C1 Set Theory

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1 Primitive Notions

Let there be *sets*.

Let there be a binary relation called *membership*, $\in$. When $x \in y$ holds, we say x is a *member* or *element* of y. We write $x \notin y$ iff x is not a member of y.

2 The Axioms

Axiom 1 (Extensionality)**.** *If two sets have exactly the same members, then they are equal.*

As a consequence of this axiom, we may identify a set A with the class $\{x : x \in A\}$. The use of the symbols $\in$ and $=$ is consistent.

Definition 2. We say that a class $\mathbf{A}$ *is a set* iff there exists a set A such that $A = \mathbf{A}$. That is, the class $\{x : P(x)\}$ is a set iff

$$\exists A. \forall x (x \in A \leftrightarrow P(x)) \ .$$

Otherwise, $\mathbf{A}$ is a *proper class.*

Definition 3 (Subset)**.** If A is a set and $\mathbf{B}$ is a class, we say A is a *subset* of $\mathbf{B}$ iff $A \subseteq \mathbf{B}$.

Axiom 4 (Empty Set)**.** *The empty class is a set, called the* empty set*.*

Axiom 5 (Pairing)**.** *For any objects a and b, the class $\{a, b\}$ is a set, called a* pair set*.*

Definition 6 (Union)**.** For any class of sets $\mathbf{A}$, the *union* $\bigcup \mathbf{A}$ is the class $\{x : \exists A \in \mathbf{A}. x \in A\}$.

We write $\bigcup_{P[x_1,\ldots,x_n]} t[x_1, \ldots, x_n]$ for $\bigcup \{t[x_1, \ldots, x_n] : P[x_1, \ldots, x_n]\}$.

Proposition 7. *If $\mathbf{A} \subseteq \mathbf{B}$ then $\bigcup \mathbf{A} \subseteq \bigcup \mathbf{B}$.*

PROOF: Easy. □

Axiom 8 (Union)**.** *For any set A, the union $\bigcup A$ is a set.*

Proposition 9. *For any sets A and B, the class $A \cup B$ is a set.*

PROOF: It is $\bigcup\{A, B\}$. $\square$

Proposition Schema 10. *For any objects a_1, ..., a_n, the class $\{a_1, \ldots, a_n\}$ is a set.*

PROOF: By repeated application of the Pairing and Union axioms. $\square$

Definition 11 (Power Set)**.** For any set A, the *power set* of A, $\mathcal{P}A$, is the class of all subsets of A.

Axiom 12 (Power Set)**.** *For any set A, the class $\mathcal{P}A$ is a set.*

Axiom 13 (Subset, Aussonderung)**.** *For any class $\mathbf{A}$ and set B, if $\mathbf{A} \subseteq B$ then $\mathbf{A}$ is a set.*

Proposition 14. *For any set A and class $\mathbf{B}$, the intersection $A \cap \mathbf{B}$ is a set.*

PROOF: By the Subset Axiom since it is a subclass of A. $\square$

Proposition 15. *For any set A and class $\mathbf{B}$, the relative complement $A - \mathbf{B}$ is a set.*

PROOF: By the Subset Axiom since it is a subclass of A. $\square$

Theorem 16. *The universal class $\mathbf{V}$ is a proper class.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $\mathbf{V}$ is a set.
$\langle 1 \rangle 2$. LET: $R = \{x : x \notin x\}$
$\langle 1 \rangle 3$. R is a set.
PROOF: By the Subset Axiom.
$\langle 1 \rangle 4$. $R \in R$ if and only if $R \notin R$
$\langle 1 \rangle 5$. Q.E.D.
PROOF: This is a contradiction.
$\square$

Definition 17 (Intersection)**.** For any class of sets $\mathbf{A}$, the *intersection* $\bigcap \mathbf{A}$ is the class $\{x : \forall A \in \mathbf{A}. x \in A\}$.

We write $\bigcap_{P[x_1,\ldots,x_n]} t[x_1, \ldots, x_n]$ for $\bigcap\{t[x_1, \ldots, x_n] : P[x_1, \ldots, x_n]\}$.

Proposition 18. *For any nonempty class of sets $\mathbf{A}$, the class $\bigcap \mathbf{A}$ is a set.*

PROOF: Pick $A \in \mathbf{A}$. Then $\bigcap \mathbf{A} \subseteq A$. $\square$

Proposition 19. *If $\mathbf{A} \subseteq \mathbf{B}$ then $\bigcap \mathbf{B} \subseteq \bigcap \mathbf{A}$.*

PROOF: Easy. $\square$

Proposition 20. *For any set A and class of sets $\mathbf{B}$, we have*

$$A \cup \bigcap \mathbf{B} = \bigcap \{A \cup X \mid X \in \mathbf{B}\}$$

PROOF: Easy. □

Proposition 21. *For any set A and class of sets $\mathbf{B}$, we have*

$$A \cap \bigcup \mathbf{B} = \bigcup \{A \cap X \mid X \in \mathbf{B}\}$$

PROOF: Easy. □

Proposition 22. *For any set C and class of sets $\mathbf{A}$, we have*

$$C - \bigcup \mathbf{A} = \bigcap \{C - X \mid X \in \mathbf{A}\} \ .$$

PROOF: Easy. □

Proposition 23. *For any set C and class of sets $\mathbf{A}$, we have*

$$C - \bigcap \mathbf{A} = \bigcup \{C - X \mid X \in \mathbf{A}\} \ .$$

PROOF: Easy. □

3 Ordered Pairs

Definition 24 (Ordered Pair)**.** For any objects a and b, the *ordered pair* (a, b) is $\{\{a\}, \{a, b\}\}$.

Theorem 25. *For any objects (a, b), we have $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.*

PROOF:
$\langle 1 \rangle 1$. If $(a, b) = (c, d)$ then $a = c$ and $b = d$
  $\langle 2 \rangle 1$. ASSUME: $(a, b) = (c, d)$
  $\langle 2 \rangle 2$. $a = c$
    PROOF: Since $\{a\} = \bigcap (a, b) = \bigcap (c, d) = \{c\}$.
  $\langle 2 \rangle 3$. $\{a, b\} = \{c, d\}$
    PROOF: $\{a, b\} = \bigcup (a, b) = \bigcup (c, d) = \{c, d\}$.
  $\langle 2 \rangle 4$. $b = c$ or $b = d$
  $\langle 2 \rangle 5$. CASE: $b = c$
    $\langle 3 \rangle 1$. $a = b$
    $\langle 3 \rangle 2$. $\{c, d\} = \{a\}$
    $\langle 3 \rangle 3$. $b = d$
  $\langle 2 \rangle 6$. CASE: $b = d$
    PROOF: We have $a = c$ and $b = d$ as required.
$\langle 1 \rangle 2$. If $a = c$ and $b = d$ then $(a, b) = (c, d)$
  PROOF: Trivial.
□