

Solutions Manual for Enderton *Elements of Set
Theory*

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Contents

1	Chapter 1 — Introduction	3
1.1	Baby Set Theory	3
1.2	Sets — An Informal View	4
2	Chapter 2 — Axioms and Operations	6
2.1	Arbitrary Unions and Intersections	6
2.2	Algebra of Sets	8
2.3	Review Exercises	12
3	Chapter 3 — Relations and Functions	17
3.1	Ordered Pairs	17
3.2	Relations	18
3.3	n -ary Relations	19
3.4	Functions	19
3.5	Infinite Cartesian Products	26
3.6	Equivalence Relations	26
3.7	Ordering Relations	31
3.8	Review Exercises	32
4	Chapter 4 — Natural Numbers	37
4.1	Inductive Sets	37
4.2	Peano's Postulates	37
4.3	Recursion on ω	38
4.4	Arithmetic	40
4.5	Ordering on ω	41
4.6	Review Exercises	43
5	Chapter 5 — Construction of the Real Numbers	46
5.1	Integers	46
5.2	Rational Numbers	47
5.3	Real Numbers	48

6	Chapter 6 — Cardinal Numbers and the Axiom of Choice	51
6.1	Equinumerosity	51
6.2	Finite Sets	53
6.3	Cardinal Arithmetic	54
6.4	Ordering Cardinal Numbers	55
6.5	Axiom of Choice	56
6.6	Countable Sets	58
6.7	Arithmetic of Infinite Cardinals	60
7	Chapter 7 — Orderings and Ordinals	62
7.1	Partial Orderings	62
7.2	Well Orderings	63
7.3	Replacement Axioms	64
7.4	Epsilon-Images	65
7.5	Isomorphisms	67
7.6	Ordinal Numbers	68
7.7	Debts Paid	70
7.8	Rank	71

Chapter 1

Chapter 1 — Introduction

1.1 Baby Set Theory

Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

Exercise 2 We have $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ has an element (namely \emptyset) while \emptyset has no elements.

We have $\emptyset \neq \{\{\emptyset\}\}$ because $\{\{\emptyset\}\}$ has an element (namely $\{\emptyset\}$) while \emptyset has no elements.

We have $\{\emptyset\} \neq \{\{\emptyset\}\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$. This last fact is true because $\emptyset \neq \{\emptyset\}$ as we proved in the first paragraph.

Exercise 3 Assume $B \subseteq C$. Let $A \in \mathcal{P}B$; we must show that $A \in \mathcal{P}C$.

We have $A \subseteq B$ (since $A \in \mathcal{P}B$) and $B \subseteq C$. From this it follows that $A \subseteq C$ (every element of A is an element of B ; every element of B is an element of C ; therefore every element of A is an element of C). Hence $A \in \mathcal{P}C$ as required.

Exercise 4 Since $x \in B$, we have $\{x\} \subseteq B$ and so $\{x\} \in \mathcal{P}B$.

Since $x \in B$ and $y \in B$, we have $\{x, y\} \subseteq B$ and so $\{x, y\} \in \mathcal{P}B$.

From these two facts, it follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$ and so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

1.2 Sets — An Informal View

Exercise 5 We have

$$\begin{aligned} V_0 &= A \\ V_1 &= V_0 \cup \mathcal{P}V_0 \\ &= A \cup \mathcal{P}A \\ V_2 &= V_1 \cup \mathcal{P}V_1 \\ &= \{\emptyset, \{\emptyset\}\} \\ V_3 &= \mathcal{P}V_2 \\ &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \end{aligned}$$

We have $\emptyset \subseteq V_0$ and so $\emptyset \in V_1$. Therefore $\{\emptyset\} \subseteq V_1$ and so $\{\emptyset\} \in V_2$. Hence $\{\{\emptyset\}\} \subseteq V_2$.

We also have $\{\{\emptyset\}\} \not\subseteq V_0$ because $\{\emptyset\}$ is not an atom, and $\{\{\emptyset\}\} \not\subseteq V_1$ since $\{\emptyset\} \notin V_1$ because \emptyset is not an atom.

Thus the rank of $\{\{\emptyset\}\}$ is 2.

Likewise we have \emptyset and $\{\emptyset\}$ are both subsets of V_1 , hence

$$\emptyset \in V_2, \quad \{\emptyset\} \in V_2$$

Thus $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are all subsets of V_2 , hence elements of V_3 . Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is not a subset of V_0 (because \emptyset is not an atom.) It is not a subset of V_1 ($\{\emptyset\} \notin V_1$ because \emptyset is not an atom.) It is not a subset of V_2 (we have $\{\emptyset, \{\emptyset\}\} \notin V_2$ since $\{\emptyset\} \notin V_1$).

Therefore the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

Exercise 6

$$\begin{aligned}
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}V_0 && (\text{since } V_0 = A) \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_1 && (\text{since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3}) \\
V_3 &= V_2 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_2 && (\text{since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3}) \\
V_4 &= V_3 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_3 && (\text{since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3})
\end{aligned}$$

Exercise 7 In Exercise 5 we calculated $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Hence

$$\begin{aligned}
V_4 &= \mathcal{P}V_3 \\
&= \{\emptyset, \\
&\quad \{\emptyset\}, \\
&\quad \{\{\emptyset\}\}, \\
&\quad \{\{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
&\quad \}
\end{aligned}$$

Chapter 2

Chapter 2 — Axioms and Operations

2.1 Arbitrary Unions and Intersections

Exercise 1 $A \cap B \cap C$ is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

Exercise 2 Take $A = \emptyset$ and $B = \{\emptyset\}$. Then $\bigcup A = \bigcup B = \emptyset$ but $A \neq B$. (There are many other possible answers.)

Exercise 3 Let $b \in A$. We must show that $b \subseteq \bigcup A$.

Let x be any element of b . We must show that $x \in \bigcup A$. We know that $x \in b$ and $b \in A$, and so $x \in \bigcup A$ by the definition of $\bigcup A$.

Exercise 4 Suppose $A \subseteq B$. Let $x \in \bigcup A$. We must show that $x \in \bigcup B$.

Pick an element $a \in A$ such that $x \in a$. Then $a \in B$ because $A \subseteq B$. Since we know $x \in a$ and $a \in B$, we know that $x \in \bigcup B$.

Exercise 5 Assume that every member of \mathcal{A} is a subset of B . Let $x \in \bigcup \mathcal{A}$. We must show that $x \in B$.

Pick $A \in \mathcal{A}$ such that $x \in A$. By our assumption, we have $A \subseteq B$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$ as required.

Exercise 6

(a) We will show that $\bigcup \mathcal{P}A \subseteq A$ and $A \subseteq \bigcup \mathcal{P}A$.

To show $\bigcup \mathcal{P}A \subseteq A$: This follows from Exercise 5, since every member of $\mathcal{P}A$ is a subset of A .

To show $A \subseteq \bigcup \mathcal{P}A$: Let $a \in A$. Then we have $a \in \{a\}$ and $\{a\} \in \mathcal{P}A$ so $a \in \bigcup \mathcal{P}A$.

(b) To show $A \subseteq \mathcal{P} \bigcup A$: This holds because every element of A is a subset of $\bigcup A$, as we proved in Exercise 3.

Equality holds if and only if $A = \mathcal{P}X$ for some set X .

Proof: If $A = \mathcal{P} \bigcup A$ then of course $A = \mathcal{P}X$ for some X .

Conversely, if $A = \mathcal{P}X$, then we have

$$\begin{aligned} \mathcal{P} \bigcup A &= \mathcal{P} \bigcup \mathcal{P}X \\ &= \mathcal{P}X && \text{(by part (a))} \\ &= A \end{aligned}$$

Exercise 7

(a) For any set X ,

$$\begin{aligned} X &\in \mathcal{P}A \cap \mathcal{P}B \\ \Leftrightarrow X &\subseteq A \text{ and } X \subseteq B \\ \Leftrightarrow \text{Every member of } X &\text{ is a member of } A \text{ and a member of } B \\ \Leftrightarrow X &\subseteq A \cap B \\ \Leftrightarrow X &\in \mathcal{P}(A \cap B) \end{aligned}$$

(b) Let $X \in \mathcal{P}A \cup \mathcal{P}B$. Then either $X \in \mathcal{P}A$ or $X \in \mathcal{P}B$ (or both). If $X \in \mathcal{P}A$, then we have $X \subseteq A$ and so $X \subseteq A \cup B$ (because $A \subseteq A \cup B$). Similarly if $X \in \mathcal{P}B$ then we have $X \subseteq A \cup B$. So in either case $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

Equality holds if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof: Suppose $A \subseteq B$. Then $\mathcal{P}A \subseteq \mathcal{P}B$ (Chapter 1 Exercise 3) and so $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$. Also $A \cup B = B$ so $\mathcal{P}(A \cup B) = \mathcal{P}B$. Thus $\mathcal{P}A \cup \mathcal{P}B$ and $\mathcal{P}(A \cup B)$ are equal.

Similarly if $B \subseteq A$ then $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$.

Conversely, suppose $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$. We have $A \cup B \in \mathcal{P}(A \cup B)$, so $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$. If $A \cup B \in \mathcal{P}A$, then we have $B \subseteq A \cup B \subseteq A$. And if $A \cup B \in \mathcal{P}B$, then we have $A \subseteq A \cup B \subseteq B$.

Exercise 8 If A is a set such that every singleton belongs to A , then every set belongs to $\bigcup A$, contradicting Theorem 2A.

Exercise 9 Let $a = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$. Then $a \in B$ but $\mathcal{P}a$ is not a subset of B because $\emptyset \in \mathcal{P}a$ and $\emptyset \notin B$.

Exercise 10 We must show that $\mathcal{P}a \subseteq \mathcal{P} \bigcup B$. So let $X \in \mathcal{P}a$. Then $X \subseteq a$; we must show that $X \subseteq \bigcup B$.

Let $x \in X$; we must show that $x \in \bigcup B$. We have $x \in a$ (because $x \in X$ and $X \subseteq a$) and $a \in B$, hence $x \in \bigcup B$ as required.

2.2 Algebra of Sets

Exercise 11 For any x we have

$$\begin{aligned} x \in (A \cap B) \cup (A - B) &\Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B) \\ &\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B) \\ &\Leftrightarrow x \in A \end{aligned}$$

Hence $A = (A \cap B) \cup (A - B)$.

For any x we have

$$\begin{aligned} x \in A \cup (B - A) &\Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A) \\ &\Leftrightarrow x \in A \text{ or } x \in B \\ &\Leftrightarrow x \in A \cup B \end{aligned}$$

Hence $A \cup (B - A) = A \cup B$.

Exercise 12 For any x ,

$$\begin{aligned} x \in C - (A \cap B) &\Leftrightarrow x \in C \& \neg(x \in A \& x \in B) \\ &\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B) \\ &\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B) \\ &\Leftrightarrow x \in (C - A) \cup (C - B) \end{aligned}$$

Exercise 13 Suppose $A \subseteq B$. Let $x \in C - B$; we must show $x \in C - A$. We have $x \in C$ and $x \notin B$. Therefore $x \notin A$, since every member of A is a member of B . And so we have $x \in C - A$ as required.

Exercise 14 Let $A = \{\emptyset\}$, $B = \emptyset$ and $C = \{\emptyset\}$. Then $A - (B - C) = A - \emptyset = \{\emptyset\}$ while $(A - B) - C = \{\emptyset\} - C = \emptyset$.

Exercise 15

(a) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have $x \in A \cap (B + C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$.

(b) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$

In every case, we have $x \in A + (B + C) \Leftrightarrow x \in (A + B) + C$.

Exercise 16

$$\begin{aligned} [(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A] &= (A \cup B) - A \\ &= B - A \end{aligned}$$

Exercise 17

(a) \Leftrightarrow (b)

$A \subseteq B \Leftrightarrow$ Every element of A is an element of B

\Leftrightarrow There is no element of A that is not an element of B

$\Leftrightarrow A - B = \emptyset$

(a) \Rightarrow (c) Suppose $A \subseteq B$. We have $B \subseteq A \cup B$ from the definition of $A \cup B$; we must prove that $A \cup B \subseteq B$. So let $x \in A \cup B$. Then $x \in A$ or $x \in B$. But in either case $x \in B$, since $x \in A \Rightarrow x \in B$. Thus we have $x \in B$ as required.

(c) \Rightarrow (a) We always have $A \subseteq A \cup B$. So if $A \cup B = B$ then we have $A \subseteq B$.

(a) \Rightarrow (d) Suppose $A \subseteq B$. We have $A \cap B \subseteq A$ from the definition of $A \cap B$; we must prove that $A \subseteq A \cap B$. So let $x \in A$. Then $x \in B$ since $A \subseteq B$, hence $x \in A \cap B$ as required.

(d) \Rightarrow (a) We always have $A \cap B \subseteq B$. So if $A \cap B = A$ then $A \subseteq B$.

Exercise 18 We can make the following 16 sets:

- $\emptyset (= A - A)$
- $A - B$
- $A \cap B$
- $B - A$
- $S - (A \cup B)$
- A
- $A + B$
- $S - B$
- B
- $S - (A + B)$
- $S - A$
- $A \cup B$
- $S - (B - A)$
- $S - (A \cap B)$
- $S - (A - B)$

Exercise 19 They are never equal, because for all A, B , we have $\emptyset \in \mathcal{P}(A - B)$ but $\emptyset \notin \mathcal{P}A - \mathcal{P}B$ since $\emptyset \in \mathcal{P}B$.

Exercise 20 Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

We first show $B \subseteq C$. Let $x \in B$; we show $x \in C$. We have $x \in A \cup B = A \cup C$, so either $x \in A$ or $x \in C$. If $x \in C$, we are done. If $x \in A$, then we have $x \in A \cap B = A \cap C$, and so $x \in C$ in this case too.

We can show $C \subseteq B$ similarly. Hence $B = C$.

Exercise 21 For any x , we have

$$\begin{aligned}
 x \in \bigcup (A \cup B) &\Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C \\
 &\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C, \text{ or there exists } C \in B \text{ such that } x \in C \\
 &\Leftrightarrow x \in \bigcup A \cup \bigcup B
 \end{aligned}$$

Exercise 22 For any x , we have

$$\begin{aligned} x \in \bigcap (A \cup B) &\Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C \\ &\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C \\ &\Leftrightarrow x \in \bigcap A \cap \bigcap B \end{aligned}$$

Exercise 23 PROOF:

- $\langle 1 \rangle 1. A \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 1. \text{ LET: } x \in A$
- $\langle 2 \rangle 2. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 3. x \in A \cup X$
- $\langle 1 \rangle 2. \bigcap \mathcal{B} \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 1. \text{ LET: } x \in \bigcap \mathcal{B}$
- $\langle 2 \rangle 2. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 3. x \in X$
- $\langle 2 \rangle 4. x \in A \cup X$
- $\langle 1 \rangle 3. \bigcap \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap \mathcal{B}$
- $\langle 2 \rangle 1. \text{ LET: } x \in \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 2. \text{ ASSUME: } x \notin A$
- PROVE: $x \in \bigcap \mathcal{B}$
- $\langle 2 \rangle 3. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 4. x \in A \cup X$
- $\langle 2 \rangle 5. x \in X$

□

Exercise 24

(a)

$$\begin{aligned} Y \in \mathcal{P} \bigcap \mathcal{A} &\Leftrightarrow Y \subseteq \bigcap \mathcal{A} \\ &\Leftrightarrow \forall y \in Y. \forall X \in \mathcal{A}. y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A}. \forall y \in Y. y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A}. Y \in \mathcal{P}X \\ &\Leftrightarrow Y \in \bigcap \{\mathcal{P}X \mid X \in \mathcal{A}\} \end{aligned}$$

(b) $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} \subseteq \mathcal{P} \bigcup \mathcal{A}$

PROOF:

- $\langle 1 \rangle 1. \text{ LET: } Y \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
- $\langle 1 \rangle 2. \text{ PICK } X \in \mathcal{A} \text{ such that } Y \in \mathcal{P}X$
- $\langle 1 \rangle 3. Y \subseteq X$
- $\langle 1 \rangle 4. Y \subseteq \bigcup \mathcal{A}$
- $\langle 1 \rangle 5. Y \in \mathcal{P} \bigcup \mathcal{A}$

Equality holds if and only if $\bigcup \mathcal{A} \in \mathcal{A}$.

- ⟨1⟩1. If $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$ then $\bigcup \mathcal{A} \in \mathcal{A}$
 - ⟨2⟩1. ASSUME: $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
 - ⟨2⟩2. $\bigcup \mathcal{A} \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
 - ⟨2⟩3. PICK $X \in \mathcal{A}$ such that $\bigcup \mathcal{A} \in \mathcal{P}X$
 - ⟨2⟩4. $X = \bigcup \mathcal{A}$
 - ⟨1⟩2. If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
- PROOF: If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\mathcal{P} \bigcup \mathcal{A} \in \{\mathcal{P}X \mid X \in \mathcal{A}\}$.
 \square

Exercise 25 We have $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ if and only if $A = \emptyset$ or $\mathcal{B} \neq \emptyset$

- ⟨1⟩1. If $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ then $A = \emptyset$ or $\mathcal{B} \neq \emptyset$
- PROOF: If $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ and $\mathcal{B} = \emptyset$ then
- $$A \cup \bigcup \emptyset = \bigcup \emptyset$$
- $$\therefore A = \emptyset$$
- ⟨1⟩2. If $A = \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
- PROOF: Both sides are equal to $\bigcup \mathcal{B}$
- ⟨1⟩3. If $\mathcal{B} \neq \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨2⟩1. ASSUME: $\mathcal{B} \neq \emptyset$
 - ⟨2⟩2. $A \cup \bigcup \mathcal{B} \subseteq \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨3⟩1. LET: $x \in A \cup \bigcup \mathcal{B}$
 - PROVE: $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨3⟩2. CASE: $x \in A$
 - ⟨4⟩1. PICK $X \in \mathcal{B}$
 - PROOF: By ⟨2⟩1
 - ⟨4⟩2. $x \in A \cup X$
 - ⟨3⟩3. CASE: $x \in \bigcup \mathcal{B}$
 - ⟨4⟩1. PICK $X \in \mathcal{B}$ such that $x \in X$
 - ⟨4⟩2. $x \in A \cup X$
 - ⟨2⟩3. $\bigcup \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcup \mathcal{B}$
 - ⟨3⟩1. LET: $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨3⟩2. PICK $X \in \mathcal{B}$ such that $x \in A \cup X$
 - ⟨3⟩3. $X \subseteq \bigcup \mathcal{B}$
 - ⟨3⟩4. $A \cup X \subseteq A \cup \bigcup \mathcal{B}$
 - ⟨3⟩5. $x \in A \cup \bigcup \mathcal{B}$

2.3 Review Exercises

Exercise 26 Sets A , B , D and F are all equal to each other. Sets C , E and G are equal to each other. None of the first list is equal to any of the second list.

