

Solutions Manual for Enderton *Elements of Set  
Theory*

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# Chapter 1

## Chapter 1 — Introduction

### 1.1 Baby Set Theory

#### Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$  — true
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$  — true
- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$  — false
- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$  — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$  — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$  — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$  — true
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$  — false
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$  — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$  — false

**Exercise 2** We have  $\emptyset \neq \{\emptyset\}$  because  $\{\emptyset\}$  has an element (namely  $\emptyset$ ) while  $\emptyset$  has no elements.

We have  $\emptyset \neq \{\{\emptyset\}\}$  because  $\{\{\emptyset\}\}$  has an element (namely  $\{\emptyset\}$ ) while  $\emptyset$  has no elements.

We have  $\{\emptyset\} \neq \{\{\emptyset\}\}$  because  $\emptyset \in \{\emptyset\}$  but  $\emptyset \notin \{\{\emptyset\}\}$ . This last fact is true because  $\emptyset \neq \{\emptyset\}$  as we proved in the first paragraph.

**Exercise 3** Assume  $B \subseteq C$ . Let  $A \in \mathcal{P}B$ ; we must show that  $A \in \mathcal{P}C$ .

We have  $A \subseteq B$  (since  $A \in \mathcal{P}B$ ) and  $B \subseteq C$ . From this it follows that  $A \subseteq C$  (every element of  $A$  is an element of  $B$ ; every element of  $B$  is an element of  $C$ ; therefore every element of  $A$  is an element of  $C$ ). Hence  $A \in \mathcal{P}C$  as required.

**Exercise 4** Since  $x \in B$ , we have  $\{x\} \subseteq B$  and so  $\{x\} \in \mathcal{P}B$ .

Since  $x \in B$  and  $y \in B$ , we have  $\{x, y\} \subseteq B$  and so  $\{x, y\} \in \mathcal{P}B$ .

From these two facts, it follows that  $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$  and so  $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$ .

## 1.2 Sets — An Informal View

**Exercise 5** We have

$$\begin{aligned} V_0 &= A \\ V_1 &= V_0 \cup \mathcal{P}V_0 \\ &= A \cup \mathcal{P}A \\ V_2 &= V_1 \cup \mathcal{P}V_1 \\ &= \{\emptyset, \{\emptyset\}\} \\ V_3 &= \mathcal{P}V_2 \\ &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \end{aligned}$$

We have  $\emptyset \subseteq V_0$  and so  $\emptyset \in V_1$ . Therefore  $\{\emptyset\} \subseteq V_1$  and so  $\{\emptyset\} \in V_2$ . Hence  $\{\{\emptyset\}\} \subseteq V_2$ .

We also have  $\{\{\emptyset\}\} \not\subseteq V_0$  because  $\{\emptyset\}$  is not an atom, and  $\{\{\emptyset\}\} \not\subseteq V_1$  since  $\{\emptyset\} \notin V_1$  because  $\emptyset$  is not an atom.

Thus the rank of  $\{\{\emptyset\}\}$  is 2.

Likewise we have  $\emptyset$  and  $\{\emptyset\}$  are both subsets of  $V_1$ , hence

$$\emptyset \in V_2, \quad \{\emptyset\} \in V_2$$

Thus  $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$  are all subsets of  $V_2$ , hence elements of  $V_3$ . Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now,  $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$  is not a subset of  $V_0$  (because  $\emptyset$  is not an atom.) It is not a subset of  $V_1$  ( $\{\emptyset\} \notin V_1$  because  $\emptyset$  is not an atom.) It is not a subset of  $V_2$  (we have  $\{\emptyset, \{\emptyset\}\} \notin V_2$  since  $\{\emptyset\} \notin V_1$ ).

Therefore the rank of  $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$  is 3.

**Exercise 6**

$$\begin{aligned}
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}V_0 && (\text{since } V_0 = A) \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_1 && (\text{since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3}) \\
V_3 &= V_2 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_2 && (\text{since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3}) \\
V_4 &= V_3 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_3 && (\text{since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3})
\end{aligned}$$

**Exercise 7** In Exercise 5 we calculated  $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$   
Hence

$$\begin{aligned}
V_4 &= \mathcal{P}V_3 \\
&= \{\emptyset, \\
&\quad \{\emptyset\}, \\
&\quad \{\{\emptyset\}\}, \\
&\quad \{\{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
&\quad \}
\end{aligned}$$

## Chapter 2

# Chapter 2 — Axioms and Operations

### 2.1 Arbitrary Unions and Intersections

**Exercise 1**  $A \cap B \cap C$  is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

**Exercise 2** Take  $A = \emptyset$  and  $B = \{\emptyset\}$ . Then  $\bigcup A = \bigcup B = \emptyset$  but  $A \neq B$ . (There are many other possible answers.)

**Exercise 3** Let  $b \in A$ . We must show that  $b \subseteq \bigcup A$ .

Let  $x$  be any element of  $b$ . We must show that  $x \in \bigcup A$ . We know that  $x \in b$  and  $b \in A$ , and so  $x \in \bigcup A$  by the definition of  $\bigcup A$ .

**Exercise 4** Suppose  $A \subseteq B$ . Let  $x \in \bigcup A$ . We must show that  $x \in \bigcup B$ .

Pick an element  $a \in A$  such that  $x \in a$ . Then  $a \in B$  because  $A \subseteq B$ . Since we know  $x \in a$  and  $a \in B$ , we know that  $x \in \bigcup B$ .

**Exercise 5** Assume that every member of  $\mathcal{A}$  is a subset of  $B$ . Let  $x \in \bigcup \mathcal{A}$ . We must show that  $x \in B$ .

Pick  $A \in \mathcal{A}$  such that  $x \in A$ . By our assumption, we have  $A \subseteq B$ . Since  $x \in A$  and  $A \subseteq B$ , we have  $x \in B$  as required.

**Exercise 6**

(a) We will show that  $\bigcup \mathcal{P}A \subseteq A$  and  $A \subseteq \bigcup \mathcal{P}A$ .

To show  $\bigcup \mathcal{P}A \subseteq A$ : This follows from Exercise 5, since every member of  $\mathcal{P}A$  is a subset of  $A$ .

To show  $A \subseteq \bigcup \mathcal{P}A$ : Let  $a \in A$ . Then we have  $a \in \{a\}$  and  $\{a\} \in \mathcal{P}A$  so  $a \in \bigcup \mathcal{P}A$ .

(b) To show  $A \subseteq \mathcal{P} \bigcup A$ : This holds because every element of  $A$  is a subset of  $\bigcup A$ , as we proved in Exercise 3.

Equality holds if and only if  $A = \mathcal{P}X$  for some set  $X$ .

Proof: If  $A = \mathcal{P} \bigcup A$  then of course  $A = \mathcal{P}X$  for some  $X$ .

Conversely, if  $A = \mathcal{P}X$ , then we have

$$\begin{aligned} \mathcal{P} \bigcup A &= \mathcal{P} \bigcup \mathcal{P}X \\ &= \mathcal{P}X && \text{(by part (a))} \\ &= A \end{aligned}$$

### Exercise 7

(a) For any set  $X$ ,

$$\begin{aligned} X &\in \mathcal{P}A \cap \mathcal{P}B \\ \Leftrightarrow X &\subseteq A \text{ and } X \subseteq B \\ \Leftrightarrow \text{Every member of } X &\text{ is a member of } A \text{ and a member of } B \\ \Leftrightarrow X &\subseteq A \cap B \\ \Leftrightarrow X &\in \mathcal{P}(A \cap B) \end{aligned}$$

(b) Let  $X \in \mathcal{P}A \cup \mathcal{P}B$ . Then either  $X \in \mathcal{P}A$  or  $X \in \mathcal{P}B$  (or both). If  $X \in \mathcal{P}A$ , then we have  $X \subseteq A$  and so  $X \subseteq A \cup B$  (because  $A \subseteq A \cup B$ ). Similarly if  $X \in \mathcal{P}B$  then we have  $X \subseteq A \cup B$ . So in either case  $X \subseteq A \cup B$ , hence  $X \in \mathcal{P}(A \cup B)$ .

Equality holds if and only if either  $A \subseteq B$  or  $B \subseteq A$ .

Proof: Suppose  $A \subseteq B$ . Then  $\mathcal{P}A \subseteq \mathcal{P}B$  (Chapter 1 Exercise 3) and so  $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$ . Also  $A \cup B = B$  so  $\mathcal{P}(A \cup B) = \mathcal{P}B$ . Thus  $\mathcal{P}A \cup \mathcal{P}B$  and  $\mathcal{P}(A \cup B)$  are equal.

Similarly if  $B \subseteq A$  then  $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$ .

Conversely, suppose  $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$ . We have  $A \cup B \in \mathcal{P}(A \cup B)$ , so  $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$ . If  $A \cup B \in \mathcal{P}A$ , then we have  $B \subseteq A \cup B \subseteq A$ . And if  $A \cup B \in \mathcal{P}B$ , then we have  $A \subseteq A \cup B \subseteq B$ .

**Exercise 8** If  $A$  is a set such that every singleton belongs to  $A$ , then every set belongs to  $\bigcup A$ , contradicting Theorem 2A.

**Exercise 9** Let  $a = \{\emptyset\}$  and  $B = \{\{\emptyset\}\}$ . Then  $a \in B$  but  $\mathcal{P}a$  is not a subset of  $B$  because  $\emptyset \in \mathcal{P}a$  and  $\emptyset \notin B$ .



**Exercise 10** We must show that  $\mathcal{P}a \subseteq \mathcal{P} \bigcup B$ . So let  $X \in \mathcal{P}a$ . Then  $X \subseteq a$ ; we must show that  $X \subseteq \bigcup B$ .

Let  $x \in X$ ; we must show that  $x \in \bigcup B$ . We have  $x \in a$  (because  $x \in X$  and  $X \subseteq a$ ) and  $a \in B$ , hence  $x \in \bigcup B$  as required.

## 2.2 Algebra of Sets

**Exercise 11** For any  $x$  we have

$$\begin{aligned} x \in (A \cap B) \cup (A - B) &\Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B) \\ &\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B) \\ &\Leftrightarrow x \in A \end{aligned}$$

Hence  $A = (A \cap B) \cup (A - B)$ .

For any  $x$  we have

$$\begin{aligned} x \in A \cup (B - A) &\Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A) \\ &\Leftrightarrow x \in A \text{ or } x \in B \\ &\Leftrightarrow x \in A \cup B \end{aligned}$$

Hence  $A \cup (B - A) = A \cup B$ .

**Exercise 12** For any  $x$ ,

$$\begin{aligned} x \in C - (A \cap B) &\Leftrightarrow x \in C \& \neg(x \in A \& x \in B) \\ &\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B) \\ &\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B) \\ &\Leftrightarrow x \in (C - A) \cup (C - B) \end{aligned}$$

**Exercise 13** Suppose  $A \subseteq B$ . Let  $x \in C - B$ ; we must show  $x \in C - A$ . We have  $x \in C$  and  $x \notin B$ . Therefore  $x \notin A$ , since every member of  $A$  is a member of  $B$ . And so we have  $x \in C - A$  as required.

**Exercise 14** Let  $A = \{\emptyset\}$ ,  $B = \emptyset$  and  $C = \{\emptyset\}$ . Then  $A - (B - C) = A - \emptyset = \{\emptyset\}$  while  $(A - B) - C = \{\emptyset\} - C = \emptyset$ .

**Exercise 15**

(a) For any  $x$  we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have  $x \in A \cap (B + C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$ .

(b) For any  $x$  we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$

In every case, we have  $x \in A + (B + C) \Leftrightarrow x \in (A + B) + C$ .

#### Exercise 16

$$\begin{aligned} [(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A] &= (A \cup B) - A \\ &= B - A \end{aligned}$$

#### Exercise 17

(a)  $\Leftrightarrow$  (b)

$$\begin{aligned} A \subseteq B &\Leftrightarrow \text{Every element of } A \text{ is an element of } B \\ &\Leftrightarrow \text{There is no element of } A \text{ that is not an element of } B \\ &\Leftrightarrow A - B = \emptyset \end{aligned}$$

(a)  $\Rightarrow$  (c) Suppose  $A \subseteq B$ . We have  $B \subseteq A \cup B$  from the definition of  $A \cup B$ ; we must prove that  $A \cup B \subseteq B$ . So let  $x \in A \cup B$ . Then  $x \in A$  or  $x \in B$ . But in either case  $x \in B$ , since  $x \in A \Rightarrow x \in B$ . Thus we have  $x \in B$  as required.

(c)  $\Rightarrow$  (a) We always have  $A \subseteq A \cup B$ . So if  $A \cup B = B$  then we have  $A \subseteq B$ .

(a)  $\Rightarrow$  (d) Suppose  $A \subseteq B$ . We have  $A \cap B \subseteq A$  from the definition of  $A \cap B$ ; we must prove that  $A \subseteq A \cap B$ . So let  $x \in A$ . Then  $x \in B$  since  $A \subseteq B$ , hence  $x \in A \cap B$  as required.

(d)  $\Rightarrow$  (a) We always have  $A \cap B \subseteq B$ . So if  $A \cap B = A$  then  $A \subseteq B$ .

**Exercise 18** We can make the following 16 sets:

- $\emptyset (= A - A)$
- $A - B$
- $A \cap B$
- $B - A$
- $S - (A \cup B)$
- $A$
- $A + B$
- $S - B$
- $B$
- $S - (A + B)$
- $S - A$
- $A \cup B$
- $S - (B - A)$
- $S - (A \cap B)$
- $S - (A - B)$

**Exercise 19** They are never equal, because for all  $A, B$ , we have  $\emptyset \in \mathcal{P}(A - B)$  but  $\emptyset \notin \mathcal{P}A - \mathcal{P}B$  since  $\emptyset \in \mathcal{P}B$ .

**Exercise 20** Assume  $A \cup B = A \cup C$  and  $A \cap B = A \cap C$ .

We first show  $B \subseteq C$ . Let  $x \in B$ ; we show  $x \in C$ . We have  $x \in A \cup B = A \cup C$ , so either  $x \in A$  or  $x \in C$ . If  $x \in C$ , we are done. If  $x \in A$ , then we have  $x \in A \cap B = A \cap C$ , and so  $x \in C$  in this case too.

We can show  $C \subseteq B$  similarly. Hence  $B = C$ .

**Exercise 21** For any  $x$ , we have

$$\begin{aligned}
 x \in \bigcup (A \cup B) &\Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C \\
 &\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C, \text{ or there exists } C \in B \text{ such that } x \in C \\
 &\Leftrightarrow x \in \bigcup A \cup \bigcup B
 \end{aligned}$$

**Exercise 22** For any  $x$ , we have

$$\begin{aligned} x \in \bigcap (A \cup B) &\Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C \\ &\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C \\ &\Leftrightarrow x \in \bigcap A \cap \bigcap B \end{aligned}$$

**Exercise 23** PROOF:

- $\langle 1 \rangle 1. A \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 1. \text{ LET: } x \in A$
- $\langle 2 \rangle 2. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 3. x \in A \cup X$
- $\langle 1 \rangle 2. \bigcap \mathcal{B} \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 1. \text{ LET: } x \in \bigcap \mathcal{B}$
- $\langle 2 \rangle 2. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 3. x \in X$
- $\langle 2 \rangle 4. x \in A \cup X$
- $\langle 1 \rangle 3. \bigcap \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap \mathcal{B}$
- $\langle 2 \rangle 1. \text{ LET: } x \in \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 2. \text{ ASSUME: } x \notin A$
- PROVE:  $x \in \bigcap \mathcal{B}$
- $\langle 2 \rangle 3. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 4. x \in A \cup X$
- $\langle 2 \rangle 5. x \in X$

□

**Exercise 24**

(a)

$$\begin{aligned} Y \in \mathcal{P} \bigcap \mathcal{A} &\Leftrightarrow Y \subseteq \bigcap \mathcal{A} \\ &\Leftrightarrow \forall y \in Y. \forall X \in \mathcal{A}. y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A}. \forall y \in Y. y \in X \\ &\Leftrightarrow \forall X \in \mathcal{A}. Y \in \mathcal{P}X \\ &\Leftrightarrow Y \in \bigcap \{\mathcal{P}X \mid X \in \mathcal{A}\} \end{aligned}$$

(b)  $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} \subseteq \mathcal{P} \bigcup \mathcal{A}$

PROOF:

- $\langle 1 \rangle 1. \text{ LET: } Y \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
- $\langle 1 \rangle 2. \text{ PICK } X \in \mathcal{A} \text{ such that } Y \in \mathcal{P}X$
- $\langle 1 \rangle 3. Y \subseteq X$
- $\langle 1 \rangle 4. Y \subseteq \bigcup \mathcal{A}$
- $\langle 1 \rangle 5. Y \in \mathcal{P} \bigcup \mathcal{A}$

Equality holds if and only if  $\bigcup \mathcal{A} \in \mathcal{A}$ .

- ⟨1⟩1. If  $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$  then  $\bigcup \mathcal{A} \in \mathcal{A}$
  - ⟨2⟩1. ASSUME:  $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
  - ⟨2⟩2.  $\bigcup \mathcal{A} \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
  - ⟨2⟩3. PICK  $X \in \mathcal{A}$  such that  $\bigcup \mathcal{A} \in \mathcal{P}X$
  - ⟨2⟩4.  $X = \bigcup \mathcal{A}$
  - ⟨1⟩2. If  $\bigcup \mathcal{A} \in \mathcal{A}$  then  $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P} \bigcup \mathcal{A}$
- PROOF: If  $\bigcup \mathcal{A} \in \mathcal{A}$  then  $\mathcal{P} \bigcup \mathcal{A} \in \{\mathcal{P}X \mid X \in \mathcal{A}\}$ .  
 $\square$

**Exercise 25** We have  $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$  if and only if  $A = \emptyset$  or  $\mathcal{B} \neq \emptyset$

- ⟨1⟩1. If  $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$  then  $A = \emptyset$  or  $\mathcal{B} \neq \emptyset$
- PROOF: If  $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$  and  $\mathcal{B} = \emptyset$  then
- $$A \cup \bigcup \emptyset = \bigcup \emptyset$$
- $$\therefore A = \emptyset$$
- ⟨1⟩2. If  $A = \emptyset$  then  $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
- PROOF: Both sides are equal to  $\bigcup \mathcal{B}$
- ⟨1⟩3. If  $\mathcal{B} \neq \emptyset$  then  $A \cup \bigcup \mathcal{B} = \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
  - ⟨2⟩1. ASSUME:  $\mathcal{B} \neq \emptyset$
  - ⟨2⟩2.  $A \cup \bigcup \mathcal{B} \subseteq \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
  - ⟨3⟩1. LET:  $x \in A \cup \bigcup \mathcal{B}$
  - PROVE:  $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
  - ⟨3⟩2. CASE:  $x \in A$
  - ⟨4⟩1. PICK  $X \in \mathcal{B}$
  - PROOF: By ⟨2⟩1
  - ⟨4⟩2.  $x \in A \cup X$
  - ⟨3⟩3. CASE:  $x \in \bigcup \mathcal{B}$
  - ⟨4⟩1. PICK  $X \in \mathcal{B}$  such that  $x \in X$
  - ⟨4⟩2.  $x \in A \cup X$
  - ⟨2⟩3.  $\bigcup \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcup \mathcal{B}$
  - ⟨3⟩1. LET:  $x \in \bigcup \{A \cup X \mid X \in \mathcal{B}\}$
  - ⟨3⟩2. PICK  $X \in \mathcal{B}$  such that  $x \in A \cup X$
  - ⟨3⟩3.  $X \subseteq \bigcup \mathcal{B}$
  - ⟨3⟩4.  $A \cup X \subseteq A \cup \bigcup \mathcal{B}$
  - ⟨3⟩5.  $x \in A \cup \bigcup \mathcal{B}$

## 2.3 Review Exercises

**Exercise 26** Sets  $A$ ,  $B$ ,  $D$  and  $F$  are all equal to each other. Sets  $C$ ,  $E$  and  $G$  are equal to each other. None of the first list is equal to any of the second list.

**Exercise 27** Take  $A = \{\{0\}, \{1\}\}$  and  $B = \{\{1\}\}$ . Then  $A \cap B = \{\{1\}\}$  and

$$\begin{aligned}\bigcap A \cap \bigcap B &= \emptyset \cap \{1\} \\ &= \emptyset \\ \bigcap (A \cap B) &= \bigcap \{\{1\}\} \\ &= \{1\}\end{aligned}$$

**Exercise 28**

$$\bigcup \{\{3, 4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\} = \{3, 4, \{3\}, \{4\}\}$$

**Exercise 29**

(a)  $\emptyset$

(b) We have

$$\begin{aligned}\{\emptyset\} &\subseteq \mathcal{P}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{PP}\{\emptyset\} \\ \{\emptyset\} &\subseteq \mathcal{PP}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{PPP}\{\emptyset\} \\ \therefore \bigcap \{\mathcal{PPP}\{\emptyset\}, \mathcal{PP}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} &= \mathcal{PPP}\{\emptyset\} \cap \mathcal{PP}\{\emptyset\} \cap \mathcal{P}\{\emptyset\} \\ &= \mathcal{P}\{\emptyset\} \\ &= \{\emptyset, \{\emptyset\}\}\end{aligned}$$

**Exercise 30**

(a)  $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}$

(b)  $\{\emptyset, \{\emptyset\}\}$

(c)  $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d)  $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

**Exercise 31**

(a)  $\{1, 2, 3, \emptyset\}$

(b)  $\emptyset$

(c)  $\emptyset$

(d)  $\emptyset$

**Exercise 32**

(a)  $a \cup b$

(b)  $a$

(c)

$$\begin{aligned} \bigcap \bigcup S \cup (\bigcup \bigcup S - \bigcup \bigcap S) &= (a \cap b) \cup ((a \cup b) - a) \\ &= (a \cap b) \cup (b - a) \\ &= b \end{aligned}$$

**Exercise 33** When  $a \neq b$ :

$$\begin{aligned} \bigcup (\bigcup S - \bigcap S) &= \bigcup (\{a, b\} - \{a\}) \\ &= \bigcup \{b\} \\ &= b \end{aligned}$$

When  $a = b$ :

$$\begin{aligned} \bigcup (\bigcup S - \bigcap S) &= \bigcup (\{a, b\} - \{a\}) \\ &= \bigcup \emptyset \\ &= \emptyset \end{aligned}$$

**Exercise 34** For any set  $S$ , we have

$$\begin{aligned} \emptyset &\subseteq \mathcal{P}S \\ \therefore \emptyset &\in \mathcal{P}\mathcal{P}S \\ \emptyset &\subseteq S \\ \therefore \emptyset &\in \mathcal{P}S \\ \therefore \{\emptyset\} &\subseteq \mathcal{P}S \\ \therefore \{\emptyset\} &\in \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} &\subseteq \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} &\in \mathcal{P}\mathcal{P}\mathcal{P}S \end{aligned}$$

**Exercise 35** Assume  $\mathcal{P}A = \mathcal{P}B$ . Then we have

$$\begin{aligned}
 A &\in \mathcal{P}A \\
 \therefore A &\in \mathcal{P}B \\
 \therefore A &\subseteq B \\
 B &\in \mathcal{P}B \\
 \therefore B &\in \mathcal{P}A \\
 \therefore B &\subseteq A \\
 \therefore A &= B
 \end{aligned}$$

**Exercise 36**

(a)

$$\begin{aligned}
 x \in A - (A \cap B) &\Leftrightarrow x \in A \ \& \neg(x \in A \ \& \ x \in B) \\
 &\Leftrightarrow x \in A \ \& \ x \notin B \\
 &\Leftrightarrow x \in A - B
 \end{aligned}$$

(b)

$$\begin{aligned}
 x \in A - (A - B) &\Leftrightarrow x \in A \ \& \neg(x \in A \ \& \ x \notin B) \\
 &\Leftrightarrow x \in A \ \& \ x \in B \\
 &\Leftrightarrow x \in A \cap B
 \end{aligned}$$

**Exercise 37**

(a)

$$\begin{aligned}
 x \in (A \cup B) - C &\Leftrightarrow (x \in A \text{ or } x \in B) \ \& \ x \notin C \\
 &\Leftrightarrow (x \in A \ \& \ x \notin C) \text{ or } (x \in B \ \& \ x \notin C) \\
 &\Leftrightarrow x \in (A - C) \cup (B - C)
 \end{aligned}$$

(b)

$$\begin{aligned}
 x \in A - (B - C) &\Leftrightarrow x \in A \ \& \neg(x \in B \ \& \ x \notin C) \\
 &\Leftrightarrow x \in A \ \& \ (x \notin B \text{ or } x \in C) \\
 &\Leftrightarrow (x \in A \ \& \ x \notin B) \text{ or } (x \in A \ \& \ x \in C) \\
 &\Leftrightarrow x \in (A - B) \cup (A \cap C)
 \end{aligned}$$

(c)

$$\begin{aligned}
 x \in (A - B) - C &\Leftrightarrow x \in A \ \& \ x \notin B \ \& \ x \notin C \\
 &\Leftrightarrow x \in A \ \& \neg(x \in B \vee x \in C) \\
 &\Leftrightarrow x \in A - (B \cup C)
 \end{aligned}$$



**Exercise 38**

(a) If every element of  $A$  is an element of  $C$ , and every element of  $B$  is an element of  $C$ , then everything that is an element of either  $A$  or  $B$  is an element of  $C$ .

(b) If every element of  $C$  is an element of  $A$ , and every element of  $C$  is an element of  $B$ , then every element of  $C$  is an element of both  $A$  and  $B$ .

