

Solutions Manual for Enderton *Elements of Set
Theory*

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Chapter 1

Chapter 1 — Introduction

1.1 Baby Set Theory

Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

Exercise 2 We have $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ has an element (namely \emptyset) while \emptyset has no elements.

We have $\emptyset \neq \{\{\emptyset\}\}$ because $\{\{\emptyset\}\}$ has an element (namely $\{\emptyset\}$) while \emptyset has no elements.

We have $\{\emptyset\} \neq \{\{\emptyset\}\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$. This last fact is true because $\emptyset \neq \{\emptyset\}$ as we proved in the first paragraph.

Exercise 3 Assume $B \subseteq C$. Let $A \in \mathcal{P}B$; we must show that $A \in \mathcal{P}C$.

We have $A \subseteq B$ (since $A \in \mathcal{P}B$) and $B \subseteq C$. From this it follows that $A \subseteq C$ (every element of A is an element of B ; every element of B is an element of C ; therefore every element of A is an element of C). Hence $A \in \mathcal{P}C$ as required.

Exercise 4 Since $x \in B$, we have $\{x\} \subseteq B$ and so $\{x\} \in \mathcal{P}B$.

Since $x \in B$ and $y \in B$, we have $\{x, y\} \subseteq B$ and so $\{x, y\} \in \mathcal{P}B$.

From these two facts, it follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$ and so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

1.2 Sets — An Informal View

Exercise 5 We have

$$\begin{aligned} V_0 &= A \\ V_1 &= V_0 \cup \mathcal{P}V_0 \\ &= A \cup \mathcal{P}A \\ V_2 &= V_1 \cup \mathcal{P}V_1 \\ &= \{\emptyset, \{\emptyset\}\} \\ V_3 &= \mathcal{P}V_2 \\ &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \end{aligned}$$

We have $\emptyset \subseteq V_0$ and so $\emptyset \in V_1$. Therefore $\{\emptyset\} \subseteq V_1$ and so $\{\emptyset\} \in V_2$. Hence $\{\{\emptyset\}\} \subseteq V_2$.

We also have $\{\{\emptyset\}\} \not\subseteq V_0$ because $\{\emptyset\}$ is not an atom, and $\{\{\emptyset\}\} \not\subseteq V_1$ since $\{\emptyset\} \notin V_1$ because \emptyset is not an atom.

Thus the rank of $\{\{\emptyset\}\}$ is 2.

Likewise we have \emptyset and $\{\emptyset\}$ are both subsets of V_1 , hence

$$\emptyset \in V_2, \quad \{\emptyset\} \in V_2$$

Thus $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are all subsets of V_2 , hence elements of V_3 . Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is not a subset of V_0 (because \emptyset is not an atom.) It is not a subset of V_1 ($\{\emptyset\} \notin V_1$ because \emptyset is not an atom.) It is not a subset of V_2 (we have $\{\emptyset, \{\emptyset\}\} \notin V_2$ since $\{\emptyset\} \notin V_1$).

Therefore the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

Exercise 6

$$\begin{aligned}
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}V_0 && (\text{since } V_0 = A) \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_1 && (\text{since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3}) \\
V_3 &= V_2 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_2 && (\text{since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3}) \\
V_4 &= V_3 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_3 && (\text{since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3})
\end{aligned}$$

Exercise 7 In Exercise 5 we calculated $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Hence

$$\begin{aligned}
V_4 &= \mathcal{P}V_3 \\
&= \{\emptyset, \\
&\quad \{\emptyset\}, \\
&\quad \{\{\emptyset\}\}, \\
&\quad \{\{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
&\quad \}
\end{aligned}$$

Chapter 2

Chapter 2 — Axioms and Operations

2.1 Arbitrary Unions and Intersections

Exercise 1 $A \cap B \cap C$ is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

Exercise 2 Take $A = \emptyset$ and $B = \{\emptyset\}$. Then $\bigcup A = \bigcup B = \emptyset$ but $A \neq B$. (There are many other possible answers.)

Exercise 3 Let $b \in A$. We must show that $b \subseteq \bigcup A$.

Let x be any element of b . We must show that $x \in \bigcup A$. We know that $x \in b$ and $b \in A$, and so $x \in \bigcup A$ by the definition of $\bigcup A$.

Exercise 4 Suppose $A \subseteq B$. Let $x \in \bigcup A$. We must show that $x \in \bigcup B$.

Pick an element $a \in A$ such that $x \in a$. Then $a \in B$ because $A \subseteq B$. Since we know $x \in a$ and $a \in B$, we know that $x \in \bigcup B$.

Exercise 5 Assume that every member of \mathcal{A} is a subset of B . Let $x \in \bigcup \mathcal{A}$. We must show that $x \in B$.

Pick $A \in \mathcal{A}$ such that $x \in A$. By our assumption, we have $A \subseteq B$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$ as required.

Exercise 6

(a) We will show that $\bigcup \mathcal{P}A \subseteq A$ and $A \subseteq \bigcup \mathcal{P}A$.

To show $\bigcup \mathcal{P}A \subseteq A$: This follows from Exercise 5, since every member of $\mathcal{P}A$ is a subset of A .

To show $A \subseteq \bigcup \mathcal{P}A$: Let $a \in A$. Then we have $a \in \{a\}$ and $\{a\} \in \mathcal{P}A$ so $a \in \bigcup \mathcal{P}A$.

(b) To show $A \subseteq \mathcal{P} \bigcup A$: This holds because every element of A is a subset of $\bigcup A$, as we proved in Exercise 3.

Equality holds if and only if $A = \mathcal{P}X$ for some set X .

Proof: If $A = \mathcal{P} \bigcup A$ then of course $A = \mathcal{P}X$ for some X .

