Solutions Manual for Enderton *Elements of Set Theory*

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Contents

Chapter 1

Chapter 1 — Introduction

1.1 Baby Set Theory

Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

Exercise 2   We have $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ has an element (namely $\emptyset$) while $\emptyset$ has no elements.

We have $\emptyset \neq \{\{\emptyset\}\}$ because $\{\{\emptyset\}\}$ has an element (namely $\{\emptyset\}$) while $\emptyset$ has no elements.

We have $\{\emptyset\} \neq \{\{\emptyset\}\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$. This last fact is true because $\emptyset \neq \{\emptyset\}$ as we proved in the first paragraph.

Exercise 3   Assume $B \subseteq C$. Let $A \in \mathcal{P}B$; we must show that $A \in \mathcal{P}C$.

We have $A \subseteq B$ (since $A \in \mathcal{P}B$) and $B \subseteq C$. From this it follows that $A \subseteq C$ (every element of A is an element of B; every element of B is an element of C; therefore every element of A is an element of C). Hence $A \in \mathcal{P}C$ as required.

Exercise 4   Since $x \in B$, we have $\{x\} \subseteq B$ and so $\{x\} \in \mathcal{P}B$.

Since $x \in B$ and $y \in B$, we have $\{x, y\} \subseteq B$ and so $\{x, y\} \in \mathcal{P}B$.

From these two facts, it follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$ and so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

1.2   Sets — An Informal View

Exercise 5   We have

$$
\begin{aligned}
V_0 &= A \\
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}A \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= \{\emptyset, \{\emptyset\}\} \\
V_3 &= \mathcal{P}V_2 \\
&= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}
$$

We have $\emptyset \subseteq V_0$ and so $\emptyset \in V_1$. Therefore $\{\emptyset\} \subseteq V_1$ and so $\{\emptyset\} \in V_2$. Hence $\{\{\emptyset\}\} \subseteq V_2$.

We also have $\{\{\emptyset\}\} \not\subseteq V_0$ because $\{\emptyset\}$ is not an atom, and $\{\{\emptyset\}\} \not\subseteq V_1$ since $\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.

Thus the rank of $\{\{\emptyset\}\}$ is 2.

Likewise we have $\emptyset$ and $\{\emptyset\}$ are both subsets of V_1, hence

$$\emptyset \in V_2, \qquad \{\emptyset\} \in V_2$$

Thus $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are all subsets of V_2, hence elements of V_3. Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is not a subset of V_0 (because $\emptyset$ is not an atom.) It is not a subset of V_1 ($\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.) It is not a subset of V_2 (we have $\{\emptyset, \{\emptyset\}\} \notin V_2$ since $\{\emptyset\} \notin V_1$).

Therefore the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

Exercise 6

$$V_1 = V_0 \cup \mathcal{P}V_0$$
$$= A \cup \mathcal{P}V_0 \qquad\qquad (\text{since } V_0 = A)$$
$$V_2 = V_1 \cup \mathcal{P}V_1$$
$$= A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1$$
$$= A \cup \mathcal{P}V_1 \qquad\qquad (\text{since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3})$$
$$V_3 = V_2 \cup \mathcal{P}V_2$$
$$= A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2$$
$$= A \cup \mathcal{P}V_2 \qquad\qquad (\text{since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3})$$
$$V_4 = V_3 \cup \mathcal{P}V_3$$
$$= A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3$$
$$= A \cup \mathcal{P}V_3 \qquad\qquad (\text{since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3})$$

Exercise 7 In Exercise 5 we calculated $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Hence

$$V_4 = \mathcal{P}V_3$$
$$= \{\emptyset,$$
$$\{\emptyset\},$$
$$\{\{\emptyset\}\},$$
$$\{\{\{\emptyset\}\}\},$$
$$\{\{\emptyset, \{\emptyset\}\}\},$$
$$\{\emptyset, \{\emptyset\}\},$$
$$\{\emptyset, \{\{\emptyset\}\}\},$$
$$\{\emptyset, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\{\emptyset\}, \{\{\emptyset\}\}\},$$
$$\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\},$$
$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\},$$
$$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$
$$\}$$

Chapter 2

Chapter 2 — Axioms and Operations

2.1 Arbitrary Unions and Intersections

Exercise 1 $A \cap B \cap C$ is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

Exercise 2   Take $A = \emptyset$ and $B = \{\emptyset\}$. Then $\bigcup A = \bigcup B = \emptyset$ but $A \neq B$. (There are many other possible answers.)

Exercise 3   Let $b \in A$. We must show that $b \subseteq \bigcup A$.
   Let x be any element of b. We must show that $x \in \bigcup A$. We know that $x \in b$ and $b \in A$, and so $x \in \bigcup A$ by the definition of $\bigcup A$.

Exercise 4   Suppose $A \subseteq B$. Let $x \in \bigcup A$. We must show that $x \in \bigcup B$.
   Pick an element $a \in A$ such that $x \in a$. Then $a \in B$ because $A \subseteq B$. Since we know $x \in a$ and $a \in B$, we know that $x \in \bigcup B$.

Exercise 5   Assume that every member of $\mathcal{A}$ is a subset of B. Let $x \in \bigcup \mathcal{A}$. We must show that $x \in B$.
   Pick $A \in \mathcal{A}$ such that $x \in A$. By our assumption, we have $A \subseteq B$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$ as required.

Exercise 6

   (a)   We will show that $\bigcup \mathcal{P}A \subseteq A$ and $A \subseteq \bigcup \mathcal{P}A$.
   To show $\bigcup \mathcal{P}A \subseteq A$: This follows from Exercise 5, since every member of $\mathcal{P}A$ is a subset of A.

To show $A \subseteq \bigcup \mathcal{P}A$: Let $a \in A$. Then we have $a \in \{a\}$ and $\{a\} \in \mathcal{P}A$ so $a \in \bigcup \mathcal{P}A$.

(b) To show $A \subseteq \mathcal{P} \bigcup A$: This holds because every element of A is a subset of $\bigcup A$, as we proved is Exercise 3.

Equality holds if and only if $A = \mathcal{P}X$ for some set X.

Proof: If $A = \mathcal{P} \bigcup A$ then of course $A = \mathcal{P}X$ for some X.

Conversely, if $A = \mathcal{P}X$, then we have

$$\mathcal{P} \bigcup A = \mathcal{P} \bigcup \mathcal{P}X$$
$$= \mathcal{P}X \qquad \qquad \text{(by part (a))}$$
$$= A$$

Exercise 7

(a) For any set X,

$$X \in \mathcal{P}A \cap \mathcal{P}B$$
$$\Leftrightarrow X \subseteq A \text{ and } X \subseteq B$$
$$\Leftrightarrow \text{Every member of } X \text{ is a member of } A \text{ and a member of } B$$
$$\Leftrightarrow X \subseteq A \cap B$$
$$\Leftrightarrow X \in \mathcal{P}(A \cap B)$$

(b) Let $X \in \mathcal{P}A \cup \mathcal{P}B$. Then either $X \in \mathcal{P}A$ or $X \in \mathcal{P}B$ (or both). If $X \in \mathcal{P}A$, then we have $X \subseteq A$ and so $X \subseteq A \cup B$ (because $A \subseteq A \cup B$). Similarly if $X \in \mathcal{P}B$ then we have $X \subseteq A \cup B$. So in either case $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

Equality holds if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof: Suppose $A \subseteq B$. Then $\mathcal{P}A \subseteq \mathcal{P}B$ (Chapter 1 Exercise 3) and so $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$. Also $A \cup B = B$ so $\mathcal{P}(A \cup B) = \mathcal{P}B$. Thus $\mathcal{P}A \cup \mathcal{P}B$ and $\mathcal{P}(A \cup B)$ are equal.

Similarly if $B \subseteq A$ then $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$.

Conversely, suppose $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$. We have $A \cup B \in \mathcal{P}(A \cup B)$, so $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$. If $A \cup B \in \mathcal{P}A$, then we have $B \subseteq A \cup B \subseteq A$. And if $A \cup B \in \mathcal{P}B$, then we have $A \subseteq A \cup B \subseteq B$.

Exercise 8 If A is a set such that every singleton belongs to A, then every set belongs to $\bigcup A$, contradicting Theorem 2A.

Exercise 9 Let $a = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$. Then $a \in B$ but $\mathcal{P}a$ is not a subset of B because $\emptyset \in \mathcal{P}a$ and $\emptyset \notin B$.

Exercise 10   We must show that $\mathcal{P}a \subseteq \mathcal{P}\bigcup B$. So let $X \in \mathcal{P}a$. Then $X \subseteq a$; we must show that $X \subseteq \bigcup B$.

Let $x \in X$; we must show that $x \in \bigcup B$. We have $x \in a$ (because $x \in X$ and $X \subseteq a$) and $a \in B$, hence $x \in \bigcup B$ as required.

2.2   Algebra of Sets

Exercise 11   For any x we have

$$x \in (A \cap B) \cup (A - B) \Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B)$$
$$\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B)$$
$$\Leftrightarrow x \in A$$

Hence $A = (A \cap B) \cup (A - B)$.

For any x we have

$$x \in A \cup (B - A) \Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A)$$
$$\Leftrightarrow x \in A \text{ or } x \in B$$
$$\Leftrightarrow x \in A \cup B$$

Hence $A \cup (B - A) = A \cup B$.

Exercise 12   For any x,

$$x \in C - (A \cap B) \Leftrightarrow x \in C \& \neg(x \in A \& x \in B)$$
$$\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B)$$
$$\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B)$$
$$\Leftrightarrow x \in (C - A) \cup (C - B)$$

Exercise 13   Suppose $A \subseteq B$. Let $x \in C - B$; we must show $x \in C - A$. We have $x \in C$ and $x \notin B$. Therefore $x \notin A$, since every member of A is a member of B. And so we have $x \in C - A$ as required.

Exercise 14   Let $A = \{\emptyset\}$, $B = \emptyset$ and $C = \{\emptyset\}$. Then $A - (B - C) = A - \emptyset = \{\emptyset\}$ while $(A - B) - C = \{\emptyset\} - C = \emptyset$.

Exercise 15

   (a)   For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B+C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B+C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B+C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have $x \in A \cap (B+C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$.

(b) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B+C)$	$x \in (A+B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B+C)$	$x \notin (A+B) + C$

In every case, we have $x \in A + (B+C) \Leftrightarrow x \in (A+B) + C$.

Exercise 16

$$[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B-C)) \cap A] = (A \cup B) - A$$
$$= B - A$$

Exercise 17

(a) $\Leftrightarrow$ **(b)**

$A \subseteq B \Leftrightarrow$ Every element of A is an element of B

$\quad\quad\quad \Leftrightarrow$ There is no element of A that is not an element of B

$\quad\quad\quad \Leftrightarrow A - B = \emptyset$

(a) $\Rightarrow$ **(c)** Suppose $A \subseteq B$. We have $B \subseteq A \cup B$ from the definition of $A \cup B$; we must prove that $A \cup B \subseteq B$. So let $x \in A \cup B$. Then $x \in A$ or $x \in B$. But in either case $x \in B$, since $x \in A \Rightarrow x \in B$. Thus we have $x \in B$ as required.

