Solutions Manual for Enderton *Elements of Set Theory*

Robin Adams

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Contents

Chapter 1

Chapter 1 — Introduction

1.1 Baby Set Theory

Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

Exercise 2 We have $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ has an element (namely $\emptyset$) while $\emptyset$ has no elements.

We have $\emptyset \neq \{\{\emptyset\}\}$ because $\{\{\emptyset\}\}$ has an element (namely $\{\emptyset\}$) while $\emptyset$ has no elements.

We have $\{\emptyset\} \neq \{\{\emptyset\}\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$. This last fact is true because $\emptyset \neq \{\emptyset\}$ as we proved in the first paragraph.

Exercise 3 Assume $B \subseteq C$. Let $A \in \mathcal{P}B$; we must show that $A \in \mathcal{P}C$.

We have $A \subseteq B$ (since $A \in \mathcal{P}B$) and $B \subseteq C$. From this it follows that $A \subseteq C$ (every element of A is an element of B; every element of B is an element of C; therefore every element of A is an element of C). Hence $A \in \mathcal{P}C$ as required.

Exercise 4 Since $x \in B$, we have $\{x\} \subseteq B$ and so $\{x\} \in \mathcal{P}B$.

Since $x \in B$ and $y \in B$, we have $\{x, y\} \subseteq B$ and so $\{x, y\} \in \mathcal{P}B$.

From these two facts, it follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$ and so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

1.2 Sets — An Informal View

Exercise 5 We have

$$\begin{aligned} V_0 &= A \\ V_1 &= V_0 \cup \mathcal{P}V_0 \\ &= A \cup \mathcal{P}A \\ V_2 &= V_1 \cup \mathcal{P}V_1 \\ &= \{\emptyset, \{\emptyset\}\} \\ V_3 &= \mathcal{P}V_2 \\ &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \end{aligned}$$

We have $\emptyset \subseteq V_0$ and so $\emptyset \in V_1$. Therefore $\{\emptyset\} \subseteq V_1$ and so $\{\emptyset\} \in V_2$. Hence $\{\{\emptyset\}\} \subseteq V_2$.

We also have $\{\{\emptyset\}\} \nsubseteq V_0$ because $\{\emptyset\}$ is not an atom, and $\{\{\emptyset\}\} \nsubseteq V_1$ since $\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.

Thus the rank of $\{\{\emptyset\}\}$ is 2.

Likewise we have $\emptyset$ and $\{\emptyset\}$ are both subsets of V_1, hence

$$\emptyset \in V_2, \qquad \{\emptyset\} \in V_2$$

Thus $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are all subsets of V_2, hence elements of V_3. Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is not a subset of V_0 (because $\emptyset$ is not an atom.) It is not a subset of V_1 ($\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.) It is not a subset of V_2 (we have $\{\emptyset, \{\emptyset\}\} \notin V_2$ since $\{\emptyset\} \notin V_1$).

Therefore the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

Exercise 6

$$\begin{aligned}
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}V_0 && \text{(since } V_0 = A\text{)} \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_1 && \text{(since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3)} \\
V_3 &= V_2 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_2 && \text{(since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3)} \\
V_4 &= V_3 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_3 && \text{(since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3)}
\end{aligned}$$

Exercise 7 In Exercise 5 we calculated $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Hence

$$\begin{aligned}
V_4 =& \mathcal{P}V_3 \\
=& \{\emptyset, \\
&\quad \{\emptyset\}, \\
&\quad \{\{\emptyset\}\}, \\
&\quad \{\{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
&\}
\end{aligned}$$

Chapter 2

Chapter 2 — Axioms and Operations

2.1 Arbitrary Unions and Intersections

Exercise 1 $A \cap B \cap C$ is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

Exercise 2 Take $A = \emptyset$ and $B = \{\emptyset\}$. Then $\bigcup A = \bigcup B = \emptyset$ but $A \neq B$. (There are many other possible answers.)

Exercise 3 Let $b \in A$. We must show that $b \subseteq \bigcup A$.

Let x be any element of b. We must show that $x \in \bigcup A$. We know that $x \in b$ and $b \in A$, and so $x \in \bigcup A$ by the definition of $\bigcup A$.

