

# C1 Set Theory

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## 1 Primitive Notions

Let there be *sets*.

Let there be a binary relation called *membership*,  $\in$ . When  $x \in y$  holds, we say  $x$  is a *member* or *element* of  $y$ . We write  $x \notin y$  iff  $x$  is not a member of  $y$ .

**Definition 1** (Transitive Set). A set  $A$  is a *transitive set* iff every member of a member of  $A$  is a member of  $A$ .

## 2 The Axioms

**Axiom 2** (Extensionality). *If two sets have exactly the same members, then they are equal.*

As a consequence of this axiom, we may identify a set  $A$  with the class  $\{x : x \in A\}$ . The use of the symbols  $\in$  and  $=$  is consistent.

**Definition 3.** We say that a class  $\mathbf{A}$  is a *set* iff there exists a set  $A$  such that  $A = \mathbf{A}$ . That is, the class  $\{x : P(x)\}$  is a set iff

$$\exists A. \forall x (x \in A \leftrightarrow P(x)) .$$

Otherwise,  $\mathbf{A}$  is a *proper class*.

**Definition 4** (Subset). If  $A$  is a set and  $\mathbf{B}$  is a class, we say  $A$  is a *subset* of  $\mathbf{B}$  iff  $A \subseteq \mathbf{B}$ .

**Axiom 5** (Empty Set). *The empty class is a set, called the empty set.*

**Axiom 6** (Replacement). *For any property  $P(x, y)$ , the following is an axiom:*

*Let  $A$  be a set. Assume that, for all  $x \in A$ , there is at most one  $y$  such that  $P(x, y)$ . Then  $\{y : \exists x \in A. P(x, y)\}$  is a set.*

**Definition 7** (Power Set). For any set  $A$ , the *power set* of  $A$ ,  $\mathcal{P}A$ , is the class of all subsets of  $A$ .

**Axiom 8** (Power Set). *For any set  $A$ , the class  $\mathcal{P}A$  is a set.*

**Theorem 9** (Pairing). *For any objects  $a$  and  $b$ , the class  $\{a, b\}$  is a set, called a pair set.*

PROOF: Let  $a$  and  $b$  be sets. Let  $P(x, y)$  be the formula  $(x = \emptyset \ \& \ y = a)$  or  $(x = \mathcal{P}\emptyset \ \& \ y = b)$ . Then we have  $(\forall x \in \mathcal{P}\mathcal{P}\emptyset) \forall y_1 \forall y_2 (P(x, y_1) \ \& \ P(x, y_2) \Rightarrow y_1 = y_2)$ , hence there exists a set  $c$  such that

$$\forall y (y \in c \Leftrightarrow (\exists x \in \mathcal{P}\mathcal{P}\emptyset) P(x, y))$$

The members of  $c$  are just  $a$  and  $b$ .  $\square$

**Definition 10** (Union). For any class of sets  $\mathbf{A}$ , the *union*  $\bigcup \mathbf{A}$  is the class  $\{x : \exists A \in \mathbf{A}. x \in A\}$ .

We write  $\bigcup_{P[x_1, \dots, x_n]} t[x_1, \dots, x_n]$  for  $\bigcup \{t[x_1, \dots, x_n] : P[x_1, \dots, x_n]\}$ .

**Proposition 11.** *If  $\mathbf{A} \subseteq \mathbf{B}$  then  $\bigcup \mathbf{A} \subseteq \bigcup \mathbf{B}$ .*

PROOF: Easy.  $\square$

**Axiom 12** (Union). *For any set  $A$ , the union  $\bigcup A$  is a set.*

**Proposition 13.** *For any sets  $A$  and  $B$ , the class  $A \cup B$  is a set.*

PROOF: It is  $\bigcup \{A, B\}$ .  $\square$

**Proposition Schema 14.** *For any objects  $a_1, \dots, a_n$ , the class  $\{a_1, \dots, a_n\}$  is a set.*

PROOF: By repeated application of the Pairing and Union axioms.  $\square$

**Theorem 15** (Subset Axioms, Aussonderung). *For any class  $\mathbf{A}$  and set  $B$ , if  $\mathbf{A} \subseteq B$  then  $\mathbf{A}$  is a set.*

PROOF: Let  $Q(x, y)$  be the formula  $x \in \mathbf{A} \wedge y = x$ . Now we reason as follows. Let  $c$  be any set. Then we have

$$(\forall x \in B) \forall y_1 \forall y_2 (Q(x, y_1) \ \& \ Q(x, y_2) \Rightarrow y_1 = y_2)$$

Then, by a Replacement Axiom, there exists a set  $c$  such that

$$\forall y (y \in c \Leftrightarrow (\exists x \in B) Q(x, y)) \ .$$

This is equivalent to  $\forall x (x \in c \Leftrightarrow x \in \mathbf{A})$ .  $\square$

**Proposition 16.** *For any set  $A$  and class  $\mathbf{B}$ , the intersection  $A \cap \mathbf{B}$  is a set.*

PROOF: By the Subset Axiom since it is a subclass of  $A$ .  $\square$

**Proposition 17.** *For any set  $A$  and class  $\mathbf{B}$ , the relative complement  $A - \mathbf{B}$  is a set.*

PROOF: By the Subset Axiom since it is a subclass of  $A$ .  $\square$

**Theorem 18.** *The universal class  $\mathbf{V}$  is a proper class.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\mathbf{V}$  is a set.

$\langle 1 \rangle 2$ . LET:  $R = \{x : x \notin x\}$

⟨1⟩3.  $R$  is a set.

PROOF: By the Subset Axiom.

⟨1⟩4.  $R \in R$  if and only if  $R \notin R$

⟨1⟩5. Q.E.D.

PROOF: This is a contradiction.

□

**Definition 19** (Intersection). For any class of sets  $\mathbf{A}$ , the *intersection*  $\bigcap \mathbf{A}$  is the class  $\{x : \forall A \in \mathbf{A}. x \in A\}$ .

We write  $\bigcap_{P[x_1, \dots, x_n]} t[x_1, \dots, x_n]$  for  $\bigcap \{t[x_1, \dots, x_n] : P[x_1, \dots, x_n]\}$ .

**Proposition 20.** For any nonempty class of sets  $\mathbf{A}$ , the class  $\bigcap \mathbf{A}$  is a set.

PROOF: Pick  $A \in \mathbf{A}$ . Then  $\bigcap \mathbf{A} \subseteq A$ . □

**Proposition 21.** If  $\mathbf{A} \subseteq \mathbf{B}$  then  $\bigcap \mathbf{B} \subseteq \bigcap \mathbf{A}$ .

PROOF: Easy. □

**Proposition 22.** For any set  $A$  and class of sets  $\mathbf{B}$ , we have

$$A \cup \bigcap \mathbf{B} = \bigcap \{A \cup X \mid X \in \mathbf{B}\}$$

PROOF: Easy. □

**Proposition 23.** For any set  $A$  and class of sets  $\mathbf{B}$ , we have

$$A \cap \bigcup \mathbf{B} = \bigcup \{A \cap X \mid X \in \mathbf{B}\}$$

PROOF: Easy. □

**Proposition 24.** For any set  $C$  and class of sets  $\mathbf{A}$ , we have

$$C - \bigcup \mathbf{A} = \bigcap \{C - X \mid X \in \mathbf{A}\} .$$

PROOF: Easy. □

**Proposition 25.** For any set  $C$  and class of sets  $\mathbf{A}$ , we have

$$C - \bigcap \mathbf{A} = \bigcup \{C - X \mid X \in \mathbf{A}\} .$$

PROOF: Easy. □

**Axiom 26** (Regularity). For every nonempty set  $A$ , there exists  $m \in A$  such that  $m \cap A = \emptyset$ .

**Theorem 27.** No set is a member of itself.

PROOF: If  $A \in A$  then there is no  $m \in \{A\}$  such that  $m \cap \{A\} = \emptyset$ . □

**Theorem 28.** There are no sets  $a$  and  $b$  with  $a \in b$  and  $b \in a$ .

PROOF: If there were, then there would be no  $m \in \{a, b\}$  such that  $m \cap \{a, b\} = \emptyset$ .

□

### 3 Ordered Pairs

**Definition 29** (Ordered Pair). For any objects  $a$  and  $b$ , the *ordered pair*  $(a, b)$  is  $\{\{a\}, \{a, b\}\}$ . We call  $a$  its *first coordinate* and  $b$  its *second coordinate*.

**Theorem 30.** For any objects  $(a, b)$ , we have  $(a, b) = (c, d)$  if and only if  $a = c$  and  $b = d$ .

PROOF:

$\langle 1 \rangle 1$ . If  $(a, b) = (c, d)$  then  $a = c$  and  $b = d$

$\langle 2 \rangle 1$ . ASSUME:  $(a, b) = (c, d)$

$\langle 2 \rangle 2$ .  $a = c$

PROOF: Since  $\{a\} = \bigcap(a, b) = \bigcap(c, d) = \{c\}$ .

$\langle 2 \rangle 3$ .  $\{a, b\} = \{c, d\}$

PROOF:  $\{a, b\} = \bigcup(a, b) = \bigcup(c, d) = \{c, d\}$ .

$\langle 2 \rangle 4$ .  $b = c$  or  $b = d$

$\langle 2 \rangle 5$ . CASE:  $b = c$

$\langle 3 \rangle 1$ .  $a = b$

$\langle 3 \rangle 2$ .  $\{c, d\} = \{a\}$

$\langle 3 \rangle 3$ .  $b = d$

$\langle 2 \rangle 6$ . CASE:  $b = d$

PROOF: We have  $a = c$  and  $b = d$  as required.

$\langle 1 \rangle 2$ . If  $a = c$  and  $b = d$  then  $(a, b) = (c, d)$

PROOF: Trivial.

□

**Definition 31** (Cartesian Product). The *Cartesian product* of classes  $\mathbf{A}$  and  $\mathbf{B}$  is the class

$$\mathbf{A} \times \mathbf{B} = \{(x, y) : x \in \mathbf{A}, y \in \mathbf{B}\} .$$

**Lemma 32.** For any objects  $x$  and  $y$  and set  $C$ , if  $x \in C$  and  $y \in C$  then  $(x, y) \in \mathcal{PPC}$ .

PROOF: Easy. □

**Corollary 32.1.** For any sets  $A$  and  $B$ , the Cartesian product  $A \times B$  is a set.

PROOF: By the Subset Axiom applied to  $\mathcal{PP}(A \cup B)$ . □

**Lemma 33.** If  $(x, y) \in \mathbf{A}$  then  $x, y \in \bigcup \bigcup \mathbf{A}$ .

PROOF: Easy. □

### 4 Relations

**Definition 34** (Relation). A *relation* is a class of ordered pairs. It is *small* iff it is a set.

When  $\mathbf{R}$  is a relation, we write  $x\mathbf{R}y$  for  $(x, y) \in \mathbf{R}$ .

**Definition 35** (Domain). The *domain* of a class  $\mathbf{R}$  is  $\text{dom } \mathbf{R} = \{x : \exists y.(x, y) \in \mathbf{R}\}$ .

**Definition 36** (Range). The *range* of a class  $\mathbf{R}$  is  $\text{ran } \mathbf{R} = \{y : \exists x.(x, y) \in \mathbf{R}\}$ .

**Definition 37** (Field). The *field* of a class  $\mathbf{R}$  is  $\text{fld } \mathbf{R} = \text{dom } \mathbf{R} \cup \text{ran } \mathbf{R}$ .

**Proposition 38.** *If  $R$  is a set then  $\text{dom } R$ ,  $\text{ran } R$  and  $\text{fld } R$  are sets.*

PROOF: Apply the Subset Axiom to  $\bigcup \bigcup R$ .  $\square$

**Definition 39** (Single-Rooted). A class  $\mathbf{R}$  is *single-rooted* iff, for all  $y \in \text{ran } \mathbf{R}$ , there is only one  $x$  such that  $x\mathbf{R}y$ .

**Definition 40** (Inverse). The *inverse* of a class  $\mathbf{F}$  is the class  $\mathbf{F}^{-1} = \{(y, x) \mid (x, y) \in \mathbf{F}\}$ .

**Theorem 41.** *For any class  $\mathbf{F}$ , we have  $\text{dom } \mathbf{F}^{-1} = \text{ran } \mathbf{F}$  and  $\text{ran } \mathbf{F}^{-1} = \text{dom } \mathbf{F}$ .*

PROOF: Easy.  $\square$

**Theorem 42.** *For a relation  $\mathbf{F}$ ,  $(\mathbf{F}^{-1})^{-1} = \mathbf{F}$ .*

PROOF: Easy.  $\square$

**Definition 43** (Composition). The *composition* of classes  $\mathbf{F}$  and  $\mathbf{G}$  is the class  $\mathbf{G} \circ \mathbf{F} = \{(x, z) \mid \exists y.(x, y) \in \mathbf{F} \wedge (y, z) \in \mathbf{G}\}$ .

**Theorem 44.** *For any classes  $\mathbf{F}$  and  $\mathbf{G}$ ,  $(\mathbf{F} \circ \mathbf{G})^{-1} = \mathbf{G}^{-1} \circ \mathbf{F}^{-1}$ .*

PROOF: Easy.  $\square$

**Definition 45** (Restriction). The *restriction* of the class  $\mathbf{F}$  to the class  $\mathbf{A}$  is the class  $\mathbf{F} \upharpoonright \mathbf{A} = \{(x, y) : x \in \mathbf{A} \wedge (x, y) \in \mathbf{F}\}$ .

**Definition 46** (Image). The *image* of the class  $\mathbf{A}$  under the class  $\mathbf{F}$  is the class  $\mathbf{F}(\mathbf{A}) = \{y : \exists x \in \mathbf{A}.(x, y) \in \mathbf{F}\}$ .

**Theorem 47.**

$$\mathbf{F}(\mathbf{A} \cup \mathbf{B}) = \mathbf{F}(\mathbf{A}) \cup \mathbf{F}(\mathbf{B})$$

PROOF: Easy.  $\square$

**Theorem 48.**

$$\mathbf{F}\left(\bigcup \mathbf{A}\right) = \bigcup \{\mathbf{F}(X) : X \in \mathbf{A}\}$$

PROOF: Easy.  $\square$

**Theorem 49.**

$$\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$$

*Equality holds if  $\mathbf{F}$  is single-rooted.*

PROOF: Easy.  $\square$

**Theorem 50.**

$$\mathbf{F}(\bigcap \mathbf{A}) \subseteq \bigcap \{\mathbf{F}(X) : X \in \mathbf{A}\}$$

Equality holds if  $\mathbf{F}$  is single-rooted.

PROOF: Easy.  $\square$

**Theorem 51.**

$$\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$$

Equality holds if  $\mathbf{F}$  is single-rooted.

PROOF: Easy.  $\square$

**Definition 52** (Reflexive). A binary relation  $\mathbf{R}$  on  $\mathbf{A}$  is *reflexive* on  $\mathbf{A}$  if and only if  $\forall x \in \mathbf{A}. x\mathbf{R}x$ .

**Definition 53** (Symmetric). A binary relation  $\mathbf{R}$  is *symmetric* iff, whenever  $x\mathbf{R}y$ , then  $y\mathbf{R}x$ .

**Definition 54** (Transitive). A binary relation  $\mathbf{R}$  is *transitive* iff, whenever  $x\mathbf{R}y$  and  $y\mathbf{R}z$ , then  $x\mathbf{R}z$ .

**Definition 55** (Minimal). Let  $R$  be a relation on  $D$ . An element  $m \in D$  is *R-minimal* iff there is no  $x \in D$  such that  $x < m$ .

## 5 $n$ -ary Relations

**Definition 56.** Given objects  $a, b, c$ , define the *ordered triple*  $(a, b, c)$  to be  $((a, b), c)$ .

Define  $(a, b, c, d) = ((a, b, c), d)$ , etc.

Define the *1-tuple*  $(a)$  to be  $a$ .

**Definition 57** ( $n$ -ary Relation). Given a class  $\mathbf{A}$ , an  *$n$ -ary relation* on  $\mathbf{A}$  is a class of ordered  $n$ -tuples, all of whose components are in  $\mathbf{A}$ .

## 6 Functions

**Definition 58** (Function). A *function* is a relation  $\mathbf{F}$  such that, for all  $x \in \text{dom } \mathbf{F}$ , there is only one  $y$  such that  $x\mathbf{F}y$ . We call this unique  $y$  the *value* of  $\mathbf{F}$  at  $x$  and denote it by  $\mathbf{F}(x)$ .

We say  $\mathbf{F}$  is a function *from*  $\mathbf{A}$  *into*  $\mathbf{B}$ , or  $\mathbf{F}$  *maps*  $\mathbf{A}$  *into*  $\mathbf{B}$ , and write  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ , iff  $\mathbf{F}$  is a function,  $\text{dom } \mathbf{F} = \mathbf{A}$ , and  $\text{ran } \mathbf{F} \subseteq \mathbf{B}$ .

If, in addition,  $\text{ran } \mathbf{F} = \mathbf{B}$ , we say  $\mathbf{F}$  is a function from  $\mathbf{A}$  *onto*  $\mathbf{B}$ .

**Theorem 59.** For a class  $\mathbf{F}$ ,  $\mathbf{F}^{-1}$  is a function if and only if  $\mathbf{F}$  is single-rooted.

PROOF: Easy.  $\square$

**Theorem 60.** A relation  $\mathbf{F}$  is a function if and only if  $\mathbf{F}^{-1}$  is single-rooted.

PROOF: Easy.  $\square$

**Theorem 61.** For any function  $\mathbf{G}$  and classes  $\mathbf{A}$  and  $\mathbf{B}$ ,

$$\begin{aligned}\mathbf{G}^{-1}(\bigcup \mathbf{A}) &= \bigcup \{\mathbf{G}^{-1}(X) : X \in \mathbf{A}\} \\ \mathbf{G}^{-1}(\bigcap \mathbf{A}) &= \bigcap \{\mathbf{G}^{-1}(X) : X \in \mathbf{A}\} \quad (\text{if } \mathbf{A} \neq \emptyset) \\ \mathbf{G}^{-1}(\mathbf{A} - \mathbf{B}) &= \mathbf{G}^{-1}(\mathbf{A}) - \mathbf{G}^{-1}(\mathbf{B})\end{aligned}$$

PROOF: Easy.  $\square$

**Theorem 62.** Assume that  $\mathbf{F}$  and  $\mathbf{G}$  are functions. Then  $\mathbf{F} \circ \mathbf{G}$  is a function, its domain is  $\{x \in \text{dom } \mathbf{G} : \mathbf{G}(x) \in \text{dom } \mathbf{F}\}$ , and for  $x$  in its domain,

$$(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x)) .$$

PROOF: Easy.  $\square$

**Definition 63** (One-to-one). A function  $\mathbf{F}$  is *one-to-one* or an *injection* iff it is single-rooted.

**Theorem 64.** Let  $\mathbf{F}$  be a one-to-one function. For  $x \in \text{dom } \mathbf{F}$ ,  $\mathbf{F}^{-1}(\mathbf{F}(x)) = x$ .

PROOF: Easy.  $\square$

**Theorem 65.** Let  $\mathbf{F}$  be a one-to-one function. For  $y \in \text{ran } \mathbf{F}$ ,  $\mathbf{F}(\mathbf{F}^{-1}(y)) = y$ .

PROOF: Easy.  $\square$

**Definition 66** (Identity Function). For any class  $\mathbf{A}$ , the *identity* function on  $\mathbf{A}$  is  $\text{id}_{\mathbf{A}} = \{(x, x) \mid x \in \mathbf{A}\}$ .

**Theorem 67.** Let  $F : A \rightarrow B$ . Assume  $A \neq \emptyset$ . Then  $F$  has a left inverse (i.e. there exists  $G : B \rightarrow A$  such that  $G \circ F = \text{id}_A$ ) if and only if  $F$  is one-to-one.

PROOF:

$\langle 1 \rangle 1$ . If  $F$  is one-to-one then  $F$  has a left inverse.

$\langle 2 \rangle 1$ . ASSUME:  $F$  is one-to-one.

