Definition 0.1 (Open Set)**.** Let F be an ordered field. Let $A \subseteq F$. Then A is *open* iff every element of A belongs to an open interval that is included in A.

Proposition 0.2. *The union of a set of open sets is open.*

Proposition 0.3. *The intersection of two open sets is open.*

Definition 0.4 (Accumulation Point)**.** Let F be an ordered field. Let $A \subseteq F$. Let $l \in F$. Then l is an *accumulation point* of A if and only if every open interval containing l intersects $A - \{l\}$.

Proposition 0.5. *If l is an accumulation point of A then every open interval containing l contains infinitely many points of A.*

Corollary 0.5.1. *A finite set has no accumulation points.*

Definition 0.6 (Closed Set)**.** Let F be an ordered field and $A \subseteq F$. Then A is *closed* iff it contains all its accumulation points.

Proposition 0.7. *A set A is open iff $F - A$ is closed.*

Proposition 0.8. *A set A is closed iff $F - A$ is open.*

Corollary 0.8.1. *The intersection of a nonempty set of closed sets is closed.*

Corollary 0.8.2. *The union of two closed sets is closed.*

Definition 0.9 (Closure)**.** Let F be an ordered field and $A \subseteq F$. Then the *closure* of A is

$$\overline{A} = A \cup \{l \in F : l \text{ is an accumulation point of } A\} \ .$$

Proposition 0.10. *A set A is closed iff $A = \overline{A}$.*

Proposition 0.11. *For any set A, we have $\overline{A} = \{x \in F : \text{every open interval containing } x \text{ intersects } A\}$.*

Proposition 0.12. *For any set A, we have $\overline{A}$ is closed.*

Proposition 0.13. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

Proposition 0.14.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proposition 0.15. *For any set A, if s is the supremum of A then $s \in \overline{A}$.*

Definition 0.16 (Open Covering)**.** Let F be an ordered field. Let $A \subseteq F$ and $\mathcal{B}$ be a set of open sets. Then $\mathcal{B}$ is an *open covering* of A, or *covers* A, iff $A \subseteq \bigcup \mathcal{B}$.

Definition 0.17 (Compact)**.** Let F be an ordered field and $A \subseteq F$. Then A is *compact* iff every open covering of A has a finite subcovering.

Theorem 0.18. *Let F be an ordered field. Then the following are equivalent.*

1. *F is isomorphic to $\mathbb{R}$*

2. *Every closed interval in F is compact.*

3. *Every bounded infinite set in F has an accumulation point.*

PROOF:

$\langle 1\rangle 1.$ $1 \Rightarrow 2$

$\langle 2\rangle 1.$ LET: $[c_0, d_0]$ be a closed interval in $\mathbb{R}$.

$\langle 2\rangle 2.$ LET: $\mathcal{B}$ be an open covering of $[c_0, d_0]$.

$\langle 2\rangle 3.$ ASSUME: for a contradiction no finite subset of $\mathcal{B}$ covers $[c_0, d_0]$.

$\langle 2\rangle 4.$ LET: $([c_n, d_n])$ be the nested sequence of closed intervals defined by: $[c_{n+1}, d_{n+1}] = [c_n, (c_n + d_n)/2]$ if this interval is not covered by any finite subset of $\mathcal{B}$, otherwise $[(c_n + d_n)/2, d_n]$.

$\langle 2\rangle 5.$ For all n, $[c_n, d_n]$ is not covered by any finite subset of $\mathcal{B}$.

$\langle 2\rangle 6.$ $\forall n. d_n - c_n = (d_0 - c_0)/2^n$

$\langle 2\rangle 7.$ $d_n - c_n \to 0$ as $n \to \infty$

$\langle 2\rangle 8.$ LET: $\bigcap_n [c_n, d_n] = \{z\}$

$\langle 2\rangle 9.$ PICK $B \in \mathcal{B}$ such that $z \in B$

$\langle 2\rangle 10.$ PICK $\epsilon > 0$ such that $(z - \epsilon, z + \epsilon) \subseteq B$

$\langle 2\rangle 11.$ PICK N such that $d_N - c_N < \epsilon$

$\langle 2\rangle 12.$ $\{B\}$ covers $[c_N, d_N]$

$\langle 2\rangle 13.$ Q.E.D.

PROOF: This contradicts $\langle 2\rangle 5$.