Exercise 4 Suppose $A \subseteq B$. Let $x \in \bigcup A$. We must show that $x \in \bigcup B$.

Pick an element $a \in A$ such that $x \in a$. Then $a \in B$ because $A \subseteq B$. Since we know $x \in a$ and $a \in B$, we know that $x \in \bigcup B$.

Exercise 5 Assume that every member of $\mathcal{A}$ is a subset of B. Let $x \in \bigcup \mathcal{A}$. We must show that $x \in B$.

Pick $A \in \mathcal{A}$ such that $x \in A$. By our assumption, we have $A \subseteq B$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$ as required.

Exercise 6

(a) We will show that $\bigcup \mathcal{P}A \subseteq A$ and $A \subseteq \bigcup \mathcal{P}A$.

To show $\bigcup \mathcal{P}A \subseteq A$: This follows from Exercise 5, since every member of $\mathcal{P}A$ is a subset of A.

To show $A \subseteq \bigcup \mathcal{P}A$: Let $a \in A$. Then we have $a \in \{a\}$ and $\{a\} \in \mathcal{P}A$ so $a \in \bigcup \mathcal{P}A$.

(b) To show $A \subseteq \mathcal{P} \bigcup A$: This holds because every element of A is a subset of $\bigcup A$, as we proved is Exercise 3.

Equality holds if and only if $A = \mathcal{P}X$ for some set X.

Proof: If $A = \mathcal{P} \bigcup A$ then of course $A = \mathcal{P}X$ for some X.

Conversely, if $A = \mathcal{P}X$, then we have

$$\begin{aligned} \mathcal{P} \bigcup A &= \mathcal{P} \bigcup \mathcal{P}X \\ &= \mathcal{P}X && \text{(by part (a))} \\ &= A \end{aligned}$$

Exercise 7

(a) For any set X,

$$\begin{aligned} & X \in \mathcal{P}A \cap \mathcal{P}B \\ \Leftrightarrow & X \subseteq A \text{ and } X \subseteq B \\ \Leftrightarrow & \text{Every member of } X \text{ is a member of } A \text{ and a member of } B \\ \Leftrightarrow & X \subseteq A \cap B \\ \Leftrightarrow & X \in \mathcal{P}(A \cap B) \end{aligned}$$

(b) Let $X \in \mathcal{P}A \cup \mathcal{P}B$. Then either $X \in \mathcal{P}A$ or $X \in \mathcal{P}B$ (or both). If $X \in \mathcal{P}A$, then we have $X \subseteq A$ and so $X \subseteq A \cup B$ (because $A \subseteq A \cup B$). Similarly if $X \in \mathcal{P}B$ then we have $X \subseteq A \cup B$. So in either case $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

Equality holds if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof: Suppose $A \subseteq B$. Then $\mathcal{P}A \subseteq \mathcal{P}B$ (Chapter 1 Exercise 3) and so $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$. Also $A \cup B = B$ so $\mathcal{P}(A \cup B) = \mathcal{P}B$. Thus $\mathcal{P}A \cup \mathcal{P}B$ and $\mathcal{P}(A \cup B)$ are equal.

Similarly if $B \subseteq A$ then $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$.

Conversely, suppose $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$. We have $A \cup B \in \mathcal{P}(A \cup B)$, so $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$. If $A \cup B \in \mathcal{P}A$, then we have $B \subseteq A \cup B \subseteq A$. And if $A \cup B \in \mathcal{P}B$, then we have $A \subseteq A \cup B \subseteq B$.

Exercise 8 If A is a set such that every singleton belongs to A, then every set belongs to $\bigcup A$, contradicting Theorem 2A.

Exercise 9 Let $a = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$. Then $a \in B$ but $\mathcal{P}a$ is not a subset of B because $\emptyset \in \mathcal{P}a$ and $\emptyset \notin B$.

Exercise 10 We must show that $\mathcal{P}a \subseteq \mathcal{P}\bigcup B$. So let $X \in \mathcal{P}a$. Then $X \subseteq a$; we must show that $X \subseteq \bigcup B$.

