# C4 Analysis

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**Definition 0.1** (Limit of a Function)**.** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $a$ be an accumulation point of $A$ and $b \in \mathbb{R}$. Then we say $b$ is the *limit* of $f$ at $a$, and write $f(x) \to b$ as $x \to a$ or $\lim_{x \to a} f(x) = b$, iff for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta$ then $|f(x) - b| < \epsilon$.

**Proposition 0.2.** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $a$ be an accumulation point of $A$ and $b, c \in \mathbb{R}$. If $f(x) \to b$ as $x \to a$ and $f(x) \to c$ as $x \to a$ then $b = c$.*

PROOF:
$\langle 1 \rangle 1.$ $\forall \epsilon > 0. |b - c| < \epsilon$
  $\langle 2 \rangle 1.$ LET: $\epsilon > 0$
  $\langle 2 \rangle 2.$ PICK $\delta > 0$ such that $\forall x \in A - \{a\}. |x - a| < \delta \Rightarrow |f(x) - b| < \epsilon/2 \wedge |f(x) - c| < \epsilon/2$
  $\langle 2 \rangle 3.$ PICK $x \in (A - \{a\}) \cap (a - \delta, a + \delta)$
  $\langle 2 \rangle 4.$ $|f(x) - b| < \epsilon/2$
  $\langle 2 \rangle 5.$ $|f(x) - c| < \epsilon/2$
  $\langle 2 \rangle 6.$ $|b - c| < \epsilon$
$\square$

**Proposition 0.3** (Choice)**.** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $a$ be an accumulation point of $A$. Let $b \in \mathbb{R}$. Then $f(x) \to b$ as $x \to a$ if and only if, for any sequence $(x_n)$ in $A - \{a\}$, if $x_n \to a$ as $n \to \infty$ then $f(x_n) \to b$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1.$ If $f(x) \to b$ as $x \to a$ then, for any sequence $(x_n)$ is $A - \{a\}$, if $x_n \to a$ as $n \to \infty$ then $f(x_n) \to b$ as $n \to \infty$.
  $\langle 2 \rangle 1.$ ASSUME: $f(x) \to b$ as $x \to a$
  $\langle 2 \rangle 2.$ LET: $(x_n)$ be a sequence in $A - \{a\}$
  $\langle 2 \rangle 3.$ ASSUME: $x_n \to a$ as $n \to \infty$
  $\langle 2 \rangle 4.$ LET: $\epsilon > 0$
  $\langle 2 \rangle 5.$ PICK $\delta > 0$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta$, then $|f(x) - b| < \epsilon$
  $\langle 2 \rangle 6.$ PICK $N$ such that $\forall n \geq N. |x - a| < \delta$
  $\langle 2 \rangle 7.$ $\forall n \geq N. |f(x) - b| < \epsilon$
$\langle 1 \rangle 2.$ If, for any sequence $(x_n)$ is $A - \{a\}$, if $x_n \to a$ as $n \to \infty$ then $f(x_n) \to b$ as $n \to \infty$, then $f(x) \to b$ as $x \to a$.
  $\langle 2 \rangle 1.$ ASSUME: $f(x) \not\to b$ as $x \to a$

⟨2⟩2. PICK $\epsilon > 0$ such that, for all $\delta > 0$, there exists $x \in A - \{a\}$ such that $|x - a| < \delta$ and $|f(x) - b| \geq \epsilon$

⟨2⟩3. For all $n \in \mathbb{Z}^+$, PICK $x_n \in A - \{a\}$ such that $|x_n - a| < 1/n$ and $|f(x_n) - b| \geq \epsilon$

⟨2⟩4. $x_n \to a$ as $n \to \infty$

⟨2⟩5. $f(x_n) \not\to b$ as $n \to \infty$

□

**Proposition 0.4.** *Let $A, B \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Let $a$ be an accumulation point of $A \cap B$. Let $b, c \in \mathbb{R}$. Assume $f(x) \to b$ as $x \to a$ and $g(x) \to c$ as $x \to a$. Then $f(x) + g(x) \to b + c$ as $x \to a$.*

PROOF:

⟨1⟩1. LET: $\epsilon > 0$

⟨1⟩2. PICK $\delta > 0$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta$ then $|f(x) - b| < \epsilon/2$, and for all $x \in B - \{a\}$, if $|x - a| < \delta$ then $|g(x) - c| < \epsilon/2$

