

Solutions Manual for Enderton *Elements of Set  
Theory*

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# Chapter 1

## Chapter 1 — Introduction

### 1.1 Baby Set Theory

#### Exercise 1

- $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$  — true
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$  — true
- $\{\emptyset\} \in \{\emptyset, \{\{\emptyset\}\}\}$  — false
- $\{\emptyset\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$  — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset\}\}$  — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$  — true
- $\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$  — true
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\{\emptyset\}\}\}$  — false
- $\{\{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$  — false
- $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$  — false

**Exercise 2** We have  $\emptyset \neq \{\emptyset\}$  because  $\{\emptyset\}$  has an element (namely  $\emptyset$ ) while  $\emptyset$  has no elements.

We have  $\emptyset \neq \{\{\emptyset\}\}$  because  $\{\{\emptyset\}\}$  has an element (namely  $\{\emptyset\}$ ) while  $\emptyset$  has no elements.

We have  $\{\emptyset\} \neq \{\{\emptyset\}\}$  because  $\emptyset \in \{\emptyset\}$  but  $\emptyset \notin \{\{\emptyset\}\}$ . This last fact is true because  $\emptyset \neq \{\emptyset\}$  as we proved in the first paragraph.

**Exercise 3** Assume  $B \subseteq C$ . Let  $A \in \mathcal{P}B$ ; we must show that  $A \in \mathcal{P}C$ .

We have  $A \subseteq B$  (since  $A \in \mathcal{P}B$ ) and  $B \subseteq C$ . From this it follows that  $A \subseteq C$  (every element of  $A$  is an element of  $B$ ; every element of  $B$  is an element of  $C$ ; therefore every element of  $A$  is an element of  $C$ ). Hence  $A \in \mathcal{P}C$  as required.

**Exercise 4** Since  $x \in B$ , we have  $\{x\} \subseteq B$  and so  $\{x\} \in \mathcal{P}B$ .

Since  $x \in B$  and  $y \in B$ , we have  $\{x, y\} \subseteq B$  and so  $\{x, y\} \in \mathcal{P}B$ .

From these two facts, it follows that  $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}B$  and so  $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}B$ .

## 1.2 Sets — An Informal View

**Exercise 5** We have

$$\begin{aligned} V_0 &= A \\ V_1 &= V_0 \cup \mathcal{P}V_0 \\ &= A \cup \mathcal{P}A \\ V_2 &= V_1 \cup \mathcal{P}V_1 \\ &= \{\emptyset, \{\emptyset\}\} \\ V_3 &= \mathcal{P}V_2 \\ &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \end{aligned}$$

We have  $\emptyset \subseteq V_0$  and so  $\emptyset \in V_1$ . Therefore  $\{\emptyset\} \subseteq V_1$  and so  $\{\emptyset\} \in V_2$ . Hence  $\{\{\emptyset\}\} \subseteq V_2$ .

We also have  $\{\{\emptyset\}\} \not\subseteq V_0$  because  $\{\emptyset\}$  is not an atom, and  $\{\{\emptyset\}\} \not\subseteq V_1$  since  $\{\emptyset\} \notin V_1$  because  $\emptyset$  is not an atom.

Thus the rank of  $\{\{\emptyset\}\}$  is 2.

Likewise we have  $\emptyset$  and  $\{\emptyset\}$  are both subsets of  $V_1$ , hence

$$\emptyset \in V_2, \quad \{\emptyset\} \in V_2$$

Thus  $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$  are all subsets of  $V_2$ , hence elements of  $V_3$ . Therefore,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq V_3$$

Now,  $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$  is not a subset of  $V_0$  (because  $\emptyset$  is not an atom.) It is not a subset of  $V_1$  ( $\{\emptyset\} \notin V_1$  because  $\emptyset$  is not an atom.) It is not a subset of  $V_2$  (we have  $\{\emptyset, \{\emptyset\}\} \notin V_2$  since  $\{\emptyset\} \notin V_1$ ).

Therefore the rank of  $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$  is 3.

**Exercise 6**

$$\begin{aligned}
V_1 &= V_0 \cup \mathcal{P}V_0 \\
&= A \cup \mathcal{P}V_0 && (\text{since } V_0 = A) \\
V_2 &= V_1 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_0 \cup \mathcal{P}V_1 \\
&= A \cup \mathcal{P}V_1 && (\text{since } \mathcal{P}V_0 \subseteq \mathcal{P}V_1 \text{ by Exercise 3}) \\
V_3 &= V_2 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_1 \cup \mathcal{P}V_2 \\
&= A \cup \mathcal{P}V_2 && (\text{since } \mathcal{P}V_1 \subseteq \mathcal{P}V_2 \text{ by Exercise 3}) \\
V_4 &= V_3 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_2 \cup \mathcal{P}V_3 \\
&= A \cup \mathcal{P}V_3 && (\text{since } \mathcal{P}V_2 \subseteq \mathcal{P}V_3 \text{ by Exercise 3})
\end{aligned}$$

**Exercise 7** In Exercise 5 we calculated  $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$   
Hence

$$\begin{aligned}
V_4 &= \mathcal{P}V_3 \\
&= \{\emptyset, \\
&\quad \{\emptyset\}, \\
&\quad \{\{\emptyset\}\}, \\
&\quad \{\{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
&\quad \}
\end{aligned}$$

## Chapter 2

# Chapter 2 — Axioms and Operations

### 2.1 Arbitrary Unions and Intersections

**Exercise 1**  $A \cap B \cap C$  is the set of all integers that are divisible by 4, 9 and 10, which is the same as the set of all integers that are divisible by 180.

**Exercise 2** Take  $A = \emptyset$  and  $B = \{\emptyset\}$ . Then  $\bigcup A = \bigcup B = \emptyset$  but  $A \neq B$ . (There are many other possible answers.)

**Exercise 3** Let  $b \in A$ . We must show that  $b \subseteq \bigcup A$ .

Let  $x$  be any element of  $b$ . We must show that  $x \in \bigcup A$ . We know that  $x \in b$  and  $b \in A$ , and so  $x \in \bigcup A$  by the definition of  $\bigcup A$ .

**Exercise 4** Suppose  $A \subseteq B$ . Let  $x \in \bigcup A$ . We must show that  $x \in \bigcup B$ .

Pick an element  $a \in A$  such that  $x \in a$ . Then  $a \in B$  because  $A \subseteq B$ . Since we know  $x \in a$  and  $a \in B$ , we know that  $x \in \bigcup B$ .

**Exercise 5** Assume that every member of  $\mathcal{A}$  is a subset of  $B$ . Let  $x \in \bigcup \mathcal{A}$ . We must show that  $x \in B$ .

Pick  $A \in \mathcal{A}$  such that  $x \in A$ . By our assumption, we have  $A \subseteq B$ . Since  $x \in A$  and  $A \subseteq B$ , we have  $x \in B$  as required.

**Exercise 6**

(a) We will show that  $\bigcup \mathcal{P}A \subseteq A$  and  $A \subseteq \bigcup \mathcal{P}A$ .

To show  $\bigcup \mathcal{P}A \subseteq A$ : This follows from Exercise 5, since every member of  $\mathcal{P}A$  is a subset of  $A$ .

To show  $A \subseteq \bigcup \mathcal{P}A$ : Let  $a \in A$ . Then we have  $a \in \{a\}$  and  $\{a\} \in \mathcal{P}A$  so  $a \in \bigcup \mathcal{P}A$ .

(b) To show  $A \subseteq \mathcal{P} \bigcup A$ : This holds because every element of  $A$  is a subset of  $\bigcup A$ , as we proved in Exercise 3.

Equality holds if and only if  $A = \mathcal{P}X$  for some set  $X$ .

Proof: If  $A = \mathcal{P} \bigcup A$  then of course  $A = \mathcal{P}X$  for some  $X$ .

Conversely, if  $A = \mathcal{P}X$ , then we have

$$\begin{aligned} \mathcal{P} \bigcup A &= \mathcal{P} \bigcup \mathcal{P}X \\ &= \mathcal{P}X && \text{(by part (a))} \\ &= A \end{aligned}$$

### Exercise 7

(a) For any set  $X$ ,

$$\begin{aligned} X &\in \mathcal{P}A \cap \mathcal{P}B \\ \Leftrightarrow X &\subseteq A \text{ and } X \subseteq B \\ \Leftrightarrow \text{Every member of } X &\text{ is a member of } A \text{ and a member of } B \\ \Leftrightarrow X &\subseteq A \cap B \\ \Leftrightarrow X &\in \mathcal{P}(A \cap B) \end{aligned}$$

(b) Let  $X \in \mathcal{P}A \cup \mathcal{P}B$ . Then either  $X \in \mathcal{P}A$  or  $X \in \mathcal{P}B$  (or both). If  $X \in \mathcal{P}A$ , then we have  $X \subseteq A$  and so  $X \subseteq A \cup B$  (because  $A \subseteq A \cup B$ ). Similarly if  $X \in \mathcal{P}B$  then we have  $X \subseteq A \cup B$ . So in either case  $X \subseteq A \cup B$ , hence  $X \in \mathcal{P}(A \cup B)$ .

Equality holds if and only if either  $A \subseteq B$  or  $B \subseteq A$ .

Proof: Suppose  $A \subseteq B$ . Then  $\mathcal{P}A \subseteq \mathcal{P}B$  (Chapter 1 Exercise 3) and so  $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}B$ . Also  $A \cup B = B$  so  $\mathcal{P}(A \cup B) = \mathcal{P}B$ . Thus  $\mathcal{P}A \cup \mathcal{P}B$  and  $\mathcal{P}(A \cup B)$  are equal.

Similarly if  $B \subseteq A$  then  $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$ .

Conversely, suppose  $\mathcal{P}A \cup \mathcal{P}B = \mathcal{P}(A \cup B)$ . We have  $A \cup B \in \mathcal{P}(A \cup B)$ , so  $A \cup B \in \mathcal{P}A \cup \mathcal{P}B$ . If  $A \cup B \in \mathcal{P}A$ , then we have  $B \subseteq A \cup B \subseteq A$ . And if  $A \cup B \in \mathcal{P}B$ , then we have  $A \subseteq A \cup B \subseteq B$ .

**Exercise 8** If  $A$  is a set such that every singleton belongs to  $A$ , then every set belongs to  $\bigcup A$ , contradicting Theorem 2A.

**Exercise 9** Let  $a = \{\emptyset\}$  and  $B = \{\{\emptyset\}\}$ . Then  $a \in B$  but  $\mathcal{P}a$  is not a subset of  $B$  because  $\emptyset \in \mathcal{P}a$  and  $\emptyset \notin B$ .

**Exercise 10** We must show that  $\mathcal{P}a \subseteq \mathcal{P} \bigcup B$ . So let  $X \in \mathcal{P}a$ . Then  $X \subseteq a$ ; we must show that  $X \subseteq \bigcup B$ .

Let  $x \in X$ ; we must show that  $x \in \bigcup B$ . We have  $x \in a$  (because  $x \in X$  and  $X \subseteq a$ ) and  $a \in B$ , hence  $x \in \bigcup B$  as required.

## 2.2 Algebra of Sets

**Exercise 11** For any  $x$  we have

$$\begin{aligned} x \in (A \cap B) \cup (A - B) &\Leftrightarrow (x \in A \& x \in B) \text{ or } (x \in A \& x \notin B) \\ &\Leftrightarrow x \in A \& (x \in B \text{ or } x \notin B) \\ &\Leftrightarrow x \in A \end{aligned}$$

Hence  $A = (A \cap B) \cup (A - B)$ .

For any  $x$  we have

$$\begin{aligned} x \in A \cup (B - A) &\Leftrightarrow x \in A \text{ or } (x \in B \& x \notin A) \\ &\Leftrightarrow x \in A \text{ or } x \in B \\ &\Leftrightarrow x \in A \cup B \end{aligned}$$

Hence  $A \cup (B - A) = A \cup B$ .

