Solutions Manual for Enderton *Elements of Set Theory*

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Contents

Chapter 1

Chapter 1 — Introduction

1.1 Baby Set Theory

Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

Exercise 2 We have $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ has an element (namely $\emptyset$) while $\emptyset$ has no elements.

We have $\emptyset \neq \{\{\emptyset\}\}$ because $\{\{\emptyset\}\}$ has an element (namely $\{\emptyset\}$) while $\emptyset$ has no elements.

We have $\{\emptyset\} \neq \{\{\emptyset\}\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$. This last fact is true because $\emptyset \neq \{\emptyset\}$ as we proved in the first paragraph.

Exercise 3 Assume $B \subseteq C$. Let $A \in \mathcal{P}B$; we must show that $A \in \mathcal{P}C$.

We have $A \subseteq B$ (since $A \in \mathcal{P}B$) and $B \subseteq C$. From this it follows that $A \subseteq C$ (every element of A is an element of B; every element of B is an element of C; therefore every element of A is an element of C). Hence $A \in \mathcal{P}C$ as required.

Exercise 4 Since $x \in B$, we have $\{x\} \subseteq B$ and so $\{x\} \in \mathcal{P}B$.

Since $x \in B$ and $y \in B$, we have $\{x, y\} \subseteq B$ and so $\{x, y\} \in \mathcal{P}B$.

From these two facts, it follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$ and so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

1.2 Sets — An Informal View

Exercise 5 We have

$$\begin{aligned}
V_0 &= A \\
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}A \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= \{\emptyset, \{\emptyset\}\} \\
V_3 &= \mathcal{P}V_2 \\
&= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}$$

We have $\emptyset \subseteq V_0$ and so $\emptyset \in V_1$. Therefore $\{\emptyset\} \subseteq V_1$ and so $\{\emptyset\} \in V_2$. Hence $\{\{\emptyset\}\} \subseteq V_2$.

We also have $\{\{\emptyset\}\} \nsubseteq V_0$ because $\{\emptyset\}$ is not an atom, and $\{\{\emptyset\}\} \nsubseteq V_1$ since $\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.

Thus the rank of $\{\{\emptyset\}\}$ is 2.

Likewise we have $\emptyset$ and $\{\emptyset\}$ are both subsets of V_1, hence

$$\emptyset \in V_2, \qquad \{\emptyset\} \in V_2$$

Thus $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are all subsets of V_2, hence elements of V_3. Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is not a subset of V_0 (because $\emptyset$ is not an atom.) It is not a subset of V_1 ($\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.) It is not a subset of V_2 (we have $\{\emptyset, \{\emptyset\}\} \notin V_2$ since $\{\emptyset\} \notin V_1$).

Therefore the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

Exercise 6

$$
\begin{aligned}
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}V_0 && \text{(since } V_0 = A\text{)} \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_1 && \text{(since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3)} \\
V_3 &= V_2 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_2 && \text{(since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3)} \\
V_4 &= V_3 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_3 && \text{(since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3)}
\end{aligned}
$$

Exercise 7 In Exercise 5 we calculated $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Hence

$$
\begin{aligned}
V_4 =& \mathcal{P}V_3 \\
=& \{\emptyset, \\
& \quad \{\emptyset\}, \\
& \quad \{\{\emptyset\}\}, \\
& \quad \{\{\{\emptyset\}\}\}, \\
& \quad \{\{\emptyset, \{\emptyset\}\}\}, \\
& \quad \{\emptyset, \{\emptyset\}\}, \\
& \quad \{\emptyset, \{\{\emptyset\}\}\}, \\
& \quad \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \\
& \quad \{\{\emptyset\}, \{\{\emptyset\}\}\}, \\
& \quad \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
& \quad \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
& \quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
& \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
& \quad \{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
& \quad \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
& \quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
& \}
\end{aligned}
$$

Chapter 2

Chapter 2 — Axioms and Operations

2.1 Arbitrary Unions and Intersections

Exercise 1 $A \cap B \cap C$ is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

Exercise 2 Take $A = \emptyset$ and $B = \{\emptyset\}$. Then $\bigcup A = \bigcup B = \emptyset$ but $A \neq B$. (There are many other possible answers.)

Exercise 3 Let $b \in A$. We must show that $b \subseteq \bigcup A$.

Let x be any element of b. We must show that $x \in \bigcup A$. We know that $x \in b$ and $b \in A$, and so $x \in \bigcup A$ by the definition of $\bigcup A$.

Exercise 4 Suppose $A \subseteq B$. Let $x \in \bigcup A$. We must show that $x \in \bigcup B$.

Pick an element $a \in A$ such that $x \in a$. Then $a \in B$ because $A \subseteq B$. Since we know $x \in a$ and $a \in B$, we know that $x \in \bigcup B$.

Exercise 5 Assume that every member of $\mathcal{A}$ is a subset of B. Let $x \in \bigcup \mathcal{A}$. We must show that $x \in B$.

Pick $A \in \mathcal{A}$ such that $x \in A$. By our assumption, we have $A \subseteq B$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$ as required.

Exercise 6

(a) We will show that $\bigcup \mathcal{P}A \subseteq A$ and $A \subseteq \bigcup \mathcal{P}A$.

To show $\bigcup \mathcal{P}A \subseteq A$: This follows from Exercise 5, since every member of $\mathcal{P}A$ is a subset of A.

To show $A \subseteq \bigcup \mathcal{P}A$: Let $a \in A$. Then we have $a \in \{a\}$ and $\{a\} \in \mathcal{P}A$ so $a \in \bigcup \mathcal{P}A$.

(b) To show $A \subseteq \mathcal{P} \bigcup A$: This holds because every element of A is a subset of $\bigcup A$, as we proved is Exercise 3.

Equality holds if and only if $A = \mathcal{P}X$ for some set X.

Proof: If $A = \mathcal{P} \bigcup A$ then of course $A = \mathcal{P}X$ for some X.

Conversely, if $A = \mathcal{P}X$, then we have

$$\begin{aligned} \mathcal{P} \bigcup A &= \mathcal{P} \bigcup \mathcal{P}X \\ &= \mathcal{P}X && \text{(by part (a))} \\ &= A \end{aligned}$$

Exercise 7

(a) For any set X,

$$\begin{aligned} & X \in \mathcal{P}A \cap \mathcal{P}B \\ \Leftrightarrow & X \subseteq A \text{ and } X \subseteq B \\ \Leftrightarrow & \text{Every member of } X \text{ is a member of } A \text{ and a member of } B \\ \Leftrightarrow & X \subseteq A \cap B \\ \Leftrightarrow & X \in \mathcal{P}(A \cap B) \end{aligned}$$

(b) Let $X \in \mathcal{P}A \cup \mathcal{P}B$. Then either $X \in \mathcal{P}A$ or $X \in \mathcal{P}B$ (or both). If $X \in \mathcal{P}A$, then we have $X \subseteq A$ and so $X \subseteq A \cup B$ (because $A \subseteq A \cup B$). Similarly if $X \in \mathcal{P}B$ then we have $X \subseteq A \cup B$. So in either case $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

Equality holds if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof: Suppose $A \subseteq B$. Then $\mathcal{P}A \subseteq \mathcal{P}B$ (Chapter 1 Exercise 3) and so $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$. Also $A \cup B = B$ so $\mathcal{P}(A \cup B) = \mathcal{P}B$. Thus $\mathcal{P}A \cup \mathcal{P}B$ and $\mathcal{P}(A \cup B)$ are equal.

Similarly if $B \subseteq A$ then $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$.

Conversely, suppose $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$. We have $A \cup B \in \mathcal{P}(A \cup B)$, so $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$. If $A \cup B \in \mathcal{P}A$, then we have $B \subseteq A \cup B \subseteq A$. And if $A \cup B \in \mathcal{P}B$, then we have $A \subseteq A \cup B \subseteq B$.

Exercise 8 If A is a set such that every singleton belongs to A, then every set belongs to $\bigcup A$, contradicting Theorem 2A.

Exercise 9 Let $a = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$. Then $a \in B$ but $\mathcal{P}a$ is not a subset of B because $\emptyset \in \mathcal{P}a$ and $\emptyset \notin B$.

Exercise 10 We must show that $\mathcal{P}a \subseteq \mathcal{P}\bigcup B$. So let $X \in \mathcal{P}a$. Then $X \subseteq a$; we must show that $X \subseteq \bigcup B$.

Let $x \in X$; we must show that $x \in \bigcup B$. We have $x \in a$ (because $x \in X$ and $X \subseteq a$) and $a \in B$, hence $x \in \bigcup B$ as required.

2.2 Algebra of Sets

Exercise 11 For any x we have

$$\begin{aligned} x \in (A \cap B) \cup (A - B) &\Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B) \\ &\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B) \\ &\Leftrightarrow x \in A \end{aligned}$$

Hence $A = (A \cap B) \cup (A - B)$.

For any x we have

$$\begin{aligned} x \in A \cup (B - A) &\Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A) \\ &\Leftrightarrow x \in A \text{ or } x \in B \\ &\Leftrightarrow x \in A \cup B \end{aligned}$$

Hence $A \cup (B - A) = A \cup B$.

Exercise 12 For any x,

$$\begin{aligned} x \in C - (A \cap B) &\Leftrightarrow x \in C \& \neg(x \in A \& x \in B) \\ &\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B) \\ &\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B) \\ &\Leftrightarrow x \in (C - A) \cup (C - B) \end{aligned}$$

Exercise 13 Suppose $A \subseteq B$. Let $x \in C - B$; we must show $x \in C - A$. We have $x \in C$ and $x \notin B$. Therefore $x \notin A$, since every member of A is a member of B. And so we have $x \in C - A$ as required.

Exercise 14 Let $A = \{\emptyset\}$, $B = \emptyset$ and $C = \{\emptyset\}$. Then $A - (B - C) = A - \emptyset = \{\emptyset\}$ while $(A - B) - C = \{\emptyset\} - C = \emptyset$.

Exercise 15

(a) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have $x \in A \cap (B + C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$.

(b) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$

In every case, we have $x \in A + (B + C) \Leftrightarrow x \in (A + B) + C$.

Exercise 16

$$\begin{aligned}[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A] &= (A \cup B) - A \\ &= B - A\end{aligned}$$

Exercise 17

(a) $\Leftrightarrow$ (b)

$$\begin{aligned}A \subseteq B &\Leftrightarrow \text{Every element of } A \text{ is an element of } B \\ &\Leftrightarrow \text{There is no element of } A \text{ that is not an element of } B \\ &\Leftrightarrow A - B = \emptyset\end{aligned}$$

(a) $\Rightarrow$ (c) Suppose $A \subseteq B$. We have $B \subseteq A \cup B$ from the definition of $A \cup B$; we must prove that $A \cup B \subseteq B$. So let $x \in A \cup B$. Then $x \in A$ or $x \in B$. But in either case $x \in B$, since $x \in A \Rightarrow x \in B$. Thus we have $x \in B$ as required.

(c) $\Rightarrow$ (a) We always have $A \subseteq A \cup B$. So if $A \cup B = B$ then we have $A \subseteq B$.

(a) $\Rightarrow$ (d) Suppose $A \subseteq B$. We have $A \cap B \subseteq A$ from the definition of $A \cap B$; we must prove that $A \subseteq A \cap B$. So let $x \in A$. Then $x \in B$ since $A \subseteq B$, hence $x \in A \cap B$ as required.

(d) $\Rightarrow$ (a) We always have $A \cap B \subseteq B$. So if $A \cap B = A$ then $A \subseteq B$.

Exercise 18 We can make the following 16 sets:

- $\emptyset$ $(= A - A)$
- $A - B$
- $A \cap B$
- $B - A$
- $S - (A \cup B)$
- A
- $A + B$
- $S - B$
- B
- $S - (A + B)$
- $S - A$
- $A \cup B$
- $S - (B - A)$
- $S - (A \cap B)$
- $S - (A - B)$

Exercise 19 They are never equal, because for all A, B, we have $\emptyset \in \mathcal{P}(A-B)$ but $\emptyset \notin \mathcal{P}A - \mathcal{P}B$ since $\emptyset \in \mathcal{P}B$.

Exercise 20 Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

We first show $B \subseteq C$. Let $x \in B$; we show $x \in C$. We have $x \in A \cup B = A \cup C$, so either $x \in A$ or $x \in C$. If $x \in C$, we are done. If $x \in A$, then we have $x \in A \cap B = A \cap C$, and so $x \in C$ in this case too.

We can show $C \subseteq B$ similarly. Hence $B = C$.

Exercise 21 For any x, we have

$$\begin{aligned} x \in \bigcup (A \cup B) &\Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C \\ &\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C, \text{ or there exists } C \in B \text{ such that } x \in C \\ &\Leftrightarrow x \in \bigcup A \cup \bigcup B \end{aligned}$$

Exercise 22 For any x, we have

$$\begin{aligned} x \in \bigcap(A \cup B) &\Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C \\ &\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C \\ &\Leftrightarrow x \in \bigcap A \cap \bigcap B \end{aligned}$$

Exercise 23 PROOF:
$\langle 1\rangle 1.$ $A \subseteq \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
$\langle 2\rangle 1.$ LET: $x \in A$
$\langle 2\rangle 2.$ LET: $X \in \mathcal{B}$
$\langle 2\rangle 3.$ $x \in A \cup X$
$\langle 1\rangle 2.$ $\bigcap \mathcal{B} \subseteq \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
$\langle 2\rangle 1.$ LET: $x \in \bigcap \mathcal{B}$
$\langle 2\rangle 2.$ LET: $X \in \mathcal{B}$
$\langle 2\rangle 3.$ $x \in X$
$\langle 2\rangle 4.$ $x \in A \cup X$
$\langle 1\rangle 3.$ $\bigcap\{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap \mathcal{B}$
$\langle 2\rangle 1.$ LET: $x \in \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
$\langle 2\rangle 2.$ ASSUME: $x \notin A$
PROVE: $x \in \bigcap \mathcal{B}$
$\langle 2\rangle 3.$ LET: $X \in \mathcal{B}$
$\langle 2\rangle 4.$ $x \in A \cup X$
$\langle 2\rangle 5.$ $x \in X$
$\square$

Exercise 24

(a)

$$\begin{aligned} Y \in \mathcal{P}\bigcap \mathcal{A} &\Leftrightarrow Y \subseteq \bigcap \mathcal{A} \\ &\Leftrightarrow \forall y \in Y . \forall X \in \mathcal{A} . y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A} . \forall y \in Y . y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A} . Y \in \mathcal{P} X \\ &\Leftrightarrow Y \in \bigcap\{\mathcal{P} X \mid X \in \mathcal{A}\} \end{aligned}$$

(b) $\bigcup\{\mathcal{P} X \mid X \in \mathcal{A}\} \subseteq \mathcal{P} \bigcup \mathcal{A}$

PROOF:
$\langle 1\rangle 1.$ LET: $Y \in \bigcup\{\mathcal{P} X \mid X \in \mathcal{A}\}$
$\langle 1\rangle 2.$ PICK $X \in \mathcal{A}$ such that $Y \in \mathcal{P} X$
$\langle 1\rangle 3.$ $Y \subseteq X$
$\langle 1\rangle 4.$ $Y \subseteq \bigcup \mathcal{A}$
$\langle 1\rangle 5.$ $Y \in \mathcal{P} \bigcup \mathcal{A}$

Equality holds if and only if $\bigcup \mathcal{A} \in \mathcal{A}$.

$\langle 1 \rangle 1$. If $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P}\bigcup\mathcal{A}$ then $\bigcup\mathcal{A} \in \mathcal{A}$
  $\langle 2 \rangle 1$. Assume: $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P}\bigcup\mathcal{A}$
  $\langle 2 \rangle 2$. $\bigcup\mathcal{A} \in \bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\}$
  $\langle 2 \rangle 3$. Pick $X \in \mathcal{A}$ such that $\bigcup\mathcal{A} \in \mathcal{P}X$
  $\langle 2 \rangle 4$. $X = \bigcup\mathcal{A}$
$\langle 1 \rangle 2$. If $\bigcup\mathcal{A} \in \mathcal{A}$ then $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P}\bigcup\mathcal{A}$
  Proof: If $\bigcup\mathcal{A} \in \mathcal{A}$ then $\mathcal{P}\bigcup\mathcal{A} \in \{\mathcal{P}X \mid X \in \mathcal{A}\}$.
□

Exercise 25 We have $A \cup \bigcup\mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$ if and only if $A = \emptyset$ or $\mathcal{B} \neq \emptyset$

$\langle 1 \rangle 1$. If $A \cup \bigcup\mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$ then $A = \emptyset$ or $\mathcal{B} \neq \emptyset$
  Proof: If $A \cup \bigcup\mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$ and $\mathcal{B} = \emptyset$ then

$$A \cup \bigcup\emptyset = \bigcup\emptyset$$
$$\therefore A = \emptyset$$

$\langle 1 \rangle 2$. If $A = \emptyset$ then $A \cup \bigcup\mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
  Proof: Both sides are equal to $\bigcup\mathcal{B}$
$\langle 1 \rangle 3$. If $\mathcal{B} \neq \emptyset$ then $A \cup \bigcup\mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 1$. Assume: $\mathcal{B} \neq \emptyset$
  $\langle 2 \rangle 2$. $A \cup \bigcup\mathcal{B} \subseteq \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 1$. Let: $x \in A \cup \bigcup\mathcal{B}$
      Prove: $x \in \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 2$. Case: $x \in A$
      $\langle 4 \rangle 1$. Pick $X \in \mathcal{B}$
        Proof: By $\langle 2 \rangle 1$
      $\langle 4 \rangle 2$. $x \in A \cup X$
    $\langle 3 \rangle 3$. Case: $x \in \bigcup\mathcal{B}$
      $\langle 4 \rangle 1$. Pick $X \in \mathcal{B}$ such that $x \in X$
      $\langle 4 \rangle 2$. $x \in A \cup X$
  $\langle 2 \rangle 3$. $\bigcup\{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcup\mathcal{B}$
    $\langle 3 \rangle 1$. Let: $x \in \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 2$. Pick $X \in \mathcal{B}$ such that $x \in A \cup X$
    $\langle 3 \rangle 3$. $X \subseteq \bigcup\mathcal{B}$
    $\langle 3 \rangle 4$. $A \cup X \subseteq A \cup \bigcup\mathcal{B}$
    $\langle 3 \rangle 5$. $x \in A \cup \bigcup\mathcal{B}$

2.3 Review Exercises

Exercise 26 Sets A, B, D and F are all equal to each other. Sets C, E and G are equal to each other. None of the first list is equal to any of the second list.

Exercise 27 Take $A = \{\{0\}, \{1\}\}$ and $B = \{\{1\}\}$. Then $A \cap B = \{\{1\}\}$ and

$$\begin{aligned} \bigcap A \cap \bigcap B &= \emptyset \cap \{1\} \\ &= \emptyset \\ \bigcap (A \cap B) &= \bigcap \{\{1\}\} \\ &= \{1\} \end{aligned}$$

Exercise 28

$$\bigcup \{\{3,4\}, \{\{3\},\{4\}\}, \{3,\{4\}\}, \{\{3\},4\}\} = \{3, 4, \{3\}, \{4\}\}$$

Exercise 29

(a) $\emptyset$

(b) We have

$$\begin{aligned} \{\emptyset\} &\subseteq \mathcal{P}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{P}\mathcal{P}\{\emptyset\} \\ \{\emptyset\} &\subseteq \mathcal{P}\mathcal{P}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{P}\mathcal{P}\mathcal{P}\{\emptyset\} \\ \therefore \bigcap \{\mathcal{P}\mathcal{P}\mathcal{P}\{\emptyset\}, \mathcal{P}\mathcal{P}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} &= \mathcal{P}\mathcal{P}\mathcal{P}\{\emptyset\} \cap \mathcal{P}\mathcal{P}\{\emptyset\} \cap \mathcal{P}\{\emptyset\} \\ &= \mathcal{P}\{\emptyset\} \\ &= \{\emptyset, \{\emptyset\}\} \end{aligned}$$

Exercise 30

(a) $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$

(b) $\{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d) $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

Exercise 31

(a) $\{1, 2, 3, \emptyset\}$

(b) $\emptyset$

(c) $\emptyset$

(d) $\emptyset$

Exercise 32

(a) $a \cup b$

(b) a

(c)

$$\begin{aligned}\bigcap\bigcup S \cup (\bigcup\bigcup S - \bigcup\bigcap S) &= (a \cap b) \cup ((a \cup b) - a) \\ &= (a \cap b) \cup (b - a) \\ &= b\end{aligned}$$

Exercise 33 When $a \neq b$:

$$\begin{aligned}\bigcup(\bigcup S - \bigcap S) &= \bigcup(\{a, b\} - \{a\}) \\ &= \bigcup\{b\} \\ &= b\end{aligned}$$

When $a = b$:

$$\begin{aligned}\bigcup(\bigcup S - \bigcap S) &= \bigcup(\{a, b\} - \{a\}) \\ &= \bigcup\emptyset \\ &= \emptyset\end{aligned}$$

Exercise 34 For any set S, we have

$$\begin{gathered}\emptyset \subseteq \mathcal{P}S \\ \therefore \emptyset \in \mathcal{P}\mathcal{P}S \\ \emptyset \subseteq S \\ \therefore \emptyset \in \mathcal{P}S \\ \therefore \{\emptyset\} \subseteq \mathcal{P}S \\ \therefore \{\emptyset\} \in \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} \in \mathcal{P}\mathcal{P}\mathcal{P}S\end{gathered}$$

Exercise 35 Assume $\mathcal{P}A = \mathcal{P}B$. Then we have

$$\begin{aligned}
& A \in \mathcal{P}A \\
\therefore\ & A \in \mathcal{P}B \\
\therefore\ & A \subseteq B \\
& B \in \mathcal{P}B \\
\therefore\ & B \in \mathcal{P}A \\
\therefore\ & B \subseteq A \\
\therefore\ & A = B
\end{aligned}$$

Exercise 36

(a)

$$\begin{aligned}
x \in A - (A \cap B) &\Leftrightarrow x \in A \ \&\neg(x \in A \ \&\ x \in B) \\
&\Leftrightarrow x \in A \ \&\ x \notin B \\
&\Leftrightarrow x \in A - B
\end{aligned}$$

(b)

$$\begin{aligned}
x \in A - (A - B) &\Leftrightarrow x \in A \ \&\neg(x \in A \ \&\ x \notin B) \\
&\Leftrightarrow x \in A \ \&\ x \in B \\
&\Leftrightarrow x \in A \cap B
\end{aligned}$$

Exercise 37

(a)

$$\begin{aligned}
x \in (A \cup B) - C &\Leftrightarrow (x \in A \text{ or } x \in B) \ \&\ x \notin C \\
&\Leftrightarrow (x \in A \ \&\ x \notin C) \text{ or } (x \in B \ \&\ x \notin C) \\
&\Leftrightarrow x \in (A - C) \cup (B - C)
\end{aligned}$$

(b)

$$\begin{aligned}
x \in A - (B - C) &\Leftrightarrow x \in A \ \&\neg(x \in B \ \&\ x \notin C) \\
&\Leftrightarrow x \in A \ \&(x \notin B \text{ or } x \in C) \\
&\Leftrightarrow (x \in A \ \&\ x \notin B) \text{ or } (x \in A \ \&\ x \in C) \\
&\Leftrightarrow x \in (A - B) \cup (A \cap C)
\end{aligned}$$

(c)

$$\begin{aligned}
x \in (A - B) - C &\Leftrightarrow x \in A \ \&\ x \notin B \ \&\ x \notin C \\
&\Leftrightarrow x \in A \ \&\neg(x \in B \vee x \in C) \\
&\Leftrightarrow x \in A - (B \cup C)
\end{aligned}$$

Exercise 38

(a) If every element of A is an element of C, and every element of B is an element of C, then everything that is an element of either A or B is an element of C.

(b) If every element of C is an element of A, and every element of C is an element of B, then every element of C is an element of both A and B.

Chapter 3

Chapter 3 — Relations and Functions

3.1 Ordered Pairs

Exercise 1 We have $\langle 0,1,0\rangle^* = \langle 0,1,1\rangle^* = \{\{0\},\{0,1\}\}$.

