

Solutions Manual for Enderton *Elements of Set
Theory*

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Chapter 1

Chapter 1 — Introduction

1.1 Baby Set Theory

Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

Exercise 2 We have $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ has an element (namely \emptyset) while \emptyset has no elements.

We have $\emptyset \neq \{\{\emptyset\}\}$ because $\{\{\emptyset\}\}$ has an element (namely $\{\emptyset\}$) while \emptyset has no elements.

We have $\{\emptyset\} \neq \{\{\emptyset\}\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$. This last fact is true because $\emptyset \neq \{\emptyset\}$ as we proved in the first paragraph.

Exercise 3 Assume $B \subseteq C$. Let $A \in \mathcal{P}B$; we must show that $A \in \mathcal{P}C$.

We have $A \subseteq B$ (since $A \in \mathcal{P}B$) and $B \subseteq C$. From this it follows that $A \subseteq C$ (every element of A is an element of B ; every element of B is an element of C ; therefore every element of A is an element of C). Hence $A \in \mathcal{P}C$ as required.

Exercise 4 Since $x \in B$, we have $\{x\} \subseteq B$ and so $\{x\} \in \mathcal{P}B$.

Since $x \in B$ and $y \in B$, we have $\{x, y\} \subseteq B$ and so $\{x, y\} \in \mathcal{P}B$.

From these two facts, it follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$ and so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

1.2 Sets — An Informal View

Exercise 5 We have

$$\begin{aligned} V_0 &= A \\ V_1 &= V_0 \cup \mathcal{P}V_0 \\ &= A \cup \mathcal{P}A \\ V_2 &= V_1 \cup \mathcal{P}V_1 \\ &= \{\emptyset, \{\emptyset\}\} \\ V_3 &= \mathcal{P}V_2 \\ &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \end{aligned}$$

We have $\emptyset \subseteq V_0$ and so $\emptyset \in V_1$. Therefore $\{\emptyset\} \subseteq V_1$ and so $\{\emptyset\} \in V_2$. Hence $\{\{\emptyset\}\} \subseteq V_2$.

We also have $\{\{\emptyset\}\} \not\subseteq V_0$ because $\{\emptyset\}$ is not an atom, and $\{\{\emptyset\}\} \not\subseteq V_1$ since $\{\emptyset\} \notin V_1$ because \emptyset is not an atom.

Thus the rank of $\{\{\emptyset\}\}$ is 2.

Likewise we have \emptyset and $\{\emptyset\}$ are both subsets of V_1 , hence

$$\emptyset \in V_2, \quad \{\emptyset\} \in V_2$$

Thus $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are all subsets of V_2 , hence elements of V_3 . Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is not a subset of V_0 (because \emptyset is not an atom.) It is not a subset of V_1 ($\{\emptyset\} \notin V_1$ because \emptyset is not an atom.) It is not a subset of V_2 (we have $\{\emptyset, \{\emptyset\}\} \notin V_2$ since $\{\emptyset\} \notin V_1$).

Therefore the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

Exercise 6

$$\begin{aligned}
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}V_0 && (\text{since } V_0 = A) \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_1 && (\text{since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3}) \\
V_3 &= V_2 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_2 && (\text{since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3}) \\
V_4 &= V_3 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_3 && (\text{since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3})
\end{aligned}$$

Exercise 7 In Exercise 5 we calculated $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Hence

$$\begin{aligned}
V_4 &= \mathcal{P}V_3 \\
&= \{\emptyset, \\
&\quad \{\emptyset\}, \\
&\quad \{\{\emptyset\}\}, \\
&\quad \{\{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
&\quad \}
\end{aligned}$$

Chapter 2

Chapter 2 — Axioms and Operations

2.1 Arbitrary Unions and Intersections

Exercise 1 $A \cap B \cap C$ is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

Exercise 2 Take $A = \emptyset$ and $B = \{\emptyset\}$. Then $\bigcup A = \bigcup B = \emptyset$ but $A \neq B$. (There are many other possible answers.)

Exercise 3 Let $b \in A$. We must show that $b \subseteq \bigcup A$.

Let x be any element of b . We must show that $x \in \bigcup A$. We know that $x \in b$ and $b \in A$, and so $x \in \bigcup A$ by the definition of $\bigcup A$.

Exercise 4 Suppose $A \subseteq B$. Let $x \in \bigcup A$. We must show that $x \in \bigcup B$.

Pick an element $a \in A$ such that $x \in a$. Then $a \in B$ because $A \subseteq B$. Since we know $x \in a$ and $a \in B$, we know that $x \in \bigcup B$.

Exercise 5 Assume that every member of \mathcal{A} is a subset of B . Let $x \in \bigcup \mathcal{A}$. We must show that $x \in B$.

Pick $A \in \mathcal{A}$ such that $x \in A$. By our assumption, we have $A \subseteq B$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$ as required.

Exercise 6

(a) We will show that $\bigcup \mathcal{P}A \subseteq A$ and $A \subseteq \bigcup \mathcal{P}A$.

To show $\bigcup \mathcal{P}A \subseteq A$: This follows from Exercise 5, since every member of $\mathcal{P}A$ is a subset of A .

To show $A \subseteq \bigcup \mathcal{P}A$: Let $a \in A$. Then we have $a \in \{a\}$ and $\{a\} \in \mathcal{P}A$ so $a \in \bigcup \mathcal{P}A$.

(b) To show $A \subseteq \mathcal{P} \bigcup A$: This holds because every element of A is a subset of $\bigcup A$, as we proved in Exercise 3.

Equality holds if and only if $A = \mathcal{P}X$ for some set X .

Proof: If $A = \mathcal{P} \bigcup A$ then of course $A = \mathcal{P}X$ for some X .

Conversely, if $A = \mathcal{P}X$, then we have

$$\begin{aligned} \mathcal{P} \bigcup A &= \mathcal{P} \bigcup \mathcal{P}X \\ &= \mathcal{P}X && \text{(by part (a))} \\ &= A \end{aligned}$$

Exercise 7

(a) For any set X ,

$$\begin{aligned} X &\in \mathcal{P}A \cap \mathcal{P}B \\ \Leftrightarrow X &\subseteq A \text{ and } X \subseteq B \\ \Leftrightarrow \text{Every member of } X &\text{ is a member of } A \text{ and a member of } B \\ \Leftrightarrow X &\subseteq A \cap B \\ \Leftrightarrow X &\in \mathcal{P}(A \cap B) \end{aligned}$$

(b) Let $X \in \mathcal{P}A \cup \mathcal{P}B$. Then either $X \in \mathcal{P}A$ or $X \in \mathcal{P}B$ (or both). If $X \in \mathcal{P}A$, then we have $X \subseteq A$ and so $X \subseteq A \cup B$ (because $A \subseteq A \cup B$). Similarly if $X \in \mathcal{P}B$ then we have $X \subseteq A \cup B$. So in either case $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

Equality holds if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof: Suppose $A \subseteq B$. Then $\mathcal{P}A \subseteq \mathcal{P}B$ (Chapter 1 Exercise 3) and so $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$. Also $A \cup B = B$ so $\mathcal{P}(A \cup B) = \mathcal{P}B$. Thus $\mathcal{P}A \cup \mathcal{P}B$ and $\mathcal{P}(A \cup B)$ are equal.

Similarly if $B \subseteq A$ then $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$.

Conversely, suppose $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$. We have $A \cup B \in \mathcal{P}(A \cup B)$, so $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$. If $A \cup B \in \mathcal{P}A$, then we have $B \subseteq A \cup B \subseteq A$. And if $A \cup B \in \mathcal{P}B$, then we have $A \subseteq A \cup B \subseteq B$.

Exercise 8 If A is a set such that every singleton belongs to A , then every set belongs to $\bigcup A$, contradicting Theorem 2A.

Exercise 9 Let $a = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$. Then $a \in B$ but $\mathcal{P}a$ is not a subset of B because $\emptyset \in \mathcal{P}a$ and $\emptyset \notin B$.

Exercise 10 We must show that $\mathcal{P}a \subseteq \mathcal{P} \bigcup B$. So let $X \in \mathcal{P}a$. Then $X \subseteq a$; we must show that $X \subseteq \bigcup B$.

Let $x \in X$; we must show that $x \in \bigcup B$. We have $x \in a$ (because $x \in X$ and $X \subseteq a$) and $a \in B$, hence $x \in \bigcup B$ as required.

2.2 Algebra of Sets

Exercise 11 For any x we have

$$\begin{aligned} x \in (A \cap B) \cup (A - B) &\Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B) \\ &\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B) \\ &\Leftrightarrow x \in A \end{aligned}$$

Hence $A = (A \cap B) \cup (A - B)$.

For any x we have

$$\begin{aligned} x \in A \cup (B - A) &\Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A) \\ &\Leftrightarrow x \in A \text{ or } x \in B \\ &\Leftrightarrow x \in A \cup B \end{aligned}$$

Hence $A \cup (B - A) = A \cup B$.

Exercise 12 For any x ,

$$\begin{aligned} x \in C - (A \cap B) &\Leftrightarrow x \in C \& \neg(x \in A \& x \in B) \\ &\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B) \\ &\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B) \\ &\Leftrightarrow x \in (C - A) \cup (C - B) \end{aligned}$$

Exercise 13 Suppose $A \subseteq B$. Let $x \in C - B$; we must show $x \in C - A$. We have $x \in C$ and $x \notin B$. Therefore $x \notin A$, since every member of A is a member of B . And so we have $x \in C - A$ as required.

Exercise 14 Let $A = \{\emptyset\}$, $B = \emptyset$ and $C = \{\emptyset\}$. Then $A - (B - C) = A - \emptyset = \{\emptyset\}$ while $(A - B) - C = \{\emptyset\} - C = \emptyset$.

Exercise 15

(a) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have $x \in A \cap (B + C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$.

(b) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$

In every case, we have $x \in A + (B + C) \Leftrightarrow x \in (A + B) + C$.

Exercise 16

$$\begin{aligned} [(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A] &= (A \cup B) - A \\ &= B - A \end{aligned}$$

Exercise 17

(a) \Leftrightarrow (b)

$A \subseteq B \Leftrightarrow$ Every element of A is an element of B

\Leftrightarrow There is no element of A that is not an element of B

$\Leftrightarrow A - B = \emptyset$

(a) \Rightarrow (c) Suppose $A \subseteq B$. We have $B \subseteq A \cup B$ from the definition of $A \cup B$; we must prove that $A \cup B \subseteq B$. So let $x \in A \cup B$. Then $x \in A$ or $x \in B$. But in either case $x \in B$, since $x \in A \Rightarrow x \in B$. Thus we have $x \in B$ as required.

(c) \Rightarrow (a) We always have $A \subseteq A \cup B$. So if $A \cup B = B$ then we have $A \subseteq B$.

(a) \Rightarrow (d) Suppose $A \subseteq B$. We have $A \cap B \subseteq A$ from the definition of $A \cap B$; we must prove that $A \subseteq A \cap B$. So let $x \in A$. Then $x \in B$ since $A \subseteq B$, hence $x \in A \cap B$ as required.

(d) \Rightarrow (a) We always have $A \cap B \subseteq B$. So if $A \cap B = A$ then $A \subseteq B$.

Exercise 18 We can make the following 16 sets:

- $\emptyset (= A - A)$
- $A - B$
- $A \cap B$
- $B - A$
- $S - (A \cup B)$
- A
- $A + B$
- $S - B$
- B
- $S - (A + B)$
- $S - A$
- $A \cup B$
- $S - (B - A)$
- $S - (A \cap B)$
- $S - (A - B)$

Exercise 19 They are never equal, because for all A, B , we have $\emptyset \in \mathcal{P}(A - B)$ but $\emptyset \notin \mathcal{P}A - \mathcal{P}B$ since $\emptyset \in \mathcal{P}B$.

Exercise 20 Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

We first show $B \subseteq C$. Let $x \in B$; we show $x \in C$. We have $x \in A \cup B = A \cup C$, so either $x \in A$ or $x \in C$. If $x \in C$, we are done. If $x \in A$, then we have $x \in A \cap B = A \cap C$, and so $x \in C$ in this case too.

We can show $C \subseteq B$ similarly. Hence $B = C$.

Exercise 21 For any x , we have

$$\begin{aligned}
 x \in \bigcup (A \cup B) &\Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C \\
 &\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C, \text{ or there exists } C \in B \text{ such that } x \in C \\
 &\Leftrightarrow x \in \bigcup A \cup \bigcup B
 \end{aligned}$$

Exercise 22 For any x , we have

$$\begin{aligned} x \in \bigcap (A \cup B) &\Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C \\ &\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C \\ &\Leftrightarrow x \in \bigcap A \cap \bigcap B \end{aligned}$$

Exercise 23 PROOF:

- $\langle 1 \rangle 1. A \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 1. \text{ LET: } x \in A$
- $\langle 2 \rangle 2. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 3. x \in A \cup X$
- $\langle 1 \rangle 2. \bigcap \mathcal{B} \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 1. \text{ LET: } x \in \bigcap \mathcal{B}$
- $\langle 2 \rangle 2. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 3. x \in X$
- $\langle 2 \rangle 4. x \in A \cup X$
- $\langle 1 \rangle 3. \bigcap \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap \mathcal{B}$
- $\langle 2 \rangle 1. \text{ LET: } x \in \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 2. \text{ ASSUME: } x \notin A$
- PROVE: $x \in \bigcap \mathcal{B}$
- $\langle 2 \rangle 3. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 4. x \in A \cup X$
- $\langle 2 \rangle 5. x \in X$

□

Exercise 24

(a)

$$\begin{aligned} Y \in \mathcal{P} \bigcap \mathcal{A} &\Leftrightarrow Y \subseteq \bigcap \mathcal{A} \\ &\Leftrightarrow \forall y \in Y. \forall X \in \mathcal{A}. y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A}. \forall y \in Y. y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A}. Y \in \mathcal{P}X \\ &\Leftrightarrow Y \in \bigcap \{\mathcal{P}X \mid X \in \mathcal{A}\} \end{aligned}$$

(b) $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} \subseteq \mathcal{P} \bigcup \mathcal{A}$

PROOF:

- $\langle 1 \rangle 1. \text{ LET: } Y \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
- $\langle 1 \rangle 2. \text{ PICK } X \in \mathcal{A} \text{ such that } Y \in \mathcal{P}X$
- $\langle 1 \rangle 3. Y \subseteq X$
- $\langle 1 \rangle 4. Y \subseteq \bigcup \mathcal{A}$
- $\langle 1 \rangle 5. Y \in \mathcal{P} \bigcup \mathcal{A}$

Equality holds if and only if $\bigcup \mathcal{A} \in \mathcal{A}$.

- ⟨1⟩1. If $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$ then $\bigcup \mathcal{A} \in \mathcal{A}$
 - ⟨2⟩1. ASSUME: $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
 - ⟨2⟩2. $\bigcup \mathcal{A} \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
 - ⟨2⟩3. PICK $X \in \mathcal{A}$ such that $\bigcup \mathcal{A} \in \mathcal{P}X$
 - ⟨2⟩4. $X = \bigcup \mathcal{A}$
 - ⟨1⟩2. If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
- PROOF: If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\mathcal{P} \bigcup \mathcal{A} \in \{\mathcal{P}X \mid X \in \mathcal{A}\}$.
 \square

Exercise 25 We have $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ if and only if $A = \emptyset$ or $\mathcal{B} \neq \emptyset$

- ⟨1⟩1. If $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ then $A = \emptyset$ or $\mathcal{B} \neq \emptyset$
- PROOF: If $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ and $\mathcal{B} = \emptyset$ then
- $$A \cup \bigcup \emptyset = \bigcup \emptyset$$
- $$\therefore A = \emptyset$$
- ⟨1⟩2. If $A = \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
- PROOF: Both sides are equal to $\bigcup \mathcal{B}$
- ⟨1⟩3. If $\mathcal{B} \neq \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨2⟩1. ASSUME: $\mathcal{B} \neq \emptyset$
 - ⟨2⟩2. $A \cup \bigcup \mathcal{B} \subseteq \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨3⟩1. LET: $x \in A \cup \bigcup \mathcal{B}$
 - PROVE: $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨3⟩2. CASE: $x \in A$
 - ⟨4⟩1. PICK $X \in \mathcal{B}$
 - PROOF: By ⟨2⟩1
 - ⟨4⟩2. $x \in A \cup X$
 - ⟨3⟩3. CASE: $x \in \bigcup \mathcal{B}$
 - ⟨4⟩1. PICK $X \in \mathcal{B}$ such that $x \in X$
 - ⟨4⟩2. $x \in A \cup X$
 - ⟨2⟩3. $\bigcup \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcup \mathcal{B}$
 - ⟨3⟩1. LET: $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨3⟩2. PICK $X \in \mathcal{B}$ such that $x \in A \cup X$
 - ⟨3⟩3. $X \subseteq \bigcup \mathcal{B}$
 - ⟨3⟩4. $A \cup X \subseteq A \cup \bigcup \mathcal{B}$
 - ⟨3⟩5. $x \in A \cup \bigcup \mathcal{B}$

2.3 Review Exercises

Exercise 26 Sets A, B, D and F are all equal to each other. Sets C, E and G are equal to each other. None of the first list is equal to any of the second list.

Exercise 27 Take $A = \{\{0\}, \{1\}\}$ and $B = \{\{1\}\}$. Then $A \cap B = \{\{1\}\}$ and

$$\begin{aligned}\bigcap A \cap \bigcap B &= \emptyset \cap \{1\} \\ &= \emptyset \\ \bigcap (A \cap B) &= \bigcap \{\{1\}\} \\ &= \{1\}\end{aligned}$$

Exercise 28

$$\bigcup \{\{3, 4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\} = \{3, 4, \{3\}, \{4\}\}$$

Exercise 29

(a) \emptyset

(b) We have

$$\begin{aligned}\{\emptyset\} &\subseteq \mathcal{P}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{PP}\{\emptyset\} \\ \{\emptyset\} &\subseteq \mathcal{PP}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{PPP}\{\emptyset\} \\ \therefore \bigcap \{\mathcal{PPP}\{\emptyset\}, \mathcal{PP}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} &= \mathcal{PPP}\{\emptyset\} \cap \mathcal{PP}\{\emptyset\} \cap \mathcal{P}\{\emptyset\} \\ &= \mathcal{P}\{\emptyset\} \\ &= \{\emptyset, \{\emptyset\}\}\end{aligned}$$

Exercise 30

(a) $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}$

(b) $\{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d) $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

Exercise 31

(a) $\{1, 2, 3, \emptyset\}$

(b) \emptyset

(c) \emptyset

(d) \emptyset

Exercise 32

(a) $a \cup b$

(b) a

(c)

$$\begin{aligned} \bigcap \bigcup S \cup (\bigcup \bigcup S - \bigcup \bigcap S) &= (a \cap b) \cup ((a \cup b) - a) \\ &= (a \cap b) \cup (b - a) \\ &= b \end{aligned}$$

Exercise 33 When $a \neq b$:

$$\begin{aligned} \bigcup (\bigcup S - \bigcap S) &= \bigcup (\{a, b\} - \{a\}) \\ &= \bigcup \{b\} \\ &= b \end{aligned}$$

When $a = b$:

$$\begin{aligned} \bigcup (\bigcup S - \bigcap S) &= \bigcup (\{a, b\} - \{a\}) \\ &= \bigcup \emptyset \\ &= \emptyset \end{aligned}$$

Exercise 34 For any set S , we have

$$\begin{aligned} \emptyset &\subseteq \mathcal{P}S \\ \therefore \emptyset &\in \mathcal{P}\mathcal{P}S \\ \emptyset &\subseteq S \\ \therefore \emptyset &\in \mathcal{P}S \\ \therefore \{\emptyset\} &\subseteq \mathcal{P}S \\ \therefore \{\emptyset\} &\in \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} &\subseteq \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} &\in \mathcal{P}\mathcal{P}\mathcal{P}S \end{aligned}$$

Exercise 35 Assume $\mathcal{P}A = \mathcal{P}B$. Then we have

$$\begin{aligned}
 A &\in \mathcal{P}A \\
 \therefore A &\in \mathcal{P}B \\
 \therefore A &\subseteq B \\
 B &\in \mathcal{P}B \\
 \therefore B &\in \mathcal{P}A \\
 \therefore B &\subseteq A \\
 \therefore A &= B
 \end{aligned}$$

Exercise 36

(a)

$$\begin{aligned}
 x \in A - (A \cap B) &\Leftrightarrow x \in A \ \& \neg(x \in A \ \& \ x \in B) \\
 &\Leftrightarrow x \in A \ \& \ x \notin B \\
 &\Leftrightarrow x \in A - B
 \end{aligned}$$

(b)

$$\begin{aligned}
 x \in A - (A - B) &\Leftrightarrow x \in A \ \& \neg(x \in A \ \& \ x \notin B) \\
 &\Leftrightarrow x \in A \ \& \ x \in B \\
 &\Leftrightarrow x \in A \cap B
 \end{aligned}$$

Exercise 37

(a)

$$\begin{aligned}
 x \in (A \cup B) - C &\Leftrightarrow (x \in A \text{ or } x \in B) \ \& \ x \notin C \\
 &\Leftrightarrow (x \in A \ \& \ x \notin C) \text{ or } (x \in B \ \& \ x \notin C) \\
 &\Leftrightarrow x \in (A - C) \cup (B - C)
 \end{aligned}$$

(b)

$$\begin{aligned}
 x \in A - (B - C) &\Leftrightarrow x \in A \ \& \neg(x \in B \ \& \ x \notin C) \\
 &\Leftrightarrow x \in A \ \& (x \notin B \text{ or } x \in C) \\
 &\Leftrightarrow (x \in A \ \& \ x \notin B) \text{ or } (x \in A \ \& \ x \in C) \\
 &\Leftrightarrow x \in (A - B) \cup (A \cap C)
 \end{aligned}$$

(c)

$$\begin{aligned}
 x \in (A - B) - C &\Leftrightarrow x \in A \ \& \ x \notin B \ \& \ x \notin C \\
 &\Leftrightarrow x \in A \ \& \neg(x \in B \vee x \in C) \\
 &\Leftrightarrow x \in A - (B \cup C)
 \end{aligned}$$

Exercise 38

(a) If every element of A is an element of C , and every element of B is an element of C , then everything that is an element of either A or B is an element of C .

(b) If every element of C is an element of A , and every element of C is an element of B , then every element of C is an element of both A and B .

Chapter 3

Chapter 3 — Relations and Functions

3.1 Ordered Pairs

Exercise 1 We have $\langle 0, 1, 0 \rangle^* = \langle 0, 1, 1 \rangle^* = \{\{0\}, \{0, 1\}\}$.

Exercise 2

(a)

$$\begin{aligned} z &\in A \times (B \cup C) \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ (y \in B \text{ or } y \in C)) \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in B) \text{ or } (z = (x, y) \ \& \ x \in A \ \& \ y \in C) \\ \Leftrightarrow z &\in (A \times B) \cup (A \times C) \end{aligned}$$

(b)

$\langle 1 \rangle 1$. ASSUME: $A \times B = A \times C$ and $A \neq \emptyset$

$\langle 1 \rangle 2$. PICK $a \in A$

$\langle 1 \rangle 3$. For all x , $x \in B \Leftrightarrow x \in C$

PROOF: $x \in B$ iff $(a, x) \in A \times B$ iff $(a, x) \in A \times C$ iff $x \in C$.

□

Exercise 3

$$\begin{aligned} z &\in A \times \bigcup \mathcal{B} \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ \exists X \in \mathcal{B}. y \in X) \\ \Leftrightarrow \exists X \in \mathcal{B}. \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in X) \\ \Leftrightarrow z &\in \bigcup \{A \times X : X \in \mathcal{B}\} \end{aligned}$$

Exercise 4 If every ordered pair belongs to A then every set belongs to $\bigcup\bigcup A$ contradicting Theorem 2A.

Exercise 5

(a) Apply a Subset Axiom to $\mathcal{P}(A \times B)$: we have $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A. z = \{x\} \times B\}$.