$\langle 1\rangle 2.$ $2 \Rightarrow 3$

$\langle 2\rangle 1.$ ASSUME: 2

$\langle 2\rangle 2.$ LET: $A \subseteq F$ be bounded and infinite.

$\langle 2\rangle 3.$ PICK $c, d \in F$ such that $A \subseteq [c, d]$

$\langle 2\rangle 4.$ ASSUME: for a contradiction A has no accumulation point.

$\langle 2\rangle 5.$ LET: $\mathcal{B}$ be the set of open intervals I such that I intersects $[c, d]$ and $I \cap A$ has at most one element.

$\langle 2\rangle 6.$ $\mathcal{B}$ is an open covering of $[c, d]$.

PROOF: From $\langle 2\rangle 4$.

$\langle 2\rangle 7.$ PICK a finite subcovering $\{B_1, \ldots, B_n\}$ of $[c, d]$.

$\langle 2\rangle 8.$ A is finite.

$\langle 2\rangle 9.$ Q.E.D.

PROOF: This contradicts $\langle 2\rangle 2$.

$\langle 1\rangle 3.$ $3 \Rightarrow 1$

$\langle 2\rangle 1.$ ASSUME: 3

$\langle 2\rangle 2.$ F is Archimedean.

$\langle 3\rangle 1.$ ASSUME: for a contradiction $\mathbb{Z}$ is bounded in F.

$\langle 3\rangle 2.$ PICK an accumulation point z of $\mathbb{Z}$.

$\langle 3\rangle 3.$ PICK $n \in (z - 1/2, z + 1/2) \cap (\mathbb{Z} - \{z\})$

$\langle 3\rangle 4.$ LET: $c = |n - z|$

$\langle 3\rangle 5.$ PICK $k \in (z - c, z + c) \cap (\mathbb{Z} - \{z\})$

$\langle 3\rangle 6.$ $k \neq n$

$\langle 3\rangle 7.$ $(z - c, z + c) \subseteq (z - 1/2, z + 1/2)$

$\langle 3\rangle 8.$ $k \in (z - 1/2, z + 1/2)$

$\langle 3\rangle 9.$ $|k - n| < 1$

$\langle 3 \rangle 10$. Q.E.D.

PROOF: This contradicts the fact that k and n are distinct integers.

$\langle 2 \rangle 3$. F is Cauchy complete.

$\langle 3 \rangle 1$. LET: (x_n) be a Cauchy sequence in F.

$\langle 3 \rangle 2$. (x_n) is bounded.

$\langle 3 \rangle 3$. LET: $A = \{x_n : n \in \mathbb{N}\}$

$\langle 3 \rangle 4$. CASE: A is finite.

$\langle 4 \rangle 1$. There is a subsequence of (x_n) that is constant.

$\langle 4 \rangle 2$. (x_n) converges.

PROOF: Proposition **??**.

$\langle 3 \rangle 5$. CASE: A is infinite.

$\langle 4 \rangle 1$. PICK an accumulation point z of A.

PROVE: $x_n \to z$ as $n \to \infty$

$\langle 4 \rangle 2$. LET: $\epsilon > 0$

$\langle 4 \rangle 3$. PICK N such that $\forall m, n \geq N . |x_m - x_n| < \epsilon/2$

$\langle 4 \rangle 4$. LET: c be the least positive element among $\epsilon/2$, $|z - x_0|$, $|z - x_1|$, $\ldots$, $|z - x_{N-1}|$

$\langle 4 \rangle 5$. PICK $w \in (z - c, z + c) \cap (A - \{z\})$

$\langle 4 \rangle 6$. PICK n such that $w = a_n$

$\langle 4 \rangle 7$. $n \geq N$

$\langle 4 \rangle 8$. $\forall m \geq N . |x_m - z| < \epsilon$

PROOF:

$$\begin{aligned} |x_m - z| &\leq |x_m - w| + |w - z| \\ &< \epsilon/2 + c \\ &\leq \epsilon \end{aligned}$$

□

Proposition 0.19 (Choice)**.** *Let F be an ordered field. Then $F \cong \mathbb{R}$ if and only if every bounded sequence in F has a convergent subsequence.*

PROOF:

$\langle 1 \rangle 1$. Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

$\langle 2 \rangle 1$. LET: (a_n) be a bounded sequence in $\mathbb{R}$.

$\langle 2 \rangle 2$. LET: $A = \{a_n : n \in \mathbb{N}\}$

$\langle 2 \rangle 3$. CASE: A is finite.