Exercise 2

(a)

$$\begin{aligned}
&z \in A \times (B \cup C) \\
\Leftrightarrow &\exists x, y(z = (x,y) \ \& \ x \in A \ \& \ (y \in B \text{ or } y \in C)) \\
\Leftrightarrow &\exists x, y(z = (x,y) \ \& \ x \in A \ \& \ y \in B) \text{ or } (z = (x,y) \ \& \ x \in A \ \& \ y \in C) \\
\Leftrightarrow &z \in (A \times B) \cup (A \times C)
\end{aligned}$$

(b)
$\langle 1\rangle 1.$ Assume: $A \times B = A \times C$ and $A \neq \emptyset$
$\langle 1\rangle 2.$ Pick $a \in A$
$\langle 1\rangle 3.$ For all x, $x \in B \Leftrightarrow x \in C$
Proof: $x \in B$ iff $(a,x) \in A \times B$ iff $(a,x) \in A \times C$ iff $x \in C$.
□

Exercise 3

$$\begin{aligned}
&z \in A \times \bigcup \mathcal{B} \\
\Leftrightarrow &\exists x, y(z = (x,y) \ \& \ x \in A \ \& \ \exists X \in \mathcal{B}. y \in X) \\
\Leftrightarrow &\exists X \in \mathcal{B}. \exists x, y(z = (x,y) \ \& \ x \in A \ \& \ y \in X) \\
\Leftrightarrow &z \in \bigcup \{A \times X : X \in \mathcal{B}\}
\end{aligned}$$

Exercise 4 If every ordered pair belongs to A then every set belongs to $\bigcup\bigcup A$ contradicting Theorem 2A.

Exercise 5

(a) Apply a Subset Axiom to $\mathcal{P}(A \times B)$: we have $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A . z = \{x\} \times B\}$.

(b)

$$\begin{aligned}
& z \in \bigcup C \\
\Leftrightarrow & \exists x \in A . z \in \{x\} \times B \\
\Leftrightarrow & \exists x \in A . \exists y \in B . z = (x, y) \\
\Leftrightarrow & z \in A \times B
\end{aligned}$$

3.2 Relations

Exercise 6 If $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$ then A is a set of ordered pairs, i.e. a relation.

Conversely, suppose A is a relation. Let $z \in A$. Then z is an ordered pair; let $z = (x, y)$. We have $x \in \operatorname{dom} A$ and $y \in \operatorname{ran} A$ and so $z \in \operatorname{dom} A \times \operatorname{ran} A$ as required.

Exercise 7 We have $\operatorname{fld} R \subseteq \bigcup\bigcup R$ by Lemma 3D.

Conversely, let $x \in \bigcup\bigcup R$. Pick a and b such that $x \in a$, $a \in b$ and $b \in R$. Then b is an ordered pair; let $b = (y, z)$. We have $a = \{y\}$ or $\{y, z\}$, hence $x = y$ or $x = z$. In either case, $x \in \operatorname{fld} R$.

Exercise 8

(a)

$$\begin{aligned}
& x \in \operatorname{dom} \bigcup \mathcal{A} \\
\Leftrightarrow & \exists y . \exists R \in \mathcal{A} . (x, y) \in R \\
\Leftrightarrow & \exists R \in \mathcal{A} . \exists y . (x, y) \in R \\
\Leftrightarrow & x \in \bigcup \{\operatorname{dom} R : R \in \mathcal{A}\}
\end{aligned}$$

(b)

$$\begin{aligned}
&y \in \operatorname{ran} \bigcup \mathcal{A} \\
\Leftrightarrow &\exists x. \exists R \in \mathcal{A}.(x, y) \in R \\
\Leftrightarrow &\exists R \in \mathcal{A}. \exists x.(x, y) \in R \\
\Leftrightarrow &y \in \bigcup \{\operatorname{ran} R : R \in \mathcal{A}\}
\end{aligned}$$

Exercise 9 Assume $\mathcal{A}$ is nonempty. We have $\operatorname{dom} \bigcap \mathcal{A} \subseteq \bigcap \{\operatorname{dom} R : R \in \mathcal{A}\}$.

PROOF:

$$\begin{aligned}
&x \in \operatorname{dom} \bigcap \mathcal{A} \\
\Leftrightarrow &\exists y. \forall R \in \mathcal{A}.(x, y) \in R \\
\Rightarrow &\forall R \in \mathcal{A}. \exists y.(x, y) \in R \\
\Leftrightarrow &x \in \bigcap \{\operatorname{dom} R : R \in \mathcal{A}\}
\end{aligned}$$

Equality holds iff the middle '$\Rightarrow$' can be reversed, i.e. iff for all x, if $\forall R \in \mathcal{A}. \exists y.(x, y) \in R$ then $\exists y. \forall R \in \mathcal{A}.(x, y) \in R$. I haven't found a simpler condition than this. The condition does not always hold, for example if $\mathcal{A} = \{\{(1, 2)\}, \{(1, 3)\}\}$ then $\operatorname{dom} \bigcap \mathcal{A} = \emptyset$ while $\bigcap \{\operatorname{dom} R : R \in \mathcal{A}\} = \{1\}$.

Similarly, $\operatorname{ran} \bigcap \mathcal{A} \subseteq \bigcap \{\operatorname{ran} R : R \in \mathcal{A}\}$, and equality holds iff, for any y, if $\forall R \in \mathcal{A}. \exists x.(x, y) \in R$ then $\exists x. \forall R \in \mathcal{A}.(x, y) \in R$.

3.3 n-ary Relations

Exercise 10 This follows from the equations at the top of page 42. An ordered 4-tuple $\langle a, b, c, d\rangle$ is also an ordered 1-tuple (because every set is), and the ordered pair $\langle\langle a, b, c\rangle, d\rangle$, and the ordered triple $\langle\langle a, b\rangle, c, d\rangle$.

3.4 Functions

Exercise 11 We prove $F \subseteq G$. Let $z \in F$. Since F is a relation, then z is an ordered pair; let $z = \langle x, y\rangle$. We have $x \in \operatorname{dom} F$ and $y = F(x)$. Therefore $x \in \operatorname{dom} G$ and $y = G(x)$ (because $\operatorname{dom} F = \operatorname{dom} G$ and $F(x) = G(x)$). Hence $\langle x, y\rangle \in G$, i.e. $z \in G$.

We have proved $F \subseteq G$. We can prove $G \subseteq F$ similarly. Thus $F = G$.

Exercise 12 PROOF:

$\langle 1\rangle 1$. If $f \subseteq g$ then $\operatorname{dom} f \subseteq \operatorname{dom} g$ and $\forall x \in \operatorname{dom} f. f(x) = g(x)$

  $\langle 2\rangle 1$. ASSUME: $f \subseteq g$

  $\langle 2\rangle 2$. LET: $x \in \operatorname{dom} f$

  $\langle 2\rangle 3$. $(x, f(x)) \in f$

  $\langle 2\rangle 4$. $(x, f(x)) \in g$

  $\langle 2\rangle 5$. $x \in \operatorname{dom} g$ and $g(x) = f(x)$

$\langle 1\rangle 2$. If $\mathrm{dom}\, f = \mathrm{dom}\, g$ and $\forall x \in \mathrm{dom}\, f . f(x) = g(x)$ then $f \subseteq g$
$\langle 2\rangle 1$. Assume: $\mathrm{dom}\, f = \mathrm{dom}\, g$ and $\forall x \in \mathrm{dom}\, f . f(x) = g(x)$
$\langle 2\rangle 2$. Let: $z \in f$
$\langle 2\rangle 3$. Let: $z = (x, y)$
$\langle 2\rangle 4$. $x \in \mathrm{dom}\, f$ and $y = f(x)$
$\langle 2\rangle 5$. $x \in \mathrm{dom}\, g$ and $y = g(x)$
$\langle 2\rangle 6$. $z = (x, y) \in g$
$\square$

Exercise 13 Proof:
$\langle 1\rangle 1$. Assume: f and g are functions
$\langle 1\rangle 2$. Assume: $f \subseteq g$
$\langle 1\rangle 3$. Assume: $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$
$\langle 1\rangle 4$. $\mathrm{dom}\, f = \mathrm{dom}\, g$
Proof: We have $\mathrm{dom}\, f \subseteq \mathrm{dom}\, g$ from $\langle 1\rangle 2$ and $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ from $\langle 1\rangle 3$
$\langle 1\rangle 5$. For $x \in \mathrm{dom}\, f$ we have $f(x) = g(x)$
Proof: From $\langle 1\rangle 2$ and Exercise 12
$\langle 1\rangle 6$. Q.E.D.
Proof: From Exercise 11.
$\square$

Exercise 14

(a) If (x, y) and (x, z) are members of $f \cap g$ then they are both members of f, hence $y = z$.

(b) Proof:
$\langle 1\rangle 1$. If $f \cup g$ is a function then, for all $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$, we have $f(x) = g(x)$.
$\langle 2\rangle 1$. Assume: $f \cup g$ is a function.
$\langle 2\rangle 2$. Let: $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$
$\langle 2\rangle 3$. $(x, f(x))$ and $(x, g(x))$ are both elements of $f \cup g$
$\langle 2\rangle 4$. $f(x) = g(x)$
$\langle 1\rangle 2$. If, for all $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$, we have $f(x) = g(x)$, then $f \cup g$ is a function.
$\langle 2\rangle 1$. Assume: For all $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$, we have $f(x) = g(x)$
$\langle 2\rangle 2$. $f \cup g$ is a relation.
Proof: Since every element of either f or g is an ordered pair.
$\langle 2\rangle 3$. Whenever (x, y) and (x, z) are elements of $f \cup g$ we have $y = z$
$\langle 3\rangle 1$. Let: $(x, y), (x, z) \in f \cup g$
$\langle 3\rangle 2$. Case: $(x, y), (x, z) \in f$
Proof: Then $y = z$ since f is a function.
$\langle 3\rangle 3$. Case: $(x, y) \in f$,$(x, z) \in g$
Proof: Then $y = z$ by $\langle 2\rangle 1$
$\langle 3\rangle 4$. Case: $(x, y) \in g$,$(x, z) \in f$
Proof: Then $y = z$ by $\langle 2\rangle 1$
$\langle 3\rangle 5$. Case: $(x, y), (x, z) \in g$

PROOF: Then $y = z$ since g is a function.

$\square$

Exercise 15 PROOF:
$\langle 1 \rangle 1.$ $\bigcup \mathcal{A}$ is a relation.
PROOF: Since every member of $\mathcal{A}$ is a relation.
$\langle 1 \rangle 2.$ Whenever (x, y) and (x, z) are elements of $\bigcup \mathcal{A}$ then $y = z$
$\langle 2 \rangle 1.$ LET: $(x, y), (x, z) \in \bigcup \mathcal{A}$
$\langle 2 \rangle 2.$ PICK $f, g \in \mathcal{A}$ such that $(x, y) \in f$ and $(x, z) \in g$
$\langle 2 \rangle 3.$ ASSUME: w.l.o.g. $f \subseteq g$
$\langle 2 \rangle 4.$ $(x, y), (x, z) \in g$
$\langle 2 \rangle 5.$ $y = z$
PROOF: Since g is a function.

$\square$

Exercise 16 If every function belongs to A then every set belongs to $\operatorname{dom} \bigcup A$ contradiction Theorem 2A.

Exercise 17 PROOF:
$\langle 1 \rangle 1.$ LET: R and S be single-rooted.
$\langle 1 \rangle 2.$ LET: $(x, z), (y, z) \in R \circ S$
$\langle 1 \rangle 3.$ PICK t and t' such that $(x, t) \in S$, $(t, z) \in R$, $(y, t') \in S$ and $(t', z) \in R$
$\langle 1 \rangle 4.$ $t = t'$
PROOF: Since R is single-rooted.
$\langle 1 \rangle 5.$ $x = y$
PROOF: Since S is single-rooted.

Thus if F and G are one-to-one functions then $F \circ G$ is single-rooted and a function by Theorem 3H, hence a one-to-one function.

Exercise 18

$$\begin{aligned}
R \circ R &= \{\langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle\} \\
R \restriction \{1\} &= \{\langle 1, 2 \rangle, \langle 1, 3 \rangle\} \\
R^{-1} \restriction \{1\} &= \{\langle 1, 0 \rangle\} \\
R[\![\{1\}]\!] &= \{2, 3\} \\
R^{-1}[\![\{1\}]\!] &= \{0\}
\end{aligned}$$

Exercise 19

$$\begin{aligned}
A(\emptyset) &= \{\emptyset, \{\emptyset\}\} \\
A[\![\emptyset]\!] &= \emptyset \\
A[\![\{\emptyset\}]\!] &= \{\{\emptyset, \{\emptyset\}\}\} \\
A[\![\{\emptyset, \{\emptyset\}\}]\!] &= \{\{\emptyset, \{\emptyset\}\}, \emptyset\} \\
A^{-1} &= \{\langle\{\emptyset, \{\emptyset\}\}, \emptyset\rangle, \langle\emptyset, \{\emptyset\}\rangle\} \\
A \circ A &= \{\langle\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\rangle\} \\
A \upharpoonright \emptyset &= \emptyset \\
A \upharpoonright \{\emptyset\} &= \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle\} \\
A \upharpoonright \{\emptyset, \{\emptyset\}\} &= \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle, \langle\{\emptyset\}, \emptyset\rangle\} \\
&= A \\
\bigcup\bigcup A &= \{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}
\end{aligned}$$

Exercise 20

$$\begin{aligned}
z \in F \upharpoonright A &\Leftrightarrow z \in F \ \& \ \exists x, y.(z = \langle x, y\rangle \ \& \ x \in A) \\
&\Leftrightarrow z \in F \ \& \ \exists x, y(z = \langle x, y\rangle \ \& \ x \in A \ \& \ y \in \operatorname{ran} F) \\
&\Leftrightarrow z \in F \cap (A \times \operatorname{ran} F)
\end{aligned}$$

Exercise 21 Both are equal to $\{\langle x, w\rangle \mid \exists y, z.xTy \ \& \ ySz \ \& \ zRw\}$.

Exercise 22

(a) Proof:
$\langle 1\rangle 1.$ Assume: $A \subseteq B$
$\langle 1\rangle 2.$ Let: $y \in F[\![A]\!]$
$\langle 1\rangle 3.$ Pick $x \in A$ such that xFy
$\langle 1\rangle 4.$ $x \in B$ and xFy
□

(b) Both are equal to $\{z : \exists x, y.x \in A \ \& \ xGy \ \& \ yFz\}$

(c) Both are equal to $\{\langle x, y\rangle : (x \in A \text{ or } x \in B) \ \& \ xQy\}$

Exercise 23

$$\begin{aligned}
B \circ I_A &= \{\langle x, z\rangle : \exists y(xI_Ay \;\&\; yBz)\} \\
&= \{\langle x, z\rangle : \exists y(x \in A \;\&\; x = y \;\&\; yBz)\} \\
&= \{\langle x, z\rangle : x \in A \;\&\; xBz\} \\
&= B \restriction A \\
I_A[\![C]\!] &= \{y : \exists x \in C.xI_Ay\} \\
&= \{y : \exists x \in C(x \in A \;\&\; x = y)\} \\
&= \{y : y \in C \;\&\; y \in A\} \\
&= A \cap C
\end{aligned}$$

Exercise 24

$$\begin{aligned}
F^{-1}[\![A]\!] &= \{x : \exists y \in A.yF^{-1}x\} \\
&= \{x : \exists y \in A.xFy\} \\
&= \{x \in \operatorname{dom} F : F(x) \in A\}
\end{aligned}$$

Exercise 25

(a) PROOF:
$\langle 1\rangle$1. LET: G be a one-to-one function.
$\langle 1\rangle$2. G^{-1} is a function.
PROOF: Theorem 3F.
$\langle 1\rangle$3. $G \circ G^{-1}$ is a function.
PROOF: Theorem 3H.
$\langle 1\rangle$4. $\operatorname{dom}(G \circ G^{-1}) = \operatorname{ran} G$
PROOF:

$$\begin{aligned}
\operatorname{dom}(G \circ G^{-1}) &= \{x \in \operatorname{dom} G^{-1} : G^{-1}(x) \in \operatorname{dom} G\} && \text{(Theorem 3H)} \\
&= \{x \in \operatorname{ran} G : G^{-1}(x) \in \operatorname{dom} G\} && \text{(Theorem 3E)} \\
&= \operatorname{ran} G
\end{aligned}$$

$\langle 1\rangle$5. $\forall x \in \operatorname{ran} G.(G \circ G^{-1})(x) = x$
PROOF: Theorem 3G.
$\square$

(b) Let G be a function. Then

$$\begin{aligned}
G \circ G^{-1} &= \{\langle x, z\rangle : \exists y(xG^{-1}y \;\&\; yGz)\} \\
&= \{\langle x, z\rangle : \exists y(yGx \;\&\; yGz)\} \\
&= \{\langle x, x\rangle : \exists y.yGx\} && \text{(G is a function)} \\
&= I_{\operatorname{ran} G}
\end{aligned}$$

Exercise 26

(a)

$$\begin{aligned} F[\![\bigcup \mathcal{A}]\!] &= \{y : \exists x . \exists A \in \mathcal{A}(x \in A \ \& \ xFy)\} \\ &= \{y : \exists A \in \mathcal{A} . \exists x(x \in A \ \& \ xFy)\} \\ &= \bigcup \{F[\![A]\!] : A \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned} F[\![\bigcup \mathcal{A}]\!] &= \{y : \exists x . \forall A \in \mathcal{A}(x \in A \ \& \ xFy)\} \\ &\subseteq \{y : \forall A \in \mathcal{A} . \exists x(x \in A \ \& \ xFy)\} \\ &= \bigcap \{F[\![A]\!] : A \in \mathcal{A}\} \end{aligned}$$

Exercise 27

$$\begin{aligned} \operatorname{dom}(F \circ G) &= \{x : \exists y . x(F \circ G)y\} \\ &= \{x : \exists y \exists z(xGz \ \& \ zFy)\} \\ &= \{x : \exists z(zG^{-1}x \ \& \ z \in \operatorname{dom} F)\} \\ &= G^{-1}[\![\operatorname{dom} F]\!] \end{aligned}$$

Exercise 28 Proof:
$\langle 1 \rangle 1.$ $G : \mathcal{P}A \to \mathcal{P}B$
  Proof: Since $f[\![X]\!] \subseteq \operatorname{ran} f \subseteq B$
$\langle 1 \rangle 2.$ For all $X, Y \in \mathcal{P}A$, if $G(X) = G(Y)$ then $X = Y$
  $\langle 2 \rangle 1.$ Let: $X, Y \in \mathcal{P}A$
  $\langle 2 \rangle 2.$ Assume: $f[\![X]\!] = f[\![Y]\!]$
  $\langle 2 \rangle 3.$ $X \subseteq Y$
    $\langle 3 \rangle 1.$ Let: $x \in X$
    $\langle 3 \rangle 2.$ $f(x) \in f[\![X]\!]$
    $\langle 3 \rangle 3.$ $f(x) \in f[\![Y]\!]$
    $\langle 3 \rangle 4.$ Pick $y \in Y$ such that $f(x) = f(y)$
    $\langle 3 \rangle 5.$ $x = y$
      Proof: Because f is one-to-one.
    $\langle 3 \rangle 6.$ $x \in Y$
      Proof: Similar.
  $\langle 2 \rangle 4.$ $Y \subseteq X$
$\square$

Example 29 Proof:
$\langle 1 \rangle 1.$ Assume: f maps A onto B
$\langle 1 \rangle 2.$ Let: $b, b' \in B$
$\langle 1 \rangle 3.$ Assume: $G(b) = G(b')$
$\langle 1 \rangle 4.$ Pick $x \in A$ such that $f(x) = b$

PROOF: By $\langle 1 \rangle 1$.
$\langle 1 \rangle 5.$ $x \in G(b)$
$\langle 1 \rangle 6.$ $x \in G(b')$
$\langle 1 \rangle 7.$ $f(x) = b'$
$\langle 1 \rangle 8.$ $b = b'$
$\square$

The converse does not hold. Let $A = \{0\}$ and $B = \{0, 1\}$. Let f be the function that maps 0 to 0. Then

$$G(0) = \{0\}$$
$$G(1) = \emptyset$$

Thus G is one-to-one but f does not map A onto B.

Exercise 30

(a) PROOF:
$\langle 1 \rangle 1.$ $F(B) = B$
$\langle 2 \rangle 1.$ $F(B) \subseteq B$
$\langle 3 \rangle 1.$ LET: $X \in \mathcal{P}A$ be such that $F(X) \subseteq X$
PROVE: $F(B) \subseteq X$
$\langle 3 \rangle 2.$ $B \subseteq X$
$\langle 3 \rangle 3.$ $F(B) \subseteq F(X)$
$\langle 3 \rangle 4.$ $F(B) \subseteq X$
PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$.
$\langle 2 \rangle 2.$ $B \subseteq F(B)$
PROOF: From $\langle 2 \rangle 1$ and the definition of B, since B is one of the sets X such that $F(X) \subseteq X$
$\langle 1 \rangle 2.$ $F(C) = C$
$\langle 2 \rangle 1.$ $C \subseteq F(C)$
$\langle 3 \rangle 1.$ LET: $X \in \mathcal{P}A$ with $X \subseteq F(X)$
PROVE: $X \subseteq F(C)$
$\langle 3 \rangle 2.$ $X \subseteq C$
$\langle 3 \rangle 3.$ $F(X) \subseteq F(C)$
$\langle 3 \rangle 4.$ $X \subseteq F(C)$
PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$
$\langle 2 \rangle 2.$ $F(C) \subseteq C$
PROOF: From $\langle 2 \rangle 1$ and the definition of C.
$\square$

(b) If $F(X) = X$ then we have $B \subseteq X$ (because $F(X) \subseteq X$) and $X \subseteq C$ (because $X \subseteq F(X)$).

3.5 Infinite Cartesian Products

Exercise 31 PROOF:

$\langle 1 \rangle 1$. If the Axiom of Choice is true then, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$.

$\langle 2 \rangle 1$. ASSUME: The Axiom of Choice.

$\langle 2 \rangle 2$. LET: I be a set.

$\langle 2 \rangle 3$. LET: H be a function with domain I.

$\langle 2 \rangle 4$. ASSUME: $H(i) \neq \emptyset$ for all $i \in I$.

$\langle 2 \rangle 5$. LET: $R = \{(i, x) : i \in I, x \in H(i)\}$

$\langle 2 \rangle 6$. PICK a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R$

PROVE: $F \in \prod_{i \in I} H(i)$

PROOF: By the Axiom of Choice.

$\langle 2 \rangle 7$. $\operatorname{dom} H = I$

PROOF: We have $\operatorname{dom} R = I$ since for all $i \in I$ there exists x such that $x \in H(i)$.

$\langle 2 \rangle 8$. $\forall i \in I. F(i) \in H(i)$

PROOF: Since $iRF(i)$.

$\langle 1 \rangle 2$. If, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$, then the Axiom of Choice is true.

$\langle 2 \rangle 1$. ASSUME: For any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$

$\langle 2 \rangle 2$. LET: R be a relation

$\langle 2 \rangle 3$. LET: $I = \operatorname{dom} R$

$\langle 2 \rangle 4$. Define the function H with domain I by: for $i \in I$, $H(i) = \{y : iRy\}$

$\langle 2 \rangle 5$. $H(i) \neq \emptyset$ for all $i \in I$

$\langle 2 \rangle 6$. PICK $F \in \prod_{i \in I} H(i)$

PROOF: By $\langle 2 \rangle 1$

$\langle 2 \rangle 7$. F is a function

$\langle 2 \rangle 8$. $F \subseteq R$

PROOF: For all $i \in I$ we have $F(i) \in H(i)$ and so $iRF(i)$.