## Chapter 3

# Chapter 3 — Relations and Functions

### 3.1 Ordered Pairs

**Exercise 1** We have  $\langle 0, 1, 0 \rangle^* = \langle 0, 1, 1 \rangle^* = \{\{0\}, \{0, 1\}\}$ .

**Exercise 2**

(a)

$$\begin{aligned} z &\in A \times (B \cup C) \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ (y \in B \text{ or } y \in C)) \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in B) \text{ or } (z = (x, y) \ \& \ x \in A \ \& \ y \in C) \\ \Leftrightarrow z &\in (A \times B) \cup (A \times C) \end{aligned}$$

(b)

$\langle 1 \rangle 1$ . ASSUME:  $A \times B = A \times C$  and  $A \neq \emptyset$

$\langle 1 \rangle 2$ . PICK  $a \in A$

$\langle 1 \rangle 3$ . For all  $x$ ,  $x \in B \Leftrightarrow x \in C$

PROOF:  $x \in B$  iff  $(a, x) \in A \times B$  iff  $(a, x) \in A \times C$  iff  $x \in C$ .

□

**Exercise 3**

$$\begin{aligned} z &\in A \times \bigcup \mathcal{B} \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ \exists X \in \mathcal{B}. y \in X) \\ \Leftrightarrow \exists X \in \mathcal{B}. \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in X) \\ \Leftrightarrow z &\in \bigcup \{A \times X : X \in \mathcal{B}\} \end{aligned}$$

**Exercise 4** If every ordered pair belongs to  $A$  then every set belongs to  $\bigcup \bigcup A$  contradicting Theorem 2A.

**Exercise 5**

(a) Apply a Subset Axiom to  $\mathcal{P}(A \times B)$ : we have  $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A. z = \{x\} \times B\}$ .

(b)

$$\begin{aligned} z &\in \bigcup C \\ \Leftrightarrow \exists x \in A. z &\in \{x\} \times B \\ \Leftrightarrow \exists x \in A. \exists y \in B. z &= (x, y) \\ \Leftrightarrow z &\in A \times B \end{aligned}$$

## 3.2 Relations

**Exercise 6** If  $A \subseteq \text{dom } A \times \text{ran } A$  then  $A$  is a set of ordered pairs, i.e. a relation.

Conversely, suppose  $A$  is a relation. Let  $z \in A$ . Then  $z$  is an ordered pair; let  $z = (x, y)$ . We have  $x \in \text{dom } A$  and  $y \in \text{ran } A$  and so  $z \in \text{dom } A \times \text{ran } A$  as required.

**Exercise 7** We have  $\text{fld } R \subseteq \bigcup \bigcup R$  by Lemma 3D.

Conversely, let  $x \in \bigcup \bigcup R$ . Pick  $a$  and  $b$  such that  $x \in a$ ,  $a \in b$  and  $b \in R$ . Then  $b$  is an ordered pair; let  $b = (y, z)$ . We have  $a = \{y\}$  or  $\{y, z\}$ , hence  $x = y$  or  $x = z$ . In either case,  $x \in \text{fld } R$ .

**Exercise 8**

(a)

$$\begin{aligned} x &\in \text{dom } \bigcup \mathcal{A} \\ \Leftrightarrow \exists y. \exists R \in \mathcal{A}. (x, y) &\in R \\ \Leftrightarrow \exists R \in \mathcal{A}. \exists y. (x, y) &\in R \\ \Leftrightarrow x &\in \bigcup \{\text{dom } R : R \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned}
y &\in \text{ran} \bigcup \mathcal{A} \\
&\Leftrightarrow \exists x. \exists R \in \mathcal{A}. (x, y) \in R \\
&\Leftrightarrow \exists R \in \mathcal{A}. \exists x. (x, y) \in R \\
&\Leftrightarrow y \in \bigcup \{\text{ran } R : R \in \mathcal{A}\}
\end{aligned}$$

**Exercise 9** Assume  $\mathcal{A}$  is nonempty. We have  $\text{dom} \bigcap \mathcal{A} \subseteq \bigcap \{\text{dom } R : R \in \mathcal{A}\}$ .

PROOF:

$$\begin{aligned}
x &\in \text{dom} \bigcap \mathcal{A} \\
&\Leftrightarrow \exists y. \forall R \in \mathcal{A}. (x, y) \in R \\
&\Rightarrow \forall R \in \mathcal{A}. \exists y. (x, y) \in R \\
&\Leftrightarrow x \in \bigcap \{\text{dom } R : R \in \mathcal{A}\}
\end{aligned}$$

Equality holds iff the middle ' $\Rightarrow$ ' can be reversed, i.e. iff for all  $x$ , if  $\forall R \in \mathcal{A}. \exists y. (x, y) \in R$  then  $\exists y. \forall R \in \mathcal{A}. (x, y) \in R$ . I haven't found a simpler condition than this. The condition does not always hold, for example if  $\mathcal{A} = \{\{(1, 2)\}, \{(1, 3)\}\}$  then  $\text{dom} \bigcap \mathcal{A} = \emptyset$  while  $\bigcap \{\text{dom } R : R \in \mathcal{A}\} = \{1\}$ .

Similarly,  $\text{ran} \bigcap \mathcal{A} \subseteq \bigcap \{\text{ran } R : R \in \mathcal{A}\}$ , and equality holds iff, for any  $y$ , if  $\forall R \in \mathcal{A}. \exists x. (x, y) \in R$  then  $\exists x. \forall R \in \mathcal{A}. (x, y) \in R$ .

### 3.3 $n$ -ary Relations

**Exercise 10** This follows from the equations at the top of page 42. An ordered 4-tuple  $\langle a, b, c, d \rangle$  is also an ordered 1-tuple (because every set is), and the ordered pair  $\langle \langle a, b, c \rangle, d \rangle$ , and the ordered triple  $\langle \langle a, b \rangle, c, d \rangle$ .

### 3.4 Functions

**Exercise 11** We prove  $F \subseteq G$ . Let  $z \in F$ . Since  $F$  is a relation, then  $z$  is an ordered pair; let  $z = \langle x, y \rangle$ . We have  $x \in \text{dom } F$  and  $y = F(x)$ . Therefore  $x \in \text{dom } G$  and  $y = G(x)$  (because  $\text{dom } F = \text{dom } G$  and  $F(x) = G(x)$ ). Hence  $\langle x, y \rangle \in G$ , i.e.  $z \in G$ .

We have proved  $F \subseteq G$ . We can prove  $G \subseteq F$  similarly. Thus  $F = G$ .

**Exercise 12** PROOF:

- $\langle 1 \rangle 1$ . If  $f \subseteq g$  then  $\text{dom } f \subseteq \text{dom } g$  and  $\forall x \in \text{dom } f. f(x) = g(x)$
- $\langle 2 \rangle 1$ . ASSUME:  $f \subseteq g$
- $\langle 2 \rangle 2$ . LET:  $x \in \text{dom } f$
- $\langle 2 \rangle 3$ .  $(x, f(x)) \in f$
- $\langle 2 \rangle 4$ .  $(x, f(x)) \in g$
- $\langle 2 \rangle 5$ .  $x \in \text{dom } g$  and  $g(x) = f(x)$

- ⟨1⟩2. If  $\text{dom } f = \text{dom } g$  and  $\forall x \in \text{dom } f. f(x) = g(x)$  then  $f \subseteq g$
- ⟨2⟩1. ASSUME:  $\text{dom } f = \text{dom } g$  and  $\forall x \in \text{dom } f. f(x) = g(x)$
- ⟨2⟩2. LET:  $z \in f$
- ⟨2⟩3. LET:  $z = (x, y)$
- ⟨2⟩4.  $x \in \text{dom } f$  and  $y = f(x)$
- ⟨2⟩5.  $x \in \text{dom } g$  and  $y = g(x)$
- ⟨2⟩6.  $z = (x, y) \in g$

□

**Exercise 13** PROOF:

- ⟨1⟩1. ASSUME:  $f$  and  $g$  are functions
- ⟨1⟩2. ASSUME:  $f \subseteq g$
- ⟨1⟩3. ASSUME:  $\text{dom } g \subseteq \text{dom } f$
- ⟨1⟩4.  $\text{dom } f = \text{dom } g$
- PROOF: We have  $\text{dom } f \subseteq \text{dom } g$  from ⟨1⟩2 and  $\text{dom } g \subseteq \text{dom } f$  from ⟨1⟩3
- ⟨1⟩5. For  $x \in \text{dom } f$  we have  $f(x) = g(x)$
- PROOF: From ⟨1⟩2 and Exercise 12
- ⟨1⟩6. Q.E.D.
- PROOF: From Exercise 11.

□

**Exercise 14**

(a) If  $(x, y)$  and  $(x, z)$  are members of  $f \cap g$  then they are both members of  $f$ , hence  $y = z$ .

(b) PROOF:

- ⟨1⟩1. If  $f \cup g$  is a function then, for all  $x \in \text{dom } f \cap \text{dom } g$ , we have  $f(x) = g(x)$ .
- ⟨2⟩1. ASSUME:  $f \cup g$  is a function.
- ⟨2⟩2. LET:  $x \in \text{dom } f \cap \text{dom } g$
- ⟨2⟩3.  $(x, f(x))$  and  $(x, g(x))$  are both elements of  $f \cup g$
- ⟨2⟩4.  $f(x) = g(x)$
- ⟨1⟩2. If, for all  $x \in \text{dom } f \cap \text{dom } g$ , we have  $f(x) = g(x)$ , then  $f \cup g$  is a function.
- ⟨2⟩1. ASSUME: For all  $x \in \text{dom } f \cap \text{dom } g$ , we have  $f(x) = g(x)$
- ⟨2⟩2.  $f \cup g$  is a relation.
- PROOF: Since every element of either  $f$  or  $g$  is an ordered pair.
- ⟨2⟩3. Whenever  $(x, y)$  and  $(x, z)$  are elements of  $f \cup g$  we have  $y = z$
- ⟨3⟩1. LET:  $(x, y), (x, z) \in f \cup g$
- ⟨3⟩2. CASE:  $(x, y), (x, z) \in f$
- PROOF: Then  $y = z$  since  $f$  is a function.
- ⟨3⟩3. CASE:  $(x, y) \in f, (x, z) \in g$
- PROOF: Then  $y = z$  by ⟨2⟩1
- ⟨3⟩4. CASE:  $(x, y) \in g, (x, z) \in f$
- PROOF: Then  $y = z$  by ⟨2⟩1
- ⟨3⟩5. CASE:  $(x, y), (x, z) \in g$

PROOF: Then  $y = z$  since  $g$  is a function.

□

**Exercise 15** PROOF:

⟨1⟩1.  $\bigcup \mathcal{A}$  is a relation.

PROOF: Since every member of  $\mathcal{A}$  is a relation.

⟨1⟩2. Whenever  $(x, y)$  and  $(x, z)$  are elements of  $\bigcup \mathcal{A}$  then  $y = z$

⟨2⟩1. LET:  $(x, y), (x, z) \in \bigcup \mathcal{A}$

⟨2⟩2. PICK  $f, g \in \mathcal{A}$  such that  $(x, y) \in f$  and  $(x, z) \in g$

⟨2⟩3. ASSUME: w.l.o.g.  $f \subseteq g$

⟨2⟩4.  $(x, y), (x, z) \in g$

⟨2⟩5.  $y = z$

PROOF: Since  $g$  is a function.

□

**Exercise 16** If every function belongs to  $\mathcal{A}$  then every set belongs to  $\text{dom} \bigcup \mathcal{A}$  contradiction Theorem 2A.

**Exercise 17** PROOF:

⟨1⟩1. LET:  $R$  and  $S$  be single-rooted.

⟨1⟩2. LET:  $(x, z), (y, z) \in R \circ S$

⟨1⟩3. PICK  $t$  and  $t'$  such that  $(x, t) \in S$ ,  $(t, z) \in R$ ,  $(y, t') \in S$  and  $(t', z) \in R$

⟨1⟩4.  $t = t'$

PROOF: Since  $R$  is single-rooted.

⟨1⟩5.  $x = y$

PROOF: Since  $S$  is single-rooted.

Thus if  $F$  and  $G$  are one-to-one functions then  $F \circ G$  is single-rooted and a function by Theorem 3H, hence a one-to-one function.

**Exercise 18**

$$R \circ R = \{\langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle\}$$

$$R \upharpoonright \{1\} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle\}$$

$$R^{-1} \upharpoonright \{1\} = \{\langle 1, 0 \rangle\}$$

$$R[\{1\}] = \{2, 3\}$$

$$R^{-1}[\{1\}] = \{0\}$$

**Exercise 19**

$$\begin{aligned}
A(\emptyset) &= \{\emptyset, \{\emptyset\}\} \\
A[\emptyset] &= \emptyset \\
A[\{\emptyset\}] &= \{\{\emptyset, \{\emptyset\}\}\} \\
A[\{\emptyset, \{\emptyset\}\}] &= \{\{\emptyset, \{\emptyset\}\}, \emptyset\} \\
A^{-1} &= \{\langle \{\emptyset, \{\emptyset\}\}, \emptyset \rangle, \langle \emptyset, \{\emptyset\} \rangle\} \\
A \circ A &= \{\langle \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \rangle\} \\
A \upharpoonright \emptyset &= \emptyset \\
A \upharpoonright \{\emptyset\} &= \{\langle \emptyset, \{\emptyset, \{\emptyset\}\} \rangle\} \\
A \upharpoonright \{\emptyset, \{\emptyset\}\} &= \{\langle \emptyset, \{\emptyset, \{\emptyset\}\} \rangle, \langle \{\emptyset\}, \emptyset \rangle\} \\
&= A \\
\bigcup A &= \{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}
\end{aligned}$$

**Exercise 20**

$$\begin{aligned}
z \in F \upharpoonright A &\Leftrightarrow z \in F \ \& \ \exists x, y. (z = \langle x, y \rangle \ \& \ x \in A) \\
&\Leftrightarrow z \in F \ \& \ \exists x, y. (z = \langle x, y \rangle \ \& \ x \in A \ \& \ y \in \text{ran } F) \\
&\Leftrightarrow z \in F \cap (A \times \text{ran } F)
\end{aligned}$$

**Exercise 21** Both are equal to  $\{\langle x, w \rangle \mid \exists y, z. xTy \ \& \ ySz \ \& \ zRw\}$ .

**Exercise 22**

(a) PROOF:  
 $\langle 1 \rangle 1$ . ASSUME:  $A \subseteq B$   
 $\langle 1 \rangle 2$ . LET:  $y \in F[A]$   
 $\langle 1 \rangle 3$ . PICK  $x \in A$  such that  $xFy$   
 $\langle 1 \rangle 4$ .  $x \in B$  and  $xFy$   
 $\square$

(b) Both are equal to  $\{z : \exists x, y. x \in A \ \& \ xGy \ \& \ yFz\}$

(c) Both are equal to  $\{\langle x, y \rangle : (x \in A \text{ or } x \in B) \ \& \ xQy\}$

**Exercise 23**

$$\begin{aligned}
B \circ I_A &= \{\langle x, z \rangle : \exists y(xI_A y \ \& \ yBz)\} \\
&= \{\langle x, z \rangle : \exists y(x \in A \ \& \ x = y \ \& \ yBz)\} \\
&= \{\langle x, z \rangle : x \in A \ \& \ xBz\} \\
&= B \upharpoonright A \\
I_A[C] &= \{y : \exists x \in C.xI_A y\} \\
&= \{y : \exists x \in C(x \in A \ \& \ x = y)\} \\
&= \{y : y \in C \ \& \ y \in A\} \\
&= A \cap C
\end{aligned}$$

**Exercise 24**

$$\begin{aligned}
F^{-1}[A] &= \{x : \exists y \in A.yF^{-1}x\} \\
&= \{x : \exists y \in A.xFy\} \\
&= \{x \in \text{dom } F : F(x) \in A\}
\end{aligned}$$

**Exercise 25****(a)** PROOF: $\langle 1 \rangle 1.$  LET:  $G$  be a one-to-one function. $\langle 1 \rangle 2.$   $G^{-1}$  is a function.

PROOF: Theorem 3F.

 $\langle 1 \rangle 3.$   $G \circ G^{-1}$  is a function.

PROOF: Theorem 3H.

 $\langle 1 \rangle 4.$   $\text{dom}(G \circ G^{-1}) = \text{ran } G$ 

PROOF:

$$\text{dom}(G \circ G^{-1}) = \{x \in \text{dom } G^{-1} : G^{-1}(x) \in \text{dom } G\} \quad (\text{Theorem 3H})$$

$$= \{x \in \text{ran } G : G^{-1}(x) \in \text{dom } G\} \quad (\text{Theorem 3E})$$

$$= \text{ran } G$$

 $\langle 1 \rangle 5.$   $\forall x \in \text{ran } G.(G \circ G^{-1})(x) = x$ 

PROOF: Theorem 3G.

□

**(b)** Let  $G$  be a function. Then

$$\begin{aligned}
G \circ G^{-1} &= \{\langle x, z \rangle : \exists y(xG^{-1}y \ \& \ yGz)\} \\
&= \{\langle x, z \rangle : \exists y(yGx \ \& \ yGz)\} \\
&= \{\langle x, x \rangle : \exists y.yGx\} & (G \text{ is a function}) \\
&= I_{\text{ran } G}
\end{aligned}$$

**Exercise 26**



(a)

$$\begin{aligned} F[\bigcup \mathcal{A}] &= \{y : \exists x. \exists A \in \mathcal{A} (x \in A \ \& \ xFy)\} \\ &= \{y : \exists A \in \mathcal{A}. \exists x (x \in A \ \& \ xFy)\} \\ &= \bigcup \{F[A] : A \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned} F[\bigcup \mathcal{A}] &= \{y : \exists x. \forall A \in \mathcal{A} (x \in A \ \& \ xFy)\} \\ &\subseteq \{y : \forall A \in \mathcal{A}. \exists x (x \in A \ \& \ xFy)\} \\ &= \bigcap \{F[A] : A \in \mathcal{A}\} \end{aligned}$$

**Exercise 27**

$$\begin{aligned} \text{dom}(F \circ G) &= \{x : \exists y. x(F \circ G)y\} \\ &= \{x : \exists y \exists z (xGz \ \& \ zFy)\} \\ &= \{x : \exists z (zG^{-1}x \ \& \ z \in \text{dom } F)\} \\ &= G^{-1}[\text{dom } F] \end{aligned}$$

**Exercise 28** PROOF:

$\langle 1 \rangle 1.$   $G : \mathcal{P}A \rightarrow \mathcal{P}B$

PROOF: Since  $f[X] \subseteq \text{ran } f \subseteq B$

$\langle 1 \rangle 2.$  For all  $X, Y \in \mathcal{P}A$ , if  $G(X) = G(Y)$  then  $X = Y$

$\langle 2 \rangle 1.$  LET:  $X, Y \in \mathcal{P}A$

$\langle 2 \rangle 2.$  ASSUME:  $f[X] = f[Y]$

$\langle 2 \rangle 3.$   $X \subseteq Y$

$\langle 3 \rangle 1.$  LET:  $x \in X$

$\langle 3 \rangle 2.$   $f(x) \in f[X]$

$\langle 3 \rangle 3.$   $f(x) \in f[Y]$

$\langle 3 \rangle 4.$  PICK  $y \in Y$  such that  $f(x) = f(y)$

$\langle 3 \rangle 5.$   $x = y$

PROOF: Because  $f$  is one-to-one.

$\langle 3 \rangle 6.$   $x \in Y$

PROOF: Similar.

$\langle 2 \rangle 4.$   $Y \subseteq X$

□

**Example 29** PROOF:

$\langle 1 \rangle 1.$  ASSUME:  $f$  maps  $A$  onto  $B$

$\langle 1 \rangle 2.$  LET:  $b, b' \in B$

$\langle 1 \rangle 3.$  ASSUME:  $G(b) = G(b')$

$\langle 1 \rangle 4.$  PICK  $x \in A$  such that  $f(x) = b$

PROOF: By  $\langle 1 \rangle 1$ .

$\langle 1 \rangle 5$ .  $x \in G(b)$

$\langle 1 \rangle 6$ .  $x \in G(b')$

$\langle 1 \rangle 7$ .  $f(x) = b'$

$\langle 1 \rangle 8$ .  $b = b'$

□

The converse does not hold. Let  $A = \{0\}$  and  $B = \{0, 1\}$ . Let  $f$  be the function that maps 0 to 0. Then

$$G(0) = \{0\}$$

$$G(1) = \emptyset$$

Thus  $G$  is one-to-one but  $f$  does not map  $A$  onto  $B$ .

### Exercise 30

(a) PROOF:

$\langle 1 \rangle 1$ .  $F(B) = B$

$\langle 2 \rangle 1$ .  $F(B) \subseteq B$

$\langle 3 \rangle 1$ . LET:  $X \in \mathcal{P}A$  be such that  $F(X) \subseteq X$

PROVE:  $F(B) \subseteq X$

$\langle 3 \rangle 2$ .  $B \subseteq X$

$\langle 3 \rangle 3$ .  $F(B) \subseteq F(X)$

$\langle 3 \rangle 4$ .  $F(B) \subseteq X$

PROOF: From  $\langle 3 \rangle 1$  and  $\langle 3 \rangle 3$ .

$\langle 2 \rangle 2$ .  $B \subseteq F(B)$

PROOF: From  $\langle 2 \rangle 1$  and the definition of  $B$ , since  $B$  is one of the sets  $X$  such that  $F(X) \subseteq X$

$\langle 1 \rangle 2$ .  $F(C) = C$

$\langle 2 \rangle 1$ .  $C \subseteq F(C)$

$\langle 3 \rangle 1$ . LET:  $X \in \mathcal{P}A$  with  $X \subseteq F(X)$

PROVE:  $X \subseteq F(C)$

$\langle 3 \rangle 2$ .  $X \subseteq C$

$\langle 3 \rangle 3$ .  $F(X) \subseteq F(C)$

$\langle 3 \rangle 4$ .  $X \subseteq F(C)$

PROOF: From  $\langle 3 \rangle 1$  and  $\langle 3 \rangle 3$

$\langle 2 \rangle 2$ .  $F(C) \subseteq C$

PROOF: From  $\langle 2 \rangle 1$  and the definition of  $C$ .

□

(b) If  $F(X) = X$  then we have  $B \subseteq X$  (because  $F(X) \subseteq X$ ) and  $X \subseteq C$  (because  $X \subseteq F(X)$ ).

### 3.5 Infinite Cartesian Products

**Exercise 31** PROOF:

- ⟨1⟩1. If the Axiom of Choice is true then, for any set  $I$  and any function  $H$  with domain  $I$ , if  $H(i) \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} H(i) \neq \emptyset$ .
- ⟨2⟩1. ASSUME: The Axiom of Choice.
- ⟨2⟩2. LET:  $I$  be a set.
- ⟨2⟩3. LET:  $H$  be a function with domain  $I$ .
- ⟨2⟩4. ASSUME:  $H(i) \neq \emptyset$  for all  $i \in I$ .
- ⟨2⟩5. LET:  $R = \{(i, x) : i \in I, x \in H(i)\}$
- ⟨2⟩6. PICK a function  $F \subseteq R$  with  $\text{dom } F = \text{dom } R$   
PROVE:  $F \in \prod_{i \in I} H(i)$   
PROOF: By the Axiom of Choice.
- ⟨2⟩7.  $\text{dom } H = I$   
PROOF: We have  $\text{dom } R = I$  since for all  $i \in I$  there exists  $x$  such that  $x \in H(i)$ .
- ⟨2⟩8.  $\forall i \in I. F(i) \in H(i)$   
PROOF: Since  $iRF(i)$ .
- ⟨1⟩2. If, for any set  $I$  and any function  $H$  with domain  $I$ , if  $H(i) \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} H(i) \neq \emptyset$ , then the Axiom of Choice is true.
- ⟨2⟩1. ASSUME: For any set  $I$  and any function  $H$  with domain  $I$ , if  $H(i) \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} H(i) \neq \emptyset$
- ⟨2⟩2. LET:  $R$  be a relation
- ⟨2⟩3. LET:  $I = \text{dom } R$
- ⟨2⟩4. Define the function  $H$  with domain  $I$  by: for  $i \in I$ ,  $H(i) = \{y : iRy\}$
- ⟨2⟩5.  $H(i) \neq \emptyset$  for all  $i \in I$
- ⟨2⟩6. PICK  $F \in \prod_{i \in I} H(i)$   
PROOF: By ⟨2⟩1
- ⟨2⟩7.  $F$  is a function
- ⟨2⟩8.  $F \subseteq R$   
PROOF: For all  $i \in I$  we have  $F(i) \in H(i)$  and so  $iRF(i)$ .
- ⟨2⟩9.  $\text{dom } F = \text{dom } R$

□

### 3.6 Equivalence Relations

**Exercise 32**

(a)

$$\begin{aligned}
 & R \text{ is symmetric} \\
 \Leftrightarrow & \forall x, y (xRy \Rightarrow yRx) \\
 \Leftrightarrow & \forall x, y (\langle y, x \rangle \in R^{-1} \Rightarrow \langle y, x \rangle \in R) \\
 \Leftrightarrow & R^{-1} \subseteq R
 \end{aligned}$$

(b)

$$\begin{aligned}
& R \text{ is transitive} \\
& \Leftrightarrow \forall x, y, z (xRy \ \& \ yRz \Rightarrow xRz) \\
& \Leftrightarrow \forall x, z (\exists y (xRy \ \& \ yRz) \Rightarrow xRz) \\
& \Leftrightarrow \forall x, z (\langle x, z \rangle \in R \circ R \Rightarrow \langle x, z \rangle \in R) \\
& \Leftrightarrow R \circ R \subseteq R
\end{aligned}$$

**Exercise 33** PROOF:

$\langle 1 \rangle 1$ . If  $R$  is a symmetric and transitive relation then  $R = R^{-1} \circ R$ .

$\langle 2 \rangle 1$ . ASSUME:  $R$  is a symmetric and transitive relation.

