Solutions Manual for Enderton *Elements of Set Theory*

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Contents

Chapter 1

Chapter 1 — Introduction

1.1   Baby Set Theory

Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ — true

- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true

- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — false

- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — true

- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$ — false

- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true

- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — true

- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — false

- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

Exercise 2   We have $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ has an element (namely $\emptyset$) while $\emptyset$ has no elements.

We have $\emptyset \neq \{\{\emptyset\}\}$ because $\{\{\emptyset\}\}$ has an element (namely $\{\emptyset\}$) while $\emptyset$ has no elements.

We have $\{\emptyset\} \neq \{\{\emptyset\}\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$. This last fact is true because $\emptyset \neq \{\emptyset\}$ as we proved in the first paragraph.

Exercise 3   Assume $B \subseteq C$. Let $A \in \mathcal{P}B$; we must show that $A \in \mathcal{P}C$.

We have $A \subseteq B$ (since $A \in \mathcal{P}B$) and $B \subseteq C$. From this it follows that $A \subseteq C$ (every element of A is an element of B; every element of B is an element of C; therefore every element of A is an element of C). Hence $A \in \mathcal{P}C$ as required.

Exercise 4   Since $x \in B$, we have $\{x\} \subseteq B$ and so $\{x\} \in \mathcal{P}B$.

Since $x \in B$ and $y \in B$, we have $\{x, y\} \subseteq B$ and so $\{x, y\} \in \mathcal{P}B$.

From these two facts, it follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$ and so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

1.2   Sets — An Informal View

Exercise 5   We have

$$
\begin{aligned}
V_0 &= A \\
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}A \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= \{\emptyset, \{\emptyset\}\} \\
V_3 &= \mathcal{P}V_2 \\
&= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}
$$

We have $\emptyset \subseteq V_0$ and so $\emptyset \in V_1$. Therefore $\{\emptyset\} \subseteq V_1$ and so $\{\emptyset\} \in V_2$. Hence $\{\{\emptyset\}\} \subseteq V_2$.

We also have $\{\{\emptyset\}\} \nsubseteq V_0$ because $\{\emptyset\}$ is not an atom, and $\{\{\emptyset\}\} \nsubseteq V_1$ since $\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.

Thus the rank of $\{\{\emptyset\}\}$ is 2.

Likewise we have $\emptyset$ and $\{\emptyset\}$ are both subsets of V_1, hence

$$\emptyset \in V_2, \qquad \{\emptyset\} \in V_2$$

Thus $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are all subsets of V_2, hence elements of V_3. Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is not a subset of V_0 (because $\emptyset$ is not an atom.) It is not a subset of V_1 ($\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.) It is not a subset of V_2 (we have $\{\emptyset, \{\emptyset\}\} \notin V_2$ since $\{\emptyset\} \notin V_1$).

Therefore the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

Exercise 6

$$V_1 = V_0 \cup \mathcal{P}V_0$$
$$\quad = A \cup \mathcal{P}V_0 \qquad\qquad\qquad \text{(since } V_0 = A)$$
$$V_2 = V_1 \cup \mathcal{P}V_1$$
$$\quad = A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1$$
$$\quad = A \cup \mathcal{P}V_1 \qquad\qquad \text{(since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3)}$$
$$V_3 = V_2 \cup \mathcal{P}V_2$$
$$\quad = A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2$$
$$\quad = A \cup \mathcal{P}V_2 \qquad\qquad \text{(since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3)}$$
$$V_4 = V_3 \cup \mathcal{P}V_3$$
$$\quad = A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3$$
$$\quad = A \cup \mathcal{P}V_3 \qquad\qquad \text{(since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3)}$$

Exercise 7   In Exercise 5 we calculated $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Hence

$$
\begin{aligned}
V_4 = &\,\mathcal{P}V_3 \\
= \{&\emptyset, \\
&\{\emptyset\}, \\
&\{\{\emptyset\}\}, \\
&\{\{\{\emptyset\}\}\}, \\
&\{\{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}\}, \\
&\{\emptyset, \{\{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
\}&
\end{aligned}
$$

Chapter 2

Chapter 2 — Axioms and Operations

2.1 Arbitrary Unions and Intersections

Exercise 1 $A \cap B \cap C$ is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

Exercise 2 Take $A = \emptyset$ and $B = \{\emptyset\}$. Then $\bigcup A = \bigcup B = \emptyset$ but $A \neq B$. (There are many other possible answers.)

Exercise 3 Let $b \in A$. We must show that $b \subseteq \bigcup A$.
 Let x be any element of b. We must show that $x \in \bigcup A$. We know that $x \in b$ and $b \in A$, and so $x \in \bigcup A$ by the definition of $\bigcup A$.

Exercise 4 Suppose $A \subseteq B$. Let $x \in \bigcup A$. We must show that $x \in \bigcup B$.
 Pick an element $a \in A$ such that $x \in a$. Then $a \in B$ because $A \subseteq B$. Since we know $x \in a$ and $a \in B$, we know that $x \in \bigcup B$.

Exercise 5 Assume that every member of $\mathcal{A}$ is a subset of B. Let $x \in \bigcup \mathcal{A}$. We must show that $x \in B$.
 Pick $A \in \mathcal{A}$ such that $x \in A$. By our assumption, we have $A \subseteq B$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$ as required.

Exercise 6

 (a) We will show that $\bigcup \mathcal{P}A \subseteq A$ and $A \subseteq \bigcup \mathcal{P}A$.
 To show $\bigcup \mathcal{P}A \subseteq A$: This follows from Exercise 5, since every member of $\mathcal{P}A$ is a subset of A.

To show $A \subseteq \bigcup \mathcal{P}A$: Let $a \in A$. Then we have $a \in \{a\}$ and $\{a\} \in \mathcal{P}A$ so $a \in \bigcup \mathcal{P}A$.

(b) To show $A \subseteq \mathcal{P}\bigcup A$: This holds because every element of A is a subset of $\bigcup A$, as we proved is Exercise 3.

Equality holds if and only if $A = \mathcal{P}X$ for some set X.

Proof: If $A = \mathcal{P}\bigcup A$ then of course $A = \mathcal{P}X$ for some X.

Conversely, if $A = \mathcal{P}X$, then we have

$$\mathcal{P}\bigcup A = \mathcal{P}\bigcup \mathcal{P}X$$
$$= \mathcal{P}X \qquad \text{(by part (a))}$$
$$= A$$

Exercise 7

(a) For any set X,

$$X \in \mathcal{P}A \cap \mathcal{P}B$$
$$\Leftrightarrow X \subseteq A \text{ and } X \subseteq B$$
$$\Leftrightarrow \text{Every member of } X \text{ is a member of } A \text{ and a member of } B$$
$$\Leftrightarrow X \subseteq A \cap B$$
$$\Leftrightarrow X \in \mathcal{P}(A \cap B)$$

(b) Let $X \in \mathcal{P}A \cup \mathcal{P}B$. Then either $X \in \mathcal{P}A$ or $X \in \mathcal{P}B$ (or both). If $X \in \mathcal{P}A$, then we have $X \subseteq A$ and so $X \subseteq A \cup B$ (because $A \subseteq A \cup B$). Similarly if $X \in \mathcal{P}B$ then we have $X \subseteq A \cup B$. So in either case $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

Equality holds if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof: Suppose $A \subseteq B$. Then $\mathcal{P}A \subseteq \mathcal{P}B$ (Chapter 1 Exercise 3) and so $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$. Also $A \cup B = B$ so $\mathcal{P}(A \cup B) = \mathcal{P}B$. Thus $\mathcal{P}A \cup \mathcal{P}B$ and $\mathcal{P}(A \cup B)$ are equal.

Similarly if $B \subseteq A$ then $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$.

Conversely, suppose $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$. We have $A \cup B \in \mathcal{P}(A \cup B)$, so $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$. If $A \cup B \in \mathcal{P}A$, then we have $B \subseteq A \cup B \subseteq A$. And if $A \cup B \in \mathcal{P}B$, then we have $A \subseteq A \cup B \subseteq B$.

Exercise 8 If A is a set such that every singleton belongs to A, then every set belongs to $\bigcup A$, contradicting Theorem 2A.

Exercise 9 Let $a = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$. Then $a \in B$ but $\mathcal{P}a$ is not a subset of B because $\emptyset \in \mathcal{P}a$ and $\emptyset \notin B$.

Exercise 10 We must show that $\mathcal{P}a \subseteq \mathcal{P}\bigcup B$. So let $X \in \mathcal{P}a$. Then $X \subseteq a$; we must show that $X \subseteq \bigcup B$.

Let $x \in X$; we must show that $x \in \bigcup B$. We have $x \in a$ (because $x \in X$ and $X \subseteq a$) and $a \in B$, hence $x \in \bigcup B$ as required.

2.2   Algebra of Sets

Exercise 11 For any x we have

$$x \in (A \cap B) \cup (A - B) \Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B)$$
$$\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B)$$
$$\Leftrightarrow x \in A$$

Hence $A = (A \cap B) \cup (A - B)$.

For any x we have

$$x \in A \cup (B - A) \Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A)$$
$$\Leftrightarrow x \in A \text{ or } x \in B$$
$$\Leftrightarrow x \in A \cup B$$

Hence $A \cup (B - A) = A \cup B$.

Exercise 12 For any x,

$$x \in C - (A \cap B) \Leftrightarrow x \in C \& \neg(x \in A \& x \in B)$$
$$\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B)$$
$$\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B)$$
$$\Leftrightarrow x \in (C - A) \cup (C - B)$$

Exercise 13 Suppose $A \subseteq B$. Let $x \in C - B$; we must show $x \in C - A$. We have $x \in C$ and $x \notin B$. Therefore $x \notin A$, since every member of A is a member of B. And so we have $x \in C - A$ as required.

Exercise 14 Let $A = \{\emptyset\}$, $B = \emptyset$ and $C = \{\emptyset\}$. Then $A - (B - C) = A - \emptyset = \{\emptyset\}$ while $(A - B) - C = \{\emptyset\} - C = \emptyset$.

Exercise 15

(a)   For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B+C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B+C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have $x \in A \cap (B+C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$.

(b) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$

In every case, we have $x \in A + (B+C) \Leftrightarrow x \in (A+B) + C$.

Exercise 16

$$[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B-C)) \cap A] = (A \cup B) - A$$
$$= B - A$$

Exercise 17

(a) $\Leftrightarrow$ **(b)**

$A \subseteq B \Leftrightarrow$ Every element of A is an element of B

$\quad\quad \Leftrightarrow$ There is no element of A that is not an element of B

$\quad\quad \Leftrightarrow A - B = \emptyset$

(a) $\Rightarrow$ **(c)** Suppose $A \subseteq B$. We have $B \subseteq A \cup B$ from the definition of $A \cup B$; we must prove that $A \cup B \subseteq B$. So let $x \in A \cup B$. Then $x \in A$ or $x \in B$. But in either case $x \in B$, since $x \in A \Rightarrow x \in B$. Thus we have $x \in B$ as required.

(c) $\Rightarrow$ **(a)** We always have $A \subseteq A \cup B$. So if $A \cup B = B$ then we have $A \subseteq B$.

(a) $\Rightarrow$ **(d)** Suppose $A \subseteq B$. We have $A \cap B \subseteq A$ from the definition of $A \cap B$; we must prove that $A \subseteq A \cap B$. So let $x \in A$. Then $x \in B$ since $A \subseteq B$, hence $x \in A \cap B$ as required.

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(d) ⇒ (a)   We always have $A \cap B \subseteq B$. So if $A \cap B = A$ then $A \subseteq B$.

Exercise 18   We can make the following 16 sets:

- $\emptyset \ (= A - A)$

- $A - B$

- $A \cap B$

- $B - A$

- $S - (A \cup B)$

- A

- $A + B$

- $S - B$

- B

- $S - (A + B)$

- $S - A$

- $A \cup B$

- $S - (B - A)$

- $S - (A \cap B)$

- $S - (A - B)$

Exercise 19   They are never equal, because for all A, B, we have $\emptyset \in \mathcal{P}(A - B)$ but $\emptyset \notin \mathcal{P}A - \mathcal{P}B$ since $\emptyset \in \mathcal{P}B$.

Exercise 20   Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$.
We first show $B \subseteq C$. Let $x \in B$; we show $x \in C$. We have $x \in A \cup B = A \cup C$, so either $x \in A$ or $x \in C$. If $x \in C$, we are done. If $x \in A$, then we have $x \in A \cap B = A \cap C$, and so $x \in C$ in this case too.
We can show $C \subseteq B$ similarly. Hence $B = C$.