**Exercise 12** For any  $x$ ,

$$\begin{aligned} x \in C - (A \cap B) &\Leftrightarrow x \in C \& \neg(x \in A \& x \in B) \\ &\Leftrightarrow x \in C \& (x \notin A \text{ or } x \notin B) \\ &\Leftrightarrow (x \in C \& x \notin A) \text{ or } (x \in C \& x \notin B) \\ &\Leftrightarrow x \in (C - A) \cup (C - B) \end{aligned}$$

**Exercise 13** Suppose  $A \subseteq B$ . Let  $x \in C - B$ ; we must show  $x \in C - A$ . We have  $x \in C$  and  $x \notin B$ . Therefore  $x \notin A$ , since every member of  $A$  is a member of  $B$ . And so we have  $x \in C - A$  as required.

**Exercise 14** Let  $A = \{\emptyset\}$ ,  $B = \emptyset$  and  $C = \{\emptyset\}$ . Then  $A - (B - C) = A - \emptyset = \{\emptyset\}$  while  $(A - B) - C = \{\emptyset\} - C = \emptyset$ .

**Exercise 15**

(a) For any  $x$  we have the following eight possibilities:



$x \in A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \in A$	$x \in B$	$x \notin C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \in C$	$x \in A \cap (B + C)$	$x \in (A \cap B) + (A \cap C)$
$x \in A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \in B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \in C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A \cap (B + C)$	$x \notin (A \cap B) + (A \cap C)$

In every case, we have  $x \in A \cap (B + C) \Leftrightarrow x \in (A \cap B) + (A \cap C)$ .

(b) For any  $x$  we have the following eight possibilities:

$x \in A$	$x \in B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \in A$	$x \in B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \in A$	$x \notin B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \in B$	$x \in C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$
$x \notin A$	$x \in B$	$x \notin C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \in C$	$x \in A + (B + C)$	$x \in (A + B) + C$
$x \notin A$	$x \notin B$	$x \notin C$	$x \notin A + (B + C)$	$x \notin (A + B) + C$

In every case, we have  $x \in A + (B + C) \Leftrightarrow x \in (A + B) + C$ .

#### Exercise 16

$$[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A] = (A \cup B) - A \\ = B - A$$

#### Exercise 17

(a)  $\Leftrightarrow$  (b)

$A \subseteq B \Leftrightarrow$  Every element of  $A$  is an element of  $B$

$\Leftrightarrow$  There is no element of  $A$  that is not an element of  $B$

$\Leftrightarrow A - B = \emptyset$

(a)  $\Rightarrow$  (c) Suppose  $A \subseteq B$ . We have  $B \subseteq A \cup B$  from the definition of  $A \cup B$ ; we must prove that  $A \cup B \subseteq B$ . So let  $x \in A \cup B$ . Then  $x \in A$  or  $x \in B$ . But in either case  $x \in B$ , since  $x \in A \Rightarrow x \in B$ . Thus we have  $x \in B$  as required.

(c)  $\Rightarrow$  (a) We always have  $A \subseteq A \cup B$ . So if  $A \cup B = B$  then we have  $A \subseteq B$ .

(a)  $\Rightarrow$  (d) Suppose  $A \subseteq B$ . We have  $A \cap B \subseteq A$  from the definition of  $A \cap B$ ; we must prove that  $A \subseteq A \cap B$ . So let  $x \in A$ . Then  $x \in B$  since  $A \subseteq B$ , hence  $x \in A \cap B$  as required.

(d)  $\Rightarrow$  (a) We always have  $A \cap B \subseteq B$ . So if  $A \cap B = A$  then  $A \subseteq B$ .

**Exercise 18** We can make the following 16 sets:

- $\emptyset (= A - A)$
- $A - B$
- $A \cap B$
- $B - A$
- $S - (A \cup B)$
- $A$
- $A + B$
- $S - B$
- $B$
- $S - (A + B)$
- $S - A$
- $A \cup B$
- $S - (B - A)$
- $S - (A \cap B)$
- $S - (A - B)$

**Exercise 19** They are never equal, because for all  $A, B$ , we have  $\emptyset \in \mathcal{P}(A - B)$  but  $\emptyset \notin \mathcal{P}A - \mathcal{P}B$  since  $\emptyset \in \mathcal{P}B$ .

**Exercise 20** Assume  $A \cup B = A \cup C$  and  $A \cap B = A \cap C$ .

We first show  $B \subseteq C$ . Let  $x \in B$ ; we show  $x \in C$ . We have  $x \in A \cup B = A \cup C$ , so either  $x \in A$  or  $x \in C$ . If  $x \in C$ , we are done. If  $x \in A$ , then we have  $x \in A \cap B = A \cap C$ , and so  $x \in C$  in this case too.

We can show  $C \subseteq B$  similarly. Hence  $B = C$ .

**Exercise 21** For any  $x$ , we have

$$\begin{aligned}
 x \in \bigcup (A \cup B) &\Leftrightarrow \text{there exists } C \text{ such that } C \in A \cup B \text{ and } x \in C \\
 &\Leftrightarrow \text{there exists } C \in A \text{ such that } x \in C, \text{ or there exists } C \in B \text{ such that } x \in C \\
 &\Leftrightarrow x \in \bigcup A \cup \bigcup B
 \end{aligned}$$

**Exercise 22** For any  $x$ , we have

$$\begin{aligned}
 x \in \bigcap (A \cup B) &\Leftrightarrow \text{for all } C, \text{ if } C \in A \text{ or } C \in B \text{ then } x \in C \\
 &\Leftrightarrow \text{for all } C \in A \text{ we have } x \in C, \text{ and for all } C \in B \text{ we have } x \in C \\
 &\Leftrightarrow x \in \bigcap A \cap \bigcap B
 \end{aligned}$$

**Exercise 23** PROOF:

- $\langle 1 \rangle 1. A \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 1. \text{ LET: } x \in A$
- $\langle 2 \rangle 2. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 3. x \in A \cup X$
- $\langle 1 \rangle 2. \bigcap \mathcal{B} \subseteq \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 1. \text{ LET: } x \in \bigcap \mathcal{B}$
- $\langle 2 \rangle 2. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 3. x \in X$
- $\langle 2 \rangle 4. x \in A \cup X$
- $\langle 1 \rangle 3. \bigcap \{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcap \mathcal{B}$
- $\langle 2 \rangle 1. \text{ LET: } x \in \bigcap \{A \cup X \mid X \in \mathcal{B}\}$
- $\langle 2 \rangle 2. \text{ ASSUME: } x \notin A$
- PROVE:  $x \in \bigcap \mathcal{B}$
- $\langle 2 \rangle 3. \text{ LET: } X \in \mathcal{B}$
- $\langle 2 \rangle 4. x \in A \cup X$
- $\langle 2 \rangle 5. x \in X$

□

**Exercise 24**

(a)

$$\begin{aligned}
 Y \in \mathcal{P} \bigcap \mathcal{A} &\Leftrightarrow Y \subseteq \bigcap \mathcal{A} \\
 &\Leftrightarrow \forall y \in Y. \forall X \in \mathcal{A}. y \in X \\
 &\Leftrightarrow \forall X \in \mathcal{A}. \forall y \in Y. y \in X \\
 &\Leftrightarrow \forall X \in \mathcal{A}. Y \in \mathcal{P}X \\
 &\Leftrightarrow Y \in \bigcap \{\mathcal{P}X \mid X \in \mathcal{A}\}
 \end{aligned}$$

(b)  $\bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\} \subseteq \mathcal{P} \bigcup \mathcal{A}$

PROOF:

- $\langle 1 \rangle 1. \text{ LET: } Y \in \bigcup \{\mathcal{P}X \mid X \in \mathcal{A}\}$
- $\langle 1 \rangle 2. \text{ PICK } X \in \mathcal{A} \text{ such that } Y \in \mathcal{P}X$
- $\langle 1 \rangle 3. Y \subseteq X$
- $\langle 1 \rangle 4. Y \subseteq \bigcup \mathcal{A}$
- $\langle 1 \rangle 5. Y \in \mathcal{P} \bigcup \mathcal{A}$

Equality holds if and only if  $\bigcup \mathcal{A} \in \mathcal{A}$ .

- ⟨1⟩1. If  $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P}\bigcup \mathcal{A}$  then  $\bigcup \mathcal{A} \in \mathcal{A}$
  - ⟨2⟩1. ASSUME:  $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P}\bigcup \mathcal{A}$
  - ⟨2⟩2.  $\bigcup \mathcal{A} \in \bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\}$
  - ⟨2⟩3. PICK  $X \in \mathcal{A}$  such that  $\bigcup \mathcal{A} \in \mathcal{P}X$
  - ⟨2⟩4.  $X = \bigcup \mathcal{A}$
  - ⟨1⟩2. If  $\bigcup \mathcal{A} \in \mathcal{A}$  then  $\bigcup\{\mathcal{P}X \mid X \in \mathcal{A}\} = \mathcal{P}\bigcup \mathcal{A}$
- PROOF: If  $\bigcup \mathcal{A} \in \mathcal{A}$  then  $\mathcal{P}\bigcup \mathcal{A} \in \{\mathcal{P}X \mid X \in \mathcal{A}\}$ .  
 $\square$

**Exercise 25** We have  $A \cup \bigcup \mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$  if and only if  $A = \emptyset$  or  $\mathcal{B} \neq \emptyset$

- ⟨1⟩1. If  $A \cup \bigcup \mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$  then  $A = \emptyset$  or  $\mathcal{B} \neq \emptyset$
- PROOF: If  $A \cup \bigcup \mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$  and  $\mathcal{B} = \emptyset$  then
- $$A \cup \bigcup \emptyset = \bigcup \emptyset$$
- $$\therefore A = \emptyset$$
- ⟨1⟩2. If  $A = \emptyset$  then  $A \cup \bigcup \mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
- PROOF: Both sides are equal to  $\bigcup \mathcal{B}$
- ⟨1⟩3. If  $\mathcal{B} \neq \emptyset$  then  $A \cup \bigcup \mathcal{B} = \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
  - ⟨2⟩1. ASSUME:  $\mathcal{B} \neq \emptyset$
  - ⟨2⟩2.  $A \cup \bigcup \mathcal{B} \subseteq \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
  - ⟨3⟩1. LET:  $x \in A \cup \bigcup \mathcal{B}$
  - PROVE:  $x \in \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
  - ⟨3⟩2. CASE:  $x \in A$
  - ⟨4⟩1. PICK  $X \in \mathcal{B}$
  - PROOF: By ⟨2⟩1
  - ⟨4⟩2.  $x \in A \cup X$
  - ⟨3⟩3. CASE:  $x \in \bigcup \mathcal{B}$
  - ⟨4⟩1. PICK  $X \in \mathcal{B}$  such that  $x \in X$
  - ⟨4⟩2.  $x \in A \cup X$
  - ⟨2⟩3.  $\bigcup\{A \cup X \mid X \in \mathcal{B}\} \subseteq A \cup \bigcup \mathcal{B}$
  - ⟨3⟩1. LET:  $x \in \bigcup\{A \cup X \mid X \in \mathcal{B}\}$
  - ⟨3⟩2. PICK  $X \in \mathcal{B}$  such that  $x \in A \cup X$
  - ⟨3⟩3.  $X \subseteq \bigcup \mathcal{B}$
  - ⟨3⟩4.  $A \cup X \subseteq A \cup \bigcup \mathcal{B}$
  - ⟨3⟩5.  $x \in A \cup \bigcup \mathcal{B}$

## 2.3 Review Exercises

**Exercise 26** Sets  $A, B, D$  and  $F$  are all equal to each other. Sets  $C, E$  and  $G$  are equal to each other. None of the first list is equal to any of the second list.