Conversely, if $A = \mathcal{P}X$, then we have

$$\begin{aligned} \mathcal{P} \bigcup A &= \mathcal{P} \bigcup \mathcal{P}X \\ &= \mathcal{P}X && \text{(by part (a))} \\ &= A \end{aligned}$$

Exercise 7

(a) For any set X ,

$$\begin{aligned} X &\in \mathcal{P}A \cap \mathcal{P}B \\ \Leftrightarrow X &\subseteq A \text{ and } X \subseteq B \\ \Leftrightarrow \text{Every member of } X &\text{ is a member of } A \text{ and a member of } B \\ \Leftrightarrow X &\subseteq A \cap B \\ \Leftrightarrow X &\in \mathcal{P}(A \cap B) \end{aligned}$$

(b) Let $X \in \mathcal{P}A \cup \mathcal{P}B$. Then either $X \in \mathcal{P}A$ or $X \in \mathcal{P}B$ (or both). If $X \in \mathcal{P}A$, then we have $X \subseteq A$ and so $X \subseteq A \cup B$ (because $A \subseteq A \cup B$). Similarly if $X \in \mathcal{P}B$ then we have $X \subseteq A \cup B$. So in either case $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

Equality holds if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof: Suppose $A \subseteq B$. Then $\mathcal{P}A \subseteq \mathcal{P}B$ (Chapter 1 Exercise 3) and so $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$. Also $A \cup B = B$ so $\mathcal{P}(A \cup B) = \mathcal{P}B$. Thus $\mathcal{P}A \cup \mathcal{P}B$ and $\mathcal{P}(A \cup B)$ are equal.

Similarly if $B \subseteq A$ then $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$.

Conversely, suppose $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$. We have $A \cup B \in \mathcal{P}(A \cup B)$, so $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$. If $A \cup B \in \mathcal{P}A$, then we have $B \subseteq A \cup B \subseteq A$. And if $A \cup B \in \mathcal{P}B$, then we have $A \subseteq A \cup B \subseteq B$.

Exercise 8 If A is a set such that every singleton belongs to A , then every set belongs to $\bigcup A$, contradicting Theorem 2A.

Exercise 9 Let $a = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$. Then $a \in B$ but $\mathcal{P}a$ is not a subset of B because $\emptyset \in \mathcal{P}a$ and $\emptyset \notin B$.

Exercise 10 We must show that $\mathcal{P}a \subseteq \mathcal{P} \bigcup B$. So let $X \in \mathcal{P}a$. Then $X \subseteq a$; we must show that $X \subseteq \bigcup B$.

Let $x \in X$; we must show that $x \in \bigcup B$. We have $x \in a$ (because $x \in X$ and $X \subseteq a$) and $a \in B$, hence $x \in \bigcup B$ as required.

2.2 Algebra of Sets

Exercise 11 For any x we have

$$\begin{aligned} x \in (A \cap B) \cup (A - B) &\Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B) \\ &\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B) \\ &\Leftrightarrow x \in A \end{aligned}$$

Hence $A = (A \cap B) \cup (A - B)$.

For any x we have

$$\begin{aligned} x \in A \cup (B - A) &\Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A) \\ &\Leftrightarrow x \in A \text{ or } x \in B \\ &\Leftrightarrow x \in A \cup B \end{aligned}$$

Hence $A \cup (B - A) = A \cup B$.

Exercise 12 For any x ,

$$\begin{aligned} x \in C - (A \cap B) &\Leftrightarrow x \in C \& \neg(x \in A \& x \in B) \\ &\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B) \\ &\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B) \\ &\Leftrightarrow x \in (C - A) \cup (C - B) \end{aligned}$$

Exercise 13 Suppose $A \subseteq B$. Let $x \in C - B$; we must show $x \in C - A$. We have $x \in C$ and $x \notin B$. Therefore $x \notin A$, since every member of A is a member of B . And so we have $x \in C - A$ as required.

Exercise 14 Let $A = \{\emptyset\}$, $B = \emptyset$ and $C = \{\emptyset\}$. Then $A - (B - C) = A - \emptyset = \{\emptyset\}$ while $(A - B) - C = \{\emptyset\} - C = \emptyset$.

Exercise 15

(a) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have $x \in A \cap (B + C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$.

(b) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$

In every case, we have $x \in A + (B + C) \Leftrightarrow x \in (A + B) + C$.

Exercise 16

$$[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A] = (A \cup B) - A \\ = B - A$$

Exercise 17

(a) \Leftrightarrow (b)

$A \subseteq B \Leftrightarrow$ Every element of A is an element of B

\Leftrightarrow There is no element of A that is not an element of B

$\Leftrightarrow A - B = \emptyset$

(a) \Rightarrow (c) Suppose $A \subseteq B$. We have $B \subseteq A \cup B$ from the definition of $A \cup B$; we must prove that $A \cup B \subseteq B$. So let $x \in A \cup B$. Then $x \in A$ or $x \in B$. But in either case $x \in B$, since $x \in A \Rightarrow x \in B$. Thus we have $x \in B$ as required.

(c) \Rightarrow (a) We always have $A \subseteq A \cup B$. So if $A \cup B = B$ then we have $A \subseteq B$.

(a) \Rightarrow (d) Suppose $A \subseteq B$. We have $A \cap B \subseteq A$ from the definition of $A \cap B$; we must prove that $A \subseteq A \cap B$. So let $x \in A$. Then $x \in B$ since $A \subseteq B$, hence $x \in A \cap B$ as required.

(d) \Rightarrow (a) We always have $A \cap B \subseteq B$. So if $A \cap B = A$ then $A \subseteq B$.