Exercise 21   For any x, we have

$$x \in \bigcup(A \cup B) \Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C$$

$$\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C, \text{ or there exists } C \in B \text{ such that } x \in C$$

$$\Leftrightarrow x \in \bigcup A \cup \bigcup B$$

Exercise 22   For any x, we have

$$x \in \bigcap(A \cup B) \Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C$$

$$\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C$$

$$\Leftrightarrow x \in \bigcap A \cap \bigcap B$$

Exercise 23   PROOF:
$\langle 1\rangle 1.$ $A \subseteq \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2\rangle 1.$ LET: $x \in A$
  $\langle 2\rangle 2.$ LET: $X \in \mathcal{B}$
  $\langle 2\rangle 3.$ $x \in A \cup X$
$\langle 1\rangle 2.$ $\bigcap\mathcal{B} \subseteq \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2\rangle 1.$ LET: $x \in \bigcap\mathcal{B}$
  $\langle 2\rangle 2.$ LET: $X \in \mathcal{B}$
  $\langle 2\rangle 3.$ $x \in X$
  $\langle 2\rangle 4.$ $x \in A \cup X$
$\langle 1\rangle 3.$ $\bigcap\{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap\mathcal{B}$
  $\langle 2\rangle 1.$ LET: $x \in \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2\rangle 2.$ ASSUME: $x \notin A$
       PROVE: $x \in \bigcap\mathcal{B}$
  $\langle 2\rangle 3.$ LET: $X \in \mathcal{B}$
  $\langle 2\rangle 4.$ $x \in A \cup X$
  $\langle 2\rangle 5.$ $x \in X$
$\square$

Exercise 24

   (a)

$$Y \in \mathcal{P}\bigcap\mathcal{A} \Leftrightarrow Y \subseteq \bigcap\mathcal{A}$$

$$\Leftrightarrow \forall y \in Y. \forall X \in \mathcal{A}. y \in X$$

$$\Leftrightarrow \forall X \in \mathcal{A}. \forall y \in Y. y \in X$$

$$\Leftrightarrow \forall X \in \mathcal{A}. Y \in \mathcal{P}X$$

$$\Leftrightarrow Y \in \bigcap\{\mathcal{P}X \mid X \in \mathcal{A}\}$$

   (b) $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} \subseteq \mathcal{P}\bigcup\mathcal{A}$

PROOF:
$\langle 1\rangle 1.$ LET: $Y \in \bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\}$
$\langle 1\rangle 2.$ PICK $X \in \mathcal{A}$ such that $Y \in \mathcal{P}X$
$\langle 1\rangle 3.$ $Y \subseteq X$
$\langle 1\rangle 4.$ $Y \subseteq \bigcup\mathcal{A}$
$\langle 1\rangle 5.$ $Y \in \mathcal{P}\bigcup\mathcal{A}$

Equality holds if and only if $\bigcup \mathcal{A} \in \mathcal{A}$.

$\langle 1 \rangle 1$. If $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$ then $\bigcup \mathcal{A} \in \mathcal{A}$
  $\langle 2 \rangle 1$. ASSUME: $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
  $\langle 2 \rangle 2$. $\bigcup \mathcal{A} \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
  $\langle 2 \rangle 3$. PICK $X \in \mathcal{A}$ such that $\bigcup \mathcal{A} \in \mathcal{P}X$
  $\langle 2 \rangle 4$. $X = \bigcup \mathcal{A}$
$\langle 1 \rangle 2$. If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
  PROOF: If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\mathcal{P} \bigcup \mathcal{A} \in \{\mathcal{P}X \mid X \in \mathcal{A}\}$.
⬚

Exercise 25   We have $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ if and only if $A = \emptyset$ or $\mathcal{B} \neq \emptyset$

$\langle 1 \rangle 1$. If $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ then $A = \emptyset$ or $\mathcal{B} \neq \emptyset$
  PROOF: If $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ and $\mathcal{B} = \emptyset$ then
$$A \cup \bigcup \emptyset = \bigcup \emptyset$$
$$\therefore A = \emptyset$$
$\langle 1 \rangle 2$. If $A = \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
  PROOF: Both sides are equal to $\bigcup \mathcal{B}$
$\langle 1 \rangle 3$. If $\mathcal{B} \neq \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 1$. ASSUME: $\mathcal{B} \neq \emptyset$
  $\langle 2 \rangle 2$. $A \cup \bigcup \mathcal{B} \subseteq \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 1$. LET: $x \in A \cup \bigcup \mathcal{B}$
      PROVE: $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 2$. CASE: $x \in A$
      $\langle 4 \rangle 1$. PICK $X \in \mathcal{B}$
      PROOF: By $\langle 2 \rangle 1$
      $\langle 4 \rangle 2$. $x \in A \cup X$
    $\langle 3 \rangle 3$. CASE: $x \in \bigcup \mathcal{B}$
      $\langle 4 \rangle 1$. PICK $X \in \mathcal{B}$ such that $x \in X$
      $\langle 4 \rangle 2$. $x \in A \cup X$
  $\langle 2 \rangle 3$. $\bigcup \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcup \mathcal{B}$
    $\langle 3 \rangle 1$. LET: $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 2$. PICK $X \in \mathcal{B}$ such that $x \in A \cup X$
    $\langle 3 \rangle 3$. $X \subseteq \bigcup \mathcal{B}$
    $\langle 3 \rangle 4$. $A \cup X \subseteq A \cup \bigcup \mathcal{B}$
    $\langle 3 \rangle 5$. $x \in A \cup \bigcup \mathcal{B}$

2.3   Review Exercises

Exercise 26   Sets A, B, D and F are all equal to each other. Sets C, E and G are equal to each other. None of the first list is equal to any of the second list.

Exercise 27   Take $A = \{\{0\}, \{1\}\}$ and $B = \{\{1\}\}$. Then $A \cap B = \{\{1\}\}$ and

$$\bigcap A \cap \bigcap B = \emptyset \cap \{1\}$$
$$= \emptyset$$
$$\bigcap(A \cap B) = \bigcap\{\{1\}\}$$
$$= \{1\}$$

Exercise 28

$$\bigcup\{\{3, 4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\} = \{3, 4, \{3\}, \{4\}\}$$

Exercise 29

(a) $\emptyset$

(b)   We have

$$\{\emptyset\} \subseteq \mathcal{P}\{\emptyset\}$$
$$\therefore \mathcal{P}\{\emptyset\} \subseteq \mathcal{PP}\{\emptyset\}$$
$$\{\emptyset\} \subseteq \mathcal{PP}\{\emptyset\}$$
$$\therefore \mathcal{P}\{\emptyset\} \subseteq \mathcal{PPP}\{\emptyset\}$$
$$\therefore \bigcap\{\mathcal{PPP}\{\emptyset\}, \mathcal{PP}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} = \mathcal{PPP}\{\emptyset\} \cap \mathcal{PP}\{\emptyset\} \cap \mathcal{P}\{\emptyset\}$$
$$= \mathcal{P}\{\emptyset\}$$
$$= \{\emptyset, \{\emptyset\}\}$$

Exercise 30

(a) $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$

(b) $\{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d) $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

Exercise 31

(a) $\{1, 2, 3, \emptyset\}$

(b) $\emptyset$

(c) $\emptyset$

(d) $\emptyset$

Exercise 32

(a) $a \cup b$

(b) a

(c)

$$\bigcap\bigcup S \cup \left(\bigcup\bigcup S - \bigcup\bigcap S\right) = (a \cap b) \cup ((a \cup b) - a)$$
$$= (a \cap b) \cup (b - a)$$
$$= b$$

Exercise 33 When $a \neq b$:

$$\bigcup\left(\bigcup S - \bigcap S\right) = \bigcup(\{a, b\} - \{a\})$$
$$= \bigcup\{b\}$$
$$= b$$

When $a = b$:

$$\bigcup\left(\bigcup S - \bigcap S\right) = \bigcup(\{a, b\} - \{a\})$$
$$= \bigcup \emptyset$$
$$= \emptyset$$

Exercise 34 For any set S, we have

$$\emptyset \subseteq \mathcal{P}S$$
$$\therefore \emptyset \in \mathcal{P}\mathcal{P}S$$
$$\emptyset \subseteq S$$
$$\therefore \emptyset \in \mathcal{P}S$$
$$\therefore \{\emptyset\} \subseteq \mathcal{P}S$$
$$\therefore \{\emptyset\} \in \mathcal{P}\mathcal{P}S$$
$$\therefore \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}\mathcal{P}S$$
$$\therefore \{\emptyset, \{\emptyset\}\} \in \mathcal{P}\mathcal{P}\mathcal{P}S$$

Exercise 35   Assume $\mathcal{P}A = \mathcal{P}B$. Then we have

$$A \in \mathcal{P}A$$
$$\therefore A \in \mathcal{P}B$$
$$\therefore A \subseteq B$$
$$B \in \mathcal{P}B$$
$$\therefore B \in \mathcal{P}A$$
$$\therefore B \subseteq A$$
$$\therefore A = B$$

Exercise 36

(a)

$$x \in A - (A \cap B) \Leftrightarrow x \in A \,\&\,\neg(x \in A \,\&\, x \in B)$$
$$\Leftrightarrow x \in A \,\&\, x \notin B$$
$$\Leftrightarrow x \in A - B$$

(b)

$$x \in A - (A - B) \Leftrightarrow x \in A \,\&\,\neg(x \in A \,\&\, x \notin B)$$
$$\Leftrightarrow x \in A \,\&\, x \in B$$
$$\Leftrightarrow x \in A \cap B$$

Exercise 37

(a)

$$x \in (A \cup B) - C \Leftrightarrow (x \in A \text{ or } x \in B) \,\&\, x \notin C$$
$$\Leftrightarrow (x \in A \,\&\, x \notin C) \text{ or } (x \in B \,\&\, x \notin C)$$
$$\Leftrightarrow x \in (A - C) \cup (B - C)$$

(b)

$$x \in A - (B - C) \Leftrightarrow x \in A \,\&\,\neg(x \in B \,\&\, x \notin C)$$
$$\Leftrightarrow x \in A \,\&\,(x \notin B \text{ or } x \in C)$$
$$\Leftrightarrow (x \in A \,\&\, x \notin B) \text{ or } (x \in A \,\&\, x \in C)$$
$$\Leftrightarrow x \in (A - B) \cup (A \cap C)$$

(c)

$$x \in (A - B) - C \Leftrightarrow x \in A \,\&\, x \notin B \,\&\, x \notin C$$
$$\Leftrightarrow x \in A \,\&\,\neg(x \in B \vee x \in C)$$
$$\Leftrightarrow x \in A - (B \cup C)$$

Exercise 38

(a)   If every element of A is an element of C, and every element of B is an element of C, then everything that is an element of either A or B is an element of C.

(b)   If every element of C is an element of A, and every element of C is an element of B, then every element of C is an element of both A and B.

Chapter 3

Chapter 3 — Relations and Functions

3.1 Ordered Pairs

Exercise 1 We have $\langle 0, 1, 0\rangle^* = \langle 0, 1, 1\rangle^* = \{\{0\}, \{0, 1\}\}$.

Exercise 2

   (a)

$$z \in A \times (B \cup C)$$
$$\Leftrightarrow \exists x, y(z = (x, y) \;\&\; x \in A \;\&\; (y \in B \text{ or } y \in C))$$
$$\Leftrightarrow \exists x, y(z = (x, y) \;\&\; x \in A \;\&\; y \in B) \text{ or } (z = (x, y) \;\&\; x \in A \;\&\; y \in C)$$
$$\Leftrightarrow z \in (A \times B) \cup (A \times C)$$

(b)
$\langle 1\rangle 1.$ ASSUME: $A \times B = A \times C$ and $A \neq \emptyset$
$\langle 1\rangle 2.$ PICK $a \in A$
$\langle 1\rangle 3.$ For all x, $x \in B \Leftrightarrow x \in C$
   PROOF: $x \in B$ iff $(a, x) \in A \times B$ iff $(a, x) \in A \times C$ iff $x \in C$.
☐

Exercise 3

$$z \in A \times \bigcup \mathcal{B}$$
$$\Leftrightarrow \exists x, y(z = (x, y) \;\&\; x \in A \;\&\; \exists X \in \mathcal{B}.y \in X)$$
$$\Leftrightarrow \exists X \in \mathcal{B}.\exists x, y(z = (x, y) \;\&\; x \in A \;\&\; y \in X)$$
$$\Leftrightarrow z \in \bigcup \{A \times X : X \in \mathcal{B}\}$$

Exercise 4   If every ordered pair belongs to A then every set belongs to $\bigcup\bigcup A$ contradicting Theorem 2A.

Exercise 5

(a)   Apply a Subset Axiom to $\mathcal{P}(A \times B)$: we have $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A.z = \{x\} \times B\}$.

(b)

$$z \in \bigcup C$$
$$\Leftrightarrow \exists x \in A.z \in \{x\} \times B$$
$$\Leftrightarrow \exists x \in A.\exists y \in B.z = (x, y)$$
$$\Leftrightarrow z \in A \times B$$

3.2   Relations

Exercise 6   If $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$ then A is a set of ordered pairs, i.e. a relation.

Conversely, suppose A is a relation. Let $z \in A$. Then z is an ordered pair; let $z = (x, y)$. We have $x \in \operatorname{dom} A$ and $y \in \operatorname{ran} A$ and so $z \in \operatorname{dom} A \times \operatorname{ran} A$ as required.

Exercise 7   We have $\operatorname{fld} R \subseteq \bigcup\bigcup R$ by Lemma 3D.

Conversely, let $x \in \bigcup\bigcup R$. Pick a and b such that $x \in a$, $a \in b$ and $b \in R$. Then b is an ordered pair; let $b = (y, z)$. We have $a = \{y\}$ or $\{y, z\}$, hence $x = y$ or $x = z$. In either case, $x \in \operatorname{fld} R$.

Exercise 8

(a)

$$x \in \operatorname{dom} \bigcup \mathcal{A}$$
$$\Leftrightarrow \exists y.\exists R \in \mathcal{A}.(x, y) \in R$$
$$\Leftrightarrow \exists R \in \mathcal{A}.\exists y.(x, y) \in R$$
$$\Leftrightarrow x \in \bigcup\{\operatorname{dom} R : R \in \mathcal{A}\}$$

(b)

$$y \in \operatorname{ran}\bigcup\mathcal{A}$$
$$\Leftrightarrow \exists x.\exists R \in \mathcal{A}.(x,y) \in R$$
$$\Leftrightarrow \exists R \in \mathcal{A}.\exists x.(x,y) \in R$$
$$\Leftrightarrow y \in \bigcup\{\operatorname{ran} R : R \in \mathcal{A}\}$$

Exercise 9 Assume $\mathcal{A}$ is nonempty. We have $\operatorname{dom}\bigcap\mathcal{A} \subseteq \bigcap\{\operatorname{dom} R : R \in \mathcal{A}\}$.