(b)

$$\begin{aligned} z &\in \bigcup C \\ \Leftrightarrow \exists x \in A. z &\in \{x\} \times B \\ \Leftrightarrow \exists x \in A. \exists y \in B. z &= (x, y) \\ \Leftrightarrow z &\in A \times B \end{aligned}$$

3.2 Relations

Exercise 6 If $A \subseteq \text{dom } A \times \text{ran } A$ then A is a set of ordered pairs, i.e. a relation.

Conversely, suppose A is a relation. Let $z \in A$. Then z is an ordered pair; let $z = (x, y)$. We have $x \in \text{dom } A$ and $y \in \text{ran } A$ and so $z \in \text{dom } A \times \text{ran } A$ as required.

Exercise 7 We have $\text{fld } R \subseteq \bigcup\bigcup R$ by Lemma 3D.

Conversely, let $x \in \bigcup\bigcup R$. Pick a and b such that $x \in a$, $a \in b$ and $b \in R$. Then b is an ordered pair; let $b = (y, z)$. We have $a = \{y\}$ or $\{y, z\}$, hence $x = y$ or $x = z$. In either case, $x \in \text{fld } R$.

Exercise 8

(a)

$$\begin{aligned} x &\in \text{dom } \bigcup \mathcal{A} \\ \Leftrightarrow \exists y. \exists R \in \mathcal{A}. (x, y) &\in R \\ \Leftrightarrow \exists R \in \mathcal{A}. \exists y. (x, y) &\in R \\ \Leftrightarrow x &\in \bigcup \{\text{dom } R : R \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned}
y &\in \text{ran} \bigcup \mathcal{A} \\
&\Leftrightarrow \exists x. \exists R \in \mathcal{A}. (x, y) \in R \\
&\Leftrightarrow \exists R \in \mathcal{A}. \exists x. (x, y) \in R \\
&\Leftrightarrow y \in \bigcup \{\text{ran } R : R \in \mathcal{A}\}
\end{aligned}$$

Exercise 9 Assume \mathcal{A} is nonempty. We have $\text{dom} \bigcap \mathcal{A} \subseteq \bigcap \{\text{dom } R : R \in \mathcal{A}\}$.

PROOF:

$$\begin{aligned}
x &\in \text{dom} \bigcap \mathcal{A} \\
&\Leftrightarrow \exists y. \forall R \in \mathcal{A}. (x, y) \in R \\
&\Rightarrow \forall R \in \mathcal{A}. \exists y. (x, y) \in R \\
&\Leftrightarrow x \in \bigcap \{\text{dom } R : R \in \mathcal{A}\}
\end{aligned}$$

Equality holds iff the middle ' \Rightarrow ' can be reversed, i.e. iff for all x , if $\forall R \in \mathcal{A}. \exists y. (x, y) \in R$ then $\exists y. \forall R \in \mathcal{A}. (x, y) \in R$. I haven't found a simpler condition than this. The condition does not always hold, for example if $\mathcal{A} = \{\{(1, 2)\}, \{(1, 3)\}\}$ then $\text{dom} \bigcap \mathcal{A} = \emptyset$ while $\bigcap \{\text{dom } R : R \in \mathcal{A}\} = \{1\}$.

Similarly, $\text{ran} \bigcap \mathcal{A} \subseteq \bigcap \{\text{ran } R : R \in \mathcal{A}\}$, and equality holds iff, for any y , if $\forall R \in \mathcal{A}. \exists x. (x, y) \in R$ then $\exists x. \forall R \in \mathcal{A}. (x, y) \in R$.

3.3 n -ary Relations

Exercise 10 This follows from the equations at the top of page 42. An ordered 4-tuple $\langle a, b, c, d \rangle$ is also an ordered 1-tuple (because every set is), and the ordered pair $\langle \langle a, b, c \rangle, d \rangle$, and the ordered triple $\langle \langle a, b \rangle, c, d \rangle$.

3.4 Functions

Exercise 11 We prove $F \subseteq G$. Let $z \in F$. Since F is a relation, then z is an ordered pair; let $z = \langle x, y \rangle$. We have $x \in \text{dom } F$ and $y = F(x)$. Therefore $x \in \text{dom } G$ and $y = G(x)$ (because $\text{dom } F = \text{dom } G$ and $F(x) = G(x)$). Hence $\langle x, y \rangle \in G$, i.e. $z \in G$.

We have proved $F \subseteq G$. We can prove $G \subseteq F$ similarly. Thus $F = G$.

Exercise 12 PROOF:

- $\langle 1 \rangle 1$. If $f \subseteq g$ then $\text{dom } f \subseteq \text{dom } g$ and $\forall x \in \text{dom } f. f(x) = g(x)$
- $\langle 2 \rangle 1$. ASSUME: $f \subseteq g$
- $\langle 2 \rangle 2$. LET: $x \in \text{dom } f$
- $\langle 2 \rangle 3$. $(x, f(x)) \in f$
- $\langle 2 \rangle 4$. $(x, f(x)) \in g$
- $\langle 2 \rangle 5$. $x \in \text{dom } g$ and $g(x) = f(x)$

- ⟨1⟩2. If $\text{dom } f = \text{dom } g$ and $\forall x \in \text{dom } f. f(x) = g(x)$ then $f \subseteq g$
- ⟨2⟩1. ASSUME: $\text{dom } f = \text{dom } g$ and $\forall x \in \text{dom } f. f(x) = g(x)$
- ⟨2⟩2. LET: $z \in f$
- ⟨2⟩3. LET: $z = (x, y)$
- ⟨2⟩4. $x \in \text{dom } f$ and $y = f(x)$
- ⟨2⟩5. $x \in \text{dom } g$ and $y = g(x)$
- ⟨2⟩6. $z = (x, y) \in g$

□

Exercise 13 PROOF:

- ⟨1⟩1. ASSUME: f and g are functions
- ⟨1⟩2. ASSUME: $f \subseteq g$
- ⟨1⟩3. ASSUME: $\text{dom } g \subseteq \text{dom } f$
- ⟨1⟩4. $\text{dom } f = \text{dom } g$
- PROOF: We have $\text{dom } f \subseteq \text{dom } g$ from ⟨1⟩2 and $\text{dom } g \subseteq \text{dom } f$ from ⟨1⟩3
- ⟨1⟩5. For $x \in \text{dom } f$ we have $f(x) = g(x)$
- PROOF: From ⟨1⟩2 and Exercise 12
- ⟨1⟩6. Q.E.D.
- PROOF: From Exercise 11.

□

Exercise 14

(a) If (x, y) and (x, z) are members of $f \cap g$ then they are both members of f , hence $y = z$.

(b) PROOF:

- ⟨1⟩1. If $f \cup g$ is a function then, for all $x \in \text{dom } f \cap \text{dom } g$, we have $f(x) = g(x)$.
- ⟨2⟩1. ASSUME: $f \cup g$ is a function.
- ⟨2⟩2. LET: $x \in \text{dom } f \cap \text{dom } g$
- ⟨2⟩3. $(x, f(x))$ and $(x, g(x))$ are both elements of $f \cup g$
- ⟨2⟩4. $f(x) = g(x)$
- ⟨1⟩2. If, for all $x \in \text{dom } f \cap \text{dom } g$, we have $f(x) = g(x)$, then $f \cup g$ is a function.
- ⟨2⟩1. ASSUME: For all $x \in \text{dom } f \cap \text{dom } g$, we have $f(x) = g(x)$
- ⟨2⟩2. $f \cup g$ is a relation.
- PROOF: Since every element of either f or g is an ordered pair.
- ⟨2⟩3. Whenever (x, y) and (x, z) are elements of $f \cup g$ we have $y = z$
- ⟨3⟩1. LET: $(x, y), (x, z) \in f \cup g$
- ⟨3⟩2. CASE: $(x, y), (x, z) \in f$
- PROOF: Then $y = z$ since f is a function.
- ⟨3⟩3. CASE: $(x, y) \in f, (x, z) \in g$
- PROOF: Then $y = z$ by ⟨2⟩1
- ⟨3⟩4. CASE: $(x, y) \in g, (x, z) \in f$
- PROOF: Then $y = z$ by ⟨2⟩1
- ⟨3⟩5. CASE: $(x, y), (x, z) \in g$

PROOF: Then $y = z$ since g is a function.

□

Exercise 15 PROOF:

⟨1⟩1. $\bigcup \mathcal{A}$ is a relation.

PROOF: Since every member of \mathcal{A} is a relation.

⟨1⟩2. Whenever (x, y) and (x, z) are elements of $\bigcup \mathcal{A}$ then $y = z$

⟨2⟩1. LET: $(x, y), (x, z) \in \bigcup \mathcal{A}$

⟨2⟩2. PICK $f, g \in \mathcal{A}$ such that $(x, y) \in f$ and $(x, z) \in g$

⟨2⟩3. ASSUME: w.l.o.g. $f \subseteq g$

⟨2⟩4. $(x, y), (x, z) \in g$

⟨2⟩5. $y = z$

PROOF: Since g is a function.

□

Exercise 16 If every function belongs to \mathcal{A} then every set belongs to $\text{dom} \bigcup \mathcal{A}$ contradiction Theorem 2A.

Exercise 17 PROOF:

⟨1⟩1. LET: R and S be single-rooted.

⟨1⟩2. LET: $(x, z), (y, z) \in R \circ S$

⟨1⟩3. PICK t and t' such that $(x, t) \in S$, $(t, z) \in R$, $(y, t') \in S$ and $(t', z) \in R$

⟨1⟩4. $t = t'$

PROOF: Since R is single-rooted.

⟨1⟩5. $x = y$

PROOF: Since S is single-rooted.

Thus if F and G are one-to-one functions then $F \circ G$ is single-rooted and a function by Theorem 3H, hence a one-to-one function.

Exercise 18

$$R \circ R = \{\langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle\}$$

$$R \upharpoonright \{1\} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle\}$$

$$R^{-1} \upharpoonright \{1\} = \{\langle 1, 0 \rangle\}$$

$$R[\{1\}] = \{2, 3\}$$

$$R^{-1}[\{1\}] = \{0\}$$

Exercise 19

$$\begin{aligned}
A(\emptyset) &= \{\emptyset, \{\emptyset\}\} \\
A[\emptyset] &= \emptyset \\
A[\{\emptyset\}] &= \{\{\emptyset, \{\emptyset\}\}\} \\
A[\{\emptyset, \{\emptyset\}\}] &= \{\{\emptyset, \{\emptyset\}\}, \emptyset\} \\
A^{-1} &= \{\langle \{\emptyset, \{\emptyset\}\}, \emptyset \rangle, \langle \emptyset, \{\emptyset\} \rangle\} \\
A \circ A &= \{\langle \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \rangle\} \\
A \upharpoonright \emptyset &= \emptyset \\
A \upharpoonright \{\emptyset\} &= \{\langle \emptyset, \{\emptyset, \{\emptyset\}\} \rangle\} \\
A \upharpoonright \{\emptyset, \{\emptyset\}\} &= \{\langle \emptyset, \{\emptyset, \{\emptyset\}\} \rangle, \langle \{\emptyset\}, \emptyset \rangle\} \\
&= A \\
\bigcup A &= \{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}
\end{aligned}$$

Exercise 20

$$\begin{aligned}
z \in F \upharpoonright A &\Leftrightarrow z \in F \ \& \ \exists x, y. (z = \langle x, y \rangle \ \& \ x \in A) \\
&\Leftrightarrow z \in F \ \& \ \exists x, y. (z = \langle x, y \rangle \ \& \ x \in A \ \& \ y \in \text{ran } F) \\
&\Leftrightarrow z \in F \cap (A \times \text{ran } F)
\end{aligned}$$

Exercise 21 Both are equal to $\{\langle x, w \rangle \mid \exists y, z. xTy \ \& \ ySz \ \& \ zRw\}$.

Exercise 22

(a) PROOF:
 $\langle 1 \rangle 1$. ASSUME: $A \subseteq B$
 $\langle 1 \rangle 2$. LET: $y \in F[A]$
 $\langle 1 \rangle 3$. PICK $x \in A$ such that xFy
 $\langle 1 \rangle 4$. $x \in B$ and xFy
 \square

(b) Both are equal to $\{z : \exists x, y. x \in A \ \& \ xGy \ \& \ yFz\}$

(c) Both are equal to $\{\langle x, y \rangle : (x \in A \text{ or } x \in B) \ \& \ xQy\}$

Exercise 23

$$\begin{aligned}
B \circ I_A &= \{\langle x, z \rangle : \exists y(x I_A y \ \& \ y B z)\} \\
&= \{\langle x, z \rangle : \exists y(x \in A \ \& \ x = y \ \& \ y B z)\} \\
&= \{\langle x, z \rangle : x \in A \ \& \ x B z\} \\
&= B \upharpoonright A \\
I_A[C] &= \{y : \exists x \in C. x I_A y\} \\
&= \{y : \exists x \in C(x \in A \ \& \ x = y)\} \\
&= \{y : y \in C \ \& \ y \in A\} \\
&= A \cap C
\end{aligned}$$

Exercise 24

$$\begin{aligned}
F^{-1}[A] &= \{x : \exists y \in A. y F^{-1} x\} \\
&= \{x : \exists y \in A. x F y\} \\
&= \{x \in \text{dom } F : F(x) \in A\}
\end{aligned}$$

Exercise 25

(a) PROOF:

$\langle 1 \rangle 1$. LET: G be a one-to-one function.

$\langle 1 \rangle 2$. G^{-1} is a function.

PROOF: Theorem 3F.

$\langle 1 \rangle 3$. $G \circ G^{-1}$ is a function.

PROOF: Theorem 3H.

$\langle 1 \rangle 4$. $\text{dom}(G \circ G^{-1}) = \text{ran } G$

PROOF:

$$\begin{aligned}
\text{dom}(G \circ G^{-1}) &= \{x \in \text{dom } G^{-1} : G^{-1}(x) \in \text{dom } G\} && \text{(Theorem 3H)} \\
&= \{x \in \text{ran } G : G^{-1}(x) \in \text{dom } G\} && \text{(Theorem 3E)} \\
&= \text{ran } G
\end{aligned}$$

$\langle 1 \rangle 5$. $\forall x \in \text{ran } G. (G \circ G^{-1})(x) = x$

PROOF: Theorem 3G.

□

(b) Let G be a function. Then

$$\begin{aligned}
G \circ G^{-1} &= \{\langle x, z \rangle : \exists y(x G^{-1} y \ \& \ y G z)\} \\
&= \{\langle x, z \rangle : \exists y(y G x \ \& \ y G z)\} \\
&= \{\langle x, x \rangle : \exists y. y G x\} && (G \text{ is a function}) \\
&= I_{\text{ran } G}
\end{aligned}$$

Exercise 26

(a)

$$\begin{aligned} F[\bigcup \mathcal{A}] &= \{y : \exists x. \exists A \in \mathcal{A} (x \in A \ \& \ xFy)\} \\ &= \{y : \exists A \in \mathcal{A}. \exists x (x \in A \ \& \ xFy)\} \\ &= \bigcup \{F[A] : A \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned} F[\bigcup \mathcal{A}] &= \{y : \exists x. \forall A \in \mathcal{A} (x \in A \ \& \ xFy)\} \\ &\subseteq \{y : \forall A \in \mathcal{A}. \exists x (x \in A \ \& \ xFy)\} \\ &= \bigcap \{F[A] : A \in \mathcal{A}\} \end{aligned}$$

Exercise 27

$$\begin{aligned} \text{dom}(F \circ G) &= \{x : \exists y. x(F \circ G)y\} \\ &= \{x : \exists y \exists z (xGz \ \& \ zFy)\} \\ &= \{x : \exists z (zG^{-1}x \ \& \ z \in \text{dom } F)\} \\ &= G^{-1}[\text{dom } F] \end{aligned}$$

Exercise 28 PROOF:

$\langle 1 \rangle 1.$ $G : \mathcal{P}A \rightarrow \mathcal{P}B$

PROOF: Since $f[X] \subseteq \text{ran } f \subseteq B$

$\langle 1 \rangle 2.$ For all $X, Y \in \mathcal{P}A$, if $G(X) = G(Y)$ then $X = Y$

$\langle 2 \rangle 1.$ LET: $X, Y \in \mathcal{P}A$

$\langle 2 \rangle 2.$ ASSUME: $f[X] = f[Y]$

$\langle 2 \rangle 3.$ $X \subseteq Y$

$\langle 3 \rangle 1.$ LET: $x \in X$

$\langle 3 \rangle 2.$ $f(x) \in f[X]$

$\langle 3 \rangle 3.$ $f(x) \in f[Y]$

$\langle 3 \rangle 4.$ PICK $y \in Y$ such that $f(x) = f(y)$

$\langle 3 \rangle 5.$ $x = y$

PROOF: Because f is one-to-one.

$\langle 3 \rangle 6.$ $x \in Y$

PROOF: Similar.

$\langle 2 \rangle 4.$ $Y \subseteq X$

□

Example 29 PROOF:

$\langle 1 \rangle 1.$ ASSUME: f maps A onto B

$\langle 1 \rangle 2.$ LET: $b, b' \in B$

$\langle 1 \rangle 3.$ ASSUME: $G(b) = G(b')$

$\langle 1 \rangle 4.$ PICK $x \in A$ such that $f(x) = b$

PROOF: By $\langle 1 \rangle 1$.

$\langle 1 \rangle 5$. $x \in G(b)$

$\langle 1 \rangle 6$. $x \in G(b')$

$\langle 1 \rangle 7$. $f(x) = b'$

$\langle 1 \rangle 8$. $b = b'$

□

The converse does not hold. Let $A = \{0\}$ and $B = \{0, 1\}$. Let f be the function that maps 0 to 0. Then

$$G(0) = \{0\}$$

$$G(1) = \emptyset$$

Thus G is one-to-one but f does not map A onto B .

Exercise 30

(a) PROOF:

$\langle 1 \rangle 1$. $F(B) = B$

$\langle 2 \rangle 1$. $F(B) \subseteq B$

$\langle 3 \rangle 1$. LET: $X \in \mathcal{P}A$ be such that $F(X) \subseteq X$

PROVE: $F(B) \subseteq X$

$\langle 3 \rangle 2$. $B \subseteq X$

$\langle 3 \rangle 3$. $F(B) \subseteq F(X)$

$\langle 3 \rangle 4$. $F(B) \subseteq X$

PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$.

$\langle 2 \rangle 2$. $B \subseteq F(B)$

PROOF: From $\langle 2 \rangle 1$ and the definition of B , since B is one of the sets X such that $F(X) \subseteq X$

$\langle 1 \rangle 2$. $F(C) = C$

$\langle 2 \rangle 1$. $C \subseteq F(C)$

$\langle 3 \rangle 1$. LET: $X \in \mathcal{P}A$ with $X \subseteq F(X)$

PROVE: $X \subseteq F(C)$

$\langle 3 \rangle 2$. $X \subseteq C$

$\langle 3 \rangle 3$. $F(X) \subseteq F(C)$

$\langle 3 \rangle 4$. $X \subseteq F(C)$

PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$

$\langle 2 \rangle 2$. $F(C) \subseteq C$

PROOF: From $\langle 2 \rangle 1$ and the definition of C .

□

(b) If $F(X) = X$ then we have $B \subseteq X$ (because $F(X) \subseteq X$) and $X \subseteq C$ (because $X \subseteq F(X)$).

3.5 Infinite Cartesian Products

Exercise 31 PROOF:

- ⟨1⟩1. If the Axiom of Choice is true then, for any set I and any function H with domain I , if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$.
- ⟨2⟩1. ASSUME: The Axiom of Choice.
- ⟨2⟩2. LET: I be a set.
- ⟨2⟩3. LET: H be a function with domain I .
- ⟨2⟩4. ASSUME: $H(i) \neq \emptyset$ for all $i \in I$.
- ⟨2⟩5. LET: $R = \{(i, x) : i \in I, x \in H(i)\}$
- ⟨2⟩6. PICK a function $F \subseteq R$ with $\text{dom } F = \text{dom } R$
PROVE: $F \in \prod_{i \in I} H(i)$
PROOF: By the Axiom of Choice.
- ⟨2⟩7. $\text{dom } H = I$
PROOF: We have $\text{dom } R = I$ since for all $i \in I$ there exists x such that $x \in H(i)$.
- ⟨2⟩8. $\forall i \in I. F(i) \in H(i)$
PROOF: Since $iRF(i)$.
- ⟨1⟩2. If, for any set I and any function H with domain I , if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$, then the Axiom of Choice is true.
- ⟨2⟩1. ASSUME: For any set I and any function H with domain I , if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$
- ⟨2⟩2. LET: R be a relation
- ⟨2⟩3. LET: $I = \text{dom } R$
- ⟨2⟩4. Define the function H with domain I by: for $i \in I$, $H(i) = \{y : iRy\}$
- ⟨2⟩5. $H(i) \neq \emptyset$ for all $i \in I$
- ⟨2⟩6. PICK $F \in \prod_{i \in I} H(i)$
PROOF: By ⟨2⟩1
- ⟨2⟩7. F is a function
- ⟨2⟩8. $F \subseteq R$
PROOF: For all $i \in I$ we have $F(i) \in H(i)$ and so $iRF(i)$.
- ⟨2⟩9. $\text{dom } F = \text{dom } R$

□

3.6 Equivalence Relations

Exercise 32

(a)

$$\begin{aligned}
 & R \text{ is symmetric} \\
 \Leftrightarrow & \forall x, y (xRy \Rightarrow yRx) \\
 \Leftrightarrow & \forall x, y (\langle y, x \rangle \in R^{-1} \Rightarrow \langle y, x \rangle \in R) \\
 \Leftrightarrow & R^{-1} \subseteq R
 \end{aligned}$$

(b)

$$\begin{aligned}
& R \text{ is transitive} \\
& \Leftrightarrow \forall x, y, z (xRy \ \& \ yRz \Rightarrow xRz) \\
& \Leftrightarrow \forall x, z (\exists y (xRy \ \& \ yRz) \Rightarrow xRz) \\
& \Leftrightarrow \forall x, z (\langle x, z \rangle \in R \circ R \Rightarrow \langle x, z \rangle \in R) \\
& \Leftrightarrow R \circ R \subseteq R
\end{aligned}$$

Exercise 33 PROOF:

$\langle 1 \rangle 1$. If R is a symmetric and transitive relation then $R = R^{-1} \circ R$.

$\langle 2 \rangle 1$. ASSUME: R is a symmetric and transitive relation.

$\langle 2 \rangle 2$. $R \subseteq R^{-1} \circ R$

$\langle 3 \rangle 1$. LET: xRy

$\langle 3 \rangle 2$. yRy

PROOF: By Theorem 3M.

$\langle 3 \rangle 3$. xRy and $yR^{-1}y$

$\langle 3 \rangle 4$. $x(R^{-1} \circ R)y$

$\langle 2 \rangle 3$. $R^{-1} \circ R \subseteq R$

PROOF:

$$R^{-1} \circ R \subseteq R \circ R \quad (\text{Exercise 32(a)})$$

$$\subseteq R \quad (\text{Exercise 32(b)})$$

$\langle 1 \rangle 2$. If $R = R^{-1} \circ R$ then R is a symmetric and transitive relation.

$\langle 2 \rangle 1$. ASSUME: $R = R^{-1} \circ R$

$\langle 2 \rangle 2$. R is a relation.

$\langle 2 \rangle 3$. R is symmetric.

$\langle 3 \rangle 1$. LET: xRy

$\langle 3 \rangle 2$. PICK z such that xRz and $zR^{-1}y$

$\langle 3 \rangle 3$. yRz and $zR^{-1}x$

$\langle 3 \rangle 4$. $y(R^{-1} \circ R)x$

$\langle 3 \rangle 5$. yRx

$\langle 2 \rangle 4$. R is transitive.

$\langle 3 \rangle 1$. LET: xRy and yRz

$\langle 3 \rangle 2$. zRy

PROOF: By $\langle 2 \rangle 3$

$\langle 3 \rangle 3$. xRy and $yR^{-1}z$

$\langle 3 \rangle 4$. $x(R^{-1} \circ R)z$

$\langle 3 \rangle 5$. xRz

□

Exercise 34

(a) $\bigcap \mathcal{A}$ is a transitive relation.