$\langle 2 \rangle 2$ .  $R \subseteq R^{-1} \circ R$

$\langle 3 \rangle 1$ . LET:  $xRy$

$\langle 3 \rangle 2$ .  $yRy$

PROOF: By Theorem 3M.

$\langle 3 \rangle 3$ .  $xRy$  and  $yR^{-1}y$

$\langle 3 \rangle 4$ .  $x(R^{-1} \circ R)y$

$\langle 2 \rangle 3$ .  $R^{-1} \circ R \subseteq R$

PROOF:

$$R^{-1} \circ R \subseteq R \circ R \quad (\text{Exercise 32(a)})$$

$$\subseteq R \quad (\text{Exercise 32(b)})$$

$\langle 1 \rangle 2$ . If  $R = R^{-1} \circ R$  then  $R$  is a symmetric and transitive relation.

$\langle 2 \rangle 1$ . ASSUME:  $R = R^{-1} \circ R$

$\langle 2 \rangle 2$ .  $R$  is a relation.

$\langle 2 \rangle 3$ .  $R$  is symmetric.

$\langle 3 \rangle 1$ . LET:  $xRy$

$\langle 3 \rangle 2$ . PICK  $z$  such that  $xRz$  and  $zR^{-1}y$

$\langle 3 \rangle 3$ .  $yRz$  and  $zR^{-1}x$

$\langle 3 \rangle 4$ .  $y(R^{-1} \circ R)x$

$\langle 3 \rangle 5$ .  $yRx$

$\langle 2 \rangle 4$ .  $R$  is transitive.

$\langle 3 \rangle 1$ . LET:  $xRy$  and  $yRz$

$\langle 3 \rangle 2$ .  $zRy$

PROOF: By  $\langle 2 \rangle 3$

$\langle 3 \rangle 3$ .  $xRy$  and  $yR^{-1}z$

$\langle 3 \rangle 4$ .  $x(R^{-1} \circ R)z$

$\langle 3 \rangle 5$ .  $xRz$

□

**Exercise 34**

(a)  $\bigcap \mathcal{A}$  is a transitive relation.

PROOF:

$\langle 1 \rangle 1$ .  $\bigcap \mathcal{A}$  is a relation.

PROOF: Every member of a member of  $\mathcal{A}$  is an ordered pair.

$\langle 1 \rangle 2$ .  $\bigcap \mathcal{A}$  is transitive.

$\langle 2 \rangle 1$ . LET:  $\langle x, y \rangle$  and  $\langle y, z \rangle$  be in  $\bigcap \mathcal{A}$

PROVE:  $\langle x, z \rangle \in \bigcap \mathcal{A}$

$\langle 2 \rangle 2$ . LET:  $R \in \mathcal{A}$

$\langle 2 \rangle 3$ .  $xRy$  and  $yRz$

$\langle 2 \rangle 4$ .  $xRz$

PROOF: Since  $R$  is transitive.

□

(b) Not necessarily. If  $\mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$  then each member of  $\mathcal{A}$  is transitive but  $\bigcup \mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$  is not.

### Example 35

$$\begin{aligned} R[\{x\}] &= \{y : \exists z(z \in \{x\} \ \& \ zRy)\} \\ &= \{y : \exists z(z = x \ \& \ zRy)\} \\ &= \{y : xRy\} \\ &= [x]_R \end{aligned}$$

### Example 36 PROOF:

$\langle 1 \rangle 1$ .  $Q$  is a relation on  $A$ .

PROOF: By definition.

$\langle 1 \rangle 2$ .  $Q$  is reflexive on  $A$ .

$\langle 2 \rangle 1$ . LET:  $x \in A$

$\langle 2 \rangle 2$ .  $f(x)Rf(x)$

PROOF: Since  $R$  is reflexive on  $B$ .

$\langle 2 \rangle 3$ .  $xQx$

$\langle 1 \rangle 3$ .  $Q$  is symmetric.

$\langle 2 \rangle 1$ . ASSUME:  $xQy$

$\langle 2 \rangle 2$ .  $f(x)Rf(y)$

$\langle 2 \rangle 3$ .  $f(y)Rf(x)$

PROOF:  $R$  is symmetric.

$\langle 2 \rangle 4$ .  $yQx$

$\langle 1 \rangle 4$ .  $Q$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $xQy$  and  $yQz$

$\langle 2 \rangle 2$ .  $f(x)Rf(y)$  and  $f(y)Rf(z)$

$\langle 2 \rangle 3$ .  $f(x)Rf(z)$

PROOF:  $R$  is transitive.

$\langle 2 \rangle 4$ .  $xQz$

□

### Exercise 37 PROOF:

$\langle 1 \rangle 1$ .  $R_\Pi$  is a relation on  $A$ .

PROOF: If  $B \in \Pi$ ,  $x \in B$  and  $y \in B$  then  $x, y \in A$ .

$\langle 1 \rangle 2$ .  $R_\Pi$  is reflexive on  $A$ .

$\langle 2 \rangle 1$ . LET:  $x \in A$

$\langle 2 \rangle 2$ . PICK  $B \in \Pi$  such that  $x \in B$

PROOF: Because  $\Pi$  is exhaustive.

$\langle 2 \rangle 3$ .  $x \in B$  and  $x \in B$

$\langle 2 \rangle 4$ .  $xR_\Pi x$

$\langle 1 \rangle 3$ .  $R_\Pi$  is symmetric.

$\langle 2 \rangle 1$ . ASSUME:  $xR_\Pi y$

$\langle 2 \rangle 2$ . PICK  $B \in \Pi$  such that  $x \in B$  and  $y \in B$

$\langle 2 \rangle 3$ .  $y \in B$  and  $x \in B$

$\langle 2 \rangle 4$ .  $yR_\Pi x$

$\langle 1 \rangle 4$ .  $R_\Pi$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $xR_\Pi y$  and  $yR_\Pi z$

$\langle 2 \rangle 2$ . PICK  $B \in \Pi$  such that  $x \in B$  and  $y \in B$

$\langle 2 \rangle 3$ . PICK  $C \in \Pi$  such that  $y \in C$  and  $z \in C$

$\langle 2 \rangle 4$ .  $B = C$

PROOF: Since  $y \in B$  and  $y \in C$

$\langle 2 \rangle 5$ .  $x \in B$  and  $z \in B$

$\langle 2 \rangle 6$ .  $xR_\Pi z$

□

**Exercise 38** PROOF:

$\langle 1 \rangle 1$ . If  $B \in \Pi$  and  $x \in B$  then  $B = [x]_{R_\Pi}$

$\langle 2 \rangle 1$ . LET:  $B \in \Pi$

$\langle 2 \rangle 2$ . LET:  $x \in B$

$\langle 2 \rangle 3$ .  $[x]_{R_\Pi} \subseteq B$

$\langle 3 \rangle 1$ . LET:  $y \in [x]_{R_\Pi}$

$\langle 3 \rangle 2$ .  $xR_\Pi y$

$\langle 3 \rangle 3$ . PICK  $C \in \Pi$  such that  $x \in C$  and  $y \in C$

$\langle 3 \rangle 4$ .  $B = C$

PROOF: Since  $x \in B$  and  $x \in C$ .

$\langle 3 \rangle 5$ .  $y \in B$

$\langle 2 \rangle 4$ .  $B \subseteq [x]_{R_\Pi}$

PROOF: For all  $y \in B$ , we have  $x \in B$  and  $y \in B$  hence  $xR_\Pi y$ .

$\langle 1 \rangle 2$ .  $A/R_\Pi \subseteq \Pi$

$\langle 2 \rangle 1$ . LET:  $x \in A$

PROVE:  $[x]_{R_\Pi} \in \Pi$

$\langle 2 \rangle 2$ . PICK  $B \in \Pi$  such that  $x \in B$

$\langle 2 \rangle 3$ .  $[x]_{R_\Pi} = B$

PROOF: By  $\langle 1 \rangle 1$

$\langle 2 \rangle 4$ .  $[x]_{R_\Pi} \in \Pi$

$\langle 1 \rangle 3$ .  $\Pi \subseteq A/R_\Pi$

$\langle 2 \rangle 1$ . LET:  $B \in \Pi$

$\langle 2 \rangle 2$ . PICK  $x \in B$

PROOF: Since every member of  $\Pi$  is nonempty.

$\langle 2 \rangle 3$ .  $B = [x]_{R_\Pi}$

PROOF: By  $\langle 1 \rangle 1$ .

$\langle 2 \rangle 4$ .  $B \in A/R_\Pi$

□

**Exercise 39** PROOF:

$\langle 1 \rangle 1$ .  $R_\Pi \subseteq R$

$\langle 2 \rangle 1$ . LET:  $xR_\Pi y$

$\langle 2 \rangle 2$ . PICK  $B \in \Pi$  such that  $x \in B$  and  $y \in B$

$\langle 2 \rangle 3$ . PICK  $z \in A$  such that  $B = [z]_R$

$\langle 2 \rangle 4$ .  $zRx$

$\langle 2 \rangle 5$ .  $zRy$

$\langle 2 \rangle 6$ .  $xRy$

PROOF: Since  $R$  is symmetric and transitive.

$\langle 1 \rangle 2$ .  $R \subseteq R_\Pi$

$\langle 2 \rangle 1$ . LET:  $xRy$

$\langle 2 \rangle 2$ .  $x \in [x]_R$

$\langle 2 \rangle 3$ .  $y \in [x]_R$

$\langle 2 \rangle 4$ .  $xR_\Pi y$

□

**Exercise 40** We have  $[2]_R = [3]_R$  but  $[6]_R \neq [9]_R$  so there is no such function  $f$ .

**Exercise 41**

(a) PROOF:

$\langle 1 \rangle 1$ .  $Q$  is reflexive on  $\mathbb{R} \times \mathbb{R}$ .

PROOF: For any  $x, y \in \mathbb{R}$ , we have  $x + y = x + y$ , hence  $\langle x, y \rangle Q \langle x, y \rangle$

$\langle 1 \rangle 2$ .  $Q$  is symmetric.

$\langle 2 \rangle 1$ . ASSUME:  $\langle u, v \rangle Q \langle x, y \rangle$

$\langle 2 \rangle 2$ .  $u + y = x + v$

$\langle 2 \rangle 3$ .  $x + v = u + y$

$\langle 2 \rangle 4$ .  $\langle x, y \rangle Q \langle u, v \rangle$

$\langle 1 \rangle 3$ .  $Q$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $\langle a, b \rangle Q \langle u, v \rangle$  and  $\langle u, v \rangle Q \langle x, y \rangle$

$\langle 2 \rangle 2$ .  $a + v = u + b$

$\langle 2 \rangle 3$ .  $u + y = x + v$

$\langle 2 \rangle 4$ .  $a + y + x + b$

PROOF: Adding  $\langle 2 \rangle 2$  and  $\langle 2 \rangle 3$  gives  $a + u + v + y = b + u + v + x$ .

$\langle 2 \rangle 5$ .  $\langle a, b \rangle Q \langle x, y \rangle$

□

(b) We prove that, if  $\langle u, v \rangle Q \langle x, y \rangle$  then  $\langle u + 2v, v + 2u \rangle Q \langle x + 2y, y + 2x \rangle$ . It follows from Theorem 3Q that the function  $G$  exists.

If  $u + y = v + x$  then  $u + 2v + y + 2x = v + 2u + x + 2y$  by adding  $u + v + y + x$  to both sides.

**Exercise 42** Assume that  $R$  is an equivalence relation on  $A$  and that  $F : A \times A \rightarrow A$ . Let us say that  $F$  is *compatible* with  $R$  iff, whenever  $xRx'$  and  $yRy'$ , then  $F(\langle x, y \rangle)RF(\langle x', y' \rangle)$ . If  $F$  is compatible with  $R$  then there exists a unique  $\hat{F} : (A/R) \times (A/R) \rightarrow A/R$  such that

$$\hat{F}(\langle [x]_R, [y]_R \rangle) = [F(\langle x, y \rangle)]_R \text{ for all } x, y \in A .$$

If  $F$  is not compatible with  $R$  then no such  $\hat{F}$  exists.

## 3.7 Ordering Relations

**Exercise 43** PROOF:

- $\langle 1 \rangle 1.$   $R^{-1}$  is transitive.
- $\langle 2 \rangle 1.$  ASSUME:  $xR^{-1}y$  and  $yR^{-1}z$
- $\langle 2 \rangle 2.$   $zRy$  and  $yRx$
- $\langle 2 \rangle 3.$   $zRx$
- PROOF: Since  $R$  is transitive.
- $\langle 2 \rangle 4.$   $xR^{-1}z$
- $\langle 1 \rangle 2.$   $R^{-1}$  satisfies trichotomy on  $A$ .
- $\langle 2 \rangle 1.$  LET:  $x, y \in A$
- $\langle 2 \rangle 2.$  Exactly one of  $xRy$ ,  $x = y$ ,  $yRx$  holds.
- $\langle 2 \rangle 3.$  Exactly one of  $yR^{-1}x$ ,  $x = y$ ,  $xR^{-1}y$  holds.

□

**Exercise 44** PROOF:

- $\langle 1 \rangle 1.$   $f$  is one-to-one.
- $\langle 2 \rangle 1.$  LET:  $x, y \in A$  with  $f(x) = f(y)$
- $\langle 2 \rangle 2.$   $f(x) < f(y)$  and  $f(y) < f(x)$  do not hold.
- PROOF: By trichotomy.
- $\langle 2 \rangle 3.$   $x < y$  and  $y < x$  do not hold.
- $\langle 2 \rangle 4.$   $x = y$
- PROOF: By trichotomy.
- $\langle 1 \rangle 2.$  Whenever  $f(x) < f(y)$  then  $x < y$
- $\langle 2 \rangle 1.$  LET:  $x, y \in A$  with  $f(x) < f(y)$
- $\langle 2 \rangle 2.$   $f(x) = f(y)$  and  $f(y) < f(x)$  do not hold.
- PROOF: By trichotomy.
- $\langle 2 \rangle 3.$   $x = y$  and  $y < x$  do not hold.
- $\langle 2 \rangle 4.$   $x < y$
- PROOF: By trichotomy.

□



**Exercise 45** PROOF:

$\langle 1 \rangle 1.$   $<_L$  is transitive.

$\langle 2 \rangle 1.$  LET:  $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$  and  $\langle a_2, b_2 \rangle <_L \langle a_3, b_3 \rangle$

PROVE:  $\langle a_1, b_1 \rangle < \langle a_3, b_3 \rangle$

$\langle 2 \rangle 2.$  CASE:  $a_1 <_A a_2$  and  $a_2 <_A a_3$

PROOF: Then  $a_1 <_A a_3$

$\langle 2 \rangle 3.$  CASE:  $a_1 <_A a_2$ ,  $a_2 = a_3$ ,  $b_2 <_B b_3$

PROOF: Then  $a_1 <_A a_3$

$\langle 2 \rangle 4.$  CASE:  $a_1 = a_2$ ,  $b_1 <_B b_2$  and  $a_2 <_A a_3$

PROOF: Then  $a_1 <_A a_3$

$\langle 2 \rangle 5.$  CASE:  $a_1 = a_2$ ,  $b_1 <_B b_2$ ,  $a_2 = a_3$ ,  $b_2 <_B b_3$

PROOF: Then  $a_1 = a_3$  and  $b_1 <_B b_3$

$\langle 1 \rangle 2.$   $<_L$  satisfies trichotomy on  $A \times B$ .

$\langle 2 \rangle 1.$  LET:  $\langle a_1, b_1 \rangle$  and  $\langle a_2, b_2 \rangle$  be elements of  $A \times B$

$\langle 2 \rangle 2.$  Exactly one of  $a_1 <_A a_2$ ,  $a_1 = a_2$ ,  $a_2 <_A a_1$  holds.

$\langle 2 \rangle 3.$  Exactly one of  $b_1 <_B b_2$ ,  $b_1 = b_2$ ,  $b_2 <_B b_1$  holds.

$\langle 2 \rangle 4.$  Exactly one of  $a_1 <_A a_2$ ,  $(a_1 = a_2 \text{ and } b_1 <_B b_2)$ ,  $(a_1 = a_2 \text{ and } b_1 = b_2)$ ,  $(a_1 = a_2 \text{ and } b_2 <_L b_1)$ ,  $a_2 <_A a_1$  holds.

$\langle 2 \rangle 5.$  Exactly one of  $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$ ,  $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$ ,  $\langle a_2, b_2 \rangle <_L \langle a_1, b_1 \rangle$  holds.

□

### 3.8 Review Exercises

**Exercise 46**

(a)

$$\bigcap \bigcap \langle x, y \rangle = \bigcap \{x\} \\ = x$$

(b)

$$\begin{aligned} \bigcap \bigcap \bigcap \{\langle x, y \rangle\}^{-1} &= \bigcap \bigcap \bigcap \{\langle y, x \rangle\} \\ &= \bigcap \bigcap \langle y, x \rangle \\ &= y \end{aligned} \quad \text{(by part (a))}$$

**Exercise 47**

(a) There are eight:

$$\begin{aligned} &\{\langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle\} \end{aligned}$$

(b) There are six:

$$\begin{aligned} &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 5 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 5 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 5 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 5 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 5 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 5 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\} \end{aligned}$$

#### Exercise 48

(a) The only ordered pair in  $\mathcal{PT}$  is  $\langle \emptyset, \emptyset \rangle = \{\{\emptyset\}\}$ .

(b)

$$\begin{aligned} (\mathcal{PT})^{-1} \circ (\mathcal{PT} \upharpoonright \{\emptyset\}) &= \{\langle \emptyset, \emptyset \rangle\} \circ \{\langle \emptyset, \emptyset \rangle\} \\ &= \{\langle \emptyset, \emptyset \rangle\} \end{aligned}$$

#### Exercise 49

There are six:

$$\begin{aligned} &\{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 1 \rangle, \langle 2, 0 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\} \end{aligned}$$

#### Exercise 50

(a)  $\{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 3 \rangle\}$

$$(b) \quad \{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}$$

**Exercise 51** There are three:

$$\begin{aligned} &\{\langle 1, 0 \rangle, \langle 1, 2 \rangle, \langle 2, 0 \rangle\}, \\ &\{\langle 1, 0 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle\}, \\ &\{\langle 0, 1 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle\} \end{aligned}$$

**Exercise 52** We can conclude this if we know that  $A$  and  $B$  are nonempty, or that  $C$  and  $D$  are nonempty.

Suppose  $A$  and  $B$  are nonempty. Then  $A \times B = C \times D \neq \emptyset$  so  $C$  and  $D$  are nonempty. We now prove  $A \subseteq C$ .

Let  $a \in A$ . Pick some  $b \in B$ . Then  $\langle a, b \rangle \in A \times B = C \times D$  and so  $a \in C$ .

We can similarly prove  $C \subseteq A$ ,  $B \subseteq D$  and  $D \subseteq B$ .

**Exercise 53**

$$\begin{aligned} x(R \cup S)^{-1}y &\Leftrightarrow y(R \cup S)x \\ &\Leftrightarrow yRx \text{ or } ySx \\ &\Leftrightarrow xR^{-1}y \text{ or } xS^{-1}y \\ &\Leftrightarrow x(R^{-1} \cup S^{-1})y \\ x(R \cap S)^{-1}y &\Leftrightarrow y(R \cap S)x \\ &\Leftrightarrow yRx \text{ and } ySx \\ &\Leftrightarrow xR^{-1}y \text{ and } xS^{-1}y \\ &\Leftrightarrow x(R^{-1} \cap S^{-1})y \\ x(R - S)^{-1}y &\Leftrightarrow y(R - S)x \\ &\Leftrightarrow yRx \text{ and } \neg ySx \\ &\Leftrightarrow xR^{-1}y \text{ and } \neg xS^{-1}y \\ &\Leftrightarrow x(R^{-1} - S^{-1})y \end{aligned}$$

**Exercise 54**

(a)

$$\begin{aligned} \langle x, y \rangle \in A \times (B \cap C) &\Leftrightarrow x \in A \text{ \& } y \in B \text{ \& } y \in C \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) \cap (A \times C) \end{aligned}$$

(b)

$$\begin{aligned} \langle x, y \rangle \in A \times (B \cup C) &\Leftrightarrow x \in A \text{ \& } (y \in B \text{ or } y \in C) \\ &\Leftrightarrow (x \in A \text{ \& } y \in B) \text{ or } (x \in A \text{ \& } y \in C) \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) \cup (A \times C) \end{aligned}$$

(c)

$$\begin{aligned}\langle x, y \rangle \in A \times (B - C) &\Leftrightarrow x \in A \ \& \ y \in B \ \& \ y \notin C \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) - (A \times C)\end{aligned}$$

**Exercise 55**

(a) No. Take  $A = \{0\}$ ,  $B = \{1\}$ ,  $C = \{2\}$ . Then  $(A \times A) \cup (B \times C) = \{\langle 0, 0 \rangle, \langle 1, 2 \rangle\}$  while  $(A \cup B) \times (A \cup C) = \{\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 2 \rangle\}$ .

(b) Yes.

$$\begin{aligned}\langle x, y \rangle \in (A \times A) \cap (B \times C) &\Leftrightarrow x \in A \ \& \ y \in A \ \& \ x \in B \ \& \ y \in C \\ &\Leftrightarrow \langle x, y \rangle \in (A \cap B) \times (A \cap C)\end{aligned}$$

**Exercise 56**

(a) Yes.

$$\begin{aligned}x \in \text{dom}(R \cup S) &\Leftrightarrow \exists y(xRy \text{ or } xSy) \\ &\Leftrightarrow \exists y.xRy \text{ or } \exists y.xSy \\ &\Leftrightarrow x \in \text{dom } R \cup \text{dom } S\end{aligned}$$

(b) No. Take  $R = \{\langle 0, 0 \rangle\}$  and  $S = \{\langle 0, 1 \rangle\}$ . Then  $\text{dom}(R \cap S) = \text{dom } \emptyset = \emptyset$  while  $\text{dom } R \cap \text{dom } S = \{0\} \cap \{0\} = \{0\}$ .

**Exercise 57**

(a) Yes.

$$\begin{aligned}x(R \circ (S \cup T))y &\Leftrightarrow \exists z(x(S \cup T)z \ \& \ zRy) \\ &\Leftrightarrow \exists z(xSz \ \& \ zRy) \text{ or } \exists z(xTz \ \& \ zRy) \\ &\Leftrightarrow x((R \circ S) \cup (R \circ T))y\end{aligned}$$

(b) No. Take  $R = \{\langle 0, 0 \rangle, \langle 1, 0 \rangle\}$ ,  $S = \{\langle 0, 0 \rangle\}$  and  $T = \{\langle 0, 1 \rangle\}$ . Then

$$\begin{aligned}R \circ (S \cap T) &= R \circ \emptyset \\ &= \emptyset \\ (R \circ S) \cap (R \circ T) &= \{\langle 0, 0 \rangle\} \cap \{\langle 0, 0 \rangle\} \\ &= \{\langle 0, 0 \rangle\}\end{aligned}$$

**Exercise 58** Take  $F = \emptyset$  and  $S = \{\emptyset\}$ . Then  $F[F^{-1}[[S]]] = \emptyset \neq S$ .

**Exercise 59**

$$\begin{aligned}
x(Q \upharpoonright (A \cap B))y &\Leftrightarrow xQy \ \& \ x \in A \ \& \ x \in B \\
&\Leftrightarrow x((Q \upharpoonright A) \cap (Q \upharpoonright B))y \\
x(Q \upharpoonright (A - B))y &\Leftrightarrow xQy \ \& \ x \in A \ \& \ x \notin B \\
&\Leftrightarrow (xQy \ \& \ x \in A) \ \& \ \neg(xQy \ \& \ x \in B) \\
&\Leftrightarrow x((Q \upharpoonright A) - (Q \upharpoonright B))y
\end{aligned}$$

**Exercise 60**

$$\begin{aligned}
x((R \circ S) \upharpoonright A)y &\Leftrightarrow \exists z(xRz \ \& \ zSy \ \& \ x \in A) \\
&\Leftrightarrow x(R \circ (S \upharpoonright A))y
\end{aligned}$$

## Chapter 4

# Chapter 4 — Natural Numbers

### 4.1 Inductive Sets

**Exercise 1** We have

$$3 = 2 \cup \{2\} = 1 \cup \{1, 2\}$$

and so  $1 \in 3$ . But  $1 \notin 1$  (since  $1 = \{\emptyset\}$  and we know  $\{\emptyset\} \neq \emptyset$  hence  $\{\emptyset\} \notin \{\emptyset\}$ ). Therefore  $1 \neq 3$ .

### 4.2 Peano's Postulates

**Exercise 2** If  $a$  is a transitive set then

$$\begin{aligned} \bigcup (a^+) &= a && \text{(Theorem 4E)} \\ &\subseteq a^+ \end{aligned}$$

**Exercise 3**

(a) Suppose  $a$  is a transitive set. Then  $a \subseteq \mathcal{P}a$ . Hence we have  $\bigcup \mathcal{P}a = a \subseteq \mathcal{P}a$  and so  $\mathcal{P}a$ .

(b) Suppose  $\mathcal{P}a$  is a transitive set. Then  $a = \bigcup \mathcal{P}a \subseteq \mathcal{P}a$  hence  $a$  is transitive.

**Exercise 4** If  $a$  is a transitive set then  $\bigcup a \subseteq a$  so  $\bigcup \bigcup a \subseteq \bigcup a$ . Hence  $\bigcup a$  is transitive.