(c) $\Rightarrow$ **(a)** We always have $A \subseteq A \cup B$. So if $A \cup B = B$ then we have $A \subseteq B$.

(a) $\Rightarrow$ **(d)** Suppose $A \subseteq B$. We have $A \cap B \subseteq A$ from the definition of $A \cap B$; we must prove that $A \subseteq A \cap B$. So let $x \in A$. Then $x \in B$ since $A \subseteq B$, hence $x \in A \cap B$ as required.

(d) ⇒ (a)   We always have $A \cap B \subseteq B$. So if $A \cap B = A$ then $A \subseteq B$.

Exercise 18   We can make the following 16 sets:

- $\emptyset \ (= A - A)$
- $A - B$
- $A \cap B$
- $B - A$
- $S - (A \cup B)$
- A
- $A + B$
- $S - B$
- B
- $S - (A + B)$
- $S - A$
- $A \cup B$
- $S - (B - A)$
- $S - (A \cap B)$
- $S - (A - B)$

Exercise 19   They are never equal, because for all A, B, we have $\emptyset \in \mathcal{P}(A - B)$ but $\emptyset \notin \mathcal{P}A - \mathcal{P}B$ since $\emptyset \in \mathcal{P}B$.

Exercise 20   Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

We first show $B \subseteq C$. Let $x \in B$; we show $x \in C$. We have $x \in A \cup B = A \cup C$, so either $x \in A$ or $x \in C$. If $x \in C$, we are done. If $x \in A$, then we have $x \in A \cap B = A \cap C$, and so $x \in C$ in this case too.

We can show $C \subseteq B$ similarly. Hence $B = C$.

Exercise 21   For any x, we have

$$x \in \bigcup (A \cup B) \Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C$$

$$\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C, \text{ or there exists } C \in B \text{ such that } x \in C$$

$$\Leftrightarrow x \in \bigcup A \cup \bigcup B$$

Exercise 22   For any x, we have

$$x \in \bigcap(A \cup B) \Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C$$
$$\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C$$
$$\Leftrightarrow x \in \bigcap A \cap \bigcap B$$

Exercise 23   Proof:
$\langle 1 \rangle 1.$ $A \subseteq \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 1.$ LET: $x \in A$
  $\langle 2 \rangle 2.$ LET: $X \in \mathcal{B}$
  $\langle 2 \rangle 3.$ $x \in A \cup X$
$\langle 1 \rangle 2.$ $\bigcap \mathcal{B} \subseteq \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 1.$ LET: $x \in \bigcap \mathcal{B}$
  $\langle 2 \rangle 2.$ LET: $X \in \mathcal{B}$
  $\langle 2 \rangle 3.$ $x \in X$
  $\langle 2 \rangle 4.$ $x \in A \cup X$
$\langle 1 \rangle 3.$ $\bigcap\{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap \mathcal{B}$
  $\langle 2 \rangle 1.$ LET: $x \in \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 2.$ ASSUME: $x \notin A$
       PROVE: $x \in \bigcap \mathcal{B}$
  $\langle 2 \rangle 3.$ LET: $X \in \mathcal{B}$
  $\langle 2 \rangle 4.$ $x \in A \cup X$
  $\langle 2 \rangle 5.$ $x \in X$
□

Exercise 24

  (a)

$$Y \in \mathcal{P}\bigcap \mathcal{A} \Leftrightarrow Y \subseteq \bigcap \mathcal{A}$$
$$\Leftrightarrow \forall y \in Y. \forall X \in \mathcal{A}. y \in X$$
$$\Leftrightarrow \forall X \in \mathcal{A}. \forall y \in Y. y \in X$$
$$\Leftrightarrow \forall X \in \mathcal{A}. Y \in \mathcal{P}X$$
$$\Leftrightarrow Y \in \bigcap\{\mathcal{P}X \mid X \in \mathcal{A}\}$$

  (b) $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} \subseteq \mathcal{P}\bigcup \mathcal{A}$

Proof:
$\langle 1 \rangle 1.$ LET: $Y \in \bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\}$
$\langle 1 \rangle 2.$ PICK $X \in \mathcal{A}$ such that $Y \in \mathcal{P}X$
$\langle 1 \rangle 3.$ $Y \subseteq X$
$\langle 1 \rangle 4.$ $Y \subseteq \bigcup \mathcal{A}$
$\langle 1 \rangle 5.$ $Y \in \mathcal{P}\bigcup \mathcal{A}$

Equality holds if and only if $\bigcup \mathcal{A} \in \mathcal{A}$.

$\langle 1 \rangle 1$. If $\bigcup \{ \mathcal{P} X \mid X \in \mathcal{A} \} = \mathcal{P} \bigcup \mathcal{A}$ then $\bigcup \mathcal{A} \in \mathcal{A}$
   $\langle 2 \rangle 1$. ASSUME: $\bigcup \{ \mathcal{P} X \mid X \in \mathcal{A} \} = \mathcal{P} \bigcup \mathcal{A}$
   $\langle 2 \rangle 2$. $\bigcup \mathcal{A} \in \bigcup \{ \mathcal{P} X \mid X \in \mathcal{A} \}$
   $\langle 2 \rangle 3$. PICK $X \in \mathcal{A}$ such that $\bigcup \mathcal{A} \in \mathcal{P} X$
   $\langle 2 \rangle 4$. $X = \bigcup \mathcal{A}$
$\langle 1 \rangle 2$. If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\bigcup \{ \mathcal{P} X \mid X \in \mathcal{A} \} = \mathcal{P} \bigcup \mathcal{A}$
   PROOF: If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\mathcal{P} \bigcup \mathcal{A} \in \{ \mathcal{P} X \mid X \in \mathcal{A} \}$.
∎

Exercise 25   We have $A \cup \bigcup \mathcal{B} = \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$ if and only if $A = \emptyset$ or $\mathcal{B} \neq \emptyset$

$\langle 1 \rangle 1$. If $A \cup \bigcup \mathcal{B} = \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$ then $A = \emptyset$ or $\mathcal{B} \neq \emptyset$
   PROOF: If $A \cup \bigcup \mathcal{B} = \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$ and $\mathcal{B} = \emptyset$ then
$$A \cup \bigcup \emptyset = \bigcup \emptyset$$
$$\therefore A = \emptyset$$
$\langle 1 \rangle 2$. If $A = \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$
   PROOF: Both sides are equal to $\bigcup \mathcal{B}$
$\langle 1 \rangle 3$. If $\mathcal{B} \neq \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$
   $\langle 2 \rangle 1$. ASSUME: $\mathcal{B} \neq \emptyset$
   $\langle 2 \rangle 2$. $A \cup \bigcup \mathcal{B} \subseteq \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$
     $\langle 3 \rangle 1$. LET: $x \in A \cup \bigcup \mathcal{B}$
       PROVE: $x \in \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$
     $\langle 3 \rangle 2$. CASE: $x \in A$
       $\langle 4 \rangle 1$. PICK $X \in \mathcal{B}$
       PROOF: By $\langle 2 \rangle 1$
       $\langle 4 \rangle 2$. $x \in A \cup X$
     $\langle 3 \rangle 3$. CASE: $x \in \bigcup \mathcal{B}$
       $\langle 4 \rangle 1$. PICK $X \in \mathcal{B}$ such that $x \in X$
       $\langle 4 \rangle 2$. $x \in A \cup X$
   $\langle 2 \rangle 3$. $\bigcup \{ A \cup X \mid X \in \mathcal{B} \} \subseteq A \cup \bigcup \mathcal{B}$
     $\langle 3 \rangle 1$. LET: $x \in \bigcup \{ A \cup X \mid X \in \mathcal{B} \}$
     $\langle 3 \rangle 2$. PICK $X \in \mathcal{B}$ such that $x \in A \cup X$
     $\langle 3 \rangle 3$. $X \subseteq \bigcup \mathcal{B}$
     $\langle 3 \rangle 4$. $A \cup X \subseteq A \cup \bigcup \mathcal{B}$
     $\langle 3 \rangle 5$. $x \in A \cup \bigcup \mathcal{B}$

2.3   Review Exercises

Exercise 26   Sets A, B, D and F are all equal to each other. Sets C, E and G are equal to each other. None of the first list is equal to any of the second list.

Exercise 27 Take $A = \{\{0\}, \{1\}\}$ and $B = \{\{1\}\}$. Then $A \cap B = \{\{1\}\}$ and

$$\bigcap A \cap \bigcap B = \emptyset \cap \{1\}$$
$$= \emptyset$$
$$\bigcap (A \cap B) = \bigcap \{\{1\}\}$$
$$= \{1\}$$

Exercise 28

$$\bigcup \{\{3, 4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\} = \{3, 4, \{3\}, \{4\}\}$$

Exercise 29

(a) $\emptyset$

(b) We have

$$\{\emptyset\} \subseteq \mathcal{P}\{\emptyset\}$$
$$\therefore \mathcal{P}\{\emptyset\} \subseteq \mathcal{PP}\{\emptyset\}$$
$$\{\emptyset\} \subseteq \mathcal{PP}\{\emptyset\}$$
$$\therefore \mathcal{P}\{\emptyset\} \subseteq \mathcal{PPP}\{\emptyset\}$$
$$\therefore \bigcap \{\mathcal{PPP}\{\emptyset\}, \mathcal{PP}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} = \mathcal{PPP}\{\emptyset\} \cap \mathcal{PP}\{\emptyset\} \cap \mathcal{P}\{\emptyset\}$$
$$= \mathcal{P}\{\emptyset\}$$
$$= \{\emptyset, \{\emptyset\}\}$$

Exercise 30

(a) $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$

(b) $\{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d) $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

Exercise 31

(a) $\{1, 2, 3, \emptyset\}$

(b) $\emptyset$

(c) $\emptyset$

(d) $\emptyset$

Exercise 32

(a) $a \cup b$

(b) a

(c)

$$\bigcap\bigcup S \cup \left(\bigcup\bigcup S - \bigcup\bigcap S\right) = (a \cap b) \cup ((a \cup b) - a)$$
$$= (a \cap b) \cup (b - a)$$
$$= b$$

Exercise 33 When $a \neq b$:

$$\bigcup\left(\bigcup S - \bigcap S\right) = \bigcup(\{a, b\} - \{a\})$$
$$= \bigcup\{b\}$$
$$= b$$

When $a = b$:

$$\bigcup\left(\bigcup S - \bigcap S\right) = \bigcup(\{a, b\} - \{a\})$$
$$= \bigcup \emptyset$$
$$= \emptyset$$

Exercise 34 For any set S, we have

$$\emptyset \subseteq \mathcal{P}S$$
$$\therefore \emptyset \in \mathcal{P}\mathcal{P}S$$
$$\emptyset \subseteq S$$
$$\therefore \emptyset \in \mathcal{P}S$$
$$\therefore \{\emptyset\} \subseteq \mathcal{P}S$$
$$\therefore \{\emptyset\} \in \mathcal{P}\mathcal{P}S$$
$$\therefore \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}\mathcal{P}S$$
$$\therefore \{\emptyset, \{\emptyset\}\} \in \mathcal{P}\mathcal{P}\mathcal{P}S$$