PROOF:

$\langle 1 \rangle 1$. $\bigcap \mathcal{A}$ is a relation.

PROOF: Every member of a member of \mathcal{A} is an ordered pair.

$\langle 1 \rangle 2$. $\bigcap \mathcal{A}$ is transitive.

$\langle 2 \rangle 1$. LET: $\langle x, y \rangle$ and $\langle y, z \rangle$ be in $\bigcap \mathcal{A}$

PROVE: $\langle x, z \rangle \in \bigcap \mathcal{A}$

$\langle 2 \rangle 2$. LET: $R \in \mathcal{A}$

$\langle 2 \rangle 3$. xRy and yRz

$\langle 2 \rangle 4$. xRz

PROOF: Since R is transitive.

□

(b) Not necessarily. If $\mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$ then each member of \mathcal{A} is transitive but $\bigcup \mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$ is not.

Example 35

$$\begin{aligned} R[\{x\}] &= \{y : \exists z(z \in \{x\} \ \& \ zRy)\} \\ &= \{y : \exists z(z = x \ \& \ zRy)\} \\ &= \{y : xRy\} \\ &= [x]_R \end{aligned}$$

Example 36 PROOF:

$\langle 1 \rangle 1$. Q is a relation on A .

PROOF: By definition.

$\langle 1 \rangle 2$. Q is reflexive on A .

$\langle 2 \rangle 1$. LET: $x \in A$

$\langle 2 \rangle 2$. $f(x)Rf(x)$

PROOF: Since R is reflexive on B .

$\langle 2 \rangle 3$. xQx

$\langle 1 \rangle 3$. Q is symmetric.

$\langle 2 \rangle 1$. ASSUME: xQy

$\langle 2 \rangle 2$. $f(x)Rf(y)$

$\langle 2 \rangle 3$. $f(y)Rf(x)$

PROOF: R is symmetric.

$\langle 2 \rangle 4$. yQx

$\langle 1 \rangle 4$. Q is transitive.

$\langle 2 \rangle 1$. ASSUME: xQy and yQz

$\langle 2 \rangle 2$. $f(x)Rf(y)$ and $f(y)Rf(z)$

$\langle 2 \rangle 3$. $f(x)Rf(z)$

PROOF: R is transitive.

$\langle 2 \rangle 4$. xQz

□

Exercise 37 PROOF:

$\langle 1 \rangle 1$. R_Π is a relation on A .

PROOF: If $B \in \Pi$, $x \in B$ and $y \in B$ then $x, y \in A$.

$\langle 1 \rangle 2$. R_Π is reflexive on A .

$\langle 2 \rangle 1$. LET: $x \in A$

$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$

PROOF: Because Π is exhaustive.

$\langle 2 \rangle 3$. $x \in B$ and $x \in B$

$\langle 2 \rangle 4$. $xR_\Pi x$

$\langle 1 \rangle 3$. R_Π is symmetric.

$\langle 2 \rangle 1$. ASSUME: $xR_\Pi y$

$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$

$\langle 2 \rangle 3$. $y \in B$ and $x \in B$

$\langle 2 \rangle 4$. $yR_\Pi x$

$\langle 1 \rangle 4$. R_Π is transitive.

$\langle 2 \rangle 1$. ASSUME: $xR_\Pi y$ and $yR_\Pi z$

$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$

$\langle 2 \rangle 3$. PICK $C \in \Pi$ such that $y \in C$ and $z \in C$

$\langle 2 \rangle 4$. $B = C$

PROOF: Since $y \in B$ and $y \in C$

$\langle 2 \rangle 5$. $x \in B$ and $z \in B$

$\langle 2 \rangle 6$. $xR_\Pi z$

□

Exercise 38 PROOF:

$\langle 1 \rangle 1$. If $B \in \Pi$ and $x \in B$ then $B = [x]_{R_\Pi}$

$\langle 2 \rangle 1$. LET: $B \in \Pi$

$\langle 2 \rangle 2$. LET: $x \in B$

$\langle 2 \rangle 3$. $[x]_{R_\Pi} \subseteq B$

$\langle 3 \rangle 1$. LET: $y \in [x]_{R_\Pi}$

$\langle 3 \rangle 2$. $xR_\Pi y$

$\langle 3 \rangle 3$. PICK $C \in \Pi$ such that $x \in C$ and $y \in C$

$\langle 3 \rangle 4$. $B = C$

PROOF: Since $x \in B$ and $x \in C$.

$\langle 3 \rangle 5$. $y \in B$

$\langle 2 \rangle 4$. $B \subseteq [x]_{R_\Pi}$

PROOF: For all $y \in B$, we have $x \in B$ and $y \in B$ hence $xR_\Pi y$.

$\langle 1 \rangle 2$. $A/R_\Pi \subseteq \Pi$

$\langle 2 \rangle 1$. LET: $x \in A$

PROVE: $[x]_{R_\Pi} \in \Pi$

$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$

$\langle 2 \rangle 3$. $[x]_{R_\Pi} = B$

PROOF: By $\langle 1 \rangle 1$

$\langle 2 \rangle 4$. $[x]_{R_\Pi} \in \Pi$

$\langle 1 \rangle 3$. $\Pi \subseteq A/R_\Pi$

$\langle 2 \rangle 1$. LET: $B \in \Pi$

$\langle 2 \rangle 2$. PICK $x \in B$

PROOF: Since every member of Π is nonempty.

$\langle 2 \rangle 3$. $B = [x]_{R_\Pi}$

PROOF: By $\langle 1 \rangle 1$.

$\langle 2 \rangle 4$. $B \in A/R_\Pi$

□

Exercise 39 PROOF:

$\langle 1 \rangle 1$. $R_\Pi \subseteq R$

$\langle 2 \rangle 1$. LET: $xR_\Pi y$

$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$

$\langle 2 \rangle 3$. PICK $z \in A$ such that $B = [z]_R$

$\langle 2 \rangle 4$. zRx

$\langle 2 \rangle 5$. zRy

$\langle 2 \rangle 6$. xRy

PROOF: Since R is symmetric and transitive.

$\langle 1 \rangle 2$. $R \subseteq R_\Pi$

$\langle 2 \rangle 1$. LET: xRy

$\langle 2 \rangle 2$. $x \in [x]_R$

$\langle 2 \rangle 3$. $y \in [x]_R$

$\langle 2 \rangle 4$. $xR_\Pi y$

□

Exercise 40 We have $[2]_R = [3]_R$ but $[6]_R \neq [9]_R$ so there is no such function f .

Exercise 41

(a) PROOF:

$\langle 1 \rangle 1$. Q is reflexive on $\mathbb{R} \times \mathbb{R}$.

PROOF: For any $x, y \in \mathbb{R}$, we have $x + y = x + y$, hence $\langle x, y \rangle Q \langle x, y \rangle$

$\langle 1 \rangle 2$. Q is symmetric.

$\langle 2 \rangle 1$. ASSUME: $\langle u, v \rangle Q \langle x, y \rangle$

$\langle 2 \rangle 2$. $u + y = x + v$

$\langle 2 \rangle 3$. $x + v = u + y$

$\langle 2 \rangle 4$. $\langle x, y \rangle Q \langle u, v \rangle$

$\langle 1 \rangle 3$. Q is transitive.

$\langle 2 \rangle 1$. ASSUME: $\langle a, b \rangle Q \langle u, v \rangle$ and $\langle u, v \rangle Q \langle x, y \rangle$

$\langle 2 \rangle 2$. $a + v = u + b$

$\langle 2 \rangle 3$. $u + y = x + v$

$\langle 2 \rangle 4$. $a + y + x + b$

PROOF: Adding $\langle 2 \rangle 2$ and $\langle 2 \rangle 3$ gives $a + u + v + y = b + u + v + x$.

$\langle 2 \rangle 5$. $\langle a, b \rangle Q \langle x, y \rangle$

□

(b) We prove that, if $\langle u, v \rangle Q \langle x, y \rangle$ then $\langle u + 2v, v + 2u \rangle Q \langle x + 2y, y + 2x \rangle$. It follows from Theorem 3Q that the function G exists.

If $u + y = v + x$ then $u + 2v + y + 2x = v + 2u + x + 2y$ by adding $u + v + y + x$ to both sides.

Exercise 42 Assume that R is an equivalence relation on A and that $F : A \times A \rightarrow A$. Let us say that F is *compatible* with R iff, whenever xRx' and yRy' , then $F(\langle x, y \rangle)RF(\langle x', y' \rangle)$. If F is compatible with R then there exists a unique $\hat{F} : (A/R) \times (A/R) \rightarrow A/R$ such that

$$\hat{F}(\langle [x]_R, [y]_R \rangle) = [F(\langle x, y \rangle)]_R \text{ for all } x, y \in A.$$

If F is not compatible with R then no such \hat{F} exists.

3.7 Ordering Relations

Exercise 43 PROOF:

- $\langle 1 \rangle 1.$ R^{-1} is transitive.
- $\langle 2 \rangle 1.$ ASSUME: $xR^{-1}y$ and $yR^{-1}z$
- $\langle 2 \rangle 2.$ zRy and yRx
- $\langle 2 \rangle 3.$ zRx
- PROOF: Since R is transitive.
- $\langle 2 \rangle 4.$ $xR^{-1}z$
- $\langle 1 \rangle 2.$ R^{-1} satisfies trichotomy on A .
- $\langle 2 \rangle 1.$ LET: $x, y \in A$
- $\langle 2 \rangle 2.$ Exactly one of xRy , $x = y$, yRx holds.
- $\langle 2 \rangle 3.$ Exactly one of $yR^{-1}x$, $x = y$, $xR^{-1}y$ holds.

□

Exercise 44 PROOF:

- $\langle 1 \rangle 1.$ f is one-to-one.
- $\langle 2 \rangle 1.$ LET: $x, y \in A$ with $f(x) = f(y)$
- $\langle 2 \rangle 2.$ $f(x) < f(y)$ and $f(y) < f(x)$ do not hold.
- PROOF: By trichotomy.
- $\langle 2 \rangle 3.$ $x < y$ and $y < x$ do not hold.
- $\langle 2 \rangle 4.$ $x = y$
- PROOF: By trichotomy.
- $\langle 1 \rangle 2.$ Whenever $f(x) < f(y)$ then $x < y$
- $\langle 2 \rangle 1.$ LET: $x, y \in A$ with $f(x) < f(y)$
- $\langle 2 \rangle 2.$ $f(x) = f(y)$ and $f(y) < f(x)$ do not hold.
- PROOF: By trichotomy.
- $\langle 2 \rangle 3.$ $x = y$ and $y < x$ do not hold.
- $\langle 2 \rangle 4.$ $x < y$
- PROOF: By trichotomy.

□

Exercise 45 PROOF:

$\langle 1 \rangle 1.$ $<_L$ is transitive.

$\langle 2 \rangle 1.$ LET: $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$ and $\langle a_2, b_2 \rangle <_L \langle a_3, b_3 \rangle$

PROVE: $\langle a_1, b_1 \rangle < \langle a_3, b_3 \rangle$

$\langle 2 \rangle 2.$ CASE: $a_1 <_A a_2$ and $a_2 <_A a_3$

PROOF: Then $a_1 <_A a_3$

$\langle 2 \rangle 3.$ CASE: $a_1 <_A a_2$, $a_2 = a_3$, $b_2 <_B b_3$

PROOF: Then $a_1 <_A a_3$

$\langle 2 \rangle 4.$ CASE: $a_1 = a_2$, $b_1 <_B b_2$ and $a_2 <_A a_3$

PROOF: Then $a_1 <_A a_3$

$\langle 2 \rangle 5.$ CASE: $a_1 = a_2$, $b_1 <_B b_2$, $a_2 = a_3$, $b_2 <_B b_3$

PROOF: Then $a_1 = a_3$ and $b_1 <_B b_3$

$\langle 1 \rangle 2.$ $<_L$ satisfies trichotomy on $A \times B$.

$\langle 2 \rangle 1.$ LET: $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle$ be elements of $A \times B$

$\langle 2 \rangle 2.$ Exactly one of $a_1 <_A a_2$, $a_1 = a_2$, $a_2 <_A a_1$ holds.

$\langle 2 \rangle 3.$ Exactly one of $b_1 <_B b_2$, $b_1 = b_2$, $b_2 <_B b_1$ holds.

$\langle 2 \rangle 4.$ Exactly one of $a_1 <_A a_2$, $(a_1 = a_2 \text{ and } b_1 <_B b_2)$, $(a_1 = a_2 \text{ and } b_1 = b_2)$, $(a_1 = a_2 \text{ and } b_2 <_L b_1)$, $a_2 <_A a_1$ holds.

$\langle 2 \rangle 5.$ Exactly one of $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$, $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$, $\langle a_2, b_2 \rangle <_L \langle a_1, b_1 \rangle$ holds.

□

3.8 Review Exercises

Exercise 46

(a)

$$\bigcap \bigcap \langle x, y \rangle = \bigcap \{x\} \\ = x$$

(b)

$$\begin{aligned} \bigcap \bigcap \bigcap \{ \langle x, y \rangle \}^{-1} &= \bigcap \bigcap \bigcap \{ \langle y, x \rangle \} \\ &= \bigcap \bigcap \langle y, x \rangle \\ &= y \end{aligned} \quad \text{(by part (a))}$$

Exercise 47

(a) There are eight:

$$\begin{aligned} &\{\langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle\} \end{aligned}$$

(b) There are six:

$$\begin{aligned} &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 5 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 5 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 5 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 5 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 5 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 5 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\} \end{aligned}$$

Exercise 48

(a) The only ordered pair in \mathcal{PT} is $\langle \emptyset, \emptyset \rangle = \{\{\emptyset\}\}$.

(b)

$$\begin{aligned} (\mathcal{PT})^{-1} \circ (\mathcal{PT} \upharpoonright \{\emptyset\}) &= \{\langle \emptyset, \emptyset \rangle\} \circ \{\langle \emptyset, \emptyset \rangle\} \\ &= \{\langle \emptyset, \emptyset \rangle\} \end{aligned}$$

Exercise 49 There are six:

$$\begin{aligned} &\{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 1 \rangle, \langle 2, 0 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\} \end{aligned}$$

Exercise 50

(a) $\{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 3 \rangle\}$

$$(b) \quad \{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}$$

Exercise 51 There are three:

$$\begin{aligned} &\{\langle 1, 0 \rangle, \langle 1, 2 \rangle, \langle 2, 0 \rangle\}, \\ &\{\langle 1, 0 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle\}, \\ &\{\langle 0, 1 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle\} \end{aligned}$$

Exercise 52 We can conclude this if we know that A and B are nonempty, or that C and D are nonempty.

Suppose A and B are nonempty. Then $A \times B = C \times D \neq \emptyset$ so C and D are nonempty. We now prove $A \subseteq C$.

Let $a \in A$. Pick some $b \in B$. Then $\langle a, b \rangle \in A \times B = C \times D$ and so $a \in C$.

We can similarly prove $C \subseteq A$, $B \subseteq D$ and $D \subseteq B$.

Exercise 53

$$\begin{aligned} x(R \cup S)^{-1}y &\Leftrightarrow y(R \cup S)x \\ &\Leftrightarrow yRx \text{ or } ySx \\ &\Leftrightarrow xR^{-1}y \text{ or } xS^{-1}y \\ &\Leftrightarrow x(R^{-1} \cup S^{-1})y \\ x(R \cap S)^{-1}y &\Leftrightarrow y(R \cap S)x \\ &\Leftrightarrow yRx \text{ and } ySx \\ &\Leftrightarrow xR^{-1}y \text{ and } xS^{-1}y \\ &\Leftrightarrow x(R^{-1} \cap S^{-1})y \\ x(R - S)^{-1}y &\Leftrightarrow y(R - S)x \\ &\Leftrightarrow yRx \text{ and } \neg ySx \\ &\Leftrightarrow xR^{-1}y \text{ and } \neg xS^{-1}y \\ &\Leftrightarrow x(R^{-1} - S^{-1})y \end{aligned}$$

Exercise 54

(a)

$$\begin{aligned} \langle x, y \rangle \in A \times (B \cap C) &\Leftrightarrow x \in A \text{ \& } y \in B \text{ \& } y \in C \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) \cap (A \times C) \end{aligned}$$

(b)

$$\begin{aligned} \langle x, y \rangle \in A \times (B \cup C) &\Leftrightarrow x \in A \text{ \& } (y \in B \text{ or } y \in C) \\ &\Leftrightarrow (x \in A \text{ \& } y \in B) \text{ or } (x \in A \text{ \& } y \in C) \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) \cup (A \times C) \end{aligned}$$

(c)

$$\begin{aligned}\langle x, y \rangle \in A \times (B - C) &\Leftrightarrow x \in A \ \& \ y \in B \ \& \ y \notin C \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) - (A \times C)\end{aligned}$$

Exercise 55

(a) No. Take $A = \{0\}$, $B = \{1\}$, $C = \{2\}$. Then $(A \times A) \cup (B \times C) = \{\langle 0, 0 \rangle, \langle 1, 2 \rangle\}$ while $(A \cup B) \times (A \cup C) = \{\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 2 \rangle\}$.

(b) Yes.

$$\begin{aligned}\langle x, y \rangle \in (A \times A) \cap (B \times C) &\Leftrightarrow x \in A \ \& \ y \in A \ \& \ x \in B \ \& \ y \in C \\ &\Leftrightarrow \langle x, y \rangle \in (A \cap B) \times (A \cap C)\end{aligned}$$

Exercise 56

(a) Yes.

$$\begin{aligned}x \in \text{dom}(R \cup S) &\Leftrightarrow \exists y(xRy \text{ or } xSy) \\ &\Leftrightarrow \exists y.xRy \text{ or } \exists y.xSy \\ &\Leftrightarrow x \in \text{dom } R \cup \text{dom } S\end{aligned}$$

(b) No. Take $R = \{\langle 0, 0 \rangle\}$ and $S = \{\langle 0, 1 \rangle\}$. Then $\text{dom}(R \cap S) = \text{dom } \emptyset = \emptyset$ while $\text{dom } R \cap \text{dom } S = \{0\} \cap \{0\} = \{0\}$.

Exercise 57

(a) Yes.

$$\begin{aligned}x(R \circ (S \cup T))y &\Leftrightarrow \exists z(x(S \cup T)z \ \& \ zRy) \\ &\Leftrightarrow \exists z(xSz \ \& \ zRy) \text{ or } \exists z(xTz \ \& \ zRy) \\ &\Leftrightarrow x((R \circ S) \cup (R \circ T))y\end{aligned}$$

(b) No. Take $R = \{\langle 0, 0 \rangle, \langle 1, 0 \rangle\}$, $S = \{\langle 0, 0 \rangle\}$ and $T = \{\langle 0, 1 \rangle\}$. Then

$$\begin{aligned}R \circ (S \cap T) &= R \circ \emptyset \\ &= \emptyset \\ (R \circ S) \cap (R \circ T) &= \{\langle 0, 0 \rangle\} \cap \{\langle 0, 0 \rangle\} \\ &= \{\langle 0, 0 \rangle\}\end{aligned}$$

Exercise 58 Take $F = \emptyset$ and $S = \{\emptyset\}$. Then $F[F^{-1}[\![S]\!]] = \emptyset \neq S$.

Exercise 59

$$\begin{aligned}
x(Q \upharpoonright (A \cap B))y &\Leftrightarrow xQy \ \& \ x \in A \ \& \ x \in B \\
&\Leftrightarrow x((Q \upharpoonright A) \cap (Q \upharpoonright B))y \\
x(Q \upharpoonright (A - B))y &\Leftrightarrow xQy \ \& \ x \in A \ \& \ x \notin B \\
&\Leftrightarrow (xQy \ \& \ x \in A) \ \& \ \neg(xQy \ \& \ x \in B) \\
&\Leftrightarrow x((Q \upharpoonright A) - (Q \upharpoonright B))y
\end{aligned}$$

Exercise 60

$$\begin{aligned}
x((R \circ S) \upharpoonright A)y &\Leftrightarrow \exists z(xRz \ \& \ zSy \ \& \ x \in A) \\
&\Leftrightarrow x(R \circ (S \upharpoonright A))y
\end{aligned}$$

Chapter 4

Chapter 4 — Natural Numbers

4.1 Inductive Sets

Exercise 1 We have

$$3 = 2 \cup \{2\} = 1 \cup \{1, 2\}$$

and so $1 \in 3$. But $1 \notin 1$ (since $1 = \{\emptyset\}$ and we know $\{\emptyset\} \neq \emptyset$ hence $\{\emptyset\} \notin \{\emptyset\}$). Therefore $1 \neq 3$.

4.2 Peano's Postulates

Exercise 2 If a is a transitive set then

$$\begin{aligned} \bigcup (a^+) &= a & (\text{Theorem 4E}) \\ &\subseteq a^+ \end{aligned}$$

Exercise 3

(a) Suppose a is a transitive set. Then $a \subseteq \mathcal{P}a$. Hence we have $\bigcup \mathcal{P}a = a \subseteq \mathcal{P}a$ and so $\mathcal{P}a$.

(b) Suppose $\mathcal{P}a$ is a transitive set. Then $a = \bigcup \mathcal{P}a \subseteq \mathcal{P}a$ hence a is transitive.

Exercise 4 If a is a transitive set then $\bigcup a \subseteq a$ so $\bigcup \bigcup a \subseteq \bigcup a$. Hence $\bigcup a$ is transitive.

Exercise 5

(a) PROOF:
 $\langle 1 \rangle 1.$ LET: $b \in \bigcup \mathcal{A}$
 $\langle 1 \rangle 2.$ PICK $A \in \mathcal{A}$ such that $b \in A$
 $\langle 1 \rangle 3.$ $b \subseteq A$
PROOF: Since A is transitive.
 $\langle 1 \rangle 4.$ $b \subseteq \bigcup \mathcal{A}$
 \square

(b) PROOF:
 $\langle 1 \rangle 1.$ LET: $b \in \bigcap \mathcal{A}$
 $\langle 1 \rangle 2.$ For all $A \in \mathcal{A}$ we have $b \subseteq A$
PROOF: Since $b \in A$ and A is transitive.
 $\langle 1 \rangle 3.$ $b \subseteq \bigcap \mathcal{A}$
 \square

Exercise 6 We have $\bigcup(a^+) = \bigcup a \cup a$ (see the proof of Theorem 4E). So if $\bigcup(a^+) = a$ we have $\bigcup a \cup a = a$ and so $\bigcup a \subseteq a$.

4.3 Recursion on ω

Exercise 7 We have $h_1(0) = h_2(0) = a$ so $0 \in S$.
Now let $n \in S$; we prove $n^+ \in S$. We have $h_1(n) = h_2(n)$ and therefore

$$\begin{aligned} h_1(n^+) &= F(h_1(n)) \\ &= F(h_2(n)) \\ &= h_2(n^+) \end{aligned}$$

Exercise 8 PROOF:
 $\langle 2 \rangle 1.$ $\forall m, n \in \omega. h(n) = h(m) \Rightarrow n = m$
 $\langle 2 \rangle 1.$ $\forall n \in \omega. h(n) = h(0) \Rightarrow n = 0$
 $\langle 3 \rangle 1.$ LET: $n \in \omega$
 $\langle 3 \rangle 2.$ ASSUME: $h(n) = h(0)$
 $\langle 3 \rangle 3.$ $h(n) = c$
 $\langle 3 \rangle 4.$ $\forall p \in \omega. n \neq p^+$
PROOF: Otherwise $f(h(p)) = c$ contradicting the fact that $c \in A - \text{ran } f$.
 $\langle 3 \rangle 5.$ $n = 0$
PROOF: Theorem 4C.
 $\langle 2 \rangle 2.$ For all $m \in \omega$, if $\forall n \in \omega. h(n) = h(m) \Rightarrow n = m$, then $\forall n \in \omega. h(n) = h(m^+) \Rightarrow n = m^+$
 $\langle 3 \rangle 1.$ LET: $m \in \omega$
 $\langle 3 \rangle 2.$ ASSUME: $\forall n \in \omega. h(n) = h(m) \Rightarrow n = m$
 $\langle 3 \rangle 3.$ LET: $n \in \omega$
 $\langle 3 \rangle 4.$ ASSUME: $h(n) = h(m^+)$
 $\langle 3 \rangle 5.$ $h(n) = f(h(m))$

$\langle 3 \rangle 6. n \neq 0$

PROOF: Otherwise $c = f(h(m))$ contradicting the fact that $c \in A - \text{ran } f$.

