Solutions Manual for Enderton *Elements of Set Theory*

Robin Adams

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Contents

Chapter 1

Chapter 1 — Introduction

1.1 Baby Set Theory

Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

Exercise 2 We have $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ has an element (namely $\emptyset$) while $\emptyset$ has no elements.

We have $\emptyset \neq \{\{\emptyset\}\}$ because $\{\{\emptyset\}\}$ has an element (namely $\{\emptyset\}$) while $\emptyset$ has no elements.

We have $\{\emptyset\} \neq \{\{\emptyset\}\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$. This last fact is true because $\emptyset \neq \{\emptyset\}$ as we proved in the first paragraph.

Exercise 3 Assume $B \subseteq C$. Let $A \in \mathcal{P}B$; we must show that $A \in \mathcal{P}C$.

We have $A \subseteq B$ (since $A \in \mathcal{P}B$) and $B \subseteq C$. From this it follows that $A \subseteq C$ (every element of A is an element of B; every element of B is an element of C; therefore every element of A is an element of C). Hence $A \in \mathcal{P}C$ as required.

Exercise 4 Since $x \in B$, we have $\{x\} \subseteq B$ and so $\{x\} \in \mathcal{P}B$.

Since $x \in B$ and $y \in B$, we have $\{x, y\} \subseteq B$ and so $\{x, y\} \in \mathcal{P}B$.

From these two facts, it follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$ and so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

1.2 Sets — An Informal View

Exercise 5 We have

$$\begin{aligned}
V_0 &= A \\
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}A \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= \{\emptyset, \{\emptyset\}\} \\
V_3 &= \mathcal{P}V_2 \\
&= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}$$

We have $\emptyset \subseteq V_0$ and so $\emptyset \in V_1$. Therefore $\{\emptyset\} \subseteq V_1$ and so $\{\emptyset\} \in V_2$. Hence $\{\{\emptyset\}\} \subseteq V_2$.

We also have $\{\{\emptyset\}\} \nsubseteq V_0$ because $\{\emptyset\}$ is not an atom, and $\{\{\emptyset\}\} \nsubseteq V_1$ since $\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.

Thus the rank of $\{\{\emptyset\}\}$ is 2.

Likewise we have $\emptyset$ and $\{\emptyset\}$ are both subsets of V_1, hence

$$\emptyset \in V_2, \qquad \{\emptyset\} \in V_2$$

Thus $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are all subsets of V_2, hence elements of V_3. Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is not a subset of V_0 (because $\emptyset$ is not an atom.) It is not a subset of V_1 ($\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.) It is not a subset of V_2 (we have $\{\emptyset, \{\emptyset\}\} \notin V_2$ since $\{\emptyset\} \notin V_1$).

Therefore the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

Exercise 6

$$
\begin{aligned}
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}V_0 && \text{(since } V_0 = A\text{)} \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_1 && \text{(since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3)} \\
V_3 &= V_2 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_2 && \text{(since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3)} \\
V_4 &= V_3 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_3 && \text{(since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3)}
\end{aligned}
$$

Exercise 7 In Exercise 5 we calculated $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Hence

$$
\begin{aligned}
V_4 =& \mathcal{P}V_3 \\
= \{&\emptyset, \\
&\{\emptyset\}, \\
&\{\{\emptyset\}\}, \\
&\{\{\{\emptyset\}\}\}, \\
&\{\{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}\}, \\
&\{\emptyset, \{\{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
\}&
\end{aligned}
$$

Chapter 2

Chapter 2 — Axioms and Operations

2.1 Arbitrary Unions and Intersections

Exercise 1 $A \cap B \cap C$ is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

Exercise 2 Take $A = \emptyset$ and $B = \{\emptyset\}$. Then $\bigcup A = \bigcup B = \emptyset$ but $A \neq B$. (There are many other possible answers.)

Exercise 3 Let $b \in A$. We must show that $b \subseteq \bigcup A$.

Let x be any element of b. We must show that $x \in \bigcup A$. We know that $x \in b$ and $b \in A$, and so $x \in \bigcup A$ by the definition of $\bigcup A$.

Exercise 4 Suppose $A \subseteq B$. Let $x \in \bigcup A$. We must show that $x \in \bigcup B$.

Pick an element $a \in A$ such that $x \in a$. Then $a \in B$ because $A \subseteq B$. Since we know $x \in a$ and $a \in B$, we know that $x \in \bigcup B$.

Exercise 5 Assume that every member of $\mathcal{A}$ is a subset of B. Let $x \in \bigcup \mathcal{A}$. We must show that $x \in B$.

Pick $A \in \mathcal{A}$ such that $x \in A$. By our assumption, we have $A \subseteq B$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$ as required.

Exercise 6

(a) We will show that $\bigcup \mathcal{P}A \subseteq A$ and $A \subseteq \bigcup \mathcal{P}A$.

To show $\bigcup \mathcal{P}A \subseteq A$: This follows from Exercise 5, since every member of $\mathcal{P}A$ is a subset of A.

To show $A \subseteq \bigcup \mathcal{P}A$: Let $a \in A$. Then we have $a \in \{a\}$ and $\{a\} \in \mathcal{P}A$ so $a \in \bigcup \mathcal{P}A$.

(b) To show $A \subseteq \mathcal{P} \bigcup A$: This holds because every element of A is a subset of $\bigcup A$, as we proved is Exercise 3.

Equality holds if and only if $A = \mathcal{P}X$ for some set X.

Proof: If $A = \mathcal{P} \bigcup A$ then of course $A = \mathcal{P}X$ for some X.

Conversely, if $A = \mathcal{P}X$, then we have

$$\begin{aligned}
\mathcal{P} \bigcup A &= \mathcal{P} \bigcup \mathcal{P}X \\
&= \mathcal{P}X && \text{(by part (a))} \\
&= A
\end{aligned}$$

Exercise 7

(a) For any set X,

$$\begin{aligned}
& X \in \mathcal{P}A \cap \mathcal{P}B \\
\Leftrightarrow & X \subseteq A \text{ and } X \subseteq B \\
\Leftrightarrow & \text{Every member of } X \text{ is a member of } A \text{ and a member of } B \\
\Leftrightarrow & X \subseteq A \cap B \\
\Leftrightarrow & X \in \mathcal{P}(A \cap B)
\end{aligned}$$

(b) Let $X \in \mathcal{P}A \cup \mathcal{P}B$. Then either $X \in \mathcal{P}A$ or $X \in \mathcal{P}B$ (or both). If $X \in \mathcal{P}A$, then we have $X \subseteq A$ and so $X \subseteq A \cup B$ (because $A \subseteq A \cup B$). Similarly if $X \in \mathcal{P}B$ then we have $X \subseteq A \cup B$. So in either case $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

Equality holds if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof: Suppose $A \subseteq B$. Then $\mathcal{P}A \subseteq \mathcal{P}B$ (Chapter 1 Exercise 3) and so $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$. Also $A \cup B = B$ so $\mathcal{P}(A \cup B) = \mathcal{P}B$. Thus $\mathcal{P}A \cup \mathcal{P}B$ and $\mathcal{P}(A \cup B)$ are equal.

Similarly if $B \subseteq A$ then $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$.

Conversely, suppose $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$. We have $A \cup B \in \mathcal{P}(A \cup B)$, so $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$. If $A \cup B \in \mathcal{P}A$, then we have $B \subseteq A \cup B \subseteq A$. And if $A \cup B \in \mathcal{P}B$, then we have $A \subseteq A \cup B \subseteq B$.

Exercise 8 If A is a set such that every singleton belongs to A, then every set belongs to $\bigcup A$, contradicting Theorem 2A.

Exercise 9 Let $a = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$. Then $a \in B$ but $\mathcal{P}a$ is not a subset of B because $\emptyset \in \mathcal{P}a$ and $\emptyset \notin B$.

Exercise 10 We must show that $\mathcal{P}a \subseteq \mathcal{P}\bigcup B$. So let $X \in \mathcal{P}a$. Then $X \subseteq a$; we must show that $X \subseteq \bigcup B$.

Let $x \in X$; we must show that $x \in \bigcup B$. We have $x \in a$ (because $x \in X$ and $X \subseteq a$) and $a \in B$, hence $x \in \bigcup B$ as required.

2.2 Algebra of Sets

Exercise 11 For any x we have

$$\begin{aligned} x \in (A \cap B) \cup (A - B) &\Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B) \\ &\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B) \\ &\Leftrightarrow x \in A \end{aligned}$$

Hence $A = (A \cap B) \cup (A - B)$.

For any x we have

$$\begin{aligned} x \in A \cup (B - A) &\Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A) \\ &\Leftrightarrow x \in A \text{ or } x \in B \\ &\Leftrightarrow x \in A \cup B \end{aligned}$$

Hence $A \cup (B - A) = A \cup B$.

Exercise 12 For any x,

$$\begin{aligned} x \in C - (A \cap B) &\Leftrightarrow x \in C \& \neg(x \in A \& x \in B) \\ &\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B) \\ &\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B) \\ &\Leftrightarrow x \in (C - A) \cup (C - B) \end{aligned}$$

Exercise 13 Suppose $A \subseteq B$. Let $x \in C - B$; we must show $x \in C - A$. We have $x \in C$ and $x \notin B$. Therefore $x \notin A$, since every member of A is a member of B. And so we have $x \in C - A$ as required.

Exercise 14 Let $A = \{\emptyset\}$, $B = \emptyset$ and $C = \{\emptyset\}$. Then $A - (B - C) = A - \emptyset = \{\emptyset\}$ while $(A - B) - C = \{\emptyset\} - C = \emptyset$.

Exercise 15

(a) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have $x \in A \cap (B + C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$.

(b) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$

In every case, we have $x \in A + (B + C) \Leftrightarrow x \in (A + B) + C$.

Exercise 16

$$\begin{aligned}[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A] &= (A \cup B) - A \\ &= B - A\end{aligned}$$

Exercise 17

(a) $\Leftrightarrow$ (b)

$$\begin{aligned}A \subseteq B &\Leftrightarrow \text{Every element of } A \text{ is an element of } B \\ &\Leftrightarrow \text{There is no element of } A \text{ that is not an element of } B \\ &\Leftrightarrow A - B = \emptyset\end{aligned}$$

(a) $\Rightarrow$ (c) Suppose $A \subseteq B$. We have $B \subseteq A \cup B$ from the definition of $A \cup B$; we must prove that $A \cup B \subseteq B$. So let $x \in A \cup B$. Then $x \in A$ or $x \in B$. But in either case $x \in B$, since $x \in A \Rightarrow x \in B$. Thus we have $x \in B$ as required.

(c) $\Rightarrow$ (a) We always have $A \subseteq A \cup B$. So if $A \cup B = B$ then we have $A \subseteq B$.

(a) $\Rightarrow$ (d) Suppose $A \subseteq B$. We have $A \cap B \subseteq A$ from the definition of $A \cap B$; we must prove that $A \subseteq A \cap B$. So let $x \in A$. Then $x \in B$ since $A \subseteq B$, hence $x \in A \cap B$ as required.

(d) ⇒ (a) We always have $A \cap B \subseteq B$. So if $A \cap B = A$ then $A \subseteq B$.

Exercise 18 We can make the following 16 sets:

- $\emptyset$ $(= A - A)$
- $A - B$
- $A \cap B$
- $B - A$
- $S - (A \cup B)$
- A
- $A + B$
- $S - B$
- B
- $S - (A + B)$
- $S - A$
- $A \cup B$
- $S - (B - A)$
- $S - (A \cap B)$
- $S - (A - B)$

Exercise 19 They are never equal, because for all A, B, we have $\emptyset \in \mathcal{P}(A-B)$ but $\emptyset \notin \mathcal{P}A - \mathcal{P}B$ since $\emptyset \in \mathcal{P}B$.

Exercise 20 Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

We first show $B \subseteq C$. Let $x \in B$; we show $x \in C$. We have $x \in A \cup B = A \cup C$, so either $x \in A$ or $x \in C$. If $x \in C$, we are done. If $x \in A$, then we have $x \in A \cap B = A \cap C$, and so $x \in C$ in this case too.

We can show $C \subseteq B$ similarly. Hence $B = C$.

Exercise 21 For any x, we have

$$\begin{aligned}
x \in \bigcup(A \cup B) &\Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C \\
&\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C\text{, or there exists } C \in B \text{ such that } x \in C \\
&\Leftrightarrow x \in \bigcup A \cup \bigcup B
\end{aligned}$$

Exercise 22 For any x, we have

$$\begin{aligned} x \in \bigcap(A \cup B) &\Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C \\ &\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C \\ &\Leftrightarrow x \in \bigcap A \cap \bigcap B \end{aligned}$$

Exercise 23 PROOF:
⟨1⟩1. $A \subseteq \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  ⟨2⟩1. LET: $x \in A$
  ⟨2⟩2. LET: $X \in \mathcal{B}$
  ⟨2⟩3. $x \in A \cup X$
⟨1⟩2. $\bigcap \mathcal{B} \subseteq \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  ⟨2⟩1. LET: $x \in \bigcap \mathcal{B}$
  ⟨2⟩2. LET: $X \in \mathcal{B}$
  ⟨2⟩3. $x \in X$
  ⟨2⟩4. $x \in A \cup X$
⟨1⟩3. $\bigcap\{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap \mathcal{B}$
  ⟨2⟩1. LET: $x \in \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  ⟨2⟩2. ASSUME: $x \notin A$
  PROVE: $x \in \bigcap \mathcal{B}$
  ⟨2⟩3. LET: $X \in \mathcal{B}$
  ⟨2⟩4. $x \in A \cup X$
  ⟨2⟩5. $x \in X$
□

Exercise 24

(a)

$$\begin{aligned} Y \in \mathcal{P}\bigcap \mathcal{A} &\Leftrightarrow Y \subseteq \bigcap \mathcal{A} \\ &\Leftrightarrow \forall y \in Y. \forall X \in \mathcal{A}. y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A}. \forall y \in Y. y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A}. Y \in \mathcal{P}X \\ &\Leftrightarrow Y \in \bigcap\{\mathcal{P}X \mid X \in \mathcal{A}\} \end{aligned}$$

(b) $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} \subseteq \mathcal{P}\bigcup \mathcal{A}$

PROOF:
⟨1⟩1. LET: $Y \in \bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\}$
⟨1⟩2. PICK $X \in \mathcal{A}$ such that $Y \in \mathcal{P}X$
⟨1⟩3. $Y \subseteq X$
⟨1⟩4. $Y \subseteq \bigcup \mathcal{A}$
⟨1⟩5. $Y \in \mathcal{P}\bigcup \mathcal{A}$

Equality holds if and only if $\bigcup \mathcal{A} \in \mathcal{A}$.

$\langle 1 \rangle 1.$ If $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$ then $\bigcup \mathcal{A} \in \mathcal{A}$
  $\langle 2 \rangle 1.$ Assume: $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
  $\langle 2 \rangle 2.$ $\bigcup \mathcal{A} \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
  $\langle 2 \rangle 3.$ Pick $X \in \mathcal{A}$ such that $\bigcup \mathcal{A} \in \mathcal{P}X$
  $\langle 2 \rangle 4.$ $X = \bigcup \mathcal{A}$
$\langle 1 \rangle 2.$ If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
  Proof: If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\mathcal{P} \bigcup \mathcal{A} \in \{\mathcal{P}X \mid X \in \mathcal{A}\}$.
$\square$

Exercise 25 We have $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ if and only if $A = \emptyset$ or $\mathcal{B} \neq \emptyset$

$\langle 1 \rangle 1.$ If $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ then $A = \emptyset$ or $\mathcal{B} \neq \emptyset$
  Proof: If $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ and $\mathcal{B} = \emptyset$ then

$$A \cup \bigcup \emptyset = \bigcup \emptyset$$
$$\therefore A = \emptyset$$

$\langle 1 \rangle 2.$ If $A = \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
  Proof: Both sides are equal to $\bigcup \mathcal{B}$
$\langle 1 \rangle 3.$ If $\mathcal{B} \neq \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 1.$ Assume: $\mathcal{B} \neq \emptyset$
  $\langle 2 \rangle 2.$ $A \cup \bigcup \mathcal{B} \subseteq \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 1.$ Let: $x \in A \cup \bigcup \mathcal{B}$
      Prove: $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 2.$ Case: $x \in A$
      $\langle 4 \rangle 1.$ Pick $X \in \mathcal{B}$
        Proof: By $\langle 2 \rangle 1$
      $\langle 4 \rangle 2.$ $x \in A \cup X$
    $\langle 3 \rangle 3.$ Case: $x \in \bigcup \mathcal{B}$
      $\langle 4 \rangle 1.$ Pick $X \in \mathcal{B}$ such that $x \in X$
      $\langle 4 \rangle 2.$ $x \in A \cup X$
  $\langle 2 \rangle 3.$ $\bigcup \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcup \mathcal{B}$
    $\langle 3 \rangle 1.$ Let: $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 2.$ Pick $X \in \mathcal{B}$ such that $x \in A \cup X$
    $\langle 3 \rangle 3.$ $X \subseteq \bigcup \mathcal{B}$
    $\langle 3 \rangle 4.$ $A \cup X \subseteq A \cup \bigcup \mathcal{B}$
    $\langle 3 \rangle 5.$ $x \in A \cup \bigcup \mathcal{B}$

2.3 Review Exercises

Exercise 26 Sets A, B, D and F are all equal to each other. Sets C, E and G are equal to each other. None of the first list is equal to any of the second list.