**Exercise 5**

(a) PROOF:  
 $\langle 1 \rangle 1.$  LET:  $b \in \bigcup \mathcal{A}$   
 $\langle 1 \rangle 2.$  PICK  $A \in \mathcal{A}$  such that  $b \in A$   
 $\langle 1 \rangle 3.$   $b \subseteq A$   
PROOF: Since  $A$  is transitive.  
 $\langle 1 \rangle 4.$   $b \subseteq \bigcup \mathcal{A}$   
 $\square$

(b) PROOF:  
 $\langle 1 \rangle 1.$  LET:  $b \in \bigcap \mathcal{A}$   
 $\langle 1 \rangle 2.$  For all  $A \in \mathcal{A}$  we have  $b \subseteq A$   
PROOF: Since  $b \in A$  and  $A$  is transitive.  
 $\langle 1 \rangle 3.$   $b \subseteq \bigcap \mathcal{A}$   
 $\square$

**Exercise 6** We have  $\bigcup(a^+) = \bigcup a \cup a$  (see the proof of Theorem 4E). So if  $\bigcup(a^+) = a$  we have  $\bigcup a \cup a = a$  and so  $\bigcup a \subseteq a$ .

### 4.3 Recursion on $\omega$

**Exercise 7** We have  $h_1(0) = h_2(0) = a$  so  $0 \in S$ .  
Now let  $n \in S$ ; we prove  $n^+ \in S$ . We have  $h_1(n) = h_2(n)$  and therefore

$$\begin{aligned} h_1(n^+) &= F(h_1(n)) \\ &= F(h_2(n)) \\ &= h_2(n^+) \end{aligned}$$

**Exercise 8** PROOF:  
 $\langle 2 \rangle 1.$   $\forall m, n \in \omega. h(n) = h(m) \Rightarrow n = m$   
 $\langle 2 \rangle 1.$   $\forall n \in \omega. h(n) = h(0) \Rightarrow n = 0$   
 $\langle 3 \rangle 1.$  LET:  $n \in \omega$   
 $\langle 3 \rangle 2.$  ASSUME:  $h(n) = h(0)$   
 $\langle 3 \rangle 3.$   $h(n) = c$   
 $\langle 3 \rangle 4.$   $\forall p \in \omega. n \neq p^+$   
PROOF: Otherwise  $f(h(p)) = c$  contradicting the fact that  $c \in A - \text{ran } f$ .  
 $\langle 3 \rangle 5.$   $n = 0$   
PROOF: Theorem 4C.  
 $\langle 2 \rangle 2.$  For all  $m \in \omega$ , if  $\forall n \in \omega. h(n) = h(m) \Rightarrow n = m$ , then  $\forall n \in \omega. h(n) = h(m^+) \Rightarrow n = m^+$   
 $\langle 3 \rangle 1.$  LET:  $m \in \omega$   
 $\langle 3 \rangle 2.$  ASSUME:  $\forall n \in \omega. h(n) = h(m) \Rightarrow n = m$   
 $\langle 3 \rangle 3.$  LET:  $n \in \omega$   
 $\langle 3 \rangle 4.$  ASSUME:  $h(n) = h(m^+)$   
 $\langle 3 \rangle 5.$   $h(n) = f(h(m))$

$\langle 3 \rangle 6. n \neq 0$

PROOF: Otherwise  $c = f(h(m))$  contradicting the fact that  $c \in A - \text{ran } f$ .

$\langle 3 \rangle 7. \text{ PICK } p \text{ such that } n = p^+$

$\langle 3 \rangle 8. f(h(p)) = f(h(m))$

$\langle 3 \rangle 9. h(p) = h(m)$

PROOF:  $f$  is one-to-one.

$\langle 3 \rangle 10. p = m$

PROOF: By  $\langle 3 \rangle 2$ .

$\langle 3 \rangle 11. n = p^+ = m^+$

□

**Exercise 9** PROOF:

$\langle 1 \rangle 1. C^* \subseteq C_*$

$\langle 2 \rangle 1. f[C_*] \subseteq C_*$

$\langle 3 \rangle 1. \text{ LET: } x \in C_*$

PROVE:  $f(x) \in C_*$

$\langle 3 \rangle 2. \text{ PICK } n \text{ such that } x \in h(n)$

$\langle 3 \rangle 3. f(x) \in h(n^+)$

$\langle 3 \rangle 4. f(x) \in C_*$

$\langle 1 \rangle 2. C_* \subseteq C^*$

$\langle 2 \rangle 1. \forall n \in \omega. h(n) \subseteq C^*$

$\langle 3 \rangle 1. h(0) \subseteq C^*$

PROOF: If  $A \subseteq X \subseteq B$  and  $f[X] \subseteq X$  then  $A \subseteq X$ .

$\langle 3 \rangle 2. \forall n \in \omega. h(n) \subseteq C^* \Rightarrow h(n^+) \subseteq C^*$

$\langle 4 \rangle 1. \text{ LET: } n \in \omega$

$\langle 4 \rangle 2. \text{ ASSUME: } h(n) \subseteq C^*$

$\langle 4 \rangle 3. f[h(n)] \subseteq C^*$

$\langle 5 \rangle 1. \text{ LET: } X \text{ be such that } A \subseteq X \subseteq B \text{ and } f[X] \subseteq X$

PROVE:  $f[h(n)] \subseteq X$

$\langle 5 \rangle 2. h(n) \subseteq X$

$\langle 5 \rangle 3. f[h(n)] \subseteq f[X]$

$\langle 5 \rangle 4. f[h(n)] \subseteq X$

$\langle 4 \rangle 4. h(n^+) \subseteq C^*$

□

**Exercise 10**  $C^* = C_* = (0, 1]$

**Exercise 11**  $\{n \in \mathbb{Z} \mid n \leq 0\}$

**Exercise 12** Let  $f : B \times B \rightarrow B$  and  $A \subseteq B$ . Let

$$C^* = \bigcap \{X \mid A \subseteq X \subseteq B \text{ \& } f[X \times X] \subseteq X\} .$$



Define the function  $h : \omega \rightarrow \mathcal{P}B$  by

$$\begin{aligned} h(0) &= A \\ h(n^+) &= h(n) \cup f[h(n) \times h(n)] \end{aligned} \quad (n \in \omega)$$

Define  $C_* = \bigcup \text{ran } h$ . Then  $C^* = C_*$ .

## 4.4 Arithmetic

**Exercise 13** We prove the contrapositive. Assume  $m \neq 0$  and  $n \neq 0$ . Then by Theorem 4C there are natural numbers  $p, q$  such that  $m = p^+$  and  $n = q^+$ . Hence  $mn = p^+q^+ = (p^+q + p)^+ \neq 0$ .

**Exercise 14** We prove the following facts for any natural number  $n$ :

1.  $n$  is even if and only if  $n^+$  is odd.

PROOF: If  $n$  is even, say  $n = 2p$ , then  $n^+ = 2p + 1$  is odd.

If  $n^+$  is odd, say  $n^+ = 2p + 1$ , then  $n = 2p$  is even.

2.  $n$  is odd if and only if  $n^+$  is even.

PROOF: If  $n$  is odd, say  $n = 2p + 1$ , then  $n^+ = 2(p + 1)$  is even.

If  $n^+$  is even, say  $n^+ = 2p$ , then we cannot have  $p = 0$  (since  $n^+ \neq 0$ ). So  $p = q + 1$  for some  $q$ . But then  $n^+ = 2q + 2$  so  $n = 2q + 1$  and  $n$  is odd.

Now, 0 is even and 0 is not odd. By the two facts above, if  $n$  is either even or odd but not both, then  $n^+$  is either odd or even but not both. The result follows by induction.

**Exercise 15** We have

$$\begin{aligned} m + (n + 0) &= m + n && \text{by (A1)} \\ &= (m + n) + 0 && \text{by (A1)} \end{aligned}$$

If  $m + (n + p) = (m + n) + p$  then

$$\begin{aligned} m + (n + p^+) &= m + (n + p)^+ && \text{by (A2)} \\ &= (m + (n + p))^+ && \text{by (A2)} \\ &= ((m + n) + p)^+ && \text{by induction hypothesis} \\ &= (m + n) + p^+ && \text{by (A2)} \end{aligned}$$

**Exercise 16** We first prove that  $0 \cdot n = 0$  for all  $n$ . We have  $0 \cdot 0 = 0$  by (M1), and if  $0 \cdot n = 0$  then

$$\begin{aligned} 0 \cdot n^+ &= 0 \cdot n + 0 && \text{by (M2)} \\ &= 0 \cdot n && \text{by (A1)} \\ &= 0 && \text{by induction hypothesis} \end{aligned}$$

Now we prove that  $m^+ \cdot n = m \cdot n + n$  for all  $m, n$ . We have

$$\begin{aligned} m^+ \cdot 0 &= 0 && \text{by (M1)} \\ m \cdot 0 + 0 &= m \cdot 0 && \text{by (A1)} \\ &= 0 && \text{by (M1)} \end{aligned}$$

Thus,  $m^+ \cdot 0 = m \cdot 0 + 0$ .

If  $m^+ \cdot n = m \cdot n + n$  then

$$\begin{aligned} m^+ \cdot n^+ &= m^+ \cdot n + m^+ && \text{by (M2)} \\ &= (m^+ \cdot n + m)^+ && \text{by (A2)} \\ &= ((m \cdot n + n) + m)^+ && \text{by induction hypothesis} \\ &= ((m \cdot n + m) + n)^+ && \text{by associativity and commutativity of addition} \\ &= (m \cdot n^+ + n)^+ && \text{by (M2)} \\ &= m \cdot n^+ + n^+ && \text{by (A2)} \end{aligned}$$

**Exercise 17** The proof is by induction on  $p$ . We have

$$\begin{aligned} m^{n+0} &= m^n && \text{by (A1)} \\ &= 0 + m^n && \text{by Theorem 4K(2)} \\ &= m^n \cdot 0 + m^n && \text{by (M1)} \\ &= m^n \cdot 1 && \text{by (M2)} \\ &= m^n \cdot m^0 && \text{by (E1)} \end{aligned}$$

If  $m^{n+p} = m^n \cdot m^p$  then

$$\begin{aligned} m^{n+p^+} &= m^{(n+p)^+} && \text{by (A2)} \\ &= m^{n+p} m && \text{by (E2)} \\ &= (m^n m^p) m && \text{by induction hypothesis} \\ &= m^n (m^p m) && \text{by Theorem 4K (4)} \\ &= m^n m^{p^+} && \text{by (E2)} \end{aligned}$$

## 4.5 Ordering on $\omega$

**Exercise 18**

$$\begin{aligned} \in_{\omega}^{-1} \llbracket \{7, 8\} \rrbracket &= \{x \in \omega \mid x \in 7 \text{ or } x \in 8\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7\} \end{aligned}$$

**Exercise 19** The proof is by induction on  $m$ .

For  $m = 0$ , take  $q = r = 0$ . Then  $m = d \cdot 0 + 0$  and  $0 \in d$ .

Suppose  $m = dq + r$  and  $r < d$ . Then  $r + 1 \leq d$ . If  $r + 1 < d$ , then we have  $m + 1 = dq + (r + 1)$  as required. If  $r + 1 = d$ , then we have  $m + 1 = dq + d = d(q + 1) + 0$ .

**Exercise 20** We first prove  $A$  is closed downwards; that is, if  $n \in A$  and  $m \in n$  then  $m \in A$ . This holds because if  $n \in A$  and  $m \in n$  then  $m \in \bigcup A$  and  $\bigcup A = A$ .

Now, we prove  $\forall n \in \omega. n \in A$  by induction on  $n$ .

To prove  $0 \in A$ : we are given that  $A$  is nonempty. Pick some  $a \in A$ . Then  $0 \in a$  so  $0 \in A$  since  $A$  is closed downwards.

Now let  $n \in A$ ; we prove  $n^+ \in A$ . We have  $n \in \bigcup A$ ; pick some  $k \in A$  such that  $n \in k$ . Then  $n^+ \in k$  so  $n^+ \in A$  since  $A$  is closed downwards.

This completes the induction. We have  $\forall n \in \omega. n \in A$ , i.e.  $A = \omega$ .

**Exercise 21** Suppose  $n$  is a natural number,  $k \in n$  and  $n \subseteq k$ . Then  $k \in k$ , contradicting Lemma 4L(b).

**Exercise 22** We have  $0 \in p^+$  (by trichotomy since  $p^+ \not\subseteq 0$  because  $0$  is empty, and  $p^+ \not\neq 0$  by Peano's First Postulate.) Hence  $n = n + 0 \in n + p^+$  by Theorem 4N.

**Exercise 23** The proof is by induction on  $n$ . The statement is vacuously true for  $n = 0$ .

Suppose the statement is true for  $n$ . Let  $m \in n^+$ . Then  $m \subseteq n$ .

If  $m = n$ , then we have  $m + 0^+ = n^+$ .

If  $m \in n$ , pick  $p$  such that  $m + p^+ = n$  by the induction hypothesis. Then  $m + p^{++} = n^+$ .

**Exercise 24** Suppose  $m \in p$ . Then we cannot have  $n \in q$  or  $n = q$ , as either of these would imply  $m + n \in p + q$ . Hence  $q \in n$  by trichotomy.

We prove  $q \in n \Rightarrow m \in p$  similarly.

**Exercise 25** By Exercise 23, pick natural numbers  $a$  and  $b$  such that  $m = n + a^+$  and  $p = q + b^+$ . Then

$$\begin{aligned} mp + nq &= (n + a^+)(q + b^+) + nq \\ &= nq + nq + a^+q + nb^+ + a^+b^+ \\ &= (n + a^+)q + n(q + b^+) + a^+b^+ \\ &= mq + np + (a^+ + b^+)^+ \end{aligned}$$

Hence  $mq + np \in mp + nq$  by Exercise 22.

**Exercise 26** The proof is by induction on  $n$ .

If  $n = 0$  then  $\text{ran } f$  is a singleton and its sole element is the largest element.

Suppose the result is true for  $n$ . Let  $f : n^{++} \rightarrow A$ . Then  $f \llbracket n^+ \rrbracket$  has a largest element  $f(k)$ , say. If  $f(k) \subseteq f(n^+)$  then  $f(n^+)$  is greatest in  $\text{ran } f$ ; otherwise  $f(k)$  is greatest.

**Exercise 27** We prove  $f_1(n) = f_2(n)$  for all  $n \in \omega$  by strong induction on  $n$ . Assume that  $(\forall m \in n) f_1(m) = f_2(m)$ . Then  $f_1 \upharpoonright n = f_2 \upharpoonright n$ . So

$$\begin{aligned} f_1(n) &= G(f_1 \upharpoonright n) \\ &= G(f_2 \upharpoonright n) \\ &= f_2(n) \end{aligned}$$

**Exercise 28** Suppose  $\omega$  is not transitive. Then there exists a natural number  $n$  such that  $n \not\subseteq \omega$ . Let  $n$  be the least such number. There exists  $x \in n$  such that  $x \not\subseteq \omega$ . Now,  $n \neq 0$  (because it is nonempty) so  $n = p^+$  for some natural number  $p$ . We have  $x \in p^+$  so  $x \in p$  or  $x = p$ . We cannot have  $x = p$  (because  $x$  is not a natural number) so we have  $x \in p$ . But this contradicts the minimality of  $n$ .

## 4.6 Review Exercises

**Exercise 29**  $4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

**Exercise 30**  $\bigcup 4 = 0 \cup 1 \cup 2 \cup 3 = 3$  since 0, 1 and 2 are all subsets of 3.  
 $\bigcap 4 = 0 \cap 1 \cap 2 \cap 3 = 0 (= \emptyset)$ .

**Exercise 31** Similarly to Exercise 30 we have  $\bigcup \bigcup 7 = \bigcup 6 = 5$ .

**Exercise 32**

$$\begin{aligned} \text{(a)} \quad A^+ &= A \cup \{A\} = \{1, A\} = \{1, \{1\}\} \\ \text{So } \bigcup A^+ &= 1 \cup \{1\} = \{0, 1\} = 2 \end{aligned}$$

$$\text{(b)} \quad \bigcup (\{2\}^+) = \bigcup \{2, \{2\}\} = \{0, 1, 2\} = 3$$

**Exercise 33**

**(a)** Yes - if  $x \in y \in \{0, 1, \{1\}\}$  then  $x$  is either 0 or 1, and in either case  $x \in \{0, 1, \{1\}\}$

**(b)** No -  $0 \in 1 \in \{1\}$  but  $0 \notin \{1\}$

**(c)** No -  $0 \in \{0\} \in \langle 0, 1 \rangle$  but  $0 \notin \langle 0, 1 \rangle$ .

**Exercise 34**

**(a)** Let  $a = \{\emptyset\}$  and  $b = \emptyset$

(b) Let  $c = \{\{\emptyset\}\}$ ,  $d = \{\emptyset\}$  and  $e = \emptyset$

**Exercise 35**

(a) Let  $T_1 = \{\{1\}, \{1, 0\}, 0, 1\}$

(b) Let  $T_2 = \{\langle 1, 0 \rangle, \{1\}, \{1, 0\}, 0, 1\}$ .

**Exercise 36**

$$\begin{aligned} h(4) &= 2h(3) \\ &= 4h(2) \\ &= 8h(1) \\ &= 16h(0) \\ &= 48 \end{aligned}$$

**Exercise 37**

(a) Let  $f : m \rightarrow A$  and  $g : n \rightarrow B$  be bijections. Define  $h : m + n \rightarrow A \cup B$  by

$$\begin{aligned} h(p) &= f(p) && \text{if } p \in m \\ h(m + q) &= g(q) && \text{if } q \in n \end{aligned}$$

To show that this is well-defined, we must prove two things:

1. For all  $p \in m + n$ , then either  $p \in m$  or there exists  $q \in n$  such that  $p = m + q$ .
2. We never have  $p \in m$  and  $p = m + q$  for some  $q \in n$ .

We prove 1 by induction on  $n$ . For all  $p \in m + 0$  we have  $p \in m$ , so the result holds for  $n = 0$ .

Now, suppose the result holds for  $n$ . Let  $p \in m + n^+ = (m + n)^+$  so  $p \in m + n$ . If  $p \in m + n$ , we simply apply the induction hypothesis. If  $p = m + n$  then  $p = m + q$  where  $q = n \in n^+$ .

To prove 2, if  $p = m + q$  then  $m = m + 0$  and  $m + q = p$  by Theorem 4N, hence  $p \notin m$  by trichotomy.

It remains to show that  $h$  is a bijection.

To prove  $h$  is injective, we consider three cases. If  $h(p) = h(p')$  where  $p, p' \in m$ , then  $f(p) = f(p')$  so  $p = p'$ . If  $h(m + q) = h(m + q')$  where  $q, q' \in n$ , then  $g(q) = g(q')$  so  $q = q'$ . And we cannot have  $h(p) = h(m + q)$  for  $p \in m$  and  $q \in n$  since  $h(p) \in A$ ,  $h(m + q) \in B$ , and  $A \cap B = \emptyset$ .

To prove  $h$  is surjective, let  $x \in A \cup B$ . If  $x \in A$ , there is some  $p \in m$  with  $f(p) = x$ , so  $h(p) = x$ . If  $x \in B$ , there is some  $q \in n$  with  $g(q) = x$ , so  $h(m + q) = x$ .

(b) Let  $f : m \rightarrow A$  and  $g : n \rightarrow B$  be bijections.

We first show that, for any  $p \in mn$ , there exist unique  $i \in m$  and  $j \in n$  such that  $p = mj + i$ .

By Exercise 19, there exist  $j$  and  $i \in m$  such that  $p = mj + i$ . We have  $j \in n$  since otherwise  $p = mj + i \in mn$ .

For uniqueness, suppose  $mj + i = mj' + i'$  where  $i, i' \in m$  and  $j, j' \in n$ . Then we have

$$mj \in mj + i = mj' + i' \in mj' + m = m(j')^+$$

so  $j \in (j')^+$  and  $j \in j'$ . Similarly  $j' \in j$ , and so  $j = j'$ . Therefore  $i = i'$  by the cancellation law for addition.

Now define  $h : mn \rightarrow A \times B$  by

$$h(mj + i) = \langle f(i), g(j) \rangle$$

where  $i \in m$  and  $j \in n$ . It is easy to check that  $h$  is bijective.

**Exercise 38**  $h(n) = 3n + 1$

**Exercise 39**  $h(n) = n^2$

**Exercise 40**  $h(n^+) = h(n) + 5$

## Chapter 5

# Chapter 5 — Construction of the Real Numbers

### 5.1 Integers

**Exercise 1** No, because  $[\langle 0, 0 \rangle] = [\langle 1, 1 \rangle]$  but  $[\langle 0, 0 \rangle] \neq [\langle 2, 1 \rangle]$ .

**Exercise 2** Yes, because if  $[\langle m, n \rangle] = [\langle p, q \rangle]$  then  $[\langle m, m \rangle] = [\langle p, p \rangle]$  because  $m + p = m + p$ .

**Exercise 3** Yes, because if  $[\langle m, n \rangle] = [\langle p, q \rangle]$  then  $[\langle n, m \rangle] = [\langle q, p \rangle]$  because  $n + p = m + q$ .

**Exercise 4** Let  $a = [\langle m, n \rangle]$ ,  $b = [\langle p, q \rangle]$  and  $c = [\langle r, s \rangle]$ . Then

$$\begin{aligned} a +_Z (b +_Z c) &= [\langle m, n \rangle] +_Z [\langle p + r, q + s \rangle] \\ &= [\langle m + (p + r), n + (q + s) \rangle] \\ &= [\langle (m + p) + r, (n + q) + s \rangle] \\ &= [\langle m + p, n + q \rangle] +_Z [\langle r, s \rangle] \\ &= (a +_Z b) +_Z c \end{aligned}$$

**Exercise 5**

$$[\langle m, n \rangle] - [\langle p, q \rangle] = [\langle m, n \rangle] + [\langle q, p \rangle] = [\langle m + q, n + p \rangle]$$

**Exercise 6** Let  $a = [\langle m, n \rangle]$ . Then

$$\begin{aligned} a \cdot_Z 0_Z &= [\langle m, n \rangle] \cdot_Z [\langle 0, 0 \rangle] \\ &= [\langle m0 + n0, m0 + n0 \rangle] \\ &= [\langle 0, 0 \rangle] \\ &= 0_Z \end{aligned}$$

**Exercise 7** We have  $a \cdot_Z b +_Z a \cdot_Z (-b) = a \cdot_Z (b +_Z (-b)) = a \cdot_Z 0_Z = 0_Z$ , hence  $a \cdot_Z (-b) = -(a \cdot_Z b)$  by the uniqueness of inverses.

We prove  $(-a) \cdot_Z b = -(a \cdot_Z b)$  similarly.

**Exercise 8**

(a) This says  $[\langle m + n, 0 \rangle] = [\langle m, 0 \rangle] +_Z [\langle n, 0 \rangle]$ , which is true from the definition of  $+_Z$ .

(b) We have

$$\begin{aligned} E(m) \cdot_Z E(n) &= [\langle m, 0 \rangle] \cdot_Z [\langle n, 0 \rangle] \\ &= [\langle mn + 0 \cdot 0, m0 + n0 \rangle] \\ &= E(mn) \end{aligned}$$

(c)

$$\begin{aligned} E(m) <_Z E(n) &\Leftrightarrow [\langle m, 0 \rangle] <_Z [\langle n, 0 \rangle] \\ &\Leftrightarrow m + 0 \in n + 0 \\ &\Leftrightarrow m \in n \end{aligned}$$

**Exercise 9**

$$\begin{aligned} E(m) - E(n) &= [\langle m, 0 \rangle] - [\langle n, 0 \rangle] \\ &= [\langle m, n \rangle] \end{aligned}$$

by Exercise 5.

## 5.2 Rational Numbers

**Exercise 10** Let  $r = [\langle a, b \rangle]$ . Then

$$\begin{aligned} r \cdot_Q 0_Q &= [\langle a, b \rangle] \cdot_Q [\langle 0, 1 \rangle] \\ &= [\langle a \cdot_Z 0, b \cdot_Z 1 \rangle] \\ &= [\langle 0, b \rangle] \\ &= [\langle 0, 1 \rangle] \end{aligned}$$

since  $\langle 0, b \rangle \sim \langle 0, 1 \rangle$  because  $0 \cdot_Z 1 = 0 \cdot_Z b = 0$ .



**Exercise 11** Let  $r = [\langle a, b \rangle]$  and  $s = [\langle c, d \rangle]$ . Suppose  $r \cdot_Q s = 0_Q$ . Then

$$[\langle ac, bd \rangle] = [\langle 0, 1 \rangle]$$

that is,  $ac = 0$ . Hence  $a = 0$  or  $c = 0$ , which means  $r = 0_Q$  or  $s = 0_Q$ .

**Exercise 12** This follows from Theorem 5QJ(a) with  $s = 0_Q$  and  $t = -r$ .

**Exercise 13** Let  $a, b, c \in \mathbb{Z}$ . If  $a +_Z c = b +_Z c$  then

$$\begin{aligned} a +_Z c +_Z (-c) &= b +_Z c +_Z (-c) \\ \therefore a +_Z 0 &= b +_Z 0 && \text{(Theorem 5ZD(b))} \\ \therefore a &= b && \text{(Theorem 5ZD(a))} \end{aligned}$$

**Exercise 14** Suppose  $p <_Q s$ . Let  $r = (p +_Q s)/2$ . Then

$$\begin{aligned} p &<_Q s \\ \therefore 2p &<_Q p +_Q s \\ \therefore p &<_Q (p +_Q s)/2 \\ &= r \\ p &<_Q s \\ \therefore p +_Q s &<_Q 2s \\ \therefore (p +_Q s)/2 &<_Q s \\ \therefore r &<_Q s \end{aligned}$$

## 5.3 Real Numbers

**Exercise 15** PROOF:

$\langle 1 \rangle 1$ .  $\bigcup A$  is closed downwards.

$\langle 2 \rangle 1$ . LET:  $q \in \bigcup A$  and  $p < q$

$\langle 2 \rangle 2$ . PICK  $x \in A$  such that  $q \in x$

$\langle 2 \rangle 3$ .  $p \in x$

PROOF: Since  $x$  is closed downwards.

