Solutions Manual for Lawvere and Schanuel *Conceptual Mathematics*

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Contents

Part I

Preview

Chapter 1

Session 1 — Galileo and Multiplication of Objects

Exercise 1 Many examples — every instance of a product in a category gives an example. I will not list them.

Exercise 2 I am not entirely sure what solution the authors had in mind. Here are some that come to my mind:

Place a spirit level between the two points and see if it reads as level.

Place a smooth plank between the two points and see if a ball placed at one point rolls to the other, or *vice versa*.

Hang a plumbline at each point and see if they form a right angle with the line joining the two points.

Of these, the third is my favourite.

Part II

Part I — The category of sets

Chapter 2

Article I — Sets, maps, composition

Exercise 1 Easy.

Exercise 2 There are 8 maps from A to B.

Exercise 3 There are 27 maps from A to A.

Exercise 4 There are 9 maps from B to A.

Exercise 5 There are 4 maps from B to B.

Exercise 6 There are 10 such maps from A to A.

Exercise 7 There are 3 such maps from B to B.

Exercise 8 There is no such pair of maps.

Exercise 9 There are 12 such pairs of maps.

Chapter 3

Session 3 — Composing maps and counting maps

Exercise 1 (a) and (c) make sense.

Exercise 2 (a) and (c) still make sense.

Part III

Part II — The algebra of composition

Chapter 4

Article II — Isomorphisms

4.1 1. Isomorphisms

Exercise 1

(R) We have $1_A \circ 1_A = 1_A$ by the Identity Laws, so 1_A is an isomorphism with inverse 1_A.

(S) We have $g \circ f = 1_A$ and $f \circ g = 1_B$ (this is what it means for g to be an inverse for f). This says exactly that f is an inverse for g.

(T) Let $f^{-1} : B \to A$ be an inverse for f and $k^{-1} : C \to B$ be an inverse for k. We prove $f^{-1} \circ k^{-1}$ is an inverse for $k \circ f$. We have

$$\begin{aligned} f^{-1} \circ k^{-1} \circ k \circ f &= f^{-1} \circ 1_B \circ f && \text{(definition of inverse)} \\ &= f^{-1} \circ f && \text{(Identity Law)} \\ &= 1_A && \text{(definition of inverse)} \end{aligned}$$

and $k \circ f \circ f^{-1} \circ k^{-1} = 1_C$ similarly.

Exercise 2 We have

$$\begin{aligned} g &= g \circ 1_B && \text{(Identity Law)} \\ &= g \circ f \circ k && (k \text{ is an inverse of } f) \\ &= 1_A \circ k && (g \text{ is an inverse of } f) \\ &= k && \text{(Identity Law)} \end{aligned}$$

Exercise 3

(a) Let $f : A \to B$. Let $h, k : C \to A$.
Suppose $f \circ h = f \circ k$. Then

$$\begin{aligned} f^{-1} \circ f \circ h &= f^{-1} \circ f \circ k \\ \therefore 1_A \circ h &= 1_A \circ k && \text{(Definition of inverse)} \\ \therefore h &= k && \text{(Identity Law)}\square \end{aligned}$$

(b) Let $f : A \to B$. Let $h, k : B \to C$.
Suppose $h \circ f = k \circ f$. Then

$$\begin{aligned} h \circ f \circ f^{-1} &= k \circ f \circ f^{-1} \\ \therefore h \circ 1_B &= k \circ 1_B && \text{(Definition of inverse)} \\ \therefore h &= k && \text{(Identity Law)}\square \end{aligned}$$

(c) Let $A = \{0, 1\}$. Define $f : A \to A$ by $f(0) = 1$ and $f(1) = 0$. Define $h : A \to A$ by $h(x) = 0$ for all x. Define $k : A \to A$ by $k(x) = 1$ for all x.
f is invertible, and is its own inverse.
We have $h \circ f = f \circ k = h$.
We do not have $h = k$.

Exercise 4

(1) This function is invertible with inverse $f^{-1}(x) = (x - 7)/3$.

(2) This function is invertible with inverse $g^{-1}(x) = \sqrt{x}$.

(3) This function is not invertible because $h(1) = h(-1) = 1$.