Exercise 27 Take $A = \{\{0\}, \{1\}\}$ and $B = \{\{1\}\}$. Then $A \cap B = \{\{1\}\}$ and

$$\begin{aligned}
\bigcap A \cap \bigcap B &= \emptyset \cap \{1\} \\
&= \emptyset \\
\bigcap (A \cap B) &= \bigcap \{\{1\}\} \\
&= \{1\}
\end{aligned}$$

Exercise 28

$$\bigcup \{\{3,4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\} = \{3, 4, \{3\}, \{4\}\}$$

Exercise 29

(a) $\emptyset$

(b) We have

$$\begin{aligned}
\{\emptyset\} &\subseteq \mathcal{P}\{\emptyset\} \\
\therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{P}\mathcal{P}\{\emptyset\} \\
\{\emptyset\} &\subseteq \mathcal{P}\mathcal{P}\{\emptyset\} \\
\therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{P}\mathcal{P}\mathcal{P}\{\emptyset\} \\
\therefore \bigcap \{\mathcal{P}\mathcal{P}\mathcal{P}\{\emptyset\}, \mathcal{P}\mathcal{P}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} &= \mathcal{P}\mathcal{P}\mathcal{P}\{\emptyset\} \cap \mathcal{P}\mathcal{P}\{\emptyset\} \cap \mathcal{P}\{\emptyset\} \\
&= \mathcal{P}\{\emptyset\} \\
&= \{\emptyset, \{\emptyset\}\}
\end{aligned}$$

Exercise 30

(a) $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$

(b) $\{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d) $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

Exercise 31

(a) $\{1, 2, 3, \emptyset\}$

(b) $\emptyset$

(c) $\emptyset$

(d) $\emptyset$

Exercise 32

(a) $a \cup b$

(b) a

(c)

$$\begin{aligned}
\bigcap\bigcup S \cup (\bigcup\bigcup S - \bigcup\bigcap S) &= (a \cap b) \cup ((a \cup b) - a) \\
&= (a \cap b) \cup (b - a) \\
&= b
\end{aligned}$$

Exercise 33 When $a \neq b$:

$$\begin{aligned}
\bigcup(\bigcup S - \bigcap S) &= \bigcup(\{a, b\} - \{a\}) \\
&= \bigcup\{b\} \\
&= b
\end{aligned}$$

When $a = b$:

$$\begin{aligned}
\bigcup(\bigcup S - \bigcap S) &= \bigcup(\{a, b\} - \{a\}) \\
&= \bigcup\emptyset \\
&= \emptyset
\end{aligned}$$

Exercise 34 For any set S, we have

$$\begin{gathered}
\emptyset \subseteq \mathcal{P}S \\
\therefore \emptyset \in \mathcal{P}\mathcal{P}S \\
\emptyset \subseteq S \\
\therefore \emptyset \in \mathcal{P}S \\
\therefore \{\emptyset\} \subseteq \mathcal{P}S \\
\therefore \{\emptyset\} \in \mathcal{P}\mathcal{P}S \\
\therefore \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}\mathcal{P}S \\
\therefore \{\emptyset, \{\emptyset\}\} \in \mathcal{P}\mathcal{P}\mathcal{P}S
\end{gathered}$$

Exercise 35 Assume $\mathcal{P}A = \mathcal{P}B$. Then we have

$$\begin{aligned}
&A \in \mathcal{P}A \\
\therefore\ &A \in \mathcal{P}B \\
\therefore\ &A \subseteq B \\
&B \in \mathcal{P}B \\
\therefore\ &B \in \mathcal{P}A \\
\therefore\ &B \subseteq A \\
\therefore\ &A = B
\end{aligned}$$

Exercise 36

(a)

$$\begin{aligned}
x \in A - (A \cap B) &\Leftrightarrow x \in A \ \&\neg(x \in A \ \&\ x \in B) \\
&\Leftrightarrow x \in A \ \&\ x \notin B \\
&\Leftrightarrow x \in A - B
\end{aligned}$$

(b)

$$\begin{aligned}
x \in A - (A - B) &\Leftrightarrow x \in A \ \&\neg(x \in A \ \&\ x \notin B) \\
&\Leftrightarrow x \in A \ \&\ x \in B \\
&\Leftrightarrow x \in A \cap B
\end{aligned}$$

Exercise 37

(a)

$$\begin{aligned}
x \in (A \cup B) - C &\Leftrightarrow (x \in A \text{ or } x \in B) \ \&\ x \notin C \\
&\Leftrightarrow (x \in A \ \&\ x \notin C) \text{ or } (x \in B \ \&\ x \notin C) \\
&\Leftrightarrow x \in (A - C) \cup (B - C)
\end{aligned}$$

(b)

$$\begin{aligned}
x \in A - (B - C) &\Leftrightarrow x \in A \ \&\neg(x \in B \ \&\ x \notin C) \\
&\Leftrightarrow x \in A \ \&(x \notin B \text{ or } x \in C) \\
&\Leftrightarrow (x \in A \ \&\ x \notin B) \text{ or } (x \in A \ \&\ x \in C) \\
&\Leftrightarrow x \in (A - B) \cup (A \cap C)
\end{aligned}$$

(c)

$$\begin{aligned}
x \in (A - B) - C &\Leftrightarrow x \in A \ \&\ x \notin B \ \&\ x \notin C \\
&\Leftrightarrow x \in A \ \&\neg(x \in B \vee x \in C) \\
&\Leftrightarrow x \in A - (B \cup C)
\end{aligned}$$

Exercise 38

(a) If every element of A is an element of C, and every element of B is an element of C, then everything that is an element of either A or B is an element of C.

(b) If every element of C is an element of A, and every element of C is an element of B, then every element of C is an element of both A and B.

Chapter 3

Chapter 3 — Relations and Functions

3.1 Ordered Pairs

Exercise 1 We have $\langle 0,1,0\rangle^* = \langle 0,1,1\rangle^* = \{\{0\},\{0,1\}\}$.

Exercise 2

(a)

$$\begin{aligned}
&z \in A \times (B \cup C) \\
\Leftrightarrow &\exists x, y(z = (x,y) \ \& \ x \in A \ \& \ (y \in B \text{ or } y \in C)) \\
\Leftrightarrow &\exists x, y(z = (x,y) \ \& \ x \in A \ \& \ y \in B) \text{ or } (z = (x,y) \ \& \ x \in A \ \& \ y \in C) \\
\Leftrightarrow &z \in (A \times B) \cup (A \times C)
\end{aligned}$$

(b)
$\langle 1\rangle 1$. Assume: $A \times B = A \times C$ and $A \neq \emptyset$
$\langle 1\rangle 2$. Pick $a \in A$
$\langle 1\rangle 3$. For all x, $x \in B \Leftrightarrow x \in C$
Proof: $x \in B$ iff $(a,x) \in A \times B$ iff $(a,x) \in A \times C$ iff $x \in C$.
$\square$

Exercise 3

$$\begin{aligned}
&z \in A \times \bigcup \mathcal{B} \\
\Leftrightarrow &\exists x, y(z = (x,y) \ \& \ x \in A \ \& \ \exists X \in \mathcal{B}. y \in X) \\
\Leftrightarrow &\exists X \in \mathcal{B}. \exists x, y(z = (x,y) \ \& \ x \in A \ \& \ y \in X) \\
\Leftrightarrow &z \in \bigcup \{A \times X : X \in \mathcal{B}\}
\end{aligned}$$

Exercise 4 If every ordered pair belongs to A then every set belongs to $\bigcup\bigcup A$ contradicting Theorem 2A.

Exercise 5

(a) Apply a Subset Axiom to $\mathcal{P}(A \times B)$: we have $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A.z = \{x\} \times B\}$.

(b)

$$\begin{aligned} & z \in \bigcup C \\ \Leftrightarrow & \exists x \in A.z \in \{x\} \times B \\ \Leftrightarrow & \exists x \in A.\exists y \in B.z = (x, y) \\ \Leftrightarrow & z \in A \times B \end{aligned}$$

3.2 Relations

Exercise 6 If $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$ then A is a set of ordered pairs, i.e. a relation.

Conversely, suppose A is a relation. Let $z \in A$. Then z is an ordered pair; let $z = (x, y)$. We have $x \in \operatorname{dom} A$ and $y \in \operatorname{ran} A$ and so $z \in \operatorname{dom} A \times \operatorname{ran} A$ as required.

Exercise 7 We have $\operatorname{fld} R \subseteq \bigcup\bigcup R$ by Lemma 3D.

Conversely, let $x \in \bigcup\bigcup R$. Pick a and b such that $x \in a$, $a \in b$ and $b \in R$. Then b is an ordered pair; let $b = (y, z)$. We have $a = \{y\}$ or $\{y, z\}$, hence $x = y$ or $x = z$. In either case, $x \in \operatorname{fld} R$.

Exercise 8

(a)

$$\begin{aligned} & x \in \operatorname{dom} \bigcup \mathcal{A} \\ \Leftrightarrow & \exists y.\exists R \in \mathcal{A}.(x, y) \in R \\ \Leftrightarrow & \exists R \in \mathcal{A}.\exists y.(x, y) \in R \\ \Leftrightarrow & x \in \bigcup\{\operatorname{dom} R : R \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned}
&y \in \operatorname{ran} \bigcup \mathcal{A} \\
\Leftrightarrow &\exists x. \exists R \in \mathcal{A}.(x, y) \in R \\
\Leftrightarrow &\exists R \in \mathcal{A}. \exists x.(x, y) \in R \\
\Leftrightarrow &y \in \bigcup \{\operatorname{ran} R : R \in \mathcal{A}\}
\end{aligned}$$

Exercise 9 Assume $\mathcal{A}$ is nonempty. We have $\operatorname{dom} \bigcap \mathcal{A} \subseteq \bigcap \{\operatorname{dom} R : R \in \mathcal{A}\}$.

PROOF:

$$\begin{aligned}
&x \in \operatorname{dom} \bigcap \mathcal{A} \\
\Leftrightarrow &\exists y. \forall R \in \mathcal{A}.(x, y) \in R \\
\Rightarrow &\forall R \in \mathcal{A}. \exists y.(x, y) \in R \\
\Leftrightarrow &x \in \bigcap \{\operatorname{dom} R : R \in \mathcal{A}\}
\end{aligned}$$

Equality holds iff the middle '$\Rightarrow$' can be reversed, i.e. iff for all x, if $\forall R \in \mathcal{A}. \exists y.(x, y) \in R$ then $\exists y. \forall R \in \mathcal{A}.(x, y) \in R$. I haven't found a simpler condition than this. The condition does not always hold, for example if $\mathcal{A} = \{\{(1, 2)\}, \{(1, 3)\}\}$ then $\operatorname{dom} \bigcap \mathcal{A} = \emptyset$ while $\bigcap \{\operatorname{dom} R : R \in \mathcal{A}\} = \{1\}$.

Similarly, $\operatorname{ran} \bigcap \mathcal{A} \subseteq \bigcap \{\operatorname{ran} R : R \in \mathcal{A}\}$, and equality holds iff, for any y, if $\forall R \in \mathcal{A}. \exists x.(x, y) \in R$ then $\exists x. \forall R \in \mathcal{A}.(x, y) \in R$.

3.3 n-ary Relations

Exercise 10 This follows from the equations at the top of page 42. An ordered 4-tuple $\langle a, b, c, d\rangle$ is also an ordered 1-tuple (because every set is), and the ordered pair $\langle\langle a, b, c\rangle, d\rangle$, and the ordered triple $\langle\langle a, b\rangle, c, d\rangle$.

3.4 Functions

Exercise 11 We prove $F \subseteq G$. Let $z \in F$. Since F is a relation, then z is an ordered pair; let $z = \langle x, y\rangle$. We have $x \in \operatorname{dom} F$ and $y = F(x)$. Therefore $x \in \operatorname{dom} G$ and $y = G(x)$ (because $\operatorname{dom} F = \operatorname{dom} G$ and $F(x) = G(x)$). Hence $\langle x, y\rangle \in G$, i.e. $z \in G$.

We have proved $F \subseteq G$. We can prove $G \subseteq F$ similarly. Thus $F = G$.

Exercise 12 PROOF:

$\langle 1\rangle 1$. If $f \subseteq g$ then $\operatorname{dom} f \subseteq \operatorname{dom} g$ and $\forall x \in \operatorname{dom} f. f(x) = g(x)$
- $\langle 2\rangle 1$. ASSUME: $f \subseteq g$
- $\langle 2\rangle 2$. LET: $x \in \operatorname{dom} f$
- $\langle 2\rangle 3$. $(x, f(x)) \in f$
- $\langle 2\rangle 4$. $(x, f(x)) \in g$
- $\langle 2\rangle 5$. $x \in \operatorname{dom} g$ and $g(x) = f(x)$

$\langle 1\rangle 2$. If $\operatorname{dom} f = \operatorname{dom} g$ and $\forall x \in \operatorname{dom} f . f(x) = g(x)$ then $f \subseteq g$
$\langle 2\rangle 1$. ASSUME: $\operatorname{dom} f = \operatorname{dom} g$ and $\forall x \in \operatorname{dom} f . f(x) = g(x)$
$\langle 2\rangle 2$. LET: $z \in f$
$\langle 2\rangle 3$. LET: $z = (x, y)$
$\langle 2\rangle 4$. $x \in \operatorname{dom} f$ and $y = f(x)$
$\langle 2\rangle 5$. $x \in \operatorname{dom} g$ and $y = g(x)$
$\langle 2\rangle 6$. $z = (x, y) \in g$
$\square$

Exercise 13 PROOF:
$\langle 1\rangle 1$. ASSUME: f and g are functions
$\langle 1\rangle 2$. ASSUME: $f \subseteq g$
$\langle 1\rangle 3$. ASSUME: $\operatorname{dom} g \subseteq \operatorname{dom} f$
$\langle 1\rangle 4$. $\operatorname{dom} f = \operatorname{dom} g$
PROOF: We have $\operatorname{dom} f \subseteq \operatorname{dom} g$ from $\langle 1\rangle 2$ and $\operatorname{dom} g \subseteq \operatorname{dom} f$ from $\langle 1\rangle 3$
$\langle 1\rangle 5$. For $x \in \operatorname{dom} f$ we have $f(x) = g(x)$
PROOF: From $\langle 1\rangle 2$ and Exercise 12
$\langle 1\rangle 6$. Q.E.D.
PROOF: From Exercise 11.
$\square$

Exercise 14

(a) If (x, y) and (x, z) are members of $f \cap g$ then they are both members of f, hence $y = z$.

(b) PROOF:
$\langle 1\rangle 1$. If $f \cup g$ is a function then, for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, we have $f(x) = g(x)$.
$\langle 2\rangle 1$. ASSUME: $f \cup g$ is a function.
$\langle 2\rangle 2$. LET: $x \in \operatorname{dom} f \cap \operatorname{dom} g$
$\langle 2\rangle 3$. $(x, f(x))$ and $(x, g(x))$ are both elements of $f \cup g$
$\langle 2\rangle 4$. $f(x) = g(x)$
$\langle 1\rangle 2$. If, for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, we have $f(x) = g(x)$, then $f \cup g$ is a function.
$\langle 2\rangle 1$. ASSUME: For all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, we have $f(x) = g(x)$
$\langle 2\rangle 2$. $f \cup g$ is a relation.
PROOF: Since every element of either f or g is an ordered pair.
$\langle 2\rangle 3$. Whenever (x, y) and (x, z) are elements of $f \cup g$ we have $y = z$
$\langle 3\rangle 1$. LET: $(x, y), (x, z) \in f \cup g$
$\langle 3\rangle 2$. CASE: $(x, y), (x, z) \in f$
PROOF: Then $y = z$ since f is a function.
$\langle 3\rangle 3$. CASE: $(x, y) \in f, (x, z) \in g$
PROOF: Then $y = z$ by $\langle 2\rangle 1$
$\langle 3\rangle 4$. CASE: $(x, y) \in g, (x, z) \in f$
PROOF: Then $y = z$ by $\langle 2\rangle 1$
$\langle 3\rangle 5$. CASE: $(x, y), (x, z) \in g$

PROOF: Then $y = z$ since g is a function.

∎

Exercise 15 PROOF:
$\langle 1\rangle 1.$ $\bigcup \mathcal{A}$ is a relation.
PROOF: Since every member of $\mathcal{A}$ is a relation.
$\langle 1\rangle 2.$ Whenever (x, y) and (x, z) are elements of $\bigcup \mathcal{A}$ then $y = z$
$\langle 2\rangle 1.$ LET: $(x, y), (x, z) \in \bigcup \mathcal{A}$
$\langle 2\rangle 2.$ PICK $f, g \in \mathcal{A}$ such that $(x, y) \in f$ and $(x, z) \in g$
$\langle 2\rangle 3.$ ASSUME: w.l.o.g. $f \subseteq g$
$\langle 2\rangle 4.$ $(x, y), (x, z) \in g$
$\langle 2\rangle 5.$ $y = z$
PROOF: Since g is a function.

∎

Exercise 16 If every function belongs to A then every set belongs to $\operatorname{dom} \bigcup A$ contradiction Theorem 2A.

