M0 Categories

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1 Categories

Definition 1 (Category)**.** A *category* consists of:

- a collection of *objects*.
- for any objects A and B, a collection of *maps* from A to B. We write $f : A \to B$ iff f is a map from A to B.
- for any object A, an *identity map* $1_A : A \to A$
- for any maps $f : A \to B$ and $g : B \to C$, a map $g \circ f : A \to C$

such that:

Identity Laws For any map $f : A \to B$, we have $1_B \circ f = f \circ 1_A = f : A \to B$

Associative Law For any maps $f : A \to B$, $g : B \to C$ and $h : C \to D$, we hav $h \circ (g \circ f) = (h \circ g) \circ f : A \to D$

Definition 2. A map $f : A \to B$ is *monic* or a *monomorphism*, $f : A \rightarrowtail B$, iff, for every object T and morphisms $x_1, x_2 : T \to B$, if $f \circ x_1 = f \circ x_2$ then $x_1 = x_2$.

Definition 3. A map $f : A \to B$ is *epi* or an *epimorphism*, $f : A \twoheadrightarrow B$, iff, for every object T and morphisms $x_1, x_2 : B \to T$, if $x_1 \circ f = x_2 \circ f$ then $x_1 = x_2$.

Definition 4 (Retraction, Section)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* for s, and s is a *section* for r, iff $r \circ s = 1_B$.

The object A is a *retract* of B iff there exists a retraction $r : B \to A$, i.e. there exist maps $s : A \to B$ and $r : B \to A$ such that $r \circ s = 1_A$.

Proposition 5. *If a map $f : A \to B$ has a section, then for any object T and any map $y : T \to B$, there exists a map $x : T \to A$ such that $f \circ x = y$.*

PROOF: If $s : B \to A$ is a section of f, then we take $x = s \circ y$. We have $f \circ x = f \circ s \circ y = 1_B \circ y = y$. □

Proposition 6. *If a map $f : A \to B$ has a retraction, then for any object T and any map $g : A \to T$, there exists a map $t : B \to T$ such that $t \circ f = g$.*

PROOF: If $r : B \to A$ is a section for f, then we take $t = g \circ r$. We have $t \circ f = g \circ r \circ f = g \circ 1_A = g$. □

Proposition 7. *Every section is monic.*

PROOF: Let $r : B \to A$ be a retraction for f. Then, if $f \circ x_1 = f \circ x_2$, then

$$\begin{aligned} r \circ f \circ x_1 &= r \circ f \circ x_2 \\ \therefore 1_A \circ x_1 &= 1_A \circ x_2 \\ \therefore x_1 &= x_2 \end{aligned} \qquad \square$$

Proposition 8. *Every retraction is epi.*

PROOF: Let $s : B \to A$ be a section for $f : A \to B$. Let T be any set and $t_1, t_2 : T \to B$. Suppose $t_1 \circ f = t_2 \circ f$. Then

$$\begin{aligned} t_1 \circ f \circ s &= t_2 \circ f \circ s \\ \therefore t_1 \circ 1_B &= t_2 \circ 1_B \\ \therefore t_1 &= t_2 \end{aligned}$$

Proposition 9. *For any object A, the identity map 1_A is a section and a retraction of itself.*

PROOF: The Unit Laws give $1_A \circ 1_A = 1_A$. □

Corollary 9.1. *Every object is a retract of itself.*

Proposition 10. *If $r_1 : B \to A$ is a retraction of $s_1 : A \to B$ and $r_2 : C \to B$ is a retraction of $s_2 : B \to C$ then $r_1 \circ r_2$ is a retraction of $s_2 \circ s_1$.*

PROOF:

$$\begin{aligned} r_1 \circ r_2 \circ s_2 \circ s_1 &= r_1 \circ 1_B \circ s_1 \\ &= r_1 \circ s_1 \\ &= 1_A \qquad \square \end{aligned}$$

Corollary 10.1. *If the object A is a retract of B and B is a retract of C then A is a retract of C.*

Theorem 11. *If r is a retraction of f and s is a section of f then $r = s$.*

PROOF: Let $f : A \to B$ and $r, s : B \to A$. Then

$$\begin{aligned} r &= r \circ 1_B \\ &= r \circ f \circ s \\ &= 1_A \circ s \\ &= s \qquad \square \end{aligned}$$

Definition 12 (Isomorphism)**.** A map $f : A \to B$ is an *isomorphism* or *invertible*, $f : A \cong B$, iff there exists a map $f^{-1} : B \to A$, the *inverse* for f, such that $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

Two objects A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between them.