$\langle 3 \rangle 7. \text{ PICK } p \text{ such that } n = p^+$

$\langle 3 \rangle 8. f(h(p)) = f(h(m))$

$\langle 3 \rangle 9. h(p) = h(m)$

PROOF: f is one-to-one.

$\langle 3 \rangle 10. p = m$

PROOF: By $\langle 3 \rangle 2$.

$\langle 3 \rangle 11. n = p^+ = m^+$

□

Exercise 9 PROOF:

$\langle 1 \rangle 1. C^* \subseteq C_*$

$\langle 2 \rangle 1. f[C_*] \subseteq C_*$

$\langle 3 \rangle 1. \text{ LET: } x \in C_*$

PROVE: $f(x) \in C_*$

$\langle 3 \rangle 2. \text{ PICK } n \text{ such that } x \in h(n)$

$\langle 3 \rangle 3. f(x) \in h(n^+)$

$\langle 3 \rangle 4. f(x) \in C_*$

$\langle 1 \rangle 2. C_* \subseteq C^*$

$\langle 2 \rangle 1. \forall n \in \omega. h(n) \subseteq C^*$

$\langle 3 \rangle 1. h(0) \subseteq C^*$

PROOF: If $A \subseteq X \subseteq B$ and $f[X] \subseteq X$ then $A \subseteq X$.

$\langle 3 \rangle 2. \forall n \in \omega. h(n) \subseteq C^* \Rightarrow h(n^+) \subseteq C^*$

$\langle 4 \rangle 1. \text{ LET: } n \in \omega$

$\langle 4 \rangle 2. \text{ ASSUME: } h(n) \subseteq C^*$

$\langle 4 \rangle 3. f[h(n)] \subseteq C^*$

$\langle 5 \rangle 1. \text{ LET: } X \text{ be such that } A \subseteq X \subseteq B \text{ and } f[X] \subseteq X$

PROVE: $f[h(n)] \subseteq X$

$\langle 5 \rangle 2. h(n) \subseteq X$

$\langle 5 \rangle 3. f[h(n)] \subseteq f[X]$

$\langle 5 \rangle 4. f[h(n)] \subseteq X$

$\langle 4 \rangle 4. h(n^+) \subseteq C^*$

□

Exercise 10 $C^* = C_* = (0, 1]$

Exercise 11 $\{n \in \mathbb{Z} \mid n \leq 0\}$

Exercise 12 Let $f : B \times B \rightarrow B$ and $A \subseteq B$. Let

$$C^* = \bigcap \{X \mid A \subseteq X \subseteq B \text{ \& } f[X \times X] \subseteq X\} .$$

Define the function $h : \omega \rightarrow \mathcal{P}B$ by

$$\begin{aligned} h(0) &= A \\ h(n^+) &= h(n) \cup f[h(n) \times h(n)] \end{aligned} \quad (n \in \omega)$$

Define $C_* = \bigcup \text{ran } h$. Then $C^* = C_*$.

4.4 Arithmetic

Exercise 13 We prove the contrapositive. Assume $m \neq 0$ and $n \neq 0$. Then by Theorem 4C there are natural numbers p, q such that $m = p^+$ and $n = q^+$. Hence $mn = p^+q^+ = (p^+q + p)^+ \neq 0$.

Exercise 14 We prove the following facts for any natural number n :

1. n is even if and only if n^+ is odd.

PROOF: If n is even, say $n = 2p$, then $n^+ = 2p + 1$ is odd.

If n^+ is odd, say $n^+ = 2p + 1$, then $n = 2p$ is even.

2. n is odd if and only if n^+ is even.

PROOF: If n is odd, say $n = 2p + 1$, then $n^+ = 2(p + 1)$ is even.

If n^+ is even, say $n^+ = 2p$, then we cannot have $p = 0$ (since $n^+ \neq 0$). So $p = q + 1$ for some q . But then $n^+ = 2q + 2$ so $n = 2q + 1$ and n is odd.

Now, 0 is even and 0 is not odd. By the two facts above, if n is either even or odd but not both, then n^+ is either odd or even but not both. The result follows by induction.

Exercise 15 We have

$$\begin{aligned} m + (n + 0) &= m + n && \text{by (A1)} \\ &= (m + n) + 0 && \text{by (A1)} \end{aligned}$$

If $m + (n + p) = (m + n) + p$ then

$$\begin{aligned} m + (n + p^+) &= m + (n + p)^+ && \text{by (A2)} \\ &= (m + (n + p))^+ && \text{by (A2)} \\ &= ((m + n) + p)^+ && \text{by induction hypothesis} \\ &= (m + n) + p^+ && \text{by (A2)} \end{aligned}$$

Exercise 16 We first prove that $0 \cdot n = 0$ for all n . We have $0 \cdot 0 = 0$ by (M1), and if $0 \cdot n = 0$ then

$$\begin{aligned} 0 \cdot n^+ &= 0 \cdot n + 0 && \text{by (M2)} \\ &= 0 \cdot n && \text{by (A1)} \\ &= 0 && \text{by induction hypothesis} \end{aligned}$$

Now we prove that $m^+ \cdot n = m \cdot n + n$ for all m, n . We have

$$\begin{aligned} m^+ \cdot 0 &= 0 && \text{by (M1)} \\ m \cdot 0 + 0 &= m \cdot 0 && \text{by (A1)} \\ &= 0 && \text{by (M1)} \end{aligned}$$

Thus, $m^+ \cdot 0 = m \cdot 0 + 0$.

If $m^+ \cdot n = m \cdot n + n$ then

$$\begin{aligned} m^+ \cdot n^+ &= m^+ \cdot n + m^+ && \text{by (M2)} \\ &= (m^+ \cdot n + m)^+ && \text{by (A2)} \\ &= ((m \cdot n + n) + m)^+ && \text{by induction hypothesis} \\ &= ((m \cdot n + m) + n)^+ && \text{by associativity and commutativity of addition} \\ &= (m \cdot n^+ + n)^+ && \text{by (M2)} \\ &= m \cdot n^+ + n^+ && \text{by (A2)} \end{aligned}$$

Exercise 17 The proof is by induction on p . We have

$$\begin{aligned} m^{n+0} &= m^n && \text{by (A1)} \\ &= 0 + m^n && \text{by Theorem 4K(2)} \\ &= m^n \cdot 0 + m^n && \text{by (M1)} \\ &= m^n \cdot 1 && \text{by (M2)} \\ &= m^n \cdot m^0 && \text{by (E1)} \end{aligned}$$

If $m^{n+p} = m^n \cdot m^p$ then

$$\begin{aligned} m^{n+p^+} &= m^{(n+p)^+} && \text{by (A2)} \\ &= m^{n+p} m && \text{by (E2)} \\ &= (m^n m^p) m && \text{by induction hypothesis} \\ &= m^n (m^p m) && \text{by Theorem 4K (4)} \\ &= m^n m^{p^+} && \text{by (E2)} \end{aligned}$$

4.5 Ordering on ω

Exercise 18

$$\begin{aligned} \in_{\omega}^{-1} \llbracket \{7, 8\} \rrbracket &= \{x \in \omega \mid x \in 7 \text{ or } x \in 8\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7\} \end{aligned}$$

Exercise 19 The proof is by induction on m .

For $m = 0$, take $q = r = 0$. Then $m = d \cdot 0 + 0$ and $0 \in d$.

Suppose $m = dq + r$ and $r < d$. Then $r + 1 \leq d$. If $r + 1 < d$, then we have $m + 1 = dq + (r + 1)$ as required. If $r + 1 = d$, then we have $m + 1 = dq + d = d(q + 1) + 0$.

Exercise 20 We first prove A is closed downwards; that is, if $n \in A$ and $m \in n$ then $m \in A$. This holds because if $n \in A$ and $m \in n$ then $m \in \bigcup A$ and $\bigcup A = A$.

Now, we prove $\forall n \in \omega. n \in A$ by induction on n .

To prove $0 \in A$: we are given that A is nonempty. Pick some $a \in A$. Then $0 \in a$ so $0 \in A$ since A is closed downwards.

Now let $n \in A$; we prove $n^+ \in A$. We have $n \in \bigcup A$; pick some $k \in A$ such that $n \in k$. Then $n^+ \in k$ so $n^+ \in A$ since A is closed downwards.

This completes the induction. We have $\forall n \in \omega. n \in A$, i.e. $A = \omega$.

Exercise 21 Suppose n is a natural number, $k \in n$ and $n \subseteq k$. Then $k \in k$, contradicting Lemma 4L(b).

Exercise 22 We have $0 \in p^+$ (by trichotomy since $p^+ \not\subseteq 0$ because 0 is empty, and $p^+ \not\neq 0$ by Peano's First Postulate.) Hence $n = n + 0 \in n + p^+$ by Theorem 4N.

Exercise 23 The proof is by induction on n . The statement is vacuously true for $n = 0$.

Suppose the statement is true for n . Let $m \in n^+$. Then $m \subseteq n$.

If $m = n$, then we have $m + 0^+ = n^+$.

If $m \in n$, pick p such that $m + p^+ = n$ by the induction hypothesis. Then $m + p^{++} = n^+$.

Exercise 24 Suppose $m \in p$. Then we cannot have $n \in q$ or $n = q$, as either of these would imply $m + n \in p + q$. Hence $q \in n$ by trichotomy.

We prove $q \in n \Rightarrow m \in p$ similarly.

Exercise 25 By Exercise 23, pick natural numbers a and b such that $m = n + a^+$ and $p = q + b^+$. Then

$$\begin{aligned} mp + nq &= (n + a^+)(q + b^+) + nq \\ &= nq + nq + a^+q + nb^+ + a^+b^+ \\ &= (n + a^+)q + n(q + b^+) + a^+b^+ \\ &= mq + np + (a^+ + b^+)^+ \end{aligned}$$

Hence $mq + np \in mp + nq$ by Exercise 22.

Exercise 26 The proof is by induction on n .

If $n = 0$ then $\text{ran } f$ is a singleton and its sole element is the largest element.

Suppose the result is true for n . Let $f : n^{++} \rightarrow A$. Then $f \llbracket n^+ \rrbracket$ has a largest element $f(k)$, say. If $f(k) \subseteq f(n^+)$ then $f(n^+)$ is greatest in $\text{ran } f$; otherwise $f(k)$ is greatest.

Exercise 27 We prove $f_1(n) = f_2(n)$ for all $n \in \omega$ by strong induction on n . Assume that $(\forall m \in n) f_1(m) = f_2(m)$. Then $f_1 \upharpoonright n = f_2 \upharpoonright n$. So

$$\begin{aligned} f_1(n) &= G(f_1 \upharpoonright n) \\ &= G(f_2 \upharpoonright n) \\ &= f_2(n) \end{aligned}$$

Exercise 28 Suppose ω is not transitive. Then there exists a natural number n such that $n \not\subseteq \omega$. Let n be the least such number. There exists $x \in n$ such that $x \not\subseteq \omega$. Now, $n \neq \emptyset$ (because it is nonempty) so $n = p^+$ for some natural number p . We have $x \in p^+$ so $x \in p$ or $x = p$. We cannot have $x = p$ (because x is not a natural number) so we have $x \in p$. But this contradicts the minimality of n .

4.6 Review Exercises

Exercise 29 $4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Exercise 30 $\bigcup 4 = 0 \cup 1 \cup 2 \cup 3 = 3$ since 0, 1 and 2 are all subsets of 3.
 $\bigcap 4 = 0 \cap 1 \cap 2 \cap 3 = 0 (= \emptyset)$.

Exercise 31 Similarly to Exercise 30 we have $\bigcup \bigcup 7 = \bigcup 6 = 5$.

Exercise 32

$$\begin{aligned} \text{(a)} \quad A^+ &= A \cup \{A\} = \{1, A\} = \{1, \{1\}\} \\ \text{So } \bigcup A^+ &= 1 \cup \{1\} = \{0, 1\} = 2 \end{aligned}$$

$$\text{(b)} \quad \bigcup (\{2\}^+) = \bigcup \{2, \{2\}\} = \{0, 1, 2\} = 3$$

Exercise 33

(a) Yes - if $x \in y \in \{0, 1, \{1\}\}$ then x is either 0 or 1, and in either case $x \in \{0, 1, \{1\}\}$

(b) No - $0 \in 1 \in \{1\}$ but $0 \notin \{1\}$

(c) No - $0 \in \{0\} \in \langle 0, 1 \rangle$ but $0 \notin \langle 0, 1 \rangle$.

Exercise 34

(a) Let $a = \{\emptyset\}$ and $b = \emptyset$

(b) Let $c = \{\{\emptyset\}\}$, $d = \{\emptyset\}$ and $e = \emptyset$

Exercise 35

(a) Let $T_1 = \{\{1\}, \{1, 0\}, 0, 1\}$

(b) Let $T_2 = \{\langle 1, 0 \rangle, \{1\}, \{1, 0\}, 0, 1\}$.

Exercise 36

$$\begin{aligned} h(4) &= 2h(3) \\ &= 4h(2) \\ &= 8h(1) \\ &= 16h(0) \\ &= 48 \end{aligned}$$

Exercise 37

(a) Let $f : m \rightarrow A$ and $g : n \rightarrow B$ be bijections. Define $h : m + n \rightarrow A \cup B$ by

$$\begin{aligned} h(p) &= f(p) && \text{if } p \in m \\ h(m + q) &= g(q) && \text{if } q \in n \end{aligned}$$

To show that this is well-defined, we must prove two things:

1. For all $p \in m + n$, then either $p \in m$ or there exists $q \in n$ such that $p = m + q$.
2. We never have $p \in m$ and $p = m + q$ for some $q \in n$.

We prove 1 by induction on n . For all $p \in m + 0$ we have $p \in m$, so the result holds for $n = 0$.

Now, suppose the result holds for n . Let $p \in m + n^+ = (m + n)^+$ so $p \in m + n$. If $p \in m + n$, we simply apply the induction hypothesis. If $p = m + n$ then $p = m + q$ where $q = n \in n^+$.

To prove 2, if $p = m + q$ then $m = m + 0$ and $m + q = p$ by Theorem 4N, hence $p \notin m$ by trichotomy.

It remains to show that h is a bijection.

To prove h is injective, we consider three cases. If $h(p) = h(p')$ where $p, p' \in m$, then $f(p) = f(p')$ so $p = p'$. If $h(m + q) = h(m + q')$ where $q, q' \in n$, then $g(q) = g(q')$ so $q = q'$. And we cannot have $h(p) = h(m + q)$ for $p \in m$ and $q \in n$ since $h(p) \in A$, $h(m + q) \in B$, and $A \cap B = \emptyset$.

To prove h is surjective, let $x \in A \cup B$. If $x \in A$, there is some $p \in m$ with $f(p) = x$, so $h(p) = x$. If $x \in B$, there is some $q \in n$ with $g(q) = x$, so $h(m + q) = x$.

(b) Let $f : m \rightarrow A$ and $g : n \rightarrow B$ be bijections.

We first show that, for any $p \in mn$, there exist unique $i \in m$ and $j \in n$ such that $p = mj + i$.

By Exercise 19, there exist j and $i \in m$ such that $p = mj + i$. We have $j \in n$ since otherwise $p = mj + i \in mn$.

For uniqueness, suppose $mj + i = mj' + i'$ where $i, i' \in m$ and $j, j' \in n$. Then we have

$$mj \in mj + i = mj' + i' \in mj' + m = m(j')^+$$

so $j \in (j')^+$ and $j \in j'$. Similarly $j' \in j$, and so $j = j'$. Therefore $i = i'$ by the cancellation law for addition.

Now define $h : mn \rightarrow A \times B$ by

$$h(mj + i) = \langle f(i), g(j) \rangle$$

where $i \in m$ and $j \in n$. It is easy to check that h is bijective.

Exercise 38 $h(n) = 3n + 1$

Exercise 39 $h(n) = n^2$

Exercise 40 $h(n^+) = h(n) + 5$

Chapter 5

Chapter 5 — Construction of the Real Numbers

5.1 Integers

Exercise 1 No, because $[\langle 0, 0 \rangle] = [\langle 1, 1 \rangle]$ but $[\langle 0, 0 \rangle] \neq [\langle 2, 1 \rangle]$.

Exercise 2 Yes, because if $[\langle m, n \rangle] = [\langle p, q \rangle]$ then $[\langle m, m \rangle] = [\langle p, p \rangle]$ because $m + p = m + p$.

Exercise 3 Yes, because if $[\langle m, n \rangle] = [\langle p, q \rangle]$ then $[\langle n, m \rangle] = [\langle q, p \rangle]$ because $n + p = m + q$.

Exercise 4 Let $a = [\langle m, n \rangle]$, $b = [\langle p, q \rangle]$ and $c = [\langle r, s \rangle]$. Then

$$\begin{aligned} a +_Z (b +_Z c) &= [\langle m, n \rangle] +_Z [\langle p + r, q + s \rangle] \\ &= [\langle m + (p + r), n + (q + s) \rangle] \\ &= [\langle (m + p) + r, (n + q) + s \rangle] \\ &= [\langle m + p, n + q \rangle] +_Z [\langle r, s \rangle] \\ &= (a +_Z b) +_Z c \end{aligned}$$

Exercise 5

$$[\langle m, n \rangle] - [\langle p, q \rangle] = [\langle m, n \rangle] + [\langle q, p \rangle] = [\langle m + q, n + p \rangle]$$

Exercise 6 Let $a = [\langle m, n \rangle]$. Then

$$\begin{aligned} a \cdot_Z 0_Z &= [\langle m, n \rangle] \cdot_Z [\langle 0, 0 \rangle] \\ &= [\langle m0 + n0, m0 + n0 \rangle] \\ &= [\langle 0, 0 \rangle] \\ &= 0_Z \end{aligned}$$

Exercise 7 We have $a \cdot_Z b +_Z a \cdot_Z (-b) = a \cdot_Z (b +_Z (-b)) = a \cdot_Z 0_Z = 0_Z$, hence $a \cdot_Z (-b) = -(a \cdot_Z b)$ by the uniqueness of inverses.

We prove $(-a) \cdot_Z b = -(a \cdot_Z b)$ similarly.

Exercise 8

(a) This says $[\langle m + n, 0 \rangle] = [\langle m, 0 \rangle] +_Z [\langle n, 0 \rangle]$, which is true from the definition of $+_Z$.

(b) We have

$$\begin{aligned} E(m) \cdot_Z E(n) &= [\langle m, 0 \rangle] \cdot_Z [\langle n, 0 \rangle] \\ &= [\langle mn + 0 \cdot 0, m0 + n0 \rangle] \\ &= E(mn) \end{aligned}$$

(c)

$$\begin{aligned} E(m) <_Z E(n) &\Leftrightarrow [\langle m, 0 \rangle] <_Z [\langle n, 0 \rangle] \\ &\Leftrightarrow m + 0 \in n + 0 \\ &\Leftrightarrow m \in n \end{aligned}$$

Exercise 9

$$\begin{aligned} E(m) - E(n) &= [\langle m, 0 \rangle] - [\langle n, 0 \rangle] \\ &= [\langle m, n \rangle] \end{aligned}$$

by Exercise 5.

5.2 Rational Numbers

Exercise 10 Let $r = [\langle a, b \rangle]$. Then

$$\begin{aligned} r \cdot_Q 0_Q &= [\langle a, b \rangle] \cdot_Q [\langle 0, 1 \rangle] \\ &= [\langle a \cdot_Z 0, b \cdot_Z 1 \rangle] \\ &= [\langle 0, b \rangle] \\ &= [\langle 0, 1 \rangle] \end{aligned}$$

since $\langle 0, b \rangle \sim \langle 0, 1 \rangle$ because $0 \cdot_Z 1 = 0 \cdot_Z b = 0$.

Exercise 11 Let $r = [\langle a, b \rangle]$ and $s = [\langle c, d \rangle]$. Suppose $r \cdot_Q s = 0_Q$. Then

$$[\langle ac, bd \rangle] = [\langle 0, 1 \rangle]$$

that is, $ac = 0$. Hence $a = 0$ or $c = 0$, which means $r = 0_Q$ or $s = 0_Q$.

Exercise 12 This follows from Theorem 5QJ(a) with $s = 0_Q$ and $t = -r$.

Exercise 13 Let $a, b, c \in \mathbb{Z}$. If $a +_Z c = b +_Z c$ then

$$\begin{aligned} a +_Z c +_Z (-c) &= b +_Z c +_Z (-c) \\ \therefore a +_Z 0 &= b +_Z 0 && \text{(Theorem 5ZD(b))} \\ \therefore a &= b && \text{(Theorem 5ZD(a))} \end{aligned}$$

Exercise 14 Suppose $p <_Q s$. Let $r = (p +_Q s)/2$. Then

$$\begin{aligned} p &<_Q s \\ \therefore 2p &<_Q p +_Q s \\ \therefore p &<_Q (p +_Q s)/2 \\ &= r \\ p &<_Q s \\ \therefore p +_Q s &<_Q 2s \\ \therefore (p +_Q s)/2 &<_Q s \\ \therefore r &<_Q s \end{aligned}$$

5.3 Real Numbers

Exercise 15 PROOF:

$\langle 1 \rangle 1$. $\bigcup A$ is closed downwards.

$\langle 2 \rangle 1$. LET: $q \in \bigcup A$ and $p < q$

$\langle 2 \rangle 2$. PICK $x \in A$ such that $q \in x$

$\langle 2 \rangle 3$. $p \in x$

PROOF: Since x is closed downwards.

$\langle 2 \rangle 4$. $p \in \bigcup A$

$\langle 1 \rangle 2$. $\bigcup A$ has no largest element.

$\langle 2 \rangle 1$. LET: $q \in \bigcup A$

$\langle 2 \rangle 2$. PICK $x \in A$ such that $q \in x$

$\langle 2 \rangle 3$. PICK $r \in x$ such that $q < r$

PROOF: Since x has no largest element.

$\langle 2 \rangle 4$. $r \in \bigcup A$

□

Exercise 16 PROOF:

- ⟨1⟩1. LET: $q \in x +_R y$
- ⟨1⟩2. PICK rationals $a \in x$ and $b \in y$ such that $q = a + b$
- ⟨1⟩3. PICK $a' \in x$ and $b' \in y$ such that $a < a'$ and $b < b'$

PROOF: Since x and y each have no largest element.

- ⟨1⟩4. $q < a' + b' \in x +_R y$

□

Exercise 17 If $b < 0$ we can take $k = 0$. If $b \geq 0$ then there is a natural number n such that $b = E(n)$; take $k = n^+$. Then $b < ak$ since $1 \leq a$ and $b < k$.

Exercise 18 Let $p = [\langle a, b \rangle]$ and $r = [\langle c, d \rangle]$ where a, b and d are positive. By Exercise 17, there exists a natural number k such that $bc < adE(k)$. Therefore $r < p \cdot E(E(k))$.

Exercise 19 Pick a rational $a \in x$ (which we can do since $x \neq \emptyset$). We first prove that there exists a natural number k such that $a + kp \notin x$.

Pick a rational $b \notin x$ (which we can do since $x \neq \mathbb{Q}$). We have $a < b$ (since x is closed downwards). By Exercise 18, there exists a natural number k such that

$$\begin{aligned} b - a &< kp \\ \therefore a + kp &> b \\ \therefore a + kp &\notin x \end{aligned}$$

Now, let k be the least natural number such that $a + kp \notin x$ (by the Well-Ordering Principle). We have $k \neq 0$ (since $a \in x$); let $k = n^+$. Then we have

$$a + np \in x \quad a + np + p \notin x$$

Take $q = a + np$.

Exercise 20 We must prove $0 \subseteq x \cup -x$. Let $q \in 0$ and assume $q \notin x$. Then $q < 0$ and $-0 = 0 \notin x$, so $q \in -x$.