Exercise 17 PROOF:
$\langle 1\rangle 1.$ LET: R and S be single-rooted.
$\langle 1\rangle 2.$ LET: $(x, z), (y, z) \in R \circ S$
$\langle 1\rangle 3.$ PICK t and t' such that $(x, t) \in S$, $(t, z) \in R$, $(y, t') \in S$ and $(t', z) \in R$
$\langle 1\rangle 4.$ $t = t'$
PROOF: Since R is single-rooted.
$\langle 1\rangle 5.$ $x = y$
PROOF: Since S is single-rooted.

Thus if F and G are one-to-one functions then $F \circ G$ is single-rooted and a function by Theorem 3H, hence a one-to-one function.

Exercise 18

$$\begin{aligned}
R \circ R &= \{\langle 0, 2\rangle, \langle 0, 3\rangle, \langle 1, 3\rangle\} \\
R \restriction \{1\} &= \{\langle 1, 2\rangle, \langle 1, 3\rangle\} \\
R^{-1} \restriction \{1\} &= \{\langle 1, 0\rangle\} \\
R[\![\{1\}]\!] &= \{2, 3\} \\
R^{-1}[\![\{1\}]\!] &= \{0\}
\end{aligned}$$

Exercise 19

$$\begin{aligned}
A(\emptyset) &= \{\emptyset, \{\emptyset\}\} \\
A[\![\emptyset]\!] &= \emptyset \\
A[\![\{\emptyset\}]\!] &= \{\{\emptyset, \{\emptyset\}\}\} \\
A[\![\{\emptyset, \{\emptyset\}\}]\!] &= \{\{\emptyset, \{\emptyset\}\}, \emptyset\} \\
A^{-1} &= \{\langle\{\emptyset, \{\emptyset\}\}, \emptyset\rangle, \langle\emptyset, \{\emptyset\}\rangle\} \\
A \circ A &= \{\langle\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\rangle\} \\
A \restriction \emptyset &= \emptyset \\
A \restriction \{\emptyset\} &= \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle\} \\
A \restriction \{\emptyset, \{\emptyset\}\} &= \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle, \langle\{\emptyset\}, \emptyset\rangle\} \\
&= A \\
\bigcup\bigcup A &= \{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}
\end{aligned}$$

Exercise 20

$$\begin{aligned}
z \in F \restriction A &\Leftrightarrow z \in F \;\&\; \exists x, y.(z = \langle x, y\rangle \;\&\; x \in A) \\
&\Leftrightarrow z \in F \;\&\; \exists x, y(z = \langle x, y\rangle \;\&\; x \in A \;\&\; y \in \operatorname{ran} F) \\
&\Leftrightarrow z \in F \cap (A \times \operatorname{ran} F)
\end{aligned}$$

Exercise 21 Both are equal to $\{\langle x, w\rangle \mid \exists y, z.xTy \;\&\; ySz \;\&\; zRw\}$.

Exercise 22

(a) PROOF:
$\langle 1\rangle$1. ASSUME: $A \subseteq B$
$\langle 1\rangle$2. LET: $y \in F[\![A]\!]$
$\langle 1\rangle$3. PICK $x \in A$ such that xFy
$\langle 1\rangle$4. $x \in B$ and xFy
$\square$

(b) Both are equal to $\{z : \exists x, y.x \in A \;\&\; xGy \;\&\; yFz\}$

(c) Both are equal to $\{\langle x, y\rangle : (x \in A \text{ or } x \in B) \;\&\; xQy\}$

Exercise 23

$$\begin{aligned} B \circ I_A &= \{\langle x, z\rangle : \exists y(x I_A y \ \&\ y B z)\} \\ &= \{\langle x, z\rangle : \exists y(x \in A \ \&\ x = y \ \&\ y B z)\} \\ &= \{\langle x, z\rangle : x \in A \ \&\ x B z\} \\ &= B \restriction A \\ I_A[\![C]\!] &= \{y : \exists x \in C. x I_A y\} \\ &= \{y : \exists x \in C(x \in A \ \&\ x = y)\} \\ &= \{y : y \in C \ \&\ y \in A\} \\ &= A \cap C \end{aligned}$$

Exercise 24

$$\begin{aligned} F^{-1}[\![A]\!] &= \{x : \exists y \in A. y F^{-1} x\} \\ &= \{x : \exists y \in A. x F y\} \\ &= \{x \in \operatorname{dom} F : F(x) \in A\} \end{aligned}$$

Exercise 25

(a) PROOF:
⟨1⟩1. LET: G be a one-to-one function.
⟨1⟩2. G^{-1} is a function.
PROOF: Theorem 3F.
⟨1⟩3. $G \circ G^{-1}$ is a function.
PROOF: Theorem 3H.
⟨1⟩4. $\operatorname{dom}(G \circ G^{-1}) = \operatorname{ran} G$
PROOF:

$$\begin{aligned} \operatorname{dom}(G \circ G^{-1}) &= \{x \in \operatorname{dom} G^{-1} : G^{-1}(x) \in \operatorname{dom} G\} && \text{(Theorem 3H)} \\ &= \{x \in \operatorname{ran} G : G^{-1}(x) \in \operatorname{dom} G\} && \text{(Theorem 3E)} \\ &= \operatorname{ran} G \end{aligned}$$

⟨1⟩5. $\forall x \in \operatorname{ran} G. (G \circ G^{-1})(x) = x$
PROOF: Theorem 3G.
□

(b) Let G be a function. Then

$$\begin{aligned} G \circ G^{-1} &= \{\langle x, z\rangle : \exists y(x G^{-1} y \ \&\ y G z)\} \\ &= \{\langle x, z\rangle : \exists y(y G x \ \&\ y G z)\} \\ &= \{\langle x, x\rangle : \exists y. y G x\} && (G \text{ is a function}) \\ &= I_{\operatorname{ran} G} \end{aligned}$$

Exercise 26

(a)

$$\begin{aligned} F[\![\bigcup \mathcal{A}]\!] &= \{y : \exists x . \exists A \in \mathcal{A}(x \in A \;\&\; xFy)\} \\ &= \{y : \exists A \in \mathcal{A} . \exists x(x \in A \;\&\; xFy)\} \\ &= \bigcup \{F[\![A]\!] : A \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned} F[\![\bigcup \mathcal{A}]\!] &= \{y : \exists x . \forall A \in \mathcal{A}(x \in A \;\&\; xFy)\} \\ &\subseteq \{y : \forall A \in \mathcal{A} . \exists x(x \in A \;\&\; xFy)\} \\ &= \bigcap \{F[\![A]\!] : A \in \mathcal{A}\} \end{aligned}$$

Exercise 27

$$\begin{aligned} \operatorname{dom}(F \circ G) &= \{x : \exists y . x(F \circ G)y\} \\ &= \{x : \exists y \exists z(xGz \;\&\; zFy)\} \\ &= \{x : \exists z(zG^{-1}x \;\&\; z \in \operatorname{dom} F)\} \\ &= G^{-1}[\![\operatorname{dom} F]\!] \end{aligned}$$

Exercise 28 PROOF:
$\langle 1 \rangle 1.$ $G : \mathcal{P}A \to \mathcal{P}B$
  PROOF: Since $f[\![X]\!] \subseteq \operatorname{ran} f \subseteq B$
$\langle 1 \rangle 2.$ For all $X, Y \in \mathcal{P}A$, if $G(X) = G(Y)$ then $X = Y$
  $\langle 2 \rangle 1.$ LET: $X, Y \in \mathcal{P}A$
  $\langle 2 \rangle 2.$ ASSUME: $f[\![X]\!] = f[\![Y]\!]$
  $\langle 2 \rangle 3.$ $X \subseteq Y$
    $\langle 3 \rangle 1.$ LET: $x \in X$
    $\langle 3 \rangle 2.$ $f(x) \in f[\![X]\!]$
    $\langle 3 \rangle 3.$ $f(x) \in f[\![Y]\!]$
    $\langle 3 \rangle 4.$ PICK $y \in Y$ such that $f(x) = f(y)$
    $\langle 3 \rangle 5.$ $x = y$
      PROOF: Because f is one-to-one.
    $\langle 3 \rangle 6.$ $x \in Y$
      PROOF: Similar.
  $\langle 2 \rangle 4.$ $Y \subseteq X$
$\square$

Example 29 PROOF:
$\langle 1 \rangle 1.$ ASSUME: f maps A onto B
$\langle 1 \rangle 2.$ LET: $b, b' \in B$
$\langle 1 \rangle 3.$ ASSUME: $G(b) = G(b')$
$\langle 1 \rangle 4.$ PICK $x \in A$ such that $f(x) = b$

PROOF: By $\langle 1\rangle 1$.
$\langle 1\rangle 5$. $x \in G(b)$
$\langle 1\rangle 6$. $x \in G(b')$
$\langle 1\rangle 7$. $f(x) = b'$
$\langle 1\rangle 8$. $b = b'$
$\square$

The converse does not hold. Let $A = \{0\}$ and $B = \{0, 1\}$. Let f be the function that maps 0 to 0. Then

$$G(0) = \{0\}$$
$$G(1) = \emptyset$$

Thus G is one-to-one but f does not map A onto B.

Exercise 30

(a) PROOF:
$\langle 1\rangle 1$. $F(B) = B$
  $\langle 2\rangle 1$. $F(B) \subseteq B$
    $\langle 3\rangle 1$. LET: $X \in \mathcal{P}A$ be such that $F(X) \subseteq X$
      PROVE: $F(B) \subseteq X$
    $\langle 3\rangle 2$. $B \subseteq X$
    $\langle 3\rangle 3$. $F(B) \subseteq F(X)$
    $\langle 3\rangle 4$. $F(B) \subseteq X$
      PROOF: From $\langle 3\rangle 1$ and $\langle 3\rangle 3$.
  $\langle 2\rangle 2$. $B \subseteq F(B)$
    PROOF: From $\langle 2\rangle 1$ and the definition of B, since B is one of the sets X such that $F(X) \subseteq X$
$\langle 1\rangle 2$. $F(C) = C$
  $\langle 2\rangle 1$. $C \subseteq F(C)$
    $\langle 3\rangle 1$. LET: $X \in \mathcal{P}A$ with $X \subseteq F(X)$
      PROVE: $X \subseteq F(C)$
    $\langle 3\rangle 2$. $X \subseteq C$
    $\langle 3\rangle 3$. $F(X) \subseteq F(C)$
    $\langle 3\rangle 4$. $X \subseteq F(C)$
      PROOF: From $\langle 3\rangle 1$ and $\langle 3\rangle 3$
  $\langle 2\rangle 2$. $F(C) \subseteq C$
    PROOF: From $\langle 2\rangle 1$ and the definition of C.
$\square$

(b) If $F(X) = X$ then we have $B \subseteq X$ (because $F(X) \subseteq X$) and $X \subseteq C$ (because $X \subseteq F(X)$).

3.5 Infinite Cartesian Products

Exercise 31 PROOF:

$\langle 1 \rangle 1$. If the Axiom of Choice is true then, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$.

$\langle 2 \rangle 1$. ASSUME: The Axiom of Choice.

$\langle 2 \rangle 2$. LET: I be a set.

$\langle 2 \rangle 3$. LET: H be a function with domain I.

$\langle 2 \rangle 4$. ASSUME: $H(i) \neq \emptyset$ for all $i \in I$.

$\langle 2 \rangle 5$. LET: $R = \{(i, x) : i \in I, x \in H(i)\}$

$\langle 2 \rangle 6$. PICK a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R$

PROVE: $F \in \prod_{i \in I} H(i)$

PROOF: By the Axiom of Choice.

$\langle 2 \rangle 7$. $\operatorname{dom} H = I$

PROOF: We have $\operatorname{dom} R = I$ since for all $i \in I$ there exists x such that $x \in H(i)$.

$\langle 2 \rangle 8$. $\forall i \in I . F(i) \in H(i)$

PROOF: Since $iRF(i)$.

$\langle 1 \rangle 2$. If, for any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$, then the Axiom of Choice is true.

$\langle 2 \rangle 1$. ASSUME: For any set I and any function H with domain I, if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$

$\langle 2 \rangle 2$. LET: R be a relation

$\langle 2 \rangle 3$. LET: $I = \operatorname{dom} R$

$\langle 2 \rangle 4$. Define the function H with domain I by: for $i \in I$, $H(i) = \{y : iRy\}$

$\langle 2 \rangle 5$. $H(i) \neq \emptyset$ for all $i \in I$

$\langle 2 \rangle 6$. PICK $F \in \prod_{i \in I} H(i)$

PROOF: By $\langle 2 \rangle 1$

$\langle 2 \rangle 7$. F is a function

$\langle 2 \rangle 8$. $F \subseteq R$

PROOF: For all $i \in I$ we have $F(i) \in H(i)$ and so $iRF(i)$.

$\langle 2 \rangle 9$. $\operatorname{dom} F = \operatorname{dom} R$

$\square$

3.6 Equivalence Relations

Exercise 32

(a)

$$
\begin{aligned}
& R \text{ is symmetric} \\
\Leftrightarrow & \forall x, y (xRy \Rightarrow yRx) \\
\Leftrightarrow & \forall x, y (\langle y, x \rangle \in R^{-1} \Rightarrow \langle y, x \rangle \in R) \\
\Leftrightarrow & R^{-1} \subseteq R
\end{aligned}
$$

(b)

$$\begin{aligned}
&R \text{ is transitive} \\
\Leftrightarrow &\forall x, y, z(xRy \ \& \ yRz \Rightarrow xRz) \\
\Leftrightarrow &\forall x, z(\exists y(xRy \ \& \ yRz) \Rightarrow xRz) \\
\Leftrightarrow &\forall x, z(\langle x, z\rangle \in R \circ R \Rightarrow \langle x, z\rangle \in R) \\
\Leftrightarrow &R \circ R \subseteq R
\end{aligned}$$

Exercise 33 PROOF:
⟨1⟩1. If R is a symmetric and transitive relation then $R = R^{-1} \circ R$.
  ⟨2⟩1. ASSUME: R is a symmetric and transitive relation.
  ⟨2⟩2. $R \subseteq R^{-1} \circ R$
    ⟨3⟩1. LET: xRy
    ⟨3⟩2. yRy
      PROOF: By Theorem 3M.
    ⟨3⟩3. xRy and $yR^{-1}y$
    ⟨3⟩4. $x(R^{-1} \circ R)y$
  ⟨2⟩3. $R^{-1} \circ R \subseteq R$
    PROOF:

$$\begin{aligned}
R^{-1} \circ R &\subseteq R \circ R && \text{(Exercise 32(a))} \\
&\subseteq R && \text{(Exercise 32(b))}
\end{aligned}$$

⟨1⟩2. If $R = R^{-1} \circ R$ then R is a symmetric and transitive relation.
  ⟨2⟩1. ASSUME: $R = R^{-1} \circ R$
  ⟨2⟩2. R is a relation.
  ⟨2⟩3. R is symmetric.
    ⟨3⟩1. LET: xRy
    ⟨3⟩2. PICK z such that xRz and $zR^{-1}y$
    ⟨3⟩3. yRz and $zR^{-1}x$
    ⟨3⟩4. $y(R^{-1} \circ R)x$
    ⟨3⟩5. yRx
  ⟨2⟩4. R is transitive.
    ⟨3⟩1. LET: xRy and yRz
    ⟨3⟩2. zRy
      PROOF: By ⟨2⟩3
    ⟨3⟩3. xRy and $yR^{-1}z$
    ⟨3⟩4. $x(R^{-1} \circ R)z$
    ⟨3⟩5. xRz

□

Exercise 34

(a) $\bigcap \mathcal{A}$ is a transitive relation.

PROOF:
⟨1⟩1. $\bigcap \mathcal{A}$ is a relation.

PROOF: Every member of a member of $\mathcal{A}$ is an ordered pair.
$\langle 1 \rangle 2.$ $\bigcap \mathcal{A}$ is transitive.
$\langle 2 \rangle 1.$ LET: $\langle x, y \rangle$ and $\langle y, z \rangle$ be in $\bigcap \mathcal{A}$
PROVE: $\langle x, z \rangle \in \bigcap \mathcal{A}$
$\langle 2 \rangle 2.$ LET: $R \in \mathcal{A}$
$\langle 2 \rangle 3.$ xRy and yRz
$\langle 2 \rangle 4.$ xRz
PROOF: Since R is transitive.
$\square$

(b) Not necessarily. If $\mathcal{A} = \{\{\langle 0, 1 \rangle\}, \{\langle 1, 2 \rangle\}\}$ then each member of $\mathcal{A}$ is transitive but $\bigcup \mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$ is not.

Example 35

$$\begin{aligned} R[\![\{x\}]\!] &= \{y : \exists z (z \in \{x\} \ \& \ zRy)\} \\ &= \{y : \exists z (z = x \ \& \ zRy)\} \\ &= \{y : xRy\} \\ &= [x]_R \end{aligned}$$

Example 36 PROOF:
$\langle 1 \rangle 1.$ Q is a relation on A.
PROOF: By definition.
$\langle 1 \rangle 2.$ Q is reflexive on A.
$\langle 2 \rangle 1.$ LET: $x \in A$
$\langle 2 \rangle 2.$ $f(x)Rf(x)$
PROOF: Since R is reflexive on B.
$\langle 2 \rangle 3.$ xQx
$\langle 1 \rangle 3.$ Q is symmetric.
$\langle 2 \rangle 1.$ ASSUME: xQy
$\langle 2 \rangle 2.$ $f(x)Rf(y)$
$\langle 2 \rangle 3.$ $f(y)Rf(x)$
PROOF: R is symmetric.
$\langle 2 \rangle 4.$ yQx
$\langle 1 \rangle 4.$ Q is transitive.
$\langle 2 \rangle 1.$ ASSUME: xQy and yQz
$\langle 2 \rangle 2.$ $f(x)Rf(y)$ and $f(y)Rf(z)$
$\langle 2 \rangle 3.$ $f(x)Rf(z)$
PROOF: R is transitive.
$\langle 2 \rangle 4.$ xQz
$\square$

Exercise 37 PROOF:
$\langle 1 \rangle 1.$ R_Π is a relation on A.

