C1 Set Theory

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August 14, 2022

1 Primitive Notions

Let there be *sets*.

Let there be a binary relation called *membership*, $\in$. When $x \in y$ holds, we say x is a *member* or *element* of y. We write $x \notin y$ iff x is not a member of y.

2 The Axioms

Axiom 1 (Extensionality)**.** *If two sets have exactly the same members, then they are equal.*

As a consequence of this axiom, we may identify a set A with the class $\{x : x \in A\}$. The use of the symbols $\in$ and $=$ is consistent.

Definition 2. We say that a class $\mathbf{A}$ *is a set* iff there exists a set A such that $A = \mathbf{A}$. That is, the class $\{x : P(x)\}$ is a set iff

$$\exists A. \forall x (x \in A \leftrightarrow P(x)) \enspace .$$

Otherwise, $\mathbf{A}$ is a *proper class*.

Definition 3 (Subset)**.** If A is a set and $\mathbf{B}$ is a class, we say A is a *subset* of $\mathbf{B}$ iff $A \subseteq \mathbf{B}$.

Axiom 4 (Empty Set)**.** *The empty class is a set, called the* empty set*.*

Axiom 5 (Pairing)**.** *For any objects a and b, the class $\{a, b\}$ is a set, called a* pair set*.*

Definition 6 (Union)**.** For any class of sets $\mathbf{A}$, the *union* $\bigcup \mathbf{A}$ is the class $\{x : \exists A \in \mathbf{A}. x \in A\}$.

We write $\bigcup_{P[x_1, \ldots, x_n]} t[x_1, \ldots, x_n]$ for $\bigcup \{t[x_1, \ldots, x_n] : P[x_1, \ldots, x_n]\}$.

Proposition 7. *If $\mathbf{A} \subseteq \mathbf{B}$ then $\bigcup \mathbf{A} \subseteq \bigcup \mathbf{B}$.*

PROOF: Easy. $\square$

Axiom 8 (Union)**.** *For any set A, the union $\bigcup A$ is a set.*

Proposition 9. *For any sets A and B, the class $A \cup B$ is a set.*

PROOF: It is $\bigcup\{A, B\}$. $\square$

Proposition Schema 10. *For any objects a_1, $\ldots$, a_n, the class $\{a_1, \ldots, a_n\}$ is a set.*

PROOF: By repeated application of the Pairing and Union axioms. $\square$

Definition 11 (Power Set)**.** For any set A, the *power set* of A, $\mathcal{P}A$, is the class of all subsets of A.

Axiom 12 (Power Set)**.** *For any set A, the class $\mathcal{P}A$ is a set.*

Axiom 13 (Subset, Aussonderung)**.** *For any class $\mathbf{A}$ and set B, if $\mathbf{A} \subseteq B$ then $\mathbf{A}$ is a set.*

Proposition 14. *For any set A and class $\mathbf{B}$, the intersection $A \cap \mathbf{B}$ is a set.*

PROOF: By the Subset Axiom since it is a subclass of A. $\square$

Proposition 15. *For any set A and class $\mathbf{B}$, the relative complement $A - \mathbf{B}$ is a set.*

PROOF: By the Subset Axiom since it is a subclass of A. $\square$

Theorem 16. *The universal class $\mathbf{V}$ is a proper class.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $\mathbf{V}$ is a set.
$\langle 1 \rangle 2$. LET: $R = \{x : x \notin x\}$
$\langle 1 \rangle 3$. R is a set.
  PROOF: By the Subset Axiom.
$\langle 1 \rangle 4$. $R \in R$ if and only if $R \notin R$
$\langle 1 \rangle 5$. Q.E.D.
  PROOF: This is a contradiction.
$\square$

Definition 17 (Intersection)**.** For any class of sets $\mathbf{A}$, the *intersection* $\bigcap \mathbf{A}$ is the class $\{x : \forall A \in \mathbf{A}. x \in A\}$.

We write $\bigcap_{P[x_1,\ldots,x_n]} t[x_1, \ldots, x_n]$ for $\bigcap\{t[x_1, \ldots, x_n] : P[x_1, \ldots, x_n]\}$.