Exercise 21 PROOF:

- ⟨1⟩1. LET: x, y be real numbers with $x < y$
- ⟨1⟩2. PICK $r \in y$ such that $r \notin x$
- ⟨1⟩3. PICK $s \in y$ such that $r < s$

PROVE: $x < E(s) < y$

- ⟨1⟩4. $x \subseteq E(s)$

PROOF: If $p \in x$ then $p < r < s$

- ⟨1⟩5. $x \neq E(s)$

PROOF: Since $r \in E(s)$ and $r \notin x$

- ⟨1⟩6. $E(s) \subseteq y$

PROOF: Since y is closed downwards.

$\langle 1 \rangle 7.$ $E(s) \neq y$

PROOF: Since $s \in y$ but $s \notin E(s)$.

Exercise 22 $|x|$ is either x or $-x$, and they are both real numbers.

Chapter 6

Chapter 6 — Cardinal Numbers and the Axiom of Choice

6.1 Equinumerosity

Exercise 1 PROOF:

- ⟨1⟩1. f is injective.
 - ⟨2⟩1. ASSUME: $f(m, n) = f(m', n')$
 - ⟨2⟩2. $2^m(2n + 1) = 2^{m'}(2n' + 1)$
 - ⟨2⟩3. $m = m'$
 - ⟨3⟩1. ASSUME: w.l.o.g. $m \leq m'$
 - ⟨3⟩2. $2n + 1 = 2^{m'-m}(2n' + 1)$
PROOF: From ⟨2⟩2 dividing by 2^m .
 - ⟨3⟩3. $m' - m = 0$
PROOF: Since $2^{m'-m}(2n' + 1)$ is odd.
 - ⟨2⟩4. $2n + 1 = 2n' + 1$
 - ⟨2⟩5. $n = n'$
- ⟨1⟩2. f is surjective.
 - ⟨2⟩1. LET: $n \in \omega$
ASSUME: $\forall m < n. m \in \text{ran } f$
PROVE: $n \in \text{ran } f$
 - ⟨2⟩2. CASE: n is even
 - ⟨3⟩1. LET: k be such that $n = 2k$
 - ⟨3⟩2. $n = f(0, k)$
 - ⟨2⟩3. CASE: n is odd
 - ⟨3⟩1. LET: k be such that $n = 2k + 1$
 - ⟨3⟩2. LET: $k = f(i, j)$
 - ⟨3⟩3. $n = f(i + 1, j)$

PROOF:

$$\begin{aligned}
 n &= 2k + 1 \\
 &= 2(2^i(2j + 1) - 1) + 1 \\
 &= 2^{i+1}(2j + 1) - 2 + 1 \\
 &= 2^{i+1}(2j + 1) - 1
 \end{aligned}$$

□

Exercise 2 Let us call (0) the 0th diagonal, $(1, 2)$ the 1st diagonal, $(3, 4, 5)$ the 2nd diagonal, etc. Then the k th is the set of all positions with coordinates (m, n) such that $m + n = k$.

Therefore, the number $J(m, n)$ at position (m, n) is the $m + 1$ st number in the $(m + n)$ th diagonal. So the number of numbers that come before $J(m, n)$ is

$$(1 + 2 + \cdots + (m + n)) + m$$

Therefore, since the natural numbers start at 0,

$$J(m, n) = (1 + 2 + \cdots + (m + n)) + m$$

We know $1 + 2 + \cdots + k = k(k + 1)/2$. Therefore,

$$\begin{aligned}
 J(m, n) &= 1/2(m + n)(m + n + 1) + m \\
 &= 1/2(m^2 + 2mn + m + n + n^2) + m \\
 &= 1/2(m^2 + 2mn + 3m + n + n^2) \\
 &= 1/2((m + n)^2 + 3m + n)
 \end{aligned}$$

Exercise 3 Define $f : (0, 1) \rightarrow \mathbb{R}$ by: $f(x) = 1/x - 2$ if $0 < x \leq 1/2$; $f(x) = 2 - 1/(1 - x)$ if $1/2 < x < 1$.

Exercise 4 Define $f : [0, 1] \rightarrow (0, 1)$ by

$$\begin{aligned}
 f(1/2 - 1/2^n) &= 1/2 - 1/2^{n-1} && \text{(for } n \text{ a positive integer)} \\
 f(1/2 + 1/2^n) &= 1/2 + 1/2^{n-1} && \text{(for } n \text{ a positive integer)} \\
 f(x) &= x && \text{(for all other } x)
 \end{aligned}$$

Exercise 5

(a) For any set A , the identity function I_A is a bijection between A and A . It is injective because, if $I_A(x) = I_A(y)$ then $x = y$ immediately. It is surjective because for any $y \in I_A$ we have $y = I_A(y)$.

(b) We prove that, if f is a bijection between A and B , then f^{-1} is a bijection between B and A . It is an injective function by Theorem 3F, and maps B onto A by Theorem 3E.

(c) Let f be a bijection between A and B , and g a bijection between A and C . We prove $g \circ f$ is a bijection between A and C .

It is a function from A to C by Theorem 3H.

We prove it is injective. Let $x, y \in A$ and assume $(g \circ f)(x) = (g \circ f)(y)$. Then

$$\begin{aligned} g(f(x)) &= g(f(y)) \\ \therefore f(x) &= f(y) && (g \text{ is injective}) \\ \therefore x &= y && (f \text{ is injective}) \end{aligned}$$

Now we prove it maps A onto C . Let $c \in C$. Pick $b \in B$ such that $g(b) = c$ (since g is surjective). Pick $a \in A$ such that $f(a) = b$ (since f is injective). Then $(g \circ f)(a) = c$.

6.2 Finite Sets

Exercise 6 Suppose every set of cardinality κ belongs to A . We will prove that every set belongs to $\bigcup A$.

Let x be any set. Pick a set y of cardinality κ . If $x \in y$ then $x \in y \in A$ so $x \in \bigcup A$.

Assume $x \notin y$. Pick an element $z \in y$ (we know y is nonempty because $\kappa \neq 0$). Then $y - \{z\} \cup \{x\}$ has cardinality κ , and so $x \in (y - \{z\} \cup \{x\}) \in A$ hence $x \in \bigcup A$.

Thus, every set is in $\bigcup A$, which we know is impossible by Theorem 2A.

Exercise 7 If f is one-to-one then f is a bijection between A and $\text{ran } f$. So we must have $\text{ran } f = A$, otherwise f would be a bijection between A and a proper subset of A , contradicting the Pigeonhole Principle.

Conversely, suppose $\text{ran } f = A$. Pick a right inverse $h : A \rightarrow A$ for f (by Theorem 3J(b). Note: Theorem 3J(b) can in fact be proved for the case B is finite without using the Axiom of Choice.). Now, h is one-to-one by Theorem 3J(a). So $\text{ran } h = A$ by the first paragraph.

We prove f is one-to-one. Let $x, y \in A$ and assume $f(x) = f(y)$. Pick $a, b \in A$ such that $h(a) = x$ and $h(b) = y$. Then

$$\begin{aligned} f(h(a)) &= f(h(b)) \\ \therefore a &= b \\ \therefore x &= y \end{aligned}$$

Exercise 8 PROOF:

$\langle 1 \rangle 1$. For any sets A and x , if A is finite then $A \cup \{x\}$ is finite.

$\langle 2 \rangle 1$. CASE: $x \in A$

PROOF: In this case $A \cup \{x\} = A$.

$\langle 2 \rangle 2$. CASE: $x \notin A$

PROOF: Then $|A \cup \{x\}| = |A|^+$.

$\langle 1 \rangle 2$. LET: A be a finite set.

$\langle 1 \rangle 3$. For any set B , if $B \approx 0$ then $A \cup B$ is finite.

PROOF: Because $B = \emptyset$ so $A \cup B = A$.

$\langle 1 \rangle 4$. Let n be a natural number. Assume that, for any set B , if $B \approx n$ then $A \cup B$ is finite. Then for any set B , if $B \approx n^+$ then $A \cup B$ is finite.

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: For any set B , if $B \approx n$ then $A \cup B$ is finite.

$\langle 2 \rangle 3$. LET: B be a set.

$\langle 2 \rangle 4$. ASSUME: $B \approx n^+$

$\langle 2 \rangle 5$. PICK a bijection $f : n^+ \rightarrow B$

$\langle 2 \rangle 6$. $B - \{f(n)\} \approx n$

$\langle 2 \rangle 7$. $A \cup (B - \{f(n)\})$ is finite.

$\langle 2 \rangle 8$. $A \cup B$ is finite.

PROOF: By $\langle 1 \rangle 1$ since $A \cup B = (A \cup (B - \{f(n)\})) \cup \{f(n)\}$.

□

Exercise 9 PROOF:

$\langle 1 \rangle 1$. LET: A be a finite set.

$\langle 1 \rangle 2$. For any set B , if $B \approx 0$ then $A \times B$ is finite.

PROOF: In this case $A \times B = \emptyset$.

$\langle 1 \rangle 3$. Let n be a natural number. Suppose that, for any set B , if $B \approx n$ then $A \times B$ is finite. Then for any set B , if $B \approx n^+$ then $A \times B$ is finite.

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. ASSUME: For any set B , if $B \approx n$ then $A \times B$ is finite.

$\langle 2 \rangle 3$. LET: B be a set.

$\langle 2 \rangle 4$. ASSUME: $B \approx n^+$

$\langle 2 \rangle 5$. PICK a bijection $f : n^+ \approx B$

$\langle 2 \rangle 6$. $A \times (B - \{f(n)\})$ is finite.

PROOF: By the induction hypothesis $\langle 2 \rangle 2$.

$\langle 2 \rangle 7$. $A \times B$ is finite.

PROOF: By Exercise 8 since $A \times B = (A \times (B - \{f(n)\})) \cup (A \times \{f(n)\})$ and $A \times \{f(n)\}$ is finite because it is equinumerous with A .

□

6.3 Cardinal Arithmetic

Exercise 10 We must show that $(L \cup M)K \approx^L K \times^M K$ where $L \cap M = \emptyset$.

Define $\Phi : (L \cup M)K \rightarrow^L K \times^M K$ by: $\Phi(f) = \langle f \upharpoonright L, f \upharpoonright M \rangle$.

To show Φ is one-to-one: suppose $\Phi(f) = \Phi(g)$. Then $f \upharpoonright L = g \upharpoonright L$ and $f \upharpoonright M = g \upharpoonright M$. Hence $f(x) = g(x)$ for all $x \in L$ and $f(x) = g(x)$ for all $x \in M$, so $f(x) = g(x)$ for all x , i.e. $f = g$.

To show Φ is surjective: given a function $g : L \rightarrow K$ and $h : M \rightarrow K$, we have $g \cup h : L \cup M \rightarrow K$ and $\Phi(g \cup h) = \langle g, h \rangle$.

Exercise 11 We must show that ${}^M(K \times L) \approx^M K \times^M L$.

Define $\Phi : {}^M(K \times L) \rightarrow {}^M K \times^M L$ by: $\Phi(f) = \langle \pi_1 \circ f, \pi_2 \circ f \rangle$, where $\pi_1 : K \times L \rightarrow K$ is the function defined by

$$\pi_1(\langle x, y \rangle) = x$$

and $\pi_2 : K \times L \rightarrow L$ is the function defined by

$$\pi_2(\langle x, y \rangle) = y .$$

To show Φ is one-to-one: suppose $\Phi(f) = \Phi(g)$. For any $x \in M$, we have $\pi_1(f(x)) = \pi_1(g(x))$ and $\pi_2(f(x)) = \pi_2(g(x))$, so $f(x) = g(x)$ by Theorem 3A.

To show Φ is surjective: given $g : M \rightarrow K$ and $h : M \rightarrow L$, define $f : M \rightarrow K \times L$ by $f(x) = \langle g(x), h(x) \rangle$ for $x \in M$. Then $\Phi(f) = \langle g, h \rangle$.

Exercise 12 We have:

$$\begin{aligned} K \cup L &= L \cup K \\ K \cup (L \cup M) &= (K \cup L) \cup M \\ K \times (L \cup M) &= (K \times L) \cup (K \times M) \end{aligned}$$

Exercise 13 Now that we have shown the union of two finite sets is finite, this follows by an easy induction on $|B|$.

Exercise 14 For any set A , let $Perm(A)$ be the set of all permutations of A .

Assume $K \approx L$: we must show $Perm(K) \approx Perm(L)$. Pick a bijection $f : K \rightarrow L$. Define $\Phi : Perm(K) \rightarrow Perm(L)$ by: $\Phi(g) = f \circ g \circ f^{-1}$. It is easy to show $\Phi(g)$ is a permutation of L whenever g is a permutation of K , and Φ is a bijection.

6.4 Ordering Cardinal Numbers

Exercise 15 Suppose for a contradiction \mathcal{A} is a set and, for every set x , there exists $y \in \mathcal{A}$ such that $x \preccurlyeq y$. Pick $y \in \mathcal{A}$ such that $\mathcal{P} \cup \mathcal{A} \preccurlyeq y$. But $y \subseteq \bigcup \mathcal{A}$ so $\mathcal{P} \cup \mathcal{A} \preccurlyeq \bigcup \mathcal{A}$, contradicting Cantor's Theorem.

Exercise 16 Define $G : S \rightarrow^S 2$ by

$$G(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Then G is injective.

Now, assume for a contradiction $F : S \rightarrow^S 2$ is bijective. Define $g : S \rightarrow 2$ by $g(x) = 1 - F(x)(x)$. Then $g(x) \neq F(x)(x)$ for all $x \in S$, so $g \neq F(x)$ for all $x \in S$. Hence $g \notin \text{ran } F$. This contradicts the assumption that F is surjective.

Exercise 17 We have $1 < 2$ but $\aleph_0 + 1 = \aleph_0 + 2 = \aleph_0$.

We have $1 < 2$ but $\aleph_0 \cdot 1 = \aleph_0 \cdot 2 = \aleph_0$.

We have $2 < 3$ but $2^{\aleph_0} = 3^{\aleph_0}$.

We have $2 < 3$ but $\aleph_0^2 = \aleph_0^3 = \aleph_0$.

6.5 Axiom of Choice

Exercise 18 PROOF:

$\langle 1 \rangle 1$. If the Axiom of Choice is true then the statement is true.

PROOF: The statement is a special case of the multiplicative axiom, taking $I = \mathcal{A}$ and $H(X) = X$ for each $X \in \mathcal{A}$.

$\langle 1 \rangle 2$. If the statement is true then the Axiom of Choice is true.

$\langle 2 \rangle 1$. ASSUME: The statement is true.

PROVE: Axiom of choice IV

$\langle 2 \rangle 2$. LET: \mathcal{A} be a set such that each member of \mathcal{A} is a nonempty set, and any two distinct members of \mathcal{A} are disjoint.

$\langle 2 \rangle 3$. PICK a function f with domain \mathcal{A} such that $f(X) \in X$ for all $X \in \mathcal{A}$

$\langle 2 \rangle 4$. LET: $C = \text{ran } f$

$\langle 2 \rangle 5$. $\forall B \in \mathcal{A}. C \cap B = \{f(B)\}$

□

Exercise 19 PROOF:

$\langle 1 \rangle 1$. For $n \in \omega$, let $P(n)$ be the statement: for every set I with $\text{card } I = n$ and function H with domain I such that $H(i)$ is nonempty for each $i \in I$, there exists a function f with domain I such that $\forall i \in I. f(i) \in H(i)$.

$\langle 1 \rangle 2$. $P(0)$ is true

PROOF: Take $f = \emptyset$

$\langle 1 \rangle 3$. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: $P(n)$

$\langle 2 \rangle 3$. LET: I be a set with $\text{card } I = n+1$

$\langle 2 \rangle 4$. LET: H be a function with domain I such that $H(i)$ is nonempty for each $i \in I$

$\langle 2 \rangle 5$. PICK a bijection $g : n+1 \approx I$

$\langle 2 \rangle 6$. PICK a function h with domain $g[n]$ such that $\forall i \in g[n]. h(i) \in H(i)$

$\langle 2 \rangle 7$. PICK $a \in H(g(n))$

$\langle 2 \rangle 8$. LET: $f = h \cup \{(g(n), a)\}$

$\langle 2 \rangle 9$. f is a function with domain I such that $\forall i \in I. f(i) \in H(i)$

□

Exercise 20 PROOF:

$\langle 1 \rangle 1$. PICK a choice function F for A

$\langle 1 \rangle 2$. PICK $a \in A$

⟨1⟩3. Define the function $f : \omega \rightarrow A$ by:

$$f(0) = a$$

$$f(n^+) = F(R^{-1}(f(n)))$$

PROOF: We know $R^{-1}(x)$ is nonempty for all $x \in A$ because $\forall x \in A. \exists y \in A. yRx$.

⟨1⟩4. $\forall n \in \omega. f(n^+)Rf(n)$

□

Exercise 21 PROOF:

⟨1⟩1. For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcup \mathcal{B} \in \mathcal{A}$

⟨2⟩1. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain.

⟨2⟩2. Every finite subset of $\bigcup \mathcal{B}$ is a member of \mathcal{A} .

⟨3⟩1. LET: $\{x_1, \dots, x_n\} \subseteq \bigcup \mathcal{B}$ be finite.

⟨3⟩2. For $1 \leq i \leq n$, PICK $B_i \in \mathcal{B}_i$ such that $x_i \in B_i$

⟨3⟩3. PICK m such that $B_1, \dots, B_n \subseteq B_m$

PROOF: Since \mathcal{B} is a chain.

⟨3⟩4. $\{x_1, \dots, x_n\}$ is a finite subset of B_m .

⟨3⟩5. $\{x_1, \dots, x_n\} \in \mathcal{A}$

PROOF: Since $B_m \in \mathcal{A}$ so every finite subset of B_m is a member of \mathcal{A} .

⟨2⟩3. $\bigcup \mathcal{B} \in \mathcal{A}$

⟨1⟩2. Q.E.D.

PROOF: By Zorn's Lemma.

□

Exercise 22 PROOF:

⟨1⟩1. If the Axiom of Choice is true then the statement is true.

⟨2⟩1. ASSUME: The Axiom of Choice

⟨2⟩2. LET: A be a set.

⟨2⟩3. LET: $R = \{\langle x, y \rangle : y \in A, x \in t\}$

⟨2⟩4. PICK a function $F \subseteq R$ such that $\text{dom } F = \text{dom } R$

⟨2⟩5. $\text{dom } R = \bigcup A$

⟨2⟩6. $\forall x \in \bigcup A. x \in F(x) \in A$

⟨1⟩2. If the statement is true then the Axiom of Choice is true.

⟨2⟩1. ASSUME: the statement

⟨2⟩2. LET: R be a relation

⟨2⟩3. LET: $A = \{\{\langle 0, x \rangle, \langle 1, y \rangle\} : xRy\}$

⟨2⟩4. PICK a function F with domain $\bigcup A$ such that $\text{dom } F = \bigcup A$ and $\forall x \in \bigcup A. x \in F(x) \in A$

⟨2⟩5. LET: $H = \{\langle x, y \rangle \mid x \in \text{dom } R, F(x) = \{\langle 0, x \rangle, \langle 1, y \rangle\}\}$

⟨2⟩6. H is a function, $H \subseteq R$, $\text{dom } H = \text{dom } R$

□

Exercise 23

⟨1⟩1. $g[0] = h(0)$

PROOF: Both are equal to \emptyset .

⟨1⟩2. $\forall n \in \omega. g[n] = h(n) \Rightarrow g[n^+] = h(n^+)$

⟨2⟩1. LET: $n \in \omega$

⟨2⟩2. ASSUME: $g[n] = h(n)$

⟨2⟩3. $g[n^+] = h(n^+)$

PROOF:

$$\begin{aligned} h(n^+) &= h(n) \cup \{F(A - h(n))\} \\ &= g[n] \cup \{g(n)\} \\ &= g[n^+] \end{aligned}$$

Exercise 24 Let $\{\kappa_i\}_{i \in I}$ be a family of cardinal numbers. For $i \in I$, let K_i be a set such that $\text{card } K_i = \kappa_i$.

We define $\sum_{i \in I} \kappa_i$ to be $\text{card}\{\langle i, x \rangle : i \in I, x \in K_i\}$

We define $\prod_{i \in I} \kappa_i$ to be $\text{card}\{f : f \text{ is a function, } \text{dom } f = I, \forall i \in I. f(i) \in K_i\}$.

Exercise 25 PROOF:

⟨1⟩1. ASSUME: for a contradiction $\forall n \in \omega. B \not\subseteq S(n)$

⟨1⟩2. PICK a function $b : \omega \rightarrow B$ such that $\forall n \in \omega. b(n) \notin S(n)$

PROOF: By the Axiom of Choice.

⟨1⟩3. LET: $B' = \{b(n) : n \in \omega\}$

⟨1⟩4. B' is infinite.

⟨2⟩1. ASSUME: for a contradiction B' is finite.

⟨2⟩2. There exists N such that $\forall n > N. \exists k \leq N. b(n) = b(k)$

⟨2⟩3. PICK $M > N$ such that $\forall k \leq N. b(k) \in S(M)$

PROOF: For $k \leq N$ there exists n_k such that $b(k) \in S(n_k)$. Take M to be the largest of these numbers and $N + 1$.

⟨2⟩4. $b(M) \in S(M)$

PROOF: Since $b(M) = b(k)$ for some $k \leq N$.

⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

⟨1⟩5. PICK n such that $B' \cap S(n)$ is infinite.

⟨1⟩6. PICK $m > n$ such that $b(m) \in B' \cap S(n)$

PROOF: There must be some m otherwise $B' \cap S(n) \subseteq \{b(0), b(1), \dots, b(n)\}$ would be finite.

⟨1⟩7. $b(m) \in S(m)$

PROOF: Since $S(n) \subseteq S(m)$.

⟨1⟩8. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

6.6 Countable Sets

Exercise 26 PROOF:

⟨1⟩1. PICK a set K of cardinality κ

- ⟨1⟩2. For all $X \in \mathcal{A}$, there exists an injective function $X \rightarrow K$
 ⟨1⟩3. PICK a function F with domain \mathcal{A} such that, for all $X \in \mathcal{A}$, $F(X)$ is an injective function $X \rightarrow K$
 PROOF: By the Axiom of Choice.
 ⟨1⟩4. PICK a function G with domain $\bigcup \mathcal{A}$ such that, for all $x \in \bigcup \mathcal{A}$, we have $x \in G(x) \in \mathcal{A}$
 PROOF: By Exercise 22.
 ⟨1⟩5. Define $f : \bigcup \mathcal{A} \rightarrow \mathcal{A} \times K$ by $f(x) = \langle G(x), F(G(x))(x) \rangle$
 ⟨1⟩6. f is injective.
 ⟨2⟩1. LET: $x, y \in \bigcup \mathcal{A}$
 ⟨2⟩2. ASSUME: $f(x) = f(y)$
 ⟨2⟩3. $G(x) = G(y)$ and $F(G(x))(x) = F(G(y))(y)$
 ⟨2⟩4. $F(G(x))(x) = F(G(x))(y)$
 ⟨2⟩5. $x = y$
 PROOF: Since $F(G(x))$ is injective.

□

Exercise 27

(a) Pick a function $f : A \rightarrow \mathbb{Q}^2$ such that $f(c) \in c$ for all $c \in A$. Then f is an injection, so $A \preccurlyeq \mathbb{Q}^2$ which is countable.

(b) No: the set of all circles with center $(0, 0)$ is an uncountable set of circles no two of which intersect.

(c) Yes. Pick a function $f : C \rightarrow \mathbb{Q}^4$ such that $f(x)$ is a pair of points with rational coordinates, one in each circle of x , for all $x \in C$. Then f is an injection; it is not possible for two points to be in separate circles of two non-intersecting figure-eights. Hence $C \preccurlyeq \mathbb{Q}^4$.

Exercise 28 Let $\mathcal{A} = \{(a, \sqrt{2}) : a < \sqrt{2}\} \cup \{(\sqrt{2}, b) : b > \sqrt{2}\}$. Then every rational is in some member of \mathcal{A} but $\bigcup \mathcal{A} = \mathbb{R} - \{\sqrt{2}\}$.

(Enderton's hint suggests he had a different solution in mind, but I am not sure what it is.)