$\langle 2 \rangle 2$ .  $F^{-1} : \text{ran } F \rightarrow A$

$\langle 2 \rangle 3$ . PICK  $a \in A$

$\langle 2 \rangle 4$ . Define  $G : B \rightarrow A$  by:

$$G(x) = \begin{cases} F^{-1}(x) & \text{if } x \in \text{ran } F \\ a & \text{if } x \in B - \text{ran } F \end{cases}$$

$\langle 2 \rangle 5$ .  $\forall x \in A. G(F(x)) = x$

$\langle 1 \rangle 2$ . If  $F$  has a left inverse then  $F$  is one-to-one.

$\langle 2 \rangle 1$ . ASSUME:  $F$  has a left inverse  $G$ .

$\langle 2 \rangle 2$ . LET:  $x, y \in A$  with  $F(x) = F(y)$

$\langle 2 \rangle 3$ .  $x = y$

PROOF:  $x = G(F(x)) = G(F(y)) = y$ .

$\square$

**Definition 68** (Binary Operation). A *binary operation* on a set  $A$  is a function from  $A \times A$  into  $A$ .

## 7 The Axiom of Choice

**Axiom 69** (Choice). *For any relation  $R$  there exists a function  $H \subseteq R$  with  $\text{dom } H = \text{dom } R$ .*

**Theorem 70.** *Let  $F : A \rightarrow B$ . Then  $F$  has a right inverse if and only if  $F$  maps  $A$  onto  $B$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $F$  has a right inverse then  $F$  maps  $A$  onto  $B$ .

PROOF: If  $H : B \rightarrow A$  is a right inverse, then for any  $y$  in  $B$ , we have  $y = F(H(y))$ .

$\langle 1 \rangle 2$ . If  $F$  maps  $A$  onto  $B$  then  $F$  has a right inverse.

$\langle 2 \rangle 1$ . ASSUME:  $F$  maps  $A$  onto  $B$ .

$\langle 2 \rangle 2$ . PICK a function  $H$  with  $H \subseteq F^{-1}$  and  $\text{dom } H = \text{dom } F^{-1}$

PROOF: By the Axiom of Choice.

$\langle 2 \rangle 3$ .  $\text{dom } H = B$

PROOF:  $\text{dom } H = \text{dom } F^{-1} = \text{ran } F = B$  by  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 4$ . For all  $y \in B$  we have  $F(H(y)) = y$

$\langle 3 \rangle 1$ . LET:  $y \in B$

$\langle 3 \rangle 2$ .  $(y, H(y)) \in F^{-1}$

$\langle 3 \rangle 3$ .  $F(H(y)) = y$

□

## 8 Sets of Functions

**Definition 71.** Let  $A$  be a set and  $\mathbf{B}$  be a class. Then  $\mathbf{B}^A$  is the class of all functions  $A \rightarrow \mathbf{B}$ .

## 9 Dependent Products

**Definition 72.** Let  $I$  be a set and  $H_i$  a set for all  $i \in I$ . Define

$$\prod_{i \in I} H_i = \{f : f \text{ is a function, } \text{dom } f = I, \forall i \in I. f(i) \in H_i\} .$$

**Theorem 73.** *The Axiom of Choice is equivalent to the statement: For any set  $I$  and any function  $H$  with domain  $I$ , if  $H(i) \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} H(i) \neq \emptyset$*

PROOF:

$\langle 1 \rangle 1$ . If the Axiom of Choice is true then, for any set  $I$  and any function  $H$  with domain  $I$ , if  $H(i) \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} H(i) \neq \emptyset$ .

$\langle 2 \rangle 1$ . ASSUME: The Axiom of Choice.

$\langle 2 \rangle 2$ . LET:  $I$  be a set.

$\langle 2 \rangle 3$ . LET:  $H$  be a function with domain  $I$ .



- ⟨2⟩4. ASSUME:  $H(i) \neq \emptyset$  for all  $i \in I$ .  
 ⟨2⟩5. LET:  $R = \{(i, x) : i \in I, x \in H(i)\}$   
 ⟨2⟩6. PICK a function  $F \subseteq R$  with  $\text{dom } F = \text{dom } R$   
 PROVE:  $F \in \prod_{i \in I} H(i)$   
 PROOF: By the Axiom of Choice.  
 ⟨2⟩7.  $\text{dom } H = I$   
 PROOF: We have  $\text{dom } R = I$  since for all  $i \in I$  there exists  $x$  such that  $x \in H(i)$ .  
 ⟨2⟩8.  $\forall i \in I. F(i) \in H(i)$   
 PROOF: Since  $iRF(i)$ .  
 ⟨1⟩2. If, for any set  $I$  and any function  $H$  with domain  $I$ , if  $H(i) \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} H(i) \neq \emptyset$ , then the Axiom of Choice is true.  
 ⟨2⟩1. ASSUME: For any set  $I$  and any function  $H$  with domain  $I$ , if  $H(i) \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} H(i) \neq \emptyset$   
 ⟨2⟩2. LET:  $R$  be a relation  
 ⟨2⟩3. LET:  $I = \text{dom } R$   
 ⟨2⟩4. Define the function  $H$  with domain  $I$  by: for  $i \in I$ ,  $H(i) = \{y : iRy\}$   
 ⟨2⟩5.  $H(i) \neq \emptyset$  for all  $i \in I$   
 ⟨2⟩6. PICK  $F \in \prod_{i \in I} H(i)$   
 PROOF: By ⟨2⟩1  
 ⟨2⟩7.  $F$  is a function  
 ⟨2⟩8.  $F \subseteq R$   
 PROOF: For all  $i \in I$  we have  $F(i) \in H(i)$  and so  $iRF(i)$ .  
 ⟨2⟩9.  $\text{dom } F = \text{dom } R$

□

**Theorem 74.** *The following are equivalent.*

1. *The Axiom of Choice.*
2. *Let  $\mathcal{A}$  be a set such that (a) every member of  $\mathcal{A}$  is a nonempty set, and (b) any two distinct members of  $\mathcal{A}$  are disjoint. Then there exists a set  $C$  such that, for all  $B \in \mathcal{A}$ , we have  $C \cap B$  is a singleton.*
3. *For any set  $A$ , there exists a function  $F : \mathcal{P}A - \{\emptyset\} \rightarrow A$  such that  $F(X) \in X$  for all  $X \in \mathcal{P}A - \{\emptyset\}$ .*

PROOF:

⟨1⟩1.  $1 \Rightarrow 2$

PROOF: Let  $\mathcal{A}$  be a set matching the two conditions. By the Multiplicative Axiom, pick a function  $f \in \prod_{B \in \mathcal{A}} B$ . Let  $C = \text{ran } f$ . Then  $C \cap B = \{f(B)\}$  for all  $B \in \mathcal{A}$ .

⟨1⟩2.  $2 \Rightarrow 3$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET:  $\mathcal{A}$  be a set.

⟨2⟩3. LET:  $\mathcal{A} = \{\{B\} \times B : B \in \mathcal{P}A - \{\emptyset\}\}$

⟨2⟩4. PICK a set  $C$  such that  $C \cap (\{B\} \times B)$  is a singleton for all  $B \in \mathcal{P}A - \{\emptyset\}$

⟨2⟩5. LET:  $F = C \cap \bigcup \mathcal{A}$



$\langle 3 \rangle 3. x \mathbf{R} z$

PROOF: Since  $\mathbf{R}$  is transitive.

$\langle 3 \rangle 4. z \in [x]_{\mathbf{R}}$

$\langle 2 \rangle 3. y \mathbf{R} x$

PROOF: Since  $\mathbf{R}$  is symmetric.

$\langle 2 \rangle 4. [x]_{\mathbf{R}} \subseteq [y]_{\mathbf{R}}$

PROOF: Similar.

□

**Definition 79** (Partition). A *partition* of a set  $A$  is a set  $P \subseteq \mathcal{P}A$  such that:

- Every member of  $P$  is nonempty.
- Any two distinct members of  $P$  are disjoint.
- $A = \bigcup P$

**Theorem 80.** Let  $R$  be an equivalence relation on the set  $A$ . Then the set of all equivalence classes is a partition of  $A$ .

PROOF:

$\langle 1 \rangle 1.$  Every equivalence class is nonempty.

PROOF: For any  $x \in A$  we have  $x \in [x]_R$ .

$\langle 1 \rangle 2.$  Any two distinct equivalence classes are disjoint.

$\langle 2 \rangle 1.$  LET:  $x, y \in A$

$\langle 2 \rangle 2.$  ASSUME:  $z \in [x]_R \cap [y]_R$

PROVE:  $[x]_R = [y]_R$

$\langle 2 \rangle 3. x R y$

$\langle 3 \rangle 1. x R z$

$\langle 3 \rangle 2. y R z$

$\langle 3 \rangle 3. z R y$

PROOF: By  $\langle 3 \rangle 2$  and symmetry.

$\langle 3 \rangle 4. x R y$

PROOF: By  $\langle 3 \rangle 1, \langle 3 \rangle 3$  and transitivity.

$\langle 2 \rangle 4. [x]_R = [y]_R$

PROOF: By Lemma 3N.

$\langle 1 \rangle 3.$   $A$  is the union of all the equivalence classes.

PROOF: For any  $x \in A$  we have  $x \in [x]_R$ .

□

**Definition 81** (Quotient Set). If  $R$  is an equivalence relation on the set  $A$ , then the *quotient set*  $A/R$  is the set of all equivalence classes, and the *natural map* or *canonical map*  $\phi : A \rightarrow A/R$  is defined by  $\phi(x) = [x]_R$ .

**Theorem 82.** Assume that  $R$  is an equivalence relation on  $A$  and that  $F : A \rightarrow B$ . Assume that  $F$  is compatible with  $R$ ; that is, whenever  $x R y$ , then  $F(x) = F(y)$ . Then there exists a unique  $\bar{F} : A/R \rightarrow B$  such that  $F = \bar{F} \circ \phi$ .

PROOF: The unique such  $\bar{F}$  is  $\{([x], F(x)) : x \in A\}$ . □

## 11 Well-Founded Relations

**Definition 83** (Well Founded). A relation  $R$  on a class  $D$  is *well-founded* iff every nonempty subset of  $D$  has an  $R$ -minimal element.

**Theorem 84** (Transfinite Induction). *Let  $R$  be a well-founded relation on  $A$ . Let  $B \subseteq A$ . Assume that, for all  $x \in A$ , if  $\forall y \in A (yRx \Rightarrow y \in B)$ , then  $x \in B$ . Then  $B = A$ .*

PROOF: If not,  $A - B$  has an  $R$ -minimal element  $a_0$ , say. But then we have  $\forall y. (yRa_0 \Rightarrow y \in B)$  and  $a_0 \notin B$ , which is a contradiction.  $\square$

**Theorem 85** (Transfinite Recursion Theorem Schema). *For any property  $P(x, y, z)$  the following is a theorem:*

*Assume that  $<$  is a well-founded relation on  $A$ . Assume that  $\forall x, y \exists! z P(x, y, z)$ . Then there exists a unique function  $F$  with domain  $A$  such that*

$$\forall t \in A. P(F \upharpoonright \text{seg } t, t, F(t)) .$$

PROOF:

- $\langle 1 \rangle 1$ . Given  $t \in A$ , let us say that a function  $v$  is *P-constructed up to  $t$*  iff  $\text{dom } v = \{x \in A : x \leq t\}$  and  $\forall x \in \text{dom } v. P(v \upharpoonright \text{seg } x, x, v(x))$
- $\langle 1 \rangle 2$ . Let  $t_1, t_2 \in A$  with  $t_1 \leq t_2$ . Let  $v_1$  be a function that is *P-constructed up to  $t_1$* , and  $v_2$  a function that is *P-constructed up to  $t_2$* . Then  $\forall x \leq t_1. v_1(x) = v_2(x)$

$\langle 2 \rangle 1$ . LET:  $x \leq t_1$

$\langle 2 \rangle 2$ . ASSUME:  $\forall y < x. v_1(y) = v_2(y)$

$\langle 2 \rangle 3$ .  $v_1 \upharpoonright \text{seg } x = v_2 \upharpoonright \text{seg } x$

$\langle 2 \rangle 4$ .  $P(v_1 \upharpoonright \text{seg } x, v_1(x))$

$\langle 2 \rangle 5$ .  $P(v_2 \upharpoonright \text{seg } x, v_2(x))$

$\langle 2 \rangle 6$ .  $v_1(x) = v_2(x)$

PROOF: Since there is only one  $y$  such that  $P(v_1 \upharpoonright \text{seg } x, x, y)$ .

$\langle 2 \rangle 7$ . Q.E.D.

PROOF: By transfinite induction.

- $\langle 1 \rangle 3$ . LET:  $\mathcal{H} = \{v : \exists t \in A. v \text{ is } P\text{-constructed up to } t\}$

- $\langle 1 \rangle 4$ .  $\mathcal{H}$  is a set.

PROOF: By a Replacement Axiom since, if  $v_1$  and  $v_2$  are both *P-constructed up to  $t$*  then  $v_1 = v_2$  by  $\langle 1 \rangle 2$ .

- $\langle 1 \rangle 5$ . LET:  $F = \bigcup \mathcal{H}$

- $\langle 1 \rangle 6$ .  $F$  is a function

$\langle 2 \rangle 1$ . ASSUME:  $tFx$  and  $tFy$

$\langle 2 \rangle 2$ . PICK  $v_1, v_2 \in \mathcal{H}$  such that  $v_1(t) = x$  and  $v_2(t) = y$

$\langle 2 \rangle 3$ . PICK  $t_1, t_2 \in A$  such that  $v_1$  is *P-constructed up to  $t_1$*  and  $v_2$  is *P-constructed up to  $t_2$*

$\langle 2 \rangle 4$ . ASSUME: w.l.o.g.  $t_1 \leq t_2$

$\langle 2 \rangle 5$ .  $v_1(t) = v_2(t)$

PROOF: By  $\langle 1 \rangle 2$

$\langle 2 \rangle 6. x = y$   
 $\langle 1 \rangle 7. \forall x \in \text{dom } F. P(F \upharpoonright \text{seg } x, x, F(x))$   
 $\langle 2 \rangle 1. \text{LET: } x \in \text{dom } F$   
 $\langle 2 \rangle 2. \text{PICK } v \in \mathcal{H} \text{ such that } x \in \text{dom } v$   
 $\langle 2 \rangle 3. P(v \upharpoonright \text{seg } x, x, v(x))$   
 $\langle 2 \rangle 4. v \upharpoonright \text{seg } x = F \upharpoonright \text{seg } x$   
 $\text{PROOF: } \forall y < x. (y, v(y)) \in \bigcup \mathcal{H} = F$   
 $\langle 2 \rangle 5. v(x) = F(x)$   
 $\text{PROOF: } (x, v(x)) \in \bigcup \mathcal{H} = F$   
 $\langle 1 \rangle 8. \text{dom } F = A$   
 $\langle 2 \rangle 1. \text{LET: } x \in A$   
 $\langle 2 \rangle 2. \text{ASSUME: } \forall y < x. y \in \text{dom } F$   
 $\langle 2 \rangle 3. \text{LET: } z \text{ be the object such that } P(F \upharpoonright \text{seg } x, z)$   
 $\langle 2 \rangle 4. F \upharpoonright \text{seg } x \cup \{(x, z)\} \text{ is } P\text{-constructed up to } x$   
 $\langle 2 \rangle 5. x \in \text{dom } F$   
 $\langle 2 \rangle 6. \text{Q.E.D.}$   
 $\text{PROOF: By transfinite induction, this proves } \forall x \in A. x \in \text{dom } F.$   
 $\langle 1 \rangle 9. F \text{ is unique.}$   
 $\langle 2 \rangle 1. \text{LET: } G \text{ be a function with domain } A \text{ such that } \forall x \in A. P(G \upharpoonright \text{seg } x, x, G(x))$   
 $\text{PROVE: } \forall x \in A. F(x) = G(x)$   
 $\langle 2 \rangle 2. \text{LET: } x \in A$   
 $\langle 2 \rangle 3. \text{ASSUME: } \forall y < x. F(y) = G(y)$   
 $\langle 2 \rangle 4. F \upharpoonright \text{seg } x = G \upharpoonright \text{seg } x$   
 $\langle 2 \rangle 5. F(x) = G(x)$   
 $\langle 2 \rangle 6. \text{Q.E.D.}$   
 $\text{PROOF: This completes the proof by transfinite induction.}$

□

## 12 Transitive Closure

**Theorem 86.** *For any relation  $R$  on a set  $A$ , there exists a least transitive relation  $R^t$  such that  $R \subseteq R^t$ .*

**PROOF:** Define  $R^t$  to be the intersection of all the transitive relations  $Q$  such that  $R \subseteq Q$ . □

**Theorem 87.** *The transitive closure of a well-founded relation is well-founded.*

**PROOF:** The  $R$ -minimal element of a nonempty set  $B$  is also the  $R^t$ -minimal element. □

## 13 Partial Orders

**Definition 88** (Strict Partial Order). A *strict partial order* is an irreflexive, transitive relation.

If  $<$  is a strict partial order, we write  $x \leq y$  for  $x < y \vee x = y$ .

**Theorem 89.** Assume that  $<$  is a partial order. Then for any  $x, y$  and  $z$ :

1. At most one of the three alternatives,

$$x < y, x = y, y < x,$$

can hold.

2.  $x \leq y \leq x \Rightarrow x = y$ .

PROOF: Easy.  $\square$

**Definition 90** (Maximal). Let  $<$  be a partial order on  $D$ . An element  $m \in D$  is *maximal* iff there is no  $x \in D$  such that  $m < x$ .

**Definition 91** (Least). Let  $<$  be a partial order on  $D$ . An element  $m \in D$  is *least*, *smallest* or the *minimum* iff  $\forall x \in D. m \leq x$ .

**Definition 92** (Greatest). Let  $<$  be a partial order on  $D$ . An element  $m \in D$  is *greatest*, *largest* or the *maximum* iff  $\forall x \in D. x \leq m$ .

**Proposition 93.** If  $R$  is a partial ordering on  $D$  then so is  $R^{-1}$ .

PROOF: Easy.  $\square$

**Definition 94** (Upper Bound). Let  $<$  be a partial order on  $A$  and  $C \subseteq A$ . An *upper bound* for  $C$  is an element  $b \in A$  such that  $\forall x \in C. x \leq b$ .

**Definition 95** (Least Upper Bound). Let  $<$  be a partial order on  $A$  and  $C \subseteq A$ . The *least upper bound* or *supremum* for  $C$  is the least element in the set of upper bounds for  $C$ .

**Definition 96** (Lower Bound). Let  $<$  be a partial order on  $A$  and  $C \subseteq A$ . A *lower bound* for  $C$  is an element  $b \in A$  such that  $\forall x \in C. b \leq x$ .

**Definition 97** (Greatest Lower Bound). Let  $<$  be a partial order on  $A$  and  $C \subseteq A$ . The *greatest lower bound* or *infimum* for  $C$  is the greatest element in the set of lower bounds for  $C$ .

**Definition 98** (Initial Segment). Let  $<$  be a partial order on  $A$  and  $t \in A$ . The *initial segment* up to  $t$  is

$$\text{seg } t = \{x \in A : x < t\} .$$

**Definition 99** (Isomorphism). Let  $A$  and  $B$  be posets. An *isomorphism* between  $A$  and  $B$  is a bijection  $f$  between  $A$  and  $B$  such that, for all  $x, y \in A$ , we have  $x < y$  if and only if  $f(x) < f(y)$ .

**Proposition 100.** Isomorphism is an equivalence relation on the class of posets.

PROOF: Easy.  $\square$

**Proposition 101.** Let  $(A, <)$  be a poset and  $B \subseteq A$ . Then  $< \cap B^2$  is a partial order on  $B$ .

PROOF: Easy.  $\square$

**Theorem 102.** Let  $R$  be a well-founded relation on  $A$ . The transitive closure of  $R$  is a partial order on  $A$ .

PROOF: It is well founded, hence irreflexive.  $\square$

## 14 Linear Orders

**Definition 103** (Linear Ordering). Let  $\mathbf{A}$  be a class. A *linear ordering* or *total ordering* on  $\mathbf{A}$  is a relation  $\mathbf{R}$  on  $\mathbf{A}$  such that:

- $\mathbf{R}$  is transitive.
- $\mathbf{R}$  satisfies *trichotomy* on  $\mathbf{A}$ ; i.e. for any  $x, y \in \mathbf{A}$ , exactly one of

$$x\mathbf{R}y, x = y, y\mathbf{R}x$$

holds.