Let $x \in X$; we must show that $x \in \bigcup B$. We have $x \in a$ (because $x \in X$ and $X \subseteq a$) and $a \in B$, hence $x \in \bigcup B$ as required.

2.2 Algebra of Sets

Exercise 11 For any x we have

$$\begin{aligned} x \in (A \cap B) \cup (A - B) &\Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B) \\ &\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B) \\ &\Leftrightarrow x \in A \end{aligned}$$

Hence $A = (A \cap B) \cup (A - B)$.

For any x we have

$$\begin{aligned} x \in A \cup (B - A) &\Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A) \\ &\Leftrightarrow x \in A \text{ or } x \in B \\ &\Leftrightarrow x \in A \cup B \end{aligned}$$

Hence $A \cup (B - A) = A \cup B$.

Exercise 12 For any x,

$$\begin{aligned} x \in C - (A \cap B) &\Leftrightarrow x \in C \& \neg(x \in A \& x \in B) \\ &\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B) \\ &\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B) \\ &\Leftrightarrow x \in (C - A) \cup (C - B) \end{aligned}$$

Exercise 13 Suppose $A \subseteq B$. Let $x \in C - B$; we must show $x \in C - A$. We have $x \in C$ and $x \notin B$. Therefore $x \notin A$, since every member of A is a member of B. And so we have $x \in C - A$ as required.

Exercise 14 Let $A = \{\emptyset\}$, $B = \emptyset$ and $C = \{\emptyset\}$. Then $A - (B - C) = A - \emptyset = \{\emptyset\}$ while $(A - B) - C = \{\emptyset\} - C = \emptyset$.

Exercise 15

(a) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have $x \in A \cap (B + C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$.

(b) For any x we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$

In every case, we have $x \in A + (B + C) \Leftrightarrow x \in (A + B) + C$.

Exercise 16

$$\begin{aligned} [(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A] &= (A \cup B) - A \\ &= B - A \end{aligned}$$

Exercise 17

(a) $\Leftrightarrow$ **(b)**

$$\begin{aligned} A \subseteq B &\Leftrightarrow \text{Every element of } A \text{ is an element of } B \\ &\Leftrightarrow \text{There is no element of } A \text{ that is not an element of } B \\ &\Leftrightarrow A - B = \emptyset \end{aligned}$$

(a) $\Rightarrow$ **(c)** Suppose $A \subseteq B$. We have $B \subseteq A \cup B$ from the definition of $A \cup B$; we must prove that $A \cup B \subseteq B$. So let $x \in A \cup B$. Then $x \in A$ or $x \in B$. But in either case $x \in B$, since $x \in A \Rightarrow x \in B$. Thus we have $x \in B$ as required.

(c) $\Rightarrow$ **(a)** We always have $A \subseteq A \cup B$. So if $A \cup B = B$ then we have $A \subseteq B$.

(a) $\Rightarrow$ **(d)** Suppose $A \subseteq B$. We have $A \cap B \subseteq A$ from the definition of $A \cap B$; we must prove that $A \subseteq A \cap B$. So let $x \in A$. Then $x \in B$ since $A \subseteq B$, hence $x \in A \cap B$ as required.

(d) $\Rightarrow$ (a) We always have $A \cap B \subseteq B$. So if $A \cap B = A$ then $A \subseteq B$.

Exercise 18 We can make the following 16 sets:

- $\emptyset$ $(= A - A)$
- $A - B$
- $A \cap B$
- $B - A$
- $S - (A \cup B)$
- A
- $A + B$
- $S - B$
- B
- $S - (A + B)$
- $S - A$
- $A \cup B$
- $S - (B - A)$
- $S - (A \cap B)$
- $S - (A - B)$

Exercise 19 They are never equal, because for all A, B, we have $\emptyset \in \mathcal{P}(A - B)$ but $\emptyset \notin \mathcal{P}A - \mathcal{P}B$ since $\emptyset \in \mathcal{P}B$.