$\langle 2 \rangle 9$. $\operatorname{dom} F = \operatorname{dom} R$

$\square$

3.6 Equivalence Relations

Exercise 32

(a)

$$\begin{aligned}
& R \text{ is symmetric} \\
\Leftrightarrow & \forall x, y (xRy \Rightarrow yRx) \\
\Leftrightarrow & \forall x, y (\langle y, x \rangle \in R^{-1} \Rightarrow \langle y, x \rangle \in R) \\
\Leftrightarrow & R^{-1} \subseteq R
\end{aligned}$$

(b)

$$\begin{aligned}
&R \text{ is transitive}\\
\Leftrightarrow &\forall x, y, z(xRy \;\&\; yRz \Rightarrow xRz)\\
\Leftrightarrow &\forall x, z(\exists y(xRy \;\&\; yRz) \Rightarrow xRz)\\
\Leftrightarrow &\forall x, z(\langle x, z\rangle \in R \circ R \Rightarrow \langle x, z\rangle \in R)\\
\Leftrightarrow &R \circ R \subseteq R
\end{aligned}$$

Exercise 33 PROOF:
$\langle 1\rangle$1. If R is a symmetric and transitive relation then $R = R^{-1} \circ R$.
  $\langle 2\rangle$1. ASSUME: R is a symmetric and transitive relation.
  $\langle 2\rangle$2. $R \subseteq R^{-1} \circ R$
    $\langle 3\rangle$1. LET: xRy
    $\langle 3\rangle$2. yRy
      PROOF: By Theorem 3M.
    $\langle 3\rangle$3. xRy and $yR^{-1}y$
    $\langle 3\rangle$4. $x(R^{-1} \circ R)y$
  $\langle 2\rangle$3. $R^{-1} \circ R \subseteq R$
    PROOF:

$$\begin{aligned}
R^{-1} \circ R &\subseteq R \circ R && \text{(Exercise 32(a))}\\
&\subseteq R && \text{(Exercise 32(b))}
\end{aligned}$$

$\langle 1\rangle$2. If $R = R^{-1} \circ R$ then R is a symmetric and transitive relation.
  $\langle 2\rangle$1. ASSUME: $R = R^{-1} \circ R$
  $\langle 2\rangle$2. R is a relation.
  $\langle 2\rangle$3. R is symmetric.
    $\langle 3\rangle$1. LET: xRy
    $\langle 3\rangle$2. PICK z such that xRz and $zR^{-1}y$
    $\langle 3\rangle$3. yRz and $zR^{-1}x$
    $\langle 3\rangle$4. $y(R^{-1} \circ R)x$
    $\langle 3\rangle$5. yRx
  $\langle 2\rangle$4. R is transitive.
    $\langle 3\rangle$1. LET: xRy and yRz
    $\langle 3\rangle$2. zRy
      PROOF: By $\langle 2\rangle$3
    $\langle 3\rangle$3. xRy and $yR^{-1}z$
    $\langle 3\rangle$4. $x(R^{-1} \circ R)z$
    $\langle 3\rangle$5. xRz
$\square$

Exercise 34

(a) $\bigcap \mathcal{A}$ is a transitive relation.

PROOF:
$\langle 1\rangle$1. $\bigcap \mathcal{A}$ is a relation.

PROOF: Every member of a member of $\mathcal{A}$ is an ordered pair.
$\langle 1\rangle 2$. $\bigcap \mathcal{A}$ is transitive.
  $\langle 2\rangle 1$. LET: $\langle x, y\rangle$ and $\langle y, z\rangle$ be in $\bigcap \mathcal{A}$
    PROVE: $\langle x, z\rangle \in \bigcap \mathcal{A}$
  $\langle 2\rangle 2$. LET: $R \in \mathcal{A}$
  $\langle 2\rangle 3$. xRy and yRz
  $\langle 2\rangle 4$. xRz
    PROOF: Since R is transitive.
$\square$

(b) Not necessarily. If $\mathcal{A} = \{\{\langle 0, 1\rangle\}, \{\langle 1, 2\rangle\}\}$ then each member of $\mathcal{A}$ is transitive but $\bigcup \mathcal{A} = \{\langle 0, 1\rangle, \langle 1, 2\rangle\}$ is not.

Example 35

$$\begin{aligned} R[\![\{x\}]\!] &= \{y : \exists z(z \in \{x\} \ \&\ zRy)\} \\ &= \{y : \exists z(z = x \ \&\ zRy)\} \\ &= \{y : xRy\} \\ &= [x]_R \end{aligned}$$

Example 36 PROOF:
$\langle 1\rangle 1$. Q is a relation on A.
  PROOF: By definition.
$\langle 1\rangle 2$. Q is reflexive on A.
  $\langle 2\rangle 1$. LET: $x \in A$
  $\langle 2\rangle 2$. $f(x)Rf(x)$
    PROOF: Since R is reflexive on B.
  $\langle 2\rangle 3$. xQx
$\langle 1\rangle 3$. Q is symmetric.
  $\langle 2\rangle 1$. ASSUME: xQy
  $\langle 2\rangle 2$. $f(x)Rf(y)$
  $\langle 2\rangle 3$. $f(y)Rf(x)$
    PROOF: R is symmetric.
  $\langle 2\rangle 4$. yQx
$\langle 1\rangle 4$. Q is transitive.
  $\langle 2\rangle 1$. ASSUME: xQy and yQz
  $\langle 2\rangle 2$. $f(x)Rf(y)$ and $f(y)Rf(z)$
  $\langle 2\rangle 3$. $f(x)Rf(z)$
    PROOF: R is transitive.
  $\langle 2\rangle 4$. xQz
$\square$

Exercise 37 PROOF:
$\langle 1\rangle 1$. R_Π is a relation on A.

PROOF: If $B \in \Pi$, $x \in B$ and $y \in B$ then $x, y \in A$.
$\langle 1 \rangle 2.$ R_Π is reflexive on A.
$\langle 2 \rangle 1.$ LET: $x \in A$
$\langle 2 \rangle 2.$ PICK $B \in \Pi$ such that $x \in B$
PROOF: Because Π is exhaustive.
$\langle 2 \rangle 3.$ $x \in B$ and $x \in B$
$\langle 2 \rangle 4.$ $x R_\Pi x$
$\langle 1 \rangle 3.$ R_Π is symmetric.
$\langle 2 \rangle 1.$ ASSUME: $x R_\Pi y$
$\langle 2 \rangle 2.$ PICK $B \in \Pi$ such that $x \in B$ and $y \in B$
$\langle 2 \rangle 3.$ $y \in B$ and $x \in B$
$\langle 2 \rangle 4.$ $y R_\Pi x$
$\langle 1 \rangle 4.$ R_Π is transitive.
$\langle 2 \rangle 1.$ ASSUME: $x R_\Pi y$ and $y R_\Pi z$
$\langle 2 \rangle 2.$ PICK $B \in \Pi$ such that $x \in B$ and $y \in B$
$\langle 2 \rangle 3.$ PICK $C \in \Pi$ such that $y \in C$ and $z \in C$
$\langle 2 \rangle 4.$ $B = C$
PROOF: Since $y \in B$ and $y \in C$
$\langle 2 \rangle 5.$ $x \in B$ and $z \in B$
$\langle 2 \rangle 6.$ $x R_\Pi z$
$\square$

Exercise 38 PROOF:
$\langle 1 \rangle 1.$ If $B \in \Pi$ and $x \in B$ then $B = [x]_{R_\Pi}$
$\langle 2 \rangle 1.$ LET: $B \in \Pi$
$\langle 2 \rangle 2.$ LET: $x \in B$
$\langle 2 \rangle 3.$ $[x]_{R_\Pi} \subseteq B$
$\langle 3 \rangle 1.$ LET: $y \in [x]_{R_\Pi}$
$\langle 3 \rangle 2.$ $x R_\Pi y$
$\langle 3 \rangle 3.$ PICK $C \in \Pi$ such that $x \in C$ and $y \in C$
$\langle 3 \rangle 4.$ $B = C$
PROOF: Since $x \in B$ and $x \in C$.
$\langle 3 \rangle 5.$ $y \in B$
$\langle 2 \rangle 4.$ $B \subseteq [x]_{R_\Pi}$
PROOF: For all $y \in B$, we have $x \in B$ and $y \in B$ hence $x R_\Pi y$.
$\langle 1 \rangle 2.$ $A/R_\Pi \subseteq \Pi$
$\langle 2 \rangle 1.$ LET: $x \in A$
PROVE: $[x]_{R_\Pi} \in \Pi$
$\langle 2 \rangle 2.$ PICK $B \in \Pi$ such that $x \in B$
$\langle 2 \rangle 3.$ $[x]_{R_\Pi} = B$
PROOF: By $\langle 1 \rangle 1$
$\langle 2 \rangle 4.$ $[x]_{R_\Pi} \in \Pi$
$\langle 1 \rangle 3.$ $\Pi \subseteq A/R_\Pi$
$\langle 2 \rangle 1.$ LET: $B \in \Pi$
$\langle 2 \rangle 2.$ PICK $x \in B$

PROOF: Since every member of Π is nonempty.
$\langle 2 \rangle 3.$ $B = [x]_{R_\Pi}$
PROOF: By $\langle 1 \rangle 1$.
$\langle 2 \rangle 4.$ $B \in A/R_\Pi$
$\square$

Exercise 39 PROOF:
$\langle 1 \rangle 1.$ $R_\Pi \subseteq R$
$\langle 2 \rangle 1.$ LET: $xR_\Pi y$
$\langle 2 \rangle 2.$ PICK $B \in \Pi$ such that $x \in B$ and $y \in B$
$\langle 2 \rangle 3.$ PICK $z \in A$ such that $B = [z]_R$
$\langle 2 \rangle 4.$ zRx
$\langle 2 \rangle 5.$ zRy
$\langle 2 \rangle 6.$ xRy
PROOF: Since R is symmetric and transitive.
$\langle 1 \rangle 2.$ $R \subseteq R_\Pi$
$\langle 2 \rangle 1.$ LET: xRy
$\langle 2 \rangle 2.$ $x \in [x]_R$
$\langle 2 \rangle 3.$ $y \in [x]_R$
$\langle 2 \rangle 4.$ $xR_\Pi y$
$\square$

Exercise 40 We have $[2]_R = [3]_R$ but $[6]_R \neq [9]_R$ so there is no such function f.

Exercise 41

(a) PROOF:
$\langle 1 \rangle 1.$ Q is reflexive on $\mathbb{R} \times \mathbb{R}$.
PROOF: For any $x, y \in \mathbb{R}$, we have $x + y = x + y$, hence $\langle x, y \rangle Q \langle x, y \rangle$
$\langle 1 \rangle 2.$ Q is symmetric.
$\langle 2 \rangle 1.$ ASSUME: $\langle u, v \rangle Q \langle x, y \rangle$
$\langle 2 \rangle 2.$ $u + y = x + v$
$\langle 2 \rangle 3.$ $x + v = u + y$
$\langle 2 \rangle 4.$ $\langle x, y \rangle Q \langle u, v \rangle$
$\langle 1 \rangle 3.$ Q is transitive.
$\langle 2 \rangle 1.$ ASSUME: $\langle a, b \rangle Q \langle u, v \rangle$ and $\langle u, v \rangle Q \langle x, y \rangle$
$\langle 2 \rangle 2.$ $a + v = u + b$
$\langle 2 \rangle 3.$ $u + y = x + v$
$\langle 2 \rangle 4.$ $a + y + x + b$
PROOF: Adding $\langle 2 \rangle 2$ and $\langle 2 \rangle 3$ gives $a + u + v + y = b + u + v + x$.
$\langle 2 \rangle 5.$ $\langle a, b \rangle Q \langle x, y \rangle$
$\square$

(b) We prove that, if $\langle u, v\rangle Q\langle x, y\rangle$ then $\langle u+2v, v+2u\rangle Q\langle x+2y, y+2x\rangle$. It follows from Theorem 3Q that the function G exists.

If $u+y=v+x$ then $u+2v+y+2x=v+2u+x+2y$ by adding $u+v+y+x$ to both sides.

Exercise 42 Assume that R is an equivalence relation on A and that $F : A \times A \to A$. Let us say that F is *compatible* with R iff, whenever xRx' and yRy', then $F(\langle x, y\rangle)RF(\langle x', y'\rangle)$. If F is compatible with R then there exists a unique $\hat{F} : (A/R) \times (A/R) \to A/R$ such that

$$\hat{F}(\langle [x]_R, [y]_R\rangle) = [F(\langle x, y\rangle)]_R \text{ for all } x, y \in A \enspace .$$

If F is not compatible with R then no such $\hat{F}$ exists.

3.7 Ordering Relations

Exercise 43 Proof:
$\langle 1\rangle 1.$ R^{-1} is transitive.
  $\langle 2\rangle 1.$ Assume: $xR^{-1}y$ and $yR^{-1}z$
  $\langle 2\rangle 2.$ zRy and yRx
  $\langle 2\rangle 3.$ zRx
    Proof: Since R is transitive.
  $\langle 2\rangle 4.$ $xR^{-1}z$
$\langle 1\rangle 2.$ R^{-1} satisfies trichotomy on A.
  $\langle 2\rangle 1.$ Let: $x, y \in A$
  $\langle 2\rangle 2.$ Exactly one of xRy, $x=y$, yRx holds.
  $\langle 2\rangle 3.$ Exactly one of $yR^{-1}x$, $x=y$, $xR^{-1}y$ holds.
□

Exercise 44 Proof:
$\langle 1\rangle 1.$ f is one-to-one.
  $\langle 2\rangle 1.$ Let: $x, y \in A$ with $f(x) = f(y)$
  $\langle 2\rangle 2.$ $f(x) < f(y)$ and $f(y) < f(x)$ do not hold.
    Proof: By trichotomy.
  $\langle 2\rangle 3.$ $x < y$ and $y < x$ do not hold.
  $\langle 2\rangle 4.$ $x = y$
    Proof: By trichotomy.
$\langle 1\rangle 2.$ Whenever $f(x) < f(y)$ then $x < y$
  $\langle 2\rangle 1.$ Let: $x, y \in A$ with $f(x) < f(y)$
  $\langle 2\rangle 2.$ $f(x) = f(y)$ and $f(y) < f(x)$ do not hold.
    Proof: By trichotomy.
  $\langle 2\rangle 3.$ $x = y$ and $y < x$ do not hold.
  $\langle 2\rangle 4.$ $x < y$
    Proof: By trichotomy.
□

Exercise 45 PROOF:
$\langle 1\rangle 1.$ $<_L$ is transitive.
$\langle 2\rangle 1.$ LET: $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle$ and $\langle a_2, b_2\rangle <_L \langle a_3, b_3\rangle$
PROVE: $\langle a_1, b_1\rangle < \langle a_3, b_3\rangle$
$\langle 2\rangle 2.$ CASE: $a_1 <_A a_2$ and $a_2 <_A a_3$
PROOF: Then $a_1 <_A <_3$
$\langle 2\rangle 3.$ CASE: $a_1 <_A a_2$, $a_2 = a_3$, $b_2 <_B b_3$
PROOF: Then $a_1 <_A <_3$
$\langle 2\rangle 4.$ CASE: $a_1 = a_2$, $b_1 <_B b_2$ and $a_2 <_A a_3$
PROOF: Then $a_1 <_A <_3$
$\langle 2\rangle 5.$ CASE: $a_1 = a_2$, $b_1 <_B b_2$, $a_2 = a_3$, $b_2 <_B b_3$
PROOF: Then $a_1 = a_3$ and $b_1 <_B b_3$
$\langle 1\rangle 2.$ $<_L$ satisfies trichotomy on $A \times B$.
$\langle 2\rangle 1.$ LET: $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ be elements of $A \times B$
$\langle 2\rangle 2.$ Exactly one of $a_1 <_A a_2$, $a_1 = a_2$, $a_2 <_A a_1$ holds.
$\langle 2\rangle 3.$ Exactly one of $b_1 <_B b_2$, $b_1 = b_2$, $b_2 <_B b_1$ holds.
$\langle 2\rangle 4.$ Exactly one of $a_1 <_A a_2$, ($a_1 = a_2$ and $b_1 <_B b_2$), ($a_1 = a_2$ and $b_1 = b_2$), ($a_1 = a_2$ and $b_2 <_L b_1$), $a_2 <_A a_1$ holds.
$\langle 2\rangle 5.$ Exactly one of $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle$, $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$, $\langle a_2, b_2\rangle <_L \langle a_1, b_1\rangle$ holds.
$\square$

3.8 Review Exercises

Exercise 46

(a)

$$\begin{aligned} \bigcap\bigcap\langle x, y\rangle &= \bigcap\{x\} \\ &= x \end{aligned}$$

(b)

$$\begin{aligned} \bigcap\bigcap\bigcap\{\langle x, y\rangle\}^{-1} &= \bigcap\bigcap\bigcap\{\langle y, x\rangle\} \\ &= \bigcap\bigcap\langle y, x\rangle \\ &= y && \text{(by part (a))} \end{aligned}$$

Exercise 47

(a) There are eight:

$$\begin{gathered}
\{\langle 0,3\rangle,\langle 1,3\rangle,\langle 2,3\rangle\},\\
\{\langle 0,3\rangle,\langle 1,3\rangle,\langle 2,4\rangle\},\\
\{\langle 0,3\rangle,\langle 1,4\rangle,\langle 2,3\rangle\},\\
\{\langle 0,3\rangle,\langle 1,4\rangle,\langle 2,4\rangle\},\\
\{\langle 0,4\rangle,\langle 1,3\rangle,\langle 2,3\rangle\},\\
\{\langle 0,4\rangle,\langle 1,3\rangle,\langle 2,4\rangle\},\\
\{\langle 0,4\rangle,\langle 1,4\rangle,\langle 2,3\rangle\},\\
\{\langle 0,4\rangle,\langle 1,4\rangle,\langle 2,4\rangle\}
\end{gathered}$$

(b) There are six:

$$\begin{gathered}
\{\langle 0,3\rangle,\langle 1,4\rangle,\langle 2,5\rangle\},\\
\{\langle 0,3\rangle,\langle 1,5\rangle,\langle 2,4\rangle\},\\
\{\langle 0,4\rangle,\langle 1,3\rangle,\langle 2,5\rangle\},\\
\{\langle 0,4\rangle,\langle 1,5\rangle,\langle 2,3\rangle\},\\
\{\langle 0,5\rangle,\langle 1,3\rangle,\langle 2,4\rangle\},\\
\{\langle 0,5\rangle,\langle 1,4\rangle,\langle 2,3\rangle\}
\end{gathered}$$

Exercise 48

(a) The only ordered pair in $\mathcal{P}T$ is $\langle\emptyset,\emptyset\rangle = \{\{\emptyset\}\}$.

(b)

$$\begin{aligned}
(\mathcal{P}T)^{-1} \circ (\mathcal{P}T \restriction \{\emptyset\}) &= \{\langle\emptyset,\emptyset\rangle\} \circ \{\langle\emptyset,\emptyset\rangle\}\\
&= \{\langle\emptyset,\emptyset\rangle\}
\end{aligned}$$

Exercise 49 There are six:

$$\begin{aligned}
\{\langle 0,0\rangle,\langle 1,1\rangle,\langle 2,2\rangle\},\\
\{\langle 0,0\rangle,\langle 0,1\rangle,\langle 1,0\rangle,\langle 1,1\rangle,\langle 2,2\rangle\},\\
\{\langle 0,0\rangle,\langle 0,2\rangle,\langle 1,1\rangle,\langle 2,0\rangle,\langle 2,2\rangle\},\\
\{\langle 0,0\rangle,\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,1\rangle,\langle 2,2\rangle\},\\
\{\langle 0,0\rangle,\langle 0,1\rangle,\langle 0,2\rangle,\langle 1,0\rangle,\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,0\rangle,\langle 2,1\rangle,\langle 2,2\rangle\}
\end{aligned}$$

Exercise 50

(a) $\{\langle 0,1\rangle,\langle 0,2\rangle,\langle 0,3\rangle,\langle 1,3\rangle,\langle 2,1\rangle,\langle 2,3\rangle\}$

(b) $\{\langle 0,1\rangle,\langle 0,2\rangle,\langle 0,3\rangle,\langle 2,1\rangle,\langle 3,1\rangle,\langle 3,2\rangle\}$

Exercise 51 There are three:

$$\begin{gathered}\{\langle 1,0\rangle,\langle 1,2\rangle,\langle 2,0\rangle\},\\ \{\langle 1,0\rangle,\langle 2,0\rangle,\langle 2,1\rangle\},\\ \{\langle 0,1\rangle,\langle 2,0\rangle,\langle 2,1\rangle\}\end{gathered}$$

Exercise 52 We can conclude this if we know that A and B are nonempty, or that C and D are nonempty.

Suppose A and B are nonempty. Then $A \times B = C \times D \neq \emptyset$ so C and D are nonempty. We now prove $A \subseteq C$.

Let $a \in A$. Pick some $b \in B$. Then $\langle a,b\rangle \in A \times B = C \times D$ and so $a \in C$.

We can similarly prove $C \subseteq A$, $B \subseteq D$ and $D \subseteq B$.

Exercise 53

$$\begin{aligned}
x(R\cup S)^{-1}y &\Leftrightarrow y(R\cup S)x\\
&\Leftrightarrow yRx \text{ or } ySx\\
&\Leftrightarrow xR^{-1}y \text{ or } xS^{-1}y\\
&\Leftrightarrow x(R^{-1}\cup S^{-1})y\\
x(R\cap S)^{-1}y &\Leftrightarrow y(R\cap S)x\\
&\Leftrightarrow yRx \text{ and } ySx\\
&\Leftrightarrow xR^{-1}y \text{ and } xS^{-1}y\\
&\Leftrightarrow x(R^{-1}\cap S^{-1})y\\
x(R-S)^{-1}y &\Leftrightarrow y(R-S)x\\
&\Leftrightarrow yRx \text{ and } \neg ySx\\
&\Leftrightarrow xR^{-1}y \text{ and } \neg xS^{-1}y\\
&\Leftrightarrow x(R^{-1}-S^{-1})y
\end{aligned}$$

Exercise 54

(a)

$$\begin{aligned}
\langle x,y\rangle \in A\times(B\cap C) &\Leftrightarrow x\in A \;\&\; y\in B \;\&\; y\in C\\
&\Leftarrow \langle x,y\rangle \in (A\times B)\cap(A\times C)
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x,y\rangle \in A\times(B\cup C) &\Leftrightarrow x\in A \;\&(y\in B \text{ or } y\in C)\\
&\Leftrightarrow (x\in A \;\&\; y\in B) \text{ or } (x\in A \;\&\; y\in C)\\
&\Leftrightarrow \langle x,y\rangle \in (A\times B)\cup(A\times C)
\end{aligned}$$

(c)

$$\begin{aligned}\langle x,y\rangle \in A\times(B-C) &\Leftrightarrow x\in A \;\&\; y\in B \;\&\; y\notin C\\ &\Leftrightarrow \langle x,y\rangle \in (A\times B)-(A\times C)\end{aligned}$$

Exercise 55

(a) No. Take $A=\{0\}$, $B=\{1\}$, $C=\{2\}$. Then $(A\times A)\cup(B\times C)=\{\langle 0,0\rangle,\langle 1,2\rangle\}$ while $(A\cup B)\times(A\cup C)=\{\langle 0,0\rangle,\langle 0,2\rangle,\langle 1,0\rangle,\langle 1,2\rangle\}$.

(b) Yes.

$$\begin{aligned}\langle x,y\rangle \in (A\times A)\cap(B\times C) &\Leftrightarrow x\in A \;\&\; y\in A \;\&\; x\in B \;\&\; y\in C\\ &\Leftrightarrow \langle x,y\rangle \in (A\cap B)\times(A\cap C)\end{aligned}$$

Exercise 56

(a) Yes.

$$\begin{aligned}x\in \operatorname{dom}(R\cup S) &\Leftrightarrow \exists y(xRy \text{ or } xSy)\\ &\Leftrightarrow \exists y.xRy \text{ or } \exists y.xSy\\ &\Leftrightarrow x\in \operatorname{dom} R\cup \operatorname{dom} S\end{aligned}$$

(b) No. Take $R=\{\langle 0,0\rangle\}$ and $S=\{\langle 0,1\rangle\}$. Then $\operatorname{dom}(R\cap S)=\operatorname{dom}\emptyset=\emptyset$ while $\operatorname{dom} R\cap\operatorname{dom} S=\{0\}\cap\{0\}=\{0\}$.

Exercise 57

(a) Yes.

$$\begin{aligned}x(R\circ(S\cup T))y &\Leftrightarrow \exists z(x(S\cup T)z \;\&\; zRy)\\ &\Leftrightarrow \exists z(xSz \;\&\; zRy) \text{ or } \exists z(xTz \;\&\; zRy)\\ &\Leftrightarrow x((R\circ S)\cup(R\circ T))y\end{aligned}$$

(b) No. Take $R=\{\langle 0,0\rangle,\langle 1,0\rangle\}$, $S=\{\langle 0,0\rangle\}$ and $T=\{\langle 0,1\rangle\}$. Then

$$\begin{aligned}R\circ(S\cap T) &= R\circ\emptyset\\ &=\emptyset\\ (R\circ S)\cap(R\circ T) &= \{\langle 0,0\rangle\}\cap\{\langle 0,0\rangle\}\\ &=\{\langle 0,0\rangle\}\end{aligned}$$

Exercise 58 Take $F=\emptyset$ and $S=\{\emptyset\}$. Then $F[\![F^{-1}[\![S]\!]]\!]=\emptyset\neq S$.