(4) This function is not invertible because $k(1) = k(-1) = 1$.

(5) This function is not invertible because there is no x such that $l(x) = 2$.

4.2 2 — General division problems: Determination and choice

Exercise 5 There are 6 maps f such that $g \circ f = 1_{\{0,1\}}$; we can map 0 to any of b, p or q, and 1 to either of r or s.

Given any one of these maps f, there are 8 maps g such that $g \circ f = 1_{\{0,1\}}$. We must map $f(0)$ to 0, $f(1)$ to 1, and the other three elements to any of 0 or 1.

Exercise 6 If $r : B \to A$ is a section for f, then we take $t = g \circ r$. We have $t \circ f = g \circ r \circ f = g \circ 1_A = g$.

Exercise 7 Let $s : B \to A$ be a section for f. Let T be any set and $t_1, t_2 : T \to B$. Suppose $t_1 \circ f = t_2 \circ f$. Then

$$\begin{aligned} t_1 \circ f \circ s &= t_2 \circ f \circ s \\ \therefore t_1 \circ 1_B &= t_2 \circ 1_B \\ \therefore t_1 &= t_2 \end{aligned}$$

Exercise 8 If $s_1 : B \to A$ is a section for $r_1 : A \to B$ and $s_2 : C \to B$ is a section for $r_2 : B \to C$, then $s_1 \circ s_2$ is a section for $r_2 \circ r_1$ since

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ 1_B \circ s_2 \\ &= r_2 \circ s_2 \\ &= 1_C \end{aligned}$$

Exercise 9 We have

$$\begin{aligned} e \circ e &= f \circ r \circ f \circ r \\ &= f \circ 1 \circ r && (r \text{ is a retraction of } f) \\ &= f \circ r \\ &= e \end{aligned}$$

Exercise 10 From the proof of Proposition 3, $f^{-1} \circ g^{-1}$ is both a section and a retraction for $g \circ f$.

Exercise 11 Set $f(Fatima) = coffee$, $f(Omer) = tea$ and $f(Alysia) = cocoa$. Then f is an isomorphism.

There is no isomorphism $g : A \to C$. For if $g(Fatima) = true$ then $g(Omer)$ must be $false$, and then it is impossible to choose a value for $g(Alysia)$ without having $g(Alysia) = g(Fatima)$ or $g(Alysia) = g(Omer)$. Similarly if $g(Fatima) = false$ then $g(Omer)$ must be $true$, and then again we cannot choose a value for $g(Alysia)$.

Chapter 5

Session 4 — Division of Maps: Isomorphisms

5.1 4. A small zoo of isomorphisms in other categories

Exercise 1 We have $h(d(x)) = h(2x) = x$ and $d(h(x)) = d(x/2) = x$ for any x.

Exercise 2 $f(odd) = negative$ and $f(even) = positive$

Exercise 3

(a) This is not an isomorphism because $p(0 + 0) = 1$ but $p(0) + p(0) = 2$

(b) This is not an isomorphism because it is not surjective; there is no x such that $sq(x) = -1$.

(c) This is not an isomorphism because it is not injective. We have $sq(1) = sq(-1) = 1$.

(d) This is an isomorphism; it is bijective and $-(x + y) = (-x) + (-y)$.

(e) This is not an isomorphism because $m(1\times1) = -1$ but $m(1)\times m(1) = 1$.

(f) This is not a well-defined map because $c(-1) = -1 \notin \mathbb{R}_{>0}$.

Chapter 6

Session 5 — Division of Maps: Sections and Retractions

6.1 1. Determination Problems

Exercise 1

(a) Suppose $h = g \circ f$ and $fa_1 = fa_2$. Then $ha_1 = g(fa_1) = g(fa_2) = ha_2$.

(b) No. Take $A = C = \emptyset$ and $B = \{*\}$. Let $f : A \to B$ and $h : A \to C$ be the unique such maps. Vacuously, if $fa_1 = fa_2$ then $ha_1 = ha_2$. But there is no map $g : B \to C$.

6.2 3. Choice Problems

Exercise 2

(a) Suppose $g \circ f = h$. Let $a \in A$. Let $b = f(a)$. Then $h(a) = g(f(a)) = g(b)$.

(b) This is equivalent to the Axiom of Choice.