Theorem 13. *The inverse of an isomorphism is unique.*

PROOF: From Theorem 11. $\square$

Theorem 14. *For any object A, the identity map $1_A : A \cong A$ is an isomorphism with $1_A^{-1} = 1_A$.*

PROOF: We have $1_A \circ 1_A = 1_A$ by the Identity Laws. $\square$

Theorem 15. *If $f : A \cong B$ then $f^{-1} : B \cong A$ and $(f^{-1})^{-1} = f$.*

PROOF: Since $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$ by the definition of inverse. $\square$

Theorem 16. *If $f : A \cong B$ and $g : B \cong C$ then $g \circ f : A \cong C$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF: From Proposition 10. $\square$

Proposition 17. *Every monomorphic retraction is an isomorphism.*

PROOF: Let $f : A \rightarrowtail B$ be a monomorphism with section $s : B \to A$. Then

$$\begin{aligned} f \circ s \circ f &= f \\ \therefore s \circ f &= 1_A \end{aligned}$$

Thus s is also a retraction for f, hence an inverse. $\square$

Proposition 18. *Every epimorphic section is an isomorphism.*

PROOF: Dual. □

Definition 19 (Idempotent)**.** A map $e : A \to A$ is *idempotent* iff $e \circ e = e$.

Definition 20 (Split Idempotent)**.** Let $e : A \to A$ be idempotent. A *splitting* of e consists of an object B and maps $s : B \to A$, $r : A \to B$ such that $r \circ s = 1_B$ and $s \circ r = e$.

Definition 21 (Automorphism)**.** An *automorphism* on an object A is an isomorphism $A \cong A$.

2 Terminal Objects

Definition 22 (Terminal Object)**.** An object 1 is *terminal* iff, for every object X, there exists exactly one morphism $X \to 1$.

Theorem 23. *If T_1 and T_2 are terminal objects, then the unique map $T_1 \to T_2$ is iso.*

PROOF: Let $f : T_1 \to T_2$ be the unique such map, and $g : T_2 \to T_1$ the unique map in the other direction. The $g \circ f = 1_{T_1}$ since there is only one map $T_1 \to T_1$, and $f \circ g = 1_{T_2}$ since there is only one map $T_2 \to T_2$. □

3 Initial Objects

Definition 24 (Initial Object)**.** An object 0 is *initial* iff, for every object X, there exists exactly one morphism $0 \to X$.

4 Products

Definition 25 (Product)**.** Let A and B be objects. A *product* of A and B consists of an object $A \times B$ and morphisms $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, the *projections*, such that, for any object X and morphisms $f : X \to A$ and $g : X \to B$, there exists exactly one map $\langle f, g \rangle : X \to A \times B$ such that $\pi_1 \circ \langle f, g \rangle = f$ and $\pi_2 \circ \langle f, g \rangle = g$.

Theorem 26 (Lawvere Diagonal Theorem)**.** *Let $\mathcal{C}$ be a category with finite products.*

Let Y and T be objects and $f : T \times T \to Y$. Suppose that, for every $g : T \to Y$, there exists a point $\overline{g} : 1 \to T$ such that $f \circ \langle 1_T, \overline{g} \circ ! \rangle = g$. Then, for every $\alpha : Y \to Y$, there exists $y : 1 \to Y$ such that $\alpha y = y$.