Exercise 27 Take $A = \{\{0\}, \{1\}\}$ and $B = \{\{1\}\}$. Then $A \cap B = \{\{1\}\}$ and

$$\begin{aligned}\bigcap A \cap \bigcap B &= \emptyset \cap \{1\} \\ &= \emptyset \\ \bigcap (A \cap B) &= \bigcap \{\{1\}\} \\ &= \{1\}\end{aligned}$$

Exercise 28

$$\bigcup \{\{3, 4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\} = \{3, 4, \{3\}, \{4\}\}$$

Exercise 29

(a) \emptyset

(b) We have

$$\begin{aligned}\{\emptyset\} &\subseteq \mathcal{P}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{PP}\{\emptyset\} \\ \{\emptyset\} &\subseteq \mathcal{PP}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{PPP}\{\emptyset\} \\ \therefore \bigcap \{\mathcal{PPP}\{\emptyset\}, \mathcal{PP}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} &= \mathcal{PPP}\{\emptyset\} \cap \mathcal{PP}\{\emptyset\} \cap \mathcal{P}\{\emptyset\} \\ &= \mathcal{P}\{\emptyset\} \\ &= \{\emptyset, \{\emptyset\}\}\end{aligned}$$

Exercise 30

(a) $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$

(b) $\{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d) $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

Exercise 31

(a) $\{1, 2, 3, \emptyset\}$

(b) \emptyset

(c) \emptyset

(d) \emptyset

Exercise 32

(a) $a \cup b$

(b) a

(c)

$$\begin{aligned} \bigcap \bigcup S \cup (\bigcup \bigcup S - \bigcup \bigcap S) &= (a \cap b) \cup ((a \cup b) - a) \\ &= (a \cap b) \cup (b - a) \\ &= b \end{aligned}$$

Exercise 33 When $a \neq b$:

$$\begin{aligned} \bigcup (\bigcup S - \bigcap S) &= \bigcup (\{a, b\} - \{a\}) \\ &= \bigcup \{b\} \\ &= b \end{aligned}$$

When $a = b$:

$$\begin{aligned} \bigcup (\bigcup S - \bigcap S) &= \bigcup (\{a, b\} - \{a\}) \\ &= \bigcup \emptyset \\ &= \emptyset \end{aligned}$$

Exercise 34 For any set S , we have

$$\begin{aligned} \emptyset &\subseteq \mathcal{P}S \\ \therefore \emptyset &\in \mathcal{P}\mathcal{P}S \\ \emptyset &\subseteq S \\ \therefore \emptyset &\in \mathcal{P}S \\ \therefore \{\emptyset\} &\subseteq \mathcal{P}S \\ \therefore \{\emptyset\} &\in \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} &\subseteq \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} &\in \mathcal{P}\mathcal{P}\mathcal{P}S \end{aligned}$$

Exercise 35 Assume $\mathcal{P}A = \mathcal{P}B$. Then we have

$$\begin{aligned}
 A &\in \mathcal{P}A \\
 \therefore A &\in \mathcal{P}B \\
 \therefore A &\subseteq B \\
 B &\in \mathcal{P}B \\
 \therefore B &\in \mathcal{P}A \\
 \therefore B &\subseteq A \\
 \therefore A &= B
 \end{aligned}$$

Exercise 36

(a)

$$\begin{aligned}
 x \in A - (A \cap B) &\Leftrightarrow x \in A \ \& \neg(x \in A \ \& \ x \in B) \\
 &\Leftrightarrow x \in A \ \& \ x \notin B \\
 &\Leftrightarrow x \in A - B
 \end{aligned}$$

(b)

$$\begin{aligned}
 x \in A - (A - B) &\Leftrightarrow x \in A \ \& \neg(x \in A \ \& \ x \notin B) \\
 &\Leftrightarrow x \in A \ \& \ x \in B \\
 &\Leftrightarrow x \in A \cap B
 \end{aligned}$$

Exercise 37

(a)

$$\begin{aligned}
 x \in (A \cup B) - C &\Leftrightarrow (x \in A \text{ or } x \in B) \ \& \ x \notin C \\
 &\Leftrightarrow (x \in A \ \& \ x \notin C) \text{ or } (x \in B \ \& \ x \notin C) \\
 &\Leftrightarrow x \in (A - C) \cup (B - C)
 \end{aligned}$$

(b)

$$\begin{aligned}
 x \in A - (B - C) &\Leftrightarrow x \in A \ \& \neg(x \in B \ \& \ x \notin C) \\
 &\Leftrightarrow x \in A \ \& \ (x \notin B \text{ or } x \in C) \\
 &\Leftrightarrow (x \in A \ \& \ x \notin B) \text{ or } (x \in A \ \& \ x \in C) \\
 &\Leftrightarrow x \in (A - B) \cup (A \cap C)
 \end{aligned}$$

(c)

$$\begin{aligned}
 x \in (A - B) - C &\Leftrightarrow x \in A \ \& \ x \notin B \ \& \ x \notin C \\
 &\Leftrightarrow x \in A \ \& \neg(x \in B \vee x \in C) \\
 &\Leftrightarrow x \in A - (B \cup C)
 \end{aligned}$$

Exercise 38

(a) If every element of A is an element of C , and every element of B is an element of C , then everything that is an element of either A or B is an element of C .

(b) If every element of C is an element of A , and every element of C is an element of B , then every element of C is an element of both A and B .

Chapter 3

Chapter 3 — Relations and Functions

3.1 Ordered Pairs

Exercise 1 We have $\langle 0, 1, 0 \rangle^* = \langle 0, 1, 1 \rangle^* = \{\{0\}, \{0, 1\}\}$.

Exercise 2

(a)

$$\begin{aligned} z &\in A \times (B \cup C) \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ (y \in B \text{ or } y \in C)) \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in B) \text{ or } (z = (x, y) \ \& \ x \in A \ \& \ y \in C) \\ \Leftrightarrow z &\in (A \times B) \cup (A \times C) \end{aligned}$$

(b)

$\langle 1 \rangle 1$. ASSUME: $A \times B = A \times C$ and $A \neq \emptyset$

$\langle 1 \rangle 2$. PICK $a \in A$

$\langle 1 \rangle 3$. For all x , $x \in B \Leftrightarrow x \in C$

PROOF: $x \in B$ iff $(a, x) \in A \times B$ iff $(a, x) \in A \times C$ iff $x \in C$.

□

Exercise 3

$$\begin{aligned} z &\in A \times \bigcup \mathcal{B} \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ \exists X \in \mathcal{B}. y \in X) \\ \Leftrightarrow \exists X \in \mathcal{B}. \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in X) \\ \Leftrightarrow z &\in \bigcup \{A \times X : X \in \mathcal{B}\} \end{aligned}$$

Exercise 4 If every ordered pair belongs to A then every set belongs to $\bigcup\bigcup A$ contradicting Theorem 2A.

Exercise 5

(a) Apply a Subset Axiom to $\mathcal{P}(A \times B)$: we have $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A. z = \{x\} \times B\}$.

(b)

$$\begin{aligned} z &\in \bigcup C \\ \Leftrightarrow \exists x \in A. z &\in \{x\} \times B \\ \Leftrightarrow \exists x \in A. \exists y \in B. z &= (x, y) \\ \Leftrightarrow z &\in A \times B \end{aligned}$$

3.2 Relations

Exercise 6 If $A \subseteq \text{dom } A \times \text{ran } A$ then A is a set of ordered pairs, i.e. a relation.

Conversely, suppose A is a relation. Let $z \in A$. Then z is an ordered pair; let $z = (x, y)$. We have $x \in \text{dom } A$ and $y \in \text{ran } A$ and so $z \in \text{dom } A \times \text{ran } A$ as required.

Exercise 7 We have $\text{fld } R \subseteq \bigcup\bigcup R$ by Lemma 3D.

Conversely, let $x \in \bigcup\bigcup R$. Pick a and b such that $x \in a$, $a \in b$ and $b \in R$. Then b is an ordered pair; let $b = (y, z)$. We have $a = \{y\}$ or $\{y, z\}$, hence $x = y$ or $x = z$. In either case, $x \in \text{fld } R$.

Exercise 8

(a)

$$\begin{aligned} x &\in \text{dom } \bigcup \mathcal{A} \\ \Leftrightarrow \exists y. \exists R \in \mathcal{A}. (x, y) &\in R \\ \Leftrightarrow \exists R \in \mathcal{A}. \exists y. (x, y) &\in R \\ \Leftrightarrow x &\in \bigcup \{\text{dom } R : R \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned}
y &\in \text{ran} \bigcup \mathcal{A} \\
&\Leftrightarrow \exists x. \exists R \in \mathcal{A}. (x, y) \in R \\
&\Leftrightarrow \exists R \in \mathcal{A}. \exists x. (x, y) \in R \\
&\Leftrightarrow y \in \bigcup \{\text{ran } R : R \in \mathcal{A}\}
\end{aligned}$$

Exercise 9 Assume \mathcal{A} is nonempty. We have $\text{dom} \bigcap \mathcal{A} \subseteq \bigcap \{\text{dom } R : R \in \mathcal{A}\}$.

PROOF:

$$\begin{aligned}
x &\in \text{dom} \bigcap \mathcal{A} \\
&\Leftrightarrow \exists y. \forall R \in \mathcal{A}. (x, y) \in R \\
&\Rightarrow \forall R \in \mathcal{A}. \exists y. (x, y) \in R \\
&\Leftrightarrow x \in \bigcap \{\text{dom } R : R \in \mathcal{A}\}
\end{aligned}$$

Equality holds iff the middle ' \Rightarrow ' can be reversed, i.e. iff for all x , if $\forall R \in \mathcal{A}. \exists y. (x, y) \in R$ then $\exists y. \forall R \in \mathcal{A}. (x, y) \in R$. I haven't found a simpler condition than this. The condition does not always hold, for example if $\mathcal{A} = \{\{(1, 2)\}, \{(1, 3)\}\}$ then $\text{dom} \bigcap \mathcal{A} = \emptyset$ while $\bigcap \{\text{dom } R : R \in \mathcal{A}\} = \{1\}$.

Similarly, $\text{ran} \bigcap \mathcal{A} \subseteq \bigcap \{\text{ran } R : R \in \mathcal{A}\}$, and equality holds iff, for any y , if $\forall R \in \mathcal{A}. \exists x. (x, y) \in R$ then $\exists x. \forall R \in \mathcal{A}. (x, y) \in R$.

3.3 n -ary Relations

Exercise 10 This follows from the equations at the top of page 42. An ordered 4-tuple $\langle a, b, c, d \rangle$ is also an ordered 1-tuple (because every set is), and the ordered pair $\langle \langle a, b, c \rangle, d \rangle$, and the ordered triple $\langle \langle a, b \rangle, c, d \rangle$.

3.4 Functions

Exercise 11 We prove $F \subseteq G$. Let $z \in F$. Since F is a relation, then z is an ordered pair; let $z = \langle x, y \rangle$. We have $x \in \text{dom } F$ and $y = F(x)$. Therefore $x \in \text{dom } G$ and $y = G(x)$ (because $\text{dom } F = \text{dom } G$ and $F(x) = G(x)$). Hence $\langle x, y \rangle \in G$, i.e. $z \in G$.

We have proved $F \subseteq G$. We can prove $G \subseteq F$ similarly. Thus $F = G$.

Exercise 12 PROOF:

- $\langle 1 \rangle 1$. If $f \subseteq g$ then $\text{dom } f \subseteq \text{dom } g$ and $\forall x \in \text{dom } f. f(x) = g(x)$
- $\langle 2 \rangle 1$. ASSUME: $f \subseteq g$
- $\langle 2 \rangle 2$. LET: $x \in \text{dom } f$
- $\langle 2 \rangle 3$. $(x, f(x)) \in f$
- $\langle 2 \rangle 4$. $(x, f(x)) \in g$
- $\langle 2 \rangle 5$. $x \in \text{dom } g$ and $g(x) = f(x)$

- ⟨1⟩2. If $\text{dom } f = \text{dom } g$ and $\forall x \in \text{dom } f. f(x) = g(x)$ then $f \subseteq g$
- ⟨2⟩1. ASSUME: $\text{dom } f = \text{dom } g$ and $\forall x \in \text{dom } f. f(x) = g(x)$
- ⟨2⟩2. LET: $z \in f$
- ⟨2⟩3. LET: $z = (x, y)$
- ⟨2⟩4. $x \in \text{dom } f$ and $y = f(x)$
- ⟨2⟩5. $x \in \text{dom } g$ and $y = g(x)$
- ⟨2⟩6. $z = (x, y) \in g$

□

Exercise 13 PROOF:

- ⟨1⟩1. ASSUME: f and g are functions
- ⟨1⟩2. ASSUME: $f \subseteq g$
- ⟨1⟩3. ASSUME: $\text{dom } g \subseteq \text{dom } f$
- ⟨1⟩4. $\text{dom } f = \text{dom } g$
- PROOF: We have $\text{dom } f \subseteq \text{dom } g$ from ⟨1⟩2 and $\text{dom } g \subseteq \text{dom } f$ from ⟨1⟩3
- ⟨1⟩5. For $x \in \text{dom } f$ we have $f(x) = g(x)$
- PROOF: From ⟨1⟩2 and Exercise 12
- ⟨1⟩6. Q.E.D.
- PROOF: From Exercise 11.

□

Exercise 14

(a) If (x, y) and (x, z) are members of $f \cap g$ then they are both members of f , hence $y = z$.

(b) PROOF:

- ⟨1⟩1. If $f \cup g$ is a function then, for all $x \in \text{dom } f \cap \text{dom } g$, we have $f(x) = g(x)$.
- ⟨2⟩1. ASSUME: $f \cup g$ is a function.
- ⟨2⟩2. LET: $x \in \text{dom } f \cap \text{dom } g$
- ⟨2⟩3. $(x, f(x))$ and $(x, g(x))$ are both elements of $f \cup g$
- ⟨2⟩4. $f(x) = g(x)$
- ⟨1⟩2. If, for all $x \in \text{dom } f \cap \text{dom } g$, we have $f(x) = g(x)$, then $f \cup g$ is a function.
- ⟨2⟩1. ASSUME: For all $x \in \text{dom } f \cap \text{dom } g$, we have $f(x) = g(x)$
- ⟨2⟩2. $f \cup g$ is a relation.
- PROOF: Since every element of either f or g is an ordered pair.
- ⟨2⟩3. Whenever (x, y) and (x, z) are elements of $f \cup g$ we have $y = z$
- ⟨3⟩1. LET: $(x, y), (x, z) \in f \cup g$
- ⟨3⟩2. CASE: $(x, y), (x, z) \in f$
- PROOF: Then $y = z$ since f is a function.
- ⟨3⟩3. CASE: $(x, y) \in f, (x, z) \in g$
- PROOF: Then $y = z$ by ⟨2⟩1
- ⟨3⟩4. CASE: $(x, y) \in g, (x, z) \in f$
- PROOF: Then $y = z$ by ⟨2⟩1
- ⟨3⟩5. CASE: $(x, y), (x, z) \in g$

PROOF: Then $y = z$ since g is a function.

□

Exercise 15 PROOF:

⟨1⟩1. $\bigcup \mathcal{A}$ is a relation.

PROOF: Since every member of \mathcal{A} is a relation.

⟨1⟩2. Whenever (x, y) and (x, z) are elements of $\bigcup \mathcal{A}$ then $y = z$

⟨2⟩1. LET: $(x, y), (x, z) \in \bigcup \mathcal{A}$

⟨2⟩2. PICK $f, g \in \mathcal{A}$ such that $(x, y) \in f$ and $(x, z) \in g$

⟨2⟩3. ASSUME: w.l.o.g. $f \subseteq g$

⟨2⟩4. $(x, y), (x, z) \in g$

⟨2⟩5. $y = z$

PROOF: Since g is a function.

□

Exercise 16 If every function belongs to \mathcal{A} then every set belongs to $\text{dom} \bigcup \mathcal{A}$ contradiction Theorem 2A.

Exercise 17 PROOF:

⟨1⟩1. LET: R and S be single-rooted.

⟨1⟩2. LET: $(x, z), (y, z) \in R \circ S$

⟨1⟩3. PICK t and t' such that $(x, t) \in S$, $(t, z) \in R$, $(y, t') \in S$ and $(t', z) \in R$

⟨1⟩4. $t = t'$

PROOF: Since R is single-rooted.

⟨1⟩5. $x = y$

PROOF: Since S is single-rooted.

Thus if F and G are one-to-one functions then $F \circ G$ is single-rooted and a function by Theorem 3H, hence a one-to-one function.

Exercise 18

$$R \circ R = \{\langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle\}$$

$$R \upharpoonright \{1\} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle\}$$

$$R^{-1} \upharpoonright \{1\} = \{\langle 1, 0 \rangle\}$$

$$R[\{1\}] = \{2, 3\}$$

$$R^{-1}[\{1\}] = \{0\}$$

Exercise 19

$$\begin{aligned}
A(\emptyset) &= \{\emptyset, \{\emptyset\}\} \\
A[\emptyset] &= \emptyset \\
A[\{\emptyset\}] &= \{\{\emptyset, \{\emptyset\}\}\} \\
A[\{\emptyset, \{\emptyset\}\}] &= \{\{\emptyset, \{\emptyset\}\}, \emptyset\} \\
A^{-1} &= \{\langle \{\emptyset, \{\emptyset\}\}, \emptyset \rangle, \langle \emptyset, \{\emptyset\} \rangle\} \\
A \circ A &= \{\langle \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \rangle\} \\
A \upharpoonright \emptyset &= \emptyset \\
A \upharpoonright \{\emptyset\} &= \{\langle \emptyset, \{\emptyset, \{\emptyset\}\} \rangle\} \\
A \upharpoonright \{\emptyset, \{\emptyset\}\} &= \{\langle \emptyset, \{\emptyset, \{\emptyset\}\} \rangle, \langle \{\emptyset\}, \emptyset \rangle\} \\
&= A \\
\bigcup A &= \{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}
\end{aligned}$$

Exercise 20

$$\begin{aligned}
z \in F \upharpoonright A &\Leftrightarrow z \in F \ \& \ \exists x, y. (z = \langle x, y \rangle \ \& \ x \in A) \\
&\Leftrightarrow z \in F \ \& \ \exists x, y. (z = \langle x, y \rangle \ \& \ x \in A \ \& \ y \in \text{ran } F) \\
&\Leftrightarrow z \in F \cap (A \times \text{ran } F)
\end{aligned}$$

Exercise 21 Both are equal to $\{\langle x, w \rangle \mid \exists y, z. xTy \ \& \ ySz \ \& \ zRw\}$.

Exercise 22

(a) PROOF:
 $\langle 1 \rangle 1$. ASSUME: $A \subseteq B$
 $\langle 1 \rangle 2$. LET: $y \in F[A]$
 $\langle 1 \rangle 3$. PICK $x \in A$ such that xFy
 $\langle 1 \rangle 4$. $x \in B$ and xFy
 \square

(b) Both are equal to $\{z : \exists x, y. x \in A \ \& \ xGy \ \& \ yFz\}$

(c) Both are equal to $\{\langle x, y \rangle : (x \in A \text{ or } x \in B) \ \& \ xQy\}$

Exercise 23

$$\begin{aligned}
B \circ I_A &= \{\langle x, z \rangle : \exists y(xI_A y \ \& \ yBz)\} \\
&= \{\langle x, z \rangle : \exists y(x \in A \ \& \ x = y \ \& \ yBz)\} \\
&= \{\langle x, z \rangle : x \in A \ \& \ xBz\} \\
&= B \upharpoonright A \\
I_A[C] &= \{y : \exists x \in C.xI_A y\} \\
&= \{y : \exists x \in C(x \in A \ \& \ x = y)\} \\
&= \{y : y \in C \ \& \ y \in A\} \\
&= A \cap C
\end{aligned}$$

Exercise 24

$$\begin{aligned}
F^{-1}[A] &= \{x : \exists y \in A.yF^{-1}x\} \\
&= \{x : \exists y \in A.xFy\} \\
&= \{x \in \text{dom } F : F(x) \in A\}
\end{aligned}$$

Exercise 25**(a)** PROOF: $\langle 1 \rangle 1.$ LET: G be a one-to-one function. $\langle 1 \rangle 2.$ G^{-1} is a function.

PROOF: Theorem 3F.

 $\langle 1 \rangle 3.$ $G \circ G^{-1}$ is a function.

PROOF: Theorem 3H.

 $\langle 1 \rangle 4.$ $\text{dom}(G \circ G^{-1}) = \text{ran } G$

PROOF:

$$\text{dom}(G \circ G^{-1}) = \{x \in \text{dom } G^{-1} : G^{-1}(x) \in \text{dom } G\} \quad (\text{Theorem 3H})$$

$$= \{x \in \text{ran } G : G^{-1}(x) \in \text{dom } G\} \quad (\text{Theorem 3E})$$

$$= \text{ran } G$$

 $\langle 1 \rangle 5.$ $\forall x \in \text{ran } G.(G \circ G^{-1})(x) = x$

PROOF: Theorem 3G.

□

(b) Let G be a function. Then

$$\begin{aligned}
G \circ G^{-1} &= \{\langle x, z \rangle : \exists y(xG^{-1}y \ \& \ yGz)\} \\
&= \{\langle x, z \rangle : \exists y(yGx \ \& \ yGz)\} \\
&= \{\langle x, x \rangle : \exists y.yGx\} & (G \text{ is a function}) \\
&= I_{\text{ran } G}
\end{aligned}$$

Exercise 26

(a)

$$\begin{aligned} F[\bigcup \mathcal{A}] &= \{y : \exists x. \exists A \in \mathcal{A} (x \in A \ \& \ xFy)\} \\ &= \{y : \exists A \in \mathcal{A}. \exists x (x \in A \ \& \ xFy)\} \\ &= \bigcup \{F[A] : A \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned} F[\bigcup \mathcal{A}] &= \{y : \exists x. \forall A \in \mathcal{A} (x \in A \ \& \ xFy)\} \\ &\subseteq \{y : \forall A \in \mathcal{A}. \exists x (x \in A \ \& \ xFy)\} \\ &= \bigcap \{F[A] : A \in \mathcal{A}\} \end{aligned}$$

Exercise 27

$$\begin{aligned} \text{dom}(F \circ G) &= \{x : \exists y. x(F \circ G)y\} \\ &= \{x : \exists y \exists z (xGz \ \& \ zFy)\} \\ &= \{x : \exists z (zG^{-1}x \ \& \ z \in \text{dom } F)\} \\ &= G^{-1}[\text{dom } F] \end{aligned}$$

Exercise 28 PROOF:

$\langle 1 \rangle 1.$ $G : \mathcal{P}A \rightarrow \mathcal{P}B$

PROOF: Since $f[X] \subseteq \text{ran } f \subseteq B$

$\langle 1 \rangle 2.$ For all $X, Y \in \mathcal{P}A$, if $G(X) = G(Y)$ then $X = Y$

$\langle 2 \rangle 1.$ LET: $X, Y \in \mathcal{P}A$

$\langle 2 \rangle 2.$ ASSUME: $f[X] = f[Y]$

$\langle 2 \rangle 3.$ $X \subseteq Y$

$\langle 3 \rangle 1.$ LET: $x \in X$

$\langle 3 \rangle 2.$ $f(x) \in f[X]$

$\langle 3 \rangle 3.$ $f(x) \in f[Y]$

$\langle 3 \rangle 4.$ PICK $y \in Y$ such that $f(x) = f(y)$

$\langle 3 \rangle 5.$ $x = y$

PROOF: Because f is one-to-one.