6.3 5. Stacking or Sorting

Exercise 3 I'm not going to draw all of them, but there are 8 of them.

Chapter 7

Session 9 — Retracts and Idempotents

7.1 1. Retracts and Comparisons

Exercise 1 If A is empty, then the nowhere-defined function is a map $A \to B$.
If B has a point, say b, then the constant map with value b is a map $A \to B$.

7.2 2. Idempotents as records of retracts

Exercise 3 Suppose $s : A \to B$, $r : B \to A$ and $s' : A' \to B$, $r' : B \to A'$ are splittings of $e : B \to B$. Let

$$\begin{aligned} f &= r' \circ s & &: A \to A' \\ f^{-1} &= r \circ s' & &: A' \to A \end{aligned}$$

Then we have

$$\begin{aligned} f \circ f^{-1} &= r' \circ s \circ r \circ s' \\ &= r' \circ e \circ s' \\ &= r' \circ s' \circ r' \circ s' \\ &= 1 \\ f^{-1} \circ f &= r \circ s' \circ r' \circ s \\ &= r \circ e \circ s \\ &= r \circ s \circ r \circ s \\ &= 1 \end{aligned}$$

Chapter 8

Quiz

Question 1 Let $A = \{*\}$ and $B = \{0, 1\}$. Define $f : A \to B$ by $f(*) = 0$. Then the unique function $r : B \to A$ is a retraction for f (since $r(f(*)) = *$) but not a section for f (since $f(s(1)) = 0$). Therefore there is no section for f, since there is only one map $B \to A$.

Question 2

(a) Yes: if $pqp = p$ then $pqpq = pq$

(b) Yes: if $pqp = p$ then $qpqp = qp$

Question 2* Let $q' = qpq$ Then we have

$$\begin{aligned}
pq'p &= pqpqp \\
&= pqp \\
&= p \\
q'pq' &= qpqpqpq \\
&= qpqpq \\
&= qpq \\
&= q'
\end{aligned}$$

Question 1* Take $A = B = \mathbb{N}$ and define $f : A \to B$ by $f(x) = 2x$. Then f has a retraction r given by

$$r(y) = \begin{cases} y/2 & \text{if } y \text{ is even} \\ 0 & \text{if } y \text{ is odd} \end{cases}$$

It has no section since it is not surjective (Article II Proposition 1).

Chapter 9

Summary / quiz on pairs of 'opposed' maps

Question 1 Given two maps f, g with domains and codomains as above, we can always form the composites $g \circ f$ and $f \circ g$. All we can say about $g \circ f$ and $f \circ g$ as maps in themselves is that they are endomaps.

Question 2 If we know that g is a retraction for f, that means $g \circ f$ is actually the identity map 1_A; then we can prove that $f \circ g$ is not only an endomap, but actually an idempotent. The latter means that the equation $f \circ g \circ f \circ g = f \circ g$ is true.

Question 3 If we even know that f is an isomorphism *and* that $g \circ f = 1_A$, then $f \circ g$ is not only an idempotent, but is the identity map 1_B. If, moreover, s is a map for which $f \circ s = 1_B$, we can conclude that $s = g$.

Question 4 Going back to 0, i.e. assuming no equations, but only the domain and codomain statements about f and g, the composite $f \circ g \circ f$ could be different from f. Likewise $f \circ g \circ f \circ g$ could be different from $f \circ g$.

Chapter 10

Test 1

Question 1

(a) Let $f(Mara) = Aurelio$, $f(Aurelio) = Mara$ and $f(Andrea) = Andrea$.

(b) Let $e(Mara) = Aurelio$, $e(Aurelio) = Aurelio$ and $e(Andrea) = Andrea$.

(c) Let $B = \{0, 1\}$. Define $s : B \to A$ by $s(0) = Aurelio$ and $s(1) = Andrea$. Define $r : A \to B$ by $r(Mara) = 0$, $r(Aurelio) = 0$ and $r(Andrea) = 1$.

Question 2 Define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = (y + 7)/4$.