**Theorem 104.** Let  $\mathbf{R}$  be a linear ordering on  $\mathbf{A}$ .

1. There is no  $x$  such that  $x\mathbf{R}x$ .
2. For distinct  $x$  and  $y$  in  $\mathbf{A}$ , either  $x\mathbf{R}y$  or  $y\mathbf{R}x$ .

PROOF: Immediate from trichotomy.  $\square$

**Definition 105** (Strictly Monotone Functions). Let  $A$  and  $B$  be linearly ordered sets. A function  $f : A \rightarrow B$  is *strictly monotone* iff, for all  $x, y \in A$ , if  $x < y$  then  $f(x) < f(y)$ .

**Theorem 106.** Let  $A$  and  $B$  be linearly ordered sets and  $f : A \rightarrow B$  be strictly monotone. For all  $x, y \in A$ , if  $f(x) < f(y)$  then  $x < y$ .

PROOF: We have  $f(x) \neq f(y)$  and  $f(y) \not< f(x)$  by trichotomy, hence  $x \neq y$  and  $y \not< x$  since  $f$  is strictly monotone, hence  $x < y$  by trichotomy.  $\square$

**Theorem 107.** Every strictly monotone function is injective.

PROOF: If  $f(x) = f(y)$ , then we have  $f(x) \not< f(y)$  and  $f(y) \not< f(x)$  by trichotomy, hence  $x \not< y$  and  $y \not< x$  since  $f$  is strictly monotone, hence  $x = y$  by trichotomy.  $\square$

**Proposition 108.** Let  $(A, <)$  be a linearly ordered set and  $B \subseteq A$ . Then  $< \cap B^2$  is a linear order on  $B$ .

PROOF: Easy.  $\square$

**Definition 109.** Let  $A$  and  $B$  be disjoint linearly ordered sets. The *concatenation* of  $A$  and  $B$ ,  $A \oplus B$ , is the set  $A \cup B$  under the order given by:  $x < y$  iff

- $x, y \in A$  and  $x < y$ ; or
- $x, y \in B$  and  $x < y$ ; or
- $x \in A$  and  $y \in B$ .

It is easy to check this is a linear ordering.

**Proposition 110.**

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

PROOF: Easy.  $\square$

**Proposition 111.**

$$A \oplus \emptyset = \emptyset \oplus A = A$$

PROOF: Easy.  $\square$

**Definition 112.** Let  $A$  and  $B$  be linearly ordered sets. The *lexicographic order* on  $A \times B$  is defined by:  $(a_1, b_1) < (a_2, b_2)$  iff  $a_1 < a_2$  or  $(a_1 = a_2 \text{ and } b_1 < b_2)$ .

**Proposition 113.** *These two orders on  $A \times B \times C$  are equal:*

- *lexicographic order formed from (lexicographic order on  $A \times B$ ) and order on  $C$*
- *lexicographic order formed from order on  $A$  and (lexicographic order on  $B \times C$ )*

PROOF: Easy.  $\square$

**Proposition 114.**

$$A \times 1 = 1 \times A = A$$

PROOF: Easy.  $\square$

**Proposition 115.**  $A \times (B \oplus C) = (A \times B) \oplus (A \times C)$

PROOF: Easy.  $\square$

## 15 Well Orderings

**Definition 116** (Well Ordering). A *well ordering* on a set  $A$  is a linear ordering on  $A$  such that every nonempty subset of  $A$  has a least element.

**Theorem 117.** *Assume that  $<$  is a linear ordering on  $A$ . Assume that the only  $<$ -inductive subset of  $A$  is  $A$  itself. Then  $<$  is a well ordering on  $A$ .*



PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $B \subseteq A$  has no least element.

$\langle 1 \rangle 2$ .  $A - B$  is  $<$ -inductive.

$\langle 2 \rangle 1$ . LET:  $t \in A$

$\langle 2 \rangle 2$ . ASSUME:  $\text{seg } t \subseteq A - B$

$\langle 2 \rangle 3$ .  $t \notin B$

PROOF: If it were, it would be the least element of  $B$ .

$\langle 2 \rangle 4$ .  $t \in A - B$

$\langle 1 \rangle 3$ .  $A - B = A$

$\langle 1 \rangle 4$ .  $B = \emptyset$

□

**Proposition 118.** *Let  $(A, <)$  be a well ordered set and  $B \subseteq A$ . Then  $< \cap B^2$  is a well order on  $B$ .*

PROOF: Easy. □

**Theorem 119.** *Let  $A$  and  $B$  be well-ordered sets. Then one of the following holds:*

- $A \cong B$
- $\exists b \in B. A \cong \text{seg } b$
- $\exists a \in A. \text{seg } a \cong B$

PROOF:

$\langle 1 \rangle 1$ . PICK  $e$  that is not a member of  $A$  or  $B$

$\langle 1 \rangle 2$ . Define  $F : A \rightarrow B \cup \{e\}$  by:

$$F(t) = \begin{cases} \text{the least element of } B - F(\text{seg } t) & \text{if } B - F(\text{seg } t) \neq \emptyset \\ e & \text{if } B - F(\text{seg } t) = \emptyset \end{cases}$$

$\langle 1 \rangle 3$ . CASE:  $e \in \text{ran } F$

$\langle 2 \rangle 1$ . LET:  $a \in A$  be least such that  $B - F(\text{seg } a) = \emptyset$

$\langle 2 \rangle 2$ .  $F \upharpoonright \text{seg } a : \text{seg } a \cong B$

$\langle 1 \rangle 4$ . CASE:  $\text{ran } F = B$

PROOF: In this case  $F : A \cong B$ .

$\langle 1 \rangle 5$ . CASE:  $\text{ran } F \subset B$

$\langle 2 \rangle 1$ . LET:  $b \in B$  be least such that  $b \notin \text{ran } F$

$\langle 2 \rangle 2$ .  $F : A \cong \text{seg } b$

□

**Theorem 120.** *The concatenation of two well-orderings is a well ordering.*

PROOF: Easy. □

**Theorem 121.** *The lexicographic ordering on the product of two well-ordered sets is a well ordering.*

PROOF: Easy. □

## 16 Epsilon-Images

**Lemma 122.** *Let  $<$  be a well ordering on  $A$ . Let  $E$  be the function on  $A$  defined by transfinite recursion thus:*

$$E(t) = \{E(x) : x < t\} \quad (t \in A) .$$

Let  $\alpha = \text{ran } E$ . Then:

1.  $\forall t \in A. E(t) \notin E(t)$
2.  $E$  is injective.
3.  $\forall s, t \in A. (s < t \Leftrightarrow E(s) \in E(t))$
4.  $\alpha$  is a transitive set.

PROOF:

$\langle 1 \rangle 1.$   $\forall t \in A. E(t) \notin E(t)$

$\langle 2 \rangle 1.$  LET:  $t \in A$

$\langle 2 \rangle 2.$  ASSUME:  $\forall s < t. E(s) \notin E(s)$

$\langle 2 \rangle 3.$  ASSUME: for a contradiction  $E(t) \in E(t)$

$\langle 2 \rangle 4.$  PICK  $x < t$  such that  $E(t) = E(x)$

$\langle 2 \rangle 5.$   $E(x) \in E(x)$

$\langle 2 \rangle 6.$  Q.E.D.

PROOF: This is a contradiction. The result follows by transfinite induction.

$\langle 1 \rangle 2.$   $E$  is injective.

$\langle 2 \rangle 1.$  ASSUME: for a contradiction  $E(x) = E(y)$  where  $x \neq y$

$\langle 2 \rangle 2.$  ASSUME: w.l.o.g.  $x < y$

$\langle 2 \rangle 3.$   $E(x) \in E(y)$

$\langle 2 \rangle 4.$  Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 1.$

$\langle 1 \rangle 3.$   $\forall s, t \in A. (s < t \Leftrightarrow E(s) \in E(t))$

$\langle 2 \rangle 1.$  LET:  $s, t \in A$

$\langle 2 \rangle 2.$  If  $s < t$  then  $E(s) \in E(t)$

PROOF: Immediate from definition of  $E$ .

$\langle 2 \rangle 3.$  If  $E(s) \in E(t)$  then  $s < t$

$\langle 3 \rangle 1.$  ASSUME:  $E(s) \in E(t)$

$\langle 3 \rangle 2.$  PICK  $x < t$  such that  $E(s) = E(x)$

$\langle 3 \rangle 3.$   $s = x$

PROOF:  $\langle 1 \rangle 2.$

$\langle 3 \rangle 4.$   $s < t$

$\langle 1 \rangle 4.$   $\alpha$  is a transitive set.

PROOF: From definition of  $E$ .

**Corollary 122.1.** *For any well-ordered set  $(A, <)$ , if  $\alpha$  is its epsilon-image, then  $(A, <)$  is isomorphic to  $(\alpha, \in)$ .*

**Corollary 122.2.** *The epsilon-image of any well-ordered set is well ordered by  $\in$ .*

**Theorem 123.** *Two well-ordered sets are isomorphic iff they have the same  $\epsilon$ -image.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $B$  be well-ordered sets.

$\langle 1 \rangle 2$ . If  $A$  and  $B$  have the same  $\epsilon$ -image then they are isomorphic.

PROOF: From Corollary 122.1.

$\langle 1 \rangle 3$ . If  $A \cong B$  then  $A$  and  $B$  have the same epsilon-image.

$\langle 2 \rangle 1$ . LET:  $f : A \cong B$

$\langle 2 \rangle 2$ . LET:  $E : A \cong \alpha$  and  $F : B \cong \beta$  be the canonical isomorphisms between  $A$  and  $B$  and their epsilon-images.

$\langle 2 \rangle 3$ .  $\forall x \in A. E(x) = F(f(x))$

$\langle 3 \rangle 1$ . LET:  $x \in A$

$\langle 3 \rangle 2$ . ASSUME:  $\forall y < x. E(y) = F(f(y))$

$\langle 3 \rangle 3$ .  $E(x) = F(f(x))$

PROOF:

$$\begin{aligned} E(x) &= \{E(y) : y < x\} \\ &= \{F(f(y)) : y < x\} \\ &= \{F(z) : z < f(x)\} \\ &= F(f(x)) \end{aligned}$$

$\langle 2 \rangle 4$ .  $\alpha = \beta$

□

## 17 Ordinal Numbers

**Definition 124** (Ordinal Number). The *ordinal number* of a well-ordered set is its epsilon-image.

**Definition 125** (Well-ordered by Epsilon). A set  $A$  is *well-ordered by epsilon* iff  $\{(x, y) : x, y \in A, x \in y\}$  is a well ordering on  $A$ .

**Theorem 126.** *A set is an ordinal number if and only if it is a transitive set that is well-ordered by epsilon.*

PROOF:

$\langle 1 \rangle 1$ . Every ordinal number is a transitive set.

PROOF: Lemma 122.

$\langle 1 \rangle 2$ . Every ordinal number is well-ordered by epsilon.

PROOF: Corollary 122.2.

$\langle 1 \rangle 3$ . Every transitive set that is well-ordered by epsilon is an ordinal number.

$\langle 2 \rangle 1$ . LET:  $\alpha$  be a transitive set well-ordered by epsilon.

$\langle 2 \rangle 2$ . LET:  $\beta$  be the epsilon-image of  $(\alpha, \in)$  with  $E : \alpha \cong \beta$  the canonical isomorphism.

$\langle 2 \rangle 3$ .  $\forall x \in \alpha. E(x) = x$

$\langle 3 \rangle 1$ . LET:  $x \in \alpha$

$\langle 3 \rangle 2$ . ASSUME:  $\forall y < x. E(y) = y$

$\langle 3 \rangle 3. E(x) = x$

PROOF:

$$\begin{aligned}
 E(x) &= \{E(y) : y \in \alpha, y \in x\} \\
 &= \{E(y) : y \in x\} && (\alpha \text{ is a transitive set}) \\
 &= \{y : y \in x\} && (\langle 3 \rangle 2) \\
 &= x
 \end{aligned}$$

$\langle 2 \rangle 4. \alpha = \beta$

□

**Theorem 127.** *Every member of an ordinal number is an ordinal number.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $\alpha$  be an ordinal number.

$\langle 1 \rangle 2.$  LET:  $\beta \in \alpha$

$\langle 1 \rangle 3.$   $\beta$  is a transitive set.

$\langle 2 \rangle 1.$  LET:  $x \in y \in \beta$

$\langle 2 \rangle 2.$   $y \in \alpha$

PROOF: Since  $\alpha$  is a transitive set.

$\langle 2 \rangle 3.$   $x \in \alpha$

PROOF: Since  $\alpha$  is a transitive set.

$\langle 2 \rangle 4.$   $x \in \beta$

PROOF: Since  $\alpha$  is a partially ordered by epsilon.

$\langle 1 \rangle 4.$   $\beta$  is well-ordered by epsilon.

PROOF: Since  $\{(x, y) : x, y \in \beta, x \in y\}$  is the restriction of  $\{(x, y) : x, y \in \alpha, x \in y\}$  to  $\beta$ .

$\langle 1 \rangle 5.$   $\beta$  is an ordinal number.

PROOF: Theorem 126.

□

**Proposition 128.** *The class of ordinals is well-ordered by epsilon.*

PROOF:

$\langle 1 \rangle 1.$  For any ordinals  $\alpha, \beta, \gamma$ , if  $\alpha \in \beta \in \gamma$  then  $\alpha \in \gamma$ .

PROOF: Since  $\gamma$  is a transitive set (Lemma 122).

$\langle 1 \rangle 2.$  For any ordinal  $\alpha$  we have  $\alpha \notin \alpha$ .

PROOF: Since  $\alpha$  is well-ordered by epsilon.

$\langle 1 \rangle 3.$  For any ordinals  $\alpha, \beta$ , exactly one of  $\alpha \in \beta, \beta \in \alpha, \alpha = \beta$  holds.

$\langle 2 \rangle 1.$  LET:  $\alpha, \beta$  be ordinals.

$\langle 2 \rangle 2.$  Either  $\alpha \cong \beta$  or  $\exists \gamma \in \beta. \alpha \cong \gamma$  or  $\exists \gamma \in \alpha. \gamma \cong \alpha$

PROOF: Theorem 119.

$\langle 2 \rangle 3.$  Either  $\alpha = \beta$  or  $\exists \gamma \in \beta. \alpha = \gamma$  or  $\exists \gamma \in \alpha. \gamma = \alpha$

PROOF: Since any ordinal is its own epsilon-image, and isomorphic well-orderings have equal epsilon-images.

$\langle 1 \rangle 4.$  Any nonempty set of ordinals has a least element.

$\langle 2 \rangle 1.$  LET:  $A$  be a nonempty set of ordinals.

$\langle 2 \rangle 2.$  PICK  $\alpha \in A$

$\langle 2 \rangle 3.$  CASE:  $A \cap \alpha = \emptyset$

PROOF: In this case,  $\alpha$  is least in  $A$ .

$\langle 2 \rangle 4$ . CASE:  $A \cap \alpha \neq \emptyset$

PROOF: In this case, the least element of  $A \cap \alpha$  is the least element in  $A$ .

□

**Corollary 128.1.** *Any transitive set of ordinal numbers is an ordinal number.*

**Corollary 128.2.**  $\emptyset$  is an ordinal number.

We write 0 for  $\emptyset$  considered as an ordinal number.

**Definition 129** (Successor). The *successor* of a set  $a$  is the set  $a^+ = a \cup \{a\}$ .

**Corollary 129.1.** *The successor of an ordinal number is an ordinal number.*

**Corollary 129.2.** *For any set  $A$  of ordinal numbers, the set  $\bigcup A$  is an ordinal number.*

**Theorem 130** (Burali-Forti). *The class of ordinal numbers is not a set.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction the class **On** is a set.

$\langle 1 \rangle 2$ . **On** is an ordinal number.

PROOF: Corollary 128.1.

$\langle 1 \rangle 3$ . **On**  $\in$  **On**

$\langle 1 \rangle 4$ . Q.E.D.

PROOF: This contradicts Lemma 122.

□

**Theorem 131** (Hartogs). *For any set  $A$ , there exists an ordinal not dominated by  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be a set.

$\langle 1 \rangle 2$ . LET:  $\alpha = \{\beta : \beta \text{ is an ordinal, } \beta \preceq A\}$ .

$\langle 1 \rangle 3$ . LET:  $W = \{(B, <) : B \subseteq A, < \text{ is a well ordering on } B\}$

$\langle 1 \rangle 4$ .  $\forall \beta \in \alpha. \exists (B, <) \in W. \beta$  is the epsilon-image of  $(B, <)$

$\langle 2 \rangle 1$ . LET:  $\beta \in \alpha$

$\langle 2 \rangle 2$ . PICK an injection  $f : \beta \rightarrow A$

$\langle 2 \rangle 3$ . Define  $<$  on  $f(\beta)$  by:  $f(\gamma) < f(\delta)$  iff  $\gamma \in \delta$

$\langle 2 \rangle 4$ .  $<$  well orders  $f(\beta)$

$\langle 2 \rangle 5$ .  $\beta$  is the epsilon-image of  $(f(\beta), <)$  with  $f^{-1}$  the canonical isomorphism.

$\langle 1 \rangle 5$ .  $\alpha$  is a set.

PROOF: By a Replacement Axiom applied to  $W$ .

$\langle 1 \rangle 6$ .  $\alpha$  is an ordinal.

$\langle 2 \rangle 1$ .  $\alpha$  is a transitive set.

$\langle 3 \rangle 1$ . LET:  $\beta \in \gamma \in \alpha$

$\langle 3 \rangle 2$ .  $\beta \subseteq \gamma \preceq A$

$\langle 3 \rangle 3$ .  $\beta \preceq A$

$\langle 3 \rangle 4. \beta \in \alpha$   
 $\langle 2 \rangle 2. \text{ Q.E.D.}$   
 PROOF: By Corollary 128.1.  
 $\langle 1 \rangle 7. \alpha \not\subseteq A$   
 PROOF: Because  $\alpha \notin \alpha$ .  
 $\square$

**Theorem 132** (Zorn's Lemma). *The following statements are equivalent:*

1. *The Axiom of Choice*

*Well-Ordering Theorem* For any set  $A$ , there exists a well ordering on  $A$ .

*Zorn's Lemma* Let  $\mathcal{A}$  be a set such that, for every chain  $\mathcal{B} \subseteq \mathcal{A}$ , we have  $\bigcup \mathcal{B} \in \mathcal{A}$ . Then  $\mathcal{A}$  has a maximal element.