Proposition 18. *For any nonempty class of sets $\mathbf{A}$, the class $\bigcap \mathbf{A}$ is a set.*

PROOF: Pick $A \in \mathbf{A}$. Then $\bigcap \mathbf{A} \subseteq A$. $\square$

Proposition 19. *If $\mathbf{A} \subseteq \mathbf{B}$ then $\bigcap \mathbf{B} \subseteq \bigcap \mathbf{A}$.*

PROOF: Easy. $\square$

Proposition 20. *For any set A and class of sets $\mathbf{B}$, we have*

$$A \cup \bigcap \mathbf{B} = \bigcap\{A \cup X \mid X \in \mathbf{B}\}$$

PROOF: Easy. ∎

Proposition 21. *For any set A and class of sets $\mathbf{B}$, we have*

$$A \cap \bigcup \mathbf{B} = \bigcup \{A \cap X \mid X \in \mathbf{B}\}$$

PROOF: Easy. ∎

Proposition 22. *For any set C and class of sets $\mathbf{A}$, we have*

$$C - \bigcup \mathbf{A} = \bigcap \{C - X \mid X \in \mathbf{A}\} \enspace .$$

PROOF: Easy. ∎

Proposition 23. *For any set C and class of sets $\mathbf{A}$, we have*

$$C - \bigcap \mathbf{A} = \bigcup \{C - X \mid X \in \mathbf{A}\} \enspace .$$

PROOF: Easy. ∎

3 Ordered Pairs

Definition 24 (Ordered Pair)**.** For any objects a and b, the *ordered pair* (a, b) is $\{\{a\}, \{a, b\}\}$. We call a its *first coordinate* and b its *second coordinate.*

Theorem 25. *For any objects (a, b), we have $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.*

PROOF:
⟨1⟩1. If $(a, b) = (c, d)$ then $a = c$ and $b = d$
  ⟨2⟩1. ASSUME: $(a, b) = (c, d)$
  ⟨2⟩2. $a = c$
    PROOF: Since $\{a\} = \bigcap (a, b) = \bigcap (c, d) = \{c\}$.
  ⟨2⟩3. $\{a, b\} = \{c, d\}$
    PROOF: $\{a, b\} = \bigcup (a, b) = \bigcup (c, d) = \{c, d\}$.
  ⟨2⟩4. $b = c$ or $b = d$
  ⟨2⟩5. CASE: $b = c$
    ⟨3⟩1. $a = b$
    ⟨3⟩2. $\{c, d\} = \{a\}$
    ⟨3⟩3. $b = d$
  ⟨2⟩6. CASE: $b = d$
    PROOF: We have $a = c$ and $b = d$ as required.
⟨1⟩2. If $a = c$ and $b = d$ then $(a, b) = (c, d)$
  PROOF: Trivial.
∎

Definition 26 (Cartesian Product)**.** The *Cartesian product* of classes $\mathbf{A}$ and $\mathbf{B}$ is the class

$$\mathbf{A} \times \mathbf{B} = \{(x, y) : x \in \mathbf{A}, y \in \mathbf{B}\} \enspace .$$

Lemma 27. *For any objects x and y and set C, if $x \in C$ and $y \in C$ then $(x, y) \in \mathcal{PP}C$.*

Proof: Easy. $\square$

Corollary 27.1. *For any sets A and B, the Cartesian product $A \times B$ is a set.*

Proof: By the Subset Axiom applied to $\mathcal{PP}(A \cup B)$. $\square$

Lemma 28. *If $(x, y) \in \mathbf{A}$ then $x, y \in \bigcup\bigcup \mathbf{A}$.*

Proof: Easy. $\square$

4 Relations

Definition 29 (Relation)**.** A *relation* is a class of ordered pairs. It is *small* iff it is a set.

When $\mathbf{R}$ is a relation, we write $x\mathbf{R}y$ for $(x, y) \in \mathbf{R}$.

Definition 30 (Domain)**.** The *domain* of a class $\mathbf{R}$ is $\operatorname{dom} \mathbf{R} = \{x : \exists y.(x, y) \in \mathbf{R}\}$.