**Exercise 27** Take  $A = \{\{0\}, \{1\}\}$  and  $B = \{\{1\}\}$ . Then  $A \cap B = \{\{1\}\}$  and

$$\begin{aligned}\bigcap A \cap \bigcap B &= \emptyset \cap \{1\} \\ &= \emptyset \\ \bigcap (A \cap B) &= \bigcap \{\{1\}\} \\ &= \{1\}\end{aligned}$$

**Exercise 28**

$$\bigcup \{\{3, 4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\} = \{3, 4, \{3\}, \{4\}\}$$

**Exercise 29**

(a)  $\emptyset$

(b) We have

$$\begin{aligned}\{\emptyset\} &\subseteq \mathcal{P}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{PP}\{\emptyset\} \\ \{\emptyset\} &\subseteq \mathcal{PP}\{\emptyset\} \\ \therefore \mathcal{P}\{\emptyset\} &\subseteq \mathcal{PPP}\{\emptyset\} \\ \therefore \bigcap \{\mathcal{PPP}\{\emptyset\}, \mathcal{PP}\{\emptyset\}, \mathcal{P}\{\emptyset\}\} &= \mathcal{PPP}\{\emptyset\} \cap \mathcal{PP}\{\emptyset\} \cap \mathcal{P}\{\emptyset\} \\ &= \mathcal{P}\{\emptyset\} \\ &= \{\emptyset, \{\emptyset\}\}\end{aligned}$$

**Exercise 30**

(a)  $\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$

(b)  $\{\emptyset, \{\emptyset\}\}$

(c)  $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(d)  $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

**Exercise 31**

(a)  $\{1, 2, 3, \emptyset\}$

(b)  $\emptyset$

(c)  $\emptyset$

(d)  $\emptyset$

**Exercise 32**

(a)  $a \cup b$

(b)  $a$

(c)

$$\begin{aligned} \bigcap \bigcup S \cup (\bigcup \bigcup S - \bigcup \bigcap S) &= (a \cap b) \cup ((a \cup b) - a) \\ &= (a \cap b) \cup (b - a) \\ &= b \end{aligned}$$

**Exercise 33** When  $a \neq b$ :

$$\begin{aligned} \bigcup (\bigcup S - \bigcap S) &= \bigcup (\{a, b\} - \{a\}) \\ &= \bigcup \{b\} \\ &= b \end{aligned}$$

When  $a = b$ :

$$\begin{aligned} \bigcup (\bigcup S - \bigcap S) &= \bigcup (\{a, b\} - \{a\}) \\ &= \bigcup \emptyset \\ &= \emptyset \end{aligned}$$

**Exercise 34** For any set  $S$ , we have

$$\begin{aligned} \emptyset &\subseteq \mathcal{P}S \\ \therefore \emptyset &\in \mathcal{P}\mathcal{P}S \\ \emptyset &\subseteq S \\ \therefore \emptyset &\in \mathcal{P}S \\ \therefore \{\emptyset\} &\subseteq \mathcal{P}S \\ \therefore \{\emptyset\} &\in \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} &\subseteq \mathcal{P}\mathcal{P}S \\ \therefore \{\emptyset, \{\emptyset\}\} &\in \mathcal{P}\mathcal{P}\mathcal{P}S \end{aligned}$$

**Exercise 35** Assume  $\mathcal{P}A = \mathcal{P}B$ . Then we have

$$\begin{aligned} A &\in \mathcal{P}A \\ \therefore A &\in \mathcal{P}B \\ \therefore A &\subseteq B \\ B &\in \mathcal{P}B \\ \therefore B &\in \mathcal{P}A \\ \therefore B &\subseteq A \\ \therefore A &= B \end{aligned}$$

**Exercise 36**

(a)

$$\begin{aligned} x \in A - (A \cap B) &\Leftrightarrow x \in A \ \& \ \neg(x \in A \ \& \ x \in B) \\ &\Leftrightarrow x \in A \ \& \ x \notin B \\ &\Leftrightarrow x \in A - B \end{aligned}$$

(b)

$$\begin{aligned} x \in A - (A - B) &\Leftrightarrow x \in A \ \& \ \neg(x \in A \ \& \ x \notin B) \\ &\Leftrightarrow x \in A \ \& \ x \in B \\ &\Leftrightarrow x \in A \cap B \end{aligned}$$

**Exercise 37**

(a)

$$\begin{aligned} x \in (A \cup B) - C &\Leftrightarrow (x \in A \text{ or } x \in B) \ \& \ x \notin C \\ &\Leftrightarrow (x \in A \ \& \ x \notin C) \text{ or } (x \in B \ \& \ x \notin C) \\ &\Leftrightarrow x \in (A - C) \cup (B - C) \end{aligned}$$

(b)

$$\begin{aligned} x \in A - (B - C) &\Leftrightarrow x \in A \ \& \ \neg(x \in B \ \& \ x \notin C) \\ &\Leftrightarrow x \in A \ \& \ (x \notin B \text{ or } x \in C) \\ &\Leftrightarrow (x \in A \ \& \ x \notin B) \text{ or } (x \in A \ \& \ x \in C) \\ &\Leftrightarrow x \in (A - B) \cup (A \cap C) \end{aligned}$$

(c)

$$\begin{aligned} x \in (A - B) - C &\Leftrightarrow x \in A \ \& \ x \notin B \ \& \ x \notin C \\ &\Leftrightarrow x \in A \ \& \ \neg(x \in B \vee x \in C) \\ &\Leftrightarrow x \in A - (B \cup C) \end{aligned}$$

**Exercise 38**

(a) If every element of  $A$  is an element of  $C$ , and every element of  $B$  is an element of  $C$ , then everything that is an element of either  $A$  or  $B$  is an element of  $C$ .

(b) If every element of  $C$  is an element of  $A$ , and every element of  $C$  is an element of  $B$ , then every element of  $C$  is an element of both  $A$  and  $B$ .



## Chapter 3

# Chapter 3 — Relations and Functions

### 3.1 Ordered Pairs

**Exercise 1** We have  $\langle 0, 1, 0 \rangle^* = \langle 0, 1, 1 \rangle^* = \{\{0\}, \{0, 1\}\}$ .

**Exercise 2**

(a)

$$\begin{aligned} z &\in A \times (B \cup C) \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ (y \in B \text{ or } y \in C)) \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in B) \text{ or } (z = (x, y) \ \& \ x \in A \ \& \ y \in C) \\ \Leftrightarrow z &\in (A \times B) \cup (A \times C) \end{aligned}$$

(b)

$\langle 1 \rangle 1$ . ASSUME:  $A \times B = A \times C$  and  $A \neq \emptyset$

$\langle 1 \rangle 2$ . PICK  $a \in A$

$\langle 1 \rangle 3$ . For all  $x$ ,  $x \in B \Leftrightarrow x \in C$

PROOF:  $x \in B$  iff  $(a, x) \in A \times B$  iff  $(a, x) \in A \times C$  iff  $x \in C$ .

□

**Exercise 3**

$$\begin{aligned} z &\in A \times \bigcup \mathcal{B} \\ \Leftrightarrow \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ \exists X \in \mathcal{B}. y \in X) \\ \Leftrightarrow \exists X \in \mathcal{B}. \exists x, y (z = (x, y) \ \& \ x \in A \ \& \ y \in X) \\ \Leftrightarrow z &\in \bigcup \{A \times X : X \in \mathcal{B}\} \end{aligned}$$

**Exercise 4** If every ordered pair belongs to  $A$  then every set belongs to  $\bigcup\bigcup A$  contradicting Theorem 2A.

**Exercise 5**

(a) Apply a Subset Axiom to  $\mathcal{P}(A \times B)$ : we have  $C = \{z \in \mathcal{P}(A \times B) \mid \exists x \in A. z = \{x\} \times B\}$ .

(b)

$$\begin{aligned} z &\in \bigcup C \\ \Leftrightarrow \exists x \in A. z &\in \{x\} \times B \\ \Leftrightarrow \exists x \in A. \exists y \in B. z &= (x, y) \\ \Leftrightarrow z &\in A \times B \end{aligned}$$

## 3.2 Relations

**Exercise 6** If  $A \subseteq \text{dom } A \times \text{ran } A$  then  $A$  is a set of ordered pairs, i.e. a relation.

Conversely, suppose  $A$  is a relation. Let  $z \in A$ . Then  $z$  is an ordered pair; let  $z = (x, y)$ . We have  $x \in \text{dom } A$  and  $y \in \text{ran } A$  and so  $z \in \text{dom } A \times \text{ran } A$  as required.

**Exercise 7** We have  $\text{fld } R \subseteq \bigcup\bigcup R$  by Lemma 3D.

Conversely, let  $x \in \bigcup\bigcup R$ . Pick  $a$  and  $b$  such that  $x \in a$ ,  $a \in b$  and  $b \in R$ . Then  $b$  is an ordered pair; let  $b = (y, z)$ . We have  $a = \{y\}$  or  $\{y, z\}$ , hence  $x = y$  or  $x = z$ . In either case,  $x \in \text{fld } R$ .

**Exercise 8**

(a)

$$\begin{aligned} x &\in \text{dom } \bigcup \mathcal{A} \\ \Leftrightarrow \exists y. \exists R \in \mathcal{A}. (x, y) &\in R \\ \Leftrightarrow \exists R \in \mathcal{A}. \exists y. (x, y) &\in R \\ \Leftrightarrow x &\in \bigcup \{\text{dom } R : R \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned}
y &\in \text{ran} \bigcup \mathcal{A} \\
&\Leftrightarrow \exists x. \exists R \in \mathcal{A}. (x, y) \in R \\
&\Leftrightarrow \exists R \in \mathcal{A}. \exists x. (x, y) \in R \\
&\Leftrightarrow y \in \bigcup \{\text{ran } R : R \in \mathcal{A}\}
\end{aligned}$$

**Exercise 9** Assume  $\mathcal{A}$  is nonempty. We have  $\text{dom} \bigcap \mathcal{A} \subseteq \bigcap \{\text{dom } R : R \in \mathcal{A}\}$ .

PROOF:

$$\begin{aligned}
x &\in \text{dom} \bigcap \mathcal{A} \\
&\Leftrightarrow \exists y. \forall R \in \mathcal{A}. (x, y) \in R \\
&\Rightarrow \forall R \in \mathcal{A}. \exists y. (x, y) \in R \\
&\Leftrightarrow x \in \bigcap \{\text{dom } R : R \in \mathcal{A}\}
\end{aligned}$$

Equality holds iff the middle ' $\Rightarrow$ ' can be reversed, i.e. iff for all  $x$ , if  $\forall R \in \mathcal{A}. \exists y. (x, y) \in R$  then  $\exists y. \forall R \in \mathcal{A}. (x, y) \in R$ . I haven't found a simpler condition than this. The condition does not always hold, for example if  $\mathcal{A} = \{\{(1, 2)\}, \{(1, 3)\}\}$  then  $\text{dom} \bigcap \mathcal{A} = \emptyset$  while  $\bigcap \{\text{dom } R : R \in \mathcal{A}\} = \{1\}$ .

Similarly,  $\text{ran} \bigcap \mathcal{A} \subseteq \bigcap \{\text{ran } R : R \in \mathcal{A}\}$ , and equality holds iff, for any  $y$ , if  $\forall R \in \mathcal{A}. \exists x. (x, y) \in R$  then  $\exists x. \forall R \in \mathcal{A}. (x, y) \in R$ .

### 3.3 $n$ -ary Relations

**Exercise 10** This follows from the equations at the top of page 42. An ordered 4-tuple  $\langle a, b, c, d \rangle$  is also an ordered 1-tuple (because every set is), and the ordered pair  $\langle \langle a, b, c \rangle, d \rangle$ , and the ordered triple  $\langle \langle a, b \rangle, c, d \rangle$ .

### 3.4 Functions

**Exercise 11** We prove  $F \subseteq G$ . Let  $z \in F$ . Since  $F$  is a relation, then  $z$  is an ordered pair; let  $z = \langle x, y \rangle$ . We have  $x \in \text{dom } F$  and  $y = F(x)$ . Therefore  $x \in \text{dom } G$  and  $y = G(x)$  (because  $\text{dom } F = \text{dom } G$  and  $F(x) = G(x)$ ). Hence  $\langle x, y \rangle \in G$ , i.e.  $z \in G$ .

We have proved  $F \subseteq G$ . We can prove  $G \subseteq F$  similarly. Thus  $F = G$ .