Exercise 20 Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

We first show $B \subseteq C$. Let $x \in B$; we show $x \in C$. We have $x \in A \cup B = A \cup C$, so either $x \in A$ or $x \in C$. If $x \in C$, we are done. If $x \in A$, then we have $x \in A \cap B = A \cap C$, and so $x \in C$ in this case too.

We can show $C \subseteq B$ similarly. Hence $B = C$.

Exercise 21 For any x, we have

$$\begin{aligned} x \in \bigcup(A \cup B) &\Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C \\ &\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C, \text{ or there exists } C \in B \text{ such that } x \in C \\ &\Leftrightarrow x \in \bigcup A \cup \bigcup B \end{aligned}$$

Exercise 22 For any x, we have

$$\begin{aligned} x \in \bigcap(A \cup B) &\Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C \\ &\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C \\ &\Leftrightarrow x \in \bigcap A \cap \bigcap B \end{aligned}$$

Exercise 23 PROOF:
$\langle 1 \rangle 1.$ $A \subseteq \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 1.$ LET: $x \in A$
  $\langle 2 \rangle 2.$ LET: $X \in \mathcal{B}$
  $\langle 2 \rangle 3.$ $x \in A \cup X$
$\langle 1 \rangle 2.$ $\bigcap \mathcal{B} \subseteq \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 1.$ LET: $x \in \bigcap \mathcal{B}$
  $\langle 2 \rangle 2.$ LET: $X \in \mathcal{B}$
  $\langle 2 \rangle 3.$ $x \in X$
  $\langle 2 \rangle 4.$ $x \in A \cup X$
$\langle 1 \rangle 3.$ $\bigcap\{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap \mathcal{B}$
  $\langle 2 \rangle 1.$ LET: $x \in \bigcap\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 2.$ ASSUME: $x \notin A$
  PROVE: $x \in \bigcap \mathcal{B}$
  $\langle 2 \rangle 3.$ LET: $X \in \mathcal{B}$
  $\langle 2 \rangle 4.$ $x \in A \cup X$
  $\langle 2 \rangle 5.$ $x \in X$
$\square$

Exercise 24

(a)

$$\begin{aligned} Y \in \mathcal{P} \bigcap \mathcal{A} &\Leftrightarrow Y \subseteq \bigcap \mathcal{A} \\ &\Leftrightarrow \forall y \in Y . \forall X \in \mathcal{A} . y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A} . \forall y \in Y . y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A} . Y \in \mathcal{P} X \\ &\Leftrightarrow Y \in \bigcap\{\mathcal{P} X \mid X \in \mathcal{A}\} \end{aligned}$$

(b) $\bigcup\{\mathcal{P} X \mid X \in \mathcal{A}\} \subseteq \mathcal{P} \bigcup \mathcal{A}$

PROOF:
$\langle 1 \rangle 1.$ LET: $Y \in \bigcup\{\mathcal{P} X \mid X \in \mathcal{A}\}$
$\langle 1 \rangle 2.$ PICK $X \in \mathcal{A}$ such that $Y \in \mathcal{P} X$
$\langle 1 \rangle 3.$ $Y \subseteq X$
$\langle 1 \rangle 4.$ $Y \subseteq \bigcup \mathcal{A}$
$\langle 1 \rangle 5.$ $Y \in \mathcal{P} \bigcup \mathcal{A}$

Equality holds if and only if $\bigcup \mathcal{A} \in \mathcal{A}$.

$\langle 1 \rangle 1.$ If $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P}\bigcup \mathcal{A}$ then $\bigcup \mathcal{A} \in \mathcal{A}$
  $\langle 2 \rangle 1.$ Assume: $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P}\bigcup \mathcal{A}$
  $\langle 2 \rangle 2.$ $\bigcup \mathcal{A} \in \bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\}$
  $\langle 2 \rangle 3.$ Pick $X \in \mathcal{A}$ such that $\bigcup \mathcal{A} \in \mathcal{P}X$
  $\langle 2 \rangle 4.$ $X = \bigcup \mathcal{A}$
$\langle 1 \rangle 2.$ If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P}\bigcup \mathcal{A}$
  Proof: If $\bigcup \mathcal{A} \in \mathcal{A}$ then $\mathcal{P}\bigcup \mathcal{A} \in \{\mathcal{P}X \mid X \in \mathcal{A}\}$.
$\square$