Definition 31 (Range)**.** The *range* of a class $\mathbf{R}$ is $\operatorname{ran} \mathbf{R} = \{y : \exists x.(x, y) \in \mathbf{R}\}$.

Definition 32 (Field)**.** The *field* of a class $\mathbf{R}$ is $\operatorname{fld} \mathbf{R} = \operatorname{dom} \mathbf{R} \cup \operatorname{ran} \mathbf{R}$.

Proposition 33. *If R is a set then $\operatorname{dom} R$, $\operatorname{ran} R$ and $\operatorname{fld} R$ are sets.*

Proof: Apply the Subset Axiom to $\bigcup\bigcup R$. $\square$

Definition 34 (Single-Rooted)**.** A class $\mathbf{R}$ is *single-rooted* iff, for all $y \in \operatorname{ran} \mathbf{R}$, there is only one x such that $x\mathbf{R}y$.

Definition 35 (Inverse)**.** The *inverse* of a class $\mathbf{F}$ is the class $\mathbf{F}^{-1} = \{(y, x) \mid (x, y) \in \mathbf{F}\}$.

Theorem 36. *For any class $\mathbf{F}$, we have $\operatorname{dom} \mathbf{F}^{-1} = \operatorname{ran} \mathbf{F}$ and $\operatorname{ran} \mathbf{F}^{-1} = \operatorname{dom} \mathbf{F}$.*

Proof: Easy. $\square$

Theorem 37. *For a relation $\mathbf{F}$, $(\mathbf{F}^{-1})^{-1} = \mathbf{F}$.*

Proof: Easy. $\square$

Definition 38 (Composition)**.** The *composition* of classes $\mathbf{F}$ and $\mathbf{G}$ is the class $\mathbf{G} \circ \mathbf{F} = \{(x, z) \mid \exists y.(x, y) \in \mathbf{F} \wedge (y, z) \in \mathbf{G}\}$.

Theorem 39. *For any classes $\mathbf{F}$ and $\mathbf{G}$, $(\mathbf{F} \circ \mathbf{G})^{-1} = \mathbf{G}^{-1} \circ \mathbf{F}^{-1}$.*

Proof: Easy. $\square$

Definition 40 (Restriction)**.** The *restriction* of the class $\mathbf{F}$ to the class $\mathbf{A}$ is the class $\mathbf{F} \upharpoonright \mathbf{A} = \{(x, y) : x \in A \wedge (x, y) \in \mathbf{F}\}$.

Definition 41 (Image)**.** The *image* of the class $\mathbf{A}$ under the class $\mathbf{F}$ is the class $\mathbf{F}(\mathbf{A}) = \{y : \exists x \in \mathbf{A}.(x, y) \in \mathbf{F}\}$.

Theorem 42.

$$\mathbf{F}(\mathbf{A} \cup \mathbf{B}) = \mathbf{F}(\mathbf{A}) \cup \mathbf{F}(\mathbf{B})$$

PROOF: Easy. □

Theorem 43.

$$\mathbf{F}(\bigcup \mathbf{A}) = \bigcup\{\mathbf{F}(X) : X \in \mathbf{A}\}$$

PROOF: Easy. □

Theorem 44.

$$\mathbf{F}(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{F}(\mathbf{A}) \cap \mathbf{F}(\mathbf{B})$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

PROOF: Easy. □

Theorem 45.

$$\mathbf{F}(\bigcap \mathbf{A}) \subseteq \bigcap\{\mathbf{F}(X) : X \in \mathbf{A}\}$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

PROOF: Easy. □

Theorem 46.

$$\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

PROOF: Easy. □

5 n-ary Relations

Definition 47. Given objects a, b, c, define the *ordered triple* (a, b, c) to be $((a, b), c)$.

Define $(a, b, c, d) = ((a, b, c), d)$, etc.

Define the *1-tuple* (a) to be a.

Definition 48 (n-ary Relation)**.** Given a class $\mathbf{A}$, an *n-ary relation* on $\mathbf{A}$ is a class of ordered n-tuples, all of whose components are in $\mathbf{A}$.