Exercise 59

$$
\begin{aligned}
x(Q \restriction (A \cap B))y &\Leftrightarrow xQy \ \&\ x \in A \ \&\ x \in B \\
&\Leftrightarrow x((Q \restriction A) \cap (Q \restriction B))y \\
x(Q \restriction (A - B))y &\Leftrightarrow xQy \ \&\ x \in A \ \&\ x \notin B \\
&\Leftrightarrow (xQy \ \&\ x \in A) \ \&\ \neg(xQy \ \&\ x \in B) \\
&\Leftrightarrow x((Q \restriction A) - (Q \restriction B))y
\end{aligned}
$$

Exercise 60

$$
\begin{aligned}
x((R \circ S) \restriction A)y &\Leftrightarrow \exists z(xRz \ \&\ zSy \ \&\ x \in A) \\
&\Leftrightarrow x(R \circ (S \restriction A))y
\end{aligned}
$$

Chapter 4

Chapter 4 — Natural Numbers

4.1 Inductive Sets

Exercise 1 We have

$$3 = 2 \cup \{2\} = 1 \cup \{1, 2\}$$

and so $1 \in 3$. But $1 \notin 1$ (since $1 = \{\emptyset\}$ and we know $\{\emptyset\} \neq \emptyset$ hence $\{\emptyset\} \notin \{\emptyset\}$). Therefore $1 \neq 3$.

4.2 Peano's Postulates

Exercise 2 If a is a transitive set then

$$\begin{aligned} \bigcup(a^+) &= a && \text{(Theorem 4E)} \\ &\subseteq a^+ \end{aligned}$$

Exercise 3

(a) Suppose a is a transitive set. Then $a \subseteq \mathcal{P}a$. Hence we have $\bigcup \mathcal{P}a = a \subseteq \mathcal{P}a$ and so $\mathcal{P}a$.

(b) Suppose $\mathcal{P}a$ is a transitive set. Then $a = \bigcup \mathcal{P}a \subseteq \mathcal{P}a$ hence a is transitive.

Exercise 4 If a is a transitive set then $\bigcup a \subseteq a$ so $\bigcup \bigcup a \subseteq \bigcup a$. Hence $\bigcup a$ is transitive.

Exercise 5

(a) Proof:
⟨1⟩1. Let: $b \in \bigcup \mathcal{A}$
⟨1⟩2. Pick $A \in \mathcal{A}$ such that $b \in A$
⟨1⟩3. $b \subseteq A$
Proof: Since A is transitive.
⟨1⟩4. $b \subseteq \bigcup \mathcal{A}$
□

(b) Proof:
⟨1⟩1. Let: $b \in \bigcap \mathcal{A}$
⟨1⟩2. For all $A \in \mathcal{A}$ we have $b \subseteq A$
Proof: Since $b \in A$ and A is transitive.
⟨1⟩3. $b \subseteq \bigcap \mathcal{A}$
□

Exercise 6 We have $\bigcup(a^+) = \bigcup a \cup a$ (see the proof of Theorem 4E). So if $\bigcup(a^+) = a$ we have $\bigcup a \cup a = a$ and so $\bigcup a \subseteq a$.

4.3 Recursion on ω

Exercise 7 We have $h_1(0) = h_2(0) = a$ so $0 \in S$.

Now let $n \in S$; we prove $n^+ \in S$. We have $h_1(n) = h_2(n)$ and therefore

$$\begin{aligned} h_1(n^+) &= F(h_1(n)) \\ &= F(h_2(n)) \\ &= h_2(n^+) \end{aligned}$$

Exercise 8 Proof:
⟨1⟩1. $\forall m, n \in \omega . h(n) = h(m) \Rightarrow n = m$
⟨2⟩1. $\forall n \in \omega . h(n) = h(0) \Rightarrow n = 0$
⟨3⟩1. Let: $n \in \omega$
⟨3⟩2. Assume: $h(n) = h(0)$
⟨3⟩3. $h(n) = c$
⟨3⟩4. $\forall p \in \omega . n \neq p^+$
Proof: Otherwise $f(h(p)) = c$ contradicting the fact that $c \in A - \operatorname{ran} f$.
⟨3⟩5. $n = 0$
Proof: Theorem 4C.
⟨2⟩2. For all $m \in \omega$, if $\forall n \in \omega . h(n) = h(m) \Rightarrow n = m$, then $\forall n \in \omega . h(n) = h(m^+) \Rightarrow n = m^+$
⟨3⟩1. Let: $m \in \omega$
⟨3⟩2. Assume: $\forall n \in \omega . h(n) = h(m) \Rightarrow n = m$
⟨3⟩3. Let: $n \in \omega$
⟨3⟩4. Assume: $h(n) = h(m^+)$
⟨3⟩5. $h(n) = f(h(m))$

$\langle 3\rangle 6.$ $n \neq 0$
PROOF: Otherwise $c = f(h(m))$ contradicting the fact that $c \in A - \operatorname{ran} f$.
$\langle 3\rangle 7.$ PICK p such that $n = p^+$
$\langle 3\rangle 8.$ $f(h(p)) = f(h(m))$
$\langle 3\rangle 9.$ $h(p) = h(m)$
PROOF: f is one-to-one.
$\langle 3\rangle 10.$ $p = m$
PROOF: By $\langle 3\rangle 2$.
$\langle 3\rangle 11.$ $n = p^+ = m^+$
□

Exercise 9 PROOF:
$\langle 1\rangle 1.$ $C^* \subseteq C_*$
$\langle 2\rangle 1.$ $f[\![C_*]\!] \subseteq C_*$
$\langle 3\rangle 1.$ LET: $x \in C_*$
PROVE: $f(x) \in C_*$
$\langle 3\rangle 2.$ PICK n such that $x \in h(n)$
$\langle 3\rangle 3.$ $f(x) \in h(n^+)$
$\langle 3\rangle 4.$ $f(x) \in C_*$
$\langle 1\rangle 2.$ $C_* \subseteq C^*$
$\langle 2\rangle 1.$ $\forall n \in \omega . h(n) \subseteq C^*$
$\langle 3\rangle 1.$ $h(0) \subseteq C^*$
PROOF: If $A \subseteq X \subseteq B$ and $f[\![X]\!] \subseteq X$ then $A \subseteq X$.
$\langle 3\rangle 2.$ $\forall n \in \omega (h(n) \subseteq C^* \Rightarrow h(n^+) \subseteq C^*)$
$\langle 4\rangle 1.$ LET: $n \in \omega$
$\langle 4\rangle 2.$ ASSUME: $h(n) \subseteq C^*$
$\langle 4\rangle 3.$ $f[\![h(n)]\!] \subseteq C^*$
$\langle 5\rangle 1.$ LET: X be such that $A \subseteq X \subseteq B$ and $f[\![X]\!] \subseteq X$
PROVE: $f[\![h(n)]\!] \subseteq X$
$\langle 5\rangle 2.$ $h(n) \subseteq X$
$\langle 5\rangle 3.$ $f[\![h(n)]\!] \subseteq f[\![X]\!]$
$\langle 5\rangle 4.$ $f[\![h(n)]\!] \subseteq X$
$\langle 4\rangle 4.$ $h(n^+) \subseteq C^*$
□

Exercise 10 $C^* = C_* = (0,1]$

Exercise 11 $\{n \in \mathbb{Z} \mid n \leq 0\}$

Exercise 12 Let $f : B \times B \to B$ and $A \subseteq B$. Let

$$C^* = \bigcap \{X \mid A \subseteq X \subseteq B \ \& \ f[\![X \times X]\!] \subseteq X\} \ .$$

Define the function $h : \omega \to \mathcal{P}B$ by

$$\begin{aligned} h(0) &= A \\ h(n^+) &= h(n) \cup f[\![h(n) \times h(n)]\!] \qquad\qquad (n \in \omega) \end{aligned}$$

Define $C_* = \bigcup \operatorname{ran} h$. Then $C^* = C_*$.

4.4 Arithmetic

Exercise 13 We prove the contrapositive. Assume $m \neq 0$ and $n \neq 0$. Then by Theorem 4C there are natural numbers p, q such that $m = p^+$ and $n = q^+$. Hence $mn = p^+q^+ = (p^+q + p)^+ \neq 0$.

Exercise 14 We prove the following facts for any natural number n:

1. n is even if and only if n^+ is odd.

   PROOF: If n is even, say $n = 2p$, then $n^+ = 2p + 1$ is odd.
   If n^+ is odd, say $n^+ = 2p + 1$, then $n = 2p$ is even.

2. n is odd if and only if n^+ is even.

   PROOF: If n is odd, say $n = 2p + 1$, then $n^+ = 2(p + 1)$ is even.
   If n^+ is even, say $n^+ = 2p$, then we cannot have $p = 0$ (since $n^+ \neq 0$). So $p = q + 1$ for some q. But then $n^+ = 2q + 2$ so $n = 2q + 1$ and n is odd.

Now, 0 is even and 0 is not odd. By the two facts above, if n is either even or odd but not both, then n^+ is either odd or even but not both. The result follows by induction.

Exercise 15 We have

$$\begin{aligned} m + (n + 0) &= m + n && \text{by (A1)} \\ &= (m + n) + 0 && \text{by (A1)} \end{aligned}$$

If $m + (n + p) = (m + n) + p$ then

$$\begin{aligned} m + (n + p^+) &= m + (n + p)^+ && \text{by (A2)} \\ &= (m + (n + p))^+ && \text{by (A2)} \\ &= ((m + n) + p)^+ && \text{by induction hypothesis} \\ &= (m + n) + p^+ && \text{by (A2)} \end{aligned}$$

Exercise 16 We first prove that $0 \cdot n = 0$ for all n. We have $0 \cdot 0 = 0$ by (M1), and if $0 \cdot n = 0$ then

$$\begin{aligned} 0 \cdot n^+ &= 0 \cdot n + 0 && \text{by (M2)} \\ &= 0 \cdot n && \text{by (A1)} \\ &= 0 && \text{by induction hypothesis} \end{aligned}$$

Now we prove that $m^+ \cdot n = m \cdot n + n$ for all m, n. We have

$$\begin{aligned} m^+ \cdot 0 &= 0 && \text{by (M1)} \\ m \cdot 0 + 0 &= m \cdot 0 && \text{by (A1)} \\ &= 0 && \text{by (M1)} \end{aligned}$$

Thus, $m^+ \cdot 0 = m \cdot 0 + 0$.

If $m^+ \cdot n = m \cdot n + n$ then

$$\begin{aligned} m^+ \cdot n^+ &= m^+ \cdot n + m^+ && \text{by (M2)} \\ &= (m^+ \cdot n + m)^+ && \text{by (A2)} \\ &= ((m \cdot n + n) + m)^+ && \text{by induction hypothesis} \\ &= ((m \cdot n + m) + n)^+ && \text{by associativity and commutativity of addition} \\ &= (m \cdot n^+ + n)^+ && \text{by (M2)} \\ &= m \cdot n^+ + n^+ && \text{by (A2)} \end{aligned}$$

Exercise 17 The proof is by induction on p. We have

$$\begin{aligned} m^{n+0} &= m^n && \text{by (A1)} \\ &= 0 + m^n && \text{by Theorem 4K(2)} \\ &= m^n \cdot 0 + m^n && \text{by (M1)} \\ &= m^n \cdot 1 && \text{by (M2)} \\ &= m^n \cdot m^0 && \text{by (E1)} \end{aligned}$$

If $m^{n+p} = m^n \cdot m^p$ then

$$\begin{aligned} m^{n+p^+} &= m^{(n+p)^+} && \text{by (A2)} \\ &= m^{n+p}m && \text{by (E2)} \\ &= (m^n m^p)m && \text{by induction hypothesis} \\ &= m^n(m^p m) && \text{by Theorem 4K (4)} \\ &= m^n m^{p^+} && \text{by (E2)} \end{aligned}$$

4.5 Ordering on ω

Exercise 18

$$\begin{aligned} \in_\omega^{-1} [\![\{7, 8\}]\!] &= \{x \in \omega \mid x \in 7 \text{ or } x \in 8\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7\} \end{aligned}$$

Exercise 19 The proof is by induction on m.

For $m = 0$, take $q = r = 0$. Then $m = d \cdot 0 + 0$ and $0 \in d$.

Suppose $m = dq + r$ and $r < d$. Then $r + 1 \leq d$. If $r + 1 < d$, then we have $m + 1 = dq + (r + 1)$ as required. If $r + 1 = d$, then we have $m + 1 = dq + d = d(q + 1) + 0$.

Exercise 20 We first prove A is closed downwards; that is, if $n \in A$ and $m \in n$ then $m \in A$. This holds because if $n \in A$ and $m \in n$ then $m \in \bigcup A$ and $\bigcup A = A$.

Now, we prove $\forall n \in \omega . n \in A$ by induction on n.

To prove $0 \in A$: we are given that A is nonempty. Pick some $a \in A$. Then $0 \underline{in} a$ so $0 \in A$ since A is closed downwards.

Now let $n \in A$; we prove $n^+ \in A$. We have $n \in \bigcup A$; pick some $k \in A$ such that $n \in k$. Then $n^+ \underline{\in} k$ so $n^+ \in A$ since A is closed downwards.

This completes the induction. We have $\forall n \in \omega . n \in A$, i.e. $A = \omega$.

Exercise 21 Suppose n is a natural number, $k \in n$ and $n \subseteq k$. Then $k \in k$, contradicting Lemma 4L(b).

Exercise 22 We have $0 \in p^+$ (by trichotomy since $p^+ \notin 0$ because 0 is empty, and $p^+ \neq 0$ by Peano's First Postulate.) Hence $n = n + 0 \in n + p^+$ by Theorem 4N.

Exercise 23 The proof is by induction on n. The statement is vacuously true for $n = 0$.

Suppose the statement is true for n. Let $m \in n^+$. Then $m \underline{\in} n$.

If $m = n$, then we have $m + 0^+ = n^+$.

If $m \in n$, pick p such that $m + p^+ = n$ by the induction hypothesis. Then $m + p^{++} = n^+$.

Exercise 24 Suppose $m \in p$. Then we cannot have $n \in q$ or $n = q$, as either of these would imply $m + n \in p + q$. Hence $q \in n$ by trichotomy.

We prove $q \in n \Rightarrow m \in p$ similarly.

Exercise 25 By Exercise 23, pick natural numbers a and b such that $m = n + a^+$ and $p = q + b^+$. Then

$$\begin{aligned} mp + nq &= (n + a^+)(q + b^+) + nq \\ &= nq + nq + a^+ q + nb^+ + a^+ b^+ \\ &= (n + a^+)q + n(q + b^+) + a^+ b^+ \\ &= mq + np + (a^+ + b)^+ \end{aligned}$$

Hence $mq + np \in mp + nq$ by Exercise 22.

Exercise 26 The proof is by induction on n.

If $n = 0$ then $\operatorname{ran} f$ is a singleton and its sole element is the largest element.

Suppose the result is true for n. Let $f : n^{++} \to A$. Then $f[\![n^+]\!]$ has a largest element $f(k)$, say. If $f(k) \underline{\in} f(n^+)$ then $f(n^+)$ is greatest in $\operatorname{ran} f$; otherwise $f(k)$ is greatest.

Exercise 27 We prove $f_1(n) = f_2(n)$ for all $n \in \omega$ by strong induction on n.
Assume that $(\forall m \in n) f_1(m) = f_2(m)$. Then $f_1 \restriction n = f_2 \restriction n$. So

$$\begin{aligned} f_1(n) &= G(f_1 \restriction n) \\ &= G(f_2 \restriction n) \\ &= f_2(n) \end{aligned}$$

Exercise 28 Suppose ω is not transitive. Then there exists a natural number n such that $n \nsubseteq \omega$. Let n be the least such number. There exists $x \in n$ such that $x \notin \omega$. Now, $n \neq 0$ (because it is nonempty) so $n = p^+$ for some natural number p. We have $x \in p^+$ so $x \in p$ or $x = p$. We cannot have $x = p$ (because x is not a natural number) so we have $x \in p$. But this contradicts the minimality of n.

4.6 Review Exercises

Exercise 29 $4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

Exercise 30 $\bigcup 4 = 0 \cup 1 \cup 2 \cup 3 = 3$ since 0, 1 and 2 are all subsets of 3.
$\bigcap 4 = 0 \cap 1 \cap 2 \cap 3 = 0 (= \emptyset)$.

Exercise 31 Similarly to Exercise 30 we have $\bigcup \bigcup 7 = \bigcup 6 = 5$.

Exercise 32

(a) $A^+ = A \cup \{A\} = \{1, A\} = \{1, \{1\}\}$
So $\bigcup A^+ = 1 \cup \{1\} = \{0, 1\} = 2$

(b) $\bigcup(\{2\}^+) = \bigcup\{2, \{2\}\} = \{0, 1, 2\} = 3$

Exercise 33

(a) Yes - if $x \in y \in \{0, 1, \{1\}\}$ then x is either 0 or 1, and in either case $x \in \{0, 1, \{1\}\}$

(b) No - $0 \in 1 \in \{1\}$ but $0 \notin \{1\}$

(c) No - $0 \in \{0\} \in \langle 0, 1 \rangle$ but $0 \notin \langle 0, 1 \rangle$.

Exercise 34

(a) Let $a = \{\emptyset\}$ and $b = \emptyset$

(b) Let $c = \{\{\emptyset\}\}$, $d = \{\emptyset\}$ and $e = \emptyset$

Exercise 35

(a) Let $T_1 = \{\{1\}, \{1, 0\}, 0, 1\}$

(b) Let $T_2 = \{\langle 1, 0\rangle, \{1\}, \{1, 0\}, 0, 1\}$.

Exercise 36

$$\begin{aligned} h(4) &= 2h(3) \\ &= 4h(2) \\ &= 8h(1) \\ &= 16h(0) \\ &= 48 \end{aligned}$$

Exercise 37

(a) Let $f : m \to A$ and $g : n \to B$ be bijections. Define $h : m+n \to A \cup B$ by

$$\begin{aligned} h(p) &= f(p) && \text{if } p \in m \\ h(m+q) &= g(q) && \text{if } q \in n \end{aligned}$$

To show that this is well-defined, we must prove two things:

1. For all $p \in m + n$, then either $p \in m$ or there exists $q \in n$ such that $p = m + n$.
2. We never have $p \in m$ and $p = m + q$ for some $q \in n$.

We prove 1 by induction on n. For all $p \in m + 0$ we have $p \in m$, so the result holds for $n = 0$.

Now, suppose the result holds for n. Let $p \in m+n^+ = (m+n)^+$ so $p\underline{in}m+n$. If $p \in m + n$, we simply apply the induction hypothesis. If $p = m + n$ then $p = m + q$ where $q = n \in n^+$.

To prove 2, if $p = m + q$ then $m = m + 0\underline{in}m + q = p$ by Theorem 4N, hence $p \notin m$ by trichotomy.

It remains to show that h is a bijection.

To prove h is injective, we consider three cases. If $h(p) = h(p')$ where $p, p' \in m$, then $f(p) = f(p')$ so $p = p'$. If $h(m + q) = h(m + q')$ where $q, q' \in n$, then $g(q) = g(q')$ so $q = q'$. And we cannot have $h(p) = h(m + q)$ for $p \in m$ and $q \in n$ since $h(p) \in A$, $h(m + q) \in B$, and $A \cap B = \emptyset$.

To prove h is surjective, let $x \in A \cup B$. If $x \in A$, there is some $p \in m$ with $f(p) = x$, so $h(p) = x$. If $x \in B$, there is some $q \in n$ with $g(q) = x$, so $h(m + q) = x$.

(b) Let $f : m \to A$ and $g : n \to B$ be bijections.

We first show that, for any $p \in mn$, there exist unique $i \in m$ and $j \in n$ such that $p = mj + i$.

By Exercise 19, there exist j and $i \in m$ such that $p = mj + i$. We have $j \in n$ since otherwise $p = mj + i \underline{\ni} mj \underline{\ni} mn$.

For uniqueness, suppose $mj + i = mj' + i'$ where $i, i' \in m$ and $j, j' \in n$. Then we have

$$mj \underline{\in} mj + i = mj' + i' \in mj' + m = m(j')^+$$

so $j \in (j')^+$ and $j \underline{\in} j'$. Similarly $j' \underline{\in} j$, and so $j = j'$. Therefore $i = i'$ by the cancellation law for addition.

Now define $h : mn \to A \times B$ by

$$h(mj + i) = \langle f(i), g(j) \rangle$$

where $i \in m$ and $j \in n$. It is easy to check that h is bijective.

Exercise 38 $h(n) = 3n + 1$

Exercise 39 $h(n) = n^2$

Exercise 40 $h(n^+) = h(n) + 5$

Chapter 5

Chapter 5 — Construction of the Real Numbers

5.1 Integers

Exercise 1 No, because $[\langle 0,0\rangle] = [\langle 1,1\rangle]$ but $[\langle 0,0\rangle] \neq [\langle 2,1\rangle]$.

Exercise 2 Yes, because if $[\langle m,n\rangle] = [\langle p,q\rangle]$ then $[\langle m,m\rangle] = [\langle p,p\rangle]$ because $m+p = m+p$.

Exercise 3 Yes, because if $[\langle m,n\rangle] = [\langle p,q\rangle]$ then $[\langle n,m\rangle] = [\langle q,p\rangle]$ because $n+p = m+q$.

Exercise 4 Let $a = [\langle m,n\rangle]$, $b = [\langle p,q\rangle]$ and $c = [\langle r,s\rangle]$. Then

$$\begin{aligned} a +_Z (b +_Z c) &= [\langle m,n\rangle] +_Z [\langle p+r, q+s\rangle] \\ &= [\langle m+(p+r), n+(q+s)\rangle] \\ &= [\langle (m+p)+r, (n+q)+s\rangle] \\ &= [\langle m+p, n+q\rangle] +_Z [\langle r,s\rangle] \\ &= (a +_Z b) +_Z c \end{aligned}$$

Exercise 5

$$[\langle m,n\rangle] - [\langle p,q\rangle] = [\langle m,n\rangle] + [\langle q,p\rangle] = [\langle m+q, n+p\rangle]$$

Exercise 6 Let $a = [\langle m, n\rangle]$. Then

$$\begin{aligned} a \cdot_Z 0_Z &= [\langle m, n\rangle] \cdot_Z [\langle 0, 0\rangle] \\ &= [\langle m0 + n0, m0 + n0\rangle] \\ &= [\langle 0, 0\rangle] \\ &= 0_Z \end{aligned}$$

Exercise 7 We have $a \cdot_Z b +_Z a \cdot_Z (-b) = a \cdot_Z (b +_Z (-b)) = a \cdot_Z 0_Z = 0_Z$, hence $a \cdot_Z (-b) = -(a \cdot_Z b)$ by the uniqueness of inverses.

We prove $(-a) \cdot_Z b = -(a \cdot_Z b)$ similarly.

Exercise 8

(a) This says $[\langle m + n, 0\rangle] = [\langle m, 0\rangle] +_Z [\langle n, 0\rangle]$, which is true from the definition of $+_Z$.

(b) We have

$$\begin{aligned} E(m) \cdot_Z E(n) &= [\langle m, 0\rangle] \cdot_Z [\langle n, 0\rangle] \\ &= [\langle mn + 0 \cdot 0, m0 + n0\rangle] \\ &= E(mn) \end{aligned}$$

(c)

$$\begin{aligned} E(m) <_Z E(n) &\Leftrightarrow [\langle m, 0\rangle] <_Z [\langle n, 0\rangle] \\ &\Leftrightarrow m + 0 \in n + 0 \\ &\Leftrightarrow m \in n \end{aligned}$$

Exercise 9

$$\begin{aligned} E(m) - E(n) &= [\langle m, 0\rangle] - [\langle n, 0\rangle] \\ &= [\langle m, n\rangle] \end{aligned}$$

by Exercise 5.

5.2 Rational Numbers

Exercise 10 Let $r = [\langle a, b\rangle]$. Then

$$\begin{aligned} r \cdot_Q 0_Q &= [\langle a, b\rangle] \cdot_Q [\langle 0, 1\rangle] \\ &= [\langle a \cdot_Z 0, b \cdot_Z 1\rangle] \\ &= [\langle 0, b\rangle] \\ &= [\langle 0, 1\rangle] \end{aligned}$$

since $\langle 0, b\rangle \sim \langle 0, 1\rangle$ because $0 \cdot_Z 1 = 0 \cdot_Z b = 0$.

Exercise 11 Let $r = [\langle a, b\rangle]$ and $s = [\langle c, d\rangle]$. Suppose $r \cdot_Q s = 0_Q$. Then

$$[\langle ac, bd\rangle] = [\langle 0, 1\rangle]$$

that is, $ac = 0$. Hence $a = 0$ or $c = 0$, which means $r = 0_Q$ or $s = 0_Q$.