**Exercise 12** PROOF:

- $\langle 1 \rangle 1$ . If  $f \subseteq g$  then  $\text{dom } f \subseteq \text{dom } g$  and  $\forall x \in \text{dom } f. f(x) = g(x)$
- $\langle 2 \rangle 1$ . ASSUME:  $f \subseteq g$
- $\langle 2 \rangle 2$ . LET:  $x \in \text{dom } f$
- $\langle 2 \rangle 3$ .  $(x, f(x)) \in f$
- $\langle 2 \rangle 4$ .  $(x, f(x)) \in g$
- $\langle 2 \rangle 5$ .  $x \in \text{dom } g$  and  $g(x) = f(x)$

- ⟨1⟩2. If  $\text{dom } f = \text{dom } g$  and  $\forall x \in \text{dom } f. f(x) = g(x)$  then  $f \subseteq g$
- ⟨2⟩1. ASSUME:  $\text{dom } f = \text{dom } g$  and  $\forall x \in \text{dom } f. f(x) = g(x)$
- ⟨2⟩2. LET:  $z \in f$
- ⟨2⟩3. LET:  $z = (x, y)$
- ⟨2⟩4.  $x \in \text{dom } f$  and  $y = f(x)$
- ⟨2⟩5.  $x \in \text{dom } g$  and  $y = g(x)$
- ⟨2⟩6.  $z = (x, y) \in g$

□

**Exercise 13** PROOF:

- ⟨1⟩1. ASSUME:  $f$  and  $g$  are functions
- ⟨1⟩2. ASSUME:  $f \subseteq g$
- ⟨1⟩3. ASSUME:  $\text{dom } g \subseteq \text{dom } f$
- ⟨1⟩4.  $\text{dom } f = \text{dom } g$
- PROOF: We have  $\text{dom } f \subseteq \text{dom } g$  from ⟨1⟩2 and  $\text{dom } g \subseteq \text{dom } f$  from ⟨1⟩3
- ⟨1⟩5. For  $x \in \text{dom } f$  we have  $f(x) = g(x)$
- PROOF: From ⟨1⟩2 and Exercise 12
- ⟨1⟩6. Q.E.D.
- PROOF: From Exercise 11.

□

**Exercise 14**

(a) If  $(x, y)$  and  $(x, z)$  are members of  $f \cap g$  then they are both members of  $f$ , hence  $y = z$ .

(b) PROOF:

- ⟨1⟩1. If  $f \cup g$  is a function then, for all  $x \in \text{dom } f \cap \text{dom } g$ , we have  $f(x) = g(x)$ .
- ⟨2⟩1. ASSUME:  $f \cup g$  is a function.
- ⟨2⟩2. LET:  $x \in \text{dom } f \cap \text{dom } g$
- ⟨2⟩3.  $(x, f(x))$  and  $(x, g(x))$  are both elements of  $f \cup g$
- ⟨2⟩4.  $f(x) = g(x)$
- ⟨1⟩2. If, for all  $x \in \text{dom } f \cap \text{dom } g$ , we have  $f(x) = g(x)$ , then  $f \cup g$  is a function.
- ⟨2⟩1. ASSUME: For all  $x \in \text{dom } f \cap \text{dom } g$ , we have  $f(x) = g(x)$
- ⟨2⟩2.  $f \cup g$  is a relation.
- PROOF: Since every element of either  $f$  or  $g$  is an ordered pair.
- ⟨2⟩3. Whenever  $(x, y)$  and  $(x, z)$  are elements of  $f \cup g$  we have  $y = z$
- ⟨3⟩1. LET:  $(x, y), (x, z) \in f \cup g$
- ⟨3⟩2. CASE:  $(x, y), (x, z) \in f$
- PROOF: Then  $y = z$  since  $f$  is a function.
- ⟨3⟩3. CASE:  $(x, y) \in f, (x, z) \in g$
- PROOF: Then  $y = z$  by ⟨2⟩1
- ⟨3⟩4. CASE:  $(x, y) \in g, (x, z) \in f$
- PROOF: Then  $y = z$  by ⟨2⟩1
- ⟨3⟩5. CASE:  $(x, y), (x, z) \in g$

PROOF: Then  $y = z$  since  $g$  is a function.

□

**Exercise 15** PROOF:

⟨1⟩1.  $\bigcup \mathcal{A}$  is a relation.

PROOF: Since every member of  $\mathcal{A}$  is a relation.

⟨1⟩2. Whenever  $(x, y)$  and  $(x, z)$  are elements of  $\bigcup \mathcal{A}$  then  $y = z$

⟨2⟩1. LET:  $(x, y), (x, z) \in \bigcup \mathcal{A}$

⟨2⟩2. PICK  $f, g \in \mathcal{A}$  such that  $(x, y) \in f$  and  $(x, z) \in g$

⟨2⟩3. ASSUME: w.l.o.g.  $f \subseteq g$

⟨2⟩4.  $(x, y), (x, z) \in g$

⟨2⟩5.  $y = z$

PROOF: Since  $g$  is a function.

□

**Exercise 16** If every function belongs to  $\mathcal{A}$  then every set belongs to  $\text{dom} \bigcup \mathcal{A}$  contradiction Theorem 2A.

**Exercise 17** PROOF:

⟨1⟩1. LET:  $R$  and  $S$  be single-rooted.

⟨1⟩2. LET:  $(x, z), (y, z) \in R \circ S$

⟨1⟩3. PICK  $t$  and  $t'$  such that  $(x, t) \in S$ ,  $(t, z) \in R$ ,  $(y, t') \in S$  and  $(t', z) \in R$

⟨1⟩4.  $t = t'$

PROOF: Since  $R$  is single-rooted.

⟨1⟩5.  $x = y$

PROOF: Since  $S$  is single-rooted.

Thus if  $F$  and  $G$  are one-to-one functions then  $F \circ G$  is single-rooted and a function by Theorem 3H, hence a one-to-one function.

**Exercise 18**

$$R \circ R = \{\langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle\}$$

$$R \upharpoonright \{1\} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle\}$$

$$R^{-1} \upharpoonright \{1\} = \{\langle 1, 0 \rangle\}$$

$$R[\{1\}] = \{2, 3\}$$

$$R^{-1}[\{1\}] = \{0\}$$

**Exercise 19**

$$\begin{aligned}
A(\emptyset) &= \{\emptyset, \{\emptyset\}\} \\
A[\emptyset] &= \emptyset \\
A[\{\emptyset\}] &= \{\{\emptyset, \{\emptyset\}\}\} \\
A[\{\emptyset, \{\emptyset\}\}] &= \{\{\emptyset, \{\emptyset\}\}, \emptyset\} \\
A^{-1} &= \{\langle \{\emptyset, \{\emptyset\}\}, \emptyset \rangle, \langle \emptyset, \{\emptyset\} \rangle\} \\
A \circ A &= \{\langle \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \rangle\} \\
A \upharpoonright \emptyset &= \emptyset \\
A \upharpoonright \{\emptyset\} &= \{\langle \emptyset, \{\emptyset, \{\emptyset\}\} \rangle\} \\
A \upharpoonright \{\emptyset, \{\emptyset\}\} &= \{\langle \emptyset, \{\emptyset, \{\emptyset\}\} \rangle, \langle \{\emptyset\}, \emptyset \rangle\} \\
&= A \\
\bigcup A &= \{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}
\end{aligned}$$

**Exercise 20**

$$\begin{aligned}
z \in F \upharpoonright A &\Leftrightarrow z \in F \ \& \ \exists x, y. (z = \langle x, y \rangle \ \& \ x \in A) \\
&\Leftrightarrow z \in F \ \& \ \exists x, y. (z = \langle x, y \rangle \ \& \ x \in A \ \& \ y \in \text{ran } F) \\
&\Leftrightarrow z \in F \cap (A \times \text{ran } F)
\end{aligned}$$

**Exercise 21** Both are equal to  $\{\langle x, w \rangle \mid \exists y, z. xTy \ \& \ ySz \ \& \ zRw\}$ .

**Exercise 22**

(a) PROOF:  
 $\langle 1 \rangle 1$ . ASSUME:  $A \subseteq B$   
 $\langle 1 \rangle 2$ . LET:  $y \in F[A]$   
 $\langle 1 \rangle 3$ . PICK  $x \in A$  such that  $xFy$   
 $\langle 1 \rangle 4$ .  $x \in B$  and  $xFy$   
 $\square$

(b) Both are equal to  $\{z : \exists x, y. x \in A \ \& \ xGy \ \& \ yFz\}$

(c) Both are equal to  $\{\langle x, y \rangle : (x \in A \text{ or } x \in B) \ \& \ xQy\}$

**Exercise 23**

$$\begin{aligned}
B \circ I_A &= \{\langle x, z \rangle : \exists y(xI_A y \ \& \ yBz)\} \\
&= \{\langle x, z \rangle : \exists y(x \in A \ \& \ x = y \ \& \ yBz)\} \\
&= \{\langle x, z \rangle : x \in A \ \& \ xBz\} \\
&= B \upharpoonright A \\
I_A[C] &= \{y : \exists x \in C.xI_A y\} \\
&= \{y : \exists x \in C(x \in A \ \& \ x = y)\} \\
&= \{y : y \in C \ \& \ y \in A\} \\
&= A \cap C
\end{aligned}$$

**Exercise 24**

$$\begin{aligned}
F^{-1}[A] &= \{x : \exists y \in A.yF^{-1}x\} \\
&= \{x : \exists y \in A.xFy\} \\
&= \{x \in \text{dom } F : F(x) \in A\}
\end{aligned}$$

**Exercise 25****(a)** PROOF: $\langle 1 \rangle 1$ . LET:  $G$  be a one-to-one function. $\langle 1 \rangle 2$ .  $G^{-1}$  is a function.

PROOF: Theorem 3F.

 $\langle 1 \rangle 3$ .  $G \circ G^{-1}$  is a function.

PROOF: Theorem 3H.

 $\langle 1 \rangle 4$ .  $\text{dom}(G \circ G^{-1}) = \text{ran } G$ 

PROOF:

$$\text{dom}(G \circ G^{-1}) = \{x \in \text{dom } G^{-1} : G^{-1}(x) \in \text{dom } G\} \quad (\text{Theorem 3H})$$

$$= \{x \in \text{ran } G : G^{-1}(x) \in \text{dom } G\} \quad (\text{Theorem 3E})$$

$$= \text{ran } G$$

 $\langle 1 \rangle 5$ .  $\forall x \in \text{ran } G.(G \circ G^{-1})(x) = x$ 

PROOF: Theorem 3G.

□

**(b)** Let  $G$  be a function. Then

$$\begin{aligned}
G \circ G^{-1} &= \{\langle x, z \rangle : \exists y(xG^{-1}y \ \& \ yGz)\} \\
&= \{\langle x, z \rangle : \exists y(yGx \ \& \ yGz)\} \\
&= \{\langle x, x \rangle : \exists y.yGx\} & (G \text{ is a function}) \\
&= I_{\text{ran } G}
\end{aligned}$$

**Exercise 26**

(a)

$$\begin{aligned} F[\bigcup \mathcal{A}] &= \{y : \exists x. \exists A \in \mathcal{A} (x \in A \ \& \ xFy)\} \\ &= \{y : \exists A \in \mathcal{A}. \exists x (x \in A \ \& \ xFy)\} \\ &= \bigcup \{F[A] : A \in \mathcal{A}\} \end{aligned}$$

(b)

$$\begin{aligned} F[\bigcup \mathcal{A}] &= \{y : \exists x. \forall A \in \mathcal{A} (x \in A \ \& \ xFy)\} \\ &\subseteq \{y : \forall A \in \mathcal{A}. \exists x (x \in A \ \& \ xFy)\} \\ &= \bigcap \{F[A] : A \in \mathcal{A}\} \end{aligned}$$

**Exercise 27**

$$\begin{aligned} \text{dom}(F \circ G) &= \{x : \exists y. x(F \circ G)y\} \\ &= \{x : \exists y \exists z (xGz \ \& \ zFy)\} \\ &= \{x : \exists z (zG^{-1}x \ \& \ z \in \text{dom } F)\} \\ &= G^{-1}[\text{dom } F] \end{aligned}$$

**Exercise 28** PROOF:

$\langle 1 \rangle 1.$   $G : \mathcal{P}A \rightarrow \mathcal{P}B$

PROOF: Since  $f[X] \subseteq \text{ran } f \subseteq B$

$\langle 1 \rangle 2.$  For all  $X, Y \in \mathcal{P}A$ , if  $G(X) = G(Y)$  then  $X = Y$

$\langle 2 \rangle 1.$  LET:  $X, Y \in \mathcal{P}A$

$\langle 2 \rangle 2.$  ASSUME:  $f[X] = f[Y]$

$\langle 2 \rangle 3.$   $X \subseteq Y$

$\langle 3 \rangle 1.$  LET:  $x \in X$

$\langle 3 \rangle 2.$   $f(x) \in f[X]$

$\langle 3 \rangle 3.$   $f(x) \in f[Y]$

$\langle 3 \rangle 4.$  PICK  $y \in Y$  such that  $f(x) = f(y)$

$\langle 3 \rangle 5.$   $x = y$

PROOF: Because  $f$  is one-to-one.

