C4 Analysis

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Definition 0.1 (Limit of a Function). Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let a be an accumulation point of A and $b \in \mathbb{R}$. Then we say b is the *limit* of f at a, and write $f(x) \to b$ as $x \to a$ or $\lim_{x \to a} f(x) = b$, iff for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta$ then $|f(x) - b| < \epsilon$.

Proposition 0.2. *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let a be an accumulation point of A and $b, c \in \mathbb{R}$. If $f(x) \to b$ as $x \to a$ and $f(x) \to c$ as $x \to a$ then $b = c$.*

PROOF:
$\langle 1 \rangle 1.$ $\forall \epsilon > 0. |b - c| < \epsilon$
  $\langle 2 \rangle 1.$ LET: $\epsilon > 0$
  $\langle 2 \rangle 2.$ PICK $\delta > 0$ such that $\forall x \in A - \{a\}. |x - a| < \delta \Rightarrow |f(x) - b| < \epsilon/2 \wedge |f(x) - c| < \epsilon/2$
  $\langle 2 \rangle 3.$ PICK $x \in (A - \{a\}) \cap (a - \delta, a + \delta)$
  $\langle 2 \rangle 4.$ $|f(x) - b| < \epsilon/2$
  $\langle 2 \rangle 5.$ $|f(x) - c| < \epsilon/2$
  $\langle 2 \rangle 6.$ $|b - c| < \epsilon$
□

Proposition 0.3 (Choice). *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let a be an accumulation point of A. Let $b \in \mathbb{R}$. Then $f(x) \to b$ as $x \to a$ if and only if, for any sequence (x_n) in $A - \{a\}$, if $x_n \to a$ as $n \to \infty$ then $f(x_n) \to b$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1.$ If $f(x) \to b$ as $x \to a$ then, for any sequence (x_n) is $A - \{a\}$, if $x_n \to a$ as $n \to \infty$ then $f(x_n) \to b$ as $n \to \infty$.
  $\langle 2 \rangle 1.$ ASSUME: $f(x) \to b$ as $x \to a$
  $\langle 2 \rangle 2.$ LET: (x_n) be a sequence in $A - \{a\}$
  $\langle 2 \rangle 3.$ ASSUME: $x_n \to a$ as $n \to \infty$
  $\langle 2 \rangle 4.$ LET: $\epsilon > 0$
  $\langle 2 \rangle 5.$ PICK $\delta > 0$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta$, then $|f(x) - b| < \epsilon$
  $\langle 2 \rangle 6.$ PICK N such that $\forall n \geq N. |x - a| < \delta$
  $\langle 2 \rangle 7.$ $\forall n \geq N. |f(x) - b| < \epsilon$
$\langle 1 \rangle 2.$ If, for any sequence (x_n) is $A - \{a\}$, if $x_n \to a$ as $n \to \infty$ then $f(x_n) \to b$ as $n \to \infty$, then $f(x) \to b$ as $x \to a$.
  $\langle 2 \rangle 1.$ ASSUME: $f(x) \not\to b$ as $x \to a$

⟨2⟩2. Pick $\epsilon > 0$ such that, for all $\delta > 0$, there exists $x \in A - \{a\}$ such that $|x - a| < \delta$ and $|f(x) - b| \geq \epsilon$

⟨2⟩3. For all $n \in \mathbb{Z}^+$, Pick $x_n \in A - \{a\}$ such that $|x_n - a| < 1/n$ and $|f(x_n) - b| \geq \epsilon$

⟨2⟩4. $x_n \to a$ as $n \to \infty$

⟨2⟩5. $f(x_n) \not\to b$ as $n \to \infty$

□

Proposition 0.4. *Let $A, B \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Let a be an accumulation point of $A \cap B$. Let $b, c \in \mathbb{R}$. Assume $f(x) \to b$ as $x \to a$ and $g(x) \to c$ as $x \to a$. Then $f(x) + g(x) \to b + c$ as $x \to a$.*

Proof:

⟨1⟩1. Let: $\epsilon > 0$

⟨1⟩2. Pick $\delta > 0$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta$ then $|f(x) - b| < \epsilon/2$, and for all $x \in B - \{a\}$, if $|x - a| < \delta$ then $|g(x) - c| < \epsilon/2$