$\langle 3 \rangle 6.$ $x \in Y$

PROOF: Similar.

$\langle 2 \rangle 4.$ $Y \subseteq X$

□

Example 29 PROOF:

$\langle 1 \rangle 1.$ ASSUME: f maps A onto B

$\langle 1 \rangle 2.$ LET: $b, b' \in B$

$\langle 1 \rangle 3.$ ASSUME: $G(b) = G(b')$

$\langle 1 \rangle 4.$ PICK $x \in A$ such that $f(x) = b$

PROOF: By $\langle 1 \rangle 1$.

$\langle 1 \rangle 5$. $x \in G(b)$

$\langle 1 \rangle 6$. $x \in G(b')$

$\langle 1 \rangle 7$. $f(x) = b'$

$\langle 1 \rangle 8$. $b = b'$

□

The converse does not hold. Let $A = \{0\}$ and $B = \{0, 1\}$. Let f be the function that maps 0 to 0. Then

$$G(0) = \{0\}$$

$$G(1) = \emptyset$$

Thus G is one-to-one but f does not map A onto B .

Exercise 30

(a) PROOF:

$\langle 1 \rangle 1$. $F(B) = B$

$\langle 2 \rangle 1$. $F(B) \subseteq B$

$\langle 3 \rangle 1$. LET: $X \in \mathcal{P}A$ be such that $F(X) \subseteq X$

PROVE: $F(B) \subseteq X$

$\langle 3 \rangle 2$. $B \subseteq X$

$\langle 3 \rangle 3$. $F(B) \subseteq F(X)$

$\langle 3 \rangle 4$. $F(B) \subseteq X$

PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$.

$\langle 2 \rangle 2$. $B \subseteq F(B)$

PROOF: From $\langle 2 \rangle 1$ and the definition of B , since B is one of the sets X such that $F(X) \subseteq X$

$\langle 1 \rangle 2$. $F(C) = C$

$\langle 2 \rangle 1$. $C \subseteq F(C)$

$\langle 3 \rangle 1$. LET: $X \in \mathcal{P}A$ with $X \subseteq F(X)$

PROVE: $X \subseteq F(C)$

$\langle 3 \rangle 2$. $X \subseteq C$

$\langle 3 \rangle 3$. $F(X) \subseteq F(C)$

$\langle 3 \rangle 4$. $X \subseteq F(C)$

PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$

$\langle 2 \rangle 2$. $F(C) \subseteq C$

PROOF: From $\langle 2 \rangle 1$ and the definition of C .

□

(b) If $F(X) = X$ then we have $B \subseteq X$ (because $F(X) \subseteq X$) and $X \subseteq C$ (because $X \subseteq F(X)$).

3.5 Infinite Cartesian Products

Exercise 31 PROOF:

- ⟨1⟩1. If the Axiom of Choice is true then, for any set I and any function H with domain I , if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$.
- ⟨2⟩1. ASSUME: The Axiom of Choice.
- ⟨2⟩2. LET: I be a set.
- ⟨2⟩3. LET: H be a function with domain I .
- ⟨2⟩4. ASSUME: $H(i) \neq \emptyset$ for all $i \in I$.
- ⟨2⟩5. LET: $R = \{(i, x) : i \in I, x \in H(i)\}$
- ⟨2⟩6. PICK a function $F \subseteq R$ with $\text{dom } F = \text{dom } R$
PROVE: $F \in \prod_{i \in I} H(i)$
PROOF: By the Axiom of Choice.
- ⟨2⟩7. $\text{dom } H = I$
PROOF: We have $\text{dom } R = I$ since for all $i \in I$ there exists x such that $x \in H(i)$.
- ⟨2⟩8. $\forall i \in I. F(i) \in H(i)$
PROOF: Since $iRF(i)$.
- ⟨1⟩2. If, for any set I and any function H with domain I , if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$, then the Axiom of Choice is true.
- ⟨2⟩1. ASSUME: For any set I and any function H with domain I , if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$
- ⟨2⟩2. LET: R be a relation
- ⟨2⟩3. LET: $I = \text{dom } R$
- ⟨2⟩4. Define the function H with domain I by: for $i \in I$, $H(i) = \{y : iRy\}$
- ⟨2⟩5. $H(i) \neq \emptyset$ for all $i \in I$
- ⟨2⟩6. PICK $F \in \prod_{i \in I} H(i)$
PROOF: By ⟨2⟩1
- ⟨2⟩7. F is a function
- ⟨2⟩8. $F \subseteq R$
PROOF: For all $i \in I$ we have $F(i) \in H(i)$ and so $iRF(i)$.
- ⟨2⟩9. $\text{dom } F = \text{dom } R$
-

3.6 Equivalence Relations

Exercise 32

(a)

$$\begin{aligned}
 & R \text{ is symmetric} \\
 \Leftrightarrow & \forall x, y (xRy \Rightarrow yRx) \\
 \Leftrightarrow & \forall x, y (\langle y, x \rangle \in R^{-1} \Rightarrow \langle y, x \rangle \in R) \\
 \Leftrightarrow & R^{-1} \subseteq R
 \end{aligned}$$

(b)

$$\begin{aligned}
& R \text{ is transitive} \\
& \Leftrightarrow \forall x, y, z (xRy \ \& \ yRz \Rightarrow xRz) \\
& \Leftrightarrow \forall x, z (\exists y (xRy \ \& \ yRz) \Rightarrow xRz) \\
& \Leftrightarrow \forall x, z (\langle x, z \rangle \in R \circ R \Rightarrow \langle x, z \rangle \in R) \\
& \Leftrightarrow R \circ R \subseteq R
\end{aligned}$$

Exercise 33 PROOF:

$\langle 1 \rangle 1$. If R is a symmetric and transitive relation then $R = R^{-1} \circ R$.

$\langle 2 \rangle 1$. ASSUME: R is a symmetric and transitive relation.

$\langle 2 \rangle 2$. $R \subseteq R^{-1} \circ R$

$\langle 3 \rangle 1$. LET: xRy

$\langle 3 \rangle 2$. yRy

PROOF: By Theorem 3M.

$\langle 3 \rangle 3$. xRy and $yR^{-1}y$

$\langle 3 \rangle 4$. $x(R^{-1} \circ R)y$

$\langle 2 \rangle 3$. $R^{-1} \circ R \subseteq R$

PROOF:

$$R^{-1} \circ R \subseteq R \circ R \quad (\text{Exercise 32(a)})$$

$$\subseteq R \quad (\text{Exercise 32(b)})$$

$\langle 1 \rangle 2$. If $R = R^{-1} \circ R$ then R is a symmetric and transitive relation.

$\langle 2 \rangle 1$. ASSUME: $R = R^{-1} \circ R$

$\langle 2 \rangle 2$. R is a relation.

$\langle 2 \rangle 3$. R is symmetric.

$\langle 3 \rangle 1$. LET: xRy

$\langle 3 \rangle 2$. PICK z such that xRz and $zR^{-1}y$

$\langle 3 \rangle 3$. yRz and $zR^{-1}x$

$\langle 3 \rangle 4$. $y(R^{-1} \circ R)x$

$\langle 3 \rangle 5$. yRx

$\langle 2 \rangle 4$. R is transitive.

$\langle 3 \rangle 1$. LET: xRy and yRz

$\langle 3 \rangle 2$. zRy

PROOF: By $\langle 2 \rangle 3$

$\langle 3 \rangle 3$. xRy and $yR^{-1}z$

$\langle 3 \rangle 4$. $x(R^{-1} \circ R)z$

$\langle 3 \rangle 5$. xRz

□

Exercise 34

(a) $\bigcap \mathcal{A}$ is a transitive relation.

PROOF:

$\langle 1 \rangle 1$. $\bigcap \mathcal{A}$ is a relation.

PROOF: Every member of a member of \mathcal{A} is an ordered pair.

$\langle 1 \rangle 2$. $\bigcap \mathcal{A}$ is transitive.

$\langle 2 \rangle 1$. LET: $\langle x, y \rangle$ and $\langle y, z \rangle$ be in $\bigcap \mathcal{A}$

PROVE: $\langle x, z \rangle \in \bigcap \mathcal{A}$

$\langle 2 \rangle 2$. LET: $R \in \mathcal{A}$

$\langle 2 \rangle 3$. xRy and yRz

$\langle 2 \rangle 4$. xRz

PROOF: Since R is transitive.

□

(b) Not necessarily. If $\mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$ then each member of \mathcal{A} is transitive but $\bigcup \mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$ is not.

Example 35

$$\begin{aligned} R[\{x\}] &= \{y : \exists z(z \in \{x\} \ \& \ zRy)\} \\ &= \{y : \exists z(z = x \ \& \ zRy)\} \\ &= \{y : xRy\} \\ &= [x]_R \end{aligned}$$

Example 36 PROOF:

$\langle 1 \rangle 1$. Q is a relation on A .

PROOF: By definition.

$\langle 1 \rangle 2$. Q is reflexive on A .

$\langle 2 \rangle 1$. LET: $x \in A$

$\langle 2 \rangle 2$. $f(x)Rf(x)$

PROOF: Since R is reflexive on B .

$\langle 2 \rangle 3$. xQx

$\langle 1 \rangle 3$. Q is symmetric.

$\langle 2 \rangle 1$. ASSUME: xQy

$\langle 2 \rangle 2$. $f(x)Rf(y)$

$\langle 2 \rangle 3$. $f(y)Rf(x)$

PROOF: R is symmetric.

$\langle 2 \rangle 4$. yQx

$\langle 1 \rangle 4$. Q is transitive.

$\langle 2 \rangle 1$. ASSUME: xQy and yQz

$\langle 2 \rangle 2$. $f(x)Rf(y)$ and $f(y)Rf(z)$

$\langle 2 \rangle 3$. $f(x)Rf(z)$

PROOF: R is transitive.

$\langle 2 \rangle 4$. xQz

□

Exercise 37 PROOF:

$\langle 1 \rangle 1$. R_Π is a relation on A .

PROOF: If $B \in \Pi$, $x \in B$ and $y \in B$ then $x, y \in A$.

$\langle 1 \rangle 2$. R_Π is reflexive on A .

$\langle 2 \rangle 1$. LET: $x \in A$

$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$

PROOF: Because Π is exhaustive.

$\langle 2 \rangle 3$. $x \in B$ and $x \in B$

$\langle 2 \rangle 4$. $xR_\Pi x$

$\langle 1 \rangle 3$. R_Π is symmetric.

$\langle 2 \rangle 1$. ASSUME: $xR_\Pi y$

$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$

$\langle 2 \rangle 3$. $y \in B$ and $x \in B$

$\langle 2 \rangle 4$. $yR_\Pi x$

$\langle 1 \rangle 4$. R_Π is transitive.

$\langle 2 \rangle 1$. ASSUME: $xR_\Pi y$ and $yR_\Pi z$

$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$

$\langle 2 \rangle 3$. PICK $C \in \Pi$ such that $y \in C$ and $z \in C$

$\langle 2 \rangle 4$. $B = C$

PROOF: Since $y \in B$ and $y \in C$

$\langle 2 \rangle 5$. $x \in B$ and $z \in B$

$\langle 2 \rangle 6$. $xR_\Pi z$

□

Exercise 38 PROOF:

$\langle 1 \rangle 1$. If $B \in \Pi$ and $x \in B$ then $B = [x]_{R_\Pi}$

$\langle 2 \rangle 1$. LET: $B \in \Pi$

$\langle 2 \rangle 2$. LET: $x \in B$

$\langle 2 \rangle 3$. $[x]_{R_\Pi} \subseteq B$

$\langle 3 \rangle 1$. LET: $y \in [x]_{R_\Pi}$

$\langle 3 \rangle 2$. $xR_\Pi y$

$\langle 3 \rangle 3$. PICK $C \in \Pi$ such that $x \in C$ and $y \in C$

$\langle 3 \rangle 4$. $B = C$

PROOF: Since $x \in B$ and $x \in C$.

$\langle 3 \rangle 5$. $y \in B$

$\langle 2 \rangle 4$. $B \subseteq [x]_{R_\Pi}$

PROOF: For all $y \in B$, we have $x \in B$ and $y \in B$ hence $xR_\Pi y$.

$\langle 1 \rangle 2$. $A/R_\Pi \subseteq \Pi$

$\langle 2 \rangle 1$. LET: $x \in A$

PROVE: $[x]_{R_\Pi} \in \Pi$

$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$

$\langle 2 \rangle 3$. $[x]_{R_\Pi} = B$

PROOF: By $\langle 1 \rangle 1$

$\langle 2 \rangle 4$. $[x]_{R_\Pi} \in \Pi$

$\langle 1 \rangle 3$. $\Pi \subseteq A/R_\Pi$

$\langle 2 \rangle 1$. LET: $B \in \Pi$

$\langle 2 \rangle 2$. PICK $x \in B$

PROOF: Since every member of Π is nonempty.

$\langle 2 \rangle 3$. $B = [x]_{R_\Pi}$

PROOF: By $\langle 1 \rangle 1$.

$\langle 2 \rangle 4$. $B \in A/R_\Pi$

□

Exercise 39 PROOF:

$\langle 1 \rangle 1$. $R_\Pi \subseteq R$

$\langle 2 \rangle 1$. LET: $xR_\Pi y$

$\langle 2 \rangle 2$. PICK $B \in \Pi$ such that $x \in B$ and $y \in B$

$\langle 2 \rangle 3$. PICK $z \in A$ such that $B = [z]_R$

$\langle 2 \rangle 4$. zRx

$\langle 2 \rangle 5$. zRy

$\langle 2 \rangle 6$. xRy

PROOF: Since R is symmetric and transitive.

$\langle 1 \rangle 2$. $R \subseteq R_\Pi$

$\langle 2 \rangle 1$. LET: xRy

$\langle 2 \rangle 2$. $x \in [x]_R$

$\langle 2 \rangle 3$. $y \in [x]_R$

$\langle 2 \rangle 4$. $xR_\Pi y$

□

Exercise 40 We have $[2]_R = [3]_R$ but $[6]_R \neq [9]_R$ so there is no such function f .

Exercise 41

(a) PROOF:

$\langle 1 \rangle 1$. Q is reflexive on $\mathbb{R} \times \mathbb{R}$.

PROOF: For any $x, y \in \mathbb{R}$, we have $x + y = x + y$, hence $\langle x, y \rangle Q \langle x, y \rangle$

$\langle 1 \rangle 2$. Q is symmetric.

$\langle 2 \rangle 1$. ASSUME: $\langle u, v \rangle Q \langle x, y \rangle$

$\langle 2 \rangle 2$. $u + y = x + v$

$\langle 2 \rangle 3$. $x + v = u + y$

$\langle 2 \rangle 4$. $\langle x, y \rangle Q \langle u, v \rangle$

$\langle 1 \rangle 3$. Q is transitive.

$\langle 2 \rangle 1$. ASSUME: $\langle a, b \rangle Q \langle u, v \rangle$ and $\langle u, v \rangle Q \langle x, y \rangle$

$\langle 2 \rangle 2$. $a + v = u + b$

$\langle 2 \rangle 3$. $u + y = x + v$

$\langle 2 \rangle 4$. $a + y + x + b$

PROOF: Adding $\langle 2 \rangle 2$ and $\langle 2 \rangle 3$ gives $a + u + v + y = b + u + v + x$.

$\langle 2 \rangle 5$. $\langle a, b \rangle Q \langle x, y \rangle$

□

(b) We prove that, if $\langle u, v \rangle Q \langle x, y \rangle$ then $\langle u + 2v, v + 2u \rangle Q \langle x + 2y, y + 2x \rangle$. It follows from Theorem 3Q that the function G exists.

If $u + y = v + x$ then $u + 2v + y + 2x = v + 2u + x + 2y$ by adding $u + v + y + x$ to both sides.

Exercise 42 Assume that R is an equivalence relation on A and that $F : A \times A \rightarrow A$. Let us say that F is *compatible* with R iff, whenever xRx' and yRy' , then $F(\langle x, y \rangle)RF(\langle x', y' \rangle)$. If F is compatible with R then there exists a unique $\hat{F} : (A/R) \times (A/R) \rightarrow A/R$ such that

$$\hat{F}(\langle [x]_R, [y]_R \rangle) = [F(\langle x, y \rangle)]_R \text{ for all } x, y \in A.$$

If F is not compatible with R then no such \hat{F} exists.

3.7 Ordering Relations

Exercise 43 PROOF:

- $\langle 1 \rangle 1.$ R^{-1} is transitive.
- $\langle 2 \rangle 1.$ ASSUME: $xR^{-1}y$ and $yR^{-1}z$
- $\langle 2 \rangle 2.$ zRy and yRx
- $\langle 2 \rangle 3.$ zRx
- PROOF: Since R is transitive.
- $\langle 2 \rangle 4.$ $xR^{-1}z$
- $\langle 1 \rangle 2.$ R^{-1} satisfies trichotomy on A .
- $\langle 2 \rangle 1.$ LET: $x, y \in A$
- $\langle 2 \rangle 2.$ Exactly one of xRy , $x = y$, yRx holds.
- $\langle 2 \rangle 3.$ Exactly one of $yR^{-1}x$, $x = y$, $xR^{-1}y$ holds.

□

Exercise 44 PROOF:

- $\langle 1 \rangle 1.$ f is one-to-one.
- $\langle 2 \rangle 1.$ LET: $x, y \in A$ with $f(x) = f(y)$
- $\langle 2 \rangle 2.$ $f(x) < f(y)$ and $f(y) < f(x)$ do not hold.
- PROOF: By trichotomy.
- $\langle 2 \rangle 3.$ $x < y$ and $y < x$ do not hold.
- $\langle 2 \rangle 4.$ $x = y$
- PROOF: By trichotomy.
- $\langle 1 \rangle 2.$ Whenever $f(x) < f(y)$ then $x < y$
- $\langle 2 \rangle 1.$ LET: $x, y \in A$ with $f(x) < f(y)$
- $\langle 2 \rangle 2.$ $f(x) = f(y)$ and $f(y) < f(x)$ do not hold.
- PROOF: By trichotomy.
- $\langle 2 \rangle 3.$ $x = y$ and $y < x$ do not hold.
- $\langle 2 \rangle 4.$ $x < y$
- PROOF: By trichotomy.

□

Exercise 45 PROOF:

$\langle 1 \rangle 1.$ $<_L$ is transitive.

$\langle 2 \rangle 1.$ LET: $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$ and $\langle a_2, b_2 \rangle <_L \langle a_3, b_3 \rangle$

PROVE: $\langle a_1, b_1 \rangle < \langle a_3, b_3 \rangle$

$\langle 2 \rangle 2.$ CASE: $a_1 <_A a_2$ and $a_2 <_A a_3$

PROOF: Then $a_1 <_A a_3$

$\langle 2 \rangle 3.$ CASE: $a_1 <_A a_2$, $a_2 = a_3$, $b_2 <_B b_3$

PROOF: Then $a_1 <_A a_3$

$\langle 2 \rangle 4.$ CASE: $a_1 = a_2$, $b_1 <_B b_2$ and $a_2 <_A a_3$

PROOF: Then $a_1 <_A a_3$

$\langle 2 \rangle 5.$ CASE: $a_1 = a_2$, $b_1 <_B b_2$, $a_2 = a_3$, $b_2 <_B b_3$

PROOF: Then $a_1 = a_3$ and $b_1 <_B b_3$

$\langle 1 \rangle 2.$ $<_L$ satisfies trichotomy on $A \times B$.

