

Solutions Manual for Lawvere and Schanuel
Conceptual Mathematics

Robin Adams

September 20, 2022

Contents

I	Preview	3
1	Session 1 — Galileo and Multiplication of Objects	4
II	Part I — The category of sets	5
2	Article I — Sets, maps, composition	6
3	Session 3 — Composing maps and counting maps	7
III	Part II — The algebra of composition	8
4	Article II — Isomorphisms	9
4.1	1. Isomorphisms	9
4.2	2 — General division problems: Determination and choice	10
5	Session 4 — Division of Maps: Isomorphisms	12
5.1	4. A small zoo of isomorphisms in other categories	12
6	Session 5 — Division of Maps: Sections and Retractions	13
6.1	1. Determination Problems	13
6.2	3. Choice Problems	13
6.3	5. Stacking or Sorting	13
7	Session 9 — Retracts and Idempotents	14
7.1	1. Retracts and Comparisons	14
7.2	2. Idempotents as records of retracts	14
8	Quiz	15
9	Summary / quiz on pairs of 'opposed' maps	16
10	Test 1	17

11 Session 10 — Brouwer’s Theorems	18
11.1 4. Relation between fixed point and retraction theorems	18
11.2 7. Using maps to formulate guesses	19
 IV Part III — Categories of Structured Sets	 20
12 Article III — Examples of Categories	21
12.1 1. The category \mathcal{S}° of endomaps of sets	21
12.2 4. Categories of endomaps	21
12.3 5. Irreflexive graphs	23

Part I

Preview

Chapter 1

Session 1 — Galileo and Multiplication of Objects

Exercise 1 Many examples — every instance of a product in a category gives an example. I will not list them.

Exercise 2 I am not entirely sure what solution the authors had in mind. Here are some that come to my mind:

Place a spirit level between the two points and see if it reads as level.

Place a smooth plank between the two points and see if a ball placed at one point rolls to the other, or *vice versa*.

Hang a plumbline at each point and see if they form a right angle with the line joining the two points.

Of these, the third is my favourite.

Part II

Part I — The category of
sets

Chapter 2

Article I — Sets, maps, composition

Exercise 1 Easy.

Exercise 2 There are 8 maps from A to B .

Exercise 3 There are 27 maps from A to A .

Exercise 4 There are 9 maps from B to A .

Exercise 5 There are 4 maps from B to B .

Exercise 6 There are 10 such maps from A to A .

Exercise 7 There are 3 such maps from B to B .

Exercise 8 There is no such pair of maps.

Exercise 9 There are 12 such pairs of maps.

Chapter 3

Session 3 — Composing maps and counting maps

Exercise 1 (a) and (c) make sense.

Exercise 2 (a) and (c) still make sense.

Part III

Part II — The algebra of composition

Chapter 4

Article II — Isomorphisms

4.1 1. Isomorphisms

Exercise 1

(R) We have $1_A \circ 1_A = 1_A$ by the Identity Laws, so 1_A is an isomorphism with inverse 1_A .

(S) We have $g \circ f = 1_A$ and $f \circ g = 1_B$ (this is what it means for g to be an inverse for f). This says exactly that f is an inverse for g .

(T) Let $f^{-1} : B \rightarrow A$ be an inverse for f and $k^{-1} : C \rightarrow B$ be an inverse for k . We prove $f^{-1} \circ k^{-1}$ is an inverse for $k \circ f$. We have

$$\begin{aligned} f^{-1} \circ k^{-1} \circ k \circ f &= f^{-1} \circ 1_B \circ f && \text{(definition of inverse)} \\ &= f^{-1} \circ f && \text{(Identity Law)} \\ &= 1_A && \text{(definition of inverse)} \end{aligned}$$

and $k \circ f \circ f^{-1} \circ k^{-1} = 1_C$ similarly.