⟨1⟩3. Let: $x \in (A \cap B) - \{a\}$

⟨1⟩4. Assume: $|x - a| < \delta$

⟨1⟩5. $|(f(x) + g(x)) - (b + c)| < \epsilon$

□

Proposition 0.5. *Let $A, B \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Let a be an accumulation point of $A \cap B$. Let $b, c \in \mathbb{R}$. Assume $f(x) \to b$ as $x \to a$ and $g(x) \to c$ as $x \to a$. Then $f(x)g(x) \to bc$ as $x \to a$.*

Proof:

⟨1⟩1. Let: $\epsilon > 0$

⟨1⟩2. Let: $d = \epsilon/2|b|$ if $b \neq 0$, or $d = 1$ if $b = 0$

⟨1⟩3. Pick $\delta > 0$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta$ then $|f(x) - b| < \epsilon/2(d + |c|)$, and for all $x \in B - \{a\}$, if $|x - a| < \delta$ then $|g(x) - c| < d$

⟨1⟩4. Let: $x \in (A \cap B) - \{a\}$

⟨1⟩5. Assume: $|x - a| < \delta$

⟨1⟩6. $|f(x)g(x) - bc| < \epsilon$

Proof:

$$\begin{aligned} |f(x)g(x) - bc| &\leq |f(x) - b||g(x)| + |b||g(x) - c| \\ &\epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

Proposition 0.6. *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let a be an accumulation point of A and $b > 0$ Suppose $\lim_{x \to a} f(x) = b$. Then there exists δ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta$ then $f(x) > b/2$.*

Proof: Take $\epsilon = b/2$ in the definition of limit. □

Proposition 0.7. *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let a be an accumulation point of A. Let $b \in \mathbb{R} - \{0\}$. Suppose $f(x) \to b$ as $x \to a$. Then a is an accumulation point of $\{x \in A : f(x) \neq 0\}$ and $1/f(x) \to 1/b$ as $x \to a$.*

PROOF:
$\langle 1\rangle 1.$ a is an accumulation point of $\{x \in A : f(x) \neq 0\}$.
$\langle 2\rangle 1.$ LET: $\delta > 0$
$\langle 2\rangle 2.$ ASSUME: w.l.o.g. $\forall x \in A - \{a\}. |x - a| < \delta \Rightarrow f(x) \neq 0$
$\langle 2\rangle 3.$ PICK $x \in (a - \delta, a + \delta) \cap (A - \{a\})$
$\langle 2\rangle 4.$ $x \in (a - \delta, a + \delta) \cap (\{x \in A : f(x) \neq 0\} - \{a\})$
$\langle 1\rangle 2.$ For all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in A - \{a\}$, if $f(x) \neq 0$ and $|x - a| < \delta$ then $|1/f(x) - 1/b| < \epsilon$
$\langle 2\rangle 1.$ LET: $\epsilon > 0$
$\langle 2\rangle 2.$ PICK $\delta > 0$ such that $\forall x \in A - \{a\}. |x - a| < \delta \Rightarrow |f(x) - b| < \epsilon |b|^2/2$ and $\forall \in A - \{a\}. |x - a| < \delta \Rightarrow |f(x)| > |b|/2$
PROOF: Proposition 0.6.
$\langle 2\rangle 3.$ LET: $x \in A - \{a\}$ satisfy $f(x) \neq 0$ and $|x - a| < \delta$
$\langle 2\rangle 4.$ $|1/f(x) - 1/b| < \epsilon$
PROOF:

$$\begin{aligned} |1/f(x) - 1/b| &= |f(x) - b|/|f(x)||b| \\ &< \epsilon && (\langle 2\rangle 2) \end{aligned}$$

□

Definition 0.8 (Continuity at a Point)**.** Let $A \subseteq \mathbb{R}$. Let $a \in A$ be an accumulation point of A. Then f is *continuous* at a if and only if $f(x) \to f(a)$ as $x \to a$.

f is *continuous* if and only if every point of A is an accumulation point of A and f is continuous at every point of A.