PROOF:

$$x \in \operatorname{dom}\bigcap\mathcal{A}$$
$$\Leftrightarrow \exists y.\forall R \in \mathcal{A}.(x,y) \in R$$
$$\Rightarrow \forall R \in \mathcal{A}.\exists y.(x,y) \in R$$
$$\Leftrightarrow x \in \bigcap\{\operatorname{dom} R : R \in \mathcal{A}\}$$

Equality holds iff the middle '$\Rightarrow$' can be reversed, i.e. iff for all x, if $\forall R \in \mathcal{A}.\exists y.(x,y) \in R$ then $\exists y.\forall R \in \mathcal{A}.(x,y) \in R$. I haven't found a simpler condition than this. The condition does not always hold, for example if $\mathcal{A} = \{\{(1,2)\}, \{(1,3)\}\}$ then $\operatorname{dom}\bigcap\mathcal{A} = \emptyset$ while $\bigcap\{\operatorname{dom} R : R \in \mathcal{A}\} = \{1\}$.

Similarly, $\operatorname{ran}\bigcap\mathcal{A} \subseteq \bigcap\{\operatorname{ran} R : R \in \mathcal{A}\}$, and equality holds iff, for any y, if $\forall R \in \mathcal{A}.\exists x.(x,y) \in R$ then $\exists x.\forall R \in \mathcal{A}.(x,y) \in R$.

3.3 n-ary Relations

Exercise 10 This follows from the equations at the top of page 42. An ordered 4-tuple $\langle a, b, c, d\rangle$ is also an ordered 1-tuple (because every set is), and the ordered pair $\langle\langle a, b, c\rangle, d\rangle$, and the ordered triple $\langle\langle a, b\rangle, c, d\rangle$.

3.4 Functions

Exercise 11 We prove $F \subseteq G$. Let $z \in F$. Since F is a relation, then z is an ordered pair; let $z = \langle x, y\rangle$. We have $x \in \operatorname{dom} F$ and $y = F(x)$. Therefore $x \in \operatorname{dom} G$ and $y = G(x)$ (because $\operatorname{dom} F = \operatorname{dom} G$ and $F(x) = G(x)$). Hence $\langle x, y\rangle \in G$, i.e. $z \in G$.

We have proved $F \subseteq G$. We can prove $G \subseteq F$ similarly. Thus $F = G$.

Exercise 12 PROOF:
$\langle 1\rangle 1$. If $f \subseteq g$ then $\operatorname{dom} f \subseteq \operatorname{dom} g$ and $\forall x \in \operatorname{dom} f.f(x) = g(x)$
  $\langle 2\rangle 1$. ASSUME: $f \subseteq g$
  $\langle 2\rangle 2$. LET: $x \in \operatorname{dom} f$
  $\langle 2\rangle 3$. $(x, f(x)) \in f$
  $\langle 2\rangle 4$. $(x, f(x)) \in g$
  $\langle 2\rangle 5$. $x \in \operatorname{dom} g$ and $g(x) = f(x)$

$\langle 1 \rangle 2$. If $\operatorname{dom} f = \operatorname{dom} g$ and $\forall x \in \operatorname{dom} f.f(x) = g(x)$ then $f \subseteq g$
  $\langle 2 \rangle 1$. ASSUME: $\operatorname{dom} f = \operatorname{dom} g$ and $\forall x \in \operatorname{dom} f.f(x) = g(x)$
  $\langle 2 \rangle 2$. LET: $z \in f$
  $\langle 2 \rangle 3$. LET: $z = (x, y)$
  $\langle 2 \rangle 4$. $x \in \operatorname{dom} f$ and $y = f(x)$
  $\langle 2 \rangle 5$. $x \in \operatorname{dom} g$ and $y = g(x)$
  $\langle 2 \rangle 6$. $z = (x, y) \in g$
☐

Exercise 13   PROOF:

$\langle 1 \rangle 1$. ASSUME: f and g are functions
$\langle 1 \rangle 2$. ASSUME: $f \subseteq g$
$\langle 1 \rangle 3$. ASSUME: $\operatorname{dom} g \subseteq \operatorname{dom} f$
$\langle 1 \rangle 4$. $\operatorname{dom} f = \operatorname{dom} g$
  PROOF: We have $\operatorname{dom} f \subseteq \operatorname{dom} g$ from $\langle 1 \rangle 2$ and $\operatorname{dom} g \subseteq \operatorname{dom} f$ from $\langle 1 \rangle 3$
$\langle 1 \rangle 5$. For $x \in \operatorname{dom} f$ we have $f(x) = g(x)$
  PROOF: From $\langle 1 \rangle 2$ and Exercise 12
$\langle 1 \rangle 6$. Q.E.D.
  PROOF: From Exercise 11.
☐

Exercise 14

(a)   If (x, y) and (x, z) are members of $f \cap g$ then they are both members of f, hence $y = z$.

(b)   PROOF:
$\langle 1 \rangle 1$. If $f \cup g$ is a function then, for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, we have $f(x) = g(x)$.
  $\langle 2 \rangle 1$. ASSUME: $f \cup g$ is a function.
  $\langle 2 \rangle 2$. LET: $x \in \operatorname{dom} f \cap \operatorname{dom} g$
  $\langle 2 \rangle 3$. $(x, f(x))$ and $(x, g(x))$ are both elements of $f \cup g$
  $\langle 2 \rangle 4$. $f(x) = g(x)$
$\langle 1 \rangle 2$. If, for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, we have $f(x) = g(x)$, then $f \cup g$ is a function.
  $\langle 2 \rangle 1$. ASSUME: For all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, we have $f(x) = g(x)$
  $\langle 2 \rangle 2$. $f \cup g$ is a relation.
    PROOF: Since every element of either f or g is an ordered pair.
  $\langle 2 \rangle 3$. Whenever (x, y) and (x, z) are elements of $f \cup g$ we have $y = z$
    $\langle 3 \rangle 1$. LET: $(x, y), (x, z) \in f \cup g$
    $\langle 3 \rangle 2$. CASE: $(x, y), (x, z) \in f$
      PROOF: Then $y = z$ since f is a function.
    $\langle 3 \rangle 3$. CASE: $(x, y) \in f, (x, z) \in g$
      PROOF: Then $y = z$ by $\langle 2 \rangle 1$
    $\langle 3 \rangle 4$. CASE: $(x, y) \in g, (x, z) \in f$
      PROOF: Then $y = z$ by $\langle 2 \rangle 1$
    $\langle 3 \rangle 5$. CASE: $(x, y), (x, z) \in g$

PROOF: Then $y = z$ since g is a function.

□

Exercise 15 PROOF:
$\langle 1 \rangle 1$. $\bigcup \mathcal{A}$ is a relation.
  PROOF: Since every member of $\mathcal{A}$ is a relation.
$\langle 1 \rangle 2$. Whenever (x, y) and (x, z) are elements of $\bigcup \mathcal{A}$ then $y = z$
  $\langle 2 \rangle 1$. LET: $(x, y), (x, z) \in \bigcup \mathcal{A}$
  $\langle 2 \rangle 2$. PICK $f, g \in \mathcal{A}$ such that $(x, y) \in f$ and $(x, z) \in g$
  $\langle 2 \rangle 3$. ASSUME: w.l.o.g. $f \subseteq g$
  $\langle 2 \rangle 4$. $(x, y), (x, z) \in g$
  $\langle 2 \rangle 5$. $y = z$
    PROOF: Since g is a function.

□

Exercise 16 If every function belongs to A then every set belongs to $\operatorname{dom} \bigcup A$ contradiction Theorem 2A.

Exercise 17 PROOF:
$\langle 1 \rangle 1$. LET: R and S be single-rooted.
$\langle 1 \rangle 2$. LET: $(x, z), (y, z) \in R \circ S$
$\langle 1 \rangle 3$. PICK t and t' such that $(x, t) \in S$, $(t, z) \in R$, $(y, t') \in S$ and $(t', z) \in R$
$\langle 1 \rangle 4$. $t = t'$
  PROOF: Since R is single-rooted.
$\langle 1 \rangle 5$. $x = y$
  PROOF: Since S is single-rooted.

Thus if F and G are one-to-one functions then $F \circ G$ is single-rooted and a function by Theorem 3H, hence a one-to-one function.

Exercise 18

$$R \circ R = \{\langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle\}$$
$$R \upharpoonright \{1\} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle\}$$
$$R^{-1} \upharpoonright \{1\} = \{\langle 1, 0 \rangle\}$$
$$R[\![\{1\}]\!] = \{2, 3\}$$
$$R^{-1}[\![\{1\}]\!] = \{0\}$$

Exercise 19

$$A(\emptyset) = \{\emptyset, \{\emptyset\}\}$$
$$A[\![\emptyset]\!] = \emptyset$$
$$A[\![\{\emptyset\}]\!] = \{\{\emptyset, \{\emptyset\}\}\}$$
$$A[\![\{\emptyset, \{\emptyset\}\}]\!] = \{\{\emptyset, \{\emptyset\}\}, \emptyset\}$$
$$A^{-1} = \{\langle\{\emptyset, \{\emptyset\}\}, \emptyset\rangle, \langle\emptyset, \{\emptyset\}\rangle\}$$
$$A \circ A = \{\langle\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\rangle\}$$
$$A \upharpoonright \emptyset = \emptyset$$
$$A \upharpoonright \{\emptyset\} = \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle\}$$
$$A \upharpoonright \{\emptyset, \{\emptyset\}\} = \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle, \langle\{\emptyset\}, \emptyset\rangle\}$$
$$= A$$
$$\bigcup\bigcup A = \{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}$$

Exercise 20

$$z \in F \upharpoonright A \Leftrightarrow z \in F \ \& \ \exists x, y.(z = \langle x, y\rangle \ \& \ x \in A)$$
$$\Leftrightarrow z \in F \ \& \ \exists x, y(z = \langle x, y\rangle \ \& \ x \in A \ \& \ y \in \operatorname{ran} F)$$
$$\Leftrightarrow z \in F \cap (A \times \operatorname{ran} F)$$

Exercise 21 Both are equal to $\{\langle x, w\rangle \mid \exists y, z.xTy \ \& \ ySz \ \& \ zRw\}$.

Exercise 22

(a) Proof:
$\langle 1\rangle 1.$ Assume: $A \subseteq B$
$\langle 1\rangle 2.$ Let: $y \in F[\![A]\!]$
$\langle 1\rangle 3.$ Pick $x \in A$ such that xFy
$\langle 1\rangle 4.$ $x \in B$ and xFy
☐

(b) Both are equal to $\{z : \exists x, y.x \in A \ \& \ xGy \ \& \ yFz\}$

(c) Both are equal to $\{\langle x, y\rangle : (x \in A \ \text{or} \ x \in B) \ \& \ xQy\}$

Exercise 23

$$B \circ I_A = \{\langle x, z \rangle : \exists y (x I_A y \ \& \ y B z)\}$$
$$= \{\langle x, z \rangle : \exists y (x \in A \ \& \ x = y \ \& \ y B z)\}$$
$$= \{\langle x, z \rangle : x \in A \ \& \ x B z\}$$
$$= B \restriction A$$
$$I_A[\![C]\!] = \{y : \exists x \in C. x I_A y\}$$
$$= \{y : \exists x \in C (x \in A \ \& \ x = y)\}$$
$$= \{y : y \in C \ \& \ y \in A\}$$
$$= A \cap C$$

Exercise 24

$$F^{-1}[\![A]\!] = \{x : \exists y \in A. y F^{-1} x\}$$
$$= \{x : \exists y \in A. x F y\}$$
$$= \{x \in \operatorname{dom} F : F(x) \in A\}$$

Exercise 25

(a) Proof:
$\langle 1 \rangle 1.$ Let: G be a one-to-one function.
$\langle 1 \rangle 2.$ G^{-1} is a function.
   Proof: Theorem 3F.
$\langle 1 \rangle 3.$ $G \circ G^{-1}$ is a function.
   Proof: Theorem 3H.
$\langle 1 \rangle 4.$ $\operatorname{dom}(G \circ G^{-1}) = \operatorname{ran} G$
   Proof:
   $$\operatorname{dom}(G \circ G^{-1}) = \{x \in \operatorname{dom} G^{-1} : G^{-1}(x) \in \operatorname{dom} G\} \quad \text{(Theorem 3H)}$$
   $$= \{x \in \operatorname{ran} G : G^{-1}(x) \in \operatorname{dom} G\} \quad \text{(Theorem 3E)}$$
   $$= \operatorname{ran} G$$
$\langle 1 \rangle 5.$ $\forall x \in \operatorname{ran} G. (G \circ G^{-1})(x) = x$
   Proof: Theorem 3G.
□

(b) Let G be a function. Then

$$G \circ G^{-1} = \{\langle x, z \rangle : \exists y (x G^{-1} y \ \& \ y G z)\}$$
$$= \{\langle x, z \rangle : \exists y (y G x \ \& \ y G z)\}$$
$$= \{\langle x, x \rangle : \exists y. y G x\} \qquad (G \text{ is a function})$$
$$= I_{\operatorname{ran} G}$$

Exercise 26

(a)

$$F[\![\bigcup \mathcal{A}]\!] = \{y : \exists x. \exists A \in \mathcal{A}(x \in A \ \& \ xFy)\}$$
$$= \{y : \exists A \in \mathcal{A}. \exists x(x \in A \ \& \ xFy)\}$$
$$= \bigcup \{F[\![A]\!] : A \in \mathcal{A}\}$$

(b)

$$F[\![\bigcup \mathcal{A}]\!] = \{y : \exists x. \forall A \in \mathcal{A}(x \in A \ \& \ xFy)\}$$
$$\subseteq \{y : \forall A \in \mathcal{A}. \exists x(x \in A \ \& \ xFy)\}$$
$$= \bigcap \{F[\![A]\!] : A \in \mathcal{A}\}$$

Exercise 27

$$\mathrm{dom}(F \circ G) = \{x : \exists y. x(F \circ G)y\}$$
$$= \{x : \exists y \exists z(xGz \ \& \ zFy)\}$$
$$= \{x : \exists z(zG^{-1}x \ \& \ z \in \mathrm{dom}\, F)\}$$
$$= G^{-1}[\![\mathrm{dom}\, F]\!]$$

Exercise 28   Proof:
⟨1⟩1. $G : \mathcal{P}A \to \mathcal{P}B$
   Proof: Since $f[\![X]\!] \subseteq \mathrm{ran}\, f \subseteq B$
⟨1⟩2. For all $X, Y \in \mathcal{P}A$, if $G(X) = G(Y)$ then $X = Y$
   ⟨2⟩1. Let: $X, Y \in \mathcal{P}A$
   ⟨2⟩2. Assume: $f[\![X]\!] = f[\![Y]\!]$
   ⟨2⟩3. $X \subseteq Y$
      ⟨3⟩1. Let: $x \in X$
      ⟨3⟩2. $f(x) \in f[\![X]\!]$
      ⟨3⟩3. $f(x) \in f[\![Y]\!]$
      ⟨3⟩4. Pick $y \in Y$ such that $f(x) = f(y)$
      ⟨3⟩5. $x = y$
        Proof: Because f is one-to-one.
      ⟨3⟩6. $x \in Y$
        Proof: Similar.
   ⟨2⟩4. $Y \subseteq X$
☐

Example 29   Proof:
⟨1⟩1. Assume: f maps A onto B
⟨1⟩2. Let: $b, b' \in B$
⟨1⟩3. Assume: $G(b) = G(b')$
⟨1⟩4. Pick $x \in A$ such that $f(x) = b$

PROOF: By ⟨1⟩1.
⟨1⟩5. $x \in G(b)$
⟨1⟩6. $x \in G(b')$
⟨1⟩7. $f(x) = b'$
⟨1⟩8. $b = b'$
☐

The converse does not hold. Let $A = \{0\}$ and $B = \{0,1\}$. Let f be the function that maps 0 to 0. Then

$$G(0) = \{0\}$$
$$G(1) = \emptyset$$

Thus G is one-to-one but f does not map A onto B.