Exercise 2 We have

$$\begin{aligned} g &= g \circ 1_B && \text{(Identity Law)} \\ &= g \circ f \circ k && (k \text{ is an inverse of } f) \\ &= 1_A \circ k && (g \text{ is an inverse of } f) \\ &= k && \text{(Identity Law)} \end{aligned}$$

Exercise 3

(a) Let $f : A \rightarrow B$. Let $h, k : C \rightarrow A$.
Suppose $f \circ h = f \circ k$. Then

$$\begin{aligned} f^{-1} \circ f \circ h &= f^{-1} \circ f \circ k \\ \therefore 1_A \circ h &= 1_A \circ k && \text{(Definition of inverse)} \\ \therefore h &= k && \text{(Identity Law)} \square \end{aligned}$$

(b) Let $f : A \rightarrow B$. Let $h, k : B \rightarrow C$.
Suppose $h \circ f = k \circ f$. Then

$$\begin{aligned} h \circ f \circ f^{-1} &= k \circ f \circ f^{-1} \\ \therefore h \circ 1_B &= k \circ 1_B && \text{(Definition of inverse)} \\ \therefore h &= k && \text{(Identity Law)} \square \end{aligned}$$

(c) Let $A = \{0, 1\}$. Define $f : A \rightarrow A$ by $f(0) = 1$ and $f(1) = 0$. Define $h : A \rightarrow A$ by $h(x) = 0$ for all x . Define $k : A \rightarrow A$ by $k(x) = 1$ for all x .
 f is invertible, and is its own inverse.
We have $h \circ f = f \circ k = h$.
We do not have $h = k$.

Exercise 4

- (1) This function is invertible with inverse $f^{-1}(x) = (x - 7)/3$.
- (2) This function is invertible with inverse $g^{-1}(x) = \sqrt{x}$.
- (3) This function is not invertible because $h(1) = h(-1) = 1$.
- (4) This function is not invertible because $k(1) = k(-1) = 1$.
- (5) This function is not invertible because there is no x such that $l(x) = 2$.

4.2 2 — General division problems: Determination and choice

Exercise 5 There are 6 maps f such that $g \circ f = 1_{\{0,1\}}$; we can map 0 to any of b, p or q , and 1 to either of r or s .

Given any one of these maps f , there are 8 maps g such that $g \circ f = 1_{\{0,1\}}$. We must map $f(0)$ to 0, $f(1)$ to 1, and the other three elements to any of 0 or 1.

Exercise 6 If $r : B \rightarrow A$ is a section for f , then we take $t = g \circ r$. We have $t \circ f = g \circ r \circ f = g \circ 1_A = g$.

Exercise 7 Let $s : B \rightarrow A$ be a section for f . Let T be any set and $t_1, t_2 : T \rightarrow B$. Suppose $t_1 \circ f = t_2 \circ f$. Then

$$\begin{aligned} t_1 \circ f \circ s &= t_2 \circ f \circ s \\ \therefore t_1 \circ 1_B &= t_2 \circ 1_B \\ \therefore t_1 &= t_2 \end{aligned}$$

Exercise 8 If $s_1 : B \rightarrow A$ is a section for $r_1 : A \rightarrow B$ and $s_2 : C \rightarrow B$ is a section for $r_2 : B \rightarrow C$, then $s_1 \circ s_2$ is a section for $r_2 \circ r_1$ since

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ 1_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= 1_C \end{aligned}$$

Exercise 9 We have

$$\begin{aligned} e \circ e &= f \circ r \circ f \circ r \\ &= f \circ 1 \circ r && (r \text{ is a retraction of } f) \\ &= f \circ r \\ &= e \end{aligned}$$

Exercise 10 From the proof of Proposition 3, $f^{-1} \circ g^{-1}$ is both a section and a retraction for $g \circ f$.

Exercise 11 Set $f(\textit{Fatima}) = \textit{coffee}$, $f(\textit{Omer}) = \textit{tea}$ and $f(\textit{Alysia}) = \textit{cocoa}$. Then f is an isomorphism.

There is no isomorphism $g : A \rightarrow C$. For if $g(\textit{Fatima}) = \textit{true}$ then $g(\textit{Omer})$ must be *false*, and then it is impossible to choose a value for $g(\textit{Alysia})$ without having $g(\textit{Alysia}) = g(\textit{Fatima})$ or $g(\textit{Alysia}) = g(\textit{Omer})$. Similarly if $g(\textit{Fatima}) = \textit{false}$ then $g(\textit{Omer})$ must be *true*, and then again we cannot choose a value for $g(\textit{Alysia})$.

Chapter 5

Session 4 — Division of Maps: Isomorphisms

5.1 4. A small zoo of isomorphisms in other categories

Exercise 1 We have $h(d(x)) = h(2x) = x$ and $d(h(x)) = d(x/2) = x$ for any x .