$\langle 3 \rangle 6.$   $x \in Y$

PROOF: Similar.

$\langle 2 \rangle 4.$   $Y \subseteq X$

□

**Example 29** PROOF:

$\langle 1 \rangle 1.$  ASSUME:  $f$  maps  $A$  onto  $B$

$\langle 1 \rangle 2.$  LET:  $b, b' \in B$

$\langle 1 \rangle 3.$  ASSUME:  $G(b) = G(b')$

$\langle 1 \rangle 4.$  PICK  $x \in A$  such that  $f(x) = b$



PROOF: By  $\langle 1 \rangle 1$ .

$\langle 1 \rangle 5$ .  $x \in G(b)$

$\langle 1 \rangle 6$ .  $x \in G(b')$

$\langle 1 \rangle 7$ .  $f(x) = b'$

$\langle 1 \rangle 8$ .  $b = b'$

□

The converse does not hold. Let  $A = \{0\}$  and  $B = \{0, 1\}$ . Let  $f$  be the function that maps 0 to 0. Then

$$G(0) = \{0\}$$

$$G(1) = \emptyset$$

Thus  $G$  is one-to-one but  $f$  does not map  $A$  onto  $B$ .

### Exercise 30

(a) PROOF:

$\langle 1 \rangle 1$ .  $F(B) = B$

$\langle 2 \rangle 1$ .  $F(B) \subseteq B$

$\langle 3 \rangle 1$ . LET:  $X \in \mathcal{P}A$  be such that  $F(X) \subseteq X$

PROVE:  $F(B) \subseteq X$

$\langle 3 \rangle 2$ .  $B \subseteq X$

$\langle 3 \rangle 3$ .  $F(B) \subseteq F(X)$

$\langle 3 \rangle 4$ .  $F(B) \subseteq X$

PROOF: From  $\langle 3 \rangle 1$  and  $\langle 3 \rangle 3$ .

$\langle 2 \rangle 2$ .  $B \subseteq F(B)$

PROOF: From  $\langle 2 \rangle 1$  and the definition of  $B$ , since  $B$  is one of the sets  $X$  such that  $F(X) \subseteq X$

$\langle 1 \rangle 2$ .  $F(C) = C$

$\langle 2 \rangle 1$ .  $C \subseteq F(C)$

$\langle 3 \rangle 1$ . LET:  $X \in \mathcal{P}A$  with  $X \subseteq F(X)$

PROVE:  $X \subseteq F(C)$

$\langle 3 \rangle 2$ .  $X \subseteq C$

$\langle 3 \rangle 3$ .  $F(X) \subseteq F(C)$

$\langle 3 \rangle 4$ .  $X \subseteq F(C)$

PROOF: From  $\langle 3 \rangle 1$  and  $\langle 3 \rangle 3$

$\langle 2 \rangle 2$ .  $F(C) \subseteq C$

PROOF: From  $\langle 2 \rangle 1$  and the definition of  $C$ .

□

(b) If  $F(X) = X$  then we have  $B \subseteq X$  (because  $F(X) \subseteq X$ ) and  $X \subseteq C$  (because  $X \subseteq F(X)$ ).

### 3.5 Infinite Cartesian Products

**Exercise 31** PROOF:

- ⟨1⟩1. If the Axiom of Choice is true then, for any set  $I$  and any function  $H$  with domain  $I$ , if  $H(i) \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} H(i) \neq \emptyset$ .
- ⟨2⟩1. ASSUME: The Axiom of Choice.
- ⟨2⟩2. LET:  $I$  be a set.
- ⟨2⟩3. LET:  $H$  be a function with domain  $I$ .
- ⟨2⟩4. ASSUME:  $H(i) \neq \emptyset$  for all  $i \in I$ .
- ⟨2⟩5. LET:  $R = \{(i, x) : i \in I, x \in H(i)\}$
- ⟨2⟩6. PICK a function  $F \subseteq R$  with  $\text{dom } F = \text{dom } R$   
PROVE:  $F \in \prod_{i \in I} H(i)$   
PROOF: By the Axiom of Choice.
- ⟨2⟩7.  $\text{dom } H = I$   
PROOF: We have  $\text{dom } R = I$  since for all  $i \in I$  there exists  $x$  such that  $x \in H(i)$ .
- ⟨2⟩8.  $\forall i \in I. F(i) \in H(i)$   
PROOF: Since  $iRF(i)$ .
- ⟨1⟩2. If, for any set  $I$  and any function  $H$  with domain  $I$ , if  $H(i) \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} H(i) \neq \emptyset$ , then the Axiom of Choice is true.
- ⟨2⟩1. ASSUME: For any set  $I$  and any function  $H$  with domain  $I$ , if  $H(i) \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} H(i) \neq \emptyset$
- ⟨2⟩2. LET:  $R$  be a relation
- ⟨2⟩3. LET:  $I = \text{dom } R$
- ⟨2⟩4. Define the function  $H$  with domain  $I$  by: for  $i \in I$ ,  $H(i) = \{y : iRy\}$
- ⟨2⟩5.  $H(i) \neq \emptyset$  for all  $i \in I$
- ⟨2⟩6. PICK  $F \in \prod_{i \in I} H(i)$   
PROOF: By ⟨2⟩1
- ⟨2⟩7.  $F$  is a function
- ⟨2⟩8.  $F \subseteq R$   
PROOF: For all  $i \in I$  we have  $F(i) \in H(i)$  and so  $iRF(i)$ .
- ⟨2⟩9.  $\text{dom } F = \text{dom } R$
- 

### 3.6 Equivalence Relations

**Exercise 32**

(a)

$$\begin{aligned}
 &R \text{ is symmetric} \\
 &\Leftrightarrow \forall x, y (xRy \Rightarrow yRx) \\
 &\Leftrightarrow \forall x, y (\langle y, x \rangle \in R^{-1} \Rightarrow \langle y, x \rangle \in R) \\
 &\Leftrightarrow R^{-1} \subseteq R
 \end{aligned}$$

(b)

$$\begin{aligned}
& R \text{ is transitive} \\
& \Leftrightarrow \forall x, y, z (xRy \ \& \ yRz \Rightarrow xRz) \\
& \Leftrightarrow \forall x, z (\exists y (xRy \ \& \ yRz) \Rightarrow xRz) \\
& \Leftrightarrow \forall x, z (\langle x, z \rangle \in R \circ R \Rightarrow \langle x, z \rangle \in R) \\
& \Leftrightarrow R \circ R \subseteq R
\end{aligned}$$

**Exercise 33** PROOF:

$\langle 1 \rangle 1$ . If  $R$  is a symmetric and transitive relation then  $R = R^{-1} \circ R$ .

$\langle 2 \rangle 1$ . ASSUME:  $R$  is a symmetric and transitive relation.

$\langle 2 \rangle 2$ .  $R \subseteq R^{-1} \circ R$

$\langle 3 \rangle 1$ . LET:  $xRy$

$\langle 3 \rangle 2$ .  $yRy$

PROOF: By Theorem 3M.

$\langle 3 \rangle 3$ .  $xRy$  and  $yR^{-1}y$

$\langle 3 \rangle 4$ .  $x(R^{-1} \circ R)y$

$\langle 2 \rangle 3$ .  $R^{-1} \circ R \subseteq R$

PROOF:

$$R^{-1} \circ R \subseteq R \circ R \quad (\text{Exercise 32(a)})$$

$$\subseteq R \quad (\text{Exercise 32(b)})$$

$\langle 1 \rangle 2$ . If  $R = R^{-1} \circ R$  then  $R$  is a symmetric and transitive relation.

$\langle 2 \rangle 1$ . ASSUME:  $R = R^{-1} \circ R$

$\langle 2 \rangle 2$ .  $R$  is a relation.

$\langle 2 \rangle 3$ .  $R$  is symmetric.

$\langle 3 \rangle 1$ . LET:  $xRy$

$\langle 3 \rangle 2$ . PICK  $z$  such that  $xRz$  and  $zR^{-1}y$

$\langle 3 \rangle 3$ .  $yRz$  and  $zR^{-1}x$

$\langle 3 \rangle 4$ .  $y(R^{-1} \circ R)x$

$\langle 3 \rangle 5$ .  $yRx$

$\langle 2 \rangle 4$ .  $R$  is transitive.

$\langle 3 \rangle 1$ . LET:  $xRy$  and  $yRz$

$\langle 3 \rangle 2$ .  $zRy$

PROOF: By  $\langle 2 \rangle 3$

$\langle 3 \rangle 3$ .  $xRy$  and  $yR^{-1}z$

$\langle 3 \rangle 4$ .  $x(R^{-1} \circ R)z$

$\langle 3 \rangle 5$ .  $xRz$

□

**Exercise 34**

(a)  $\bigcap \mathcal{A}$  is a transitive relation.

PROOF:

$\langle 1 \rangle 1$ .  $\bigcap \mathcal{A}$  is a relation.

PROOF: Every member of a member of  $\mathcal{A}$  is an ordered pair.

$\langle 1 \rangle 2$ .  $\bigcap \mathcal{A}$  is transitive.

$\langle 2 \rangle 1$ . LET:  $\langle x, y \rangle$  and  $\langle y, z \rangle$  be in  $\bigcap \mathcal{A}$

PROVE:  $\langle x, z \rangle \in \bigcap \mathcal{A}$

$\langle 2 \rangle 2$ . LET:  $R \in \mathcal{A}$

$\langle 2 \rangle 3$ .  $xRy$  and  $yRz$

$\langle 2 \rangle 4$ .  $xRz$

PROOF: Since  $R$  is transitive.

□

(b) Not necessarily. If  $\mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$  then each member of  $\mathcal{A}$  is transitive but  $\bigcup \mathcal{A} = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle\}$  is not.

### Example 35

$$\begin{aligned} R[\{x\}] &= \{y : \exists z(z \in \{x\} \ \& \ zRy)\} \\ &= \{y : \exists z(z = x \ \& \ zRy)\} \\ &= \{y : xRy\} \\ &= [x]_R \end{aligned}$$

### Example 36 PROOF:

$\langle 1 \rangle 1$ .  $Q$  is a relation on  $A$ .

PROOF: By definition.

$\langle 1 \rangle 2$ .  $Q$  is reflexive on  $A$ .

$\langle 2 \rangle 1$ . LET:  $x \in A$

$\langle 2 \rangle 2$ .  $f(x)Rf(x)$

PROOF: Since  $R$  is reflexive on  $B$ .

$\langle 2 \rangle 3$ .  $xQx$

$\langle 1 \rangle 3$ .  $Q$  is symmetric.

$\langle 2 \rangle 1$ . ASSUME:  $xQy$

$\langle 2 \rangle 2$ .  $f(x)Rf(y)$

$\langle 2 \rangle 3$ .  $f(y)Rf(x)$

PROOF:  $R$  is symmetric.

