C1 Set Theory

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1 Primitive Notions

Let there be *sets*.

Let there be a binary relation called *membership*, $\in$. When $x \in y$ holds, we say x is a *member* or *element* of y. We write $x \notin y$ iff x is not a member of y.

2 The Axioms

Axiom 1 (Extensionality)**.** *If two sets have exactly the same members, then they are equal.*

As a consequence of this axiom, we may identify a set A with the class $\{x : x \in A\}$. The use of the symbols $\in$ and $=$ is consistent.

Definition 2. We say that a class $\mathbf{A}$ *is a set* iff there exists a set A such that $A = \mathbf{A}$. That is, the class $\{x : P(x)\}$ is a set iff

$$\exists A. \forall x (x \in A \leftrightarrow P(x)) \enspace .$$

Otherwise, $\mathbf{A}$ is a *proper class*.

Definition 3 (Subset)**.** If A is a set and $\mathbf{B}$ is a class, we say A is a *subset* of $\mathbf{B}$ iff $A \subseteq \mathbf{B}$.

Axiom 4 (Empty Set)**.** *The empty class is a set, called the* empty set*.*

Axiom 5 (Pairing)**.** *For any objects a and b, the class $\{a, b\}$ is a set, called a* pair set*.*

Axiom 6 (Union)**.** *For any sets A and B, the class $A \cup B$ is a set.*

Proposition Schema 7. *For any objects a_1, ..., a_n, the class $\{a_1, \ldots, a_n\}$ is a set.*

PROOF: By repeated application of the Pairing and Union axioms. □

Definition 8 (Power Set)**.** For any set A, the *power set* of A, $\mathcal{P}A$, is the class of all subsets of A.

Axiom 9 (Power Set)**.** *For any set A, the class $\mathcal{P}A$ is a set.*

Axiom 10 (Subset, Aussonderung)**.** *For any class $\mathbf{A}$ and set B, if $\mathbf{A} \subseteq B$ then $\mathbf{A}$ is a set.*

Proposition 11. *For any set A and class $\mathbf{B}$, the intersection $A \cap \mathbf{B}$ is a set.*

PROOF: By the Subset Axiom since it is a subclass of A. □

Proposition 12. *For any set A and class $\mathbf{B}$, the relative complement $A - \mathbf{B}$ is a set.*

PROOF: By the Subset Axiom since it is a subclass of A. □

Theorem 13. *The universal class $\mathbf{V}$ is a proper class.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $\mathbf{V}$ is a set.
$\langle 1 \rangle 2$. LET: $R = \{x : x \notin x\}$
$\langle 1 \rangle 3$. R is a set.
  PROOF: By the Subset Axiom.
$\langle 1 \rangle 4$. $R \in R$ if and only if $R \notin R$
$\langle 1 \rangle 5$. Q.E.D.
  PROOF: This is a contradiction.
□