Proposition 0.9. *Let $A, B \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Let $a \in A \cap B$ be an accumulation point of $A \cap B$. Assume f and g are continuous at a. Then $f + g$ and fg are continuous at a.*

PROOF: Propositions 0.4 and 0.5. □

Corollary 0.9.1. *Every polynomial is continuous on $\mathbb{R}$.*

Proposition 0.10. *Let $A \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$. Let $a \in A$ be an accumulation point of A. Assume f is continuous at a and $f(a) \neq 0$. Then $1/f$ is continuous at a.*

PROOF: Proposition 0.7. □

Proposition 0.11. *Let $A, B \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Let $a \in A$ be an accumulation point of A. Assume $f(a) \in B$ and $f(a)$ is an accumulation point of B. If f is continuous at a and g is continuous at $f(a)$ then $g \circ f$ is continuous at a.*

PROOF:
$\langle 1\rangle 1.$ LET: $\epsilon > 0$
$\langle 1\rangle 2.$ PICK $\delta_1 > 0$ such that, for all $y \in B - \{f(a)\}$, if $|y - f(a)| < \delta_1$ then $|g(y) - g(f(a))| < \epsilon$

⟨1⟩3. PICK $\delta_2 > 0$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta_2$ then $|f(x) - f(a)| < \delta_1$

⟨1⟩4. For all $x \in A - \{a\}$, if $|x - a| < \delta_2$ then $|g(f(x)) - g(f(a))| < \epsilon$

□

Definition 0.12 (Relatively Open)**.** Let $A \subseteq \mathbb{R}$ and $B \subseteq A$. Then B is *relatively open* in A iff there exists an open set $V \subseteq \mathbb{R}$ such that $B = A \cap V$.

Lemma 0.13. *Let $B \subseteq A \subseteq \mathbb{R}$. Then B is relatively open in A iff, for all $x \in B$, there exists an open interval I containing x such that $I \cap A \subseteq B$.*

PROOF:

⟨1⟩1. If B is relatively open in A then, for all $x \in B$, there exists an open interval I containing x such that $I \cap A \subseteq B$

  ⟨2⟩1. ASSUME: B is relatively open in A.

  ⟨2⟩2. PICK an open set V such that $B = A \cap V$

  ⟨2⟩3. LET: $x \in B$

  ⟨2⟩4. PICK an open interval I such that $x \in I \subseteq V$

  ⟨2⟩5. $I \cap A \subseteq B$

⟨1⟩2. If, for all $x \in B$, there exists an open interval I containing x such that $I \cap A \subseteq B$, then B is relatively open in A.

  ⟨2⟩1. ASSUME: For all $x \in B$, there exists an open interval I containing x such that $I \cap A \subseteq B$

  ⟨2⟩2. LET: V be the union of all the open intervals I such that $I \cap A \subseteq B$

  ⟨2⟩3. $B = A \cap V$

□

Theorem 0.14. *Let $A \subseteq \mathbb{R}$ be a set such that every point in A is an accumulation point of A. Let $f : A \to \mathbb{R}$. Then f is continuous if and only if, for every open set W, we have $f^{-1}(W)$ relatively open in A.*

PROOF:

⟨1⟩1. If f is continuous then, for every open set W, we have $f^{-1}(W)$ is relatively open in A.

  ⟨2⟩1. ASSUME: f is continuous.

  ⟨2⟩2. LET: W be an open set.

  ⟨2⟩3. For all $x \in f^{-1}(W)$, there exists an open interval containing I such that $I \cap A \subseteq f^{-1}(W)$

    ⟨3⟩1. LET: $x \in f^{-1}(W)$

    ⟨3⟩2. PICK $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subseteq W$

    ⟨3⟩3. PICK $\delta > 0$ such that, for all $y \in A - \{x\}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$

    ⟨3⟩4. LET: $I = (x - \delta, x + \delta)$
    PROVE: $I \cap A \subseteq f^{-1}(W)$

    ⟨3⟩5. LET: $y \in I \cap A$

    ⟨3⟩6. $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$

    ⟨3⟩7. $f(y) \in W$

  ⟨2⟩4. $f^{-1}(W)$ is relatively open in A.

PROOF: Lemma 0.13.

⟨1⟩2. If, for every open set W, we have $f^{-1}(W)$ is relatively open in A, then f is continuous.

⟨2⟩1. ASSUME: For every open set W, we have $f^{-1}(W)$ is relatively open in A.

⟨2⟩2. LET: $x \in A$

⟨2⟩3. LET: $\epsilon > 0$

⟨2⟩4. $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is relatively open in A.

⟨2⟩5. PICK $\delta > 0$ such that $(x - \delta, x + \delta) \cap A \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$

PROOF: Lemma 0.13.