PROOF: In this case, (a_n) has a subsequence that is constant, hence convergent.

$\langle 2 \rangle 4$. CASE: A is infinite.

$\langle 3 \rangle 1$. PICK an accumulation point l for A.

$\langle 3 \rangle 2$. For each n, PICK $r_n > r_{n-1}$ such that $a_{r_n} \in (l - 1/n, l + 1/n)$

PROOF: This is possible because $(l - 1/n, l + 1/n) \cap A$ is infinite.

$\langle 3 \rangle 3$. $a_{r_n} \to l$ as $n \to \infty$

$\langle 1 \rangle 2$. For any ordered field F, if every bounded sequence in F has a convergent subsequence, then $F \cong \mathbb{R}$.

$\langle 2 \rangle 1$. ASSUME: Every bounded sequence in F has a convergent subsequence.

PROVE: Every bounded infinite set in F has an accumulation point.

$\langle 2 \rangle 2.$ LET: A be a bounded infinite set in F.
$\langle 2 \rangle 3.$ PICK an infinite sequence (a_n) in A, all distinct.
$\langle 2 \rangle 4.$ PICK a convergent subsequence (a_{n_r}) with limit l.
PROVE: l is an accumulation point for A
$\langle 2 \rangle 5.$ LET: $\epsilon > 0$
PROVE: $(l - \epsilon, l + \epsilon)$ intersects A in a point other than l
$\langle 2 \rangle 6.$ PICK R such that $\forall r \geq R. a_{n_r} \in (l - \epsilon, l + \epsilon)$
$\langle 2 \rangle 7.$ Either a_{n_R} or $a_{n_{R+1}}$ is in $(l - \epsilon, l + \epsilon) \cap (A - \{l\})$
□

Proposition 0.20. *Let (a_n) be a bounded sequence in $\mathbb{R}$. Assume that any two convergent subsequences of (a_n) have the same limit l. Then $a_n \to l$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: for a contradiction a_n does not converge to l.
$\langle 1 \rangle 2.$ PICK $\epsilon > 0$ such that, for all N, there exists $n \geq N$ such that $|a_n - l| > \epsilon$
$\langle 1 \rangle 3.$ PICK an increasing sequence (n_r) such that $|a_{n_r} - l| > \epsilon$
$\langle 1 \rangle 4.$ PICK a convergent subsequence s of (a_{n_r})
$\langle 1 \rangle 5.$ s converges to l
$\langle 1 \rangle 6.$ Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 3$.
□

Proposition 0.21. *Let F be an ordered field. Then $F \cong \mathbb{R}$ if and only if the compact subsets of F are exactly the closed bounded subsets of F.*

PROOF:
$\langle 1 \rangle 1.$ Every compact subset of $\mathbb{R}$ is closed.
$\langle 2 \rangle 1.$ LET: $A \subseteq \mathbb{R}$ be compact.
PROVE: $F - A$ is open.
$\langle 2 \rangle 2.$ LET: $z \in F - A$
$\langle 2 \rangle 3.$ For $n \in \mathbb{Z}^+$,
LET: $I_n = \{w \in F : |w - z| > 1/n\}$
$\langle 2 \rangle 4.$ LET: $\mathcal{B} = \{I_n : n \in \mathbb{Z}^+\}$
$\langle 2 \rangle 5.$ $\mathcal{B}$ is an open covering of A
$\langle 2 \rangle 6.$ PICK a finite subcovering $\{I_{n_1}, \ldots, I_{n_k}\}$
$\langle 2 \rangle 7.$ LET: $m = \max(n_1, \ldots, n_k)$
$\langle 2 \rangle 8.$ $\forall w \in A. |w - z| > 1/m$
$\langle 2 \rangle 9.$ $(z - 1/m, z + 1/m) \subseteq F - A$
$\langle 1 \rangle 2.$ Every compact subset of $\mathbb{R}$ is bounded.
$\langle 2 \rangle 1.$ LET: $A \subseteq \mathbb{R}$ be compact.
$\langle 2 \rangle 2.$ $\{(-n, n) : n \in \mathbb{Z}^+\}$ is an open covering of A
$\langle 2 \rangle 3.$ PICK a finite subcovering $\{(-n_1, n_1), \ldots, (-n_k, n_k)\}$
$\langle 2 \rangle 4.$ LET: $m = \max(n_1, \ldots, n_k)$
$\langle 2 \rangle 5.$ $A \subseteq (-m, m)$
$\langle 1 \rangle 3.$ Every closed bounded subset of $\mathbb{R}$ is compact.
$\langle 2 \rangle 1.$ LET: $A \subseteq \mathbb{R}$ be closed and bounded.

