A set A is a *transitive set* iff every member of a member of A is a member of A.

Theorem 87. *For any transitive set a, $\bigcup(a^+) = a$.*

PROOF:
$$\begin{aligned}
\bigcup(a^+) &= \bigcup(a \cup \{a\}) \\
&= \bigcup a \cup \bigcup\{a\} \\
&= \bigcup a \cup a \\
&= a
\end{aligned}$$
since $\bigcup a \subseteq a$. □

Theorem 88. *Every natural number is a transitive set.*

Proof:
⟨1⟩1. 0 is a transitive set.
  Proof: Vacuous.
⟨1⟩2. For any natural number n, if n is a transitive set then n^+ is a transitive
    set.
  ⟨2⟩1. LET: n be a natural number that is a transitive set.
  ⟨2⟩2. $\bigcup(n^+) \subseteq n^+$
    Proof: Theorem 87.
◻

Theorem 89. *$\langle \omega, \sigma, 0 \rangle$ is a Peano system, where $0 = \emptyset$ and $\sigma = \{\langle n, n^+ \rangle : n \in \omega\}$.*

Proof:
⟨1⟩1. $0 \notin \operatorname{ran}\sigma$
  Proof: For any $n \in \omega$ we have $0 \neq n^+$ since $n \in n^+$ and $n \notin 0$.
⟨1⟩2. σ is one-to-one.
  Proof: If $m^+ = n^+$ then $m = \bigcup(m^+) = \bigcup(n^+) = n$ using Theorems 87 and
  88.
⟨1⟩3. Any subset $A \subseteq \omega$ that contains 0 and is closed under σ equals ω.
◻

Theorem 90. *The set ω is a transitive set.*

Proof:
⟨1⟩1. For every natural number n we have $\forall m \in n$. m is a natural number.
  ⟨2⟩1. $\forall m \in 0$. m is a natural number.
    Proof: Vacuous.
  ⟨2⟩2. If n is a natural number and $\forall m \in n$. m is a natural number, then
      $\forall m \in n^+$. m is a natural number.
    Proof: Since if $m \in n^+$ we have either $m \in n$ or $m = n$, and m is a natural
    number in either case.
◻

Theorem 91 (Recursion Theorem on ω)**.** *Let A be a set, $a \in A$ and $F : A \to A$. Then there exists a unique function $h : \omega \to A$ such that*

$$h(0) = a \ ,$$

and for every n in ω,
$$h(n^+) = F(h(n)) \ .$$

Proof:
⟨1⟩1. Let us call a function v *acceptable* iff $\operatorname{dom} v \subseteq \omega$, $\operatorname{ran} v \subseteq A$ and:

    1. If $0 \in \operatorname{dom} v$ then $v(0) = a$

    2. For all $n \in \omega$, if $n^+ \in \operatorname{dom} v$ then $n \in \operatorname{dom} v$ and $v(n^+) = F(v(n))$.

$\langle 1\rangle 2$. LET: $\mathcal{K}$ be the set of acceptable functions.
$\langle 1\rangle 3$. LET: $h = \bigcup \mathcal{K}$
$\langle 1\rangle 4$. h is a function.
  $\langle 2\rangle 1$. LET: $S = \{n \in \omega : \text{for at most one } y, (n, y) \in h\}$
  $\langle 2\rangle 2$. S is inductive.
    $\langle 3\rangle 1$. $0 \in S$
      $\langle 4\rangle 1$. LET: $\langle 0, y_1\rangle, \langle 0, y_2\rangle \in h$
      $\langle 4\rangle 2$. PICK acceptable v_1 and v_2 such that $v_1(0) = y_1$ and $v_2(0) = y_2$
      $\langle 4\rangle 3$. $y_1 = a$
      $\langle 4\rangle 4$. $y_2 = a$
      $\langle 4\rangle 5$. $y_1 = y_2$
    $\langle 3\rangle 2$. $\forall k \in S.k^+ \in S$
      $\langle 4\rangle 1$. LET: $k \in S$
      $\langle 4\rangle 2$. LET: $(k^+, y_1), (k^+, y_2) \in h$
      $\langle 4\rangle 3$. PICK acceptable v_1, v_2 such that $v_1(k^+) = y_1$ and $v_2(k^+) = y_2$
      $\langle 4\rangle 4$. $y_1 = F(v_1(k))$
      $\langle 4\rangle 5$. $f_2 = F(v_2(k))$
      $\langle 4\rangle 6$. $v_1(k) = v_2(k)$
        $\langle 5\rangle 1$. $(k, v_1(k)), (k, v_2(k)) \in h$
        $\langle 5\rangle 2$. Q.E.D.
          PROOF: By $\langle 4\rangle 1$
      $\langle 4\rangle 7$. $y_1 = y_2$
  $\langle 2\rangle 3$. $S = \omega$
$\langle 1\rangle 5$. h is acceptable.
  $\langle 2\rangle 1$. If $0 \in \operatorname{dom} h$ then $h(0) = a$
    $\langle 3\rangle 1$. ASSUME: $0 \in \operatorname{dom} h$
    $\langle 3\rangle 2$. PICK v acceptable with $v(0) = h(0)$
    $\langle 3\rangle 3$. $v(0) = a$
    $\langle 3\rangle 4$. $h(0) = a$
  $\langle 2\rangle 2$. For all $n \in \omega$, if $n^+ \in \operatorname{dom} h$ then $n \in \operatorname{dom} h$ and $h(n^+) = F(h(n))$
    $\langle 3\rangle 1$. LET: $n \in \omega$ with $n^+ \in \operatorname{dom} h$
    $\langle 3\rangle 2$. PICK v acceptable with $v(n^+) = h(n^+)$
    $\langle 3\rangle 3$. $n \in \operatorname{dom} v$
    $\langle 3\rangle 4$. $v(n) = h(n)$
    $\langle 3\rangle 5$. $h(n^+) = F(h(n))$
      PROOF:
$$
\begin{aligned}
h(n^+) &= v(n^+) \\
&= F(v(n)) \\
&= F(h(n))
\end{aligned}
$$
$\langle 1\rangle 6$. $\operatorname{dom} h = \omega$
  $\langle 2\rangle 1$. $0 \in \operatorname{dom} h$
    PROOF: Since $\{(0, a)\}$ is an acceptable function.
  $\langle 2\rangle 2$. $\forall n \in \operatorname{dom} h.n^+ \in \operatorname{dom} h$
    $\langle 3\rangle 1$. LET: $n \in \operatorname{dom} h$
    $\langle 3\rangle 2$. PICK an acceptable v such that $n \in \operatorname{dom} v$