Exercise 18 We can make the following 16 sets:

- $\emptyset (= A - A)$
- $A - B$
- $A \cap B$
- $B - A$
- $S - (A \cup B)$
- A
- $A + B$
- $S - B$
- B
- $S - (A + B)$
- $S - A$
- $A \cup B$
- $S - (B - A)$
- $S - (A \cap B)$
- $S - (A - B)$

Exercise 19 They are never equal, because for all A, B , we have $\emptyset \in \mathcal{P}(A - B)$ but $\emptyset \notin \mathcal{P}A - \mathcal{P}B$ since $\emptyset \in \mathcal{P}B$.

Exercise 20 Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

We first show $B \subseteq C$. Let $x \in B$; we show $x \in C$. We have $x \in A \cup B = A \cup C$, so either $x \in A$ or $x \in C$. If $x \in C$, we are done. If $x \in A$, then we have $x \in A \cap B = A \cap C$, and so $x \in C$ in this case too.

We can show $C \subseteq B$ similarly. Hence $B = C$.

Exercise 21 For any x , we have

$$\begin{aligned}
 x \in \bigcup (A \cup B) &\Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C \\
 &\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C, \text{ or there exists } C \in B \text{ such that } x \in C \\
 &\Leftrightarrow x \in \bigcup A \cup \bigcup B
 \end{aligned}$$

Exercise 22 For any x , we have

$$\begin{aligned} x \in \bigcap (A \cup B) &\Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C \\ &\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C \\ &\Leftrightarrow x \in \bigcap A \cap \bigcap B \end{aligned}$$

Exercise 23 PROOF:

- $\langle 1 \rangle 1. A \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 1. \text{ LET: } x \in A$
- $\langle 2 \rangle 2. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 3. x \in A \cup X$
- $\langle 1 \rangle 2. \bigcap \mathcal{B} \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 1. \text{ LET: } x \in \bigcap \mathcal{B}$
- $\langle 2 \rangle 2. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 3. x \in X$
- $\langle 2 \rangle 4. x \in A \cup X$
- $\langle 1 \rangle 3. \bigcap \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap \mathcal{B}$
- $\langle 2 \rangle 1. \text{ LET: } x \in \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 2. \text{ ASSUME: } x \notin A$
- PROVE: $x \in \bigcap \mathcal{B}$
- $\langle 2 \rangle 3. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 4. x \in A \cup X$
- $\langle 2 \rangle 5. x \in X$

□

Exercise 24

(a)

$$\begin{aligned} Y \in \mathcal{P} \bigcap \mathcal{A} &\Leftrightarrow Y \subseteq \bigcap \mathcal{A} \\ &\Leftrightarrow \forall y \in Y. \forall X \in \mathcal{A}. y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A}. \forall y \in Y. y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A}. Y \in \mathcal{P}X \\ &\Leftrightarrow Y \in \bigcap \{\mathcal{P}X \mid X \in \mathcal{A}\} \end{aligned}$$

(b) $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} \subseteq \mathcal{P} \bigcup \mathcal{A}$

PROOF:

- $\langle 1 \rangle 1. \text{ LET: } Y \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
- $\langle 1 \rangle 2. \text{ PICK } X \in \mathcal{A} \text{ such that } Y \in \mathcal{P}X$
- $\langle 1 \rangle 3. Y \subseteq X$
- $\langle 1 \rangle 4. Y \subseteq \bigcup \mathcal{A}$
- $\langle 1 \rangle 5. Y \in \mathcal{P} \bigcup \mathcal{A}$

Equality holds if and only if $\bigcup \mathcal{A} \in \mathcal{A}$.

- ⟨1⟩1. If $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$ then $\bigcup \mathcal{A} \in \mathcal{A}$
 - ⟨2⟩1. ASSUME: $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
 - ⟨2⟩2. $\bigcup \mathcal{A} \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
 - ⟨2⟩3. PICK $X \in \mathcal{A}$ such that $\bigcup \mathcal{A} \in \mathcal{P}X$
 - ⟨2⟩4. $X = \bigcup \mathcal{A}$
 - ⟨1⟩2. If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
- PROOF: If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\mathcal{P} \bigcup \mathcal{A} \in \{\mathcal{P}X \mid X \in \mathcal{A}\}$.
 \square

Exercise 25 We have $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ if and only if $A = \emptyset$ or $\mathcal{B} \neq \emptyset$

- ⟨1⟩1. If $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ then $A = \emptyset$ or $\mathcal{B} \neq \emptyset$
- PROOF: If $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$ and $\mathcal{B} = \emptyset$ then
- $$A \cup \bigcup \emptyset = \bigcup \emptyset$$
- $$\therefore A = \emptyset$$
- ⟨1⟩2. If $A = \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
- PROOF: Both sides are equal to $\bigcup \mathcal{B}$
- ⟨1⟩3. If $\mathcal{B} \neq \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨2⟩1. ASSUME: $\mathcal{B} \neq \emptyset$
 - ⟨2⟩2. $A \cup \bigcup \mathcal{B} \subseteq \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨3⟩1. LET: $x \in A \cup \bigcup \mathcal{B}$
 - PROVE: $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨3⟩2. CASE: $x \in A$
 - ⟨4⟩1. PICK $X \in \mathcal{B}$
 - PROOF: By ⟨2⟩1
 - ⟨4⟩2. $x \in A \cup X$
 - ⟨3⟩3. CASE: $x \in \bigcup \mathcal{B}$
 - ⟨4⟩1. PICK $X \in \mathcal{B}$ such that $x \in X$
 - ⟨4⟩2. $x \in A \cup X$
 - ⟨2⟩3. $\bigcup \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcup \mathcal{B}$
 - ⟨3⟩1. LET: $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
 - ⟨3⟩2. PICK $X \in \mathcal{B}$ such that $x \in A \cup X$
 - ⟨3⟩3. $X \subseteq \bigcup \mathcal{B}$
 - ⟨3⟩4. $A \cup X \subseteq A \cup \bigcup \mathcal{B}$
 - ⟨3⟩5. $x \in A \cup \bigcup \mathcal{B}$

2.3 Review Exercises

Exercise 26 Sets A, B, D and F are all equal to each other. Sets C, E and G are equal to each other. None of the first list is equal to any of the second list.