PROOF: If $B \in \Pi$, $x \in B$ and $y \in B$ then $x, y \in A$.
$\langle 1 \rangle 2$. R_Π is reflexive on A.
$\langle 2 \rangle 1$. LET: $x \in A$
$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$
PROOF: Because Π is exhaustive.
$\langle 2 \rangle 3$. $x \in B$ and $x \in B$
$\langle 2 \rangle 4$. $x R_\Pi x$
$\langle 1 \rangle 3$. R_Π is symmetric.
$\langle 2 \rangle 1$. ASSUME: $x R_\Pi y$
$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$
$\langle 2 \rangle 3$. $y \in B$ and $x \in B$
$\langle 2 \rangle 4$. $y R_\Pi x$
$\langle 1 \rangle 4$. R_Π is transitive.
$\langle 2 \rangle 1$. ASSUME: $x R_\Pi y$ and $y R_\Pi z$
$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$
$\langle 2 \rangle 3$. PICK $C \in \Pi$ such that $y \in C$ and $z \in C$
$\langle 2 \rangle 4$. $B = C$
PROOF: Since $y \in B$ and $y \in C$
$\langle 2 \rangle 5$. $x \in B$ and $z \in B$
$\langle 2 \rangle 6$. $x R_\Pi z$
$\square$

Exercise 38 PROOF:
$\langle 1 \rangle 1$. If $B \in \Pi$ and $x \in B$ then $B = [x]_{R_\Pi}$
$\langle 2 \rangle 1$. LET: $B \in \Pi$
$\langle 2 \rangle 2$. LET: $x \in B$
$\langle 2 \rangle 3$. $[x]_{R_\Pi} \subseteq B$
$\langle 3 \rangle 1$. LET: $y \in [x]_{R_\Pi}$
$\langle 3 \rangle 2$. $x R_\Pi y$
$\langle 3 \rangle 3$. PICK $C \in \Pi$ such that $x \in C$ and $y \in C$
$\langle 3 \rangle 4$. $B = C$
PROOF: Since $x \in B$ and $x \in C$.
$\langle 3 \rangle 5$. $y \in B$
$\langle 2 \rangle 4$. $B \subseteq [x]_{R_\Pi}$
PROOF: For all $y \in B$, we have $x \in B$ and $y \in B$ hence $x R_\Pi y$.
$\langle 1 \rangle 2$. $A / R_\Pi \subseteq \Pi$
$\langle 2 \rangle 1$. LET: $x \in A$
PROVE: $[x]_{R_\Pi} \in \Pi$
$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$
$\langle 2 \rangle 3$. $[x]_{R_\Pi} = B$
PROOF: By $\langle 1 \rangle 1$
$\langle 2 \rangle 4$. $[x]_{R_\Pi} \in \Pi$
$\langle 1 \rangle 3$. $\Pi \subseteq A / R_\Pi$
$\langle 2 \rangle 1$. LET: $B \in \Pi$
$\langle 2 \rangle 2$. PICK $x \in B$

PROOF: Since every member of Π is nonempty.
$\langle 2\rangle 3.$ $B = [x]_{R_\Pi}$
PROOF: By $\langle 1\rangle 1$.
$\langle 2\rangle 4.$ $B \in A/R_\Pi$
$\square$

Exercise 39 PROOF:
$\langle 1\rangle 1.$ $R_\Pi \subseteq R$
$\langle 2\rangle 1.$ LET: $xR_\Pi y$
$\langle 2\rangle 2.$ PICK $B \in \Pi$ such that $x \in B$ and $y \in B$
$\langle 2\rangle 3.$ PICK $z \in A$ such that $B = [z]_R$
$\langle 2\rangle 4.$ zRx
$\langle 2\rangle 5.$ zRy
$\langle 2\rangle 6.$ xRy
PROOF: Since R is symmetric and transitive.
$\langle 1\rangle 2.$ $R \subseteq R_\Pi$
$\langle 2\rangle 1.$ LET: xRy
$\langle 2\rangle 2.$ $x \in [x]_R$
$\langle 2\rangle 3.$ $y \in [x]_R$
$\langle 2\rangle 4.$ $xR_\Pi y$
$\square$

Exercise 40 We have $[2]_R = [3]_R$ but $[6]_R \neq [9]_R$ so there is no such function f.

Exercise 41

(a) PROOF:
$\langle 1\rangle 1.$ Q is reflexive on $\mathbb{R} \times \mathbb{R}$.
PROOF: For any $x, y \in \mathbb{R}$, we have $x + y = x + y$, hence $\langle x, y\rangle Q \langle x, y\rangle$
$\langle 1\rangle 2.$ Q is symmetric.
$\langle 2\rangle 1.$ ASSUME: $\langle u, v\rangle Q \langle x, y\rangle$
$\langle 2\rangle 2.$ $u + y = x + v$
$\langle 2\rangle 3.$ $x + v = u + y$
$\langle 2\rangle 4.$ $\langle x, y\rangle Q \langle u, v\rangle$
$\langle 1\rangle 3.$ Q is transitive.
$\langle 2\rangle 1.$ ASSUME: $\langle a, b\rangle Q \langle u, v\rangle$ and $\langle u, v\rangle Q \langle x, y\rangle$
$\langle 2\rangle 2.$ $a + v = u + b$
$\langle 2\rangle 3.$ $u + y = x + v$
$\langle 2\rangle 4.$ $a + y + x + b$
PROOF: Adding $\langle 2\rangle 2$ and $\langle 2\rangle 3$ gives $a + u + v + y = b + u + v + x$.
$\langle 2\rangle 5.$ $\langle a, b\rangle Q \langle x, y\rangle$
$\square$

(b) We prove that, if $\langle u, v\rangle Q\langle x, y\rangle$ then $\langle u + 2v, v + 2u\rangle Q\langle x + 2y, y + 2x\rangle$. It follows from Theorem 3Q that the function G exists.

If $u+y = v+x$ then $u+2v+y+2x = v+2u+x+2y$ by adding $u+v+y+x$ to both sides.

Exercise 42 Assume that R is an equivalence relation on A and that $F : A \times A \to A$. Let us say that F is *compatible* with R iff, whenever xRx' and yRy', then $F(\langle x, y\rangle)RF(\langle x', y'\rangle)$. If F is compatible with R then there exists a unique $\hat{F} : (A/R) \times (A/R) \to A/R$ such that

$$\hat{F}(\langle [x]_R, [y]_R\rangle) = [F(\langle x, y\rangle)]_R \text{ for all } x, y \in A \ .$$

If F is not compatible with R then no such $\hat{F}$ exists.

3.7 Ordering Relations

Exercise 43 Proof:
$\langle 1\rangle 1.$ R^{-1} is transitive.
  $\langle 2\rangle 1.$ Assume: $xR^{-1}y$ and $yR^{-1}z$
  $\langle 2\rangle 2.$ zRy and yRx
  $\langle 2\rangle 3.$ zRx
    Proof: Since R is transitive.
  $\langle 2\rangle 4.$ $xR^{-1}z$
$\langle 1\rangle 2.$ R^{-1} satisfies trichotomy on A.
  $\langle 2\rangle 1.$ Let: $x, y \in A$
  $\langle 2\rangle 2.$ Exactly one of xRy, $x = y$, yRx holds.
  $\langle 2\rangle 3.$ Exactly one of $yR^{-1}x$, $x = y$, $xR^{-1}y$ holds.
$\square$

Exercise 44 Proof:
$\langle 1\rangle 1.$ f is one-to-one.
  $\langle 2\rangle 1.$ Let: $x, y \in A$ with $f(x) = f(y)$
  $\langle 2\rangle 2.$ $f(x) < f(y)$ and $f(y) < f(x)$ do not hold.
    Proof: By trichotomy.
  $\langle 2\rangle 3.$ $x < y$ and $y < x$ do not hold.
  $\langle 2\rangle 4.$ $x = y$
    Proof: By trichotomy.
$\langle 1\rangle 2.$ Whenever $f(x) < f(y)$ then $x < y$
  $\langle 2\rangle 1.$ Let: $x, y \in A$ with $f(x) < f(y)$
  $\langle 2\rangle 2.$ $f(x) = f(y)$ and $f(y) < f(x)$ do not hold.
    Proof: By trichotomy.
  $\langle 2\rangle 3.$ $x = y$ and $y < x$ do not hold.
  $\langle 2\rangle 4.$ $x < y$
    Proof: By trichotomy.
$\square$

Exercise 45 PROOF:

$\langle 1\rangle 1.$ $<_L$ is transitive.

  $\langle 2\rangle 1.$ LET: $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle$ and $\langle a_2, b_2\rangle <_L \langle a_3, b_3\rangle$
    PROVE: $\langle a_1, b_1\rangle < \langle a_3, b_3\rangle$

  $\langle 2\rangle 2.$ CASE: $a_1 <_A a_2$ and $a_2 <_A a_3$
    PROOF: Then $a_1 <_A <_3$

  $\langle 2\rangle 3.$ CASE: $a_1 <_A a_2$, $a_2 = a_3$, $b_2 <_B b_3$
    PROOF: Then $a_1 <_A <_3$

  $\langle 2\rangle 4.$ CASE: $a_1 = a_2$, $b_1 <_B b_2$ and $a_2 <_A a_3$
    PROOF: Then $a_1 <_A <_3$

  $\langle 2\rangle 5.$ CASE: $a_1 = a_2$, $b_1 <_B b_2$, $a_2 = a_3$, $b_2 <_B b_3$
    PROOF: Then $a_1 = a_3$ and $b_1 <_B b_3$

$\langle 1\rangle 2.$ $<_L$ satisfies trichotomy on $A \times B$.

  $\langle 2\rangle 1.$ LET: $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ be elements of $A \times B$

  $\langle 2\rangle 2.$ Exactly one of $a_1 <_A a_2$, $a_1 = a_2$, $a_2 <_A a_1$ holds.

  $\langle 2\rangle 3.$ Exactly one of $b_1 <_B b_2$, $b_1 = b_2$, $b_2 <_B b_1$ holds.

  $\langle 2\rangle 4.$ Exactly one of $a_1 <_A a_2$, ($a_1 = a_2$ and $b_1 <_B b_2$), ($a_1 = a_2$ and $b_1 = b_2$), ($a_1 = a_2$ and $b_2 <_L b_1$), $a_2 <_A a_1$ holds.

  $\langle 2\rangle 5.$ Exactly one of $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle$, $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$, $\langle a_2, b_2\rangle <_L \langle a_1, b_1\rangle$ holds.

□

3.8 Review Exercises

Exercise 46

(a)

$$\begin{aligned} \bigcap\bigcap\langle x, y\rangle &= \bigcap\{x\} \\ &= x \end{aligned}$$

(b)

$$\begin{aligned} \bigcap\bigcap\bigcap\{\langle x, y\rangle\}^{-1} &= \bigcap\bigcap\bigcap\{\langle y, x\rangle\} \\ &= \bigcap\bigcap\langle y, x\rangle \\ &= y && \text{(by part (a))} \end{aligned}$$

Exercise 47

(a) There are eight:

$$\begin{gathered}
\{\langle 0,3\rangle,\langle 1,3\rangle,\langle 2,3\rangle\},\\
\{\langle 0,3\rangle,\langle 1,3\rangle,\langle 2,4\rangle\},\\
\{\langle 0,3\rangle,\langle 1,4\rangle,\langle 2,3\rangle\},\\
\{\langle 0,3\rangle,\langle 1,4\rangle,\langle 2,4\rangle\},\\
\{\langle 0,4\rangle,\langle 1,3\rangle,\langle 2,3\rangle\},\\
\{\langle 0,4\rangle,\langle 1,3\rangle,\langle 2,4\rangle\},\\
\{\langle 0,4\rangle,\langle 1,4\rangle,\langle 2,3\rangle\},\\
\{\langle 0,4\rangle,\langle 1,4\rangle,\langle 2,4\rangle\}
\end{gathered}$$

(b) There are six:

$$\begin{gathered}
\{\langle 0,3\rangle,\langle 1,4\rangle,\langle 2,5\rangle\},\\
\{\langle 0,3\rangle,\langle 1,5\rangle,\langle 2,4\rangle\},\\
\{\langle 0,4\rangle,\langle 1,3\rangle,\langle 2,5\rangle\},\\
\{\langle 0,4\rangle,\langle 1,5\rangle,\langle 2,3\rangle\},\\
\{\langle 0,5\rangle,\langle 1,3\rangle,\langle 2,4\rangle\},\\
\{\langle 0,5\rangle,\langle 1,4\rangle,\langle 2,3\rangle\}
\end{gathered}$$

Exercise 48

(a) The only ordered pair in $\mathcal{P}T$ is $\langle\emptyset,\emptyset\rangle = \{\{\emptyset\}\}$.

(b)

$$\begin{aligned}
(\mathcal{P}T)^{-1}\circ(\mathcal{P}T\restriction\{\emptyset\}) &= \{\langle\emptyset,\emptyset\rangle\}\circ\{\langle\emptyset,\emptyset\rangle\}\\
&= \{\langle\emptyset,\emptyset\rangle\}
\end{aligned}$$

Exercise 49 There are six:

$$\begin{aligned}
\{\langle 0,0\rangle,\langle 1,1\rangle,\langle 2,2\rangle\},\\
\{\langle 0,0\rangle,\langle 0,1\rangle,\langle 1,0\rangle,\langle 1,1\rangle,\langle 2,2\rangle\},\\
\{\langle 0,0\rangle,\langle 0,2\rangle,\langle 1,1\rangle,\langle 2,0\rangle,\langle 2,2\rangle\},\\
\{\langle 0,0\rangle,\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,1\rangle,\langle 2,2\rangle\},\\
\{\langle 0,0\rangle,\langle 0,1\rangle,\langle 0,2\rangle,\langle 1,0\rangle,\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,0\rangle,\langle 2,1\rangle,\langle 2,2\rangle\}
\end{aligned}$$

Exercise 50

(a) $\{\langle 0,1\rangle,\langle 0,2\rangle,\langle 0,3\rangle,\langle 1,3\rangle,\langle 2,1\rangle,\langle 2,3\rangle\}$

(b) $\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}$

Exercise 51 There are three:

$$\{\langle 1,0\rangle, \langle 1,2\rangle, \langle 2,0\rangle\},$$
$$\{\langle 1,0\rangle, \langle 2,0\rangle, \langle 2,1\rangle\},$$
$$\{\langle 0,1\rangle, \langle 2,0\rangle, \langle 2,1\rangle\}$$

Exercise 52 We can conclude this if we know that A and B are nonempty, or that C and D are nonempty.

Suppose A and B are nonempty. Then $A \times B = C \times D \neq \emptyset$ so C and D are nonempty. We now prove $A \subseteq C$.

Let $a \in A$. Pick some $b \in B$. Then $\langle a, b\rangle \in A \times B = C \times D$ and so $a \in C$.

We can similarly prove $C \subseteq A$, $B \subseteq D$ and $D \subseteq B$.

Exercise 53

$$\begin{aligned}
x(R \cup S)^{-1}y &\Leftrightarrow y(R \cup S)x \\
&\Leftrightarrow yRx \text{ or } ySx \\
&\Leftrightarrow xR^{-1}y \text{ or } xS^{-1}y \\
&\Leftrightarrow x(R^{-1} \cup S^{-1})y \\
x(R \cap S)^{-1}y &\Leftrightarrow y(R \cap S)x \\
&\Leftrightarrow yRx \text{ and } ySx \\
&\Leftrightarrow xR^{-1}y \text{ and } xS^{-1}y \\
&\Leftrightarrow x(R^{-1} \cap S^{-1})y \\
x(R - S)^{-1}y &\Leftrightarrow y(R - S)x \\
&\Leftrightarrow yRx \text{ and } \neg ySx \\
&\Leftrightarrow xR^{-1}y \text{ and } \neg xS^{-1}y \\
&\Leftrightarrow x(R^{-1} - S^{-1})y
\end{aligned}$$

Exercise 54

(a)

$$\begin{aligned}
\langle x, y\rangle \in A \times (B \cap C) &\Leftrightarrow x \in A \;\&\; y \in B \;\&\; y \in C \\
&\Leftarrow \langle x, y\rangle \in (A \times B) \cap (A \times C)
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x, y\rangle \in A \times (B \cup C) &\Leftrightarrow x \in A \;\&(y \in B \text{ or } y \in C) \\
&\Leftrightarrow (x \in A \;\&\; y \in B) \text{ or } (x \in A \;\&\; y \in C) \\
&\Leftrightarrow \langle x, y\rangle \in (A \times B) \cup (A \times C)
\end{aligned}$$

(c)

$$\begin{aligned}\langle x,y\rangle \in A\times(B-C) &\Leftrightarrow x\in A \;\&\; y\in B \;\&\; y\notin C\\ &\Leftrightarrow \langle x,y\rangle \in (A\times B)-(A\times C)\end{aligned}$$

Exercise 55

(a) No. Take $A=\{0\}$, $B=\{1\}$, $C=\{2\}$. Then $(A\times A)\cup(B\times C)=\{\langle 0,0\rangle,\langle 1,2\rangle\}$ while $(A\cup B)\times(A\cup C)=\{\langle 0,0\rangle,\langle 0,2\rangle,\langle 1,0\rangle,\langle 1,2\rangle\}$.