$\langle 3 \rangle$3. ASSUME: w.l.o.g. $n^+ \notin \operatorname{dom} v$

$\langle 3 \rangle$4. $v \cup \{(n^+, F(v(n)))\}$ is acceptable.

$\langle 1 \rangle$7. For any acceptable function $h' : \omega \to A$ we have $h' = h$

  $\langle 2 \rangle$1. LET: $h' : \omega \to A$ be acceptable.

  $\langle 2 \rangle$2. $h'(0) = h(0)$

    PROOF: $h'(0) = h(0) = a$

  $\langle 2 \rangle$3. $\forall n \in \omega . h'(n) = h(n) \Rightarrow h'(n^+) = h(n^+)$

    PROOF: We have $h'(n^+) = F(h'(n)) = F(h(n)) = h(n^+)$.

$\square$

Theorem 92. *Let (N, S, e) be a Peano system. Then $(\omega, \sigma, 0)$ is isomorphic to (N, S, e), i.e. there is a function h mapping ω one-to-one onto N in a way that preserves the successor operation*

$$h(\sigma(n)) = S(h(n))$$

and the zero element

$$h(0) = e \ .$$

PROOF:

$\langle 1 \rangle$1. There exists a function h that satisfies those two conditions.

  PROOF: By the Recursion Theorem.

$\langle 1 \rangle$2. For all $m, n \in \omega$, if $m \neq n$ then $h(m) \neq h(n)$

  $\langle 2 \rangle$1. For all $n \in \omega$, if $n \neq 0$ then $h(n) \neq h(0)$

    $\langle 3 \rangle$1. LET: $n \in \omega$

    $\langle 3 \rangle$2. ASSUME: $n \neq 0$

    $\langle 3 \rangle$3. PICK p such that $n = p^+$

    $\langle 3 \rangle$4. $h(n) \neq h(0)$

      PROOF: $h(n) = S(h(p)) \neq e = h(0)$.

  $\langle 2 \rangle$2. For all $m \in \omega$, if $\forall n (m \neq n \Rightarrow h(m) \neq h(n))$ then $\forall n (m^+ \neq n \Rightarrow h(m^+) \neq h(n))$

    $\langle 3 \rangle$1. LET: $m \in \omega$

    $\langle 3 \rangle$2. ASSUME: $\forall n (m \neq n \Rightarrow h(m) \neq h(n))$

    $\langle 3 \rangle$3. LET: $n \in \omega$

    $\langle 3 \rangle$4. ASSUME: $m^+ \neq n$

      PROVE: $h(m^+) \neq h(n)$

    $\langle 3 \rangle$5. CASE: $n = 0$

      PROOF: $h(m^+) = S(h(m)) \neq e = h(n)$

    $\langle 3 \rangle$6. CASE: $n = p^+$

      $\langle 4 \rangle$1. $m \neq p$

      $\langle 4 \rangle$2. $h(m) \neq h(p)$

      $\langle 4 \rangle$3. $S(h(m)) \neq S(h(p))$

      $\langle 4 \rangle$4. $h(m^+) \neq h(p^+)$

$\langle 1 \rangle$3. For all $x \in N$, there exists $n \in \omega$ such that $h(n) = x$

  PROOF: An easy induction on x.

$\square$