⟨2⟩2. LET: $\mathcal{B}$ be an open covering of A.
⟨2⟩3. PICK $c, d \in \mathbb{R}$ such that $A \subseteq [c, d]$
⟨2⟩4. $\mathcal{B} \cup \{F - A\}$ is an open covering of $[c, d]$
⟨2⟩5. PICK a finite subcovering $\mathcal{B}_1 \cup \{F - A\}$
⟨2⟩6. $\mathcal{B}_1$ is a finite subset of $\mathcal{B}$ that covers A.
⟨1⟩4. If the compact subsets of F are exactly the closed bounded subsets then $F \cong \mathbb{R}$.
PROOF: By Theorem 0.18 since the closed intervals in F are compact.
□

Proposition 0.22. *In any ordered field, any nested sequence of nonempty compact sets has nonempty intersection.*

PROOF:
⟨1⟩1. LET: F be an ordered field.
⟨1⟩2. LET: (B_n) be a nested sequence of nonempty compact sets.
⟨1⟩3. ASSUME: $\bigcap_n B_n = \emptyset$
⟨1⟩4. $\{F - B_n : n \geq 2\}$ is an open covering of B_1.
⟨1⟩5. PICK a finite subcovering $\{F - B_{n_1}, \dots, F - B_{n_k}\}$
⟨1⟩6. LET: $m = \max(n_1, \dots, n_k)$
⟨1⟩7. $B_{m+1} = \emptyset$
⟨1⟩8. Q.E.D.
PROOF: This contradicts ⟨1⟩2.
□

Definition 0.23 (Connected)**.** Let F be an ordered field and $A \subseteq F$. Then A is *connected* iff, whenever $A = B \cup C$ with B and C nonempty and disjoint, then either B contains an accumulation point of C or C contains an accumulation point of B.

Proposition 0.24. *Let F be an ordered field. Then $F \cong \mathbb{R}$ if and only if every closed interval in F is connected.*

PROOF:
⟨1⟩1. Every closed interval in $\mathbb{R}$ is connected.
⟨2⟩1. LET: $[u, v] = B \cup C$ where B and C are nonempty and disjoint.
⟨2⟩2. ASSUME: for a contradiction B contains no accumulation point of C and C contains no accumulation point of B.
⟨2⟩3. ASSUME: w.l.o.g. $u \in B$
⟨2⟩4. u is not an accumulation point of C.
⟨2⟩5. PICK an open interval (w, z) containing u that is disjoint from C such that $z \leq v$.
⟨2⟩6. $[u, z) \subseteq B$
⟨2⟩7. LET: $W = \{y \in [u, v] : [u, y) \subseteq B\}$
⟨2⟩8. $W \neq \emptyset$
⟨2⟩9. W is bounded above by v.
⟨2⟩10. LET: $d = \sup W$
⟨2⟩11. $d \in [u, v]$

⟨2⟩12. $[u, d) \subseteq B$
⟨2⟩13. $d \notin B$
⟨2⟩14. $d \in C$
⟨2⟩15. d is not an accumulation point of B
⟨2⟩16. PICK an open interval (w_2, v_2) containing d and disjoint from B
⟨2⟩17. (w_2, v_2) intersects $[u, d)$
⟨2⟩18. Q.E.D.
⟨1⟩2. If every closed interval in F is connected then $F \cong \mathbb{R}$.
⟨2⟩1. ASSUME: Every closed interval in F is connected.
⟨2⟩2. LET: (A_1, A_2) be a cut in F.
⟨2⟩3. PICK $u \in A_1$ and $v \in A_2$.
⟨2⟩4. ASSUME: w.l.o.g. u is not the maximum of A_1 and v is not the minimum of A_2.
⟨2⟩5. LET: $B = A_1 \cap [u, v]$
⟨2⟩6. LET: $C = A_2 \cap [u, v]$
⟨2⟩7. $[u, v] = B \cup C$
⟨2⟩8. $B \neq \emptyset$
⟨2⟩9. $C \neq \emptyset$
⟨2⟩10. $B \cap C = \emptyset$
⟨2⟩11. ASSUME: w.l.o.g. B contains an accumulation point of C.
⟨2⟩12. PICK $z \in B$ that is an accumulation point of C.
⟨2⟩13. z is the maximum of A_1