(a) $g(f(x)) = g(4x - 7) = (4x - 7 + 7)/4 = 4x/4 = x$

(b) $f(g(x)) = f((x + 7)/4) = 4((x + 7)/4) - 7 = x + 7 - 7 = x$

Chapter 11

Session 10 — Brouwer's Theorems

11.1 4. Relation between fixed point and retraction theorems

Exercise 1 Suppose for a contradiction there is no point x such that $f(x) = g(x)$. Define $r : D \to C$ as follows: for $x \in D$, $r(x)$ is the point on C that is pointed at by the arrow with tail at $f(x)$ and head at $g(x)$. For $x \in C$, we have $g(j(x)) = j(x)$, so the point that is pointed at by any arrow with head at $g(j(x))$ is x. Hence

$$r(j(x)) = x$$

and so r is a retraction for j, contradicting the retraction theorem.

Exercise 2 Let $f : A \to A$ be any endomap. Then $s \circ f \circ r : X \to X$ is an endomap on X. Hence there exists $x : T \to X$ such that $sfrx = x$. But then we have

$$rsfrx = rx$$
$$\therefore frx = rx$$

and so $r \circ x : T \to A$ is a fixed point of f.

Exercise 3 Let A be either E, C or S, and X be I, D or B respectively. Assume that every endomap $X \to X$ has a fixed point.

Assume for a contradiction that X is a retract of A. By Exercise 2, every endomap on A has a fixed point. This is a contradiction, as the antipodal map on A has no fixed point.

11.2 7. Using maps to formulate guesses

Exercise 1

(a) We can express 'I start in Buffalo and end in Rochester' as $m \circ j = i \circ j$.

We can express 'You start and finish anywhere between Buffalo and Rochester' as: there exists $f : I \to E$ such that $y \circ j = i \circ f$.

(b) There exists $t : 1 \to I$ such that $mt = yt$.

(c) Let C be the circle, D the disk and P the plane. Let $j : C \to D$ and $i : D \to P$ be the inclusions.

For any maps $m, y : D \to P$ such that:

- $mj = ij$
- there exists $f : C \to D$ such that $yj = if$

then there exists $t : 1 \to D$ such that $mt = yt$.

(d) I have not been able to find any smooth maps for which it is not true.

Part IV

Part III — Categories of Structured Sets

Chapter 12

Article III — Examples of Categories

12.1 1. The category $\mathcal{S}^{\circ}$ of endomaps of sets

Exercise 1 Let $f : (X, \alpha) \to (Y, \beta)$ and $g : (Y, \beta) \to (Z, \gamma)$. Then

$$g \circ f \circ \alpha = g \circ \beta \circ f = \gamma \circ g \circ f$$

and so $g \circ f : (X, \alpha) \to (Z, \gamma)$.

12.2 4. Categories of endomaps

Exercise 2 Suppose $e : A \to A$ is idempotent and has a retraction $r : A \to A$. Then

$$1_A = r \circ e = r \circ e \circ e = 1_A \circ e = e$$

so $e = 1_A$. Thus, the identities are the only idempotents that have retractions.

Exercise 3 Suppose A has an even number of elements, say $\{a_1, a_2, \ldots, a_{2n}\}$. Define $\theta : A \to A$ by $\theta(a_{2k+1}) = a_{2k+2}$ and $\theta(a_{2k+2}) = a_{2k+1}$ $(0 \leq k < n)$. Then θ is an involution with no fixed point.

Conversely, suppose $\theta : A \to A$ is an involution with no fixed point. Enumerate the elements of A as follows: Pick any element $a_1 \in A$. Let $a_2 = \theta(a_1)$; then $a_1 = \theta(a_2)$.

Assuming we have picked $a_1, \ldots, a_{2m}$ such that $\{a_1, \ldots, a_{2m}\}$ is closed under θ and $A \neq \{a_1, \ldots, a_{2m}\}$, pick $a_{2m+1} \in A - \{a_1, \ldots, a_{2m}\}$. Then $\theta(a_{2m+1}) \notin \{a_1, \ldots, a_{2m}\}$ (since $\theta(\theta(a_{2m+1})) = a_{2m+1}$) and $\theta(a_{2m+1}) \neq a_{2m+1}$ (since θ has no fixed point). So let $a_{2m+2} = \theta(a_{2m+1})$.

This process must end because A is finite. So $A = \{a_1, \ldots, a_{2n}\}$ for some n.