Exercise 12 This follows from Theorem 5QJ(a) with $s = 0_Q$ and $t = -r$.

Exercise 13 Let $a, b, c \in \mathbb{Z}$. If $a +_Z c = b +_Z c$ then

$$\begin{aligned}
a +_Z c +_Z (-c) &= b +_Z c +_Z (-c) \\
\therefore a +_Z 0 &= b +_Z 0 && \text{(Theorem 5ZD(b))} \\
\therefore a &= b && \text{(Theorem 5ZD(a))}
\end{aligned}$$

Exercise 14 Suppose $p <_Q s$. Let $r = (p +_Q s)/2$. Then

$$\begin{aligned}
p &<_Q s \\
\therefore 2p &<_Q p +_Q s \\
\therefore p &<_Q (p +_Q s)/2 \\
&= r \\
p &<_Q s \\
\therefore p +_Q s &<_Q 2s \\
\therefore (p +_Q s)/2 &<_Q s \\
\therefore r &<_Q s
\end{aligned}$$

5.3 Real Numbers

Exercise 15 PROOF:
⟨1⟩1. $\bigcup A$ is closed downwards.
  ⟨2⟩1. LET: $q \in \bigcup A$ and $p < q$
  ⟨2⟩2. PICK $x \in A$ such that $q \in x$
  ⟨2⟩3. $p \in x$
    PROOF: Since x is closed downwards.
  ⟨2⟩4. $p \in \bigcup A$
⟨1⟩2. $\bigcup A$ has no largest element.
  ⟨2⟩1. LET: $q \in \bigcup A$
  ⟨2⟩2. PICK $x \in A$ such that $q \in x$
  ⟨2⟩3. PICK $r \in x$ such that $q < r$
    PROOF: Since x has no largest element.
  ⟨2⟩4. $r \in \bigcup A$
□

Exercise 16 PROOF:
$\langle 1 \rangle 1$. LET: $q \in x +_R y$
$\langle 1 \rangle 2$. PICK rationals $a \in x$ and $b \in y$ such that $q = a + b$
$\langle 1 \rangle 3$. PICK $a' \in x$ and $b' \in y$ such that $a < a'$ and $b < b'$
PROOF: Since x and y each have no largest element.
$\langle 1 \rangle 4$. $q < a' + b' \in x +_R y$
$\square$

Exercise 17 If $b < 0$ we can take $k = 0$. If $b \geq 0$ then there is a natural number n such that $b = E(n)$; take $k = n^+$. Then $b < ak$ since $1 \leq a$ and $b < k$.

Exercise 18 Let $p = [\langle a, b \rangle]$ and $r = [\langle c, d \rangle]$ where a, b and d are positive. By Exercise 17, there exists a natural number k such that $bc < adE(k)$. Therefore $r < p \cdot E(E(k))$.

Exercise 19 Pick a rational $a \in x$ (which we can do since $x \neq \emptyset$). We first prove that there exists a natural number k such that $a + kp \notin x$.

Pick a rational $b \notin x$ (which we can do since $x \neq \mathbb{Q}$). We have $a < b$ (since x is closed downwards). By Exercise 18, there exists a natural number k such that

$$\begin{aligned} b - a &< kp \\ \therefore a + kp &> b \\ \therefore a + kp &\notin x \end{aligned}$$

Now, let k be the least natural number such that $a + kp \notin x$ (by the Well-Ordering Principle). We have $k \neq 0$ (since $a \in x$); let $k = n^+$. Then we have

$$a + np \in x \qquad a + np + p \notin x$$

Take $q = a + np$.

Exercise 20 We must prove $0 \subseteq x \cup -x$. Let $q \in 0$ and assume $q \notin x$. Then $q < 0$ and $-0 = 0 \notin x$, so $q \in -x$.

Exercise 21 PROOF:
$\langle 1 \rangle 1$. LET: x, y be real numbers with $x < y$
$\langle 1 \rangle 2$. PICK $r \in y$ such that $r \notin x$
$\langle 1 \rangle 3$. PICK $s \in y$ such that $r < s$
PROVE: $x < E(s) < y$
$\langle 1 \rangle 4$. $x \subseteq E(s)$
PROOF: If $p \in x$ then $p < r < s$
$\langle 1 \rangle 5$. $x \neq E(s)$
PROOF: Since $r \in E(s)$ and $r \notin x$
$\langle 1 \rangle 6$. $E(s) \subseteq y$

PROOF: Since y is closed downwards.
$\langle 1 \rangle 7.$ $E(s) \neq y$
PROOF: Since $s \in y$ but $s \notin E(s)$.

Exercise 22 $|x|$ is either x or $-x$, and they are both real numbers.

Chapter 6

Chapter 6 — Cardinal Numbers and the Axiom of Choice

6.1 Equinumerosity

Exercise 1 PROOF:
⟨1⟩1. f is injective.
  ⟨2⟩1. ASSUME: $f(m,n) = f(m',n')$
  ⟨2⟩2. $2^m(2n+1) = 2^{m'}(2n'+1)$
  ⟨2⟩3. $m = m'$
    ⟨3⟩1. ASSUME: w.l.o.g. $m \leq m'$
    ⟨3⟩2. $2n+1 = 2^{m'-m}(2n'+1)$
      PROOF: From ⟨2⟩2 dividing by 2^m.
    ⟨3⟩3. $m' - m = 0$
      PROOF: Since $2^{m'-m}(2n'+1)$ is odd.
  ⟨2⟩4. $2n+1 = 2n'+1$
  ⟨2⟩5. $n = n'$
⟨1⟩2. f is surjective.
  ⟨2⟩1. LET: $n \in \omega$
    ASSUME: $\forall m < n . m \in \operatorname{ran} f$
    PROVE: $n \in \operatorname{ran} f$
  ⟨2⟩2. CASE: n is even
    ⟨3⟩1. LET: k be such that $n = 2k$
    ⟨3⟩2. $n = f(0,k)$
  ⟨2⟩3. CASE: n is odd
    ⟨3⟩1. LET: k be such that $n = 2k+1$
    ⟨3⟩2. LET: $k = f(i,j)$
    ⟨3⟩3. $n = f(i+1,j)$

Proof:

$$\begin{aligned} n &= 2k+1 \\ &= 2(2^i(2j+1)-1)+1 \\ &= 2^{i+1}(2j+1)-2+1 \\ &= 2^{i+1}(2j+1)-1 \end{aligned}$$

□

Exercise 2 Let us call (0) the 0th diagonal, $(1,2)$ the 1st diagonal, $(3,4,5)$ the 2nd diagonal, etc. Then the kth is the set of all positions with coordinates (m,n) such that $m+n=k$.

Therefore, the number $J(m,n)$ at position (m,n) is the $m+1$st number in the $(m+n)$th diagonal. So the number of numbers that come before $J(m,n)$ is

$$(1+2+\cdots+(m+n))+m$$

Therefore, since the natural numbers start at 0,

$$J(m,n) = (1+2+\cdots+(m+n))+m$$

We know $1+2+\cdots+k = k(k+1)/2$. Therefore,

$$\begin{aligned} J(m,n) &= 1/2(m+n)(m+n+1)+m \\ &= 1/2(m^2+2mn+m+n+n^2)+m \\ &= 1/2(m^2+2mn+3m+n+n^2) \\ &= 1/2((m+n)^2+3m+n) \end{aligned}$$

Exercise 3 Define $f:(0,1)\to\mathbb{R}$ by: $f(x)=1/x-2$ if $0<x\leq 1/2$; $f(x)=2-1/(1-x)$ if $1/2<x<1$.

Exercise 4 Define $f:[0,1]\to(0,1)$ by

$$\begin{aligned} f(1/2-1/2^n) &= 1/2-1/2^{n-1} && \text{(for } n \text{ a positive integer)} \\ f(1/2+1/2^n) &= 1/2+1/2^{n-1} && \text{(for } n \text{ a positive integer)} \\ f(x) &= x && \text{(for all other } x\text{)} \end{aligned}$$

Exercise 5

(a) For any set A, the identity function I_A is a bijection between A and A. It is injective because, if $I_A(x)=I_A(y)$ then $x=y$ immediately. It is surjective because for any $y\in I_A$ we have $y=I_A(y)$.

(b) We prove that, if f is a bijection between A and B, then f^{-1} is a bijection between B and A. It is an injective function by Theorem 3F, and maps B onto A by Theorem 3E.

(c) Let f be a bijection between A and B, and g a bijection between A and C. We prove $g \circ f$ is a bijection between A and C.

It is a function from A to C by Theorem 3H.

We prove it is injective. Let $x, y \in A$ and assume $(g \circ f)(x) = (g \circ f)(y)$. Then

$$\begin{aligned} g(f(x)) &= g(f(y)) & \\ \therefore f(x) &= f(y) & (g \text{ is injective}) \\ \therefore x &= y & (f \text{ is injective}) \end{aligned}$$

Now we prove it maps A onto C. Let $c \in C$. Pick $b \in B$ such that $g(b) = c$ (since g is surjective). Pick $a \in A$ such that $f(a) = b$ (since f is injective). Then $(g \circ f)(a) = c$.

6.2 Finite Sets

Exercise 6 Suppose every set of cardinality κ belongs to A. We will prove that every set belongs to $\bigcup A$.

Let x be any set. Pick a set y of cardinality κ. If $x \in y$ then $x \in y \in A$ so $x \in \bigcup A$.

Assume $x \notin y$. Pick an element $z \in y$ (we know y is nonempty because $\kappa \neq 0$). Then $y - \{z\} \cup \{x\}$ has cardinality κ, and so $x \in (y - \{z\} \cup \{x\}) \in A$ hence $x \in \bigcup A$.

Thus, every set is in $\bigcup A$, which we know is impossible by Theorem 2A.

Exercise 7 If f is one-to-one then f is a bijection between A and $\operatorname{ran} f$. So we must have $\operatorname{ran} f = A$, otherwise f would be a bijection between A and a proper subset of A, contradicting the Pigeonhole Principle.

Conversely, suppose $\operatorname{ran} f = A$. Pick a right inverse $h : A \to A$ for f (by Theorem 3J(b). Note: Theorem 3J(b) can in fact be proved for the case B is finite without using the Axiom of Choice.). Now, h is one-to-one by Theorem 3J(a). So $\operatorname{ran} h = A$ by the first paragraph.

We prove f is one-to-one. Let $x, y \in A$ and assume $f(x) = f(y)$. Pick $a, b \in A$ such that $h(a) = x$ and $h(b) = y$. Then

$$\begin{gathered} f(h(a)) = f(h(b)) \\ \therefore a = b \\ \therefore x = y \end{gathered}$$

Exercise 8 Proof:
⟨1⟩1. For any sets A and x, if A is finite then $A \cup \{x\}$ is finite.
  ⟨2⟩1. Case: $x \in A$
    Proof: In this case $A \cup \{x\} = A$.
  ⟨2⟩2. Case: $x \notin A$

PROOF: Then $|A \cup \{x\}| = |A|^+$.

$\langle 1 \rangle 2$. LET: A be a finite set.

$\langle 1 \rangle 3$. For any set B, if $B \approx 0$ then $A \cup B$ is finite.

PROOF: Because $B = \emptyset$ so $A \cup B = A$.

$\langle 1 \rangle 4$. Let n be a natural number. Assume that, for any set B, if $B \approx n$ then $A \cup B$ is finite. Then for any set B, if $B \approx n^+$ then $A \cup B$ is finite.

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: For any set B, if $B \approx n$ then $A \cup B$ is finite.

$\langle 2 \rangle 3$. LET: B be a set.

$\langle 2 \rangle 4$. ASSUME: $B \approx n^+$

$\langle 2 \rangle 5$. PICK a bijection $f : n^+ \to B$

$\langle 2 \rangle 6$. $B - \{f(n)\} \approx n$

$\langle 2 \rangle 7$. $A \cup (B - \{f(n)\})$ is finite.

$\langle 2 \rangle 8$. $A \cup B$ is finite.

PROOF: By $\langle 1 \rangle 1$ since $A \cup B = (A \cup (B - \{f(n)\})) \cup \{f(n)\}$.

$\square$

Exercise 9 PROOF:

$\langle 1 \rangle 1$. LET: A be a finite set.

$\langle 1 \rangle 2$. For any set B, if $B \approx 0$ then $A \times B$ is finite.

PROOF: In this case $A \times B = \emptyset$.

$\langle 1 \rangle 3$. Let n be a natural number. Suppose that, for any set B, if $B \approx n$ then $A \times B$ is finite. Then for any set B, if $B \approx n^+$ then $A \times B$ is finite.

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. ASSUME: For any set B, if $B \approx n$ then $A \times B$ is finite.

$\langle 2 \rangle 3$. LET: B be a set.

$\langle 2 \rangle 4$. ASSUME: $B \approx n^+$

$\langle 2 \rangle 5$. PICK a bijection $f : n^+ \approx B$

$\langle 2 \rangle 6$. $A \times (B - \{f(n)\})$ is finite.

PROOF: By the induction hypothesis $\langle 2 \rangle 2$.

$\langle 2 \rangle 7$. $A \times B$ is finite.

PROOF: By Exercise 8 since $A \times B = (A \times (B - \{f(n)\})) \cup (A \times \{f(n)\})$ and $A \times \{f(n)\}$ is finite because it is equinumerous with A.

$\square$

6.3 Cardinal Arithmetic

Exercise 10 We must show that ${}^{(L \cup M)}K \approx {}^{L}K \times {}^{M}K$ where $L \cap M = \emptyset$.

Define $\Phi : {}^{(L \cup M)}K \to {}^{L}K \times {}^{M}K$ by: $\Phi(f) = \langle f \upharpoonright L, f \upharpoonright M \rangle$.

To show Φ is one-to-one: suppose $\Phi(f) = \Phi(g)$. Then $f \upharpoonright L = g \upharpoonright L$ and $f \upharpoonright M = g \upharpoonright M$. Hence $f(x) = g(x)$ for all $x \in L$ and $f(x) = g(x)$ for all $x \in M$, so $f(x) = g(x)$ for all x, i.e. $f = g$.

To show Φ is surjective: given a function $g : L \to K$ and $h : M \to K$, we have $g \cup h : L \cup M \to K$ and $\Phi(g \cup h) = \langle g, h \rangle$.

Exercise 11 We must show that ${}^M(K \times L) \approx {}^M K \times {}^M L$.

Define $\Phi : {}^M (K \times L) \to {}^M K \times {}^M L$ by: $\Phi(f) = \langle \pi_1 \circ f, \pi_2 \circ f \rangle$, where $\pi_1 : K \times L \to K$ is the function defined by

$$\pi_1(\langle x, y \rangle) = x$$

and $\pi_2 : K \times L \to L$ is the function defined by

$$\pi_2(\langle x, y \rangle) = y \ .$$

To show Φ is one-to-one: suppose $\Phi(f) = \Phi(g)$. For any $x \in M$, we have $\pi_1(f(x)) = \pi_1(g(x))$ and $\pi_2(f(x)) = \pi_2(g(x))$, so $f(x) = g(x)$ by Theorem 3A.

To show Φ is surjective: given $g : M \to K$ and $h : M \to L$, define $f : M \to K \times L$ by $f(x) = \langle g(x), h(x) \rangle$ for $x \in M$. Then $\Phi(f) = \langle g, h \rangle$.

Exercise 12 We have:

$$\begin{aligned} K \cup L &= L \cup K \\ K \cup (L \cup M) &= (K \cup L) \cup M \\ K \times (L \cup M) &= (K \times L) \cup (K \times M) \end{aligned}$$

Exercise 13 Now that we have shown the union of two finite sets is finite, this follows by an easy induction on $|B|$.

Exercise 14 For any set A, let $Perm(A)$ be the set of all permutations of A.

Assume $K \approx L$: we must show $Perm(K) \approx Perm(L)$. Pick a bijection $f : K \to L$. Define $\Phi : Perm(K) \to Perm(L)$ by: $\Phi(g) = f \circ g \circ f^{-1}$. It is easy to show $\Phi(g)$ is a permutation of L whenever g is a permutation of K, and Φ is a bijection.

6.4 Ordering Cardinal Numbers

Exercise 15 Suppose for a contradiction $\mathcal{A}$ is a set and, for every set x, there exists $y \in \mathcal{A}$ such that $x \preccurlyeq y$. Pick $y \in \mathcal{A}$ such that $\mathcal{P} \bigcup \mathcal{A} \preccurlyeq y$. But $y \subseteq \bigcup \mathcal{A}$ so $\mathcal{P} \bigcup \mathcal{A} \preccurlyeq \bigcup \mathcal{A}$, contradicting Cantor's Theorem.

Exercise 16 Define $G : S \to {}^S 2$ by

$$G(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Then G is injective.

Now, assume for a contradiction $F : S \to {}^S 2$ is bijective. Define $g : S \to 2$ by $g(x) = 1 - F(x)(x)$. Then $g(x) \neq F(x)(x)$ for all $x \in S$, so $g \neq F(x)$ for all $x \in S$. Hence $g \notin \operatorname{ran} F$. This contradicts the assumption that F is surjective.

Exercise 17 We have $1 < 2$ but $\aleph_0 + 1 = \aleph_0 + 2 = \aleph_0$.

We have $1 < 2$ but $\aleph_0 \cdot 1 = \aleph_0 \cdot 2 = \aleph_0$.

We have $2 < 3$ but $2^{\aleph_0} = 3^{\aleph_0}$.

We have $2 < 3$ but $\aleph_0^2 = \aleph_0^3 = \aleph_0$.

6.5 Axiom of Choice

Exercise 18 PROOF:

$\langle 1 \rangle 1$. If the Axiom of Choice is true then the statement is true.

PROOF: The statement is a special case of the multiplicative axiom, taking $I = \mathcal{A}$ and $H(X) = X$ for each $X \in \mathcal{A}$.

$\langle 1 \rangle 2$. If the statement is true then the Axiom of Choice is true.

$\langle 2 \rangle 1$. ASSUME: The statement is true.
PROVE: Axiom of choice IV

$\langle 2 \rangle 2$. LET: $\mathcal{A}$ be a set such that each member of $\mathcal{A}$ is a nonempty set, and any two distinct members of $\mathcal{A}$ are disjoint.

$\langle 2 \rangle 3$. PICK a function f with domain $\mathcal{A}$ such that $f(X) \in X$ for all $X \in \mathcal{A}$

$\langle 2 \rangle 4$. LET: $C = \operatorname{ran} f$

$\langle 2 \rangle 5$. $\forall B \in \mathcal{A}. C \cap B = \{f(B)\}$

□

Exercise 19 PROOF:

$\langle 1 \rangle 1$. For $n \in \omega$, let $P(n)$ be the statement: for every set I with $\operatorname{card} I = n$ and function H with domain I such that $H(i)$ is nonempty for each $i \in I$, there exists a function f with domain I such that $\forall i \in I. f(i) \in H(i)$.

$\langle 1 \rangle 2$. $P(0)$ is true

PROOF: Take $f = \emptyset$

$\langle 1 \rangle 3$. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: $P(n)$

$\langle 2 \rangle 3$. LET: I be a set with $\operatorname{card} I = n + 1$

$\langle 2 \rangle 4$. LET: H be a function with domain I such that $H(i)$ is nonempty for each $i \in I$

$\langle 2 \rangle 5$. PICK a bijection $g : n + 1 \approx I$

$\langle 2 \rangle 6$. PICK a function h with domain $g[\![n]\!]$ such that $\forall i \in g[\![n]\!]. h(i) \in H(i)$

$\langle 2 \rangle 7$. PICK $a \in H(g(n))$

$\langle 2 \rangle 8$. LET: $f = h \cup \{(g(n), a)\}$

$\langle 2 \rangle 9$. f is a function with domain I such that $\forall i \in I. f(i) \in H(i)$

□

Exercise 20 PROOF:

$\langle 1 \rangle 1$. PICK a choice function F for A

$\langle 1 \rangle 2$. PICK $a \in A$

$\langle 1 \rangle 3.$ Define the function $f : \omega \to A$ by:

$$f(0) = a$$

$$f(n^+) = F(R^{-1}(f(n)))$$

PROOF: We know $R^{-1}(x)$ is nonempty for all $x \in A$ because $\forall x \in A . \exists y \in A . y R x$.

$\langle 1 \rangle 4.$ $\forall n \in \omega . f(n^+) R f(n)$

□

Exercise 21 PROOF:

$\langle 1 \rangle 1.$ For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 2 \rangle 1.$ LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain.

$\langle 2 \rangle 2.$ Every finite subset of $\bigcup \mathcal{B}$ is a member of $\mathcal{A}$.

$\langle 3 \rangle 1.$ LET: $\{x_1, \ldots, x_n\} \subseteq \bigcup \mathcal{B}$ be finite.

$\langle 3 \rangle 2.$ For $1 \leq i \leq n$, PICK $B_i \in \mathcal{B}_i$ such that $x_i \in B_i$

$\langle 3 \rangle 3.$ PICK m such that $B_1, \ldots, B_n \subseteq B_m$

PROOF: Since $\mathcal{B}$ is a chain.

$\langle 3 \rangle 4.$ $\{x_1, \ldots, x_n\}$ is a finite subset of B_m.

$\langle 3 \rangle 5.$ $\{x_1, \ldots, x_n\} \in \mathcal{A}$

PROOF: Since $B_m \in \mathcal{A}$ so every finite subset of B_m is a member of $\mathcal{A}$.

$\langle 2 \rangle 3.$ $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 1 \rangle 2.$ Q.E.D.

PROOF: By Zorn's Lemma.

□

Exercise 22 PROOF:

$\langle 1 \rangle 1.$ If the Axiom of Choice is true then the statement is true.

$\langle 2 \rangle 1.$ ASSUME: The Axiom of Choice

$\langle 2 \rangle 2.$ LET: A be a set.

$\langle 2 \rangle 3.$ LET: $R = \{\langle x, y \rangle : y \in A, x \in t\}$

$\langle 2 \rangle 4.$ PICK a function $F \subseteq R$ such that $\operatorname{dom} F = \operatorname{dom} R$

$\langle 2 \rangle 5.$ $\operatorname{dom} R = \bigcup A$

$\langle 2 \rangle 6.$ $\forall x \in \bigcup A . x \in F(x) \in A$

$\langle 1 \rangle 2.$ If the statement is true then the Axiom of Choice is true.

$\langle 2 \rangle 1.$ ASSUME: the statement

$\langle 2 \rangle 2.$ LET: R be a relation

$\langle 2 \rangle 3.$ LET: $A = \{\{\langle 0, x \rangle, \langle 1, y \rangle\} : x R y\}$

$\langle 2 \rangle 4.$ PICK a function F with domain $\bigcup A$ such that $\operatorname{dom} F = \bigcup A$ and $\forall x \in \bigcup A . x \in F(x) \in A$

$\langle 2 \rangle 5.$ LET: $H = \{\langle x, y \rangle \mid x \in \operatorname{dom} R, F(x) = \{\langle 0, x \rangle, \langle 1, y \rangle\}\}$

$\langle 2 \rangle 6.$ H is a function, $H \subseteq R$, $\operatorname{dom} H = \operatorname{dom} R$

□

Exercise 23

$\langle 1 \rangle 1.$ $g[\![0]\!] = h(0)$

PROOF: Both are equal to $\emptyset$.

$\langle 1\rangle 2.$ $\forall n \in \omega . g[\![n]\!] = h(n) \Rightarrow g[\![n^+]\!] = h(n^+)$
$\langle 2\rangle 1.$ Let: $n \in \omega$
$\langle 2\rangle 2.$ Assume: $g[\![n]\!] = h(n)$
$\langle 2\rangle 3.$ $g[\![n^+]\!] = h(n^+)$
Proof:

$$\begin{aligned} h(^+) &= h(n) \cup \{F(A - h(n))\} \\ &= g[\![n]\!] \cup \{g(n)\} \\ &= g[\![n^+]\!] \end{aligned}$$

Exercise 24 Let $\{\kappa_i\}_{i\in I}$ be a family of cardinal numbers. For $i \in I$, let K_i be a set such that $\operatorname{card} K_i = \kappa_i$.

We define $\sum_{i\in I} \kappa_i$ to be $\operatorname{card}\{\langle i, x\rangle : i \in I, x \in K_i\}$

We define $\prod_{i\in I} \kappa_i$ to be $\operatorname{card}\{f : f \text{ is a function}, \operatorname{dom} f = I, \forall i \in I. f(i) \in K_i\}$.