⟨2⟩6. For all $y \in A - \{x\}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$

□

Proposition 0.15. *Let $C \subseteq \mathbb{R}$ be compact and be such that every element of C is an accumulation point of C. Let $f : C \to \mathbb{R}$ be continuous. Then $f(C)$ is compact.*

PROOF:

⟨1⟩1. LET: $\mathcal{A}$ be an open covering of $f(C)$.

⟨1⟩2. $\{W \in \mathcal{P}\mathbb{R} : W \text{ is open}, \exists A \in \mathcal{A}. f^{-1}(A) = W \cap C\}$ is an open covering of C.

PROOF: Theorem 0.14.

⟨1⟩3. PICK a finite subcover $\{W_1, \ldots, W_n\}$ of C.

⟨1⟩4. For $i = 1, \ldots, n$, PICK $A_i \in \mathcal{A}$ such that $f^{-1}(A_i) = W_i \cap C$

⟨1⟩5. $\{A_1, \ldots, A_n\}$ covers $f(C)$.

□

Corollary 0.15.1. *Let $C \subseteq \mathbb{R}$ be compact and be such that every element of C is an accumulation point of C. Let $f : C \to \mathbb{R}$ be continuous. Then $f(C)$ has a maximum and a minimum value.*

Lemma 0.16. *Let $A \subseteq \mathbb{R}$. Then A is connected if and only if there do not exist nonempty disjoint sets B, C relatively open in A such that $A = B \cup C$.*

PROOF:

⟨1⟩1. If $A = B \cup C$ where B and C are nonempty, disjoint and relatively open in A, then A is disconnected.

⟨2⟩1. ASSUME: $A = B \cup C$ where B and C are nonempty, disjoint and relatively open in A.

⟨2⟩2. PICK open sets B_1 and C_1 such that $B = B_1 \cap A$ and $C = C_1 \cap A$

⟨2⟩3. B contains no accumulation point of C.

⟨3⟩1. ASSUME: for a contradiction $b \in B$ and b is an accumulation point of C

⟨3⟩2. b is an accumulation point of $\mathbb{R} - B_1$

⟨3⟩3. $b \in \mathbb{R} - B_1$

PROOF: Since $\mathbb{R} - B_1$ is closed.

⟨3⟩4. Q.E.D.

PROOF: This contradicts the fact that $b \in B$.

⟨2⟩4. C contains no accumulation point of B.
PROOF: Similar.
⟨1⟩2. If A is disconnected then there exist nonempty, disjoint sets B and C relatively open in A such that $A = B \cup C$.
⟨2⟩1. ASSUME: A is disconnected
⟨2⟩2. PICK disjoint nonempty sets B and C such that $A = B \cup C$ and neither of B and C contains an accumulation point of the other.
⟨2⟩3. B is relatively open in A
PROOF: $B = A \cap (\mathbb{R} - \overline{C})$
⟨2⟩4. C is relatively open in A
PROOF: Similar.
□

Theorem 0.17. *Let $C \subseteq \mathbb{R}$ be connected and such that every element of C is an accumulation point of C. Let $f : C \to \mathbb{R}$ be continuous. Then $f(C)$ is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $f(C) = B \cup D$ where B and D are nonempty, disjoint and relatively open in $f(C)$
PROOF: Lemma 0.16.
⟨1⟩2. PICK open sets B', D' such that $B = f(C) \cap B'$ and $D = f(C) \cap D'$
⟨1⟩3. $C = f^{-1}(B') \cup f^{-1}(D')$
⟨1⟩4. $f^{-1}(B')$ and $f^{-1}(D')$ are relatively open in C
PROOF: Theorem 0.14
⟨1⟩5. $f^{-1}(B')$ and $f^{-1}(D')$ are nonempty and disjoint
⟨1⟩6. Q.E.D.
PROOF: This contradicts the fact that C is connected by Lemma 0.16.
□

Corollary 0.17.1. *The continuous image of a closed interval is a closed interval.*

Corollary 0.17.2 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Let c be between $f(a)$ and $f(b)$. Then there exists $x \in [a, b]$ such that $f(x) = c$.*

Proposition 0.18. *Let $f : [a, b] \to \mathbb{R}$ be continuous and injective. Then f^{-1} is continuous.*

