M5 Categories

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Definition 0.1 (Category)**.** A *category* $\mathcal{C}$ is a sextuple $(Ar, Ob, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, m)$ such that:

- $\mathrm{dom} : Ar \to Ob$
- $\mathrm{cod} : Ar \to Ob$
- $\mathrm{id} : Ob \to Ar$
- $m : \{(f, g) \in Ar^2 : \mathrm{dom}\, f = \mathrm{cod}\, g\} \to Ar$

such that:

- $\forall A \in Ob.\, \mathrm{dom}(\mathrm{id}_A) = A$
- $\forall A \in Ob.\, \mathrm{cod}(\mathrm{id}_A) = A$
- $\forall f, g \in Ar.(\mathrm{dom}\, f = \mathrm{cod}\, g \Rightarrow \mathrm{dom}\, m(f, g) = \mathrm{dom}\, g)$
- $\forall f, g \in Ar.(\mathrm{dom}\, f = \mathrm{cod}\, g \Rightarrow \mathrm{cod}\, m(f, g) = \mathrm{cod}\, f)$
- $\forall f \in Ar.m(\mathrm{id}_(\mathrm{cod}\, f), f) = f$
- $\forall f \in Ar.m(f, \mathrm{id}_(\mathrm{dom}\, f)) = f$
- $\forall f, g, h \in Ar.(\mathrm{dom}\, f = \mathrm{cod}\, g \wedge \mathrm{dom}\, g = \mathrm{cod}\, h \Rightarrow m(f, m(g, h)) = m(m(f, g), h))$

We call Ar the *arrows* of $\mathcal{C}$ and Ob the *objects*. We call $\mathrm{dom}\, f$ the *domain* of the arrow f, and $\mathrm{cod}\, f$ the *codomain*. We write $f : A \to B$ for $\mathrm{dom}\, f = A \wedge \mathrm{cod}\, f = B$.

We say arrows f and g are *composable* iff $\mathrm{dom}\, f = \mathrm{cod}\, g$, in which case $m(f, g)$ is called their *composite*, written $f \circ g$.

We call id_A the *identity arrow* on A.

Definition 0.2 (Category of Sets)**.** The *category of sets* **Set** is the category with set of objects $\mathbf{V}$ and with $\mathbf{Set}[A, B]$ the set of all functions from A to B.

Definition 0.3 (Preordered Sets as Categories)**.** Identify any preordered set $(P, \leq)$ with the category P with set of objects P and a morphism $(a, b) : a \to b$ iff $a \leq b$.

Definition 0.4 (Discrete Category)**.** For any set A, the *discrete category* A is the preordered set $(A, =)$ considered as a category.

Definition 0.5 (Slice Category)**.** Let $\mathbb{C}$ be a category and $A \in \mathbb{C}$. The *slice category* $\mathbb{C}/A$ is the category with:

- objects all pairs (B, f) where $B \in \mathbb{C}$ and $f : B \to A$ in $\mathbb{C}$
- morphisms $(B, f) \to (C, g)$ all morphisms $h : B \to C$ in $\mathbb{C}$ such that $g \circ h = f$.

Definition 0.6 (Slice Category)**.** Let $\mathbb{C}$ be a category and $A \in \mathbb{C}$. The *coslice category* $\mathbb{C}\backslash A$ is the category with:

- objects all pairs (B, f) where $B \in \mathbb{C}$ and $f : A \to B$ in $\mathbb{C}$
- morphisms $(B, f) \to (C, g)$ all morphisms $h : B \to C$ in $\mathbb{C}$ such that $g = h \circ f$.

Definition 0.7 (Pointed Sets)**.** The category $\mathbf{Set}_*$ of *pointed sets* is the coslice category $\mathbf{Set}\backslash 1$.

Definition 0.8 (Subcategory)**.** A category $\mathbb{C}$ is a *subcategory* of $\mathbb{D}$ iff every object of $\mathbb{C}$ is an object of $\mathbb{D}$, and every morphism $A \to B$ in $\mathbb{C}$ is a morphism $A \to B$ in $\mathbb{D}$.

It is *full* iff, for all $A, B \in \mathbb{C}$, every morphism $A \to B$ in $\mathbb{D}$ is a morphism $A \to B$ in $\mathbb{C}$.

Proposition 0.9. *Let $f, g : A \to B$. Then $f = g$ if and only if, for every object X and arrow $x : X \to A$, we have $f \circ x = g \circ x$.*

Proof: If the right-hand side holds then $f = f \circ \mathrm{id}_A = g \circ \mathrm{id}_A = g$. $\square$

Definition 0.10 (Monic)**.** An arrow $f : A \to B$ is *monic*, $f : A \rightarrowtail B$, iff, for every object X and morphisms $x, y : X \to A$, if $f \circ x = f \circ y$ then $x = y$.