□

Corollary 0.24.1. *Let F be an ordered field. Then the following are equivalent:*

1. *$F \cong \mathbb{R}$*
2. *Every interval in F is connected.*
3. *The connected subsets of F are exactly the intervals.*

Proposition 0.25. *Let F be an ordered field. Let $\mathcal{A}$ be a set of connected subsets of F such that any two elements of $\mathcal{A}$ intersect. Then $\bigcup \mathcal{A}$ is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $\bigcup \mathcal{A} = B \cup C$ where B and C are nonempty, disjoint, and neither contains an accumulation point of the other.
⟨1⟩2. PICK $b \in B$ and $c \in C$
⟨1⟩3. PICK $A_1, A_2 \in \mathcal{A}$ such that $b \in A_1$ and $c \in A_2$.
⟨1⟩4. PICK $w \in A_1 \cap A_2$
⟨1⟩5. ASSUME: w.l.o.g. $w \in B$
⟨1⟩6. LET: $B_1 = B \cap A_2$
⟨1⟩7. LET: $C_1 = C \cap A_2$
⟨1⟩8. $A_2 = B_1 \cup C_1$
⟨1⟩9. $B_1 \neq \emptyset$
PROOF: Since $w \in B_1$.
⟨1⟩10. $C_1 \neq \emptyset$

PROOF: Since $c \in C_1$.
$\langle 1 \rangle 11.$ $B_1 \cap C_1 = \emptyset$
$\langle 1 \rangle 12.$ Neither of B_1 and C_1 contains an accumulation point of the other.
$\langle 1 \rangle 13.$ Q.E.D.
PROOF: This contradicts the fact that A_2 is connected.
$\square$

Proposition 0.26. *The closure of a connected set is connected.*

PROOF:
$\langle 1 \rangle 1.$ LET: F be an ordered field.
$\langle 1 \rangle 2.$ LET: $A \subseteq F$ be connected.
$\langle 1 \rangle 3.$ LET: $\overline{A} = B \cup C$ where B and C are nonempty and disjoint.
$\langle 1 \rangle 4.$ LET: $B_1 = A \cap B$
$\langle 1 \rangle 5.$ LET: $C_1 = A \cap C$
$\langle 1 \rangle 6.$ $A = B_1 \cup C_1$ and B_1 and C_1 are disjoint.
$\langle 1 \rangle 7.$ CASE: B_1 and C_1 are both nonempty.
$\langle 2 \rangle 1.$ ASSUME: w.l.o.g. B_1 contains an accumulation point of C_1
$\langle 2 \rangle 2.$ PICK $z \in B_1$ that is an accumulation point of C_1
$\langle 2 \rangle 3.$ $z \in B$ and z is an accumulation point of C
$\langle 1 \rangle 8.$ CASE: $B_1 = \emptyset$
$\langle 2 \rangle 1.$ PICK $z \in B$
$\langle 2 \rangle 2.$ $z \in \overline{A} - A$
$\langle 2 \rangle 3.$ z is an accumulation point of A.
$\langle 2 \rangle 4.$ z is an accumulation point of C.
$\langle 1 \rangle 9.$ CASE: $C_1 = \emptyset$
PROOF: Similar.
$\square$

Definition 0.27 (Connected Component)**.** A *connected component* of an ordered field is a maximal connected subset.

Proposition 0.28. *Two distinct connected components of an ordered field are disjoint.*

PROOF:
$\langle 1 \rangle 1.$ LET: F be an ordered field.
$\langle 1 \rangle 2.$ LET: A and B be connected components of F.
$\langle 1 \rangle 3.$ ASSUME: $A \cap B \neq \emptyset$
$\langle 1 \rangle 4.$ $A \cup B$ is connected.
PROOF: Proposition 0.25.
$\langle 1 \rangle 5.$ $A = A \cup B = B$
$\square$

Proposition 0.29. *Connected components are closed.*

PROOF:
$\langle 1 \rangle 1.$ LET: F be an ordered field.

$\langle 1 \rangle 2.$ LET: $C \subseteq F$ be a connected component.
$\langle 1 \rangle 3.$ $\overline{C}$ is connected.
$\langle 1 \rangle 4.$ $C = \overline{C}$
$\langle 1 \rangle 5.$ C is closed.
$\square$