Exercise 25 Proof:
$\langle 1\rangle 1.$ Assume: for a contradiction $\forall n \in \omega . B \not\subseteq S(n)$
$\langle 1\rangle 2.$ Pick a function $b : \omega \to B$ such that $\forall n \in \omega . b(n) \notin S(n)$
Proof: By the Axiom of Choice.
$\langle 1\rangle 3.$ Let: $B' = \{b(n) : n \in \omega\}$
$\langle 1\rangle 4.$ B' is infinite.
$\langle 2\rangle 1.$ Assume: for a contradiction B' is finite.
$\langle 2\rangle 2.$ There exists N such that $\forall n > N . \exists k \leq N . b(n) = b(k)$
$\langle 2\rangle 3.$ Pick $M > N$ such that $\forall k \leq N . b(k) \in S(M)$
Proof: For $k \leq N$ there exists n_k such that $b(k) \in S(n_k)$. Take M to be the largest of these numbers and $N + 1$.
$\langle 2\rangle 4.$ $b(M) \in S(M)$
Proof: Since $b(M) = b(k)$ for some $k \leq N$.
$\langle 2\rangle 5.$ Q.E.D.
Proof: This contradicts $\langle 1\rangle 2$.
$\langle 1\rangle 5.$ Pick n such that $B' \cap S(n)$ is infinite.
$\langle 1\rangle 6.$ Pick $m > n$ such that $b(m) \in B' \cap S(n)$
Proof: There must be some m otherwise $B' \cap S(n) \subseteq \{b(0), b(1), \ldots, b(n)\}$ would be finite.
$\langle 1\rangle 7.$ $b(m) \in S(m)$
Proof: Since $S(n) \subseteq S(m)$.
$\langle 1\rangle 8.$ Q.E.D.
Proof: This contradicts $\langle 1\rangle 2$.
□

6.6 Countable Sets

Exercise 26 Proof:
$\langle 1\rangle 1.$ Pick a set K of cardinality κ

$\langle 1 \rangle 2$. For all $X \in \mathcal{A}$, there exists an injective function $X \to K$

$\langle 1 \rangle 3$. PICK a function F with domain $\mathcal{A}$ such that, for all $X \in A$, $F(X)$ is an injective function $X \to K$

PROOF: By the Axiom of Choice.

$\langle 1 \rangle 4$. PICK a function G with domain $\bigcup \mathcal{A}$ such that, for all $x \in \bigcup \mathcal{A}$, we have $x \in G(x) \in \mathcal{A}$

PROOF: By Exercise 22.

$\langle 1 \rangle 5$. Define $f : \bigcup \mathcal{A} \to \mathcal{A} \times K$ by $f(x) = \langle G(x), F(G(x))(x) \rangle$

$\langle 1 \rangle 6$. f is injective.

$\langle 2 \rangle 1$. LET: $x, y \in \bigcup \mathcal{A}$

$\langle 2 \rangle 2$. ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 3$. $G(x) = G(y)$ and $F(G(x))(x) = F(G(y))(y)$

$\langle 2 \rangle 4$. $F(G(x))(x) = F(G(x))(y)$

$\langle 2 \rangle 5$. $x = y$

PROOF: Since $F(G(x))$ is injective.

$\square$

Exercise 27

(a) Pick a function $f : A \to \mathbb{Q}^2$ such that $f(c) \in c$ for all $c \in A$. Then f is an injection, so $A \preccurlyeq \mathbb{Q}^2$ which is countable.

(b) No: the set of all circles with center $(0, 0)$ is an uncountable set of circles no two of which intersect.

(c) Yes. Pick a function $f : C \to \mathbb{Q}^4$ such that $f(x)$ is a pair of points with rational coordinates, one in each circle of x, for all $x \in C$. Then f is an injection; it is not possible for two points to be in separate circles of two non-intersecting figure-eights. Hence $C \preccurlyeq \mathbb{Q}^4$.

Exercise 28 Let $\mathcal{A} = \{(a, \sqrt{2}) : a < \sqrt{2}\} \cup \{(\sqrt{2}, b) : b > \sqrt{2}\}$. Then every rational is in some member of $\mathcal{A}$ but $\bigcup \mathcal{A} = \mathbb{R} - \{\sqrt{2}\}$.

(Enderton's hint suggests he had a different solution in mind, but I am not sure what it is.)

Exercise 29 For each integer $n \geq 2$, let $B_n = \{x \in A : x > b/n\}$. Then each B_n is finite (B_n cannot have more than $n - 1$ elements because n elements in B_n would have a sum $> b$) and $A = \bigcup_n B_n$. So A is a countable union of finite sets, and therefore countable.

Exercise 30 PROOF:

$\langle 1 \rangle 1$. PICK $a \in A$

$\langle 1 \rangle 2$. Define $f : Sq(A) \to \omega \times^\omega A$ by $f(s) = \langle n, g \rangle$, where n is the length pf s, and $g(i) = s(i)$ for $i < n$, $g(i) = a$ for $i \geq n$

$\langle 1\rangle 3.$ f is injective.
$\langle 1\rangle 4.$ $Sq(A) \preccurlyeq \omega \times^\omega A$
$\langle 1\rangle 5.$ $\operatorname{card} Sq(A) \leq (\operatorname{card} A)^{\aleph_0}$
PROOF:

$$\begin{aligned}
\operatorname{card} Sq(A) &\leq \aleph_0 \cdot (\operatorname{card} A)^{\aleph_0} && (\langle 1\rangle 4) \\
&\leq (\operatorname{card} A)^{\aleph_0} \cdot (\operatorname{card} A)^{\aleph_0} && \text{(Cantor's Theorem)} \\
&= (\operatorname{card} A)^{\aleph_0 + \aleph_0} && \text{(Theorem 6I)} \\
&= (\operatorname{card} A)^{\aleph_0}
\end{aligned}$$

6.7 Arithmetic of Infinite Cardinals

Exercise 31 If f is a one-to-one correspondence between $A \times A$ and A, where $A \subseteq B$, then

$$f \subseteq (A \times A) \times A \subseteq (B \times B) \times B \ .$$

Also $\emptyset \subseteq (B \times B) \times B$. So we can form $\mathcal{H}$ by applying a Subset Axiom to $\mathcal{P}((B \times B) \times B)$.

Exercise 32 The function that maps x to $\{x\}$ is an injection $A \to \mathcal{F}A$, so we have $A \approx \mathcal{F}A$.

For the converse, let $F_n = \{X \in \mathcal{F}A : \operatorname{card} X \leq n\}$ for $n \in \omega$. The function that sends $\langle a_1, \ldots, a_n \rangle$ to $\{a_1, \ldots, a_n\}$ is a surjection $A^n \to F_n$, so we have

$$\operatorname{card} F_n \leq (\operatorname{card} A)^n = \operatorname{card} A$$

by Lemma 6R. Now, $\mathcal{F}A = \bigcup_n F_n$, so

$$\operatorname{card} \mathcal{F}A \leq \aleph_0 \cdot \operatorname{card} A = \operatorname{card} A$$

by the Absorption Law.

Exercise 33 The function that maps a to the sequence of length 1 containing a is an injection $A \to Sq(A)$, so $A \preccurlyeq Sq(A)$.

For the converse, we have $\operatorname{card}({}^n A) = (\operatorname{card} A)^n = \operatorname{card} A$ for any natural number n

$$\begin{aligned}
\operatorname{card} Sq(A) &= \operatorname{card}({}^0 A \cup {}^1 A \cup {}^2 A \cup \cdots) \\
&= \aleph_0 \cdot \operatorname{card} A \\
&= \operatorname{card} A
\end{aligned}$$

by the Absorption Law.

Exercise 34

$$\begin{aligned}
2^\lambda &\leq \kappa^\lambda \\
&\leq (2^\kappa)^\lambda \\
&= 2^{\kappa \cdot \lambda} \\
&= 2^\lambda && \text{(Absorption Law)}
\end{aligned}$$

Exercise 35 For any infinite set of primes A and natural number n, let $f(A,n) = \prod\{p \in A : p \leq n\}$. Let $P(A) = \{f(A,n) : n \in \omega\}$. Let $\mathcal{A}$ be the set of all sets of the form $P(A)$.

The number of infinite sets of primes is $2^{\aleph_0}$ (there are $2^{\aleph_0}$ sets of primes and $\aleph_0$ finite sets of primes by Exercise 32.)

If $P(A) = P(B)$ then $A = B$. (If $p \in A - B$ then $p \mid f(A,p)$ but p does not divide any member of $P(B)$.) So P is an injection from the set of infinite sets of primes into $\mathcal{A}$. Hence $\operatorname{card}\mathcal{A} = 2^{\aleph_0}$.

We now prove that, if $A \neq B$, then $P(A) \cap P(B)$ is finite. Let $p \in A - B$. For $n \geq p$ we have $f(A,n) \notin P(B)$ since $p \mid f(A,n)$ but p does not divide any member of B. Hence $A \cap B \subseteq \{f(A,0), f(A,1), \ldots, f(A,p-1)\}$.

Exercise 36 PROOF:

$\langle 1\rangle 1$. For any set A, there exists a permutation of A with no fixed points.

$\langle 2\rangle 1$. For every natural number n, there exists a permuatation of n with no fixed points.

PROOF: Map i to $i+1$ if $i+1 < n$, and map $n-1$ to 0.

$\langle 2\rangle 2$. For every infinite set A, there exists a permutation of A with no fixed points.

$\langle 3\rangle 1$. PICK a bijection $f : A \approx A \times 2$

$\langle 3\rangle 2$. Define $\pi : A \times 2 \to A \times 2$ by $\pi(x,0) = (x,1)$ and $\pi(x,1) = (x,0)$

$\langle 3\rangle 3$. $f^{-1} \circ \pi \circ f$ is a permutation of A with no fixed point.

$\langle 1\rangle 2$. $\kappa! \leq 2^\kappa$

PROOF: Because the set of permutations of K is a subset of ${}^K K$, where K is a set of cardinality κ.

$\langle 1\rangle 3$. $2^\kappa \leq \kappa!$

$\langle 2\rangle 1$. PICK a set K of cardinality κ

$\langle 2\rangle 2$. LET: $Perm(K)$ be the set of permutations of K.

$\langle 2\rangle 3$. Define $f : \mathcal{P}K \to Perm(K)$ as follows. Given $A \subseteq \mathcal{P}K$, pick a permutation π_{K-A} of $K - A$ with no fixed point. Then $f(A) = I_A \cup \pi_{K-A}$

$\langle 2\rangle 4$. f is injective

PROOF: The function that maps a permutation to its set of fixed points is a left inverse.

$\langle 2\rangle 5$. $2^\kappa \leq \kappa!$

$\square$

Chapter 7

Chapter 7 — Orderings and Ordinals

7.1 Partial Orderings

Exercise 1

(a) No we cannot. Let $A = \mathcal{P}3$ and $B = \omega$. Let $<_A = \subset_3$ and $<_B$ be the usual ordering on ω. Define $f : A \to B$ by: $f(X) = \operatorname{card} X$. Then $X \subset_2 Y \Rightarrow \operatorname{card} X < \operatorname{card} Y$ but f is not one-to-one because $f(\{0\}) = f(\{1\}) = 1$.

(b) No we cannot. With the same example, we have $f(\{0\}) < f(\{1,2\})$ but $\{0\} \not\subset \{1,2\}$.

Exercise 2 We show R^{-1} is transitive. Suppose $xR^{-1}y$ and $yR^{-1}z$. Then zRy and yRx, so zRx because R is transitive. Hence $xR^{-1}z$.

We now show R^{-1} is irreflexive. For any x, we have $\langle x, x\rangle \notin R$, so $\langle x, x\rangle \notin R^{-1}$.

Exercise 3 The proof is by induction on n.

The only linear ordering on $\emptyset$ is $\emptyset$, which has 0 pairs.

Suppose that, whenever $\operatorname{card} S = n$, then every linear ordering on S has $1/2n(n-1)$ pairs. Let S be a set of cardinality $n+1$. Let $<$ be a linear ordering on S.

Pick an element $a \in S$ and let $T = S - \{a\}$. Then $< \cap (T \times T)$ is a linear ordering on T, hence has $1/2n(n-1)$ pairs. Now, for every $x \in T$, exactly one of $\langle x, a\rangle$ and $\langle a, x\rangle$ is in $<$. Hence $<$ has n pairs that are not in $< \cap (T \times T)$. So

$$\operatorname{card} <= 1/2n(n-1) + n = 1/2n(n+1) \ .$$

7.2 Well Orderings

Exercise 4 PROOF:
⟨1⟩1. R is transitive.
⟨2⟩1. ASSUME: mRn and nRp.
⟨2⟩2. CASE: $f(m) < f(n)$
PROOF: In this case $f(m) < f(p)$ so mRp.
⟨2⟩3. CASE: $f(m) = f(n)$ and $m < n$.
⟨3⟩1. CASE: $f(n) < f(p)$
PROOF: In this case $f(m) < f(p)$ so mRp.
⟨3⟩2. CASE: $f(n) = f(p)$ and $n < p$.
PROOF: In this case $f(m) = f(p)$ and $m < p$ so mRp.
⟨1⟩2. R satisfies trichotomy on P.
⟨2⟩1. LET: $m, n \in P$
⟨2⟩2. Exactly one of $f(m) < f(n)$, $f(n) < f(m)$, $f(n) = f(m)$ holds.
⟨2⟩3. Exactly one of $m < n$, $n < m$, $n = m$ holds.
⟨2⟩4. Exactly one of $f(m) < f(n)$, $(f(m) = f(n)\ \&\ m < n)$, $(f(m) = f(n)\ \&\ m = n)$, $(f(m) = f(n)\ \&\ n < m)$, $f(n) < f(m)$ holds.
⟨2⟩5. Exactly one of mRn, $m = n$, nRm holds.
⟨1⟩3. Every nonempty subset of P has an R-least element.
⟨2⟩1. LET: $A \subseteq P$ be nonempty.
⟨2⟩2. LET: k be the least element of $f(A)$.
⟨2⟩3. LET: n be the least element of $f^{-1}(k) \cap A$.
⟨2⟩4. n is the R-least element of A.
□

$\langle P, R\rangle$ resembles Fig. 45 (d).

Exercise 5 PROOF:
⟨1⟩1. LET: $x \in A$
⟨1⟩2. ASSUME: for a contradiction $f(x) < x$
⟨1⟩3. Define $g : \omega \to A$ by $g(0) = x$ and $g(n^+) = f(g(n))$ for all $n \in \omega$
⟨1⟩4. $\forall n \in \omega . g(n^+) < g(n)$
PROOF: By induction on n using ⟨1⟩2 and the hypothesis.
⟨1⟩5. Q.E.D.
PROOF: This contradicts Theorem 7B.
□

Exercise 6 PROOF:
⟨1⟩1. For all $x \in S$ that is not greatest, there exists $y \in S$ and $q \in \mathbb{Q}$ such that $x < q < y$ and there is no $z \in S$ such that $x < z < y$
⟨1⟩2. PICK a function $f : S \to \mathbb{Q}$ such that $\forall x \in S . x < f(x)$ and, if x is not greatest, then $f(x) < y$ where y is the next element in S.
⟨1⟩3. f is injective.
⟨1⟩4. $S \preccurlyeq \mathbb{Q}$
□

Exercise 7

(a) We have $F(t) = C \cup \bigcup \bigcup \mathrm{ran}(F \upharpoonright t)$ for all $t \in \omega$. So:

$$\begin{aligned} F(0) &= C \cup \bigcup \bigcup \mathrm{ran}\, \emptyset \\ &= C \\ F(1) &= C \cup \bigcup \bigcup \mathrm{ran}(F \upharpoonright 0) \\ &= C \cup \bigcup \bigcup \{C\} \\ &= C \cup \bigcup C \\ F(2) &= C \cup \bigcup \bigcup \{C, C \cup \bigcup C\} \\ &= C \cup \bigcup (C \cup \bigcup C) \\ &= C \cup \bigcup C \cup \bigcup \bigcup C \end{aligned}$$

We guess:

$$F(n) = C \cup \bigcup C \cup \cdots \cup \overbrace{n}{} \bigcup \bigcup \cdots \bigcup C$$

(b) Proof:
⟨1⟩1. Let: $a \in F(n)$
⟨1⟩2. $a \in \bigcup \mathrm{ran}(F \upharpoonright n^+)$
⟨1⟩3. $a \subseteq \bigcup \bigcup \mathrm{ran}(F \upharpoonright n^+)$
⟨1⟩4. $a \subseteq F(n^+)$
□

(c) Proof:
⟨1⟩1. $\overline{C}$ is a transitive set.
  ⟨2⟩1. Let: $x \in y \in \overline{C}$
  ⟨2⟩2. Pick $n \in \omega$ such that $y \in F(n)$
  ⟨2⟩3. $x \in F(n^+)$
    Proof: By (b).
  ⟨2⟩4. $x \in \overline{C}$
⟨1⟩2. $C \subseteq \overline{C}$
  ⟨2⟩1. Since $C = F(0)$
□

7.3 Replacement Axioms

Exercise 8 Let $P(x)$ be a formula not containing B. We prove the statement

$$\forall c \exists B \forall x (x \in B \Leftrightarrow x \in c \;\&\; P(x)) \; .$$

Let $Q(x, y)$ be the formula $P(x) \wedge y = x$. Now we reason as follows.

Let c be any set. Then we have

$$(\forall x \in c)\forall y_1 \forall y_2 (Q(x, y_1) \ \&\ Q(x, y_2) \Rightarrow y_1 = y_2)$$

Then, by a Replacement Axiom, there exists a set B such that

$$\forall y (y \in B \Leftrightarrow (\exists x \in c) Q(x, y)) \ .$$

This is equivalent to $\forall x(x \in B \Leftrightarrow x \in c \ \&\ P(x))$.

Exercise 9 Let a and b be sets. Let $P(x, y)$ be the formula $(x = \emptyset \ \&\ y = a)$ or $(x = \mathcal{P}\emptyset \ \&\ y = b)$. Then we have $(\forall x \in \mathcal{P}\mathcal{P}\emptyset)\forall y_1 \forall y_2 (P(x, y_1) \ \&\ P(x, y_2) \Rightarrow y_1 = y_2)$, hence there exists a set c such that

$$\forall y (y \in c \Leftrightarrow (\exists x \in \mathcal{P}\mathcal{P}\emptyset) P(x, y))$$

The members of c are just a and b.

7.4 Epsilon-Images

Exercise 10

(a) Let n be a natural number. Let α be its epsilon-image, and $E : n \to \alpha$ be as in the definition of epsilon-image.

We prove $\forall x \in n.E(x) = x$ by strong induction on x. Let $x \in n$ and assume $\forall y \in x.E(y) = y$. Then

$$\begin{aligned} E(x) &= \{E(y) : y \in x\} \\ &= \{y : y \in x\} \\ &= x \end{aligned}$$

Hence

$$\begin{aligned} \alpha &= \{E(x) : x \in n\} \\ &= \{x : x \in n\} \\ &= n \end{aligned}$$

(b) Similarly the ϵ-image of ω is ω.

Exercise 11

(a) Let R be the ordering given in the question. Thus xRy iff (x and y are nonnegative and $x < y$) or (x and y are both negative and $y < x$) or (x is nonnegative and y is negative).

PROOF:
⟨1⟩1. R is transitive
  ⟨2⟩1. ASSUME: xRy and yRz
  ⟨2⟩2. CASE: x and y are nonnegative and $x < y$
    ⟨3⟩1. CASE: z is nonnegative and $y < z$
      PROOF: In this case x and z are nonnegative and $x < z$.
    ⟨3⟩2. CASE: z is negative
      PROOF: In this case x is nonnegative and z is negative.
  ⟨2⟩3. CASE: x and y are both negative and $y < x$
    PROOF: We must have z is negative and $z < y$, hence $z < x$.
  ⟨2⟩4. CASE: x is nonnegative and y is negative
    PROOF: We must have z is negative.
⟨1⟩2. R satisfies trichotomy on $\mathbb{Z}$
  ⟨2⟩1. LET: $x, y \in \mathbb{Z}$
  ⟨2⟩2. CASE: x and y are nonnegative.
    PROOF: Exactly one of $x < y$, $x = y$, $y < x$ holds.
  ⟨2⟩3. CASE: x is nonnegative and y is negative.
    PROOF: In this case $x < y$.
  ⟨2⟩4. CASE: x is negative and y is nonnegative.
    PROOF: In this case $y < x$.
  ⟨2⟩5. CASE: x and y are negative.
    PROOF: Exactly one of $x < y$, $x = y$, $y < x$ holds.
⟨1⟩3. R is well-founded
  ⟨2⟩1. LET: $A \subseteq \mathbb{Z}$ be nonempty.
  ⟨2⟩2. CASE: There exists a nonnegative integer in A.
    PROOF: Let n be the least nonnegative element of A. Then n is the R-least element of A.
  ⟨2⟩3. CASE: All elements of A are negative.
    PROOF: Let n be least such that $-n \in A$. Then $-n$ is the R-least element of A.
□

(b)

$$\begin{aligned}
E(3) &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
&= 3 \\
E(-1) &= \omega \\
E(-2) &= \omega^+ \\
\operatorname{ran} E &= \omega \cup \{\omega, \omega^+, \omega^{++}, \ldots\}
\end{aligned}$$

7.5 Isomorphisms

Exercise 12

(a) PROOF:
⟨1⟩1. $<_A$ is irreflexive.
PROOF: For any $x \in A$ we have $f(x) \not<_B f(x)$ so $x \not<_A x$.
⟨1⟩2. $<_A$ is transitive.
PROOF: If $x <_A y$ and $y <_A z$ then $f(x) <_B f(y) <_B f(z)$ hence $f(x) <_B f(z)$ and so $x <_A z$.

(b) For any $x, y \in A$ we have that exactly one of $f(x) <_B f(y)$, $f(x) = f(y)$, $f(y) <_B f(x)$ holds. Hence exactly one of $x <_A y$, $x = y$, $y <_A x$ holds. (Using the fact that $x = y$ iff $f(x) = f(y)$ since f is one-to-one.)

Exercise 13 PROOF:
⟨1⟩1. LET: $\langle A, <_A\rangle$ and $\langle B, <_B\rangle$ be two well-ordered structures.
⟨1⟩2. LET: $f, g : A \to B$ be isomorphisms.
PROVE: $\forall x \in A.f(x) = g(x)$
⟨1⟩3. LET: $x \in A$
⟨1⟩4. ASSUME: $\forall y < x.f(y) = g(y)$
⟨1⟩5. $f(x)$ is the least element in $B - f[\![\operatorname{seg} x]\!]$
⟨2⟩1. $f(x) \notin f[\![\operatorname{seg} x]\!]$
PROOF: Since f is one-to-one.
⟨2⟩2. $\forall b \in B - f[\![\operatorname{seg} x]\!].f(x) \leq b$
⟨3⟩1. LET: $b \in B - f[\![\operatorname{seg} x]\!]$
⟨3⟩2. LET: $a \in A$ be such that $f(a) = b$
PROOF: f is surjective.
⟨3⟩3. $a \notin \operatorname{seg} x$
⟨3⟩4. $x \leq a$
PROOF: By trichotomy
⟨3⟩5. $f(x) \leq b$
⟨1⟩6. $g(x)$ is the least element in $B - g[\![\operatorname{seg} x]\!]$
PROOF: Similar.
⟨1⟩7. $f[\![\operatorname{seg} x]\!] = g[\![\operatorname{seg} x]\!]$
PROOF: By ⟨1⟩4
⟨1⟩8. $f(x) = g(x)$
□

Exercise 14 PROOF:
⟨1⟩1. $\forall a, b \in A.a < b \Rightarrow F(a) \subset F(b)$
⟨2⟩1. LET: $a, b \in A$
⟨2⟩2. ASSUME: $a < b$
⟨2⟩3. $F(a) \subseteq F(b)$
PROOF: If $x \leq a$ then $x \leq b$

$\langle 2\rangle 4.$ $F(a) \neq F(b)$
PROOF: Since $b \in F(b)$ but $b \notin F(a)$
$\langle 1\rangle 2.$ $\forall a, b \in A. F(a) \subset F(b) \Rightarrow a < b$
PROOF: We cannot have $b < a$ or $b = a$ (as then $F(b) \subset F(a)$ or $F(b) = F(a)$ by $\langle 1\rangle 1$), so $a < b$ by trichotomy.
$\langle 1\rangle 3.$ F is one-to-one
PROOF: If $F(a) = F(b)$ then we cannot have $a < b$ or $b < a$ by $\langle 1\rangle 1$, so $a = b$ by trichotomy.
$\langle 1\rangle 4.$ F maps A onto $\operatorname{ran} F$
PROOF: By definition of $\operatorname{ran} F$.
$\square$

7.6 Ordinal Numbers

Exercise 15

(a) PROOF:
$\langle 1\rangle 1.$ ASSUME: $f : A \to \operatorname{seg} t$ is an isomorphism
$\langle 1\rangle 2.$ Define $g : \omega \to A$ by recursion:

$$g(0) = t$$
$$g(n^+) = f(g(n)) \qquad (n \in \omega)$$

$\langle 1\rangle 3.$ $\forall n \in \omega. g(n^+) < g(n)$
$\langle 2\rangle 1.$ $g(0^+) < g(0)$
PROOF: Since $g(0^+) = f(t) \in \operatorname{seg} t$ so $g(0^+) < t = g(0)$.
$\langle 2\rangle 2.$ $\forall n \in \omega. (g(n^+) < g(n) \Rightarrow g(n^{++}) < g(n^+))$
$\langle 3\rangle 1.$ LET: $n \in \omega$
$\langle 3\rangle 2.$ ASSUME: $g(n^+) < g(n)$
$\langle 3\rangle 3.$ $f(g(n^+)) < f(g(n))$
PROOF: Since f is an isomorphism.
$\langle 3\rangle 4.$ $g(n^{++}) < g(n^+)$
$\langle 1\rangle 4.$ Q.E.D.
PROOF: This contradicts Theorem 7B.
$\square$

(b) If two of them hold then we have a well-ordered set isomorphic with an initial segment, contradicting part (a):

If $A \cong B$ and $A \cong \operatorname{seg} b$ then $B \cong \operatorname{seg} b$.