Suppose now A has an odd number of elements, say $A = \{a_1, a_2, \ldots, a_{2n+1}\}$. Define $\theta : A \to A$ by

$$\begin{aligned} \theta(a_{2k+1}) &= a_{2k+2} && (0 \leq k < n) \\ \theta(a_{2k+2}) &= a_{2k+1} && (0 \leq k < n) \\ \theta(a_{2n+1}) &= a_{2n+1} \end{aligned}$$

Then θ is an involution whose only fixed point is a_{2n+1}.

Conversely, suppose $\theta : A \to A$ is an involution with one fixed point f. Then $\theta \restriction (A - \{f\})$ is an involution on $A - \{f\}$ with no fixed point. So $A - \{f\}$ has an even number of elements, and so A has an odd number of elements.

Exercise 4 The map α is an involution because $-(-x) = x$. It is not idempotent because $-(-1) \neq -1$. Its only fixed point is 0.

Exercise 5 The map α is not an involution because $||-1|| = 1 \neq -1$. It is idempotent because $||x|| = |x|$. Its fixed points are the non-negative integers.

Exercise 6 The map α is an automorphism with inverse $\alpha^{-1}(x) = x - 3$.

Exercise 7 The map α is not an automorphism because there is no integer x with $\alpha(x) = 1$.

Exercise 8 If α is idempotent then $\alpha \circ \alpha \circ \alpha = \alpha \circ \alpha = \alpha$.

If α is an involution then $\alpha \circ \alpha \circ \alpha = 1 \circ \alpha = \alpha$.

Exercise 9 Label the elements in the diagram 0, 1, 2 from top to bottom. Then

$$\begin{aligned} \alpha^3(0) &= \alpha^2(1) = \alpha(2) = 1 \\ &= \alpha(0) \\ \alpha^3(1) &= \alpha^2(2) = \alpha(1) = 2 \\ &= \alpha(1) \\ \alpha^3(2) &= \alpha^2(1) = \alpha(2) = 1 \\ &= \alpha(2) \end{aligned}$$

Thus, $\alpha^3 = \alpha$.

However, α is not idempotent because $\alpha^2(0) = 2 \neq \alpha(0)$. And α is not an involution because $\alpha^2(0) = 2 \neq 0$.

12.3 5. Irreflexive graphs

Exercise 10

$$s(a) = k, s(b) = m, s(c) = k, s(d) = p, s(e) = m$$

$$t(a) = m, t(b) = m, t(c) = m, t(d) = q, t(e) = r$$

The arrow b has $s(b) = t(b)$. There is no arrow x with $t(x) = k$.

Exercise 11 We have

$$s'' \circ g \circ f = g \circ s' \circ f = g \circ f \circ s$$

$$t'' \circ g \circ f = g \circ t' \circ f = g \circ f \circ t$$

and so $g \circ f : (X, P, s, t) \to (Z, R, s'', t'')$.

12.4 6. Endomaps as special graphs

Exericse 12

$$I(g \circ f) = (g \circ f, g \circ f) = (g, g) \circ (f, f) = I(g) \circ I(f)$$

Exercise 13 For any $x \in X$ we have $f_A(x) = 1_Y(f_A(x)) = f_D(1_X(x)) = f_D(x)$, and so $f_A = f_D$. Thus $(f_A, f_D) = I(f_A)$.

12.5 7. The simpler category $\mathcal{S}^{\downarrow}$: Objects are just maps of sets

Exercise 14 Let $X = \{*\}$ and $Y = \{0, 1\}$. Let α be the only map $X \to X$, and $\beta : Y \to Y$ be the map with $\beta(0) = 1$ and $\beta(1) = 0$. Let $f_A(*) = 0$ and $f_D(*) = 1$. Then $f_D \circ \alpha = \beta \circ f_A$ but $f_A \neq f_D$.

12.6 8. Reflexive graphs

Exercise 15 Let $x_1 = s$ and $x_2 = t$, so $e_i = ix_i$ for each i. Then

$$\begin{aligned} e_k e_j &= ix_k ix_j \\ &= i1_P x_j \\ &= ix_j \\ &= e_j \end{aligned}$$

In particular, $e_j e_j = e_j$, so each e_j is idempotent.