Exercise 35 Assume $\mathcal{P}A = \mathcal{P}B$. Then we have

$$A \in \mathcal{P}A$$
$$\therefore A \in \mathcal{P}B$$
$$\therefore A \subseteq B$$
$$B \in \mathcal{P}B$$
$$\therefore B \in \mathcal{P}A$$
$$\therefore B \subseteq A$$
$$\therefore A = B$$

Exercise 36

(a)

$$x \in A - (A \cap B) \Leftrightarrow x \in A \,\&\, \neg(x \in A \,\&\, x \in B)$$
$$\Leftrightarrow x \in A \,\&\, x \notin B$$
$$\Leftrightarrow x \in A - B$$

(b)

$$x \in A - (A - B) \Leftrightarrow x \in A \,\&\, \neg(x \in A \,\&\, x \notin B)$$
$$\Leftrightarrow x \in A \,\&\, x \in B$$
$$\Leftrightarrow x \in A \cap B$$

Exercise 37

(a)

$$x \in (A \cup B) - C \Leftrightarrow (x \in A \text{ or } x \in B) \,\&\, x \notin C$$
$$\Leftrightarrow (x \in A \,\&\, x \notin C) \text{ or } (x \in B \,\&\, x \notin C)$$
$$\Leftrightarrow x \in (A - C) \cup (B - C)$$

(b)

$$x \in A - (B - C) \Leftrightarrow x \in A \,\&\, \neg(x \in B \,\&\, x \notin C)$$
$$\Leftrightarrow x \in A \,\&\, (x \notin B \text{ or } x \in C)$$
$$\Leftrightarrow (x \in A \,\&\, x \notin B) \text{ or } (x \in A \,\&\, x \in C)$$
$$\Leftrightarrow x \in (A - B) \cup (A \cap C)$$

(c)

$$x \in (A - B) - C \Leftrightarrow x \in A \,\&\, x \notin B \,\&\, x \notin C$$
$$\Leftrightarrow x \in A \,\&\, \neg(x \in B \vee x \in C)$$
$$\Leftrightarrow x \in A - (B \cup C)$$

Exercise 38

    (a)   If every element of A is an element of C, and every element of B is an element of C, then everything that is an element of either A or B is an element of C.

    (b)   If every element of C is an element of A, and every element of C is an element of B, then every element of C is an element of both A and B.

Chapter 3

Chapter 3 — Relations and Functions

3.1 Ordered Pairs

Exercise 1   We have $\langle 0, 1, 0 \rangle^* = \langle 0, 1, 1 \rangle^* = \{\{0\}, \{0, 1\}\}$.

Exercise 2

    (a)

$$z \in A \times (B \cup C)$$
$$\Leftrightarrow \exists x, y(z = (x, y) \ \& \ x \in A \ \& \ (y \in B \text{ or } y \in C))$$
$$\Leftrightarrow \exists x, y(z = (x, y) \ \& \ x \in A \ \& \ y \in B) \text{ or } (z = (x, y) \ \& \ x \in A \ \& \ y \in C)$$
$$\Leftrightarrow z \in (A \times B) \cup (A \times C)$$

(b)
$\langle 1 \rangle 1.$ ASSUME: $A \times B = A \times C$ and $A \neq \emptyset$
$\langle 1 \rangle 2.$ PICK $a \in A$
$\langle 1 \rangle 3.$ For all x, $x \in B \Leftrightarrow x \in C$
  PROOF: $x \in B$ iff $(a, x) \in A \times B$ iff $(a, x) \in A \times C$ iff $x \in C$.
$\square$

Exercise 3

$$z \in A \times \bigcup \mathcal{B}$$
$$\Leftrightarrow \exists x, y(z = (x, y) \ \& \ x \in A \ \& \ \exists X \in \mathcal{B}.y \in X)$$
$$\Leftrightarrow \exists X \in \mathcal{B}.\exists x, y(z = (x, y) \ \& \ x \in A \ \& \ y \in X)$$
$$\Leftrightarrow z \in \bigcup \{A \times X : X \in \mathcal{B}\}$$

Exercise 4   If every ordered pair belongs to A then every set belongs to $\bigcup\bigcup A$ contradicting Theorem 2A.

Exercise 5

(a)   Apply a Subset Axiom to $\mathcal{P}(A \times B)$: we have $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A.z = \{x\} \times B\}$.

(b)

$$z \in \bigcup C$$
$$\Leftrightarrow \exists x \in A.z \in \{x\} \times B$$
$$\Leftrightarrow \exists x \in A.\exists y \in B.z = (x, y)$$
$$\Leftrightarrow z \in A \times B$$

3.2   Relations

Exercise 6   If $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$ then A is a set of ordered pairs, i.e. a relation.

Conversely, suppose A is a relation. Let $z \in A$. Then z is an ordered pair; let $z = (x, y)$. We have $x \in \operatorname{dom} A$ and $y \in \operatorname{ran} A$ and so $z \in \operatorname{dom} A \times \operatorname{ran} A$ as required.

Exercise 7   We have $\operatorname{fld} R \subseteq \bigcup\bigcup R$ by Lemma 3D.

Conversely, let $x \in \bigcup\bigcup R$. Pick a and b such that $x \in a$, $a \in b$ and $b \in R$. Then b is an ordered pair; let $b = (y, z)$. We have $a = \{y\}$ or $\{y, z\}$, hence $x = y$ or $x = z$. In either case, $x \in \operatorname{fld} R$.

Exercise 8

(a)

$$x \in \operatorname{dom} \bigcup \mathcal{A}$$
$$\Leftrightarrow \exists y.\exists R \in \mathcal{A}.(x, y) \in R$$
$$\Leftrightarrow \exists R \in \mathcal{A}.\exists y.(x, y) \in R$$
$$\Leftrightarrow x \in \bigcup\{\operatorname{dom} R : R \in \mathcal{A}\}$$

(b)

$$y \in \text{ran}\bigcup\mathcal{A}$$
$$\Leftrightarrow \exists x.\exists R \in \mathcal{A}.(x,y) \in R$$
$$\Leftrightarrow \exists R \in \mathcal{A}.\exists x.(x,y) \in R$$
$$\Leftrightarrow y \in \bigcup\{\text{ran}\,R : R \in \mathcal{A}\}$$

Exercise 9 Assume $\mathcal{A}$ is nonempty. We have $\text{dom}\bigcap\mathcal{A} \subseteq \bigcap\{\text{dom}\,R : R \in \mathcal{A}\}$.

PROOF:

$$x \in \text{dom}\bigcap\mathcal{A}$$
$$\Leftrightarrow \exists y.\forall R \in \mathcal{A}.(x,y) \in R$$
$$\Rightarrow \forall R \in \mathcal{A}.\exists y.(x,y) \in R$$
$$\Leftrightarrow x \in \bigcap\{\text{dom}\,R : R \in \mathcal{A}\}$$

Equality holds iff the middle '$\Rightarrow$' can be reversed, i.e. iff for all x, if $\forall R \in \mathcal{A}.\exists y.(x,y) \in R$ then $\exists y.\forall R \in \mathcal{A}.(x,y) \in R$. I haven't found a simpler condition than this. The condition does not always hold, for example if $\mathcal{A} = \{\{(1,2)\}, \{(1,3)\}\}$ then $\text{dom}\bigcap\mathcal{A} = \emptyset$ while $\bigcap\{\text{dom}\,R : R \in \mathcal{A}\} = \{1\}$.

Similarly, $\text{ran}\bigcap\mathcal{A} \subseteq \bigcap\{\text{ran}\,R : R \in \mathcal{A}\}$, and equality holds iff, for any y, if $\forall R \in \mathcal{A}.\exists x.(x,y) \in R$ then $\exists x.\forall R \in \mathcal{A}.(x,y) \in R$.

3.3 n-ary Relations

Exercise 10 This follows from the equations at the top of page 42. An ordered 4-tuple $\langle a, b, c, d\rangle$ is also an ordered 1-tuple (because every set is), and the ordered pair $\langle\langle a, b, c\rangle, d\rangle$, and the ordered triple $\langle\langle a, b\rangle, c, d\rangle$.

3.4   Functions

Exercise 11 We prove $F \subseteq G$. Let $z \in F$. Since F is a relation, then z is an ordered pair; let $z = \langle x, y\rangle$. We have $x \in \text{dom}\,F$ and $y = F(x)$. Therefore $x \in \text{dom}\,G$ and $y = G(x)$ (because $\text{dom}\,F = \text{dom}\,G$ and $F(x) = G(x)$). Hence $\langle x, y\rangle \in G$, i.e. $z \in G$.

We have proved $F \subseteq G$. We can prove $G \subseteq F$ similarly. Thus $F = G$.

Exercise 12   PROOF:
$\langle 1\rangle 1$. If $f \subseteq g$ then $\text{dom}\,f \subseteq \text{dom}\,g$ and $\forall x \in \text{dom}\,f.f(x) = g(x)$
  $\langle 2\rangle 1$. ASSUME: $f \subseteq g$
  $\langle 2\rangle 2$. LET: $x \in \text{dom}\,f$
  $\langle 2\rangle 3$. $(x, f(x)) \in f$
  $\langle 2\rangle 4$. $(x, f(x)) \in g$
  $\langle 2\rangle 5$. $x \in \text{dom}\,g$ and $g(x) = f(x)$

$\langle 1 \rangle 2$. If $\operatorname{dom} f = \operatorname{dom} g$ and $\forall x \in \operatorname{dom} f . f(x) = g(x)$ then $f \subseteq g$
  $\langle 2 \rangle 1$. ASSUME: $\operatorname{dom} f = \operatorname{dom} g$ and $\forall x \in \operatorname{dom} f . f(x) = g(x)$
  $\langle 2 \rangle 2$. LET: $z \in f$
  $\langle 2 \rangle 3$. LET: $z = (x, y)$
  $\langle 2 \rangle 4$. $x \in \operatorname{dom} f$ and $y = f(x)$
  $\langle 2 \rangle 5$. $x \in \operatorname{dom} g$ and $y = g(x)$
  $\langle 2 \rangle 6$. $z = (x, y) \in g$
▯

Exercise 13   PROOF:
$\langle 1 \rangle 1$. ASSUME: f and g are functions
$\langle 1 \rangle 2$. ASSUME: $f \subseteq g$
$\langle 1 \rangle 3$. ASSUME: $\operatorname{dom} g \subseteq \operatorname{dom} f$
$\langle 1 \rangle 4$. $\operatorname{dom} f = \operatorname{dom} g$
  PROOF: We have $\operatorname{dom} f \subseteq \operatorname{dom} g$ from $\langle 1 \rangle 2$ and $\operatorname{dom} g \subseteq \operatorname{dom} f$ from $\langle 1 \rangle 3$
$\langle 1 \rangle 5$. For $x \in \operatorname{dom} f$ we have $f(x) = g(x)$
  PROOF: From $\langle 1 \rangle 2$ and Exercise 12
$\langle 1 \rangle 6$. Q.E.D.
  PROOF: From Exercise 11.
▯

Exercise 14

(a)   If (x, y) and (x, z) are members of $f \cap g$ then they are both members of f, hence $y = z$.

