C0 Classes

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We speak informally of *classes*. A class is determined by a unary predicate. We write $\{x : P(x)\}$ or $\{x \mid P(x)\}$ for the class determined by the predicate $P(x)$.

We define what it means for an object a to be an element of the class $\mathbf{A}$, $a \in \mathbf{A}$, by: $a \in \{x : P(x)\}$ means $P(a)$.

We write $\{x \in \mathbf{A} : P(x)\}$ for $\{x : x \in \mathbf{A} \wedge P(x)\}$, and $\{t[x_1, \ldots, x_n] : P[x_1, \ldots, x_n]\}$ for $\{y : \exists x_1 \cdots \exists x_n (y = t[x_1, \ldots, x_n] \wedge P[x_1, \ldots, x_n])\}$.

Definition 1 (Equality of Classes)**.** Two classes $\mathbf{A}$ and $\mathbf{B}$ are *equal*, $\mathbf{A} = \mathbf{B}$, iff they have exactly the same members.

Proposition 2. *For any class* $\mathbf{A}$ *we have* $\mathbf{A} = \mathbf{A}$.

Since $\mathbf{A}$ and $\mathbf{A}$ have exactly the same members.

Proposition 3. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, if* $\mathbf{A} = \mathbf{B}$ *then* $\mathbf{B} = \mathbf{A}$.

PROOF: If $\mathbf{A}$ and $\mathbf{B}$ have exactly the same members, then $\mathbf{B}$ and $\mathbf{A}$ have exactly the same members.

Proposition 4. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, if* $\mathbf{A} = \mathbf{B}$ *and* $\mathbf{B} = \mathbf{C}$ *then* $\mathbf{A} = \mathbf{C}$.

PROOF: If $\mathbf{A}$ and $\mathbf{B}$ have exactly the same members, and $\mathbf{B}$ and $\mathbf{C}$ have exactly the same members, then $\mathbf{A}$ and $\mathbf{C}$ have exactly the same members. □

Definition 5 (Subclass)**.** A class $\mathbf{A}$ is a *subclass* of a class $\mathbf{B}$, $\mathbf{A} \subseteq \mathbf{B}$, iff every member of $\mathbf{A}$ is a member of $\mathbf{B}$.

Proposition 6. *For any class* $\mathbf{A}$ *we have* $\mathbf{A} \subseteq \mathbf{A}$.

PROOF: Every member of $\mathbf{A}$ is a member of $\mathbf{A}$. □

Proposition 7. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, if* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{C}$ *then* $\mathbf{A} \subseteq \mathbf{C}$.

PROOF: If every member of $\mathbf{A}$ is a member of $\mathbf{B}$, and every member of $\mathbf{B}$ is a member of $\mathbf{C}$, then every member of $\mathbf{A}$ is a member of $\mathbf{C}$. □

Proposition 8. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, if* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{A}$ *then* $\mathbf{A} = \mathbf{B}$.

PROOF: If every member of $\mathbf{A}$ is a member of $\mathbf{B}$, and every member of $\mathbf{B}$ is a member of $\mathbf{A}$, then $\mathbf{A}$ and $\mathbf{B}$ have exactly the same members. □

Definition 9 (Empty Class)**.** The *empty class*, $\emptyset$, is $\{x : \bot\}$.

Proposition 10. *For any class* $\mathbf{A}$*, we have* $\emptyset \subseteq \mathbf{A}$*.*

PROOF: Vacuously, every member of $\emptyset$ is a member of $\mathbf{A}$.

Definition 11 (Universal Class)**.** The *universal class* $\mathbf{V}$ is the class $\{x : \top\}$.

Proposition 12. *For any class* $\mathbf{A}$*, we have* $\mathbf{A} \subseteq \mathbf{V}$*.*

PROOF: Every member of $\mathbf{A}$ is a member of $\mathbf{V}$. □

Definition 13. For any objects $a_1, \ldots, a_n$, we write $\{a_1, \ldots, a_n\}$ for the class $\{x : x = a_1 \vee \cdots \vee x = a_n\}$.

A class of the form $\{a\}$ is called a *singleton*.

A class of the form $\{a, b\}$ is called a *pair class*.

Definition 14 (Union)**.** The *union* of classes $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \cup \mathbf{B}$, is the set whose elements are exactly the things that are members of $\mathbf{A}$ or members of $\mathbf{B}$.

