Solutions Manual for Enderton *Elements of Set Theory*

Robin Adams

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Contents

Chapter 1

Chapter 1 — Introduction

1.1 Baby Set Theory

1.1.1 Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ — true
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ — false

1.1.2 Exercise 2

We have $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ has an element (namely $\emptyset$) while $\emptyset$ has no elements.

We have $\emptyset \neq \{\{\emptyset\}\}$ because $\{\{\emptyset\}\}$ has an element (namely $\{\emptyset\}$) while $\emptyset$ has no elements.

We have $\{\emptyset\} \neq \{\{\emptyset\}\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$. This last fact is true because $\emptyset \neq \{\emptyset\}$ as we proved in the first paragraph.

1.1.3 Exercise 3

Assume $B \subseteq C$. Let $A \in \mathcal{P}B$; we must show that $A \in \mathcal{P}C$.

We have $A \subseteq B$ (since $A \in \mathcal{P}B$) and $B \subseteq C$. From this it follows that $A \subseteq C$ (every element of A is an element of B; every element of B is an element of C; therefore every element of A is an element of C). Hence $A \in \mathcal{P}C$ as required.

1.1.4 Exercise 4

Since $x \in B$, we have $\{x\} \subseteq B$ and so $\{x\} \in \mathcal{P}B$.

Since $x \in B$ and $y \in B$, we have $\{x, y\} \subseteq B$ and so $\{x, y\} \in \mathcal{P}B$.

From these two facts, it follows that $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$ and so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$.

1.2 Sets — An Informal View

1.2.1 Exercise 5

We have

$$
\begin{aligned}
V_0 &= A \\
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}A \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= \{\emptyset, \{\emptyset\}\} \\
V_3 &= \mathcal{P}V_2 \\
&= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}
$$

We have $\emptyset \subseteq V_0$ and so $\emptyset \in V_1$. Therefore $\{\emptyset\} \subseteq V_1$ and so $\{\emptyset\} \in V_2$. Hence $\{\{\emptyset\}\} \subseteq V_2$.

We also have $\{\{\emptyset\}\} \nsubseteq V_0$ because $\{\emptyset\}$ is not an atom, and $\{\{\emptyset\}\} \nsubseteq V_1$ since $\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.

Thus the rank of $\{\{\emptyset\}\}$ is 2.

Likewise we have $\emptyset$ and $\{\emptyset\}$ are both subsets of V_1, hence

$$\emptyset \in V_2, \qquad \{\emptyset\} \in V_2$$

Thus $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are all subsets of V_2, hence elements of V_3. Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is not a subset of V_0 (because $\emptyset$ is not an atom.) It is not a subset of V_1 ($\{\emptyset\} \notin V_1$ because $\emptyset$ is not an atom.) It is not a subset of V_2 (we have $\{\emptyset, \{\emptyset\}\} \notin V_2$ since $\{\emptyset\} \notin V_1$).

Therefore the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

1.2.2 Exercise 6

$$\begin{aligned}
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}V_0 && \text{(since } V_0 = A\text{)} \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_1 && \text{(since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3)} \\
V_3 &= V_2 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_2 && \text{(since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3)} \\
V_4 &= V_3 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_3 && \text{(since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3)}
\end{aligned}$$

1.2.3 Exercise 7

In Exercise 5 we calculated $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

Hence

$$\begin{aligned}
V_4 =& \mathcal{P}V_3 \\
= \{&\emptyset, \\
&\{\emptyset\}, \\
&\{\{\emptyset\}\}, \\
&\{\{\{\emptyset\}\}\}, \\
&\{\{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}\}, \\
&\{\emptyset, \{\{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
&\}
\end{aligned}$$