Exercise 29 For each integer $n \geq 2$, let $B_n = \{x \in A : x > b/n\}$. Then each B_n is finite (B_n cannot have more than $n - 1$ elements because n elements in B_n would have a sum $> b$) and $A = \bigcup_n B_n$. So A is a countable union of finite sets, and therefore countable.

Exercise 30 PROOF:

- ⟨1⟩1. PICK $a \in A$
 ⟨1⟩2. Define $f : Sq(A) \rightarrow \omega \times^\omega A$ by $f(s) = \langle n, g \rangle$, where n is the length of s , and $g(i) = s(i)$ for $i < n$, $g(i) = a$ for $i \geq n$

- $\langle 1 \rangle 3.$ f is injective.
 $\langle 1 \rangle 4.$ $Sq(A) \preceq \omega \times^\omega A$
 $\langle 1 \rangle 5.$ $\text{card } Sq(A) \leq (\text{card } A)^{\aleph_0}$

PROOF:

$$\begin{aligned}
 \text{card } Sq(A) &\leq \aleph_0 \cdot (\text{card } A)^{\aleph_0} && (\langle 1 \rangle 4) \\
 &\leq (\text{card } A)^{\aleph_0} \cdot (\text{card } A)^{\aleph_0} && (\text{Cantor's Theorem}) \\
 &= (\text{card } A)^{\aleph_0 + \aleph_0} && (\text{Theorem 6I}) \\
 &= (\text{card } A)^{\aleph_0}
 \end{aligned}$$

6.7 Arithmetic of Infinite Cardinals

Exercise 31 If f is a one-to-one correspondence between $A \times A$ and A , where $A \subseteq B$, then

$$f \subseteq (A \times A) \times A \subseteq (B \times B) \times B .$$

Also $\emptyset \subseteq (B \times B) \times B$. So we can form \mathcal{H} by applying a Subset Axiom to $\mathcal{P}((B \times B) \times B)$.

Exercise 32 The function that maps x to $\{x\}$ is an injection $A \rightarrow \mathcal{F}A$, so we have $A \approx \mathcal{F}A$.

For the converse, let $F_n = \{X \in \mathcal{F}A : \text{card } X \leq n\}$ for $n \in \omega$. The function that sends $\langle a_1, \dots, a_n \rangle$ to $\{a_1, \dots, a_n\}$ is a surjection $A^n \rightarrow F_n$, so we have

$$\text{card } F_n \leq (\text{card } A)^n = \text{card } A$$

by Lemma 6R. Now, $\mathcal{F}A = \bigcup_n F_n$, so

$$\text{card } \mathcal{F}A \leq \aleph_0 \cdot \text{card } A = \text{card } A$$

by the Absorption Law.

Exercise 33 The function that maps a to the sequence of length 1 containing a is an injection $A \rightarrow Sq(A)$, so $A \preceq Sq(A)$.

For the converse, we have $\text{card}({}^n A) = (\text{card } A)^n = \text{card } A$ for any natural number n

$$\begin{aligned}
 \text{card } Sq(A) &= \text{card}({}^0 A \cup {}^1 A \cup {}^2 A \cup \dots) \\
 &= \aleph_0 \cdot \text{card } A \\
 &= \text{card } A
 \end{aligned}$$

by the Absorption Law.

Exercise 34

$$\begin{aligned}
 2^\lambda &\leq \kappa^\lambda \\
 &\leq (2^\kappa)^\lambda \\
 &= 2^{\kappa \cdot \lambda} \\
 &= 2^\lambda \qquad \qquad \qquad (\text{Absorption Law})
 \end{aligned}$$

Exercise 35 For any infinite set of primes A and natural number n , let $f(A, n) = \prod \{p \in A : p \leq n\}$. Let $P(A) = \{f(A, n) : n \in \omega\}$. Let \mathcal{A} be the set of all sets of the form $P(A)$.

The number of infinite sets of primes is 2^{\aleph_0} (there are 2^{\aleph_0} sets of primes and \aleph_0 finite sets of primes by Exercise 32.)

If $P(A) = P(B)$ then $A = B$. (If $p \in A - B$ then $p \mid f(A, p)$ but p does not divide any member of $P(B)$.) So P is an injection from the set of infinite sets of primes into \mathcal{A} . Hence $\text{card } \mathcal{A} = 2^{\aleph_0}$.

We now prove that, if $A \neq B$, then $P(A) \cap P(B)$ is finite. Let $p \in A - B$. For $n \geq p$ we have $f(A, n) \notin P(B)$ since $p \mid f(A, n)$ but p does not divide any member of B . Hence $A \cap B \subseteq \{f(A, 0), f(A, 1), \dots, f(A, p-1)\}$.

Exercise 36 PROOF:

- $\langle 1 \rangle 1$. For any set A , there exists a permutation of A with no fixed points.
- $\langle 2 \rangle 1$. For every natural number n , there exists a permutation of n with no fixed points.
PROOF: Map i to $i + 1$ if $i + 1 < n$, and map $n - 1$ to 0.
- $\langle 2 \rangle 2$. For every infinite set A , there exists a permutation of A with no fixed points.
- $\langle 3 \rangle 1$. PICK a bijection $f : A \approx A \times 2$
- $\langle 3 \rangle 2$. Define $\pi : A \times 2 \rightarrow A \times 2$ by $\pi(x, 0) = (x, 1)$ and $\pi(x, 1) = (x, 0)$
- $\langle 3 \rangle 3$. $f^{-1} \circ \pi \circ f$ is a permutation of A with no fixed point.
- $\langle 1 \rangle 2$. $\kappa! \leq 2^\kappa$
PROOF: Because the set of permutations of K is a subset of ${}^K K$, where K is a set of cardinality κ .
- $\langle 1 \rangle 3$. $2^\kappa \leq \kappa!$
- $\langle 2 \rangle 1$. PICK a set K of cardinality κ
- $\langle 2 \rangle 2$. LET: $\text{Perm}(K)$ be the set of permutations of K .
- $\langle 2 \rangle 3$. Define $f : \mathcal{P}K \rightarrow \text{Perm}(K)$ as follows. Given $A \subseteq \mathcal{P}K$, pick a permutation π_{K-A} of $K - A$ with no fixed point. Then $f(A) = I_A \cup \pi_{K-A}$
- $\langle 2 \rangle 4$. f is injective
PROOF: The function that maps a permutation to its set of fixed points is a left inverse.
- $\langle 2 \rangle 5$. $2^\kappa \leq \kappa!$

□

Chapter 7

Chapter 7 — Orderings and Ordinals

7.1 Partial Orderings

Exercise 1

(a) No we cannot. Let $A = \mathcal{P}3$ and $B = \omega$. Let $<_A = \subset_3$ and $<_B$ be the usual ordering on ω . Define $f : A \rightarrow B$ by: $f(X) = \text{card } X$. Then $X \subset_2 Y \Rightarrow \text{card } X < \text{card } Y$ but f is not one-to-one because $f(\{0\}) = f(\{1\}) = 1$.

(b) No we cannot. With the same example, we have $f(\{0\}) < f(\{1, 2\})$ but $\{0\} \not\subset \{1, 2\}$.

Exercise 2 We show R^{-1} is transitive. Suppose $xR^{-1}y$ and $yR^{-1}z$. Then zRx and yRx , so zRx because R is transitive. Hence $xR^{-1}z$.

We now show R^{-1} is irreflexive. For any x , we have $\langle x, x \rangle \notin R$, so $\langle x, x \rangle \notin R^{-1}$.

Exercise 3 The proof is by induction on n .

The only linear ordering on \emptyset is \emptyset , which has 0 pairs.

Suppose that, whenever $\text{card } S = n$, then every linear ordering on S has $1/2n(n-1)$ pairs. Let S be a set of cardinality $n+1$. Let $<$ be a linear ordering on S .

Pick an element $a \in S$ and let $T = S - \{a\}$. Then $< \cap (T \times T)$ is a linear ordering on T , hence has $1/2n(n-1)$ pairs. Now, for every $x \in T$, exactly one of $\langle x, a \rangle$ and $\langle a, x \rangle$ is in $<$. Hence $<$ has n pairs that are not in $< \cap (T \times T)$. So

$$\text{card } < = 1/2n(n-1) + n = 1/2n(n+1) .$$

7.2 Well Orderings

Exercise 4 PROOF:

- ⟨1⟩1. R is transitive.
- ⟨2⟩1. ASSUME: mRn and nRp .
- ⟨2⟩2. CASE: $f(m) < f(n)$
PROOF: In this case $f(m) < f(p)$ so mRp .
- ⟨2⟩3. CASE: $f(m) = f(n)$ and $m < n$.
- ⟨3⟩1. CASE: $f(n) < f(p)$
PROOF: In this case $f(m) < f(p)$ so mRp .
- ⟨3⟩2. CASE: $f(n) = f(p)$ and $n < p$.
PROOF: In this case $f(m) = f(p)$ and $m < p$ so mRp .
- ⟨1⟩2. R satisfies trichotomy on P .
- ⟨2⟩1. LET: $m, n \in P$
- ⟨2⟩2. Exactly one of $f(m) < f(n)$, $f(n) < f(m)$, $f(n) = f(m)$ holds.
- ⟨2⟩3. Exactly one of $m < n$, $n < m$, $n = m$ holds.
- ⟨2⟩4. Exactly one of $f(m) < f(n)$, $(f(m) = f(n) \ \& \ m < n)$, $(f(m) = f(n) \ \& \ m = n)$, $(f(m) = f(n) \ \& \ n < m)$, $f(n) < f(m)$ holds.
- ⟨2⟩5. Exactly one of mRn , $m = n$, nRm holds.
- ⟨1⟩3. Every nonempty subset of P has an R -least element.
- ⟨2⟩1. LET: $A \subseteq P$ be nonempty.
- ⟨2⟩2. LET: k be the least element of $f(A)$.
- ⟨2⟩3. LET: n be the least element of $f^{-1}(k) \cap A$.
- ⟨2⟩4. n is the R -least element of A .

□

$\langle P, R \rangle$ resembles Fig. 45 (d).

Exercise 5 PROOF:

- ⟨1⟩1. LET: $x \in A$
- ⟨1⟩2. ASSUME: for a contradiction $f(x) < x$
- ⟨1⟩3. Define $g : \omega \rightarrow A$ by $g(0) = x$ and $g(n^+) = f(g(n))$ for all $n \in \omega$
- ⟨1⟩4. $\forall n \in \omega. g(n^+) < g(n)$
PROOF: By induction on n using ⟨1⟩2 and the hypothesis.
- ⟨1⟩5. Q.E.D.
PROOF: This contradicts Theorem 7B.

□

Exercise 6 PROOF:

- ⟨1⟩1. For all $x \in S$ that is not greatest, there exists $y \in S$ and $q \in \mathbb{Q}$ such that $x < q < y$ and there is no $z \in S$ such that $x < z < y$
- ⟨1⟩2. PICK a function $f : S \rightarrow \mathbb{Q}$ such that $\forall x \in S. x < f(x)$ and, if x is not greatest, then $f(x) < y$ where y is the next element in S .
- ⟨1⟩3. f is injective.
- ⟨1⟩4. $S \preccurlyeq \mathbb{Q}$

□

Exercise 7

(a) We have $F(t) = C \cup \bigcup \text{ran}(F \upharpoonright t)$ for all $t \in \omega$. So:

$$\begin{aligned}
 F(0) &= C \cup \bigcup \text{ran} \emptyset \\
 &= C \\
 F(1) &= C \cup \bigcup \text{ran}(F \upharpoonright 0) \\
 &= C \cup \bigcup \{C\} \\
 &= C \cup C \\
 F(2) &= C \cup \bigcup \{C, C \cup C\} \\
 &= C \cup (C \cup C) \\
 &= C \cup C \cup C
 \end{aligned}$$

We guess:

$$F(n) = C \cup C \cup \dots \cup \underbrace{C}_n \cup \dots \cup C$$

(b) PROOF:

- $\langle 1 \rangle 1.$ LET: $a \in F(n)$
- $\langle 1 \rangle 2.$ $a \in \bigcup \text{ran}(F \upharpoonright n^+)$
- $\langle 1 \rangle 3.$ $a \subseteq \bigcup \text{ran}(F \upharpoonright n^+)$
- $\langle 1 \rangle 4.$ $a \subseteq F(n^+)$

□

(c) PROOF:

- $\langle 1 \rangle 1.$ \overline{C} is a transitive set.
- $\langle 2 \rangle 1.$ LET: $x \in y \in \overline{C}$
- $\langle 2 \rangle 2.$ PICK $n \in \omega$ such that $y \in F(n)$
- $\langle 2 \rangle 3.$ $x \in F(n^+)$

PROOF: By (b).

- $\langle 2 \rangle 4.$ $x \in \overline{C}$
- $\langle 1 \rangle 2.$ $C \subseteq \overline{C}$
- $\langle 2 \rangle 1.$ Since $C = F(0)$

□

7.3 Replacement Axioms

Exercise 8 Let $P(x)$ be a formula not containing B . We prove the statement

$$\forall c \exists B \forall x (x \in B \Leftrightarrow x \in c \ \& \ P(x)) .$$

Let $Q(x, y)$ be the formula $P(x) \wedge y = x$. Now we reason as follows.

Let c be any set. Then we have

$$(\forall x \in c) \forall y_1 \forall y_2 (Q(x, y_1) \& Q(x, y_2) \Rightarrow y_1 = y_2)$$

Then, by a Replacement Axiom, there exists a set B such that

$$\forall y (y \in B \Leftrightarrow (\exists x \in c) Q(x, y)) \text{ .}$$

This is equivalent to $\forall x (x \in B \Leftrightarrow x \in c \& P(x))$.

Exercise 9 Let a and b be sets. Let $P(x, y)$ be the formula $(x = \emptyset \& y = a)$ or $(x = \mathcal{P}\emptyset \& y = b)$. Then we have $(\forall x \in \mathcal{P}\mathcal{P}\emptyset) \forall y_1 \forall y_2 (P(x, y_1) \& P(x, y_2) \Rightarrow y_1 = y_2)$, hence there exists a set c such that

$$\forall y (y \in c \Leftrightarrow (\exists x \in \mathcal{P}\mathcal{P}\emptyset) P(x, y))$$

The members of c are just a and b .

7.4 Epsilon-Images

Exercise 10

(a) Let n be a natural number. Let α be its epsilon-image, and $E : n \rightarrow \alpha$ be as in the definition of epsilon-image.

We prove $\forall x \in n. E(x) = x$ by strong induction on x . Let $x \in n$ and assume $\forall y \in x. E(y) = y$. Then

$$\begin{aligned} E(x) &= \{E(y) : y \in x\} \\ &= \{y : y \in x\} \\ &= x \end{aligned}$$

Hence

$$\begin{aligned} \alpha &= \{E(x) : x \in n\} \\ &= \{x : x \in n\} \\ &= n \end{aligned}$$

(b) Similarly the ϵ -image of ω is ω .

Exercise 11

(a) Let R be the ordering given in the question. Thus xRy iff (x and y are nonnegative and $x < y$) or (x and y are both negative and $y < x$) or (x is nonnegative and y is negative).

PROOF:

- ⟨1⟩1. R is transitive
 - ⟨2⟩1. ASSUME: xRy and yRz
 - ⟨2⟩2. CASE: x and y are nonnegative and $x < y$
 - ⟨3⟩1. CASE: z is nonnegative and $y < z$
 - PROOF: In this case x and z are nonnegative and $x < z$.
 - ⟨3⟩2. CASE: z is negative
 - PROOF: In this case x is nonnegative and z is negative.
 - ⟨2⟩3. CASE: x and y are both negative and $y < x$
 - PROOF: We must have z is negative and $z < y$, hence $z < x$.
 - ⟨2⟩4. CASE: x is nonnegative and y is negative
 - PROOF: We must have z is negative.
- ⟨1⟩2. R satisfies trichotomy on \mathbb{Z}
 - ⟨2⟩1. LET: $x, y \in \mathbb{Z}$
 - ⟨2⟩2. CASE: x and y are nonnegative.
 - PROOF: Exactly one of $x < y$, $x = y$, $y < x$ holds.
 - ⟨2⟩3. CASE: x is nonnegative and y is negative.
 - PROOF: In this case $x < y$.
 - ⟨2⟩4. CASE: x is negative and y is nonnegative.
 - PROOF: In this case $y < x$.
 - ⟨2⟩5. CASE: x and y are negative.
 - PROOF: Exactly one of $x < y$, $x = y$, $y < x$ holds.
- ⟨1⟩3. R is well-founded
 - ⟨2⟩1. LET: $A \subseteq \mathbb{Z}$ be nonempty.
 - ⟨2⟩2. CASE: There exists a nonnegative integer in A .
 - PROOF: Let n be the least nonnegative element of A . Then n is the R -least element of A .
 - ⟨2⟩3. CASE: All elements of A are negative.
 - PROOF: Let n be least such that $-n \in A$. Then $-n$ is the R -least element of A .

□

(b)

$$\begin{aligned}
 E(3) &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
 &= 3 \\
 E(-1) &= \omega \\
 E(-2) &= \omega^+ \\
 \text{ran } E &= \omega \cup \{\omega, \omega^+, \omega^{++}, \dots\}
 \end{aligned}$$

7.5 Isomorphisms

Exercise 12

(a) PROOF:

$\langle 1 \rangle 1$. $<_A$ is irreflexive.

PROOF: For any $x \in A$ we have $f(x) \not<_B f(x)$ so $x \not<_A x$.

$\langle 1 \rangle 2$. $<_A$ is transitive.

PROOF: If $x <_A y$ and $y <_A z$ then $f(x) <_B f(y) <_B f(z)$ hence $f(x) <_B f(z)$ and so $x <_A z$.

(b) For any $x, y \in A$ we have that exactly one of $f(x) <_B f(y)$, $f(x) = f(y)$, $f(y) <_B f(x)$ holds. Hence exactly one of $x <_A y$, $x = y$, $y <_A x$ holds. (Using the fact that $x = y$ iff $f(x) = f(y)$ since f is one-to-one.)

Exercise 13 PROOF:

$\langle 1 \rangle 1$. LET: $\langle A, <_A \rangle$ and $\langle B, <_B \rangle$ be two well-ordered structures.

$\langle 1 \rangle 2$. LET: $f, g : A \rightarrow B$ be isomorphisms.

PROVE: $\forall x \in A. f(x) = g(x)$

$\langle 1 \rangle 3$. LET: $x \in A$

$\langle 1 \rangle 4$. ASSUME: $\forall y < x. f(y) = g(y)$

$\langle 1 \rangle 5$. $f(x)$ is the least element in $B - f[\text{seg } x]$

$\langle 2 \rangle 1$. $f(x) \notin f[\text{seg } x]$

PROOF: Since f is one-to-one.

$\langle 2 \rangle 2$. $\forall b \in B - f[\text{seg } x]. f(x) \leq b$

$\langle 3 \rangle 1$. LET: $b \in B - f[\text{seg } x]$

$\langle 3 \rangle 2$. LET: $a \in A$ be such that $f(a) = b$

PROOF: f is surjective.

$\langle 3 \rangle 3$. $a \notin \text{seg } x$

$\langle 3 \rangle 4$. $x \leq a$

PROOF: By trichotomy

$\langle 3 \rangle 5$. $f(x) \leq b$

$\langle 1 \rangle 6$. $g(x)$ is the least element in $B - g[\text{seg } x]$

PROOF: Similar.

$\langle 1 \rangle 7$. $f[\text{seg } x] = g[\text{seg } x]$

PROOF: By $\langle 1 \rangle 4$

$\langle 1 \rangle 8$. $f(x) = g(x)$

□

Exercise 14 PROOF:

$\langle 1 \rangle 1$. $\forall a, b \in A. a < b \Rightarrow F(a) \subset F(b)$

$\langle 2 \rangle 1$. LET: $a, b \in A$

$\langle 2 \rangle 2$. ASSUME: $a < b$

$\langle 2 \rangle 3$. $F(a) \subseteq F(b)$

PROOF: If $x \leq a$ then $x \leq b$

$\langle 2 \rangle 4. F(a) \neq F(b)$
 PROOF: Since $b \in F(b)$ but $b \notin F(a)$
 $\langle 1 \rangle 2. \forall a, b \in A. F(a) \subset F(b) \Rightarrow a < b$
 PROOF: We cannot have $b < a$ or $b = a$ (as then $F(b) \subset F(a)$ or $F(b) = F(a)$ by $\langle 1 \rangle 1$), so $a < b$ by trichotomy.
 $\langle 1 \rangle 3. F$ is one-to-one
 PROOF: If $F(a) = F(b)$ then we cannot have $a < b$ or $b < a$ by $\langle 1 \rangle 1$, so $a = b$ by trichotomy.
 $\langle 1 \rangle 4. F$ maps A onto $\text{ran } F$
 PROOF: By definition of $\text{ran } F$.
 \square

7.6 Ordinal Numbers

Exercise 15

(a) PROOF:
 $\langle 1 \rangle 1.$ ASSUME: $f : A \rightarrow \text{seg } t$ is an isomorphism
 $\langle 1 \rangle 2.$ Define $g : \omega \rightarrow A$ by recursion:

$$g(0) = t$$

$$g(n^+) = f(g(n)) \quad (n \in \omega)$$
 $\langle 1 \rangle 3. \forall n \in \omega. g(n^+) < g(n)$
 $\langle 2 \rangle 1. g(0^+) < g(0)$
 PROOF: Since $g(0^+) = f(t) \in \text{seg } t$ so $g(0^+) < t = g(0)$.
 $\langle 2 \rangle 2. \forall n \in \omega. (g(n^+) < g(n) \Rightarrow g(n^{++}) < g(n^+))$
 $\langle 3 \rangle 1.$ LET: $n \in \omega$
 $\langle 3 \rangle 2.$ ASSUME: $g(n^+) < g(n)$
 $\langle 3 \rangle 3. f(g(n^+)) < f(g(n))$
 PROOF: Since f is an isomorphism.
 $\langle 3 \rangle 4. g(n^{++}) < g(n^+)$
 $\langle 1 \rangle 4.$ Q.E.D.
 PROOF: This contradicts Theorem 7B.
 \square

(b) If two of them hold then we have a well-ordered set isomorphic with an initial segment, contradicting part (a):
 If $A \cong B$ and $A \cong \text{seg } b$ then $B \cong \text{seg } b$.
 If $A \cong B$ and $\text{seg } a \cong B$ then $A \cong \text{seg } a$.
 Now assume $A \cong \text{seg } b$ and $\text{seg } a \cong B$. Let $f : A \cong \text{seg } b$ and $g : \text{seg } a \cong B$ be isomorphisms. Let $b_0 = f(a)$. Then $f \upharpoonright \text{seg } a : \text{seg } a \cong \text{seg } b_0$ and so $B \cong \text{seg } b_0$.

Exercise 16 Suppose $\alpha \in \beta$. We first prove that $\beta \notin \alpha^+$.

If $\beta \in \alpha^+$ then $\beta \in \alpha$ or $\beta = \alpha$. In either case we have $\alpha \in \alpha$, which is impossible.

So $\beta \notin \alpha^+$. Therefore $\alpha^+ \subseteq \beta$, and so $\alpha^+ \in \beta^+$.

Now, suppose $\alpha \neq \beta$. Then $\alpha \in \beta$ or $\beta \in \alpha$. Hence $\alpha^+ \in \beta^+$ or $\beta^+ \in \alpha^+$, and in either case $\alpha^+ \neq \beta^+$.

Exercise 17 Suppose for a contradiction $\alpha \in \beta$. Then A is isomorphic to $\text{seg}_B b$ for some $b \in B$. Let $f : A \rightarrow \text{seg}_B b$ be an isomorphism.

We have $f \upharpoonright B : B \rightarrow \text{seg}_B b$. Now, define $g : \omega \rightarrow B$ by

$$\begin{aligned} g(0) &= b \\ g(n^+) &= f(g(n)) \end{aligned}$$

Then $g(n^+) < g(n)$ for all $n \in \omega$, contradicting Theorem 7B.

Exercise 18 Suppose first $\bigcup S \in S$. For all $\alpha \in S$ we have $\alpha \subseteq \bigcup S$ and so $\alpha \in \bigcup S$, and so $\bigcup S$ is the greatest element of S .