Proposition 15. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, we have:*

1. $\mathbf{A} \subseteq \mathbf{A} \cup \mathbf{B}$
2. $\mathbf{B} \subseteq \mathbf{A} \cup \mathbf{B}$
3. *If* $\mathbf{A} \subseteq \mathbf{C}$ *and* $\mathbf{B} \subseteq \mathbf{C}$ *then* $\mathbf{A} \cup \mathbf{B} \subseteq \mathbf{C}$

PROOF: Immediate from definitions. □

Proposition 16. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$ *we have* $\mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A}$*.*

PROOF: They are each the class of objects that belong to either $\mathbf{A}$ or $\mathbf{B}$. □

Proposition 17. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$ *we have* $\mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C}$*.*

PROOF: They are each the class of objects that belong to at least one of $\mathbf{A}$, $\mathbf{B}$ or $\mathbf{C}$. □

Proposition 18. *For any class* $\mathbf{A}$ *we have* $\mathbf{A} \cup \emptyset = \mathbf{A}$*.*

PROOF: Immediate from definitions. □

Proposition 19. *If* $\mathbf{A} \subseteq \mathbf{B}$ *then* $\mathbf{A} \cup \mathbf{C} \subseteq \mathbf{B} \cup \mathbf{C}$*.*

PROOF: Easy. □

Definition 20 (Intersection)**.** The *intersection* of classes $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \cap \mathbf{B}$, is the set whose elements are exactly the things that are members of both $\mathbf{A}$ and $\mathbf{B}$.

Proposition 21. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, we have:*

1. $\mathbf{A} \cap \mathbf{B} \subseteq \mathbf{A}$
2. $\mathbf{A} \cap \mathbf{B} \subseteq \mathbf{B}$
3. *If* $\mathbf{C} \subseteq \mathbf{A}$ *and* $\mathbf{C} \subseteq \mathbf{B}$ *then* $\mathbf{C} \subseteq \mathbf{A} \cap \mathbf{B}$

PROOF: Immediate from definitions. □

Proposition 22. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$ *we have* $\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A}$*.*

PROOF: They are each the class of objects that belong to both $\mathbf{A}$ and $\mathbf{B}$. □

Proposition 23. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$ *we have* $\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$*.*

PROOF: They are each the class of objects that belong to all of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. □

Proposition 24. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, we have* $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$*.*

PROOF:

$$\begin{aligned} x \in \mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) &\Leftrightarrow x \in \mathbf{A} \vee (x \in \mathbf{B} \wedge x \in \mathbf{C}) \\ &\Leftrightarrow (x \in \mathbf{A} \vee x \in \mathbf{B}) \wedge (x \in \mathbf{A} \vee x \in \mathbf{C}) \\ &\Leftrightarrow x \in (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{B}) \end{aligned}$$

Proposition 25. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, we have* $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$*.*

PROOF: Dual. □

Proposition 26. *For any class* $\mathbf{A}$ *we have* $\mathbf{A} \cap \emptyset = \emptyset$*.*

PROOF: Immediate from definitions. □

Proposition 27. *If* $\mathbf{A} \subseteq \mathbf{B}$ *then* $\mathbf{A} \cap \mathbf{C} \subseteq \mathbf{B} \cap \mathbf{C}$*.*

PROOF: Easy. □

Definition 28 (Disjoint)**.** Two classes $\mathbf{A}$ and $\mathbf{B}$ are *disjoint* iff they have no common members.

Proposition 29. *Two classes* $\mathbf{A}$ *and* $\mathbf{B}$ *are* disjoint *iff* $\mathbf{A} \cap \mathbf{B} = \emptyset$*.*

PROOF: Immediate from definitions. □

Definition 30 (Relative Complement)**.** Given classes $\mathbf{A}$ and $\mathbf{B}$, the *relative complement* $\mathbf{A} - \mathbf{B}$ is $\{x \in \mathbf{A} : x \notin \mathbf{B}\}$.

Proposition 31 (De Morgan's Law)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, we have* $\mathbf{C} - (\mathbf{A} \cup \mathbf{B}) = (\mathbf{C} - \mathbf{A}) \cap (\mathbf{C} - \mathbf{B})$*.*

PROOF: They are each the set of objects that belong to $\mathbf{C}$ and not to $\mathbf{A}$ nor $\mathbf{B}$. $\square$

Proposition 32 (De Morgan's Law)**.** *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, we have* $\mathbf{C} - (\mathbf{A} \cap \mathbf{B}) = (\mathbf{C} - \mathbf{A}) \cup (\mathbf{C} - \mathbf{B})$*.*

PROOF: Dual. $\square$

Proposition 33. *For any classes* $\mathbf{A}$ *and* $\mathbf{C}$ *we have* $\mathbf{A} \cap (\mathbf{C} - \mathbf{A}) = \emptyset$*.*

PROOF: Immediate from definitions. $\square$

Proposition 34. *If* $\mathbf{A} \subseteq \mathbf{B}$ *then* $\mathbf{C} - \mathbf{B} \subseteq \mathbf{C} - \mathbf{A}$*.*

PROOF: Easy. $\square$