Exercise 25 We have $A \cup \bigcup \mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$ if and only if $A = \emptyset$ or $\mathcal{B} \neq \emptyset$

$\langle 1 \rangle 1.$ If $A \cup \bigcup \mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$ then $A = \emptyset$ or $\mathcal{B} \neq \emptyset$
  Proof: If $A \cup \bigcup \mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$ and $\mathcal{B} = \emptyset$ then

$$A \cup \bigcup \emptyset = \bigcup \emptyset$$
$$\therefore A = \emptyset$$

$\langle 1 \rangle 2.$ If $A = \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
  Proof: Both sides are equal to $\bigcup \mathcal{B}$
$\langle 1 \rangle 3.$ If $\mathcal{B} \neq \emptyset$ then $A \cup \bigcup \mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
  $\langle 2 \rangle 1.$ Assume: $\mathcal{B} \neq \emptyset$
  $\langle 2 \rangle 2.$ $A \cup \bigcup \mathcal{B} \subseteq \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 1.$ Let: $x \in A \cup \bigcup \mathcal{B}$
      Prove: $x \in \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 2.$ Case: $x \in A$
      $\langle 4 \rangle 1.$ Pick $X \in \mathcal{B}$
        Proof: By $\langle 2 \rangle 1$
      $\langle 4 \rangle 2.$ $x \in A \cup X$
    $\langle 3 \rangle 3.$ Case: $x \in \bigcup \mathcal{B}$
      $\langle 4 \rangle 1.$ Pick $X \in \mathcal{B}$ such that $x \in X$
      $\langle 4 \rangle 2.$ $x \in A \cup X$
  $\langle 2 \rangle 3.$ $\bigcup\{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcup \mathcal{B}$
    $\langle 3 \rangle 1.$ Let: $x \in \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
    $\langle 3 \rangle 2.$ Pick $X \in \mathcal{B}$ such that $x \in A \cup X$
    $\langle 3 \rangle 3.$ $X \subseteq \bigcup \mathcal{B}$
    $\langle 3 \rangle 4.$ $A \cup X \subseteq A \cup \bigcup \mathcal{B}$
    $\langle 3 \rangle 5.$ $x \in A \cup \bigcup \mathcal{B}$

2.3 Review Exercises

Exercise 26 Sets A, B, D and F are all equal to each other. Sets C, E and G are equal to each other. None of the first list is equal to any of the second list.

Exercise 27 Take $A = \{\{0\}, \{1\}\}$ and $B = \{\{1\}\}$. Then $A \cap B = \{\{1\}\}$ and

$$\begin{aligned} \bigcap A \cap \bigcap B &= \emptyset \cap \{1\} \\ &= \emptyset \\ \bigcap (A \cap B) &= \bigcap \{\{1\}\} \\ &= \{1\} \end{aligned}$$

Exercise 28

$$\bigcup \{\{3,4\}, \{\{3\},\{4\}\}, \{3,\{4\}\}, \{\{3\},4\}\} = \{3, 4, \{3\}, \{4\}\}$$

Exercise 29

(a) $\emptyset$

(b) We have

$$\begin{aligned} \{\emptyset\} &\subseteq \mathcal{P}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{P}\mathcal{P}\{\emptyset\} \\ \{\emptyset\} &\subseteq \mathcal{P}\mathcal{P}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{P}\mathcal{P}\mathcal{P}\{\emptyset\} \\ \therefore \bigcap \{\mathcal{P}\mathcal{P}\mathcal{P}\{\emptyset\}, \mathcal{P}\mathcal{P}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} &= \mathcal{P}\mathcal{P}\mathcal{P}\{\emptyset\} \cap \mathcal{P}\mathcal{P}\{\emptyset\} \cap \mathcal{P}\{\emptyset\} \\ &= \mathcal{P}\{\emptyset\} \\ &= \{\emptyset, \{\emptyset\}\} \end{aligned}$$

Exercise 30

(a) $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$

(b) $\{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d) $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