(b)   PROOF:
$\langle 1 \rangle 1$. If $f \cup g$ is a function then, for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, we have $f(x) = g(x)$.
  $\langle 2 \rangle 1$. ASSUME: $f \cup g$ is a function.
  $\langle 2 \rangle 2$. LET: $x \in \operatorname{dom} f \cap \operatorname{dom} g$
  $\langle 2 \rangle 3$. $(x, f(x))$ and $(x, g(x))$ are both elements of $f \cup g$
  $\langle 2 \rangle 4$. $f(x) = g(x)$
$\langle 1 \rangle 2$. If, for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, we have $f(x) = g(x)$, then $f \cup g$ is a function.
  $\langle 2 \rangle 1$. ASSUME: For all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, we have $f(x) = g(x)$
  $\langle 2 \rangle 2$. $f \cup g$ is a relation.
    PROOF: Since every element of either f or g is an ordered pair.
  $\langle 2 \rangle 3$. Whenever (x, y) and (x, z) are elements of $f \cup g$ we have $y = z$
    $\langle 3 \rangle 1$. LET: $(x, y), (x, z) \in f \cup g$
    $\langle 3 \rangle 2$. CASE: $(x, y), (x, z) \in f$
      PROOF: Then $y = z$ since f is a function.
    $\langle 3 \rangle 3$. CASE: $(x, y) \in f, (x, z) \in g$
      PROOF: Then $y = z$ by $\langle 2 \rangle 1$
    $\langle 3 \rangle 4$. CASE: $(x, y) \in g, (x, z) \in f$
      PROOF: Then $y = z$ by $\langle 2 \rangle 1$
    $\langle 3 \rangle 5$. CASE: $(x, y), (x, z) \in g$

PROOF: Then $y = z$ since g is a function.

$\square$

Exercise 15 PROOF:

$\langle 1 \rangle 1$. $\bigcup \mathcal{A}$ is a relation.
   PROOF: Since every member of $\mathcal{A}$ is a relation.
$\langle 1 \rangle 2$. Whenever (x, y) and (x, z) are elements of $\bigcup \mathcal{A}$ then $y = z$
  $\langle 2 \rangle 1$. LET: $(x, y), (x, z) \in \bigcup \mathcal{A}$
  $\langle 2 \rangle 2$. PICK $f, g \in \mathcal{A}$ such that $(x, y) \in f$ and $(x, z) \in g$
  $\langle 2 \rangle 3$. ASSUME: w.l.o.g. $f \subseteq g$
  $\langle 2 \rangle 4$. $(x, y), (x, z) \in g$
  $\langle 2 \rangle 5$. $y = z$
    PROOF: Since g is a function.

$\square$

Exercise 16 If every function belongs to A then every set belongs to $\operatorname{dom} \bigcup A$ contradiction Theorem 2A.

Exercise 17 PROOF:

$\langle 1 \rangle 1$. LET: R and S be single-rooted.
$\langle 1 \rangle 2$. LET: $(x, z), (y, z) \in R \circ S$
$\langle 1 \rangle 3$. PICK t and t' such that $(x, t) \in S$, $(t, z) \in R$, $(y, t') \in S$ and $(t', z) \in R$
$\langle 1 \rangle 4$. $t = t'$
  PROOF: Since R is single-rooted.
$\langle 1 \rangle 5$. $x = y$
  PROOF: Since S is single-rooted.

Thus if F and G are one-to-one functions then $F \circ G$ is single-rooted and a function by Theorem 3H, hence a one-to-one function.

Exercise 18

$$R \circ R = \{\langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle\}$$
$$R \restriction \{1\} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle\}$$
$$R^{-1} \restriction \{1\} = \{\langle 1, 0 \rangle\}$$
$$R[\![\{1\}]\!] = \{2, 3\}$$
$$R^{-1}[\![\{1\}]\!] = \{0\}$$

Exercise 19

$$A(\emptyset) = \{\emptyset, \{\emptyset\}\}$$
$$A[\![\emptyset]\!] = \emptyset$$
$$A[\![\{\emptyset\}]\!] = \{\{\emptyset, \{\emptyset\}\}\}$$
$$A[\![\{\emptyset, \{\emptyset\}\}]\!] = \{\{\emptyset, \{\emptyset\}\}, \emptyset\}$$
$$A^{-1} = \{\langle\{\emptyset, \{\emptyset\}\}, \emptyset\rangle, \langle\emptyset, \{\emptyset\}\rangle\}$$
$$A \circ A = \{\langle\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\rangle\}$$
$$A \upharpoonright \emptyset = \emptyset$$
$$A \upharpoonright \{\emptyset\} = \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle\}$$
$$A \upharpoonright \{\emptyset, \{\emptyset\}\} = \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle, \langle\{\emptyset\}, \emptyset\rangle\}$$
$$= A$$
$$\bigcup\bigcup A = \{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}$$

Exercise 20

$$z \in F \upharpoonright A \Leftrightarrow z \in F \ \& \ \exists x, y.(z = \langle x, y\rangle \ \& \ x \in A)$$
$$\Leftrightarrow z \in F \ \& \ \exists x, y(z = \langle x, y\rangle \ \& \ x \in A \ \& \ y \in \operatorname{ran} F)$$
$$\Leftrightarrow z \in F \cap (A \times \operatorname{ran} F)$$

Exercise 21 Both are equal to $\{\langle x, w\rangle \mid \exists y, z.xTy \ \& \ ySz \ \& \ zRw\}$.

Exercise 22

(a) Proof:
$\langle 1\rangle 1$. Assume: $A \subseteq B$
$\langle 1\rangle 2$. Let: $y \in F[\![A]\!]$
$\langle 1\rangle 3$. Pick $x \in A$ such that xFy
$\langle 1\rangle 4$. $x \in B$ and xFy
$\square$

(b) Both are equal to $\{z : \exists x, y.x \in A \ \& \ xGy \ \& \ yFz\}$

(c) Both are equal to $\{\langle x, y\rangle : (x \in A \text{ or } x \in B) \ \& \ xQy\}$

Exercise 23

$$B \circ I_A = \{\langle x, z \rangle : \exists y(xI_A y \ \& \ yBz)\}$$
$$= \{\langle x, z \rangle : \exists y(x \in A \ \& \ x = y \ \& \ yBz)\}$$
$$= \{\langle x, z \rangle : x \in A \ \& \ xBz\}$$
$$= B \restriction A$$
$$I_A[\![C]\!] = \{y : \exists x \in C.xI_A y\}$$
$$= \{y : \exists x \in C(x \in A \ \& \ x = y)\}$$
$$= \{y : y \in C \ \& \ y \in A\}$$
$$= A \cap C$$

Exercise 24

$$F^{-1}[\![A]\!] = \{x : \exists y \in A.yF^{-1}x\}$$
$$= \{x : \exists y \in A.xFy\}$$
$$= \{x \in \mathrm{dom}\, F : F(x) \in A\}$$

Exercise 25

(a) PROOF:
$\langle 1 \rangle 1.$ LET: G be a one-to-one function.
$\langle 1 \rangle 2.$ G^{-1} is a function.
  PROOF: Theorem 3F.
$\langle 1 \rangle 3.$ $G \circ G^{-1}$ is a function.
  PROOF: Theorem 3H.
$\langle 1 \rangle 4.$ $\mathrm{dom}(G \circ G^{-1}) = \mathrm{ran}\, G$
  PROOF:
$$\mathrm{dom}(G \circ G^{-1}) = \{x \in \mathrm{dom}\, G^{-1} : G^{-1}(x) \in \mathrm{dom}\, G\} \quad \text{(Theorem 3H)}$$
$$= \{x \in \mathrm{ran}\, G : G^{-1}(x) \in \mathrm{dom}\, G\} \quad \text{(Theorem 3E)}$$
$$= \mathrm{ran}\, G$$
$\langle 1 \rangle 5.$ $\forall x \in \mathrm{ran}\, G.(G \circ G^{-1})(x) = x$
  PROOF: Theorem 3G.
∎

(b) Let G be a function. Then

$$G \circ G^{-1} = \{\langle x, z \rangle : \exists y(xG^{-1}y \ \& \ yGz)\}$$
$$= \{\langle x, z \rangle : \exists y(yGx \ \& \ yGz)\}$$
$$= \{\langle x, x \rangle : \exists y.yGx\} \quad \quad (G \text{ is a function})$$
$$= I_{\mathrm{ran}\, G}$$

Exercise 26

22

(a)

$$F[\![\bigcup \mathcal{A}]\!] = \{y : \exists x.\exists A \in \mathcal{A}(x \in A \ \& \ xFy)\}$$
$$= \{y : \exists A \in \mathcal{A}.\exists x(x \in A \ \& \ xFy)\}$$
$$= \bigcup\{F[\![A]\!] : A \in \mathcal{A}\}$$

(b)

$$F[\![\bigcup \mathcal{A}]\!] = \{y : \exists x.\forall A \in \mathcal{A}(x \in A \ \& \ xFy)\}$$
$$\subseteq \{y : \forall A \in \mathcal{A}.\exists x(x \in A \ \& \ xFy)\}$$
$$= \bigcap\{F[\![A]\!] : A \in \mathcal{A}\}$$

Exercise 27

$$\mathrm{dom}(F \circ G) = \{x : \exists y.x(F \circ G)y\}$$
$$= \{x : \exists y \exists z(xGz \ \& \ zFy)\}$$
$$= \{x : \exists z(zG^{-1}x \ \& \ z \in \mathrm{dom}\, F)\}$$
$$= G^{-1}[\![\mathrm{dom}\, F]\!]$$

Exercise 28   Proof:
⟨1⟩1. $G : \mathcal{P}A \to \mathcal{P}B$
 Proof: Since $f[\![X]\!] \subseteq \mathrm{ran}\, f \subseteq B$
⟨1⟩2. For all $X, Y \in \mathcal{P}A$, if $G(X) = G(Y)$ then $X = Y$
  ⟨2⟩1. Let: $X, Y \in \mathcal{P}A$
  ⟨2⟩2. Assume: $f[\![X]\!] = f[\![Y]\!]$
  ⟨2⟩3. $X \subseteq Y$
   ⟨3⟩1. Let: $x \in X$
   ⟨3⟩2. $f(x) \in f[\![X]\!]$
   ⟨3⟩3. $f(x) \in f[\![Y]\!]$
   ⟨3⟩4. Pick $y \in Y$ such that $f(x) = f(y)$
   ⟨3⟩5. $x = y$
     Proof: Because f is one-to-one.
   ⟨3⟩6. $x \in Y$
     Proof: Similar.
  ⟨2⟩4. $Y \subseteq X$
☐

Example 29   Proof:
⟨1⟩1. Assume: f maps A onto B
⟨1⟩2. Let: $b, b' \in B$
⟨1⟩3. Assume: $G(b) = G(b')$
⟨1⟩4. Pick $x \in A$ such that $f(x) = b$

PROOF: By $\langle 1 \rangle 1$.
$\langle 1 \rangle 5.$ $x \in G(b)$
$\langle 1 \rangle 6.$ $x \in G(b')$
$\langle 1 \rangle 7.$ $f(x) = b'$
$\langle 1 \rangle 8.$ $b = b'$
☐

The converse does not hold. Let $A = \{0\}$ and $B = \{0, 1\}$. Let f be the function that maps 0 to 0. Then

$$G(0) = \{0\}$$
$$G(1) = \emptyset$$

Thus G is one-to-one but f does not map A onto B.