$\langle 2 \rangle 4$ .  $p \in \bigcup A$

$\langle 1 \rangle 2$ .  $\bigcup A$  has no largest element.

$\langle 2 \rangle 1$ . LET:  $q \in \bigcup A$

$\langle 2 \rangle 2$ . PICK  $x \in A$  such that  $q \in x$

$\langle 2 \rangle 3$ . PICK  $r \in x$  such that  $q < r$

PROOF: Since  $x$  has no largest element.

$\langle 2 \rangle 4$ .  $r \in \bigcup A$

□

**Exercise 16** PROOF:

- ⟨1⟩1. LET:  $q \in x +_R y$
- ⟨1⟩2. PICK rationals  $a \in x$  and  $b \in y$  such that  $q = a + b$
- ⟨1⟩3. PICK  $a' \in x$  and  $b' \in y$  such that  $a < a'$  and  $b < b'$

PROOF: Since  $x$  and  $y$  each have no largest element.

- ⟨1⟩4.  $q < a' + b' \in x +_R y$

□

**Exercise 17** If  $b < 0$  we can take  $k = 0$ . If  $b \geq 0$  then there is a natural number  $n$  such that  $b = E(n)$ ; take  $k = n^+$ . Then  $b < ak$  since  $1 \leq a$  and  $b < k$ .

**Exercise 18** Let  $p = [\langle a, b \rangle]$  and  $r = [\langle c, d \rangle]$  where  $a, b$  and  $d$  are positive. By Exercise 17, there exists a natural number  $k$  such that  $bc < adE(k)$ . Therefore  $r < p \cdot E(E(k))$ .

**Exercise 19** Pick a rational  $a \in x$  (which we can do since  $x \neq \emptyset$ ). We first prove that there exists a natural number  $k$  such that  $a + kp \notin x$ .

Pick a rational  $b \notin x$  (which we can do since  $x \neq \mathbb{Q}$ ). We have  $a < b$  (since  $x$  is closed downwards). By Exercise 18, there exists a natural number  $k$  such that

$$\begin{aligned} b - a &< kp \\ \therefore a + kp &> b \\ \therefore a + kp &\notin x \end{aligned}$$

Now, let  $k$  be the least natural number such that  $a + kp \notin x$  (by the Well-Ordering Principle). We have  $k \neq 0$  (since  $a \in x$ ); let  $k = n^+$ . Then we have

$$a + np \in x \quad a + np + p \notin x$$

Take  $q = a + np$ .

**Exercise 20** We must prove  $0 \subseteq x \cup -x$ . Let  $q \in 0$  and assume  $q \notin x$ . Then  $q < 0$  and  $-0 = 0 \notin x$ , so  $q \in -x$ .

**Exercise 21** PROOF:

- ⟨1⟩1. LET:  $x, y$  be real numbers with  $x < y$
- ⟨1⟩2. PICK  $r \in y$  such that  $r \notin x$
- ⟨1⟩3. PICK  $s \in y$  such that  $r < s$

PROVE:  $x < E(s) < y$

- ⟨1⟩4.  $x \subseteq E(s)$

PROOF: If  $p \in x$  then  $p < r < s$

- ⟨1⟩5.  $x \neq E(s)$

PROOF: Since  $r \in E(s)$  and  $r \notin x$

- ⟨1⟩6.  $E(s) \subseteq y$

PROOF: Since  $y$  is closed downwards.

$\langle 1 \rangle 7.$   $E(s) \neq y$

PROOF: Since  $s \in y$  but  $s \notin E(s)$ .

**Exercise 22**  $|x|$  is either  $x$  or  $-x$ , and they are both real numbers.

## Chapter 6

# Chapter 6 — Cardinal Numbers and the Axiom of Choice

### 6.1 Equinumerosity

**Exercise 1** PROOF:

- ⟨1⟩1.  $f$  is injective.
  - ⟨2⟩1. ASSUME:  $f(m, n) = f(m', n')$
  - ⟨2⟩2.  $2^m(2n + 1) = 2^{m'}(2n' + 1)$
  - ⟨2⟩3.  $m = m'$ 
    - ⟨3⟩1. ASSUME: w.l.o.g.  $m \leq m'$
    - ⟨3⟩2.  $2n + 1 = 2^{m'-m}(2n' + 1)$ 
      - PROOF: From ⟨2⟩2 dividing by  $2^m$ .
    - ⟨3⟩3.  $m' - m = 0$ 
      - PROOF: Since  $2^{m'-m}(2n' + 1)$  is odd.
  - ⟨2⟩4.  $2n + 1 = 2n' + 1$
  - ⟨2⟩5.  $n = n'$
- ⟨1⟩2.  $f$  is surjective.
  - ⟨2⟩1. LET:  $n \in \omega$ 
    - ASSUME:  $\forall m < n. m \in \text{ran } f$
    - PROVE:  $n \in \text{ran } f$
  - ⟨2⟩2. CASE:  $n$  is even
    - ⟨3⟩1. LET:  $k$  be such that  $n = 2k$
    - ⟨3⟩2.  $n = f(0, k)$
  - ⟨2⟩3. CASE:  $n$  is odd
    - ⟨3⟩1. LET:  $k$  be such that  $n = 2k + 1$
    - ⟨3⟩2. LET:  $k = f(i, j)$
    - ⟨3⟩3.  $n = f(i + 1, j)$

PROOF:

$$\begin{aligned}
 n &= 2k + 1 \\
 &= 2(2^i(2j + 1) - 1) + 1 \\
 &= 2^{i+1}(2j + 1) - 2 + 1 \\
 &= 2^{i+1}(2j + 1) - 1
 \end{aligned}$$

□

**Exercise 2** Let us call  $(0)$  the 0th diagonal,  $(1, 2)$  the 1st diagonal,  $(3, 4, 5)$  the 2nd diagonal, etc. Then the  $k$ th is the set of all positions with coordinates  $(m, n)$  such that  $m + n = k$ .

Therefore, the number  $J(m, n)$  at position  $(m, n)$  is the  $m + 1$ st number in the  $(m + n)$ th diagonal. So the number of numbers that come before  $J(m, n)$  is

$$(1 + 2 + \cdots + (m + n)) + m$$

Therefore, since the natural numbers start at 0,

$$J(m, n) = (1 + 2 + \cdots + (m + n)) + m$$

We know  $1 + 2 + \cdots + k = k(k + 1)/2$ . Therefore,

$$\begin{aligned}
 J(m, n) &= 1/2(m + n)(m + n + 1) + m \\
 &= 1/2(m^2 + 2mn + m + n + n^2) + m \\
 &= 1/2(m^2 + 2mn + 3m + n + n^2) \\
 &= 1/2((m + n)^2 + 3m + n)
 \end{aligned}$$

**Exercise 3** Define  $f : (0, 1) \rightarrow \mathbb{R}$  by:  $f(x) = 1/x - 2$  if  $0 < x \leq 1/2$ ;  $f(x) = 2 - 1/(1 - x)$  if  $1/2 < x < 1$ .

**Exercise 4** Define  $f : [0, 1] \rightarrow (0, 1)$  by

$$\begin{aligned}
 f(1/2 - 1/2^n) &= 1/2 - 1/2^{n-1} && \text{(for } n \text{ a positive integer)} \\
 f(1/2 + 1/2^n) &= 1/2 + 1/2^{n-1} && \text{(for } n \text{ a positive integer)} \\
 f(x) &= x && \text{(for all other } x)
 \end{aligned}$$

**Exercise 5**

(a) For any set  $A$ , the identity function  $I_A$  is a bijection between  $A$  and  $A$ . It is injective because, if  $I_A(x) = I_A(y)$  then  $x = y$  immediately. It is surjective because for any  $y \in I_A$  we have  $y = I_A(y)$ .

(b) We prove that, if  $f$  is a bijection between  $A$  and  $B$ , then  $f^{-1}$  is a bijection between  $B$  and  $A$ . It is an injective function by Theorem 3F, and maps  $B$  onto  $A$  by Theorem 3E.

(c) Let  $f$  be a bijection between  $A$  and  $B$ , and  $g$  a bijection between  $A$  and  $C$ . We prove  $g \circ f$  is a bijection between  $A$  and  $C$ .

It is a function from  $A$  to  $C$  by Theorem 3H.

We prove it is injective. Let  $x, y \in A$  and assume  $(g \circ f)(x) = (g \circ f)(y)$ . Then

$$\begin{aligned} g(f(x)) &= g(f(y)) \\ \therefore f(x) &= f(y) && (g \text{ is injective}) \\ \therefore x &= y && (f \text{ is injective}) \end{aligned}$$

Now we prove it maps  $A$  onto  $C$ . Let  $c \in C$ . Pick  $b \in B$  such that  $g(b) = c$  (since  $g$  is surjective). Pick  $a \in A$  such that  $f(a) = b$  (since  $f$  is injective). Then  $(g \circ f)(a) = c$ .

## 6.2 Finite Sets

**Exercise 6** Suppose every set of cardinality  $\kappa$  belongs to  $A$ . We will prove that every set belongs to  $\bigcup A$ .

Let  $x$  be any set. Pick a set  $y$  of cardinality  $\kappa$ . If  $x \in y$  then  $x \in y \in A$  so  $x \in \bigcup A$ .

Assume  $x \notin y$ . Pick an element  $z \in y$  (we know  $y$  is nonempty because  $\kappa \neq 0$ ). Then  $y - \{z\} \cup \{x\}$  has cardinality  $\kappa$ , and so  $x \in (y - \{z\} \cup \{x\}) \in A$  hence  $x \in \bigcup A$ .

Thus, every set is in  $\bigcup A$ , which we know is impossible by Theorem 2A.

**Exercise 7** If  $f$  is one-to-one then  $f$  is a bijection between  $A$  and  $\text{ran } f$ . So we must have  $\text{ran } f = A$ , otherwise  $f$  would be a bijection between  $A$  and a proper subset of  $A$ , contradicting the Pigeonhole Principle.

Conversely, suppose  $\text{ran } f = A$ . Pick a right inverse  $h : A \rightarrow A$  for  $f$  (by Theorem 3J(b). Note: Theorem 3J(b) can in fact be proved for the case  $B$  is finite without using the Axiom of Choice.). Now,  $h$  is one-to-one by Theorem 3J(a). So  $\text{ran } h = A$  by the first paragraph.

We prove  $f$  is one-to-one. Let  $x, y \in A$  and assume  $f(x) = f(y)$ . Pick  $a, b \in A$  such that  $h(a) = x$  and  $h(b) = y$ . Then

$$\begin{aligned} f(h(a)) &= f(h(b)) \\ \therefore a &= b \\ \therefore x &= y \end{aligned}$$

**Exercise 8** PROOF:

$\langle 1 \rangle 1$ . For any sets  $A$  and  $x$ , if  $A$  is finite then  $A \cup \{x\}$  is finite.

$\langle 2 \rangle 1$ . CASE:  $x \in A$

PROOF: In this case  $A \cup \{x\} = A$ .

$\langle 2 \rangle 2$ . CASE:  $x \notin A$

PROOF: Then  $|A \cup \{x\}| = |A|^+$ .

$\langle 1 \rangle 2$ . LET:  $A$  be a finite set.

$\langle 1 \rangle 3$ . For any set  $B$ , if  $B \approx 0$  then  $A \cup B$  is finite.

PROOF: Because  $B = \emptyset$  so  $A \cup B = A$ .

$\langle 1 \rangle 4$ . Let  $n$  be a natural number. Assume that, for any set  $B$ , if  $B \approx n$  then  $A \cup B$  is finite. Then for any set  $B$ , if  $B \approx n^+$  then  $A \cup B$  is finite.

$\langle 2 \rangle 1$ . LET:  $n \in \omega$

$\langle 2 \rangle 2$ . ASSUME: For any set  $B$ , if  $B \approx n$  then  $A \cup B$  is finite.

$\langle 2 \rangle 3$ . LET:  $B$  be a set.

$\langle 2 \rangle 4$ . ASSUME:  $B \approx n^+$

$\langle 2 \rangle 5$ . PICK a bijection  $f : n^+ \rightarrow B$

$\langle 2 \rangle 6$ .  $B - \{f(n)\} \approx n$

$\langle 2 \rangle 7$ .  $A \cup (B - \{f(n)\})$  is finite.

$\langle 2 \rangle 8$ .  $A \cup B$  is finite.

PROOF: By  $\langle 1 \rangle 1$  since  $A \cup B = (A \cup (B - \{f(n)\})) \cup \{f(n)\}$ .

□

**Exercise 9** PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  be a finite set.

$\langle 1 \rangle 2$ . For any set  $B$ , if  $B \approx 0$  then  $A \times B$  is finite.

PROOF: In this case  $A \times B = \emptyset$ .

$\langle 1 \rangle 3$ . Let  $n$  be a natural number. Suppose that, for any set  $B$ , if  $B \approx n$  then  $A \times B$  is finite. Then for any set  $B$ , if  $B \approx n^+$  then  $A \times B$  is finite.

$\langle 2 \rangle 1$ . LET:  $n$  be a natural number.

$\langle 2 \rangle 2$ . ASSUME: For any set  $B$ , if  $B \approx n$  then  $A \times B$  is finite.

$\langle 2 \rangle 3$ . LET:  $B$  be a set.

$\langle 2 \rangle 4$ . ASSUME:  $B \approx n^+$

$\langle 2 \rangle 5$ . PICK a bijection  $f : n^+ \approx B$

$\langle 2 \rangle 6$ .  $A \times (B - \{f(n)\})$  is finite.

PROOF: By the induction hypothesis  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 7$ .  $A \times B$  is finite.

PROOF: By Exercise 8 since  $A \times B = (A \times (B - \{f(n)\})) \cup (A \times \{f(n)\})$  and  $A \times \{f(n)\}$  is finite because it is equinumerous with  $A$ .

□

## 6.3 Cardinal Arithmetic

**Exercise 10** We must show that  $(L \cup M)K \approx^L K \times^M K$  where  $L \cap M = \emptyset$ .

Define  $\Phi : (L \cup M)K \rightarrow^L K \times^M K$  by:  $\Phi(f) = \langle f \upharpoonright L, f \upharpoonright M \rangle$ .

To show  $\Phi$  is one-to-one: suppose  $\Phi(f) = \Phi(g)$ . Then  $f \upharpoonright L = g \upharpoonright L$  and  $f \upharpoonright M = g \upharpoonright M$ . Hence  $f(x) = g(x)$  for all  $x \in L$  and  $f(x) = g(x)$  for all  $x \in M$ , so  $f(x) = g(x)$  for all  $x$ , i.e.  $f = g$ .

To show  $\Phi$  is surjective: given a function  $g : L \rightarrow K$  and  $h : M \rightarrow K$ , we have  $g \cup h : L \cup M \rightarrow K$  and  $\Phi(g \cup h) = \langle g, h \rangle$ .

**Exercise 11** We must show that  ${}^M(K \times L) \approx^M K \times^M L$ .

Define  $\Phi : {}^M(K \times L) \rightarrow {}^M K \times^M L$  by:  $\Phi(f) = \langle \pi_1 \circ f, \pi_2 \circ f \rangle$ , where  $\pi_1 : K \times L \rightarrow K$  is the function defined by

$$\pi_1(\langle x, y \rangle) = x$$

and  $\pi_2 : K \times L \rightarrow L$  is the function defined by

$$\pi_2(\langle x, y \rangle) = y .$$

To show  $\Phi$  is one-to-one: suppose  $\Phi(f) = \Phi(g)$ . For any  $x \in M$ , we have  $\pi_1(f(x)) = \pi_1(g(x))$  and  $\pi_2(f(x)) = \pi_2(g(x))$ , so  $f(x) = g(x)$  by Theorem 3A.

To show  $\Phi$  is surjective: given  $g : M \rightarrow K$  and  $h : M \rightarrow L$ , define  $f : M \rightarrow K \times L$  by  $f(x) = \langle g(x), h(x) \rangle$  for  $x \in M$ . Then  $\Phi(f) = \langle g, h \rangle$ .

**Exercise 12** We have:

$$\begin{aligned} K \cup L &= L \cup K \\ K \cup (L \cup M) &= (K \cup L) \cup M \\ K \times (L \cup M) &= (K \times L) \cup (K \times M) \end{aligned}$$

**Exercise 13** Now that we have shown the union of two finite sets is finite, this follows by an easy induction on  $|B|$ .

**Exercise 14** For any set  $A$ , let  $Perm(A)$  be the set of all permutations of  $A$ .

Assume  $K \approx L$ : we must show  $Perm(K) \approx Perm(L)$ . Pick a bijection  $f : K \rightarrow L$ . Define  $\Phi : Perm(K) \rightarrow Perm(L)$  by:  $\Phi(g) = f \circ g \circ f^{-1}$ . It is easy to show  $\Phi(g)$  is a permutation of  $L$  whenever  $g$  is a permutation of  $K$ , and  $\Phi$  is a bijection.

## 6.4 Ordering Cardinal Numbers

**Exercise 15** Suppose for a contradiction  $\mathcal{A}$  is a set and, for every set  $x$ , there exists  $y \in \mathcal{A}$  such that  $x \preccurlyeq y$ . Pick  $y \in \mathcal{A}$  such that  $\mathcal{P} \cup \mathcal{A} \preccurlyeq y$ . But  $y \subseteq \bigcup \mathcal{A}$  so  $\mathcal{P} \cup \mathcal{A} \preccurlyeq \bigcup \mathcal{A}$ , contradicting Cantor's Theorem.

**Exercise 16** Define  $G : S \rightarrow^S 2$  by

$$G(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Then  $G$  is injective.

Now, assume for a contradiction  $F : S \rightarrow^S 2$  is bijective. Define  $g : S \rightarrow 2$  by  $g(x) = 1 - F(x)(x)$ . Then  $g(x) \neq F(x)(x)$  for all  $x \in S$ , so  $g \neq F(x)$  for all  $x \in S$ . Hence  $g \notin \text{ran } F$ . This contradicts the assumption that  $F$  is surjective.



**Exercise 17** We have  $1 < 2$  but  $\aleph_0 + 1 = \aleph_0 + 2 = \aleph_0$ .

We have  $1 < 2$  but  $\aleph_0 \cdot 1 = \aleph_0 \cdot 2 = \aleph_0$ .

We have  $2 < 3$  but  $2^{\aleph_0} = 3^{\aleph_0}$ .

We have  $2 < 3$  but  $\aleph_0^2 = \aleph_0^3 = \aleph_0$ .

## 6.5 Axiom of Choice

**Exercise 18** PROOF:

$\langle 1 \rangle 1$ . If the Axiom of Choice is true then the statement is true.

PROOF: The statement is a special case of the multiplicative axiom, taking  $I = \mathcal{A}$  and  $H(X) = X$  for each  $X \in \mathcal{A}$ .

$\langle 1 \rangle 2$ . If the statement is true then the Axiom of Choice is true.

$\langle 2 \rangle 1$ . ASSUME: The statement is true.

PROVE: Axiom of choice IV

$\langle 2 \rangle 2$ . LET:  $\mathcal{A}$  be a set such that each member of  $\mathcal{A}$  is a nonempty set, and any two distinct members of  $\mathcal{A}$  are disjoint.

$\langle 2 \rangle 3$ . PICK a function  $f$  with domain  $\mathcal{A}$  such that  $f(X) \in X$  for all  $X \in \mathcal{A}$

$\langle 2 \rangle 4$ . LET:  $C = \text{ran } f$

$\langle 2 \rangle 5$ .  $\forall B \in \mathcal{A}. C \cap B = \{f(B)\}$

□

**Exercise 19** PROOF:

$\langle 1 \rangle 1$ . For  $n \in \omega$ , let  $P(n)$  be the statement: for every set  $I$  with  $\text{card } I = n$  and function  $H$  with domain  $I$  such that  $H(i)$  is nonempty for each  $i \in I$ , there exists a function  $f$  with domain  $I$  such that  $\forall i \in I. f(i) \in H(i)$ .

$\langle 1 \rangle 2$ .  $P(0)$  is true

PROOF: Take  $f = \emptyset$

$\langle 1 \rangle 3$ .  $\forall n \in \omega. P(n) \Rightarrow P(n+1)$

$\langle 2 \rangle 1$ . LET:  $n \in \omega$

$\langle 2 \rangle 2$ . ASSUME:  $P(n)$

$\langle 2 \rangle 3$ . LET:  $I$  be a set with  $\text{card } I = n+1$

$\langle 2 \rangle 4$ . LET:  $H$  be a function with domain  $I$  such that  $H(i)$  is nonempty for each  $i \in I$

$\langle 2 \rangle 5$ . PICK a bijection  $g : n+1 \approx I$

$\langle 2 \rangle 6$ . PICK a function  $h$  with domain  $g[n]$  such that  $\forall i \in g[n]. h(i) \in H(i)$

$\langle 2 \rangle 7$ . PICK  $a \in H(g(n))$

$\langle 2 \rangle 8$ . LET:  $f = h \cup \{(g(n), a)\}$

$\langle 2 \rangle 9$ .  $f$  is a function with domain  $I$  such that  $\forall i \in I. f(i) \in H(i)$

□

**Exercise 20** PROOF:

$\langle 1 \rangle 1$ . PICK a choice function  $F$  for  $A$

$\langle 1 \rangle 2$ . PICK  $a \in A$

⟨1⟩3. Define the function  $f : \omega \rightarrow A$  by:

$$f(0) = a$$

$$f(n^+) = F(R^{-1}(f(n)))$$

PROOF: We know  $R^{-1}(x)$  is nonempty for all  $x \in A$  because  $\forall x \in A. \exists y \in A. yRx$ .

⟨1⟩4.  $\forall n \in \omega. f(n^+)Rf(n)$

□

**Exercise 21** PROOF:

⟨1⟩1. For every chain  $\mathcal{B} \subseteq \mathcal{A}$  we have  $\bigcup \mathcal{B} \in \mathcal{A}$

⟨2⟩1. LET:  $\mathcal{B} \subseteq \mathcal{A}$  be a chain.

⟨2⟩2. Every finite subset of  $\bigcup \mathcal{B}$  is a member of  $\mathcal{A}$ .

⟨3⟩1. LET:  $\{x_1, \dots, x_n\} \subseteq \bigcup \mathcal{B}$  be finite.

⟨3⟩2. For  $1 \leq i \leq n$ , PICK  $B_i \in \mathcal{B}_i$  such that  $x_i \in B_i$

⟨3⟩3. PICK  $m$  such that  $B_1, \dots, B_n \subseteq B_m$

PROOF: Since  $\mathcal{B}$  is a chain.

⟨3⟩4.  $\{x_1, \dots, x_n\}$  is a finite subset of  $B_m$ .

⟨3⟩5.  $\{x_1, \dots, x_n\} \in \mathcal{A}$

PROOF: Since  $B_m \in \mathcal{A}$  so every finite subset of  $B_m$  is a member of  $\mathcal{A}$ .

⟨2⟩3.  $\bigcup \mathcal{B} \in \mathcal{A}$

⟨1⟩2. Q.E.D.

PROOF: By Zorn's Lemma.

□

**Exercise 22** PROOF:

⟨1⟩1. If the Axiom of Choice is true then the statement is true.

⟨2⟩1. ASSUME: The Axiom of Choice

⟨2⟩2. LET:  $A$  be a set.

⟨2⟩3. LET:  $R = \{\langle x, y \rangle : y \in A, x \in t\}$

⟨2⟩4. PICK a function  $F \subseteq R$  such that  $\text{dom } F = \text{dom } R$

⟨2⟩5.  $\text{dom } R = \bigcup A$

⟨2⟩6.  $\forall x \in \bigcup A. x \in F(x) \in A$

⟨1⟩2. If the statement is true then the Axiom of Choice is true.

⟨2⟩1. ASSUME: the statement

⟨2⟩2. LET:  $R$  be a relation

⟨2⟩3. LET:  $A = \{\{\langle 0, x \rangle, \langle 1, y \rangle\} : xRy\}$

⟨2⟩4. PICK a function  $F$  with domain  $\bigcup A$  such that  $\text{dom } F = \bigcup A$  and  $\forall x \in \bigcup A. x \in F(x) \in A$

⟨2⟩5. LET:  $H = \{\langle x, y \rangle \mid x \in \text{dom } R, F(x) = \{\langle 0, x \rangle, \langle 1, y \rangle\}\}$

⟨2⟩6.  $H$  is a function,  $H \subseteq R$ ,  $\text{dom } H = \text{dom } R$

□

**Exercise 23**

⟨1⟩1.  $g[0] = h(0)$

PROOF: Both are equal to  $\emptyset$ .

⟨1⟩2.  $\forall n \in \omega. g[n] = h(n) \Rightarrow g[n^+] = h(n^+)$

⟨2⟩1. LET:  $n \in \omega$

⟨2⟩2. ASSUME:  $g[n] = h(n)$

⟨2⟩3.  $g[n^+] = h(n^+)$

PROOF:

$$\begin{aligned} h(n^+) &= h(n) \cup \{F(A - h(n))\} \\ &= g[n] \cup \{g(n)\} \\ &= g[n^+] \end{aligned}$$

**Exercise 24** Let  $\{\kappa_i\}_{i \in I}$  be a family of cardinal numbers. For  $i \in I$ , let  $K_i$  be a set such that  $\text{card } K_i = \kappa_i$ .

We define  $\sum_{i \in I} \kappa_i$  to be  $\text{card}\{\langle i, x \rangle : i \in I, x \in K_i\}$

We define  $\prod_{i \in I} \kappa_i$  to be  $\text{card}\{f : f \text{ is a function, } \text{dom } f = I, \forall i \in I. f(i) \in K_i\}$ .

**Exercise 25** PROOF:

⟨1⟩1. ASSUME: for a contradiction  $\forall n \in \omega. B \not\subseteq S(n)$

⟨1⟩2. PICK a function  $b : \omega \rightarrow B$  such that  $\forall n \in \omega. b(n) \notin S(n)$

PROOF: By the Axiom of Choice.