PROOF:
⟨1⟩1. LET: $y \in f([a, b])$
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. LET: x be the point such that $f(x) = y$
⟨1⟩4. $f([x - \epsilon/2, x + \epsilon/2] \cap [a, b])$ is a closed interval.
⟨1⟩5. PICK $\delta > 0$ such that $(y - \delta, y + \delta) \subseteq f([x - \epsilon/2, x + \epsilon/2] \cap [a, b])$
⟨1⟩6. LET: $z \in f([a, b]) - \{y\}$ be such that $|y - z| < \delta$
⟨1⟩7. $|f^{-1}(z) - x| < \epsilon$

□

Definition 0.19 (Uniformly Continuous)**.** Let $A \subseteq \mathbb{R}$ be such that every point of A is an accumulation point of A. Let $f : A \to \mathbb{R}$. Then f is *uniformly continuous* iff, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in A$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Theorem 0.20. *Let $A \subseteq \mathbb{R}$ be compact and such that every point of A is an accumulation point of A. Let $f : A \to \mathbb{R}$. If f is continuous then f is uniformly continuous.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\epsilon > 0$
$\langle 1 \rangle 2.$ LET: $\mathcal{B}$ be the set of all sets of the form $\{(z-\delta, z+\delta) : z \in A, \delta > 0, \forall u \in A.|z - u| < 2\delta \Rightarrow |f(z) - f(u)| < \epsilon/2\}$
$\langle 1 \rangle 3.$ $\mathcal{B}$ covers A.
$\langle 1 \rangle 4.$ PICK a finite subcover $\{(z_1 - \delta_1, z_1 + \delta_1), \dots, (z_n - \delta_n, z_n + \delta_n)\}$
$\langle 1 \rangle 5.$ LET: $\delta = \min(\delta_1, \dots, \delta_n)$
$\langle 1 \rangle 6.$ LET: $x, y \in A$ with $|x - y| < \delta$
$\langle 1 \rangle 7.$ PICK i such that $x \in (z_i - \delta_i, z_i + \delta_i)$
$\langle 1 \rangle 8.$ $|f(x) - f(z_i)| < \epsilon/2$
$\langle 1 \rangle 9.$ $|y - z_i| < 2\delta_i$
$\langle 1 \rangle 10.$ $|f(y) - f(z_i)| < \epsilon/2$
$\langle 1 \rangle 11.$ $|f(y) - f(x)| < \epsilon$
□

1 Infinite Series

Definition 1.1 (Infinite Series)**.** Let (a_n) be a sequence of real numbers. The *infinite series* $\sum_n a_n$ is the sequence $(\sum_{i=0}^n a_i)$. The term $\sum_{i=0}^n a_i$ is the *nth partial sum* of the series. If the series converges, its limit is called the *sum* of the series and denoted $\sum_{i=0}^\infty a_i$.

Theorem 1.2 (Cauchy Criterion)**.** *Let (a_n) be a sequence of real numbers. Then $\sum_{n=1}^\infty a_n$ converges if and only if, for any $\epsilon > 0$, there exists N such that, for all $m \geq n \geq N$, we have $|\sum_{i=m}^n a_i| < \epsilon$.*

PROOF: Since the reals are Cauchy complete. □

Corollary 1.2.1. *If $\sum_{n=1}^\infty a_n$ converges then $a_n \to 0$ as $n \to \infty$.*

Corollary 1.2.2 (Comparison Test)**.** *Let (a_n) be a sequence of non-negative real numbers, and (b_n) a sequence of real numbers. If $\sum_{n=1}^\infty a_n$ converges and $\forall n.|b_n| \leq a_n$ then $\sum_{n=1}^\infty b_n$ converges.*

PROOF: Since $|sum_{i=m}^n b_i| \leq \sum_{i=m}^n a_i$. □

Proposition 1.3. *For $k \geq 2$ an integer, the series $\sum_{n=1}^\infty 1/n^k$ converges.*

PROOF:
⟨1⟩1. The series $\sum_{n=1}^{\infty} 1/n(n+1)$ converges.
PROOF: The Nth partial sum is

$$\begin{aligned}\sum_{n=1}^{N} 1/n(n+1) &= \sum_{n=1}^{N} (1/n - 1/(n+1)) \\ &= 1 - 1/(N+1) \\ &\to 1 & \text{as } N \to \infty\end{aligned}$$

⟨1⟩2. The series $\sum_{n=1}^{\infty} 1/n^2$ converges.
PROOF: By the Comparison Test, we have $\sum_{n=1}^{\infty} 1/(n+1)^2$ converges.
⟨1⟩3. For $k \geq 2$ an integer, the series $\sum_{n=1}^{\infty} 1/n^k$ converges.
PROOF: By the Comparison Test.
□