Exercise 2 $f(\text{odd}) = \text{negative}$ and $f(\text{even}) = \text{positive}$

Exercise 3

(a) This is not an isomorphism because $p(0 + 0) = 1$ but $p(0) + p(0) = 2$

(b) This is not an isomorphism because it is not surjective; there is no x such that $sq(x) = -1$.

(c) This is not an isomorphism because it is not injective. We have $sq(1) = sq(-1) = 1$.

(d) This is an isomorphism; it is bijective and $-(x + y) = (-x) + (-y)$.

(e) This is not an isomorphism because $m(1 \times 1) = -1$ but $m(1) \times m(1) = 1$.

(f) This is not a well-defined map because $c(-1) = -1 \notin \mathbb{R}_{>0}$.

Chapter 6

Session 5 — Division of Maps: Sections and Retractions

6.1 1. Determination Problems

Exercise 1

- (a) Suppose $h = g \circ f$ and $fa_1 = fa_2$. Then $ha_1 = g(fa_1) = g(fa_2) = ha_2$.
- (b) No. Take $A = C = \emptyset$ and $B = \{*\}$. Let $f : A \rightarrow B$ and $h : A \rightarrow C$ be the unique such maps. Vacuously, if $fa_1 = fa_2$ then $ha_1 = ha_2$. But there is no map $g : B \rightarrow C$.

6.2 3. Choice Problems

Exercise 2

- (a) Suppose $g \circ f = h$. Let $a \in A$. Let $b = f(a)$. Then $h(a) = g(f(a)) = g(b)$.
- (b) This is equivalent to the Axiom of Choice.

6.3 5. Stacking or Sorting

Exercise 3 I'm not going to draw all of them, but there are 8 of them.

Chapter 7

Session 9 — Retracts and Idempotents

7.1 1. Retracts and Comparisons

Exercise 1 If A is empty, then the nowhere-defined function is a map $A \rightarrow B$.
If B has a point, say b , then the constant map with value b is a map $A \rightarrow B$.

7.2 2. Idempotents as records of retracts

Exercise 3 Suppose $s : A \rightarrow B$, $r : B \rightarrow A$ and $s' : A' \rightarrow B$, $r' : B \rightarrow A'$ are splittings of $e : B \rightarrow B$. Let

$$\begin{aligned} f &= r' \circ s & : A &\rightarrow A' \\ f^{-1} &= r \circ s' & : A' &\rightarrow A \end{aligned}$$

Then we have

$$\begin{aligned} f \circ f^{-1} &= r' \circ s \circ r \circ s' \\ &= r' \circ e \circ s' \\ &= r' \circ s' \circ r' \circ s' \\ &= 1 \\ f^{-1} \circ f &= r \circ s' \circ r' \circ s \\ &= r \circ e \circ s \\ &= r \circ s \circ r \circ s \\ &= 1 \end{aligned}$$

Chapter 8

Quiz

Question 1 Let $A = \{*\}$ and $B = \{0, 1\}$. Define $f : A \rightarrow B$ by $f(*) = 0$. Then the unique function $r : B \rightarrow A$ is a retraction for f (since $r(f(*)) = *$) but not a section for f (since $f(r(1)) = 0$). Therefore there is no section for f , since there is only one map $B \rightarrow A$.

Question 2

(a) Yes: if $ppp = p$ then $pqpq = pq$

(b) Yes: if $ppp = p$ then $qpqp = qp$

Question 2* Let $q' = qpq$ Then we have

$$\begin{aligned}pq'p &= pqpqp \\&= pqp \\&= p \\q'pq' &= qpqpqpq \\&= qpqpq \\&= qpq \\&= q'\end{aligned}$$

Question 1* Take $A = B = \mathbb{N}$ and define $f : A \rightarrow B$ by $f(x) = 2x$. Then f has a retraction r given by

$$r(y) = \begin{cases} y/2 & \text{if } y \text{ is even} \\ 0 & \text{if } y \text{ is odd} \end{cases}$$

It has no section since it is not surjective (Article II Proposition 1).

Chapter 9

Summary / quiz on pairs of 'opposed' maps

Question 1 Given two maps f, g with domains and codomains as above, we can always form the composites $g \circ f$ and $f \circ g$. All we can say about $g \circ f$ and $f \circ g$ as maps in themselves is that they are endomaps.

Question 2 If we know that g is a retraction for f , that means $g \circ f$ is actually the identity map 1_A ; then we can prove that $f \circ g$ is not only an endomap, but actually an idempotent. The latter means that the equation $f \circ g \circ f \circ g = f \circ g$ is true.

