C0 Classes

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We speak informally of *classes*. A class is determined by a unary predicate. We write $\{x : P(x)\}$ or $\{x \mid P(x)\}$ for the class determined by the predicate $P(x)$.

We define what it means for an object a to be an element of the class $\mathbf{A}$, $a \in \mathbf{A}$, by: $a \in \{x : P(x)\}$ means $P(a)$.

Definition 1 (Equality of Classes)**.** Two classes $\mathbf{A}$ and $\mathbf{B}$ are *equal*, $\mathbf{A} = \mathbf{B}$, iff they have exactly the same members.

Proposition 2. *For any class* $\mathbf{A}$ *we have* $\mathbf{A} = \mathbf{A}$.

Since $\mathbf{A}$ and $\mathbf{A}$ have exactly the same members.

Proposition 3. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, if* $\mathbf{A} = \mathbf{B}$ *then* $\mathbf{B} = \mathbf{A}$.

PROOF: If $\mathbf{A}$ and $\mathbf{B}$ have exactly the same members, then $\mathbf{B}$ and $\mathbf{A}$ have exactly the same members.

Proposition 4. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, if* $\mathbf{A} = \mathbf{B}$ *and* $\mathbf{B} = \mathbf{C}$ *then* $\mathbf{A} = \mathbf{C}$.

PROOF: If $\mathbf{A}$ and $\mathbf{B}$ have exactly the same members, and $\mathbf{B}$ and $\mathbf{C}$ have exactly the same members, then $\mathbf{A}$ and $\mathbf{C}$ have exactly the same members. □

Definition 5 (Subclass)**.** A class $\mathbf{A}$ is a *subclass* of a class $\mathbf{B}$, $\mathbf{A} \subseteq \mathbf{B}$, iff every member of $\mathbf{A}$ is a member of $\mathbf{B}$.

Proposition 6. *For any class* $\mathbf{A}$ *we have* $\mathbf{A} \subseteq \mathbf{A}$.

PROOF: Every member of $\mathbf{A}$ is a member of $\mathbf{A}$. □

Proposition 7. *For any classes* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, if* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{C}$ *then* $\mathbf{A} \subseteq \mathbf{C}$.

PROOF: If every member of $\mathbf{A}$ is a member of $\mathbf{B}$, and every member of $\mathbf{B}$ is a member of $\mathbf{C}$, then every member of $\mathbf{A}$ is a member of $\mathbf{C}$. □

Proposition 8. *For any classes* $\mathbf{A}$ *and* $\mathbf{B}$*, if* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{A}$ *then* $\mathbf{A} = \mathbf{B}$.

PROOF: If every member of $\mathbf{A}$ is a member of $\mathbf{B}$, and every member of $\mathbf{B}$ is a member of $\mathbf{A}$, then $\mathbf{A}$ and $\mathbf{B}$ have exactly the same members. □

Definition 9 (Empty Class)**.** The *empty class*, $\emptyset$, is $\{x : \bot\}$.

Proposition 10. *For any class* $\mathbf{A}$*, we have* $\emptyset \subseteq \mathbf{A}$*.*

PROOF: Vacuously, every member of $\emptyset$ is a member of $\mathbf{A}$.

Definition 11 (Universal Class)**.** The *universal class* $\mathbf{V}$ is the class $\{x : \top\}$.

Proposition 12. *For any class* $\mathbf{A}$*, we have* $\mathbf{A} \subseteq \mathbf{V}$*.*

PROOF: Every member of $\mathbf{A}$ is a member of $\mathbf{V}$. □

Definition 13. For any objects $a_1, \ldots, a_n$, we write $\{a_1, \ldots, a_n\}$ for the class $\{x : x = a_1 \vee \cdots \vee x = a_n\}$.

Definition 14 (Union)**.** The *union* of classes $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \cup \mathbf{B}$, is the set whose elements are exactly the things that are members of $\mathbf{A}$ or members of $\mathbf{B}$.

Proposition 15. *For any classes* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$*, we have:*

1. $\mathbf{A} \subseteq \mathbf{A} \cup \mathbf{B}$
2. $\mathbf{B} \subseteq \mathbf{A} \cup \mathbf{B}$
3. *If* $\mathbf{A} \subseteq \mathbf{C}$ *and* $\mathbf{B} \subseteq \mathbf{C}$ *then* $\mathbf{A} \cup \mathbf{B} \subseteq \mathbf{C}$

PROOF: Immediate from definitions. □

Definition 16 (Intersection)**.** The *intersection* of classes $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \cap \mathbf{B}$, is the set whose elements are exactly the things that are members of both $\mathbf{A}$ and $\mathbf{B}$.

Proposition 17. *For any classes* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$*, we have:*

1. $\mathbf{A} \cap \mathbf{B} \subseteq \mathbf{A}$
2. $\mathbf{A} \cap \mathbf{B} \subseteq \mathbf{B}$
3. *If* $\mathbf{C} \subseteq \mathbf{A}$ *and* $\mathbf{C} \subseteq \mathbf{B}$ *then* $\mathbf{C} \subseteq \mathbf{A} \cap \mathbf{B}$

PROOF: Immediate from definitions. □

Definition 18 (Disjoint)**.** Two classes $\mathbf{A}$ and $\mathbf{B}$ are *disjoint* iff they have no common members.

Proposition 19. *Two classes* $\mathbf{A}$ *and* $\mathbf{B}$ *are* disjoint *iff* $\mathbf{A} \cap \mathbf{B} = \emptyset$*.*

PROOF: Immediate from definitions. □