Exercise 30

(a) PROOF:
⟨1⟩1. $F(B) = B$
  ⟨2⟩1. $F(B) \subseteq B$
    ⟨3⟩1. LET: $X \in \mathcal{P}A$ be such that $F(X) \subseteq X$
          PROVE: $F(B) \subseteq X$
    ⟨3⟩2. $B \subseteq X$
    ⟨3⟩3. $F(B) \subseteq F(X)$
    ⟨3⟩4. $F(B) \subseteq X$
       PROOF: From ⟨3⟩1 and ⟨3⟩3.
  ⟨2⟩2. $B \subseteq F(B)$
     PROOF: From ⟨2⟩1 and the definition of B, since B is one of the sets X such that $F(X) \subseteq X$
⟨1⟩2. $F(C) = C$
  ⟨2⟩1. $C \subseteq F(C)$
    ⟨3⟩1. LET: $X \in \mathcal{P}A$ with $X \subseteq F(X)$
          PROVE: $X \subseteq F(C)$
    ⟨3⟩2. $X \subseteq C$
    ⟨3⟩3. $F(X) \subseteq F(C)$
    ⟨3⟩4. $X \subseteq F(C)$
       PROOF: From ⟨3⟩1 and ⟨3⟩3
  ⟨2⟩2. $F(C) \subseteq C$
     PROOF: From ⟨2⟩1 and the definition of C.
☐

(b) If $F(X) = X$ then we have $B \subseteq X$ (because $F(X) \subseteq X$) and $X \subseteq C$ (because $X \subseteq F(X)$).

3.5   Infinite Cartesian Products

Exercise 31   PROOF:

⟨1⟩1. If the Axiom of Choice is true then, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$.

  ⟨2⟩1. ASSUME: The Axiom of Choice.

  ⟨2⟩2. LET: I be a set.

  ⟨2⟩3. LET: H be a function with domain I.

  ⟨2⟩4. ASSUME: $H(i) \neq \emptyset$ for all $i \in I$.

  ⟨2⟩5. LET: $R = \{(i, x) : i \in I, x \in H(i)\}$

  ⟨2⟩6. PICK a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R$

      PROVE: $F \in \prod_{i \in I} H(i)$

   PROOF: By the Axiom of Choice.

  ⟨2⟩7. $\operatorname{dom} H = I$

   PROOF: We have $\operatorname{dom} R = I$ since for all $i \in I$ there exists x such that $x \in H(i)$.

  ⟨2⟩8. $\forall i \in I.F(i) \in H(i)$

   PROOF: Since $iRF(i)$.

⟨1⟩2. If, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$, then the Axiom of Choice is true.

  ⟨2⟩1. ASSUME: For any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$

  ⟨2⟩2. LET: R be a relation

  ⟨2⟩3. LET: $I = \operatorname{dom} R$

  ⟨2⟩4. Define the function H with domain I by: for $i \in I$, $H(i) = \{y : iRy\}$

  ⟨2⟩5. $H(i) \neq \emptyset$ for all $i \in I$

  ⟨2⟩6. PICK $F \in \prod_{i \in I} H(i)$

   PROOF: By ⟨2⟩1

  ⟨2⟩7. F is a function

  ⟨2⟩8. $F \subseteq R$

   PROOF: For all $i \in I$ we have $F(i) \in H(i)$ and so $iRF(i)$.

  ⟨2⟩9. $\operatorname{dom} F = \operatorname{dom} R$

□

3.6   Equivalence Relations

Exercise 32

  (a)

$$R \text{ is symmetric}$$
$$\Leftrightarrow \forall x, y(xRy \Rightarrow yRx)$$
$$\Leftrightarrow \forall x, y(\langle y, x \rangle \in R^{-1} \Rightarrow \langle y, x \rangle \in R)$$
$$\Leftrightarrow R^{-1} \subseteq R$$

(b)

$$R \text{ is transitive}$$
$$\Leftrightarrow \forall x, y, z(xRy \ \& \ yRz \Rightarrow xRz)$$
$$\Leftrightarrow \forall x, z(\exists y(xRy \ \& \ yRz) \Rightarrow xRz)$$
$$\Leftrightarrow \forall x, z(\langle x, z \rangle \in R \circ R \Rightarrow \langle x, z \rangle \in R)$$
$$\Leftrightarrow R \circ R \subseteq R$$

Exercise 33   PROOF:
⟨1⟩1. If R is a symmetric and transitive relation then $R = R^{-1} \circ R$.
  ⟨2⟩1. ASSUME: R is a symmetric and transitive relation.
  ⟨2⟩2. $R \subseteq R^{-1} \circ R$
    ⟨3⟩1. LET: xRy
    ⟨3⟩2. yRy
       PROOF: By Theorem 3M.
    ⟨3⟩3. xRy and $yR^{-1}y$
    ⟨3⟩4. $x(R^{-1} \circ R)y$
  ⟨2⟩3. $R^{-1} \circ R \subseteq R$
    PROOF:
$$R^{-1} \circ R \subseteq R \circ R \qquad \text{(Exercise 32(a))}$$
$$\subseteq R \qquad \text{(Exercise 32(b))}$$
⟨1⟩2. If $R = R^{-1} \circ R$ then R is a symmetric and transitive relation.
  ⟨2⟩1. ASSUME: $R = R^{-1} \circ R$
  ⟨2⟩2. R is a relation.
  ⟨2⟩3. R is symmetric.
    ⟨3⟩1. LET: xRy
    ⟨3⟩2. PICK z such that xRz and $zR^{-1}y$
    ⟨3⟩3. yRz and $zR^{-1}x$
    ⟨3⟩4. $y(R^{-1} \circ R)x$
    ⟨3⟩5. yRx
  ⟨2⟩4. R is transitive.
    ⟨3⟩1. LET: xRy and yRz
    ⟨3⟩2. zRy
       PROOF: By ⟨2⟩3
    ⟨3⟩3. xRy and $yR^{-1}z$
    ⟨3⟩4. $x(R^{-1} \circ R)z$
    ⟨3⟩5. xRz
☐

Exercise 34

(a) $\bigcap \mathcal{A}$ is a transitive relation.

PROOF:
⟨1⟩1. $\bigcap \mathcal{A}$ is a relation.

PROOF: Every member of a member of $\mathcal{A}$ is an ordered pair.

$\langle 1 \rangle 2$. $\bigcap \mathcal{A}$ is transitive.

   $\langle 2 \rangle 1$. LET: $\langle x, y \rangle$ and $\langle y, z \rangle$ be in $\bigcap \mathcal{A}$

       PROVE: $\langle x, z \rangle \in \bigcap \mathcal{A}$

   $\langle 2 \rangle 2$. LET: $R \in \mathcal{A}$

   $\langle 2 \rangle 3$. xRy and yRz

   $\langle 2 \rangle 4$. xRz

     PROOF: Since R is transitive.

$\square$

(b) Not necessarily. If $\mathcal{A} = \{\{\langle 0, 1 \rangle\}, \{\langle 1, 2 \rangle\}\}$ then each member of $\mathcal{A}$ is transitive but $\bigcup \mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$ is not.

Example 35

$$\begin{aligned}
R[\![\{x\}]\!] &= \{y : \exists z (z \in \{x\} \ \& \ zRy)\} \\
&= \{y : \exists z (z = x \ \& \ zRy)\} \\
&= \{y : xRy\} \\
&= [x]_R
\end{aligned}$$

Example 36 PROOF:

$\langle 1 \rangle 1$. Q is a relation on A.

  PROOF: By definition.

$\langle 1 \rangle 2$. Q is reflexive on A.

   $\langle 2 \rangle 1$. LET: $x \in A$

   $\langle 2 \rangle 2$. $f(x)Rf(x)$

     PROOF: Since R is reflexive on B.

   $\langle 2 \rangle 3$. xQx

$\langle 1 \rangle 3$. Q is symmetric.

   $\langle 2 \rangle 1$. ASSUME: xQy

   $\langle 2 \rangle 2$. $f(x)Rf(y)$

   $\langle 2 \rangle 3$. $f(y)Rf(x)$

     PROOF: R is symmetric.

   $\langle 2 \rangle 4$. yQx

$\langle 1 \rangle 4$. Q is transitive.

   $\langle 2 \rangle 1$. ASSUME: xQy and yQz

   $\langle 2 \rangle 2$. $f(x)Rf(y)$ and $f(y)Rf(z)$

   $\langle 2 \rangle 3$. $f(x)Rf(z)$

     PROOF: R is transitive.

   $\langle 2 \rangle 4$. xQz

$\square$

Exercise 37 PROOF:

$\langle 1 \rangle 1$. R_Π is a relation on A.

PROOF: If $B \in \Pi$, $x \in B$ and $y \in B$ then $x, y \in A$.

$\langle 1 \rangle 2$. R_Π is reflexive on A.

   $\langle 2 \rangle 1$. LET: $x \in A$

   $\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$

     PROOF: Because Π is exhaustive.

   $\langle 2 \rangle 3$. $x \in B$ and $x \in B$

   $\langle 2 \rangle 4$. $x R_\Pi x$

$\langle 1 \rangle 3$. R_Π is symmetric.

   $\langle 2 \rangle 1$. ASSUME: $x R_\Pi y$

   $\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$

   $\langle 2 \rangle 3$. $y \in B$ and $x \in B$

   $\langle 2 \rangle 4$. $y R_\Pi x$

$\langle 1 \rangle 4$. R_Π is transitive.

   $\langle 2 \rangle 1$. ASSUME: $x R_\Pi y$ and $y R_\Pi z$

   $\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$

   $\langle 2 \rangle 3$. PICK $C \in \Pi$ such that $y \in C$ and $z \in C$

   $\langle 2 \rangle 4$. $B = C$

     PROOF: Since $y \in B$ and $y \in C$

   $\langle 2 \rangle 5$. $x \in B$ and $z \in B$

   $\langle 2 \rangle 6$. $x R_\Pi z$

☐

Exercise 38 PROOF:

$\langle 1 \rangle 1$. If $B \in \Pi$ and $x \in B$ then $B = [x]_{R_\Pi}$

   $\langle 2 \rangle 1$. LET: $B \in \Pi$

   $\langle 2 \rangle 2$. LET: $x \in B$

   $\langle 2 \rangle 3$. $[x]_{R_\Pi} \subseteq B$

     $\langle 3 \rangle 1$. LET: $y \in [x]_{R_\Pi}$

     $\langle 3 \rangle 2$. $x R_\Pi y$

     $\langle 3 \rangle 3$. PICK $C \in \Pi$ such that $x \in C$ and $y \in C$

     $\langle 3 \rangle 4$. $B = C$

       PROOF: Since $x \in B$ and $x \in C$.

     $\langle 3 \rangle 5$. $y \in B$

   $\langle 2 \rangle 4$. $B \subseteq [x]_{R_\Pi}$

     PROOF: For all $y \in B$, we have $x \in B$ and $y \in B$ hence $x R_\Pi y$.

$\langle 1 \rangle 2$. $A / R_\Pi \subseteq \Pi$

   $\langle 2 \rangle 1$. LET: $x \in A$

      PROVE: $[x]_{R_\Pi} \in \Pi$

   $\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$

   $\langle 2 \rangle 3$. $[x]_{R_\Pi} = B$

     PROOF: By $\langle 1 \rangle 1$

   $\langle 2 \rangle 4$. $[x]_{R_\Pi} \in \Pi$

$\langle 1 \rangle 3$. $\Pi \subseteq A / R_\Pi$

   $\langle 2 \rangle 1$. LET: $B \in \Pi$

   $\langle 2 \rangle 2$. PICK $x \in B$

PROOF: Since every member of Π is nonempty.
$\langle 2 \rangle 3.$ $B = [x]_{R_\Pi}$
   PROOF: By $\langle 1 \rangle 1.$
$\langle 2 \rangle 4.$ $B \in A/R_\Pi$
$\square$

Exercise 39   PROOF:

$\langle 1 \rangle 1.$ $R_\Pi \subseteq R$
   $\langle 2 \rangle 1.$ LET: $xR_\Pi y$
   $\langle 2 \rangle 2.$ PICK $B \in \Pi$ such that $x \in B$ and $y \in B$
   $\langle 2 \rangle 3.$ PICK $z \in A$ such that $B = [z]_R$
   $\langle 2 \rangle 4.$ zRx
   $\langle 2 \rangle 5.$ zRy
   $\langle 2 \rangle 6.$ xRy
   PROOF: Since R is symmetric and transitive.
$\langle 1 \rangle 2.$ $R \subseteq R_\Pi$
   $\langle 2 \rangle 1.$ LET: xRy
   $\langle 2 \rangle 2.$ $x \in [x]_R$
   $\langle 2 \rangle 3.$ $y \in [x]_R$
   $\langle 2 \rangle 4.$ $xR_\Pi y$
$\square$

Exercise 40   We have $[2]_R = [3]_R$ but $[6]_R \neq [9]_R$ so there is no such function f.