⟨1⟩3. LET:  $B' = \{b(n) : n \in \omega\}$

⟨1⟩4.  $B'$  is infinite.

⟨2⟩1. ASSUME: for a contradiction  $B'$  is finite.

⟨2⟩2. There exists  $N$  such that  $\forall n > N. \exists k \leq N. b(n) = b(k)$

⟨2⟩3. PICK  $M > N$  such that  $\forall k \leq N. b(k) \in S(M)$

PROOF: For  $k \leq N$  there exists  $n_k$  such that  $b(k) \in S(n_k)$ . Take  $M$  to be the largest of these numbers and  $N + 1$ .

⟨2⟩4.  $b(M) \in S(M)$

PROOF: Since  $b(M) = b(k)$  for some  $k \leq N$ .

⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

⟨1⟩5. PICK  $n$  such that  $B' \cap S(n)$  is infinite.

⟨1⟩6. PICK  $m > n$  such that  $b(m) \in B' \cap S(n)$

PROOF: There must be some  $m$  otherwise  $B' \cap S(n) \subseteq \{b(0), b(1), \dots, b(n)\}$  would be finite.

⟨1⟩7.  $b(m) \in S(m)$

PROOF: Since  $S(n) \subseteq S(m)$ .

⟨1⟩8. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

## 6.6 Countable Sets

**Exercise 26** PROOF:

⟨1⟩1. PICK a set  $K$  of cardinality  $\kappa$

- ⟨1⟩2. For all  $X \in \mathcal{A}$ , there exists an injective function  $X \rightarrow K$   
 ⟨1⟩3. PICK a function  $F$  with domain  $\mathcal{A}$  such that, for all  $X \in \mathcal{A}$ ,  $F(X)$  is an injective function  $X \rightarrow K$   
 PROOF: By the Axiom of Choice.  
 ⟨1⟩4. PICK a function  $G$  with domain  $\bigcup \mathcal{A}$  such that, for all  $x \in \bigcup \mathcal{A}$ , we have  $x \in G(x) \in \mathcal{A}$   
 PROOF: By Exercise 22.  
 ⟨1⟩5. Define  $f : \bigcup \mathcal{A} \rightarrow \mathcal{A} \times K$  by  $f(x) = \langle G(x), F(G(x))(x) \rangle$   
 ⟨1⟩6.  $f$  is injective.  
 ⟨2⟩1. LET:  $x, y \in \bigcup \mathcal{A}$   
 ⟨2⟩2. ASSUME:  $f(x) = f(y)$   
 ⟨2⟩3.  $G(x) = G(y)$  and  $F(G(x))(x) = F(G(y))(y)$   
 ⟨2⟩4.  $F(G(x))(x) = F(G(x))(y)$   
 ⟨2⟩5.  $x = y$   
 PROOF: Since  $F(G(x))$  is injective.

□

### Exercise 27

(a) Pick a function  $f : A \rightarrow \mathbb{Q}^2$  such that  $f(c) \in c$  for all  $c \in A$ . Then  $f$  is an injection, so  $A \preceq \mathbb{Q}^2$  which is countable.

(b) No: the set of all circles with center  $(0, 0)$  is an uncountable set of circles no two of which intersect.

(c) Yes. Pick a function  $f : C \rightarrow \mathbb{Q}^4$  such that  $f(x)$  is a pair of points with rational coordinates, one in each circle of  $x$ , for all  $x \in C$ . Then  $f$  is an injection; it is not possible for two points to be in separate circles of two non-intersecting figure-eights. Hence  $C \preceq \mathbb{Q}^4$ .

**Exercise 28** Let  $\mathcal{A} = \{(a, \sqrt{2}) : a < \sqrt{2}\} \cup \{(\sqrt{2}, b) : b > \sqrt{2}\}$ . Then every rational is in some member of  $\mathcal{A}$  but  $\bigcup \mathcal{A} = \mathbb{R} - \{\sqrt{2}\}$ .

(Enderton's hint suggests he had a different solution in mind, but I am not sure what it is.)

**Exercise 29** For each integer  $n \geq 2$ , let  $B_n = \{x \in A : x > b/n\}$ . Then each  $B_n$  is finite ( $B_n$  cannot have more than  $n - 1$  elements because  $n$  elements in  $B_n$  would have a sum  $> b$ ) and  $A = \bigcup_n B_n$ . So  $A$  is a countable union of finite sets, and therefore countable.

**Exercise 30** PROOF:

- ⟨1⟩1. PICK  $a \in A$   
 ⟨1⟩2. Define  $f : Sq(A) \rightarrow \omega \times^\omega A$  by  $f(s) = \langle n, g \rangle$ , where  $n$  is the length of  $s$ , and  $g(i) = s(i)$  for  $i < n$ ,  $g(i) = a$  for  $i \geq n$

- $\langle 1 \rangle 3.$   $f$  is injective.  
 $\langle 1 \rangle 4.$   $Sq(A) \preceq \omega \times^\omega A$   
 $\langle 1 \rangle 5.$   $\text{card } Sq(A) \leq (\text{card } A)^{\aleph_0}$

PROOF:

$$\begin{aligned}
 \text{card } Sq(A) &\leq \aleph_0 \cdot (\text{card } A)^{\aleph_0} && (\langle 1 \rangle 4) \\
 &\leq (\text{card } A)^{\aleph_0} \cdot (\text{card } A)^{\aleph_0} && (\text{Cantor's Theorem}) \\
 &= (\text{card } A)^{\aleph_0 + \aleph_0} && (\text{Theorem 6I}) \\
 &= (\text{card } A)^{\aleph_0}
 \end{aligned}$$

## 6.7 Arithmetic of Infinite Cardinals

**Exercise 31** If  $f$  is a one-to-one correspondence between  $A \times A$  and  $A$ , where  $A \subseteq B$ , then

$$f \subseteq (A \times A) \times A \subseteq (B \times B) \times B.$$

Also  $\emptyset \subseteq (B \times B) \times B$ . So we can form  $\mathcal{H}$  by applying a Subset Axiom to  $\mathcal{P}((B \times B) \times B)$ .

**Exercise 32** The function that maps  $x$  to  $\{x\}$  is an injection  $A \rightarrow \mathcal{F}A$ , so we have  $A \approx \mathcal{F}A$ .

For the converse, let  $F_n = \{X \in \mathcal{F}A : \text{card } X \leq n\}$  for  $n \in \omega$ . The function that sends  $\langle a_1, \dots, a_n \rangle$  to  $\{a_1, \dots, a_n\}$  is a surjection  $A^n \rightarrow F_n$ , so we have

$$\text{card } F_n \leq (\text{card } A)^n = \text{card } A$$

by Lemma 6R. Now,  $\mathcal{F}A = \bigcup_n F_n$ , so

$$\text{card } \mathcal{F}A \leq \aleph_0 \cdot \text{card } A = \text{card } A$$

by the Absorption Law.

**Exercise 33** The function that maps  $a$  to the sequence of length 1 containing  $a$  is an injection  $A \rightarrow Sq(A)$ , so  $A \preceq Sq(A)$ .

For the converse, we have  $\text{card}(^n A) = (\text{card } A)^n = \text{card } A$  for any natural number  $n$

$$\begin{aligned}
 \text{card } Sq(A) &= \text{card}(^0 A \cup ^1 A \cup ^2 A \cup \dots) \\
 &= \aleph_0 \cdot \text{card } A \\
 &= \text{card } A
 \end{aligned}$$

by the Absorption Law.

**Exercise 34**

$$\begin{aligned}
 2^\lambda &\leq \kappa^\lambda \\
 &\leq (2^\kappa)^\lambda \\
 &= 2^{\kappa \cdot \lambda} \\
 &= 2^\lambda \qquad \text{(Absorption Law)}
 \end{aligned}$$

**Exercise 35** For any infinite set of primes  $A$  and natural number  $n$ , let  $f(A, n) = \prod \{p \in A : p \leq n\}$ . Let  $P(A) = \{f(A, n) : n \in \omega\}$ . Let  $\mathcal{A}$  be the set of all sets of the form  $P(A)$ .

The number of infinite sets of primes is  $2^{\aleph_0}$  (there are  $2^{\aleph_0}$  sets of primes and  $\aleph_0$  finite sets of primes by Exercise 32.)

If  $P(A) = P(B)$  then  $A = B$ . (If  $p \in A - B$  then  $p \mid f(A, p)$  but  $p$  does not divide any member of  $P(B)$ .) So  $P$  is an injection from the set of infinite sets of primes into  $\mathcal{A}$ . Hence  $\text{card } \mathcal{A} = 2^{\aleph_0}$ .

We now prove that, if  $A \neq B$ , then  $P(A) \cap P(B)$  is finite. Let  $p \in A - B$ . For  $n \geq p$  we have  $f(A, n) \notin P(B)$  since  $p \mid f(A, n)$  but  $p$  does not divide any member of  $B$ . Hence  $A \cap B \subseteq \{f(A, 0), f(A, 1), \dots, f(A, p-1)\}$ .

**Exercise 36** PROOF:

$\langle 1 \rangle 1$ . For any set  $A$ , there exists a permutation of  $A$  with no fixed points.

$\langle 2 \rangle 1$ . For every natural number  $n$ , there exists a permutation of  $n$  with no fixed points.

PROOF: Map  $i$  to  $i + 1$  if  $i + 1 < n$ , and map  $n - 1$  to 0.

$\langle 2 \rangle 2$ . For every infinite set  $A$ , there exists a permutation of  $A$  with no fixed points.

$\langle 3 \rangle 1$ . PICK a bijection  $f : A \approx A \times 2$

$\langle 3 \rangle 2$ . Define  $\pi : A \times 2 \rightarrow A \times 2$  by  $\pi(x, 0) = (x, 1)$  and  $\pi(x, 1) = (x, 0)$

$\langle 3 \rangle 3$ .  $f^{-1} \circ \pi \circ f$  is a permutation of  $A$  with no fixed point.

$\langle 1 \rangle 2$ .  $\kappa! \leq 2^\kappa$

PROOF: Because the set of permutations of  $K$  is a subset of  ${}^K K$ , where  $K$  is a set of cardinality  $\kappa$ .

$\langle 1 \rangle 3$ .  $2^\kappa \leq \kappa!$

$\langle 2 \rangle 1$ . PICK a set  $K$  of cardinality  $\kappa$

$\langle 2 \rangle 2$ . LET:  $\text{Perm}(K)$  be the set of permutations of  $K$ .

$\langle 2 \rangle 3$ . Define  $f : \mathcal{P}K \rightarrow \text{Perm}(K)$  as follows. Given  $A \subseteq \mathcal{P}K$ , pick a permutation  $\pi_{K-A}$  of  $K - A$  with no fixed point. Then  $f(A) = I_A \cup \pi_{K-A}$

$\langle 2 \rangle 4$ .  $f$  is injective

PROOF: The function that maps a permutation to its set of fixed points is a left inverse.

$\langle 2 \rangle 5$ .  $2^\kappa \leq \kappa!$

□

## Chapter 7

# Chapter 7 — Orderings and Ordinals

### 7.1 Partial Orderings

#### Exercise 1

(a) No we cannot. Let  $A = \mathcal{P}3$  and  $B = \omega$ . Let  $<_A = \subset_3$  and  $<_B$  be the usual ordering on  $\omega$ . Define  $f : A \rightarrow B$  by:  $f(X) = \text{card } X$ . Then  $X \subset_2 Y \Rightarrow \text{card } X < \text{card } Y$  but  $f$  is not one-to-one because  $f(\{0\}) = f(\{1\}) = 1$ .

(b) No we cannot. With the same example, we have  $f(\{0\}) < f(\{1, 2\})$  but  $\{0\} \not\subset \{1, 2\}$ .

**Exercise 2** We show  $R^{-1}$  is transitive. Suppose  $xR^{-1}y$  and  $yR^{-1}z$ . Then  $zRx$  and  $yRx$ , so  $zRx$  because  $R$  is transitive. Hence  $xR^{-1}z$ .

We now show  $R^{-1}$  is irreflexive. For any  $x$ , we have  $\langle x, x \rangle \notin R$ , so  $\langle x, x \rangle \notin R^{-1}$ .

**Exercise 3** The proof is by induction on  $n$ .

The only linear ordering on  $\emptyset$  is  $\emptyset$ , which has 0 pairs.

Suppose that, whenever  $\text{card } S = n$ , then every linear ordering on  $S$  has  $1/2n(n-1)$  pairs. Let  $S$  be a set of cardinality  $n+1$ . Let  $<$  be a linear ordering on  $S$ .

Pick an element  $a \in S$  and let  $T = S - \{a\}$ . Then  $< \cap (T \times T)$  is a linear ordering on  $T$ , hence has  $1/2n(n-1)$  pairs. Now, for every  $x \in T$ , exactly one of  $\langle x, a \rangle$  and  $\langle a, x \rangle$  is in  $<$ . Hence  $<$  has  $n$  pairs that are not in  $< \cap (T \times T)$ . So

$$\text{card } < = 1/2n(n-1) + n = 1/2n(n+1) .$$

## 7.2 Well Orderings

**Exercise 4** PROOF:

- ⟨1⟩1.  $R$  is transitive.
- ⟨2⟩1. ASSUME:  $mRn$  and  $nRp$ .
- ⟨2⟩2. CASE:  $f(m) < f(n)$   
PROOF: In this case  $f(m) < f(p)$  so  $mRp$ .
- ⟨2⟩3. CASE:  $f(m) = f(n)$  and  $m < n$ .
- ⟨3⟩1. CASE:  $f(n) < f(p)$   
PROOF: In this case  $f(m) < f(p)$  so  $mRp$ .
- ⟨3⟩2. CASE:  $f(n) = f(p)$  and  $n < p$ .  
PROOF: In this case  $f(m) = f(p)$  and  $m < p$  so  $mRp$ .
- ⟨1⟩2.  $R$  satisfies trichotomy on  $P$ .
- ⟨2⟩1. LET:  $m, n \in P$
- ⟨2⟩2. Exactly one of  $f(m) < f(n)$ ,  $f(n) < f(m)$ ,  $f(n) = f(m)$  holds.
- ⟨2⟩3. Exactly one of  $m < n$ ,  $n < m$ ,  $n = m$  holds.
- ⟨2⟩4. Exactly one of  $f(m) < f(n)$ ,  $(f(m) = f(n) \ \& \ m < n)$ ,  $(f(m) = f(n) \ \& \ m = n)$ ,  $(f(m) = f(n) \ \& \ n < m)$ ,  $f(n) < f(m)$  holds.
- ⟨2⟩5. Exactly one of  $mRn$ ,  $m = n$ ,  $nRm$  holds.
- ⟨1⟩3. Every nonempty subset of  $P$  has an  $R$ -least element.
- ⟨2⟩1. LET:  $A \subseteq P$  be nonempty.
- ⟨2⟩2. LET:  $k$  be the least element of  $f(A)$ .
- ⟨2⟩3. LET:  $n$  be the least element of  $f^{-1}(k) \cap A$ .
- ⟨2⟩4.  $n$  is the  $R$ -least element of  $A$ .

□

$\langle P, R \rangle$  resembles Fig. 45 (d).

**Exercise 5** PROOF:

- ⟨1⟩1. LET:  $x \in A$
- ⟨1⟩2. ASSUME: for a contradiction  $f(x) < x$
- ⟨1⟩3. Define  $g : \omega \rightarrow A$  by  $g(0) = x$  and  $g(n^+) = f(g(n))$  for all  $n \in \omega$
- ⟨1⟩4.  $\forall n \in \omega. g(n^+) < g(n)$   
PROOF: By induction on  $n$  using ⟨1⟩2 and the hypothesis.
- ⟨1⟩5. Q.E.D.  
PROOF: This contradicts Theorem 7B.

□

**Exercise 6** PROOF:

- ⟨1⟩1. For all  $x \in S$  that is not greatest, there exists  $y \in S$  and  $q \in \mathbb{Q}$  such that  $x < q < y$  and there is no  $z \in S$  such that  $x < z < y$
- ⟨1⟩2. PICK a function  $f : S \rightarrow \mathbb{Q}$  such that  $\forall x \in S. x < f(x)$  and, if  $x$  is not greatest, then  $f(x) < y$  where  $y$  is the next element in  $S$ .
- ⟨1⟩3.  $f$  is injective.
- ⟨1⟩4.  $S \preccurlyeq \mathbb{Q}$

□



### Exercise 7

(a) We have  $F(t) = C \cup \bigcup \text{ran}(F \upharpoonright t)$  for all  $t \in \omega$ . So:

$$\begin{aligned}
 F(0) &= C \cup \bigcup \text{ran} \emptyset \\
 &= C \\
 F(1) &= C \cup \bigcup \text{ran}(F \upharpoonright 0) \\
 &= C \cup \bigcup \{C\} \\
 &= C \cup C \\
 F(2) &= C \cup \bigcup \{C, C \cup C\} \\
 &= C \cup (C \cup C) \\
 &= C \cup C \cup C
 \end{aligned}$$

We guess:

$$F(n) = C \cup C \cup \dots \cup \underbrace{C}_n \cup \dots \cup C$$

(b) PROOF:

- $\langle 1 \rangle 1$ . LET:  $a \in F(n)$
- $\langle 1 \rangle 2$ .  $a \in \bigcup \text{ran}(F \upharpoonright n^+)$
- $\langle 1 \rangle 3$ .  $a \subseteq \bigcup \text{ran}(F \upharpoonright n^+)$
- $\langle 1 \rangle 4$ .  $a \subseteq F(n^+)$

□

(c) PROOF:

- $\langle 1 \rangle 1$ .  $\overline{C}$  is a transitive set.
- $\langle 2 \rangle 1$ . LET:  $x \in y \in \overline{C}$
- $\langle 2 \rangle 2$ . PICK  $n \in \omega$  such that  $y \in F(n)$
- $\langle 2 \rangle 3$ .  $x \in F(n^+)$
- PROOF: By (b).
- $\langle 2 \rangle 4$ .  $x \in \overline{C}$
- $\langle 1 \rangle 2$ .  $C \subseteq \overline{C}$
- $\langle 2 \rangle 1$ . Since  $C = F(0)$

□

## 7.3 Replacement Axioms

**Exercise 8** Let  $P(x)$  be a formula not containing  $B$ . We prove the statement

$$\forall c \exists B \forall x (x \in B \Leftrightarrow x \in c \ \& \ P(x)) .$$

Let  $Q(x, y)$  be the formula  $P(x) \wedge y = x$ . Now we reason as follows.

Let  $c$  be any set. Then we have

$$(\forall x \in c) \forall y_1 \forall y_2 (Q(x, y_1) \& Q(x, y_2) \Rightarrow y_1 = y_2)$$

Then, by a Replacement Axiom, there exists a set  $B$  such that

$$\forall y (y \in B \Leftrightarrow (\exists x \in c) Q(x, y)) \text{ .}$$

This is equivalent to  $\forall x (x \in B \Leftrightarrow x \in c \& P(x))$ .

**Exercise 9** Let  $a$  and  $b$  be sets. Let  $P(x, y)$  be the formula  $(x = \emptyset \& y = a)$  or  $(x = \mathcal{P}\emptyset \& y = b)$ . Then we have  $(\forall x \in \mathcal{P}\mathcal{P}\emptyset) \forall y_1 \forall y_2 (P(x, y_1) \& P(x, y_2) \Rightarrow y_1 = y_2)$ , hence there exists a set  $c$  such that

$$\forall y (y \in c \Leftrightarrow (\exists x \in \mathcal{P}\mathcal{P}\emptyset) P(x, y))$$

The members of  $c$  are just  $a$  and  $b$ .

## 7.4 Epsilon-Images

### Exercise 10

(a) Let  $n$  be a natural number. Let  $\alpha$  be its epsilon-image, and  $E : n \rightarrow \alpha$  be as in the definition of epsilon-image.

We prove  $\forall x \in n. E(x) = x$  by strong induction on  $x$ . Let  $x \in n$  and assume  $\forall y \in x. E(y) = y$ . Then

$$\begin{aligned} E(x) &= \{E(y) : y \in x\} \\ &= \{y : y \in x\} \\ &= x \end{aligned}$$

Hence

$$\begin{aligned} \alpha &= \{E(x) : x \in n\} \\ &= \{x : x \in n\} \\ &= n \end{aligned}$$

(b) Similarly the  $\epsilon$ -image of  $\omega$  is  $\omega$ .

### Exercise 11

(a) Let  $R$  be the ordering given in the question. Thus  $xRy$  iff ( $x$  and  $y$  are nonnegative and  $x < y$ ) or ( $x$  and  $y$  are both negative and  $y < x$ ) or ( $x$  is nonnegative and  $y$  is negative).

PROOF:

- ⟨1⟩1.  $R$  is transitive
  - ⟨2⟩1. ASSUME:  $xRy$  and  $yRz$
  - ⟨2⟩2. CASE:  $x$  and  $y$  are nonnegative and  $x < y$ 
    - ⟨3⟩1. CASE:  $z$  is nonnegative and  $y < z$ 
      - PROOF: In this case  $x$  and  $z$  are nonnegative and  $x < z$ .
    - ⟨3⟩2. CASE:  $z$  is negative
      - PROOF: In this case  $x$  is nonnegative and  $z$  is negative.
  - ⟨2⟩3. CASE:  $x$  and  $y$  are both negative and  $y < x$ 
    - PROOF: We must have  $z$  is negative and  $z < y$ , hence  $z < x$ .
  - ⟨2⟩4. CASE:  $x$  is nonnegative and  $y$  is negative
    - PROOF: We must have  $z$  is negative.
- ⟨1⟩2.  $R$  satisfies trichotomy on  $\mathbb{Z}$ 
  - ⟨2⟩1. LET:  $x, y \in \mathbb{Z}$
  - ⟨2⟩2. CASE:  $x$  and  $y$  are nonnegative.
    - PROOF: Exactly one of  $x < y$ ,  $x = y$ ,  $y < x$  holds.
  - ⟨2⟩3. CASE:  $x$  is nonnegative and  $y$  is negative.
    - PROOF: In this case  $x < y$ .
  - ⟨2⟩4. CASE:  $x$  is negative and  $y$  is nonnegative.
    - PROOF: In this case  $y < x$ .
  - ⟨2⟩5. CASE:  $x$  and  $y$  are negative.
    - PROOF: Exactly one of  $x < y$ ,  $x = y$ ,  $y < x$  holds.
- ⟨1⟩3.  $R$  is well-founded
  - ⟨2⟩1. LET:  $A \subseteq \mathbb{Z}$  be nonempty.
  - ⟨2⟩2. CASE: There exists a nonnegative integer in  $A$ .
    - PROOF: Let  $n$  be the least nonnegative element of  $A$ . Then  $n$  is the  $R$ -least element of  $A$ .
  - ⟨2⟩3. CASE: All elements of  $A$  are negative.
    - PROOF: Let  $n$  be least such that  $-n \in A$ . Then  $-n$  is the  $R$ -least element of  $A$ .

□

(b)

$$\begin{aligned}
 E(3) &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
 &= 3 \\
 E(-1) &= \omega \\
 E(-2) &= \omega^+ \\
 \text{ran } E &= \omega \cup \{\omega, \omega^+, \omega^{++}, \dots\}
 \end{aligned}$$

## 7.5 Isomorphisms

### Exercise 12

(a) PROOF:

$\langle 1 \rangle 1$ .  $<_A$  is irreflexive.

PROOF: For any  $x \in A$  we have  $f(x) \not<_B f(x)$  so  $x \not<_A x$ .

$\langle 1 \rangle 2$ .  $<_A$  is transitive.

PROOF: If  $x <_A y$  and  $y <_A z$  then  $f(x) <_B f(y) <_B f(z)$  hence  $f(x) <_B f(z)$  and so  $x <_A z$ .

(b) For any  $x, y \in A$  we have that exactly one of  $f(x) <_B f(y)$ ,  $f(x) = f(y)$ ,  $f(y) <_B f(x)$  holds. Hence exactly one of  $x <_A y$ ,  $x = y$ ,  $y <_A x$  holds. (Using the fact that  $x = y$  iff  $f(x) = f(y)$  since  $f$  is one-to-one.)

### Exercise 13 PROOF:

$\langle 1 \rangle 1$ . LET:  $\langle A, <_A \rangle$  and  $\langle B, <_B \rangle$  be two well-ordered structures.

$\langle 1 \rangle 2$ . LET:  $f, g : A \rightarrow B$  be isomorphisms.

PROVE:  $\forall x \in A. f(x) = g(x)$

$\langle 1 \rangle 3$ . LET:  $x \in A$

$\langle 1 \rangle 4$ . ASSUME:  $\forall y < x. f(y) = g(y)$

$\langle 1 \rangle 5$ .  $f(x)$  is the least element in  $B - f[\text{seg } x]$

$\langle 2 \rangle 1$ .  $f(x) \notin f[\text{seg } x]$

PROOF: Since  $f$  is one-to-one.

$\langle 2 \rangle 2$ .  $\forall b \in B - f[\text{seg } x]. f(x) \leq b$

$\langle 3 \rangle 1$ . LET:  $b \in B - f[\text{seg } x]$

$\langle 3 \rangle 2$ . LET:  $a \in A$  be such that  $f(a) = b$

PROOF:  $f$  is surjective.

$\langle 3 \rangle 3$ .  $a \notin \text{seg } x$

$\langle 3 \rangle 4$ .  $x \leq a$

PROOF: By trichotomy

$\langle 3 \rangle 5$ .  $f(x) \leq b$

$\langle 1 \rangle 6$ .  $g(x)$  is the least element in  $B - g[\text{seg } x]$

PROOF: Similar.