Suppose now $\bigcup S \notin S$. Suppose for a contradiction $\alpha \in S$ is the greatest element of S . We have $\alpha \subseteq \bigcup S$ (because $\alpha \in S$). Also for all $\beta \in S$ we have $\beta \subseteq \alpha$, hence $\bigcup S \subseteq \alpha$. Thus $\bigcup S = \alpha \in S$, which is a contradiction.

So if $\bigcup S \notin S$ then S has no greatest element. Therefore S cannot be the successor of any ordinal, because α is the greatest element of α^+ for any α .

Exercise 19 By Theorem 7B, every linear ordering on a finite set is a well ordering.

If $<$ and \prec are two linear orderings on the same set A , we cannot have that $(A, <)$ is isomorphic to $(\text{seg } a, \prec)$ for any $a \in A$, because then we would have a finite set bijective with a proper subset of itself.

So by Theorem 7E we must have $\langle A, < \rangle \cong \langle A, \prec \rangle$.

Exercise 20 Let R be a well ordering on the set S . Assume S is infinite; we will prove R^{-1} is not a well-ordering on S .

Define $g : \omega \rightarrow S$ by: $g(n)$ is the least element of $S - g[[n]]$. For each n , we know $S - g[[n]]$ is nonempty because S is infinite.

Then $g[[\omega]]$ is a nonempty subset of S that has no R^{-1} -least element (no R -greatest element), so R^{-1} is not a well ordering on S .

Exercise 21 Let $\mathcal{A} = \{C \in \mathcal{P}A : <^\circ \text{ is a linear ordering on } C\}$.

We prove that, for every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$.

Let $\mathcal{B} \subseteq \mathcal{A}$ be a chain. Let $x, y \in \bigcup \mathcal{B}$. Pick $C, D \in \mathcal{B}$ such that $x \in C$ and $y \in D$. Then either $C \subseteq D$ or $D \subseteq C$; assume without loss of generality $C \subseteq D$. We have $x, y \in D$, and so exactly one of $x < y$, $x = y$, $y < x$ holds. Thus, $<^\circ$ linearly orders $\bigcup \mathcal{B}$, i.e. $\bigcup \mathcal{B} \in \mathcal{A}$.

Hence by Zorn's Lemma \mathcal{A} has a maximal element C , say. Now, by hypothesis, C has an upper bound m . We prove m is maximal in A .

Let $x \in A$ and suppose $m \leq x$. Then $C \cup \{m, x\}$ is linearly ordered by $<^\circ$, and so $C = C \cup \{m, x\}$ by maximality of C . Hence $x \in C$ and so $x \leq m$, hence $x = m$. Thus, m is maximal in A .

7.7 Debts Paid

Exercise 22 Let A be any set. Let \mathcal{A} be the set of all pairs $\langle B, R \rangle$ where $B \subseteq A$ and R is a well ordering on B , and define $<$ on \mathcal{A} by: $\langle B, R \rangle < \langle C, S \rangle$ iff B is an initial segment of C and $R = S \cap B^2$.

It is easy to see that $<$ is a partial ordering on \mathcal{A}

We prove that, if $\mathcal{C} \subseteq \mathcal{A}$ and $<$ is a linear ordering on \mathcal{C} , then \mathcal{C} has an upper bound in \mathcal{A} . Let $B = \bigcup \{C : \exists S. \langle C, S \rangle \in \mathcal{C}\}$ and $R = \bigcup \{S : \exists C. \langle C, S \rangle \in \mathcal{C}\}$. We prove that R well orders B . It is then easy to see that $\langle B, R \rangle$ is an upper bound for \mathcal{C} in \mathcal{A} .

PROOF:

- $\langle 1 \rangle 1.$ R is transitive.
 - $\langle 2 \rangle 1.$ ASSUME: xRy and yRz
 - $\langle 2 \rangle 2.$ PICK $\langle C, S \rangle, \langle D, T \rangle \in \mathcal{C}$ such that xSy and yTz
 - $\langle 2 \rangle 3.$ $\langle C, S \rangle \leq \langle D, T \rangle$ or $\langle D, T \rangle \leq \langle C, S \rangle$
 - $\langle 2 \rangle 4.$ ASSUME: w.l.o.g. $\langle C, S \rangle \leq \langle D, T \rangle$
 - $\langle 2 \rangle 5.$ xTy and yTz
 - $\langle 2 \rangle 6.$ xTz
 - $\langle 2 \rangle 7.$ xRz
- $\langle 1 \rangle 2.$ R is irreflexive.
 - $\langle 2 \rangle 1.$ ASSUME: for a contradiction xRx
 - $\langle 2 \rangle 2.$ PICK $\langle C, S \rangle \in \mathcal{C}$ such that xSx
 - $\langle 2 \rangle 3.$ This is a contradiction.
- $\langle 1 \rangle 3.$ R satisfies trichotomy.
 - $\langle 2 \rangle 1.$ LET: $x, y \in B$
 - $\langle 2 \rangle 2.$ PICK $\langle C, S \rangle, \langle D, T \rangle \in \mathcal{C}$ such that $x \in C$ and $y \in D$
 - $\langle 2 \rangle 3.$ ASSUME: w.l.o.g. $\langle C, S \rangle \leq \langle D, T \rangle$
 - $\langle 2 \rangle 4.$ $x, y \in D$
 - $\langle 2 \rangle 5.$ xTy or yTx
 - $\langle 2 \rangle 6.$ xRy or yRx
- $\langle 1 \rangle 4.$ Every non-empty subset of B has an R -least element.
 - $\langle 2 \rangle 1.$ LET: $C \subseteq B$ be nonempty
 - $\langle 2 \rangle 2.$ PICK $c \in C$
 - $\langle 2 \rangle 3.$ PICK $\langle D, T \rangle \in \mathcal{C}$ such that $c \in D$
 - $\langle 2 \rangle 4.$ LET: x be the T -least element of $C \cap D$
 PROVE: x is R -least in C
 - $\langle 2 \rangle 5.$ LET: $y \in C$
 - $\langle 2 \rangle 6.$ PICK $\langle E, U \rangle \in \mathcal{C}$ such that $y \in E$
 - $\langle 2 \rangle 7.$ $\langle D, T \rangle \leq \langle E, U \rangle$ or $\langle E, U \rangle \leq \langle D, T \rangle$
 - $\langle 2 \rangle 8.$ CASE: $\langle D, T \rangle \leq \langle E, U \rangle$
 - $\langle 3 \rangle 1.$ xUy or $x = y$

PROOF:

- $\langle 4 \rangle 1.$ ASSUME: for a contradiction yUx
- $\langle 4 \rangle 2.$ $y \in D$ and yTx

PROOF: Since D is an initial segment of E and $T = U \cap D^2$

⟨4⟩3. Q.E.D.

PROOF: This contradicts the T -minimality of x .

⟨2⟩9. CASE: $\langle E, U \rangle \leq \langle D, T \rangle$

PROOF: xTy or $x = y$, so xRy or $x = y$.

Hence by Exercise 21 there is a maximal element $\langle B, R \rangle$ in \mathcal{A} . We must have $B = A$; for if $a \in A - B$ then $\langle B \cup \{a\}, R \cup \{\langle x, a \rangle : x \in B\} \rangle$ would be a larger element. Hence R is a well ordering on A .

Exercise 23

(i) We must show that α is an initial ordinal. So let $\beta \in \alpha$. Then $\beta \prec A$ but $\alpha \not\prec A$. Hence $\alpha \not\approx \beta$.

(ii) We know that $\alpha \not\prec A$, so $\alpha \not\leq \text{card } A$.

(iii) Let κ be any cardinal greater than $\text{card } A$. Then κ is not dominated by A , so $\kappa \notin \alpha$, and so $\alpha \subseteq \kappa$.

Exercise 24 The cardinal number of $\mathcal{P}\alpha$ is larger than α (both as a cardinal and as an ordinal).

Exercise 25 Suppose there exists an ordinal α such that $\neg\phi(\alpha)$. Let α_0 be the least such ordinal. Then we have $\forall x \in \alpha_0. \phi(x)$ but $\neg\phi(\alpha_0)$. This contradicts the hypothesis.

7.8 Rank

Exercise 26 The proof is by transfinite induction on α . Suppose that α is an ordinal and, for all $\beta \in \alpha$, we have β is grounded and $\text{rank } \beta = \beta$. Then by Theorem 7V(b) we have that α is grounded and

$$\begin{aligned} \text{rank } \alpha &= \bigcup \{(\text{rank } \beta)^+ \mid \beta \in \alpha\} \\ &= \bigcup \{\beta^+ \mid \beta \in \alpha\} \quad (\text{induction hypothesis}) \end{aligned}$$

So we must show that $\bigcup \{\beta^+ \mid \beta \in \alpha\} = \alpha$.

If $\beta \in \alpha$ then $\beta^+ \subseteq \alpha$ so $\beta^+ \subseteq \alpha$. This shows that $\bigcup \{\beta^+ \mid \beta \in \alpha\} \subseteq \alpha$.

If $\beta \in \alpha$ then $\beta \in \beta^+$ so $\beta \in \bigcup \{\beta^+ \mid \beta \in \alpha\}$. This shows that $\alpha \subseteq \bigcup \{\beta^+ \mid \beta \in \alpha\}$

Exercise 27 PROOF:

⟨1⟩1. For natural numbers m and n , we have $\text{rank } \langle m, n \rangle = \max(m, n)^{+++}$

PROOF:

$$\begin{aligned}
\text{rank}\{\{m\}, \{m, n\}\} &= (\text{rank}\{m\})^+ \cup (\text{rank}\{m, n\})^+ \\
&= (\text{rank } m)^{++} \cup ((\text{rank } m)^+ \cup (\text{rank } n)^+)^+ \\
&= m^{++} \cup (m^+ \cup n^+)^+ \quad (\text{Exercise 26}) \\
&= \max(m, n)^{++}
\end{aligned}$$

$\langle 1 \rangle 2$. For any integer a we have $\text{rank } a = \omega$

PROOF: For any natural numbers m and n , we have

$$\begin{aligned}
\text{rank}[\langle m, n \rangle] &= \bigcup \{(\text{rank}\langle p, q \rangle)^+ : m + q = n + p\} \\
&= \bigcup \{\max(p, q)^+ : m + q = n + p\} \\
&= \omega
\end{aligned}$$

since for any natural number $p > m$ there exists q such that $m + q = n + p$.

$\langle 1 \rangle 3$. For any integers a and b we have $\text{rank}\langle a, b \rangle = \omega^{++}$

PROOF:

$$\begin{aligned}
\text{rank}\{\{a\}, \{a, b\}\} &= (\text{rank}\{a\})^+ \cup (\text{rank}\{a, b\})^+ \\
&= (\text{rank } a)^{++} \cup ((\text{rank } a)^+ \cup (\text{rank } b)^+)^+ \\
&= \omega^{++} \cup (\omega^+ \cup \omega^+)^+ \\
&= \omega^{++}
\end{aligned}$$

$\langle 1 \rangle 4$. For any rational q we have $\text{rank } q = \omega^{+++}$

PROOF: Since every element of q has rank ω^{++}

$\langle 1 \rangle 5$. For any real number r we have $\text{rank } r = \omega^{++++}$

PROOF: Since every element of r has rank ω^{+++} .

$\langle 1 \rangle 6$. $\text{rank } \mathbb{R} = \omega^{+++++}$

□

Exercise 28 If $X \in V_\alpha$ then $X \subseteq V_\beta$ for some $\beta \in \alpha$. Hence $\text{rank } X \subseteq \beta$ and so $\text{rank } X \in \alpha$.

Conversely, if $\text{rank } X \in \alpha$ then $X \in V_{(\text{rank } X)^+} \subseteq V_\alpha$.

Exercise 29 Direct proofs:

For any set a , there exists $m \in \{a\}$ such that $m \cap \{a\} = \emptyset$. This m must be the set a , so $a \cap \{a\} = \emptyset$, meaning $a \notin a$.

For any sets a and b , there exists $m \in \{a, b\}$ such that $m \cap \{a, b\} = \emptyset$. Now, m is either a or b . If $m = a$ then $a \cap \{a, b\} = \emptyset$ so $b \notin a$. And if $m = b$ then $b \cap \{a, b\} = \emptyset$ so $a \notin b$.

Consequences of part (c):

Assume $a \in a$. Define $f : \omega \rightarrow \{a\}$ by $f(n) = a$ for all $n \in \omega$. Then $f(n^+) \in f(n)$ for all n , contradicting (c).

Assume now $a \in b$ and $b \in a$. Define $f : \omega \rightarrow \{a, b\}$ by $f(n) = a$ if n is even, $f(n) = b$ if n is odd. Then $f(n^+) \in f(n)$ for all n , contradicting (c).

Exercise 30

$$\begin{aligned}
 \text{rank}\{a, b\} &= (\text{rank } a)^+ \cup (\text{rank } b)^+ && (\text{Theorem 7V(b)}) \\
 &= \max((\text{rank } a)^+, (\text{rank } b)^+) \\
 &= \max(\text{rank } a, \text{rank } b)^+
 \end{aligned}$$

We have

$$\begin{aligned}
 a &\subseteq V_{\text{rank } a} \\
 \therefore \mathcal{P}a &\subseteq \mathcal{P}V_{\text{rank } a} \\
 &= V_{(\text{rank } a)^+} \\
 \therefore \text{rank } \mathcal{P}a &\subseteq (\text{rank } a)^+ \\
 a &\in \mathcal{P}a \\
 \therefore \text{rank } a &\in \text{rank } \mathcal{P}a \\
 \therefore \text{rank } \mathcal{P}a &= (\text{rank } a)^+
 \end{aligned}$$

Now, for all $x \in \bigcup a$, there exists y such that $x \in y \in a$. Hence

$$\begin{aligned}
 \text{rank } x &\in \text{rank } y \in \text{rank } a \text{ .} \\
 \therefore (\text{rank } x)^+ &\in \text{rank } a \text{ .}
 \end{aligned}$$

So $\text{rank } a$ is an upper bound for $\{(\text{rank } x)^+ : x \in \bigcup a\}$, and so

$$\text{rank } \bigcup a \subseteq \text{rank } a \text{ .}$$

Exercise 31

(a) If $A \approx B$ and nothing of rank less than $\text{rank } B$ is equinumerous to B , then $\text{rank } B \in \text{rank } A$, and so $B \in V_{(\text{rank } A)^+}$. So we can construct the set $\text{kard } A$ by applying a Subset Axiom to $V_{(\text{rank } A)^+}$.

(b) There exists a set of rank $\text{rank } A$ that is equinumerous with A (namely A !). Let μ be the least ordinal $\leq \text{rank } A$ such that there exists a set of rank μ that is equinumerous with A . Pick a set B of rank μ such that $B \approx A$. Then $B \in \text{kard } A$.

(c) Suppose $\text{kard } A = \text{kard } B$. Pick $C \in \text{kard } A$. Then $C \approx A$ and $C \approx B$, so $A \approx B$.

Conversely, suppose $A \approx B$. Then we have $(A \approx C \text{ and nothing of rank less than } \text{rank } C \text{ is equinumerous with } C) \text{ iff } (B \approx C \text{ and nothing of rank less than } \text{rank } C \text{ is equinumerous with } C)$, i.e. $\text{kard } A = \text{kard } B$.

Exercise 32 Similar to Exercise 31.

Exercise 33 Suppose for a contradiction D is not a subset of B . Then $D - B$ is nonempty. So by the Regularity Axiom, there exists $m \in D - B$ such that $m \cap (D - B) = \emptyset$. Now, for all $x \in m$, we have $x \in D$ (since D is a transitive set) and $x \notin D - B$, so we must have $x \in B$; that is, $m \subseteq B$. But then $m \in B$, which is a contradiction.

Exercise 34 PROOF:

$\langle 1 \rangle 1$. ASSUME: $\{x, \{x, y\}\} = \{u, \{u, v\}\}$

$\langle 1 \rangle 2$. $x = u$ or $x = \{u, v\}$

$\langle 1 \rangle 3$. $u = x$ or $u = \{x, y\}$

$\langle 1 \rangle 4$. $x \neq \{u, v\}$

$\langle 2 \rangle 1$. ASSUME: for a contradiction $x = \{u, v\}$

$\langle 2 \rangle 2$. $u = x$ or $u = \{x, y\}$

$\langle 2 \rangle 3$. CASE: $u = x$

PROOF: In this case $x = u \in \{u, v\} = x$ contradicting Theorem 7X(a).

$\langle 2 \rangle 4$. CASE: $u = \{x, y\}$

PROOF: In this case $u \in x$ and $x \in u$ contradicting Theorem 7X(b).

$\langle 1 \rangle 5$. $x = u$

$\langle 1 \rangle 6$. $\{x, y\} = \{u, v\}$

PROOF: We cannot have $\{x, y\} = u$ because then we would have $x \in x$ contradicting Theorem 7X(a).

$\langle 1 \rangle 7$. $y = u$ or $y = v$

$\langle 1 \rangle 8$. $v = x$ or $v = y$

$\langle 1 \rangle 9$. If $y = u$ and $v = x$ then $y = v$

$\langle 1 \rangle 10$. $y = v$

PROOF: Checking all the cases in $\langle 1 \rangle 7$ and $\langle 1 \rangle 8$.

□

Exercise 35 Suppose $a^+ = b^+$. Then $a \in b^+$ so $a = b$ or $a \in b$. Likewise $b \in a^+$ so $b = a$ or $b \in a$. We cannot have both $a \in b$ and $b \in a$ (Theorem 7X(b)), so we must have $a = b$.

Exercise 36 We have that $V_{\text{rank } S}$ is a transitive set and $S \subseteq V_{\text{rank } S}$, so $TC S \subseteq V_{\text{rank } S}$. Thus, $\text{rank}(TC S) \leq \text{rank } S$.

We also have $S \subseteq TC S$ so $\text{rank } S \leq \text{rank}(TC S)$. Thus, $\text{rank}(TC S) = \text{rank } S$.

Exercise 37 If α is an ordinal then it is a transitive set and, for any distinct $x, y \in \alpha$, we have $x \in y$ or $y \in x$ (Theorem 7M).

Conversely, let α be a transitive set such that, for any distinct $x, y \in \alpha$, we have $x \in y$ or $y \in x$. We will prove that α is well ordered by epsilon. It will follow by Theorem 7L that α is an ordinal.

PROOF:

$\langle 1 \rangle 1$. ϵ_α is transitive.

$\langle 2 \rangle 1.$ LET: $x, y, z \in \alpha$ with $x \in y$ and $y \in z$
 $\langle 2 \rangle 2.$ $x \neq z$
 PROOF: Otherwise we would have $x \in y \in x$ contradicting the Axiom of Regularity.
 $\langle 2 \rangle 3.$ $x \in z$ or $z \in x$
 $\langle 2 \rangle 4.$ $z \notin x$
 PROOF: By the Axiom of Regularity we cannot have $x \in y \in z \in x$.
 $\langle 2 \rangle 5.$ $x \in z$
 $\langle 1 \rangle 2.$ ϵ_α is irreflexive.
 PROOF: By the Axiom of Regularity.
 $\langle 1 \rangle 3.$ For any $x, y \in \alpha$ we have $x \in y$ or $x = y$ or $y \in x$.
 PROOF: By assumption.
 $\langle 1 \rangle 4.$ Any nonempty subset of α has an ϵ_α -least element.
 $\langle 2 \rangle 1.$ LET: $A \subseteq \alpha$ be nonempty.
 $\langle 2 \rangle 2.$ PICK $m \in A$ such that $m \cap A = \emptyset$
 $\langle 2 \rangle 3.$ For all $x \in A$ we have $m \subseteq x$
 PROOF: Since $x \notin m$.
 \square

Exercise 38 Let λ be a limit ordinal. We have $\bigcup \lambda \subseteq \lambda$ because λ is a transitive set. Conversely, for all $\alpha \in \lambda$ we have $\alpha \in \alpha^+ \in \lambda$ so $\alpha \in \bigcup \lambda$.

Exercise 39 An ordinal number is a transitive set of ordinals, hence a transitive set of transitive sets.

Conversely, let α be a transitive set of transitive sets. We prove that α is a set of ordinals. The result will follow by Corollary 7N (a).

So suppose for a contradiction that not every element in α is an ordinal. Let $A = \{x \in \alpha : x \text{ is not an ordinal}\}$. Then A is nonempty. Pick $m \in A$ such that $m \cap A = \emptyset$. Then m is a transitive set of ordinals, hence an ordinal. This is a contradiction.

Chapter 8

Chapter 8 — Ordinals and Order Types

8.1 Alephs

Exercise 1 Let $\gamma(f, y)$ be the formula:

Either

1. f is a function with domain 0 and $y = 5$; or
2. f is a function whose domain is a successor ordinal α^+ and $y = f(\alpha)^+$; or
3. f is a function whose domain is a limit ordinal λ and $y = \bigcup(\text{ran } f)$; or
4. none of the above and $y = \emptyset$.

By transfinite recursion, construct a formula $\phi(u, v)$ such that:

- for every ordinal α there exists a unique y such that $\phi(\alpha, y)$;
- whenever f is a function whose domain is an ordinal α and $\phi(\beta, f(\beta))$ for all $\beta \in \alpha$, then we have $\phi(\alpha, y)$ iff $\gamma(f, y)$ for all y .

For α an ordinal, let t_α be the unique set such that $\phi(\alpha, t_\alpha)$.

Exercise 2 We prove that $\forall \alpha \in \omega. t_\alpha = 5 + \alpha$ by induction on α . We have $t_0 = 5$ and if $t_\alpha = 5 + \alpha$ then $t_{\alpha^+} = (5 + \alpha)^+ = 5 + \alpha^+$.

We now prove that if $\omega \subseteq \alpha$ then $t_\alpha = \alpha$ by transfinite induction on α . We have

$$t_\omega = \bigcup_{n \in \omega} (5 + n) = \omega$$

If $\omega \subseteq \alpha$ and $t_\alpha = \alpha$ then $t_{\alpha^+} = \alpha^+$.

If λ is a limit ordinal and $t_\alpha = \alpha$ for all α with $\omega \subseteq \alpha \in \lambda$ then

$$\begin{aligned} t_\lambda &= \bigcup_{\alpha \in \lambda} t_\alpha \\ &= \bigcup_{\omega \subseteq \alpha \in \lambda} t_\alpha \\ &= \bigcup_{\omega \subseteq \alpha \in \lambda} \alpha \\ &= \lambda \end{aligned}$$

Exercise 3 If $\beta \in \gamma$ then $t_\beta \in t_\gamma$ by the definition of monotonicity.

Conversely, suppose $t_\beta \in t_\gamma$. Then $t_\beta \neq t_\gamma$ and $t_\gamma \notin t_\beta$, so $\beta \neq \gamma$ and $\gamma \notin \beta$. Hence $\beta \in \gamma$ by trichotomy.

Now suppose $t_\beta = t_\gamma$. Then $t_\beta \notin t_\gamma$ and $t_\gamma \notin t_\beta$, hence $\beta \notin \gamma$ and $\gamma \notin \beta$, and therefore $\beta = \gamma$ by trichotomy.

Exercise 4 We have $t_\lambda \neq 0$ because $t_0 \in t_\lambda$.

Now, suppose for a contradiction $t_\lambda = \alpha^+$ for some α . Then we have $\alpha \in t_\lambda = \bigcup_{\beta \in \lambda} t_\beta$. Hence $\alpha \in t_\beta$ for some $\beta \in \lambda$. Therefore,

$$\begin{aligned} \alpha^+ &\subseteq t_\beta \\ \therefore \alpha^+ &\in t_{\beta^+} \\ \therefore \alpha^{++} &\subseteq t_{\beta^+} \\ \therefore \alpha^{++} &\subseteq t_\lambda \end{aligned}$$

which is a contradiction.

Exercise 5 The proof is by transfinite induction on β .

We have $0 \subseteq t_0$.

If $\beta \subseteq t_\beta$ then $\beta \in t_{\beta^+}$, hence $\beta^+ \subseteq t_{\beta^+}$.