PROOF:
$\langle 1 \rangle 1.$ LET: $\alpha : Y \to Y$
$\langle 1 \rangle 2.$ LET: $g = \alpha \circ f \circ \langle 1_T, 1_T \rangle$

$\langle 1\rangle 3.$ PICK $t_0 : 1 \to T$ such that $f \circ \langle 1_T, t_0 \circ !\rangle = \alpha \circ f \circ \langle 1_T, 1_T\rangle$
$\langle 1\rangle 4.$ LET: $y = f \circ \langle t_0, t_0\rangle$
$\langle 1\rangle 5.$ $\alpha y = y$
PROOF:

$$\begin{aligned} \alpha y &= \alpha f \langle t_0, t_0\rangle \\ &= \alpha f \langle 1_T, 1_T\rangle t_0 \\ &= f\langle 1_T, t!\rangle t_0 \\ &= f\langle t_0, t_0\rangle \\ &= y \end{aligned}$$

□

5 Sums

Definition 27 (Sum)**.** Let A and B be objects. A *sum* of A and B consists of an object $A + B$ and morphisms $\kappa_1 : A \to A + B$, $\kappa_2 : B \to A + B$, the *injections*, such that, for any object X and morphisms $f : A \to X$ and $g : B \to X$, there exists exactly one map $[f, g] : A + B \to X$ such that $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$.

6 Distributive Categories

Definition 28 (Distributive Category)**.** Let $\mathcal{C}$ be a category with binary products and binary coproducts. Then $\mathcal{C}$ is *distributive* iff, for any objects A, B, C, the map

$$[1_A \times \kappa_1, 1_A \times \kappa_2] : (A \times B) + (A \times C) \to A \times (B + C)$$

is an isomorphism.

7 Equalizers

Definition 29 (Equalizer)**.** Let $f, g : A \to B$. An *equalizer* of f and g consists of an object E and a morphism $e : E \to A$ such that $f \circ e = g \circ e$ and, for any object X and morphism $x : X \to A$ such that $fx = gx$, there exists a unique $\overline{x} : X \to E$ such that $x = e\overline{x}$.

8 Map Objects

Definition 30 (Map Objects)**.** Let $\mathcal{C}$ be a category with binary products.

Let A and B be objects. A *map object* from A to B consists of an object B^A and a morphism $e : B^A \times A \to B$ such that, for any object X and morphism $f : X \times A \to B$, there exists a unique morphism $\lambda f : X \to B^A$ such that $e \circ (\lambda f \times 1_A) = f$.

Proposition 31. *Let $\mathcal{C}$ be a category with binary products and coproducts. Let T be an object such that, for every object A, the map object A^T exists. Then binary products with T distribute over binary coproducts.*

PROOF:
$\langle 1\rangle 1.$ LET: $A, B \in \mathcal{C}$
$\langle 1\rangle 2.$ LET: $c = [\kappa_1 \times 1_T, \kappa_2 \times 1_T] : (A \times T) + (B \times T) \to (A + B) \times T$
$\langle 1\rangle 3.$ $\lambda\kappa_1 : A \to ((A \times T) + (B \times T))^T$, $\lambda\kappa_2 : B \to ((A \times T) + (B \times T))^T$, and they are unique such that $e \circ (\lambda\kappa_1 \times 1_T) = \kappa_1$ and $e \circ (\lambda\kappa_2 \times 1_T) = \kappa_2$
$\langle 1\rangle 4.$ $[\lambda\kappa_1, \lambda\kappa_2] : A + B \to ((A \times T) + (B \times T))^T$ is unique such that $e \circ ([\lambda\kappa_1, \lambda\kappa_2]\kappa_1 \times 1_T) = \kappa_1$ and $e \circ ([\lambda\kappa_1, \lambda\kappa_2]\kappa_2 \times 1_T) = \kappa_2$
$\langle 1\rangle 5.$ LET: $c^{-1} = e \circ ([\lambda\kappa_1, \lambda\kappa_2] \times 1_T) : (A + B) \times T \to (A \times T) + (B \times T)$
$\langle 1\rangle 6.$ $cc^{-1} = 1_{(A+B)\times T}$
$\langle 2\rangle 1.$ $\pi_1 cc^{-1} = \pi_1 : (A + B) \times T \to A + B$
$\langle 3\rangle 1.$ $\lambda(\pi_1 cc^{-1}) = \lambda\pi_1 : A + B \to (A + B)^T$
$\langle 4\rangle 1.$ $\lambda(\pi_1 cc^{-1})\kappa_1 = (\lambda\pi_1)\kappa_1 : A \to (A + B)^T$
$\langle 5\rangle 1.$ $e \circ (\lambda(\pi_1 cc^{-1})\kappa_1 \times 1_T) = e \circ ((\lambda\pi_1)\kappa_1 \times 1_T) : A \times T \to A + B$
PROOF:

$$
\begin{aligned}
e \circ (\lambda(\pi_1 cc^{-1})\kappa_1 \times 1_T) &= (e \circ (\lambda(\pi_1 cc^{-1}) \times 1_T)(\kappa_1 \times 1_T) \\
&= \pi_1 cc^{-1}(\kappa_1 \times 1_T) \\
&= \pi_1 c\kappa_1 && (\langle 1\rangle 4) \\
&= \pi_1(\kappa_1 \times 1_T) \\
&= \kappa_1\pi_1 \\
&= \pi_1(\kappa_1 \times 1_T) \\
&= e(\lambda\pi_1 \times 1_T)(\kappa_1 \times 1_T) \\
&= e((\lambda\pi_1)\kappa_1 \times 1_T))
\end{aligned}
$$

$\langle 4\rangle 2.$ $\lambda(\pi_1 cc^{-1})\kappa_2 = (\lambda\pi_1)\kappa_2 : B \to (A + B)^T$
PROOF: Similar.
$\langle 2\rangle 2.$ $\pi_2 cc^{-1} = \pi_2$
PROOF: Similar
$\langle 1\rangle 7.$ $c^{-1}c = 1_{(A\times T)+(B\times T)}$
PROOF: $c^{-1}c = [\kappa_1, \kappa_2] = 1$
□

Definition 32 (Cartesian closed category)**.** A *Cartesian closed category* is a category with finite products and map objects.

Theorem 33 (Cantor's Diagonal Argument)**.** *Let $\mathcal{C}$ be a Cartesian closed category. Let $T, Y \in \mathcal{C}$ and $f : T \to Y^T$. Suppose that, for every map $g : T \to Y$, there exists $x : 1 \to T$ such that $\lambda g = fx$. Then every endomorphism on Y has a fixed point.*

PROOF:
$\langle 1\rangle 1.$ LET: $\alpha : Y \to Y$
$\langle 1\rangle 2.$ LET: $g = \alpha \circ e \circ \langle f, 1_T\rangle : T \to Y$

$\langle 1 \rangle 3$. PICK $x : 1 \to T$ such that $\lambda(g\pi_2) = fx$
$\langle 1 \rangle 4$. LET: $y = e \circ \langle fx, x \rangle : 1 \to Y$
$\langle 1 \rangle 5$. $\alpha y = y$
PROOF:

$$\begin{aligned} \alpha y &= \alpha e \langle f, 1_T \rangle x \\ &= gx \\ &= g\pi_2 \langle 1_1, x \rangle \\ &= e(\lambda(g\pi_1) \times 1) \langle 1_1, x \rangle \\ &= e(fx \times 1) \langle 1_1, x \rangle \\ &= e \langle fx, x \rangle \\ &= y \end{aligned}$$

□

9 Pullbacks

Definition 34 (Pullback)**.** The diagram below is a *pullback* iff $fp = gq$ and, for any object X and morphisms $x : X \to B$ and $y : X \to C$ such that $fx = gy$, there exists a unique morphism $m : X \to A$ such that $pm = x$ and $qm = y$.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

10 Subobject Classifier

Definition 35 (Subobject Classifier)**.** Let $\mathcal{C}$ be a category with a terminal object 1. A *subobject classifier* consists of an object Ω and morphism $\top : 1 \to \Omega$ such that, for every monomorphism $m : A \rightarrowtail B$, there exists a unique morphism $\chi_m : B \to \Omega$, the *characterisetic morphism* of m, such that the following diagram is a pullback

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, "m"'] & 1 \arrow[d, "\top"] \\ B \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

11 Toposes

Definition 36 (Topos)**.** A *topos* is a Cartesian closed category $\mathcal{C}$ with finite coproducts and a subobject classifier such that, for every object X, the slice category $\mathcal{C}/X$ has finite products.