$\langle 2 \rangle 1.$ LET: $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle$ be elements of $A \times B$

$\langle 2 \rangle 2.$ Exactly one of $a_1 <_A a_2$, $a_1 = a_2$, $a_2 <_A a_1$ holds.

$\langle 2 \rangle 3.$ Exactly one of $b_1 <_B b_2$, $b_1 = b_2$, $b_2 <_B b_1$ holds.

$\langle 2 \rangle 4.$ Exactly one of $a_1 <_A a_2$, $(a_1 = a_2 \text{ and } b_1 <_B b_2)$, $(a_1 = a_2 \text{ and } b_1 = b_2)$, $(a_1 = a_2 \text{ and } b_2 <_L b_1)$, $a_2 <_A a_1$ holds.

$\langle 2 \rangle 5.$ Exactly one of $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$, $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$, $\langle a_2, b_2 \rangle <_L \langle a_1, b_1 \rangle$ holds.

□

3.8 Review Exercises

Exercise 46

(a)

$$\bigcap \bigcap \langle x, y \rangle = \bigcap \{x\} \\ = x$$

(b)

$$\begin{aligned} \bigcap \bigcap \bigcap \{\langle x, y \rangle\}^{-1} &= \bigcap \bigcap \bigcap \{\langle y, x \rangle\} \\ &= \bigcap \bigcap \langle y, x \rangle \\ &= y \end{aligned} \quad \text{(by part (a))}$$

Exercise 47

(a) There are eight:

$$\begin{aligned} &\{\langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle\} \end{aligned}$$

(b) There are six:

$$\begin{aligned} &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 5 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 5 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 5 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 5 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 5 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 5 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\} \end{aligned}$$

Exercise 48

(a) The only ordered pair in \mathcal{PT} is $\langle \emptyset, \emptyset \rangle = \{\{\emptyset\}\}$.

(b)

$$\begin{aligned} (\mathcal{PT})^{-1} \circ (\mathcal{PT} \upharpoonright \{\emptyset\}) &= \{\langle \emptyset, \emptyset \rangle\} \circ \{\langle \emptyset, \emptyset \rangle\} \\ &= \{\langle \emptyset, \emptyset \rangle\} \end{aligned}$$

Exercise 49

There are six:

$$\begin{aligned} &\{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 1 \rangle, \langle 2, 0 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\} \end{aligned}$$

Exercise 50

(a) $\{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 3 \rangle\}$

$$(b) \quad \{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}$$

Exercise 51 There are three:

$$\begin{aligned} & \{\langle 1, 0 \rangle, \langle 1, 2 \rangle, \langle 2, 0 \rangle\}, \\ & \{\langle 1, 0 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle\}, \\ & \{\langle 0, 1 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle\} \end{aligned}$$

Exercise 52 We can conclude this if we know that A and B are nonempty, or that C and D are nonempty.

Suppose A and B are nonempty. Then $A \times B = C \times D \neq \emptyset$ so C and D are nonempty. We now prove $A \subseteq C$.

Let $a \in A$. Pick some $b \in B$. Then $\langle a, b \rangle \in A \times B = C \times D$ and so $a \in C$.

We can similarly prove $C \subseteq A$, $B \subseteq D$ and $D \subseteq B$.

Exercise 53

$$\begin{aligned} x(R \cup S)^{-1}y &\Leftrightarrow y(R \cup S)x \\ &\Leftrightarrow yRx \text{ or } ySx \\ &\Leftrightarrow xR^{-1}y \text{ or } xS^{-1}y \\ &\Leftrightarrow x(R^{-1} \cup S^{-1})y \\ x(R \cap S)^{-1}y &\Leftrightarrow y(R \cap S)x \\ &\Leftrightarrow yRx \text{ and } ySx \\ &\Leftrightarrow xR^{-1}y \text{ and } xS^{-1}y \\ &\Leftrightarrow x(R^{-1} \cap S^{-1})y \\ x(R - S)^{-1}y &\Leftrightarrow y(R - S)x \\ &\Leftrightarrow yRx \text{ and } \neg ySx \\ &\Leftrightarrow xR^{-1}y \text{ and } \neg xS^{-1}y \\ &\Leftrightarrow x(R^{-1} - S^{-1})y \end{aligned}$$

Exercise 54

(a)

$$\begin{aligned} \langle x, y \rangle \in A \times (B \cap C) &\Leftrightarrow x \in A \text{ \& } y \in B \text{ \& } y \in C \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) \cap (A \times C) \end{aligned}$$

(b)

$$\begin{aligned} \langle x, y \rangle \in A \times (B \cup C) &\Leftrightarrow x \in A \text{ \& } (y \in B \text{ or } y \in C) \\ &\Leftrightarrow (x \in A \text{ \& } y \in B) \text{ or } (x \in A \text{ \& } y \in C) \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) \cup (A \times C) \end{aligned}$$

(c)

$$\begin{aligned}\langle x, y \rangle \in A \times (B - C) &\Leftrightarrow x \in A \ \& \ y \in B \ \& \ y \notin C \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) - (A \times C)\end{aligned}$$

Exercise 55

(a) No. Take $A = \{0\}$, $B = \{1\}$, $C = \{2\}$. Then $(A \times A) \cup (B \times C) = \{\langle 0, 0 \rangle, \langle 1, 2 \rangle\}$ while $(A \cup B) \times (A \cup C) = \{\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 2 \rangle\}$.

(b) Yes.

$$\begin{aligned}\langle x, y \rangle \in (A \times A) \cap (B \times C) &\Leftrightarrow x \in A \ \& \ y \in A \ \& \ x \in B \ \& \ y \in C \\ &\Leftrightarrow \langle x, y \rangle \in (A \cap B) \times (A \cap C)\end{aligned}$$

Exercise 56

(a) Yes.

$$\begin{aligned}x \in \text{dom}(R \cup S) &\Leftrightarrow \exists y(xRy \text{ or } xSy) \\ &\Leftrightarrow \exists y.xRy \text{ or } \exists y.xSy \\ &\Leftrightarrow x \in \text{dom } R \cup \text{dom } S\end{aligned}$$

(b) No. Take $R = \{\langle 0, 0 \rangle\}$ and $S = \{\langle 0, 1 \rangle\}$. Then $\text{dom}(R \cap S) = \text{dom } \emptyset = \emptyset$ while $\text{dom } R \cap \text{dom } S = \{0\} \cap \{0\} = \{0\}$.

Exercise 57

(a) Yes.

$$\begin{aligned}x(R \circ (S \cup T))y &\Leftrightarrow \exists z(x(S \cup T)z \ \& \ zRy) \\ &\Leftrightarrow \exists z(xSz \ \& \ zRy) \text{ or } \exists z(xTz \ \& \ zRy) \\ &\Leftrightarrow x((R \circ S) \cup (R \circ T))y\end{aligned}$$

(b) No. Take $R = \{\langle 0, 0 \rangle, \langle 1, 0 \rangle\}$, $S = \{\langle 0, 0 \rangle\}$ and $T = \{\langle 0, 1 \rangle\}$. Then

$$\begin{aligned}R \circ (S \cap T) &= R \circ \emptyset \\ &= \emptyset \\ (R \circ S) \cap (R \circ T) &= \{\langle 0, 0 \rangle\} \cap \{\langle 0, 0 \rangle\} \\ &= \{\langle 0, 0 \rangle\}\end{aligned}$$

Exercise 58 Take $F = \emptyset$ and $S = \{\emptyset\}$. Then $F[F^{-1}[[S]]] = \emptyset \neq S$.

Exercise 59

$$\begin{aligned}
x(Q \upharpoonright (A \cap B))y &\Leftrightarrow xQy \ \& \ x \in A \ \& \ x \in B \\
&\Leftrightarrow x((Q \upharpoonright A) \cap (Q \upharpoonright B))y \\
x(Q \upharpoonright (A - B))y &\Leftrightarrow xQy \ \& \ x \in A \ \& \ x \notin B \\
&\Leftrightarrow (xQy \ \& \ x \in A) \ \& \ \neg(xQy \ \& \ x \in B) \\
&\Leftrightarrow x((Q \upharpoonright A) - (Q \upharpoonright B))y
\end{aligned}$$

Exercise 60

$$\begin{aligned}
x((R \circ S) \upharpoonright A)y &\Leftrightarrow \exists z(xRz \ \& \ zSy \ \& \ x \in A) \\
&\Leftrightarrow x(R \circ (S \upharpoonright A))y
\end{aligned}$$

Chapter 4

Chapter 4 — Natural Numbers

4.1 Inductive Sets

Exercise 1 We have

$$3 = 2 \cup \{2\} = 1 \cup \{1, 2\}$$

and so $1 \in 3$. But $1 \notin 1$ (since $1 = \{\emptyset\}$ and we know $\{\emptyset\} \neq \emptyset$ hence $\{\emptyset\} \notin \{\emptyset\}$). Therefore $1 \neq 3$.

4.2 Peano's Postulates

Exercise 2 If a is a transitive set then

$$\begin{aligned} \bigcup (a^+) &= a && \text{(Theorem 4E)} \\ &\subseteq a^+ \end{aligned}$$

Exercise 3

(a) Suppose a is a transitive set. Then $a \subseteq \mathcal{P}a$. Hence we have $\bigcup \mathcal{P}a = a \subseteq \mathcal{P}a$ and so $\mathcal{P}a$.

(b) Suppose $\mathcal{P}a$ is a transitive set. Then $a = \bigcup \mathcal{P}a \subseteq \mathcal{P}a$ hence a is transitive.

Exercise 4 If a is a transitive set then $\bigcup a \subseteq a$ so $\bigcup \bigcup a \subseteq \bigcup a$. Hence $\bigcup a$ is transitive.

Exercise 5

(a) PROOF:
 $\langle 1 \rangle 1.$ LET: $b \in \bigcup \mathcal{A}$
 $\langle 1 \rangle 2.$ PICK $A \in \mathcal{A}$ such that $b \in A$
 $\langle 1 \rangle 3.$ $b \subseteq A$
 PROOF: Since A is transitive.
 $\langle 1 \rangle 4.$ $b \subseteq \bigcup \mathcal{A}$
 \square

(b) PROOF:
 $\langle 1 \rangle 1.$ LET: $b \in \bigcap \mathcal{A}$
 $\langle 1 \rangle 2.$ For all $A \in \mathcal{A}$ we have $b \subseteq A$
 PROOF: Since $b \in A$ and A is transitive.
 $\langle 1 \rangle 3.$ $b \subseteq \bigcap \mathcal{A}$
 \square

Exercise 6 We have $\bigcup(a^+) = \bigcup a \cup a$ (see the proof of Theorem 4E). So if $\bigcup(a^+) = a$ we have $\bigcup a \cup a = a$ and so $\bigcup a \subseteq a$.

4.3 Recursion on ω

Exercise 7 We have $h_1(0) = h_2(0) = a$ so $0 \in S$.
 Now let $n \in S$; we prove $n^+ \in S$. We have $h_1(n) = h_2(n)$ and therefore

$$\begin{aligned} h_1(n^+) &= F(h_1(n)) \\ &= F(h_2(n)) \\ &= h_2(n^+) \end{aligned}$$

Exercise 8 PROOF:
 $\langle 2 \rangle 1.$ $\forall m, n \in \omega. h(n) = h(m) \Rightarrow n = m$
 $\langle 2 \rangle 1.$ $\forall n \in \omega. h(n) = h(0) \Rightarrow n = 0$
 $\langle 3 \rangle 1.$ LET: $n \in \omega$
 $\langle 3 \rangle 2.$ ASSUME: $h(n) = h(0)$
 $\langle 3 \rangle 3.$ $h(n) = c$
 $\langle 3 \rangle 4.$ $\forall p \in \omega. n \neq p^+$
 PROOF: Otherwise $f(h(p)) = c$ contradicting the fact that $c \in A - \text{ran } f$.
 $\langle 3 \rangle 5.$ $n = 0$
 PROOF: Theorem 4C.
 $\langle 2 \rangle 2.$ For all $m \in \omega$, if $\forall n \in \omega. h(n) = h(m) \Rightarrow n = m$, then $\forall n \in \omega. h(n) = h(m^+) \Rightarrow n = m^+$
 $\langle 3 \rangle 1.$ LET: $m \in \omega$
 $\langle 3 \rangle 2.$ ASSUME: $\forall n \in \omega. h(n) = h(m) \Rightarrow n = m$
 $\langle 3 \rangle 3.$ LET: $n \in \omega$
 $\langle 3 \rangle 4.$ ASSUME: $h(n) = h(m^+)$
 $\langle 3 \rangle 5.$ $h(n) = f(h(m))$

$\langle 3 \rangle 6. n \neq 0$

PROOF: Otherwise $c = f(h(m))$ contradicting the fact that $c \in A - \text{ran } f$.

$\langle 3 \rangle 7. \text{ PICK } p \text{ such that } n = p^+$

$\langle 3 \rangle 8. f(h(p)) = f(h(m))$

$\langle 3 \rangle 9. h(p) = h(m)$

PROOF: f is one-to-one.

$\langle 3 \rangle 10. p = m$

PROOF: By $\langle 3 \rangle 2$.

$\langle 3 \rangle 11. n = p^+ = m^+$

□

Exercise 9 PROOF:

$\langle 1 \rangle 1. C^* \subseteq C_*$

$\langle 2 \rangle 1. f[C_*] \subseteq C_*$

$\langle 3 \rangle 1. \text{ LET: } x \in C_*$

PROVE: $f(x) \in C_*$

$\langle 3 \rangle 2. \text{ PICK } n \text{ such that } x \in h(n)$

$\langle 3 \rangle 3. f(x) \in h(n^+)$

$\langle 3 \rangle 4. f(x) \in C_*$

$\langle 1 \rangle 2. C_* \subseteq C^*$

$\langle 2 \rangle 1. \forall n \in \omega. h(n) \subseteq C^*$

$\langle 3 \rangle 1. h(0) \subseteq C^*$

PROOF: If $A \subseteq X \subseteq B$ and $f[X] \subseteq X$ then $A \subseteq X$.

$\langle 3 \rangle 2. \forall n \in \omega. h(n) \subseteq C^* \Rightarrow h(n^+) \subseteq C^*$

$\langle 4 \rangle 1. \text{ LET: } n \in \omega$

$\langle 4 \rangle 2. \text{ ASSUME: } h(n) \subseteq C^*$

$\langle 4 \rangle 3. f[h(n)] \subseteq C^*$

$\langle 5 \rangle 1. \text{ LET: } X \text{ be such that } A \subseteq X \subseteq B \text{ and } f[X] \subseteq X$

PROVE: $f[h(n)] \subseteq X$

$\langle 5 \rangle 2. h(n) \subseteq X$

$\langle 5 \rangle 3. f[h(n)] \subseteq f[X]$

$\langle 5 \rangle 4. f[h(n)] \subseteq X$

$\langle 4 \rangle 4. h(n^+) \subseteq C^*$

□

Exercise 10 $C^* = C_* = (0, 1]$

Exercise 11 $\{n \in \mathbb{Z} \mid n \leq 0\}$

Exercise 12 Let $f : B \times B \rightarrow B$ and $A \subseteq B$. Let

$$C^* = \bigcap \{X \mid A \subseteq X \subseteq B \text{ \& } f[X \times X] \subseteq X\} .$$

Define the function $h : \omega \rightarrow \mathcal{P}B$ by

$$\begin{aligned} h(0) &= A \\ h(n^+) &= h(n) \cup f[h(n) \times h(n)] \end{aligned} \quad (n \in \omega)$$

Define $C_* = \bigcup \text{ran } h$. Then $C^* = C_*$.

4.4 Arithmetic

Exercise 13 We prove the contrapositive. Assume $m \neq 0$ and $n \neq 0$. Then by Theorem 4C there are natural numbers p, q such that $m = p^+$ and $n = q^+$. Hence $mn = p^+q^+ = (p^+q + p)^+ \neq 0$.

Exercise 14 We prove the following facts for any natural number n :

1. n is even if and only if n^+ is odd.

PROOF: If n is even, say $n = 2p$, then $n^+ = 2p + 1$ is odd.

If n^+ is odd, say $n^+ = 2p + 1$, then $n = 2p$ is even.

2. n is odd if and only if n^+ is even.

PROOF: If n is odd, say $n = 2p + 1$, then $n^+ = 2(p + 1)$ is even.

If n^+ is even, say $n^+ = 2p$, then we cannot have $p = 0$ (since $n^+ \neq 0$). So $p = q + 1$ for some q . But then $n^+ = 2q + 2$ so $n = 2q + 1$ and n is odd.

Now, 0 is even and 0 is not odd. By the two facts above, if n is either even or odd but not both, then n^+ is either odd or even but not both. The result follows by induction.

Exercise 15 We have

$$\begin{aligned} m + (n + 0) &= m + n && \text{by (A1)} \\ &= (m + n) + 0 && \text{by (A1)} \end{aligned}$$

If $m + (n + p) = (m + n) + p$ then

$$\begin{aligned} m + (n + p^+) &= m + (n + p)^+ && \text{by (A2)} \\ &= (m + (n + p))^+ && \text{by (A2)} \\ &= ((m + n) + p)^+ && \text{by induction hypothesis} \\ &= (m + n) + p^+ && \text{by (A2)} \end{aligned}$$

Exercise 16 We first prove that $0 \cdot n = 0$ for all n . We have $0 \cdot 0 = 0$ by (M1), and if $0 \cdot n = 0$ then

$$\begin{aligned} 0 \cdot n^+ &= 0 \cdot n + 0 && \text{by (M2)} \\ &= 0 \cdot n && \text{by (A1)} \\ &= 0 && \text{by induction hypothesis} \end{aligned}$$

Now we prove that $m^+ \cdot n = m \cdot n + n$ for all m, n . We have

$$\begin{aligned} m^+ \cdot 0 &= 0 && \text{by (M1)} \\ m \cdot 0 + 0 &= m \cdot 0 && \text{by (A1)} \\ &= 0 && \text{by (M1)} \end{aligned}$$

Thus, $m^+ \cdot 0 = m \cdot 0 + 0$.

If $m^+ \cdot n = m \cdot n + n$ then

$$\begin{aligned} m^+ \cdot n^+ &= m^+ \cdot n + m^+ && \text{by (M2)} \\ &= (m^+ \cdot n + m)^+ && \text{by (A2)} \\ &= ((m \cdot n + n) + m)^+ && \text{by induction hypothesis} \\ &= ((m \cdot n + m) + n)^+ && \text{by associativity and commutativity of addition} \\ &= (m \cdot n^+ + n)^+ && \text{by (M2)} \\ &= m \cdot n^+ + n^+ && \text{by (A2)} \end{aligned}$$

Exercise 17 The proof is by induction on p . We have

$$\begin{aligned} m^{n+0} &= m^n && \text{by (A1)} \\ &= 0 + m^n && \text{by Theorem 4K(2)} \\ &= m^n \cdot 0 + m^n && \text{by (M1)} \\ &= m^n \cdot 1 && \text{by (M2)} \\ &= m^n \cdot m^0 && \text{by (E1)} \end{aligned}$$

If $m^{n+p} = m^n \cdot m^p$ then

$$\begin{aligned} m^{n+p^+} &= m^{(n+p)^+} && \text{by (A2)} \\ &= m^{n+p} m && \text{by (E2)} \\ &= (m^n m^p) m && \text{by induction hypothesis} \\ &= m^n (m^p m) && \text{by Theorem 4K (4)} \\ &= m^n m^{p^+} && \text{by (E2)} \end{aligned}$$

4.5 Ordering on ω

Exercise 18

$$\begin{aligned} \in_{\omega}^{-1} \llbracket \{7, 8\} \rrbracket &= \{x \in \omega \mid x \in 7 \text{ or } x \in 8\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7\} \end{aligned}$$

Exercise 19 The proof is by induction on m .

For $m = 0$, take $q = r = 0$. Then $m = d \cdot 0 + 0$ and $0 \in d$.

Suppose $m = dq + r$ and $r < d$. Then $r + 1 \leq d$. If $r + 1 < d$, then we have $m + 1 = dq + (r + 1)$ as required. If $r + 1 = d$, then we have $m + 1 = dq + d = d(q + 1) + 0$.

Exercise 20 We first prove A is closed downwards; that is, if $n \in A$ and $m \in n$ then $m \in A$. This holds because if $n \in A$ and $m \in n$ then $m \in \bigcup A$ and $\bigcup A = A$.

Now, we prove $\forall n \in \omega. n \in A$ by induction on n .

To prove $0 \in A$: we are given that A is nonempty. Pick some $a \in A$. Then $0 \in a$ so $0 \in A$ since A is closed downwards.