(b) Yes.

$$\begin{aligned}\langle x,y\rangle \in (A\times A)\cap(B\times C) &\Leftrightarrow x\in A \;\&\; y\in A \;\&\; x\in B \;\&\; y\in C\\ &\Leftrightarrow \langle x,y\rangle \in (A\cap B)\times(A\cap C)\end{aligned}$$

Exercise 56

(a) Yes.

$$\begin{aligned}x\in \operatorname{dom}(R\cup S) &\Leftrightarrow \exists y(xRy \text{ or } xSy)\\ &\Leftrightarrow \exists y.xRy \text{ or } \exists y.xSy\\ &\Leftrightarrow x\in \operatorname{dom} R\cup \operatorname{dom} S\end{aligned}$$

(b) No. Take $R=\{\langle 0,0\rangle\}$ and $S=\{\langle 0,1\rangle\}$. Then $\operatorname{dom}(R\cap S)=\operatorname{dom}\emptyset=\emptyset$ while $\operatorname{dom} R\cap\operatorname{dom} S=\{0\}\cap\{0\}=\{0\}$.

Exercise 57

(a) Yes.

$$\begin{aligned}x(R\circ(S\cup T))y &\Leftrightarrow \exists z(x(S\cup T)z \;\&\; zRy)\\ &\Leftrightarrow \exists z(xSz \;\&\; zRy) \text{ or } \exists z(xTz \;\&\; zRy)\\ &\Leftrightarrow x((R\circ S)\cup(R\circ T))y\end{aligned}$$

(b) No. Take $R=\{\langle 0,0\rangle,\langle 1,0\rangle\}$, $S=\{\langle 0,0\rangle\}$ and $T=\{\langle 0,1\rangle\}$. Then

$$\begin{aligned}R\circ(S\cap T) &= R\circ\emptyset\\ &=\emptyset\\ (R\circ S)\cap(R\circ T) &= \{\langle 0,0\rangle\}\cap\{\langle 0,0\rangle\}\\ &=\{\langle 0,0\rangle\}\end{aligned}$$

Exercise 58 Take $F=\emptyset$ and $S=\{\emptyset\}$. Then $F[\![F^{-1}[\![S]\!]]\!]=\emptyset\neq S$.

Exercise 59

$$\begin{aligned}
x(Q \restriction (A \cap B))y &\Leftrightarrow xQy \ \& \ x \in A \ \& \ x \in B \\
&\Leftrightarrow x((Q \restriction A) \cap (Q \restriction B))y \\
x(Q \restriction (A - B))y &\Leftrightarrow xQy \ \& \ x \in A \ \& \ x \notin B \\
&\Leftrightarrow (xQy \ \& \ x \in A) \ \& \ \neg(xQy \ \& \ x \in B) \\
&\Leftrightarrow x((Q \restriction A) - (Q \restriction B))y
\end{aligned}$$

Exercise 60

$$\begin{aligned}
x((R \circ S) \restriction A)y &\Leftrightarrow \exists z(xRz \ \& \ zSy \ \& \ x \in A) \\
&\Leftrightarrow x(R \circ (S \restriction A))y
\end{aligned}$$

Chapter 4

Chapter 4 — Natural Numbers

4.1 Inductive Sets

Exercise 1 We have
$$3 = 2 \cup \{2\} = 1 \cup \{1, 2\}$$
and so $1 \in 3$. But $1 \notin 1$ (since $1 = \{\emptyset\}$ and we know $\{\emptyset\} \neq \emptyset$ hence $\{\emptyset\} \notin \{\emptyset\}$). Therefore $1 \neq 3$.

4.2 Peano's Postulates

Exercise 2 If a is a transitive set then
$$\begin{aligned} \bigcup(a^+) &= a && \text{(Theorem 4E)} \\ &\subseteq a^+ \end{aligned}$$

Exercise 3

(a) Suppose a is a transitive set. Then $a \subseteq \mathcal{P}a$. Hence we have $\bigcup \mathcal{P}a = a \subseteq \mathcal{P}a$ and so $\mathcal{P}a$.

(b) Suppose $\mathcal{P}a$ is a transitive set. Then $a = \bigcup \mathcal{P}a \subseteq \mathcal{P}a$ hence a is transitive.

Exercise 4 If a is a transitive set then $\bigcup a \subseteq a$ so $\bigcup\bigcup a \subseteq \bigcup a$. Hence $\bigcup a$ is transitive.

Exercise 5

(a) PROOF:
⟨1⟩1. LET: $b \in \bigcup \mathcal{A}$
⟨1⟩2. PICK $A \in \mathcal{A}$ such that $b \in A$
⟨1⟩3. $b \subseteq A$
PROOF: Since A is transitive.
⟨1⟩4. $b \subseteq \bigcup \mathcal{A}$
□

(b) PROOF:
⟨1⟩1. LET: $b \in \bigcap \mathcal{A}$
⟨1⟩2. For all $A \in \mathcal{A}$ we have $b \subseteq A$
PROOF: Since $b \in A$ and A is transitive.
⟨1⟩3. $b \subseteq \bigcap \mathcal{A}$
□

Exercise 6 We have $\bigcup(a^+) = \bigcup a \cup a$ (see the proof of Theorem 4E). So if $\bigcup(a^+) = a$ we have $\bigcup a \cup a = a$ and so $\bigcup a \subseteq a$.

4.3 Recursion on ω

Exercise 7 We have $h_1(0) = h_2(0) = a$ so $0 \in S$.

Now let $n \in S$; we prove $n^+ \in S$. We have $h_1(n) = h_2(n)$ and therefore

$$\begin{aligned} h_1(n^+) &= F(h_1(n)) \\ &= F(h_2(n)) \\ &= h_2(n^+) \end{aligned}$$

Exercise 8 PROOF:
⟨1⟩1. $\forall m, n \in \omega . h(n) = h(m) \Rightarrow n = m$
⟨2⟩1. $\forall n \in \omega . h(n) = h(0) \Rightarrow n = 0$
⟨3⟩1. LET: $n \in \omega$
⟨3⟩2. ASSUME: $h(n) = h(0)$
⟨3⟩3. $h(n) = c$
⟨3⟩4. $\forall p \in \omega . n \neq p^+$
PROOF: Otherwise $f(h(p)) = c$ contradicting the fact that $c \in A - \operatorname{ran} f$.
⟨3⟩5. $n = 0$
PROOF: Theorem 4C.
⟨2⟩2. For all $m \in \omega$, if $\forall n \in \omega . h(n) = h(m) \Rightarrow n = m$, then $\forall n \in \omega . h(n) = h(m^+) \Rightarrow n = m^+$
⟨3⟩1. LET: $m \in \omega$
⟨3⟩2. ASSUME: $\forall n \in \omega . h(n) = h(m) \Rightarrow n = m$
⟨3⟩3. LET: $n \in \omega$
⟨3⟩4. ASSUME: $h(n) = h(m^+)$
⟨3⟩5. $h(n) = f(h(m))$

⟨3⟩6. $n \neq 0$

Proof: Otherwise $c = f(h(m))$ contradicting the fact that $c \in A - \operatorname{ran} f$.

⟨3⟩7. Pick p such that $n = p^+$

⟨3⟩8. $f(h(p)) = f(h(m))$

⟨3⟩9. $h(p) = h(m)$

Proof: f is one-to-one.

⟨3⟩10. $p = m$

Proof: By ⟨3⟩2.

⟨3⟩11. $n = p^+ = m^+$

$\square$

Exercise 9 Proof:

⟨1⟩1. $C^* \subseteq C_*$

⟨2⟩1. $f[\![C_*]\!] \subseteq C_*$

⟨3⟩1. Let: $x \in C_*$

Prove: $f(x) \in C_*$

⟨3⟩2. Pick n such that $x \in h(n)$

⟨3⟩3. $f(x) \in h(n^+)$

⟨3⟩4. $f(x) \in C_*$

⟨1⟩2. $C_* \subseteq C^*$

⟨2⟩1. $\forall n \in \omega . h(n) \subseteq C^*$

⟨3⟩1. $h(0) \subseteq C^*$

Proof: If $A \subseteq X \subseteq B$ and $f[\![X]\!] \subseteq X$ then $A \subseteq X$.

⟨3⟩2. $\forall n \in \omega (h(n) \subseteq C^* \Rightarrow h(n^+) \subseteq C^*)$

⟨4⟩1. Let: $n \in \omega$

⟨4⟩2. Assume: $h(n) \subseteq C^*$

⟨4⟩3. $f[\![h(n)]\!] \subseteq C^*$

⟨5⟩1. Let: X be such that $A \subseteq X \subseteq B$ and $f[\![X]\!] \subseteq X$

Prove: $f[\![h(n)]\!] \subseteq X$

⟨5⟩2. $h(n) \subseteq X$

⟨5⟩3. $f[\![h(n)]\!] \subseteq f[\![X]\!]$

⟨5⟩4. $f[\![h(n)]\!] \subseteq X$

⟨4⟩4. $h(n^+) \subseteq C^*$

$\square$

Exercise 10 $C^* = C_* = (0, 1]$

Exercise 11 $\{n \in \mathbb{Z} \mid n \leq 0\}$

Exercise 12 Let $f : B \times B \to B$ and $A \subseteq B$. Let

$$C^* = \bigcap \{X \mid A \subseteq X \subseteq B \;\&\; f[\![X \times X]\!] \subseteq X\} \;.$$

Define the function $h : \omega \to \mathcal{P}B$ by

$$\begin{aligned} h(0) &= A \\ h(n^+) &= h(n) \cup f[\![h(n) \times h(n)]\!] \qquad\qquad (n \in \omega) \end{aligned}$$

Define $C_* = \bigcup \operatorname{ran} h$. Then $C^* = C_*$.

4.4 Arithmetic

Exercise 13 We prove the contrapositive. Assume $m \neq 0$ and $n \neq 0$. Then by Theorem 4C there are natural numbers p, q such that $m = p^+$ and $n = q^+$. Hence $mn = p^+q^+ = (p^+q + p)^+ \neq 0$.

Exercise 14 We prove the following facts for any natural number n:

1. n is even if and only if n^+ is odd.

   PROOF: If n is even, say $n = 2p$, then $n^+ = 2p + 1$ is odd.
   If n^+ is odd, say $n^+ = 2p + 1$, then $n = 2p$ is even.

2. n is odd if and only if n^+ is even.

   PROOF: If n is odd, say $n = 2p + 1$, then $n^+ = 2(p + 1)$ is even.
   If n^+ is even, say $n^+ = 2p$, then we cannot have $p = 0$ (since $n^+ \neq 0$). So $p = q + 1$ for some q. But then $n^+ = 2q + 2$ so $n = 2q + 1$ and n is odd.

Now, 0 is even and 0 is not odd. By the two facts above, if n is either even or odd but not both, then n^+ is either odd or even but not both. The result follows by induction.

Exercise 15 We have

$$\begin{aligned} m + (n + 0) &= m + n && \text{by (A1)} \\ &= (m + n) + 0 && \text{by (A1)} \end{aligned}$$

If $m + (n + p) = (m + n) + p$ then

$$\begin{aligned} m + (n + p^+) &= m + (n + p)^+ && \text{by (A2)} \\ &= (m + (n + p))^+ && \text{by (A2)} \\ &= ((m + n) + p)^+ && \text{by induction hypothesis} \\ &= (m + n) + p^+ && \text{by (A2)} \end{aligned}$$

Exercise 16 We first prove that $0 \cdot n = 0$ for all n. We have $0 \cdot 0 = 0$ by (M1), and if $0 \cdot n = 0$ then

$$\begin{aligned} 0 \cdot n^+ &= 0 \cdot n + 0 && \text{by (M2)} \\ &= 0 \cdot n && \text{by (A1)} \\ &= 0 && \text{by induction hypothesis} \end{aligned}$$

Now we prove that $m^+ \cdot n = m \cdot n + n$ for all m, n. We have

$$\begin{aligned} m^+ \cdot 0 &= 0 && \text{by (M1)} \\ m \cdot 0 + 0 &= m \cdot 0 && \text{by (A1)} \\ &= 0 && \text{by (M1)} \end{aligned}$$

Thus, $m^+ \cdot 0 = m \cdot 0 + 0$.

If $m^+ \cdot n = m \cdot n + n$ then

$$\begin{aligned} m^+ \cdot n^+ &= m^+ \cdot n + m^+ && \text{by (M2)} \\ &= (m^+ \cdot n + m)^+ && \text{by (A2)} \\ &= ((m \cdot n + n) + m)^+ && \text{by induction hypothesis} \\ &= ((m \cdot n + m) + n)^+ && \text{by associativity and commutativity of addition} \\ &= (m \cdot n^+ + n)^+ && \text{by (M2)} \\ &= m \cdot n^+ + n^+ && \text{by (A2)} \end{aligned}$$

Exercise 17 The proof is by induction on p. We have

$$\begin{aligned} m^{n+0} &= m^n && \text{by (A1)} \\ &= 0 + m^n && \text{by Theorem 4K(2)} \\ &= m^n \cdot 0 + m^n && \text{by (M1)} \\ &= m^n \cdot 1 && \text{by (M2)} \\ &= m^n \cdot m^0 && \text{by (E1)} \end{aligned}$$

If $m^{n+p} = m^n \cdot m^p$ then

$$\begin{aligned} m^{n+p^+} &= m^{(n+p)^+} && \text{by (A2)} \\ &= m^{n+p} m && \text{by (E2)} \\ &= (m^n m^p) m && \text{by induction hypothesis} \\ &= m^n (m^p m) && \text{by Theorem 4K (4)} \\ &= m^n m^{p^+} && \text{by (E2)} \end{aligned}$$

4.5 Ordering on ω

Exercise 18

$$\begin{aligned} \in_\omega^{-1} [\![\{7, 8\}]\!] &= \{x \in \omega \mid x \in 7 \text{ or } x \in 8\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7\} \end{aligned}$$

Exercise 19 The proof is by induction on m.

For $m = 0$, take $q = r = 0$. Then $m = d \cdot 0 + 0$ and $0 \in d$.

Suppose $m = dq + r$ and $r < d$. Then $r + 1 \leq d$. If $r + 1 < d$, then we have $m + 1 = dq + (r + 1)$ as required. If $r + 1 = d$, then we have $m + 1 = dq + d = d(q + 1) + 0$.

Exercise 20 We first prove A is closed downwards; that is, if $n \in A$ and $m \in n$ then $m \in A$. This holds because if $n \in A$ and $m \in n$ then $m \in \bigcup A$ and $\bigcup A = A$.

Now, we prove $\forall n \in \omega . n \in A$ by induction on n.

To prove $0 \in A$: we are given that A is nonempty. Pick some $a \in A$. Then $0 \underline{ina}$ so $0 \in A$ since A is closed downwards.

Now let $n \in A$; we prove $n^+ \in A$. We have $n \in \bigcup A$; pick some $k \in A$ such that $n \in k$. Then $n^+ \underline{\in} k$ so $n^+ \in A$ since A is closed downwards.

This completes the induction. We have $\forall n \in \omega . n \in A$, i.e. $A = \omega$.

Exercise 21 Suppose n is a natural number, $k \in n$ and $n \subseteq k$. Then $k \in k$, contradicting Lemma 4L(b).

Exercise 22 We have $0 \in p^+$ (by trichotomy since $p^+ \notin 0$ because 0 is empty, and $p^+ \neq 0$ by Peano's First Postulate.) Hence $n = n + 0 \in n + p^+$ by Theorem 4N.

Exercise 23 The proof is by induction on n. The statement is vacuously true for $n = 0$.

Suppose the statement is true for n. Let $m \in n^+$. Then $m \underline{\in} n$.

If $m = n$, then we have $m + 0^+ = n^+$.

If $m \in n$, pick p such that $m + p^+ = n$ by the induction hypothesis. Then $m + p^{++} = n^+$.

Exercise 24 Suppose $m \in p$. Then we cannot have $n \in q$ or $n = q$, as either of these would imply $m + n \in p + q$. Hence $q \in n$ by trichotomy.

We prove $q \in n \Rightarrow m \in p$ similarly.

Exercise 25 By Exercise 23, pick natural numbers a and b such that $m = n + a^+$ and $p = q + b^+$. Then

$$\begin{aligned}
mp + nq &= (n + a^+)(q + b^+) + nq \\
&= nq + nq + a^+ q + nb^+ + a^+ b^+ \\
&= (n + a^+)q + n(q + b^+) + a^+ b^+ \\
&= mq + np + (a^+ + b)^+
\end{aligned}$$

Hence $mq + np \in mp + nq$ by Exercise 22.

Exercise 26 The proof is by induction on n.

If $n = 0$ then $\operatorname{ran} f$ is a singleton and its sole element is the largest element.

Suppose the result is true for n. Let $f : n^{++} \to A$. Then $f[\![n^+]\!]$ has a largest element $f(k)$, say. If $f(k) \underline{\in} f(n^+)$ then $f(n^+)$ is greatest in $\operatorname{ran} f$; otherwise $f(k)$ is greatest.