$\langle 1 \rangle 7$ .  $f[\text{seg } x] = g[\text{seg } x]$

PROOF: By  $\langle 1 \rangle 4$

$\langle 1 \rangle 8$ .  $f(x) = g(x)$

□

### Exercise 14 PROOF:

$\langle 1 \rangle 1$ .  $\forall a, b \in A. a < b \Rightarrow F(a) \subset F(b)$

$\langle 2 \rangle 1$ . LET:  $a, b \in A$

$\langle 2 \rangle 2$ . ASSUME:  $a < b$

$\langle 2 \rangle 3$ .  $F(a) \subseteq F(b)$

PROOF: If  $x \leq a$  then  $x \leq b$

$\langle 2 \rangle 4. F(a) \neq F(b)$   
 PROOF: Since  $b \in F(b)$  but  $b \notin F(a)$   
 $\langle 1 \rangle 2. \forall a, b \in A. F(a) \subset F(b) \Rightarrow a < b$   
 PROOF: We cannot have  $b < a$  or  $b = a$  (as then  $F(b) \subset F(a)$  or  $F(b) = F(a)$  by  $\langle 1 \rangle 1$ ), so  $a < b$  by trichotomy.  
 $\langle 1 \rangle 3. F$  is one-to-one  
 PROOF: If  $F(a) = F(b)$  then we cannot have  $a < b$  or  $b < a$  by  $\langle 1 \rangle 1$ , so  $a = b$  by trichotomy.  
 $\langle 1 \rangle 4. F$  maps  $A$  onto  $\text{ran } F$   
 PROOF: By definition of  $\text{ran } F$ .  
 $\square$

## 7.6 Ordinal Numbers

### Exercise 15

**(a)** PROOF:  
 $\langle 1 \rangle 1.$  ASSUME:  $f : A \rightarrow \text{seg } t$  is an isomorphism  
 $\langle 1 \rangle 2.$  Define  $g : \omega \rightarrow A$  by recursion:  

$$g(0) = t$$

$$g(n^+) = f(g(n)) \quad (n \in \omega)$$
 $\langle 1 \rangle 3. \forall n \in \omega. g(n^+) < g(n)$   
 $\langle 2 \rangle 1. g(0^+) < g(0)$   
 PROOF: Since  $g(0^+) = f(t) \in \text{seg } t$  so  $g(0^+) < t = g(0)$ .  
 $\langle 2 \rangle 2. \forall n \in \omega. (g(n^+) < g(n) \Rightarrow g(n^{++}) < g(n^+))$   
 $\langle 3 \rangle 1.$  LET:  $n \in \omega$   
 $\langle 3 \rangle 2.$  ASSUME:  $g(n^+) < g(n)$   
 $\langle 3 \rangle 3. f(g(n^+)) < f(g(n))$   
 PROOF: Since  $f$  is an isomorphism.  
 $\langle 3 \rangle 4. g(n^{++}) < g(n^+)$   
 $\langle 1 \rangle 4.$  Q.E.D.  
 PROOF: This contradicts Theorem 7B.  
 $\square$

**(b)** If two of them hold then we have a well-ordered set isomorphic with an initial segment, contradicting part (a):  
 If  $A \cong B$  and  $A \cong \text{seg } b$  then  $B \cong \text{seg } b$ .  
 If  $A \cong B$  and  $\text{seg } a \cong B$  then  $A \cong \text{seg } a$ .  
 Now assume  $A \cong \text{seg } b$  and  $\text{seg } a \cong B$ . Let  $f : A \cong \text{seg } b$  and  $g : \text{seg } a \cong B$  be isomorphisms. Let  $b_0 = f(a)$ . Then  $f \upharpoonright \text{seg } a : \text{seg } a \cong \text{seg } b_0$  and so  $B \cong \text{seg } b_0$ .

**Exercise 16** Suppose  $\alpha \in \beta$ . We first prove that  $\beta \notin \alpha^+$ .

If  $\beta \in \alpha^+$  then  $\beta \in \alpha$  or  $\beta = \alpha$ . In either case we have  $\alpha \in \alpha$ , which is impossible.

So  $\beta \notin \alpha^+$ . Therefore  $\alpha^+ \subseteq \beta$ , and so  $\alpha^+ \in \beta^+$ .

Now, suppose  $\alpha \neq \beta$ . Then  $\alpha \in \beta$  or  $\beta \in \alpha$ . Hence  $\alpha^+ \in \beta^+$  or  $\beta^+ \in \alpha^+$ , and in either case  $\alpha^+ \neq \beta^+$ .

**Exercise 17** Suppose for a contradiction  $\alpha \in \beta$ . Then  $A$  is isomorphic to  $\text{seg}_B b$  for some  $b \in B$ . Let  $f : A \rightarrow \text{seg}_B b$  be an isomorphism.

We have  $f \upharpoonright B : B \rightarrow \text{seg}_B b$ . Now, define  $g : \omega \rightarrow B$  by

$$\begin{aligned} g(0) &= b \\ g(n^+) &= f(g(n)) \end{aligned}$$

Then  $g(n^+) < g(n)$  for all  $n \in \omega$ , contradicting Theorem 7B.

**Exercise 18** Suppose first  $\bigcup S \in S$ . For all  $\alpha \in S$  we have  $\alpha \subseteq \bigcup S$  and so  $\alpha \in \bigcup S$ , and so  $\bigcup S$  is the greatest element of  $S$ .

Suppose now  $\bigcup S \notin S$ . Suppose for a contradiction  $\alpha \in S$  is the greatest element of  $S$ . We have  $\alpha \subseteq \bigcup S$  (because  $\alpha \in S$ ). Also for all  $\beta \in S$  we have  $\beta \subseteq \alpha$ , hence  $\bigcup S \subseteq \alpha$ . Thus  $\bigcup S = \alpha \in S$ , which is a contradiction.

So if  $\bigcup S \notin S$  then  $S$  has no greatest element. Therefore  $S$  cannot be the successor of any ordinal, because  $\alpha$  is the greatest element of  $\alpha^+$  for any  $\alpha$ .

**Exercise 19** By Theorem 7B, every linear ordering on a finite set is a well ordering.

If  $<$  and  $\prec$  are two linear orderings on the same set  $A$ , we cannot have that  $(A, <)$  is isomorphic to  $(\text{seg } a, \prec)$  for any  $a \in A$ , because then we would have a finite set bijective with a proper subset of itself.

So by Theorem 7E we must have  $\langle A, < \rangle \cong \langle A, \prec \rangle$ .

**Exercise 20** Let  $R$  be a well ordering on the set  $S$ . Assume  $S$  is infinite; we will prove  $R^{-1}$  is not a well-ordering on  $S$ .

Define  $g : \omega \rightarrow S$  by:  $g(n)$  is the least element of  $S - g[[n]]$ . For each  $n$ , we know  $S - g[[n]]$  is nonempty because  $S$  is infinite.

Then  $g[[\omega]]$  is a nonempty subset of  $S$  that has no  $R^{-1}$ -least element (no  $R$ -greatest element), so  $R^{-1}$  is not a well ordering on  $S$ .

**Exercise 21** Let  $\mathcal{A} = \{C \in \mathcal{P}A : <^\circ \text{ is a linear ordering on } C\}$ .

We prove that, for every chain  $\mathcal{B} \subseteq \mathcal{A}$ , we have  $\bigcup \mathcal{B} \in \mathcal{A}$ .

Let  $\mathcal{B} \subseteq \mathcal{A}$  be a chain. Let  $x, y \in \bigcup \mathcal{B}$ . Pick  $C, D \in \mathcal{B}$  such that  $x \in C$  and  $y \in D$ . Then either  $C \subseteq D$  or  $D \subseteq C$ ; assume without loss of generality  $C \subseteq D$ . We have  $x, y \in D$ , and so exactly one of  $x < y$ ,  $x = y$ ,  $y < x$  holds. Thus,  $<^\circ$  linearly orders  $\bigcup \mathcal{B}$ , i.e.  $\bigcup \mathcal{B} \in \mathcal{A}$ .

Hence by Zorn's Lemma  $\mathcal{A}$  has a maximal element  $C$ , say. Now, by hypothesis,  $C$  has an upper bound  $m$ . We prove  $m$  is maximal in  $A$ .

Let  $x \in A$  and suppose  $m \leq x$ . Then  $C \cup \{m, x\}$  is linearly ordered by  $<^\circ$ , and so  $C = C \cup \{m, x\}$  by maximality of  $C$ . Hence  $x \in C$  and so  $x \leq m$ , hence  $x = m$ . Thus,  $m$  is maximal in  $A$ .

## 7.7 Debts Paid

**Exercise 22** Let  $A$  be any set. Let  $\mathcal{A}$  be the set of all pairs  $\langle B, R \rangle$  where  $B \subseteq A$  and  $R$  is a well ordering on  $B$ , and define  $<$  on  $\mathcal{A}$  by:  $\langle B, R \rangle < \langle C, S \rangle$  iff  $B$  is an initial segment of  $C$  and  $R = S \cap B^2$ .

It is easy to see that  $<$  is a partial ordering on  $\mathcal{A}$

We prove that, if  $\mathcal{C} \subseteq \mathcal{A}$  and  $<$  is a linear ordering on  $\mathcal{C}$ , then  $\mathcal{C}$  has an upper bound in  $\mathcal{A}$ . Let  $B = \bigcup \{C : \exists S. \langle C, S \rangle \in \mathcal{C}\}$  and  $R = \bigcup \{S : \exists C. \langle C, S \rangle \in \mathcal{C}\}$ . We prove that  $R$  well orders  $B$ . It is then easy to see that  $\langle B, R \rangle$  is an upper bound for  $\mathcal{C}$  in  $\mathcal{A}$ .

PROOF:

- $\langle 1 \rangle 1.$   $R$  is transitive.
  - $\langle 2 \rangle 1.$  ASSUME:  $xRy$  and  $yRz$
  - $\langle 2 \rangle 2.$  PICK  $\langle C, S \rangle, \langle D, T \rangle \in \mathcal{C}$  such that  $xSy$  and  $yTz$
  - $\langle 2 \rangle 3.$   $\langle C, S \rangle \leq \langle D, T \rangle$  or  $\langle D, T \rangle \leq \langle C, S \rangle$
  - $\langle 2 \rangle 4.$  ASSUME: w.l.o.g.  $\langle C, S \rangle \leq \langle D, T \rangle$
  - $\langle 2 \rangle 5.$   $xTy$  and  $yTz$
  - $\langle 2 \rangle 6.$   $xTz$
  - $\langle 2 \rangle 7.$   $xRz$
- $\langle 1 \rangle 2.$   $R$  is irreflexive.
  - $\langle 2 \rangle 1.$  ASSUME: for a contradiction  $xRx$
  - $\langle 2 \rangle 2.$  PICK  $\langle C, S \rangle \in \mathcal{C}$  such that  $xSx$
  - $\langle 2 \rangle 3.$  This is a contradiction.
- $\langle 1 \rangle 3.$   $R$  satisfies trichotomy.
  - $\langle 2 \rangle 1.$  LET:  $x, y \in B$
  - $\langle 2 \rangle 2.$  PICK  $\langle C, S \rangle, \langle D, T \rangle \in \mathcal{C}$  such that  $x \in C$  and  $y \in D$
  - $\langle 2 \rangle 3.$  ASSUME: w.l.o.g.  $\langle C, S \rangle \leq \langle D, T \rangle$
  - $\langle 2 \rangle 4.$   $x, y \in D$
  - $\langle 2 \rangle 5.$   $xTy$  or  $yTx$
  - $\langle 2 \rangle 6.$   $xRy$  or  $yRx$
- $\langle 1 \rangle 4.$  Every non-empty subset of  $B$  has an  $R$ -least element.
  - $\langle 2 \rangle 1.$  LET:  $C \subseteq B$  be nonempty
  - $\langle 2 \rangle 2.$  PICK  $c \in C$
  - $\langle 2 \rangle 3.$  PICK  $\langle D, T \rangle \in \mathcal{C}$  such that  $c \in D$
  - $\langle 2 \rangle 4.$  LET:  $x$  be the  $T$ -least element of  $C \cap D$   
 PROVE:  $x$  is  $R$ -least in  $C$
  - $\langle 2 \rangle 5.$  LET:  $y \in C$
  - $\langle 2 \rangle 6.$  PICK  $\langle E, U \rangle \in \mathcal{C}$  such that  $y \in E$
  - $\langle 2 \rangle 7.$   $\langle D, T \rangle \leq \langle E, U \rangle$  or  $\langle E, U \rangle \leq \langle D, T \rangle$
  - $\langle 2 \rangle 8.$  CASE:  $\langle D, T \rangle \leq \langle E, U \rangle$ 
    - $\langle 3 \rangle 1.$   $xUy$  or  $x = y$

PROOF:

- $\langle 4 \rangle 1.$  ASSUME: for a contradiction  $yUx$
- $\langle 4 \rangle 2.$   $y \in D$  and  $yTx$

PROOF: Since  $D$  is an initial segment of  $E$  and  $T = U \cap D^2$

⟨4⟩3. Q.E.D.

PROOF: This contradicts the  $T$ -minimality of  $x$ .

⟨2⟩9. CASE:  $\langle E, U \rangle \leq \langle D, T \rangle$

PROOF:  $xTy$  or  $x = y$ , so  $xRy$  or  $x = y$ .

Hence by Exercise 21 there is a maximal element  $\langle B, R \rangle$  in  $\mathcal{A}$ . We must have  $B = A$ ; for if  $a \in A - B$  then  $\langle B \cup \{a\}, R \cup \{\langle x, a \rangle : x \in B\} \rangle$  would be a larger element. Hence  $R$  is a well ordering on  $A$ .

### Exercise 23

(i) We must show that  $\alpha$  is an initial ordinal. So let  $\beta \in \alpha$ . Then  $\beta \prec A$  but  $\alpha \not\prec A$ . Hence  $\alpha \not\approx \beta$ .

(ii) We know that  $\alpha \not\prec A$ , so  $\alpha \not\leq \text{card } A$ .

(iii) Let  $\kappa$  be any cardinal greater than  $\text{card } A$ . Then  $\kappa$  is not dominated by  $A$ , so  $\kappa \notin \alpha$ , and so  $\alpha \subseteq \kappa$ .

**Exercise 24** The cardinal number of  $\mathcal{P}\alpha$  is larger than  $\alpha$  (both as a cardinal and as an ordinal).

**Exercise 25** Suppose there exists an ordinal  $\alpha$  such that  $\neg\phi(\alpha)$ . Let  $\alpha_0$  be the least such ordinal. Then we have  $\forall x \in \alpha_0. \phi(x)$  but  $\neg\phi(\alpha_0)$ . This contradicts the hypothesis.

## 7.8 Rank

**Exercise 26** The proof is by transfinite induction on  $\alpha$ . Suppose that  $\alpha$  is an ordinal and, for all  $\beta \in \alpha$ , we have  $\beta$  is grounded and  $\text{rank } \beta = \beta$ . Then by Theorem 7V(b) we have that  $\alpha$  is grounded and

$$\begin{aligned} \text{rank } \alpha &= \bigcup \{(\text{rank } \beta)^+ \mid \beta \in \alpha\} \\ &= \bigcup \{\beta^+ \mid \beta \in \alpha\} \quad (\text{induction hypothesis}) \end{aligned}$$

So we must show that  $\bigcup \{\beta^+ \mid \beta \in \alpha\} = \alpha$ .

If  $\beta \in \alpha$  then  $\beta^+ \subseteq \alpha$  so  $\beta^+ \subseteq \alpha$ . This shows that  $\bigcup \{\beta^+ \mid \beta \in \alpha\} \subseteq \alpha$ .

If  $\beta \in \alpha$  then  $\beta \in \beta^+$  so  $\beta \in \bigcup \{\beta^+ \mid \beta \in \alpha\}$ . This shows that  $\alpha \subseteq \bigcup \{\beta^+ \mid \beta \in \alpha\}$

**Exercise 27** PROOF:

⟨1⟩1. For natural numbers  $m$  and  $n$ , we have  $\text{rank } \langle m, n \rangle = \max(m, n)^{+++}$



PROOF:

$$\begin{aligned}
\text{rank}\{\{m\}, \{m, n\}\} &= (\text{rank}\{m\})^+ \cup (\text{rank}\{m, n\})^+ \\
&= (\text{rank } m)^{++} \cup ((\text{rank } m)^+ \cup (\text{rank } n)^+)^+ \\
&= m^{++} \cup (m^+ \cup n^+)^+ \quad (\text{Exercise 26}) \\
&= \max(m, n)^{++}
\end{aligned}$$

$\langle 1 \rangle 2$ . For any integer  $a$  we have  $\text{rank } a = \omega$

PROOF: For any natural numbers  $m$  and  $n$ , we have

$$\begin{aligned}
\text{rank}[\langle m, n \rangle] &= \bigcup \{(\text{rank}\langle p, q \rangle)^+ : m + q = n + p\} \\
&= \bigcup \{\max(p, q)^+ : m + q = n + p\} \\
&= \omega
\end{aligned}$$

since for any natural number  $p > m$  there exists  $q$  such that  $m + q = n + p$ .

$\langle 1 \rangle 3$ . For any integers  $a$  and  $b$  we have  $\text{rank}\langle a, b \rangle = \omega^{++}$

PROOF:

$$\begin{aligned}
\text{rank}\{\{a\}, \{a, b\}\} &= (\text{rank}\{a\})^+ \cup (\text{rank}\{a, b\})^+ \\
&= (\text{rank } a)^{++} \cup ((\text{rank } a)^+ \cup (\text{rank } b)^+)^+ \\
&= \omega^{++} \cup (\omega^+ \cup \omega^+)^+ \\
&= \omega^{++}
\end{aligned}$$

$\langle 1 \rangle 4$ . For any rational  $q$  we have  $\text{rank } q = \omega^{+++}$

PROOF: Since every element of  $q$  has rank  $\omega^{++}$

$\langle 1 \rangle 5$ . For any real number  $r$  we have  $\text{rank } r = \omega^{++++}$

PROOF: Since every element of  $r$  has rank  $\omega^{+++}$ .

$\langle 1 \rangle 6$ .  $\text{rank } \mathbb{R} = \omega^{+++++}$

□

**Exercise 28** If  $X \in V_\alpha$  then  $X \subseteq V_\beta$  for some  $\beta \in \alpha$ . Hence  $\text{rank } X \subseteq \beta$  and so  $\text{rank } X \in \alpha$ .

Conversely, if  $\text{rank } X \in \alpha$  then  $X \in V_{(\text{rank } X)^+} \subseteq V_\alpha$ .

**Exercise 29** Direct proofs:

For any set  $a$ , there exists  $m \in \{a\}$  such that  $m \cap \{a\} = \emptyset$ . This  $m$  must be the set  $a$ , so  $a \cap \{a\} = \emptyset$ , meaning  $a \notin a$ .

For any sets  $a$  and  $b$ , there exists  $m \in \{a, b\}$  such that  $m \cap \{a, b\} = \emptyset$ . Now,  $m$  is either  $a$  or  $b$ . If  $m = a$  then  $a \cap \{a, b\} = \emptyset$  so  $b \notin a$ . And if  $m = b$  then  $b \cap \{a, b\} = \emptyset$  so  $a \notin b$ .

Consequences of part (c):

Assume  $a \in a$ . Define  $f : \omega \rightarrow \{a\}$  by  $f(n) = a$  for all  $n \in \omega$ . Then  $f(n^+) \in f(n)$  for all  $n$ , contradicting (c).

Assume now  $a \in b$  and  $b \in a$ . Define  $f : \omega \rightarrow \{a, b\}$  by  $f(n) = a$  if  $n$  is even,  $f(n) = b$  if  $n$  is odd. Then  $f(n^+) \in f(n)$  for all  $n$ , contradicting (c).

**Exercise 30**

$$\begin{aligned}
 \text{rank}\{a, b\} &= (\text{rank } a)^+ \cup (\text{rank } b)^+ && (\text{Theorem 7V(b)}) \\
 &= \max((\text{rank } a)^+, (\text{rank } b)^+) \\
 &= \max(\text{rank } a, \text{rank } b)^+
 \end{aligned}$$

We have

$$\begin{aligned}
 a &\subseteq V_{\text{rank } a} \\
 \therefore \mathcal{P}a &\subseteq \mathcal{P}V_{\text{rank } a} \\
 &= V_{(\text{rank } a)^+} \\
 \therefore \text{rank } \mathcal{P}a &\subseteq (\text{rank } a)^+ \\
 a &\in \mathcal{P}a \\
 \therefore \text{rank } a &\in \text{rank } \mathcal{P}a \\
 \therefore \text{rank } \mathcal{P}a &= (\text{rank } a)^+
 \end{aligned}$$

Now, for all  $x \in \bigcup a$ , there exists  $y$  such that  $x \in y \in a$ . Hence

$$\begin{aligned}
 \text{rank } x &\in \text{rank } y \in \text{rank } a \text{ .} \\
 \therefore (\text{rank } x)^+ &\in \text{rank } a \text{ .}
 \end{aligned}$$

So  $\text{rank } a$  is an upper bound for  $\{(\text{rank } x)^+ : x \in \bigcup a\}$ , and so

$$\text{rank } \bigcup a \subseteq \text{rank } a \text{ .}$$

**Exercise 31**

(a) If  $A \approx B$  and nothing of rank less than  $\text{rank } B$  is equinumerous to  $B$ , then  $\text{rank } B \in \text{rank } A$ , and so  $B \in V_{(\text{rank } A)^+}$ . So we can construct the set  $\text{kard } A$  by applying a Subset Axiom to  $V_{(\text{rank } A)^+}$ .

(b) There exists a set of rank  $\text{rank } A$  that is equinumerous with  $A$  (namely  $A$ !). Let  $\mu$  be the least ordinal  $\leq \text{rank } A$  such that there exists a set of rank  $\mu$  that is equinumerous with  $A$ . Pick a set  $B$  of rank  $\mu$  such that  $B \approx A$ . Then  $B \in \text{kard } A$ .

(c) Suppose  $\text{kard } A = \text{kard } B$ . Pick  $C \in \text{kard } A$ . Then  $C \approx A$  and  $C \approx B$ , so  $A \approx B$ .

Conversely, suppose  $A \approx B$ . Then we have  $(A \approx C \text{ and nothing of rank less than } \text{rank } C \text{ is equinumerous with } C) \text{ iff } (B \approx C \text{ and nothing of rank less than } \text{rank } C \text{ is equinumerous with } C)$ , i.e.  $\text{kard } A = \text{kard } B$ .

**Exercise 32** Similar to Exercise 31.

**Exercise 33** Suppose for a contradiction  $D$  is not a subset of  $B$ . Then  $D - B$  is nonempty. So by the Regularity Axiom, there exists  $m \in D - B$  such that  $m \cap (D - B) = \emptyset$ . Now, for all  $x \in m$ , we have  $x \in D$  (since  $D$  is a transitive set) and  $x \notin D - B$ , so we must have  $x \in B$ ; that is,  $m \subseteq B$ . But then  $m \in B$ , which is a contradiction.

**Exercise 34** PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\{x, \{x, y\}\} = \{u, \{u, v\}\}$

$\langle 1 \rangle 2$ .  $x = u$  or  $x = \{u, v\}$

$\langle 1 \rangle 3$ .  $u = x$  or  $u = \{x, y\}$

$\langle 1 \rangle 4$ .  $x \neq \{u, v\}$

$\langle 2 \rangle 1$ . ASSUME: for a contradiction  $x = \{u, v\}$

$\langle 2 \rangle 2$ .  $u = x$  or  $u = \{x, y\}$

$\langle 2 \rangle 3$ . CASE:  $u = x$

PROOF: In this case  $x = u \in \{u, v\} = x$  contradicting Theorem 7X(a).

$\langle 2 \rangle 4$ . CASE:  $u = \{x, y\}$

PROOF: In this case  $u \in x$  and  $x \in u$  contradicting Theorem 7X(b).

$\langle 1 \rangle 5$ .  $x = u$

$\langle 1 \rangle 6$ .  $\{x, y\} = \{u, v\}$

PROOF: We cannot have  $\{x, y\} = u$  because then we would have  $x \in x$  contradicting Theorem 7X(a).

$\langle 1 \rangle 7$ .  $y = u$  or  $y = v$

$\langle 1 \rangle 8$ .  $v = x$  or  $v = y$

$\langle 1 \rangle 9$ . If  $y = u$  and  $v = x$  then  $y = v$

$\langle 1 \rangle 10$ .  $y = v$

PROOF: Checking all the cases in  $\langle 1 \rangle 7$  and  $\langle 1 \rangle 8$ .

□

**Exercise 35** Suppose  $a^+ = b^+$ . Then  $a \in b^+$  so  $a = b$  or  $a \in b$ . Likewise  $b \in a^+$  so  $b = a$  or  $b \in a$ . We cannot have both  $a \in b$  and  $b \in a$  (Theorem 7X(b)), so we must have  $a = b$ .

**Exercise 36** We have that  $V_{\text{rank } S}$  is a transitive set and  $S \subseteq V_{\text{rank } S}$ , so  $TC S \subseteq V_{\text{rank } S}$ . Thus,  $\text{rank}(TC S) \leq \text{rank } S$ .

We also have  $S \subseteq TC S$  so  $\text{rank } S \leq \text{rank}(TC S)$ . Thus,  $\text{rank}(TC S) = \text{rank } S$ .

**Exercise 37** If  $\alpha$  is an ordinal then it is a transitive set and, for any distinct  $x, y \in \alpha$ , we have  $x \in y$  or  $y \in x$  (Theorem 7M).

Conversely, let  $\alpha$  be a transitive set such that, for any distinct  $x, y \in \alpha$ , we have  $x \in y$  or  $y \in x$ . We will prove that  $\alpha$  is well ordered by epsilon. It will follow by Theorem 7L that  $\alpha$  is an ordinal.