Now let $n \in A$; we prove $n^+ \in A$. We have $n \in \bigcup A$; pick some $k \in A$ such that $n \in k$. Then $n^+ \in k$ so $n^+ \in A$ since A is closed downwards.

This completes the induction. We have $\forall n \in \omega. n \in A$, i.e. $A = \omega$.

Exercise 21 Suppose n is a natural number, $k \in n$ and $n \subseteq k$. Then $k \in k$, contradicting Lemma 4L(b).

Exercise 22 We have $0 \in p^+$ (by trichotomy since $p^+ \not\subseteq 0$ because 0 is empty, and $p^+ \neq 0$ by Peano's First Postulate.) Hence $n = n + 0 \in n + p^+$ by Theorem 4N.

Exercise 23 The proof is by induction on n . The statement is vacuously true for $n = 0$.

Suppose the statement is true for n . Let $m \in n^+$. Then $m \subseteq n$.

If $m = n$, then we have $m + 0^+ = n^+$.

If $m \in n$, pick p such that $m + p^+ = n$ by the induction hypothesis. Then $m + p^{++} = n^+$.

Exercise 24 Suppose $m \in p$. Then we cannot have $n \in q$ or $n = q$, as either of these would imply $m + n \in p + q$. Hence $q \in n$ by trichotomy.

We prove $q \in n \Rightarrow m \in p$ similarly.

Exercise 25 By Exercise 23, pick natural numbers a and b such that $m = n + a^+$ and $p = q + b^+$. Then

$$\begin{aligned} mp + nq &= (n + a^+)(q + b^+) + nq \\ &= nq + nq + a^+q + nb^+ + a^+b^+ \\ &= (n + a^+)q + n(q + b^+) + a^+b^+ \\ &= mq + np + (a^+ + b^+)^+ \end{aligned}$$

Hence $mq + np \in mp + nq$ by Exercise 22.

Exercise 26 The proof is by induction on n .

If $n = 0$ then $\text{ran } f$ is a singleton and its sole element is the largest element.

Suppose the result is true for n . Let $f : n^{++} \rightarrow A$. Then $f \llbracket n^+ \rrbracket$ has a largest element $f(k)$, say. If $f(k) \subseteq f(n^+)$ then $f(n^+)$ is greatest in $\text{ran } f$; otherwise $f(k)$ is greatest.

Exercise 27 We prove $f_1(n) = f_2(n)$ for all $n \in \omega$ by strong induction on n . Assume that $(\forall m \in n) f_1(m) = f_2(m)$. Then $f_1 \upharpoonright n = f_2 \upharpoonright n$. So

$$\begin{aligned} f_1(n) &= G(f_1 \upharpoonright n) \\ &= G(f_2 \upharpoonright n) \\ &= f_2(n) \end{aligned}$$

Exercise 28 Suppose ω is not transitive. Then there exists a natural number n such that $n \not\subseteq \omega$. Let n be the least such number. There exists $x \in n$ such that $x \not\subseteq \omega$. Now, $n \neq \emptyset$ (because it is nonempty) so $n = p^+$ for some natural number p . We have $x \in p^+$ so $x \in p$ or $x = p$. We cannot have $x = p$ (because x is not a natural number) so we have $x \in p$. But this contradicts the minimality of n .

4.6 Review Exercises

Exercise 29 $4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Exercise 30 $\bigcup 4 = 0 \cup 1 \cup 2 \cup 3 = 3$ since 0, 1 and 2 are all subsets of 3.
 $\bigcap 4 = 0 \cap 1 \cap 2 \cap 3 = 0 (= \emptyset)$.

Exercise 31 Similarly to Exercise 30 we have $\bigcup \bigcup 7 = \bigcup 6 = 5$.

Exercise 32

$$\begin{aligned} \text{(a)} \quad A^+ &= A \cup \{A\} = \{1, A\} = \{1, \{1\}\} \\ \text{So } \bigcup A^+ &= 1 \cup \{1\} = \{0, 1\} = 2 \end{aligned}$$

$$\text{(b)} \quad \bigcup (\{2\}^+) = \bigcup \{2, \{2\}\} = \{0, 1, 2\} = 3$$

Exercise 33

(a) Yes - if $x \in y \in \{0, 1, \{1\}\}$ then x is either 0 or 1, and in either case $x \in \{0, 1, \{1\}\}$

(b) No - $0 \in 1 \in \{1\}$ but $0 \notin \{1\}$

(c) No - $0 \in \{0\} \in \langle 0, 1 \rangle$ but $0 \notin \langle 0, 1 \rangle$.

Exercise 34

(a) Let $a = \{\emptyset\}$ and $b = \emptyset$

(b) Let $c = \{\{\emptyset\}\}$, $d = \{\emptyset\}$ and $e = \emptyset$

Exercise 35

(a) Let $T_1 = \{\{1\}, \{1, 0\}, 0, 1\}$

(b) Let $T_2 = \{\langle 1, 0 \rangle, \{1\}, \{1, 0\}, 0, 1\}$.

Exercise 36

$$\begin{aligned} h(4) &= 2h(3) \\ &= 4h(2) \\ &= 8h(1) \\ &= 16h(0) \\ &= 48 \end{aligned}$$

Exercise 37

(a) Let $f : m \rightarrow A$ and $g : n \rightarrow B$ be bijections. Define $h : m + n \rightarrow A \cup B$ by

$$\begin{aligned} h(p) &= f(p) && \text{if } p \in m \\ h(m + q) &= g(q) && \text{if } q \in n \end{aligned}$$

To show that this is well-defined, we must prove two things:

1. For all $p \in m + n$, then either $p \in m$ or there exists $q \in n$ such that $p = m + n$.
2. We never have $p \in m$ and $p = m + q$ for some $q \in n$.

We prove 1 by induction on n . For all $p \in m + 0$ we have $p \in m$, so the result holds for $n = 0$.

Now, suppose the result holds for n . Let $p \in m + n^+ = (m + n)^+$ so $p \in m + n$. If $p \in m + n$, we simply apply the induction hypothesis. If $p = m + n$ then $p = m + q$ where $q = n \in n^+$.

To prove 2, if $p = m + q$ then $m = m + 0$ and $m + q = p$ by Theorem 4N, hence $p \notin m$ by trichotomy.

It remains to show that h is a bijection.

To prove h is injective, we consider three cases. If $h(p) = h(p')$ where $p, p' \in m$, then $f(p) = f(p')$ so $p = p'$. If $h(m + q) = h(m + q')$ where $q, q' \in n$, then $g(q) = g(q')$ so $q = q'$. And we cannot have $h(p) = h(m + q)$ for $p \in m$ and $q \in n$ since $h(p) \in A$, $h(m + q) \in B$, and $A \cap B = \emptyset$.

To prove h is surjective, let $x \in A \cup B$. If $x \in A$, there is some $p \in m$ with $f(p) = x$, so $h(p) = x$. If $x \in B$, there is some $q \in n$ with $g(q) = x$, so $h(m + q) = x$.

(b) Let $f : m \rightarrow A$ and $g : n \rightarrow B$ be bijections.

We first show that, for any $p \in mn$, there exist unique $i \in m$ and $j \in n$ such that $p = mj + i$.

By Exercise 19, there exist j and $i \in m$ such that $p = mj + i$. We have $j \in n$ since otherwise $p = mj + i \in mn$.

For uniqueness, suppose $mj + i = mj' + i'$ where $i, i' \in m$ and $j, j' \in n$. Then we have

$$mj \in mj + i = mj' + i' \in mj' + m = m(j')^+$$

so $j \in (j')^+$ and $j \in j'$. Similarly $j' \in j$, and so $j = j'$. Therefore $i = i'$ by the cancellation law for addition.

Now define $h : mn \rightarrow A \times B$ by

$$h(mj + i) = \langle f(i), g(j) \rangle$$

where $i \in m$ and $j \in n$. It is easy to check that h is bijective.

Exercise 38 $h(n) = 3n + 1$

Exercise 39 $h(n) = n^2$

Exercise 40 $h(n^+) = h(n) + 5$

Chapter 5

Chapter 5 — Construction of the Real Numbers

5.1 Integers

Exercise 1 No, because $[\langle 0, 0 \rangle] = [\langle 1, 1 \rangle]$ but $[\langle 0, 0 \rangle] \neq [\langle 2, 1 \rangle]$.

Exercise 2 Yes, because if $[\langle m, n \rangle] = [\langle p, q \rangle]$ then $[\langle m, m \rangle] = [\langle p, p \rangle]$ because $m + p = m + p$.

Exercise 3 Yes, because if $[\langle m, n \rangle] = [\langle p, q \rangle]$ then $[\langle n, m \rangle] = [\langle q, p \rangle]$ because $n + p = m + q$.

Exercise 4 Let $a = [\langle m, n \rangle]$, $b = [\langle p, q \rangle]$ and $c = [\langle r, s \rangle]$. Then

$$\begin{aligned} a +_Z (b +_Z c) &= [\langle m, n \rangle] +_Z [\langle p + r, q + s \rangle] \\ &= [\langle m + (p + r), n + (q + s) \rangle] \\ &= [\langle (m + p) + r, (n + q) + s \rangle] \\ &= [\langle m + p, n + q \rangle] +_Z [\langle r, s \rangle] \\ &= (a +_Z b) +_Z c \end{aligned}$$

Exercise 5

$$[\langle m, n \rangle] - [\langle p, q \rangle] = [\langle m, n \rangle] + [\langle q, p \rangle] = [\langle m + q, n + p \rangle]$$

Exercise 6 Let $a = [\langle m, n \rangle]$. Then

$$\begin{aligned} a \cdot_Z 0_Z &= [\langle m, n \rangle] \cdot_Z [\langle 0, 0 \rangle] \\ &= [\langle m0 + n0, m0 + n0 \rangle] \\ &= [\langle 0, 0 \rangle] \\ &= 0_Z \end{aligned}$$

Exercise 7 We have $a \cdot_Z b +_Z a \cdot_Z (-b) = a \cdot_Z (b +_Z (-b)) = a \cdot_Z 0_Z = 0_Z$, hence $a \cdot_Z (-b) = -(a \cdot_Z b)$ by the uniqueness of inverses.

We prove $(-a) \cdot_Z b = -(a \cdot_Z b)$ similarly.

Exercise 8

(a) This says $[\langle m + n, 0 \rangle] = [\langle m, 0 \rangle] +_Z [\langle n, 0 \rangle]$, which is true from the definition of $+_Z$.

(b) We have

$$\begin{aligned} E(m) \cdot_Z E(n) &= [\langle m, 0 \rangle] \cdot_Z [\langle n, 0 \rangle] \\ &= [\langle mn + 0 \cdot 0, m0 + n0 \rangle] \\ &= E(mn) \end{aligned}$$

(c)

$$\begin{aligned} E(m) <_Z E(n) &\Leftrightarrow [\langle m, 0 \rangle] <_Z [\langle n, 0 \rangle] \\ &\Leftrightarrow m + 0 \in n + 0 \\ &\Leftrightarrow m \in n \end{aligned}$$

Exercise 9

$$\begin{aligned} E(m) - E(n) &= [\langle m, 0 \rangle] - [\langle n, 0 \rangle] \\ &= [\langle m, n \rangle] \end{aligned}$$

by Exercise 5.

5.2 Rational Numbers

Exercise 10 Let $r = [\langle a, b \rangle]$. Then

$$\begin{aligned} r \cdot_Q 0_Q &= [\langle a, b \rangle] \cdot_Q [\langle 0, 1 \rangle] \\ &= [\langle a \cdot_Z 0, b \cdot_Z 1 \rangle] \\ &= [\langle 0, b \rangle] \\ &= [\langle 0, 1 \rangle] \end{aligned}$$

since $\langle 0, b \rangle \sim \langle 0, 1 \rangle$ because $0 \cdot_Z 1 = 0 \cdot_Z b = 0$.

Exercise 11 Let $r = [\langle a, b \rangle]$ and $s = [\langle c, d \rangle]$. Suppose $r \cdot_Q s = 0_Q$. Then

$$[\langle ac, bd \rangle] = [\langle 0, 1 \rangle]$$

that is, $ac = 0$. Hence $a = 0$ or $c = 0$, which means $r = 0_Q$ or $s = 0_Q$.

Exercise 12 This follows from Theorem 5QJ(a) with $s = 0_Q$ and $t = -r$.

Exercise 13 Let $a, b, c \in \mathbb{Z}$. If $a +_Z c = b +_Z c$ then

$$\begin{aligned} a +_Z c +_Z (-c) &= b +_Z c +_Z (-c) \\ \therefore a +_Z 0 &= b +_Z 0 && \text{(Theorem 5ZD(b))} \\ \therefore a &= b && \text{(Theorem 5ZD(a))} \end{aligned}$$

Exercise 14 Suppose $p <_Q s$. Let $r = (p +_Q s)/2$. Then

$$\begin{aligned} p &<_Q s \\ \therefore 2p &<_Q p +_Q s \\ \therefore p &<_Q (p +_Q s)/2 \\ &= r \\ p &<_Q s \\ \therefore p +_Q s &<_Q 2s \\ \therefore (p +_Q s)/2 &<_Q s \\ \therefore r &<_Q s \end{aligned}$$

5.3 Real Numbers

Exercise 15 PROOF:

$\langle 1 \rangle 1$. $\bigcup A$ is closed downwards.

$\langle 2 \rangle 1$. LET: $q \in \bigcup A$ and $p < q$

$\langle 2 \rangle 2$. PICK $x \in A$ such that $q \in x$

$\langle 2 \rangle 3$. $p \in x$

PROOF: Since x is closed downwards.

$\langle 2 \rangle 4$. $p \in \bigcup A$

$\langle 1 \rangle 2$. $\bigcup A$ has no largest element.

$\langle 2 \rangle 1$. LET: $q \in \bigcup A$

$\langle 2 \rangle 2$. PICK $x \in A$ such that $q \in x$

$\langle 2 \rangle 3$. PICK $r \in x$ such that $q < r$

PROOF: Since x has no largest element.

$\langle 2 \rangle 4$. $r \in \bigcup A$

□

Exercise 16 PROOF:

- ⟨1⟩1. LET: $q \in x +_R y$
- ⟨1⟩2. PICK rationals $a \in x$ and $b \in y$ such that $q = a + b$
- ⟨1⟩3. PICK $a' \in x$ and $b' \in y$ such that $a < a'$ and $b < b'$

PROOF: Since x and y each have no largest element.

- ⟨1⟩4. $q < a' + b' \in x +_R y$

□

Exercise 17 If $b < 0$ we can take $k = 0$. If $b \geq 0$ then there is a natural number n such that $b = E(n)$; take $k = n^+$. Then $b < ak$ since $1 \leq a$ and $b < k$.

Exercise 18 Let $p = [\langle a, b \rangle]$ and $r = [\langle c, d \rangle]$ where a, b and d are positive. By Exercise 17, there exists a natural number k such that $bc < adE(k)$. Therefore $r < p \cdot E(E(k))$.

Exercise 19 Pick a rational $a \in x$ (which we can do since $x \neq \emptyset$). We first prove that there exists a natural number k such that $a + kp \notin x$.

Pick a rational $b \notin x$ (which we can do since $x \neq \mathbb{Q}$). We have $a < b$ (since x is closed downwards). By Exercise 18, there exists a natural number k such that

$$\begin{aligned} b - a &< kp \\ \therefore a + kp &> b \\ \therefore a + kp &\notin x \end{aligned}$$

Now, let k be the least natural number such that $a + kp \notin x$ (by the Well-Ordering Principle). We have $k \neq 0$ (since $a \in x$); let $k = n^+$. Then we have

$$a + np \in x \quad a + np + p \notin x$$

Take $q = a + np$.

Exercise 20 We must prove $0 \subseteq x \cup -x$. Let $q \in 0$ and assume $q \notin x$. Then $q < 0$ and $-0 = 0 \notin x$, so $q \in -x$.

Exercise 21 PROOF:

- ⟨1⟩1. LET: x, y be real numbers with $x < y$
- ⟨1⟩2. PICK $r \in y$ such that $r \notin x$
- ⟨1⟩3. PICK $s \in y$ such that $r < s$

PROVE: $x < E(s) < y$

- ⟨1⟩4. $x \subseteq E(s)$

PROOF: If $p \in x$ then $p < r < s$

- ⟨1⟩5. $x \neq E(s)$

PROOF: Since $r \in E(s)$ and $r \notin x$

- ⟨1⟩6. $E(s) \subseteq y$

PROOF: Since y is closed downwards.

$\langle 1 \rangle 7.$ $E(s) \neq y$

PROOF: Since $s \in y$ but $s \notin E(s)$.

Exercise 22 $|x|$ is either x or $-x$, and they are both real numbers.

Chapter 6

Chapter 6 — Cardinal Numbers and the Axiom of Choice

6.1 Equinumerosity

Exercise 1 PROOF:

- ⟨1⟩1. f is injective.
 - ⟨2⟩1. ASSUME: $f(m, n) = f(m', n')$
 - ⟨2⟩2. $2^m(2n + 1) = 2^{m'}(2n' + 1)$
 - ⟨2⟩3. $m = m'$
 - ⟨3⟩1. ASSUME: w.l.o.g. $m \leq m'$
 - ⟨3⟩2. $2n + 1 = 2^{m'-m}(2n' + 1)$
 - PROOF: From ⟨2⟩2 dividing by 2^m .
 - ⟨3⟩3. $m' - m = 0$
 - PROOF: Since $2^{m'-m}(2n' + 1)$ is odd.
 - ⟨2⟩4. $2n + 1 = 2n' + 1$
 - ⟨2⟩5. $n = n'$
- ⟨1⟩2. f is surjective.
 - ⟨2⟩1. LET: $n \in \omega$
 - ASSUME: $\forall m < n. m \in \text{ran } f$
 - PROVE: $n \in \text{ran } f$
 - ⟨2⟩2. CASE: n is even
 - ⟨3⟩1. LET: k be such that $n = 2k$
 - ⟨3⟩2. $n = f(0, k)$
 - ⟨2⟩3. CASE: n is odd
 - ⟨3⟩1. LET: k be such that $n = 2k + 1$
 - ⟨3⟩2. LET: $k = f(i, j)$
 - ⟨3⟩3. $n = f(i + 1, j)$

PROOF:

$$\begin{aligned}
 n &= 2k + 1 \\
 &= 2(2^i(2j + 1) - 1) + 1 \\
 &= 2^{i+1}(2j + 1) - 2 + 1 \\
 &= 2^{i+1}(2j + 1) - 1
 \end{aligned}$$

□

Exercise 2 Let us call (0) the 0th diagonal, $(1, 2)$ the 1st diagonal, $(3, 4, 5)$ the 2nd diagonal, etc. Then the k th is the set of all positions with coordinates (m, n) such that $m + n = k$.

Therefore, the number $J(m, n)$ at position (m, n) is the $m + 1$ st number in the $(m + n)$ th diagonal. So the number of numbers that come before $J(m, n)$ is

$$(1 + 2 + \cdots + (m + n)) + m$$

Therefore, since the natural numbers start at 0,

$$J(m, n) = (1 + 2 + \cdots + (m + n)) + m$$

We know $1 + 2 + \cdots + k = k(k + 1)/2$. Therefore,

$$\begin{aligned}
 J(m, n) &= 1/2(m + n)(m + n + 1) + m \\
 &= 1/2(m^2 + 2mn + m + n + n^2) + m \\
 &= 1/2(m^2 + 2mn + 3m + n + n^2) \\
 &= 1/2((m + n)^2 + 3m + n)
 \end{aligned}$$

Exercise 3 Define $f : (0, 1) \rightarrow \mathbb{R}$ by: $f(x) = 1/x - 2$ if $0 < x \leq 1/2$; $f(x) = 2 - 1/(1 - x)$ if $1/2 < x < 1$.

Exercise 4 Define $f : [0, 1] \rightarrow (0, 1)$ by

$$\begin{aligned}
 f(1/2 - 1/2^n) &= 1/2 - 1/2^{n-1} && \text{(for } n \text{ a positive integer)} \\
 f(1/2 + 1/2^n) &= 1/2 + 1/2^{n-1} && \text{(for } n \text{ a positive integer)} \\
 f(x) &= x && \text{(for all other } x)
 \end{aligned}$$

Exercise 5

(a) For any set A , the identity function I_A is a bijection between A and A . It is injective because, if $I_A(x) = I_A(y)$ then $x = y$ immediately. It is surjective because for any $y \in I_A$ we have $y = I_A(y)$.

(b) We prove that, if f is a bijection between A and B , then f^{-1} is a bijection between B and A . It is an injective function by Theorem 3F, and maps B onto A by Theorem 3E.