Exercise 27 We prove $f_1(n) = f_2(n)$ for all $n \in \omega$ by strong induction on n.
Assume that $(\forall m \in n) f_1(m) = f_2(m)$. Then $f_1 \restriction n = f_2 \restriction n$. So

$$\begin{aligned} f_1(n) &= G(f_1 \restriction n) \\ &= G(f_2 \restriction n) \\ &= f_2(n) \end{aligned}$$

Exercise 28 Suppose ω is not transitive. Then there exists a natural number n such that $n \nsubseteq \omega$. Let n be the least such number. There exists $x \in n$ such that $x \notin \omega$. Now, $n \neq 0$ (because it is nonempty) so $n = p^+$ for some natural number p. We have $x \in p^+$ so $x \in p$ or $x = p$. We cannot have $x = p$ (because x is not a natural number) so we have $x \in p$. But this contradicts the minimality of n.

4.6 Review Exercises

Exercise 29 $4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

Exercise 30 $\bigcup 4 = 0 \cup 1 \cup 2 \cup 3 = 3$ since 0, 1 and 2 are all subsets of 3.
$\bigcap 4 = 0 \cap 1 \cap 2 \cap 3 = 0 (= \emptyset)$.

Exercise 31 Similarly to Exercise 30 we have $\bigcup \bigcup 7 = \bigcup 6 = 5$.

Exercise 32

(a) $A^+ = A \cup \{A\} = \{1, A\} = \{1, \{1\}\}$
So $\bigcup A^+ = 1 \cup \{1\} = \{0, 1\} = 2$

(b) $\bigcup(\{2\}^+) = \bigcup\{2, \{2\}\} = \{0, 1, 2\} = 3$

Exercise 33

(a) Yes - if $x \in y \in \{0, 1, \{1\}\}$ then x is either 0 or 1, and in either case $x \in \{0, 1, \{1\}\}$

(b) No - $0 \in 1 \in \{1\}$ but $0 \notin \{1\}$

(c) No - $0 \in \{0\} \in \langle 0, 1 \rangle$ but $0 \notin \langle 0, 1 \rangle$.

Exercise 34

(a) Let $a = \{\emptyset\}$ and $b = \emptyset$

(b) Let $c = \{\{\emptyset\}\}$, $d = \{\emptyset\}$ and $e = \emptyset$

Exercise 35

(a) Let $T_1 = \{\{1\}, \{1,0\}, 0, 1\}$

(b) Let $T_2 = \{\langle 1,0\rangle, \{1\}, \{1,0\}, 0, 1\}$.

Exercise 36

$$\begin{aligned} h(4) &= 2h(3) \\ &= 4h(2) \\ &= 8h(1) \\ &= 16h(0) \\ &= 48 \end{aligned}$$

Exercise 37

(a) Let $f : m \to A$ and $g : n \to B$ be bijections. Define $h : m+n \to A \cup B$ by

$$\begin{aligned} h(p) &= f(p) && \text{if } p \in m \\ h(m+q) &= g(q) && \text{if } q \in n \end{aligned}$$

To show that this is well-defined, we must prove two things:

1. For all $p \in m+n$, then either $p \in m$ or there exists $q \in n$ such that $p = m+n$.
2. We never have $p \in m$ and $p = m+q$ for some $q \in n$.

We prove 1 by induction on n. For all $p \in m+0$ we have $p \in m$, so the result holds for $n = 0$.

Now, suppose the result holds for n. Let $p \in m+n^+ = (m+n)^+$ so $p\underline{in}m+n$. If $p \in m+n$, we simply apply the induction hypothesis. If $p = m+n$ then $p = m+q$ where $q = n \in n^+$.

To prove 2, if $p = m+q$ then $m = m+0\underline{in}m+q = p$ by Theorem 4N, hence $p \notin m$ by trichotomy.

It remains to show that h is a bijection.

To prove h is injective, we consider three cases. If $h(p) = h(p')$ where $p, p' \in m$, then $f(p) = f(p')$ so $p = p'$. If $h(m+q) = h(m+q')$ where $q, q' \in n$, then $g(q) = g(q')$ so $q = q'$. And we cannot have $h(p) = h(m+q)$ for $p \in m$ and $q \in n$ since $h(p) \in A$, $h(m+q) \in B$, and $A \cap B = \emptyset$.

To prove h is surjective, let $x \in A \cup B$. If $x \in A$, there is some $p \in m$ with $f(p) = x$, so $h(p) = x$. If $x \in B$, there is some $q \in n$ with $g(q) = x$, so $h(m+q) = x$.

(b) Let $f : m \to A$ and $g : n \to B$ be bijections.

We first show that, for any $p \in mn$, there exist unique $i \in m$ and $j \in n$ such that $p = mj + i$.

By Exercise 19, there exist j and $i \in m$ such that $p = mj + i$. We have $j \in n$ since otherwise $p = mj + i \underline{\ni} mj \underline{\ni} mn$.

For uniqueness, suppose $mj + i = mj' + i'$ where $i, i' \in m$ and $j, j' \in n$. Then we have

$$mj \underline{\in} mj + i = mj' + i' \in mj' + m = m(j')^+$$

so $j \in (j')^+$ and $j \underline{\in} j'$. Similarly $j' \underline{\in} j$, and so $j = j'$. Therefore $i = i'$ by the cancellation law for addition.

Now define $h : mn \to A \times B$ by

$$h(mj + i) = \langle f(i), g(j) \rangle$$

where $i \in m$ and $j \in n$. It is easy to check that h is bijective.

Exercise 38 $h(n) = 3n + 1$

Exercise 39 $h(n) = n^2$

Exercise 40 $h(n^+) = h(n) + 5$

Chapter 5

Chapter 5 — Construction of the Real Numbers

5.1 Integers

Exercise 1 No, because $[\langle 0,0\rangle] = [\langle 1,1\rangle]$ but $[\langle 0,0\rangle] \neq [\langle 2,1\rangle]$.

Exercise 2 Yes, because if $[\langle m,n\rangle] = [\langle p,q\rangle]$ then $[\langle m,m\rangle] = [\langle p,p\rangle]$ because $m+p = m+p$.

Exercise 3 Yes, because if $[\langle m,n\rangle] = [\langle p,q\rangle]$ then $[\langle n,m\rangle] = [\langle q,p\rangle]$ because $n+p = m+q$.

Exercise 4 Let $a = [\langle m,n\rangle]$, $b = [\langle p,q\rangle]$ and $c = [\langle r,s\rangle]$. Then

$$\begin{aligned} a +_Z (b +_Z c) &= [\langle m,n\rangle] +_Z [\langle p+r, q+s\rangle] \\ &= [\langle m+(p+r), n+(q+s)\rangle] \\ &= [\langle (m+p)+r, (n+q)+s\rangle] \\ &= [\langle m+p, n+q\rangle] +_Z [\langle r,s\rangle] \\ &= (a +_Z b) +_Z c \end{aligned}$$

Exercise 5

$$[\langle m,n\rangle] - [\langle p,q\rangle] = [\langle m,n\rangle] + [\langle q,p\rangle] = [\langle m+q, n+p\rangle]$$

Exercise 6 Let $a = [\langle m, n \rangle]$. Then

$$\begin{aligned} a \cdot_Z 0_Z &= [\langle m, n \rangle] \cdot_Z [\langle 0, 0 \rangle] \\ &= [\langle m0 + n0, m0 + n0 \rangle] \\ &= [\langle 0, 0 \rangle] \\ &= 0_Z \end{aligned}$$

Exercise 7 We have $a \cdot_Z b +_Z a \cdot_Z (-b) = a \cdot_Z (b +_Z (-b)) = a \cdot_Z 0_Z = 0_Z$, hence $a \cdot_Z (-b) = -(a \cdot_Z b)$ by the uniqueness of inverses.

We prove $(-a) \cdot_Z b = -(a \cdot_Z b)$ similarly.

Exercise 8

(a) This says $[\langle m + n, 0 \rangle] = [\langle m, 0 \rangle] +_Z [\langle n, 0 \rangle]$, which is true from the definition of $+_Z$.

(b) We have

$$\begin{aligned} E(m) \cdot_Z E(n) &= [\langle m, 0 \rangle] \cdot_Z [\langle n, 0 \rangle] \\ &= [\langle mn + 0 \cdot 0, m0 + n0 \rangle] \\ &= E(mn) \end{aligned}$$

(c)

$$\begin{aligned} E(m) <_Z E(n) &\Leftrightarrow [\langle m, 0 \rangle] <_Z [\langle n, 0 \rangle] \\ &\Leftrightarrow m + 0 \in n + 0 \\ &\Leftrightarrow m \in n \end{aligned}$$

Exercise 9

$$\begin{aligned} E(m) - E(n) &= [\langle m, 0 \rangle] - [\langle n, 0 \rangle] \\ &= [\langle m, n \rangle] \end{aligned}$$

by Exercise 5.

5.2 Rational Numbers

Exercise 10 Let $r = [\langle a, b \rangle]$. Then

$$\begin{aligned} r \cdot_Q 0_Q &= [\langle a, b \rangle] \cdot_Q [\langle 0, 1 \rangle] \\ &= [\langle a \cdot_Z 0, b \cdot_Z 1 \rangle] \\ &= [\langle 0, b \rangle] \\ &= [\langle 0, 1 \rangle] \end{aligned}$$

since $\langle 0, b \rangle \sim \langle 0, 1 \rangle$ because $0 \cdot_Z 1 = 0 \cdot_Z b = 0$.

Exercise 11 Let $r = [\langle a, b\rangle]$ and $s = [\langle c, d\rangle]$. Suppose $r \cdot_Q s = 0_Q$. Then

$$[\langle ac, bd\rangle] = [\langle 0, 1\rangle]$$

that is, $ac = 0$. Hence $a = 0$ or $c = 0$, which means $r = 0_Q$ or $s = 0_Q$.

Exercise 12 This follows from Theorem 5QJ(a) with $s = 0_Q$ and $t = -r$.

Exercise 13 Let $a, b, c \in \mathbb{Z}$. If $a +_Z c = b +_Z c$ then

$$\begin{aligned} a +_Z c +_Z (-c) &= b +_Z c +_Z (-c) \\ \therefore a +_Z 0 &= b +_Z 0 && \text{(Theorem 5ZD(b))} \\ \therefore a &= b && \text{(Theorem 5ZD(a))} \end{aligned}$$

Exercise 14 Suppose $p <_Q s$. Let $r = (p +_Q s)/2$. Then

$$\begin{aligned} p &<_Q s \\ \therefore 2p &<_Q p +_Q s \\ \therefore p &<_Q (p +_Q s)/2 \\ &= r \\ p &<_Q s \\ \therefore p +_Q s &<_Q 2s \\ \therefore (p +_Q s)/2 &<_Q s \\ \therefore r &<_Q s \end{aligned}$$

5.3 Real Numbers

Exercise 15 PROOF:
$\langle 1\rangle 1.$ $\bigcup A$ is closed downwards.
  $\langle 2\rangle 1.$ LET: $q \in \bigcup A$ and $p < q$
  $\langle 2\rangle 2.$ PICK $x \in A$ such that $q \in x$
  $\langle 2\rangle 3.$ $p \in x$
    PROOF: Since x is closed downwards.
  $\langle 2\rangle 4.$ $p \in \bigcup A$
$\langle 1\rangle 2.$ $\bigcup A$ has no largest element.
  $\langle 2\rangle 1.$ LET: $q \in \bigcup A$
  $\langle 2\rangle 2.$ PICK $x \in A$ such that $q \in x$
  $\langle 2\rangle 3.$ PICK $r \in x$ such that $q < r$
    PROOF: Since x has no largest element.
  $\langle 2\rangle 4.$ $r \in \bigcup A$
□

Exercise 16 PROOF:
$\langle 1\rangle 1$. LET: $q \in x +_R y$
$\langle 1\rangle 2$. PICK rationals $a \in x$ and $b \in y$ such that $q = a + b$
$\langle 1\rangle 3$. PICK $a' \in x$ and $b' \in y$ such that $a < a'$ and $b < b'$
PROOF: Since x and y each have no largest element.
$\langle 1\rangle 4$. $q < a' + b' \in x +_R y$
$\square$

Exercise 17 If $b < 0$ we can take $k = 0$. If $b \geq 0$ then there is a natural number n such that $b = E(n)$; take $k = n^+$. Then $b < ak$ since $1 \leq a$ and $b < k$.

Exercise 18 Let $p = [\langle a, b\rangle]$ and $r = [\langle c, d\rangle]$ where a, b and d are positive. By Exercise 17, there exists a natural number k such that $bc < adE(k)$. Therefore $r < p \cdot E(E(k))$.

Exercise 19 Pick a rational $a \in x$ (which we can do since $x \neq \emptyset$). We first prove that there exists a natural number k such that $a + kp \notin x$.

Pick a rational $b \notin x$ (which we can do since $x \neq \mathbb{Q}$). We have $a < b$ (since x is closed downwards). By Exercise 18, there exists a natural number k such that

$$\begin{aligned} b - a &< kp \\ \therefore a + kp &> b \\ \therefore a + kp &\notin x \end{aligned}$$

Now, let k be the least natural number such that $a + kp \notin x$ (by the Well-Ordering Principle). We have $k \neq 0$ (since $a \in x$); let $k = n^+$. Then we have

$$a + np \in x \qquad a + np + p \notin x$$

Take $q = a + np$.

Exercise 20 We must prove $0 \subseteq x \cup -x$. Let $q \in 0$ and assume $q \notin x$. Then $q < 0$ and $-0 = 0 \notin x$, so $q \in -x$.

Exercise 21 PROOF:
$\langle 1\rangle 1$. LET: x, y be real numbers with $x < y$
$\langle 1\rangle 2$. PICK $r \in y$ such that $r \notin x$
$\langle 1\rangle 3$. PICK $s \in y$ such that $r < s$
PROVE: $x < E(s) < y$
$\langle 1\rangle 4$. $x \subseteq E(s)$
PROOF: If $p \in x$ then $p < r < s$
$\langle 1\rangle 5$. $x \neq E(s)$
PROOF: Since $r \in E(s)$ and $r \notin x$
$\langle 1\rangle 6$. $E(s) \subseteq y$

PROOF: Since y is closed downwards.

$\langle 1 \rangle 7.$ $E(s) \neq y$

PROOF: Since $s \in y$ but $s \notin E(s)$.

Exercise 22 $|x|$ is either x or $-x$, and they are both real numbers.

Chapter 6

Chapter 6 — Cardinal Numbers and the Axiom of Choice

6.1 Equinumerosity

Exercise 1 Proof:
$\langle 1 \rangle 1$. f is injective.
  $\langle 2 \rangle 1$. Assume: $f(m,n) = f(m',n')$
  $\langle 2 \rangle 2$. $2^m(2n+1) = 2^{m'}(2n'+1)$
  $\langle 2 \rangle 3$. $m = m'$
    $\langle 3 \rangle 1$. Assume: w.l.o.g. $m \leq m'$
    $\langle 3 \rangle 2$. $2n+1 = 2^{m'-m}(2n'+1)$
      Proof: From $\langle 2 \rangle 2$ dividing by 2^m.
    $\langle 3 \rangle 3$. $m' - m = 0$
      Proof: Since $2^{m'-m}(2n'+1)$ is odd.
  $\langle 2 \rangle 4$. $2n+1 = 2n'+1$
  $\langle 2 \rangle 5$. $n = n'$
$\langle 1 \rangle 2$. f is surjective.
  $\langle 2 \rangle 1$. Let: $n \in \omega$
    Assume: $\forall m < n. m \in \operatorname{ran} f$
    Prove: $n \in \operatorname{ran} f$
  $\langle 2 \rangle 2$. Case: n is even
    $\langle 3 \rangle 1$. Let: k be such that $n = 2k$
    $\langle 3 \rangle 2$. $n = f(0,k)$
  $\langle 2 \rangle 3$. Case: n is odd
    $\langle 3 \rangle 1$. Let: k be such that $n = 2k+1$
    $\langle 3 \rangle 2$. Let: $k = f(i,j)$
    $\langle 3 \rangle 3$. $n = f(i+1,j)$

PROOF:

$$\begin{aligned} n &= 2k + 1 \\ &= 2(2^i(2j + 1) - 1) + 1 \\ &= 2^{i+1}(2j + 1) - 2 + 1 \\ &= 2^{i+1}(2j + 1) - 1 \end{aligned}$$

$\square$

Exercise 2 Let us call (0) the 0th diagonal, $(1, 2)$ the 1st diagonal, $(3, 4, 5)$ the 2nd diagonal, etc. Then the kth is the set of all positions with coordinates (m, n) such that $m + n = k$.

Therefore, the number $J(m, n)$ at position (m, n) is the $m + 1$st number in the $(m + n)$th diagonal. So the number of numbers that come before $J(m, n)$ is

$$(1 + 2 + \cdots + (m + n)) + m$$

Therefore, since the natural numbers start at 0,

$$J(m, n) = (1 + 2 + \cdots + (m + n)) + m$$

We know $1 + 2 + \cdots + k = k(k + 1)/2$. Therefore,

$$\begin{aligned} J(m, n) &= 1/2(m + n)(m + n + 1) + m \\ &= 1/2(m^2 + 2mn + m + n + n^2) + m \\ &= 1/2(m^2 + 2mn + 3m + n + n^2) \\ &= 1/2((m + n)^2 + 3m + n) \end{aligned}$$

Exercise 3 Define $f : (0, 1) \to \mathbb{R}$ by: $f(x) = 1/x - 2$ if $0 < x \leq 1/2$; $f(x) = 2 - 1/(1 - x)$ if $1/2 < x < 1$.