⟨1⟩3. LET: $x \in (A \cap B) - \{a\}$

⟨1⟩4. ASSUME: $|x - a| < \delta$

⟨1⟩5. $|(f(x) + g(x)) - (b + c)| < \epsilon$

□

**Proposition 0.5.** *Let $A, B \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Let $a$ be an accumulation point of $A \cap B$. Let $b, c \in \mathbb{R}$. Assume $f(x) \to b$ as $x \to a$ and $g(x) \to c$ as $x \to a$. Then $f(x)g(x) \to bc$ as $x \to a$.*

PROOF:

⟨1⟩1. LET: $\epsilon > 0$

⟨1⟩2. LET: $d = \epsilon/2|b|$ if $b \neq 0$, or $d = 1$ if $b = 0$

⟨1⟩3. PICK $\delta > 0$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta$ then $|f(x) - b| < \epsilon/2(d + |c|)$, and for all $x \in B - \{a\}$, if $|x - a| < \delta$ then $|g(x) - c| < d$

⟨1⟩4. LET: $x \in (A \cap B) - \{a\}$

⟨1⟩5. ASSUME: $|x - a| < \delta$

⟨1⟩6. $|f(x)g(x) - bc| < \epsilon$

PROOF:

$$\begin{aligned} |f(x)g(x) - bc| &\leq |f(x) - b||g(x)| + |b||g(x) - c| \\ & \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

**Proposition 0.6.** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $a$ be an accumulation point of $A$ and $b > 0$ Suppose $\lim_{x \to a} f(x) = b$. Then there exists $\delta$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta$ then $f(x) > b/2$.*

PROOF: Take $\epsilon = b/2$ in the definition of limit. □

**Proposition 0.7.** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $a$ be an accumulation point of $A$. Let $b \in \mathbb{R} - \{0\}$. Suppose $f(x) \to b$ as $x \to a$. Then $a$ is an accumulation point of $\{x \in A : f(x) \neq 0\}$ and $1/f(x) \to 1/b$ as $x \to a$.*

PROOF:
$\langle 1\rangle 1.$ $a$ is an accumulation point of $\{x \in A : f(x) \neq 0\}$.
  $\langle 2\rangle 1.$ LET: $\delta > 0$
  $\langle 2\rangle 2.$ ASSUME: w.l.o.g. $\forall x \in A - \{a\}.|x - a| < \delta \Rightarrow f(x) \neq 0$
  $\langle 2\rangle 3.$ PICK $x \in (a - \delta, a + \delta) \cap (A - \{a\})$
  $\langle 2\rangle 4.$ $x \in (a - \delta, a + \delta) \cap (\{x \in A : f(x) \neq 0\} - \{a\})$
$\langle 1\rangle 2.$ For all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in A - \{a\}$, if $f(x) \neq 0$ and $|x - a| < \delta$ then $|1/f(x) - 1/b| < \epsilon$
  $\langle 2\rangle 1.$ LET: $\epsilon > 0$
  $\langle 2\rangle 2.$ PICK $\delta > 0$ such that $\forall x \in A - \{a\}.|x - a| < \delta \Rightarrow |f(x) - b| < \epsilon|b|^2/2$ and $\forall \in A - \{a\}.|x - a| < \delta \Rightarrow |f(x)| > |b|/2$
    PROOF: Proposition 0.6.
  $\langle 2\rangle 3.$ LET: $x \in A - \{a\}$ satisfy $f(x) \neq 0$ and $|x - a| < \delta$
  $\langle 2\rangle 4.$ $|1/f(x) - 1/b| < \epsilon$
    PROOF:

$$
\begin{aligned}
|1/f(x) - 1/b| &= |f(x) - b|/|f(x)||b| \\
&< \epsilon && (\langle 2\rangle 2)
\end{aligned}
$$

$\square$

**Definition 0.8** (Continuity at a Point)**.** Let $A \subseteq \mathbb{R}$. Let $a \in A$ be an accumulation point of $A$. Then $f$ is *continuous* at $a$ if and only if $f(x) \to f(a)$ as $x \to a$.

$f$ is *continuous* if and only if every point of $A$ is an accumulation point of $A$ and $f$ is continuous at every point of $A$.