Question 3 If we even know that f is an isomorphism *and* that $g \circ f = 1_A$, then $f \circ g$ is not only an idempotent, but is the identity map 1_B . If, moreover, s is a map for which $f \circ s = 1_B$, we can conclude that $s = g$.

Question 4 Going back to 0, i.e. assuming no equations, but only the domain and codomain statements about f and g , the composite $f \circ g \circ f$ could be different from f . Likewise $f \circ g \circ f \circ g$ could be different from $f \circ g$.

Chapter 10

Test 1

Question 1

(a) Let $f(Mara) = Aurelio$, $f(Aurelio) = Mara$ and $f(Andrea) = Andrea$.

(b) Let $e(Mara) = Aurelio$, $e(Aurelio) = Aurelio$ and $e(Andrea) = Andrea$.

(c) Let $B = \{0, 1\}$. Define $s : B \rightarrow A$ by $s(0) = Aurelio$ and $s(1) = Andrea$. Define $r : A \rightarrow B$ by $r(Mara) = 0$, $r(Aurelio) = 0$ and $r(Andrea) = 1$.

Question 2 Define $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(y) = (y + 7)/4$.

(a) $g(f(x)) = g(4x - 7) = (4x - 7 + 7)/4 = 4x/4 = x$

(b) $f(g(x)) = f((x + 7)/4) = 4((x + 7)/4) - 7 = x + 7 - 7 = x$

Chapter 11

Session 10 — Brouwer's Theorems

11.1 4. Relation between fixed point and retraction theorems

Exercise 1 Suppose for a contradiction there is no point x such that $f(x) = g(x)$. Define $r : D \rightarrow C$ as follows: for $x \in D$, $r(x)$ is the point on C that is pointed at by the arrow with tail at $f(x)$ and head at $g(x)$. For $x \in C$, we have $g(j(x)) = j(x)$, so the point that is pointed at by any arrow with head at $g(j(x))$ is x . Hence

$$r(j(x)) = x$$

and so r is a retraction for j , contradicting the retraction theorem.

Exercise 2 Let $f : A \rightarrow A$ be any endomap. Then $s \circ f \circ r : X \rightarrow X$ is an endomap on X . Hence there exists $x : T \rightarrow X$ such that $sfrx = x$. But then we have

$$\begin{aligned} rsfrx &= rx \\ \therefore frx &= rx \end{aligned}$$

and so $r \circ x : T \rightarrow A$ is a fixed point of f .

Exercise 3 Let A be either E , C or S , and X be I , D or B respectively. Assume that every endomap $X \rightarrow X$ has a fixed point.

Assume for a contradiction that X is a retract of A . By Exercise 2, every endomap on A has a fixed point. This is a contradiction, as the antipodal map on A has no fixed point.

11.2 7. Using maps to formulate guesses

Exercise 1

(a) We can express 'I start in Buffalo and end in Rochester' as $m \circ j = i \circ j$.
We can express 'You start and finish anywhere between Buffalo and Rochester'
as: there exists $f : I \rightarrow E$ such that $y \circ j = i \circ f$.

(b) There exists $t : 1 \rightarrow I$ such that $mt = yt$.

(c) Let C be the circle, D the disk and P the plane. Let $j : C \rightarrow D$ and
 $i : D \rightarrow P$ be the inclusions.

For any maps $m, y : D \rightarrow P$ such that:

- $mj = ij$
- there exists $f : C \rightarrow D$ such that $yj = if$

then there exists $t : 1 \rightarrow D$ such that $mt = yt$.

(d) I have not been able to find any smooth maps for which it is not true.

Part IV

Part III — Categories of Structured Sets

Chapter 12

Article III — Examples of Categories

12.1 1. The category \mathcal{S}° of endomaps of sets

Exercise 1 Let $f : (X, \alpha) \rightarrow (Y, \beta)$ and $g : (Y, \beta) \rightarrow (Z, \gamma)$. Then

$$g \circ f \circ \alpha = g \circ \beta \circ f = \gamma \circ g \circ f$$

and so $g \circ f : (X, \alpha) \rightarrow (Z, \gamma)$.

12.2 4. Categories of endomaps

Exercise 2 Suppose $e : A \rightarrow A$ is idempotent and has a retraction $r : A \rightarrow A$. Then

$$1_A = r \circ e = r \circ e \circ e = 1_A \circ e = e$$

so $e = 1_A$. Thus, the identities are the only idempotents that have retractions.