Proposition 1.4. *If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then*

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

PROOF: Apply Theorem **??** to the partial sums. □

Proposition 1.5. *If $\sum_{n=1}^{\infty} a_n$ converges then, for $\lambda \in \mathbb{R}$, we have $\sum_{n=1}^{\infty} \lambda a_n = \lambda \sum_{n=1}^{\infty} a_n$.*

PROOF: Easy. □

Proposition 1.6. *Let (a_n) and (b_n) be sequences of positive real numbers. Let c be a positive real. Assume $a_n/b_n \to c$ as $n \to \infty$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.*

PROOF:
⟨1⟩1. ASSUME: $\sum_{n=1}^{\infty} b_n$ converges.
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK N such that, for all $m, n \geq N$, we have $\sum_{i=m}^{n} b_i < \epsilon/(c+1)$ and for all $n \geq N$ we have $|a_n/b_n - c| < 1$
⟨1⟩4. LET: $m, n \geq N$
⟨1⟩5. $\sum_{i=m}^{n} a_i < \epsilon$
PROOF:

$$\begin{aligned}\sum_{i=m}^{n} a_i &< \sum_{i=m}^{n} b_i(c+1) \\ &= (c+1) \sum_{i=m}^{n} b_i \\ &< \epsilon\end{aligned}$$

□

Proposition 1.7 (Geometric Series)**.** *Let $b, r \in \mathbb{R}$ with $|r| < 1$. Then $\sum_{n=0}^{\infty} br^n = b/(1-r)$.*

PROOF:
⟨1⟩1. $\sum_{i=0}^{n} br^i = b(1 - r^{n+1})/(1 - r)$
PROOF:

$$\begin{aligned}(1-r)\sum_{i=0}^{n} br^i &= \sum_{i=0}^{n} br^i - \sum_{i=1}^{n+1} br^i \\ &= b - br^{n+1}\end{aligned}$$

⟨1⟩2. $\sum_{i=0}^{\infty} br^i = b/(1 - r)$
PROOF: Lemma **??**
□

Proposition 1.8. *Let $b, r \in \mathbb{R}$ with $|r| \geq 1$. Then $\sum_{n=0}^{\infty} br^n$ diverges.*

PROOF: Since br^n does not converge to 0. □

Proposition 1.9 (Harmonic Series)**.** *$\sum_{n=1}^{\infty} 1/n$ diverges.*

PROOF: Since $\sum_{i=1}^{2^n} 1/i \geq 1 + n/2$. □

Definition 1.10 (Absolute Convergence)**.** A series $\sum_n a_n$ *converges absolutely* iff $\sum_n |a_n|$ converges.

Proposition 1.11. *An absolutely convergent series converges.*

PROOF: By the Comparison Test. □

Theorem 1.12 (Alternating Series Test)**.** *Let (a_n) be a decreasing sequence of nonnegative real numbers that converges to 0. Then $\sum_n (-1)^n a_n$ converges.*

PROOF:
⟨1⟩1. For natural numbers $m \leq n$,
LET: $R_{mn} = \sum_{i=m}^{n} (-1)^i a_i$
⟨1⟩2. For natural numbers $m \leq n$, $(-1)^m R_{mn} \geq 0$
PROOF: It is $\sum_{0 \leq j, m+2j+1 \leq n} (a_{m+2j} - a_{m+2j+1})$ if $n - m$ is even, or $\sum_{0 \leq j, m+2j+1 \leq n} (a_{m+2j} - a_{m+2j+1}) + a_n$ if $n - m$ is odd.
⟨1⟩3. For natural numbers $m \leq n$, $(-1)^m R_{mn} \leq a_m$
PROOF: It is $a_m + \sum_j (-a_{m+2j+1} + a_{m+2j+2})$ if $n - m$ is odd, or $a_m + \sum_j (-a_{m+2j+1} + a_{m+2j+2}) - a_n$ if $n - m$ is even.
⟨1⟩4. For natural numbers $m \leq n$, $|R_{mn}| \leq a_m$
⟨1⟩5. LET: $\epsilon > 0$
⟨1⟩6. PICK N such that $\forall n \geq N. a_n < \epsilon$
⟨1⟩7. For all m, n, if $N \leq m \leq n$ then $|R_{mn}| < \epsilon$
⟨1⟩8. Q.E.D.
PROOF: By the Cauchy criterion.
□

Definition 1.13 (Remainder of a Series)**.** Let $\sum_{n=0}^{\infty} a_n$ be a series and $N \in \mathbb{N}$. The *remainder* of the series after the Nth term is the series $\sum_{n=N+1}^{\infty} a_n$.