Exercise 4 Define $f : [0, 1] \to (0, 1)$ by

$$\begin{aligned} f(1/2 - 1/2^n) &= 1/2 - 1/2^{n-1} && \text{(for } n \text{ a positive integer)} \\ f(1/2 + 1/2^n) &= 1/2 + 1/2^{n-1} && \text{(for } n \text{ a positive integer)} \\ f(x) &= x && \text{(for all other } x\text{)} \end{aligned}$$

Exercise 5

(a) For any set A, the identity function I_A is a bijection between A and A. It is injective because, if $I_A(x) = I_A(y)$ then $x = y$ immediately. It is surjective because for any $y \in I_A$ we have $y = I_A(y)$.

(b) We prove that, if f is a bijection between A and B, then f^{-1} is a bijection between B and A. It is an injective function by Theorem 3F, and maps B onto A by Theorem 3E.

(c) Let f be a bijection between A and B, and g a bijection between A and C. We prove $g \circ f$ is a bijection between A and C.

It is a function from A to C by Theorem 3H.

We prove it is injective. Let $x, y \in A$ and assume $(g \circ f)(x) = (g \circ f)(y)$. Then

$$\begin{aligned} g(f(x)) &= g(f(y)) \\ \therefore f(x) &= f(y) && (g \text{ is injective}) \\ \therefore x &= y && (f \text{ is injective}) \end{aligned}$$

Now we prove it maps A onto C. Let $c \in C$. Pick $b \in B$ such that $g(b) = c$ (since g is surjective). Pick $a \in A$ such that $f(a) = b$ (since f is injective). Then $(g \circ f)(a) = c$.

6.2 Finite Sets

Exercise 6 Suppose every set of cardinality κ belongs to A. We will prove that every set belongs to $\bigcup A$.

Let x be any set. Pick a set y of cardinality κ. If $x \in y$ then $x \in y \in A$ so $x \in \bigcup A$.

Assume $x \notin y$. Pick an element $z \in y$ (we know y is nonempty because $\kappa \neq 0$). Then $y - \{z\} \cup \{x\}$ has cardinality κ, and so $x \in (y - \{z\} \cup \{x\}) \in A$ hence $x \in \bigcup A$.

Thus, every set is in $\bigcup A$, which we know is impossible by Theorem 2A.

Exercise 7 If f is one-to-one then f is a bijection between A and $\operatorname{ran} f$. So we must have $\operatorname{ran} f = A$, otherwise f would be a bijection between A and a proper subset of A, contradicting the Pigeonhole Principle.

Conversely, suppose $\operatorname{ran} f = A$. Pick a right inverse $h : A \to A$ for f (by Theorem 3J(b). Note: Theorem 3J(b) can in fact be proved for the case B is finite without using the Axiom of Choice.). Now, h is one-to-one by Theorem 3J(a). So $\operatorname{ran} h = A$ by the first paragraph.

We prove f is one-to-one. Let $x, y \in A$ and assume $f(x) = f(y)$. Pick $a, b \in A$ such that $h(a) = x$ and $h(b) = y$. Then

$$\begin{aligned} f(h(a)) &= f(h(b)) \\ \therefore a &= b \\ \therefore x &= y \end{aligned}$$

Exercise 8 Proof:

⟨1⟩1. For any sets A and x, if A is finite then $A \cup \{x\}$ is finite.

⟨2⟩1. Case: $x \in A$

Proof: In this case $A \cup \{x\} = A$.

⟨2⟩2. Case: $x \notin A$

PROOF: Then $|A \cup \{x\}| = |A|^+$.

$\langle 1 \rangle 2$. LET: A be a finite set.

$\langle 1 \rangle 3$. For any set B, if $B \approx 0$ then $A \cup B$ is finite.

PROOF: Because $B = \emptyset$ so $A \cup B = A$.

$\langle 1 \rangle 4$. Let n be a natural number. Assume that, for any set B, if $B \approx n$ then $A \cup B$ is finite. Then for any set B, if $B \approx n^+$ then $A \cup B$ is finite.

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: For any set B, if $B \approx n$ then $A \cup B$ is finite.

$\langle 2 \rangle 3$. LET: B be a set.

$\langle 2 \rangle 4$. ASSUME: $B \approx n^+$

$\langle 2 \rangle 5$. PICK a bijection $f : n^+ \to B$

$\langle 2 \rangle 6$. $B - \{f(n)\} \approx n$

$\langle 2 \rangle 7$. $A \cup (B - \{f(n)\})$ is finite.

$\langle 2 \rangle 8$. $A \cup B$ is finite.

PROOF: By $\langle 1 \rangle 1$ since $A \cup B = (A \cup (B - \{f(n)\})) \cup \{f(n)\}$.

□

Exercise 9 PROOF:

$\langle 1 \rangle 1$. LET: A be a finite set.

$\langle 1 \rangle 2$. For any set B, if $B \approx 0$ then $A \times B$ is finite.

PROOF: In this case $A \times B = \emptyset$.

$\langle 1 \rangle 3$. Let n be a natural number. Suppose that, for any set B, if $B \approx n$ then $A \times B$ is finite. Then for any set B, if $B \approx n^+$ then $A \times B$ is finite.

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. ASSUME: For any set B, if $B \approx n$ then $A \times B$ is finite.

$\langle 2 \rangle 3$. LET: B be a set.

$\langle 2 \rangle 4$. ASSUME: $B \approx n^+$

$\langle 2 \rangle 5$. PICK a bijection $f : n^+ \approx B$

$\langle 2 \rangle 6$. $A \times (B - \{f(n)\})$ is finite.

PROOF: By the induction hypothesis $\langle 2 \rangle 2$.

$\langle 2 \rangle 7$. $A \times B$ is finite.

PROOF: By Exercise 8 since $A \times B = (A \times (B - \{f(n)\})) \cup (A \times \{f(n)\})$ and $A \times \{f(n)\}$ is finite because it is equinumerous with A.

□

6.3 Cardinal Arithmetic

Exercise 10 We must show that ${}^{(L \cup M)}K \approx {}^L K \times {}^M K$ where $L \cap M = \emptyset$.

Define $\Phi : {}^{(L \cup M)} K \to {}^L K \times {}^M K$ by: $\Phi(f) = \langle f \upharpoonright L, f \upharpoonright M \rangle$.

To show Φ is one-to-one: suppose $\Phi(f) = \Phi(g)$. Then $f \upharpoonright L = g \upharpoonright L$ and $f \upharpoonright M = g \upharpoonright M$. Hence $f(x) = g(x)$ for all $x \in L$ and $f(x) = g(x)$ for all $x \in M$, so $f(x) = g(x)$ for all x, i.e. $f = g$.

To show Φ is surjective: given a function $g : L \to K$ and $h : M \to K$, we have $g \cup h : L \cup M \to K$ and $\Phi(g \cup h) = \langle g, h \rangle$.

Exercise 11 We must show that ${}^M(K \times L) \approx {}^M K \times {}^M L$.

Define $\Phi : {}^M(K \times L) \to {}^M K \times {}^M L$ by: $\Phi(f) = \langle \pi_1 \circ f, \pi_2 \circ f \rangle$, where $\pi_1 : K \times L \to K$ is the function defined by

$$\pi_1(\langle x, y \rangle) = x$$

and $\pi_2 : K \times L \to L$ is the function defined by

$$\pi_2(\langle x, y \rangle) = y \ .$$

To show Φ is one-to-one: suppose $\Phi(f) = \Phi(g)$. For any $x \in M$, we have $\pi_1(f(x)) = \pi_1(g(x))$ and $\pi_2(f(x)) = \pi_2(g(x))$, so $f(x) = g(x)$ by Theorem 3A.

To show Φ is surjective: given $g : M \to K$ and $h : M \to L$, define $f : M \to K \times L$ by $f(x) = \langle g(x), h(x) \rangle$ for $x \in M$. Then $\Phi(f) = \langle g, h \rangle$.

Exercise 12 We have:

$$\begin{aligned} K \cup L &= L \cup K \\ K \cup (L \cup M) &= (K \cup L) \cup M \\ K \times (L \cup M) &= (K \times L) \cup (K \times M) \end{aligned}$$

Exercise 13 Now that we have shown the union of two finite sets is finite, this follows by an easy induction on $|B|$.

Exercise 14 For any set A, let $Perm(A)$ be the set of all permutations of A.

Assume $K \approx L$: we must show $Perm(K) \approx Perm(L)$. Pick a bijection $f : K \to L$. Define $\Phi : Perm(K) \to Perm(L)$ by: $\Phi(g) = f \circ g \circ f^{-1}$. It is easy to show $\Phi(g)$ is a permutation of L whenever g is a permutation of K, and Φ is a bijection.

6.4 Ordering Cardinal Numbers

Exercise 15 Suppose for a contradiction $\mathcal{A}$ is a set and, for every set x, there exists $y \in \mathcal{A}$ such that $x \preccurlyeq y$. Pick $y \in \mathcal{A}$ such that $\mathcal{P} \bigcup \mathcal{A} \preccurlyeq y$. But $y \subseteq \bigcup \mathcal{A}$ so $\mathcal{P} \bigcup \mathcal{A} \preccurlyeq \bigcup \mathcal{A}$, contradicting Cantor's Theorem.

Exercise 16 Define $G : S \to {}^S 2$ by

$$G(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Then G is injective.

Now, assume for a contradiction $F : S \to {}^S 2$ is bijective. Define $g : S \to 2$ by $g(x) = 1 - F(x)(x)$. Then $g(x) \neq F(x)(x)$ for all $x \in S$, so $g \neq F(x)$ for all $x \in S$. Hence $g \notin \operatorname{ran} F$. This contradicts the assumption that F is surjective.

Exercise 17 We have $1 < 2$ but $\aleph_0 + 1 = \aleph_0 + 2 = \aleph_0$.

We have $1 < 2$ but $\aleph_0 \cdot 1 = \aleph_0 \cdot 2 = \aleph_0$.

We have $2 < 3$ but $2^{\aleph_0} = 3^{\aleph_0}$.

We have $2 < 3$ but $\aleph_0^2 = \aleph_0^3 = \aleph_0$.

6.5 Axiom of Choice

Exercise 18 PROOF:

$\langle 1 \rangle 1$. If the Axiom of Choice is true then the statement is true.

PROOF: The statement is a special case of the multiplicative axiom, taking $I = \mathcal{A}$ and $H(X) = X$ for each $X \in \mathcal{A}$.

$\langle 1 \rangle 2$. If the statement is true then the Axiom of Choice is true.

$\langle 2 \rangle 1$. ASSUME: The statement is true.

PROVE: Axiom of choice IV

$\langle 2 \rangle 2$. LET: $\mathcal{A}$ be a set such that each member of $\mathcal{A}$ is a nonempty set, and any two distinct members of $\mathcal{A}$ are disjoint.

$\langle 2 \rangle 3$. PICK a function f with domain $\mathcal{A}$ such that $f(X) \in X$ for all $X \in \mathcal{A}$

$\langle 2 \rangle 4$. LET: $C = \operatorname{ran} f$

$\langle 2 \rangle 5$. $\forall B \in \mathcal{A}. C \cap B = \{f(B)\}$

$\square$

Exercise 19 PROOF:

$\langle 1 \rangle 1$. For $n \in \omega$, let $P(n)$ be the statement: for every set I with $\operatorname{card} I = n$ and function H with domain I such that $H(i)$ is nonempty for each $i \in I$, there exists a function f with domain I such that $\forall i \in I. f(i) \in H(i)$.

$\langle 1 \rangle 2$. $P(0)$ is true

PROOF: Take $f = \emptyset$

$\langle 1 \rangle 3$. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: $P(n)$

$\langle 2 \rangle 3$. LET: I be a set with $\operatorname{card} I = n + 1$

$\langle 2 \rangle 4$. LET: H be a function with domain I such that $H(i)$ is nonempty for each $i \in I$

$\langle 2 \rangle 5$. PICK a bijection $g : n + 1 \approx I$

$\langle 2 \rangle 6$. PICK a function h with domain $g[\![n]\!]$ such that $\forall i \in g[\![n]\!]. h(i) \in H(i)$

$\langle 2 \rangle 7$. PICK $a \in H(g(n))$

$\langle 2 \rangle 8$. LET: $f = h \cup \{(g(n), a)\}$

$\langle 2 \rangle 9$. f is a function with domain I such that $\forall i \in I. f(i) \in H(i)$

$\square$

Exercise 20 PROOF:

$\langle 1 \rangle 1$. PICK a choice function F for A

$\langle 1 \rangle 2$. PICK $a \in A$

$\langle 1 \rangle 3$. Define the function $f : \omega \to A$ by:

$$f(0) = a$$
$$f(n^+) = F(R^{-1}(f(n)))$$

PROOF: We know $R^{-1}(x)$ is nonempty for all $x \in A$ because $\forall x \in A . \exists y \in A . yRx$.

$\langle 1 \rangle 4$. $\forall n \in \omega . f(n^+) R f(n)$

$\square$

Exercise 21 PROOF:

$\langle 1 \rangle 1$. For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 2 \rangle 1$. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain.

$\langle 2 \rangle 2$. Every finite subset of $\bigcup \mathcal{B}$ is a member of $\mathcal{A}$.

$\langle 3 \rangle 1$. LET: $\{x_1, \ldots, x_n\} \subseteq \bigcup \mathcal{B}$ be finite.

$\langle 3 \rangle 2$. For $1 \leq i \leq n$, PICK $B_i \in \mathcal{B}_i$ such that $x_i \in B_i$

$\langle 3 \rangle 3$. PICK m such that $B_1, \ldots, B_n \subseteq B_m$

PROOF: Since $\mathcal{B}$ is a chain.

$\langle 3 \rangle 4$. $\{x_1, \ldots, x_n\}$ is a finite subset of B_m.

$\langle 3 \rangle 5$. $\{x_1, \ldots, x_n\} \in \mathcal{A}$

PROOF: Since $B_m \in \mathcal{A}$ so every finite subset of B_m is a member of $\mathcal{A}$.

$\langle 2 \rangle 3$. $\bigcup \mathcal{B} \in \mathcal{A}$

$\langle 1 \rangle 2$. Q.E.D.

PROOF: By Zorn's Lemma.

$\square$

Exercise 22 PROOF:

$\langle 1 \rangle 1$. If the Axiom of Choice is true then the statement is true.

$\langle 2 \rangle 1$. ASSUME: The Axiom of Choice

$\langle 2 \rangle 2$. LET: A be a set.

$\langle 2 \rangle 3$. LET: $R = \{\langle x, y \rangle : y \in A, x \in t\}$

$\langle 2 \rangle 4$. PICK a function $F \subseteq R$ such that $\operatorname{dom} F = \operatorname{dom} R$

$\langle 2 \rangle 5$. $\operatorname{dom} R = \bigcup A$

$\langle 2 \rangle 6$. $\forall x \in \bigcup A . x \in F(x) \in A$

$\langle 1 \rangle 2$. If the statement is true then the Axiom of Choice is true.

$\langle 2 \rangle 1$. ASSUME: the statement

$\langle 2 \rangle 2$. LET: R be a relation

$\langle 2 \rangle 3$. LET: $A = \{\{\langle 0, x \rangle, \langle 1, y \rangle\} : xRy\}$

$\langle 2 \rangle 4$. PICK a function F with domain $\bigcup A$ such that $\operatorname{dom} F = \bigcup A$ and $\forall x \in \bigcup A . x \in F(x) \in A$

$\langle 2 \rangle 5$. LET: $H = \{\langle x, y \rangle \mid x \in \operatorname{dom} R, F(x) = \{\langle 0, x \rangle, \langle 1, y \rangle\}\}$

$\langle 2 \rangle 6$. H is a function, $H \subseteq R$, $\operatorname{dom} H = \operatorname{dom} R$

$\square$

Exercise 23

$\langle 1 \rangle 1$. $g[\![0]\!] = h(0)$

PROOF: Both are equal to $\emptyset$.

⟨1⟩2. $\forall n \in \omega . g[\![n]\!] = h(n) \Rightarrow g[\![n^+]\!] = h(n^+)$
  ⟨2⟩1. Let: $n \in \omega$
  ⟨2⟩2. Assume: $g[\![n]\!] = h(n)$
  ⟨2⟩3. $g[\![n^+]\!] = h(n^+)$
    Proof:

$$\begin{aligned} h(^+) &= h(n) \cup \{F(A - h(n))\} \\ &= g[\![n]\!] \cup \{g(n)\} \\ &= g[\![n^+]\!] \end{aligned}$$

Exercise 24 Let $\{\kappa_i\}_{i \in I}$ be a family of cardinal numbers. For $i \in I$, let K_i be a set such that $\operatorname{card} K_i = \kappa_i$.

We define $\sum_{i \in I} \kappa_i$ to be $\operatorname{card}\{\langle i, x\rangle : i \in I, x \in K_i\}$

We define $\prod_{i \in I} \kappa_i$ to be $\operatorname{card}\{f : f \text{ is a function}, \operatorname{dom} f = I, \forall i \in I . f(i) \in K_i\}$.