**Proposition 0.9.** *Let $A, B \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Let $a \in A \cap B$ be an accumulation point of $A \cap B$. Assume $f$ and $g$ are continuous at $a$. Then $f + g$ and $fg$ are continuous at $a$.*

PROOF: Propositions 0.4 and 0.5. $\square$

**Corollary 0.9.1.** *Every polynomial is continuous on $\mathbb{R}$.*

**Proposition 0.10.** *Let $A \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$. Let $a \in A$ be an accumulation point of $A$. Assume $f$ is continuous at $a$ and $f(a) \neq 0$. Then $1/f$ is continuous at $a$.*

PROOF: Proposition 0.7. $\square$

**Proposition 0.11.** *Let $A, B \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Let $a \in A$ be an accumulation point of $A$. Assume $f(a) \in B$ and $f(a)$ is an accumulation point of $B$. If $f$ is continuous at $a$ and $g$ is continuous at $f(a)$ then $g \circ f$ is continuous at $a$.*

PROOF:
$\langle 1\rangle 1.$ LET: $\epsilon > 0$
$\langle 1\rangle 2.$ PICK $\delta_1 > 0$ such that, for all $y \in B - \{f(a)\}$, if $|y - f(a)| < \delta_1$ then $|g(y) - g(f(a))| < \epsilon$

$\langle 1 \rangle 3$. PICK $\delta_2 > 0$ such that, for all $x \in A - \{a\}$, if $|x - a| < \delta_2$ then $|f(x) - f(a)| < \delta_1$

$\langle 1 \rangle 4$. For all $x \in A - \{a\}$, if $|x - a| < \delta_2$ then $|g(f(x)) - g(f(a))| < \epsilon$

$\square$

**Definition 0.12** (Relatively Open)**.** Let $A \subseteq \mathbb{R}$ and $B \subseteq A$. Then $B$ is *relatively open* in $A$ iff there exists an open set $V \subseteq \mathbb{R}$ such that $B = A \cap V$.

**Lemma 0.13.** *Let $B \subseteq A \subseteq \mathbb{R}$. Then $B$ is relatively open in $A$ iff, for all $x \in B$, there exists an open interval $I$ containing $x$ such that $I \cap A \subseteq B$.*

PROOF:

$\langle 1 \rangle 1$. If $B$ is relatively open in $A$ then, for all $x \in B$, there exists an open interval $I$ containing $x$ such that $I \cap A \subseteq B$

  $\langle 2 \rangle 1$. ASSUME: $B$ is relatively open in $A$.

  $\langle 2 \rangle 2$. PICK an open set $V$ such that $B = A \cap V$

  $\langle 2 \rangle 3$. LET: $x \in B$

  $\langle 2 \rangle 4$. PICK an open interval $I$ such that $x \in I \subseteq V$

  $\langle 2 \rangle 5$. $I \cap A \subseteq B$

$\langle 1 \rangle 2$. If, for all $x \in B$, there exists an open interval $I$ containing $x$ such that $I \cap A \subseteq B$, then $B$ is relatively open in $A$.

  $\langle 2 \rangle 1$. ASSUME: For all $x \in B$, there exists an open interval $I$ containing $x$ such that $I \cap A \subseteq B$

  $\langle 2 \rangle 2$. LET: $V$ be the union of all the open intervals $I$ such that $I \cap A \subseteq B$

  $\langle 2 \rangle 3$. $B = A \cap V$

$\square$

**Theorem 0.14.** *Let $A \subseteq \mathbb{R}$ be a set such that every point in $A$ is an accumulation point of $A$. Let $f : A \to \mathbb{R}$. Then $f$ is continuous if and only if, for every open set $W$, we have $f^{-1}(W)$ relatively open in $A$.*

PROOF:

$\langle 1 \rangle 1$. If $f$ is continuous then, for every open set $W$, we have $f^{-1}(W)$ is relatively open in $A$.

  $\langle 2 \rangle 1$. ASSUME: $f$ is continuous.

  $\langle 2 \rangle 2$. LET: $W$ be an open set.