PROOF:

- $\langle 1 \rangle 1.$  If the Axiom of Choice is true then the Well-Ordering Theorem is true.  
 $\langle 2 \rangle 1.$  ASSUME: The Axiom of Choice  
 $\langle 2 \rangle 2.$  LET:  $A$  be any set.  
 $\langle 2 \rangle 3.$  PICK an ordinal  $\alpha$  not dominated by  $A$ .  
 $\langle 2 \rangle 4.$  PICK an object  $e$  such that  $e \notin A$ .  
 $\langle 2 \rangle 5.$  PICK a choice function  $G : \mathcal{P}A - \{\emptyset\} \rightarrow A$  for  $A$ .  
 $\langle 2 \rangle 6.$  Define the function  $F : \alpha \rightarrow A \cup \{e\}$  by transfinite recursion thus:  

$$F(\gamma) = \begin{cases} G(A - \{F(\delta) : \delta < \gamma\}) & \text{if } A - \{F(\delta) : \delta < \gamma\} \neq \emptyset \\ e & \text{if } A - \{F(\delta) : \delta < \gamma\} = \emptyset \end{cases}$$
 $\langle 2 \rangle 7.$  LET:  $\delta$  be least such that  $F(\delta) = e$   
 PROOF: There is such a  $\delta$ , otherwise  $F$  would be a bijection between  $\alpha$  and  $A$ .  
 $\langle 2 \rangle 8.$   $F \upharpoonright \delta$  is a bijection between  $\delta$  and  $A$   
 $\langle 2 \rangle 9.$  Define  $<$  on  $A$  by:  $F(\gamma) < F(\beta)$  iff  $\gamma \in \beta$  for  $\gamma, \beta \in \delta$   
 $\langle 2 \rangle 10.$   $<$  is a well ordering on  $A$ .  
 $\langle 1 \rangle 2.$  If the Well-Ordering Theorem is true then Zorn's Lemma is true.  
 $\langle 2 \rangle 1.$  ASSUME: The Well-Ordering Theorem  
 $\langle 2 \rangle 2.$  LET:  $\mathcal{A}$  be a set that is closed under unions of chains.  
 $\langle 2 \rangle 3.$  PICK a well ordering  $<$  on  $\mathcal{A}$   
 $\langle 2 \rangle 4.$  Define the function  $F : \mathcal{A} \rightarrow 2$  by transfinite recursion thus:  

$$F(A) = \begin{cases} 1 & \text{if } \forall B < A. F(B) = 1 \Rightarrow B \subseteq A \\ 0 & \text{otherwise} \end{cases}$$
 $\langle 2 \rangle 5.$  LET:  $\mathcal{C} = \{A \in \mathcal{A} : F(A) = 1\}$   
 $\langle 2 \rangle 6.$   $\mathcal{C}$  is a chain.  
 $\langle 3 \rangle 1.$  LET:  $A, B \in \mathcal{C}$   
 $\langle 3 \rangle 2.$  ASSUME: w.l.o.g.  $A < B$   
 $\langle 3 \rangle 3.$   $F(A) = 1$   
 $\langle 3 \rangle 4.$   $F(B) = 1$   
 $\langle 3 \rangle 5.$   $A \subseteq B$   
 $\langle 2 \rangle 7.$   $\bigcup \mathcal{C} \in \mathcal{A}$

PROOF: By  $\langle 2 \rangle 2$ .  
 $\langle 2 \rangle 8$ .  $\bigcup \mathcal{C}$  is maximal in  $\mathcal{A}$   
 $\langle 3 \rangle 1$ . ASSUME:  $\bigcup \mathcal{C} \subseteq D \in \mathcal{A}$   
 $\langle 3 \rangle 2$ .  $\forall B < D. F(B) = 1 \Rightarrow B \subseteq D$   
PROOF: If  $F(B) = 1$  then  $B \in \mathcal{C}$  so  $B \subseteq \bigcup \mathcal{C} \subseteq D$ .  
 $\langle 3 \rangle 3$ .  $F(D) = 1$   
 $\langle 3 \rangle 4$ .  $D \in \mathcal{C}$   
 $\langle 3 \rangle 5$ .  $D = \bigcup \mathcal{C}$   
 $\langle 1 \rangle 3$ . If Zorn's Lemma is true then the Axiom of Choice is true.  
 $\langle 2 \rangle 1$ . ASSUME: Zorn's Lemma  
 $\langle 2 \rangle 2$ . LET:  $R$  be a relation.  
 $\langle 2 \rangle 3$ . LET:  $\mathcal{A}$  be the set of all functions that are subsets of  $R$ .  
 $\langle 2 \rangle 4$ . For any chain  $\mathcal{B} \subseteq \mathcal{A}$  we have  $\bigcup \mathcal{B} \in \mathcal{A}$   
 $\langle 2 \rangle 5$ . PICK  $F \in \mathcal{A}$  maximal.  
 $\langle 2 \rangle 6$ .  $\text{dom } F = \text{dom } R$

□

**Theorem 133** (Well-Ordering Theorem (Choice)). *For any set  $A$ , there exists a well ordering on  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be a set.  
 $\langle 1 \rangle 2$ . PICK an ordinal  $\alpha$  not dominated by  $A$   
 $\langle 1 \rangle 3$ .  $A \preceq \alpha$   
 $\langle 1 \rangle 4$ . PICK an injection  $f : A \rightarrow \alpha$   
 $\langle 1 \rangle 5$ . Define  $<$  on  $A$  by:  $x < y$  iff  $f(x) \in f(y)$   
 $\langle 1 \rangle 6$ .  $<$  is a well ordering on  $A$ .

□

**Corollary 133.1** (Numeration Theorem (Choice)). *Any set is equinumerous to some ordinal number.*

**Theorem 134** (Transfinite Recursion). *Let  $\mathbf{F} : \mathbf{V} \rightarrow \mathbf{V}$ . Then there exists a function  $\mathbf{G} : \mathbf{On} \rightarrow \mathbf{V}$  such that*

$$\forall \alpha \in \mathbf{On}. \mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \upharpoonright \alpha) .$$

PROOF: Define  $\mathbf{G} = \{(\alpha, y) : \exists f : \alpha^+ \rightarrow \mathbf{V}. \forall \beta \in \alpha^+. f(\beta) = \mathbf{F}(f \upharpoonright \beta)\}$ . □

**Definition 135** (Continuous). A function  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  is *continuous* iff  $\mathbf{F}(\lambda) = \bigcup_{\beta \in \lambda} \mathbf{F}(\beta)$  for every limit ordinal  $\lambda$ .

**Theorem 136**. *Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  be continuous. Suppose  $\forall \alpha \in \mathbf{On}. \mathbf{F}(\alpha) < \mathbf{F}(\alpha + 1)$ . Then  $\mathbf{F}$  is strictly monotone.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $P(\beta)$  be the statement:  $\forall \alpha < \beta. \mathbf{F}(\alpha) < \mathbf{F}(\beta)$   
 $\langle 1 \rangle 2$ .  $P(0)$

PROOF: Vacuous.

$\langle 1 \rangle 3. \forall \beta \in \mathbf{On}. P(\beta) \Rightarrow P(\beta^+)$

PROOF: For  $\alpha < \beta^+$  we have  $\mathbf{F}(\alpha) \leq \mathbf{F}(\beta) < \mathbf{F}(\beta^+)$ .

$\langle 1 \rangle 4.$  For every limit ordinal  $\lambda$ , if  $\forall \beta < \lambda. P(\beta)$  then  $P(\lambda)$

PROOF: For  $\alpha < \lambda$  we have  $\mathbf{F}(\alpha) < \mathbf{F}(\alpha^+) \leq \mathbf{F}(\lambda)$ .

□

**Definition 137** (Normal). A function  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  is *normal* iff it is strictly monotone and continuous.

**Theorem 138.** Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  be normal. Let  $t_0 \leq \beta$ . Then there exists a greatest  $\gamma$  such that  $\mathbf{F}(\gamma) \leq \beta$ .

PROOF:

$\langle 1 \rangle 1.$  LET:  $\gamma = \{\alpha \in \mathbf{On} : \mathbf{F}(\alpha) \leq \beta\}$

$\langle 1 \rangle 2.$   $\gamma$  is an ordinal.

$\langle 2 \rangle 1.$   $\gamma$  is a set.

PROOF: We have  $\alpha \leq \mathbf{F}(\alpha)$  for all  $\alpha$ , so  $\gamma \subseteq \beta$ .

$\langle 2 \rangle 2.$   $\gamma$  is a transitive set.

PROOF: If  $\alpha < \alpha'$  and  $\mathbf{F}(\alpha') \leq \beta$  then  $\mathbf{F}(\alpha) < \beta$  by monotonicity.

$\langle 1 \rangle 3.$   $\gamma \neq 0$

PROOF: By hypothesis.

$\langle 1 \rangle 4.$  CASE:  $\gamma$  is a successor ordinal.

PROOF: Let  $\gamma = \alpha^+$ . Then  $\alpha$  is greatest such that  $\mathbf{F}(\alpha) \leq \beta$ .

$\langle 1 \rangle 5.$  CASE:  $\gamma$  is a limit ordinal.

PROOF: This is impossible since then  $\mathbf{F}(\gamma) = \bigcup_{\alpha \in \gamma} \mathbf{F}(\alpha) \leq \beta$  and so  $\gamma \in \gamma$ .

□

**Theorem 139.** Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  be normal. Let  $S$  be a nonempty set of ordinals. Then  $\mathbf{F}(\sup S) = \sup \mathbf{F}(S)$ .

PROOF:

$\langle 1 \rangle 1.$   $\mathbf{F}(\sup S) \geq \sup \mathbf{F}(S)$

PROOF: By monotonicity.

$\langle 1 \rangle 2.$   $\mathbf{F}(\sup S) \leq \sup \mathbf{F}(S)$

$\langle 2 \rangle 1.$  CASE:  $\sup S \in S$

PROOF: Immediate.

$\langle 2 \rangle 2.$  CASE:  $\sup S \notin S$

$\langle 3 \rangle 1.$   $\sup S$  is a limit ordinal.

$\langle 3 \rangle 2.$   $\mathbf{F}(\sup S) = \sup\{\mathbf{F}(\beta) : \beta < \sup S\}$

$\langle 3 \rangle 3.$   $\forall \beta < \sup S. \mathbf{F}(\beta) \leq \sup \mathbf{F}(S)$

□

**Theorem 140** (Veblen Fixed-Point Theorem (1907)). A normal operation on ordinals has arbitrarily large fixed points.

That is, let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  be normal. For all  $\alpha \in \mathbf{On}$ , there exists  $\beta \geq \alpha$  such that  $\mathbf{F}(\beta) = \beta$ .



PROOF: Let  $\beta = \sup_{n \in \omega} F^n(\alpha)$ . Then  $\alpha \leq \beta$  using monotonicity, and

$$F(\beta) = \sup_{n \in \omega} F^{n+1}(\alpha) = \beta \quad \square$$

**Definition 141** (Addition). The *sum* of two ordinal numbers is the ordinal number of their concatenation.

**Theorem 142.** *Addition is associative.*

PROOF: Easy.  $\square$

**Theorem 143.**

$$\alpha + 0 = 0 + \alpha = \alpha$$

PROOF: Easy.  $\square$

**Theorem 144.**

$$\alpha + \beta^+ = (\alpha + \beta)^+$$

PROOF: Easy.  $\square$

**Theorem 145.** *For  $\lambda$  a limit ordinal,  $\alpha + \lambda = \sup_{\beta < \lambda} (\alpha + \beta)$ .*

PROOF: Easy.  $\square$

**Theorem 146.** *For any ordinal  $\alpha$ , the function that maps  $\beta$  to  $\alpha + \beta$  is normal.*

PROOF: Easy.  $\square$

**Corollary 146.1.**

$$\beta < \gamma \Leftrightarrow \alpha + \beta < \alpha + \gamma$$

**Corollary 146.2.** *If  $\alpha + \beta = \alpha + \gamma$  then  $\beta = \gamma$ .*

**Theorem 147.** *If  $\beta \leq \gamma$  then  $\beta + \alpha \leq \gamma + \alpha$ .*

PROOF: Transfinite induction on  $\alpha$ .  $\square$

**Theorem 148** (Subtraction Theorem). *If  $\alpha \leq \beta$  then there exists a unique ordinal  $\gamma$  such that  $\alpha + \gamma = \beta$ .*

PROOF: Let  $\gamma$  be greatest such that  $\alpha + \gamma \leq \beta$ .  $\square$

**Definition 149** (Multiplication). The *product* of two ordinal numbers  $\alpha$  and  $\beta$  is the ordinal number of  $\alpha \times \beta$  under the lexicographic ordering.

**Theorem 150.** *Multiplication is associative.*

PROOF: Easy.  $\square$

**Theorem 151.**

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$

PROOF: Easy.  $\square$

**Theorem 152.**

$$\alpha 1 = 1\alpha = \alpha$$

PROOF: Easy.  $\square$

**Theorem 153.**

$$\alpha 0 = 0\alpha = 0$$

PROOF: Easy.  $\square$

**Theorem 154.**

$$\alpha\beta^+ = \alpha\beta + \alpha$$

PROOF: Easy.  $\square$

**Theorem 155.** *For  $\lambda$  a limit ordinal,  $\alpha\lambda = \sup_{\beta < \lambda}(\alpha\beta)$ .*

PROOF: Easy.  $\square$

**Theorem 156.** *For any ordinal  $\alpha > 0$ , the function that maps  $\beta$  to  $\alpha\beta$  is normal.*

PROOF: Easy.  $\square$

**Corollary 156.1.** *For  $\alpha > 0$  we have*

$$\beta < \gamma \Leftrightarrow \alpha\beta < \alpha\gamma$$

**Corollary 156.2.** *For  $\alpha > 0$ , if  $\alpha\beta = \alpha\gamma$  then  $\beta = \gamma$ .*

**Theorem 157.** *If  $\beta \leq \gamma$  then  $\beta\alpha \leq \gamma\alpha$ .*

PROOF: Transfinite induction on  $\alpha$ .  $\square$

**Theorem 158** (Division Theorem). *Let  $\delta \neq 0$ . For any  $\alpha$ , there exist unique ordinals  $\beta, \gamma$  such that  $\alpha = \delta\beta + \gamma$  and  $\gamma < \delta$ .*

PROOF: Let  $\beta$  be largest such that  $\delta\beta \leq \alpha$ , and let  $\gamma$  be as given by the Subtraction Theorem.  $\square$

PROOF: Let  $\gamma$  be greatest such that  $\alpha + \gamma \leq \beta$ .  $\square$

**Definition 159** (Exponentiation). Define  $\alpha^\beta$  by transfinite recursion thus:

$$\begin{aligned}\alpha^0 &= 1 \\ \alpha^{\beta^+} &= \alpha^\beta \alpha \\ \alpha^\lambda &= \sup_{\beta < \lambda} \alpha^\beta\end{aligned}$$

for  $\lambda$  a limit ordinal.

**Theorem 160.** For  $\alpha > 1$ , the function that maps  $\beta$  to  $\alpha^\beta$  is normal.

PROOF: Easy.  $\square$

**Corollary 160.1.** For  $\alpha > 1$  we have

$$\beta < \gamma \Leftrightarrow \alpha^\beta < \alpha^\gamma$$

**Corollary 160.2.** For  $\alpha > 1$ , if  $\alpha^\beta = \alpha^\gamma$  then  $\beta = \gamma$ .

**Theorem 161.** If  $\beta \leq \gamma$  then  $\beta^\alpha \leq \gamma^\alpha$ .

PROOF: Transfinite induction on  $\alpha$ .  $\square$

**Theorem 162** (Logarithm Theorem). Let  $\alpha \neq 0$  and  $\beta > 1$ . Then there exist unique ordinals  $\gamma, \delta, \rho$  such that  $\alpha = \beta^\gamma \delta + \rho$ ,  $0 < \delta < \beta$  and  $\rho < \beta^\gamma$ .

PROOF: Let  $\gamma$  be greatest such that  $\beta^\gamma \leq \alpha$ , and then apply the Division Theorem.  $\square$

**Theorem 163.**

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$$

PROOF: Transfinite induction on  $\gamma$ .  $\square$

**Theorem 164.**

$$\alpha^{\beta\gamma} = (\alpha^\beta)^\gamma$$

PROOF: Transfinite induction on  $\gamma$ .  $\square$

## 18 Natural Numbers

**Definition 165** (Inductive). A class  $\mathbf{A}$  is *inductive* iff  $\emptyset \in \mathbf{A}$  and  $\forall a \in \mathbf{A}. a^+ \in \mathbf{A}$ .

**Axiom 166** (Infinity). There exists an inductive set.

**Definition 167** (Natural Number). A *natural number* is a set that belongs to every inductive set.

We write  $\omega$  for the class of all natural numbers.

**Theorem 168.** The class  $\omega$  is a set.

PROOF: Pick an inductive set  $I$  (by the Axiom of Infinity), then apply a Subset Axiom to  $I$ .  $\square$

**Theorem 169.** The set  $\omega$  is inductive, and is a subset of every inductive set.

PROOF: Easy.  $\square$

**Corollary 169.1** (Proof by Induction). Any inductive subclass of  $\omega$  is equal to  $\omega$ .

**Theorem 170.** *Every natural number except 0 is the successor of some natural number.*

PROOF: Easy proof by induction.  $\square$

**Definition 171** (Peano System). A *Peano system* is a triple  $\langle N, S, e \rangle$  consisting of a set  $N$ , a function  $S : N \rightarrow N$  and an element  $e \in N$  such that:

1.  $e \notin \text{ran } S$
2.  $S$  is one-to-one
3. Any subset  $A \subseteq N$  that contains  $e$  and is closed under  $S$  equals  $N$ .

**Theorem 172.** *For any transitive set  $a$ ,  $\bigcup(a^+) = a$ .*

PROOF:

$$\begin{aligned} \bigcup(a^+) &= \bigcup(a \cup \{a\}) \\ &= \bigcup a \cup \bigcup \{a\} \\ &= \bigcup a \cup a \\ &= a \end{aligned}$$

since  $\bigcup a \subseteq a$ .  $\square$

**Theorem 173.** *Every natural number is a transitive set.*

PROOF:

$\langle 1 \rangle 1$ . 0 is a transitive set.

PROOF: Vacuous.

$\langle 1 \rangle 2$ . For any natural number  $n$ , if  $n$  is a transitive set then  $n^+$  is a transitive set.

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number that is a transitive set.

$\langle 2 \rangle 2$ .  $\bigcup(n^+) \subseteq n^+$

PROOF: Theorem 172.

$\square$

**Theorem 174.**  $\langle \omega, \sigma, 0 \rangle$  is a Peano system, where  $0 = \emptyset$  and  $\sigma = \{ \langle n, n^+ \rangle : n \in \omega \}$ .

PROOF:

$\langle 1 \rangle 1$ .  $0 \notin \text{ran } \sigma$

PROOF: For any  $n \in \omega$  we have  $0 \neq n^+$  since  $n \in n^+$  and  $0 \notin n$ .

$\langle 1 \rangle 2$ .  $\sigma$  is one-to-one.

PROOF: If  $m^+ = n^+$  then  $m = \bigcup(m^+) = \bigcup(n^+) = n$  using Theorems 172 and 173.

$\langle 1 \rangle 3$ . Any subset  $A \subseteq \omega$  that contains 0 and is closed under  $\sigma$  equals  $\omega$ .

$\square$

**Theorem 175.** *The set  $\omega$  is a transitive set.*

PROOF:

$\langle 1 \rangle 1$ . For every natural number  $n$  we have  $\forall m \in n$ .  $m$  is a natural number.

$\langle 2 \rangle 1$ .  $\forall m \in 0$ .  $m$  is a natural number.

PROOF: Vacuous.

$\langle 2 \rangle 2$ . If  $n$  is a natural number and  $\forall m \in n$ .  $m$  is a natural number, then  $\forall m \in n^+$ .  $m$  is a natural number.

PROOF: Since if  $m \in n^+$  we have either  $m \in n$  or  $m = n$ , and  $m$  is a natural number in either case.

□

**Theorem 176** (Recursion Theorem on  $\omega$ ). *Let  $A$  be a set,  $a \in A$  and  $F : A \rightarrow A$ . Then there exists a unique function  $h : \omega \rightarrow A$  such that*

$$h(0) = a ,$$

and for every  $n$  in  $\omega$ ,

$$h(n^+) = F(h(n)) .$$

PROOF:

$\langle 1 \rangle 1$ . Let us call a function  $v$  *acceptable* iff  $\text{dom } v \subseteq \omega$ ,  $\text{ran } v \subseteq A$  and:

1. If  $0 \in \text{dom } v$  then  $v(0) = a$

2. For all  $n \in \omega$ , if  $n^+ \in \text{dom } v$  then  $n \in \text{dom } v$  and  $v(n^+) = F(v(n))$ .

$\langle 1 \rangle 2$ . LET:  $\mathcal{K}$  be the set of acceptable functions.

$\langle 1 \rangle 3$ . LET:  $h = \bigcup \mathcal{K}$

$\langle 1 \rangle 4$ .  $h$  is a function.

$\langle 2 \rangle 1$ . LET:  $S = \{n \in \omega : \text{for at most one } y, (n, y) \in h\}$

$\langle 2 \rangle 2$ .  $S$  is inductive.