If $A \cong B$ and $\operatorname{seg} a \cong B$ then $A \cong \operatorname{seg} a$.

Now assume $A \cong \operatorname{seg} b$ and $\operatorname{seg} a \cong B$. Let $f : A \cong \operatorname{seg} b$ and $g : \operatorname{seg} a \cong B$ be isomorphisms. Let $b_0 = f(a)$. Then $f \restriction \operatorname{seg} a : \operatorname{seg} a \cong \operatorname{seg} b_0$ and so $B \cong \operatorname{seg} b_0$.

Exercise 16 Suppose $\alpha \in \beta$. We first prove that $\beta \notin \alpha^+$.

If $\beta \in \alpha^+$ then $\beta \in \alpha$ or $\beta = \alpha$. In either case we have $\alpha \in \alpha$, which is impossible.

So $\beta \notin \alpha^+$. Therefore $\alpha^+ \underline{\in} \beta$, and so $\alpha^+ \in \beta^+$.

Now, suppose $\alpha \neq \beta$. Then $\alpha \in \beta$ or $\beta \in \alpha$. Hence $\alpha^+ \in \beta^+$ or $\beta^+ \in \alpha^+$, and in either case $\alpha^+ \neq \beta^+$.

Exercise 17 Suppose for a contradiction $\alpha \in \beta$. Then A is isomorphic to $\operatorname{seg}_B b$ for some $b \in B$. Let $f : A \to \operatorname{seg} b$ be an isomorphism.

We have $f \restriction B : B \to \operatorname{seg}_B b$. Now, define $g : \omega \to B$ by

$$\begin{aligned} g(0) &= b \\ g(n^+) &= f(g(n)) \end{aligned}$$

Then $g(n^+) < g(n)$ for all $n \in \omega$, contradicting Theorem 7B.

Exercise 18 Suppose first $\bigcup S \in S$. For all $\alpha \in S$ we have $\alpha \subseteq \bigcup S$ and so $\alpha \in \bigcup S$, and so $\bigcup S$ is the greatest element of S.

Suppose now $\bigcup S \notin S$. Suppose for a contradiction $\alpha \in S$ is the greatest element of S. We have $\alpha \subseteq \bigcup S$ (because $\alpha \in S$). Also for all $\beta \in S$ we have $\beta \subseteq \alpha$, hence $\bigcup S \subseteq \alpha$. Thus $\bigcup S = \alpha \in S$, which is a contradiction.

So if $\bigcup S \notin S$ then S has no greatest element. Therefore S cannot be the successor of any ordinal, because α is the greatest element of α^+ for any α.

Exercise 19 By Theorem 7B, every linear ordering on a finite set is a well ordering.

If $<$ and $\prec$ are two linear orderings on the same set A, we cannot have that $(A, <)$ is isomorphic to $(\operatorname{seg} a, \prec)$ for any $a \in A$, because then we would have a finite set bijective with a proper subset of itself.

So by Theorem 7E we must have $\langle A, < \rangle \cong \langle A, \prec \rangle$.

Exercise 20 Let R be a well ordering on the set S. Assume S is infinite; we will prove R^{-1} is not a well-ordering on S.

Define $g : \omega \to S$ by: $g(n)$ is the least element of $S - g[\![n]\!]$. For each n, we know $S - g[\![n]\!]$ is nonempty because S is infinite.

Then $g[\![\omega]\!]$ is a nonempty subset of S that has no R^{-1}-least element (no R-greatest element), so R^{-1} is not a well ordering on S.

Exercise 21 Let $\mathcal{A} = \{C \in \mathcal{P}A : <^\circ \text{ is a linear ordering on } C\}$.

We prove that, for every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$.

Let $\mathcal{B} \subseteq \mathcal{A}$ be a chain. Let $x, y \in \bigcup \mathcal{B}$. Pick $C, D \in \mathcal{B}$ such that $x \in C$ and $y \in D$. Then either $C \subseteq D$ or $D \subseteq C$; assume without loss of generality $C \subseteq D$. We have $x, y \in D$, and so exactly one of $x < y$, $x = y$, $y < x$ holds. Thus, $<^\circ$ linearly orders $\bigcup \mathcal{B}$, i.e. $\bigcup \mathcal{B} \in \mathcal{A}$.

Hence by Zorn's Lemma $\mathcal{A}$ has a maximal element C, say. Now, by hypothesis, C has an upper bound m. We prove m is maximal in A.

Let $x \in A$ and suppose $m \leq x$. Then $C \cup \{m, x\}$ is linearly ordered by $<^\circ$, and so $C = C \cup \{m, x\}$ by maximality of C. Hence $x \in C$ and so $x \leq m$, hence $x = m$. Thus, m is maximal in A.

7.7 Debts Paid

Exercise 22 Let A be any set. Let $\mathcal{A}$ be the set of all pairs $\langle B, R\rangle$ where $B \subseteq A$ and R is a well ordering on B, and define $<$ on $\mathcal{A}$ by: $\langle B, R\rangle < \langle C, S\rangle$ iff B is an initial segment of C and $R = S \cap B^2$.

It is easy to see that $<$ is a partial ordering on $\mathcal{A}$

We prove that, if $\mathcal{C} \subseteq \mathcal{A}$ and $<$ is a linear ordering on $\mathcal{C}$, then $\mathcal{C}$ has an upper bound in $\mathcal{A}$. Let $B = \bigcup\{C : \exists S.\langle C, S\rangle \in \mathcal{C}\}$ and $R = \bigcup\{S : \exists C.\langle C, S\rangle \in \mathcal{C}\}$. We prove that R well orders B. It is then easy to see that $\langle B, R\rangle$ is an upper bound for $\mathcal{C}$ is $\mathcal{A}$.

PROOF:
$\langle 1\rangle 1.$ R is transitive.
  $\langle 2\rangle 1.$ ASSUME: xRy and yRz
  $\langle 2\rangle 2.$ PICK $\langle C, S\rangle, \langle D, T\rangle \in \mathcal{C}$ such that xSy and yTz
  $\langle 2\rangle 3.$ $\langle C, S\rangle \leq \langle D, T\rangle$ or $\langle D, T\rangle \leq \langle C, S\rangle$
  $\langle 2\rangle 4.$ ASSUME: w.l.o.g. $\langle C, S\rangle \leq \langle D, T\rangle$
  $\langle 2\rangle 5.$ xTy and yTz
  $\langle 2\rangle 6.$ xTz
  $\langle 2\rangle 7.$ xRz
$\langle 1\rangle 2.$ R is irreflexive.
  $\langle 2\rangle 1.$ ASSUME: for a contradiction xRx
  $\langle 2\rangle 2.$ PICK $\langle C, S\rangle \in \mathcal{C}$ such that xSx
  $\langle 2\rangle 3.$ This is a contradiction.
$\langle 1\rangle 3.$ R satisfies trichotomy.
  $\langle 2\rangle 1.$ LET: $x, y \in B$
  $\langle 2\rangle 2.$ PICK $\langle C, S\rangle, \langle D, T\rangle \in \mathcal{C}$ such that $x \in C$ and $y \in D$
  $\langle 2\rangle 3.$ ASSUME: w.l.o.g. $\langle C, S\rangle \leq \langle D, T\rangle$
  $\langle 2\rangle 4.$ $x, y \in D$
  $\langle 2\rangle 5.$ xTy or yTx
  $\langle 2\rangle 6.$ xRy or yRx
$\langle 1\rangle 4.$ Every non-empty subset of B has an R-least element.
  $\langle 2\rangle 1.$ LET: $C \subseteq B$ be nonempty
  $\langle 2\rangle 2.$ PICK $c \in C$
  $\langle 2\rangle 3.$ PICK $\langle D, T\rangle \in \mathcal{C}$ such that $c \in D$
  $\langle 2\rangle 4.$ LET: x be the T-least element of $C \cap D$
    PROVE: x is R-least in C
  $\langle 2\rangle 5.$ LET: $y \in C$
  $\langle 2\rangle 6.$ PICK $\langle E, U\rangle \in \mathcal{C}$ such that $y \in E$
  $\langle 2\rangle 7.$ $\langle D, T\rangle \leq \langle E, U\rangle$ or $\langle E, U\rangle \leq \langle D, T\rangle$
  $\langle 2\rangle 8.$ CASE: $\langle D, T\rangle \leq \langle E, U\rangle$
    $\langle 3\rangle 1.$ xUy or $x = y$
      PROOF:
      $\langle 4\rangle 1.$ ASSUME: for a contradiction yUx
      $\langle 4\rangle 2.$ $y \in D$ and yTx
        PROOF: Since D is an initial segment of E and $T = U \cap D^2$

⟨4⟩3. Q.E.D.

PROOF: This contradicts the T-minimality of x.

⟨2⟩9. CASE: $\langle E, U\rangle \leq \langle D, T\rangle$

PROOF: xTy or $x = y$, so xRy or $x = y$.

Hence by Exercise 21 there is a maximal element $\langle B, R\rangle$ in $\mathcal{A}$. We must have $B = A$; for if $a \in A - B$ then $\langle B \cup \{a\}, R \cup \{\langle x, a\rangle : x \in B\}\rangle$ would be a larger element. Hence R is a well ordering on A.

Exercise 23

(i) We must show that α is an initial ordinal. So let $\beta \in \alpha$. Then $\beta \preccurlyeq A$ but $\alpha \not\preccurlyeq A$. Hence $\alpha \not\approx \beta$.

(ii) We know that $\alpha \not\preccurlyeq A$, so $\alpha \not\leq \operatorname{card} A$.

(iii) Let κ be any cardinal greater than $\operatorname{card} A$. Then κ is not dominated by A, so $\kappa \notin \alpha$, and so $\alpha \underline{\in} \kappa$.

Exercise 24 The cardinal number of $\mathcal{P}\alpha$ is larger than α (both as a cardinal and as an ordinal).

Exercise 25 Suppose there exists an ordinal α such that $\neg\phi(\alpha)$. Let α_0 be the least such ordinal. Then we have $\forall x \in \alpha_0 . \phi(x)$ but $\neg\phi(\alpha_0)$. This contradicts the hypothesis.

7.8 Rank

Exercise 26 The proof is by transfinite induction on α. Suppose that α is an ordinal and, for all $\beta \in \alpha$, we have β is grounded and $\operatorname{rank} \beta = \beta$. Then by Theorem 7V(b) we have that α is grounded and

$$\begin{aligned} \operatorname{rank} \alpha &= \bigcup \{(\operatorname{rank} \beta)^+ \mid \beta \in \alpha\} \\ &= \bigcup \{\beta^+ \mid \beta \in \alpha\} && \text{(induction hypothesis)} \end{aligned}$$

So we must show that $\bigcup\{\beta^+ \mid \beta \in \alpha\} = \alpha$.

If $\beta \in \alpha$ then $\beta^+ \underline{\in} \alpha$ so $\beta^+ \subseteq \alpha$. This shows that $\bigcup\{\beta^+ \mid \beta \in \alpha\} \subseteq \alpha$.

If $\beta \in \alpha$ then $\beta \in \beta^+$ so $\beta \in \bigcup\{\beta^+ \mid \beta \in \alpha\}$. This shows that $\alpha \subseteq \bigcup\{\beta^+ \mid \beta \in \alpha\}$

Exercise 27 PROOF:

⟨1⟩1. For natural numbers m and n, we have $\operatorname{rank}\langle m, n\rangle = \max(m, n)^{+++}$

PROOF:

$$\begin{aligned}\operatorname{rank}\{\{m\},\{m,n\}\} &= (\operatorname{rank}\{m\})^+ \cup (\operatorname{rank}\{m,n\})^+ \\ &= (\operatorname{rank} m)^{++} \cup ((\operatorname{rank} m)^+ \cup (\operatorname{rank} n)^+)^+ \\ &= m^{++} \cup (m^+ \cup n^+)^+ && \text{(Exercise 26)} \\ &= \max(m,n)^{++}\end{aligned}$$

$\langle 1\rangle 2$. For any integer a we have $\operatorname{rank} a = \omega$

PROOF: For any natural numbers m and n, we have

$$\begin{aligned}\operatorname{rank}[\langle m,n\rangle] &= \bigcup\{(\operatorname{rank}\langle p,q\rangle)^+ : m+q=n+p\} \\ &= \bigcup\{\max(p,q)^+ : m+q=n+p\} \\ &= \omega\end{aligned}$$

since for any natural number $p > m$ there exists q such that $m+q=n+p$.

$\langle 1\rangle 3$. For any integers a and b we have $\operatorname{rank}\langle a,b\rangle = \omega^{++}$

PROOF:

$$\begin{aligned}\operatorname{rank}\{\{a\},\{a,b\}\} &= (\operatorname{rank}\{a\})^+ \cup (\operatorname{rank}\{a,b\})^+ \\ &= (\operatorname{rank} a)^{++} \cup ((\operatorname{rank} a)^+ \cup (\operatorname{rank} b)^+)^+ \\ &= \omega^{++} \cup (\omega^+ \cup \omega^+)^+ \\ &= \omega^{++}\end{aligned}$$

$\langle 1\rangle 4$. For any rational q we have $\operatorname{rank} q = \omega^{+++}$

PROOF: Since every element of q has rank ω^{++}

$\langle 1\rangle 5$. For any real number r we have $\operatorname{rank} r = \omega^{++++}$

PROOF: Since every element of r has rank ω^{+++}.

$\langle 1\rangle 6$. $\operatorname{rank}\mathbb{R} = \omega^{+++++}$

$\square$

Exercise 28 If $X \in V_\alpha$ then $X \subseteq V_\beta$ for some $\beta \in \alpha$. Hence $\operatorname{rank} X \underline{\in} \beta$ and so $\operatorname{rank} X \in \alpha$.

Conversely, if $\operatorname{rank} X \in \alpha$ then $X \in V_{(\operatorname{rank} X)^+} \subseteq V_\alpha$.

Exercise 29 Direct proofs:

For any set a, there exists $m \in \{a\}$ such that $m \cap \{a\} = \emptyset$. This m must be the set a, so $a \cap \{a\} = \emptyset$, meaning $a \notin a$.

For any sets a and b, there exists $m \in \{a,b\}$ such that $m \cap \{a,b\} = \emptyset$. Now, m is either a or b. If $m = a$ then $a \cap \{a,b\} = \emptyset$ so $b \notin a$. And if $m = b$ then $b \cap \{a,b\} = \emptyset$ so $a \notin b$.

Consequences of part (c):

Assume $a \in a$. Define $f : \omega \to \{a\}$ by $f(n) = a$ for all $n \in \omega$. Then $f(n^+) \in f(n)$ for all n, contradicting (c).

Assume now $a \in b$ and $b \in a$. Define $f : \omega \to \{a,b\}$ by $f(n) = a$ if n is even, $f(n) = b$ if n is odd. Then $f(n^+) \in f(n)$ for all n, contradicting (c).

Exercise 30

$$\begin{aligned}
\operatorname{rank}\{a,b\} &= (\operatorname{rank} a)^+ \cup (\operatorname{rank} b)^+ && \text{(Theorem 7V(b))}\\
&= \max((\operatorname{rank} a)^+, (\operatorname{rank} b)^+)\\
&= \max(\operatorname{rank} a, \operatorname{rank} b)^+
\end{aligned}$$

We have

$$\begin{aligned}
a &\subseteq V_{\operatorname{rank} a}\\
\therefore \mathcal{P}a &\subseteq \mathcal{P}V_{\operatorname{rank} a}\\
&= V_{(\operatorname{rank} a)^+}\\
\therefore \operatorname{rank} \mathcal{P}a &\underline{\in} (\operatorname{rank} a)^+\\
a &\in \mathcal{P}a\\
\therefore \operatorname{rank} a &\in \operatorname{rank} \mathcal{P}a\\
\therefore \operatorname{rank} \mathcal{P}a &= (\operatorname{rank} a)^+
\end{aligned}$$

Now, for all $x \in \bigcup a$, there exists y such that $x \in y \in a$. Hence

$$\operatorname{rank} x \in \operatorname{rank} y \in \operatorname{rank} a \quad .$$

$$\therefore (\operatorname{rank} x)^+ \in \operatorname{rank} a \quad .$$

So $\operatorname{rank} a$ is an upper bound for $\{(\operatorname{rank} x)^+ : x \in \bigcup a\}$, and so

$$\operatorname{rank} \bigcup a \underline{\in} \operatorname{rank} a \quad .$$

Exercise 31

(a) If $A \approx B$ and nothing of rank less than $\operatorname{rank} B$ is equinumerous to B, then $\operatorname{rank} B \underline{\in} \operatorname{rank} A$, and so $B \in V_{(\operatorname{rank} A)^+}$. So we can construct the set $\operatorname{kard} A$ by applying a Subset Axiom to $V_{(\operatorname{rank} A)^+}$.

(b) There exists a set of rank $\operatorname{rank} A$ that is equinumerous with A (namely A!). Let μ be the least ordinal $\leq \operatorname{rank} A$ such that there exists a set of rank μ that is equinumerous with A. Pick a set B of rank μ such that $B \approx A$. Then $B \in \operatorname{kard} A$.

(c) Suppose $\operatorname{kard} A = \operatorname{kard} B$. Pick $C \in \operatorname{kard} A$. Then $C \approx A$ and $C \approx B$, so $A \approx B$.

Conversely, suppose $A \approx B$. Then we have ($A \approx C$ and nothing of rank less than $\operatorname{rank} C$ is equinumerous with C) iff ($B \approx C$ and nothing of rank less than $\operatorname{rank} C$ is equinumerous with C), i.e. $\operatorname{kard} A = \operatorname{kard} B$.

Exercise 32 Similar to Exercise 31.

Exercise 33 Suppose for a contradiction D is not a subset of B. Then $D - B$ is nonempty. So by the Regularity Axiom, there exists $m \in D - B$ such that $m \cap (D - B) = \emptyset$. Now, for all $x \in m$, we have $x \in D$ (since D is a transitive set) and $x \notin D - B$, so we must have $x \in X$; that is, $m \subseteq B$. But then $m \in B$, which is a contradiction.

Exercise 34 PROOF:
$\langle 1 \rangle 1$. ASSUME: $\{x, \{x, y\}\} = \{u, \{u, v\}\}$
$\langle 1 \rangle 2$. $x = u$ or $x = \{u, v\}$
$\langle 1 \rangle 3$. $u = x$ or $u = \{x, y\}$
$\langle 1 \rangle 4$. $x \neq \{u, v\}$
  $\langle 2 \rangle 1$. ASSUME: for a contradiction $x = \{u, v\}$
  $\langle 2 \rangle 2$. $u = x$ or $u = \{x, y\}$
  $\langle 2 \rangle 3$. CASE: $u = x$
    PROOF: In this case $x = u \in \{u, v\} = x$ contradicting Theorem 7X(a).
  $\langle 2 \rangle 4$. CASE: $u = \{x, y\}$
    PROOF: In this case $u \in x$ and $x \in u$ contradicting Theorem 7X(b).
$\langle 1 \rangle 5$. $x = u$
$\langle 1 \rangle 6$. $\{x, y\} = \{u, v\}$
  PROOF: We cannot have $\{x, y\} = u$ because then we would have $x \in x$ contradicting Theorem 7X(a).
$\langle 1 \rangle 7$. $y = u$ or $y = v$
$\langle 1 \rangle 8$. $v = x$ or $v = y$
$\langle 1 \rangle 9$. If $y = u$ and $v = x$ then $y = v$
$\langle 1 \rangle 10$. $y = v$
  PROOF: Checking all the cases in $\langle 1 \rangle 7$ and $\langle 1 \rangle 8$.
$\square$

Exercise 35 Suppose $a^+ = b^+$. Then $a \in b^+$ so $a = b$ or $a \in b$. Likewise $b \in a^+$ so $b = a$ or $b \in a$. We cannot have both $a \in b$ and $b \in a$ (Theorem 7X(b)), so we must have $a = b$.

Exercise 36 We have that $V_{\operatorname{rank} S}$ is a transitive set and $S \subseteq V_{\operatorname{rank} S}$, so $TC\ S \subseteq V_{\operatorname{rank} S}$. Thus, $\operatorname{rank}(TC\ S) \leq \operatorname{rank} S$.

We also have $S \subseteq TC\ S$ so $\operatorname{rank} S \leq \operatorname{rank}(TC\ S)$. Thus, $\operatorname{rank}(TC\ S) = \operatorname{rank} S$.

Exercise 37 If α is an ordinal then it is a transitive set and, for any distinct $x, y \in \alpha$, we have $x \in y$ or $y \in x$ (Theorem 7M).

Conversely, let α be a transitive set such that, for any distinct $x, y \in \alpha$, we have $x \in y$ or $y \in x$. We will prove that α is well ordered by epsilon. It will follow by Theorem 7L that α is an ordinal.

PROOF:
$\langle 1 \rangle 1$. ϵ_α is transitive.

⟨2⟩1. Let: $x, y, z \in \alpha$ with $x \in y$ and $y \in z$
⟨2⟩2. $x \neq z$
Proof: Otherwise we would have $x \in y \in x$ contradicting the Axiom of Regularity.
⟨2⟩3. $x \in z$ or $z \in x$
⟨2⟩4. $z \notin x$
Proof: By the Axiom of Regularity we cannot have $x \in y \in z \in x$.s
⟨2⟩5. $x \in z$
⟨1⟩2. ϵ_α is irreflexive.
Proof: By the Axiom of Regularity.
⟨1⟩3. For any $x, y \in \alpha$ we have $x \in y$ or $x = y$ or $y \in x$.
Proof: By assumption.
⟨1⟩4. Any nonempty subset of α has an ϵ_α-least element.
⟨2⟩1. Let: $A \subseteq \alpha$ be nonempty.
⟨2⟩2. Pick $m \in A$ such that $m \cap A = \emptyset$
⟨2⟩3. For all $x \in A$ we have $m \underline{\in} x$
Proof: Since $x \notin m$.
□

Exercise 38 Let λ be a limit ordinal. We have $\bigcup \lambda \subseteq \lambda$ because λ is a transitive set. Conversely, for all $\alpha \in \lambda$ we have $\alpha \in \alpha^+ \in \lambda$ so $\alpha \in \bigcup \lambda$.

Exercise 39 An ordinal number is a transitive set of ordinals, hence a transitive set of transitive sets.

Conversely, let α be a transitive set of transitive sets. We prove that α is a set of ordinals. The result will follow by Corollary 7N (a).

So suppose for a contradiction that not every element in α is an ordinal. Let $A = \{x \in \alpha : x \text{ is not an ordinal}\}$. Then A is nonempty. Pick $m \in A$ such that $m \cap A = \emptyset$. Then m is a transitive set of ordinals, hence an ordinal. This is a contradiction.

Chapter 8

Chapter 8 — Ordinals and Order Types

8.1 Alephs

Exercise 1 Let $\gamma(f, y)$ be the formula:

Either

1. f is a function with domain 0 and $y = 5$; or
2. f is a function whose domain is a successor ordinal α^+ and $y = f(\alpha)^+$; or
3. f is a function whose domain is a limit ordinal λ and $y = \bigcup(\operatorname{ran} f)$; or
4. none of the above and $y = \emptyset$.