Exercise 25 Proof:
⟨1⟩1. Assume: for a contradiction $\forall n \in \omega . B \nsubseteq S(n)$
⟨1⟩2. Pick a function $b : \omega \to B$ such that $\forall n \in \omega . b(n) \notin S(n)$
  Proof: By the Axiom of Choice.
⟨1⟩3. Let: $B' = \{b(n) : n \in \omega\}$
⟨1⟩4. B' is infinite.
  ⟨2⟩1. Assume: for a contradiction B' is finite.
  ⟨2⟩2. There exists N such that $\forall n > N . \exists k \leq N . b(n) = b(k)$
  ⟨2⟩3. Pick $M > N$ such that $\forall k \leq N . b(k) \in S(M)$
    Proof: For $k \leq N$ there exists n_k such that $b(k) \in S(n_k)$. Take M to be the largest of these numbers and $N + 1$.
  ⟨2⟩4. $b(M) \in S(M)$
    Proof: Since $b(M) = b(k)$ for some $k \leq N$.
  ⟨2⟩5. Q.E.D.
    Proof: This contradicts ⟨1⟩2.
⟨1⟩5. Pick n such that $B' \cap S(n)$ is infinite.
⟨1⟩6. Pick $m > n$ such that $b(m) \in B' \cap S(n)$
  Proof: There must be some m otherwise $B' \cap S(n) \subseteq \{b(0), b(1), \ldots, b(n)\}$ would be finite.
⟨1⟩7. $b(m) \in S(m)$
  Proof: Since $S(n) \subseteq S(m)$.
⟨1⟩8. Q.E.D.
  Proof: This contradicts ⟨1⟩2.
□

6.6 Countable Sets

Exercise 26 Proof:
⟨1⟩1. Pick a set K of cardinality κ

$\langle 1\rangle 2.$ For all $X \in \mathcal{A}$, there exists an injective function $X \to K$

$\langle 1\rangle 3.$ PICK a function F with domain $\mathcal{A}$ such that, for all $X \in A$, $F(X)$ is an injective function $X \to K$

PROOF: By the Axiom of Choice.

$\langle 1\rangle 4.$ PICK a function G with domain $\bigcup \mathcal{A}$ such that, for all $x \in \bigcup \mathcal{A}$, we have $x \in G(x) \in \mathcal{A}$

PROOF: By Exercise 22.

$\langle 1\rangle 5.$ Define $f : \bigcup \mathcal{A} \to \mathcal{A} \times K$ by $f(x) = \langle G(x), F(G(x))(x) \rangle$

$\langle 1\rangle 6.$ f is injective.

$\langle 2\rangle 1.$ LET: $x, y \in \bigcup \mathcal{A}$

$\langle 2\rangle 2.$ ASSUME: $f(x) = f(y)$

$\langle 2\rangle 3.$ $G(x) = G(y)$ and $F(G(x))(x) = F(G(y))(y)$

$\langle 2\rangle 4.$ $F(G(x))(x) = F(G(x))(y)$

$\langle 2\rangle 5.$ $x = y$

PROOF: Since $F(G(x))$ is injective.

$\square$

Exercise 27

(a) Pick a function $f : A \to \mathbb{Q}^2$ such that $f(c) \in c$ for all $c \in A$. Then f is an injection, so $A \preccurlyeq \mathbb{Q}^2$ which is countable.

(b) No: the set of all circles with center $(0, 0)$ is an uncountable set of circles no two of which intersect.

(c) Yes. Pick a function $f : C \to \mathbb{Q}^4$ such that $f(x)$ is a pair of points with rational coordinates, one in each circle of x, for all $x \in C$. Then f is an injection; it is not possible for two points to be in separate circles of two non-intersecting figure-eights. Hence $C \preccurlyeq \mathbb{Q}^4$.

Exercise 28 Let $\mathcal{A} = \{(a, \sqrt{2}) : a < \sqrt{2}\} \cup \{(\sqrt{2}, b) : b > \sqrt{2}\}$. Then every rational is in some member of $\mathcal{A}$ but $\bigcup \mathcal{A} = \mathbb{R} - \{\sqrt{2}\}$.

(Enderton's hint suggests he had a different solution in mind, but I am not sure what it is.)

Exercise 29 For each integer $n \geq 2$, let $B_n = \{x \in A : x > b/n\}$. Then each B_n is finite (B_n cannot have more than $n - 1$ elements because n elements in B_n would have a sum $> b$) and $A = \bigcup_n B_n$. So A is a countable union of finite sets, and therefore countable.

Exercise 30 PROOF:

$\langle 1\rangle 1.$ PICK $a \in A$

$\langle 1\rangle 2.$ Define $f : Sq(A) \to \omega \times^\omega A$ by $f(s) = \langle n, g \rangle$, where n is the length pf s, and $g(i) = s(i)$ for $i < n$, $g(i) = a$ for $i \geq n$

$\langle 1\rangle 3.$ f is injective.
$\langle 1\rangle 4.$ $Sq(A) \preccurlyeq \omega \times^{\omega} A$
$\langle 1\rangle 5.$ $\operatorname{card} Sq(A) \leq (\operatorname{card} A)^{\aleph_0}$
PROOF:

$$\begin{aligned}
\operatorname{card} Sq(A) &\leq \aleph_0 \cdot (\operatorname{card} A)^{\aleph_0} && (\langle 1\rangle 4) \\
&\leq (\operatorname{card} A)^{\aleph_0} \cdot (\operatorname{card} A)^{\aleph_0} && \text{(Cantor's Theorem)} \\
&= (\operatorname{card} A)^{\aleph_0 + \aleph_0} && \text{(Theorem 6I)} \\
&= (\operatorname{card} A)^{\aleph_0}
\end{aligned}$$

6.7 Arithmetic of Infinite Cardinals

Exercise 31 If f is a one-to-one correspondence between $A \times A$ and A, where $A \subseteq B$, then

$$f \subseteq (A \times A) \times A \subseteq (B \times B) \times B \ .$$

Also $\emptyset \subseteq (B \times B) \times B$. So we can form $\mathcal{H}$ by applying a Subset Axiom to $\mathcal{P}((B \times B) \times B)$.

Exercise 32 The function that maps x to $\{x\}$ is an injection $A \to \mathcal{F}A$, so we have $A \approx \mathcal{F}A$.

For the converse, let $F_n = \{X \in \mathcal{F}A : \operatorname{card} X \leq n\}$ for $n \in \omega$. The function that sends $\langle a_1, \ldots, a_n \rangle$ to $\{a_1, \ldots, a_n\}$ is a surjection $A^n \to F_n$, so we have

$$\operatorname{card} F_n \leq (\operatorname{card} A)^n = \operatorname{card} A$$

by Lemma 6R. Now, $\mathcal{F}A = \bigcup_n F_n$, so

$$\operatorname{card} \mathcal{F}A \leq \aleph_0 \cdot \operatorname{card} A = \operatorname{card} A$$

by the Absorption Law.

Exercise 33 The function that maps a to the sequence of length 1 containing a is an injection $A \to Sq(A)$, so $A \preccurlyeq Sq(A)$.

For the converse, we have $\operatorname{card}({}^n A) = (\operatorname{card} A)^n = \operatorname{card} A$ for any natural number n

$$\begin{aligned}
\operatorname{card} Sq(A) &= \operatorname{card}({}^0 A \cup {}^1 A \cup {}^2 A \cup \cdots) \\
&= \aleph_0 \cdot \operatorname{card} A \\
&= \operatorname{card} A
\end{aligned}$$

by the Absorption Law.

Exercise 34

$$\begin{aligned} 2^\lambda &\leq \kappa^\lambda \\ &\leq (2^\kappa)^\lambda \\ &= 2^{\kappa \cdot \lambda} \\ &= 2^\lambda && \text{(Absorption Law)} \end{aligned}$$

Exercise 35 For any infinite set of primes A and natural number n, let $f(A,n) = \prod\{p \in A : p \leq n\}$. Let $P(A) = \{f(A,n) : n \in \omega\}$. Let $\mathcal{A}$ be the set of all sets of the form $P(A)$.

The number of infinite sets of primes is $2^{\aleph_0}$ (there are $2^{\aleph_0}$ sets of primes and $\aleph_0$ finite sets of primes by Exercise 32.)

If $P(A) = P(B)$ then $A = B$. (If $p \in A - B$ then $p \mid f(A,p)$ but p does not divide any member of $P(B)$.) So P is an injection from the set of infinite sets of primes into $\mathcal{A}$. Hence card $\mathcal{A} = 2^{\aleph_0}$.

We now prove that, if $A \neq B$, then $P(A) \cap P(B)$ is finite. Let $p \in A - B$. For $n \geq p$ we have $f(A,n) \notin P(B)$ since $p \mid f(A,n)$ but p does not divide any member of B. Hence $A \cap B \subseteq \{f(A,0), f(A,1), \ldots, f(A,p-1)\}$.

Exercise 36 PROOF:

$\langle 1 \rangle 1.$ For any set A, there exists a permutation of A with no fixed points.

  $\langle 2 \rangle 1.$ For every natural number n, there exists a permuatation of n with no fixed points.

  PROOF: Map i to $i+1$ if $i+1 < n$, and map $n-1$ to 0.

  $\langle 2 \rangle 2.$ For every infinite set A, there exists a permutation of A with no fixed points.

    $\langle 3 \rangle 1.$ PICK a bijection $f : A \approx A \times 2$

    $\langle 3 \rangle 2.$ Define $\pi : A \times 2 \to A \times 2$ by $\pi(x,0) = (x,1)$ and $\pi(x,1) = (x,0)$

    $\langle 3 \rangle 3.$ $f^{-1} \circ \pi \circ f$ is a permutation of A with no fixed point.

$\langle 1 \rangle 2.$ $\kappa! \leq 2^\kappa$

  PROOF: Because the set of permutations of K is a subset of ${}^K K$, where K is a set of cardinality κ.

$\langle 1 \rangle 3.$ $2^\kappa \leq \kappa!$

  $\langle 2 \rangle 1.$ PICK a set K of cardinality κ

  $\langle 2 \rangle 2.$ LET: $Perm(K)$ be the set of permutations of K.

  $\langle 2 \rangle 3.$ Define $f : \mathcal{P}K \to Perm(K)$ as follows. Given $A \subseteq \mathcal{P}K$, pick a permutation π_{K-A} of $K - A$ with no fixed point. Then $f(A) = I_A \cup \pi_{K-A}$

  $\langle 2 \rangle 4.$ f is injective

  PROOF: The function that maps a permutation to its set of fixed points is a left inverse.

  $\langle 2 \rangle 5.$ $2^\kappa \leq \kappa!$

$\square$

Chapter 7

Chapter 7 — Orderings and Ordinals

7.1 Partial Orderings

Exercise 1

(a) No we cannot. Let $A = \mathcal{P}3$ and $B = \omega$. Let $<_A = \subset_3$ and $<_B$ be the usual ordering on ω. Define $f : A \to B$ by: $f(X) = \operatorname{card} X$. Then $X \subset_2 Y \Rightarrow \operatorname{card} X < \operatorname{card} Y$ but f is not one-to-one because $f(\{0\}) = f(\{1\}) = 1$.

(b) No we cannot. With the same example, we have $f(\{0\}) < f(\{1,2\})$ but $\{0\} \not\subset \{1,2\}$.

Exercise 2 We show R^{-1} is transitive. Suppose $xR^{-1}y$ and $yR^{-1}z$. Then zRy and yRx, so zRx because R is transitive. Hence $xR^{-1}z$.

We now show R^{-1} is irreflexive. For any x, we have $\langle x, x\rangle \notin R$, so $\langle x, x\rangle \notin R^{-1}$.

Exercise 3 The proof is by induction on n.

The only linear ordering on $\emptyset$ is $\emptyset$, which has 0 pairs.

Suppose that, whenever $\operatorname{card} S = n$, then every linear ordering on S has $1/2n(n-1)$ pairs. Let S be a set of cardinality $n+1$. Let $<$ be a linear ordering on S.

Pick an element $a \in S$ and let $T = S - \{a\}$. Then $< \cap (T \times T)$ is a linear ordering on T, hence has $1/2n(n-1)$ pairs. Now, for every $x \in T$, exactly one of $\langle x, a\rangle$ and $\langle a, x\rangle$ is in $<$. Hence $<$ has n pairs that are not in $< \cap (T \times T)$. So

$$\operatorname{card} <= 1/2n(n-1) + n = 1/2n(n+1) \ .$$

7.2 Well Orderings

Exercise 4 Proof:
$\langle 1 \rangle 1$. R is transitive.
$\langle 2 \rangle 1$. Assume: mRn and nRp.
$\langle 2 \rangle 2$. Case: $f(m) < f(n)$
Proof: In this case $f(m) < f(p)$ so mRp.
$\langle 2 \rangle 3$. Case: $f(m) = f(n)$ and $m < n$.
$\langle 3 \rangle 1$. Case: $f(n) < f(p)$
Proof: In this case $f(m) < f(p)$ so mRp.
$\langle 3 \rangle 2$. Case: $f(n) = f(p)$ and $n < p$.
Proof: In this case $f(m) = f(p)$ and $m < p$ so mRp.
$\langle 1 \rangle 2$. R satisfies trichotomy on P.
$\langle 2 \rangle 1$. Let: $m, n \in P$
$\langle 2 \rangle 2$. Exactly one of $f(m) < f(n)$, $f(n) < f(m)$, $f(n) = f(m)$ holds.
$\langle 2 \rangle 3$. Exactly one of $m < n$, $n < m$, $n = m$ holds.
$\langle 2 \rangle 4$. Exactly one of $f(m) < f(n)$, $(f(m) = f(n) \;\&\; m < n)$, $(f(m) = f(n) \;\&\; m = n)$, $(f(m) = f(n) \;\&\; n < m)$, $f(n) < f(m)$ holds.
$\langle 2 \rangle 5$. Exactly one of mRn, $m = n$, nRm holds.
$\langle 1 \rangle 3$. Every nonempty subset of P has an R-least element.
$\langle 2 \rangle 1$. Let: $A \subseteq P$ be nonempty.
$\langle 2 \rangle 2$. Let: k be the least element of $f(A)$.
$\langle 2 \rangle 3$. Let: n be the least element of $f^{-1}(k) \cap A$.
$\langle 2 \rangle 4$. n is the R-least element of A.
□

$\langle P, R \rangle$ resembles Fig. 45 (d).

Exercise 5 Proof:
$\langle 1 \rangle 1$. Let: $x \in A$
$\langle 1 \rangle 2$. Assume: for a contradiction $f(x) < x$
$\langle 1 \rangle 3$. Define $g : \omega \to A$ by $g(0) = x$ and $g(n^+) = f(g(n))$ for all $n \in \omega$
$\langle 1 \rangle 4$. $\forall n \in \omega . g(n^+) < g(n)$
Proof: By induction on n using $\langle 1 \rangle 2$ and the hypothesis.
$\langle 1 \rangle 5$. Q.E.D.
Proof: This contradicts Theorem 7B.
□

Exercise 6 Proof:
$\langle 1 \rangle 1$. For all $x \in S$ that is not greatest, there exists $y \in S$ and $q \in \mathbb{Q}$ such that $x < q < y$ and there is no $z \in S$ such that $x < z < y$
$\langle 1 \rangle 2$. Pick a function $f : S \to \mathbb{Q}$ such that $\forall x \in S . x < f(x)$ and, if x is not greatest, then $f(x) < y$ where y is the next element in S.
$\langle 1 \rangle 3$. f is injective.
$\langle 1 \rangle 4$. $S \preccurlyeq \mathbb{Q}$
□

Exercise 7

(a) We have $F(t) = C \cup \bigcup\bigcup \operatorname{ran}(F \restriction t)$ for all $t \in \omega$. So:

$$\begin{aligned}
F(0) &= C \cup \bigcup\bigcup \operatorname{ran} \emptyset \\
&= C \\
F(1) &= C \cup \bigcup\bigcup \operatorname{ran}(F \restriction 0) \\
&= C \cup \bigcup\bigcup \{C\} \\
&= C \cup \bigcup C \\
F(2) &= C \cup \bigcup\bigcup \{C, C \cup \bigcup C\} \\
&= C \cup \bigcup (C \cup \bigcup C) \\
&= C \cup \bigcup C \cup \bigcup\bigcup C
\end{aligned}$$

We guess:

$$F(n) = C \cup \bigcup C \cup \cdots \cup \overbrace{\bigcup\bigcup \cdots \bigcup}^{n} C$$

(b) Proof:
⟨1⟩1. Let: $a \in F(n)$
⟨1⟩2. $a \in \bigcup \operatorname{ran}(F \restriction n^+)$
⟨1⟩3. $a \subseteq \bigcup\bigcup \operatorname{ran}(F \restriction n^+)$
⟨1⟩4. $a \subseteq F(n^+)$
□

(c) Proof:
⟨1⟩1. $\overline{C}$ is a transitive set.
  ⟨2⟩1. Let: $x \in y \in \overline{C}$
  ⟨2⟩2. Pick $n \in \omega$ such that $y \in F(n)$
  ⟨2⟩3. $x \in F(n^+)$
    Proof: By (b).
  ⟨2⟩4. $x \in \overline{C}$
⟨1⟩2. $C \subseteq \overline{C}$
  ⟨2⟩1. Since $C = F(0)$
□