  $\langle 2 \rangle 3$. For all $x \in f^{-1}(W)$, there exists an open interval containing $I$ such that $I \cap A \subseteq f^{-1}(W)$

    $\langle 3 \rangle 1$. LET: $x \in f^{-1}(W)$

    $\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subseteq W$

    $\langle 3 \rangle 3$. PICK $\delta > 0$ such that, for all $y \in A - \{x\}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$

    $\langle 3 \rangle 4$. LET: $I = (x - \delta, x + \delta)$
    PROVE: $I \cap A \subseteq f^{-1}(W)$

    $\langle 3 \rangle 5$. LET: $y \in I \cap A$

    $\langle 3 \rangle 6$. $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$

    $\langle 3 \rangle 7$. $f(y) \in W$

  $\langle 2 \rangle 4$. $f^{-1}(W)$ is relatively open in $A$.

PROOF: Lemma 0.13.

⟨1⟩2. If, for every open set $W$, we have $f^{-1}(W)$ is relatively open in $A$, then $f$ is continuous.

⟨2⟩1. ASSUME: For every open set $W$, we have $f^{-1}(W)$ is relatively open in $A$.

⟨2⟩2. LET: $x \in A$

⟨2⟩3. LET: $\epsilon > 0$

⟨2⟩4. $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is relatively open in $A$.

⟨2⟩5. PICK $\delta > 0$ such that $(x - \delta, x + \delta) \cap A \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$

PROOF: Lemma 0.13.

⟨2⟩6. For all $y \in A - \{x\}$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$

□

**Proposition 0.15.** *Let $C \subseteq \mathbb{R}$ be compact and be such that every element of $C$ is an accumulation point of $C$. Let $f : C \to \mathbb{R}$ be continuous. Then $f(C)$ is compact.*

PROOF:

⟨1⟩1. LET: $\mathcal{A}$ be an open covering of $f(C)$.

⟨1⟩2. $\{W \in \mathcal{P}\mathbb{R} : W \text{ is open}, \exists A \in \mathcal{A}. f^{-1}(A) = W \cap C\}$ is an open covering of $C$.

PROOF: Theorem 0.14.

⟨1⟩3. PICK a finite subcover $\{W_1, \ldots, W_n\}$ of $C$.

⟨1⟩4. For $i = 1, \ldots, n$, PICK $A_i \in \mathcal{A}$ such that $f^{-1}(A_i) = W_i \cap C$

⟨1⟩5. $\{A_1, \ldots, A_n\}$ covers $f(C)$.

□

**Corollary 0.15.1.** *Let $C \subseteq \mathbb{R}$ be compact and be such that every element of $C$ is an accumulation point of $C$. Let $f : C \to \mathbb{R}$ be continuous. Then $f(C)$ has a maximum and a minimum value.*

**Lemma 0.16.** *Let $A \subseteq \mathbb{R}$. Then $A$ is connected if and only if there do not exist nonempty disjoint sets $B$, $C$ relatively open in $A$ such that $A = B \cup C$.*

PROOF:

⟨1⟩1. If $A = B \cup C$ where $B$ and $C$ are nonempty, disjoint and relatively open in $A$, then $A$ is disconnected.

⟨2⟩1. ASSUME: $A = B \cup C$ where $B$ and $C$ are nonempty, disjoint and relatively open in $A$.

⟨2⟩2. PICK open sets $B_1$ and $C_1$ such that $B = B_1 \cap A$ and $C = C_1 \cap A$

⟨2⟩3. $B$ contains no accumulation point of $C$.

⟨3⟩1. ASSUME: for a contradiction $b \in B$ and $b$ is an accumulation point of $C$

⟨3⟩2. $b$ is an accumulation point of $\mathbb{R} - B_1$

⟨3⟩3. $b \in \mathbb{R} - B_1$

PROOF: Since $\mathbb{R} - B_1$ is closed.

⟨3⟩4. Q.E.D.

PROOF: This contradicts the fact that $b \in B$.

$\langle 2\rangle 4$. $C$ contains no accumulation point of $B$.
Proof: Similar.
$\langle 1\rangle 2$. If $A$ is disconnected then there exist nonempty, disjoint sets $B$ and $C$ relatively open in $A$ such that $A = B \cup C$.
$\langle 2\rangle 1$. Assume: $A$ is disconnected
$\langle 2\rangle 2$. Pick disjoint nonempty sets $B$ and $C$ such that $A = B \cup C$ and neither of $B$ and $C$ contains an accumulation point of the other.
$\langle 2\rangle 3$. $B$ is relatively open in $A$
Proof: $B = A \cap (\mathbb{R} - \overline{C})$
$\langle 2\rangle 4$. $C$ is relatively open in $A$
Proof: Similar.
□