$\langle 2 \rangle 4$ .  $yQx$

$\langle 1 \rangle 4$ .  $Q$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $xQy$  and  $yQz$

$\langle 2 \rangle 2$ .  $f(x)Rf(y)$  and  $f(y)Rf(z)$

$\langle 2 \rangle 3$ .  $f(x)Rf(z)$

PROOF:  $R$  is transitive.

$\langle 2 \rangle 4$ .  $xQz$

□

### Exercise 37 PROOF:

$\langle 1 \rangle 1$ .  $R_\Pi$  is a relation on  $A$ .

PROOF: If  $B \in \Pi$ ,  $x \in B$  and  $y \in B$  then  $x, y \in A$ .

$\langle 1 \rangle 2$ .  $R_\Pi$  is reflexive on  $A$ .

$\langle 2 \rangle 1$ . LET:  $x \in A$

$\langle 2 \rangle 2$ . PICK  $B \in \Pi$  such that  $x \in B$

PROOF: Because  $\Pi$  is exhaustive.

$\langle 2 \rangle 3$ .  $x \in B$  and  $x \in B$

$\langle 2 \rangle 4$ .  $xR_\Pi x$

$\langle 1 \rangle 3$ .  $R_\Pi$  is symmetric.

$\langle 2 \rangle 1$ . ASSUME:  $xR_\Pi y$

$\langle 2 \rangle 2$ . PICK  $B \in \Pi$  such that  $x \in B$  and  $y \in B$

$\langle 2 \rangle 3$ .  $y \in B$  and  $x \in B$

$\langle 2 \rangle 4$ .  $yR_\Pi x$

$\langle 1 \rangle 4$ .  $R_\Pi$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $xR_\Pi y$  and  $yR_\Pi z$

$\langle 2 \rangle 2$ . PICK  $B \in \Pi$  such that  $x \in B$  and  $y \in B$

$\langle 2 \rangle 3$ . PICK  $C \in \Pi$  such that  $y \in C$  and  $z \in C$

$\langle 2 \rangle 4$ .  $B = C$

PROOF: Since  $y \in B$  and  $y \in C$

$\langle 2 \rangle 5$ .  $x \in B$  and  $z \in B$

$\langle 2 \rangle 6$ .  $xR_\Pi z$

□

**Exercise 38** PROOF:

$\langle 1 \rangle 1$ . If  $B \in \Pi$  and  $x \in B$  then  $B = [x]_{R_\Pi}$

$\langle 2 \rangle 1$ . LET:  $B \in \Pi$

$\langle 2 \rangle 2$ . LET:  $x \in B$

$\langle 2 \rangle 3$ .  $[x]_{R_\Pi} \subseteq B$

$\langle 3 \rangle 1$ . LET:  $y \in [x]_{R_\Pi}$

$\langle 3 \rangle 2$ .  $xR_\Pi y$

$\langle 3 \rangle 3$ . PICK  $C \in \Pi$  such that  $x \in C$  and  $y \in C$

$\langle 3 \rangle 4$ .  $B = C$

PROOF: Since  $x \in B$  and  $x \in C$ .

$\langle 3 \rangle 5$ .  $y \in B$

$\langle 2 \rangle 4$ .  $B \subseteq [x]_{R_\Pi}$

PROOF: For all  $y \in B$ , we have  $x \in B$  and  $y \in B$  hence  $xR_\Pi y$ .

$\langle 1 \rangle 2$ .  $A/R_\Pi \subseteq \Pi$

$\langle 2 \rangle 1$ . LET:  $x \in A$

PROVE:  $[x]_{R_\Pi} \in \Pi$

$\langle 2 \rangle 2$ . PICK  $B \in \Pi$  such that  $x \in B$

$\langle 2 \rangle 3$ .  $[x]_{R_\Pi} = B$

PROOF: By  $\langle 1 \rangle 1$

$\langle 2 \rangle 4$ .  $[x]_{R_\Pi} \in \Pi$

$\langle 1 \rangle 3$ .  $\Pi \subseteq A/R_\Pi$

$\langle 2 \rangle 1$ . LET:  $B \in \Pi$

$\langle 2 \rangle 2$ . PICK  $x \in B$

PROOF: Since every member of  $\Pi$  is nonempty.

$\langle 2 \rangle 3$ .  $B = [x]_{R_\Pi}$

PROOF: By  $\langle 1 \rangle 1$ .

$\langle 2 \rangle 4$ .  $B \in A/R_\Pi$

□

**Exercise 39** PROOF:

$\langle 1 \rangle 1$ .  $R_\Pi \subseteq R$

$\langle 2 \rangle 1$ . LET:  $xR_\Pi y$

$\langle 2 \rangle 2$ . PICK  $B \in \Pi$  such that  $x \in B$  and  $y \in B$

$\langle 2 \rangle 3$ . PICK  $z \in A$  such that  $B = [z]_R$

$\langle 2 \rangle 4$ .  $zRx$

$\langle 2 \rangle 5$ .  $zRy$

$\langle 2 \rangle 6$ .  $xRy$

PROOF: Since  $R$  is symmetric and transitive.

$\langle 1 \rangle 2$ .  $R \subseteq R_\Pi$

$\langle 2 \rangle 1$ . LET:  $xRy$

$\langle 2 \rangle 2$ .  $x \in [x]_R$

$\langle 2 \rangle 3$ .  $y \in [x]_R$

$\langle 2 \rangle 4$ .  $xR_\Pi y$

□

**Exercise 40** We have  $[2]_R = [3]_R$  but  $[6]_R \neq [9]_R$  so there is no such function  $f$ .

**Exercise 41**

(a) PROOF:

$\langle 1 \rangle 1$ .  $Q$  is reflexive on  $\mathbb{R} \times \mathbb{R}$ .

PROOF: For any  $x, y \in \mathbb{R}$ , we have  $x + y = x + y$ , hence  $\langle x, y \rangle Q \langle x, y \rangle$

$\langle 1 \rangle 2$ .  $Q$  is symmetric.

$\langle 2 \rangle 1$ . ASSUME:  $\langle u, v \rangle Q \langle x, y \rangle$

$\langle 2 \rangle 2$ .  $u + y = x + v$

$\langle 2 \rangle 3$ .  $x + v = u + y$

$\langle 2 \rangle 4$ .  $\langle x, y \rangle Q \langle u, v \rangle$

$\langle 1 \rangle 3$ .  $Q$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $\langle a, b \rangle Q \langle u, v \rangle$  and  $\langle u, v \rangle Q \langle x, y \rangle$

$\langle 2 \rangle 2$ .  $a + v = u + b$

$\langle 2 \rangle 3$ .  $u + y = x + v$

$\langle 2 \rangle 4$ .  $a + y + x + b$

PROOF: Adding  $\langle 2 \rangle 2$  and  $\langle 2 \rangle 3$  gives  $a + u + v + y = b + u + v + x$ .

$\langle 2 \rangle 5$ .  $\langle a, b \rangle Q \langle x, y \rangle$

□

(b) We prove that, if  $\langle u, v \rangle Q \langle x, y \rangle$  then  $\langle u + 2v, v + 2u \rangle Q \langle x + 2y, y + 2x \rangle$ . It follows from Theorem 3Q that the function  $G$  exists.

If  $u + y = v + x$  then  $u + 2v + y + 2x = v + 2u + x + 2y$  by adding  $u + v + y + x$  to both sides.

**Exercise 42** Assume that  $R$  is an equivalence relation on  $A$  and that  $F : A \times A \rightarrow A$ . Let us say that  $F$  is *compatible* with  $R$  iff, whenever  $xRx'$  and  $yRy'$ , then  $F(\langle x, y \rangle)RF(\langle x', y' \rangle)$ . If  $F$  is compatible with  $R$  then there exists a unique  $\hat{F} : (A/R) \times (A/R) \rightarrow A/R$  such that

$$\hat{F}(\langle [x]_R, [y]_R \rangle) = [F(\langle x, y \rangle)]_R \text{ for all } x, y \in A.$$

If  $F$  is not compatible with  $R$  then no such  $\hat{F}$  exists.

## 3.7 Ordering Relations

**Exercise 43** PROOF:

- $\langle 1 \rangle 1.$   $R^{-1}$  is transitive.
  - $\langle 2 \rangle 1.$  ASSUME:  $xR^{-1}y$  and  $yR^{-1}z$
  - $\langle 2 \rangle 2.$   $zRy$  and  $yRx$
  - $\langle 2 \rangle 3.$   $zRx$
- PROOF: Since  $R$  is transitive.
- $\langle 2 \rangle 4.$   $xR^{-1}z$
- $\langle 1 \rangle 2.$   $R^{-1}$  satisfies trichotomy on  $A$ .
  - $\langle 2 \rangle 1.$  LET:  $x, y \in A$
  - $\langle 2 \rangle 2.$  Exactly one of  $xRy$ ,  $x = y$ ,  $yRx$  holds.
  - $\langle 2 \rangle 3.$  Exactly one of  $yR^{-1}x$ ,  $x = y$ ,  $xR^{-1}y$  holds.

□

**Exercise 44** PROOF:

- $\langle 1 \rangle 1.$   $f$  is one-to-one.
  - $\langle 2 \rangle 1.$  LET:  $x, y \in A$  with  $f(x) = f(y)$
  - $\langle 2 \rangle 2.$   $f(x) < f(y)$  and  $f(y) < f(x)$  do not hold.
- PROOF: By trichotomy.
- $\langle 2 \rangle 3.$   $x < y$  and  $y < x$  do not hold.
- $\langle 2 \rangle 4.$   $x = y$
- PROOF: By trichotomy.
- $\langle 1 \rangle 2.$  Whenever  $f(x) < f(y)$  then  $x < y$ 
  - $\langle 2 \rangle 1.$  LET:  $x, y \in A$  with  $f(x) < f(y)$
  - $\langle 2 \rangle 2.$   $f(x) = f(y)$  and  $f(y) < f(x)$  do not hold.
- PROOF: By trichotomy.
- $\langle 2 \rangle 3.$   $x = y$  and  $y < x$  do not hold.
- $\langle 2 \rangle 4.$   $x < y$
- PROOF: By trichotomy.

□

**Exercise 45** PROOF:

$\langle 1 \rangle 1.$   $<_L$  is transitive.

$\langle 2 \rangle 1.$  LET:  $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$  and  $\langle a_2, b_2 \rangle <_L \langle a_3, b_3 \rangle$

PROVE:  $\langle a_1, b_1 \rangle < \langle a_3, b_3 \rangle$

$\langle 2 \rangle 2.$  CASE:  $a_1 <_A a_2$  and  $a_2 <_A a_3$

PROOF: Then  $a_1 <_A a_3$

$\langle 2 \rangle 3.$  CASE:  $a_1 <_A a_2$ ,  $a_2 = a_3$ ,  $b_2 <_B b_3$

PROOF: Then  $a_1 <_A a_3$

$\langle 2 \rangle 4.$  CASE:  $a_1 = a_2$ ,  $b_1 <_B b_2$  and  $a_2 <_A a_3$

PROOF: Then  $a_1 <_A a_3$

$\langle 2 \rangle 5.$  CASE:  $a_1 = a_2$ ,  $b_1 <_B b_2$ ,  $a_2 = a_3$ ,  $b_2 <_B b_3$

PROOF: Then  $a_1 = a_3$  and  $b_1 <_B b_3$

$\langle 1 \rangle 2.$   $<_L$  satisfies trichotomy on  $A \times B$ .

$\langle 2 \rangle 1.$  LET:  $\langle a_1, b_1 \rangle$  and  $\langle a_2, b_2 \rangle$  be elements of  $A \times B$

$\langle 2 \rangle 2.$  Exactly one of  $a_1 <_A a_2$ ,  $a_1 = a_2$ ,  $a_2 <_A a_1$  holds.

$\langle 2 \rangle 3.$  Exactly one of  $b_1 <_B b_2$ ,  $b_1 = b_2$ ,  $b_2 <_B b_1$  holds.