PROOF:

$\langle 1 \rangle 1$ .  $\epsilon_\alpha$  is transitive.

$\langle 2 \rangle 1.$  LET:  $x, y, z \in \alpha$  with  $x \in y$  and  $y \in z$   
 $\langle 2 \rangle 2.$   $x \neq z$   
 PROOF: Otherwise we would have  $x \in y \in x$  contradicting the Axiom of Regularity.  
 $\langle 2 \rangle 3.$   $x \in z$  or  $z \in x$   
 $\langle 2 \rangle 4.$   $z \notin x$   
 PROOF: By the Axiom of Regularity we cannot have  $x \in y \in z \in x$ .  
 $\langle 2 \rangle 5.$   $x \in z$   
 $\langle 1 \rangle 2.$   $\epsilon_\alpha$  is irreflexive.  
 PROOF: By the Axiom of Regularity.  
 $\langle 1 \rangle 3.$  For any  $x, y \in \alpha$  we have  $x \in y$  or  $x = y$  or  $y \in x$ .  
 PROOF: By assumption.  
 $\langle 1 \rangle 4.$  Any nonempty subset of  $\alpha$  has an  $\epsilon_\alpha$ -least element.  
 $\langle 2 \rangle 1.$  LET:  $A \subseteq \alpha$  be nonempty.  
 $\langle 2 \rangle 2.$  PICK  $m \in A$  such that  $m \cap A = \emptyset$   
 $\langle 2 \rangle 3.$  For all  $x \in A$  we have  $m \subseteq x$   
 PROOF: Since  $x \notin m$ .  
 $\square$

**Exercise 38** Let  $\lambda$  be a limit ordinal. We have  $\bigcup \lambda \subseteq \lambda$  because  $\lambda$  is a transitive set. Conversely, for all  $\alpha \in \lambda$  we have  $\alpha \in \alpha^+ \in \lambda$  so  $\alpha \in \bigcup \lambda$ .

**Exercise 39** An ordinal number is a transitive set of ordinals, hence a transitive set of transitive sets.

Conversely, let  $\alpha$  be a transitive set of transitive sets. We prove that  $\alpha$  is a set of ordinals. The result will follow by Corollary 7N (a).

So suppose for a contradiction that not every element in  $\alpha$  is an ordinal. Let  $A = \{x \in \alpha : x \text{ is not an ordinal}\}$ . Then  $A$  is nonempty. Pick  $m \in A$  such that  $m \cap A = \emptyset$ . Then  $m$  is a transitive set of ordinals, hence an ordinal. This is a contradiction.

## Chapter 8

# Chapter 8 — Ordinals and Order Types

### 8.1 Alephs

**Exercise 1** Let  $\gamma(f, y)$  be the formula:

Either

1.  $f$  is a function with domain 0 and  $y = 5$ ; or
2.  $f$  is a function whose domain is a successor ordinal  $\alpha^+$  and  $y = f(\alpha)^+$ ; or
3.  $f$  is a function whose domain is a limit ordinal  $\lambda$  and  $y = \bigcup(\text{ran } f)$ ; or
4. none of the above and  $y = \emptyset$ .

By transfinite recursion, construct a formula  $\phi(u, v)$  such that:

- for every ordinal  $\alpha$  there exists a unique  $y$  such that  $\phi(\alpha, y)$ ;
- whenever  $f$  is a function whose domain is an ordinal  $\alpha$  and  $\phi(\beta, f(\beta))$  for all  $\beta \in \alpha$ , then we have  $\phi(\alpha, y)$  iff  $\gamma(f, y)$  for all  $y$ .

For  $\alpha$  an ordinal, let  $t_\alpha$  be the unique set such that  $\phi(\alpha, t_\alpha)$ .

**Exercise 2** We prove that  $\forall \alpha \in \omega. t_\alpha = 5 + \alpha$  by induction on  $\alpha$ . We have  $t_0 = 5$  and if  $t_\alpha = 5 + \alpha$  then  $t_{\alpha^+} = (5 + \alpha)^+ = 5 + \alpha^+$ .

We now prove that if  $\omega \subseteq \alpha$  then  $t_\alpha = \alpha$  by transfinite induction on  $\alpha$ . We have

$$t_\omega = \bigcup_{n \in \omega} (5 + n) = \omega$$

If  $\omega \subseteq \alpha$  and  $t_\alpha = \alpha$  then  $t_{\alpha^+} = \alpha^+$ .

If  $\lambda$  is a limit ordinal and  $t_\alpha = \alpha$  for all  $\alpha$  with  $\omega \subseteq \alpha \in \lambda$  then

$$\begin{aligned} t_\lambda &= \bigcup_{\alpha \in \lambda} t_\alpha \\ &= \bigcup_{\omega \subseteq \alpha \in \lambda} t_\alpha \\ &= \bigcup_{\omega \subseteq \alpha \in \lambda} \alpha \\ &= \lambda \end{aligned}$$

**Exercise 3** If  $\beta \in \gamma$  then  $t_\beta \in t_\gamma$  by the definition of monotonicity.

Conversely, suppose  $t_\beta \in t_\gamma$ . Then  $t_\beta \neq t_\gamma$  and  $t_\gamma \notin t_\beta$ , so  $\beta \neq \gamma$  and  $\gamma \notin \beta$ . Hence  $\beta \in \gamma$  by trichotomy.

Now suppose  $t_\beta = t_\gamma$ . Then  $t_\beta \notin t_\gamma$  and  $t_\gamma \notin t_\beta$ , hence  $\beta \notin \gamma$  and  $\gamma \notin \beta$ , and therefore  $\beta = \gamma$  by trichotomy.

**Exercise 4** We have  $t_\lambda \neq 0$  because  $t_0 \in t_\lambda$ .

Now, suppose for a contradiction  $t_\lambda = \alpha^+$  for some  $\alpha$ . Then we have  $\alpha \in t_\lambda = \bigcup_{\beta \in \lambda} t_\beta$ . Hence  $\alpha \in t_\beta$  for some  $\beta \in \lambda$ . Therefore,

$$\begin{aligned} \alpha^+ &\subseteq t_\beta \\ \therefore \alpha^+ &\in t_{\beta^+} \\ \therefore \alpha^{++} &\subseteq t_{\beta^+} \\ \therefore \alpha^{++} &\subseteq t_\lambda \end{aligned}$$

which is a contradiction.

**Exercise 5** The proof is by transfinite induction on  $\beta$ .

We have  $0 \subseteq t_0$ .

If  $\beta \subseteq t_\beta$  then  $\beta \in t_{\beta^+}$ , hence  $\beta^+ \subseteq t_{\beta^+}$ .

If  $\lambda$  is a limit ordinal and  $\forall \beta \in \lambda. \beta \subseteq t_\beta$  then

$$\begin{aligned} t_\lambda &= \sup_{\beta \in \lambda} t_\beta \\ &\supseteq \sup_{\beta \in \lambda} \beta \\ &= \lambda \end{aligned}$$

**Exercise 6** The class is closed by Theorem Schema 8E. It is unbounded because, for any ordinal  $\alpha$ , we have  $\alpha \in \alpha^+ \subseteq t_{\alpha^+}$  by Exercise 5.

**Exercise 7** Let  $\gamma$  be any fixed point of  $t$  with  $\beta \in \gamma$ . Then we have  $f(0) \in \gamma$ ; and, if  $f(n) \in \gamma$ , then

$$\begin{aligned} f(n^+) &= t_{f(n)} \\ &\subseteq t_\gamma \\ &= \gamma \end{aligned}$$

Hence by induction  $f(n) \in \gamma$  for all  $n$ , and so  $\lambda \in \gamma$ . Thus  $\lambda$  is the least fixed point of  $t$ .

**Exercise 8** Monotonicity holds by the analogue of Theorem 8A (see the second Example on page 216).

For continuity, let  $\lambda$  be a limit ordinal. We must prove that  $\bigcup_{\beta \in \lambda} t'_\beta$  is the least fixed point of  $t$  different from  $t'_\beta$  for all  $\beta \in \lambda$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\mu = \bigcup_{\beta \in \lambda} t'_\beta$   
 $\langle 1 \rangle 2$ .  $\mu$  is a fixed point of  $t$

PROOF:

$$\begin{aligned} t_\mu &= \bigcup_{\beta \in \lambda} t_{t'_\beta} && \text{(Theorem Schema 8E)} \\ &= \bigcup_{\beta \in \lambda} t'_\beta && (t'_\beta \text{ is a fixed point of } t) \\ &= \mu \end{aligned}$$

- $\langle 1 \rangle 3$ .  $\forall \beta \in \lambda. \mu \neq t'_\beta$

PROOF: Because  $t'_\beta \in t'_{\beta+} \subseteq \mu$ .

- $\langle 1 \rangle 4$ . If  $\gamma$  is a fixed point of  $t$  and  $\forall \beta \in \lambda. \gamma \neq t'_\beta$  then  $\mu \subseteq \gamma$

PROOF: We have  $\forall \beta \in \lambda. t'_\beta \in \gamma$  hence  $\mu \subseteq \gamma$ .

□

## 8.2 Isomorphism Types

**Exercise 9** Pick  $a \in A$ . For any set  $x \notin A$ , let  $A' = A - \{a\} \cup \{x\}$ , and let  $R'$  be the relation formed by replacing any pair  $\langle a, y \rangle$  with  $\langle x, y \rangle$ , any pair  $\langle y, a \rangle$  with  $\langle y, x \rangle$ , and  $\langle a, a \rangle$  with  $\langle x, x \rangle$  if  $aRa$ . Then  $\langle A, R \rangle \cong \langle A', R' \rangle$  and  $\text{rank} \langle A', R' \rangle > \text{rank } x$ .

Hence for every ordinal  $\alpha$  there is a structure isomorphic to  $\langle A, R \rangle$  with  $\text{rank} > \alpha$ . Thus the class of structures isomorphic to  $\langle A, R \rangle$  is not a set, because the ranks of its members are unbounded.

### Exercise 10

- (a) The only set equinumerous with 0 is 0, so  $\text{kard } 0 = \{0\}$ .

We have  $V_1 = \{\emptyset\} = \{0\}$  and  $V_2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$ . So 1 is the only set of rank 2 equinumerous with 1, and no set of rank  $< 2$  is equinumerous with 1. Hence  $\text{kard } 1 = \{1\}$ .

We have  $V_3 = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} = \{0, 1, \{1\}, 2\}$ . So 2 is the only set of rank 3 equinumerous with 2, and no set of rank  $< 3$  is equinumerous with 2. Thus  $\text{kard } 2 = \{2\}$ .

(b)  $\text{kard } 3$  is the set of all sets of rank 4 that are equinumerous with 3, i.e. the set of all subsets of  $V_3$  of cardinality 3. So

$$\text{kard } 3 = \{\{0, 1, \{1\}\}, 3, \{0, \{1\}, 2\}, \{1, \{1\}, 2\}\}.$$

### 8.3 Arithmetic of Order Types

**Exercise 11** Pick structures  $\langle A, R \rangle$  and  $\langle B, S \rangle$  of order type  $\rho$  and  $\sigma$  respectively. Define  $R'$  on  $A \times \{0\}$  by:  $\langle a, 0 \rangle R' \langle a', 0 \rangle$  iff  $aRa'$ . Define  $S'$  on  $B \times \{1\}$  by:  $\langle b, 1 \rangle S' \langle b', 1 \rangle$  iff  $bSb'$ . Then  $\langle A \times \{0\}, R' \rangle$  has order type  $\rho$ ,  $\langle B \times \{1\}, S' \rangle$  has order type  $\sigma$ , and  $(A \times \{0\}) \cap (B \times \{1\}) = \emptyset$ .

**Exercise 12** Since we have:

$$\begin{aligned} \langle 0, a \rangle <_L \langle 0, a' \rangle &\text{ iff } aRa' \\ \langle 1, b \rangle <_L \langle 1, b' \rangle &\text{ iff } bSb' \\ \langle 0, a \rangle <_L \langle 1, b \rangle &\text{ for all } a \in A \text{ and } b \in B \\ \langle 1, b \rangle \not<_L \langle 0, a \rangle &\text{ for all } a \in A \text{ and } b \in B \end{aligned}$$

**Exercise 13** If  $f$  is an isomorphism between  $\langle A, R \rangle$  and  $\langle A', R' \rangle$ , and  $g$  is an isomorphism between  $\langle B, S \rangle$  and  $\langle B', S' \rangle$ , and  $A \cap B = A' \cap B' = \emptyset$ , then  $f \cup g$  is an isomorphism between  $\langle A \cup B, R \oplus S \rangle$  and  $\langle A' \cup B', R' \oplus S' \rangle$ .

If  $f$  is an isomorphism between  $\langle A, R \rangle$  and  $\langle A', R' \rangle$ , and  $g$  is an isomorphism between  $\langle B, S \rangle$  and  $\langle B', S' \rangle$ , then the function  $h : A \times B \rightarrow A' \times B'$  defined by

$$h(\langle a, b \rangle) = \langle f(a), g(b) \rangle$$

is an isomorphism between  $\langle A \times B, R * S \rangle$  and  $\langle A' \times B', R' * S' \rangle$ .

**Exercise 14** Let  $\langle A, R \rangle$  be a structure of order type  $\rho$  and  $\langle B, S \rangle$  a structure of order type  $\sigma$ . Then  $A \times B \approx \emptyset$  so  $A \times B = \emptyset$ . Therefore  $A = \emptyset$  or  $B = \emptyset$ , and so  $\rho = 0$  or  $\sigma = 0$ .

**Exercise 15**

$$\begin{aligned} (\bar{\omega} + \bar{1}) \cdot \bar{2} &= \bar{\omega} + \bar{1} + \bar{\omega} + \bar{1} \\ &= \bar{\omega} + \bar{\omega} + \bar{1} \\ &\neq \bar{\omega} + \bar{\omega} + \bar{2} \\ &= (\bar{\omega} \cdot \bar{2}) + (\bar{1} \cdot \bar{2}) \end{aligned}$$



**Exercise 16** Let  $\langle A, R \rangle$  be a structure of order type  $\rho$ .

We have  $\langle A \cup \emptyset, R \oplus \emptyset \rangle = \langle \emptyset \cup A, \emptyset \oplus R \rangle = \langle A, R \rangle$  so  $\rho + \bar{0} = \bar{0} + \rho = \rho$ .

Now,  $\langle 1, \emptyset \rangle$  is a structure of order type  $\bar{1}$ . We have  $\langle A \times 1, R * \emptyset \rangle = \langle 1 \times A, \emptyset * R \rangle = \langle A, R \rangle$  so  $\rho \cdot \bar{1} = \bar{1} \cdot \rho = \rho$ .

We have  $\langle A \times \emptyset, R * \emptyset \rangle = \langle \emptyset \times A, \emptyset * R \rangle = \langle \emptyset, \emptyset \rangle$ .

**Exercise 17** Pick an enumeration  $A = \{a_0, a_1, \dots\}$  of  $A$ . Define  $f : A \rightarrow \mathbb{Q}$  by recursion as follows:

Let  $f(a_0) = 0$ .

Given  $f(a_0), f(a_1), \dots, f(a_n)$ , we have the following three possibilities:

- $a_{n+1}$  is smaller than all of  $a_0, \dots, a_n$ . In this case, let  $a_k$  be the minimum of  $a_0, \dots, a_n$ , and set  $f(a_{n+1}) = f(a_k) - 1$
- $a_{n+1}$  is larger than all of  $a_0, \dots, a_n$ . In this case, let  $a_k$  be the maximum of  $a_0, \dots, a_n$ , and set  $f(a_{n+1}) = f(a_k) + 1$
- Otherwise, let  $a_i$  be the largest element of  $a_0, \dots, a_n$  such that  $a_i < a_{n+1}$ , and  $a_j$  the smallest element such that  $a_{n+1} < a_j$ . Set  $f(a_{n+1}) = (f(a_i) + f(a_j))/2$ .

Then we have  $a_i < a_j$  iff  $f(a_i) < f(a_j)$  for all  $i, j$ . Hence  $f$  is an isomorphism between  $\langle A, R \rangle$  and  $\langle f[A], <^\circ \rangle$ .

**Exercise 18** Pick enumerations  $\{a_0, a_1, \dots\}$  of  $A$  and  $\{b_0, b_1, \dots\}$  of  $B$ .

Define isomorphisms  $F_n \subseteq A \times B$  by recursion on  $n$  in such a way that each  $F_n$  is an isomorphism between a subset of  $A_n$  of  $A$  and a subset  $B_n$  of  $B$  such that:

- For all  $n$  we have  $a_n \in A_{2n}$
- For all  $n$  we have  $b_n \in B_{2n+1}$

as follows.

$$F_0 = \{\langle a_0, b_0 \rangle\}$$

Given  $F_{2n}$ , if  $b_n \in B_{2n}$  then  $F_{2n+1} = F_{2n}$ . Otherwise:

- if  $b_n$  is greater than every element in  $B_{2n}$ , then let  $m$  be least such that  $a_m$  is larger than every element of  $A_{2n}$  (here we use the fact that  $A$  has no largest element) and set  $F_{2n+1} = F_{2n} \cup \{\langle a_m, b_n \rangle\}$
- if  $b_n$  is smaller than every element in  $B_{2n}$ , then let  $m$  be least such that  $a_m$  is smaller than every element of  $A_{2n}$  (here we use the fact that  $A$  has no smallest element) and set  $F_{2n+1} = F_{2n} \cup \{\langle a_m, b_n \rangle\}$
- otherwise let  $b$  be the greatest element in  $B_{2n}$  such that  $b < b_n$ , and  $b'$  the least element in  $B_{2n}$  such that  $b_n < b'$ . Let  $a = F_{2n}^{-1}(b)$  and  $a' = F_{2n}^{-1}(b')$ . Let  $m$  be least such that  $a < a_m < a'$  (here we use the fact that  $A$  is dense). Let  $F_{2n+1} = F_{2n} \cup \{\langle a_m, b_n \rangle\}$

In every case we have that  $F_{2n+1}$  is an isomorphism between a subset of  $A$  and a subset of  $B$  that contains  $b_n$ .

Similarly, given  $F_{2n+1}$ , we can define  $F_{2n+2}$  to be an isomorphism between a subset of  $A$  that contains  $a_n$  and a subset of  $B$ .

Now, let  $f = \bigcup_n F_n$ . Then  $f$  is an isomorphism between  $\langle A, R \rangle$  and  $\langle B, S \rangle$ .

**Exercise 19** This holds because the concatenation of  $\mathbb{Q}$  with itself, and the lexicographic ordering on  $\mathbb{Q}^2$ , are dense linear orderings on countable nonempty sets.

## 8.4 Ordinal Arithmetic

**Exercise 20** PROOF:

- ⟨1⟩1. For every ordinal  $\alpha$ , there exists an ordinal  $\lambda$  that is either a limit ordinal or 0 and a natural number  $n$  such that  $\alpha = \lambda + n$
- ⟨2⟩1.  $0 = 0 + 0$
- ⟨2⟩2. If  $\alpha = \lambda + n$  then  $\alpha^+ = \lambda + n^+$
- ⟨2⟩3. For  $\lambda$  a limit ordinal we have  $\lambda = \lambda + 0$
- ⟨1⟩2. If  $\lambda, \mu$  are either limit ordinals or 0, and  $m, n \in \omega$ , and  $\lambda + m = \mu + n$ , then  $\lambda = \mu$  and  $m = n$
- ⟨2⟩1. LET:  $P(m)$  be the property: for all  $\lambda, \mu$  and  $n \in \omega$ , if  $\lambda$  and  $\mu$  are either limit ordinals or 0 and  $\lambda + m = \mu + n$ , then  $\lambda = \mu$  and  $m = n$
- ⟨2⟩2.  $P(0)$ 
  - ⟨3⟩1. ASSUME:  $\lambda + 0 = \mu + n$
  - ⟨3⟩2.  $n = 0$
  - PROOF: Otherwise  $\lambda = \mu + n$  would be a successor ordinal.
  - ⟨3⟩3.  $\lambda = \mu$
- ⟨2⟩3.  $\forall m \in \omega. P(m) \Rightarrow P(m^+)$ 
  - ⟨3⟩1. LET:  $m \in \omega$
  - ⟨3⟩2. ASSUME:  $P(m)$
  - ⟨3⟩3. ASSUME:  $\lambda + m^+ = \mu + n$
  - ⟨3⟩4.  $n \neq 0$
  - PROOF: Otherwise  $\mu = \lambda + m^+$  is a successor ordinal.
  - ⟨3⟩5. PICK  $p$  such that  $n = p^+$
  - ⟨3⟩6.  $(\lambda + m)^+ = (\mu + p)^+$
  - ⟨3⟩7.  $\lambda + m = \mu + p$
  - ⟨3⟩8.  $\lambda = \mu$  and  $m = p$
  - PROOF: By ⟨3⟩2
  - ⟨3⟩9.  $m^+ = n$

□

**Exercise 21** 1 is the least integer in the ordering, followed by all the integers with exactly one prime factor, then all the integers with two prime factors, etc. So the ordinal is  $1 + \omega \cdot \omega = \omega^2$ .

**Exercise 22**

(a) If  $\beta \in \gamma$  then  $\beta + 0 = \beta \in \gamma = \gamma + 0$ .

If  $\beta + \alpha \in \gamma + \alpha$  then  $\beta + \alpha^+ = (\beta + \alpha)^+ \in (\gamma + \alpha)^+ = \gamma + \alpha^+$ .

For  $\lambda$  a limit ordinal, if  $\forall \alpha \in \lambda. \beta + \alpha \in \gamma + \alpha$ , then we have  $\beta + \lambda = \sup_{\alpha \in \lambda} (\beta + \alpha) \in \sup_{\alpha \in \lambda} (\gamma + \alpha) = \gamma + \lambda$ .

(b) We have  $\beta \cdot 0 = 0 = \gamma \cdot 0$ .

If  $\beta \in \gamma$  and  $\beta \cdot \alpha \in \gamma \cdot \alpha$  then  $\beta \cdot \alpha^+ = \beta \cdot \alpha + \beta \in \gamma \cdot \alpha + \gamma = \gamma \cdot \alpha^+$  using part (a).

For  $\lambda$  a limit ordinal, if  $\forall \alpha \in \lambda. \beta \cdot \alpha \in \gamma \cdot \alpha$ , then we have  $\beta \cdot \lambda = \sup_{\alpha \in \lambda} (\beta \cdot \alpha) \in \sup_{\alpha \in \lambda} (\gamma \cdot \alpha) = \gamma \cdot \lambda$ .

**Exercise 23**

(a)

$$\begin{aligned} \omega + \omega^2 &= \omega \cdot 1 + \omega \cdot \omega \\ &= \omega \cdot (1 + \omega) && \text{(Theorem 8K)} \\ &= \omega \cdot \omega && \text{(Example on page 228)} \\ &= \omega^2 \end{aligned}$$

(b) Let  $\omega^2 \in \beta$ . Let  $\gamma$  be the ordinal such that  $\beta = \omega^2 + \gamma$  (Subtraction Theorem). Then

$$\begin{aligned} \omega + \beta &= \omega + \omega^2 + \gamma \\ &= \omega + \gamma \\ &= \beta \end{aligned}$$

**Exercise 24** We prove first that  $1 + \alpha = \alpha$ . Let  $\gamma$  be the ordinal such that  $\alpha = \omega + \gamma$ . Then

$$\begin{aligned} 1 + \alpha &= 1 + \omega + \gamma \\ &= \omega + \gamma && \text{(Example on page 228)} \\ &= \alpha \end{aligned}$$

Hence

$$\begin{aligned} \alpha + \alpha^2 &= \alpha \cdot (1 + \alpha) \\ &= \alpha^2 \end{aligned}$$

Now, let  $\delta$  be the ordinal such that  $\beta = \alpha^2 + \delta$ . Then

$$\begin{aligned} \alpha + \beta &= \alpha + \alpha^2 + \delta \\ &= \alpha^2 + \delta \\ &= \beta \end{aligned}$$

**Exercise 25** Let  $\beta = \alpha \cup \{\alpha + \delta : \delta \in \theta\}$ . Then  $\beta$  is a transitive set of ordinals, hence an ordinal. We also have  $\alpha \subseteq \beta$ . By the Subtraction Theorem, let  $\gamma$  be the ordinal such that

$$\beta = \alpha + \gamma .$$

For any  $\delta \in \theta$  we have  $\alpha + \delta \in \beta$  hence  $\delta \in \gamma$  (Corollary 8P). Thus  $\theta \subseteq \gamma$ .

We have  $\alpha + \theta \notin \beta$  (since  $\alpha + \theta \notin \alpha$  and  $\alpha + \theta \neq \alpha + \delta$  for any  $\delta \in \theta$ ). So  $\theta \notin \gamma$  (Corollary 8P).

Thus  $\theta = \gamma$ , and so  $\beta = \alpha + \theta$ .

**Exercise 26** Follows just by repeated application of uniqueness in the Logarithm Theorem.

**Exercise 27**

**Theorem 8R** If  $\alpha = 0$ , then both sides are 1 if  $\beta = \gamma = 0$  and 0 otherwise. If  $\alpha = 1$  then both sides are 1.

**Theorem 8S** If  $\alpha = 0$ , and either  $\beta = 0$  or  $\gamma = 0$ , then both sides are 1. If  $\alpha = 0$  and  $\beta$  and  $\gamma$  are both non-zero, then both sides are 0. If  $\alpha = 1$  then both sides are 1.

**Exercise 28** This follows immediately from a Veblen Fixed-Point Theorem.

**Exercise 29** Let  $S$  be a nonempty set of epsilon numbers. Then

$$\begin{aligned} \omega^{\sup S} &= \sup_{\alpha \in S} \omega^\alpha && \text{(Theorem Schema 8E)} \\ &= \sup_{\alpha \in S} \alpha \\ &= \sup S \end{aligned}$$