Exercise 27 Take $A = \{\{0\}, \{1\}\}$ and $B = \{\{1\}\}$. Then $A \cap B = \{\{1\}\}$ and

$$\begin{aligned}\bigcap A \cap \bigcap B &= \emptyset \cap \{1\} \\ &= \emptyset \\ \bigcap (A \cap B) &= \bigcap \{\{1\}\} \\ &= \{1\}\end{aligned}$$

Exercise 28

$$\bigcup \{\{3, 4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\} = \{3, 4, \{3\}, \{4\}\}$$

Exercise 29

(a) \emptyset

(b) We have

$$\begin{aligned}\{\emptyset\} &\subseteq \mathcal{P}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{PP}\{\emptyset\} \\ \{\emptyset\} &\subseteq \mathcal{PP}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{PPP}\{\emptyset\} \\ \therefore \bigcap \{\mathcal{PPP}\{\emptyset\}, \mathcal{PP}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} &= \mathcal{PPP}\{\emptyset\} \cap \mathcal{PP}\{\emptyset\} \cap \mathcal{P}\{\emptyset\} \\ &= \mathcal{P}\{\emptyset\} \\ &= \{\emptyset, \{\emptyset\}\}\end{aligned}$$

Exercise 30

(a) $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$

(b) $\{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d) $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

Exercise 31

(a) $\{1, 2, 3, \emptyset\}$

(b) \emptyset

(c) \emptyset

(d) \emptyset

Exercise 32

(a) $a \cup b$

(b) a

(c)

$$\begin{aligned} \bigcap \bigcup S \cup (\bigcup \bigcup S - \bigcup \bigcap S) &= (a \cap b) \cup ((a \cup b) - a) \\ &= (a \cap b) \cup (b - a) \\ &= b \end{aligned}$$

Exercise 33 When $a \neq b$:

$$\begin{aligned} \bigcup (\bigcup S - \bigcap S) &= \bigcup (\{a, b\} - \{a\}) \\ &= \bigcup \{b\} \\ &= b \end{aligned}$$

When $a = b$:

$$\begin{aligned} \bigcup (\bigcup S - \bigcap S) &= \bigcup (\{a, b\} - \{a\}) \\ &= \bigcup \emptyset \\ &= \emptyset \end{aligned}$$

Exercise 34 For any set S , we have

$$\begin{aligned} \emptyset &\subseteq \mathcal{P}S \\ \therefore \emptyset &\in \mathcal{P}\mathcal{P}S \\ \emptyset &\subseteq S \\ \therefore \emptyset &\in \mathcal{P}S \\ \therefore \{\emptyset\} &\subseteq \mathcal{P}S \\ \therefore \{\emptyset\} &\in \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} &\subseteq \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} &\in \mathcal{P}\mathcal{P}\mathcal{P}S \end{aligned}$$

Exercise 35 Assume $\mathcal{P}A = \mathcal{P}B$. Then we have

$$\begin{aligned}
 A &\in \mathcal{P}A \\
 \therefore A &\in \mathcal{P}B \\
 \therefore A &\subseteq B \\
 B &\in \mathcal{P}B \\
 \therefore B &\in \mathcal{P}A \\
 \therefore B &\subseteq A \\
 \therefore A &= B
 \end{aligned}$$

Exercise 36

(a)

$$\begin{aligned}
 x \in A - (A \cap B) &\Leftrightarrow x \in A \ \& \ \neg(x \in A \ \& \ x \in B) \\
 &\Leftrightarrow x \in A \ \& \ x \notin B \\
 &\Leftrightarrow x \in A - B
 \end{aligned}$$

(b)

$$\begin{aligned}
 x \in A - (A - B) &\Leftrightarrow x \in A \ \& \ \neg(x \in A \ \& \ x \notin B) \\
 &\Leftrightarrow x \in A \ \& \ x \in B \\
 &\Leftrightarrow x \in A \cap B
 \end{aligned}$$

Exercise 37

(a)

$$\begin{aligned}
 x \in (A \cup B) - C &\Leftrightarrow (x \in A \text{ or } x \in B) \ \& \ x \notin C \\
 &\Leftrightarrow (x \in A \ \& \ x \notin C) \text{ or } (x \in B \ \& \ x \notin C) \\
 &\Leftrightarrow x \in (A - C) \cup (B - C)
 \end{aligned}$$

(b)

$$\begin{aligned}
 x \in A - (B - C) &\Leftrightarrow x \in A \ \& \ \neg(x \in B \ \& \ x \notin C) \\
 &\Leftrightarrow x \in A \ \& \ (x \notin B \text{ or } x \in C) \\
 &\Leftrightarrow (x \in A \ \& \ x \notin B) \text{ or } (x \in A \ \& \ x \in C) \\
 &\Leftrightarrow x \in (A - B) \cup (A \cap C)
 \end{aligned}$$

(c)

$$\begin{aligned}
 x \in (A - B) - C &\Leftrightarrow x \in A \ \& \ x \notin B \ \& \ x \notin C \\
 &\Leftrightarrow x \in A \ \& \ \neg(x \in B \vee x \in C) \\
 &\Leftrightarrow x \in A - (B \cup C)
 \end{aligned}$$

Exercise 38

(a) If every element of A is an element of C , and every element of B is an element of C , then everything that is an element of either A or B is an element of C .

(b) If every element of C is an element of A , and every element of C is an element of B , then every element of C is an element of both A and B .

Chapter 3

Chapter 3 — Relations and Functions

3.1 Ordered Pairs

Exercise 1 We have $\langle 0, 1, 0 \rangle^* = \langle 0, 1, 1 \rangle^* = \{\{0\}, \{0, 1\}\}$.

