M2 Topology

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September 18, 2022

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1 Topology

Theorem 1 (Brouwer Retraction Theorem)**.** *Let $j : \{0, 1\} \to I$ be the inclusion. Then j has no continuous retraction.*

Theorem 2 (Brouwer Retraction Theorem)**.** *Let $j : C \to D$ be the inclusion of the circle in the disk. Then j has no continuous retraction.*

Theorem 3 (Brouwer Retraction Theorem)**.** *Let $j : S \to B$ be the inclusion of the sphere in the ball. Then j has no continuous retraction.*

Theorem 4. *Let A, S and B be objects of a category $\mathcal{C}$. Let $h : A \to B$, $j : S \to B$, $p : A \to S$ with $jp = h$. Suppose that, for any $a : S \to A$ and $s : S \to S$, if $ha = js$ then $pa = s$. Let $\alpha : B \to A$ and assume $h\alpha j = j$. Then $p\alpha$ is a retraction for j.*

Proof: We have $h(\alpha j) = j1$ and so $p(\alpha j) = 1$. □

Corollary 4.1. *Let A, S and B be objects of a category $\mathcal{C}$. Let $h : A \to B$, $j : S \to B$, $p : A \to S$ with $jp = h$. Suppose that, for any $a : S \to A$ and $s : S \to S$, if $ha = js$ then $pa = s$. Let $\alpha : B \to A$ and assume $h\alpha = 1$. Then $p\alpha$ is a retraction for j.*

Theorem 5. *Let A, S and B be objects of a category $\mathcal{C}$ with a terminal object 1. Let $h : A \to B$, $j : S \to B$, $p : A \to S$ with $jp = h$. Assume:*

1. *for any $a : S \to A$ and $s : S \to S$, if $ha = js$ then $pa = s$.*
2. *For any maps $f, g : B \to B$, either there exists $t : 1 \to B$ such that $ft = gt$, or there exists $\alpha : B \to A$ such that $h\alpha = g$.*

Let $f, g : B \to B$ with $gj = j$. Then either there is a point $b : 1 \to B$ with $fb = gb$, or j has a retraction.

PROOF: By the second hypothesis, either there exists $b : 1 \to B$ such that $fb = gb$, or there exists $\alpha : B \to A$ such that $h\alpha = g$. In the latter case, we have

$$\begin{aligned} h\alpha j &= gj \\ &= j \\ \therefore p\alpha j &= 1 \qquad\qquad \text{(Hypothesis 1)} \end{aligned}$$

Thus, $p\alpha$ is a retraction for j.

Corollary 5.1. *Let A, S and B be objects of a category $\mathcal{C}$ with a terminal object 1. Let $h : A \to B$, $j : S \to B$, $p : A \to S$ with $jp = h$. Assume:*

1. *for any $a : S \to A$ and $s : S \to S$, if $ha = js$ then $pa = s$.*
2. *For any maps $f, g : B \to B$, either there exists $t : 1 \to B$ such that $ft = gt$, or there exists $\alpha : B \to A$ such that $h\alpha = g$.*

Then either j has a retraction, or any map $f : B \to B$ has a fixed point.

PROOF: Take $g = 1_B$ in the theorem.

Theorem 6 (Brouwer Fixed Point Theorem)**.** *Let I be the line segment. Every continuous endomap $I \to I$ has a fixed point.*

PROOF: Apply Corollary **??** with $S = \{0, 1\}$, $B = I$ and A the set of all directed line segments in I of length > 0. Let $h : A \to B$ map any directed line segment to its head, $j : S \to A$ be the inclusion, and $p : A \to S$ be defined by $p(a) = 0$ if a points to the left, 1 if a points to the right.
Hypothesis 1 is obvious. For hypothesis 2, let $f, g : B \to B$ and suppose there is no t such that $ft = gt$. Then define $\alpha : B \to A$ by: $\alpha(x)$ is the directed line segment from fx to gx.
By the Brouwer Retraction Theorem, j has no retraction. Therefore every endomap $B \to B$ has a fixed point. □

Theorem 7 (Brouwer Fixed Point Theorem)**.** *Let D be the closed disk. Every continuous endomap $D \to D$ has a fixed point.*

PROOF: Apply Corollary **??** with S the circle, B the disk and A the set of all directed line segments in B of length > 0. Let $h : A \to B$ map any directed line segment to its head, $j : S \to A$ be the inclusion, and $p : A \to S$ be defined by: $p(a)$ is the point on the circle that a points to.
Hypothesis 1 is obvious. For hypothesis 2, let $f, g : B \to B$ and suppose there is no t such that $ft = gt$. Then define $\alpha : B \to A$ by: $\alpha(x)$ is the directed line segment from fx to gx.
By the Brouwer Retraction Theorem, j has no retraction. Therefore every endomap $B \to B$ has a fixed point. □

Theorem 8 (Brouwer Fixed Point Theorem)**.** *Every continuous endomap from the solid ball to itself has a fixed point.*

PROOF: Apply Corollary **??** with S the sphere, B the ball and A the set of all directed line segments in B of length > 0. Let $h : A \to B$ map any directed line segment to its head, $j : S \to A$ be the inclusion, and $p : A \to S$ be defined by: $p(a)$ is the point on the sphere that a points to.

Hypothesis 1 is obvious. For hypothesis 2, let $f, g : B \to B$ and suppose there is no t such that $ft = gt$. Then define $\alpha : B \to A$ by: $\alpha(x)$ is the directed line segment from fx to gx.

By the Brouwer Retraction Theorem, j has no retraction. Therefore every endomap $B \to B$ has a fixed point. $\square$