# M0 Categories

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## Contents



## 1 Categories

**Definition 1** (Category)**.** A *category* consists of:

- a collection of *objects*.
- for any objects $A$ and $B$, a collection of *maps* from $A$ to $B$. We write $f : A \to B$ iff $f$ is a map from $A$ to $B$.
- for any object $A$, an *identity map* $1_A : A \to A$
- for any maps $f : A \to B$ and $g : B \to C$, a map $g \circ f : A \to C$

such that:

**Identity Laws** For any map $f : A \to B$, we have $1_B \circ f = f \circ 1_A = f : A \to B$

**Associative Law** For any maps $f : A \to B$, $g : B \to C$ and $h : C \to D$, we hav $h \circ (g \circ f) = (h \circ g) \circ f : A \to D$

**Definition 2.** A map $f : A \to B$ is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for every object $T$ and morphisms $x_1, x_2 : T \to B$, if $f \circ x_1 = f \circ x_2$ then $x_1 = x_2$.

**Definition 3.** A map $f : A \to B$ is *epi* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for every object $T$ and morphisms $x_1, x_2 : B \to T$, if $x_1 \circ f = x_2 \circ f$ then $x_1 = x_2$.

**Definition 4** (Retraction, Section)**.** Let $r : A \to B$ and $s : B \to A$. Then $r$ is a *retraction* for $s$, and $s$ is a *section* for $r$, iff $r \circ s = 1_B$.

The object $A$ is a *retract* of $B$ iff there exists a retraction $r : B \to A$, i.e. there exist maps $s : A \to B$ and $r : B \to A$ such that $r \circ s = 1_A$.

**Proposition 5.** *If a map $f : A \to B$ has a section, then for any object $T$ and any map $y : T \to B$, there exists a map $x : T \to A$ such that $f \circ x = y$.*

PROOF: If $s : B \to A$ is a section of $f$, then we take $x = s \circ y$. We have $f \circ x = f \circ s \circ y = 1_B \circ y = y$. $\square$

**Proposition 6.** *If a map $f : A \to B$ has a retraction, then for any object $T$ and any map $g : A \to T$, there exists a map $t : B \to T$ such that $t \circ f = g$.*

PROOF: If $r : B \to A$ is a section for $f$, then we take $t = g \circ r$. We have $t \circ f = g \circ r \circ f = g \circ 1_A = g$. $\square$

**Proposition 7.** *Every section is monic.*

PROOF: Let $r : B \to A$ be a retraction for $f$. Then, if $f \circ x_1 = f \circ x_2$, then

$$\begin{aligned} r \circ f \circ x_1 &= r \circ f \circ x_2 \\ \therefore 1_A \circ x_1 &= 1_A \circ x_2 \\ \therefore x_1 &= x_2 \end{aligned} \qquad \square$$

**Proposition 8.** *Every retraction is epi.*

PROOF: Let $s : B \to A$ be a section for $f : A \to B$. Let $T$ be any set and $t_1, t_2 : T \to B$. Suppose $t_1 \circ f = t_2 \circ f$. Then

$$\begin{aligned} t_1 \circ f \circ s &= t_2 \circ f \circ s \\ \therefore t_1 \circ 1_B &= t_2 \circ 1_B \\ \therefore t_1 &= t_2 \end{aligned}$$

**Proposition 9.** *For any object $A$, the identity map $1_A$ is a section and a retraction of itself.*

PROOF: The Unit Laws give $1_A \circ 1_A = 1_A$. $\square$

**Corollary 9.1.** *Every object is a retract of itself.*

**Proposition 10.** *If $r_1 : B \to A$ is a retraction of $s_1 : A \to B$ and $r_2 : C \to B$ is a retraction of $s_2 : B \to C$ then $r_1 \circ r_2$ is a retraction of $s_2 \circ s_1$.*

PROOF:

$$\begin{aligned} r_1 \circ r_2 \circ s_2 \circ s_1 &= r_1 \circ 1_B \circ s_1 \\ &= r_1 \circ s_1 \\ &= 1_A \end{aligned} \qquad \square$$

**Corollary 10.1.** *If the object $A$ is a retract of $B$ and $B$ is a retract of $C$ then $A$ is a retract of $C$.*

**Theorem 11.** *If $r$ is a retraction of $f$ and $s$ is a section of $f$ then $r = s$.*

PROOF: Let $f : A \to B$ and $r, s : B \to A$. Then

$$\begin{aligned} r &= r \circ 1_B \\ &= r \circ f \circ s \\ &= 1_A \circ s \\ &= s \end{aligned} \qquad \square$$

**Definition 12** (Isomorphism)**.** A map $f : A \to B$ is an *isomorphism* or *invertible*, $f : A \cong B$, iff there exists a map $f^{-1} : B \to A$, the *inverse* for $f$, such that $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

Two objects $A$ and $B$ are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

**Theorem 13.** *The inverse of an isomorphism is unique.*

PROOF: From Theorem 11. □

**Theorem 14.** *For any object $A$, the identity map $1_A : A \cong A$ is an isomorphism with $1_A^{-1} = 1_A$.*

PROOF: We have $1_A \circ 1_A = 1_A$ by the Identity Laws. □

**Theorem 15.** *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$ by the definition of inverse. □

**Theorem 16.** *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF: From Proposition 10. □

**Proposition 17.** *Every monomorphic retraction is an isomorphism.*

PROOF: Let $f : A \rightarrowtail B$ be a monomorphism with section $s : B \to A$. Then

$$f \circ s \circ f = f$$
$$\therefore s \circ f = 1_A$$

Thus $s$ is also a retraction for $f$, hence an inverse. □

**Proposition 18.** *Every epimorphic section is an isomorphism.*

PROOF: Dual. □

**Definition 19** (Idempotent)**.** A map $e : A \to A$ is *idempotent* iff $e \circ e = e$.

**Definition 20** (Automorphism)**.** An *automorphism* on an object $A$ is an isomorphism $A \cong A$.