Exercise 2

(a)

$$\begin{aligned} z &\in A \times (B \cup C) \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ (y \in B \text{ or } y \in C)) \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in B) \text{ or } (z = (x, y) \ \& \ x \in A \ \& \ y \in C) \\ \Leftrightarrow z &\in (A \times B) \cup (A \times C) \end{aligned}$$

(b)

$\langle 1 \rangle 1$. ASSUME: $A \times B = A \times C$ and $A \neq \emptyset$

$\langle 1 \rangle 2$. PICK $a \in A$

$\langle 1 \rangle 3$. For all x , $x \in B \Leftrightarrow x \in C$

PROOF: $x \in B$ iff $(a, x) \in A \times B$ iff $(a, x) \in A \times C$ iff $x \in C$.

□

Exercise 3

$$\begin{aligned} z &\in A \times \bigcup \mathcal{B} \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ \exists X \in \mathcal{B}. y \in X) \\ \Leftrightarrow \exists X \in \mathcal{B}. \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in X) \\ \Leftrightarrow z &\in \bigcup \{A \times X : X \in \mathcal{B}\} \end{aligned}$$

Exercise 4 If every ordered pair belongs to A then every set belongs to $\bigcup \bigcup A$ contradicting Theorem 2A.

Exercise 5

(a) Apply a Subset Axiom to $\mathcal{P}(A \times B)$: we have $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A. z = \{x\} \times B\}$.

(b)

$$\begin{aligned} z &\in \bigcup C \\ \Leftrightarrow \exists x \in A. z &\in \{x\} \times B \\ \Leftrightarrow \exists x \in A. \exists y \in B. z &= (x, y) \\ \Leftrightarrow z &\in A \times B \end{aligned}$$

3.2 Relations

Exercise 6 If $A \subseteq \text{dom } A \times \text{ran } A$ then A is a set of ordered pairs, i.e. a relation.

Conversely, suppose A is a relation. Let $z \in A$. Then z is an ordered pair; let $z = (x, y)$. We have $x \in \text{dom } A$ and $y \in \text{ran } A$ and so $z \in \text{dom } A \times \text{ran } A$ as required.

Exercise 7 We have $\text{fld } R \subseteq \bigcup \bigcup R$ by Lemma 3D.

Conversely, let $x \in \bigcup \bigcup R$. Pick a and b such that $x \in a$, $a \in b$ and $b \in R$. Then b is an ordered pair; let $b = (y, z)$. We have $a = \{y\}$ or $\{y, z\}$, hence $x = y$ or $x = z$. In either case, $x \in \text{fld } R$.

Exercise 8

(a)

$$\begin{aligned} x &\in \text{dom } \bigcup \mathcal{A} \\ \Leftrightarrow \exists y. \exists R \in \mathcal{A}. (x, y) &\in R \\ \Leftrightarrow \exists R \in \mathcal{A}. \exists y. (x, y) &\in R \\ \Leftrightarrow x &\in \bigcup \{\text{dom } R : R \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned}
y &\in \text{ran} \bigcup \mathcal{A} \\
&\Leftrightarrow \exists x. \exists R \in \mathcal{A}. (x, y) \in R \\
&\Leftrightarrow \exists R \in \mathcal{A}. \exists x. (x, y) \in R \\
&\Leftrightarrow y \in \bigcup \{\text{ran } R : R \in \mathcal{A}\}
\end{aligned}$$

Exercise 9 Assume \mathcal{A} is nonempty. We have $\text{dom} \bigcap \mathcal{A} \subseteq \bigcap \{\text{dom } R : R \in \mathcal{A}\}$.

PROOF:

$$\begin{aligned}
x &\in \text{dom} \bigcap \mathcal{A} \\
&\Leftrightarrow \exists y. \forall R \in \mathcal{A}. (x, y) \in R \\
&\Rightarrow \forall R \in \mathcal{A}. \exists y. (x, y) \in R \\
&\Leftrightarrow x \in \bigcap \{\text{dom } R : R \in \mathcal{A}\}
\end{aligned}$$

Equality holds iff the middle ' \Rightarrow ' can be reversed, i.e. iff for all x , if $\forall R \in \mathcal{A}. \exists y. (x, y) \in R$ then $\exists y. \forall R \in \mathcal{A}. (x, y) \in R$. I haven't found a simpler condition than this. The condition does not always hold, for example if $\mathcal{A} = \{\{(1, 2)\}, \{(1, 3)\}\}$ then $\text{dom} \bigcap \mathcal{A} = \emptyset$ while $\bigcap \{\text{dom } R : R \in \mathcal{A}\} = \{1\}$.

Similarly, $\text{ran} \bigcap \mathcal{A} \subseteq \bigcap \{\text{ran } R : R \in \mathcal{A}\}$, and equality holds iff, for any y , if $\forall R \in \mathcal{A}. \exists x. (x, y) \in R$ then $\exists x. \forall R \in \mathcal{A}. (x, y) \in R$.

3.3 n -ary Relations

Exercise 10 This follows from the equations at the top of page 42. An ordered 4-tuple $\langle a, b, c, d \rangle$ is also an ordered 1-tuple (because every set is), and the ordered pair $\langle \langle a, b, c \rangle, d \rangle$, and the ordered triple $\langle \langle a, b \rangle, c, d \rangle$.

3.4 Functions

Exercise 11 We prove $F \subseteq G$. Let $z \in F$. Since F is a relation, then z is an ordered pair; let $z = \langle x, y \rangle$. We have $x \in \text{dom } F$ and $y = F(x)$. Therefore $x \in \text{dom } G$ and $y = G(x)$ (because $\text{dom } F = \text{dom } G$ and $F(x) = G(x)$). Hence $\langle x, y \rangle \in G$, i.e. $z \in G$.

We have proved $F \subseteq G$. We can prove $G \subseteq F$ similarly. Thus $F = G$.