Definition 0.11 (Epic)**.** An arrow $f : A \to B$ is *epic*, $f : A \twoheadrightarrow B$, iff, for every object X and morphisms $x, y : B \to X$, if $x \circ f = y \circ f$ then $x = y$.

Definition 0.12 (Section, Retraction)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $r \circ s = \mathrm{id}_B$.

We also call a retraction a *split epi*, and a section a *split monic*.

Proposition 0.13. *Every section is monic.*

Proof:
$\langle 1 \rangle 1.$ Let: $s : A \to B$ be a section of $r : B \to A$
$\langle 1 \rangle 2.$ Let: X be an object and $x, y : X \to A$
$\langle 1 \rangle 3.$ Assume: $s \circ x = s \circ y$
$\langle 1 \rangle 4.$ $x = y$

PROOF:

$$\begin{aligned} x &= r \circ s \circ x \\ &= r \circ s \circ y \\ &= y \end{aligned}$$

□

Proposition 0.14. *Every retraction is epic.*

PROOF: Dual.

Lemma 0.15. *Let $f : A \to B$, $g : B \to A$ and $h : B \to A$ be arrows. If g is a retraction of f and h is a section of f then $g = h$.*

PROOF:

$$\begin{aligned} g &= g \circ \mathrm{id}_B \\ &= g \circ f \circ h \\ &= \mathrm{id}_A \circ h \\ &= h \end{aligned}$$

□

Proposition 0.16. *Let $r : A \to B$. Then r is a retraction if and only if, for every object X and morphism $y : X \to B$, there exists $x : X \to A$ such that $r \circ x = y$.*

PROOF:

⟨1⟩1. If r is a retraction then, for every object X and morphism $y : X \to B$, there exists $x : X \to A$ such that $r \circ x = y$.

⟨2⟩1. LET: $s : B \to A$ be a section of r.

⟨2⟩2. LET: X be an object and $y : X \to B$

⟨2⟩3. LET: $x = s \circ y$

⟨2⟩4. $r \circ x = y$

⟨1⟩2. If, for every object X and morphism $y : X \to B$, there exists $x : X \to A$ such that $r \circ x = y$, then r is a retraction.

PROOF: Simply take $x = \mathrm{id}_A$.

□

Definition 0.17 (Isomorphism)**.** An arrow $f : A \to B$ is an *isomorphism*, $f : A \cong B$, iff there exists an arrow $f^{-1} : B \to A$, its *inverse*, such that f^{-1} is both a section and a retraction of f.

Proposition 0.18. *The inverse of an isomorphism is unique.*

PROOF: Lemma 0.15. □

Proposition 0.19. *Every isomorphism is monic.*

PROOF: Proposition 0.13. □

Proposition 0.20. *Every isomorphism is epic.*

PROOF: Proposition 0.14. □

Proposition 0.21. *If a morphism is monic and split epi then it is an isomorphism.*

PROOF:
⟨1⟩1. LET: $f : A \to B$ be monic and have section $s : B \to A$
⟨1⟩2. $f \circ s \circ f = f$
⟨1⟩3. $s \circ f = \mathrm{id}_A$
⟨1⟩4. f is iso with inverse s.
□

Proposition 0.22. *If a morphism is epic and split monic then it is an isomorphism.*

PROOF: Dual. □

Proposition 0.23. *For any object A, we have $\mathrm{id}_A : A \cong A$ with ${\mathrm{id}_A}^{-1} = \mathrm{id}_A$.*

PROOF: Since $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$. □

Proposition 0.24. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

Proposition 0.25. *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF:

$$\begin{aligned} f^{-1} \circ g^{-1} \circ g \circ f &= f^{-1} \circ f \\ &= \mathrm{id}_A \\ g \circ f \circ f^{-1} \circ g^{-1} &= g \circ g^{-1} \\ &= \mathrm{id}_C \end{aligned}$$

□

Definition 0.26 (Balanced)**.** A category is *balanced* iff every arrow that is both monic and epic is an isomorphism.

1 Terminal Objects

Definition 1.1 (Terminal)**.** An object T is *terminal* iff, for every object X, there exists a unique morphism $X \to T$.

Example 1.2. In **Set**, 1 is terminal.

Example 1.3. $* \in 1$ is terminal in $\mathbf{Set}_*$.

Proposition 1.4. *Any two terminal objects are isomorphic, and the isomorphism between them is unique.*

PROOF:
⟨1⟩1. LET: S and T be terminal objects.