Exercise 3 Suppose A has an even number of elements, say $\{a_1, a_2, \dots, a_{2n}\}$. Define $\theta : A \rightarrow A$ by $\theta(a_{2k+1}) = a_{2k+2}$ and $\theta(a_{2k+2}) = a_{2k+1}$ ($0 \leq k < n$). Then θ is an involution with no fixed point.

Conversely, suppose $\theta : A \rightarrow A$ is an involution with no fixed point. Enumerate the elements of A as follows: Pick any element $a_1 \in A$. Let $a_2 = \theta(a_1)$; then $a_1 = \theta(a_2)$.

Assuming we have picked a_1, \dots, a_{2m} such that $\{a_1, \dots, a_{2m}\}$ is closed under θ and $A \neq \{a_1, \dots, a_{2m}\}$, pick $a_{2m+1} \in A - \{a_1, \dots, a_{2m}\}$. Then $\theta(a_{2m+1}) \notin \{a_1, \dots, a_{2m}\}$ (since $\theta(\theta(a_{2m+1})) = a_{2m+1}$) and $\theta(a_{2m+1}) \neq a_{2m+1}$ (since θ has no fixed point). So let $a_{2m+2} = \theta(a_{2m+1})$.

This process must end because A is finite. So $A = \{a_1, \dots, a_{2n}\}$ for some n .

Suppose now A has an odd number of elements, say $A = \{a_1, a_2, \dots, a_{2n+1}\}$. Define $\theta : A \rightarrow A$ by

$$\begin{aligned}\theta(a_{2k+1}) &= a_{2k+2} & (0 \leq k < n) \\ \theta(a_{2k+2}) &= a_{2k+1} & (0 \leq k < n) \\ \theta(a_{2n+1}) &= a_{2n+1}\end{aligned}$$

Then θ is an involution whose only fixed point is a_{2n+1} .

Conversely, suppose $\theta : A \rightarrow A$ is an involution with one fixed point f . Then $\theta \upharpoonright (A - \{f\})$ is an involution on $A - \{f\}$ with no fixed point. So $A - \{f\}$ has an even number of elements, and so A has an odd number of elements.

Exercise 4 The map α is an involution because $-(-x) = x$. It is not idempotent because $-(-1) \neq -1$. Its only fixed point is 0.

Exercise 5 The map α is not an involution because $\| -1 \| = 1 \neq -1$. It is idempotent because $\|x\| = |x|$. Its fixed points are the non-negative integers.

Exercise 6 The map α is an automorphism with inverse $\alpha^{-1}(x) = x - 3$.

Exercise 7 The map α is not an automorphism because there is no integer x with $\alpha(x) = 1$.

Exercise 8 If α is idempotent then $\alpha \circ \alpha \circ \alpha = \alpha \circ \alpha = \alpha$.

If α is an involution then $\alpha \circ \alpha \circ \alpha = 1 \circ \alpha = \alpha$.

Exercise 9 Label the elements in the diagram 0, 1, 2 from top to bottom. Then

$$\begin{aligned}\alpha^3(0) &= \alpha^2(1) = \alpha(2) = 1 \\ &= \alpha(0) \\ \alpha^3(1) &= \alpha^2(2) = \alpha(1) = 2 \\ &= \alpha(1) \\ \alpha^3(2) &= \alpha^2(1) = \alpha(2) = 1 \\ &= \alpha(2)\end{aligned}$$

Thus, $\alpha^3 = \alpha$.

However, α is not idempotent because $\alpha^2(0) = 2 \neq \alpha(0)$. And α is not an involution because $\alpha^2(0) = 2 \neq 0$.

12.3 5. Irreflexive graphs

Exercise 10

$$s(a) = k, s(b) = m, s(c) = k, s(d) = p, s(e) = m$$

$$t(a) = m, t(b) = m, t(c) = m, t(d) = q, t(e) = r$$

The arrow b has $s(b) = t(b)$. There is no arrow x with $t(x) = k$.

Exercise 11

 We have

$$s'' \circ g \circ f = g \circ s' \circ f = g \circ f \circ s$$

$$t'' \circ g \circ f = g \circ t' \circ f = g \circ f \circ t$$

and so $g \circ f : (X, P, s, t) \rightarrow (Z, R, s'', t'')$.