Exercise 12 PROOF:

- $\langle 1 \rangle 1$. If $f \subseteq g$ then $\text{dom } f \subseteq \text{dom } g$ and $\forall x \in \text{dom } f. f(x) = g(x)$
- $\langle 2 \rangle 1$. ASSUME: $f \subseteq g$
- $\langle 2 \rangle 2$. LET: $x \in \text{dom } f$
- $\langle 2 \rangle 3$. $(x, f(x)) \in f$
- $\langle 2 \rangle 4$. $(x, f(x)) \in g$
- $\langle 2 \rangle 5$. $x \in \text{dom } g$ and $g(x) = f(x)$

- ⟨1⟩2. If $\text{dom } f = \text{dom } g$ and $\forall x \in \text{dom } f. f(x) = g(x)$ then $f \subseteq g$
- ⟨2⟩1. ASSUME: $\text{dom } f = \text{dom } g$ and $\forall x \in \text{dom } f. f(x) = g(x)$
- ⟨2⟩2. LET: $z \in f$
- ⟨2⟩3. LET: $z = (x, y)$
- ⟨2⟩4. $x \in \text{dom } f$ and $y = f(x)$
- ⟨2⟩5. $x \in \text{dom } g$ and $y = g(x)$
- ⟨2⟩6. $z = (x, y) \in g$

□

Exercise 13 PROOF:

- ⟨1⟩1. ASSUME: f and g are functions
- ⟨1⟩2. ASSUME: $f \subseteq g$
- ⟨1⟩3. ASSUME: $\text{dom } g \subseteq \text{dom } f$
- ⟨1⟩4. $\text{dom } f = \text{dom } g$
- PROOF: We have $\text{dom } f \subseteq \text{dom } g$ from ⟨1⟩2 and $\text{dom } g \subseteq \text{dom } f$ from ⟨1⟩3
- ⟨1⟩5. For $x \in \text{dom } f$ we have $f(x) = g(x)$
- PROOF: From ⟨1⟩2 and Exercise 12
- ⟨1⟩6. Q.E.D.
- PROOF: From Exercise 11.

□

Exercise 14

(a) If (x, y) and (x, z) are members of $f \cap g$ then they are both members of f , hence $y = z$.

(b) PROOF:

- ⟨1⟩1. If $f \cup g$ is a function then, for all $x \in \text{dom } f \cap \text{dom } g$, we have $f(x) = g(x)$.
- ⟨2⟩1. ASSUME: $f \cup g$ is a function.
- ⟨2⟩2. LET: $x \in \text{dom } f \cap \text{dom } g$
- ⟨2⟩3. $(x, f(x))$ and $(x, g(x))$ are both elements of $f \cup g$
- ⟨2⟩4. $f(x) = g(x)$
- ⟨1⟩2. If, for all $x \in \text{dom } f \cap \text{dom } g$, we have $f(x) = g(x)$, then $f \cup g$ is a function.
- ⟨2⟩1. ASSUME: For all $x \in \text{dom } f \cap \text{dom } g$, we have $f(x) = g(x)$
- ⟨2⟩2. $f \cup g$ is a relation.
- PROOF: Since every element of either f or g is an ordered pair.
- ⟨2⟩3. Whenever (x, y) and (x, z) are elements of $f \cup g$ we have $y = z$
- ⟨3⟩1. LET: $(x, y), (x, z) \in f \cup g$
- ⟨3⟩2. CASE: $(x, y), (x, z) \in f$
- PROOF: Then $y = z$ since f is a function.
- ⟨3⟩3. CASE: $(x, y) \in f, (x, z) \in g$
- PROOF: Then $y = z$ by ⟨2⟩1
- ⟨3⟩4. CASE: $(x, y) \in g, (x, z) \in f$
- PROOF: Then $y = z$ by ⟨2⟩1
- ⟨3⟩5. CASE: $(x, y), (x, z) \in g$

PROOF: Then $y = z$ since g is a function.

□

Exercise 15 PROOF:

⟨1⟩1. $\bigcup \mathcal{A}$ is a relation.

PROOF: Since every member of \mathcal{A} is a relation.

⟨1⟩2. Whenever (x, y) and (x, z) are elements of $\bigcup \mathcal{A}$ then $y = z$

⟨2⟩1. LET: $(x, y), (x, z) \in \bigcup \mathcal{A}$

⟨2⟩2. PICK $f, g \in \mathcal{A}$ such that $(x, y) \in f$ and $(x, z) \in g$

⟨2⟩3. ASSUME: w.l.o.g. $f \subseteq g$

⟨2⟩4. $(x, y), (x, z) \in g$

⟨2⟩5. $y = z$

PROOF: Since g is a function.

□

Exercise 16 If every function belongs to \mathcal{A} then every set belongs to $\text{dom} \bigcup \mathcal{A}$ contradiction Theorem 2A.

Exercise 17 PROOF:

⟨1⟩1. LET: R and S be single-rooted.

⟨1⟩2. LET: $(x, z), (y, z) \in R \circ S$

⟨1⟩3. PICK t and t' such that $(x, t) \in S$, $(t, z) \in R$, $(y, t') \in S$ and $(t', z) \in R$

⟨1⟩4. $t = t'$

PROOF: Since R is single-rooted.

⟨1⟩5. $x = y$

PROOF: Since S is single-rooted.

Thus if F and G are one-to-one functions then $F \circ G$ is single-rooted and a function by Theorem 3H, hence a one-to-one function.