⟨1⟩2. LET: ϕ be the unique arrow $S \to T$
⟨1⟩3. LET: ϕ^{-1} be the unique arrow $T \to S$
⟨1⟩4. $\phi \circ \phi^{-1} = \mathrm{id}_T$
  PROOF: Each is the unique arrow $T \to T$.
⟨1⟩5. $\phi^{-1} \circ \phi = \mathrm{id}_S$
  PROOF: Each is the unique arrow $S \to S$.
□

2 Initial Objects

Definition 2.1 (Initial)**.** An object I is *initial* iff, for every object X, there exists a unique morphism $I \to X$.

Example 2.2. In a preorder, an initial object is the same as a least element.

Example 2.3. The only initial object in **Set** is $\emptyset$.

Example 2.4. $* \in 1$ is initial in $\mathbf{Set}_*$.

Proposition 2.5. *Any two initial objects are isomorphic, and the isomorphism between them is unique.*

PROOF: Dual to Proposition 1.4. □

3 Zero Objects

Definition 3.1 (Zero Object)**.** A *zero object* is an object that is both initial and terminal.

Example 3.2. $* \in 1$ is the zero object in $\mathbf{Set}_*$.

Proposition 3.3. *Let $(P, \leq)$ be a preorder and $a \in P$. The slice category P/a is isomorphic to the preorder $(\{x \in P : x \leq a\}, \leq)$ considered as a category.*

Proposition 3.4. *Let $(P, \leq)$ be a preorder and $a \in P$. The coslice category $P\backslash a$ is isomorphic to the preorder $(\{x \in P : a \leq x\}, \leq)$ considered as a category.*

4 Opposite Category

Definition 4.1 (Opposite Category)**.** Let $\mathbb{C}$ be any category. The *opposite category* $\mathbb{C}^{\mathrm{op}}$ has objects the objects of $\mathbb{C}$ and morphisms $A \to B$ the morphisms $B \to A$ in $\mathbb{C}$.

Proposition 4.2. *An initial object in* $\mathbb{C}$

5 Groupoids

Definition 5.1 (Groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

6 Automorphisms

Definition 6.1 (Automorphism)**.** Let A be an object. An *automorphism* on A is an isomorphism $A \cong A$.

7 Quotient Sets

Proposition 7.1. *Let A be a set and let $\sim$ be an equivalence relation on A. Let $\mathbb{C}$ be the subcategory of* $\mathbf{Set}\backslash A$ *consisting of pairs (Z, ϕ) such that, for all $x, y \in A$, if $x \sim y$ then $\phi(x) = \phi(y)$.*

Then $(A/\sim, \pi)$ is the initial object in $\mathbb{C}$, where $\pi : A \to A/\sim$ is the canonical projection.

PROOF: For any object $\phi : A \to Z$ in the category, the only morphism $(A/\sim, \pi) \to (Z, \phi)$ is the function $f : A/\sim \to Z$ such that $f([a]) = \phi(a)$ for all $a \in A$. $\square$

8 Products

Definition 8.1 (Product)**.** Let $\mathbb{C}$ be a category and $\{A_i\}_{i \in I}$ a family of objects in $\mathbb{C}$. A *product* of $\{A_i\}_{i \in I}$ is a terminal object in the category with:

- objects all pairs $(C, \{f_i\}_{i \in I})$ where $C \in \mathbb{C}$ and $f_i : C \to A_i$ for all $i \in I$;
- morphisms $(C, \{f_i\}_{i \in I}) \to (D, \{g_i\}_{i \in I})$ all morphisms $x : C \to D$ such that, for all $i \in I$, we have $g_i \circ x = f_i$.

Example 8.2. The Cartesian product $\prod_{i \in I} A_i$ with projections is the product of $\{A_i\}_{i \in I}$ in **Set**.

Example 8.3. Products in a preorder are meets.

Proposition 8.4. *If $A \times B$ and $B \times A$ exist then they are isomorphic.*

Proposition 8.5. *If $A \times (B \times C)$ and $(A \times B) \times C$ exist then they are isomorphic.*

9 Coproducts

Definition 9.1 (Coproduct)**.** Let $\mathbb{C}$ be a category and $A, B \in \mathbb{C}$. A *coproduct* of A and B is an initial object in the category with:

- objects all triples (C, f, g) where $C \in \mathbb{C}$, $f : A \to C$ and $g : B \to C$
- morphisms $(C, f, g) \to (D, h, k)$ all morphisms $x : C \to D$ such that $h = x \circ f$ and $k = x \circ g$

Example 9.2. Disjoint unions are coproducts in **Set**.

Example 9.3. Products in a preorder are joins.