$\langle 3 \rangle 1$ .  $0 \in S$

$\langle 4 \rangle 1$ . LET:  $\langle 0, y_1 \rangle, \langle 0, y_2 \rangle \in h$

$\langle 4 \rangle 2$ . PICK acceptable  $v_1$  and  $v_2$  such that  $v_1(0) = y_1$  and  $v_2(0) = y_2$

$\langle 4 \rangle 3$ .  $y_1 = a$

$\langle 4 \rangle 4$ .  $y_2 = a$

$\langle 4 \rangle 5$ .  $y_1 = y_2$

$\langle 3 \rangle 2$ .  $\forall k \in S. k^+ \in S$

$\langle 4 \rangle 1$ . LET:  $k \in S$

$\langle 4 \rangle 2$ . LET:  $(k^+, y_1), (k^+, y_2) \in h$

$\langle 4 \rangle 3$ . PICK acceptable  $v_1, v_2$  such that  $v_1(k^+) = y_1$  and  $v_2(k^+) = y_2$

$\langle 4 \rangle 4$ .  $y_1 = F(v_1(k))$

$\langle 4 \rangle 5$ .  $y_2 = F(v_2(k))$

$\langle 4 \rangle 6$ .  $v_1(k) = v_2(k)$

$\langle 5 \rangle 1$ .  $(k, v_1(k)), (k, v_2(k)) \in h$

$\langle 5 \rangle 2$ . Q.E.D.

PROOF: By  $\langle 4 \rangle 1$

$\langle 4 \rangle 7$ .  $y_1 = y_2$

$\langle 2 \rangle 3$ .  $S = \omega$

$\langle 1 \rangle 5$ .  $h$  is acceptable.

- ⟨2⟩1. If  $0 \in \text{dom } h$  then  $h(0) = a$ 
  - ⟨3⟩1. ASSUME:  $0 \in \text{dom } h$
  - ⟨3⟩2. PICK  $v$  acceptable with  $v(0) = h(0)$
  - ⟨3⟩3.  $v(0) = a$
  - ⟨3⟩4.  $h(0) = a$
- ⟨2⟩2. For all  $n \in \omega$ , if  $n^+ \in \text{dom } h$  then  $n \in \text{dom } h$  and  $h(n^+) = F(h(n))$ 
  - ⟨3⟩1. LET:  $n \in \omega$  with  $n^+ \in \text{dom } h$
  - ⟨3⟩2. PICK  $v$  acceptable with  $v(n^+) = h(n^+)$
  - ⟨3⟩3.  $n \in \text{dom } v$
  - ⟨3⟩4.  $v(n) = h(n)$
  - ⟨3⟩5.  $h(n^+) = F(h(n))$

PROOF:

$$\begin{aligned}
 h(n^+) &= v(n^+) \\
 &= F(v(n)) \\
 &= F(h(n))
 \end{aligned}$$

- ⟨1⟩6.  $\text{dom } h = \omega$ 
  - ⟨2⟩1.  $0 \in \text{dom } h$ 

PROOF: Since  $\{(0, a)\}$  is an acceptable function.
  - ⟨2⟩2.  $\forall n \in \text{dom } h. n^+ \in \text{dom } h$ 
    - ⟨3⟩1. LET:  $n \in \text{dom } h$
    - ⟨3⟩2. PICK an acceptable  $v$  such that  $n \in \text{dom } v$
    - ⟨3⟩3. ASSUME: w.l.o.g.  $n^+ \notin \text{dom } v$
    - ⟨3⟩4.  $v \cup \{(n^+, F(v(n)))\}$  is acceptable.
- ⟨1⟩7. For any acceptable function  $h' : \omega \rightarrow A$  we have  $h' = h$ 
  - ⟨2⟩1. LET:  $h' : \omega \rightarrow A$  be acceptable.
  - ⟨2⟩2.  $h'(0) = h(0)$ 

PROOF:  $h'(0) = h(0) = a$
  - ⟨2⟩3.  $\forall n \in \omega. h'(n) = h(n) \Rightarrow h'(n^+) = h(n^+)$ 

PROOF: We have  $h'(n^+) = F(h'(n)) = F(h(n)) = h(n^+)$ .

□

**Theorem 177.** *Let  $(N, S, e)$  be a Peano system. Then  $(\omega, \sigma, 0)$  is isomorphic to  $(N, S, e)$ , i.e. there is a function  $h$  mapping  $\omega$  one-to-one onto  $N$  in a way that preserves the successor operation*

$$h(\sigma(n)) = S(h(n))$$

and the zero element

$$h(0) = e .$$

PROOF:

- ⟨1⟩1. There exists a function  $h$  that satisfies those two conditions.

PROOF: By the Recursion Theorem.
- ⟨1⟩2. For all  $m, n \in \omega$ , if  $m \neq n$  then  $h(m) \neq h(n)$ 
  - ⟨2⟩1. For all  $n \in \omega$ , if  $n \neq 0$  then  $h(n) \neq h(0)$ 
    - ⟨3⟩1. LET:  $n \in \omega$

(3)2. ASSUME:  $n \neq 0$   
 (3)3. PICK  $p$  such that  $n = p^+$   
 (3)4.  $h(n) \neq h(0)$   
 PROOF:  $h(n) = S(h(p)) \neq e = h(0)$ .  
 (2)2. For all  $m \in \omega$ , if  $\forall n(m \neq n \Rightarrow h(m) \neq h(n))$  then  $\forall n(m^+ \neq n \Rightarrow h(m^+) \neq h(n))$   
 (3)1. LET:  $m \in \omega$   
 (3)2. ASSUME:  $\forall n(m \neq n \Rightarrow h(m) \neq h(n))$   
 (3)3. LET:  $n \in \omega$   
 (3)4. ASSUME:  $m^+ \neq n$   
 PROVE:  $h(m^+) \neq h(n)$   
 (3)5. CASE:  $n = 0$   
 PROOF:  $h(m^+) = S(h(m)) \neq e = h(n)$   
 (3)6. CASE:  $n = p^+$   
 (4)1.  $m \neq p$   
 (4)2.  $h(m) \neq h(p)$   
 (4)3.  $S(h(m)) \neq S(h(p))$   
 (4)4.  $h(m^+) \neq h(p^+)$   
 (1)3. For all  $x \in N$ , there exists  $n \in \omega$  such that  $h(n) = x$   
 PROOF: An easy induction on  $x$ .  
 □

**Theorem 178.** *There is no function  $f$  with domain  $\omega$  such that  $\dots \in f(2) \in f(1) \in f(0)$ .*

PROOF: If there were then there would be no  $m \in \text{ran } f$  such that  $m \cap \text{ran } f = \emptyset$ , contradicting the Axiom of Regularity. □

**Theorem 179 (Choice).** *Let  $R$  be a relation on  $A$ . Then  $R$  is well founded iff there does not exist any function  $f : \omega \rightarrow A$  such that  $f(n+1)Rf(n)$  for all  $n \in \omega$ .*

PROOF:

(1)1. If  $R$  is well founded then there does not exist any function  $f : \omega \rightarrow A$  such that  $f(n+1)Rf(n)$  for all  $n \in \omega$ .  
 PROOF: If there is such a function  $f$  then  $\text{ran } f$  is a nonempty subset of  $A$  with no  $R$ -minimal element.  
 (1)2. If there does not exist any function  $f : \omega \rightarrow A$  such that  $f(n+1)Rf(n)$  for all  $n \in \omega$  then  $R$  is well founded.  
 (2)1. LET:  $X \subseteq A$  be a nonempty subset of  $A$  with no  $R$ -minimal element.  
 PROVE: There exists a function  $f : \omega \rightarrow A$  such that  $f(n+1) < f(n)$  for all  $n \in \omega$   
 (2)2. PICK  $a_0 \in X$   
 (2)3.  $\forall x \in X. \exists y \in X. yRx$   
 (2)4. PICK a function  $g : X \rightarrow X$  such that  $\forall x \in X. g(x)Rx$   
 PROOF: By the Axiom of Choice.

⟨2⟩5. Define  $f : \omega \rightarrow A$  recursively by:

$$f(0) = a_0$$

$$f(n^+) = g(f(n))$$

⟨2⟩6.  $\forall n \in \omega. f(n^+) R f(n)$

□

**Alternative proof for Theorem 86** Define  $f : \omega \rightarrow \mathcal{P}A^2$  by  $f(0) = R$  and  $f(n^+) = f(n) \circ R$ . Define  $R^t = \bigcup_{n \in \omega} f(n)$ .

**Theorem 180.** *For any set  $A$ , there exists the smallest transitive set  $B$  such that  $A \subseteq B$ .*

PROOF: Define  $f : \omega \rightarrow \mathbf{V}$  by

$$f(0) = A$$

$$f(n^+) = f(n) \cup \bigcup f(n)$$

Then  $\bigcup_n f(n)$  is the smallest transitive set that includes  $A$ . □

**Definition 181** (Transitive Closure). The *transitive closure* of a set  $A$  is the least transitive set that includes  $A$ .

## 19 Finite Sets

**Definition 182** (Finite). A set is *finite* iff it is equinumerous with a natural number. Otherwise it is infinite.

**Theorem 183.** *No natural number is equinumerous with a proper subset of itself.*

PROOF:

⟨1⟩1. Any injective function  $f : 0 \rightarrow 0$  has range 0.

PROOF: Since the only such function is  $\emptyset$ .

⟨1⟩2. For any natural number  $n$ , if every injective function  $f : n \rightarrow n$  has range  $n$ , then every injective function  $f : n^+ \rightarrow n^+$  has range  $n^+$ .

⟨2⟩1. LET:  $n \in \omega$

⟨2⟩2. ASSUME: Every injective function  $f : n \rightarrow n$  has range  $n$ .

⟨2⟩3. LET:  $f : n^+ \rightarrow n^+$  be injective.

⟨2⟩4. Define  $g : n \rightarrow n$  by

$$g(k) = \begin{cases} f(k) & \text{if } f(k) \in n \\ f(n) & \text{if } f(k) = n \end{cases}$$

PROOF: If  $k \in n$  and  $f(k) = n$  then  $f(n) \in n$  since  $f$  is injective.

⟨2⟩5.  $g$  is injective.

⟨3⟩1. LET:  $i, j \in n$

⟨3⟩2. ASSUME:  $g(i) = g(j)$

⟨3⟩3. CASE:  $f(i) \in n, f(j) \in n$

PROOF: Then  $f(i) = f(j)$  so  $i = j$



$\langle 3 \rangle 4$ . CASE:  $f(i) \in n, f(j) \notin n$   
 PROOF: Then  $f(i) = f(j)$  which is impossible as  $f$  is injective.  
 $\langle 3 \rangle 5$ . CASE:  $f(i) \notin n, f(j) \in n$   
 PROOF: Then  $f(i) = f(j)$  which is impossible as  $f$  is injective.  
 $\langle 3 \rangle 6$ . CASE:  $f(i) \notin n, f(j) \notin n$   
 PROOF: Then  $f(i) = f(j) = n$  so  $i = j$ .  
 $\langle 2 \rangle 6$ .  $\text{ran } g = n$   
 PROOF: By  $\langle 2 \rangle 2$ .  
 $\langle 2 \rangle 7$ .  $\text{ran } f = n^+$   
 $\langle 3 \rangle 1$ .  $\forall k \in n. k \in \text{ran } f$   
 PROOF: Since  $\text{ran } g \subseteq \text{ran } f$ .  
 $\langle 3 \rangle 2$ .  $n \in \text{ran } f$   
 $\langle 4 \rangle 1$ . CASE:  $f(n) \in n$   
 $\langle 5 \rangle 1$ . PICK  $k$  such that  $g(k) = f(n)$   
 $\langle 5 \rangle 2$ .  $f(k) = n$   
 $\langle 4 \rangle 2$ . CASE:  $f(n) = n$   
 PROOF: Then  $n \in \text{ran } f$ .

□

**Corollary 183.1.** *No finite set is equinumerous with a proper subset of itself.*

**Corollary 183.2.** *The set  $\omega$  is infinite.*

PROOF: Since the function that maps  $n$  to  $n + 1$  is a bijection between  $\omega$  and the proper subset  $\omega - \{0\}$ . □

**Corollary 183.3.** *Every finite set is equinumerous with a unique natural number.*

**Lemma 184.** *Let  $n$  be a natural number and  $C \subseteq n$ . Then there exists  $m \in n$  such that  $C \approx m$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $C \subseteq 0$ , there exists  $m \in 0$  such that  $C \approx m$ .  
 PROOF: In this case  $C = \emptyset$  and so  $C \approx 0$ .  
 $\langle 1 \rangle 2$ . Let  $n \in \omega$ . Assume that, for all  $C \subseteq n$ , there exists  $m \in n$  such that  $C \approx m$ .  
 Let  $C \subseteq n^+$ . Then there exists  $m \in n^+$  such that  $C \approx m$ .  
 $\langle 2 \rangle 1$ . LET:  $n \in \omega$   
 $\langle 2 \rangle 2$ . ASSUME: For all  $C \subseteq n$ , there exists  $m \in n$  such that  $C \approx m$ .  
 $\langle 2 \rangle 3$ . LET:  $C \subseteq n^+$   
 $\langle 2 \rangle 4$ . CASE:  $n \in C$   
 $\langle 3 \rangle 1$ . PICK  $m \in n$  such that  $C - \{n\} \approx m$   
 $\langle 3 \rangle 2$ .  $C \approx m^+$   
 $\langle 2 \rangle 5$ . CASE:  $n \notin C$   
 PROOF: Then  $C \subseteq n$  so  $C \approx m$  for some  $m \in n$ .

□

**Corollary 184.1.** *Any subset of a finite set is finite.*

## 20 Cardinal Numbers

**Definition 185** (Cardinality (Choice)). For any set  $A$ , define the *cardinal number* of  $A$ ,  $|A|$ , to be the least ordinal that is equinumerous with  $A$ .

**Theorem 186.** For any sets  $A$  and  $B$ ,  $|A| = |B|$  if and only if  $A \approx B$ .

PROOF: Easy.  $\square$

**Theorem 187.** For any finite set  $A$ ,  $|A|$  is the natural number such that  $A \approx |A|$ .

PROOF: Immediate from definitions.  $\square$

**Definition 188.** We write  $\aleph_0$  for  $|\omega|$ .

## 21 Cardinal Arithmetic

**Definition 189** (Addition). Let  $\kappa$  and  $\lambda$  be any cardinal numbers. Then  $\kappa + \lambda = |K \cup L|$ , where  $K$  and  $L$  are any disjoint sets of cardinality  $\kappa$  and  $\lambda$  respectively.

To show this is well-defined, we must prove that, if  $K_1 \approx K_2$ ,  $L_1 \approx L_2$ , and  $K_1 \cap L_1 = K_2 \cap L_2 = \emptyset$ , then  $K_1 \cup L_1 \approx K_2 \cup L_2$ .

PROOF: Easy.

**Lemma 190.** For any cardinal number  $\kappa$  we have  $\kappa + 0 = \kappa$ .

PROOF: Since for any set  $K$  we have  $K \cup \emptyset = K$ .

**Lemma 191.** For any natural number  $n$  we have  $n + \aleph_0 = \aleph_0$ .

PROOF: Easy.  $\square$

**Lemma 192.**

$$\aleph_0 + \aleph_0 = \aleph_0$$

PROOF: Define  $f : (\omega \times \{0\}) \cup (\omega \times \{1\}) \rightarrow \omega$  by  $f(n, 0) = 2n$  and  $f(n, 1) = 2n + 1$ . Then  $f$  is a bijection.  $\square$

**Theorem 193.**

$$\kappa + \lambda = \lambda + \kappa$$

PROOF: Easy.  $\square$

**Theorem 194.**

$$\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$$

PROOF: Easy.  $\square$

**Definition 195** (Multiplication). Let  $\kappa$  and  $\lambda$  be any cardinal numbers. Then  $\kappa\lambda = |K \times L|$ , where  $K$  and  $L$  are any sets of cardinality  $\kappa$  and  $\lambda$  respectively.

It is easy to prove this well-defined.

**Lemma 196.** *For any cardinal number  $\kappa$  we have  $\kappa 0 = 0$ .*

PROOF: For any set  $K$  we have  $K \times \emptyset = \emptyset$ .  $\square$

**Lemma 197.** *For any natural number  $n$  we have  $n\aleph_0 = \aleph_0$ .*

PROOF: Induction on  $n$  using Lemma 192.  $\square$

**Lemma 198.**

$$\aleph_0 \aleph_0 = \aleph_0$$

PROOF: Define  $f : \omega \times \omega \rightarrow \omega$  by  $f(m, n) = 2^m(2n + 1) - 1$ . Then  $f$  is a bijection.  $\square$

**Lemma 199.**

$$\kappa 1 = \kappa$$

PROOF: Easy.  $\square$

**Theorem 200.**

$$\kappa \lambda = \lambda \kappa$$

PROOF: Easy.  $\square$

**Theorem 201.**

$$\kappa(\lambda \mu) = (\kappa \lambda) \mu$$

PROOF: Easy.  $\square$

**Theorem 202.**

$$\kappa(\lambda + \mu) = \kappa \lambda + \kappa \mu$$

PROOF: Easy.  $\square$

**Definition 203** (Exponentiation). Let  $\kappa$  and  $\lambda$  be any cardinal numbers. Then  $\kappa^\lambda = |K^L|$ , where  $K$  and  $L$  are any sets of cardinality  $\kappa$  and  $\lambda$  respectively.

It is easy to prove this well-defined.

**Theorem 204.** *For any cardinal  $\kappa$ ,  $\kappa^0 = 1$ .*

PROOF: For any set  $K$ , there is only one function  $\emptyset \rightarrow K$ , namely  $\emptyset$ .  $\square$

**Theorem 205.** *For any non-zero cardinal  $\kappa$ , we have  $0^\kappa = 0$ .*

PROOF: For any nonempty set  $K$ , there is no function  $K \rightarrow \emptyset$ .  $\square$

**Theorem 206.** *For any set  $A$ ,  $|\mathcal{P}A| = 2^{|A|}$ .*

PROOF: Define the bijection  $f : \mathcal{P}A \rightarrow 2^A$  by  $f(S)(a) = 1$  if  $a \in S$ , 0 if  $a \notin S$ .  $\square$

**Corollary 206.1.** *For any cardinal  $\kappa$ , we have  $\kappa \neq 2^\kappa$ .*

**Theorem 207.**

$$\kappa^{\lambda+\mu} = \kappa^\lambda \kappa^\mu$$

PROOF: Easy.  $\square$

**Theorem 208.**

$$(\kappa\lambda)^\mu = \kappa^\mu \lambda^\mu$$

PROOF: Easy.  $\square$

**Theorem 209.**

$$(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$$

PROOF: Easy.  $\square$

**Lemma 210.** *The union of a set of cardinal numbers is a cardinal number.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be a set of cardinal numbers.

$\langle 1 \rangle 2$ . LET:  $\alpha \in \bigcup A$

$\langle 1 \rangle 3$ . PICK  $\kappa \in A$  such that  $\alpha \in \kappa$

$\langle 1 \rangle 4$ .  $\alpha \prec \kappa$

$\langle 1 \rangle 5$ .  $\alpha \prec \bigcup A$

$\square$

## 22 Alephs

**Definition 211.** Define the cardinal number  $\aleph_\alpha$  for every ordinal  $\alpha$  by transfinite recursion thus:  $\aleph_\alpha$  is the least infinite cardinal different from  $\aleph_\beta$  for every  $\beta < \alpha$ .

**Theorem 212.** *If  $\alpha < \beta$  then  $\aleph_\alpha < \aleph_\beta$ .*

PROOF: By minimality of  $\aleph_\alpha$ .  $\square$

**Theorem 213.** *Every infinite cardinal is of the form  $\aleph_\alpha$  for some  $\alpha$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\kappa$  be an infinite cardinal

$\langle 1 \rangle 2$ . ASSUME: for every infinite cardinal  $\lambda < \kappa$ , there exists  $\alpha$  such that  $\lambda = \aleph_\alpha$

$\langle 1 \rangle 3$ . LET:  $\alpha = \{\beta : \aleph_\beta < \kappa\}$

$\langle 1 \rangle 4$ .  $\alpha$  is a set.

PROOF: The mapping  $\beta \mapsto \aleph_\beta$  is an injection  $\alpha \rightarrow \kappa$ .

$\langle 1 \rangle 5$ .  $\alpha$  is a transitive set.

$\langle 1 \rangle 6$ .  $\alpha$  is an ordinal.