Proposition 1.14. *If the series $\sum_{n=0}^{\infty} a_n$ converges, then*

$$\sum_{n=N}^{\infty} a_n \to 0 \text{ as } N \to \infty$$

PROOF:
$\langle 1 \rangle 1.$ For all N, k with $N \leq k$, we have $\sum_{n=0}^{k} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{k} a_n$
$\langle 1 \rangle 2.$ $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$
  PROOF: Taking limits.
$\langle 1 \rangle 3.$ $\sum_{n=N}^{\infty} a_n = \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{N-1} a_n$
$\langle 1 \rangle 4.$ $\sum_{n=N}^{\infty} a_n \to 0$ as $N \to \infty$
  PROOF:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{N-1} a_n &\to \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{\infty} a_n && \text{as } N \to \infty \\ &= 0 \end{aligned}$$

∎

Theorem 1.15 (Ratio Test)**.** *Let (a_n) be a sequence of non-zero real numbers.*

1. *If $(|a_{n+1}/a_n|)$ converges to a limit < 1, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.*
2. *If $(|a_{n+1}/a_n|)$ converges to a limit > 1 or diverges to $+\infty$, then $\sum_{n=0}^{\infty} a_n$ diverges.*

PROOF:
$\langle 1 \rangle 1.$ If $(|a_{n+1}/a_n|)$ converges to a limit < 1, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.
  $\langle 2 \rangle 1.$ ASSUME: $|a_{n+1}/a_n| \to b < 1$ as $n \to \infty$
  $\langle 2 \rangle 2.$ PICK N such that $\forall n \geq N. ||a_{n+1}/a_n| - b| < (1-b)/2$
  $\langle 2 \rangle 3.$ LET: $c = (1+b)/2$
  $\langle 2 \rangle 4.$ $\forall n \geq N. |a_{n+1}/a_n| \leq c$
  $\langle 2 \rangle 5.$ $\forall n \geq N. |a_{n+1}| < c|a_n|$
  $\langle 2 \rangle 6.$ $0 < c < 1$
  $\langle 2 \rangle 7.$ $\forall n \geq N. \sum_{i=N}^{n} |a_i| \leq |a_N|/(1-c)$
    PROOF:

$$\begin{aligned} \sum_{i=N}^{n} |a_i| &\leq |a_N| \sum_{i=N}^{n} c^{n-N} \\ &\leq |a_N|(1/(1-c)) \end{aligned}$$

  $\langle 2 \rangle 8.$ $\forall n \geq N. \sum_{i=0}^{n} |a_i| \leq \sum_{i=0}^{N-1} |a_i| + |a_N|/(1-c)$
  $\langle 2 \rangle 9.$ $\sum_{i=0}^{n} |a_i|$ converges.
$\langle 1 \rangle 2.$ If $|a_{n+1}/a_n|$ converges to a limit > 1, then $\sum_{n=0}^{\infty} a_n$ diverges.
  $\langle 2 \rangle 1.$ ASSUME: $|a_{n+1}/a_n| \to b > 1$ as $n \to \infty$
  $\langle 2 \rangle 2.$ PICK N such that $\forall n \geq N. ||a_{n+1}/a_n| - b| < (b-1)/2$
  $\langle 2 \rangle 3.$ LET: $c = (b+1)/2$
  $\langle 2 \rangle 4.$ $\forall n \geq N. |a_{n+1}/a_n| > c$
  $\langle 2 \rangle 5.$ $c > 1$

⟨2⟩6. $\forall n \geq N. |a_n| \geq |a_N|$
⟨2⟩7. $a_n \not\to 0$ as $n \to \infty$
⟨1⟩3. If $|a_{n+1}/a_n|$ diverges to $+\infty$ then $\sum_{n=0}^{\infty} a_n$ diverges.
⟨2⟩1. Assume: $|a_{n+1}/a_n|$ diverges to $+\infty$
⟨2⟩2. Pick N such that $\forall n \geq N. |a_{n+1}/a_n| > 2$
⟨2⟩3. $\forall n \geq N. |a_n| \geq |a_N|$
⟨2⟩4. $a_n \not\to 0$ as $n \to \infty$
□