Exercise 41

(a)   PROOF:
$\langle 1 \rangle 1.$ Q is reflexive on $\mathbb{R} \times \mathbb{R}$.
   PROOF: For any $x, y \in \mathbb{R}$, we have $x + y = x + y$, hence $\langle x, y \rangle Q \langle x, y \rangle$
$\langle 1 \rangle 2.$ Q is symmetric.
   $\langle 2 \rangle 1.$ ASSUME: $\langle u, v \rangle Q \langle x, y \rangle$
   $\langle 2 \rangle 2.$ $u + y = x + v$
   $\langle 2 \rangle 3.$ $x + v = u + y$
   $\langle 2 \rangle 4.$ $\langle x, y \rangle Q \langle u, v \rangle$
$\langle 1 \rangle 3.$ Q is transitive.
   $\langle 2 \rangle 1.$ ASSUME: $\langle a, b \rangle Q \langle u, v \rangle$ and $\langle u, v \rangle Q \langle x, y \rangle$
   $\langle 2 \rangle 2.$ $a + v = u + b$
   $\langle 2 \rangle 3.$ $u + y = x + v$
   $\langle 2 \rangle 4.$ $a + y + x + b$
   PROOF: Adding $\langle 2 \rangle 2$ and $\langle 2 \rangle 3$ gives $a + u + v + y = b + u + v + x$.
   $\langle 2 \rangle 5.$ $\langle a, b \rangle Q \langle x, y \rangle$
$\square$

(b)　We prove that, if $\langle u, v \rangle Q \langle x, y \rangle$ then $\langle u + 2v, v + 2u \rangle Q \langle x + 2y, y + 2x \rangle$. It follows from Theorem 3Q that the function G exists.

If $u + y = v + x$ then $u + 2v + y + 2x = v + 2u + x + 2y$ by adding $u + v + y + x$ to both sides.

Exercise 42　Assume that R is an equivalence relation on A and that $F : A \times A \to A$. Let us say that F is *compatible* with R iff, whenever xRx' and yRy', then $F(\langle x, y \rangle) R F(\langle x', y' \rangle)$. If F is compatible with R then there exists a unique $\hat{F} : (A/R) \times (A/R) \to A/R$ such that

$$\hat{F}(\langle [x]_R, [y]_R \rangle) = [F(\langle x, y \rangle)]_R \text{ for all } x, y \in A .$$

If F is not compatible with R then no such $\hat{F}$ exists.

3.7　Ordering Relations

Exercise 43　PROOF:
$\langle 1 \rangle 1.$ R^{-1} is transitive.
　$\langle 2 \rangle 1.$ ASSUME: $xR^{-1}y$ and $yR^{-1}z$
　$\langle 2 \rangle 2.$ zRy and yRx
　$\langle 2 \rangle 3.$ zRx
　　PROOF: Since R is transitive.
　$\langle 2 \rangle 4.$ $xR^{-1}z$
$\langle 1 \rangle 2.$ R^{-1} satisfies trichotomy on A.
　$\langle 2 \rangle 1.$ LET: $x, y \in A$
　$\langle 2 \rangle 2.$ Exactly one of xRy, $x = y$, yRx holds.
　$\langle 2 \rangle 3.$ Exactly one of $yR^{-1}x$, $x = y$, $xR^{-1}y$ holds.
▯

Exercise 44　PROOF:
$\langle 1 \rangle 1.$ f is one-to-one.
　$\langle 2 \rangle 1.$ LET: $x, y \in A$ with $f(x) = f(y)$
　$\langle 2 \rangle 2.$ $f(x) < f(y)$ and $f(y) < f(x)$ do not hold.
　　PROOF: By trichotomy.
　$\langle 2 \rangle 3.$ $x < y$ and $y < x$ do not hold.
　$\langle 2 \rangle 4.$ $x = y$
　　PROOF: By trichotomy.
$\langle 1 \rangle 2.$ Whenever $f(x) < f(y)$ then $x < y$
　$\langle 2 \rangle 1.$ LET: $x, y \in A$ with $f(x) < f(y)$
　$\langle 2 \rangle 2.$ $f(x) = f(y)$ and $f(y) < f(x)$ do not hold.
　　PROOF: By trichotomy.
　$\langle 2 \rangle 3.$ $x = y$ and $y < x$ do not hold.
　$\langle 2 \rangle 4.$ $x < y$
　　PROOF: By trichotomy.
▯

Exercise 45   PROOF:

$\langle 1\rangle 1$. $<_L$ is transitive.

   $\langle 2\rangle 1$. LET: $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle$ and $\langle a_2, b_2\rangle <_L \langle a_3, b_3\rangle$
           PROVE: $\langle a_1, b_1\rangle < \langle a_3, b_3\rangle$
   $\langle 2\rangle 2$. CASE: $a_1 <_A a_2$ and $a_2 <_A a_3$
      PROOF: Then $a_1 <_A <_3$
   $\langle 2\rangle 3$. CASE: $a_1 <_A a_2$, $a_2 = a_3$, $b_2 <_B b_3$
      PROOF: Then $a_1 <_A <_3$
   $\langle 2\rangle 4$. CASE: $a_1 = a_2$, $b_1 <_B b_2$ and $a_2 <_A a_3$
      PROOF: Then $a_1 <_A <_3$
   $\langle 2\rangle 5$. CASE: $a_1 = a_2$, $b_1 <_B b_2$, $a_2 = a_3$, $b_2 <_B b_3$
      PROOF: Then $a_1 = a_3$ and $b_1 <_B b_3$

$\langle 1\rangle 2$. $<_L$ satisfies trichotomy on $A \times B$.

   $\langle 2\rangle 1$. LET: $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ be elements of $A \times B$
   $\langle 2\rangle 2$. Exactly one of $a_1 <_A a_2$, $a_1 = a_2$, $a_2 <_A a_1$ holds.
   $\langle 2\rangle 3$. Exactly one of $b_1 <_B b_2$, $b_1 = b_2$, $b_2 <_B b_1$ holds.
   $\langle 2\rangle 4$. Exactly one of $a_1 <_A a_2$, $(a_1 = a_2$ and $b_1 <_B b_2)$, $(a_1 = a_2$ and $b_1 = b_2)$,
          $(a_1 = a_2$ and $b_2 <_L b_1)$, $a_2 <_A a_1$ holds.
   $\langle 2\rangle 5$. Exactly one of $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle$, $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$, $\langle a_2, b_2\rangle <_L$
          $\langle a_1, b_1\rangle$ holds.

☐

3.8   Review Exercises

Exercise 46

   (a)

$$\bigcap\bigcap\langle x, y\rangle = \bigcap\{x\}$$
$$= x$$

   (b)

$$\bigcap\bigcap\bigcap\{\langle x, y\rangle\}^{-1} = \bigcap\bigcap\bigcap\{\langle y, x\rangle\}$$
$$= \bigcap\bigcap\langle y, x\rangle$$
$$= y \qquad\qquad\text{(by part (a))}$$

Exercise 47

(a) There are eight:

$$\{\langle 0, 3\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\},$$
$$\{\langle 0, 3\rangle, \langle 1, 3\rangle, \langle 2, 4\rangle\},$$
$$\{\langle 0, 3\rangle, \langle 1, 4\rangle, \langle 2, 3\rangle\},$$
$$\{\langle 0, 3\rangle, \langle 1, 4\rangle, \langle 2, 4\rangle\},$$
$$\{\langle 0, 4\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\},$$
$$\{\langle 0, 4\rangle, \langle 1, 3\rangle, \langle 2, 4\rangle\},$$
$$\{\langle 0, 4\rangle, \langle 1, 4\rangle, \langle 2, 3\rangle\},$$
$$\{\langle 0, 4\rangle, \langle 1, 4\rangle, \langle 2, 4\rangle\}$$

(b) There are six:

$$\{\langle 0, 3\rangle, \langle 1, 4\rangle, \langle 2, 5\rangle\},$$
$$\{\langle 0, 3\rangle, \langle 1, 5\rangle, \langle 2, 4\rangle\},$$
$$\{\langle 0, 4\rangle, \langle 1, 3\rangle, \langle 2, 5\rangle\},$$
$$\{\langle 0, 4\rangle, \langle 1, 5\rangle, \langle 2, 3\rangle\},$$
$$\{\langle 0, 5\rangle, \langle 1, 3\rangle, \langle 2, 4\rangle\},$$
$$\{\langle 0, 5\rangle, \langle 1, 4\rangle, \langle 2, 3\rangle\}$$

Exercise 48

(a) The only ordered pair in $\mathcal{P}T$ is $\langle \emptyset, \emptyset\rangle = \{\{\emptyset\}\}$.

(b)

$$(\mathcal{P}T)^{-1} \circ (\mathcal{P}T \restriction \{\emptyset\}) = \{\langle \emptyset, \emptyset\rangle\} \circ \{\langle \emptyset, \emptyset\rangle\}$$
$$= \{\langle \emptyset, \emptyset\rangle\}$$

Exercise 49   There are six:

$$\{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle\},$$
$$\{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle\},$$
$$\{\langle 0, 0\rangle, \langle 0, 2\rangle, \langle 1, 1\rangle, \langle 2, 0\rangle, \langle 2, 2\rangle\},$$
$$\{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\},$$
$$\{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 0, 2\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$$

Exercise 50

(a) $\{\langle 0, 1\rangle, \langle 0, 2\rangle, \langle 0, 3\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 2, 3\rangle\}$

(b) $\{\langle 0,1 \rangle, \langle 0,2 \rangle, \langle 0,3 \rangle, \langle 2,1 \rangle, \langle 3,1 \rangle, \langle 3,2 \rangle\}$

Exercise 51   There are three:

$$\{\langle 1,0 \rangle, \langle 1,2 \rangle, \langle 2,0 \rangle\},$$
$$\{\langle 1,0 \rangle, \langle 2,0 \rangle, \langle 2,1 \rangle\},$$
$$\{\langle 0,1 \rangle, \langle 2,0 \rangle, \langle 2,1 \rangle\}$$

Exercise 52   We can conclude this if we know that A and B are nonempty, or that C and D are nonempty.

Suppose A and B are nonempty. Then $A \times B = C \times D \neq \emptyset$ so C and D are nonempty. We now prove $A \subseteq C$.

Let $a \in A$. Pick some $b \in B$. Then $\langle a,b \rangle \in A \times B = C \times D$ and so $a \in C$.

We can similarly prove $C \subseteq A$, $B \subseteq D$ and $D \subseteq B$.

Exercise 53

$$
\begin{aligned}
x(R \cup S)^{-1}y &\Leftrightarrow y(R \cup S)x \\
&\Leftrightarrow yRx \text{ or } ySx \\
&\Leftrightarrow xR^{-1}y \text{ or } xS^{-1}y \\
&\Leftrightarrow x(R^{-1} \cup S^{-1})y \\
x(R \cap S)^{-1}y &\Leftrightarrow y(R \cap S)x \\
&\Leftrightarrow yRx \text{ and } ySx \\
&\Leftrightarrow xR^{-1}y \text{ and } xS^{-1}y \\
&\Leftrightarrow x(R^{-1} \cap S^{-1})y \\
x(R - S)^{-1}y &\Leftrightarrow y(R - S)x \\
&\Leftrightarrow yRx \text{ and } \neg ySx \\
&\Leftrightarrow xR^{-1}y \text{ and } \neg xS^{-1}y \\
&\Leftrightarrow x(R^{-1} - S^{-1})y
\end{aligned}
$$

Exercise 54

(a)

$$
\begin{aligned}
\langle x,y \rangle \in A \times (B \cap C) &\Leftrightarrow x \in A \And y \in B \And y \in C \\
&\Leftarrow \langle x,y \rangle \in (A \times B) \cap (A \times C)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\langle x,y \rangle \in A \times (B \cup C) &\Leftrightarrow x \in A \And (y \in B \text{ or } y \in C) \\
&\Leftrightarrow (x \in A \And y \in B) \text{ or } (x \in A \And y \in C) \\
&\Leftrightarrow \langle x,y \rangle \in (A \times B) \cup (A \times C)
\end{aligned}
$$

(c)

$$\langle x, y \rangle \in A \times (B - C) \Leftrightarrow x \in A \ \& \ y \in B \ \& \ y \notin C$$
$$\Leftrightarrow \langle x, y \rangle \in (A \times B) - (A \times C)$$

Exercise 55

(a) No. Take $A = \{0\}$, $B = \{1\}$, $C = \{2\}$. Then $(A \times A) \cup (B \times C) = \{\langle 0, 0 \rangle, \langle 1, 2 \rangle\}$ while $(A \cup B) \times (A \cup C) = \{\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 2 \rangle\}$.