(c) Let f be a bijection between A and B , and g a bijection between A and C . We prove $g \circ f$ is a bijection between A and C .

It is a function from A to C by Theorem 3H.

We prove it is injective. Let $x, y \in A$ and assume $(g \circ f)(x) = (g \circ f)(y)$. Then

$$\begin{aligned} g(f(x)) &= g(f(y)) \\ \therefore f(x) &= f(y) && (g \text{ is injective}) \\ \therefore x &= y && (f \text{ is injective}) \end{aligned}$$

Now we prove it maps A onto C . Let $c \in C$. Pick $b \in B$ such that $g(b) = c$ (since g is surjective). Pick $a \in A$ such that $f(a) = b$ (since f is injective). Then $(g \circ f)(a) = c$.

6.2 Finite Sets

Exercise 6 Suppose every set of cardinality κ belongs to A . We will prove that every set belongs to $\bigcup A$.

Let x be any set. Pick a set y of cardinality κ . If $x \in y$ then $x \in y \in A$ so $x \in \bigcup A$.

Assume $x \notin y$. Pick an element $z \in y$ (we know y is nonempty because $\kappa \neq 0$). Then $y - \{z\} \cup \{x\}$ has cardinality κ , and so $x \in (y - \{z\} \cup \{x\}) \in A$ hence $x \in \bigcup A$.

Thus, every set is in $\bigcup A$, which we know is impossible by Theorem 2A.

Exercise 7 If f is one-to-one then f is a bijection between A and $\text{ran } f$. So we must have $\text{ran } f = A$, otherwise f would be a bijection between A and a proper subset of A , contradicting the Pigeonhole Principle.

Conversely, suppose $\text{ran } f = A$. Pick a right inverse $h : A \rightarrow A$ for f (by Theorem 3J(b). Note: Theorem 3J(b) can in fact be proved for the case B is finite without using the Axiom of Choice.). Now, h is one-to-one by Theorem 3J(a). So $\text{ran } h = A$ by the first paragraph.

We prove f is one-to-one. Let $x, y \in A$ and assume $f(x) = f(y)$. Pick $a, b \in A$ such that $h(a) = x$ and $h(b) = y$. Then

$$\begin{aligned} f(h(a)) &= f(h(b)) \\ \therefore a &= b \\ \therefore x &= y \end{aligned}$$

Exercise 8 PROOF:

$\langle 1 \rangle 1$. For any sets A and x , if A is finite then $A \cup \{x\}$ is finite.

$\langle 2 \rangle 1$. CASE: $x \in A$

PROOF: In this case $A \cup \{x\} = A$.

$\langle 2 \rangle 2$. CASE: $x \notin A$

PROOF: Then $|A \cup \{x\}| = |A|^+$.

$\langle 1 \rangle 2$. LET: A be a finite set.

$\langle 1 \rangle 3$. For any set B , if $B \approx 0$ then $A \cup B$ is finite.

PROOF: Because $B = \emptyset$ so $A \cup B = A$.

$\langle 1 \rangle 4$. Let n be a natural number. Assume that, for any set B , if $B \approx n$ then $A \cup B$ is finite. Then for any set B , if $B \approx n^+$ then $A \cup B$ is finite.

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: For any set B , if $B \approx n$ then $A \cup B$ is finite.

$\langle 2 \rangle 3$. LET: B be a set.

$\langle 2 \rangle 4$. ASSUME: $B \approx n^+$

$\langle 2 \rangle 5$. PICK a bijection $f : n^+ \rightarrow B$

$\langle 2 \rangle 6$. $B - \{f(n)\} \approx n$

$\langle 2 \rangle 7$. $A \cup (B - \{f(n)\})$ is finite.

$\langle 2 \rangle 8$. $A \cup B$ is finite.

PROOF: By $\langle 1 \rangle 1$ since $A \cup B = (A \cup (B - \{f(n)\})) \cup \{f(n)\}$.

□

Exercise 9 PROOF:

$\langle 1 \rangle 1$. LET: A be a finite set.

$\langle 1 \rangle 2$. For any set B , if $B \approx 0$ then $A \times B$ is finite.

PROOF: In this case $A \times B = \emptyset$.

$\langle 1 \rangle 3$. Let n be a natural number. Suppose that, for any set B , if $B \approx n$ then $A \times B$ is finite. Then for any set B , if $B \approx n^+$ then $A \times B$ is finite.

$\langle 2 \rangle 1$. LET: n be a natural number.

$\langle 2 \rangle 2$. ASSUME: For any set B , if $B \approx n$ then $A \times B$ is finite.

$\langle 2 \rangle 3$. LET: B be a set.

$\langle 2 \rangle 4$. ASSUME: $B \approx n^+$

$\langle 2 \rangle 5$. PICK a bijection $f : n^+ \approx B$

$\langle 2 \rangle 6$. $A \times (B - \{f(n)\})$ is finite.

PROOF: By the induction hypothesis $\langle 2 \rangle 2$.

$\langle 2 \rangle 7$. $A \times B$ is finite.

PROOF: By Exercise 8 since $A \times B = (A \times (B - \{f(n)\})) \cup (A \times \{f(n)\})$ and $A \times \{f(n)\}$ is finite because it is equinumerous with A .

□

6.3 Cardinal Arithmetic

Exercise 10 We must show that $(L \cup M)K \approx^L K \times^M K$ where $L \cap M = \emptyset$.

Define $\Phi : (L \cup M)K \rightarrow^L K \times^M K$ by: $\Phi(f) = \langle f \upharpoonright L, f \upharpoonright M \rangle$.

To show Φ is one-to-one: suppose $\Phi(f) = \Phi(g)$. Then $f \upharpoonright L = g \upharpoonright L$ and $f \upharpoonright M = g \upharpoonright M$. Hence $f(x) = g(x)$ for all $x \in L$ and $f(x) = g(x)$ for all $x \in M$, so $f(x) = g(x)$ for all x , i.e. $f = g$.

To show Φ is surjective: given a function $g : L \rightarrow K$ and $h : M \rightarrow K$, we have $g \cup h : L \cup M \rightarrow K$ and $\Phi(g \cup h) = \langle g, h \rangle$.

Exercise 11 We must show that ${}^M(K \times L) \approx^M K \times^M L$.

Define $\Phi : {}^M(K \times L) \rightarrow {}^M K \times^M L$ by: $\Phi(f) = \langle \pi_1 \circ f, \pi_2 \circ f \rangle$, where $\pi_1 : K \times L \rightarrow K$ is the function defined by

$$\pi_1(\langle x, y \rangle) = x$$

and $\pi_2 : K \times L \rightarrow L$ is the function defined by

$$\pi_2(\langle x, y \rangle) = y .$$

To show Φ is one-to-one: suppose $\Phi(f) = \Phi(g)$. For any $x \in M$, we have $\pi_1(f(x)) = \pi_1(g(x))$ and $\pi_2(f(x)) = \pi_2(g(x))$, so $f(x) = g(x)$ by Theorem 3A.

To show Φ is surjective: given $g : M \rightarrow K$ and $h : M \rightarrow L$, define $f : M \rightarrow K \times L$ by $f(x) = \langle g(x), h(x) \rangle$ for $x \in M$. Then $\Phi(f) = \langle g, h \rangle$.

Exercise 12 We have:

$$\begin{aligned} K \cup L &= L \cup K \\ K \cup (L \cup M) &= (K \cup L) \cup M \\ K \times (L \cup M) &= (K \times L) \cup (K \times M) \end{aligned}$$

Exercise 13 Now that we have shown the union of two finite sets is finite, this follows by an easy induction on $|B|$.

Exercise 14 For any set A , let $Perm(A)$ be the set of all permutations of A .

Assume $K \approx L$: we must show $Perm(K) \approx Perm(L)$. Pick a bijection $f : K \rightarrow L$. Define $\Phi : Perm(K) \rightarrow Perm(L)$ by: $\Phi(g) = f \circ g \circ f^{-1}$. It is easy to show $\Phi(g)$ is a permutation of L whenever g is a permutation of K , and Φ is a bijection.

6.4 Ordering Cardinal Numbers

Exercise 15 Suppose for a contradiction \mathcal{A} is a set and, for every set x , there exists $y \in \mathcal{A}$ such that $x \preccurlyeq y$. Pick $y \in \mathcal{A}$ such that $\mathcal{P} \cup \mathcal{A} \preccurlyeq y$. But $y \subseteq \bigcup \mathcal{A}$ so $\mathcal{P} \cup \mathcal{A} \preccurlyeq \bigcup \mathcal{A}$, contradicting Cantor's Theorem.

Exercise 16 Define $G : S \rightarrow^S 2$ by

$$G(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Then G is injective.

Now, assume for a contradiction $F : S \rightarrow^S 2$ is bijective. Define $g : S \rightarrow 2$ by $g(x) = 1 - F(x)(x)$. Then $g(x) \neq F(x)(x)$ for all $x \in S$, so $g \neq F(x)$ for all $x \in S$. Hence $g \notin \text{ran } F$. This contradicts the assumption that F is surjective.

Exercise 17 We have $1 < 2$ but $\aleph_0 + 1 = \aleph_0 + 2 = \aleph_0$.

We have $1 < 2$ but $\aleph_0 \cdot 1 = \aleph_0 \cdot 2 = \aleph_0$.

We have $2 < 3$ but $2^{\aleph_0} = 3^{\aleph_0}$.

We have $2 < 3$ but $\aleph_0^2 = \aleph_0^3 = \aleph_0$.

6.5 Axiom of Choice

Exercise 18 PROOF:

$\langle 1 \rangle 1$. If the Axiom of Choice is true then the statement is true.

PROOF: The statement is a special case of the multiplicative axiom, taking $I = \mathcal{A}$ and $H(X) = X$ for each $X \in \mathcal{A}$.

$\langle 1 \rangle 2$. If the statement is true then the Axiom of Choice is true.

$\langle 2 \rangle 1$. ASSUME: The statement is true.

PROVE: Axiom of choice IV

$\langle 2 \rangle 2$. LET: \mathcal{A} be a set such that each member of \mathcal{A} is a nonempty set, and any two distinct members of \mathcal{A} are disjoint.

$\langle 2 \rangle 3$. PICK a function f with domain \mathcal{A} such that $f(X) \in X$ for all $X \in \mathcal{A}$

$\langle 2 \rangle 4$. LET: $C = \text{ran } f$

$\langle 2 \rangle 5$. $\forall B \in \mathcal{A}. C \cap B = \{f(B)\}$

□

Exercise 19 PROOF:

$\langle 1 \rangle 1$. For $n \in \omega$, let $P(n)$ be the statement: for every set I with $\text{card } I = n$ and function H with domain I such that $H(i)$ is nonempty for each $i \in I$, there exists a function f with domain I such that $\forall i \in I. f(i) \in H(i)$.

$\langle 1 \rangle 2$. $P(0)$ is true

PROOF: Take $f = \emptyset$

$\langle 1 \rangle 3$. $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

$\langle 2 \rangle 1$. LET: $n \in \omega$

$\langle 2 \rangle 2$. ASSUME: $P(n)$

$\langle 2 \rangle 3$. LET: I be a set with $\text{card } I = n+1$

$\langle 2 \rangle 4$. LET: H be a function with domain I such that $H(i)$ is nonempty for each $i \in I$

$\langle 2 \rangle 5$. PICK a bijection $g : n+1 \approx I$

$\langle 2 \rangle 6$. PICK a function h with domain $g[n]$ such that $\forall i \in g[n]. h(i) \in H(i)$

$\langle 2 \rangle 7$. PICK $a \in H(g(n))$

$\langle 2 \rangle 8$. LET: $f = h \cup \{(g(n), a)\}$

$\langle 2 \rangle 9$. f is a function with domain I such that $\forall i \in I. f(i) \in H(i)$

□

Exercise 20 PROOF:

$\langle 1 \rangle 1$. PICK a choice function F for A

$\langle 1 \rangle 2$. PICK $a \in A$

⟨1⟩3. Define the function $f : \omega \rightarrow A$ by:

$$f(0) = a$$

$$f(n^+) = F(R^{-1}(f(n)))$$

PROOF: We know $R^{-1}(x)$ is nonempty for all $x \in A$ because $\forall x \in A. \exists y \in A. yRx$.

⟨1⟩4. $\forall n \in \omega. f(n^+)Rf(n)$

□

Exercise 21 PROOF:

⟨1⟩1. For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcup \mathcal{B} \in \mathcal{A}$

⟨2⟩1. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain.

⟨2⟩2. Every finite subset of $\bigcup \mathcal{B}$ is a member of \mathcal{A} .

⟨3⟩1. LET: $\{x_1, \dots, x_n\} \subseteq \bigcup \mathcal{B}$ be finite.

⟨3⟩2. For $1 \leq i \leq n$, PICK $B_i \in \mathcal{B}_i$ such that $x_i \in B_i$

⟨3⟩3. PICK m such that $B_1, \dots, B_n \subseteq B_m$

PROOF: Since \mathcal{B} is a chain.

⟨3⟩4. $\{x_1, \dots, x_n\}$ is a finite subset of B_m .

⟨3⟩5. $\{x_1, \dots, x_n\} \in \mathcal{A}$

PROOF: Since $B_m \in \mathcal{A}$ so every finite subset of B_m is a member of \mathcal{A} .

⟨2⟩3. $\bigcup \mathcal{B} \in \mathcal{A}$

⟨1⟩2. Q.E.D.

PROOF: By Zorn's Lemma.

□

Exercise 22 PROOF:

⟨1⟩1. If the Axiom of Choice is true then the statement is true.

⟨2⟩1. ASSUME: The Axiom of Choice

⟨2⟩2. LET: A be a set.

⟨2⟩3. LET: $R = \{\langle x, y \rangle : y \in A, x \in t\}$

⟨2⟩4. PICK a function $F \subseteq R$ such that $\text{dom } F = \text{dom } R$

⟨2⟩5. $\text{dom } R = \bigcup A$

⟨2⟩6. $\forall x \in \bigcup A. x \in F(x) \in A$

⟨1⟩2. If the statement is true then the Axiom of Choice is true.

⟨2⟩1. ASSUME: the statement

⟨2⟩2. LET: R be a relation

⟨2⟩3. LET: $A = \{\{\langle 0, x \rangle, \langle 1, y \rangle\} : xRy\}$

⟨2⟩4. PICK a function F with domain $\bigcup A$ such that $\text{dom } F = \bigcup A$ and $\forall x \in \bigcup A. x \in F(x) \in A$

⟨2⟩5. LET: $H = \{\langle x, y \rangle \mid x \in \text{dom } R, F(x) = \{\langle 0, x \rangle, \langle 1, y \rangle\}\}$

⟨2⟩6. H is a function, $H \subseteq R$, $\text{dom } H = \text{dom } R$

□

Exercise 23

⟨1⟩1. $g[0] = h(0)$

PROOF: Both are equal to \emptyset .

$\langle 1 \rangle 2. \forall n \in \omega. g[n] = h(n) \Rightarrow g[n^+] = h(n^+)$

$\langle 2 \rangle 1. \text{ LET: } n \in \omega$

$\langle 2 \rangle 2. \text{ ASSUME: } g[n] = h(n)$

$\langle 2 \rangle 3. g[n^+] = h(n^+)$

PROOF:

$$\begin{aligned} h(n^+) &= h(n) \cup \{F(A - h(n))\} \\ &= g[n] \cup \{g(n)\} \\ &= g[n^+] \end{aligned}$$

Exercise 24 Let $\{\kappa_i\}_{i \in I}$ be a family of cardinal numbers. For $i \in I$, let K_i be a set such that $\text{card } K_i = \kappa_i$.

We define $\sum_{i \in I} \kappa_i$ to be $\text{card}\{\langle i, x \rangle : i \in I, x \in K_i\}$

We define $\prod_{i \in I} \kappa_i$ to be $\text{card}\{f : f \text{ is a function, } \text{dom } f = I, \forall i \in I. f(i) \in K_i\}$.

Exercise 25 PROOF:

$\langle 1 \rangle 1. \text{ ASSUME: for a contradiction } \forall n \in \omega. B \not\subseteq S(n)$

$\langle 1 \rangle 2. \text{ PICK a function } b : \omega \rightarrow B \text{ such that } \forall n \in \omega. b(n) \notin S(n)$

PROOF: By the Axiom of Choice.

$\langle 1 \rangle 3. \text{ LET: } B' = \{b(n) : n \in \omega\}$

$\langle 1 \rangle 4. B' \text{ is infinite.}$

$\langle 2 \rangle 1. \text{ ASSUME: for a contradiction } B' \text{ is finite.}$

$\langle 2 \rangle 2. \text{ There exists } N \text{ such that } \forall n > N. \exists k \leq N. b(n) = b(k)$

$\langle 2 \rangle 3. \text{ PICK } M > N \text{ such that } \forall k \leq N. b(k) \in S(M)$

PROOF: For $k \leq N$ there exists n_k such that $b(k) \in S(n_k)$. Take M to be the largest of these numbers and $N + 1$.

$\langle 2 \rangle 4. b(M) \in S(M)$

PROOF: Since $b(M) = b(k)$ for some $k \leq N$.

$\langle 2 \rangle 5. \text{ Q.E.D.}$

PROOF: This contradicts $\langle 1 \rangle 2$.

$\langle 1 \rangle 5. \text{ PICK } n \text{ such that } B' \cap S(n) \text{ is infinite.}$

$\langle 1 \rangle 6. \text{ PICK } m > n \text{ such that } b(m) \in B' \cap S(n)$

PROOF: There must be some m otherwise $B' \cap S(n) \subseteq \{b(0), b(1), \dots, b(n)\}$ would be finite.

$\langle 1 \rangle 7. b(m) \in S(m)$

PROOF: Since $S(n) \subseteq S(m)$.

$\langle 1 \rangle 8. \text{ Q.E.D.}$

PROOF: This contradicts $\langle 1 \rangle 2$.

□

6.6 Countable Sets

Exercise 26 PROOF:

$\langle 1 \rangle 1. \text{ PICK a set } K \text{ of cardinality } \kappa$

- ⟨1⟩2. For all $X \in \mathcal{A}$, there exists an injective function $X \rightarrow K$
 ⟨1⟩3. PICK a function F with domain \mathcal{A} such that, for all $X \in \mathcal{A}$, $F(X)$ is an injective function $X \rightarrow K$
 PROOF: By the Axiom of Choice.
 ⟨1⟩4. PICK a function G with domain $\bigcup \mathcal{A}$ such that, for all $x \in \bigcup \mathcal{A}$, we have $x \in G(x) \in \mathcal{A}$
 PROOF: By Exercise 22.
 ⟨1⟩5. Define $f : \bigcup \mathcal{A} \rightarrow \mathcal{A} \times K$ by $f(x) = \langle G(x), F(G(x))(x) \rangle$
 ⟨1⟩6. f is injective.
 ⟨2⟩1. LET: $x, y \in \bigcup \mathcal{A}$
 ⟨2⟩2. ASSUME: $f(x) = f(y)$
 ⟨2⟩3. $G(x) = G(y)$ and $F(G(x))(x) = F(G(y))(y)$
 ⟨2⟩4. $F(G(x))(x) = F(G(x))(y)$
 ⟨2⟩5. $x = y$
 PROOF: Since $F(G(x))$ is injective.

□

Exercise 27

(a) Pick a function $f : A \rightarrow \mathbb{Q}^2$ such that $f(c) \in c$ for all $c \in A$. Then f is an injection, so $A \preccurlyeq \mathbb{Q}^2$ which is countable.

(b) No: the set of all circles with center $(0, 0)$ is an uncountable set of circles no two of which intersect.

(c) Yes. Pick a function $f : C \rightarrow \mathbb{Q}^4$ such that $f(x)$ is a pair of points with rational coordinates, one in each circle of x , for all $x \in C$. Then f is an injection; it is not possible for two points to be in separate circles of two non-intersecting figure-eights. Hence $C \preccurlyeq \mathbb{Q}^4$.

Exercise 28 Let $\mathcal{A} = \{(a, \sqrt{2}) : a < \sqrt{2}\} \cup \{(\sqrt{2}, b) : b > \sqrt{2}\}$. Then every rational is in some member of \mathcal{A} but $\bigcup \mathcal{A} = \mathbb{R} - \{\sqrt{2}\}$.

(Enderton's hint suggests he had a different solution in mind, but I am not sure what it is.)

Exercise 29 For each integer $n \geq 2$, let $B_n = \{x \in A : x > b/n\}$. Then each B_n is finite (B_n cannot have more than $n - 1$ elements because n elements in B_n would have a sum $> b$) and $A = \bigcup_n B_n$. So A is a countable union of finite sets, and therefore countable.