$\langle 1 \rangle 7$ .  $\aleph_\alpha$  is the least infinite cardinal different from  $\aleph_\beta$  for all  $\beta$  such that  $\aleph_\beta < \kappa$ .

$\langle 1 \rangle 8$ .  $\aleph_\alpha$  is the least infinite cardinal different from  $\lambda$  for every infinite cardinal  $\lambda < \kappa$ .

PROOF: By  $\langle 1 \rangle 2$ .

$\langle 1 \rangle 9$ .  $\aleph_\alpha = \kappa$

$\square$

## 23 Arithmetic

**Lemma 214.** *For any natural numbers  $m$  and  $n$ , we have  $m + n^+ = (m + n)^+$ .*

PROOF: Easy.  $\square$

**Corollary 214.1.** *The union of two finite sets is finite.*

**Lemma 215.** *For any natural numbers  $m$  and  $n$  we have  $mn^+ = mn + m$ .*

PROOF: Easy.  $\square$

**Corollary 215.1.** *The Cartesian product of two finite sets is finite.*

**Lemma 216.** *For any natural numbers  $m$  and  $n$  we have  $m^{n^+} = m^n m$ .*

PROOF: Easy.  $\square$

**Corollary 216.1.** *If  $A$  and  $B$  are finite sets then  $A^B$  is finite.*

## 24 Ordering on the Natural Numbers

**Lemma 217.** *For any natural numbers  $m$  and  $n$ ,  $m \in n$  if and only if  $m^+ \in n^+$ .*

PROOF:

$\langle 1 \rangle 1.$   $\forall m, n \in \omega (m \in n \Rightarrow m^+ \in n^+)$

$\langle 2 \rangle 1.$   $\forall m \in \omega (m \in 0 \Rightarrow m^+ \in 0^+)$

PROOF: Vacuous.

$\langle 2 \rangle 2.$  For all  $n \in \omega$ , if  $\forall m \in n. m^+ \in n^+$  then  $\forall m \in n^+. m^+ \in n^{++}$

$\langle 3 \rangle 1.$  LET:  $n \in \omega$

$\langle 3 \rangle 2.$  ASSUME:  $\forall m \in n. m^+ \in n^+$

$\langle 3 \rangle 3.$  LET:  $m \in n^+$

$\langle 3 \rangle 4.$  CASE:  $m \in n$

$\langle 4 \rangle 1.$   $m^+ \in n^+$

PROOF: By  $\langle 3 \rangle 2$

$\langle 4 \rangle 2.$   $m^+ \in n^{++}$

$\langle 3 \rangle 5.$  CASE:  $m = n$

PROOF:  $m^+ = n^+ \in n^{++}$

$\langle 1 \rangle 2.$   $\forall m, n \in \omega (m^+ \in n^+ \Rightarrow m \in n)$

$\langle 2 \rangle 1.$  LET:  $m, n \in \omega$

$\langle 2 \rangle 2.$  ASSUME:  $m^+ \in n^+$

$\langle 2 \rangle 3.$   $m \in m^+$

$\langle 2 \rangle 4.$   $m^+ \in n$  or  $m^+ = n$

$\langle 2 \rangle 5.$   $m \in n$

PROOF: If  $m^+ \in n$  this follows because  $n$  is transitive (Theorem 173).

$\square$

**Lemma 218.** *For any natural number  $n$  we have  $n \notin n$ .*

PROOF:

- ⟨1⟩1.  $0 \notin 0$
  - ⟨1⟩2. For all  $n \in \omega$ , if  $n \notin n$  then  $n^+ \notin n^+$ 
    - ⟨2⟩1. LET:  $n \in \omega$
    - ⟨2⟩2. ASSUME:  $n^+ \in n^+$ 
      - PROVE:  $n \in n$
    - ⟨2⟩3.  $n^+ \in n$  or  $n^+ = n$
    - ⟨2⟩4.  $n \in n^+$
    - ⟨2⟩5.  $n \in n$
- PROOF: If  $n^+ \in n$  this follows because  $n$  is transitive (Theorem 173).

□

**Theorem 219** (Trichotomy Law for  $\omega$ ). *For any natural numbers  $m$  and  $n$ , exactly one of*

$$m \in n, m = n, n \in m$$

*holds.*

PROOF:

- ⟨1⟩1. For any  $m, n \in \omega$ , at most one of  $m \in n$ ,  $m = n$ ,  $n \in m$  holds.
  - PROOF: If  $m \in n$  and  $m = n$  then  $m \in m$  contradicting Lemma 218.
  - If  $m \in n$  and  $n \in m$  then  $m \in m$  by Theorem 173, contradicting Lemma 218.
- ⟨1⟩2. For any  $m, n \in \omega$ , at least one of  $m \in n$ ,  $m = n$ ,  $n \in m$  holds.
  - ⟨2⟩1. For all  $n \in \omega$ , either  $0 \in n$  or  $0 = n$ 
    - ⟨3⟩1.  $0 = 0$
    - ⟨3⟩2. For all  $n \in \omega$ , if  $0 \in n$  or  $0 = n$  then  $0 \in n^+$
  - ⟨2⟩2. For all  $m \in \omega$ , if  $\forall n \in \omega (m \in n \vee m = n \vee n \in m)$  then  $\forall n \in \omega (m^+ \in n \vee m^+ = n \vee n \in m^+)$ 
    - ⟨3⟩1. LET:  $m \in \omega$
    - ⟨3⟩2. ASSUME:  $\forall n \in \omega (m \in n \vee m = n \vee n \in m)$
    - ⟨3⟩3. LET:  $n \in \omega$
    - ⟨3⟩4. CASE:  $m \in n$ 
      - PROOF: Then  $m \in n^+$
    - ⟨3⟩5. CASE:  $m = n$ 
      - PROOF: Then  $m \in n^+$
    - ⟨3⟩6. CASE:  $n \in m$ 
      - PROOF: Then  $n^+ \in m^+$  by Lemma 217 so  $n^+ \in m$  or  $n^+ = m$ .

□

**Corollary 219.1.** *The relation  $\in$  is a linear ordering on  $\omega$ .*

**Corollary 219.2.** *For any natural numbers  $m$  and  $n$ ,*

$$m \in n \Leftrightarrow m \subset n .$$

PROOF:

- ⟨1⟩1. LET:  $m, n \in \omega$
- ⟨1⟩2. If  $m \in n$  then  $m \subset n$ .
  - ⟨2⟩1. ASSUME:  $m \in n$

$\langle 2 \rangle 2. m \subseteq n$

PROOF: Theorem 173.

$\langle 2 \rangle 3. m \neq n$

PROOF: Lemma 218.

$\langle 1 \rangle 3. \text{ If } m \subset n \text{ then } m \in n.$

PROOF: We have  $m \neq n$  and  $n \notin m$  by  $\langle 1 \rangle 2$ , hence  $m \in n$  by trichotomy.

□

**Theorem 220.** *For any natural number  $p$ , the function that maps  $n$  to  $n + p$  is strictly monotone. For any natural numbers  $m$ ,  $n$  and  $p$ , we have  $m \in n$  if and only if  $m + p \in n + p$ .*

PROOF: We prove that  $m \in n \Rightarrow m + p \in n + p$ . This is an easy induction on  $p$  using Lemma 217. □

**Theorem 221.** *For any non-zero natural number  $p$ , the function that maps  $n$  to  $np$  is strictly monotone.*

PROOF: Easy induction on  $p$  using Theorem 220. □

**Theorem 222** (Strong Induction). *Let  $A$  be a subset of  $\omega$  and suppose that, for all  $n \in \omega$ , we have*

$$(\forall m < n. m \in A) \Rightarrow n \in A .$$

*Then  $A = \omega$ .*

PROOF: Prove  $\forall n \in \omega. \forall m < n. m \in A$  by induction on  $n$ . □

**Theorem 223** (Well-Ordering of  $\omega$ ). *The ordering  $<$  on  $\omega$  is a well-ordering.*

PROOF: If  $A$  is a subset of  $\omega$  with no least element, we prove  $\forall n \in \omega. n \notin A$  by strong induction on  $n$ . □

**Lemma 224.** *For any natural numbers  $m$  and  $n$ , we have  $m \in n$  if and only if there exists a natural number  $p$  such that  $n = m + p^+$ .*

PROOF:

$\langle 1 \rangle 1. \text{ For all } m, p, \text{ we have } m \in m + p^+$

PROOF:  $m = m + 0 \in m + p^+$

$\langle 1 \rangle 2. \text{ For all } m, n, \text{ if } m \in n \text{ then there exists } p \text{ such that } n = m + p^+$

$\langle 2 \rangle 1. \text{ For all } m, \text{ if } m \in 0 \text{ then there exists } p \text{ such that } 0 = m + p^+$

PROOF: Vacuous.

$\langle 2 \rangle 2. \text{ For all } n \in \omega, \text{ if } \forall m \in n. \exists p \in \omega. n = m + p^+ \text{ then } \forall m \in n^+. \exists p \in \omega. n^+ = m + p^+$

$\langle 3 \rangle 1. \text{ LET: } n \in \omega$

$\langle 3 \rangle 2. \text{ ASSUME: } \forall m \in n. \exists p \in \omega. n = m + p^+$

$\langle 3 \rangle 3. \text{ LET: } m \in n^+$

$\langle 3 \rangle 4. \text{ CASE: } m \in n$

$\langle 4 \rangle 1. \text{ PICK } p \text{ such that } n = m + p^+$

$\langle 4 \rangle 2. n^+ = m + p^{++}$   
 $\langle 3 \rangle 5. \text{ CASE: } m = n$   
 PROOF:  $n^+ = m + 0^+$

□

**Lemma 225.** *For natural numbers  $m, n, p$  and  $q$ , if  $m \in n$  and  $p \in q$  then  $mp + nq \in mq + np$ .*

$\langle 1 \rangle 1.$  PICK natural numbers  $a$  and  $b$  such that  $n = m + a^+$  and  $q = p + b^+$

PROOF: Lemma 224.

$\langle 1 \rangle 2. mp + nq = mq + np + (a^+ + b)^+$

$\langle 1 \rangle 3. mp + nq \in mq + np$

PROOF: Lemma 224.

## 25 The Integers

**Theorem 226.** *The relation  $\sim$  is an equivalence relation on  $\omega \times \omega$ , where  $(m, n) \sim (p, q)$  iff  $m + q = n + p$ .*

PROOF:

$\langle 1 \rangle 1.$  The relation  $\sim$  is reflexive on  $\omega^2$

PROOF: For any  $m, n$ , we have  $m + n = m + n$  and so  $(m, n) \sim (m, n)$ .

$\langle 1 \rangle 2.$  The relation  $\sim$  is symmetric.

PROOF: If  $m + q = n + p$  then  $p + n = q + m$ .

$\langle 1 \rangle 3.$  The relation  $\sim$  is transitive.

$\langle 2 \rangle 1.$  ASSUME:  $(m, n) \sim (p, q) \sim (r, s)$

$\langle 2 \rangle 2. m + q = n + p$

$\langle 2 \rangle 3. p + s = q + r$

$\langle 2 \rangle 4. m + p + q + s = n + p + q + r$

$\langle 2 \rangle 5. m + s = n + r$

PROOF: By cancellation of addition in  $\omega$ .

□

**Definition 227.** The set  $\mathbb{Z}$  of *integers* is the quotient set  $(\omega \times \omega) / \sim$ .

**Lemma 228.** *If  $(m, n) \sim (m', n')$  and  $(p, q) \sim (p', q')$  then  $(m + p, n + q) \sim (m' + p', n' + q')$ .*

PROOF: Assume  $m + n' = m' + n$  and  $p + q' = p' + q$ . Then  $m + p + n' + q' = m' + p' + n + q$ . □

**Definition 229 (Addition).** Addition  $+$  on  $\mathbb{Z}$  is the binary operation such that

$$[(m, n)] + [(p, q)] = [(m + p, n + q)]$$

**Theorem 230.** *Addition on  $\mathbb{Z}$  is commutative.*

PROOF: From the definition. □



**Theorem 231.** *Addition on  $\mathbb{Z}$  is associative.*

PROOF: Easy.  $\square$

**Definition 232** (Zero). The zero in the integers is  $0 = [(0, 0)]$ .

**Theorem 233.** *For any integer  $a$  we have  $a + 0 = a$ .*

PROOF: Easy.  $\square$

**Theorem 234.** *For any integer  $a$ , there exists an integer  $b$  such that  $a + b = 0$ .*

PROOF: If  $a = [(m, n)]$  take  $b = [(n, m)]$ .  $\square$

**Lemma 235.** *If  $(m, n) \sim (m', n')$  and  $(p, q) \sim (p', q')$  then  $(mp + nq, mq + np) \sim (m'p' + n'q', m'q' + n'p')$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $m + n' = m' + n$  and  $p + q' = p' + q$

$\langle 1 \rangle 2$ .  $mp + n'p = m'p + np$

$\langle 1 \rangle 3$ .  $m'q + nq = mq + n'q$

$\langle 1 \rangle 4$ .  $mp + mq' = m'p' + mq$

$\langle 1 \rangle 5$ .  $n'p' + n'q = n'p + n'q'$

$\langle 1 \rangle 6$ .  $mp + n'p + m'q + nq + mp + mq' + n'p' + n'q = m'p + np + mq + n'q + m'p' + mq + n'p + n'q'$

$\langle 1 \rangle 7$ .  $mp + nq + m'q' + n'p' = mq + np + m'p' + n'q'$

$\square$

**Definition 236** (Multiplication). *Multiplication  $\cdot$  is the binary operation on  $\mathbb{Z}$  such that*

$$[(m, n)][(p, q)] = [(mp + nq, mq + np)]$$

**Theorem 237.** *Multiplication is commutative.*

PROOF: Easy.  $\square$

**Theorem 238.** *Multiplication is associative.*

PROOF: Easy.  $\square$

**Theorem 239.** *Multiplication is distributive over addition.*

PROOF: Easy.  $\square$

**Definition 240.** The integer one is  $1 = [(1, 0)]$ .

**Theorem 241.** *For any integer  $a$  we have  $a1 = a$ .*

PROOF: Easy.  $\square$

**Theorem 242.**  $0 \neq 1$

PROOF: Easy.  $\square$

**Lemma 243.** *If  $(m, n) \sim (m', n')$  and  $(p, q) \sim (p', q')$  then  $m + q \in p + n$  iff  $m' + q' \in p' + n'$ .*

PROOF:

$$\begin{aligned} m + q \in p + n &\Leftrightarrow m + q + n' + q' \in p + n + n' + q' \\ &\Leftrightarrow m' + n + q + q' \in p' + n + n' + q \\ &\Leftrightarrow m' + q' \in p' + n' \end{aligned} \quad \square$$

**Definition 244** (Ordering). The ordering  $<$  on  $\mathbb{Z}$  is defined by:  $[(m, n)] < [(p, q)]$  iff  $m + q \in n + p$ .

**Theorem 245.** *The relation  $<$  is a linear ordering on  $\mathbb{Z}$ .*

PROOF:

- $\langle 1 \rangle 1.$   $<$  is transitive.
- $\langle 2 \rangle 1.$  ASSUME:  $[(m, n)] < [(p, q)]$  and  $[(p, q)] < [(r, s)]$
- $\langle 2 \rangle 2.$   $m + q \in n + p$  and  $p + s \in q + r$
- $\langle 2 \rangle 3.$   $m + q + s \in n + p + s$
- $\langle 2 \rangle 4.$   $n + p + s \in n + q + r$
- $\langle 2 \rangle 5.$   $m + q + s \in n + q + r$
- $\langle 2 \rangle 6.$   $m + s \in n + r$

- $\langle 1 \rangle 2.$   $<$  satisfies trichotomy.

PROOF: From trichotomy on  $\omega$ .

$\square$

**Theorem 246.** *For any integers  $a, b$  and  $c$ , we have  $a < b$  iff  $a + c < b + c$ .*

PROOF: An easy consequence of the corresponding property in  $\omega$ .

**Corollary 246.1.** *If  $a + c = b + c$  then  $a = b$ .*

**Theorem 247.** *If  $0 < c$ , then the function that maps an integer  $a$  to  $ac$  is strictly monotone.*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $a, b$  and  $c$  be integers.
- $\langle 1 \rangle 2.$  ASSUME:  $0 < c$  and  $a < b$
- $\langle 1 \rangle 3.$  LET:  $a = [(m, n)]$
- $\langle 1 \rangle 4.$  LET:  $b = [(p, q)]$
- $\langle 1 \rangle 5.$  LET:  $c = [(r, s)]$
- $\langle 1 \rangle 6.$   $s \in r$
- $\langle 1 \rangle 7.$   $m + q \in p + n$
- $\langle 1 \rangle 8.$   $(m + q)r + (p + n)s \in (m + q)s + (p + n)r$

PROOF: Lemma 225.

- $\langle 1 \rangle 9.$   $ac < bc$

$\square$

**Lemma 248.** *For integers  $a$  and  $b$ ,  $a(-b) = -(ab)$*

PROOF: This follows from the fact that  $ab + a(-b) = a(b + (-b)) = a0 = 0$ .  $\square$

**Theorem 249.** For integers  $a$ ,  $b$  and  $c$ , if  $a < b$  and  $c < 0$  then  $ac > bc$ .

PROOF: We have  $0 < -c$  so  $a(-c) < b(-c)$  hence  $-(ac) < -(bc)$  so  $bc < ac$ .  $\square$

**Theorem 250.** For any integers  $a$  and  $b$ , if  $ab = 0$  then  $a = 0$  or  $b = 0$ .

PROOF: We prove if  $a \neq 0$  and  $b \neq 0$  then  $ab \neq 0$ .

If  $a > 0$  and  $b > 0$  then  $ab > 0$ . Similarly for the other four cases.  $\square$

**Theorem 251.** If  $ac = bc$  and  $c \neq 0$  then  $a = b$ .

PROOF: We have  $(a - b)c = 0$  so  $a - b = 0$  hence  $a = b$ .  $\square$

**Definition 252** (Positive). An integer  $a$  is *positive* iff  $0 < a$ .

**Theorem 253.** Define  $E : \omega \rightarrow \mathbb{Z}$  by  $E(n) = [(n, 0)]$ . Then  $E$  maps  $\omega$  one-to-one into  $\mathbb{Z}$ , and:

1.  $E(m + n) = E(m) + E(n)$
2.  $E(mn) = E(m)E(n)$
3.  $m \in n$  if and only if  $E(m) < E(n)$ .

PROOF: Routine calculations.  $\square$

## 26 Equinumerosity

**Definition 254** (Equinumerous). Two sets  $A$  and  $B$  are *equinumerous*,  $A \approx B$ , iff there exists a bijection between them.

**Theorem 255.** *Equinumerosity is an equivalence relation on the class of sets.*

PROOF: Easy.  $\square$

**Theorem 256** (Cantor 1873). *No set is equinumerous with its power set.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $g : A \rightarrow \mathcal{P}A$

PROVE:  $g$  is not surjective.

$\langle 1 \rangle 2$ . LET:  $B = \{x \in A : x \notin g(x)\}$

$\langle 1 \rangle 3$ .  $\forall x \in A. g(x) \neq B$

PROOF: Because  $x \in B$  iff  $x \notin g(x)$ .

$\square$

## 27 Ordering Cardinal Numbers

**Definition 257** (Dominated). A set  $A$  is *dominated* by a set  $B$ ,  $A \preceq B$ , iff there exists an injection  $f : A \rightarrow B$ .

**Lemma 258.** *Domination is a preorder on the class of sets.*

PROOF: Easy.  $\square$

**Lemma 259.** *If  $A \subseteq B$  then  $A \preccurlyeq B$ .*

PROOF: The inclusion from  $A$  to  $B$  is an injection.  $\square$

**Lemma 260.** *If  $A \preccurlyeq B$ ,  $A \approx A'$  and  $B \approx B'$  then  $A' \preccurlyeq B'$ .*

PROOF: Easy.  $\square$

**Definition 261.** Given cardinal numbers  $\kappa$  and  $\lambda$ , we write  $\kappa \leq \lambda$  iff  $K \preccurlyeq L$ , where  $K$  is any set of cardinality  $\kappa$  and  $L$  is any set of cardinality  $\lambda$ .