6 Functions

Definition 49 (Function)**.** A *function* is a relation $\mathbf{F}$ such that, for all $x \in \operatorname{dom}\mathbf{F}$, there is only one y such that $x\mathbf{F}y$. We call this unique y the *value* of $\mathbf{F}$ at x and denote it by $\mathbf{F}(x)$.

We say $\mathbf{F}$ is a function *from* $\mathbf{A}$ *into* $\mathbf{B}$, or $\mathbf{F}$ *maps* $\mathbf{A}$ into $\mathbf{B}$, and write $\mathbf{F} : \mathbf{A} \to \mathbf{B}$, iff $\mathbf{F}$ is a function, $\operatorname{dom}\mathbf{F} = \mathbf{A}$, and $\operatorname{ran}\mathbf{F} \subseteq \mathbf{B}$.

If, in addition, $\operatorname{ran}\mathbf{F} = \mathbf{B}$, we say $\mathbf{F}$ is a function from $\mathbf{A}$ *onto* $\mathbf{B}$.

Theorem 50. *For a class $\mathbf{F}$, $\mathbf{F}^{-1}$ is a function if and only if $\mathbf{F}$ is single-rooted.*

Proof: Easy. □

Theorem 51. *A relation $\mathbf{F}$ is a function if and only if $\mathbf{F}^{-1}$ is single-rooted.*

Proof: Easy. □

Theorem 52. *For any function $\mathbf{G}$ and classes $\mathbf{A}$ and $\mathbf{B}$,*

$$\begin{aligned}
\mathbf{G}^{-1}(\bigcup \mathbf{A}) &= \bigcup\{\mathbf{G}^{-1}(X) : X \in \mathbf{A}\} \\
\mathbf{G}^{-1}(\bigcap \mathbf{A}) &= \bigcap\{\mathbf{G}^{-1}(X) : X \in \mathbf{A}\} \qquad (\textit{if } \mathbf{A} \neq \emptyset) \\
\mathbf{G}^{-1}(\mathbf{A} - \mathbf{B}) &= \mathbf{G}^{-1}(\mathbf{A}) - \mathbf{G}^{-1}(\mathbf{B})
\end{aligned}$$

Proof: Easy. □

Theorem 53. *Assume that $\mathbf{F}$ and $\mathbf{G}$ are functions. Then $\mathbf{F} \circ \mathbf{G}$ is a function, its domain is $\{x \in \operatorname{dom}\mathbf{G} : \mathbf{G}(x) \in \operatorname{dom}\mathbf{F}\}$, and for x in its domain,*

$$(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x)) \ .$$

Proof: Easy. □

Definition 54 (One-to-one)**.** A function $\mathbf{F}$ is *one-to-one* or an *injection* iff it is single-rooted.

Theorem 55. *Let $\mathbf{F}$ be a one-to-one function. For $x \in \operatorname{dom}\mathbf{F}$, $\mathbf{F}^{-1}(\mathbf{F}(x)) = x$.*

Proof: Easy. □

Theorem 56. *Let $\mathbf{F}$ be a one-to-one function. For $y \in \operatorname{ran}\mathbf{F}$, $\mathbf{F}(\mathbf{F}^{-1}(y)) = y$.*

Proof: Easy. □

Definition 57 (Identity Function)**.** For any class $\mathbf{A}$, the *identity* function on $\mathbf{A}$ is $\mathrm{id}_{\mathbf{A}} = \{(x, x) \mid x \in \mathbf{A}\}$.

Theorem 58. *Let $F : A \to B$. Assume $A \neq \emptyset$. Then F has a left inverse (i.e. there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$) if and only if F is one-to-one.*

Proof:
$\langle 1\rangle 1$. If F is one-to-one then F has a left inverse.