Exercise 30

(a) PROOF:
$\langle 1 \rangle 1.$ $F(B) = B$
  $\langle 2 \rangle 1.$ $F(B) \subseteq B$
    $\langle 3 \rangle 1.$ LET: $X \in \mathcal{P}A$ be such that $F(X) \subseteq X$
        PROVE: $F(B) \subseteq X$
    $\langle 3 \rangle 2.$ $B \subseteq X$
    $\langle 3 \rangle 3.$ $F(B) \subseteq F(X)$
    $\langle 3 \rangle 4.$ $F(B) \subseteq X$
        PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$.
  $\langle 2 \rangle 2.$ $B \subseteq F(B)$
    PROOF: From $\langle 2 \rangle 1$ and the definition of B, since B is one of the sets X
    such that $F(X) \subseteq X$
$\langle 1 \rangle 2.$ $F(C) = C$
  $\langle 2 \rangle 1.$ $C \subseteq F(C)$
    $\langle 3 \rangle 1.$ LET: $X \in \mathcal{P}A$ with $X \subseteq F(X)$
        PROVE: $X \subseteq F(C)$
    $\langle 3 \rangle 2.$ $X \subseteq C$
    $\langle 3 \rangle 3.$ $F(X) \subseteq F(C)$
    $\langle 3 \rangle 4.$ $X \subseteq F(C)$
        PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$
  $\langle 2 \rangle 2.$ $F(C) \subseteq C$
    PROOF: From $\langle 2 \rangle 1$ and the definition of C.
☐

(b) If $F(X) = X$ then we have $B \subseteq X$ (because $F(X) \subseteq X$) and $X \subseteq C$ (because $X \subseteq F(X)$).

3.5 Infinite Cartesian Products

Exercise 31   PROOF:

⟨1⟩1. If the Axiom of Choice is true then, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$.

  ⟨2⟩1. ASSUME: The Axiom of Choice.

  ⟨2⟩2. LET: I be a set.

  ⟨2⟩3. LET: H be a function with domain I.

  ⟨2⟩4. ASSUME: $H(i) \neq \emptyset$ for all $i \in I$.

  ⟨2⟩5. LET: $R = \{(i, x) : i \in I, x \in H(i)\}$

  ⟨2⟩6. PICK a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R$

      PROVE: $F \in \prod_{i \in I} H(i)$

    PROOF: By the Axiom of Choice.

  ⟨2⟩7. $\operatorname{dom} H = I$

    PROOF: We have $\operatorname{dom} R = I$ since for all $i \in I$ there exists x such that $x \in H(i)$.

  ⟨2⟩8. $\forall i \in I.F(i) \in H(i)$

    PROOF: Since $iRF(i)$.

⟨1⟩2. If, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$, then the Axiom of Choice is true.

  ⟨2⟩1. ASSUME: For any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$

  ⟨2⟩2. LET: R be a relation

  ⟨2⟩3. LET: $I = \operatorname{dom} R$

  ⟨2⟩4. Define the function H with domain I by: for $i \in I$, $H(i) = \{y : iRy\}$

  ⟨2⟩5. $H(i) \neq \emptyset$ for all $i \in I$

  ⟨2⟩6. PICK $F \in \prod_{i \in I} H(i)$

    PROOF: By ⟨2⟩1

  ⟨2⟩7. F is a function

  ⟨2⟩8. $F \subseteq R$

    PROOF: For all $i \in I$ we have $F(i) \in H(i)$ and so $iRF(i)$.

  ⟨2⟩9. $\operatorname{dom} F = \operatorname{dom} R$

☐

3.6 Equivalence Relations

Exercise 32

(a)

$$R \text{ is symmetric}$$
$$\Leftrightarrow \forall x, y (xRy \Rightarrow yRx)$$
$$\Leftrightarrow \forall x, y (\langle y, x \rangle \in R^{-1} \Rightarrow \langle y, x \rangle \in R)$$
$$\Leftrightarrow R^{-1} \subseteq R$$

(b)

$$R \text{ is transitive}$$
$$\Leftrightarrow \forall x, y, z(xRy \ \& \ yRz \Rightarrow xRz)$$
$$\Leftrightarrow \forall x, z(\exists y(xRy \ \& \ yRz) \Rightarrow xRz)$$
$$\Leftrightarrow \forall x, z(\langle x, z \rangle \in R \circ R \Rightarrow \langle x, z \rangle \in R)$$
$$\Leftrightarrow R \circ R \subseteq R$$

Exercise 33   PROOF:
$\langle 1 \rangle 1$. If R is a symmetric and transitive relation then $R = R^{-1} \circ R$.
  $\langle 2 \rangle 1$. ASSUME: R is a symmetric and transitive relation.
  $\langle 2 \rangle 2$. $R \subseteq R^{-1} \circ R$
    $\langle 3 \rangle 1$. LET: xRy
    $\langle 3 \rangle 2$. yRy
       PROOF: By Theorem 3M.
    $\langle 3 \rangle 3$. xRy and $yR^{-1}y$
    $\langle 3 \rangle 4$. $x(R^{-1} \circ R)y$
  $\langle 2 \rangle 3$. $R^{-1} \circ R \subseteq R$
     PROOF:
$$R^{-1} \circ R \subseteq R \circ R \qquad \text{(Exercise 32(a))}$$
$$\subseteq R \qquad \text{(Exercise 32(b))}$$
$\langle 1 \rangle 2$. If $R = R^{-1} \circ R$ then R is a symmetric and transitive relation.
  $\langle 2 \rangle 1$. ASSUME: $R = R^{-1} \circ R$
  $\langle 2 \rangle 2$. R is a relation.
  $\langle 2 \rangle 3$. R is symmetric.
    $\langle 3 \rangle 1$. LET: xRy
    $\langle 3 \rangle 2$. PICK z such that xRz and $zR^{-1}y$
    $\langle 3 \rangle 3$. yRz and $zR^{-1}x$
    $\langle 3 \rangle 4$. $y(R^{-1} \circ R)x$
    $\langle 3 \rangle 5$. yRx
  $\langle 2 \rangle 4$. R is transitive.
    $\langle 3 \rangle 1$. LET: xRy and yRz
    $\langle 3 \rangle 2$. zRy
       PROOF: By $\langle 2 \rangle 3$
    $\langle 3 \rangle 3$. xRy and $yR^{-1}z$
    $\langle 3 \rangle 4$. $x(R^{-1} \circ R)z$
    $\langle 3 \rangle 5$. xRz
☐

Exercise 34

(a) $\bigcap \mathcal{A}$ is a transitive relation.

PROOF:
$\langle 1 \rangle 1$. $\bigcap \mathcal{A}$ is a relation.

PROOF: Every member of a member of $\mathcal{A}$ is an ordered pair.

⟨1⟩2. $\bigcap \mathcal{A}$ is transitive.

    ⟨2⟩1. LET: $\langle x, y \rangle$ and $\langle y, z \rangle$ be in $\bigcap \mathcal{A}$

        PROVE: $\langle x, z \rangle \in \bigcap \mathcal{A}$

    ⟨2⟩2. LET: $R \in \mathcal{A}$

    ⟨2⟩3. xRy and yRz

    ⟨2⟩4. xRz

      PROOF: Since R is transitive.

∎

   (b) Not necessarily. If $\mathcal{A} = \{\{\langle 0, 1 \rangle\}, \{\langle 1, 2 \rangle\}\}$ then each member of $\mathcal{A}$ is transitive but $\bigcup \mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$ is not.

Example 35

$$
\begin{aligned}
R[\![\{x\}]\!] &= \{y : \exists z (z \in \{x\} \ \& \ zRy)\} \\
&= \{y : \exists z (z = x \ \& \ zRy)\} \\
&= \{y : xRy\} \\
&= [x]_R
\end{aligned}
$$

Example 36 PROOF:

⟨1⟩1. Q is a relation on A.

  PROOF: By definition.

⟨1⟩2. Q is reflexive on A.

    ⟨2⟩1. LET: $x \in A$

    ⟨2⟩2. $f(x)Rf(x)$

      PROOF: Since R is reflexive on B.

    ⟨2⟩3. xQx

⟨1⟩3. Q is symmetric.

    ⟨2⟩1. ASSUME: xQy

    ⟨2⟩2. $f(x)Rf(y)$

    ⟨2⟩3. $f(y)Rf(x)$

      PROOF: R is symmetric.

    ⟨2⟩4. yQx

⟨1⟩4. Q is transitive.

    ⟨2⟩1. ASSUME: xQy and yQz

    ⟨2⟩2. $f(x)Rf(y)$ and $f(y)Rf(z)$

    ⟨2⟩3. $f(x)Rf(z)$

      PROOF: R is transitive.

    ⟨2⟩4. xQz

∎

Exercise 37 PROOF:

⟨1⟩1. R_Π is a relation on A.

PROOF: If $B \in \Pi$, $x \in B$ and $y \in B$ then $x, y \in A$.