Exercise 30 PROOF:

- ⟨1⟩1. PICK $a \in A$
 ⟨1⟩2. Define $f : Sq(A) \rightarrow \omega \times^\omega A$ by $f(s) = \langle n, g \rangle$, where n is the length of s , and $g(i) = s(i)$ for $i < n$, $g(i) = a$ for $i \geq n$

- $\langle 1 \rangle 3.$ f is injective.
 $\langle 1 \rangle 4.$ $Sq(A) \preceq \omega \times^\omega A$
 $\langle 1 \rangle 5.$ $\text{card } Sq(A) \leq (\text{card } A)^{\aleph_0}$

PROOF:

$$\begin{aligned}
 \text{card } Sq(A) &\leq \aleph_0 \cdot (\text{card } A)^{\aleph_0} && (\langle 1 \rangle 4) \\
 &\leq (\text{card } A)^{\aleph_0} \cdot (\text{card } A)^{\aleph_0} && (\text{Cantor's Theorem}) \\
 &= (\text{card } A)^{\aleph_0 + \aleph_0} && (\text{Theorem 6I}) \\
 &= (\text{card } A)^{\aleph_0}
 \end{aligned}$$

6.7 Arithmetic of Infinite Cardinals

Exercise 31 If f is a one-to-one correspondence between $A \times A$ and A , where $A \subseteq B$, then

$$f \subseteq (A \times A) \times A \subseteq (B \times B) \times B .$$

Also $\emptyset \subseteq (B \times B) \times B$. So we can form \mathcal{H} by applying a Subset Axiom to $\mathcal{P}((B \times B) \times B)$.

Exercise 32 The function that maps x to $\{x\}$ is an injection $A \rightarrow \mathcal{F}A$, so we have $A \approx \mathcal{F}A$.

For the converse, let $F_n = \{X \in \mathcal{F}A : \text{card } X \leq n\}$ for $n \in \omega$. The function that sends $\langle a_1, \dots, a_n \rangle$ to $\{a_1, \dots, a_n\}$ is a surjection $A^n \rightarrow F_n$, so we have

$$\text{card } F_n \leq (\text{card } A)^n = \text{card } A$$

by Lemma 6R. Now, $\mathcal{F}A = \bigcup_n F_n$, so

$$\text{card } \mathcal{F}A \leq \aleph_0 \cdot \text{card } A = \text{card } A$$

by the Absorption Law.

Exercise 33 The function that maps a to the sequence of length 1 containing a is an injection $A \rightarrow Sq(A)$, so $A \preceq Sq(A)$.

For the converse, we have $\text{card}(^n A) = (\text{card } A)^n = \text{card } A$ for any natural number n

$$\begin{aligned}
 \text{card } Sq(A) &= \text{card}(^0 A \cup ^1 A \cup ^2 A \cup \dots) \\
 &= \aleph_0 \cdot \text{card } A \\
 &= \text{card } A
 \end{aligned}$$

by the Absorption Law.

Exercise 34

$$\begin{aligned}
 2^\lambda &\leq \kappa^\lambda \\
 &\leq (2^\kappa)^\lambda \\
 &= 2^{\kappa \cdot \lambda} \\
 &= 2^\lambda \qquad \qquad \qquad (\text{Absorption Law})
 \end{aligned}$$

Exercise 35 For any infinite set of primes A and natural number n , let $f(A, n) = \prod \{p \in A : p \leq n\}$. Let $P(A) = \{f(A, n) : n \in \omega\}$. Let \mathcal{A} be the set of all sets of the form $P(A)$.

The number of infinite sets of primes is 2^{\aleph_0} (there are 2^{\aleph_0} sets of primes and \aleph_0 finite sets of primes by Exercise 32.)

If $P(A) = P(B)$ then $A = B$. (If $p \in A - B$ then $p \mid f(A, p)$ but p does not divide any member of $P(B)$.) So P is an injection from the set of infinite sets of primes into \mathcal{A} . Hence $\text{card } \mathcal{A} = 2^{\aleph_0}$.

We now prove that, if $A \neq B$, then $P(A) \cap P(B)$ is finite. Let $p \in A - B$. For $n \geq p$ we have $f(A, n) \notin P(B)$ since $p \mid f(A, n)$ but p does not divide any member of B . Hence $A \cap B \subseteq \{f(A, 0), f(A, 1), \dots, f(A, p-1)\}$.

Exercise 36 PROOF:

- $\langle 1 \rangle 1$. For any set A , there exists a permutation of A with no fixed points.
- $\langle 2 \rangle 1$. For every natural number n , there exists a permutation of n with no fixed points.
PROOF: Map i to $i + 1$ if $i + 1 < n$, and map $n - 1$ to 0.
- $\langle 2 \rangle 2$. For every infinite set A , there exists a permutation of A with no fixed points.
- $\langle 3 \rangle 1$. PICK a bijection $f : A \approx A \times 2$
- $\langle 3 \rangle 2$. Define $\pi : A \times 2 \rightarrow A \times 2$ by $\pi(x, 0) = (x, 1)$ and $\pi(x, 1) = (x, 0)$
- $\langle 3 \rangle 3$. $f^{-1} \circ \pi \circ f$ is a permutation of A with no fixed point.
- $\langle 1 \rangle 2$. $\kappa! \leq 2^\kappa$
PROOF: Because the set of permutations of K is a subset of ${}^K K$, where K is a set of cardinality κ .
- $\langle 1 \rangle 3$. $2^\kappa \leq \kappa!$
- $\langle 2 \rangle 1$. PICK a set K of cardinality κ
- $\langle 2 \rangle 2$. LET: $\text{Perm}(K)$ be the set of permutations of K .
- $\langle 2 \rangle 3$. Define $f : \mathcal{P}K \rightarrow \text{Perm}(K)$ as follows. Given $A \subseteq \mathcal{P}K$, pick a permutation π_{K-A} of $K - A$ with no fixed point. Then $f(A) = I_A \cup \pi_{K-A}$
- $\langle 2 \rangle 4$. f is injective
PROOF: The function that maps a permutation to its set of fixed points is a left inverse.
- $\langle 2 \rangle 5$. $2^\kappa \leq \kappa!$

□

Chapter 7

Chapter 7 — Orderings and Ordinals

7.1 Partial Orderings

Exercise 1

(a) No we cannot. Let $A = \mathcal{P}3$ and $B = \omega$. Let $<_A = \subset_3$ and $<_B$ be the usual ordering on ω . Define $f : A \rightarrow B$ by: $f(X) = \text{card } X$. Then $X \subset_2 Y \Rightarrow \text{card } X < \text{card } Y$ but f is not one-to-one because $f(\{0\}) = f(\{1\}) = 1$.

(b) No we cannot. With the same example, we have $f(\{0\}) < f(\{1, 2\})$ but $\{0\} \not\subset \{1, 2\}$.

Exercise 2 We show R^{-1} is transitive. Suppose $xR^{-1}y$ and $yR^{-1}z$. Then zRx and yRx , so zRx because R is transitive. Hence $xR^{-1}z$.

We now show R^{-1} is irreflexive. For any x , we have $\langle x, x \rangle \notin R$, so $\langle x, x \rangle \notin R^{-1}$.

Exercise 3 The proof is by induction on n .

The only linear ordering on \emptyset is \emptyset , which has 0 pairs.

Suppose that, whenever $\text{card } S = n$, then every linear ordering on S has $1/2n(n-1)$ pairs. Let S be a set of cardinality $n+1$. Let $<$ be a linear ordering on S .

Pick an element $a \in S$ and let $T = S - \{a\}$. Then $< \cap (T \times T)$ is a linear ordering on T , hence has $1/2n(n-1)$ pairs. Now, for every $x \in T$, exactly one of $\langle x, a \rangle$ and $\langle a, x \rangle$ is in $<$. Hence $<$ has n pairs that are not in $< \cap (T \times T)$. So

$$\text{card } < = 1/2n(n-1) + n = 1/2n(n+1) .$$

7.2 Well Orderings

Exercise 4 PROOF:

- ⟨1⟩1. R is transitive.
- ⟨2⟩1. ASSUME: mRn and nRp .
- ⟨2⟩2. CASE: $f(m) < f(n)$
PROOF: In this case $f(m) < f(p)$ so mRp .
- ⟨2⟩3. CASE: $f(m) = f(n)$ and $m < n$.
- ⟨3⟩1. CASE: $f(n) < f(p)$
PROOF: In this case $f(m) < f(p)$ so mRp .
- ⟨3⟩2. CASE: $f(n) = f(p)$ and $n < p$.
PROOF: In this case $f(m) = f(p)$ and $m < p$ so mRp .
- ⟨1⟩2. R satisfies trichotomy on P .
- ⟨2⟩1. LET: $m, n \in P$
- ⟨2⟩2. Exactly one of $f(m) < f(n)$, $f(n) < f(m)$, $f(n) = f(m)$ holds.
- ⟨2⟩3. Exactly one of $m < n$, $n < m$, $n = m$ holds.
- ⟨2⟩4. Exactly one of $f(m) < f(n)$, $(f(m) = f(n) \ \& \ m < n)$, $(f(m) = f(n) \ \& \ m = n)$, $(f(m) = f(n) \ \& \ n < m)$, $f(n) < f(m)$ holds.
- ⟨2⟩5. Exactly one of mRn , $m = n$, nRm holds.
- ⟨1⟩3. Every nonempty subset of P has an R -least element.
- ⟨2⟩1. LET: $A \subseteq P$ be nonempty.
- ⟨2⟩2. LET: k be the least element of $f(A)$.
- ⟨2⟩3. LET: n be the least element of $f^{-1}(k) \cap A$.
- ⟨2⟩4. n is the R -least element of A .

□

$\langle P, R \rangle$ resembles Fig. 45 (d).

Exercise 5 PROOF:

- ⟨1⟩1. LET: $x \in A$
- ⟨1⟩2. ASSUME: for a contradiction $f(x) < x$
- ⟨1⟩3. Define $g : \omega \rightarrow A$ by $g(0) = x$ and $g(n^+) = f(g(n))$ for all $n \in \omega$
- ⟨1⟩4. $\forall n \in \omega. g(n^+) < g(n)$
PROOF: By induction on n using ⟨1⟩2 and the hypothesis.
- ⟨1⟩5. Q.E.D.
PROOF: This contradicts Theorem 7B.

□

Exercise 6 PROOF:

- ⟨1⟩1. For all $x \in S$ that is not greatest, there exists $y \in S$ and $q \in \mathbb{Q}$ such that $x < q < y$ and there is no $z \in S$ such that $x < z < y$
- ⟨1⟩2. PICK a function $f : S \rightarrow \mathbb{Q}$ such that $\forall x \in S. x < f(x)$ and, if x is not greatest, then $f(x) < y$ where y is the next element in S .
- ⟨1⟩3. f is injective.
- ⟨1⟩4. $S \preccurlyeq \mathbb{Q}$

□

Exercise 7

(a) We have $F(t) = C \cup \bigcup \text{ran}(F \upharpoonright t)$ for all $t \in \omega$. So:

$$\begin{aligned}
 F(0) &= C \cup \bigcup \text{ran} \emptyset \\
 &= C \\
 F(1) &= C \cup \bigcup \text{ran}(F \upharpoonright 0) \\
 &= C \cup \bigcup \{C\} \\
 &= C \cup C \\
 F(2) &= C \cup \bigcup \{C, C \cup C\} \\
 &= C \cup (C \cup C) \\
 &= C \cup C \cup C
 \end{aligned}$$

We guess:

$$F(n) = C \cup C \cup \dots \cup \underbrace{C}_n \cup \dots \cup C$$

(b) PROOF:

- $\langle 1 \rangle 1.$ LET: $a \in F(n)$
- $\langle 1 \rangle 2.$ $a \in \bigcup \text{ran}(F \upharpoonright n^+)$
- $\langle 1 \rangle 3.$ $a \subseteq \bigcup \text{ran}(F \upharpoonright n^+)$
- $\langle 1 \rangle 4.$ $a \subseteq F(n^+)$

□

(c) PROOF:

- $\langle 1 \rangle 1.$ \overline{C} is a transitive set.
- $\langle 2 \rangle 1.$ LET: $x \in y \in \overline{C}$
- $\langle 2 \rangle 2.$ PICK $n \in \omega$ such that $y \in F(n)$
- $\langle 2 \rangle 3.$ $x \in F(n^+)$

PROOF: By (b).

- $\langle 2 \rangle 4.$ $x \in \overline{C}$
- $\langle 1 \rangle 2.$ $C \subseteq \overline{C}$
- $\langle 2 \rangle 1.$ Since $C = F(0)$

□

7.3 Replacement Axioms

Exercise 8 Let $P(x)$ be a formula not containing B . We prove the statement

$$\forall c \exists B \forall x (x \in B \Leftrightarrow x \in c \ \& \ P(x)) .$$

Let $Q(x, y)$ be the formula $P(x) \wedge y = x$. Now we reason as follows.

Let c be any set. Then we have

$$(\forall x \in c) \forall y_1 \forall y_2 (Q(x, y_1) \ \& \ Q(x, y_2) \Rightarrow y_1 = y_2)$$

Then, by a Replacement Axiom, there exists a set B such that

$$\forall y (y \in B \Leftrightarrow (\exists x \in c) Q(x, y)) \ .$$

This is equivalent to $\forall x (x \in B \Leftrightarrow x \in c \ \& \ P(x))$.

Exercise 9 Let a and b be sets. Let $P(x, y)$ be the formula $(x = \emptyset \ \& \ y = a)$ or $(x = \mathcal{P}\emptyset \ \& \ y = b)$. Then we have $(\forall x \in \mathcal{P}\mathcal{P}\emptyset) \forall y_1 \forall y_2 (P(x, y_1) \ \& \ P(x, y_2) \Rightarrow y_1 = y_2)$, hence there exists a set c such that

$$\forall y (y \in c \Leftrightarrow (\exists x \in \mathcal{P}\mathcal{P}\emptyset) P(x, y))$$

The members of c are just a and b .

7.4 Epsilon-Images

Exercise 10

(a) Let n be a natural number. Let α be its epsilon-image, and $E : n \rightarrow \alpha$ be as in the definition of epsilon-image.

We prove $\forall x \in n. E(x) = x$ by strong induction on x . Let $x \in n$ and assume $\forall y \in x. E(y) = y$. Then

$$\begin{aligned} E(x) &= \{E(y) : y \in x\} \\ &= \{y : y \in x\} \\ &= x \end{aligned}$$

Hence

$$\begin{aligned} \alpha &= \{E(x) : x \in n\} \\ &= \{x : x \in n\} \\ &= n \end{aligned}$$

(b) Similarly the ϵ -image of ω is ω .

Exercise 11

(a) Let R be the ordering given in the question. Thus xRy iff (x and y are nonnegative and $x < y$) or (x and y are both negative and $y < x$) or (x is nonnegative and y is negative).

PROOF:

- ⟨1⟩1. R is transitive
 - ⟨2⟩1. ASSUME: xRy and yRz
 - ⟨2⟩2. CASE: x and y are nonnegative and $x < y$
 - ⟨3⟩1. CASE: z is nonnegative and $y < z$
 - PROOF: In this case x and z are nonnegative and $x < z$.
 - ⟨3⟩2. CASE: z is negative
 - PROOF: In this case x is nonnegative and z is negative.
 - ⟨2⟩3. CASE: x and y are both negative and $y < x$
 - PROOF: We must have z is negative and $z < y$, hence $z < x$.
 - ⟨2⟩4. CASE: x is nonnegative and y is negative
 - PROOF: We must have z is negative.
- ⟨1⟩2. R satisfies trichotomy on \mathbb{Z}
 - ⟨2⟩1. LET: $x, y \in \mathbb{Z}$
 - ⟨2⟩2. CASE: x and y are nonnegative.
 - PROOF: Exactly one of $x < y$, $x = y$, $y < x$ holds.
 - ⟨2⟩3. CASE: x is nonnegative and y is negative.
 - PROOF: In this case $x < y$.
 - ⟨2⟩4. CASE: x is negative and y is nonnegative.
 - PROOF: In this case $y < x$.
 - ⟨2⟩5. CASE: x and y are negative.
 - PROOF: Exactly one of $x < y$, $x = y$, $y < x$ holds.
- ⟨1⟩3. R is well-founded
 - ⟨2⟩1. LET: $A \subseteq \mathbb{Z}$ be nonempty.
 - ⟨2⟩2. CASE: There exists a nonnegative integer in A .
 - PROOF: Let n be the least nonnegative element of A . Then n is the R -least element of A .
 - ⟨2⟩3. CASE: All elements of A are negative.
 - PROOF: Let n be least such that $-n \in A$. Then $-n$ is the R -least element of A .

□

(b)

$$\begin{aligned}
 E(3) &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
 &= 3 \\
 E(-1) &= \omega \\
 E(-2) &= \omega^+ \\
 \text{ran } E &= \omega \cup \{\omega, \omega^+, \omega^{++}, \dots\}
 \end{aligned}$$

7.5 Isomorphisms

Exercise 12

(a) PROOF:

$\langle 1 \rangle 1$. $<_A$ is irreflexive.

PROOF: For any $x \in A$ we have $f(x) \not<_B f(x)$ so $x \not<_A x$.

$\langle 1 \rangle 2$. $<_A$ is transitive.

PROOF: If $x <_A y$ and $y <_A z$ then $f(x) <_B f(y) <_B f(z)$ hence $f(x) <_B f(z)$ and so $x <_A z$.

(b) For any $x, y \in A$ we have that exactly one of $f(x) <_B f(y)$, $f(x) = f(y)$, $f(y) <_B f(x)$ holds. Hence exactly one of $x <_A y$, $x = y$, $y <_A x$ holds. (Using the fact that $x = y$ iff $f(x) = f(y)$ since f is one-to-one.)

Exercise 13 PROOF:

$\langle 1 \rangle 1$. LET: $\langle A, <_A \rangle$ and $\langle B, <_B \rangle$ be two well-ordered structures.

$\langle 1 \rangle 2$. LET: $f, g : A \rightarrow B$ be isomorphisms.

PROVE: $\forall x \in A. f(x) = g(x)$

$\langle 1 \rangle 3$. LET: $x \in A$

$\langle 1 \rangle 4$. ASSUME: $\forall y < x. f(y) = g(y)$

$\langle 1 \rangle 5$. $f(x)$ is the least element in $B - f[\text{seg } x]$

$\langle 2 \rangle 1$. $f(x) \notin f[\text{seg } x]$

PROOF: Since f is one-to-one.

$\langle 2 \rangle 2$. $\forall b \in B - f[\text{seg } x]. f(x) \leq b$

$\langle 3 \rangle 1$. LET: $b \in B - f[\text{seg } x]$

$\langle 3 \rangle 2$. LET: $a \in A$ be such that $f(a) = b$

PROOF: f is surjective.

$\langle 3 \rangle 3$. $a \notin \text{seg } x$

$\langle 3 \rangle 4$. $x \leq a$

PROOF: By trichotomy

$\langle 3 \rangle 5$. $f(x) \leq b$

$\langle 1 \rangle 6$. $g(x)$ is the least element in $B - g[\text{seg } x]$

PROOF: Similar.