By transfinite recursion, construct a formula $\phi(u, v)$ such that:

- for every ordinal α there exists a unique y such that $\phi(\alpha, y)$;
- whenever f is a function whose domain is an ordinal α and $\phi(\beta, f(\beta))$ for all $\beta \in \alpha$, then we have $\phi(\alpha, y)$ iff $\gamma(f, y)$ for all y.

For α an ordinal, let t_α be the unique set such that $\phi(\alpha, t_\alpha)$.

Exercise 2 We prove that $\forall \alpha \in \omega . t_\alpha = 5 + \alpha$ by induction on α. We have $t_0 = 5$ and if $t_\alpha = 5 + \alpha$ then $t_{\alpha^+} = (5 + \alpha)^+ = 5 + \alpha^+$.

We now prove that if $\omega \subseteq \alpha$ then $t_\alpha = \alpha$ by transfinite induction on α. We have

$$t_\omega = \bigcup_{n \in \omega} (5 + n) = \omega$$

If $\omega \subseteq \alpha$ and $t_\alpha = \alpha$ then $t_{\alpha^+} = \alpha^+$.

If λ is a limit ordinal and $t_\alpha = \alpha$ for all α with $\omega \subseteq \alpha \in \lambda$ then

$$\begin{aligned} t_\lambda &= \bigcup_{\alpha \in \lambda} t_\alpha \\ &= \bigcup_{\omega \subseteq \alpha \in \lambda} t_\alpha \\ &= \bigcup_{\omega \subseteq \alpha \in \lambda} \alpha \\ &= \lambda \end{aligned}$$

Exercise 3 If $\beta \in \gamma$ then $t_\beta \in t_\gamma$ by the definition of monotonicity.

Conversely, suppose $t_\beta \in t_\gamma$. Then $t_\beta \neq t_\gamma$ and $t_\gamma \notin t_\beta$, so $\beta \neq \gamma$ and $\gamma \notin \beta$. Hence $\beta \in \gamma$ by trichotomy.

Now suppose $t_\beta = t_\gamma$. Then $t_\beta \notin t_\gamma$ and $t_\gamma \notin t_\beta$, hence $\beta \notin \gamma$ and $\gamma \notin \beta$, and therefore $\beta = \gamma$ by trichotomy.

Exercise 4 We have $t_\lambda \neq 0$ because $t_0 \in t_\lambda$.

Now, suppose for a contradiction $t_\lambda = \alpha^+$ for some α. Then we have $\alpha \in t_\lambda = \bigcup_{\beta \in \lambda} t_\beta$. Hence $\alpha \in t_\beta$ for some $\beta \in \lambda$. Therefore,

$$\begin{aligned} \alpha^+ &\subseteq t_\beta \\ \therefore \alpha^+ &\in t_{\beta^+} \\ \therefore \alpha^{++} &\subseteq t_{\beta^+} \\ \therefore \alpha^{++} &\subseteq t_\lambda \end{aligned}$$

which is a contradiction.

Exercise 5 The proof is by transfinite induction on β.

We have $0 \subseteq t_0$.

If $\beta \subseteq t_\beta$ then $\beta \in t_{\beta^+}$, hence $\beta^+ \subseteq t_{\beta^+}$.

If λ is a limit ordinal and $\forall \beta \in \lambda . \beta \subseteq t_\beta$ then

$$\begin{aligned} t_\lambda &= \sup_{\beta \in \lambda} t_\beta \\ &\supseteq \sup_{\beta \in \lambda} \beta \\ &= \lambda \end{aligned}$$

Exercise 6 The class is closed by Theorem Schema 8E. It is unbounded because, for any ordinal α, we have $\alpha \in \alpha^+ \subseteq t_{\alpha^+}$ by Exercise 5.

Exercise 7 Let γ be any fixed point of t with $\beta \underline{\in} \gamma$. Then we have $f(0) \underline{\in} \gamma$; and, if $f(n) \underline{\in} \gamma$, then

$$\begin{aligned} f(n^+) &= t_{f(n)} \\ &\underline{\in} t_\gamma \\ &= \gamma \end{aligned}$$

Hence by induction $f(n) \underline{\in} \gamma$ for all n, and so $\lambda \underline{\in} \gamma$. Thus λ is the least fixed point of t.

Exercise 8 Monotonicity holds by the analogue of Theorem 8A (see the second Example on page 216).

For continuity, let λ be a limit ordinal. We must prove that $\bigcup_{\beta \in \lambda} t'_\beta$ is the least fixed point of t different from t'_β for all $\beta \in \lambda$.

PROOF:
⟨1⟩1. LET: $\mu = \bigcup_{\beta \in \lambda} t'_\beta$
⟨1⟩2. μ is a fixed point of t
PROOF:

$$\begin{aligned} t_\mu &= \bigcup_{\beta \in \lambda} t_{t'_\beta} && \text{(Theorem Schema 8E)} \\ &= \bigcup_{\beta \in \lambda} t'_\beta && (t'_\beta \text{ is a fixed point of } t) \\ &= \mu \end{aligned}$$

⟨1⟩3. $\forall \beta \in \lambda . \mu \neq t'_\beta$
PROOF: Because $t'_\beta \in t'_{\beta+} \underline{\in} \mu$.
⟨1⟩4. If γ is a fixed point of t and $\forall \beta \in \lambda . \gamma \neq t'_\beta$ then $\mu \underline{\in} \gamma$
PROOF: We have $\forall \beta \in \lambda . t'_\beta \in \gamma$ hence $\mu \underline{\in} \gamma$.
□

8.2 Isomorphism Types

Exercise 9 Pick $a \in A$. For any set $x \notin A$, let $A' = A - \{a\} \cup \{x\}$, and let R' be the relation formed by replacing any pair $\langle a, y \rangle$ with $\langle x, y \rangle$, any pair $\langle y, a \rangle$ with $\langle y, x \rangle$, and $\langle a, a \rangle$ with $\langle x, x \rangle$ if aRa. Then $\langle A, R \rangle \cong \langle A', R' \rangle$ and $\operatorname{rank}\langle A', R' \rangle > \operatorname{rank} x$.

Hence for every ordinal α there is a structure isomorphic to $\langle A, R \rangle$ with rank $> \alpha$. Thus the class of structures isomorphic to $\langle A, R \rangle$ is not a set, because the ranks of its members are unbounded.

Exercise 10

(a) The only set equinumerous with 0 is 0, so $\operatorname{kard} 0 = \{0\}$.

We have $V_1 = \{\emptyset\} = \{0\}$ and $V_2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$. So 1 is the only set of rank 2 equinumerous with 1, and no set of rank < 2 is equinumerous with 1. Hence $\operatorname{kard} 1 = \{1\}$.

We have $V_3 = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} = \{0, 1, \{1\}, 2\}$. So 2 is the only set of rank 3 equinumerous with 2, and no set of rank < 3 is equinumerous with 2. Thus $\operatorname{kard} 2 = \{2\}$.

(b) kard 3 is the set of all sets of rank 4 that are equinumerous with 3, i.e. the set of all subsets of V_3 of cardinality 3. So

$$\operatorname{kard} 3 = \{\{0, 1, \{1\}\}, 3, \{0, \{1\}, 2\}, \{1, \{1\}, 2\}\} \ .$$

8.3 Arithmetic of Order Types

Exercise 11 Pick structures $\langle A, R\rangle$ and $\langle B, S\rangle$ of order type ρ and σ respectively. Define R' on $A \times \{0\}$ by: $\langle a, 0\rangle R' \langle a', 0\rangle$ iff aRa'. Define S' on $B \times \{1\}$ by: $\langle b, 1\rangle S' \langle b', 1\rangle$ iff bSb'. Then $\langle A \times \{0\}, R'\rangle$ has order type ρ, $\langle B \times \{1\}, S'\rangle$ has order type σ, and $(A \times \{0\}) \cap (B \times \{1\}) = \emptyset$.

Exercise 12 Since we have:
$\langle 0, a\rangle <_L \langle 0, a'\rangle$ iff aRa'
$\langle 1, b\rangle <_L \langle 1, b'\rangle$ iff bSb'
$\langle 0, a\rangle <_L \langle 1, b\rangle$ for all $a \in A$ and $b \in B$
$\langle 1, b\rangle \not<_L \langle 0, a\rangle$ for all $a \in A$ and $b \in B$

Exercise 13 If f is an isomorphism between $\langle A, R\rangle$ and $\langle A', R'\rangle$, snd g is an isomorphism between $\langle B, S\rangle$ and $\langle B', S'\rangle$, and $A \cap B = A' \cap B' = \emptyset$, then $f \cup g$ is an isomorphism between $\langle A \cup B, R \oplus S\rangle$ and $\langle A' \cup B', R' \oplus S'\rangle$.

If f is an isomorphism between $\langle A, R\rangle$ and $\langle A', R'\rangle$, snd g is an isomorphism between $\langle B, S\rangle$ and $\langle B', S'\rangle$, then the function $h : A \times B \to A' \times B'$ defined by

$$h(\langle a, b\rangle) = \langle f(a), g(b)\rangle$$

is an isomorphism between $\langle A \times B, R * S\rangle$ and $\langle A' \times B', R' * S'\rangle$.

Exercise 14 Let $\langle A, R\rangle$ be a structure of order type ρ and $\langle B, S\rangle$ a structure of order type σ. Then $A \times B \approx \emptyset$ so $A \times B = \emptyset$. Therefore $A = \emptyset$ or $B = \emptyset$, and so $\rho = 0$ or $\sigma = 0$.

Exercise 15

$$\begin{aligned}(\overline{\omega} + \overline{1}) \cdot \overline{2} &= \overline{\omega} + \overline{1} + \overline{\omega} + \overline{1} \\ &= \overline{\omega} + \overline{\omega} + \overline{1} \\ &\neq \overline{\omega} + \overline{\omega} + \overline{2} \\ &= (\overline{\omega} \cdot \overline{2}) + (\overline{1} \cdot \overline{2})\end{aligned}$$

Exercise 16 Let $\langle A, R\rangle$ be a structure of order type ρ.

We have $\langle A \cup \emptyset, R \oplus \emptyset\rangle = \langle \emptyset \cup A, \emptyset \oplus R\rangle = \langle A, R\rangle$ so $\rho + \overline{0} = \overline{0} + \rho = \rho$.

Now, $\langle 1, \emptyset\rangle$ is a structure of order type $\overline{1}$. We have $\langle A \times 1, R * \emptyset\rangle = \langle 1 \times A, \emptyset * R\rangle = \langle A, R\rangle$ so $\rho \cdot \overline{1} = \overline{1} \cdot \rho = \rho$.

We have $\langle A \times \emptyset, R * \emptyset\rangle = \langle \emptyset \times A, \emptyset * R\rangle = \langle \emptyset, \emptyset\rangle$.

Exercise 17 Pick an enumeration $A = \{a_0, a_1, \ldots\}$ of A. Define $f : A \to \mathbb{Q}$ by recursion as follows:

Let $f(a_0) = 0$.

Given $f(a_0)$, $f(a_1)$, $\ldots$, $f(a_n)$, we have the following three possibilities:

- a_{n+1} is smaller than all of $a_0, \ldots, a_n$. In this case, let a_k be the minimum of $a_0, \ldots, a_n$, and set $f(a_{n+1}) = f(a_k) - 1$
- a_{n+1} is larger than all of $a_0, \ldots, a_n$. In this case, let a_k be the maximum of $a_0, \ldots, a_n$, and set $f(a_{n+1}) = f(a_k) + 1$
- Otherwise, let a_i be the largest element of $a_0, \ldots, a_n$ such that $a_i < a_{n+1}$, and a_j the smallest element such that $a_{n+1} < a_j$. Set $f(a_{n+1}) = (f(a_i) + f(a_j))/2$.

Then we have $a_i < a_j$ iff $f(a_i) < f(a_j)$ for all i, j. Hence f is an isomorphism between $\langle A, R\rangle$ and $\langle f[\![A]\!], <^\circ\rangle$.

Exercise 18 Pick enumerations $\{a_0, a_1, \ldots\}$ of A and $\{b_0, b_1, \ldots\}$ of B.

Define isomorphisms $F_n \subseteq A \times B$ by recursion on n in such a way that each F_n is an isomorphism between a subset of A_n of A and a subset B_n of B such that:

- For all n we have $a_n \in A_{2n}$
- For all n we have $b_n \in B_{2n+1}$

as follows.

$$F_0 = \{\langle a_0, b_0\}$$

Given F_{2n}, if $b_n \in B_{2n}$ then $F_{2n+1} = F_{2n}$. Otherwise:

- if b_n is greater than every element in B_{2n}, then let m be least such that a_m is larger than every element of A_{2n} (here we use the fact that A has no largest element) and set $F_{2n+1} = F_{2n} \cup \{\langle a_m, b_n\rangle\}$
- if b_n is smaller than every element in B_{2n}, then let m be least such that a_m is smaller than every element of A_{2n} (here we use the fact that A has no smallest element) and set $F_{2n+1} = F_{2n} \cup \{\langle a_m, b_n\rangle\}$
- otherwise let b be the greatest element in B_{2n} such that $b < b_n$, and b' the least element in B_{2n} such that $b_n < b'$. Let $a = {F_{2n}}^{-1}(b)$ and $a' = {F_{2n}}^{-1}(b')$. Let m be least such that $a < a_m < a'$ (here we use the fact that A is dense). Let $F_{2n+1} = F_{2n} \cup \{\langle a_m, b_n\rangle\}$

In every case we have that F_{2n+1} is an isomorphism between a subset of A and a subset of B that contains b_n.

Similarly, given F_{2n+1}, we can define F_{2n+2} to be an isomorphism between a subset of A that contains a_n and a subset of B.

Now, let $f = \bigcup_n F_n$. Then f is an isomorphism between $\langle A, R\rangle$ and $\langle B, S\rangle$.

Exercise 19 This holds because the concatenation of $\mathbb{Q}$ with itself, and the lexicographic ordering on $\mathbb{Q}^2$, are dense linear orderings on countable nonempty sets.

8.4 Ordinal Arithmetic

Exercise 20 PROOF:

$\langle 1\rangle 1$. For every ordinal α, there exists an ordinal λ that is either a limit ordinal or 0 and a natural number n such that $\alpha = \lambda + n$

- $\langle 2\rangle 1$. $0 = 0 + 0$
- $\langle 2\rangle 2$. If $\alpha = \lambda + n$ then $\alpha^+ = \lambda + n^+$
- $\langle 2\rangle 3$. For λ a limit ordinal we have $\lambda = \lambda + 0$

$\langle 1\rangle 2$. If λ, μ are either limit ordinals or 0, and $m, n \in \omega$, and $\lambda + m = \mu + n$, then $\lambda = \mu$ and $m = n$

- $\langle 2\rangle 1$. LET: $P(m)$ be the property: for all λ, μ and $n \in \omega$, if λ and μ are either limit ordinals or 0 and $\lambda + m = \mu + n$, then $\lambda = \mu$ and $m = n$
- $\langle 2\rangle 2$. $P(0)$
  - $\langle 3\rangle 1$. ASSUME: $\lambda + 0 = \mu + n$
  - $\langle 3\rangle 2$. $n = 0$

    PROOF: Otherwise $\lambda = \mu + n$ would be a successor ordinal.
  - $\langle 3\rangle 3$. $\lambda = \mu$
- $\langle 2\rangle 3$. $\forall m \in \omega . P(m) \Rightarrow P(m^+)$
  - $\langle 3\rangle 1$. LET: $m \in \omega$
  - $\langle 3\rangle 2$. ASSUME: $P(m)$
  - $\langle 3\rangle 3$. ASSUME: $\lambda + m^+ = \mu + n$
  - $\langle 3\rangle 4$. $n \neq 0$

    PROOF: Otherwise $\mu = \lambda + m^+$ is a successor ordinal.
  - $\langle 3\rangle 5$. PICK p such that $n = p^+$
  - $\langle 3\rangle 6$. $(\lambda + m)^+ = (\mu + p)^+$
  - $\langle 3\rangle 7$. $\lambda + m = \mu + p$
  - $\langle 3\rangle 8$. $\lambda = \mu$ and $m = p$

    PROOF: By $\langle 3\rangle 2$
  - $\langle 3\rangle 9$. $m^+ = n$

$\square$

Exercise 21 1 is the least integer in the ordering, followed by all the integers with exactly one prime factor, then all the integers with two prime factors, etc. So the ordinal is $1 + \omega \cdot \omega = \omega^2$.

Exercise 22

(a) If $\beta \underline{\in} \gamma$ then $\beta + 0 = \beta \underline{\in} \gamma = \gamma + 0$.
If $\beta + \alpha \underline{\in} \gamma + \alpha$ then $\beta + \alpha^+ = (\beta + \alpha)^+ \underline{\in} (\gamma + \alpha)^+ = \gamma + \alpha^+$.
For λ a limit ordinal, if $\forall \alpha \in \lambda . \beta + \alpha \underline{\in} \gamma + \alpha$, then we have $\beta + \lambda = \sup_{\alpha \in \lambda}(\beta + \alpha) \underline{\in} \sup_{\alpha \in \lambda}(\gamma + \alpha) = \gamma + \lambda$.

(b) We have $\beta \cdot 0 = 0 = \gamma \cdot 0$.
If $\beta \underline{\in} \gamma$ and $\beta \cdot \alpha \underline{\in} \gamma \cdot \alpha$ then $\beta \cdot \alpha^+ = \beta \cdot \alpha + \beta \underline{\in} \gamma \cdot \alpha + \gamma = \gamma \cdot \alpha^+$ using part (a).
For λ a limit ordinal, if $\forall \alpha \in \lambda . \beta \cdot \alpha \underline{\in} \gamma \cdot \alpha$, then we have $\beta \cdot \lambda = \sup_{\alpha \in \lambda}(\beta \cdot \alpha) \underline{\in} \sup_{\alpha \in \lambda}(\gamma \cdot \alpha) = \gamma \cdot \lambda$.

Exercise 23

(a)

$$\begin{aligned}
\omega + \omega^2 &= \omega \cdot 1 + \omega \cdot \omega \\
&= \omega \cdot (1 + \omega) && \text{(Theorem 8K)} \\
&= \omega \cdot \omega && \text{(Example on page 228)} \\
&= \omega^2
\end{aligned}$$

(b) Let $\omega^2 \underline{\in} \beta$. Let γ be the ordinal such that $\beta = \omega^2 + \gamma$ (Subtraction Theorem). Then

$$\begin{aligned}
\omega + \beta &= \omega + \omega^2 + \gamma \\
&= \omega + \gamma \\
&= \beta
\end{aligned}$$

Exercise 24 We prove first that $1 + \alpha = \alpha$. Let γ be the ordinal such that $\alpha = \omega + \gamma$. Then

$$\begin{aligned}
1 + \alpha &= 1 + \omega + \gamma \\
&= \omega + \gamma && \text{(Example on page 228)} \\
&= \alpha
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha + \alpha^2 &= \alpha \cdot (1 + \alpha) \\
&= \alpha^2
\end{aligned}$$

Now, let δ be the ordinal such that $\beta = \alpha^2 + \delta$. Then

$$\begin{aligned}
\alpha + \beta &= \alpha + \alpha^2 + \delta \\
&= \alpha^2 + \delta \\
&= \beta
\end{aligned}$$

Exercise 25 Let $\beta = \alpha \cup \{\alpha + \delta : \delta \in \theta\}$. Then β is a transitive set of ordinals, hence an ordinal. We also have $\alpha \underline{\in} \beta$. By the Subtraction Theorem, let γ be the ordinal such that

$$\beta = \alpha + \gamma \ .$$

For any $\delta \in \theta$ we have $\alpha + \delta \in \beta$ hence $\delta \in \gamma$ (Corollary 8P). Thus $\theta \underline{\in} \gamma$.

We have $\alpha + \theta \notin \beta$ (since $\alpha + \theta \notin \alpha$ and $\alpha + \theta \neq \alpha + \delta$ for any $\delta \in \theta$). So $\theta \notin \gamma$ (Corollary 8P).

Thus $\theta = \gamma$, and so $\beta = \alpha + \theta$.

Exercise 26 Follows just by repeated application of uniqueness in the Logarithm Theorem.

Exercise 27

Theorem 8R If $\alpha = 0$, then both sides are 1 if $\beta = \gamma = 0$ and 0 otherwise. If $\alpha = 1$ then both sides are 1.

Theorem 8S If $\alpha = 0$, and either $\beta = 0$ or $\gamma = 0$, then both sides are 1. If $\alpha = 0$ and β and γ are both non-zero, then both sides are 0. If $\alpha = 1$ then both sides are 1.

Exercise 28 This follows immediately from a Veblen Fixed-Point Theorem.

Exercise 29 Let S be a nonempty set of epsilon numbers. Then

$$\begin{aligned} \omega^{\sup S} &= \sup_{\alpha \in S} \omega^\alpha && \text{(Theorem Schema 8E)} \\ &= \sup_{\alpha \in S} \alpha \\ &= \sup S \end{aligned}$$

8.5 Well-Founded Relations

Exercise 1 We first prove: if xR^ty then there exists z such that zRy and either xR^tz or $x = z$.

Proof:

$\langle 1 \rangle 1$. $\{\langle x, y \rangle : \exists z(zRy \ \& \ (xR^tz \text{ or } x = z))\}$ is a transitive relation that includes R.

  $\langle 2 \rangle 1$. Let: $S = \{\langle x, y \rangle : \exists z(zRy \ \& \ (xR^tz \text{ or } x = z))\}$

  $\langle 2 \rangle 2$. S is transitive

    $\langle 3 \rangle 1$. Let: xSy and ySz

    $\langle 3 \rangle 2$. Pick a and b such that aRy, (xR^ta or $x = a$), bRz and (yR^tb or $y = b$)

    $\langle 3 \rangle 3$. xR^ty

$\langle 3\rangle 4.$ $xR^t b$
$\langle 2\rangle 3.$ $R \subseteq S$
□

PROOF:
$\langle 1\rangle 1.$ LET: R be a well-founded relation.
$\langle 1\rangle 2.$ LET: A be a nonempty set.
$\langle 1\rangle 3.$ PICK an R-minimal element a of A.
$\langle 1\rangle 4.$ a is R^t-minimal
PROOF: By the lemma, if there exists x such that $xR^t a$ then there exists x such that xRa.
□

Exercise 2 The relation R^t is always transitive, so it is a partial ordering iff it is irreflexive, i.e. there is no x such that $xR^t x$. This is the same as saying there is no cycle in R, i.e. no finite sequence of elements $x_1, \ldots, x_n$ such that $x_1 R x_2$, $x_2 R x_3$, $\ldots$, $x_{n-1} R x_n$ and $x_n R x_1$.

Exercise 3 The proof is by transfinite induction on y over R. Assume $\{x : xR^t z\}$ is finite for all z such that zRy. Then

$$\{x : xR^t y\} = \bigcup \{\{z\} \cup \{x : xR^t z\} : zRy\}$$

which is a finite union of finite sets, hence finite.

Exercise 4 PROOF:
$\langle 1\rangle 1.$ LET: $T = S \cup \bigcup \{TC\ x : x \in S\}$
$\langle 1\rangle 2.$ T is a transitive set
$\langle 2\rangle 1.$ LET: $x \in y \in T$
$\langle 2\rangle 2.$ CASE: $y \in S$
$\langle 3\rangle 1.$ $x \in TC\ y$
$\langle 3\rangle 2.$ $x \in T$
$\langle 2\rangle 3.$ CASE: $y \in TC\ a$ and $a \in S$
$\langle 3\rangle 1.$ $x \in TC\ a$
$\langle 3\rangle 2.$ $x \in T$
$\langle 1\rangle 3.$ $S \subseteq T$
$\langle 1\rangle 4.$ For any transitive set T', if $S \subseteq T'$ then $T \subseteq T'$
$\langle 2\rangle 1.$ LET: T' be a transitive set.
$\langle 2\rangle 2.$ ASSUME: $S \subseteq T'$
$\langle 2\rangle 3.$ LET: $x \in T$
$\langle 2\rangle 4.$ CASE: $x \in S$
PROOF: Then $x \in T'$ by $\langle 2\rangle 2$
$\langle 2\rangle 5.$ CASE: $x \in TC\ y$ and $y \in S$
$\langle 3\rangle 1.$ $y \in T'$
$\langle 3\rangle 2.$ $y \subseteq T'$
$\langle 3\rangle 3.$ $TC\ y \subseteq T'$

$\langle 3 \rangle 4.$ $x \in T'$

$\square$