$\langle 1 \rangle 2$. R_Π is reflexive on A.
  $\langle 2 \rangle 1$. LET: $x \in A$
  $\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$
    PROOF: Because Π is exhaustive.
  $\langle 2 \rangle 3$. $x \in B$ and $x \in B$
  $\langle 2 \rangle 4$. $xR_\Pi x$
$\langle 1 \rangle 3$. R_Π is symmetric.
  $\langle 2 \rangle 1$. ASSUME: $xR_\Pi y$
  $\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$
  $\langle 2 \rangle 3$. $y \in B$ and $x \in B$
  $\langle 2 \rangle 4$. $yR_\Pi x$
$\langle 1 \rangle 4$. R_Π is transitive.
  $\langle 2 \rangle 1$. ASSUME: $xR_\Pi y$ and $yR_\Pi z$
  $\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$
  $\langle 2 \rangle 3$. PICK $C \in \Pi$ such that $y \in C$ and $z \in C$
  $\langle 2 \rangle 4$. $B = C$
    PROOF: Since $y \in B$ and $y \in C$
  $\langle 2 \rangle 5$. $x \in B$ and $z \in B$
  $\langle 2 \rangle 6$. $xR_\Pi z$
□

Exercise 38 PROOF:
$\langle 1 \rangle 1$. If $B \in \Pi$ and $x \in B$ then $B = [x]_{R_\Pi}$
  $\langle 2 \rangle 1$. LET: $B \in \Pi$
  $\langle 2 \rangle 2$. LET: $x \in B$
  $\langle 2 \rangle 3$. $[x]_{R_\Pi} \subseteq B$
    $\langle 3 \rangle 1$. LET: $y \in [x]_{R_\Pi}$
    $\langle 3 \rangle 2$. $xR_\Pi y$
    $\langle 3 \rangle 3$. PICK $C \in \Pi$ such that $x \in C$ and $y \in C$
    $\langle 3 \rangle 4$. $B = C$
      PROOF: Since $x \in B$ and $x \in C$.
    $\langle 3 \rangle 5$. $y \in B$
  $\langle 2 \rangle 4$. $B \subseteq [x]_{R_\Pi}$
    PROOF: For all $y \in B$, we have $x \in B$ and $y \in B$ hence $xR_\Pi y$.
$\langle 1 \rangle 2$. $A/R_\Pi \subseteq \Pi$
  $\langle 2 \rangle 1$. LET: $x \in A$
    PROVE: $[x]_{R_\Pi} \in \Pi$
  $\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$
  $\langle 2 \rangle 3$. $[x]_{R_\Pi} = B$
    PROOF: By $\langle 1 \rangle 1$
  $\langle 2 \rangle 4$. $[x]_{R_\Pi} \in \Pi$
$\langle 1 \rangle 3$. $\Pi \subseteq A/R_\Pi$
  $\langle 2 \rangle 1$. LET: $B \in \Pi$
  $\langle 2 \rangle 2$. PICK $x \in B$

PROOF: Since every member of Π is nonempty.
$\langle 2 \rangle 3.$ $B = [x]_{R_\Pi}$
    PROOF: By $\langle 1 \rangle 1.$
$\langle 2 \rangle 4.$ $B \in A/R_\Pi$
☐

Exercise 39   PROOF:
$\langle 1 \rangle 1.$ $R_\Pi \subseteq R$
    $\langle 2 \rangle 1.$ LET: $xR_\Pi y$
    $\langle 2 \rangle 2.$ PICK $B \in \Pi$ such that $x \in B$ and $y \in B$
    $\langle 2 \rangle 3.$ PICK $z \in A$ such that $B = [z]_R$
    $\langle 2 \rangle 4.$ zRx
    $\langle 2 \rangle 5.$ zRy
    $\langle 2 \rangle 6.$ xRy
        PROOF: Since R is symmetric and transitive.
$\langle 1 \rangle 2.$ $R \subseteq R_\Pi$
    $\langle 2 \rangle 1.$ LET: xRy
    $\langle 2 \rangle 2.$ $x \in [x]_R$
    $\langle 2 \rangle 3.$ $y \in [x]_R$
    $\langle 2 \rangle 4.$ $xR_\Pi y$
☐

Exercise 40   We have $[2]_R = [3]_R$ but $[6]_R \neq [9]_R$ so there is no such function f.

Exercise 41

(a)   PROOF:
$\langle 1 \rangle 1.$ Q is reflexive on $\mathbb{R} \times \mathbb{R}$.
    PROOF: For any $x, y \in \mathbb{R}$, we have $x + y = x + y$, hence $\langle x, y \rangle Q \langle x, y \rangle$
$\langle 1 \rangle 2.$ Q is symmetric.
    $\langle 2 \rangle 1.$ ASSUME: $\langle u, v \rangle Q \langle x, y \rangle$
    $\langle 2 \rangle 2.$ $u + y = x + v$
    $\langle 2 \rangle 3.$ $x + v = u + y$
    $\langle 2 \rangle 4.$ $\langle x, y \rangle Q \langle u, v \rangle$
$\langle 1 \rangle 3.$ Q is transitive.
    $\langle 2 \rangle 1.$ ASSUME: $\langle a, b \rangle Q \langle u, v \rangle$ and $\langle u, v \rangle Q \langle x, y \rangle$
    $\langle 2 \rangle 2.$ $a + v = u + b$
    $\langle 2 \rangle 3.$ $u + y = x + v$
    $\langle 2 \rangle 4.$ $a + y + x + b$
        PROOF: Adding $\langle 2 \rangle 2$ and $\langle 2 \rangle 3$ gives $a + u + v + y = b + u + v + x$.
    $\langle 2 \rangle 5.$ $\langle a, b \rangle Q \langle x, y \rangle$
☐

(b)   We prove that, if $\langle u, v \rangle Q \langle x, y \rangle$ then $\langle u + 2v, v + 2u \rangle Q \langle x + 2y, y + 2x \rangle$. It follows from Theorem 3Q that the function G exists.

If $u + y = v + x$ then $u + 2v + y + 2x = v + 2u + x + 2y$ by adding $u + v + y + x$ to both sides.

Exercise 42   Assume that R is an equivalence relation on A and that $F : A \times A \to A$. Let us say that F is *compatible* with R iff, whenever xRx' and yRy', then $F(\langle x, y \rangle) R F(\langle x', y' \rangle)$. If F is compatible with R then there exists a unique $\hat{F} : (A/R) \times (A/R) \to A/R$ such that

$$\hat{F}(\langle [x]_R, [y]_R \rangle) = [F(\langle x, y \rangle)]_R \text{ for all } x, y \in A \ .$$

If F is not compatible with R then no such $\hat{F}$ exists.

3.7   Ordering Relations

Exercise 43   PROOF:
$\langle 1 \rangle 1.$ R^{-1} is transitive.
    $\langle 2 \rangle 1.$ ASSUME: $xR^{-1}y$ and $yR^{-1}z$
    $\langle 2 \rangle 2.$ zRy and yRx
    $\langle 2 \rangle 3.$ zRx
       PROOF: Since R is transitive.
    $\langle 2 \rangle 4.$ $xR^{-1}z$
$\langle 1 \rangle 2.$ R^{-1} satisfies trichotomy on A.
    $\langle 2 \rangle 1.$ LET: $x, y \in A$
    $\langle 2 \rangle 2.$ Exactly one of xRy, $x = y$, yRx holds.
    $\langle 2 \rangle 3.$ Exactly one of $yR^{-1}x$, $x = y$, $xR^{-1}y$ holds.
▯

Exercise 44   PROOF:
$\langle 1 \rangle 1.$ f is one-to-one.
    $\langle 2 \rangle 1.$ LET: $x, y \in A$ with $f(x) = f(y)$
    $\langle 2 \rangle 2.$ $f(x) < f(y)$ and $f(y) < f(x)$ do not hold.
       PROOF: By trichotomy.
    $\langle 2 \rangle 3.$ $x < y$ and $y < x$ do not hold.
    $\langle 2 \rangle 4.$ $x = y$
       PROOF: By trichotomy.
$\langle 1 \rangle 2.$ Whenever $f(x) < f(y)$ then $x < y$
    $\langle 2 \rangle 1.$ LET: $x, y \in A$ with $f(x) < f(y)$
    $\langle 2 \rangle 2.$ $f(x) = f(y)$ and $f(y) < f(x)$ do not hold.
       PROOF: By trichotomy.
    $\langle 2 \rangle 3.$ $x = y$ and $y < x$ do not hold.
    $\langle 2 \rangle 4.$ $x < y$
       PROOF: By trichotomy.
▯

Exercise 45   PROOF:

$\langle 1 \rangle 1.$ $<_L$ is transitive.

   $\langle 2 \rangle 1.$ LET: $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$ and $\langle a_2, b_2 \rangle <_L \langle a_3, b_3 \rangle$
       PROVE: $\langle a_1, b_1 \rangle < \langle a_3, b_3 \rangle$

   $\langle 2 \rangle 2.$ CASE: $a_1 <_A a_2$ and $a_2 <_A a_3$
    PROOF: Then $a_1 <_A <_3$

   $\langle 2 \rangle 3.$ CASE: $a_1 <_A a_2$, $a_2 = a_3$, $b_2 <_B b_3$
    PROOF: Then $a_1 <_A <_3$

   $\langle 2 \rangle 4.$ CASE: $a_1 = a_2$, $b_1 <_B b_2$ and $a_2 <_A a_3$
    PROOF: Then $a_1 <_A <_3$

   $\langle 2 \rangle 5.$ CASE: $a_1 = a_2$, $b_1 <_B b_2$, $a_2 = a_3$, $b_2 <_B b_3$
    PROOF: Then $a_1 = a_3$ and $b_1 <_B b_3$

$\langle 1 \rangle 2.$ $<_L$ satisfies trichotomy on $A \times B$.

   $\langle 2 \rangle 1.$ LET: $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle$ be elements of $A \times B$

   $\langle 2 \rangle 2.$ Exactly one of $a_1 <_A a_2$, $a_1 = a_2$, $a_2 <_A a_1$ holds.

   $\langle 2 \rangle 3.$ Exactly one of $b_1 <_B b_2$, $b_1 = b_2$, $b_2 <_B b_1$ holds.

   $\langle 2 \rangle 4.$ Exactly one of $a_1 <_A a_2$, $(a_1 = a_2$ and $b_1 <_B b_2)$, $(a_1 = a_2$ and $b_1 = b_2)$, $(a_1 = a_2$ and $b_2 <_L b_1)$, $a_2 <_A a_1$ holds.

   $\langle 2 \rangle 5.$ Exactly one of $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$, $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$, $\langle a_2, b_2 \rangle <_L \langle a_1, b_1 \rangle$ holds.

☐

3.8   Review Exercises

Exercise 46

  (a)

$$\bigcap \bigcap \langle x, y \rangle = \bigcap \{x\}$$
$$= x$$

  (b)

$$\bigcap \bigcap \bigcap \{\langle x, y \rangle\}^{-1} = \bigcap \bigcap \bigcap \{\langle y, x \rangle\}$$
$$= \bigcap \bigcap \langle y, x \rangle$$
$$= y \qquad\qquad \text{(by part (a))}$$

Exercise 47

(a)   There are eight:

$$\{\langle 0,3\rangle, \langle 1,3\rangle, \langle 2,3\rangle\},$$
$$\{\langle 0,3\rangle, \langle 1,3\rangle, \langle 2,4\rangle\},$$
$$\{\langle 0,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle\},$$
$$\{\langle 0,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle\},$$
$$\{\langle 0,4\rangle, \langle 1,3\rangle, \langle 2,3\rangle\},$$
$$\{\langle 0,4\rangle, \langle 1,3\rangle, \langle 2,4\rangle\},$$
$$\{\langle 0,4\rangle, \langle 1,4\rangle, \langle 2,3\rangle\},$$
$$\{\langle 0,4\rangle, \langle 1,4\rangle, \langle 2,4\rangle\}$$

(b)   There are six:

$$\{\langle 0,3\rangle, \langle 1,4\rangle, \langle 2,5\rangle\},$$
$$\{\langle 0,3\rangle, \langle 1,5\rangle, \langle 2,4\rangle\},$$
$$\{\langle 0,4\rangle, \langle 1,3\rangle, \langle 2,5\rangle\},$$
$$\{\langle 0,4\rangle, \langle 1,5\rangle, \langle 2,3\rangle\},$$
$$\{\langle 0,5\rangle, \langle 1,3\rangle, \langle 2,4\rangle\},$$
$$\{\langle 0,5\rangle, \langle 1,4\rangle, \langle 2,3\rangle\}$$

Exercise 48

(a)   The only ordered pair in $\mathcal{P}T$ is $\langle \emptyset, \emptyset \rangle = \{\{\emptyset\}\}$.