Exercise 31

(a) $\{1, 2, 3, \emptyset\}$

(b) $\emptyset$

(c) $\emptyset$

(d) $\emptyset$

Exercise 32

(a) $a \cup b$

(b) a

(c)

$$\begin{aligned}\bigcap\bigcup S \cup (\bigcup\bigcup S - \bigcup\bigcap S) &= (a \cap b) \cup ((a \cup b) - a) \\ &= (a \cap b) \cup (b - a) \\ &= b\end{aligned}$$

Exercise 33 When $a \neq b$:

$$\begin{aligned}\bigcup(\bigcup S - \bigcap S) &= \bigcup(\{a, b\} - \{a\}) \\ &= \bigcup\{b\} \\ &= b\end{aligned}$$

When $a = b$:

$$\begin{aligned}\bigcup(\bigcup S - \bigcap S) &= \bigcup(\{a, b\} - \{a\}) \\ &= \bigcup\emptyset \\ &= \emptyset\end{aligned}$$

Exercise 34 For any set S, we have

$$\begin{gathered}\emptyset \subseteq \mathcal{P}S \\ \therefore \emptyset \in \mathcal{P}\mathcal{P}S \\ \emptyset \subseteq S \\ \therefore \emptyset \in \mathcal{P}S \\ \therefore \{\emptyset\} \subseteq \mathcal{P}S \\ \therefore \{\emptyset\} \in \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} \in \mathcal{P}\mathcal{P}\mathcal{P}S\end{gathered}$$

Exercise 35 Assume $\mathcal{P}A = \mathcal{P}B$. Then we have

$$\begin{aligned}
&A \in \mathcal{P}A \\
\therefore\ &A \in \mathcal{P}B \\
\therefore\ &A \subseteq B \\
&B \in \mathcal{P}B \\
\therefore\ &B \in \mathcal{P}A \\
\therefore\ &B \subseteq A \\
\therefore\ &A = B
\end{aligned}$$

Exercise 36

(a)

$$\begin{aligned}
x \in A - (A \cap B) &\Leftrightarrow x \in A \ \&\neg(x \in A \ \&\ x \in B) \\
&\Leftrightarrow x \in A \ \&\ x \notin B \\
&\Leftrightarrow x \in A - B
\end{aligned}$$

(b)

$$\begin{aligned}
x \in A - (A - B) &\Leftrightarrow x \in A \ \&\neg(x \in A \ \&\ x \notin B) \\
&\Leftrightarrow x \in A \ \&\ x \in B \\
&\Leftrightarrow x \in A \cap B
\end{aligned}$$

Exercise 37

(a)

$$\begin{aligned}
x \in (A \cup B) - C &\Leftrightarrow (x \in A \text{ or } x \in B) \ \&\ x \notin C \\
&\Leftrightarrow (x \in A \ \&\ x \notin C) \text{ or } (x \in B \ \&\ x \notin C) \\
&\Leftrightarrow x \in (A - C) \cup (B - C)
\end{aligned}$$

(b)

$$\begin{aligned}
x \in A - (B - C) &\Leftrightarrow x \in A \ \&\neg(x \in B \ \&\ x \notin C) \\
&\Leftrightarrow x \in A \ \&(x \notin B \text{ or } x \in C) \\
&\Leftrightarrow (x \in A \ \&\ x \notin B) \text{ or } (x \in A \ \&\ x \in C) \\
&\Leftrightarrow x \in (A - B) \cup (A \cap C)
\end{aligned}$$

(c)

$$\begin{aligned}
x \in (A - B) - C &\Leftrightarrow x \in A \ \&\ x \notin B \ \&\ x \notin C \\
&\Leftrightarrow x \in A \ \&\neg(x \in B \vee x \in C) \\
&\Leftrightarrow x \in A - (B \cup C)
\end{aligned}$$

Exercise 38

(a) If every element of A is an element of C, and every element of B is an element of C, then everything that is an element of either A or B is an element of C.

(b) If every element of C is an element of A, and every element of C is an element of B, then every element of C is an element of both A and B.