If λ is a limit ordinal and $\forall \beta \in \lambda. \beta \subseteq t_\beta$ then

$$\begin{aligned} t_\lambda &= \sup_{\beta \in \lambda} t_\beta \\ &\supseteq \sup_{\beta \in \lambda} \beta \\ &= \lambda \end{aligned}$$

Exercise 6 The class is closed by Theorem Schema 8E. It is unbounded because, for any ordinal α , we have $\alpha \in \alpha^+ \subseteq t_{\alpha^+}$ by Exercise 5.

Exercise 7 Let γ be any fixed point of t with $\beta \in \gamma$. Then we have $f(0) \in \gamma$; and, if $f(n) \in \gamma$, then

$$\begin{aligned} f(n^+) &= t_{f(n)} \\ &\subseteq t_\gamma \\ &= \gamma \end{aligned}$$

Hence by induction $f(n) \in \gamma$ for all n , and so $\lambda \in \gamma$. Thus λ is the least fixed point of t .

Exercise 8 Monotonicity holds by the analogue of Theorem 8A (see the second Example on page 216).

For continuity, let λ be a limit ordinal. We must prove that $\bigcup_{\beta \in \lambda} t'_\beta$ is the least fixed point of t different from t'_β for all $\beta \in \lambda$.

PROOF:

- $\langle 1 \rangle 1$. LET: $\mu = \bigcup_{\beta \in \lambda} t'_\beta$
 $\langle 1 \rangle 2$. μ is a fixed point of t

PROOF:

$$\begin{aligned} t_\mu &= \bigcup_{\beta \in \lambda} t_{t'_\beta} && \text{(Theorem Schema 8E)} \\ &= \bigcup_{\beta \in \lambda} t'_\beta && (t'_\beta \text{ is a fixed point of } t) \\ &= \mu \end{aligned}$$

- $\langle 1 \rangle 3$. $\forall \beta \in \lambda. \mu \neq t'_\beta$

PROOF: Because $t'_\beta \in t'_{\beta+} \subseteq \mu$.

- $\langle 1 \rangle 4$. If γ is a fixed point of t and $\forall \beta \in \lambda. \gamma \neq t'_\beta$ then $\mu \subseteq \gamma$

PROOF: We have $\forall \beta \in \lambda. t'_\beta \in \gamma$ hence $\mu \subseteq \gamma$.

□

8.2 Isomorphism Types

Exercise 9 Pick $a \in A$. For any set $x \notin A$, let $A' = A - \{a\} \cup \{x\}$, and let R' be the relation formed by replacing any pair $\langle a, y \rangle$ with $\langle x, y \rangle$, any pair $\langle y, a \rangle$ with $\langle y, x \rangle$, and $\langle a, a \rangle$ with $\langle x, x \rangle$ if aRa . Then $\langle A, R \rangle \cong \langle A', R' \rangle$ and $\text{rank} \langle A', R' \rangle > \text{rank } x$.

Hence for every ordinal α there is a structure isomorphic to $\langle A, R \rangle$ with $\text{rank} > \alpha$. Thus the class of structures isomorphic to $\langle A, R \rangle$ is not a set, because the ranks of its members are unbounded.

Exercise 10

- (a) The only set equinumerous with 0 is 0, so $\text{kard } 0 = \{0\}$.

We have $V_1 = \{\emptyset\} = \{0\}$ and $V_2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$. So 1 is the only set of rank 2 equinumerous with 1, and no set of rank < 2 is equinumerous with 1. Hence $\text{kard } 1 = \{1\}$.

We have $V_3 = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} = \{0, 1, \{1\}, 2\}$. So 2 is the only set of rank 3 equinumerous with 2, and no set of rank < 3 is equinumerous with 2. Thus $\text{kard } 2 = \{2\}$.

(b) $\text{kard } 3$ is the set of all sets of rank 4 that are equinumerous with 3, i.e. the set of all subsets of V_3 of cardinality 3. So

$$\text{kard } 3 = \{\{0, 1, \{1\}\}, 3, \{0, \{1\}, 2\}, \{1, \{1\}, 2\}\}.$$

8.3 Arithmetic of Order Types

Exercise 11 Pick structures $\langle A, R \rangle$ and $\langle B, S \rangle$ of order type ρ and σ respectively. Define R' on $A \times \{0\}$ by: $\langle a, 0 \rangle R' \langle a', 0 \rangle$ iff aRa' . Define S' on $B \times \{1\}$ by: $\langle b, 1 \rangle S' \langle b', 1 \rangle$ iff bSb' . Then $\langle A \times \{0\}, R' \rangle$ has order type ρ , $\langle B \times \{1\}, S' \rangle$ has order type σ , and $(A \times \{0\}) \cap (B \times \{1\}) = \emptyset$.

Exercise 12 Since we have:

$$\begin{aligned} \langle 0, a \rangle <_L \langle 0, a' \rangle &\text{ iff } aRa' \\ \langle 1, b \rangle <_L \langle 1, b' \rangle &\text{ iff } bSb' \\ \langle 0, a \rangle <_L \langle 1, b \rangle &\text{ for all } a \in A \text{ and } b \in B \\ \langle 1, b \rangle \not<_L \langle 0, a \rangle &\text{ for all } a \in A \text{ and } b \in B \end{aligned}$$

Exercise 13 If f is an isomorphism between $\langle A, R \rangle$ and $\langle A', R' \rangle$, and g is an isomorphism between $\langle B, S \rangle$ and $\langle B', S' \rangle$, and $A \cap B = A' \cap B' = \emptyset$, then $f \cup g$ is an isomorphism between $\langle A \cup B, R \oplus S \rangle$ and $\langle A' \cup B', R' \oplus S' \rangle$.

If f is an isomorphism between $\langle A, R \rangle$ and $\langle A', R' \rangle$, and g is an isomorphism between $\langle B, S \rangle$ and $\langle B', S' \rangle$, then the function $h : A \times B \rightarrow A' \times B'$ defined by

$$h(\langle a, b \rangle) = \langle f(a), g(b) \rangle$$

is an isomorphism between $\langle A \times B, R * S \rangle$ and $\langle A' \times B', R' * S' \rangle$.

Exercise 14 Let $\langle A, R \rangle$ be a structure of order type ρ and $\langle B, S \rangle$ a structure of order type σ . Then $A \times B \approx \emptyset$ so $A \times B = \emptyset$. Therefore $A = \emptyset$ or $B = \emptyset$, and so $\rho = 0$ or $\sigma = 0$.

Exercise 15

$$\begin{aligned} (\bar{\omega} + \bar{1}) \cdot \bar{2} &= \bar{\omega} + \bar{1} + \bar{\omega} + \bar{1} \\ &= \bar{\omega} + \bar{\omega} + \bar{1} \\ &\neq \bar{\omega} + \bar{\omega} + \bar{2} \\ &= (\bar{\omega} \cdot \bar{2}) + (\bar{1} \cdot \bar{2}) \end{aligned}$$

Exercise 16 Let $\langle A, R \rangle$ be a structure of order type ρ .

We have $\langle A \cup \emptyset, R \oplus \emptyset \rangle = \langle \emptyset \cup A, \emptyset \oplus R \rangle = \langle A, R \rangle$ so $\rho + \bar{0} = \bar{0} + \rho = \rho$.

Now, $\langle 1, \emptyset \rangle$ is a structure of order type $\bar{1}$. We have $\langle A \times 1, R * \emptyset \rangle = \langle 1 \times A, \emptyset * R \rangle = \langle A, R \rangle$ so $\rho \cdot \bar{1} = \bar{1} \cdot \rho = \rho$.

We have $\langle A \times \emptyset, R * \emptyset \rangle = \langle \emptyset \times A, \emptyset * R \rangle = \langle \emptyset, \emptyset \rangle$.

Exercise 17 Pick an enumeration $A = \{a_0, a_1, \dots\}$ of A . Define $f : A \rightarrow \mathbb{Q}$ by recursion as follows:

Let $f(a_0) = 0$.

Given $f(a_0), f(a_1), \dots, f(a_n)$, we have the following three possibilities:

- a_{n+1} is smaller than all of a_0, \dots, a_n . In this case, let a_k be the minimum of a_0, \dots, a_n , and set $f(a_{n+1}) = f(a_k) - 1$
- a_{n+1} is larger than all of a_0, \dots, a_n . In this case, let a_k be the maximum of a_0, \dots, a_n , and set $f(a_{n+1}) = f(a_k) + 1$
- Otherwise, let a_i be the largest element of a_0, \dots, a_n such that $a_i < a_{n+1}$, and a_j the smallest element such that $a_{n+1} < a_j$. Set $f(a_{n+1}) = (f(a_i) + f(a_j))/2$.

Then we have $a_i < a_j$ iff $f(a_i) < f(a_j)$ for all i, j . Hence f is an isomorphism between $\langle A, R \rangle$ and $\langle f[A], <^\circ \rangle$.

Exercise 18 Pick enumerations $\{a_0, a_1, \dots\}$ of A and $\{b_0, b_1, \dots\}$ of B .

Define isomorphisms $F_n \subseteq A \times B$ by recursion on n in such a way that each F_n is an isomorphism between a subset of A_n of A and a subset B_n of B such that:

- For all n we have $a_n \in A_{2n}$
- For all n we have $b_n \in B_{2n+1}$

as follows.

$$F_0 = \{\langle a_0, b_0 \rangle\}$$

Given F_{2n} , if $b_n \in B_{2n}$ then $F_{2n+1} = F_{2n}$. Otherwise:

- if b_n is greater than every element in B_{2n} , then let m be least such that a_m is larger than every element of A_{2n} (here we use the fact that A has no largest element) and set $F_{2n+1} = F_{2n} \cup \{\langle a_m, b_n \rangle\}$
- if b_n is smaller than every element in B_{2n} , then let m be least such that a_m is smaller than every element of A_{2n} (here we use the fact that A has no smallest element) and set $F_{2n+1} = F_{2n} \cup \{\langle a_m, b_n \rangle\}$
- otherwise let b be the greatest element in B_{2n} such that $b < b_n$, and b' the least element in B_{2n} such that $b_n < b'$. Let $a = F_{2n}^{-1}(b)$ and $a' = F_{2n}^{-1}(b')$. Let m be least such that $a < a_m < a'$ (here we use the fact that A is dense). Let $F_{2n+1} = F_{2n} \cup \{\langle a_m, b_n \rangle\}$

In every case we have that F_{2n+1} is an isomorphism between a subset of A and a subset of B that contains b_n .

Similarly, given F_{2n+1} , we can define F_{2n+2} to be an isomorphism between a subset of A that contains a_n and a subset of B .

Now, let $f = \bigcup_n F_n$. Then f is an isomorphism between $\langle A, R \rangle$ and $\langle B, S \rangle$.

Exercise 19 This holds because the concatenation of \mathbb{Q} with itself, and the lexicographic ordering on \mathbb{Q}^2 , are dense linear orderings on countable nonempty sets.

8.4 Ordinal Arithmetic

Exercise 20 PROOF:

- ⟨1⟩1. For every ordinal α , there exists an ordinal λ that is either a limit ordinal or 0 and a natural number n such that $\alpha = \lambda + n$
- ⟨2⟩1. $0 = 0 + 0$
- ⟨2⟩2. If $\alpha = \lambda + n$ then $\alpha^+ = \lambda + n^+$
- ⟨2⟩3. For λ a limit ordinal we have $\lambda = \lambda + 0$
- ⟨1⟩2. If λ, μ are either limit ordinals or 0, and $m, n \in \omega$, and $\lambda + m = \mu + n$, then $\lambda = \mu$ and $m = n$
- ⟨2⟩1. LET: $P(m)$ be the property: for all λ, μ and $n \in \omega$, if λ and μ are either limit ordinals or 0 and $\lambda + m = \mu + n$, then $\lambda = \mu$ and $m = n$
- ⟨2⟩2. $P(0)$
 - ⟨3⟩1. ASSUME: $\lambda + 0 = \mu + n$
 - ⟨3⟩2. $n = 0$
 - PROOF: Otherwise $\lambda = \mu + n$ would be a successor ordinal.
 - ⟨3⟩3. $\lambda = \mu$
- ⟨2⟩3. $\forall m \in \omega. P(m) \Rightarrow P(m^+)$
 - ⟨3⟩1. LET: $m \in \omega$
 - ⟨3⟩2. ASSUME: $P(m)$
 - ⟨3⟩3. ASSUME: $\lambda + m^+ = \mu + n$
 - ⟨3⟩4. $n \neq 0$
 - PROOF: Otherwise $\mu = \lambda + m^+$ is a successor ordinal.
 - ⟨3⟩5. PICK p such that $n = p^+$
 - ⟨3⟩6. $(\lambda + m)^+ = (\mu + p)^+$
 - ⟨3⟩7. $\lambda + m = \mu + p$
 - ⟨3⟩8. $\lambda = \mu$ and $m = p$
 - PROOF: By ⟨3⟩2
 - ⟨3⟩9. $m^+ = n$

□

Exercise 21 1 is the least integer in the ordering, followed by all the integers with exactly one prime factor, then all the integers with two prime factors, etc. So the ordinal is $1 + \omega \cdot \omega = \omega^2$.

Exercise 22

(a) If $\beta \in \gamma$ then $\beta + 0 = \beta \in \gamma = \gamma + 0$.

If $\beta + \alpha \in \gamma + \alpha$ then $\beta + \alpha^+ = (\beta + \alpha)^+ \in (\gamma + \alpha)^+ = \gamma + \alpha^+$.

For λ a limit ordinal, if $\forall \alpha \in \lambda. \beta + \alpha \in \gamma + \alpha$, then we have $\beta + \lambda = \sup_{\alpha \in \lambda} (\beta + \alpha) \in \sup_{\alpha \in \lambda} (\gamma + \alpha) = \gamma + \lambda$.

(b) We have $\beta \cdot 0 = 0 = \gamma \cdot 0$.

If $\beta \in \gamma$ and $\beta \cdot \alpha \in \gamma \cdot \alpha$ then $\beta \cdot \alpha^+ = \beta \cdot \alpha + \beta \in \gamma \cdot \alpha + \gamma = \gamma \cdot \alpha^+$ using part (a).

For λ a limit ordinal, if $\forall \alpha \in \lambda. \beta \cdot \alpha \in \gamma \cdot \alpha$, then we have $\beta \cdot \lambda = \sup_{\alpha \in \lambda} (\beta \cdot \alpha) \in \sup_{\alpha \in \lambda} (\gamma \cdot \alpha) = \gamma \cdot \lambda$.

Exercise 23

(a)

$$\begin{aligned} \omega + \omega^2 &= \omega \cdot 1 + \omega \cdot \omega \\ &= \omega \cdot (1 + \omega) && \text{(Theorem 8K)} \\ &= \omega \cdot \omega && \text{(Example on page 228)} \\ &= \omega^2 \end{aligned}$$

(b) Let $\omega^2 \in \beta$. Let γ be the ordinal such that $\beta = \omega^2 + \gamma$ (Subtraction Theorem). Then

$$\begin{aligned} \omega + \beta &= \omega + \omega^2 + \gamma \\ &= \omega + \gamma \\ &= \beta \end{aligned}$$

Exercise 24 We prove first that $1 + \alpha = \alpha$. Let γ be the ordinal such that $\alpha = \omega + \gamma$. Then

$$\begin{aligned} 1 + \alpha &= 1 + \omega + \gamma \\ &= \omega + \gamma && \text{(Example on page 228)} \\ &= \alpha \end{aligned}$$

Hence

$$\begin{aligned} \alpha + \alpha^2 &= \alpha \cdot (1 + \alpha) \\ &= \alpha^2 \end{aligned}$$

Now, let δ be the ordinal such that $\beta = \alpha^2 + \delta$. Then

$$\begin{aligned} \alpha + \beta &= \alpha + \alpha^2 + \delta \\ &= \alpha^2 + \delta \\ &= \beta \end{aligned}$$

Exercise 25 Let $\beta = \alpha \cup \{\alpha + \delta : \delta \in \theta\}$. Then β is a transitive set of ordinals, hence an ordinal. We also have $\alpha \subseteq \beta$. By the Subtraction Theorem, let γ be the ordinal such that

$$\beta = \alpha + \gamma .$$

For any $\delta \in \theta$ we have $\alpha + \delta \in \beta$ hence $\delta \in \gamma$ (Corollary 8P). Thus $\theta \subseteq \gamma$.

We have $\alpha + \theta \notin \beta$ (since $\alpha + \theta \notin \alpha$ and $\alpha + \theta \neq \alpha + \delta$ for any $\delta \in \theta$). So $\theta \notin \gamma$ (Corollary 8P).

Thus $\theta = \gamma$, and so $\beta = \alpha + \theta$.

Exercise 26 Follows just by repeated application of uniqueness in the Logarithm Theorem.

Exercise 27

Theorem 8R If $\alpha = 0$, then both sides are 1 if $\beta = \gamma = 0$ and 0 otherwise. If $\alpha = 1$ then both sides are 1.

Theorem 8S If $\alpha = 0$, and either $\beta = 0$ or $\gamma = 0$, then both sides are 1. If $\alpha = 0$ and β and γ are both non-zero, then both sides are 0. If $\alpha = 1$ then both sides are 1.

Exercise 28 This follows immediately from a Veblen Fixed-Point Theorem.

Exercise 29 Let S be a nonempty set of epsilon numbers. Then

$$\begin{aligned} \omega^{\sup S} &= \sup_{\alpha \in S} \omega^\alpha && \text{(Theorem Schema 8E)} \\ &= \sup_{\alpha \in S} \alpha \\ &= \sup S \end{aligned}$$

8.5 Well-Founded Relations

Exercise 1 We first prove: if $xR^t y$ then there exists z such that zRy and either $xR^t z$ or $x = z$.

PROOF:

$\langle 1 \rangle 1.$ $\{\langle x, y \rangle : \exists z(zRy \ \& \ (xR^t z \text{ or } x = z))\}$ is a transitive relation that includes R .

$\langle 2 \rangle 1.$ LET: $S = \{\langle x, y \rangle : \exists z(zRy \ \& \ (xR^t z \text{ or } x = z))\}$

$\langle 2 \rangle 2.$ S is transitive

$\langle 3 \rangle 1.$ LET: xSy and ySz

$\langle 3 \rangle 2.$ PICK a and b such that aRy , $(xR^t a \text{ or } x = a)$, bRz and $(yR^t b \text{ or } y = b)$

$\langle 3 \rangle 3.$ $xR^t y$

- ⟨3⟩4. xR^tb
- ⟨2⟩3. $R \subseteq S$

□

PROOF:

- ⟨1⟩1. LET: R be a well-founded relation.
- ⟨1⟩2. LET: A be a nonempty set.
- ⟨1⟩3. PICK an R -minimal element a of A .
- ⟨1⟩4. a is R^t -minimal

PROOF: By the lemma, if there exists x such that xR^ta then there exists x such that xRa .

□

Exercise 2 The relation R^t is always transitive, so it is a partial ordering iff it is irreflexive, i.e. there is no x such that xR^tx . This is the same as saying there is no cycle in R , i.e. no finite sequence of elements x_1, \dots, x_n such that $x_1Rx_2, x_2Rx_3, \dots, x_{n-1}Rx_n$ and x_nRx_1 .

Exercise 3 The proof is by transfinite induction on y over R . Assume $\{x : xR^tz\}$ is finite for all z such that zRy . Then

$$\{x : xR^ty\} = \bigcup \{\{z\} \cup \{x : xR^tz\} : zRy\}$$

which is a finite union of finite sets, hence finite.

Exercise 4 PROOF:

- ⟨1⟩1. LET: $T = S \cup \bigcup \{TC x : x \in S\}$
- ⟨1⟩2. T is a transitive set
 - ⟨2⟩1. LET: $x \in y \in T$
 - ⟨2⟩2. CASE: $y \in S$
 - ⟨3⟩1. $x \in TC y$
 - ⟨3⟩2. $x \in T$
 - ⟨2⟩3. CASE: $y \in TC a$ and $a \in S$
 - ⟨3⟩1. $x \in TC a$
 - ⟨3⟩2. $x \in T$
- ⟨1⟩3. $S \subseteq T$
- ⟨1⟩4. For any transitive set T' , if $S \subseteq T'$ then $T \subseteq T'$
 - ⟨2⟩1. LET: T' be a transitive set.
 - ⟨2⟩2. ASSUME: $S \subseteq T'$
 - ⟨2⟩3. LET: $x \in T$
 - ⟨2⟩4. CASE: $x \in S$

PROOF: Then $x \in T'$ by ⟨2⟩2
 - ⟨2⟩5. CASE: $x \in TC y$ and $y \in S$
 - ⟨3⟩1. $y \in T'$
 - ⟨3⟩2. $y \subseteq T'$
 - ⟨3⟩3. $TC y \subseteq T'$

□ $\langle 3 \rangle 4. x \in T'$

8.6 Natural Models

Exercise 5 Suppose $S \in V_\omega$. Pick $n \in \omega$ such that $S \subseteq V_n$. Then $TC S \subseteq V_n$ (since V_n is transitive). Therefore every member of $TC S$ is in V_n and hence finite.

Conversely, suppose $TC S$ is finite. By Theorem 9E, rank x is finite for all $x \in TC S$. Hence $\text{rank } S = \{\text{rank } x : x \in TC S\}$ is finite, and so $S \in V_\omega$.

Exercise 6 Yes, the replacement axioms are all true in V_ω by the same argument as the proof of Theorem 9L. By the arguments before Theorem 9F, all the axioms of ZFC are true in V_ω except the Axiom of Infinity.

Exercise 7 PROOF:

- $\langle 1 \rangle 1.$ Define $h : \omega \rightarrow V_\omega$ by recursion thus: $h(n) = \{h(m) : m \in g(n)\}$.
 - $\langle 1 \rangle 2.$ h is injective.
 - $\langle 2 \rangle 1.$ ASSUME: $h(m) = h(n)$
 - $\langle 2 \rangle 2.$ $g(m) = g(n)$
 - $\langle 3 \rangle 1.$ $p \in g(m)$ iff $h(p) \in h(m)$ iff $h(p) \in h(n)$ iff $p \in g(n)$
 - $\langle 2 \rangle 3.$ $m = n$
 - $\langle 1 \rangle 3.$ h is surjective.
 - $\langle 2 \rangle 1.$ We prove by ϵ_{V_ω} -induction on A that, for all $A \in V_\omega$, there exists $n \in \omega$ such that $h(n) = A$
 - $\langle 2 \rangle 2.$ LET: $A \in V_\omega$
 - $\langle 2 \rangle 3.$ ASSUME: $\forall x \in A. \exists n \in \omega. h(n) = x$
 - $\langle 2 \rangle 4.$ LET: m be such that $g(m) = \{n \in \omega : h(n) \in A\}$
 - PROOF: Since g is surjective.
 - $\langle 2 \rangle 5.$ $h(m) = A$
 - $\langle 1 \rangle 4.$ If mEn then $h(m) \in h(n)$
-

Exercise 8 Consider the structure $\langle P, R \rangle$ in Exercise 4 of Chapter 7. Then $\langle P, R \rangle \in V_\omega$ but its ordinal ω^2 is not.

Exercise 9 PROOF:

- $\langle 1 \rangle 1.$ $V_{\alpha+\lambda} = \bigcup_{\delta < \lambda} V_{\alpha+\delta}$
- $\langle 2 \rangle 1.$ LET: $x \in V_{\alpha+\lambda}$
- $\langle 2 \rangle 2.$ PICK $\beta < \alpha + \lambda$ such that $x \in V_\beta$
- $\langle 2 \rangle 3.$ CASE: $\beta < \alpha$
- PROOF: Then $x \in V_{\alpha+0}$.
- $\langle 2 \rangle 4.$ CASE: $\alpha \leq \beta$
- $\langle 3 \rangle 1.$ LET: δ be the ordinal such that $\beta = \alpha + \delta$

$\langle 3 \rangle 2.$ $x \in V_{\alpha+\delta}$ and $\delta < \lambda$
 $\langle 1 \rangle 2.$ $\bigcup_{\delta < \lambda} V_{\alpha+\delta} \subseteq V_{\alpha+\lambda}$
 \square