(b)

$$(\mathcal{P}T)^{-1} \circ (\mathcal{P}T \upharpoonright \{\emptyset\}) = \{\langle \emptyset, \emptyset \rangle\} \circ \{\langle \emptyset, \emptyset \rangle\}$$
$$= \{\langle \emptyset, \emptyset \rangle\}$$

Exercise 49   There are six:

$$\{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle\},$$
$$\{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle\},$$
$$\{\langle 0,0\rangle, \langle 0,2\rangle, \langle 1,1\rangle, \langle 2,0\rangle, \langle 2,2\rangle\},$$
$$\{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\},$$
$$\{\langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \langle 1,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,0\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$$

Exercise 50

(a) $\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,3\rangle\}$

(b) $\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}$

Exercise 51   There are three:

$$\{\langle 1,0\rangle, \langle 1,2\rangle, \langle 2,0\rangle\},$$
$$\{\langle 1,0\rangle, \langle 2,0\rangle, \langle 2,1\rangle\},$$
$$\{\langle 0,1\rangle, \langle 2,0\rangle, \langle 2,1\rangle\}$$

Exercise 52   We can conclude this if we know that A and B are nonempty, or that C and D are nonempty.

Suppose A and B are nonempty. Then $A \times B = C \times D \neq \emptyset$ so C and D are nonempty. We now prove $A \subseteq C$.

Let $a \in A$. Pick some $b \in B$. Then $\langle a,b\rangle \in A \times B = C \times D$ and so $a \in C$.

We can similarly prove $C \subseteq A$, $B \subseteq D$ and $D \subseteq B$.

Exercise 53

$$\begin{aligned}
x(R \cup S)^{-1}y &\Leftrightarrow y(R \cup S)x \\
&\Leftrightarrow yRx \text{ or } ySx \\
&\Leftrightarrow xR^{-1}y \text{ or } xS^{-1}y \\
&\Leftrightarrow x(R^{-1} \cup S^{-1})y \\
x(R \cap S)^{-1}y &\Leftrightarrow y(R \cap S)x \\
&\Leftrightarrow yRx \text{ and } ySx \\
&\Leftrightarrow xR^{-1}y \text{ and } xS^{-1}y \\
&\Leftrightarrow x(R^{-1} \cap S^{-1})y \\
x(R - S)^{-1}y &\Leftrightarrow y(R - S)x \\
&\Leftrightarrow yRx \text{ and } \neg ySx \\
&\Leftrightarrow xR^{-1}y \text{ and } \neg xS^{-1}y \\
&\Leftrightarrow x(R^{-1} - S^{-1})y
\end{aligned}$$

Exercise 54

(a)

$$\begin{aligned}
\langle x,y\rangle \in A \times (B \cap C) &\Leftrightarrow x \in A \ \& \ y \in B \ \& \ y \in C \\
&\Leftarrow \langle x,y\rangle \in (A \times B) \cap (A \times C)
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x,y\rangle \in A \times (B \cup C) &\Leftrightarrow x \in A \ \&(y \in B \text{ or } y \in C) \\
&\Leftrightarrow (x \in A \ \& \ y \in B) \text{ or } (x \in A \ \& \ y \in C) \\
&\Leftrightarrow \langle x,y\rangle \in (A \times B) \cup (A \times C)
\end{aligned}$$

33

(c)

$$\langle x, y \rangle \in A \times (B - C) \Leftrightarrow x \in A \ \& \ y \in B \ \& \ y \notin C$$
$$\Leftrightarrow \langle x, y \rangle \in (A \times B) - (A \times C)$$

Exercise 55

(a) No. Take $A = \{0\}$, $B = \{1\}$, $C = \{2\}$. Then $(A \times A) \cup (B \times C) = \{\langle 0, 0 \rangle, \langle 1, 2 \rangle\}$ while $(A \cup B) \times (A \cup C) = \{\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 2 \rangle\}$.

(b) Yes.

$$\langle x, y \rangle \in (A \times A) \cap (B \times C) \Leftrightarrow x \in A \ \& \ y \in A \ \& \ x \in B \ \& \ y \in C$$
$$\Leftrightarrow \langle x, y \rangle \in (A \cap B) \times (A \cap C)$$

Exercise 56

(a) Yes.

$$x \in \mathrm{dom}(R \cup S) \Leftrightarrow \exists y (xRy \ \text{or} \ xSy)$$
$$\Leftrightarrow \exists y.xRy \ \text{or} \ \exists y.xSy$$
$$\Leftrightarrow x \in \mathrm{dom}\, R \cup \mathrm{dom}\, S$$

(b) No. Take $R = \{\langle 0, 0 \rangle\}$ and $S = \{\langle 0, 1 \rangle\}$. Then $\mathrm{dom}(R \cap S) = \mathrm{dom}\, \emptyset = \emptyset$ while $\mathrm{dom}\, R \cap \mathrm{dom}\, S = \{0\} \cap \{0\} = \{0\}$.

Exercise 57

(a) Yes.

$$x(R \circ (S \cup T))y \Leftrightarrow \exists z (x(S \cup T)z \ \& \ zRy)$$
$$\Leftrightarrow \exists z (xSz \ \& \ zRy) \ \text{or} \ \exists z (xTz \ \& \ zRy)$$
$$\Leftrightarrow x((R \circ S) \cup (R \circ T))y$$

(b) No. Take $R = \{\langle 0, 0 \rangle, \langle 1, 0 \rangle\}$, $S = \{\langle 0, 0 \rangle\}$ and $T = \{\langle 0, 1 \rangle\}$. Then

$$R \circ (S \cap T) = R \circ \emptyset$$
$$= \emptyset$$
$$(R \circ S) \cap (R \circ T) = \{\langle 0, 0 \rangle\} \cap \{\langle 0, 0 \rangle\}$$
$$= \{\langle 0, 0 \rangle\}$$

Exercise 58 Take $F = \emptyset$ and $S = \{\emptyset\}$. Then $F[\![F^{-1}[\![S]\!]]\!] = \emptyset \neq S$.

Exercise 59

$$x(Q \upharpoonright (A \cap B))y \Leftrightarrow xQy \text{ \& } x \in A \text{ \& } x \in B$$
$$\Leftrightarrow x((Q \upharpoonright A) \cap (Q \upharpoonright B))y$$
$$x(Q \upharpoonright (A - B))y \Leftrightarrow xQy \text{ \& } x \in A \text{ \& } x \notin B$$
$$\Leftrightarrow (xQy \text{ \& } x \in A) \text{ \& } \neg(xQy \text{ \& } x \in B)$$
$$\Leftrightarrow x((Q \upharpoonright A) - (Q \upharpoonright B))y$$

Exercise 60

$$x((R \circ S) \upharpoonright A)y \Leftrightarrow \exists z(xRz \text{ \& } zSy \text{ \& } x \in A)$$
$$\Leftrightarrow x(R \circ (S \upharpoonright A))y$$

Chapter 4

Chapter 4 — Natural Numbers

4.1 Inductive Sets

Exercise 1   We have
$$3 = 2 \cup \{2\} = 1 \cup \{1, 2\}$$
and so $1 \in 3$. But $1 \notin 1$ (since $1 = \{\emptyset\}$ and we know $\{\emptyset\} \neq \emptyset$ hence $\{\emptyset\} \notin \{\emptyset\}$). Therefore $1 \neq 3$.

4.2 Peano's Postulates

Exercise 2   If a is a transitive set then
$$\bigcup(a^+) = a \qquad\qquad \text{(Theorem 4E)}$$
$$\subseteq a^+$$

Exercise 3

   (a)   Suppose a is a transitive set. Then $a \subseteq \mathcal{P}a$. Hence we have $\bigcup \mathcal{P}a = a \subseteq \mathcal{P}a$ and so $\mathcal{P}a$.

   (b)   Suppose $\mathcal{P}a$ is a transitive set. Then $a = \bigcup \mathcal{P}a \subseteq \mathcal{P}a$ hence a is transitive.

Exercise 4   If a is a transitive set then $\bigcup a \subseteq a$ so $\bigcup\bigcup a \subseteq \bigcup a$. Hence $\bigcup a$ is transitive.

Exercise 5

(a)   PROOF:
⟨1⟩1. LET: $b \in \bigcup \mathcal{A}$
⟨1⟩2. PICK $A \in \mathcal{A}$ such that $b \in A$
⟨1⟩3. $b \subseteq A$
   PROOF: Since A is transitive.
⟨1⟩4. $b \subseteq \bigcup \mathcal{A}$
□

(b)   PROOF:
⟨1⟩1. LET: $b \in \bigcap \mathcal{A}$
⟨1⟩2. For all $A \in \mathcal{A}$ we have $b \subseteq A$
   PROOF: Since $b \in A$ and A is transitive.
⟨1⟩3. $b \subseteq \bigcap \mathcal{A}$
□

Exercise 6   We have $\bigcup (a^+) = \bigcup a \cup a$ (see the proof of Theorem 4E). So if $\bigcup (a^+) = a$ we have $\bigcup a \cup a = a$ and so $\bigcup a \subseteq a$.