(b) Yes.

$$\langle x, y \rangle \in (A \times A) \cap (B \times C) \Leftrightarrow x \in A \ \& \ y \in A \ \& \ x \in B \ \& \ y \in C$$
$$\Leftrightarrow \langle x, y \rangle \in (A \cap B) \times (A \cap C)$$

Exercise 56

(a) Yes.

$$x \in \mathrm{dom}(R \cup S) \Leftrightarrow \exists y (xRy \text{ or } xSy)$$
$$\Leftrightarrow \exists y.xRy \text{ or } \exists y.xSy$$
$$\Leftrightarrow x \in \mathrm{dom}\, R \cup \mathrm{dom}\, S$$

(b) No. Take $R = \{\langle 0, 0 \rangle\}$ and $S = \{\langle 0, 1 \rangle\}$. Then $\mathrm{dom}(R \cap S) = \mathrm{dom}\, \emptyset = \emptyset$ while $\mathrm{dom}\, R \cap \mathrm{dom}\, S = \{0\} \cap \{0\} = \{0\}$.

Exercise 57

(a) Yes.

$$x(R \circ (S \cup T))y \Leftrightarrow \exists z (x(S \cup T)z \ \& \ zRy)$$
$$\Leftrightarrow \exists z (xSz \ \& \ zRy) \text{ or } \exists z (xTz \ \& \ zRy)$$
$$\Leftrightarrow x((R \circ S) \cup (R \circ T))y$$

(b) No. Take $R = \{\langle 0, 0 \rangle, \langle 1, 0 \rangle\}$, $S = \{\langle 0, 0 \rangle\}$ and $T = \{\langle 0, 1 \rangle\}$. Then

$$R \circ (S \cap T) = R \circ \emptyset$$
$$= \emptyset$$
$$(R \circ S) \cap (R \circ T) = \{\langle 0, 0 \rangle\} \cap \{\langle 0, 0 \rangle\}$$
$$= \{\langle 0, 0 \rangle\}$$

Exercise 58 Take $F = \emptyset$ and $S = \{\emptyset\}$. Then $F[\![F^{-1}[\![S]\!]]\!] = \emptyset \neq S$.

Exercise 59

$$x(Q \restriction (A \cap B))y \Leftrightarrow xQy \ \& \ x \in A \ \& \ x \in B$$
$$\Leftrightarrow x((Q \restriction A) \cap (Q \restriction B))y$$
$$x(Q \restriction (A - B))y \Leftrightarrow xQy \ \& \ x \in A \ \& \ x \notin B$$
$$\Leftrightarrow (xQy \ \& \ x \in A) \ \& \ \neg(xQy \ \& \ x \in B)$$
$$\Leftrightarrow x((Q \restriction A) - (Q \restriction B))y$$

Exercise 60

$$x((R \circ S) \restriction A)y \Leftrightarrow \exists z(xRz \ \& \ zSy \ \& \ x \in A)$$
$$\Leftrightarrow x(R \circ (S \restriction A))y$$

Chapter 4

Chapter 4 — Natural Numbers

4.1 Inductive Sets

Exercise 1 We have
$$3 = 2 \cup \{2\} = 1 \cup \{1, 2\}$$
and so $1 \in 3$. But $1 \notin 1$ (since $1 = \{\emptyset\}$ and we know $\{\emptyset\} \neq \emptyset$ hence $\{\emptyset\} \notin \{\emptyset\}$). Therefore $1 \neq 3$.

4.2 Peano's Postulates

Exercise 2 If a is a transitive set then
$$\bigcup(a^+) = a \qquad\qquad \text{(Theorem 4E)}$$
$$\subseteq a^+$$

Exercise 3

 (a) Suppose a is a transitive set. Then $a \subseteq \mathcal{P}a$. Hence we have $\bigcup \mathcal{P}a = a \subseteq \mathcal{P}a$ and so $\mathcal{P}a$.

 (b) Suppose $\mathcal{P}a$ is a transitive set. Then $a = \bigcup \mathcal{P}a \subseteq \mathcal{P}a$ hence a is transitive.

Exercise 4 If a is a transitive set then $\bigcup a \subseteq a$ so $\bigcup \bigcup a \subseteq \bigcup a$. Hence $\bigcup a$ is transitive.

Exercise 5

(a) PROOF:

$\langle 1 \rangle 1$. LET: $b \in \bigcup \mathcal{A}$

$\langle 1 \rangle 2$. PICK $A \in \mathcal{A}$ such that $b \in A$

$\langle 1 \rangle 3$. $b \subseteq A$

   PROOF: Since A is transitive.

$\langle 1 \rangle 4$. $b \subseteq \bigcup \mathcal{A}$

☐

(b) PROOF:

$\langle 1 \rangle 1$. LET: $b \in \bigcap \mathcal{A}$

$\langle 1 \rangle 2$. For all $A \in \mathcal{A}$ we have $b \subseteq A$

   PROOF: Since $b \in A$ and A is transitive.

$\langle 1 \rangle 3$. $b \subseteq \bigcap \mathcal{A}$

☐

Exercise 6 We have $\bigcup(a^+) = \bigcup a \cup a$ (see the proof of Theorem 4E). So if $\bigcup(a^+) = a$ we have $\bigcup a \cup a = a$ and so $\bigcup a \subseteq a$.

4.3   Recursion on ω

Exercise 7 We have $h_1(0) = h_2(0) = a$ so $0 \in S$.

   Now let $n \in S$; we prove $n^+ \in S$. We have $h_1(n) = h_2(n)$ and therefore

$$\begin{aligned} h_1(n^+) &= F(h_1(n)) \\ &= F(h_2(n)) \\ &= h_2(n^+) \end{aligned}$$

Exercise 8 PROOF:

$\langle 1 \rangle 1$. $\forall m, n \in \omega . h(n) = h(m) \Rightarrow n = m$

  $\langle 2 \rangle 1$. $\forall n \in \omega . h(n) = h(0) \Rightarrow n = 0$

    $\langle 3 \rangle 1$. LET: $n \in \omega$

    $\langle 3 \rangle 2$. ASSUME: $h(n) = h(0)$

    $\langle 3 \rangle 3$. $h(n) = c$

    $\langle 3 \rangle 4$. $\forall p \in \omega . n \neq p^+$

      PROOF: Otherwise $f(h(p)) = c$ contradicting the fact that $c \in A - \operatorname{ran} f$.

    $\langle 3 \rangle 5$. $n = 0$

      PROOF: Theorem 4C.

  $\langle 2 \rangle 2$. For all $m \in \omega$, if $\forall n \in \omega . h(n) = h(m) \Rightarrow n = m$, then $\forall n \in \omega . h(n) = h(m^+) \Rightarrow n = m^+$

    $\langle 3 \rangle 1$. LET: $m \in \omega$

    $\langle 3 \rangle 2$. ASSUME: $\forall n \in \omega . h(n) = h(m) \Rightarrow n = m$

    $\langle 3 \rangle 3$. LET: $n \in \omega$

    $\langle 3 \rangle 4$. ASSUME: $h(n) = h(m^+)$

    $\langle 3 \rangle 5$. $h(n) = f(h(m))$

$\langle 3 \rangle 6.$ $n \neq 0$
  PROOF: Otherwise $c = f(h(m))$ contradicting the fact that $c \in A - \mathrm{ran}\, f$.
$\langle 3 \rangle 7.$ PICK p such that $n = p^+$
$\langle 3 \rangle 8.$ $f(h(p)) = f(h(m))$
$\langle 3 \rangle 9.$ $h(p) = h(m)$
  PROOF: f is one-to-one.
$\langle 3 \rangle 10.$ $p = m$
  PROOF: By $\langle 3 \rangle 2$.
$\langle 3 \rangle 11.$ $n = p^+ = m^+$
□

Exercise 9 PROOF:
$\langle 1 \rangle 1.$ $C^* \subseteq C_*$
  $\langle 2 \rangle 1.$ $f[\![C_*]\!] \subseteq C_*$
    $\langle 3 \rangle 1.$ LET: $x \in C_*$
        PROVE: $f(x) \in C_*$
    $\langle 3 \rangle 2.$ PICK n such that $x \in h(n)$
    $\langle 3 \rangle 3.$ $f(x) \in h(n^+)$
    $\langle 3 \rangle 4.$ $f(x) \in C_*$
$\langle 1 \rangle 2.$ $C_* \subseteq C^*$
  $\langle 2 \rangle 1.$ $\forall n \in \omega . h(n) \subseteq C^*$
    $\langle 3 \rangle 1.$ $h(0) \subseteq C^*$
      PROOF: If $A \subseteq X \subseteq B$ and $f[\![X]\!] \subseteq X$ then $A \subseteq X$.
    $\langle 3 \rangle 2.$ $\forall n \in \omega (h(n) \subseteq C^* \Rightarrow h(n^+) \subseteq C^*)$
      $\langle 4 \rangle 1.$ LET: $n \in \omega$
      $\langle 4 \rangle 2.$ ASSUME: $h(n) \subseteq C^*$
      $\langle 4 \rangle 3.$ $f[\![h(n)]\!] \subseteq C^*$
        $\langle 5 \rangle 1.$ LET: X be such that $A \subseteq X \subseteq B$ and $f[\![X]\!] \subseteq X$
            PROVE: $f[\![h(n)]\!] \subseteq X$
        $\langle 5 \rangle 2.$ $h(n) \subseteq X$
        $\langle 5 \rangle 3.$ $f[\![h(n)]\!] \subseteq f[\![X]\!]$
        $\langle 5 \rangle 4.$ $f[\![h(n)]\!] \subseteq X$
      $\langle 4 \rangle 4.$ $h(n^+) \subseteq C^*$
□

Exercise 10 $C^* = C_* = (0, 1]$

Exercise 11 $\{n \in \mathbb{Z} \mid n \leq 0\}$

Exercise 12   Let $f : B \times B \to B$ and $A \subseteq B$. Let

$$C^* = \bigcap \{X \mid A \subseteq X \subseteq B \ \& \ f[\![X \times X]\!] \subseteq X\} \ .$$

Define the function $h : \omega \to \mathcal{P}B$ by

$$h(0) = A$$
$$h(n^+) = h(n) \cup f[\![h(n) \times h(n)]\!] \qquad (n \in \omega)$$

Define $C_* = \bigcup \operatorname{ran} h$. Then $C^* = C_*$.

4.4   Arithmetic

Exercise 13   We prove the contrapositive. Assume $m \neq 0$ and $n \neq 0$. Then by Theorem 4C there are natural numbers p, q such that $m = p^+$ and $n = q^+$. Hence $mn = p^+q^+ = (p^+q + p)^+ \neq 0$.

Exercise 14   We prove the following facts for any natural number n:

1. n is even if and only if n^+ is odd.

   PROOF: If n is even, say $n = 2p$, then $n^+ = 2p + 1$ is odd.
   If n^+ is odd, say $n^+ = 2p + 1$, then $n = 2p$ is even.

2. n is odd if and only if n^+ is even.

   PROOF: If n is odd, say $n = 2p + 1$, then $n^+ = 2(p + 1)$ is even.
   If n^+ is even, say $n^+ = 2p$, then we cannot have $p = 0$ (since $n^+ \neq 0$). So $p = q + 1$ for some q. But then $n^+ = 2q + 2$ so $n = 2q + 1$ and n is odd.

Now, 0 is even and 0 is not odd. By the two facts above, if n is either even or odd but not both, then n^+ is either odd or even but not both. The result follows by induction.

Exercise 15   We have

$$m + (n + 0) = m + n \qquad\qquad\qquad \text{by (A1)}$$
$$= (m + n) + 0 \qquad\qquad\qquad \text{by (A1)}$$

If $m + (n + p) = (m + n) + p$ then

$$m + (n + p^+) = m + (n + p)^+ \qquad\qquad\qquad \text{by (A2)}$$
$$= (m + (n + p))^+ \qquad\qquad\qquad \text{by (A2)}$$
$$= ((m + n) + p)^+ \qquad\qquad \text{by induction hypothesis}$$
$$= (m + n) + p^+ \qquad\qquad\qquad \text{by (A2)}$$

Exercise 16   We first prove that $0 \cdot n = 0$ for all n. We have $0 \cdot 0 = 0$ by (M1), and if $0 \cdot n = 0$ then

$$0 \cdot n^+ = 0 \cdot n + 0 \qquad\qquad\qquad \text{by (M2)}$$
$$= 0 \cdot n \qquad\qquad\qquad \text{by (A1)}$$
$$= 0 \qquad\qquad \text{by induction hypothesis}$$

Now we prove that $m^+ \cdot n = m \cdot n + n$ for all m, n. We have

$$
\begin{aligned}
m^+ \cdot 0 &= 0 && \text{by (M1)} \\
m \cdot 0 + 0 &= m \cdot 0 && \text{by (A1)} \\
&= 0 && \text{by (M1)}
\end{aligned}
$$

Thus, $m^+ \cdot 0 = m \cdot 0 + 0$.

If $m^+ \cdot n = m \cdot n + n$ then

$$
\begin{aligned}
m^+ \cdot n^+ &= m^+ \cdot n + m^+ && \text{by (M2)} \\
&= (m^+ \cdot n + m)^+ && \text{by (A2)} \\
&= ((m \cdot n + n) + m)^+ && \text{by induction hypothesis} \\
&= ((m \cdot n + m) + n)^+ && \text{by associativity and commutativity of addition} \\
&= (m \cdot n^+ + n)^+ && \text{by (M2)} \\
&= m \cdot n^+ + n^+ && \text{by (A2)}
\end{aligned}
$$

Exercise 17 The proof is by induction on p. We have

$$
\begin{aligned}
m^{n+0} &= m^n && \text{by (A1)} \\
&= 0 + m^n && \text{by Theorem 4K(2)} \\
&= m^n \cdot 0 + m^n && \text{by (M1)} \\
&= m^n \cdot 1 && \text{by (M2)} \\
&= m^n \cdot m^0 && \text{by (E1)}
\end{aligned}
$$

If $m^{n+p} = m^n \cdot m^p$ then

$$
\begin{aligned}
m^{n+p^+} &= m^{(n+p)^+} && \text{by (A2)} \\
&= m^{n+p} m && \text{by (E2)} \\
&= (m^n m^p) m && \text{by induction hypothesis} \\
&= m^n (m^p m) && \text{by Theorem 4K (4)} \\
&= m^n m^{p^+} && \text{by (E2)}
\end{aligned}
$$

4.5   Ordering on ω

Exercise 18

$$
\begin{aligned}
\in_\omega^{-1} [\![\{7, 8\}]\!] &= \{x \in \omega \mid x \in 7 \text{ or } x \in 8\} \\
&= \{0, 1, 2, 3, 4, 5, 6, 7\}
\end{aligned}
$$

Exercise 19 The proof is by induction on m.