Exercise 18

$$R \circ R = \{\langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle\}$$

$$R \upharpoonright \{1\} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle\}$$

$$R^{-1} \upharpoonright \{1\} = \{\langle 1, 0 \rangle\}$$

$$R[\{1\}] = \{2, 3\}$$

$$R^{-1}[\{1\}] = \{0\}$$

Exercise 19

$$\begin{aligned}
A(\emptyset) &= \{\emptyset, \{\emptyset\}\} \\
A[\emptyset] &= \emptyset \\
A[\{\emptyset\}] &= \{\{\emptyset, \{\emptyset\}\}\} \\
A[\{\emptyset, \{\emptyset\}\}] &= \{\{\emptyset, \{\emptyset\}\}, \emptyset\} \\
A^{-1} &= \{\langle \{\emptyset, \{\emptyset\}\}, \emptyset \rangle, \langle \emptyset, \{\emptyset\} \rangle\} \\
A \circ A &= \{\langle \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \rangle\} \\
A \upharpoonright \emptyset &= \emptyset \\
A \upharpoonright \{\emptyset\} &= \{\langle \emptyset, \{\emptyset, \{\emptyset\}\} \rangle\} \\
A \upharpoonright \{\emptyset, \{\emptyset\}\} &= \{\langle \emptyset, \{\emptyset, \{\emptyset\}\} \rangle, \langle \{\emptyset\}, \emptyset \rangle\} \\
&= A \\
\bigcup A &= \{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}
\end{aligned}$$

Exercise 20

$$\begin{aligned}
z \in F \upharpoonright A &\Leftrightarrow z \in F \ \& \ \exists x, y. (z = \langle x, y \rangle \ \& \ x \in A) \\
&\Leftrightarrow z \in F \ \& \ \exists x, y. (z = \langle x, y \rangle \ \& \ x \in A \ \& \ y \in \text{ran } F) \\
&\Leftrightarrow z \in F \cap (A \times \text{ran } F)
\end{aligned}$$

Exercise 21 Both are equal to $\{\langle x, w \rangle \mid \exists y, z. xTy \ \& \ ySz \ \& \ zRw\}$.

Exercise 22

(a) PROOF:
 $\langle 1 \rangle 1$. ASSUME: $A \subseteq B$
 $\langle 1 \rangle 2$. LET: $y \in F[A]$
 $\langle 1 \rangle 3$. PICK $x \in A$ such that xFy
 $\langle 1 \rangle 4$. $x \in B$ and xFy
 \square

(b) Both are equal to $\{z : \exists x, y. x \in A \ \& \ xGy \ \& \ yFz\}$

(c) Both are equal to $\{\langle x, y \rangle : (x \in A \text{ or } x \in B) \ \& \ xQy\}$

Exercise 23

$$\begin{aligned}
B \circ I_A &= \{\langle x, z \rangle : \exists y(xI_A y \ \& \ yBz)\} \\
&= \{\langle x, z \rangle : \exists y(x \in A \ \& \ x = y \ \& \ yBz)\} \\
&= \{\langle x, z \rangle : x \in A \ \& \ xBz\} \\
&= B \upharpoonright A \\
I_A[C] &= \{y : \exists x \in C.xI_A y\} \\
&= \{y : \exists x \in C(x \in A \ \& \ x = y)\} \\
&= \{y : y \in C \ \& \ y \in A\} \\
&= A \cap C
\end{aligned}$$

Exercise 24

$$\begin{aligned}
F^{-1}[A] &= \{x : \exists y \in A.yF^{-1}x\} \\
&= \{x : \exists y \in A.xFy\} \\
&= \{x \in \text{dom } F : F(x) \in A\}
\end{aligned}$$

Exercise 25**(a)** PROOF: $\langle 1 \rangle 1.$ LET: G be a one-to-one function. $\langle 1 \rangle 2.$ G^{-1} is a function.

PROOF: Theorem 3F.

 $\langle 1 \rangle 3.$ $G \circ G^{-1}$ is a function.

PROOF: Theorem 3H.

 $\langle 1 \rangle 4.$ $\text{dom}(G \circ G^{-1}) = \text{ran } G$

PROOF:

$$\text{dom}(G \circ G^{-1}) = \{x \in \text{dom } G^{-1} : G^{-1}(x) \in \text{dom } G\} \quad (\text{Theorem 3H})$$

$$= \{x \in \text{ran } G : G^{-1}(x) \in \text{dom } G\} \quad (\text{Theorem 3E})$$

$$= \text{ran } G$$

 $\langle 1 \rangle 5.$ $\forall x \in \text{ran } G.(G \circ G^{-1})(x) = x$

PROOF: Theorem 3G.

□

(b) Let G be a function. Then

$$\begin{aligned}
G \circ G^{-1} &= \{\langle x, z \rangle : \exists y(xG^{-1}y \ \& \ yGz)\} \\
&= \{\langle x, z \rangle : \exists y(yGx \ \& \ yGz)\} \\
&= \{\langle x, x \rangle : \exists y.yGx\} & (G \text{ is a function}) \\
&= I_{\text{ran } G}
\end{aligned}$$

Exercise 26

(a)

$$\begin{aligned} F[\bigcup \mathcal{A}] &= \{y : \exists x. \exists A \in \mathcal{A} (x \in A \ \& \ xFy)\} \\ &= \{y : \exists A \in \mathcal{A}. \exists x (x \in A \ \& \ xFy)\} \\ &= \bigcup \{F[A] : A \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned} F[\bigcup \mathcal{A}] &= \{y : \exists x. \forall A \in \mathcal{A} (x \in A \ \& \ xFy)\} \\ &\subseteq \{y : \forall A \in \mathcal{A}. \exists x (x \in A \ \& \ xFy)\} \\ &= \bigcap \{F[A] : A \in \mathcal{A}\} \end{aligned}$$

Exercise 27

$$\begin{aligned} \text{dom}(F \circ G) &= \{x : \exists y. x(F \circ G)y\} \\ &= \{x : \exists y \exists z (xGz \ \& \ zFy)\} \\ &= \{x : \exists z (zG^{-1}x \ \& \ z \in \text{dom } F)\} \\ &= G^{-1}[\text{dom } F] \end{aligned}$$