Proposition 1.16. *$u^n/n! \to 0$ as $n \to \infty$*

Proof:
⟨1⟩1. For $n \in \mathbb{N}$,
Let: $a_n = u^n/n!$
⟨1⟩2. $|a_{n+1}/a_n| \to 0$ as $n \to \infty$
Proof: Since $a_{n+1}/a_n = u/(n+1)$
⟨1⟩3. $\sum_{n=0}^{\infty} a_n$ converges absolutely.
Proof: By the Ratio Test.
⟨1⟩4. $a_n \to 0$ as $n \to \infty$
□

Definition 1.17 (Rearrangement)**.** A *rearrangement* of a series $\sum_{n=0}^{\infty} a_n$ is a series of the form $\sum_{n=0}^{\infty} a_{\sigma(n)}$ for a permutation σ of $\mathbb{N}$.

Proposition 1.18. *Let $\sum_{n=0}^{\infty} a_{\sigma(n)}$ be a rearrangement of a series $\sum_{n=0}^{\infty} a_n$. If $\sum_{n=0}^{\infty} a_n$ converges absolutely, then $\sum_{n=0}^{\infty} a_{\sigma(n)}$ converges and $\sum_{n=0}^{\infty} a_{\sigma(n)} = \sum_{n=0}^{\infty} a_n$.*

Proof:
⟨1⟩1. For all $\epsilon > 0$, there exists N such that $\forall n \geq N. |\sum_{i=0}^{n} a_i - \sum_{i=0}^{n} a_{\sigma(i)}| < \epsilon$
⟨2⟩1. Let: $\epsilon > 0$
⟨2⟩2. Pick N such that $\forall m \geq n \geq N. \sum_{i=n}^{m} |a_i| < \epsilon/2$
⟨2⟩3. Let: P be the least integer such that $0, \ldots, N \in \{\sigma(0), \ldots, \sigma(P)\}$
⟨2⟩4. Let: $n \geq P$
⟨2⟩5. $|\sum_{i=0}^{n} a_i - \sum_{i=0}^{n} a_{\sigma(i)}|$
Proof:

$$\begin{aligned}
|\sum_{i=0}^{n} a_i - \sum_{i=0}^{n} a_{\sigma(i)}| &= |\sum_{i=N+1}^{n} a_i - \sum_{0 \leq i \leq n, \sigma(i) > N} a_{\sigma(i)}| \\
&\leq \sum_{i=N+1}^{n} |a_i| + \sum_{i=N+1}^{\max(\sigma(0),\ldots,\sigma(n))} |a_i| \\
&< \epsilon
\end{aligned}$$

⟨1⟩2. $(\sum_{i=0}^{n} a_i)$ and $(\sum_{i=0}^{n} a_{\sigma(i)})$ converge to the same limit.
□

Proposition 1.19. *Let $\sum_n a_n$ be a series that converges but does not converge absolutely. Let r be any real number. Then there exists a rearrangement of $\sum_n a_n$ that converges to r.*

Proof: The series has infinitely many positive terms and infinitely many negative terms. The subseries of positive terms diverges to $+\infty$ and the subseries of negative terms diverges to $-\infty$. Select positive terms until the sum is $> r$, then negative terms until the sum is $< r$, then positive terms until the sum is $> r$, etc. $\square$

Definition 1.20 (Infinite Decimal)**.** An *infinite decimal* $a_0.a_1a_2\cdots$ consists of an integer a_0 and a sequence $(a_1, a_2, \ldots)$ of natural numbers < 10.

Theorem 1.21. *Given any infinite decimal $a_0.a_1a_2\cdots$, the series $a_0 + \sum_{n=1}^{\infty} a_n 10^{-n}$ converges.*

Proof: By comparison with $\sum_n 10^{-(n-1)}$. $\square$

Definition 1.22. The sum $a_0 + \sum_{n=1}^{\infty} a_n 10^{-n}$ is the number *represented* by the infinite decimal $a_0.a_1a_2\cdots$.

Lemma 1.23. *Every real number is represented by an infinite decimal, unique except that $a_0.a_1a_2\cdots a_n000\cdots$ and $a_0.a_1a_2\cdots a_{n-1}(a_n - 1)999\cdots$ represent the same number.*