For $m = 0$, take $q = r = 0$. Then $m = d \cdot 0 + 0$ and $0 \in d$.

Suppose $m = dq + r$ and $r < d$. Then $r + 1 \le d$. If $r + 1 < d$, then we have $m + 1 = dq + (r + 1)$ as required. If $r + 1 = d$, then we have $m + 1 = dq + d = d(q + 1) + 0$.

Exercise 20 We first prove A is closed downwards; that is, if $n \in A$ and $m \in n$ then $m \in A$. This holds because if $n \in A$ and $m \in n$ then $m \in \bigcup A$ and $\bigcup A = A$.

Now, we prove $\forall n \in \omega . n \in A$ by induction on n.

To prove $0 \in A$: we are given that A is nonempty. Pick some $a \in A$. Then $0 \underline{in} a$ so $0 \in A$ since A is closed downwards.

Now let $n \in A$; we prove $n^+ \in A$. We have $n \in \bigcup A$; pick some $k \in A$ such that $n \in k$. Then $n^+ \underline{\in} k$ so $n^+ \in A$ since A is closed downwards.

This completes the induction. We have $\forall n \in \omega . n \in A$, i.e. $A = \omega$.

Exercise 21 Suppose n is a natural number, $k \in n$ and $n \subseteq k$. Then $k \in k$, contradicting Lemma 4L(b).

Exercise 22 We have $0 \in p^+$ (by trichotomy since $p^+ \notin 0$ because 0 is empty, and $p^+ \neq 0$ by Peano's First Postulate.) Hence $n = n + 0 \in n + p^+$ by Theorem 4N.

Exercise 23 The proof is by induction on n. The statement is vacuously true for $n = 0$.

Suppose the statement is true for n. Let $m \in n^+$. Then $m \underline{\in} n$.

If $m = n$, then we have $m + 0^+ = n^+$.

If $m \in n$, pick p such that $m + p^+ = n$ by the induction hypothesis. Then $m + p^{++} = n^+$.

Exercise 24 Suppose $m \in p$. Then we cannot have $n \in q$ or $n = q$, as either of these would imply $m + n \in p + q$. Hence $q \in n$ by trichotomy.

We prove $q \in n \Rightarrow m \in p$ similarly.

Exercise 25 By Exercise 23, pick natural numbers a and b such that $m = n + a^+$ and $p = q + b^+$. Then

$$
\begin{aligned}
mp + nq &= (n + a^+)(q + b^+) + nq \\
&= nq + nq + a^+ q + nb^+ + a^+ b^+ \\
&= (n + a^+)q + n(q + b^+) + a^+ b^+ \\
&= mq + np + (a^+ + b)^+
\end{aligned}
$$

Hence $mq + np \in mp + nq$ by Exercise 22.

Exercise 26 The proof is by induction on n.

If $n = 0$ then $\operatorname{ran} f$ is a singleton and its sole element is the largest element.

Suppose the result is true for n. Let $f : n^{++} \to A$. Then $f[\![n^+]\!]$ has a largest element $f(k)$, say. If $f(k) \underline{\in} f(n^+)$ then $f(n^+)$ is greatest in $\operatorname{ran} f$; otherwise $f(k)$ is greatest.

Exercise 27   We prove $f_1(n) = f_2(n)$ for all $n \in \omega$ by strong induction on n. Assume that $(\forall m \in n)f_1(m) = f_2(m)$. Then $f_1 \restriction n = f_2 \restriction n$. So

$$\begin{aligned} f_1(n) &= G(f_1 \restriction n) \\ &= G(f_2 \restriction n) \\ &= f_2(n) \end{aligned}$$

Exercise 28   Suppose ω is not transitive. Then there exists a natural number n such that $n \nsubseteq \omega$. Let n be the least such number. There exists $x \in n$ such that $x \notin \omega$. Now, $n \neq 0$ (because it is nonempty) so $n = p^+$ for some natural number p. We have $x \in p^+$ so $x \in p$ or $x = p$. We cannot have $x = p$ (because x is not a natural number) so we have $x \in p$. But this contradicts the minimality of n.

4.6   Review Exercises

Exercise 29 $4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

Exercise 30 $\bigcup 4 = 0 \cup 1 \cup 2 \cup 3 = 3$ since 0, 1 and 2 are all subsets of 3. $\bigcap 4 = 0 \cap 1 \cap 2 \cap 3 = 0(= \emptyset)$.

Exercise 31   Similarly to Exercise 30 we have $\bigcup\bigcup 7 = \bigcup 6 = 5$.

Exercise 32

(a) $A^+ = A \cup \{A\} = \{1, A\} = \{1, \{1\}\}$
So $\bigcup A^+ = 1 \cup \{1\} = \{0, 1\} = 2$

(b) $\bigcup(\{2\}^+) = \bigcup\{2, \{2\}\} = \{0, 1, 2\} = 3$

Exercise 33

(a)   Yes - if $x \in y \in \{0, 1, \{1\}\}$ then x is either 0 or 1, and in either case $x \in \{0, 1, \{1\}\}$

(b)   No - $0 \in 1 \in \{1\}$ but $0 \notin \{1\}$

(c)   No - $0 \in \{0\} \in \langle 0, 1 \rangle$ but $0 \notin \langle 0, 1 \rangle$.

Exercise 34

(a)   Let $a = \{\emptyset\}$ and $b = \emptyset$

(b) Let $c = \{\{\emptyset\}\}$, $d = \{\emptyset\}$ and $e = \emptyset$

Exercise 35

(a) Let $T_1 = \{\{1\}, \{1, 0\}, 0, 1\}$

(b) Let $T_2 = \{\langle 1, 0\rangle, \{1\}, \{1, 0\}, 0, 1\}$.

Exercise 36

$$
\begin{aligned}
h(4) &= 2h(3) \\
&= 4h(2) \\
&= 8h(1) \\
&= 16h(0) \\
&= 48
\end{aligned}
$$

Exercise 37

(a) Let $f : m \to A$ and $g : n \to B$ be bijections. Define $h : m + n \to A \cup B$ by

$$
\begin{aligned}
h(p) &= f(p) && \text{if } p \in m \\
h(m + q) &= g(q) && \text{if } q \in n
\end{aligned}
$$

To show that this is well-defined, we must prove two things:

1. For all $p \in m + n$, then either $p \in m$ or there exists $q \in n$ such that $p = m + n$.

2. We never have $p \in m$ and $p = m + q$ for some $q \in n$.

We prove 1 by induction on n. For all $p \in m + 0$ we have $p \in m$, so the result holds for $n = 0$.

Now, suppose the result holds for n. Let $p \in m + n^+ = (m + n)^+$ so $p \underline{in} m + n$. If $p \in m + n$, we simply apply the induction hypothesis. If $p = m + n$ then $p = m + q$ where $q = n \in n^+$.

To prove 2, if $p = m + q$ then $m = m + 0 \underline{in} m + q = p$ by Theorem 4N, hence $p \notin m$ by trichotomy.

It remains to show that h is a bijection.

To prove h is injective, we consider three cases. If $h(p) = h(p')$ where $p, p' \in m$, then $f(p) = f(p')$ so $p = p'$. If $h(m + q) = h(m + q')$ where $q, q' \in n$, then $g(q) = g(q')$ so $q = q'$. And we cannot have $h(p) = h(m + q)$ for $p \in m$ and $q \in n$ since $h(p) \in A$, $h(m + q) \in B$, and $A \cap B = \emptyset$.

To prove h is surjective, let $x \in A \cup B$. If $x \in A$, there is some $p \in m$ with $f(p) = x$, so $h(p) = x$. If $x \in B$, there is some $q \in n$ with $g(q) = x$, so $h(m + q) = x$.

(b)   Let $f : m \to A$ and $g : n \to B$ be bijections.

We first show that, for any $p \in mn$, there exist unique $i \in m$ and $j \in n$ such that $p = mj + i$.

By Exercise 19, there exist j and $i \in m$ such that $p = mj + i$. We have $j \in n$ since otherwise $p = mj + i \supseteq mj \supseteq mn$.

For uniqueness, suppose $mj + i = mj' + i'$ where $i, i' \in m$ and $j, j' \in n$. Then we have

$$mj \subseteq mj + i = mj' + i' \in mj' + m = m(j')^+$$

so $j \in (j')^+$ and $j \subseteq j'$. Similarly $j' \subseteq j$, and so $j = j'$. Therefore $i = i'$ by the cancellation law for addition.

Now define $h : mn \to A \times B$ by

$$h(mj + i) = \langle f(i), g(j) \rangle$$

where $i \in m$ and $j \in n$. It is easy to check that h is bijective.

Exercise 38 $h(n) = 3n + 1$

Exercise 39 $h(n) = n^2$

Exercise 40 $h(n^+) = h(n) + 5$

Chapter 5

Chapter 5 — Construction of the Real Numbers

5.1 Integers

Exercise 1 No, because $[\langle 0, 0 \rangle] = [\langle 1, 1 \rangle]$ but $[\langle 0, 0 \rangle] \neq [\langle 2, 1 \rangle]$.

Exercise 2 Yes, because if $[\langle m, n \rangle] = [\langle p, q \rangle]$ then $[\langle m, m \rangle] = [\langle p, p \rangle]$ because $m + p = m + p$.

Exercise 3 Yes, because if $[\langle m, n \rangle] = [\langle p, q \rangle]$ then $[\langle n, m \rangle] = [\langle q, p \rangle]$ because $n + p = m + q$.

Exercise 4 Let $a = [\langle m, n \rangle]$, $b = [\langle p, q \rangle]$ and $c = [\langle r, s \rangle]$. Then

$$
\begin{aligned}
a +_Z (b +_Z c) &= [\langle m, n \rangle] +_Z [\langle p + r, q + s \rangle] \\
&= [\langle m + (p + r), n + (q + s) \rangle] \\
&= [\langle (m + p) + r, (n + q) + s \rangle] \\
&= [\langle m + p, n + q \rangle] +_Z [\langle r, s \rangle] \\
&= (a +_Z b) +_Z c
\end{aligned}
$$

Exercise 5

$$
[\langle m, n \rangle] - [\langle p, q \rangle] = [\langle m, n \rangle] + [\langle q, p \rangle] = [\langle m + q, n + p \rangle]
$$

Exercise 6 Let $a = [\langle m, n \rangle]$. Then

$$
\begin{aligned}
a \cdot_Z 0_Z &= [\langle m, n \rangle] \cdot_Z [\langle 0, 0 \rangle] \\
&= [\langle m0 + n0, m0 + n0 \rangle] \\
&= [\langle 0, 0 \rangle] \\
&= 0_Z
\end{aligned}
$$

Exercise 7 We have $a \cdot_Z b +_Z a \cdot_Z (-b) = a \cdot_Z (b +_Z (-b)) = a \cdot_Z 0_Z = 0_Z$, hence $a \cdot_Z (-b) = -(a \cdot_Z b)$ by the uniqueness of inverses.

We prove $(-a) \cdot_Z b = -(a \cdot_Z b)$ similarly.

Exercise 8

(a) This says $[\langle m + n, 0 \rangle] = [\langle m, 0 \rangle] +_Z [\langle n, 0 \rangle]$, which is true from the definition of $+_Z$.

(b) We have

$$
\begin{aligned}
E(m) \cdot_Z E(n) &= [\langle m, 0 \rangle] \cdot_Z [\langle n, 0 \rangle] \\
&= [\langle mn + 0 \cdot 0, m0 + n0 \rangle] \\
&= E(mn)
\end{aligned}
$$

(c)

$$
\begin{aligned}
E(m) <_Z E(n) &\Leftrightarrow [\langle m, 0 \rangle] <_Z [\langle n, 0 \rangle] \\
&\Leftrightarrow m + 0 \in n + 0 \\
&\Leftrightarrow m \in n
\end{aligned}
$$

Exercise 9

$$
\begin{aligned}
E(m) - E(n) &= [\langle m, 0 \rangle] - [\langle n, 0 \rangle] \\
&= [\langle m, n \rangle]
\end{aligned}
$$

by Exercise 5.

5.2 Rational Numbers

Exercise 10 Let $r = [\langle a, b \rangle]$. Then

$$
\begin{aligned}
r \cdot_Q 0_Q &= [\langle a, b \rangle] \cdot_Q [\langle 0, 1 \rangle] \\
&= [\langle a \cdot_Z 0, b \cdot_Z 1 \rangle] \\
&= [\langle 0, b \rangle] \\
&= [\langle 0, 1 \rangle]
\end{aligned}
$$

since $\langle 0, b \rangle \sim \langle 0, 1 \rangle$ because $0 \cdot_Z 1 = 0 \cdot_Z b = 0$.

Exercise 11 Let $r = [\langle a, b\rangle]$ and $s = [\langle c, d\rangle]$. Suppose $r \cdot_Q s = 0_Q$. Then

$$[\langle ac, bd\rangle] = [\langle 0, 1\rangle]$$

that is, $ac = 0$. Hence $a = 0$ or $c = 0$, which means $r = 0_Q$ or $s = 0_Q$.

Exercise 12 This follows from Theorem 5QJ(a) with $s = 0_Q$ and $t = -r$.