Exercise 16 Let $(f_A, f_D) : (X, P, s, t, i) \to (Y, Q, s', t', j)$. Then

$$\begin{aligned} f_D s &= s' f_A \\ \therefore f_D &= f_D s i \\ &= s' f_A i \end{aligned}$$

Exercise 17 A map between $(M, F, \phi, \phi', \mu, \mu')$ and $(N, G, \psi, \psi', \nu, \nu')$ is a pair of functions $f : M \to N$ and $g : F \to G$ such that

$$\begin{aligned} \psi f &= f\phi \\ \psi' g &= f\phi' \\ \nu g &= g\mu \\ \nu' f &= g\nu \end{aligned}$$

12.7 10. Retractions and Injectivity

Exercise 18 Let $a : X \to Y$ have a retraction $r : Y \to X$. Let $x_1, x_2 : T \to X$ satisfy $ax_1 = ax_2$. Then

$$x_1 = rax_1 = rax_2 = x_2 \ .$$

Exercise 19 We have

$$\begin{aligned} \beta a x &= 0 & \qquad a\alpha x &= 0 \\ \beta a 0 &= 0 & \qquad a\alpha 0 &= 0 \end{aligned}$$

So $\beta a = a\alpha$ as required.

Exercise 20 Let $x_1, x_2 : (T, \gamma) \to (X, \alpha)$ satisfy $ax_1 = ax_2$. Then, for any $t \in T$, we have $ax_1 t = ax_2 t$, hence $x_1 t = x_2 t$ (since a is injective as a function). Thus $x_1 = x_2$.

Exercise 21 The retractions are the maps that send y to x, 0 to 0, and $\overline{y}$ to either x or 0.

Exercise 22 Let $r : Y \to X$ be either of the retractions of a in $\mathcal{S}$. Then, no matter what $r\overline{y}$ is, we have $\alpha r \overline{y} = 0$. But $r\beta\overline{y} = ry = x$. Thus r is not a map $(Y, \beta) \to (X, \alpha)$ in $\mathcal{S}^\circ$.

Exercise 23 The following are maps in $\mathcal{S}^\circ$:

$$\begin{aligned} \overline{y} \mapsto xy \mapsto 0 & \qquad 0 \mapsto 0 \\ \overline{y} \mapsto 0y \mapsto 0 & \qquad 0 \mapsto 0 \end{aligned}$$

Exericse 24 Suppose $(r, s) : (Y, \beta) \to (X, \alpha)$ is a retraction of (a, a) in $\mathcal{S}^{\downarrow}$. Then $ra = sa = 1_X$ and $\alpha r = s\beta$. Now, whatever $r\overline{y}$ is, we have $\alpha r\overline{y} = 0$. But $s\beta\overline{y} = sy = x$. This is a contradiction.

Exercise 25 Suppose $f_D s = f_D t$. For $x \in X$, we have

$$s' f_A x = f_D s x = f_D t x = t' f_A x$$

and so $f_A x$ is a loop in Y.

Exercise 26 Let $f : \mathbb{Z} \to \mathbb{Q}$ be the inclusion (i.e. $f(n) = n$ for every integer n). Then

1 f is a map in $\mathcal{S}^{\circ}$ because $5f(n) = f(5n) = 5n$

2 The map $5 \times ()$ is an automorphism with inverse $()/5$

3 If $f(m) = f(n)$ then immediately $m = n$.

Exercise 27 Let $f : (X, \alpha) \to (Y, \beta)$. Let the two elements of X be 0 and 1, where $\alpha 0 = \alpha 1 = 1$. Then

$$\begin{aligned} f\alpha 0 &= f\alpha 1 \\ \therefore \beta f 0 &= \beta f 1 \\ \therefore f 0 &= f 1 \end{aligned}$$

since β is injective. Thus f is not injective.

Exercise 28 Let $x, y \in X$ and assume $\alpha x = \alpha y$. Then

$$\begin{aligned} f\alpha x &= f\alpha y \\ \therefore \beta f x &= \beta f y \\ \therefore f x &= f y && (\beta \text{ is injective}) \\ \therefore x &= y && (f \text{ is injective}) \end{aligned}$$