4.3   Recursion on ω

Exercise 7   We have $h_1(0) = h_2(0) = a$ so $0 \in S$.
   Now let $n \in S$; we prove $n^+ \in S$. We have $h_1(n) = h_2(n)$ and therefore

$$
\begin{aligned}
h_1(n^+) &= F(h_1(n)) \\
&= F(h_2(n)) \\
&= h_2(n^+)
\end{aligned}
$$

Exercise 8   PROOF:
⟨1⟩1. $\forall m, n \in \omega.h(n) = h(m) \Rightarrow n = m$
   ⟨2⟩1. $\forall n \in \omega.h(n) = h(0) \Rightarrow n = 0$
      ⟨3⟩1. LET: $n \in \omega$
      ⟨3⟩2. ASSUME: $h(n) = h(0)$
      ⟨3⟩3. $h(n) = c$
      ⟨3⟩4. $\forall p \in \omega.n \neq p^+$
         PROOF: Otherwise $f(h(p)) = c$ contradicting the fact that $c \in A - \operatorname{ran} f$.
      ⟨3⟩5. $n = 0$
         PROOF: Theorem 4C.
   ⟨2⟩2. For all $m \in \omega$, if $\forall n \in \omega.h(n) = h(m) \Rightarrow n = m$, then $\forall n \in \omega.h(n) = h(m^+) \Rightarrow n = m^+$
      ⟨3⟩1. LET: $m \in \omega$
      ⟨3⟩2. ASSUME: $\forall n \in \omega.h(n) = h(m) \Rightarrow n = m$
      ⟨3⟩3. LET: $n \in \omega$
      ⟨3⟩4. ASSUME: $h(n) = h(m^+)$
      ⟨3⟩5. $h(n) = f(h(m))$

$\langle 3 \rangle 6. \ n \neq 0$
    PROOF: Otherwise $c = f(h(m))$ contradicting the fact that $c \in A - \operatorname{ran} f$.
$\langle 3 \rangle 7.$ PICK p such that $n = p^+$
$\langle 3 \rangle 8. \ f(h(p)) = f(h(m))$
$\langle 3 \rangle 9. \ h(p) = h(m)$
    PROOF: f is one-to-one.
$\langle 3 \rangle 10. \ p = m$
    PROOF: By $\langle 3 \rangle 2.$
$\langle 3 \rangle 11. \ n = p^+ = m^+$
$\square$

Exercise 9 PROOF:
$\langle 1 \rangle 1. \ C^* \subseteq C_*$
  $\langle 2 \rangle 1. \ f[\![C_*]\!] \subseteq C_*$
    $\langle 3 \rangle 1.$ LET: $x \in C_*$
        PROVE: $f(x) \in C_*$
    $\langle 3 \rangle 2.$ PICK n such that $x \in h(n)$
    $\langle 3 \rangle 3. \ f(x) \in h(n^+)$
    $\langle 3 \rangle 4. \ f(x) \in C_*$
$\langle 1 \rangle 2. \ C_* \subseteq C^*$
  $\langle 2 \rangle 1. \ \forall n \in \omega.h(n) \subseteq C^*$
    $\langle 3 \rangle 1. \ h(0) \subseteq C^*$
      PROOF: If $A \subseteq X \subseteq B$ and $f[\![X]\!] \subseteq X$ then $A \subseteq X$.
    $\langle 3 \rangle 2. \ \forall n \in \omega(h(n) \subseteq C^* \Rightarrow h(n^+) \subseteq C^*)$
      $\langle 4 \rangle 1.$ LET: $n \in \omega$
      $\langle 4 \rangle 2.$ ASSUME: $h(n) \subseteq C^*$
      $\langle 4 \rangle 3. \ f[\![h(n)]\!] \subseteq C^*$
        $\langle 5 \rangle 1.$ LET: X be such that $A \subseteq X \subseteq B$ and $f[\![X]\!] \subseteq X$
            PROVE: $f[\![h(n)]\!] \subseteq X$
        $\langle 5 \rangle 2. \ h(n) \subseteq X$
        $\langle 5 \rangle 3. \ f[\![h(n)]\!] \subseteq f[\![X]\!]$
        $\langle 5 \rangle 4. \ f[\![h(n)]\!] \subseteq X$
      $\langle 4 \rangle 4. \ h(n^+) \subseteq C^*$
$\square$

Exercise 10 $C^* = C_* = (0, 1]$

Exercise 11 $\{n \in \mathbb{Z} \mid n \leq 0\}$

Exercise 12 Let $f : B \times B \to B$ and $A \subseteq B$. Let

$$C^* = \bigcap \{X \mid A \subseteq X \subseteq B \ \& \ f[\![X \times X]\!] \subseteq X\} \ .$$

Define the function $h : \omega \to \mathcal{P}B$ by

$$h(0) = A$$
$$h(n^+) = h(n) \cup f[\![h(n) \times h(n)]\!] \qquad\qquad (n \in \omega)$$

Define $C_* = \bigcup \operatorname{ran} h$. Then $C^* = C_*$.

4.4 Arithmetic

Exercise 13 We prove the contrapositive. Assume $m \neq 0$ and $n \neq 0$. Then by Theorem 4C there are natural numbers p, q such that $m = p^+$ and $n = q^+$. Hence $mn = p^+ q^+ = (p^+ q + p)^+ \neq 0$.

Exercise 14 We prove the following facts for any natural number n:

1. n is even if and only if n^+ is odd.

   PROOF: If n is even, say $n = 2p$, then $n^+ = 2p + 1$ is odd.
   If n^+ is odd, say $n^+ = 2p + 1$, then $n = 2p$ is even.

2. n is odd if and only if n^+ is even.

   PROOF: If n is odd, say $n = 2p + 1$, then $n^+ = 2(p + 1)$ is even.
   If n^+ is even, say $n^+ = 2p$, then we cannot have $p = 0$ (since $n^+ \neq 0$). So $p = q + 1$ for some q. But then $n^+ = 2q + 2$ so $n = 2q + 1$ and n is odd.

Now, 0 is even and 0 is not odd. By the two facts above, if n is either even or odd but not both, then n^+ is either odd or even but not both. The result follows by induction.

Exercise 15 We have

$$m + (n + 0) = m + n \qquad\qquad \text{by (A1)}$$
$$= (m + n) + 0 \qquad\qquad \text{by (A1)}$$

If $m + (n + p) = (m + n) + p$ then

$$m + (n + p^+) = m + (n + p)^+ \qquad\qquad \text{by (A2)}$$
$$= (m + (n + p))^+ \qquad\qquad \text{by (A2)}$$
$$= ((m + n) + p)^+ \qquad\qquad \text{by induction hypothesis}$$
$$= (m + n) + p^+ \qquad\qquad \text{by (A2)}$$

Exercise 16 We first prove that $0 \cdot n = 0$ for all n. We have $0 \cdot 0 = 0$ by (M1), and if $0 \cdot n = 0$ then

$$0 \cdot n^+ = 0 \cdot n + 0 \qquad\qquad \text{by (M2)}$$
$$= 0 \cdot n \qquad\qquad \text{by (A1)}$$
$$= 0 \qquad\qquad \text{by induction hypothesis}$$

Now we prove that $m^+ \cdot n = m \cdot n + n$ for all m, n. We have

$$m^+ \cdot 0 = 0 \qquad \text{by (M1)}$$
$$m \cdot 0 + 0 = m \cdot 0 \qquad \text{by (A1)}$$
$$= 0 \qquad \text{by (M1)}$$

Thus, $m^+ \cdot 0 = m \cdot 0 + 0$.

If $m^+ \cdot n = m \cdot n + n$ then

$$
\begin{aligned}
m^+ \cdot n^+ &= m^+ \cdot n + m^+ & \text{by (M2)} \\
&= (m^+ \cdot n + m)^+ & \text{by (A2)} \\
&= ((m \cdot n + n) + m)^+ & \text{by induction hypothesis} \\
&= ((m \cdot n + m) + n)^+ & \text{by associativity and commutativity of addition} \\
&= (m \cdot n^+ + n)^+ & \text{by (M2)} \\
&= m \cdot n^+ + n^+ & \text{by (A2)}
\end{aligned}
$$

Exercise 17 The proof is by induction on p. We have

$$
\begin{aligned}
m^{n+0} &= m^n & \text{by (A1)} \\
&= 0 + m^n & \text{by Theorem 4K(2)} \\
&= m^n \cdot 0 + m^n & \text{by (M1)} \\
&= m^n \cdot 1 & \text{by (M2)} \\
&= m^n \cdot m^0 & \text{by (E1)}
\end{aligned}
$$

If $m^{n+p} = m^n \cdot m^p$ then

$$
\begin{aligned}
m^{n+p^+} &= m^{(n+p)^+} & \text{by (A2)} \\
&= m^{n+p} m & \text{by (E2)} \\
&= (m^n m^p) m & \text{by induction hypothesis} \\
&= m^n (m^p m) & \text{by Theorem 4K (4)} \\
&= m^n m^{p^+} & \text{by (E2)}
\end{aligned}
$$

4.5 Ordering on ω

Exercise 18

$$\in_\omega^{-1} [\![\{7, 8\}]\!] = \{x \in \omega \mid x \in 7 \text{ or } x \in 8\}$$
$$= \{0, 1, 2, 3, 4, 5, 6, 7\}$$

Exercise 19 The proof is by induction on m.

For $m = 0$, take $q = r = 0$. Then $m = d \cdot 0 + 0$ and $0 \in d$.

Suppose $m = dq + r$ and $r < d$. Then $r + 1 \leq d$. If $r + 1 < d$, then we have $m + 1 = dq + (r + 1)$ as required. If $r + 1 = d$, then we have $m + 1 = dq + d = d(q + 1) + 0$.

40

Exercise 20 We first prove A is closed downwards; that is, if $n \in A$ and $m \in n$ then $m \in A$. This holds because if $n \in A$ and $m \in n$ then $m \in \bigcup A$ and $\bigcup A = A$.

Now, we prove $\forall n \in \omega . n \in A$ by induction on n.

To prove $0 \in A$: we are given that A is nonempty. Pick some $a \in A$. Then $0 \underline{in} a$ so $0 \in A$ since A is closed downwards.

Now let $n \in A$; we prove $n^+ \in A$. We have $n \in \bigcup A$; pick some $k \in A$ such that $n \in k$. Then $n^+ \underline{\in} k$ so $n^+ \in A$ since A is closed downwards.

This completes the induction. We have $\forall n \in \omega . n \in A$, i.e. $A = \omega$.

Exercise 21 Suppose n is a natural number, $k \in n$ and $n \subseteq k$. Then $k \in k$, contradicting Lemma 4L(b).

Exercise 22 We have $0 \in p^+$ (by trichotomy since $p^+ \notin 0$ because 0 is empty, and $p^+ \neq 0$ by Peano's First Postulate.) Hence $n = n + 0 \in n + p^+$ by Theorem 4N.

Exercise 23 The proof is by induction on n. The statement is vacuously true for $n = 0$.

Suppose the statement is true for n. Let $m \in n^+$. Then $m \underline{\in} n$.

If $m = n$, then we have $m + 0^+ = n^+$.

If $m \in n$, pick p such that $m + p^+ = n$ by the induction hypothesis. Then $m + p^{++} = n^+$.

Exercise 24 Suppose $m \in p$. Then we cannot have $n \in q$ or $n = q$, as either of these would imply $m + n \in p + q$. Hence $q \in n$ by trichotomy.

We prove $q \in n \Rightarrow m \in p$ similarly.

Exercise 25 By Exercise 23, pick natural numbers a and b such that $m = n + a^+$ and $p = q + b^+$. Then

$$
\begin{aligned}
mp + nq &= (n + a^+)(q + b^+) + nq \\
&= nq + nq + a^+ q + nb^+ + a^+ b^+ \\
&= (n + a^+)q + n(q + b^+) + a^+ b^+ \\
&= mq + np + (a^+ + b)^+
\end{aligned}
$$

Hence $mq + np \in mp + nq$ by Exercise 22.

Exercise 26 The proof is by induction on n.

If $n = 0$ then $\operatorname{ran} f$ is a singleton and its sole element is the largest element.

Suppose the result is true for n. Let $f : n^{++} \to A$. Then $f[\![n^+]\!]$ has a largest element $f(k)$, say. If $f(k) \underline{\in} f(n^+)$ then $f(n^+)$ is greatest in $\operatorname{ran} f$; otherwise $f(k)$ is greatest.

Exercise 27