⟨2⟩1. ASSUME: F is one-to-one.
⟨2⟩2. $F^{-1} : \operatorname{ran} F \to A$
⟨2⟩3. PICK $a \in A$
⟨2⟩4. Define $G : B \to A$ by:

$$G(x) = \begin{cases} F^{-1}(x) & \text{if } x \in \operatorname{ran} F \\ a & \text{if } x \in B - \operatorname{ran} F \end{cases}$$

⟨2⟩5. $\forall x \in A . G(F(x)) = x$
⟨1⟩2. If F has a left inverse then F is one-to-one.
⟨2⟩1. ASSUME: F has a left inverse G.
⟨2⟩2. LET: $x, y \in A$ with $F(x) = F(y)$
⟨2⟩3. $x = y$
PROOF: $x = G(F(x)) = G(F(y)) = y$.
□

7 The Axiom of Choice

Axiom 59 (Choice)**.** *For any relation R there exists a function $H \subseteq R$ with $\operatorname{dom} H = \operatorname{dom} R$.*

Theorem 60. *Let $F : A \to B$. Then F has a right inverse if and only if F maps A onto B.*

PROOF:
⟨1⟩1. If F has a right inverse then F maps A onto B.
PROOF: If $H : B \to A$ is a right inverse, then for any y in B, we have $y = F(H(y))$.
⟨1⟩2. If F maps A onto B then F has a right inverse.
⟨2⟩1. ASSUME: F maps A onto B.
⟨2⟩2. PICK a function H with $H \subseteq F^{-1}$ and $\operatorname{dom} H = \operatorname{dom} F^{-1}$
PROOF: By the Axiom of Choice.
⟨2⟩3. $\operatorname{dom} H = B$
PROOF: $\operatorname{dom} H = \operatorname{dom} F^{-1} = \operatorname{ran} F = B$ by ⟨2⟩1.
⟨2⟩4. For all $y \in B$ we have $F(H(y)) = y$
⟨3⟩1. LET: $y \in B$
⟨3⟩2. $(y, H(y)) \in F^{-1}$
⟨3⟩3. $F(H(y)) = y$
□

8 Sets of Functions

Definition 61. Let A be a set and $\mathbf{B}$ be a class. Then $\mathbf{B}^A$ is the class of all functions $A \to \mathbf{B}$.

9 Dependent Products

Definition 62. Let I be a set and H_i a set for all $i \in I$. Define

$$\prod_{i \in I} H_i = \{f : f \text{ is a function}, \operatorname{dom} f = I, \forall i \in I. f(i) \in H_i\} \ .$$

Theorem 63. *The Axiom of Choice is equivalent to the statement: For any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$*

PROOF:

$\langle 1 \rangle 1$. If the Axiom of Choice is true then, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$.

$\langle 2 \rangle 1$. ASSUME: The Axiom of Choice.

$\langle 2 \rangle 2$. LET: I be a set.

$\langle 2 \rangle 3$. LET: H be a function with domain I.

$\langle 2 \rangle 4$. ASSUME: $H(i) \neq \emptyset$ for all $i \in I$.

$\langle 2 \rangle 5$. LET: $R = \{(i, x) : i \in I, x \in H(i)\}$

$\langle 2 \rangle 6$. PICK a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R$
PROVE: $F \in \prod_{i \in I} H(i)$

PROOF: By the Axiom of Choice.

$\langle 2 \rangle 7$. $\operatorname{dom} H = I$

PROOF: We have $\operatorname{dom} R = I$ since for all $i \in I$ there exists x such that $x \in H(i)$.

$\langle 2 \rangle 8$. $\forall i \in I. F(i) \in H(i)$

PROOF: Since $iRF(i)$.

$\langle 1 \rangle 2$. If, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$, then the Axiom of Choice is true.

$\langle 2 \rangle 1$. ASSUME: For any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$

$\langle 2 \rangle 2$. LET: R be a relation

$\langle 2 \rangle 3$. LET: $I = \operatorname{dom} R$

$\langle 2 \rangle 4$. Define the function H with domain I by: for $i \in I$, $H(i) = \{y : iRy\}$

$\langle 2 \rangle 5$. $H(i) \neq \emptyset$ for all $i \in I$

$\langle 2 \rangle 6$. PICK $F \in \prod_{i \in I} H(i)$

PROOF: By $\langle 2 \rangle 1$

$\langle 2 \rangle 7$. F is a function

$\langle 2 \rangle 8$. $F \subseteq R$

PROOF: For all $i \in I$ we have $F(i) \in H(i)$ and so $iRF(i)$.

$\langle 2 \rangle 9$. $\operatorname{dom} F = \operatorname{dom} R$

$\square$