Exercise 13 Let $a, b, c \in \mathbb{Z}$. If $a +_Z c = b +_Z c$ then

$$
\begin{aligned}
a +_Z c +_Z (-c) &= b +_Z c +_Z (-c) & \\
\therefore a +_Z 0 &= b +_Z 0 & \text{(Theorem 5ZD(b))} \\
\therefore a &= b & \text{(Theorem 5ZD(a))}
\end{aligned}
$$

Exercise 14 Suppose $p <_Q s$. Let $r = (p +_Q s)/2$. Then

$$
\begin{aligned}
p &<_Q s \\
\therefore 2p &<_Q p +_Q s \\
\therefore p &<_Q (p +_Q s)/2 \\
&= r \\
p &<_Q s \\
\therefore p +_Q s &<_Q 2s \\
\therefore (p +_Q s)/2 &<_Q s \\
\therefore r &<_Q s
\end{aligned}
$$

5.3   Real Numbers

Exercise 15 PROOF:
$\langle 1\rangle 1.$ $\bigcup A$ is closed downwards.
  $\langle 2\rangle 1.$ LET: $q \in \bigcup A$ and $p < q$
  $\langle 2\rangle 2.$ PICK $x \in A$ such that $q \in x$
  $\langle 2\rangle 3.$ $p \in x$
    PROOF: Since x is closed downwards.
  $\langle 2\rangle 4.$ $p \in \bigcup A$
$\langle 1\rangle 2.$ $\bigcup A$ has no largest element.
  $\langle 2\rangle 1.$ LET: $q \in \bigcup A$
  $\langle 2\rangle 2.$ PICK $x \in A$ such that $q \in x$
  $\langle 2\rangle 3.$ PICK $r \in x$ such that $q < r$
    PROOF: Since x has no largest element.
  $\langle 2\rangle 4.$ $r \in \bigcup A$
□

Exercise 16 PROOF:

⟨1⟩1. LET: $q \in x +_R y$

⟨1⟩2. PICK rationals $a \in x$ and $b \in y$ such that $q = a + b$

⟨1⟩3. PICK $a' \in x$ and $b' \in y$ such that $a < a'$ and $b < b'$

  PROOF: Since x and y each have no largest element.

⟨1⟩4. $q < a' + b' \in x +_R y$

☐

Exercise 17 If $b < 0$ we can take $k = 0$. If $b \geq 0$ then there is a natural number n such that $b = E(n)$; take $k = n^+$. Then $b < ak$ since $1 \leq a$ and $b < k$.

Exercise 18 Let $p = [\langle a, b \rangle]$ and $r = [\langle c, d \rangle]$ where a, b and d are positive. By Exercise 17, there exists a natural number k such that $bc < adE(k)$. Therefore $r < p \cdot E(E(k))$.

Exercise 19 Pick a rational $a \in x$ (which we can do since $x \neq \emptyset$). We first prove that there exists a natural number k such that $a + kp \notin x$.

Pick a rational $b \notin x$ (which we can do since $x \neq \mathbb{Q}$). We have $a < b$ (since x is closed downwards). By Exercise 18, there exists a natural number k such that

$$b - a < kp$$
$$\therefore a + kp > b$$
$$\therefore a + kp \notin x$$

Now, let k be the least natural number such that $a + kp \notin x$ (by the Well-Ordering Principle). We have $k \neq 0$ (since $a \in x$); let $k = n^+$. Then we have

$$a + np \in x \qquad a + np + p \notin x$$

Take $q = a + np$.

Exercise 20 We must prove $0 \subseteq x \cup -x$. Let $q \in 0$ and assume $q \notin x$. Then $q < 0$ and $-0 = 0 \notin x$, so $q \in -x$.

Exercise 21 PROOF:

⟨1⟩1. LET: x, y be real numbers with $x < y$

⟨1⟩2. PICK $r \in y$ such that $r \notin x$

⟨1⟩3. PICK $s \in y$ such that $r < s$

    PROVE: $x < E(s) < y$

⟨1⟩4. $x \subseteq E(s)$

  PROOF: If $p \in x$ then $p < r < s$

⟨1⟩5. $x \neq E(s)$

  PROOF: Since $r \in E(s)$ and $r \notin x$

⟨1⟩6. $E(s) \subseteq y$

PROOF: Since y is closed downwards.

$\langle 1 \rangle 7$. $E(s) \neq y$

PROOF: Since $s \in y$ but $s \notin E(s)$.

Exercise 22 $|x|$ is either x or $-x$, and they are both real numbers.

Chapter 6

Chapter 6 — Cardinal Numbers and the Axiom of Choice

6.1 Equinumerosity

Exercise 1 PROOF:

$\langle 1 \rangle 1$. f is injective.

   $\langle 2 \rangle 1$. ASSUME: $f(m, n) = f(m', n')$

   $\langle 2 \rangle 2$. $2^m(2n + 1) = 2^{m'}(2n' + 1)$

   $\langle 2 \rangle 3$. $m = m'$

      $\langle 3 \rangle 1$. ASSUME: w.l.o.g. $m \leq m'$

      $\langle 3 \rangle 2$. $2n + 1 = 2^{m'-m}(2n' + 1)$

        PROOF: From $\langle 2 \rangle 2$ dividing by 2^m.

      $\langle 3 \rangle 3$. $m' - m = 0$

        PROOF: Since $2^{m'-m}(2n' + 1)$ is odd.

   $\langle 2 \rangle 4$. $2n + 1 = 2n' + 1$

   $\langle 2 \rangle 5$. $n = n'$

$\langle 1 \rangle 2$. f is surjective.

   $\langle 2 \rangle 1$. LET: $n \in \omega$

       ASSUME: $\forall m < n.m \in \operatorname{ran} f$

       PROVE: $n \in \operatorname{ran} f$

   $\langle 2 \rangle 2$. CASE: n is even

      $\langle 3 \rangle 1$. LET: k be such that $n = 2k$

      $\langle 3 \rangle 2$. $n = f(0, k)$

   $\langle 2 \rangle 3$. CASE: n is odd

      $\langle 3 \rangle 1$. LET: k be such that $n = 2k + 1$

      $\langle 3 \rangle 2$. LET: $k = f(i, j)$

      $\langle 3 \rangle 3$. $n = f(i + 1, j)$

PROOF:

$$n = 2k + 1$$
$$= 2(2^i(2j + 1) - 1) + 1$$
$$= 2^{i+1}(2j + 1) - 2 + 1$$
$$= 2^{i+1}(2j + 1) - 1$$

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Exercise 2   Let us call (0) the 0th diagonal, $(1, 2)$ the 1st diagonal, $(3, 4, 5)$ the 2nd diagonal, etc. Then the kth is the set of all positions with coordinates (m, n) such that $m + n = k$.

Therefore, the number $J(m, n)$ at position (m, n) is the $m + 1$st number in the $(m + n)$th diagonal. So the number of numbers that come before $J(m, n)$ is

$$(1 + 2 + \cdots + (m + n)) + m$$

Therefore, since the natural numbers start at 0,

$$J(m, n) = (1 + 2 + \cdots + (m + n)) + m$$

We know $1 + 2 + \cdots + k = k(k + 1)/2$. Therefore,

$$J(m, n) = 1/2(m + n)(m + n + 1) + m$$
$$= 1/2(m^2 + 2mn + m + n + n^2) + m$$
$$= 1/2(m^2 + 2mn + 3m + n + n^2)$$
$$= 1/2((m + n)^2 + 3m + n)$$

Exercise 3   Define $f : (0, 1) \to \mathbb{R}$ by: $f(x) = 1/x - 2$ if $0 < x \leq 1/2$; $f(x) = 2 - 1/(1 - x)$ if $1/2 < x < 1$.

Exercise 4   Define $f : [0, 1] \to (0, 1)$ by

$$f(1/2 - 1/2^n) = 1/2 - 1/2^{n-1} \qquad \text{(for } n \text{ a positive integer)}$$
$$f(1/2 + 1/2^n) = 1/2 + 1/2^{n-1} \qquad \text{(for } n \text{ a positive integer)}$$
$$f(x) = x \qquad \text{(for all other } x)$$

Exercise 5

(a)   For any set A, the identity function I_A is a bijection between A and A. It is injective because, if $I_A(x) = I_A(y)$ then $x = y$ immediately. It is surjective because for any $y \in I_A$ we have $y = I_A(y)$.

(b)   We prove that, if f is a bijection between A and B, then f^{-1} is a bijection between B and A. It is an injective function by Theorem 3F, and maps B onto A by Theorem 3E.

(c) Let f be a bijection between A and B, and g a bijection between A and C. We prove $g \circ f$ is a bijection between A and C.

It is a function from A to C by Theorem 3H.

We prove it is injective. Let $x, y \in A$ and assume $(g \circ f)(x) = (g \circ f)(y)$. Then

$$
\begin{aligned}
g(f(x)) &= g(f(y)) \\
\therefore f(x) &= f(y) \qquad && (g \text{ is injective}) \\
\therefore x &= y \qquad && (f \text{ is injective})
\end{aligned}
$$

Now we prove it maps A onto C. Let $c \in C$. Pick $b \in B$ such that $g(b) = c$ (since g is surjective). Pick $a \in A$ such that $f(a) = b$ (since f is injective). Then $(g \circ f)(a) = c$.

6.2   Finite Sets

Exercise 6   Suppose every set of cardinality κ belongs to A. We will prove that every set belongs to $\bigcup A$.

Let x be any set. Pick a set y of cardinality κ. If $x \in y$ then $x \in y \in A$ so $x \in \bigcup A$.

Assume $x \notin y$. Pick an element $z \in y$ (we know y is nonempty because $\kappa \neq 0$). Then $y - \{z\} \cup \{x\}$ has cardinality κ, and so $x \in (y - \{z\} \cup \{x\}) \in A$ hence $x \in \bigcup A$.

Thus, every set is in $\bigcup A$, which we know is impossible by Theorem 2A.

Exercise 7   If f is one-to-one then f is a bijection between A and $\operatorname{ran} f$. So we must have $\operatorname{ran} f = A$, otherwise f would be a bijection between A and a proper subset of A, contradicting the Pigeonhole Principle.

Conversely, suppose $\operatorname{ran} f = A$. Pick a right inverse $h : A \to A$ for f (by Theorem 3J(b). Note: Theorem 3J(b) can in fact be proved for the case B is finite without using the Axiom of Choice.). Now, h is one-to-one by Theorem 3J(a). So $\operatorname{ran} h = A$ by the first paragraph.

We prove f is one-to-one. Let $x, y \in A$ and assume $f(x) = f(y)$. Pick $a, b \in A$ such that $h(a) = x$ and $h(b) = y$. Then

$$
\begin{aligned}
f(h(a)) &= f(h(b)) \\
\therefore a &= b \\
\therefore x &= y
\end{aligned}
$$

Exercise 8   PROOF:

$\langle 1 \rangle 1$. For any sets A and x, if A is finite then $A \cup \{x\}$ is finite.

$\quad \langle 2 \rangle 1$. CASE: $x \in A$

$\qquad$ PROOF: In this case $A \cup \{x\} = A$.

$\quad \langle 2 \rangle 2$. CASE: $x \notin A$

PROOF: Then $|A \cup \{x\}| = |A|^+$.

$\langle 1 \rangle 2.$ LET: A be a finite set.

$\langle 1 \rangle 3.$ For any set B, if $B \approx 0$ then $A \cup B$ is finite.

  PROOF: Because $B = \emptyset$ so $A \cup B = A$.

$\langle 1 \rangle 4.$ Let n be a natural number. Assume that, for any set B, if $B \approx n$ then $A \cup B$ is finite. Then for any set B, if $B \approx n^+$ then $A \cup B$ is finite.

  $\langle 2 \rangle 1.$ LET: $n \in \omega$

  $\langle 2 \rangle 2.$ ASSUME: For any set B, if $B \approx n$ then $A \cup B$ is finite.

  $\langle 2 \rangle 3.$ LET: B be a set.

  $\langle 2 \rangle 4.$ ASSUME: $B \approx n^+$

  $\langle 2 \rangle 5.$ PICK a bijection $f : n^+ \to B$

  $\langle 2 \rangle 6.$ $B - \{f(n)\} \approx n$

  $\langle 2 \rangle 7.$ $A \cup (B - \{f(n)\})$ is finite.

  $\langle 2 \rangle 8.$ $A \cup B$ is finite.

    PROOF: By $\langle 1 \rangle 1$ since $A \cup B = (A \cup (B - \{f(n)\})) \cup \{f(n)\}$.

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Exercise 9 PROOF:

$\langle 1 \rangle 1.$ LET: A be a finite set.

$\langle 1 \rangle 2.$ For any set B, if $B \approx 0$ then $A \times B$ is finite.

  PROOF: In this case $A \times B = \emptyset$.

$\langle 1 \rangle 3.$ Let n be a natural number. Suppose that, for any set B, if $B \approx n$ then $A \times B$ is finite. Then for any set B, if $B \approx n^+$ then $A \times B$ is finite.

  $\langle 2 \rangle 1.$ LET: n be a natural number.

  $\langle 2 \rangle 2.$ ASSUME: For any set B, if $B \approx n$ then $A \times B$ is finite.

  $\langle 2 \rangle 3.$ LET: B be a set.

  $\langle 2 \rangle 4.$ ASSUME: $B \approx n^+$

  $\langle 2 \rangle 5.$ PICK a bijection $f : n^+ \approx B$

  $\langle 2 \rangle 6.$ $A \times (B - \{f(n)\})$ is finite.

    PROOF: By the induction hypothesis $\langle 2 \rangle 2$.

  $\langle 2 \rangle 7.$ $A \times B$ is finite.

    PROOF: By Exercise 8 since $A \times B = (A \times (B - \{f(n)\})) \cup (A \times \{f(n)\})$ and $A \times \{f(n)\}$ is finite because it is equinumerous with A.

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