$\langle 1 \rangle 7$. $f[\text{seg } x] = g[\text{seg } x]$

PROOF: By $\langle 1 \rangle 4$

$\langle 1 \rangle 8$. $f(x) = g(x)$

□

Exercise 14 PROOF:

$\langle 1 \rangle 1$. $\forall a, b \in A. a < b \Rightarrow F(a) \subset F(b)$

$\langle 2 \rangle 1$. LET: $a, b \in A$

$\langle 2 \rangle 2$. ASSUME: $a < b$

$\langle 2 \rangle 3$. $F(a) \subseteq F(b)$

PROOF: If $x \leq a$ then $x \leq b$

$\langle 2 \rangle 4. F(a) \neq F(b)$
 PROOF: Since $b \in F(b)$ but $b \notin F(a)$
 $\langle 1 \rangle 2. \forall a, b \in A. F(a) \subset F(b) \Rightarrow a < b$
 PROOF: We cannot have $b < a$ or $b = a$ (as then $F(b) \subset F(a)$ or $F(b) = F(a)$ by $\langle 1 \rangle 1$), so $a < b$ by trichotomy.
 $\langle 1 \rangle 3. F$ is one-to-one
 PROOF: If $F(a) = F(b)$ then we cannot have $a < b$ or $b < a$ by $\langle 1 \rangle 1$, so $a = b$ by trichotomy.
 $\langle 1 \rangle 4. F$ maps A onto $\text{ran } F$
 PROOF: By definition of $\text{ran } F$.
 \square

7.6 Ordinal Numbers

Exercise 15

(a) PROOF:
 $\langle 1 \rangle 1.$ ASSUME: $f : A \rightarrow \text{seg } t$ is an isomorphism
 $\langle 1 \rangle 2.$ Define $g : \omega \rightarrow A$ by recursion:

$$g(0) = t$$

$$g(n^+) = f(g(n)) \quad (n \in \omega)$$
 $\langle 1 \rangle 3. \forall n \in \omega. g(n^+) < g(n)$
 $\langle 2 \rangle 1. g(0^+) < g(0)$
 PROOF: Since $g(0^+) = f(t) \in \text{seg } t$ so $g(0^+) < t = g(0)$.
 $\langle 2 \rangle 2. \forall n \in \omega. (g(n^+) < g(n) \Rightarrow g(n^{++}) < g(n^+))$
 $\langle 3 \rangle 1.$ LET: $n \in \omega$
 $\langle 3 \rangle 2.$ ASSUME: $g(n^+) < g(n)$
 $\langle 3 \rangle 3. f(g(n^+)) < f(g(n))$
 PROOF: Since f is an isomorphism.
 $\langle 3 \rangle 4. g(n^{++}) < g(n^+)$
 $\langle 1 \rangle 4.$ Q.E.D.
 PROOF: This contradicts Theorem 7B.
 \square

(b) If two of them hold then we have a well-ordered set isomorphic with an initial segment, contradicting part (a):
 If $A \cong B$ and $A \cong \text{seg } b$ then $B \cong \text{seg } b$.
 If $A \cong B$ and $\text{seg } a \cong B$ then $A \cong \text{seg } a$.
 Now assume $A \cong \text{seg } b$ and $\text{seg } a \cong B$. Let $f : A \cong \text{seg } b$ and $g : \text{seg } a \cong B$ be isomorphisms. Let $b_0 = f(a)$. Then $f \upharpoonright \text{seg } a : \text{seg } a \cong \text{seg } b_0$ and so $B \cong \text{seg } b_0$.

Exercise 16 Suppose $\alpha \in \beta$. We first prove that $\beta \notin \alpha^+$.

If $\beta \in \alpha^+$ then $\beta \in \alpha$ or $\beta = \alpha$. In either case we have $\alpha \in \alpha$, which is impossible.

So $\beta \notin \alpha^+$. Therefore $\alpha^+ \subseteq \beta$, and so $\alpha^+ \in \beta^+$.

Now, suppose $\alpha \neq \beta$. Then $\alpha \in \beta$ or $\beta \in \alpha$. Hence $\alpha^+ \in \beta^+$ or $\beta^+ \in \alpha^+$, and in either case $\alpha^+ \neq \beta^+$.

Exercise 17 Suppose for a contradiction $\alpha \in \beta$. Then A is isomorphic to $\text{seg}_B b$ for some $b \in B$. Let $f : A \rightarrow \text{seg}_B b$ be an isomorphism.

We have $f \upharpoonright B : B \rightarrow \text{seg}_B b$. Now, define $g : \omega \rightarrow B$ by

$$\begin{aligned} g(0) &= b \\ g(n^+) &= f(g(n)) \end{aligned}$$

Then $g(n^+) < g(n)$ for all $n \in \omega$, contradicting Theorem 7B.

Exercise 18 Suppose first $\bigcup S \in S$. For all $\alpha \in S$ we have $\alpha \subseteq \bigcup S$ and so $\alpha \in \bigcup S$, and so $\bigcup S$ is the greatest element of S .

Suppose now $\bigcup S \notin S$. Suppose for a contradiction $\alpha \in S$ is the greatest element of S . We have $\alpha \subseteq \bigcup S$ (because $\alpha \in S$). Also for all $\beta \in S$ we have $\beta \subseteq \alpha$, hence $\bigcup S \subseteq \alpha$. Thus $\bigcup S = \alpha \in S$, which is a contradiction.

So if $\bigcup S \notin S$ then S has no greatest element. Therefore S cannot be the successor of any ordinal, because α is the greatest element of α^+ for any α .

Exercise 19 By Theorem 7B, every linear ordering on a finite set is a well ordering.

If $<$ and \prec are two linear orderings on the same set A , we cannot have that $(A, <)$ is isomorphic to $(\text{seg } a, \prec)$ for any $a \in A$, because then we would have a finite set bijective with a proper subset of itself.

So by Theorem 7E we must have $\langle A, < \rangle \cong \langle A, \prec \rangle$.

Exercise 20 Let R be a well ordering on the set S . Assume S is infinite; we will prove R^{-1} is not a well-ordering on S .

Define $g : \omega \rightarrow S$ by: $g(n)$ is the least element of $S - g[[n]]$. For each n , we know $S - g[[n]]$ is nonempty because S is infinite.

Then $g[[\omega]]$ is a nonempty subset of S that has no R^{-1} -least element (no R -greatest element), so R^{-1} is not a well ordering on S .

Exercise 21 Let $\mathcal{A} = \{C \in \mathcal{P}A : <^\circ \text{ is a linear ordering on } C\}$.

We prove that, for every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$.

Let $\mathcal{B} \subseteq \mathcal{A}$ be a chain. Let $x, y \in \bigcup \mathcal{B}$. Pick $C, D \in \mathcal{B}$ such that $x \in C$ and $y \in D$. Then either $C \subseteq D$ or $D \subseteq C$; assume without loss of generality $C \subseteq D$. We have $x, y \in D$, and so exactly one of $x < y$, $x = y$, $y < x$ holds. Thus, $<^\circ$ linearly orders $\bigcup \mathcal{B}$, i.e. $\bigcup \mathcal{B} \in \mathcal{A}$.

Hence by Zorn's Lemma \mathcal{A} has a maximal element C , say. Now, by hypothesis, C has an upper bound m . We prove m is maximal in A .

Let $x \in A$ and suppose $m \leq x$. Then $C \cup \{m, x\}$ is linearly ordered by $<^\circ$, and so $C = C \cup \{m, x\}$ by maximality of C . Hence $x \in C$ and so $x \leq m$, hence $x = m$. Thus, m is maximal in A .

7.7 Debts Paid

Exercise 22 Let A be any set. Let \mathcal{A} be the set of all pairs $\langle B, R \rangle$ where $B \subseteq A$ and R is a well ordering on B , and define $<$ on \mathcal{A} by: $\langle B, R \rangle < \langle C, S \rangle$ iff B is an initial segment of C and $R = S \cap B^2$.

It is easy to see that $<$ is a partial ordering on \mathcal{A}

We prove that, if $\mathcal{C} \subseteq \mathcal{A}$ and $<$ is a linear ordering on \mathcal{C} , then \mathcal{C} has an upper bound in \mathcal{A} . Let $B = \bigcup \{C : \exists S. \langle C, S \rangle \in \mathcal{C}\}$ and $R = \bigcup \{S : \exists C. \langle C, S \rangle \in \mathcal{C}\}$. We prove that R well orders B . It is then easy to see that $\langle B, R \rangle$ is an upper bound for \mathcal{C} in \mathcal{A} .

PROOF:

- $\langle 1 \rangle 1.$ R is transitive.
 - $\langle 2 \rangle 1.$ ASSUME: xRy and yRz
 - $\langle 2 \rangle 2.$ PICK $\langle C, S \rangle, \langle D, T \rangle \in \mathcal{C}$ such that xSy and yTz
 - $\langle 2 \rangle 3.$ $\langle C, S \rangle \leq \langle D, T \rangle$ or $\langle D, T \rangle \leq \langle C, S \rangle$
 - $\langle 2 \rangle 4.$ ASSUME: w.l.o.g. $\langle C, S \rangle \leq \langle D, T \rangle$
 - $\langle 2 \rangle 5.$ xTy and yTz
 - $\langle 2 \rangle 6.$ xTz
 - $\langle 2 \rangle 7.$ xRz
- $\langle 1 \rangle 2.$ R is irreflexive.
 - $\langle 2 \rangle 1.$ ASSUME: for a contradiction xRx
 - $\langle 2 \rangle 2.$ PICK $\langle C, S \rangle \in \mathcal{C}$ such that xSx
 - $\langle 2 \rangle 3.$ This is a contradiction.
- $\langle 1 \rangle 3.$ R satisfies trichotomy.
 - $\langle 2 \rangle 1.$ LET: $x, y \in B$
 - $\langle 2 \rangle 2.$ PICK $\langle C, S \rangle, \langle D, T \rangle \in \mathcal{C}$ such that $x \in C$ and $y \in D$
 - $\langle 2 \rangle 3.$ ASSUME: w.l.o.g. $\langle C, S \rangle \leq \langle D, T \rangle$
 - $\langle 2 \rangle 4.$ $x, y \in D$
 - $\langle 2 \rangle 5.$ xTy or yTx
 - $\langle 2 \rangle 6.$ xRy or yRx
- $\langle 1 \rangle 4.$ Every non-empty subset of B has an R -least element.
 - $\langle 2 \rangle 1.$ LET: $C \subseteq B$ be nonempty
 - $\langle 2 \rangle 2.$ PICK $c \in C$
 - $\langle 2 \rangle 3.$ PICK $\langle D, T \rangle \in \mathcal{C}$ such that $c \in D$
 - $\langle 2 \rangle 4.$ LET: x be the T -least element of $C \cap D$
 PROVE: x is R -least in C
 - $\langle 2 \rangle 5.$ LET: $y \in C$
 - $\langle 2 \rangle 6.$ PICK $\langle E, U \rangle \in \mathcal{C}$ such that $y \in E$
 - $\langle 2 \rangle 7.$ $\langle D, T \rangle \leq \langle E, U \rangle$ or $\langle E, U \rangle \leq \langle D, T \rangle$
 - $\langle 2 \rangle 8.$ CASE: $\langle D, T \rangle \leq \langle E, U \rangle$
 - $\langle 3 \rangle 1.$ xUy or $x = y$

PROOF:

- $\langle 4 \rangle 1.$ ASSUME: for a contradiction yUx
- $\langle 4 \rangle 2.$ $y \in D$ and yTx

PROOF: Since D is an initial segment of E and $T = U \cap D^2$

⟨4⟩3. Q.E.D.

PROOF: This contradicts the T -minimality of x .

⟨2⟩9. CASE: $\langle E, U \rangle \leq \langle D, T \rangle$

PROOF: xTy or $x = y$, so xRy or $x = y$.

Hence by Exercise 21 there is a maximal element $\langle B, R \rangle$ in \mathcal{A} . We must have $B = A$; for if $a \in A - B$ then $\langle B \cup \{a\}, R \cup \{\langle x, a \rangle : x \in B\} \rangle$ would be a larger element. Hence R is a well ordering on A .

Exercise 23

(i) We must show that α is an initial ordinal. So let $\beta \in \alpha$. Then $\beta \prec A$ but $\alpha \not\prec A$. Hence $\alpha \not\approx \beta$.

(ii) We know that $\alpha \not\prec A$, so $\alpha \not\leq \text{card } A$.

(iii) Let κ be any cardinal greater than $\text{card } A$. Then κ is not dominated by A , so $\kappa \notin \alpha$, and so $\alpha \subseteq \kappa$.

Exercise 24 The cardinal number of $\mathcal{P}\alpha$ is larger than α (both as a cardinal and as an ordinal).

Exercise 25 Suppose there exists an ordinal α such that $\neg\phi(\alpha)$. Let α_0 be the least such ordinal. Then we have $\forall x \in \alpha_0. \phi(x)$ but $\neg\phi(\alpha_0)$. This contradicts the hypothesis.

7.8 Rank

Exercise 26 The proof is by transfinite induction on α . Suppose that α is an ordinal and, for all $\beta \in \alpha$, we have β is grounded and $\text{rank } \beta = \beta$. Then by Theorem 7V(b) we have that α is grounded and

$$\begin{aligned} \text{rank } \alpha &= \bigcup \{(\text{rank } \beta)^+ \mid \beta \in \alpha\} \\ &= \bigcup \{\beta^+ \mid \beta \in \alpha\} \quad (\text{induction hypothesis}) \end{aligned}$$

So we must show that $\bigcup \{\beta^+ \mid \beta \in \alpha\} = \alpha$.

If $\beta \in \alpha$ then $\beta^+ \subseteq \alpha$ so $\beta^+ \subseteq \alpha$. This shows that $\bigcup \{\beta^+ \mid \beta \in \alpha\} \subseteq \alpha$.

If $\beta \in \alpha$ then $\beta \in \beta^+$ so $\beta \in \bigcup \{\beta^+ \mid \beta \in \alpha\}$. This shows that $\alpha \subseteq \bigcup \{\beta^+ \mid \beta \in \alpha\}$

Exercise 27 PROOF:

⟨1⟩1. For natural numbers m and n , we have $\text{rank } \langle m, n \rangle = \max(m, n)^{+++}$

PROOF:

$$\begin{aligned}
\text{rank}\{\{m\}, \{m, n\}\} &= (\text{rank}\{m\})^+ \cup (\text{rank}\{m, n\})^+ \\
&= (\text{rank } m)^{++} \cup ((\text{rank } m)^+ \cup (\text{rank } n)^+)^+ \\
&= m^{++} \cup (m^+ \cup n^+)^+ \quad (\text{Exercise 26}) \\
&= \max(m, n)^{++}
\end{aligned}$$

$\langle 1 \rangle 2$. For any integer a we have $\text{rank } a = \omega$

PROOF: For any natural numbers m and n , we have

$$\begin{aligned}
\text{rank}[\langle m, n \rangle] &= \bigcup \{(\text{rank}\langle p, q \rangle)^+ : m + q = n + p\} \\
&= \bigcup \{\max(p, q)^+ : m + q = n + p\} \\
&= \omega
\end{aligned}$$

since for any natural number $p > m$ there exists q such that $m + q = n + p$.

$\langle 1 \rangle 3$. For any integers a and b we have $\text{rank}\langle a, b \rangle = \omega^{++}$

PROOF:

$$\begin{aligned}
\text{rank}\{\{a\}, \{a, b\}\} &= (\text{rank}\{a\})^+ \cup (\text{rank}\{a, b\})^+ \\
&= (\text{rank } a)^{++} \cup ((\text{rank } a)^+ \cup (\text{rank } b)^+)^+ \\
&= \omega^{++} \cup (\omega^+ \cup \omega^+)^+ \\
&= \omega^{++}
\end{aligned}$$

$\langle 1 \rangle 4$. For any rational q we have $\text{rank } q = \omega^{+++}$

PROOF: Since every element of q has rank ω^{++}

$\langle 1 \rangle 5$. For any real number r we have $\text{rank } r = \omega^{++++}$

PROOF: Since every element of r has rank ω^{+++} .

$\langle 1 \rangle 6$. $\text{rank } \mathbb{R} = \omega^{+++++}$

□

Exercise 28 If $X \in V_\alpha$ then $X \subseteq V_\beta$ for some $\beta \in \alpha$. Hence $\text{rank } X \subseteq \beta$ and so $\text{rank } X \in \alpha$.

Conversely, if $\text{rank } X \in \alpha$ then $X \in V_{(\text{rank } X)^+} \subseteq V_\alpha$.

Exercise 29 Direct proofs:

For any set a , there exists $m \in \{a\}$ such that $m \cap \{a\} = \emptyset$. This m must be the set a , so $a \cap \{a\} = \emptyset$, meaning $a \notin a$.

For any sets a and b , there exists $m \in \{a, b\}$ such that $m \cap \{a, b\} = \emptyset$. Now, m is either a or b . If $m = a$ then $a \cap \{a, b\} = \emptyset$ so $b \notin a$. And if $m = b$ then $b \cap \{a, b\} = \emptyset$ so $a \notin b$.

Consequences of part (c):

Assume $a \in a$. Define $f : \omega \rightarrow \{a\}$ by $f(n) = a$ for all $n \in \omega$. Then $f(n^+) \in f(n)$ for all n , contradicting (c).

Assume now $a \in b$ and $b \in a$. Define $f : \omega \rightarrow \{a, b\}$ by $f(n) = a$ if n is even, $f(n) = b$ if n is odd. Then $f(n^+) \in f(n)$ for all n , contradicting (c).

Exercise 30

$$\begin{aligned}
 \text{rank}\{a, b\} &= (\text{rank } a)^+ \cup (\text{rank } b)^+ && (\text{Theorem 7V(b)}) \\
 &= \max((\text{rank } a)^+, (\text{rank } b)^+) \\
 &= \max(\text{rank } a, \text{rank } b)^+
 \end{aligned}$$

We have

$$\begin{aligned}
 a &\subseteq V_{\text{rank } a} \\
 \therefore \mathcal{P}a &\subseteq \mathcal{P}V_{\text{rank } a} \\
 &= V_{(\text{rank } a)^+} \\
 \therefore \text{rank } \mathcal{P}a &\subseteq (\text{rank } a)^+ \\
 a &\in \mathcal{P}a \\
 \therefore \text{rank } a &\in \text{rank } \mathcal{P}a \\
 \therefore \text{rank } \mathcal{P}a &= (\text{rank } a)^+
 \end{aligned}$$

Now, for all $x \in \bigcup a$, there exists y such that $x \in y \in a$. Hence

$$\begin{aligned}
 \text{rank } x &\in \text{rank } y \in \text{rank } a \text{ .} \\
 \therefore (\text{rank } x)^+ &\in \text{rank } a \text{ .}
 \end{aligned}$$

So $\text{rank } a$ is an upper bound for $\{(\text{rank } x)^+ : x \in \bigcup a\}$, and so

$$\text{rank } \bigcup a \subseteq \text{rank } a \text{ .}$$

Exercise 31

(a) If $A \approx B$ and nothing of rank less than $\text{rank } B$ is equinumerous to B , then $\text{rank } B \in \text{rank } A$, and so $B \in V_{(\text{rank } A)^+}$. So we can construct the set $\text{kard } A$ by applying a Subset Axiom to $V_{(\text{rank } A)^+}$.

(b) There exists a set of rank $\text{rank } A$ that is equinumerous with A (namely A !). Let μ be the least ordinal $\leq \text{rank } A$ such that there exists a set of rank μ that is equinumerous with A . Pick a set B of rank μ such that $B \approx A$. Then $B \in \text{kard } A$.

(c) Suppose $\text{kard } A = \text{kard } B$. Pick $C \in \text{kard } A$. Then $C \approx A$ and $C \approx B$, so $A \approx B$.

Conversely, suppose $A \approx B$. Then we have $(A \approx C \text{ and nothing of rank less than } \text{rank } C \text{ is equinumerous with } C) \text{ iff } (B \approx C \text{ and nothing of rank less than } \text{rank } C \text{ is equinumerous with } C)$, i.e. $\text{kard } A = \text{kard } B$.

Exercise 32 Similar to Exercise 31.

Exercise 33 Suppose for a contradiction D is not a subset of B . Then $D - B$ is nonempty. So by the Regularity Axiom, there exists $m \in D - B$ such that $m \cap (D - B) = \emptyset$. Now, for all $x \in m$, we have $x \in D$ (since D is a transitive set) and $x \notin D - B$, so we must have $x \in B$; that is, $m \subseteq B$. But then $m \in B$, which is a contradiction.

Exercise 34 PROOF:

$\langle 1 \rangle 1$. ASSUME: $\{x, \{x, y\}\} = \{u, \{u, v\}\}$

$\langle 1 \rangle 2$. $x = u$ or $x = \{u, v\}$

$\langle 1 \rangle 3$. $u = x$ or $u = \{x, y\}$

$\langle 1 \rangle 4$. $x \neq \{u, v\}$

$\langle 2 \rangle 1$. ASSUME: for a contradiction $x = \{u, v\}$

$\langle 2 \rangle 2$. $u = x$ or $u = \{x, y\}$

$\langle 2 \rangle 3$. CASE: $u = x$

PROOF: In this case $x = u \in \{u, v\} = x$ contradicting Theorem 7X(a).

$\langle 2 \rangle 4$. CASE: $u = \{x, y\}$

PROOF: In this case $u \in x$ and $x \in u$ contradicting Theorem 7X(b).

$\langle 1 \rangle 5$. $x = u$

$\langle 1 \rangle 6$. $\{x, y\} = \{u, v\}$

PROOF: We cannot have $\{x, y\} = u$ because then we would have $x \in x$ contradicting Theorem 7X(a).

$\langle 1 \rangle 7$. $y = u$ or $y = v$

$\langle 1 \rangle 8$. $v = x$ or $v = y$

$\langle 1 \rangle 9$. If $y = u$ and $v = x$ then $y = v$

$\langle 1 \rangle 10$. $y = v$

PROOF: Checking all the cases in $\langle 1 \rangle 7$ and $\langle 1 \rangle 8$.

□