$\langle 2 \rangle 4.$  Exactly one of  $a_1 <_A a_2$ ,  $(a_1 = a_2 \text{ and } b_1 <_B b_2)$ ,  $(a_1 = a_2 \text{ and } b_1 = b_2)$ ,  $(a_1 = a_2 \text{ and } b_2 <_L b_1)$ ,  $a_2 <_A a_1$  holds.

$\langle 2 \rangle 5.$  Exactly one of  $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$ ,  $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$ ,  $\langle a_2, b_2 \rangle <_L \langle a_1, b_1 \rangle$  holds.

□

### 3.8 Review Exercises

**Exercise 46**

(a)

$$\bigcap \bigcap \langle x, y \rangle = \bigcap \{x\} \\ = x$$

(b)

$$\begin{aligned} \bigcap \bigcap \bigcap \{\langle x, y \rangle\}^{-1} &= \bigcap \bigcap \bigcap \{\langle y, x \rangle\} \\ &= \bigcap \bigcap \langle y, x \rangle \\ &= y \end{aligned} \quad \text{(by part (a))}$$

**Exercise 47**



(a) There are eight:

$$\begin{aligned} &\{\langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle\} \end{aligned}$$

(b) There are six:

$$\begin{aligned} &\{\langle 0, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 5 \rangle\}, \\ &\{\langle 0, 3 \rangle, \langle 1, 5 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 3 \rangle, \langle 2, 5 \rangle\}, \\ &\{\langle 0, 4 \rangle, \langle 1, 5 \rangle, \langle 2, 3 \rangle\}, \\ &\{\langle 0, 5 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle\}, \\ &\{\langle 0, 5 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle\} \end{aligned}$$

**Exercise 48**

(a) The only ordered pair in  $\mathcal{PT}$  is  $\langle \emptyset, \emptyset \rangle = \{\{\emptyset\}\}$ .

(b)

$$\begin{aligned} (\mathcal{PT})^{-1} \circ (\mathcal{PT} \upharpoonright \{\emptyset\}) &= \{\langle \emptyset, \emptyset \rangle\} \circ \{\langle \emptyset, \emptyset \rangle\} \\ &= \{\langle \emptyset, \emptyset \rangle\} \end{aligned}$$

**Exercise 49** There are six:

$$\begin{aligned} &\{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 1 \rangle, \langle 2, 0 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}, \\ &\{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\} \end{aligned}$$

**Exercise 50**

(a)  $\{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 3 \rangle\}$

$$(b) \quad \{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}$$

**Exercise 51** There are three:

$$\begin{aligned} &\{\langle 1, 0 \rangle, \langle 1, 2 \rangle, \langle 2, 0 \rangle\}, \\ &\{\langle 1, 0 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle\}, \\ &\{\langle 0, 1 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle\} \end{aligned}$$

**Exercise 52** We can conclude this if we know that  $A$  and  $B$  are nonempty, or that  $C$  and  $D$  are nonempty.

Suppose  $A$  and  $B$  are nonempty. Then  $A \times B = C \times D \neq \emptyset$  so  $C$  and  $D$  are nonempty. We now prove  $A \subseteq C$ .

Let  $a \in A$ . Pick some  $b \in B$ . Then  $\langle a, b \rangle \in A \times B = C \times D$  and so  $a \in C$ .

We can similarly prove  $C \subseteq A$ ,  $B \subseteq D$  and  $D \subseteq B$ .

**Exercise 53**

$$\begin{aligned} x(R \cup S)^{-1}y &\Leftrightarrow y(R \cup S)x \\ &\Leftrightarrow yRx \text{ or } ySx \\ &\Leftrightarrow xR^{-1}y \text{ or } xS^{-1}y \\ &\Leftrightarrow x(R^{-1} \cup S^{-1})y \\ x(R \cap S)^{-1}y &\Leftrightarrow y(R \cap S)x \\ &\Leftrightarrow yRx \text{ and } ySx \\ &\Leftrightarrow xR^{-1}y \text{ and } xS^{-1}y \\ &\Leftrightarrow x(R^{-1} \cap S^{-1})y \\ x(R - S)^{-1}y &\Leftrightarrow y(R - S)x \\ &\Leftrightarrow yRx \text{ and } \neg ySx \\ &\Leftrightarrow xR^{-1}y \text{ and } \neg xS^{-1}y \\ &\Leftrightarrow x(R^{-1} - S^{-1})y \end{aligned}$$

**Exercise 54**

(a)

$$\begin{aligned} \langle x, y \rangle \in A \times (B \cap C) &\Leftrightarrow x \in A \text{ \& } y \in B \text{ \& } y \in C \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) \cap (A \times C) \end{aligned}$$

(b)

$$\begin{aligned} \langle x, y \rangle \in A \times (B \cup C) &\Leftrightarrow x \in A \text{ \& } (y \in B \text{ or } y \in C) \\ &\Leftrightarrow (x \in A \text{ \& } y \in B) \text{ or } (x \in A \text{ \& } y \in C) \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) \cup (A \times C) \end{aligned}$$

(c)

$$\begin{aligned}\langle x, y \rangle \in A \times (B - C) &\Leftrightarrow x \in A \ \& \ y \in B \ \& \ y \notin C \\ &\Leftrightarrow \langle x, y \rangle \in (A \times B) - (A \times C)\end{aligned}$$

**Exercise 55**

(a) No. Take  $A = \{0\}$ ,  $B = \{1\}$ ,  $C = \{2\}$ . Then  $(A \times A) \cup (B \times C) = \{\langle 0, 0 \rangle, \langle 1, 2 \rangle\}$  while  $(A \cup B) \times (A \cup C) = \{\langle 0, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 0 \rangle, \langle 1, 2 \rangle\}$ .

(b) Yes.

$$\begin{aligned}\langle x, y \rangle \in (A \times A) \cap (B \times C) &\Leftrightarrow x \in A \ \& \ y \in A \ \& \ x \in B \ \& \ y \in C \\ &\Leftrightarrow \langle x, y \rangle \in (A \cap B) \times (A \cap C)\end{aligned}$$

**Exercise 56**

(a) Yes.

$$\begin{aligned}x \in \text{dom}(R \cup S) &\Leftrightarrow \exists y(xRy \text{ or } xSy) \\ &\Leftrightarrow \exists y.xRy \text{ or } \exists y.xSy \\ &\Leftrightarrow x \in \text{dom } R \cup \text{dom } S\end{aligned}$$

(b) No. Take  $R = \{\langle 0, 0 \rangle\}$  and  $S = \{\langle 0, 1 \rangle\}$ . Then  $\text{dom}(R \cap S) = \text{dom } \emptyset = \emptyset$  while  $\text{dom } R \cap \text{dom } S = \{0\} \cap \{0\} = \{0\}$ .

**Exercise 57**

(a) Yes.

$$\begin{aligned}x(R \circ (S \cup T))y &\Leftrightarrow \exists z(x(S \cup T)z \ \& \ zRy) \\ &\Leftrightarrow \exists z(xSz \ \& \ zRy) \text{ or } \exists z(xTz \ \& \ zRy) \\ &\Leftrightarrow x((R \circ S) \cup (R \circ T))y\end{aligned}$$

(b) No. Take  $R = \{\langle 0, 0 \rangle, \langle 1, 0 \rangle\}$ ,  $S = \{\langle 0, 0 \rangle\}$  and  $T = \{\langle 0, 1 \rangle\}$ . Then

$$\begin{aligned}R \circ (S \cap T) &= R \circ \emptyset \\ &= \emptyset \\ (R \circ S) \cap (R \circ T) &= \{\langle 0, 0 \rangle\} \cap \{\langle 0, 0 \rangle\} \\ &= \{\langle 0, 0 \rangle\}\end{aligned}$$

**Exercise 58** Take  $F = \emptyset$  and  $S = \{\emptyset\}$ . Then  $F[F^{-1}[[S]]] = \emptyset \neq S$ .

**Exercise 59**

$$\begin{aligned}
x(Q \upharpoonright (A \cap B))y &\Leftrightarrow xQy \ \& \ x \in A \ \& \ x \in B \\
&\Leftrightarrow x((Q \upharpoonright A) \cap (Q \upharpoonright B))y \\
x(Q \upharpoonright (A - B))y &\Leftrightarrow xQy \ \& \ x \in A \ \& \ x \notin B \\
&\Leftrightarrow (xQy \ \& \ x \in A) \ \& \ \neg(xQy \ \& \ x \in B) \\
&\Leftrightarrow x((Q \upharpoonright A) - (Q \upharpoonright B))y
\end{aligned}$$

**Exercise 60**

$$\begin{aligned}
x((R \circ S) \upharpoonright A)y &\Leftrightarrow \exists z(xRz \ \& \ zSy \ \& \ x \in A) \\
&\Leftrightarrow x(R \circ (S \upharpoonright A))y
\end{aligned}$$

## Chapter 4

# Chapter 4 — Natural Numbers

### 4.1 Inductive Sets

**Exercise 1** We have

$$3 = 2 \cup \{2\} = 1 \cup \{1, 2\}$$

and so  $1 \in 3$ . But  $1 \notin 1$  (since  $1 = \{\emptyset\}$  and we know  $\{\emptyset\} \neq \emptyset$  hence  $\{\emptyset\} \notin \{\emptyset\}$ ). Therefore  $1 \neq 3$ .

### 4.2 Peano's Postulates

**Exercise 2** If  $a$  is a transitive set then

$$\begin{aligned} \bigcup (a^+) &= a && \text{(Theorem 4E)} \\ &\subseteq a^+ \end{aligned}$$

**Exercise 3**

(a) Suppose  $a$  is a transitive set. Then  $a \subseteq \mathcal{P}a$ . Hence we have  $\bigcup \mathcal{P}a = a \subseteq \mathcal{P}a$  and so  $\mathcal{P}a$ .

(b) Suppose  $\mathcal{P}a$  is a transitive set. Then  $a = \bigcup \mathcal{P}a \subseteq \mathcal{P}a$  hence  $a$  is transitive.

**Exercise 4** If  $a$  is a transitive set then  $\bigcup a \subseteq a$  so  $\bigcup \bigcup a \subseteq \bigcup a$ . Hence  $\bigcup a$  is transitive.

**Exercise 5**

(a) PROOF:  
 $\langle 1 \rangle 1.$  LET:  $b \in \bigcup \mathcal{A}$   
 $\langle 1 \rangle 2.$  PICK  $A \in \mathcal{A}$  such that  $b \in A$   
 $\langle 1 \rangle 3.$   $b \subseteq A$   
 PROOF: Since  $A$  is transitive.  
 $\langle 1 \rangle 4.$   $b \subseteq \bigcup \mathcal{A}$   
 $\square$

(b) PROOF:  
 $\langle 1 \rangle 1.$  LET:  $b \in \bigcap \mathcal{A}$   
 $\langle 1 \rangle 2.$  For all  $A \in \mathcal{A}$  we have  $b \subseteq A$   
 PROOF: Since  $b \in A$  and  $A$  is transitive.  
 $\langle 1 \rangle 3.$   $b \subseteq \bigcap \mathcal{A}$   
 $\square$

**Exercise 6** We have  $\bigcup(a^+) = \bigcup a \cup a$  (see the proof of Theorem 4E). So if  $\bigcup(a^+) = a$  we have  $\bigcup a \cup a = a$  and so  $\bigcup a \subseteq a$ .