Chapter 3

Chapter 3 — Relations and Functions

3.1 Ordered Pairs

Exercise 1 We have $\langle 0,1,0\rangle^* = \langle 0,1,1\rangle^* = \{\{0\},\{0,1\}\}$.

Exercise 2

(a)

$$\begin{aligned}
& z \in A \times (B \cup C) \\
\Leftrightarrow & \exists x, y(z = (x,y) \ \&\ x \in A \ \&\ (y \in B \text{ or } y \in C)) \\
\Leftrightarrow & \exists x, y(z = (x,y) \ \&\ x \in A \ \&\ y \in B) \text{ or } (z = (x,y) \ \&\ x \in A \ \&\ y \in C) \\
\Leftrightarrow & z \in (A \times B) \cup (A \times C)
\end{aligned}$$

(b)
$\langle 1\rangle 1.$ Assume: $A \times B = A \times C$ and $A \neq \emptyset$
$\langle 1\rangle 2.$ Pick $a \in A$
$\langle 1\rangle 3.$ For all x, $x \in B \Leftrightarrow x \in C$
  Proof: $x \in B$ iff $(a,x) \in A \times B$ iff $(a,x) \in A \times C$ iff $x \in C$.
□

Exercise 3

$$\begin{aligned}
& z \in A \times \bigcup \mathcal{B} \\
\Leftrightarrow & \exists x, y(z = (x,y) \ \&\ x \in A \ \&\ \exists X \in \mathcal{B}. y \in X) \\
\Leftrightarrow & \exists X \in \mathcal{B}. \exists x, y(z = (x,y) \ \&\ x \in A \ \&\ y \in X) \\
\Leftrightarrow & z \in \bigcup \{A \times X : X \in \mathcal{B}\}
\end{aligned}$$

Exercise 4 If every ordered pair belongs to A then every set belongs to $\bigcup\bigcup A$ contradicting Theorem 2A.

Exercise 5

(a) Apply a Subset Axiom to $\mathcal{P}(A \times B)$: we have $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A.z = \{x\} \times B\}$.

(b)

$$\begin{aligned}
& z \in \bigcup C \\
\Leftrightarrow & \exists x \in A.z \in \{x\} \times B \\
\Leftrightarrow & \exists x \in A.\exists y \in B.z = (x, y) \\
\Leftrightarrow & z \in A \times B
\end{aligned}$$

3.2 Relations

Exercise 6 If $A \subseteq \operatorname{dom} A \times \operatorname{ran} A$ then A is a set of ordered pairs, i.e. a relation.

Conversely, suppose A is a relation. Let $z \in A$. Then z is an ordered pair; let $z = (x, y)$. We have $x \in \operatorname{dom} A$ and $y \in \operatorname{ran} A$ and so $z \in \operatorname{dom} A \times \operatorname{ran} A$ as required.

Exercise 7 We have $\operatorname{fld} R \subseteq \bigcup\bigcup R$ by Lemma 3D.

Conversely, let $x \in \bigcup\bigcup R$. Pick a and b such that $x \in a$, $a \in b$ and $b \in R$. Then b is an ordered pair; let $b = (y, z)$. We have $a = \{y\}$ or $\{y, z\}$, hence $x = y$ or $x = z$. In either case, $x \in \operatorname{fld} R$.

Exercise 8

(a)

$$\begin{aligned}
& x \in \operatorname{dom} \bigcup \mathcal{A} \\
\Leftrightarrow & \exists y.\exists R \in \mathcal{A}.(x, y) \in R \\
\Leftrightarrow & \exists R \in \mathcal{A}.\exists y.(x, y) \in R \\
\Leftrightarrow & x \in \bigcup\{\operatorname{dom} R : R \in \mathcal{A}\}
\end{aligned}$$

(b)

$$
\begin{aligned}
& y \in \operatorname{ran} \bigcup \mathcal{A} \\
\Leftrightarrow & \exists x . \exists R \in \mathcal{A} .(x, y) \in R \\
\Leftrightarrow & \exists R \in \mathcal{A} . \exists x .(x, y) \in R \\
\Leftrightarrow & y \in \bigcup\{\operatorname{ran} R: R \in \mathcal{A}\}
\end{aligned}
$$