We write  $\kappa < \lambda$  iff  $\kappa \leq \lambda$  and  $\kappa \neq \lambda$ .

**Theorem 262** (Schröder-Bernstein). *If  $A \preccurlyeq B$  and  $B \preccurlyeq A$  then  $A \approx B$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : A \rightarrow B$  and  $g : B \rightarrow A$  be one-to-one.

$\langle 1 \rangle 2$ . Define the sequence of sets  $C_n \subseteq A$  by:

$$C_0 = A - \text{ran } g$$

$$C_{n+1} = g(f(C_n))$$

$\langle 1 \rangle 3$ . Define  $h : A \rightarrow B$  by

$$h(x) = \begin{cases} f(x) & \text{if } \exists n \in \mathbb{N}. x \in C_n \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

$\langle 1 \rangle 4$ .  $h$  is injective.

$\langle 2 \rangle 1$ . LET:  $x, y \in A$

$\langle 2 \rangle 2$ . ASSUME:  $h(x) = h(y)$

$\langle 2 \rangle 3$ . CASE:  $x \in C_m, y \in C_n$

PROOF: We have  $f(x) = f(y)$  so  $x = y$

$\langle 2 \rangle 4$ . CASE:  $x \in C_m, y \notin \bigcup_n C_n$

PROOF: This case is impossible because we would have  $y = g(f(x))$  and so  $y \in C_{m+1}$ .

$\langle 2 \rangle 5$ . CASE:  $x, y \notin \bigcup_n C_n$

PROOF: We have  $g^{-1}(x) = g^{-1}(y)$  so  $x = y$ .

$\langle 1 \rangle 5$ .  $h$  is surjective.

$\langle 2 \rangle 1$ . LET:  $y \in B$

$\langle 2 \rangle 2$ . ASSUME:  $y \notin f(C_n)$  for all  $n$

$\langle 2 \rangle 3$ .  $g(y) \notin C_n$  for all  $n$

$\langle 2 \rangle 4$ .  $y = h(g(y))$

$\square$

**Corollary 262.1.** *The relation  $\leq$  is a partial order on the class of cardinal numbers.*

**Theorem 263.** *Let  $\kappa, \lambda$  and  $\mu$  be cardinal numbers.*

$$1. \kappa \leq \lambda \Rightarrow \kappa + \mu \leq \lambda + \mu$$

$$2. \kappa \leq \lambda \Rightarrow \kappa \mu \leq \lambda \mu$$

3.  $\kappa \leq \lambda \Rightarrow \kappa^\mu \leq \lambda^\mu$

4.  $\kappa \leq \lambda \Rightarrow \mu^\kappa \leq \mu^\lambda$  if  $\kappa$  and  $\mu$  are not both zero.

PROOF: Parts 1–3 are easy. For part 4:

Let  $|K| = \kappa$ ,  $|L| = \lambda$  and  $|M| = \mu$  with  $K \subseteq L$ .

If  $M = \emptyset$  then  $\kappa \neq 0$  so  $\mu^\kappa = 0 \leq \mu^\lambda$ .

Otherwise, pick  $a \in M$ . Define  $\Phi : M^K \rightarrow M^L$  by:

$$\Phi(f)(x) = \begin{cases} f(x) & \text{if } x \in K \\ a & \text{if } x \notin K \end{cases}$$

Then  $\Phi$  is an injection.  $\square$

**Theorem 264** (Cardinal Comparability). *The Axiom of Choice is equivalent to the statement: for any sets  $C$  and  $D$ , either  $C \preceq D$  or  $D \preceq C$ .*

PROOF:

$\langle 1 \rangle 1$ . If Zorn's Lemma then Cardinal Comparability.

$\langle 2 \rangle 1$ . ASSUME: Zorn's Lemma

$\langle 2 \rangle 2$ . LET:  $C$  and  $D$  be sets.

$\langle 2 \rangle 3$ . LET:  $\mathcal{A}$  be the set of all injective functions  $f$  with  $\text{dom } f \subseteq C$  and  $\text{ran } f \subseteq D$

$\langle 2 \rangle 4$ . For every chain  $\mathcal{B} \subseteq \mathcal{A}$  we have  $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 2 \rangle 5$ . LET:  $f \in \mathcal{A}$  be maximal

$\langle 2 \rangle 6$ .  $\text{dom } f = C$  or  $\text{ran } f = D$

$\langle 2 \rangle 7$ .  $f$  is an injective function  $C \rightarrow D$  or  $f^{-1}$  is an injective function  $D \rightarrow C$

$\langle 1 \rangle 2$ . If Cardinal Comparability then the Well-Ordering Theorem.

$\langle 2 \rangle 1$ . ASSUME: Cardinal Comparability

$\langle 2 \rangle 2$ . LET:  $A$  be any set

$\langle 2 \rangle 3$ . PICK an ordinal  $\alpha$  not dominated by  $A$

PROOF: Hartogs' Theorem.

$\langle 2 \rangle 4$ .  $A \preceq \alpha$

$\langle 2 \rangle 5$ . PICK an injective function  $f : A \rightarrow \alpha$

$\langle 2 \rangle 6$ . Define  $<$  on  $A$  by:  $x < y$  iff  $f(x) \in f(y)$

$\langle 2 \rangle 7$ .  $<$  is a well ordering on  $A$ .

$\square$

**Theorem 265** (Choice). *For any infinite set  $A$ , we have  $\omega \preceq A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be an infinite set.

$\langle 1 \rangle 2$ . PICK a choice function  $F$  for  $A$

$\langle 1 \rangle 3$ . Define  $f : \omega \rightarrow A$  by recursion by:  $f(n) = F(A - \{f(0), f(1), \dots, f(n-1)\})$

PROOF:  $A - \{f(0), f(1), \dots, f(n-1)\}$  is nonempty because  $A$  is infinite.

$\langle 1 \rangle 4$ .  $f$  is injective.

$\square$

**Corollary 265.1** (Choice). *For any infinite cardinal  $\kappa$  we have  $\aleph_0 \leq \kappa$ .*

**Corollary 265.2** (Choice). *A set is infinite iff it is equinumerous to a proper subset of itself.*

**Proposition 266** (Choice). *If there exists a surjection  $A \rightarrow B$  then  $B \preceq A$ .*

PROOF: Any surjection  $A \rightarrow B$  has a right inverse which is an injection  $B \rightarrow A$ .

## 28 Countable Sets

**Definition 267** (Countable). A set is *countable* iff it is dominated by  $\omega$ .

**Proposition 268.** *Any subset of a countable set is countable.*

PROOF: Easy.  $\square$

The union of two countable sets is countable.

PROOF: Because  $\aleph_0 + \aleph_0 = \aleph_0$   $\square$

**Proposition 269.** *The product of two countable sets is countable.*

PROOF: Because  $\aleph_0 \aleph_0 = \aleph_0$ .  $\square$

**Proposition 270** (Choice). *For any infinite set  $A$ , the set  $\mathcal{P}A$  is uncountable.*

PROOF: If  $|A| \geq \aleph_0$  then  $|\mathcal{P}A| \geq 2^{\aleph_0}$ .  $\square$

**Theorem 271** (Choice). *A countable union of countable sets is countable.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{A}$  be a countable set of countable sets.

$\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $\mathcal{A} \neq \emptyset$  and  $\emptyset \notin \mathcal{A}$

$\langle 1 \rangle 3$ . PICK a surjection  $G : \omega \rightarrow \mathcal{A}$

$\langle 1 \rangle 4$ . PICK a function  $F$  with domain  $\omega$  such that, for all  $m$ ,  $F(m)$  is a surjection  $\omega \rightarrow G(m)$

PROOF: By the Axiom of Choice.

$\langle 1 \rangle 5$ . Define  $f : \omega \times \omega \rightarrow \bigcup \mathcal{A}$  by  $f(m, n) = F(m)(n)$

$\langle 1 \rangle 6$ .  $f$  is surjective.

$\langle 1 \rangle 7$ .  $A \preceq \omega \times \omega$

$\square$

## 29 Arithmetic of Infinite Cardinals

**Lemma 272** (Choice). *For any infinite cardinal  $\kappa$  we have  $\kappa \cdot \kappa = \kappa$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\kappa$  be an infinite cardinal.

$\langle 1 \rangle 2$ . LET:  $B$  be a set of cardinality  $\kappa$ .

$\langle 1 \rangle 3$ . LET:  $\mathcal{H} = \{f : f = \emptyset \text{ or for some infinite } A \subseteq B, f \text{ is a bijection between } A \times A \text{ and } A\}$

$\langle 1 \rangle 4$ . For any chain  $\mathcal{C} \subseteq \mathcal{H}$ , we have  $\bigcup \mathcal{C} \in \mathcal{H}$

- ⟨2⟩1. LET:  $\mathcal{C} \subseteq \mathcal{H}$  be a chain.
- ⟨2⟩2. ASSUME: w.l.o.g.  $\mathcal{C}$  has a nonempty element.  
PROOF: Otherwise  $\bigcup \mathcal{C} = \emptyset \in \mathcal{H}$ .
- ⟨2⟩3.  $\bigcup \mathcal{C}$  is an injective function.
- ⟨2⟩4. LET:  $A = \text{ran } \bigcup \mathcal{C}$
- ⟨2⟩5.  $A$  is infinite.
- ⟨2⟩6.  $\bigcup \mathcal{C}$  is a bijection between  $A \times A$  and  $A$ .
  - ⟨3⟩1. LET:  $a_1, a_2 \in A$
  - ⟨3⟩2. PICK  $f_1, f_2 \in \mathcal{C}$  such that  $a_1 \in \text{ran } f_1$  and  $a_2 \in \text{ran } f_2$
  - ⟨3⟩3. ASSUME: w.l.o.g.  $f_1 \subseteq f_2$
  - ⟨3⟩4.  $\langle a_1, a_2 \rangle \in \text{dom } f_2$
  - ⟨3⟩5.  $\langle a_1, a_2 \rangle \in \text{dom } \bigcup \mathcal{C}$
- ⟨1⟩5. PICK a maximal  $f_0 \in \mathcal{H}$   
PROOF: Zorn's Lemma.
- ⟨1⟩6.  $f_0 \neq \emptyset$   
PROOF:  $B$  has a countable subset  $A$ , say, and  $A \times A \approx A$ .
- ⟨1⟩7. PICK  $A_0 \subseteq B$  infinite such that  $f_0$  is a bijection between  $A_0 \times A_0$  and  $A_0$ .
- ⟨1⟩8. LET:  $\lambda = |A_0|$
- ⟨1⟩9.  $\lambda$  is infinite
- ⟨1⟩10.  $\lambda = \lambda \cdot \lambda$
- ⟨1⟩11.  $\lambda = \kappa$ 
  - ⟨2⟩1.  $|B - A_0| < \lambda$ 
    - ⟨3⟩1. ASSUME: for a contradiction  $\lambda \leq |B - A_0|$
    - ⟨3⟩2. PICK  $D \subseteq B - A_0$  with  $|D| = \lambda$
    - ⟨3⟩3.  $(A_0 \cup D) \times (A_0 \cup D) = (A_0 \times A_0) \cup (A_0 \times D) \cup (D \times A_0) \cup (D \times D)$
    - ⟨3⟩4.  $f_0 : A_0 \times A_0 \approx A_0$
    - ⟨3⟩5.  $|(A_0 \times D) \cup (D \times A_0) \cup (D \times D)| = \lambda$   
PROOF:  

$$\begin{aligned} |(A_0 \times D) \cup (D \times A_0) \cup (D \times D)| &= \lambda \cdot \lambda + \lambda \cdot \lambda + \lambda \cdot \lambda \\ &= \lambda + \lambda + \lambda & (\langle 1 \rangle 10) \\ &= 3 \cdot \lambda \\ &\leq \lambda \cdot \lambda \\ &= \lambda & (\langle 1 \rangle 10) \end{aligned}$$
  - ⟨3⟩6. PICK a bijection  $g : (A_0 \times D) \cup (D \times A_0) \cup (D \times D) \approx D$
  - ⟨3⟩7.  $f_0 \cup g : (A_0 \cup D) \times (A_0 \cup D) \approx A_0 \cup D$
  - ⟨3⟩8. Q.E.D.
- PROOF: This contradicts the maximality of  $f_0$ .
- ⟨2⟩2.  $\lambda = \kappa$

PROOF:

$$\begin{aligned}
\kappa &= |B| \\
&= |A_0| + |B - A_0| \\
&\leq \lambda + \lambda \\
&= 2 \cdot \lambda \\
&\leq \lambda \cdot \lambda \\
&= \lambda \\
&\leq \kappa
\end{aligned}$$

□

**Corollary 272.1** (Absorption Law of Cardinal Arithmetic (Choice)). *Let  $\kappa$  and  $\lambda$  be cardinal numbers, the larger of which is infinite and the smaller of which is nonzero. Then*

$$\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda) \ .$$

PROOF:

⟨1⟩1. ASSUME: w.l.o.g.  $\kappa \leq \lambda$

⟨1⟩2.  $\kappa + \lambda = \lambda$

PROOF:

$$\begin{aligned}
\lambda &\leq \kappa + \lambda \\
&\leq \lambda + \lambda \\
&= 2 \cdot \lambda \\
&\leq \lambda \cdot \lambda \\
&= \lambda
\end{aligned}$$

⟨1⟩3.  $\kappa \cdot \lambda = \lambda$

PROOF:

$$\begin{aligned}
\lambda &= 1 \cdot \lambda \\
&\leq \kappa \cdot \lambda \\
&\leq \lambda \cdot \lambda \\
&= \lambda
\end{aligned}$$

□

## 30 Rank

**Definition 273.** Define the set  $V_\alpha$  for every ordinal  $\alpha$  by transfinite recursion thus:

$$V_\alpha = \bigcup \{ \mathcal{P}V_\beta : \beta \in \alpha \} \ .$$

**Lemma 274.** *For any ordinal  $\alpha$ ,  $V_\alpha$  is a transitive set.*

PROOF:

⟨1⟩1. LET:  $\alpha$  be an ordinal.



- $\langle 1 \rangle 2.$  LET:  $x \in y \in V_\alpha$
  - $\langle 1 \rangle 3.$  PICK  $\beta \in \alpha$  such that  $y \in \mathcal{P}V_\beta$
  - $\langle 1 \rangle 4.$   $x \in V_\beta$
  - $\langle 1 \rangle 5.$  PICK  $\gamma \in \beta$  such that  $x \in \mathcal{P}V_\gamma$
  - $\langle 1 \rangle 6.$   $\gamma \in \alpha$  and  $x \in \mathcal{P}V_\gamma$
  - $\langle 1 \rangle 7.$   $x \in V_\alpha$
- 

**Theorem 275.** For ordinals  $\beta \in \alpha$  we have  $V_\beta \subseteq V_\alpha$ .

PROOF:

$$\begin{aligned}
 V_\beta &= \bigcup_{\gamma \in \beta} \mathcal{P}V_\gamma \\
 &\subseteq \bigcup_{\gamma \in \alpha} \mathcal{P}V_\gamma \\
 &= V_\alpha
 \end{aligned}
 \quad \square$$

**Theorem 276.**

$$V_0 = \emptyset$$

PROOF: Immediate from definitions. □

**Theorem 277.** For any ordinal  $\alpha$ ,  $V_{\alpha+} = \mathcal{P}V_\alpha$ .

PROOF:

$$\begin{aligned}
 V_{\alpha+} &= \bigcup_{\beta \leq \alpha} \mathcal{P}V_\beta \\
 &= \mathcal{P}V_\alpha
 \end{aligned}$$

by Theorem 275. □

**Theorem 278.** For  $\lambda$  a limit ordinal,  $V_\lambda = \bigcup_{\beta < \lambda} V_\beta$ .

PROOF:

$$\begin{aligned}
 V_\lambda &= \bigcup_{\beta < \lambda} \mathcal{P}V_\beta \\
 &= \bigcup_{\beta < \lambda} V_{\beta+} \\
 &= \bigcup_{\beta < \lambda} V_\beta
 \end{aligned}$$

since  $\beta < \lambda$  iff  $\beta^+ < \lambda$ . □

**Definition 279** (Grounded, Rank). A set  $A$  is *grounded* iff  $\exists \alpha. A \subseteq V_\alpha$ . The *rank* of a grounded set  $A$ ,  $\text{rank } A$ , is then the least ordinal  $\alpha$  such that  $A \subseteq V_\alpha$ .

**Theorem 280.** If  $A$  is grounded and  $a \in A$  then  $a$  is grounded and  $\text{rank } a < \text{rank } A$ .

PROOF: We have  $a \in A \subseteq V_{\text{rank } A}$ . So  $a \in \mathcal{P}V_\alpha$  for some  $\alpha < \text{rank } A$ , i.e.  $a \subseteq V_\alpha$  for some  $\alpha < \text{rank } A$ , as required.

**Theorem 281.** *If every member of  $A$  is grounded then  $A$  is grounded and*

$$\text{rank } A = \sup_{a \in A} (\text{rank } a)^+ .$$

PROOF:

- $\langle 1 \rangle 1.$  LET:  $\alpha = \sup_{a \in A} (\text{rank } a)^+$
- $\langle 1 \rangle 2.$   $A \subseteq V_\alpha$ 
  - $\langle 2 \rangle 1.$  LET:  $a \in A$
  - $\langle 2 \rangle 2.$   $a \subseteq V_{\text{rank } a}$
  - $\langle 2 \rangle 3.$   $a \in V_{(\text{rank } a)^+}$
  - $\langle 2 \rangle 4.$   $a \in V_\alpha$
- $\langle 1 \rangle 3.$  If  $A \subseteq V_\beta$  then  $\alpha \leq \beta$ 
  - $\langle 2 \rangle 1.$  ASSUME:  $A \subseteq V_\beta$
  - $\langle 2 \rangle 2.$   $\forall a \in A. a \in V_\beta$
  - $\langle 2 \rangle 3.$   $\forall a \in A. \exists \gamma < \beta. a \subseteq V_\gamma$
  - $\langle 2 \rangle 4.$   $\forall a \in A. \exists \gamma < \beta. \text{rank } a \leq \gamma$
  - $\langle 2 \rangle 5.$   $\forall a \in A. \text{rank } a < \beta$
  - $\langle 2 \rangle 6.$   $\forall a \in A. (\text{rank } a)^+ \leq \beta$
  - $\langle 2 \rangle 7.$   $\alpha \leq \beta$

□

**Theorem 282.** *Every set is grounded.*

PROOF:

- $\langle 1 \rangle 1.$  ASSUME: for a contradiction  $c$  is not grounded.
- $\langle 1 \rangle 2.$  LET:  $B$  be the transitive closure of  $\{c\}$ .
- $\langle 1 \rangle 3.$  LET:  $A = \{x \in B : x \text{ is not grounded}\}$
- $\langle 1 \rangle 4.$  PICK  $m \in A$  such that  $m \cap A = \emptyset$ 
  - PROOF: By the Axiom of Regularity.
- $\langle 1 \rangle 5.$  Every member of  $m$  is grounded.
  - PROOF: Every member of  $m$  is in  $B$  by transitivity but not in  $A$ .
- $\langle 1 \rangle 6.$   $m$  is grounded.
  - PROOF: Theorem 281.
- $\langle 1 \rangle 7.$  Q.E.D.
  - PROOF: This contradicts the fact that  $m \in A$ .

□

**Theorem 283.** *Let  $A$  be any set and  $A^t$  its transitive closure. Let  $M^t$  be the transitive closure of the relation  $\{\langle x, y \rangle : x \in y \in A^t\}$ . Define  $E : A^t \rightarrow \mathbf{V}$  by transfinite recursion thus:*

$$E(a) = \{E(x) : x M^t a\} \quad (a \in A^t) .$$

*Then  $E(a) = \text{rank } a$  for all  $a \in A^t$ , and  $\text{ran } E = \text{rank } A$ .*

PROOF:

- $\langle 1 \rangle 1.$   $M^t$  is well-founded
  - PROOF: Theorem 87.