### 4.3 Recursion on $\omega$

**Exercise 7** We have  $h_1(0) = h_2(0) = a$  so  $0 \in S$ .  
 Now let  $n \in S$ ; we prove  $n^+ \in S$ . We have  $h_1(n) = h_2(n)$  and therefore

$$\begin{aligned} h_1(n^+) &= F(h_1(n)) \\ &= F(h_2(n)) \\ &= h_2(n^+) \end{aligned}$$

**Exercise 8** PROOF:  
 $\langle 2 \rangle 1.$   $\forall m, n \in \omega. h(n) = h(m) \Rightarrow n = m$   
 $\langle 2 \rangle 1.$   $\forall n \in \omega. h(n) = h(0) \Rightarrow n = 0$   
 $\langle 3 \rangle 1.$  LET:  $n \in \omega$   
 $\langle 3 \rangle 2.$  ASSUME:  $h(n) = h(0)$   
 $\langle 3 \rangle 3.$   $h(n) = c$   
 $\langle 3 \rangle 4.$   $\forall p \in \omega. n \neq p^+$   
 PROOF: Otherwise  $f(h(p)) = c$  contradicting the fact that  $c \in A - \text{ran } f$ .  
 $\langle 3 \rangle 5.$   $n = 0$   
 PROOF: Theorem 4C.  
 $\langle 2 \rangle 2.$  For all  $m \in \omega$ , if  $\forall n \in \omega. h(n) = h(m) \Rightarrow n = m$ , then  $\forall n \in \omega. h(n) = h(m^+) \Rightarrow n = m^+$   
 $\langle 3 \rangle 1.$  LET:  $m \in \omega$   
 $\langle 3 \rangle 2.$  ASSUME:  $\forall n \in \omega. h(n) = h(m) \Rightarrow n = m$   
 $\langle 3 \rangle 3.$  LET:  $n \in \omega$   
 $\langle 3 \rangle 4.$  ASSUME:  $h(n) = h(m^+)$   
 $\langle 3 \rangle 5.$   $h(n) = f(h(m))$

$\langle 3 \rangle 6. n \neq 0$

PROOF: Otherwise  $c = f(h(m))$  contradicting the fact that  $c \in A - \text{ran } f$ .

$\langle 3 \rangle 7. \text{ PICK } p \text{ such that } n = p^+$

$\langle 3 \rangle 8. f(h(p)) = f(h(m))$

$\langle 3 \rangle 9. h(p) = h(m)$

PROOF:  $f$  is one-to-one.

$\langle 3 \rangle 10. p = m$

PROOF: By  $\langle 3 \rangle 2$ .

$\langle 3 \rangle 11. n = p^+ = m^+$

□

**Exercise 9** PROOF:

$\langle 1 \rangle 1. C^* \subseteq C_*$

$\langle 2 \rangle 1. f[C_*] \subseteq C_*$

$\langle 3 \rangle 1. \text{ LET: } x \in C_*$

PROVE:  $f(x) \in C_*$

$\langle 3 \rangle 2. \text{ PICK } n \text{ such that } x \in h(n)$

$\langle 3 \rangle 3. f(x) \in h(n^+)$

$\langle 3 \rangle 4. f(x) \in C_*$

$\langle 1 \rangle 2. C_* \subseteq C^*$

$\langle 2 \rangle 1. \forall n \in \omega. h(n) \subseteq C^*$

$\langle 3 \rangle 1. h(0) \subseteq C^*$

PROOF: If  $A \subseteq X \subseteq B$  and  $f[X] \subseteq X$  then  $A \subseteq X$ .

$\langle 3 \rangle 2. \forall n \in \omega. h(n) \subseteq C^* \Rightarrow h(n^+) \subseteq C^*$

$\langle 4 \rangle 1. \text{ LET: } n \in \omega$

$\langle 4 \rangle 2. \text{ ASSUME: } h(n) \subseteq C^*$

$\langle 4 \rangle 3. f[h(n)] \subseteq C^*$

$\langle 5 \rangle 1. \text{ LET: } X \text{ be such that } A \subseteq X \subseteq B \text{ and } f[X] \subseteq X$

PROVE:  $f[h(n)] \subseteq X$

$\langle 5 \rangle 2. h(n) \subseteq X$

$\langle 5 \rangle 3. f[h(n)] \subseteq f[X]$

$\langle 5 \rangle 4. f[h(n)] \subseteq X$

$\langle 4 \rangle 4. h(n^+) \subseteq C^*$

□

**Exercise 10**  $C^* = C_* = (0, 1]$

**Exercise 11**  $\{n \in \mathbb{Z} \mid n \leq 0\}$

**Exercise 12** Let  $f : B \times B \rightarrow B$  and  $A \subseteq B$ . Let

$$C^* = \bigcap \{X \mid A \subseteq X \subseteq B \text{ \& } f[X \times X] \subseteq X\} .$$

Define the function  $h : \omega \rightarrow \mathcal{P}B$  by

$$\begin{aligned} h(0) &= A \\ h(n^+) &= h(n) \cup f[h(n) \times h(n)] \end{aligned} \quad (n \in \omega)$$

Define  $C_* = \bigcup \text{ran } h$ . Then  $C^* = C_*$ .

## 4.4 Arithmetic

**Exercise 13** We prove the contrapositive. Assume  $m \neq 0$  and  $n \neq 0$ . Then by Theorem 4C there are natural numbers  $p, q$  such that  $m = p^+$  and  $n = q^+$ . Hence  $mn = p^+q^+ = (p^+q + p)^+ \neq 0$ .

**Exercise 14** We prove the following facts for any natural number  $n$ :

1.  $n$  is even if and only if  $n^+$  is odd.

PROOF: If  $n$  is even, say  $n = 2p$ , then  $n^+ = 2p + 1$  is odd.

If  $n^+$  is odd, say  $n^+ = 2p + 1$ , then  $n = 2p$  is even.

2.  $n$  is odd if and only if  $n^+$  is even.

PROOF: If  $n$  is odd, say  $n = 2p + 1$ , then  $n^+ = 2(p + 1)$  is even.

If  $n^+$  is even, say  $n^+ = 2p$ , then we cannot have  $p = 0$  (since  $n^+ \neq 0$ ). So  $p = q + 1$  for some  $q$ . But then  $n^+ = 2q + 2$  so  $n = 2q + 1$  and  $n$  is odd.

Now, 0 is even and 0 is not odd. By the two facts above, if  $n$  is either even or odd but not both, then  $n^+$  is either odd or even but not both. The result follows by induction.

**Exercise 15** We have

$$\begin{aligned} m + (n + 0) &= m + n && \text{by (A1)} \\ &= (m + n) + 0 && \text{by (A1)} \end{aligned}$$

If  $m + (n + p) = (m + n) + p$  then

$$\begin{aligned} m + (n + p^+) &= m + (n + p)^+ && \text{by (A2)} \\ &= (m + (n + p))^+ && \text{by (A2)} \\ &= ((m + n) + p)^+ && \text{by induction hypothesis} \\ &= (m + n) + p^+ && \text{by (A2)} \end{aligned}$$

**Exercise 16** We first prove that  $0 \cdot n = 0$  for all  $n$ . We have  $0 \cdot 0 = 0$  by (M1), and if  $0 \cdot n = 0$  then

$$\begin{aligned} 0 \cdot n^+ &= 0 \cdot n + 0 && \text{by (M2)} \\ &= 0 \cdot n && \text{by (A1)} \\ &= 0 && \text{by induction hypothesis} \end{aligned}$$



Now we prove that  $m^+ \cdot n = m \cdot n + n$  for all  $m, n$ . We have

$$\begin{aligned} m^+ \cdot 0 &= 0 && \text{by (M1)} \\ m \cdot 0 + 0 &= m \cdot 0 && \text{by (A1)} \\ &= 0 && \text{by (M1)} \end{aligned}$$

Thus,  $m^+ \cdot 0 = m \cdot 0 + 0$ .

If  $m^+ \cdot n = m \cdot n + n$  then

$$\begin{aligned} m^+ \cdot n^+ &= m^+ \cdot n + m^+ && \text{by (M2)} \\ &= (m^+ \cdot n + m)^+ && \text{by (A2)} \\ &= ((m \cdot n + n) + m)^+ && \text{by induction hypothesis} \\ &= ((m \cdot n + m) + n)^+ && \text{by associativity and commutativity of addition} \\ &= (m \cdot n^+ + n)^+ && \text{by (M2)} \\ &= m \cdot n^+ + n^+ && \text{by (A2)} \end{aligned}$$

**Exercise 17** The proof is by induction on  $p$ . We have

$$\begin{aligned} m^{n+0} &= m^n && \text{by (A1)} \\ &= 0 + m^n && \text{by Theorem 4K(2)} \\ &= m^n \cdot 0 + m^n && \text{by (M1)} \\ &= m^n \cdot 1 && \text{by (M2)} \\ &= m^n \cdot m^0 && \text{by (E1)} \end{aligned}$$

If  $m^{n+p} = m^n \cdot m^p$  then

$$\begin{aligned} m^{n+p^+} &= m^{(n+p)^+} && \text{by (A2)} \\ &= m^{n+p} m && \text{by (E2)} \\ &= (m^n m^p) m && \text{by induction hypothesis} \\ &= m^n (m^p m) && \text{by Theorem 4K (4)} \\ &= m^n m^{p^+} && \text{by (E2)} \end{aligned}$$

## 4.5 Ordering on $\omega$

**Exercise 18**

$$\begin{aligned} \in_{\omega}^{-1} \llbracket \{7, 8\} \rrbracket &= \{x \in \omega \mid x \in 7 \text{ or } x \in 8\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7\} \end{aligned}$$

**Exercise 19** The proof is by induction on  $m$ .

For  $m = 0$ , take  $q = r = 0$ . Then  $m = d \cdot 0 + 0$  and  $0 \in d$ .

Suppose  $m = dq + r$  and  $r < d$ . Then  $r + 1 \leq d$ . If  $r + 1 < d$ , then we have  $m + 1 = dq + (r + 1)$  as required. If  $r + 1 = d$ , then we have  $m + 1 = dq + d = d(q + 1) + 0$ .

**Exercise 20** We first prove  $A$  is closed downwards; that is, if  $n \in A$  and  $m \in n$  then  $m \in A$ . This holds because if  $n \in A$  and  $m \in n$  then  $m \in \bigcup A$  and  $\bigcup A = A$ .

Now, we prove  $\forall n \in \omega. n \in A$  by induction on  $n$ .

To prove  $0 \in A$ : we are given that  $A$  is nonempty. Pick some  $a \in A$ . Then  $0 \in a$  so  $0 \in A$  since  $A$  is closed downwards.

Now let  $n \in A$ ; we prove  $n^+ \in A$ . We have  $n \in \bigcup A$ ; pick some  $k \in A$  such that  $n \in k$ . Then  $n^+ \in k$  so  $n^+ \in A$  since  $A$  is closed downwards.

This completes the induction. We have  $\forall n \in \omega. n \in A$ , i.e.  $A = \omega$ .

**Exercise 21** Suppose  $n$  is a natural number,  $k \in n$  and  $n \subseteq k$ . Then  $k \in k$ , contradicting Lemma 4L(b).

**Exercise 22** We have  $0 \in p^+$  (by trichotomy since  $p^+ \not\subseteq 0$  because  $0$  is empty, and  $p^+ \neq 0$  by Peano's First Postulate.) Hence  $n = n + 0 \in n + p^+$  by Theorem 4N.

**Exercise 23** The proof is by induction on  $n$ . The statement is vacuously true for  $n = 0$ .

Suppose the statement is true for  $n$ . Let  $m \in n^+$ . Then  $m \in n$ .

If  $m = n$ , then we have  $m + 0^+ = n^+$ .

If  $m \in n$ , pick  $p$  such that  $m + p^+ = n$  by the induction hypothesis. Then  $m + p^{++} = n^+$ .

**Exercise 24** Suppose  $m \in p$ . Then we cannot have  $n \in q$  or  $n = q$ , as either of these would imply  $m + n \in p + q$ . Hence  $q \in n$  by trichotomy.

We prove  $q \in n \Rightarrow m \in p$  similarly.

**Exercise 25** By Exercise 23, pick natural numbers  $a$  and  $b$  such that  $m = n + a^+$  and  $p = q + b^+$ . Then

$$\begin{aligned} mp + nq &= (n + a^+)(q + b^+) + nq \\ &= nq + nq + a^+q + nb^+ + a^+b^+ \\ &= (n + a^+)q + n(q + b^+) + a^+b^+ \\ &= mq + np + (a^+ + b^+)^+ \end{aligned}$$

Hence  $mq + np \in mp + nq$  by Exercise 22.