**Theorem 0.17.** *Let $C \subseteq \mathbb{R}$ be connected and such that every element of $C$ is an accumulation point of $C$. Let $f : C \to \mathbb{R}$ be continuous. Then $f(C)$ is connected.*

Proof:
$\langle 1\rangle 1$. Assume: for a contradiction $f(C) = B \cup D$ where $B$ and $D$ are nonempty, disjoint and relatively open in $f(C)$
Proof: Lemma 0.16.
$\langle 1\rangle 2$. Pick open sets $B'$, $D'$ such that $B = f(C) \cap B'$ and $D = f(C) \cap D'$
$\langle 1\rangle 3$. $C = f^{-1}(B') \cup f^{-1}(D')$
$\langle 1\rangle 4$. $f^{-1}(B')$ and $f^{-1}(D')$ are relatively open in $C$
Proof: Theorem 0.14
$\langle 1\rangle 5$. $f^{-1}(B')$ and $f^{-1}(D')$ are nonempty and disjoint
$\langle 1\rangle 6$. Q.E.D.
Proof: This contradicts the fact that $C$ is connected by Lemma 0.16.
□

**Corollary 0.17.1.** *The continuous image of a closed interval is a closed interval.*

**Corollary 0.17.2** (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $c$ be between $f(a)$ and $f(b)$. Then there exists $x \in [a, b]$ such that $f(x) = c$.*

**Proposition 0.18.** *Let $f : [a, b] \to \mathbb{R}$ be continuous and injective. Then $f^{-1}$ is continuous.*

Proof:
$\langle 1\rangle 1$. Let: $y \in f([a, b])$
$\langle 1\rangle 2$. Let: $\epsilon > 0$
$\langle 1\rangle 3$. Let: $x$ be the point such that $f(x) = y$
$\langle 1\rangle 4$. $f([x - \epsilon/2, x + \epsilon/2] \cap [a, b])$ is a closed interval.
$\langle 1\rangle 5$. Pick $\delta > 0$ such that $(y - \delta, y + \delta) \subseteq f([x - \epsilon/2, x + \epsilon/2] \cap [a, b])$
$\langle 1\rangle 6$. Let: $z \in f([a, b]) - \{y\}$ be such that $|y - z| < \delta$
$\langle 1\rangle 7$. $|f^{-1}(z) - x| < \epsilon$

□

**Definition 0.19** (Uniformly Continuous)**.** Let $A \subseteq \mathbb{R}$ be such that every point of $A$ is an accumulation point of $A$. Let $f : A \to \mathbb{R}$. Then $f$ is *uniformly continuous* iff, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in A$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

**Theorem 0.20.** *Let $A \subseteq \mathbb{R}$ be compact and such that every point of $A$ is an accumulation point of $A$. Let $f : A \to \mathbb{R}$. If $f$ is continuous then $f$ is uniformly continuous.*

PROOF:
$\langle 1 \rangle$1. LET: $\epsilon > 0$
$\langle 1 \rangle$2. LET: $\mathcal{B}$ be the set of all sets of the form $\{(z - \delta, z + \delta) : z \in A, \delta > 0, \forall u \in A. |z - u| < 2\delta \Rightarrow |f(z) - f(u)| < \epsilon/2\}$
$\langle 1 \rangle$3. $\mathcal{B}$ covers $A$.
$\langle 1 \rangle$4. PICK a finite subcover $\{(z_1 - \delta_1, z_1 + \delta_1), \dots, (z_n - \delta_n, z_n + \delta_n)\}$
$\langle 1 \rangle$5. LET: $\delta = \min(\delta_1, \dots, \delta_n)$
$\langle 1 \rangle$6. LET: $x, y \in A$ with $|x - y| < \delta$
$\langle 1 \rangle$7. PICK $i$ such that $x \in (z_i - \delta_i, z_i + \delta_i)$
$\langle 1 \rangle$8. $|f(x) - f(z_i)| < \epsilon/2$
$\langle 1 \rangle$9. $|y - z_i| < 2\delta_i$
$\langle 1 \rangle$10. $|f(y) - f(z_i)| < \epsilon/2$
$\langle 1 \rangle$11. $|f(y) - f(x)| < \epsilon$
□