$\langle 1 \rangle 2.$   $\forall a \in A^t. \text{rank } a = \{\text{rank } x : x M^t a\}$   
 $\langle 2 \rangle 1.$   $\forall x, a \in A^t. x M^t a \Rightarrow \text{rank } x < \text{rank } a$   
 PROOF: Theorem 280.  
 $\langle 2 \rangle 2.$   $\forall x \in A^t. \forall \alpha < \text{rank } a. \exists x M^t a. \alpha = \text{rank } x$   
 $\langle 3 \rangle 1.$  LET:  $a \in A^t$   
 $\langle 3 \rangle 2.$  ASSUME:  $\forall b M^t a. \forall \alpha < \text{rank } b. \exists x M^t b. \alpha = \text{rank } x$   
 $\langle 3 \rangle 3.$  LET:  $\alpha < \text{rank } a$   
 $\langle 3 \rangle 4.$  PICK  $b \in a$  such that  $\alpha \leq \text{rank } b$   
 PROOF: Theorem 281.  
 $\langle 3 \rangle 5.$  CASE:  $\alpha < \text{rank } b$   
 $\langle 4 \rangle 1.$  PICK  $x M^t b$  such that  $\alpha = \text{rank } x$   
 PROOF: By  $\langle 3 \rangle 2$   
 $\langle 4 \rangle 2.$   $x M^t a$   
 $\langle 3 \rangle 6.$  CASE:  $\alpha = \text{rank } b$   
 PROOF: We have  $b M^t a$  and  $\alpha = \text{rank } b$  as required.  
 $\langle 3 \rangle 7.$  Q.E.D.  
 PROOF: This concludes the proof by transfinite induction over  $M^t$  ( $\langle 1 \rangle 1$ ).  
 $\langle 1 \rangle 3.$   $\forall a \in A^t. E(a) = \text{rank } a$   
 PROOF: By transfinite induction on  $a$ .  
 $\langle 1 \rangle 4.$   $\text{ran } E = \text{rank } A$   
 PROOF: From  $\langle 1 \rangle 3$  substituting  $\{A\}$  for  $A$ .  
 $\square$

## 31 Models of Set Theory

**Theorem 284.** *For any limit ordinal  $\lambda > \omega$ , we have  $V_\lambda$  is a model of Zermelo set theory.*

PROOF: Easy.  $\square$

**Theorem 285 (Choice).** *For any ordinal  $\alpha$ , we have  $V_\alpha$  is a model of the Axiom of Choice.*

PROOF: Easy.  $\square$

**Lemma 286 (Choice).** *There exists a well-ordered structure in  $V_{\omega_2}$  whose ordinal number is not in  $V_{\omega_2}$ .*

PROOF: Pick an uncountable set  $S \in V_{\omega_2}$ . Pick a well-ordering  $R$  on  $S$ . Then  $\langle S, R \rangle \in V_{\omega_2}$  but its ordinal is not, because every ordinal in  $V_{\omega_2}$  is  $< \omega_2$  hence countable.  $\square$

**Corollary 286.1 (Choice).** *The set  $V_{\omega_2}$  is not a model of ZFC.*

**Corollary 286.2.** *The Replacement Axioms are not provable from the Zermelo axioms.*

## 32 Cofinality

**Definition 287** (Cofinal). Let  $\lambda$  be a limit ordinal and  $S$  a set of smaller ordinals. Then  $S$  is *cofinal* in  $\lambda$  iff  $\lambda = \sup S$ .

**Definition 288** (Cofinality). The *cofinality* of a limit ordinal  $\lambda$ ,  $\text{cf } \lambda$ , is the least cardinal  $\kappa$  such that  $\lambda$  is the limit of  $\kappa$  smaller ordinals.

We also define  $\text{cf } 0 = 0$  and  $\text{cf } \alpha^+ = 1$ .

**Definition 289** (Regular Cardinal). A cardinal  $\kappa$  is *regular* iff  $\text{cf } \kappa = \kappa$ ; otherwise  $\kappa$  is *singular*.

**Theorem 290.** For every ordinal  $\alpha$ , the cardinal  $\aleph_{\alpha+1}$  is regular.

PROOF: If  $S$  is a set of fewer than  $\aleph_{\alpha+1}$  smaller ordinals then  $\forall \beta \in S. |\beta| \leq \aleph_\alpha$  and so

$$|\bigcup S| \leq |S| \cdot \aleph_\alpha \leq \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha. \square$$

**Theorem 291.** For every limit ordinal  $\lambda$ , we have  $\text{cf } \aleph_\lambda = \text{cf } \lambda$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $\lambda$  be a limit ordinal.

$\langle 1 \rangle 2$ .  $\text{cf } \aleph_\lambda \leq \text{cf } \lambda$

$\langle 2 \rangle 1$ . PICK a set  $S$  with  $|S| = \text{cf } \lambda$  and  $\bigcup S = \lambda$

$\langle 2 \rangle 2$ .  $\aleph_\lambda = \sup_{\alpha \in S} \aleph_\alpha$

PROOF: Theorem 139.

$\langle 1 \rangle 3$ .  $\text{cf } \lambda \leq \text{cf } \aleph_\lambda$

$\langle 2 \rangle 1$ . LET:  $A$  be a set of smaller ordinals such that  $\aleph_\lambda = \sup A$

PROVE:  $\text{cf } \lambda \leq |A|$

$\langle 2 \rangle 2$ . LET:  $B = \{\gamma \in \lambda : \exists \alpha \in A. |\alpha| = \aleph_\gamma\}$

$\langle 2 \rangle 3$ .  $|B| \leq |A|$

$\langle 2 \rangle 4$ .  $\sup B = \lambda$

$\langle 3 \rangle 1$ .  $\forall \alpha \in A. \alpha \in \aleph_{\sup B+1}$

$\langle 4 \rangle 1$ . LET:  $\alpha \in A$

$\langle 4 \rangle 2$ .  $|\alpha| \leq \aleph_{\sup B}$

$\langle 4 \rangle 3$ .  $\alpha \in \aleph_{\sup B+1}$

$\langle 3 \rangle 2$ .  $\lambda \subseteq \sup B + 1$

$\langle 4 \rangle 1$ .  $\aleph_\lambda \leq \aleph_{\sup B+1}$

$\langle 3 \rangle 3$ .  $\lambda = \sup B$

$\langle 4 \rangle 1$ .  $\lambda \leq \sup B$

PROOF: From  $\langle 3 \rangle 2$  since  $\lambda$  is a limit ordinal.

$\langle 4 \rangle 2$ .  $\sup B \leq \lambda$

PROOF: From  $\langle 2 \rangle 2$ .

$\square$

**Definition 292** (Weakly Inaccessible). An ordinal  $\lambda$  is *weakly inaccessible* iff  $\aleph_\lambda$  is regular.

**Lemma 293.** Let  $f$  be an  $\alpha$ -sequence of ordinals. Then there exists an increasing  $\beta$ -sequence  $g$  for some  $\beta \leq \alpha$  such that  $\sup \text{ran } f = \sup \text{ran } g$ .

PROOF:

- ⟨1⟩1. LET:  $h$  be the sequence defined by transfinite recursion thus:  $h_\xi$  is the least  $\gamma$  such that  $\forall \delta < \xi. f_{h_\delta} < f_\gamma$  if any such  $\gamma$  exists; otherwise the sequence halts.
- ⟨1⟩2. LET:  $\beta = \text{dom } h$
- ⟨1⟩3.  $g_\xi = f_{h_\xi}$  for  $\xi < \beta$
- ⟨1⟩4.  $\sup \text{ran } g \leq \sup \text{ran } f$   
 PROOF: Since  $g$  is a subsequence of  $f$ .
- ⟨1⟩5.  $\sup \text{ran } f \leq \sup \text{ran } g$ 
  - ⟨2⟩1.  $\forall \xi < \beta. \forall \delta \leq h_\xi. f_\delta \leq g_\xi$ 
    - ⟨3⟩1. LET:  $\xi < \beta$
    - ⟨3⟩2. LET:  $\delta \leq h_\xi$
    - ⟨3⟩3.  $f_\delta \leq f_{h_\xi}$ 
      - ⟨4⟩1. ASSUME:  $\delta < h_\xi$
      - ⟨4⟩2. PICK  $\alpha < \xi$  such that  $f_{\delta_\alpha} \leq f_{h_\alpha}$
      - ⟨4⟩3.  $f_\delta \leq f_{h_\alpha} \leq f_{h_\xi}$
    - ⟨3⟩4.  $f_{h_\xi} = g_\xi$
  - ⟨2⟩2.  $\forall \xi < \beta. f_\xi \leq g_\xi$
  - ⟨2⟩3. CASE:  $\beta = \alpha$   
 PROOF: Then  $\sup \text{ran } f \leq \sup \text{ran } g$  immediately.
  - ⟨2⟩4. CASE:  $\beta < \alpha$ 
    - ⟨3⟩1. There is no  $\gamma$  such that  $g_\delta < f_\gamma$  for all  $\delta < \beta$   
 PROOF: This is the condition for the sequence  $h$  to halt.
    - ⟨3⟩2. For all  $\gamma$ , there exists  $\delta$  such that  $f_\gamma < g_\delta$
    - ⟨3⟩3.  $\sup \text{ran } f \leq \sup \text{ran } g$

□

**Theorem 294.** *Let  $\lambda$  be a limit ordinal. Then there exists an increasing  $(\text{cf } \lambda)$ -sequence of ordinals that converges to  $\lambda$ .*

PROOF:

- ⟨1⟩1. PICK a set  $S$  with  $|S| = \text{cf } \lambda$  and  $\lambda = \sup S$
- ⟨1⟩2. PICK a bijection  $f : \text{cf } \lambda \approx S$
- ⟨1⟩3. PICK an increasing  $\beta$ -sequence converging to  $\lambda$  with  $\beta \leq \text{cf } \lambda$   
 PROOF: Lemma 293.
- ⟨1⟩4.  $\beta = \text{cf } \lambda$   
 PROOF: By leastness of  $\text{cf } \lambda$ .

□

**Corollary 294.1.** *For any limit ordinal  $\lambda$ , we have  $\text{cf } \lambda$  is the least ordinal  $\alpha$  such that there exists an increasing  $\alpha$ -sequence of ordinals  $< \lambda$  that converges to  $\lambda$ .*

**Theorem 295.** *For any ordinal  $\lambda$ , we have  $\text{cf } \lambda$  is a regular cardinal.*

PROOF:

- ⟨1⟩1. ASSUME: w.l.o.g.  $\lambda$  is a limit ordinal.
- ⟨1⟩2. PICK an increasing  $\text{cf } \lambda$ -sequence  $f$  of ordinals  $< \lambda$  that converges to  $\lambda$ .

- ⟨1⟩3. LET:  $S$  be a set of ordinals  $< \text{cf } \lambda$  such that  $\text{cf } \lambda = \sup S$ .
- ⟨1⟩4.  $f(S)$  is cofinal in  $\lambda$ 
  - ⟨2⟩1. LET:  $\alpha < \lambda$
  - ⟨2⟩2. PICK  $\beta < \text{cf } \lambda$  such that  $\alpha < f(\beta) < \lambda$   
PROOF: Since  $f$  converges to  $\lambda$ .
  - ⟨2⟩3. PICK  $\gamma \in S$  such that  $\beta < \gamma$   
PROOF: Since  $\sup S = \text{cf } \lambda$ .
  - ⟨2⟩4.  $\alpha < f(\gamma) \in f(S)$
- ⟨1⟩5.  $\text{cf } \lambda \leq |S|$   
PROOF: We have  $\text{cf } \lambda \leq |f(S)| = |S|$
- ⟨1⟩6.  $\text{cf } \text{cf } \lambda = \text{cf } \lambda$

□

**Theorem 296.** *Let  $\lambda$  be an infinite cardinal. Then  $\text{cf } \lambda$  is the least cardinal  $\kappa$  such that  $\lambda$  can be decomposed as the union of  $\kappa$  sets each with cardinality  $< \lambda$ .*

PROOF:

- ⟨1⟩1.  $\lambda$  can be decomposed as the union of  $\text{cf } \lambda$  sets each with cardinality  $< \lambda$   
PROOF: Since  $\lambda$  is the union of a set of  $\text{cf } \lambda$  smaller ordinals.
- ⟨1⟩2. If  $\lambda = \bigcup \mathcal{A}$  where  $\forall X \in \mathcal{A}. |X| < \lambda$  then  $\text{cf } \lambda \leq |\mathcal{A}|$ .
  - ⟨2⟩1. LET:  $\kappa = |\mathcal{A}|$
  - ⟨2⟩2. LET:  $\mathcal{A} = \{A_\xi : \xi < \kappa\}$
  - ⟨2⟩3.  $\lambda = \bigcup_{\xi < \kappa} A_\xi$
  - ⟨2⟩4.  $\forall \xi < \kappa. |A_\xi| < \lambda$
  - ⟨2⟩5. LET:  $\mu = \sup_{\xi < \kappa} |A_\xi|$
  - ⟨2⟩6.  $\lambda \leq \mu \cdot \kappa$   
PROOF: Since  $\lambda = |\bigcup_{\xi < \kappa} A_\xi|$
  - ⟨2⟩7. CASE:  $\lambda \leq \kappa$   
PROOF: Then  $\text{cf } \lambda \leq \lambda \leq \kappa$
  - ⟨2⟩8. CASE:  $\kappa < \lambda$ 
    - ⟨3⟩1.  $\lambda = \mu$   
PROOF: Since  $\lambda \leq \mu \cdot \kappa \leq \lambda \cdot \lambda = \lambda$
    - ⟨3⟩2.  $\lambda$  is the supremum of  $\kappa$  smaller ordinals.
    - ⟨3⟩3.  $\text{cf } \lambda \leq \kappa$

□

**Theorem 297** (König's Theorem (Choice)). *For any infinite cardinal  $\kappa$  we have  $\kappa < \text{cf } 2^\kappa$*

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction  $\text{cf } 2^\kappa \leq \kappa$
- ⟨1⟩2. PICK a set  $S$  with  $|S| = 2^\kappa$
- ⟨1⟩3. PICK a  $\kappa$ -sequence of sets  $A_\xi$  with  $S^\kappa = \bigcup_{\xi < \kappa} A_\xi$  and  $\forall \xi < \kappa. |A_\xi| < 2^\kappa$   
PROOF: Since  $|S^\kappa| = 2^\kappa$
- ⟨1⟩4.  $\forall \xi < \kappa. \{g(\xi) : g \in A_\xi\} \subset S$   
PROOF: Since  $|\{g(\xi) : g \in A_\xi\}| \leq |A_\xi| < 2^\kappa$
- ⟨1⟩5. For all  $\xi < \kappa$ , PICK  $s_\xi \in S - \{g(\xi) : g \in A_\xi\}$

- $\langle 1 \rangle 6. s \in S^\kappa$   
 $\langle 1 \rangle 7. \forall \xi < \kappa. s \notin A_\xi$   
 $\langle 1 \rangle 8. \text{Q.E.D.}$

PROOF: This contradicts  $\langle 1 \rangle 3$ .

□

**Corollary 297.1.**  $2^{\aleph_0} \neq \aleph_\omega$

PROOF: Since  $\text{cf } \aleph_\omega = \aleph_0$  and  $\text{cf } 2^{\aleph_0} > \aleph_0$ . □

### 33 Inaccessible Cardinals

**Definition 298** (Inaccessible Cardinal). A cardinal  $\kappa$  is *inaccessible* iff:

- $\kappa > \aleph_0$
- For every cardinal  $\lambda < \kappa$  we have  $2^\lambda < \kappa$
- $\kappa$  is regular.

**Lemma 299.** For any ordinal  $\alpha$  and limit ordinal  $\lambda$ ,

$$V_{\alpha+\lambda} = \bigcup_{\delta < \lambda} V_{\alpha+\delta}$$

PROOF:

- $\langle 1 \rangle 1. V_{\alpha+\lambda} = \bigcup_{\delta < \lambda} V_{\alpha+\delta}$   
 $\langle 2 \rangle 1. \text{LET: } x \in V_{\alpha+\lambda}$   
 $\langle 2 \rangle 2. \text{PICK } \beta < \alpha + \lambda \text{ such that } x \in V_\beta$   
 $\langle 2 \rangle 3. \text{CASE: } \beta < \alpha$   
     PROOF: Then  $x \in V_{\alpha+0}$ .  
 $\langle 2 \rangle 4. \text{CASE: } \alpha \leq \beta$   
 $\langle 3 \rangle 1. \text{LET: } \delta \text{ be the ordinal such that } \beta = \alpha + \delta$   
 $\langle 3 \rangle 2. x \in V_{\alpha+\delta} \text{ and } \delta < \lambda$   
 $\langle 1 \rangle 2. \bigcup_{\delta < \lambda} V_{\alpha+\delta} \subseteq V_{\alpha+\lambda}$

□

**Lemma 300.** For any ordinal  $\alpha$  we have  $|V_{\omega+\alpha}| = \beth_\alpha$ .

PROOF:

- $\langle 1 \rangle 1. |V_\omega| = \beth_0$   
     PROOF: Since  $V_\omega$  is the union of an  $\omega$ -sequence of finite sets of increasing size.  
 $\langle 1 \rangle 2. \text{For any ordinal } \alpha, \text{ if } |V_{\omega+\alpha}| = \beth_\alpha \text{ then } |V_{\omega+\alpha+1}| = \beth_{\alpha+1}$   
 $\langle 1 \rangle 3. \text{For any limit ordinal } \lambda, \text{ if } \forall \alpha < \lambda. |V_{\omega+\alpha}| = \beth_\alpha \text{ then } |V_{\omega+\lambda}| = \beth_\lambda$   
 $\langle 2 \rangle 1. \text{LET: } \lambda \text{ be a limit ordinal.}$   
 $\langle 2 \rangle 2. \text{ASSUME: } \forall \alpha < \lambda. |V_{\omega+\alpha}| = \beth_\alpha$   
 $\langle 2 \rangle 3. |V_{\omega+\lambda}| \geq \beth_\lambda$

PROOF:

$$\begin{aligned}
|V_{\omega+\lambda}| &= \left| \bigcup_{\delta < \lambda} V_{\omega+\delta} \right| && \text{(Lemma 299)} \\
&\geq \sup_{\delta < \lambda} |V_{\omega+\delta}| \\
&= \sup_{\delta < \lambda} \beth_{\delta} \\
&= \beth_{\lambda}
\end{aligned}$$

$\langle 2 \rangle 4. \beth_{\lambda} \leq |V_{\omega+\lambda}|$

PROOF:

$$\begin{aligned}
|V_{\omega+\lambda}| &= \left| \bigcup_{\delta < \lambda} V_{\omega+\delta} \right| \\
&\leq |\lambda| \cdot \beth_{\lambda} \\
&\leq \beth_{\lambda} \cdot \beth_{\lambda} \\
&= \beth_{\lambda}
\end{aligned}$$

□

**Lemma 301.** *Let  $\kappa$  be an inaccessible cardinal. For any ordinal  $\alpha < \kappa$ , we have  $\beth_{\alpha} < \kappa$ .*

PROOF:

$\langle 1 \rangle 1. \beth_0 < \kappa$

PROOF: By definition of inaccessible.

$\langle 1 \rangle 2. \text{ If } \beth_{\alpha} < \kappa \text{ then } \beth_{\alpha^+} < \kappa$

PROOF:  $\beth_{\alpha^+} = 2^{\beth_{\alpha}} < \kappa$

$\langle 1 \rangle 3. \text{ If } \lambda \text{ is a limit ordinal, } \lambda < \kappa \text{ and } \forall \alpha < \lambda. \beth_{\alpha} < \kappa \text{ then } \beth_{\lambda} < \kappa$

PROOF: Since  $\beth_{\lambda} = \sup_{\alpha < \lambda} \beth_{\alpha}$  is the supremum of fewer than  $\kappa$  smaller ordinals.

□

**Lemma 302.** *Let  $\kappa$  be an inaccessible cardinal. For all  $A \in V_{\kappa}$  we have  $|A| < \kappa$ .*

PROOF: Pick  $\alpha < \kappa$  such that  $A \subseteq V_{\alpha}$ . Then  $|A| \leq |V_{\alpha}| \leq \beth_{\alpha} < \kappa$ . □

**Theorem 303.** *If  $\kappa$  is an inaccessible cardinal then  $V_{\kappa}$  is a model of ZF.*

PROOF: Easy. □