Exercise 28 PROOF:

$\langle 1 \rangle 1.$ $G : \mathcal{P}A \rightarrow \mathcal{P}B$

PROOF: Since $f[X] \subseteq \text{ran } f \subseteq B$

$\langle 1 \rangle 2.$ For all $X, Y \in \mathcal{P}A$, if $G(X) = G(Y)$ then $X = Y$

$\langle 2 \rangle 1.$ LET: $X, Y \in \mathcal{P}A$

$\langle 2 \rangle 2.$ ASSUME: $f[X] = f[Y]$

$\langle 2 \rangle 3.$ $X \subseteq Y$

$\langle 3 \rangle 1.$ LET: $x \in X$

$\langle 3 \rangle 2.$ $f(x) \in f[X]$

$\langle 3 \rangle 3.$ $f(x) \in f[Y]$

$\langle 3 \rangle 4.$ PICK $y \in Y$ such that $f(x) = f(y)$

$\langle 3 \rangle 5.$ $x = y$

PROOF: Because f is one-to-one.

$\langle 3 \rangle 6.$ $x \in Y$

PROOF: Similar.

$\langle 2 \rangle 4.$ $Y \subseteq X$

□

Example 29 PROOF:

$\langle 1 \rangle 1.$ ASSUME: f maps A onto B

$\langle 1 \rangle 2.$ LET: $b, b' \in B$

$\langle 1 \rangle 3.$ ASSUME: $G(b) = G(b')$

$\langle 1 \rangle 4.$ PICK $x \in A$ such that $f(x) = b$

PROOF: By $\langle 1 \rangle 1$.

$\langle 1 \rangle 5$. $x \in G(b)$

$\langle 1 \rangle 6$. $x \in G(b')$

$\langle 1 \rangle 7$. $f(x) = b'$

$\langle 1 \rangle 8$. $b = b'$

□

The converse does not hold. Let $A = \{0\}$ and $B = \{0, 1\}$. Let f be the function that maps 0 to 0. Then

$$G(0) = \{0\}$$

$$G(1) = \emptyset$$

Thus G is one-to-one but f does not map A onto B .

Exercise 30

(a) PROOF:

$\langle 1 \rangle 1$. $F(B) = B$

$\langle 2 \rangle 1$. $F(B) \subseteq B$

$\langle 3 \rangle 1$. LET: $X \in \mathcal{P}A$ be such that $F(X) \subseteq X$

PROVE: $F(B) \subseteq X$

$\langle 3 \rangle 2$. $B \subseteq X$

$\langle 3 \rangle 3$. $F(B) \subseteq F(X)$

$\langle 3 \rangle 4$. $F(B) \subseteq X$

PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$.

$\langle 2 \rangle 2$. $B \subseteq F(B)$

PROOF: From $\langle 2 \rangle 1$ and the definition of B , since B is one of the sets X such that $F(X) \subseteq X$

$\langle 1 \rangle 2$. $F(C) = C$

$\langle 2 \rangle 1$. $C \subseteq F(C)$

$\langle 3 \rangle 1$. LET: $X \in \mathcal{P}A$ with $X \subseteq F(X)$

PROVE: $X \subseteq F(C)$

$\langle 3 \rangle 2$. $X \subseteq C$

$\langle 3 \rangle 3$. $F(X) \subseteq F(C)$

$\langle 3 \rangle 4$. $X \subseteq F(C)$

PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$

$\langle 2 \rangle 2$. $F(C) \subseteq C$

PROOF: From $\langle 2 \rangle 1$ and the definition of C .

□

(b) If $F(X) = X$ then we have $B \subseteq X$ (because $F(X) \subseteq X$) and $X \subseteq C$ (because $X \subseteq F(X)$).

3.5 Infinite Cartesian Products

Exercise 31 PROOF:

- ⟨1⟩1. If the Axiom of Choice is true then, for any set I and any function H with domain I , if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$.
- ⟨2⟩1. ASSUME: The Axiom of Choice.
- ⟨2⟩2. LET: I be a set.
- ⟨2⟩3. LET: H be a function with domain I .
- ⟨2⟩4. ASSUME: $H(i) \neq \emptyset$ for all $i \in I$.
- ⟨2⟩5. LET: $R = \{(i, x) : i \in I, x \in H(i)\}$
- ⟨2⟩6. PICK a function $F \subseteq R$ with $\text{dom } F = \text{dom } R$
PROVE: $F \in \prod_{i \in I} H(i)$
PROOF: By the Axiom of Choice.
- ⟨2⟩7. $\text{dom } H = I$
PROOF: We have $\text{dom } R = I$ since for all $i \in I$ there exists x such that $x \in H(i)$.
- ⟨2⟩8. $\forall i \in I. F(i) \in H(i)$
PROOF: Since $iRF(i)$.
- ⟨1⟩2. If, for any set I and any function H with domain I , if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$, then the Axiom of Choice is true.
- ⟨2⟩1. ASSUME: For any set I and any function H with domain I , if $H(i) \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H(i) \neq \emptyset$
- ⟨2⟩2. LET: R be a relation
- ⟨2⟩3. LET: $I = \text{dom } R$
- ⟨2⟩4. Define the function H with domain I by: for $i \in I$, $H(i) = \{y : iRy\}$
- ⟨2⟩5. $H(i) \neq \emptyset$ for all $i \in I$
- ⟨2⟩6. PICK $F \in \prod_{i \in I} H(i)$
PROOF: By ⟨2⟩1
- ⟨2⟩7. F is a function
- ⟨2⟩8. $F \subseteq R$
PROOF: For all $i \in I$ we have $F(i) \in H(i)$ and so $iRF(i)$.
- ⟨2⟩9. $\text{dom } F = \text{dom } R$

□